UNIVERSITAT DE BARCELONA

Algebraic model theory for languages without equality

Raimon Elgueta Montó

Algebraic Model Theory for Languages without Equality

Raimon Elgueta

Doctoral Thesis

Universitat de Barcelona
March 1994

Algebraic Model Theory for Languages without Equality

Raimon Elgueta

Thesis submitted for the degree of Doctor of Mathematics.
Universitat de Barcelona. March 1994.

Advisors: **Don L. Pigozzi**
Josep M. Font

Abstract

The purpose of this work is to fashion an approach to model theory for general first-order languages without equality which is aimed at providing a unified vantage point for some concepts and results common in some other fruitful areas, so far independently developed from classical model theory; namely, universal algebra and general sentential logic. The approach has its most recent precedents in two programs, not simultaneously developed but quite similar in spirit: the *model theory of Universal Horn Logic with equality* developed by Mal'cev, and the *model theory of sentential logic* based on the concept of *logical matrix*, initiated in Poland. The novelty of the approach mainly lies in the importance that we attach to the double semantics available for languages without equality (*full* and *reduced semantics*) and the parallel development we make of their model theories on the base of the *reduction operator*. We put forward the purpose of generalizing as much as possible of the *theory of varieties and quasivarieties* to the model theory of first-order logic without equality. For this, we make a widespread use of the *Leibniz operator*, a mapping between structures and congruences in terms of which several elementary theories can be distinguished by the different algebraic character of their model classes.

The main results of the work cover three topics: (1) the characterization in the style of Birkhoff's Variety Theorem of some classes axiomatized by different sorts of first-order sentences without equality; (2) the conditions under which a reduced class of general structures (either reduced or nonreduced) can retain or acquire the different algebraic properties of a quasivariety of algebras; (3) the extension to general structures without equality of properties typical of certain classes of algebras, like structure theorems, freeness and Mal'cev conditions. We also discuss in a final Chapter the way in which the developed theory relates to modern algebraic logic.

Contents

"Jo deixo als qui estimo les petites coses;
les grans coses són per a tothom".
R. Tagore

Dedicat al Dídac i l'Iris,
amb els quals he deixat de compartir moltes
hores a causa d'aquest treball, i als meus pares,
que d'alguna manera l'han fet possible.

RESUM DE LA TESI

Raimon Elgueta

El propòsit és desenvolupar alguns aspectes bàsics d'una *teoria algebraica de models* per a llenguatges de primer ordre sense igualtat que extengui el programa de A.I. Mal'cev (veure *The metamathematics of algebraic systems*, North-Holland, Amsterdam, 1971, i *Algebraic systems*, Springer-Verlag, Berlin, 1973), on desenvolupa bàsicament la teoria de models de la lògica universal de Horn amb igualtat, i el programa iniciat per l'escola polonesa sobre la teoria de models de la lògica sentencial (veure, per exemple, el treball recent de W. Blok, D. Pigozzi *Algebraic semantics for universal Horn logic without equality*, in "Universal Algebra and Quasigroup Theory", Heldermann-Verlag, Berlin, 1992, pp. 1–56). Principalment, es tracta d'anar més enllà en la generalització d'una part de l'àlgebra universal, la *teoria de varietats i quasivarietats*, a estructures amb relacions arbitràries i sense igualtat.

El treball comprèn 10 capítols. Els tres primers contenen material bàsic. El Capítol 1 introdueix alguna terminologia i notació bàsiques, i presenta alguns resultats fonamentals de la teoria clàssica de models que són també vàlids per a llenguatges sense igualtat.

El Capítol 2 proporciana el concepte bàsic d'*igualtat de Leibniz*, de la qual se'n troba una caracterització algebraica. S'enceta el capítol amb una definició de congruència en estructures de primer ordre i es prova que el conjunt de congruències en una estructura donada contitueix un reticle complet, l'element màxim del qual es precisament la *igualtat de Leibniz*. Es proven extensions dels teoremes d'isomorfia de l'àlgebra universal i s'investiga la noció d'estructura quocient, en particular la noció de *quocient de Leibniz*.

Al Capítol 3 es discuteixen les conseqüències semàntiques de factoritzar una estructura per una congruència i es prova que la lògica de primer ordre sense igualtat admet dues semàntiques completes, la *semàntica plena* i la *semàntica reduïda*. En endavant, l'objectiu és bàsicament estudiar les propietats algebraiques que mostren les classes plena i reduida de models d'una teoria de primer ordre (sense igualtat).

El Capítol 4 conté caracteritzacions a l'estil del *Teorema de Birkhoff* sobre varietats d'àlgebres de diferents tipus de classes d'estructures: elementals, universals, universals de Horn (*quasivarietats*) i universals atòmiques (*varietats*). Les demostracions es basen en una tècnica clàssica de la teoria de models, el mètode dels diagrames. És per això que al capítol anterior s'introdueix el *diagrama de Leibniz* d'una estructura com a generalització per a llenguatges sense igualtat del concepte usual de diagrama. S'inclou també l'estudi de les corresponents classes reduïdes, el qual es centra en la investigació de les propietats que l'*operador de reducció* té en composar-lo amb els diferents procediments de construcció d'estructures.

El Capítol 5 és central; examina les conseqüències bàsiques de considerar la part relacional d'una estructura com a extensió de la noció de congruència d'una àlgebra. S'introdueix l'*operador de Leibniz* (una correspondència entre estructures -o la seva part relacional- i congruències sobre una àlgebra) com a criteri principal per distingir propietats de la igualtat de Leibniz en els models d'una classe d'estructures, les quals es comprova *a posteriori* que determinen caràcters algebraics diferents. Utilitzant aquest operador, s'arriba a establir una jerarquia de classes, *protoalgebraiques, semialgebraiques, algebraiques* i *purament algebraiques*, i es comença l'estudi de les diferents propietats que tenen les classes de cadascuna d'aquestes categories. S'obtenen resultats que mostren que les categories de classes protoalgebraiques i semialgebraiques són les més àmplies mostrant un mínim caràcter algebraic.

El Capítol 7 estudia com poden millorar-se les caracteritzacions obtingudes al capítol 4 de les classes reduïdes de models en el cas de tractar-se de classes d'alguna de les categories anteriors. S'arriben a obtenir, per via purament semàntica, noves caracteritzacions de les diferents categories de classes d'estructures introduïdes al capítol anterior.

Finalment, els Capítols 6, 8 i 9 contenen generalitzacions explícites de resultats ben coneguts de l'àlgebra universal. Concretament, al Capítol 6 es desenvolupa la *teoria de la representació subdirecta* per a estructures de primer ordre sense igualtat, posant un èmfasi especial en l'obtencio de formes més generals del *teorema de Jónsson* per a varietats d'àlgebres congruent-distributives (veure B. Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. 21, 1967, pp. 110–121). El Capítol 8 s'ocupa d'investigar el concepte d'*estructura lliure.* Es prova una condició necessària i suficient per a l'existència d'estructures lliures en classes plenes, i condicions necessaries i suficients, expressades en termes de les propietats de l'operador de Leibniz, per a l'existència d'estructures lliures (en sentit feble) en classes reduïdes de models. El capítol inclou també demostracions alternatives, basades en propietats de les estructures lliures, de results ja provats al capítol 4, i generalitza la correspondència entre varietats d'àlgebras i congruències plenament invariants sobre l'àlgebra de termes. Al Capítol 9 es planteja el problema de trobar *condicions de tipus Mal'cev* per a les propietats dels reticles de congruències relatives (les obtingudes aplicant l'operador de Leibniz) i se'n proven alguns resultats.

Finalment, el Capítol 10 es una breu discussió sobre el lligam que existeix entre la teoria desenvolupada als capítols anteriors i algunes tendències de la lògica algebraica moderna.

Introduction

In our opinion, it is fair to distinguish two separate branches in the origins of model theory. The first one, the *model theory of first-order logic*, can be traced back to the pioneering work of L. Lowenheim [77], T. Skolem [107], K. Gödel [54], A. Tarski [113] and A.I. Mal'cev [83], published before the mid 30's. This branch was put forward during the 40's and 50's by several authors, including A. Tarski, L. Henkin, A. Robinson, J. Los; see [61], [62], [73], [102], [103], [116], [117]. Their contribution, however, was rather influenced by modern algebra, a discipline whose development was being truly fast at the time. Largely due to this influence, it was a very common usage among these authors to take the equality symbol as belonging to the language. Even when a few years later the algebraic methods started to be supplanted to a large extent by the set-theoretical techniques that mark present-day model theory, the consideration of the equality as a logical constant in the language still subsisted.

The second branch is the *model theory of equational logic*. It was born with the seminal papers of G. Birkhoff [2], [3], which established the first basic tools and results of what later developed into the part of universal algebra known as the *theory of varieties and quasivarieties*. The algebraic character of this other branch of model theory was clearer and stronger, for it simply emerged as the last step in the continuous process of abstraction in algebra.

Amid these two branches of model theory, which suffered a rapid growth at the time, there appeared the work done by Mal'cev in between the early 1950's and the late 60's, which early gained some influence in the future development of the discipline, at least in the old Soviet Union. During the period mentioned above, he developed a first-order model theory that retained much of the algebraic spirit of the early period and diverged openly from the model theory developed in the West[1]. In particular, in a series of papers [84], [85], [86], [87], he put forward the *model theory of universal Horn logic*[2] *with equality* along the lines of Birkhoff's theory of varieties, and showed that such logic forms a right setting for a large part

[1]Most of his work on this topic was collected in two books [88], [89]; especially this last issue contains a fairly nice systematic exposition.

[2]For us, universal Horn logic (UHL for short) will mean the fragment of first-order logic that deals with the so-called strict universal Horn sentences, i.e., universal sentences in prenex normal form whose matrix is a finite disjunction of negated atomic formulas and just one atomic formula.

of universal algebra, including the theory of presentations and free structures. The most worth-mentioning peculiarities of Mal'cev's program were the following: first, he kept on dealing with first-order languages with equality[3]; second, he adopted notions of homomorphism and congruence that had little to do with the relational part of the language.

This well-rooted tradition of developing model theory in the presence of an equality symbol to express the identity relation, which goes back to its very origin, was finally broken when logicians from the Polish School started a program similar to that of Mal'cev for another type of UHL, viz. *general sentential logic.* Indeed, in spite of the fact that the algebraic character of sentential logic was evident early in its development (chiefly because classical sentential calculus could be completely reduced to the quasi-equational theory of boolean algebras), the natural models of an arbitrary sentential calculus quickly took the form of *logical matrices,* that is, algebras endowed with a unary relation on their universe. This matrix semantics so became the first attempt of starting a systematic development of a *model theory for first-order languages without equality*[4]. Begining with the publication of a paper by Los [72] in 1949, matrix semantics was successfully developed over the next three decades by a number of different authors in Poland, including J. Los himself, R. Suszko, R. Wojcicki and J. Zygmunt; see [17], [74], [122], [127].

The present evolution of these issues points towards an effort of encompassing the theory of varieties and quasivarieties and the model theory of sentential logic, by means of the development of a program similar to Mal'cev's for UHL without equality. We recognize that this evolution has been fast and notorious in the last decade, thanks mainly to the work done by J. Czelakowski, W. Blok and D. Pigozzi among others. For example, in a series of papers [29], [30], [31], [32], [33], [34], [35], [36], the first author has been developing a model theory of sentential logic that inherits a lot of the algebraic character of Mal'cev's ideas and the theory of varieties originated by Birkhoff. On the other hand, Blok and Pigozzi, in their very recent paper [12], have succeeded in the development of a model theory -based on the *Leibniz operator* introduced by them in [8]- that does comprise for the first time both *equational logic* and *sentential logic*, and so strengthens Czelakowski's program. What enables such a simultaneous treatment in their approach is the observation that equational logic can be viewed as an example of a *2-dimensional sentential calculus* [11] and thus admits a matrix semantics, this time a matrix being an algebra together with a congruence on the algebra.

A characteristic of decisive importance in Blok and Pigozzi's approach is their apparent conviction that only reduced models really possess the algebraic character of the models of quasi-equational theories. We give up such a conviction and the restriction to particular types of languages.

[3]In some of his original papers, there is a certain ambiguity concerning this point, in the sense that he seems to allow the interpretation of the equality symbol in a structure which were not the identity relation; see e.g. his discussion about the notion of *consistent configuration* in [83, §3]. However, there is no doubt that in his posthumous book [89] the equality symbol is always intended to mean the identity.

[4]An isolated but worth-mentioning incursion on this topic was [126], a short paper published in 1957 but practically unknown.

The main purpose of this paper is to outline some basic aspects of the model theory for first-order languages that definitively do not include the equality symbol and which takes account of both the *full* and the *reduced semantics*. The theory is intended to follow as much as possible of the Mal'cev's tradition by giving it a pronounced algebraic character and mainly covering topics fairly well studied in universal algebra (that is the reason for giving the term "algebraic" to our model theory). Most of the work, that extends to general languages and fairly clarifies some recent trends in algebraic logic, constitutes the foundations of a *model theory of UHL without equality*. An important number of the results in the paper run side by side with some well-known results of either classical model theory or universal algebra; so, we make an effort to highlight the concepts and techniques only applied in these contexts although, in some sense, they find a more general setting in ours. The outgrowth of the current interest in the model theory of UHL without equality is the emergence of several applications mainly in algebraic logic and computer science. Therefore we also discuss the way that the developed theory relates to algebraic logic. Actually, we maintain that our approach provides an appropiate context to investigate the availability of nice algebraic semantics, not only for the traditional deductive systems that arise in sentential logic, but also for some other types of deductive systems that are attracting an increasing attention at the time; see, e.g., [101], [119]. The reason is that all of them admit an interpretation as universal Horn theories without equality [13], [46]. We finally mention that a distinct program that also attempts to generalize the theory of varieties to general first-order structures has been initiated very recently by N. Weaver [120].

Outline of the work

As we said before, the absence of a symbol in the language to mean the identity relation is central to this work. Traditionally, the equality in classical model theory has had a representation in the formal language and has been understood in an absolute sense, i.e., for any interpretation of the language, the interest of model-theorists has been put on the relation according to which two members of the universe are the same or has no other logical relation. We break this tradition by introducing a weak form of equality predicate and not presupposing its formal representation by a symbol of the language[5]. Then the main problem consists, broadly speaking, in the investigation of the relationship between the features of this weaker equality in a given class of structures and the fulfillment of certain properties by this class.

This is not at all a recent treatment of the equality; for instance, it underlies

[5]The idea of defining the identity relation in second order logic as $x \approx y \leftrightarrow \forall P(P(x) \leftrightarrow P(y))$, where P is a variable ranging over all properties, goes back to Leibniz. A natural relativized first-order analogue of this definition is exactly what we mean by *equality in the sense of Leibniz* all over the paper. On the other hand, we should say that the eventual presence of a symbol in the language to express the common identity relation would not be restrictive; we can also define the equality in the sense of Leibniz regardless of it, as it is done in Weaver [120]. The point is that we do want to distinguish equality from other predicates.

the old notion of *Lindenbaum-Tarski algebra* in the model theory of sentential logic [108], and more recently some contributions to the study of algebraic semantics for sentential logics. Our contribution amounts to no more than providing a broader framework for the investigation of this question in the domain of first-order logic, especially the universal Horn fragment.

Several points stand out for they govern all our approach. First, the extended use we make of two unlike notions of homomorphism, whose difference relies on the importance each one attaches to relations; this is a distinction that no longer exists in universal algebra but does exist in classical model theory. Secondly, the availability of two distinct adequate semantics easily connected through an algebraic operation, which consists in factorizing the structures in such a way that the Leibniz equality and the usual identity relation coincide. We believe this double semantics is what is mainly responsible for the interest of the model theory for languages without equality as a research topic; in spite of their equivalence from a semantical point of view, they furnish several stimulating problems regarding their comparability from an algebraic perspective (Theorems 8.1.6, 8.1.7, 8.1.8 illustrate this point thoroughly). Thirdly, the two extensions that the notion of congruence on an algebra admits when dealing with general structures over languages without equality, namely, as a special sort of binary relation associated to a structure, here called *congruence*, and as the relational part of a structure, which is embodied in the concept of *filter extension*. Finally, and not because of this less important, the nice algebraic description that our weak equality predicate has as the greatest one of the congruences on a structure. This fact allows to replace the fundamental (logical) concept of *Leibniz equality* by an entirely algebraic notion, and to put the main emphasis on the algebraic methods. Actually, it seems to us that other forms of equality without such a property hardly give rise to model theories that work out so beautifully.

The paper is organized in 10 chapters. The first three contain basic material that is essential to overcome the small inadequacies of some approaches to the topic formerly provided by other authors. Chapter 1 reviews some terminology and notation that will appear repeteadly thereafter, and presents some elementary notions and results of classical model theory that remain equal for languages without equality. Chapter 2 states and characterizes algebraically the fundamental concept of equality in the sense of Leibniz which we deal with all over the paper. Starting with the extension of the concept of congruence from algebras to general structures, we then show that the greatest one of these congruences on each structure (*Leibniz congruence*) amounts to the equality in the sense of Leibniz that is inherent in every interpretation of the language (Corollary 2.1.3). The development of the corresponding basic Isomorphism Theorems and the fundamental notion of *Leibniz quotient*, which in some sense is an extension of the above mentioned *Lindenbaum-Tarski algebras*, are also included here. Finally, in Chapter 3 we discuss the semantical consequences of factorizing a structure by a congruence and show that first-order logic without equality has two complete semantics related by a *reduction operator* (Theorem 3.2.1). Right here we pose one of the central problems to which most of the subsequent work is devoted, i.e., the investigation of the algebraic properties that the full and reduced model classes of an elementary theory exhibit.

Chapter 4 contains the first difficult results in the work. By a rather obvious generalization of proofs known from classical model theory, we obtain Birkhoff-type characterizations of full classes of structures axiomatized by certain sorts of first-order sentences without equality, and apply these results to derive analogue characterizations for the corresponding reduced classes. In particular, the Chapter provides a proof of a generalized Birkhoff's Variety Theorem entirely based in elementary model-theoretic techniques that do not involve free structures in the same way as in Birkhoff's original proof (remark to Theorem 4.5.1).

Chapter 5 is a central one; it examines the primary consequences of dealing with the relational part of a structure as the natural extension of congruences when passing from algebraic to general first-order languages without equality. A key observation in this case is that we must often restrict our attention to classes that satisfy the *filter-lattice condition*, i.e., such that the sets of structures on a given underlying algebra exhibit the structure of an algebraic complete lattice. It is proved that this classes are just the *quasivarieties* of structures (Theorem 5.1.1). The *Leibniz operator* is defined right here as a primary criterion to distinguish properties of the Leibniz equality in a class of models. Using this operator, a fundamental hierarchy of classes -*protoalgebraic, semialgebraic, algebraic* and *purely algebraic classes*- is described (Definitions 5.3.1, 5.4.1, 5.4.6 and 5.4.7). The rest of the work is almost entirely devoted to investigate the distinct algebraic character of these classes. Parcicularly, *protoalgebraic classes* and *semialgebraic classes* seem to be the more generic classes for which the corresponding reduced semantics have a reasonable algebraic character; at least this seems to be the case after their characterizations and properties are obtained here (Theorems 5.3.8, 5.3.9 and 5.4.5) and in subsequent chapters. The concept of *relative congruence* on a structure is also included in this Chapter, and the close relation between the properties of the Leibniz operator and the fact that the set of relative congruences have some lattice structure is pointed out.

Chapter 7 examines how the characterizations of reduced quasivarieties (relative varieties) obtained in Chapter 4 can be improved when we deal with the special types of classes introduced formerly. In order to achieve this, we prove characterizations of protoalgebraic, semialgebraic and algebraic classes in terms of the closure under some operators of the corresponding reduced class (Theorems 7.2.1, 7.2.4 and 7.2.7). Also, in this Chapter it is posed the problem of relating the lattice-homomorphism properties of the Leibniz operator and the formal representability of the Leibniz equality in a class of structures. This is still an open problem at this level of generality, though some results are known for some particular types of languages; see [12].

Chapters 6, 8 and 9 provide explicit generalizations of well-known results from universal algebra. Concretely, in Chapter 6 we present the main tools of a Subdirect Representation Theory for general first-order structures without equality, certifying the validity of some Mal'cev's results in this wider context (cf. e.g. Proposition 6.1.1 and Theorems 6.1.8 and 6.2.2) and proving general forms of Jónsson's Theorem for quasivarieties and relative subvarieties (Theorems 6.2.4 and 6.3.2). We also introduce here the concept of *filter-distributive class* as a natural extension of congruence-distributivity in universal algebra. Chapter 8 deals with the existence of *free structures* both in full and reduced classes. Begining with the former, we characterize quasivarieties as those classes that admit free structures in a very precise sense (Theorem 8.1.6). We then pay attention to one of the central problems in the Chapter: to find out necessary and sufficient conditions on a quasivariety so that the corresponding reduced class admits free structures in exactly the same way (Theorems 8.1.7 and 8.1.8). The basic tools for this purposes are the *Herbrand structures*, i.e., the models built up out of the same language that are minimal in the posets of filter extensions. We precisely use these structures and their relative filter extensions, the so-called *term-structures*, to supply new proofs of the characterization of the variety and quasivariety generated by a given class, proofs that are closer in spirit to the proof of Birkhoff's Variety Theorem. Such proofs rely on characterizations, in the style of Jónsson's Theorem on finitely subdirectly irreducible algebras, of the relative filter extensions of Herbrand structures (Theorem 8.2.2). This Chapter also includes the investigation of a correspondence between (quasi)varieties and some lattice structures associated with the Herbrand structures, correspondence that offers the possibility of turning the logical methods used in the theory of varieties and quasivarieties into purely algebraic ones (Theorems 8.3.3 and 8.3.6). In Chapter 9 we set the problem of finding Mal'cev-type conditions for some properties concerning posets of relative congruences or relative filter extensions of members of quasivarieties. We just pay attention to relative congruences and show that, in this case, the problem has a purely universal algebraic interpretation (Proposition 9.1.1). We prove a stronger form of Mal'cev's Theorem on congruence permutable varieties of algebras (Theorem 9.1.4) and examine the possibility of getting similar extensions of other Mal'cev-Type theorems.

Finally, Chapter 10 discusses briefly the relation between algebraic logic and the approach to model theory outlined in the previous chapters, providing thus some vindication to it. Of course, we cannot say whether this work will ultimately have a bearing on the resolution of any of the problems of algebraic logic, but for us, it could at least provide fresh insights in this exciting branch of logic.

Acknowledgements

I am indebted to Willem Blok, Janusz Czelakowski and Don Pigozzi, because their work has been a good source of inspiration to me; in particular, the influence of Blok and Pigozzi's paper [7] is apparent all over the present work. I also want to express my sincere gratitude to Katyrzyna Palasinska for the opportunity of exchanging some ideas with her; Janusz Czelakowski, Josep Maria Font and Ramon Jansana for their valuable comments on a draft version, and especially to Don Pigozzi for his encouragement and his continuous suggestions on this work, mainly done during my stay in Ames. He was the advisor I did not find to work on the topic. Finally, I acknowledge CIRIT (Generalitat de Catalunya) for its financial support during my stay at Iowa State University, USA.

1. Preliminaries

Let the triple $\mathcal{L} = \langle F, R, \rho \rangle$ be a first order language; F and R denote parwise disjoint sets of function and relation symbols of $\mathcal{L}$ respectively (R must be nonempty), and ρ is the arity function from $F \cup R$ into the set of natural numbers. We use capital Gothic letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}, \ldots$, with appropriate subscripts, to range over structures on $\mathcal{L}$, also called *$\mathcal{L}$-structures*. In order to be consistent with the notation, we denote by A the universe of $\mathfrak{A}$, and by $F_{\mathfrak{A}}$ and $R_{\mathfrak{A}}$ the interpretations on $\mathfrak{A}$ of the collections of function and relation symbols of $\mathcal{L}$, respectively, i.e., $F_{\mathfrak{A}} = \{f^{\mathfrak{A}} : f \in F\}$ and $R_{\mathfrak{A}} = \{r^{\mathfrak{A}} : r \in R\}$. The corresponding boldface letter $\mathbf{A}$ is used to understand the underlying algebra $\langle A, F_{\mathfrak{A}} \rangle$ of $\mathfrak{A}$, and we very often write $f^{\mathbf{A}}$ to mean the interpretation of f in $\mathfrak{A}$. Lowercase boldface letters $\mathbf{a}$,$\mathbf{b}$,... are used to indicate members of a cartesian product of some family of sets. If $\mathfrak{A}$ is an $\mathcal{L}$-structure, $\mathbf{a} = \langle a_1, \ldots, a_n \rangle$ belongs to A^n, $f \in F$ and $r \in R$, and h is any mapping with domain A, then $f^{\mathfrak{A}}\mathbf{a}$, $\mathbf{a} \in r^{\mathfrak{A}}$ and $h\mathbf{a}$ are short-hand notations for $f^{\mathfrak{A}}a_1 \ldots a_n$, $\langle a_1, \ldots, a_n \rangle \in r^{\mathfrak{A}}$ and $\langle ha_1, \ldots, ha_n \rangle$, respectively.

By an *$\mathcal{L}$-algebra* we mean the underlying algebra of an $\mathcal{L}$-structure; of course, if the set of function symbols is empty, an $\mathcal{L}$-algebra means simply an arbitrary set. The absolutely free $\mathcal{L}$-algebra over a set of α variables, i.e., the algebra of all $\mathcal{L}$-terms over α variables, is denoted $\mathbf{Te}_{\mathcal{L},\alpha}$. When $\alpha = \omega$ we simply write $\mathbf{Te}_{\mathcal{L}}$. A language $\mathcal{L}' = \langle F', R', \rho' \rangle$ is called an *expansion of* $\mathcal{L}$ provided that $F \subseteq F'$, $R \subseteq R'$ and $\rho = \rho' \restriction F \cup R$. In this case, the *$\mathcal{L}$-reduct* of an structure $\mathfrak{A}$ over $\mathcal{L}'$ is defined as usual, $\mathfrak{A} \restriction \mathcal{L} = \langle A, F_{\mathfrak{A}}, R_{\mathfrak{A}} \rangle$.

Some more notation of common usage is the following: $Str\ \mathcal{L}$ to mean the class of $\mathcal{L}$-structures, $For_\alpha \mathcal{L}$ the set of $\mathcal{L}$-formulas and $Atm_\alpha \mathcal{L}$ the set of atomic $\mathcal{L}$-formulas over α variables, for every cardinal α. Formulas are represented by means of lowercase greek letters $\varphi, \psi, \vartheta, \ldots$, and uppercase ones are used to denote sets of formulas. We write $\varphi(x_1, \ldots, x_n)$ to mean that the free variables that occur in φ are among $x_1, \ldots, x_n$. Unless otherwise indicated, all $\mathcal{L}$-formulas are assumed to be over the set of ω variables $Var = \{x_0, x_1, x_2, \ldots\}$; in this case, the preceding notations are abbreviated by $For\ \mathcal{L}$ and $Atm\ \mathcal{L}$. We deal with a special type of $\mathcal{L}$-formulas very often: quantifier-free formulas in prenex form whose matrix is the disjunction of a finite set (maybe empty) of negated atomic formulas and exactly one atomic formula. The most common name for them is "strict basic Horn formulas"; nevertheless, for the sake of simplicity, we call them *implicative formulas* and denote by $Imp_\alpha\mathcal{L}$ the corresponding subset of $For_\alpha\mathcal{L}$, or simply $Imp\,\mathcal{L}$ if $\alpha = \omega$.

Given an $\mathcal{L}$-structure $\mathfrak{A}$, an algebra homomorphism $g : \mathbf{Te}_{\mathcal{L},\alpha} \longrightarrow \mathbf{A}$ (called *assignment*) and an $\mathcal{L}$-formula φ, we use the notation $\mathfrak{A} \models \varphi[g]$ to refer to the satisfaction relation defined in the usual way. Following the standard convention, $\mathfrak{A} \models \varphi$ expresses that $\mathfrak{A}$ satisfies the universal closure of φ. When we write

$$\mathfrak{A} \models \varphi(x_1, \dots, x_k)\ [a_1, \dots, a_k],$$

we mean $\mathfrak{A}$ satisfies φ with respect to any assigment $g : \mathbf{Te}_{\mathcal{L}} \longrightarrow \mathbf{A}$ such that $gx_i = a_i$, for all $1 \leq i \leq k$. The notation $\mathfrak{A} \models \varphi\ [g(x/a)]$ expresses that $\mathfrak{A}$ satisfies φ with respect to the assignment that sends the variable x to a and coincides with g otherwise. Given a class K of $\mathcal{L}$-structures, $Th_\alpha\mathsf{K}$ denote the set of all $\mathcal{L}$-formulas over α variables which hold in every member of K, i.e., if $\varphi \in For_\alpha\mathcal{L}$ then $\varphi \in Th_\alpha\mathsf{K}$ iff $\mathfrak{A} \models \varphi$ for all $\mathfrak{A} \in \mathsf{K}$. When we only want to refer to the quantifier-free formulas, the implicative formulas and the atomic formulas that hold in K we put respectively $Un_\alpha\mathsf{K}$, $Imp_\alpha\mathsf{K}$ and $Atm_\alpha\mathsf{K}$ (Un stands for "universal"). Once more, the subscript is omitted if $\alpha = \omega$.

We say $\mathcal{L}$ is a *language with equality*, or simply $\mathcal{L}$ *has equality*, to mean that $\mathcal{L}$ contains a binary relation symbol $\approx$ which is always interpreted as the identity; in other terms, only the structures $\mathfrak{A}$ for which $\approx^{\mathfrak{A}}$ is the *diagonal relation*, i.e., the set $\Delta_A = \{\langle a, a\rangle : a \in A\}$, count as $\mathcal{L}$-structures. On the contrary, we say $\mathcal{L}$ is a *language without equality*, or $\mathcal{L}$ *has no equality*, provided that $\mathcal{L}$ does not contain any such binary relation symbol. Thus, if $\mathcal{L}$ is without equality and r is some binary relation symbol of $\mathcal{L}$, then r can be interpreted in the $\mathcal{L}$-structures as any binary relation whatsoever[6].

1.1. Substructures and Filter Extensions

Let $\mathfrak{A} = \langle \mathbf{A}, R_{\mathfrak{A}}\rangle$ and $\mathfrak{B} = \langle \mathbf{B}, R_{\mathfrak{B}}\rangle$ be two $\mathcal{L}$-structures. $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, in symbols $\mathfrak{A} \subseteq \mathfrak{B}$, if $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $r^{\mathfrak{A}} = r^{\mathfrak{B}} \cap A^{\rho(r)}$ for all $r \in R$. Likewise, $\mathfrak{B}$ is a *filter extension* of $\mathfrak{A}$, and in this case we write $\mathfrak{A} \preccurlyeq \mathfrak{B}$, provided the underlying algebras of $\mathfrak{A}$ and $\mathfrak{B}$ coincide and $r^{\mathfrak{A}} \subseteq r^{\mathfrak{B}}$ for every $r \in R$.

As is well known, the class of substructures of a given structure $\mathfrak{A}$ defines an inductive closure system as follows. Let $\{\mathfrak{A}_i : i \in I\}$ be a family of substructures of $\mathfrak{A}$, and define

$$\textstyle\bigcap_{i\in I} R_{\mathfrak{A}_i} = \{\bigcap_{i\in I} r^{\mathfrak{A}_i} : r \in R\}, \qquad \bigcap_{i\in I} \mathfrak{A}_i = \langle \bigcap_{i\in I} \mathbf{A}_i, \bigcap_{i\in I} R_{\mathfrak{A}_i}\rangle,$$

where $\bigcap_{i\in I} \mathbf{A}_i$ means the intersection of the algebras $\mathbf{A}_i$, $i \in I$, as it is usually defined in universal algebra; in particular, the intersection of the empty family is

[6] Usually the distinction between a language with or without equality relies on the presence or not of the equality symbol among the logical constants. For convenience, we do not follow this widely accepted convention in the preceding definition; the reason is that in this way all the results we state in the sequel amount to well-known results in the case the language has equality.

taken to be the whole structure $\mathfrak{A}$. If, moreover, $\{\mathfrak{A}_i : i \in I\}$ is a directed family, set

$$\bigcup_{i\in I} R_{\mathfrak{A}_i} = \{\bigcup_{i\in I} r^{\mathfrak{A}_i} : r \in R\}, \qquad \bigcup_{i\in I} \mathfrak{A}_i = \langle \bigcup_{i\in I} \mathbf{A}_i, \bigcup_{i\in I} R_{\mathfrak{A}_i} \rangle,$$

where $\bigcup_{i\in I} \mathbf{A}_i$ is the union of the algebras $\mathbf{A}_i$, $i \in I$. Then, it can be easily proved that $\bigcap_{i\in I} \mathfrak{A}_i$ and $\bigcup_{i\in I} \mathfrak{A}_i$ are again substructures of $\mathfrak{A}$.

The same is true for the class of all the filter extensions of $\mathfrak{A}$; in this case, when referring to the associated (algebraic) complete lattice we shall write $\mathbf{Fe}\,\mathfrak{A}$. Given a subset X of A, we use the notation $\mathfrak{A} \restriction X$ to understand the *substructure of* $\mathfrak{A}$ *generated by* X, i.e.,

$$\mathfrak{A} \restriction X = \langle [X], \{f^{\mathfrak{A}} \restriction [X] : f \in F\}, \{r^{\mathfrak{A}} \cap [X]^{\rho(r)} : r \in R\} \rangle,$$

where $[X]$ denotes the universe of the subalgebra of $\mathbf{A}$ generated by X.

1.2. Homomorphisms between Structures

A mapping $h : A \to B$ is said to be a *homomorphism* from $\mathfrak{A}$ into $\mathfrak{B}$ if h is an algebra homomorphism from $\mathbf{A}$ into $\mathbf{B}$ and the condition

$$(1.1) \qquad \langle a_1, \ldots, a_n \rangle \in r^{\mathfrak{A}} \implies \langle ha_1, \ldots, ha_n \rangle \in r^{\mathfrak{B}}.$$

holds for all n-ary relation symbol $r \in R$ and all $a_1, \ldots, a_n \in A$; we write $h : \mathfrak{A} \to \mathfrak{B}$ to mean that h is such a homomorphism. It is an *embedding* or an *epimorphism* provided it is as a homomorphism between the underlying algebras; in these cases we put $h : \mathfrak{A} \rightarrowtail \mathfrak{B}$ and $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, respectively. When h is onto we also say that $\mathfrak{B}$ is a *homomorphic image* of $\mathfrak{A}$. Finally, h is an *isomorphism* between $\mathfrak{A}$ and $\mathfrak{B}$, in symbols $h : \mathfrak{A} \cong \mathfrak{B}$, if h is one-one and onto and the inverse of h is also a homomorphism.

We call $h : \mathfrak{A} \to \mathfrak{B}$ a *strong homomorphism* from $\mathfrak{A}$ into $\mathfrak{B}$, and we write $h : \mathfrak{A} \to_s \mathfrak{B}$, if h is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ for which the reverse implication of (1.1) also holds[7]; so for all n-ary relation symbol $r \in R$ and all $a_1, \ldots, a_n \in A$,

$$(1.2) \qquad \langle a_1, \ldots, a_n \rangle \in r^{\mathfrak{A}} \iff \langle ha_1, \ldots, ha_n \rangle \in r^{\mathfrak{B}}.$$

Strong homomorphisms that are one-one are called *strong embeddings*, whereas those that are surjective are referred to as *reductive homomorphisms*; we write, respectively, $h : \mathfrak{A} \rightarrowtail_s \mathfrak{B}$ and $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. If there is a reductive homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ we also say that $\mathfrak{B}$ is a *reduction* of $\mathfrak{A}$ and $\mathfrak{A}$ an *expansion* of $\mathfrak{B}$. Note that a bijective strong homomorphism is simply an isomorphism as it is defined before and that reductive homomorphisms are the same as isomorphisms when the language has equality. Both assertions are easy consequences of the following result.

[7] Compare this notion of *strong homomorphism* with the one given by Chang and Keisler in [25, p.242]; they coincide whenever h is an onto mapping.

LEMMA 1.2.1. *The following holds for every algebra homomorphism* $h : \mathbf{A} \rightarrow \mathbf{B}$.

(i) $h : \mathfrak{A} \rightarrow \mathfrak{B}$ *iff* $r^{\mathfrak{A}} \subseteq h^{-1}r^{\mathfrak{B}}$, *for all* $r \in R$.

(ii) $h : \mathfrak{A} \rightarrow_s \mathfrak{B}$ *iff* $r^{\mathfrak{A}} = h^{-1}r^{\mathfrak{B}}$, *for all* $r \in R$.

(iii) $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ *implies* $r^{\mathfrak{A}} = h^{-1}r^{\mathfrak{B}}$ *and* $hr^{\mathfrak{A}} = r^{\mathfrak{B}}$, *for all* $r \in R$. ∎

For each $h : \mathfrak{A} \rightarrow \mathfrak{B}$, we define the *image of* $\mathfrak{A}$ *through* h as the structure $h\mathfrak{A} = \langle h\mathbf{A}, hR_{\mathfrak{A}}\rangle$, where $hR_{\mathfrak{A}} = \{hr^{\mathfrak{A}} : r \in R\}$ and $hr^{\mathfrak{A}} = \{\langle ha_1, \dots, ha_n\rangle \in B^n : \langle a_1, \dots, a_n\rangle \in r^{\mathfrak{A}}\}$. Conversely, we define $h^{-1}\mathfrak{B} = \langle h^{-1}\mathbf{B}, h^{-1}R_{\mathfrak{B}}\rangle$, with $h^{-1}R_{\mathfrak{B}} = \{h^{-1}r^{\mathfrak{B}} : r \in R\}$ and $h^{-1}r^{\mathfrak{B}} = \{\langle a_1, \dots, a_n\rangle \in A^n : \langle ha_1, \dots, ha_n\rangle \in r^{\mathfrak{B}}\}$, and call $h^{-1}\mathfrak{B}$ the *inverse image of* $\mathfrak{B}$ *through* h. Both $h\mathfrak{A}$ and $h^{-1}\mathfrak{B}$ are again structures over $\mathcal{L}$, even though $h\mathfrak{A}$ is not in general a substructure of $\mathfrak{B}$ nor $h^{-1}\mathfrak{B}$ a substructure of $\mathfrak{A}$. This is true, however, in case that h is a strong homomorphism. The next proposition states a generalized form of this property.

LEMMA 1.2.2. *Let* $h : \mathfrak{A} \rightarrow_s \mathfrak{B}$. *For each substructure* $\mathfrak{A}'$ *of* $\mathfrak{A}$ *we have that* $h\mathfrak{A}' \subseteq \mathfrak{B}$. *Conversely, if* $\mathfrak{B}'$ *is a substructure of* $\mathfrak{B}$ *then* $h^{-1}\mathfrak{B}' \subseteq \mathfrak{A}$. ∎

Observe that every surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ can be canonically decomposed through a reductive homomorphism. Concretely, if $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, then h maps strong homomorphically $h^{-1}\mathfrak{B}$ onto $\mathfrak{B}$ ($h^{-1}\mathfrak{B}$ is in fact the least filter extension of $\mathfrak{A}$ satisfying this property!). So, the composition of the identity $id : \mathfrak{A} \rightarrowtail\!\!\!\twoheadrightarrow h^{-1}\mathfrak{B}$ and $\hat{h} : h^{-1}\mathfrak{B} \twoheadrightarrow_s \mathfrak{B}$ ($\hat{h}a = ha$) coincides with $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$. As a result, every homomorphism $h : \mathfrak{A} \rightarrow \mathfrak{B}$ factorizes according to the following diagram:

$$(*) \qquad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{\;h\;} & \mathfrak{B} \\ {\scriptstyle id}\big\downarrow & & \big\uparrow{\scriptstyle j} \\ h^{-1}\mathfrak{B} & \underset{\hat{h}}{\twoheadrightarrow_s} & h\mathfrak{A} \end{array}$$

This decomposition explains why homomorphisms will not play quite as important a role in the algebraic model theory we try to develop as they do in universal algebra. As we shall see later on, such a role in this case is performed by strong homomorphisms.

1.3. Products of Structures

Assume that $\mathfrak{A}_i = \langle \mathbf{A}_i, R_{\mathfrak{A}_i}\rangle$, with $i \in I$, is a family of $\mathcal{L}$-structures. We define the *direct product* of $\{\mathfrak{A}_i : i \in I\}$ by setting

$$\prod_{i\in I} \mathfrak{A}_i := \langle \prod_{i\in I} \mathbf{A}_i, \prod_{i\in I} R_{\mathfrak{A}_i}\rangle,$$

where $\prod_{i\in I} \mathbf{A}_i$ is the usual direct product of the underlying algebras $\{\mathbf{A}_i : i \in I\}$ and $\prod_{i\in I} R_{\mathfrak{A}_i}$ denotes the interpretations on $\prod_{i\in I} \mathfrak{A}_i$ of the symbols of R defined in the obvious way: if $r \in R$ is an n-ary relation symbol and a_{ij} means the ith component of $\mathbf{a}_j$, for each $1 \leq j \leq n$,

$$r^{\prod_{i\in I} \mathfrak{A}_i} := \{\langle \mathbf{a}_1, \dots, \mathbf{a}_n\rangle \in (\prod_{i\in I} A_i)^n : \langle a_{i1}, \dots, a_{in}\rangle \in r^{\mathfrak{A}_i} \text{ for all } i \in I\}.$$

We allow I to be empty; in this case, $\prod_{i\in I}\mathfrak{A}_i$ is the trivial structure with a one element underlying algebra and all relations holding.

The direct product construction can be extended in several ways, three of which are useful for our purposes. The first one is the following. Suppose that $\mathcal{F}$ is a proper filter of $Sb(I)$ and define

$$\Theta_{\mathcal{F}} := \{\langle \mathbf{a},\mathbf{b}\rangle \in \left(\textstyle\prod_{i\in I} A_i\right)^2 : \{i \in I : a_i = b_i\} \in \mathcal{F}\}.$$

As is well known, $\Theta_{\mathcal{F}}$ is a congruence relation on the algebra $\prod_{i\in I}\mathbf{A}_i$, so that we can put

$$\textstyle\prod_{i\in I}\mathbf{A}_i/\mathcal{F} := \prod_{i\in I}\mathbf{A}_i/\Theta_{\mathcal{F}}, \qquad \prod_{i\in I} R_{\mathfrak{A}_i}/\mathcal{F} := \prod_{i\in I}^{\mathcal{F}} R_{\mathfrak{A}_i}/\Theta_{\mathcal{F}},$$

where $\prod_{i\in I}^{\mathcal{F}} R_{\mathfrak{A}_i}$ denotes the family of relations

$$\textstyle\prod_{i\in I}^{\mathcal{F}} r^{\mathfrak{A}_i} := \{\langle \mathbf{a}_1,\ldots,\mathbf{a}_n\rangle \in (\prod_{i\in I} A_i)^n : \{i \in I : \langle a_{i1},\ldots,a_{in}\rangle \in r^{\mathfrak{A}_i}\} \in \mathcal{F}\},$$

for $r \in R$ (actually, for each r, the set $\prod_{i\in I}^{\mathcal{F}} r^{\mathfrak{A}_i}$ is just the least relation that contains $r^{\prod_{i\in I}\mathfrak{A}_i}$ and is compatible with $\Theta_{\mathcal{F}}$). Then, the *filtered product* of $\{\mathfrak{A}_i : i \in I\}$ by $\mathcal{F}$ is defined as the structure

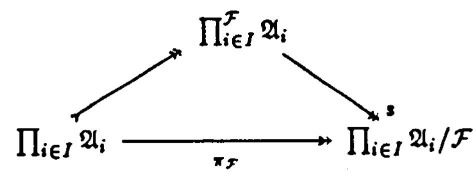
$$\textstyle\prod_{i\in I}\mathfrak{A}_i/\mathcal{F} := \langle \prod_{i\in I}\mathbf{A}_i/\mathcal{F}, \prod_{i\in I} R_{\mathfrak{A}_i}/\mathcal{F}\rangle,$$

which coincide with the direct product in the case $\mathcal{F} = \{I\}$. For simplicity, if $\mathbf{a} \in \prod_{i\in I} A_i$, the equivalence class of $\mathbf{a}$ modulo $\Theta_{\mathcal{F}}$ is denoted by $\mathbf{a}/\mathcal{F}$.

We point out that in general the direct product does not map strong homomorphically by the natural projection neither onto its components nor onto the quotient modulo $\Theta_{\mathcal{F}}$. Actually, if $\pi_{\mathcal{F}}$ means the projection from $\prod_{i\in I}\mathfrak{A}_i$ onto $\prod_{i\in I}\mathfrak{A}_i/\mathcal{F}$, the inverse image of $\prod_{i\in I}\mathfrak{A}_i/\mathcal{F}$ through $\pi_{\mathcal{F}}$ is

$$\textstyle\prod_{i\in I}^{\mathcal{F}}\mathfrak{A}_i = \langle \prod_{i\in I}\mathbf{A}_i, \prod_{i\in I}^{\mathcal{F}} R_{\mathfrak{A}_i}\rangle.$$

This fact is illustrated by the following diagram.



It is easy to show that, if $\mathcal{L}$ has equality, the filtered product $\prod_{i\in I}\mathfrak{A}_i/\mathcal{F}$ is again an $\mathcal{L}$-structure for any proper filter $\mathcal{F}$, i.e., the interpretation of $\approx$ on $\prod_{i\in I}\mathfrak{A}_i/\mathcal{F}$ is again the identity relation on $\prod_{i\in I} A_i/\Theta_{\mathcal{F}}$. This is in fact a consequence of the following result, whose proof can be found in almost every model theory textbook; see e.g. [25].

THEOREM 1.3.1. *Let $\varphi = \varphi(x_1, \dots, x_n)$ be an arbitrary Horn formula and let $\mathfrak{A}_i$, $i \in I$, be a family of $\mathcal{L}$-structures over a nonempty index set I. Let $\mathcal{F}$ be a proper filter of $Sb(I)$. If $g : \mathbf{Te}_{\mathcal{L}} \to \prod_{i\in I} \mathbf{A}_i$ is any assignment, then*

$$\{i \in I : \mathfrak{A}_i \vDash \varphi\,[\pi_i \circ g]\} \in \mathcal{F} \;\; \textit{implies} \;\; \textstyle\prod_{i\in I} \mathfrak{A}_i/\mathcal{F} \vDash \varphi\,[\pi_{\mathcal{F}} \circ g].$$

Moreover, this implication becomes an equivalence if φ is atomic. ∎

A filtered product $\prod_{i\in I} \mathfrak{A}_i/\mathcal{U}$ is called an *ultraproduct* if $\mathcal{U}$ is an ultrafilter on the index set I. The following, due to Los [73], is the main property of ultraproducts; it shows that for ultraproducts the last implication becomes an equivalence which holds for any first-order formula. Its proof can also be found, for example, in [25]; see also [127] for a treatment of ultraproducts of structures over a special type of languages without equality.

THEOREM 1.3.2. *(*Los Theorem*) Let I be a nonempty set. Assume $\mathfrak{A}_i$, $i \in I$, are $\mathcal{L}$-structures and let $\mathcal{U}$ be an ultrafilter of $Sb(I)$. If $g : \mathbf{Te}_{\mathcal{L}} \to \prod_{i\in I} \mathbf{A}_i$ and $\varphi = \varphi(x_1, \dots, x_n)$ is an arbitrary first-order formula over $\mathcal{L}$, then*

$$\{i \in I : \mathfrak{A}_i \vDash \varphi\,[\pi_i \circ g]\} \in \mathcal{U} \;\; \textit{iff} \;\; \textstyle\prod_{i\in I} \mathfrak{A}_i/\mathcal{U} \vDash \varphi\,[\pi_{\mathcal{U}} \circ g]. \;\blacksquare$$

The second generalization of the direct product construction we are interested in comes from Birkhoff's work in universal algebra. A structure $\mathfrak{A}$ is called a *subdirect product* of the system $\{\mathfrak{A}_i : i \in I\}$, in symbols $\mathfrak{A} \subseteq_{sd} \prod_{i\in I} \mathfrak{A}_i$, if $\mathfrak{A}$ is a substructure of $\prod_{i\in I} \mathfrak{A}_i$ and the restriction of the projection map π_i to A is surjective for every $i \in I$[8]. An embedding $h : \mathfrak{A} \rightarrowtail \prod_{i\in I} \mathfrak{A}_i$ is *subdirect* if $h\mathfrak{A} \subseteq_{sd} \prod_{i\in I} \mathfrak{A}_i$; we write $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i\in I} \mathfrak{A}_i$ to mean h is a subdirect embedding. Note that every subdirect embedding is strong; indeed, if $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i\in I} \mathfrak{A}_i$, then $hr^{\mathfrak{A}} = r^{\prod_{i\in I} \mathfrak{A}_i} \cap (hA)^n$ for every n-ary relation symbol r; so, being h one-one, $\langle ha_1, \dots, ha_n\rangle \in r^{\prod_{i\in I} \mathfrak{A}_i}$ implies $\langle a_1, \dots, a_n\rangle \in r^{\mathfrak{A}}$, for all $a_1, \dots, a_n \in A$.

Finally, a last generalization of direct products that combines filtered and subdirect products has been recently introduced by Czelakowski [37]; we also use it in subsequent sections. Let $\mathfrak{A}$ be a subdirect product of a system $\{\mathfrak{A}_i : i \in I\}$ of $\mathcal{L}$-structures, and let $\mathcal{F}$ be a proper filter on I. It is an easy matter to check that the restriction of $\Theta_{\mathcal{F}}$ to A,

$$\Theta_{\mathcal{F},A} := \Theta_{\mathcal{F}} \cap A^2,$$

is a congruence on the algebra $\mathbf{A}$. So we define the *filtered subdirect product* of $\mathfrak{A}$ by $\mathcal{F}$ as the structure

$$\mathfrak{A}/\mathcal{F} := \langle \mathbf{A}/\Theta_{\mathcal{F},A}, R^{\mathcal{F}}_{\mathfrak{A}}/\Theta_{\mathcal{F},A}\rangle,$$

where

$$r^{\mathfrak{A},\mathcal{F}} := \textstyle\prod^{\mathcal{F}}_{i\in I} r^{\mathfrak{A}_i} \cap A^{\rho(r)}, \quad \text{for all } r \in R,$$
$$R^{\mathcal{F}}_{\mathfrak{A}} := \{r^{\mathfrak{A},\mathcal{F}} : r \in R\}.$$

[8] This way of extending the notion of subdirect product from algebras to arbitrary structures is due to Tarski [117], even though it were Mal'cev [86] and a bit later Lyndon [79] who investigated its properties. A distinct nontrivial generalization of the concept can be found in [18] with the notion of *full subdirect product*.

Let us notice that, by the definition, $\mathfrak{A}/\mathcal{F}$ is isomorphic to a substructure of the filtered product $\prod_{i\in I}\mathfrak{A}_i/\mathcal{F}$, for in fact the quotient algebra $\mathbf{A}/\Theta_{\mathcal{F},A}$ can be embedded into $\prod_{i\in I}\mathbf{A}_i/\mathcal{F}$; the embedding is established by the mapping that assigns the equivalence class $\mathbf{a}/\mathcal{F}$ of $\mathbf{a}$ (modulo $\Theta_{\mathcal{F}}$) to each element $\mathbf{a}/\Theta_{\mathcal{F},A} \in A/\Theta_{\mathcal{F},A}$. Also, observe that the projection from $\mathfrak{A}$ to $\mathfrak{A}/\mathcal{F}$ given by $\mathbf{a} \longmapsto \mathbf{a}/\Theta_{\mathcal{F},A}$ need not be strong; the inverse image of $\mathfrak{A}$ through it is the structure $\mathfrak{A}^{\mathcal{F}} := \langle \mathbf{A}, R^{\mathcal{F}}_{\mathfrak{A}} \rangle$.

2. Congruences on Structures

The theory of congruence lattices of universal algebras is one of the most rich and developed parts of contemporary algebra, but unfortunately the rather special and purely internal definition of the congruence relation does not seem to extend in a unique successful manner to structures other than sets with operations. This explains why the notion of congruence on an algebra has been extended to general first-order structures in at least two different ways. The first one of these extensions can be found, for instance, in [76], [89], and gives rise to a theory already put forward by Mal'cev and not very different from the theory of algebras. In some sense, this first definition is not quite satisfactory, since it has relatively little to do with the relations of a structure; for example, the quotient modulo a nontrivial congruence in this sense of the linearly ordered additive group $\mathbb{Z}$ of integers is a finite group $\mathbb{Z}_n$ which cannot be linearly ordered.

The second extension also appears in the literature though implicitly and in different contexts; e.g. [14], [126] and more recently [7], [29], [120]. This second notion is the one adopted here and plays a central role in the present work; the way to deal with relations in this case is based on the notion of compatibility. A fundamental result in the Chapter is to show that this notion leads just to an algebraic description of a weak form of equality predicate, viz. the *equality in the sense of Leibniz* outlined in the introduction (Corollary 2.1.3). The very definition of this new equality predicate is given right here, and we point out the importance of relating this logical concept of equality and the algebraic concept of congruence.[9].

2.1. The Lattice of Congruences

Let $\mathfrak{A} = \langle \mathbf{A}, R_{\mathfrak{A}} \rangle$ be any $\mathcal{L}$-structure. A binary relation θ on A is said to be a *congruence on* $\mathfrak{A}$ if θ is a congruence on the underlying algebra which is compatible

[9]A third notion of congruence on an arbitrary first-order structure, which strictly speaking differs from the preceding ones, can also be found in the literature; it results from pasting together the second notion and the concept of filter extension [56]. We come back to this point later in Section 6.1. Also, other notions of congruence for particular types of structures have been used with interesting results, specially for (quasi)ordered algebras; see, e.g., [27], [70].

with all the relations belonging to $R_{\mathfrak{A}}$, i.e., for every $r \in R$ of arity n,

$$\langle a_1, \dots, a_n\rangle \in r^{\mathfrak{A}} \text{ and } a_i \equiv b_i(\theta), \text{ for } 1 \leq i \leq n \implies \langle b_1, \dots, b_n\rangle \in r^{\mathfrak{A}}.$$

For simplicity, if $\mathbf{a} = \langle a_1, \dots, a_n\rangle$, $\mathbf{b} = \langle b_1, \dots, b_n\rangle \in A^n$ we write $\mathbf{a} \equiv \mathbf{b}\ (\theta)$ to mean $a_i \equiv b_i(\theta)$ for all $1 \leq i \leq n$. The compatibility property of θ with an arbitary n-subset D of A can then be expressed as follows: if $\mathbf{a} \in D$ then $\mathbf{b} \in D$ for every $\mathbf{b} \in A^n$ such that $\mathbf{a} \equiv \mathbf{b}\ (\theta)$. In this case,

$$D = \bigcup\nolimits_{\mathbf{a}\in D}(a_1/\theta) \times (a_2/\theta) \times \dots \times (a_n/\theta).$$

Clearly the set of all congruences on $\mathfrak{A}$, denoted $Co\,\mathfrak{A}$, is a subset of $Co\,\mathbf{A}$. The following proposition provides a full description of this subset.

PROPOSITION 2.1.1. *For every $\mathcal{L}$-structure $\mathfrak{A}$, the poset* $\mathbf{Co}\,\mathfrak{A} = \langle Co\,\mathfrak{A}, \subseteq\rangle$ *is a principal ideal of* $\mathbf{Co}\,\mathbf{A}$.

Proof. Clearly $\phi \subseteq \theta$ and $\theta \in Co\,\mathfrak{A}$ implies that $\phi \in Co\,\mathfrak{A}$. Hence, it suffices to show that for each family $\{\theta_i : i \in I\}$ of congruences on $\mathfrak{A}$, $\bigvee_{i\in I}\theta_i$ belongs to $Co\,\mathfrak{A}$. Let $\mathbf{a}= \langle a_1, \dots, a_n\rangle, \mathbf{b} = \langle b_1, \dots, b_n\rangle \in A^n$. From universal algebra, we know $\mathbf{a} \equiv \mathbf{b}\ (\bigvee_{i\in I}\theta_i)$ iff there exists a sequence of elements $\mathbf{c}_1, \dots, \mathbf{c}_k \in A^n$ and $i_1, \dots, i_{k-1} \in I$ such that $\mathbf{a} = \mathbf{c}_1$, $\mathbf{c}_k = \mathbf{b}$ and

$$\mathbf{c}_j \equiv \mathbf{c}_{j+1}\ (\theta_{i_j}), \quad 1 \leq j \leq k-1.$$

Thus, the compatibility of $\bigvee_{i\in I}\theta_i$ follows immediately. ∎

We call $\mathbf{Co}\,\mathfrak{A}$ the *lattice of congruences of* $\mathfrak{A}$ and, extending the terminology and notation introduced in [8], we denote by $\Omega\mathfrak{A}$ its maximum element and call it the *Leibniz congruence of* $\mathfrak{A}$. So, by the previous lemma,

$$Co\,\mathfrak{A} = \{\theta \in Co\,\mathbf{A} : \theta \subseteq \Omega\mathfrak{A}\}.$$

Notice that $\Omega\mathfrak{A} \neq \nabla_A$ whenever $r^{\mathfrak{A}} \neq A^{\rho(r)}$ and $r^{\mathfrak{A}} \neq \emptyset$ for some relation symbol $r \in R$ (∇_A denotes the set of all ordered pairs of members of A, and is called the *all relation*). Also, if $\mathfrak{A}$ contains a binary relation that satisfies the axioms of equality, $\Omega\mathfrak{A}$ coincides with this relation. In fact, a structure $\mathfrak{A}$ is said to be *reduced* if $Co\,\mathfrak{A} = \{\Delta_A\}$ or, equivalently, if $\Omega\mathfrak{A} = \Delta_A$. So if $\mathcal{L}$ has equality, then any structure over $\mathcal{L}$ is reduced.

Examples. Let $\mathcal{L}$ be a language with some (possibly none) function symbols and a sole relation symbol, of arity 2. We are going to describe the Leibniz congruence of four types of structures over $\mathcal{L}$ which we use frequently in the sequel. For this, assume $\mathbf{A}$ is an $\mathcal{L}$-algebra. We claim that a congruence ϕ on $\mathbf{A}$ is compatible with a binary relation R on A iff $\phi \cdot R \cdot \phi \subseteq R$, where $\cdot$ denotes the relative product of any two binary relations; the proof is straightforward and is provided for instance in [12, Prop. 5.7]. Using this fact, it is easy to conclude the following: for any binary relation θ on A, the Leibniz congruence of $\mathfrak{A} = \langle \mathbf{A}, \theta\rangle$ is

(1) $\bigvee\{\phi \in Co\,\mathbf{A} : \phi \subseteq \theta\}$, if θ is an equivalence relation on A;

(2) $\bigvee\{\phi \in Co\,\mathbf{A} : \phi \cdot \theta \cdot \phi \subseteq \theta\}$, if θ is a reflexive, symmetric relation compatible with the functions on $\mathbf{A}$, i.e., such that for any funtion symbol f and any $\mathbf{a}, \mathbf{b} \in A^{\rho(f)}$, $f^{\mathbf{A}}\mathbf{a} \equiv f^{\mathbf{A}}\mathbf{b}\,(\theta)$ whenever $\mathbf{a} \equiv \mathbf{b}\,(\theta)$[10];

(3) $\theta \cap \theta^{-1}$, if θ is a *quasi-order* on $\mathbf{A}$, i.e., a reflexive, transitive relation compatible with the functions on $\mathfrak{A}$;

(4) θ, if θ is a congrunece on $\mathbf{A}$. ⊣

The next purpose is to see that the notion of Leibniz congruence of a structure is just the algebraic counterpart of a purely logical concept, viz. the concept of equality in the sense of Leibniz. To this goal, let us call *Leibniz formula* over $\mathfrak{L}$ any formula $\psi(x, y)$ with two free variables such that, for some atomic $\mathcal{L}$-formula $\varphi = \varphi(x, z_1, \ldots, z_k)$ with at least one free variable x,

$$\psi(x, y) := \forall z_1 \ldots \forall z_k(\varphi(x, z_1, \ldots, z_k) \leftrightarrow \varphi(y, z_1, \ldots, z_k)).$$

Then we have

THEOREM 2.1.2. *If $\mathfrak{A}$ is an $\mathfrak{L}$-structure, then $a \equiv b\,(\Omega\mathfrak{A})$ iff $\mathfrak{A} \vDash \psi(x, y)\ [a, b]$ for all $a, b \in A$ and all Leibniz formulas $\psi(x, y)$ over $\mathcal{L}$.*

Proof. Let θ be the set of all pairs $\langle a, b\rangle$ such that $\mathfrak{A} \vDash \psi(x, y)\ [a, b]$ for all Leibniz formulas ψ. One easily verifies that θ is an equivalence relation. In order to see that it is actually a congruence, let f be any n-ary function symbol. We have to show that, if $\mathbf{a} \equiv \mathbf{b}\ (\theta)$, where $\mathbf{a}, \mathbf{b} \in A^n$, then $f^{\mathfrak{A}}\mathbf{a} \equiv f^{\mathfrak{A}}\mathbf{b}\,(\theta)$. Since θ is transitive, it suffices to prove the condition

$$(2.1) \qquad f^{\mathfrak{A}}b_1 \ldots b_{i-1}a_i a_{i+1} \ldots a_n \equiv f^{\mathfrak{A}}b_1 \ldots b_{i-1}b_i a_{i+1} \ldots a_n\,(\theta),$$

for all $i \geq 1$. Let $\psi(x, y)$ be any Leibniz formula and select any parwise distinct variables $w_1, \ldots, w_{n-1}$ not in ψ. Let ϑ be the formula that results of simultaneously substituting $fw_1 \ldots w_{i-1}xw_i \ldots w_{n-1}$ for x and $fw_1 \ldots w_{i-1}yw_i \ldots w_{n-1}$ for y in $\psi(x, y)$. Then

$$\mathfrak{A} \vDash \psi(x, y)\ [f^{\mathfrak{A}}b_1 \ldots b_{i-1}a_i a_{i+1} \ldots a_n, f^{\mathfrak{A}}b_1 \ldots b_{i-1}b_i a_{i+1} \ldots a_n] \quad \text{iff}$$
$$\mathfrak{A} \vDash \vartheta(x, y, w_1, \ldots, w_{n-1})\ [a_i, b_i, b_1, \ldots, b_{i-1}, a_{i+1}, \ldots, a_n].$$

Hence, since $\forall w_1 \ldots \forall w_{n-1}\vartheta(x, y, w_1, \ldots, w_{n-1})$ is again a Leibniz formula over $\mathcal{L}$ and $a_i \equiv b_i(\theta)$, the second condition holds and (2.1) is proved.

Assume now that r is an n-ary relation symbol of $\mathcal{L}$ and that $\mathbf{a} \in r^{\mathfrak{A}}$, $\mathbf{a} \equiv \mathbf{b}\ (\theta)$ hold for some members $\mathbf{a} = \langle a_1, \ldots, a_n\rangle$, $\mathbf{b} = \langle b_1, \ldots, b_n\rangle$ of A^n. Take φ to be the atomic formula $rz_1 \ldots z_{i-1}xz_i \ldots z_{n-1}$. Then $\mathfrak{A} \vDash \psi(x, y)\ [a_i, b_i]$ and, consequently, we have the equivalence $\langle b_1 \ldots b_{i-1}a_i a_{i+1} \ldots a_n\rangle \in r^{\mathfrak{A}}$ iff $\langle b_1 \ldots b_{i-1}b_i a_{i+1} \ldots a_n\rangle \in r^{\mathfrak{A}}$. This is true for all $i \geq 1$; so that $\mathbf{a} \in r^{\mathfrak{A}}$ implies $\mathbf{b} \in r^{\mathfrak{A}}$, and θ is a congruence on $\mathfrak{A}$.

[10]Such relations are known in the literature as *tolerance relations*. Introduced by Zelinka in his 1970 paper [125], they have attracted some attention in the last decade; for exemple, see [21, 22, 23, 104] and other references there.

Finally, suppose Φ is another congruence on $\mathfrak{A}$, $a \equiv b\ (\Phi)$ and $c_1, \ldots, c_k \in A$. If $t_1, \ldots, t_n$ are terms over $\mathcal{L}$ whose free variables are among $x, z_1, \ldots, z_k$ then

$$t_i^{\mathfrak{A}}(a, c_1, \ldots, c_k) \equiv t_i^{\mathfrak{A}}(b, c_1, \ldots, c_k)\,(\Phi), \quad \text{for } 1 \leq i \leq n.$$

Thus, if r is any n-ary relation symbol, the compatibility of Φ with relations implies

$$\mathfrak{A} \models rt_1(x,\bar{z}) \ldots t_n(x,\bar{z}) \leftrightarrow rt_1(y,\bar{z}) \ldots t_n(y,\bar{z})\ [a, b, c_1, \ldots, c_k],$$

where $\bar{z} = \langle z_1, \ldots, z_k \rangle$. As a result, $\mathfrak{A} \models \psi(x,y)\ [a,b]$ for each Leibniz formula ψ and $a \equiv b(\theta)$. This shows that θ is the greatest element of $Co\,\mathfrak{A}$ and completes the proof. ∎

An easy induction on the complexity of the formulas allows to prove that the atomic predicate φ in the Leibniz formulas can be replaced by arbitrary elementary predicates. Therefore, we actually have the following logical description of the Leibniz congruence on a structure.

COROLLARY 2.1.3. *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and let $a, b \in A$. Then $a \equiv b\ (\Omega\mathfrak{A})$ iff for any first-order formula $\varphi := \varphi(x, z_1, \ldots, z_k)$ over $\mathcal{L}$ and any $c_1, \ldots, c_k \in A$,*

$$\mathfrak{A} \models \varphi(x, z_1, \ldots, z_k)\ [a, c_1, \ldots, c_k] \ \text{ iff } \ \mathfrak{A} \models \varphi(x, z_1, \ldots, z_k)\ [b, c_1, \ldots, c_k]. \ ∎$$

In light of the previous result, the binary relation $\Omega\mathfrak{A}$ has a double meaning; from an algebraic viewpoint, it is the greatest congruence on $\mathfrak{A}$, whereas for its logical content it represents a weak form of equality in the model $\mathfrak{A}$. So, we shall use indistinctly the expressions Leibniz congruence of $\mathfrak{A}$ and *Leibniz equality in* $\mathfrak{A}$ to mean the relation $\Omega\mathfrak{A}$[11].

The following result is quite simple but it will be used later on.

PROPOSITION 2.1.4. *Let $\mathcal{L}'$ be an expansion of the language $\mathcal{L}$ obtained by adding some new constants and relation symbols, and let $\mathfrak{A}'$ be any structure over $\mathcal{L}'$. If the interpretations in $\mathfrak{A}'$ of all the new relation symbols in $\mathcal{L}'$ are elementary definible in $\mathcal{L}$, then $\Omega\mathfrak{A}' = \Omega(\mathfrak{A}' \upharpoonright \mathcal{L})$.*

Proof. The inclusion $\Omega(\mathfrak{A}' \upharpoonright \mathcal{L}) \subseteq \Omega\mathfrak{A}'$ is clear. To see the converse we use that any atomic formula φ over $\mathcal{L}'$ has associated a first-order $\mathcal{L}$-formula φ', in the same free variables, such that

$$\mathfrak{A}' \models \forall z_1 \ldots \forall z_k(\varphi(x, z_1, \ldots, z_k) \leftrightarrow \varphi(y, z_1, \ldots, z_k))\ [a,b]$$
$$\text{iff } \ \mathfrak{A}' \upharpoonright \mathcal{L} \models \forall z_1 \ldots \forall z_k(\varphi'(x, z_1, \ldots, z_k) \leftrightarrow \varphi'(y, z_1, \ldots, z_k))\ [a,b].$$

[11] Observe that the Leibniz equality in $\mathfrak{A}$ does not coincide with another form of equality relation that naturally follows from the Leibniz Principle, namely, the relation according to which two members a, b of the universe of $\mathfrak{A}$ are equivalent if for every $\mathcal{L}$-formula with exactly one free variable, $\phi := \phi(x)$, the following holds: $\mathfrak{A} \models \phi(x)\ [a]$ iff $\mathfrak{A} \models \phi(x)\ [b]$. As it was pointed out to the author by Czelakowski, the latter is not in general a congruence on the algebra $\mathbf{A}$, and $\Omega\mathfrak{A}$ is just the least congruence on $\mathbf{A}$ that includes it.

So the inclusion follows trivially. ■

There is an interesting relationship between strong homomorphisms and congruences that arises from a further notion. For any homomorphism $h : \mathfrak{A} \to \mathfrak{B}$, define the *kernel of h* as the set $Ker\, h = h^{-1}\Delta_B$. Then we have the next result, which is the first step to provide an external (categorical) characterization of the notion of congruence.

LEMMA 2.1.5. *Let $\mathfrak{A}, \mathfrak{B}$ be two $\mathcal{L}$-structures and let $h : \mathbf{A} \to \mathbf{B}$. If $h : \mathfrak{A} \to_s \mathfrak{B}$ then $Ker\, h \in Co\,\mathfrak{A}$. Conversely, if $Ker\, h \in Co\,\mathfrak{A}$ and $r^{\mathfrak{B}} = hr^{\mathfrak{A}}$ for all $r \in R$, then $h : \mathfrak{A} \to_s \mathfrak{B}$.*

Proof. Clearly $Ker\, h \in Co\,\mathbf{A}$. Let r be any relation symbol and let $\mathbf{a}, \mathbf{a}' \in A^{\rho(r)}$ be such that $\mathbf{a} \in r^{\mathfrak{A}}$ and $\mathbf{a} \equiv \mathbf{a}'\,(Ker\, h)$. Since h is strong, $h\mathbf{a}' = h\mathbf{a} \in r^{\mathfrak{B}}$ implies $\mathbf{a}' \in r^{\mathfrak{A}}$, and so $Ker\, h$ belongs to $Co\,\mathfrak{A}$. Assume now that $Ker\, h \in Co\,\mathfrak{A}$ and $r^{\mathfrak{B}} = hr^{\mathfrak{A}}$ for all $r \in R$. If $h\mathbf{a} \in r^{\mathfrak{B}}$, there is an element $\mathbf{a}' \in r^{\mathfrak{A}}$ such that $h\mathbf{a} = h\mathbf{a}'$. Thus $\mathbf{a} \equiv \mathbf{a}'\,(Ker\, h)$ and, consequently, $\mathbf{a} \in r^{\mathfrak{A}}$ is equivalent to $h\mathbf{a} \in r^{\mathfrak{B}}$. ■

Some natural questions concerning the relation between the lattice of congruences of certain structures and those of their substructures, homomorphic images and products may arise. We shall not enter into this subject, but merely state two results that tell us something in this sense and that will become useful later on.

LEMMA 2.1.6. *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and $\mathfrak{B} \subseteq \mathfrak{A}$. For every binary relation $\theta \subseteq A^2$, define $\theta_B = \theta \cap B^2$. Then, $\theta \in Co\,\mathfrak{A}$ implies $\theta_B \in Co\,\mathfrak{B}$.*

Proof. Clearly θ_B is a congruence on the underlying algebra of $\mathfrak{B}$. The fact that θ_B is compatible with the relations of $\mathfrak{B}$ follows directly from the definition of substructure. ■

LEMMA 2.1.7. *For all $h : \mathfrak{A} \to_s \mathfrak{B}$, $\phi \in Co\,\mathfrak{B}$ implies $h^{-1}\phi \in Co\,\mathfrak{A}$. If, moreover, h is a reductive homomorphism, then $\theta \in Co\,\mathfrak{A}$ and $\theta \supseteq Ker\, h$ implies $h\theta \in Co\,\mathfrak{B}$.*

Proof. Suppose that h is strong and ϕ is a congruence on $\mathfrak{B}$. Obviously, $h^{-1}\phi$ is an equivalence relation on A. Let $\mathbf{a} = \langle a_1, \dots, a_n\rangle, \mathbf{a}' = \langle a'_1, \dots, a'_n\rangle \in A^n$ be such that $\mathbf{a} \equiv \mathbf{a}'\,(h^{-1}\phi)$. Then $h\mathbf{a} \equiv h\mathbf{a}'\,(\phi)$, so that $f^{\mathfrak{B}}ha_1, \dots, ha_n \equiv f^{\mathfrak{B}}ha'_1 \dots ha'_n\,(\phi)$ for all n-ary function symbol f and, consequently, $f^{\mathfrak{A}}a_1, \dots, a_n \equiv f^{\mathfrak{A}}a'_1 \dots a'_n\,(h^{-1}\phi)$. Hence, $h^{-1}\phi$ is a congruence on $\mathbf{A}$. Moreover, if r is an n-ary relation symbol, $\mathbf{a} \in r^{\mathfrak{A}}$ and $\mathbf{a} \equiv \mathbf{a}'\,(h^{-1}\phi)$ imply $h\mathbf{a} \in r^{\mathfrak{B}}$ and $h\mathbf{a} \equiv h\mathbf{a}'\,(\phi)$. Then, since ϕ is compatible with relations, $h\mathbf{a}' \in r^{\mathfrak{B}}$, which entails $\mathbf{a}' \in r^{\mathfrak{A}}$ by the strongness condition on h. As a result, $h^{-1}\phi$ is compatible with relations and hence a congruence on $\mathfrak{A}$.

The proof of the converse is also a straighforward consequence from the assumptions. Now the fact that h is surjective and $Ker\ h \subseteq \theta$ is used to show that $h\theta$ is still an equivalence relation. Let us see that $h\theta$ is transitive as example. Take $b_1, b_2, b_3 \in B$ such that

$$b_1 \equiv_{h\theta} b_2 \equiv_{h\theta} b_3.$$

Then $ha_1 = b_1$, $ha_2 = b_2 = ha'_2$ and $ha_3 = b_3$ for some $a_1, a_2, a'_2, a_3 \in A$ satisfying that $\langle a_1, a_2\rangle, \langle a'_2, a_3\rangle \in \theta$; hence, as $\langle a_2, a'_2\rangle \in Ker\ k \subseteq \theta$, we have $a_1 \equiv a_3\,(\theta)$ and consequently $\langle b_1, b_3\rangle \in h\theta$. So, $h\theta$ is transitive. To prove that $h\theta$ is a congruence, suppose $\mathbf{b} = \langle b_1, \ldots, b_n \rangle, \mathbf{b}' = \langle b'_1, \ldots, b'_n \rangle \in B^n$ are such that $\mathbf{b} \equiv \mathbf{b}' \ (h\theta)$. This means that for all $1 \leq i \leq n$ there exist $a_i, a'_i \in A$ satisfying

$$a_i \equiv a'_i \ (\theta), \quad ha_i = b_i, \quad ha'_i = b'_i.$$

And from here we conclude the desired compatibility condition of $h\theta$ with the functions and relations of $\mathfrak{B}$. ■

The following is an interesting consequence from the last lemma.

THEOREM 2.1.8. *Let $\mathfrak{A}, \mathfrak{B}$ be two $\mathcal{L}$-structures. If $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, the following holds.*
(i) $h^{-1}\Omega\mathfrak{B} = \Omega\mathfrak{A}$;
(ii) $h\Omega\mathfrak{A} = \Omega\mathfrak{B}$.

Proof. Evidently $h^{-1}\Omega\mathfrak{B} \in Co\,\mathfrak{A}$. Assume $\theta \in Co\,\mathfrak{A}$ and let $\theta' = \theta \vee Ker\,h$, where the supremum is taken in the lattice $Co\,\mathfrak{A}$. By Lemma 2.1.7, $h\theta' \in Co\,\mathfrak{B}$, so that $h\theta \subseteq \Omega\mathfrak{B}$. Hence, $\theta \subseteq h^{-1}h\theta \subseteq h^{-1}\Omega\mathfrak{B}$, which proves that $h^{-1}\Omega\mathfrak{B}$ is the greatest congruence on $\mathfrak{A}$. On the other hand, since h is surjective, $\phi \in Co\,\mathfrak{B}$ implies $\phi = hh^{-1}\phi \subseteq h\Omega\mathfrak{A}$. Thus, using 2.1.7, (ii) is also proved. ■

2.2. Quotient Structures and the Isomorphisms Theorems

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and θ a congruence on $\mathfrak{A}$. We construct a new $\mathcal{L}$-structure $\mathfrak{A}/\theta$ on the quotient set $A/\theta = \{a/\theta : a \in A\}$ as follows. For each n-ary function symbol f in F and each $a_1, \ldots, a_n \in A$, we put

$$f^{\mathfrak{A}/\theta} a_1/\theta \ldots a_n/\theta = f^{\mathfrak{A}} a_1 \ldots a_n/\theta;$$

similarly, for each n-ary relation symbol r in R, let

$$r^{\mathfrak{A}/\theta} = \{\langle a_1/\theta, \ldots, a_n/\theta \rangle \in (A/\theta)^n : \langle a_1, \ldots, a_n \rangle \in r^{\mathfrak{A}}\}.$$

Evidently, the interpretations of the symbols as defined above do not depend on the choosen representatives, since θ is supposed to be compatible with functions and relations of $\mathfrak{A}$. Thus $\mathfrak{A}/\theta$ is well defined; it is called the *quotient of* $\mathfrak{A}$ *modulo* θ. From now on, the notations $F_{\mathfrak{A}/\theta}$ and $R_{\mathfrak{A}/\theta}$ will mean the interpretations of the symbols of F and R in $\mathfrak{A}/\theta$, respectively.

The following result provides a converse of Lemma 2.1.5; it shows that every congruence is the kernel of a reductive homomorphism and thus completes the external characterization of congruences on a structure announced before.

PROPOSITION 2.2.1. *Assume $\mathfrak{A}$ is an $\mathcal{L}$-structure and $\theta \in Co\,\mathfrak{A}$. Then the natural mapping π_θ from $\mathfrak{A}$ into $\mathfrak{A}/\theta$ given by $\pi_\theta a = a/\theta$ is a reductive homomorphism such that $Ker\,\pi_\theta = \theta$.* ■

The last proposition can also be used to conclude that quotient structures are reductions. The converse is true again and allows us to state a homomorphism theorem similar to the one that holds in universal algebra when we extend the notion of congruence from algebras to general structures. We base its proof on a new lemma that is important by itself.

LEMMA 2.2.2. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be $\mathcal{L}$-structures and assume that $h : \mathfrak{A} \to \mathfrak{B}$ and $g : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$ satisfy $Ker\, g \subseteq Ker\, h$. Then there exists a homomorphism $k : \mathfrak{C} \to \mathfrak{B}$ such that $h = k \circ g$. Moreover, h is strong iff k is strong.*

Proof. Given $c \in C$, choose $a \in A$ such that $g(a) = c$ and define $k(c) = h(a)$. The condition $Ker\, g \subseteq Ker\, h$ says that k is an algebra homomorphism from $\mathbf{C}$ into $\mathbf{B}$. Indeed, let $c_1, \ldots, c_n \in C$ and $a, a_1, \ldots, a_n \in A$ satisfying that $g(a) = f^{\mathfrak{C}} c_1 \ldots c_n$ and $g(a_i) = c_i$, $1 \leq i \leq n$. Then $\langle a, f^{\mathfrak{A}} a_1 \ldots a_n \rangle \in Ker\, g \subseteq Ker\, h$, so that k is an algebra homomorphism. Finally, if $\mathbf{c} = \langle c_1, \ldots, c_n \rangle \in r^{\mathfrak{C}}$ for some n-ary relation symbol $r \in R$, we have that $\mathbf{a} = \langle a_1, \ldots, a_n \rangle \in r^{\mathfrak{A}}$ and hence $h\mathbf{a} = k\mathbf{c} \in r^{\mathfrak{B}}$. If, in addition, h is strong, then $k\mathbf{c} \in r^{\mathfrak{B}}$ implies $\mathbf{a} \in r^{\mathfrak{A}}$ and consequently $\mathbf{c} \in r^{\mathfrak{C}}$. ∎

THEOREM 2.2.3. (Homomorphism Theorem) *Given any two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$, if $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ then $\mathfrak{A}/Ker\, h \cong \mathfrak{B}$.*

Proof. The proof is a straighforward consequence of Proposition 2.2.1 and the preceding lemma. ∎

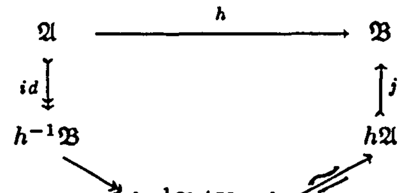
COROLLARY 2.12. *Let $\mathfrak{A}, \mathfrak{B}$ be two $\mathcal{L}$-structures, and let $h : \mathfrak{A} \to \mathfrak{B}$. There exists a descomposition of h,*

$$h = jh'\pi id,$$

where id is the identity function from $\mathfrak{A}$ onto $h^{-1}\mathfrak{B}$, π denotes the natural projection from $h^{-1}\mathfrak{B}$ onto the quotient $h^{-1}\mathfrak{B}/Ker\, h$, j is the inclusion mapping $h\mathfrak{A} \rightarrowtail \mathfrak{B}$ and $h' : h^{-1}\mathfrak{B}/Ker\, h \to h\mathfrak{A}$ is an isomorphism given by $a/Ker\, h \longmapsto ha$. ∎

The situation may be illustrated by the commutative diagram

(**)



which completes the decomposition of an arbitrary homomorphism described in diagram (*). As a result of such decomposition, homomorphic images are best thought of as quotients of filter extensions.

The following are model-theoretic versions of the First and Second Isomorphism Theorems in universal algebra; they will be used occasionally in the sequel.

THEOREM 2.2.5. (First Isomorphism Theorem) *Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, $\mathfrak{B}$ a substructure and θ a congruence on $\mathfrak{A}$. Define $B^\theta = \{a \in A : B \cap a/\theta \neq \emptyset\}$. Then B^θ is the univers of a substructure $\mathfrak{B}^\theta$ of $\mathfrak{A}$ and*

$$\mathfrak{B}/\theta_B \cong \mathfrak{B}^\theta/\theta_{B^\theta}$$

by the mapping $b/\theta_B \longmapsto b/\theta_{B^\theta}$.

Proof. Clearly the mapping $b/\theta_B \longmapsto b/\theta_{B^\theta}$ is an isomorphism between the underlying algebras. Moreover, from the definition of quotient structure, if $\langle b_1, \dots, b_n \rangle \in B^n$ then $\langle b_1/\theta_B, \dots, b_n/\theta_B \rangle \in r^{\mathfrak{B}/\theta_B}$ iff $\langle b_1, \dots, b_n \rangle \in r^{\mathfrak{B}}$. Similarly,

$$\langle b_1/\theta_{B^\theta}, \dots, b_n/\theta_{B^\theta} \rangle \in r^{\mathfrak{B}^\theta/\theta_{B^\theta}} \text{ iff } \langle b_1, \dots, b_n \rangle \in r^{\mathfrak{B}^\theta} = r^{\mathfrak{A}} \cap (b^\theta)^n.$$

Hence, $\langle b_1/\theta_B, \dots, b_n/\theta_B \rangle \in r^{\mathfrak{B}/\theta_B}$ is equivalent to $\langle b_1/\theta_{B^\theta}, \dots, b_n/\theta_{B^\theta} \rangle \in r^{\mathfrak{B}^\theta/\theta_{B^\theta}}$ and the theorem is proved. ∎

THEOREM 2.2.6. (Second Isomorphism Theorem) *Assume* $\mathfrak{A}$ *is an* $\mathcal{L}$*-structure and* $\phi, \theta \in Co\,\mathfrak{A}$ *with* $\theta \subseteq \phi$. *Let* ϕ/θ *denote the congruence* $\pi_\theta\phi$ *on* $\mathfrak{A}/\theta$. *Then*

$$(\mathfrak{A}/\theta)/(\phi/\theta) \cong \mathfrak{A}/\phi$$

by the mapping $a/\theta/\phi/\theta \longmapsto a/\phi$.

Proof. It is also an easy consequence of the definition of quotient structure. ∎

COROLLARY 2.2.7. (Correspondence Theorem) *Let* $\mathfrak{A}$ *be an* $\mathcal{L}$*-structure and* θ *a congruence on* $\mathfrak{A}$. *Let* $[\theta, \Omega\mathfrak{A}]$ *denote the sublattice of* $\mathbf{Co}\,\mathfrak{A}$ *whose carrier is* $\{\phi \in Co\,\mathfrak{A} : \theta \subseteq \phi\}$. *Then* $[\theta, \Omega\mathfrak{A}] \cong \mathbf{Co}\,\mathfrak{A}/\theta$ *by the mapping* $\phi \longmapsto \phi/\theta$. ∎

2.3. Leibniz Quotient of a Structure

For any $\mathcal{L}$-structure $\mathfrak{A} = \langle \mathbf{A}, R_{\mathfrak{A}} \rangle$, the quotient of $\mathfrak{A}$ modulo $\Omega\mathfrak{A}$ is called the *Leibniz quotient of* $\mathfrak{A}$. For simplicity, we write $\mathfrak{A}^*$ to mean $\mathfrak{A}/\Omega\mathfrak{A}$; $\mathbf{A}^*$ denotes the underlying algebra of $\mathfrak{A}^*$ and a^* is used sometimes to mean the equivalence class $a/\Omega\mathfrak{A}$, for each $a \in A$. Given two $\mathcal{L}$-structures $\mathfrak{A}, \mathfrak{B}$ and a mapping $h : A \to B$, we denote by h^* the correspondence $a^* \longmapsto (ha)^*$ induced by h between the quotient sets A^* and B^*; it is not in general a well-defined mapping.

By Proposition 2.2.1, $\mathfrak{A}^*$ is a reduction of $\mathfrak{A}$. Moreover, according to the Correspondence Theorem stated in 2.2.7, $\mathfrak{A}^*$ is a reduced structure so that $\mathfrak{A}^{**} \cong \mathfrak{A}^*$. The next results show that actually the Leibniz quotient $\mathfrak{A}^*$ is minimal in the sense that it is a reduction of any other reduction of $\mathfrak{A}$.

PROPOSITION 2.3.1. *For each* $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, *the correspondence* h^* *defines an isomorphism between* $\mathfrak{A}^*$ *and* $\mathfrak{B}^*$. *More generally, if* $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ *then* $h^* : (h^{-1}\mathfrak{B})^* \cong \mathfrak{B}^*$.

Proof. Assume $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. By 2.1.8, $a \equiv a'\,(\Omega\mathfrak{A})$ iff $ha \equiv ha'\,(\Omega\mathfrak{B})$, so that h^* is well defined and one-one. Moreover, for any $\mathbf{a} = \langle a_1, \dots, a_n \rangle \in A^n$, if $\mathbf{a}^* = \langle a_1^*, \dots, a_n^* \rangle$ then

$$h^* f^{\mathfrak{A}^*} \mathbf{a}^* = (h f^{\mathfrak{A}} \mathbf{a})^* = (f^{\mathfrak{B}} h \mathbf{a})^* = f^{\mathfrak{B}^*} h^* \mathbf{a}^*.$$

Also, since h is strong, $\mathbf{a}^* \in r^{\mathfrak{A}^*}$ iff $h^*\mathbf{a}^* \in r^{\mathfrak{B}^*}$. Hence h^* is an isomorphism.

To see that $h^* : (h^{-1}\mathfrak{B})^* \cong \mathfrak{B}^*$ if h is an onto homomorphism, it suffices to apply the decomposition stated in diagram (∗). ■

COROLLARY 2.3.2. *Let* $\mathfrak{B}$ *an* $\mathcal{L}$*-structure. If* $\mathfrak{B}$ *is a reduction of* $\mathfrak{A}$*, the Leibniz quotient* $\mathfrak{A}^*$ *is a reduction of* $\mathfrak{B}$*.*

Also, if $\mathfrak{A} \cong \mathfrak{B}$*, then* $\mathfrak{A}^* \cong \mathfrak{B}^*$*.* ■

The importance of the Leibniz quotients rests on the fact that the Leibniz equality in them coincides with the common equality relation.

3. Semantics for First-Order Logic without Equality

The development in the last Chapter shows that the quotient of a structure modulo a congruence as defined formerly entails a process of identifying some of the elements of its carrier when they have exactly the same elementary properties expressable in $\mathfrak{L}$. In particular, this identification is carried out as far as possible in the Leibniz quotient: any two elements in this case are going to be the same in the quotient if and only if they are "equal" in the above sense, i.e., they can be mutually replaced in any elementary predicate with no change on truth (see Corollary 2.1.3).

All this suggests that the class of reduced structures is enough to give a complete semantics for first-order logic. And certainly this is the case. In this Chapter we discuss the semantical meaning of reductions and expansions and state both reduced and nonreduced structures as building blocks of two distinct complete semantics for languages without equality. Also, we supply some examples of elementary theories that serve to motivate the ultimate issue that underlies the present work, and which is posed at the end of the Chapter.

3.1. Elementary Homomorphisms and Reductions

The basic logical relation between structures is provided by the notion of elementary equivalence; remember from classical model theory that two structures $\mathfrak{A}, \mathfrak{B}$ over $\mathfrak{L}$ are *elementary equivalent* iff every $\mathcal{L}$-sentence true in $\mathfrak{A}$ is also true in $\mathfrak{B}$, and viceversa. This relationship is usually denoted by $\equiv$ and it can be easily proved that $\mathfrak{A} \cong \mathfrak{B}$ entails $\mathfrak{A} \equiv \mathfrak{B}$. For our purposes, however, we need a stronger form of elementary equivalence, which also comes from classical model theory. We say that $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$, in symbols $\mathfrak{A} \subseteq_e \mathfrak{B}$, iff $\mathfrak{A} \subseteq \mathfrak{B}$ and for any formula φ and any assignment g of elements of A to $\mathcal{L}$-terms, the equivalence $\mathfrak{A} \models \varphi\, [g]$ iff $\mathfrak{B} \models \varphi\, [g]$ holds. Still extending this notion, a homomorphism $h : \mathfrak{A} \rightarrow \mathfrak{B}$ is said to be *elementary*, in symbols $h : \mathfrak{A} \rightarrow_e \mathfrak{B}$, iff for any formula φ and any assignment g, $\mathfrak{A} \models \varphi\, [g]$ iff $\mathfrak{B} \models \varphi\, [h \circ g]$. Evidently, if $h : \mathfrak{A} \rightarrow_e \mathfrak{B}$ then $\mathfrak{A} \equiv \mathfrak{B}$ and, as a result, $\mathfrak{A} \subseteq_e \mathfrak{B}$ implies $\mathfrak{A} \equiv \mathfrak{B}$.

The next proposition will be used several times in the sequel; its converse is not true but a weaker implication is contained in Corollary 3.1.4 below.

PROPOSITION 3.1.1. *Every reductive homomorphism is elementary.*

Proof. Let h be a reductive homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. We claim that $\mathfrak{A} \models \varphi\,[g]$ iff $\mathfrak{B} \models \varphi\,[h \circ g]$ for every formula φ over $\mathcal{L}$ and every assignment g. The proof goes by induction on the logical complexity of φ.

Clearly the statement is true if φ is an atomic formula. Moreover, the induction step is obvious when φ is a negation or a conjunction. Hence, suppose that φ is $\exists x\psi(x)$ for some other formula ψ. Then $\mathfrak{A} \models \exists x\psi(x)\,[g]$ iff $\mathfrak{A} \models \psi(x)\,[g(a/x)]$ for some $a \in A$, which is equivelent, by the induction hypothesis, to the condition $\mathfrak{B} \models \psi(x)\,[h \circ g(a/x)]$ for some $a \in A$. But

$$\mathfrak{B} \models \psi(x)\,[h \circ g(a/x)] \text{ iff } \mathfrak{B} \models \psi(x)\,[(h \circ g)(ha/x)].$$

Thus, the fact that h is surjective completes the proof. ∎

COROLLARY 3.1.2. *Let $\mathfrak{A}, \mathfrak{B}$ be two arbitrary $\mathcal{L}$-structures. If $\mathfrak{B}$ is a reduction of $\mathfrak{A}$, then $\mathfrak{A} \equiv \mathfrak{B}$.* ∎

Note that if the language $\mathcal{L}$ has equality then any elementary homomorphism is an embedding, so that our definition may be formulated by saying that a map $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ is elementary if h is an isomorphism of $\mathfrak{A}$ onto an elementary substructure of $\mathfrak{B}$. But this is not true if $\mathcal{L}$ does not involve the equality symbol $\approx$. We are going to examine what happens for languages without equality. To start with we need some preliminaries.

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, and let $\mathcal{L}_A$ be an A-expansion of $\mathcal{L}$, i.e., the language obtained from $\mathcal{L}$ by adding new distinct individual constants c_a for all $a \in A$. Following a common notation, all over this Section we use $\bar{a}$ to mean the sequence of elements of A according to a certain well order on A, and $\bar{c}$ to mean the corresponding sequence of constants. Structures over $\mathcal{L}_A$ are denoted $(\mathfrak{B}, b_a)_{a \in A}$, where $\mathfrak{B}$ is an structure over $\mathcal{L}$ and b_a is a member of B for each $a \in A$.

As usual, we call *diagram of* $\mathfrak{A}$, denoted $D\mathfrak{A}$, the set of all atomic sentences and negations of atomic sentences over $\mathcal{L}_A$ which hold in $(\mathfrak{A}, a)_{a \in A}$. We define the *Leibniz diagram of* $\mathfrak{A}$, and denote it by $D_l\mathfrak{A}$, as the set that results from $D\mathfrak{A}$ by adding all $\mathcal{L}_A$-sentences of the form $\psi(t, t')$, for $\psi(x, y)$ a Leibniz $\mathcal{L}$-formula and t, t' closed terms of $\mathcal{L}_A$ (i.e., terms constructed only from constants and function symbols of $\mathcal{L}_A$) such that their interpretations in $(\mathfrak{A}, a)_{a \in A}$ are congruent modulo $\Omega\mathfrak{A}$. This can be expressed as follows:

$$\begin{aligned} D_l\mathfrak{A} = D\mathfrak{A} \cup \{\psi(t, t') \in Sen\,\mathcal{L}_A :\ & \psi(x, y) \text{ is a Leibniz } \mathcal{L}\text{-formula}, \\ & t = t(\bar{c}) \text{ and } t' = t'(\bar{c}), \text{ and } t^{\mathbf{A}}(\bar{a}) \equiv t'^{\mathbf{A}}(\bar{a})\,(\Omega\mathfrak{A})\}. \end{aligned}$$

Finally, we call *elementary diagram of* $\mathfrak{A}$, $D_e\mathfrak{A}$, the set of all sentences of $\mathcal{L}_A$ which hold in $(\mathfrak{A}, a)_{a \in A}$. Note that by 2.1.2, $D_l\mathfrak{A} \subseteq D_e\mathfrak{A}$.

The following theorem shows that, whereas the nature of elementary diagrams does not depend on the presense of the equality symbol in the language, the weaker concept of diagram, as a logical expression of the notion of substructure when $\mathcal{L}$ has equality, need to be replaced by that of Leibniz diagram if $\mathcal{L}$ has no equality. This fact is largely used to prove the main results of Chapter 4.

THEOREM 3.1.3. (Diagrams' Lemma) *The following holds for all $\mathcal{L}$-structures $\mathfrak{A}, \mathfrak{B}$.*

(i) If $(\mathfrak{B}, ha)_{a\in A}$ is a model of $D_l\mathfrak{A}$ then $h^* : \mathfrak{A}^* \rightarrowtail_s \mathfrak{B}^*$.

(ii) If $(\mathfrak{B}, ha)_{a\in A}$ is a model of $D_e\mathfrak{A}$ then $h^* : \mathfrak{A}^* \rightarrowtail_e \mathfrak{B}^*$.

Moreover, implications become equivalences under the assumption that h is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$.

Proof. (i) Assume $(\mathfrak{B}, ha)_{a\in A}$ is a model of $D_l\mathfrak{A}$ and $a^* = a'^*$, for some $a, a' \in A$. By Theorem 2.1.2, $\psi(c_a, c_{a'}) \in D_l\mathfrak{A}$, for all Leibniz $\mathcal{L}$-formula $\psi(x, y)$. Thus $\mathfrak{B} \models \psi(x, y)\ [ha, ha']$ and consequently $(ha)^* = (ha')^*$. This proves that h^* is well defined.

Let us see that h^* is a strong homomorphism. For this, let f be an n-ary function symbol and $\mathbf{a} = \langle a_1, \ldots, a_n\rangle$ a member of A^n. For all Leibniz formula $\psi(x, y)$ we have $\psi(c_{f^{\mathfrak{A}} a_1 \ldots a_n}, fc_{a_1} \ldots c_{a_n}) \in D_l\mathfrak{A}$. Consequently, since $(\mathfrak{B}, ha)_{a\in A}$ is a model of $D_l\mathfrak{A}$,

$$hf^{\mathfrak{A}}a_1 \ldots a_n \equiv f^{\mathfrak{B}}ha_1 \ldots ha_n\,(\Omega\mathfrak{B}),$$

which implies that h^* is a homomorphism between the underlying quotient algebras. Similarly, if r is an n-ary relation symbol, the condition

$$rc_{a_1} \ldots c_{a_n} \in D_l\mathfrak{A} \text{ iff } (\mathfrak{B}, ha)_{a\in A} \models rc_{a_1} \ldots c_{a_n}$$

follows directly from the definition of $D_l\mathfrak{A}$ and the fact that $(\mathfrak{B}, ha)_{a\in A}$ is a model of $D_l\mathfrak{A}$. So, $\mathbf{a} \in r^{\mathfrak{A}^*}$ iff $(h\mathbf{a})^* \in r^{\mathfrak{B}^*}$ and h^* is strong. Finally, Proposition 2.1.5 implies that $Ker\, h^* \in Co\,\mathfrak{A}^*$. Hence, $Ker\, h^* = \Delta_{A^*}$ and h^* is a strong embedding from $\mathfrak{A}^*$ into $\mathfrak{B}^*$.

The reverse implication is an easy consequence from the definitions involved. Given an n-ary relation symbol r and elements $a_1, \ldots, a_n \in A$, the condition $(\mathfrak{A}^*, a^*)_{a\in A} \models rc_{a_1} \ldots c_{a_n}$ is equivalent to $(\mathfrak{B}^*, (ha)^*)_{a\in A} \models rc_{a_1} \ldots c_{a_n}$, because h^* is strong. Hence, $(\mathfrak{A}, a)_{a\in A} \models rc_{a_1} \ldots c_{a_n}$ iff $(\mathfrak{B}, ha)_{a\in A} \models rc_{a_1} \ldots c_{a_n}$. On the other hand, let t and t' be terms over $\mathcal{L}_A$ whose constants are among $c_{a_1}, \ldots, c_{a_k}$ for some $a_1, \ldots, a_k \in A$. If $\psi(t, t') \in D_l\mathfrak{A}$ then

$$t^{\mathfrak{A}}(a_1, \ldots, a_k) \equiv t'^{\mathfrak{A}}(a_1, \ldots, a_k)\,(\Omega\mathfrak{A}).$$

Since h is an homomorphism and h^* an embedding, this implies

$$t^{\mathfrak{B}}(ha_1, \ldots, ha_k) \equiv t'^{\mathfrak{B}}(ha_1, \ldots, ha_k)\,(\Omega\mathfrak{B}),$$

so that, by Theorem 2.1.2, $(\mathfrak{B}, ha)_{a\in A}$ satisfies $\psi(t, t')$. Therefore, $(\mathfrak{B}, ha)_{a\in A}$ is a model of $D_l\mathfrak{A}$. This completes the proof of (i).

(ii) The fact that h^* is elementary follows from Corollary 3.1.2. According to this corollary, $(\mathfrak{A}^*, a^*)_{a\in A} \equiv (\mathfrak{A}, a)_{a\in A}$ and $(\mathfrak{B}^*, (ha)^*)_{a\in A} \equiv (\mathfrak{B}, ha)_{a\in A}$, so that for all $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_k)$ and all $a_1, \ldots, a_k \in A$, we have

$$(\mathfrak{A}^*, a^*)_{a\in A} \models \varphi(c_{a_1}, \ldots, c_{a_k}) \text{ iff } (\mathfrak{B}^*, (ha)^*)_{a\in A} \models \varphi(c_{a_1}, \ldots, c_{a_k}).$$

Thus,

$$\mathfrak{A}^* \vDash \varphi(x_1, \ldots, x_k)\,[a_1^*, \ldots, a_k^*] \text{ iff } \mathfrak{B}^* \vDash \varphi(x_1, \ldots, x_k)\,[(ha_1)^*, \ldots, (ha_k)^*],$$

and the only-if part is proved. The converse is obtained by a similar argument. ∎

COROLLARY 3.1.4. *Let $\mathfrak{A}, \mathfrak{B}$ be $\mathcal{L}$-structures. Then:*
(i) $h : \mathfrak{A} \to_e \mathfrak{B}$ implies $h : \mathfrak{A} \to_s \mathfrak{B}$ and $h\mathfrak{A} \subseteq_e \mathfrak{B}$;
(ii) $h : \mathfrak{A} \to_e \mathfrak{B}$ implies $h^* : \mathfrak{A}^* \rightarrowtail_e \mathfrak{B}^*$ and $h^*\mathfrak{A}^* \subseteq_e \mathfrak{B}^*$.

Proof. Using Proposition 1.2.2, part (i) is easy to check. So, let us show (ii). Since h is elementary, $(\mathfrak{B}, ha)_{a \in A}$ is a model of $D_e\mathfrak{A}$. Hence, by 3.1.3(ii), h^* is an elementary embedding. Moreover, $\mathfrak{A}^* \cong h^*\mathfrak{A}^*$ and $\mathfrak{A}^* \equiv \mathfrak{B}^*$ imply that $h^*\mathfrak{A}^* \equiv \mathfrak{B}^*$. ∎

Observe that, according to the preceding corollary, if $h : \mathfrak{A} \to_e \mathfrak{B}$ then some quotient of $\mathfrak{A}$ is isomorphic to some elementary substructure of $\mathfrak{B}$, whereas the Leibniz quotient $\mathfrak{A}^*$ of $\mathfrak{A}$ is directly isomorphic, as it occurs when the language has equality, to some elementary substructure of the Leibniz quotient $\mathfrak{B}^*$ of $\mathfrak{B}$. We shall see in the next Chapter that much of the difference of the algebraic characterization of certain classes of structures as compared to the characterization of the corresponding classes defined using the equality symbol has to do with this fact.

3.2. Model Classes and Completeness Theorem

Given any elementary theory Γ over the language $\mathcal{L}$ (or more generally, any set of $\mathcal{L}$-formulas), let

$$Mod\,\Gamma = \{\mathfrak{A} \in Str\,\mathcal{L} : \mathfrak{A} \vDash \varphi \text{ for all } \varphi \in \Gamma\},$$
$$Mod^*\Gamma = \{\mathfrak{A} \in Mod\,\Gamma : \mathfrak{A} \text{ is reduced}\}.$$

$Mod\,\Gamma$ and $Mod^*\Gamma$ are called, respectively, the *full model class* and the *reduced model class* of Γ. Their relationship can be expressed as follows. If K is any class of $\mathcal{L}$-structures, define

$$L(\mathsf{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{B}^* \text{ for some } \mathfrak{B} \in \mathsf{K}\}$$

(for simplicity, we often write K^* to mean this class). Then, since $\mathfrak{A}^*$ is elementary equivalent to $\mathfrak{A}$ by Corollary 3.1.2, we have $Mod^*\Gamma = L(Mod\Gamma)$.

The operator L is called *reduction operator.* If K is an arbitrary class of $\mathcal{L}$-structures, we say K is a *full class* whenever it is closed under expansions and reductions[12]; also, we say K is a *reduced class* if it is obtained by applying the reduction operator to some other arbitrary class. In particular, the whole class of reduced $\mathcal{L}$-structures, denoted $Str^*\mathcal{L}$, is called *reduced semantics* to differentiate it

[12] This concept of full class for the case $\mathcal{L}$ has equality amounts to what Mal'cev called *abstract classes*, i.e., classes closed under isomorphims [89]; see the comments following the definition of expansion and reduction in Section 1.2.

from the class $Str\,\mathcal{L}$, named *full semantics*. Observe that, if every member $\mathfrak{A}$ of $\mathbf{K}$ satisfies that $r^{\mathfrak{A}} = \emptyset$ or $r^{\mathfrak{A}} = A^{\rho(r)}$, for all $r \in R$, then $\mathbf{K}^*$ is formed of one-element algebras endowed with empty and/or all relations. A reduced class whose elements are all of this kind is called *trivial*.

Following the standard notation, for any set Σ of $\mathcal{L}$-formulas and any single $\mathcal{L}$-formula φ, we write $\Sigma \vDash \varphi$ to mean that for all $\mathfrak{A} \in Str\,\mathcal{L}$ and all assignment g, $\mathfrak{A} \vDash \varphi\,[g]$ holds whenever $\mathfrak{A} \vDash \Sigma\,[g]$ holds. Similarly, $\Sigma \vDash^* \varphi$ will mean that for all $\mathfrak{A} \in Str^*\mathcal{L}$ and all assignment g, $\mathfrak{A} \vDash \Sigma\,[g]$ implies $\mathfrak{A} \vDash \varphi\,[g]$. At first glance, it can seem that $\vDash^*$ is weaker than $\vDash$, but the following easy result expresses that actually both full and reduced semantics are complete for first-order logic without equality.

THEOREM 3.2.1. (Completeness Theorem) *Let Σ be a set of first-order $\mathcal{L}$-sentences and φ a single first-order $\mathcal{L}$-sentence. Then $\Sigma \vdash \varphi$ iff $\Sigma \vDash \varphi$ iff $\Sigma \vDash^* \varphi$.*

Proof. The first equivalence is just the contents of Gödel's completeness theorem, and the second one is a direct consequence from Corollary 3.1.2. ∎

The same is true for fragments of first-order logic. The result for universal Horn logic (see [68], [92]) is specially interesting for our purposes, particularly in Chapter 10. Another remarkable property that is closely related to the content of the above theorem is the following.

THEOREM 3.2.2. *For all set Γ of $\mathcal{L}$-sentences, $Th(Mod\,\Gamma) = Th(Mod^*\Gamma)$.* ∎

In light of these theorems, both the full and the reduced model classes of an elementary theory are indistinguishible from a semantical viewpoint. But their comparability from an algebraic perspective appears as an interesting problem by itself, motivated by the distinctive algebraic character of the model classes of some special theories arising in algebraic logic. We shall come back to this point in the next section.

3.3. Some Exemples

We have described in Chapter 2 the Leibniz congruence of several structures (see the examples following the definition). These structures are, in fact, models of some elementary theories which have a paradigmatic character, for their model classes are amenable to the common universal algebraic methods but to different degrees. We are going to define now these theories accurately; they have been picked out for their intrinsec mathematical interest and not for their logical content.

Let $\mathcal{L}$ be any first-order language with some function symbols and a sole relation symbol r, of arity 2. Let

(Ref) $r(x, x)$;

(Sym) $r(x, y) \rightarrow r(y, x)$;

(Tra) $r(x, y) \wedge r(y, z) \rightarrow r(x, z)$;

(Com) $r(x,y) \rightarrow r(\tau(z_1,\dots,z_{i-1},x,z_{i+1},\dots,z_k),$
$\tau(z_1,\dots,z_{i-1},y,z_{i+1},\dots,z_k))$, for all $\mathcal{L}$-terms τ and all i.

Our theories consists of some of the forecoming axioms:

Theory of equivalence relations, $\Gamma_{\mathrm{eq},\mathcal{L}}$:
(Ref) + (Sym) + (Tra).

Theory of tolerance relations $\Gamma_{\mathrm{to},\mathcal{L}}$:
(Ref) + (Sym) + (Com).

Theory of quasiorders $\Gamma_{\mathrm{qo},\mathcal{L}}$:
(Ref) + (Tra) + (Com).

Theory of congruences, $\Gamma_{\mathrm{co},\mathcal{L}}$:
(Ref) + (Sym) + (Tra) + (Com).

Usually, we omit the supscript $\mathcal{L}$ and write simply Γ_{eq}, Γ_{to}, Γ_{qo} and Γ_{co}. The full model classes of these theories are denoted respectively by K_{eq}, K_{to}, K_{qo} and K_{co}, again omitting the supscript. We remark that, for the case $\mathcal{L}$ has no function symbol, Γ_{eq} and Γ_{co} coincide.

Some more general elementary theories (rather, strict universal Horn theories) that deserve a special attention are the following ones. Let $\mathcal{Q}$ be a quasivariety of $\mathcal{L}$-algebras, and let Σ be a set of quasi-identities that axiomatizes $\mathcal{Q}$. Let $\mathcal{L}'$ be the language with the same function symbols as $\mathcal{L}$ and a sole relation symbol d, of arity 2, distinct from r, and denote

$$d(\Sigma) = \{\textstyle\bigwedge_{i=1}^m d(s_i,t_i) \rightarrow d(s,t) : \bigwedge_{i=1}^m s_i \approx t_i \rightarrow s \approx t \in \Sigma\}.$$

Then we define the following theories:

Theory of tolerance $\mathcal{Q}$-algebras, $\Gamma_{\mathrm{to},\mathcal{Q}}$:

$$\Gamma_{\mathrm{to},\mathcal{L}} + \Gamma_{\mathrm{co},\mathcal{L}'} + d(\Sigma) + \{r(x,y) \wedge d(x,u) \wedge d(y,v) \rightarrow r(u,v)\}.$$

Theory of quasiordered $\mathcal{Q}$-algebras, $\Gamma_{\mathrm{qo},\mathcal{Q}}$:

$$\Gamma_{\mathrm{qo},\mathcal{L}} + \Gamma_{\mathrm{co},\mathcal{L}'} + d(\Sigma) + \{d(x,y) \rightarrow r(x,y)\}.$$

Theory of ordered $\mathcal{Q}$-algebras, $\Gamma_{\mathrm{po},\mathcal{Q}}$:

$$\Gamma_{\mathrm{qo},\mathcal{L}} + \{\textstyle\bigwedge_{i=1}^m r(s_i,t_i) \wedge r(t_i,s_i) \rightarrow r(s,t) \wedge r(t,s) : \bigwedge_{i=1}^m s_i \approx t_i \rightarrow s \approx t \in \Sigma\}.$$

Theory of $\mathcal{Q}$-algebras, $\Gamma_{\mathcal{Q}}$:

$$\Gamma_{\mathrm{co},\mathcal{L}} + \{\textstyle\bigwedge_{i=1}^m r(s_i,t_i) \rightarrow r(s,t) : \bigwedge_{i=1}^m s_i \approx t_i \rightarrow s \approx t \in \Sigma\}.$$

We follow the same conventions on notation as before; so, notations like $\mathsf{K}_{\mathrm{to},\mathcal{Q}}$ and $\mathsf{K}^*_{\mathrm{to},\mathcal{Q}}$ are selfexplanatory.

By the definition, the theories $\Gamma_{to,Q}$, $\Gamma_{po,Q}$ and Γ_Q are the same as Γ_{to}, Γ_{qo} and Γ_{co}, respectively, when Q is taken to be the whole class of $\mathcal{L}$-algebras. Also, notice that if $\mathfrak{A} = \langle \mathbf{A}, d^{\mathfrak{A}}, r^{\mathfrak{A}} \rangle$ is a member of any one of the full model classes $\mathbf{K}_{to,Q}$ or $\mathbf{K}_{qo,Q}$, then

$$\mathfrak{A}^* = \langle \mathbf{A}/\Omega\mathfrak{A}, \Delta_{A/\Omega\mathfrak{A}}, r^{\mathfrak{A}}/\Omega\mathfrak{A} \rangle,$$

where now $\mathbf{A}/\Omega\mathfrak{A}$ belongs to the quasivariety Q and $r^{\mathfrak{A}}/\Omega\mathfrak{A}$ is a tolerance relation or a quasi-order on $\mathbf{A}/\Omega\mathfrak{A}$, respectively. Also, if $\mathfrak{A} = \langle \mathbf{A}, r^{\mathfrak{A}} \rangle$ is a member of $\mathbf{K}_{po,Q}$, then $\Omega\mathfrak{A} = r^{\mathfrak{A}} \cap (r^{\mathfrak{A}})^{-1}$ and so

$$\mathfrak{A}^* = \langle \mathbf{A}/\Omega\mathfrak{A}, r^{\mathfrak{A}}/\Omega\mathfrak{A} \rangle,$$

where $\mathbf{A}/\Omega\mathfrak{A} \in Q$ and $r^{\mathfrak{A}}/\Omega\mathfrak{A}$ is a partial order on the algebra. Finally, if $\mathfrak{A} = \langle \mathbf{A}, r^{\mathfrak{A}} \rangle$ is a member of $\mathbf{K}_Q$, then

$$\mathfrak{A}^* = \langle \mathbf{A}/\Omega\mathfrak{A}, \Delta_{A/\Omega\mathfrak{A}} \rangle,$$

where $\mathbf{A}/\Omega\mathfrak{A} \in Q$. As a result, the reduced model classes $\mathbf{K}^*_{to,Q}$, $\mathbf{K}^*_{qo,Q}$, $\mathbf{K}^*_{po,Q}$ and $\mathbf{K}^*_Q$ amount essentially to quasivarieties of tolerance algebras, quasiordered algebras, ordered algebras and algebras (or equivalently, ordered algebras with the discrete order), respectively. In particular, when Q is a variety, $\mathbf{K}^*_{or,Q}$ is what Bloom calls a *variety of ordered algebras* [14].

The full model classes $\mathbf{K}_{po,Q}$ and $\mathbf{K}_Q$ are specially important, for they have some, or all, of the algebraic properties we are interested in. In this sense, they fall into two very nice categories of classes. The problem we announced at the begining of the Chapter is suggested by the following question: are there any other types of classes of structures that are still amenable of universal algebraic methods and for which we can prove general forms of certain universal algebraic results? The rest of the work is mainly devoted to answer this question; roughly speaking, we propose to identify those full model classes $\mathbf{K}$ such that $\mathbf{K}$ and $\mathbf{K}^*$ exhibit a metatheory as similar as possible to that of Γ_Q, and to investigate them. We come back to this point in Chapter 5, where we also discuss the nature of the assumptions that set such kind of classes apart.

4. Birkhoff-Type Characterization of some Model Classes

In Chapter 3 it has been established that first-order logic without equality has two complete semantics, viz., the *full* and the *reduced semantics.* Now the aim is to state characterizations, in the style of Birkhoff's Variety Theorem (see, e.g., [20, Thm. II.11.9]), of both the full and the reduced model classes of certain theories; namely, elementary, universal, universal Horn and universal atomic theories[13]. For this purpose, we prove algebraic characterizations for the full classes and examine the commutativity properties of the reduction operator when composed with the different constructions described in Chapter 1; these commutativity properties allow to derive the analogue results for reduced classes. An important aspect, closely related to this problem, is that these latter characterizations can be sharpened by assuming some properties about the Leibniz equality predicate introduced before, but we don not deal with this issue until Chapter 7 below.

4.1. Operators on Classes of Structures

In order to investigate the algebraic properties of classes of structures, let us introduce the operators that correspond to the constructions defined so far and let us state some technical lemmas. For any class K of $\mathcal{L}$-structures, define

$$\begin{aligned}
S(\mathsf{K}) &= \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } \mathfrak{C} \subseteq \mathfrak{B} \text{ for some } \mathfrak{B} \in \mathsf{K}\},\\
S_e(\mathsf{K}) &= \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } \mathfrak{C} \subseteq_e \mathfrak{B} \text{ for some } \mathfrak{B} \in \mathsf{K}\},\\
F(\mathsf{K}) &= \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } \mathfrak{B} \preccurlyeq \mathfrak{C} \text{ for some } \mathfrak{B} \in \mathsf{K}\},\\
H(\mathsf{K}) &= \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{B} \twoheadrightarrow \mathfrak{C} \text{ for some } \mathfrak{B} \in \mathsf{K} \text{ and some } h\},\\
R(\mathsf{K}) &= \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C} \text{ for some } \mathfrak{B} \in \mathsf{K} \text{ and some } h\},\\
E(\mathsf{K}) &= \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{C} \text{ and } h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B} \text{ for some } \mathfrak{B} \in \mathsf{K} \text{ and some } h\},\\
P(\mathsf{K}) &= \{\mathfrak{A} : \mathfrak{A} \cong \textstyle\prod_{i \in I} \mathfrak{A}_i \text{ and } \mathfrak{A}_i \in \mathsf{K} \text{ for all } i \in I\},
\end{aligned}$$

[13]The investigation of similar characterizations for some other theories (e.g., universal-existential theories) is an interesting problem but it is outside the purpose of the present work.

$P_f(\mathsf{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \prod_{i\in I} \mathfrak{A}_i/\mathcal{F},\ \mathfrak{A}_i \in \mathsf{K}$ for all $i \in I$ and $\mathcal{F}$ is a proper filter on $I\}$,
$P_u(\mathsf{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \prod_{i\in I} \mathfrak{A}_i/\mathcal{U},\ \mathfrak{A}_i \in \mathsf{K}$ for all $i \in I$ and $\mathcal{U}$ is an ultrafilter on $I\}$,
$P_{sd}(\mathsf{K}) = \{\mathfrak{A} : \mathfrak{A}_i \in \mathsf{K}$ for all $i \in I$ and $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i\in I} \mathfrak{A}_i$ for some $h\}$,
$P_{fs}(\mathsf{K}) = \{\mathfrak{A} : \mathfrak{A} \cong \mathfrak{B}/\mathcal{F},\ \mathfrak{B} \in P_{sd}(\mathsf{K})$ and $\mathcal{F}$ is a proper filter on $I\}$.

If $\mathcal{O}$ and $\mathcal{O}'$ are any two of these operators, we write $\mathcal{O}\mathcal{O}'$ for their composition and $\mathcal{O} \leq \mathcal{O}'$ to mean that $\mathcal{O}(\mathsf{K}) \subseteq \mathcal{O}'(\mathsf{K})$ for any class K of $\mathcal{L}$-structures. For each $\mathcal{O}$, we also use the short notation $\mathcal{O}^*$ for $L\mathcal{O}$. Note that $\mathcal{O}(\mathsf{K})$, for $\mathcal{O} \in \{P, P_f, P_u, P_{sd}, P_{fs}\}$, are always nonempty classes, even if K is empty, since one can choose $I = \emptyset$ and then they contain the trivial, one-element structure with all relations holding. When necessary, we shall write $\bar{\mathcal{O}}(\mathsf{K})$ to indicate that we only take nomempty index sets in the respective constructions.

LEMMA 4.1.1. *If $\mathcal{O}$ is any one of the operators defined above, $\mathcal{O}^2 = \mathcal{O}$.*

Proof. The equality is easy to verify except when $\mathcal{O}$ is one of the operators that corresponds to a product construction. So let us give an idea of the proof for these cases. Assume first that $\mathcal{O} = P_f$. Obviously, $P_f \leq P_f P_f$. To prove the reverse inclusion, let $\mathcal{F}_j$ be a proper filter on I_j, for all $j \in J$, and $\mathcal{F}$ a proper filter on J. Define a new index set $K = \bigcup_{j\in J}(I_j \times \{j\})$ and let

$$\mathcal{G} = \{\textstyle\bigcup_{j\in J'}(F_j \times \{j\}) : J' \in \mathcal{F} \text{ and } F_j \in \mathcal{F}_j \text{ for each } j \in J'\}.$$

It is easy to see that $\mathcal{G}$ is a proper filter of $Sb(K)$, so it suffices to show that

$$\textstyle\prod_{(i,j)\in K} \mathfrak{A}_{ij}/\mathcal{G} \cong \prod_{j\in J}(\prod_{i\in I_j} \mathfrak{A}_{ij}/\mathcal{F}_j)/\mathcal{F},$$

for all $\mathcal{L}$-structures $\mathfrak{A}_{ij}$. We shall give the precise definition of the isomorphism and omit the details. Let $\mathbf{a} = \langle a_{ij} : (i,j) \in K\rangle \in \prod_{(i,j)\in K} \mathfrak{A}_{ij}$. For each $j \in J$, let $\mathbf{a}^j := \mathbf{a} \restriction I_j \times \{j\}$. Clearly $\mathbf{a}^j \in \prod_{i\in I_j} \mathfrak{A}_{ij}$, so that $\mathbf{a}^j/\mathcal{F}_j \in \prod_{i\in I_j} \mathfrak{A}_{ij}/\mathcal{F}_j$. Define h to be the function given by

$$h(\mathbf{a}/\mathcal{G}) := \langle \mathbf{a}^j/\mathcal{F}_j : j \in J\rangle/\mathcal{F}.$$

Then h is the desired isomorphism.

The above construction specializes trivially to the case that the filters $\mathcal{F}_j$ and $\mathcal{F}$ are respectively $\{I_j\}$ and $\{J\}$, so that we also have a proof that $P = PP$. Moreover, if $\mathcal{F}_j$, for $j \in J$, and $\mathcal{F}$ are all ultrafilters, the set $\mathcal{G}$ is again an ultrafilter of $Sb(K)$, and thus the equality $P_u = P_u P_u$ follows.

Suppose now $\mathcal{O} = P_{sd}$. Let $h_j : \mathfrak{A}_j \rightarrowtail_{sd} \prod_{i\in I_j} \mathfrak{A}_{ij}$ and $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{j\in J} \mathfrak{A}_j$. If K denotes again the set $\bigcup_{j\in J}(I_j \times \{j\})$, we already know that

$$\textstyle\prod_{j\in J}(\prod_{i\in I_j} \mathfrak{A}_{ij}) \cong \prod_{(i,j)\in K} \mathfrak{A}_{ij}.$$

So it is enough to verify that there exists a subdirect embedding from $\mathfrak{A}$ into the product $\prod_{j\in J}(\prod_{i\in I_j} \mathfrak{A}_{ij})$. Indeed, the mapping given by $\mathbf{a} \longmapsto \langle h_j \pi_j h\mathbf{a} : j \in J\rangle$ satisfies the desired condition.

Finally, the idempotency of the operator P_{fs} can be derived almost immediately from the equalities already stated; we omit the details. ∎

The second of the lemmas describes the behavior of the operators E and R when composed with some other operator.

LEMMA 4.1.2. *For all $\mathcal{O} \in \{S, S_e, P, P_f, P_u, P_{sd}\}$, the following is true.*
(i) $\mathcal{O}E \leq E\mathcal{O}$;
(ii) $\mathcal{O}R \leq R\mathcal{O}$, except if $\mathcal{O} \neq P_{sd}$.

Proof. Let K be any class of $\mathcal{L}$-structures.

(i) Suppose first $\mathfrak{A} \in SE(\mathsf{K})$. Let $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{C}$ for some $\mathfrak{B} \in \mathsf{K}$. By 1.2.2, $h\mathfrak{A} \subseteq \mathfrak{B}$. Moreover, the restriction of h to $\mathfrak{A}$ defines a reductive homomorphism from $\mathfrak{A}$ onto $h\mathfrak{A}$, as $h^{-1}r^{h\mathfrak{A}} = h^{-1}hr^{\mathfrak{A}}$ by the definition of $h\mathfrak{A}$. Hence, $\mathfrak{A} \in ES(\mathsf{K})$. This gives the statement for $\mathcal{O} = S$.

To show that $S_eE(\mathsf{K}) \subseteq ES_e(\mathsf{K})$, let $\mathfrak{A}$ be such that $\mathfrak{A} \subseteq_e \mathfrak{C}$ and $h : \mathfrak{C} \twoheadrightarrow_s \mathfrak{B}$ for some $\mathfrak{B} \in \mathsf{K}$. The restriction of h to $\mathfrak{A}$ is still an elementary homomorphism, so by Corollary 3.1.4, $h\mathfrak{A} \subseteq_e \mathfrak{B}$. Therefore, $\mathfrak{A} \in ES_e(\mathsf{K})$.

Let us consider now the case $\mathcal{O} = P$. Suppose $\mathfrak{A} \cong \prod_{i\in I} \mathfrak{A}_i$ and $h_i : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i$ with $\mathfrak{B}_i \in \mathsf{K}$, for any $i \in I$. Define the mapping h from $\prod_{i\in I} A_i$ into $\prod_{i\in I} B_i$ by $h\mathbf{a} = \langle h_i a_i : i \in I\rangle$, where $\mathbf{a} = \langle a_i : i \in I\rangle$ is an arbitrary element of $\prod_{i\in I} A_i$. We already know that h defines a surjective algebra homomorphism, so let us verify the strongness condition. For any n-ary relation symbol $r \in R$, if $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \prod_{i\in I} A_i$ we have

$$\langle \mathbf{a}_1, \ldots, \mathbf{a}_n\rangle \in h^{-1}r^{\prod_{i\in I}\mathfrak{B}_i} \quad \text{iff} \quad \langle h_i a_{i1}, \ldots, h_i a_{in}\rangle \in r^{\mathfrak{B}_i} \text{ for all } i \in I.$$

Thus, since h_i is strong for each $i \in I$, the definition of the product structure implies that this is equivalent to $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n\rangle \in r^{\prod_{i\in I}\mathfrak{A}_i}$. As a result, there is a reductive homomorphism from $\prod_{i\in I} \mathfrak{A}_i$ onto $\prod_{i\in I} \mathfrak{B}_i$, and hence $\mathfrak{A} \in EP(\mathsf{K})$.

The above proof extends easily to the cases $\mathcal{O} = P_f$, P_u. Now, given any proper filter $\mathcal{F}$ (possibly an untrafilter) of $Sb(I)$ we define the canonical mapping $h_{\mathcal{F}}$ from $\prod_{i\in I} A_i/\mathcal{F}$ into $\prod_{i\in I} B_i/\mathcal{F}$ by $h_{\mathcal{F}}(\mathbf{a}/\mathcal{F}) = (h\mathbf{a})/\mathcal{F}$ and we can verify that $h_{\mathcal{F}}$ is again a reductive homomorphism from $\prod_{i\in I} \mathfrak{A}_i/\mathcal{F}$ onto $\prod_{i\in I} \mathfrak{B}_i/\mathcal{F}$. In particular, the strongness condition follows from the equality

$$\{i \in I : \langle h_i a_{i1}, \ldots, h_i a_{in}\rangle \in r^{\mathfrak{B}_i}\} = \{i \in I : \langle a_{i1}, \ldots, a_{in}\rangle \in r^{\mathfrak{A}_i}\}.$$

Finally, assume $\mathfrak{A} \in P_{sd}E(\mathsf{K})$. Let $h_i : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i$ for $\mathfrak{B}_i \in \mathsf{K}$ and $i \in I$, and let $g : \mathfrak{A} \rightarrowtail_{sd} \prod_{i\in I} \mathfrak{A}_i$. Define the map h from $\mathfrak{A}$ into $\prod_{i\in I} \mathfrak{B}_i$ by letting $ha = \langle h_i \circ g(a) : i \in I\rangle$. So defined h is the composition of two strong homomorphism (remember that any subdirect embedding is strong), so it is strong. Hence, $\mathfrak{A}/Ker\, h \cong h\mathfrak{A} \subseteq \prod_{i\in I} \mathfrak{B}_i$. On the other hand, the commutativity of the diagram involved implies that the composition of h with the projection from $\prod_{i\in I} \mathfrak{B}_i$ onto $\mathfrak{B}_i$ is surjective, for all i. Consequently, $h\mathfrak{A}$ is a subdirect product of $\{\mathfrak{B}_i : i \in I\}$ and $\mathfrak{A} \in EP_{sd}(\mathsf{K})$. This completes the proof of part (i).

(ii) Let $\mathfrak{A} \subseteq \mathfrak{C}$ and $h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$ for some $\mathfrak{B} \in \mathsf{K}$. As h is strong, Lemma 1.2.2 says that $\mathfrak{A}' = h^{-1}\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and the restriction of h to $\mathfrak{A}'$ is a reductive homorphism. So $\mathfrak{A}$ belongs to $RS(\mathsf{K})$.

Assume now that $\mathfrak{A}$ is in addition an elementary substructure of $\mathfrak{C}$ and let us see $\mathfrak{A} \in RS_e(\mathsf{K})$. We keep the previous notation up. Let $\varphi = \varphi(x_1, \dots, x_k)$ be any formula over $\mathcal{L}$ and let $a'_1, \dots, a'_k \in A'$. As $h : \mathfrak{A}' \twoheadrightarrow_s \mathfrak{A}$, by 3.1.1 we have $\mathfrak{A}' \vDash \varphi(x_1, \dots, x_k)\ [a'_1, \dots, a'_k]$ iff $\mathfrak{A} \vDash \varphi(x_1, \dots, x_k)\ [ha'_1, \dots, ha'_k]$. Similarly, $\mathfrak{B} \vDash \varphi(x_1, \dots, x_k)\ [a'_1, \dots, a'_k]$ is equivalent to $\mathfrak{C} \vDash \varphi(x_1, \dots, x_k)\ [ha'_1, \dots, ha'_k]$. So, as $\mathfrak{A} \subseteq_e \mathfrak{C}$, we conclude that $\mathfrak{A}' \subseteq_e \mathfrak{B}$ and, consequently, $\mathfrak{A} \in R(\mathfrak{A}') \subseteq RS_e(\mathsf{K})$.

The proof of the inequalities $OR \leq RO$, for $O \in \{P, P_f, P_u\}$, is again straighforward and it is omitted. ■

Algebraically, filter extensions and homomorphic images do not retain some of the nice properties of other constructions. This fact will appear obvious in the study of elementary classes axiomatized by atomic formulas. The next lemma contains some of the properties of filter extensions that we shall need in the investigation of these classes.

LEMMA 4.1.3. *(i)* $EF \leq FE$;
(ii) $FR \leq RF = H$;
(iii) $FS \leq SF$.

Proof. (i) Suppose $\mathfrak{A} \in EF(\mathsf{K})$ and let $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$, where $\mathfrak{B} \preccurlyeq \mathfrak{C}$ for some $\mathfrak{B} \in \mathsf{K}$. Then $h^{-1}r^{\mathfrak{B}} \subseteq r^{\mathfrak{A}}$ for all $r \in R$, so that $\mathfrak{A}$ is a filter extension of $\langle \mathbf{A}, h^{-1}R_{\mathfrak{B}} \rangle$. Moreover, $\langle \mathbf{A}, h^{-1}R_{\mathfrak{B}} \rangle \in E(\mathfrak{B})$. So we conclude $\mathfrak{A} \in FE(\mathsf{K})$.

(ii) Let $\mathfrak{A} \in FR(\mathsf{K})$ and let $h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$ and $\mathfrak{C} \preccurlyeq \mathfrak{A}$ for some $\mathfrak{B} \in \mathsf{K}$. The inverse image of $\mathfrak{A}$ under h is a filter extension of $\mathfrak{B}$ and, consequently, $\mathfrak{A} \in RF(\mathsf{K})$. The equality $RF = H$ has already been proved in Section 2.2.

(iii) Assume $\mathfrak{C} \preccurlyeq \mathfrak{A}$ and $\mathfrak{C} \subseteq \mathfrak{B}$ for some $\mathfrak{B} \in \mathsf{K}$ (in fact, we should suppose $\mathfrak{A}$ is isomorphic to some filter extension of $\mathfrak{C}$ and $\mathfrak{C}$ isomorphic to some substructure of $\mathfrak{B}$, but the same argument goes through). Define $\mathfrak{D} = \langle \mathbf{B}, R_B \cup R_A \rangle$. Then $\mathfrak{B} \preccurlyeq \mathfrak{D}$ and, as can be easily proved, $\mathfrak{A} \subseteq \mathfrak{D}$. Consequently, $\mathfrak{A} \in SF(\mathsf{K})$. ■

LEMMA 4.1.4. *(i)* $EL = ER = RE$;
(ii) $LE = LR = RL = L \leq EL$.

Proof. Let K be again an arbitrary class of $\mathcal{L}$-structures.

(i) Assume $\mathfrak{A} \in RE(\mathsf{K})$ and let $\mathfrak{C}$ be such that $\mathfrak{A}, \mathfrak{B} \in R(\mathfrak{C})$ for some $\mathfrak{B} \in \mathsf{K}$. By 2.3.1, $\mathfrak{C}^* \cong \mathfrak{A}^*$ and $\mathfrak{C}^* \cong \mathfrak{B}^*$. Thus, $\mathfrak{A}, \mathfrak{B} \in E(\mathfrak{C}^*)$ and hence $\mathfrak{A} \in ER(\mathsf{K})$. For the converse, let $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{C}$ and $g : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$, with $\mathfrak{B} \in \mathsf{K}$. From universal algebra we know that there exists an absolutely free algebra $\mathbf{F}$ and surjective homomorphisms $k : \mathbf{F} \twoheadrightarrow \mathbf{A}$ and $f : \mathbf{F} \twoheadrightarrow \mathbf{B}$ such that $h \circ k = g \circ f$. Then it suffices to define $\mathfrak{F} = \langle \mathbf{F}, (h \circ k)^{-1} R_{\mathfrak{C}} \rangle$; the condition $\mathfrak{A}, \mathfrak{B} \in R(\mathfrak{F})$ holds and, consequently, $\mathfrak{A} \in RE(\mathsf{K})$. This proves the equality $ER = RE$.

To show that $ER = EL$, assume as before that $\mathfrak{A}, \mathfrak{B}$ are expansions of some $\mathfrak{C}$, for $\mathfrak{B} \in \mathsf{K}$. Then $\mathfrak{A}, \mathfrak{B}$ are also expansions of $\mathfrak{C}^*$, so that $\mathfrak{A} \in EL(\mathsf{K})$. The opposite inclusion is trivial.

(ii) It is a direct consequence of the definitions involved. ■

The last result of this section states the special commutativity properties of the reduction operator L when composed with other operators. This sort of commutativity is central to derive in following sections the Birkhoff-type characterizations of reduced model classes from the analogue ones for the full classes.

PROPOSITION 4.1.5. *(i) For each operator $\mathcal{O} \in \{S, P, P_f, P_u, P_{sd}\}$, we have $L\mathcal{O} = L\mathcal{O}L$, i.e., $\mathcal{O}^* = \mathcal{O}^*L$.*
(ii) $LS_e = LS_eL = S_eL$.

Proof. (i) Assume first $\mathcal{O} = S$ and let $\mathfrak{A} \in S^*(\mathsf{K})$. Suppose $\mathfrak{A} \cong \mathfrak{C}^*$ for some $\mathfrak{C}$ such that $\mathfrak{C} \subseteq \mathfrak{B}$ and $\mathfrak{B} \in \mathsf{K}$. We need a sublemma whose proof is immediate:

> If $\mathfrak{A} \subseteq \mathfrak{B}$ and $\theta \in Co\,\mathfrak{B}$, then the mapping $a/\theta_A \longmapsto a/\theta$ defines a strong embedding from $\mathfrak{A}/\theta_A$ into $\mathfrak{B}/\theta$, where $\theta_A = \theta \cap A^2$.

In our case, the sublemma says that there is a strong embedding from $\mathfrak{C}/\theta_C$ into $\mathfrak{B}^*$, where $\theta_C = \Omega\mathfrak{B} \cap C^2$. So, by virtue of 1.2.2 and the Homomorphism Theorem stated in 2.2.3, $\mathfrak{C}/\theta_C$ is isomorphic to some substructure of $\mathfrak{B}^*$. On the other hand, Lemma 2.3.1 implies $\mathfrak{C}^* \cong (\mathfrak{C}/\theta_C)^*$ and consequently $\mathfrak{A} \cong (\mathfrak{C}/\theta_C)^*$. Thus, $\mathfrak{A} \in S^*L(\mathsf{K})$. To prove the reverse inclusion, let $\mathfrak{A} \cong \mathfrak{C}^*$ for some $\mathfrak{C}$ such that $\mathfrak{C} \subseteq \mathfrak{B}^*$ and $\mathfrak{B} \in \mathsf{K}$. If $\pi_{\mathfrak{B}}$ denotes the projection from $\mathfrak{B}$ onto $\mathfrak{B}^*$ then $\mathfrak{C}' := \pi_{\mathfrak{B}}^{-1}\mathfrak{C}$ is a substructure of $\mathfrak{B}$ and the restriction of $\pi_{\mathfrak{B}}$ to $\mathfrak{C}'$ is also a reductive homomorphism. Therefore we have $\mathfrak{A} \in S^*(\mathsf{K})$.

Let us suppose now that $\mathcal{O} = P_f$. We shall show that for each family of $\mathcal{L}$-structures $\{\mathfrak{A}_i : i \in I\}$ and each proper filter $\mathcal{F}$ over I, we have

$$(4.1) \qquad (\textstyle\prod_{i\in I} \mathfrak{A}_i/\mathcal{F})^* \cong (\prod_{i\in I} \mathfrak{A}_i^*/\mathcal{F})^*.$$

Under this assumption, the desired equality follows trivially, for $\mathfrak{A} \in P_f^*(\mathsf{K})$ iff $\mathfrak{A} \cong (\prod_{i\in I} \mathfrak{A}_i/\mathcal{F})^*$ for some $\mathfrak{A}_i \in \mathsf{K}$, $i \in I$, and $\mathfrak{A} \in P_f^*L(\mathsf{K})$ iff $\mathfrak{A} \cong (\prod_{i\in I} \mathfrak{A}_i^*/\mathcal{F})^*$ for some $\mathfrak{A}_i \in \mathsf{K}$, $i \in I$. So let us proceed to prove (4.1).

Denote by $\widehat{\mathfrak{A}}$ and $\mathfrak{A}$ respectively the products $\prod_{i\in I} \mathfrak{A}_i^*$ and $\prod_{i\in I} \mathfrak{A}_i$, and define a mapping h from $\widehat{\mathfrak{A}}/\mathcal{F}$ into $(\mathfrak{A}/\mathcal{F})^*$ by $h(\widehat{\mathrm{a}}/\mathcal{F}) = (\mathrm{a}/\mathcal{F})^*$, for every element $\widehat{\mathrm{a}} = \langle a_i^* : i \in I\rangle \in \widehat{A}$. We must first of all show that h is well defined. For this, assume $\widehat{\mathrm{a}}/\mathcal{F} = \widehat{\mathrm{b}}/\mathcal{F}$, i.e., $\{i \in I : a_i^* = b_i^*\} \in \mathcal{F}$, and let us conclude that $(\mathrm{a}/\mathcal{F})^* = (\mathrm{b}/\mathcal{F})^*$. We use Theorem 2.1.2. Given any atomic $\mathcal{L}$-formula $\varphi := \varphi(x, z_1, \dots, z_k)$ and elements $\mathrm{a}_1/\mathcal{F}, \dots, \mathrm{a}_k/\mathcal{F} \in A/\mathcal{F}$, Theorem 1.3.1 says that

$$\mathfrak{A}/\mathcal{F} \vDash \varphi(x, z_1, \dots, z_k)\,[\mathrm{a}/\mathcal{F}, \mathrm{a}_1/\mathcal{F}, \dots, \mathrm{a}_k/\mathcal{F}]$$
$$\text{iff } \{i \in I : \mathfrak{A}_i \vDash \varphi(x, z_1, \dots, z_k)\,[a_i, a_{i1}, \dots, a_{ik}]\} \in \mathcal{F}.$$

On the other hand,

$$\{i \in I : a_i^* = b_i^*\} \cap \{i \in I : \mathfrak{A}_i \vDash \varphi(x, z_1, \dots, z_k)\,[a_i, a_{i1}, \dots, a_{ik}]\}$$
$$\subseteq \{i \in I : \mathfrak{A}_i \vDash \varphi(x, z_1, \dots, z_k)\,[b_i, a_{i1}, \dots, a_{ik}]\}.$$

Therefore, since $\mathcal{F}$ is a filter, $\mathfrak{A}/\mathcal{F} \vDash \varphi(x, z_1, \dots, z_k)$ $[\mathrm{a}/\mathcal{F}, \mathrm{a}_1/\mathcal{F}, \dots, \mathrm{a}_k/\mathcal{F}]$ and $\widehat{\mathrm{a}}/\mathcal{F} = \widehat{\mathrm{b}}/\mathcal{F}$ implies that $\{i \in I : \mathfrak{A}_i \vDash \varphi(x, z_1, \dots, z_k)$ $[b_i, a_{1i}, \dots, a_{ki}]\} \in \mathcal{F}$, which is the same as $\mathfrak{A}/\mathcal{F} \models \varphi(x, z_1, \ldots, z_k)$ $[\mathrm{b}/\mathcal{F}, \mathrm{a}_1/\mathcal{F}, \ldots, \mathrm{a}_k/\mathcal{F}]$, again by 1.3.1. Consequently, under the assumption $\widehat{\mathrm{a}}/\mathcal{F} = \widehat{\mathrm{b}}/\mathcal{F}$ we conclude that

$$\mathfrak{A}/\mathcal{F} \models \varphi(x, z_1, \ldots, z_k) \rightarrow \varphi(y, z_1, \ldots, z_k)\ [\mathrm{a}/\mathcal{F}, \mathrm{b}/\mathcal{F}, \mathrm{a}_1/\mathcal{F}, \ldots, \mathrm{a}_k/\mathcal{F}].$$

The same argument proves the reverse implication. So, as $\varphi(x, z_1, \ldots, z_k)$ and $\mathrm{a}_1/\mathcal{F}, \ldots, \mathrm{a}_k/\mathcal{F}$ are arbitrary, 2.1.2 gives $(\mathrm{a}/\mathcal{F})^* = (\mathrm{b}/\mathcal{F})^*$. To verify that h is a strong homomorphism is a direct consequence of the definitions involved and the proof is omitted. Finally, since h is surjective, Proposition 2.3.1 says that

$$h^* : (\widehat{\mathfrak{A}}/\mathcal{F})^* \cong (\mathfrak{A}/\mathcal{F})^*$$

and hence h^* is the desired isomorphism. This completes the proof of (4.1) and consequently the equalities $\mathcal{O}^* = \mathcal{O}^*L$ for $\mathcal{O} \in \{P, P_f, P_u\}$.

Consider finally the case $\mathcal{O} = P_{sd}$. Let $g : \mathfrak{A} \rightarrowtail_{sd} \prod_{i \in I} \mathfrak{A}_i$ with $\mathfrak{A}_i \in \mathsf{K}$, for $i \in I$, so that $\mathfrak{A}^* \in P^*_{sd}(\mathsf{K})$. We are going to show that $\mathfrak{A}^* \in P^*_{sd}(\mathsf{K}^*)$. Indeed, consider the map h from $\mathfrak{A}$ into $\prod_{i \in I} \mathfrak{A}^*_i$ defined as follows: if $a \in A$ and $ga = \langle a_i : i \in I \rangle$, let

$$ha = \langle a^*_i : i \in I \rangle.$$

Clearly h is a strong homomorphism and its composition with the projection from $\mathfrak{A}_i$ into $\mathfrak{A}^*_i$ is surjective, for all i. Therefore, $\mathfrak{A}/Ker\ h \rightarrowtail_{sd} \prod_{i \in I} \mathfrak{A}^*_i$, and hence $(\mathfrak{A}/Ker\ h)^* \in P^*_{sd}(\mathsf{K}^*)$. Proposition 2.3.1 completes the proof.

(ii) For $\mathcal{O} = S_e$ we reason in very much the same manner as for $\mathcal{O} = S$ and then apply 3.1.2 to obtain $S^*_e = S^*_e L$. To be more precise, let us keep the same notation and assume $\mathfrak{C} \subseteq_e \mathfrak{B}$. Then, since $\mathfrak{C}/\theta \equiv \mathfrak{C}$ and $\mathfrak{B} \equiv \mathfrak{B}^*$, we have $\mathfrak{C}/\theta \equiv \mathfrak{B}^*$ and consequently the embedding from $\mathfrak{C}/\theta$ into $\mathfrak{B}^*$ is elementary. For the converse we just need to check that if $\mathfrak{C} \equiv \mathfrak{B}^*$, the inverse image of $\mathfrak{C}$ under $\pi_{\mathfrak{B}}$ is also elementary equivalent to $\mathfrak{B}$. And this is a straightforward verification

Once we have derived the equality $S^*_e = S^*_e L$, it is easy to see that $S^*_e L = S_e L$. Indeed, let us prove that if $\mathfrak{A} \subseteq_e \mathfrak{B}$ and $\mathfrak{B}$ is reduced then $\mathfrak{A}$ is reduced. Two applications of 2.1.2 give the following: for all $a, b \in A$,

$$\begin{aligned} a \equiv b\ (\Omega\mathfrak{A}) &\text{ iff } \mathfrak{A} \models \psi(x, y)\ [a, b] \text{ for each Leibniz } \mathcal{L}\text{-formula } \psi \\ &\text{ iff } \mathfrak{B} \models \psi(x, y)\ [a, b] \text{ for each Leibniz } \mathcal{L}\text{-formula } \psi \text{iff } a \equiv b\ (\Omega\mathfrak{B}). \end{aligned}$$

So $\Omega\mathfrak{A} \subseteq \Omega\mathfrak{B}$, and consequently if $\mathfrak{B}$ is reduced then $\mathfrak{A}$ is reduced as well. ∎

4.2. Elementary Classes

Remember that a class K of $\mathcal{L}$-structures is said to be elementary if there exists some set Γ of sentences over $\mathcal{L}$ such that $\mathsf{K} = Mod\Gamma$, or equivalently, if $\mathsf{K} = ModTh\,\mathsf{K}$. Thus, the following theorem is an extension to general first-order languages, with or without equality, of a well known result in classical model theory (recall that if $\mathcal{L}$ has equality, expansions and reductions are just isomorphic images, for the reductive homomorphisms are isomorphisms in this case).

THEOREM 4.2.1. *For any class* K *of* $\mathcal{L}$*-structures, the following statements are equivalent.*

(i) K *is an elementary class.*

(ii) K *is closed under* E, R, S_e *and* $\overline{P}_u$.

(iii) $\mathsf{K} = ERS_e\,\overline{P}_u(\mathsf{K}')$, *for some* K'.

Proof. The implication from (i) to (ii) follows directly from Corollary 3.1.2 and Los Theorem on ultraproducts. Moreover, (ii) implies (iii) is trivial, for let $\mathsf{K}' = \mathsf{K}$. So let us show that (iii) entails (i). We claim that K is axiomatizable by $Th\,\mathsf{K}'$, where K' is as in (iii). Note first of all that for any class L of $\mathcal{L}$-structures, $Th\,\mathsf{L} = Th\,\mathcal{O}(\mathsf{L})$ whenever $\mathcal{O} \in \{E, R, S_e, \overline{P}_u\}$, again by 3.1.2 and 1.3.2. Thus, $Th\,\mathsf{K} = Th\,\mathsf{K}'$ and the inclusion $\mathsf{K} \subseteq Mod\,Th\,\mathsf{K}'$ is clear. Assume $\mathfrak{A} \in Mod\,Th\,\mathsf{K}'$ and let us see that $\mathfrak{A} \in \mathsf{K}$. Let $\Delta = Sb_\omega(D_e\mathfrak{A})$. Given any set Φ of $\mathcal{L}_A$-formulas, we write $\Phi(c_{a_1}, \ldots, c_{a_k})$ to mean that the constants c_a, for $a \in A$, appearing in the elements of Φ are among $c_{a_1}, \ldots, c_{a_k}$. We claim that if $\Phi \in \Delta$, then there exist some $\mathfrak{B}_\Phi \in \mathsf{K}'$ and some $\{b_{a,\Phi} : a \in A\} \subseteq B_\Phi$ such that

$$(\mathfrak{B}_\Phi, b_{a,\Phi})_{a\in A} \vDash \bigwedge \Phi(c_{a_1}, \ldots, c_{a_k}).$$

Suppose not. Then, given any $\mathfrak{B} \in \mathsf{K}'$ and any $\{b_a : a \in A\}$, we have

$$(\mathfrak{B}, b_a)_{a\in A} \vDash \neg \bigwedge \Phi(c_{a_1}, \ldots, c_{a_k}).$$

Consequently, the class K' satisfies the $\mathcal{L}$-sentence $\forall x_1 \ldots \forall x_k \neg \bigwedge \Phi(x_1, \ldots, x_k)$, i.e.,

$$\forall x_1 \ldots \forall x_k \neg \bigwedge \Phi(x_1, \ldots, x_k) \in Th\,\mathsf{K}'.$$

But this implies that $\mathfrak{A} \vDash \forall x_1 \ldots \forall x_k \neg \bigwedge \Phi(x_1, \ldots, x_k)$, and hence contradicts the assumption $\Phi \in \Delta$. So the claim does hold.

As usual, define $J_\Phi = \{\Psi \in \Delta : \Phi \subseteq \Psi\}$ for $\Phi \in \Delta$. The family $\{J_\Phi : \Phi \in \Delta\}$ has the finite intersection property, so that there is an ultrafilter $\mathcal{U}$ on Δ such that $J_\Phi \in \mathcal{U}$ for every Φ. Let $\mathfrak{B} = \prod_{\Phi\in\Delta} \mathfrak{B}_\Phi/\mathcal{U}$. Clearly $\mathfrak{B} \in \overline{P}_u(\mathsf{K}')$. Let us show that if $b_a := \langle b_{a,\Phi} : \Phi \in \Delta\rangle \in \prod_{\Phi\in\Delta} B_\Phi$, for each $a \in A$, then

(4.2) $\qquad (\mathfrak{B}, b_a/\mathcal{U})_{a\in A}$ is a model of $D_e\mathfrak{A}$.

Indeed, suppose $\varphi := \varphi(c_{a_1}, \ldots, c_{a_k}) \in D_e\mathfrak{A}$. The following equivalences hold (the second one by Los Theorem):

$$\begin{aligned}
&(\mathfrak{B}, b_a/\mathcal{U})_{a\in A} \vDash \varphi(c_{a_1}, \ldots, c_{a_k}) \\
&\quad \text{iff } \mathfrak{B} \vDash \varphi(x_1, \ldots, x_k)\,[b_{a_1}/\mathcal{U}, \ldots, b_{a_k}/\mathcal{U}] \\
&\quad \text{iff } \{\Phi \in \Delta : \mathfrak{B}_\Phi \vDash \varphi(x_1, \ldots, x_k)\,[b_{a_1,\Phi}, \ldots, b_{a_k,\Phi}]\} \in \mathcal{U} \\
&\quad \text{iff } \{\Phi \in \Delta : (\mathfrak{B}_\Phi, b_{a,\Phi})_{a\in A} \vDash \varphi(c_{a_1}, \ldots, c_{a_k})\} \in \mathcal{U}.
\end{aligned}$$

Also, $J_{\{\varphi\}} \in \mathcal{U}$ and $J_{\{\varphi\}} \subseteq \{\Phi \in \Delta : (\mathfrak{B}_\Phi, b_{a,\Phi})_{a\in A} \vDash \varphi(c_{a_1}, \ldots, c_{a_k})\}$. Therefore, since $\mathcal{U}$ is an ultrafilter, the last condition above is satisfied. So $(\mathfrak{B}, b_a/\mathcal{U})_{a\in A}$ is a model of $\varphi(c_{a_1}, \ldots, c_{a_k})$ and (4.2) is proved.

We now apply Diagrams' Lemma. Then $h^* : \mathfrak{A}^* \rightarrowtail_e \mathfrak{B}^*$, where $ha = (b_a/\mathcal{U})$, and so 3.1.4 gives $\mathfrak{A} \in ES_e R\, \overline{P}_u(\mathsf{K}')$. Lemma 4.1.2(ii) and the assumption that $\mathsf{K} = ERS_e\, \overline{P}_u(\mathsf{K}')$ complete the proof. ■

Given a class K of $\mathcal{L}$-structures, we define the *full elementary class* generated by K, or simply the *elementary class* generated by K, as $\mathsf{K}^E = Mod\,Th\,\mathsf{K}$. Also, we call *reduced elementary class* generated by K the class $(\mathsf{K}^E)^* = L(Mod\,Th\,\mathsf{K})$; observe that $(\mathsf{K}^E)^*$ is not in general elementary. The next corollary describes the way to contruct K^E and $(\mathsf{K}^E)^*$ from K by applying certain operators.

COROLLARY 4.2.2. *The following holds for any class* K *of* $\mathcal{L}$*-structures.*

(i) $\mathsf{K}^E = ERS_e\, \overline{P}_u\, (\mathsf{K})$.

(ii) $(\mathsf{K}^E)^* = S_e\, \overline{P}^*_u(\mathsf{K}^*)$.

Proof. Part (i) follows immediately from the proof of the preceding theorem, for the latter states the equality $Mod\,Th\,\mathsf{K} = ERS_e\, \overline{P}_u(\mathsf{K})$. To see (ii), it suffices to show that $LERS_e\, \overline{P}_u = S_e\, \overline{P}^*_u L$. Indeed,

$$\begin{aligned} LERS_e\, \overline{P}_u &= LS_e\, \overline{P}_u, && \text{by Lemma 4.1.4,} \\ &= S_e\, \overline{P}^*_u L, && \text{by Proposition 4.1.5. ■} \end{aligned}$$

COROLLARY 4.2.3. *A class* K *of reduced* $\mathcal{L}$*-structures is a reduced elementary class (i.e.,* $\mathsf{K} = (\mathsf{K}^E)^*$*) iff it is closed under elementary substructures and reduced ultraproducts modulo ultrafilters over nonempty sets.*

Proof. It is an obvious consequence of Corollary 4.2.2(ii). ■

A reduced elementary class is not in general closed under the operator $\overline{P}_u$. A counterexample is provided by Blok and Pigozzi [12, p. 30]; actually, they give a universal Horn theory and describe an ulraproduct of reduced models which is not reduced[14].

4.3. Universal Classes

Recall that a class K of $\mathcal{L}$-structures is said to be universal if there exists some set Γ of universal sentences over $\mathcal{L}$ such that $\mathsf{K} = Mod\,\Gamma$, or equivalently, if $\mathsf{K} = Mod\,Un\,\mathsf{K}$. The following is the characterization of universal classes defined with or without equality; it simultaneously extends a well known result in classical model theory (see, e.g., [20, Thm. V.2.16]) and a more recent result of Czelakowski [32, Thm. I.7].

[14] An earlier counterexample of Malinowski [90, p. 26] shows that the reduced model class of a universal Horn theory is not in general closed under direct limits of directed systems (see e.g. [25, p. 320] for a definition of direct limit), and this actually implies that the class cannot be closed under the operator P_u.

THEOREM 4.3.1. *For any class* K *of* $\mathcal{L}$*-structures, the following statements are equivalent.*

(i) K *is a universal class.*

(ii) K *is closed under* E, R, S *and* $\overline{P}_u$.

(iii) $\mathsf{K} = ERS\,\overline{P}_u\,(\mathsf{K}')$, *for some* K'.

Proof. The implication from (i) to (ii) follows from 3.1.2, Los Theorem and the additional well-known fact that universal sentences are preserved under substructures. (ii) implies (iii) is again trivial. So, let us concentrate on the proof that (iii) entails (i). We follow a similar argument to the one given for Theorem 4.2.1. In this case, the aim is to see that K is axiomatizable by $Un\,\mathsf{K}'$. Note again the inclusion $\mathsf{K} \subseteq Mod\,Un\,\mathsf{K}'$. Assume $\mathfrak{A} \in Mod\,Un\,\mathsf{K}'$ and let us show $\mathfrak{A} \in \mathsf{K}$. Let $\Delta = Sb(D\mathfrak{A})$. For every $\Phi \in \Delta$, there exist some $\mathfrak{B}_\Phi \in \mathsf{K}'$ and some $\{b_{a,\Phi} : a \in A\} \subseteq B_\Phi$ such that

$$(\mathfrak{B}_\Phi, b_{a,\Phi})_{a\in A} \models \bigwedge \Phi(c_{a_1}, \ldots, c_{a_k});$$

otherwise, we could conclude that $\forall x_1 \ldots \forall x_k \neg \bigwedge \Phi(x_1, \ldots, x_k) \in Un\,\mathsf{K}'$, which is impossible for $\mathfrak{A} \models \exists x_1 \ldots \exists x_k \bigwedge \Phi(x_1, \ldots, x_k)$.

Define as before $J_\Phi = \{\Psi \in \Delta : \Phi \subseteq \Psi\}$ for every $\Phi \in \Delta$, and let $\mathcal{U}$ be an ultrafilter on Δ containing the family $\{J_\Phi : \Phi \in \Delta\}$. Let

$$\mathfrak{B} := \textstyle\prod_{\Phi\in\Delta} \mathfrak{B}_\Phi/\mathcal{U},$$
$$b_a := (b_{a,\Phi} : \Phi \in \Delta) \in \textstyle\prod_{\Phi\in\Delta} B_\Phi, \quad \text{for each } a \in A,$$
$$\mathfrak{C} := \mathfrak{B} \restriction \{b_a/\mathcal{U} : a \in A\}.$$

Clearly $\mathfrak{C} \in S\,\overline{P}_u(\mathsf{K}')$. Let us establish the following lemma:

$$(\mathfrak{C}, b_a/\mathcal{U})_{a\in A} \text{ is a model of } D_l\mathfrak{A}. \tag{4.3}$$

We begin by showing that $(\mathfrak{C}, b_a/\mathcal{U})_{a\in A}$ is a model of $D\mathfrak{A}$. Consider any element $\varphi := \varphi(c_{a_1}, \ldots, c_{a_k})$ of $D\mathfrak{A}$. We have

$$J_{\{\varphi\}} \in \mathcal{U} \text{ and } J_{\{\varphi\}} \subseteq \{\Phi \in \Delta : \mathfrak{B}_\Phi \models \varphi(x_1, \ldots, x_k)\,[b_{a_1,\Phi}, \ldots, b_{a_k,\Phi}]\},$$

so that, as $\mathcal{U}$ is an ultrafilter, the last set belongs to $\mathcal{U}$. So, by virtue of 1.3.2,

$$\mathfrak{B} \models \varphi(x_1, \ldots, x_k)\,[b_{a_1}/\mathcal{U}, \ldots, b_{a_k}/\mathcal{U}]$$

and consequently, since φ is an atomic or negated atomic $\mathcal{L}$-formula,

$$\mathfrak{C} \models \varphi(x_1, \ldots, x_k)\,[b_{a_1}/\mathcal{U}, \ldots, b_{a_k}/\mathcal{U}].$$

Finally, this last condition is equivalent to $(\mathfrak{C}, b_a/\mathcal{U})_{a\in A} \models \varphi(c_{a_1}, \ldots, c_{a_k})$.

Now consider any other element $\psi(t, t')$ of $D_l\mathfrak{A}$, where $t := t(c_{a_1}, \ldots, c_{a_k})$ and $t' := t'(c_{a_1}, \ldots, c_{a_k})$ for some $k > 0$ and some $a_1, \ldots, a_k \in A$. Our definition of Leibniz diagram says that we have

$$t^{\mathbf{A}}(a_1, \ldots, a_k) \cong t'^{\mathbf{A}}(a_1, \ldots, a_k)\ (\Omega\mathfrak{A}).$$

Assume $\psi(x,y) := \forall z_1 \ldots \forall z_p(\varphi(x,z_1,\ldots,z_p) \leftrightarrow \varphi(y,z_1,\ldots,z_p))$ and take arbitrary elements $b_1/\mathcal{U},\ldots, b_p/\mathcal{U}$ of C. We must prove the equivalence

$$(4.4)\quad \begin{aligned}&(\mathfrak{C}, b_a/\mathcal{U})_{a\in A} \models \varphi(t(c_{a_1},\ldots,c_{a_k}), z_1,\ldots,z_p)\,[b_1/\mathcal{U},\ldots,b_p/\mathcal{U}]\\ &\text{iff } (\mathfrak{C}, b_a/\mathcal{U})_{a\in A} \models \varphi(t'(c_{a_1},\ldots,c_{a_k}), z_1,\ldots,z_p)\,[b_1/\mathcal{U},\ldots,b_p/\mathcal{U}].\end{aligned}$$

Since $\mathfrak{C}$ is generated by $\{b_a/\mathcal{U} : a \in A\}$, there exist some $q \geq 0$, some $a'_1,\ldots,a'_q \in A$ and some $\mathcal{L}$-terms $t_1,\ldots,t_p$ in q variables such that

$$b_i/\mathcal{U} := t_i^{\mathbf{C}}(b_{a'_1}/\mathcal{U},\ldots,b_{a'_q}/\mathcal{U}), \qquad \text{for } 1 \leq i \leq p.$$

Thus we have the following chain of equivalences:

$$\begin{aligned}&(\mathfrak{C}, b_a/\mathcal{U})_{a\in A} \models \varphi(t(c_{a_1},\ldots,c_{a_k}), z_1,\ldots,z_p)\,[b_1/\mathcal{U},\ldots,b_p/\mathcal{U}]\\ &\quad\text{iff } \mathfrak{C} \models \varphi(t(x_1,\ldots,x_k), z_1,\ldots,z_p)\,[b_{a_1}/\mathcal{U},\ldots,b_{a_k}/\mathcal{U}, b_1/\mathcal{U},\ldots,b_p/\mathcal{U}]\\ &\quad\text{iff } \mathfrak{C} \models \varphi(t(x_1,\ldots,x_k), t_1(u_1,\ldots,u_q),\ldots,t_p(u_1,\ldots,u_q))\\ &\qquad\qquad [b_{a_1}/\mathcal{U},\ldots,b_{a_k}/\mathcal{U}, b_{a'_1}/\mathcal{U},\ldots,b_{a'_q}/\mathcal{U}]\end{aligned}$$

($u_1,\ldots,u_q$ are additional variables distinct from $x_1,\ldots,x_k$). Take y to be some other new variable and let σ be the atomic $\mathcal{L}$-formula given by

$$\sigma(y,u_1,\ldots,u_q) := \varphi(y, t_1(u_1,\ldots,u_q),\ldots,t_p(u_1,\ldots,u_q)).$$

Then the last condition above can be expressed as

$$\mathfrak{C} \models \sigma(t(x_1,\ldots,x_k), u_1,\ldots,u_q)\,[b_{a_1}/\mathcal{U},\ldots,b_{a_k}/\mathcal{U}, b_{a'_1}/\mathcal{U},\ldots,b_{a'_q}/\mathcal{U}].$$

Hence, since it has already been proved that $(\mathfrak{C}, b_a/\mathcal{U})_{a\in A}$ is a model of $D\mathfrak{A}$, we have $\mathfrak{A} \models \sigma(t(x_1,\ldots,x_k), u_1,\ldots,u_q)\,[a_1,\ldots,a_k,a'_1,\ldots,a'_q]$, i.e.,

$$\mathfrak{A} \models \sigma(x,u_1,\ldots,u_q)\,[t^{\mathbf{A}}(a_1,\ldots,a_k), a'_1,\ldots,a'_q].$$

We now apply the assumption $t^{\mathbf{A}}(a_1,\ldots,a_k) \cong t'^{\mathbf{A}}(a_1,\ldots,a_k)\,(\Omega\mathfrak{A})$, which says that the preceding condition is equivalent to

$$(4.5)\qquad \mathfrak{A} \models \sigma(x,u_1,\ldots,u_q)\,[t'^{\mathbf{A}}(a_1,\ldots,a_k), a'_1,\ldots,a'_q],$$

Finally, backing the argument just made we derive the equivalence of (4.5) with the right-hand side of (4.4):

$$(\mathfrak{C}, b_a/\mathcal{U})_{a\in A} \models \varphi(t'(c_{a_1},\ldots,c_{a_k}), z_1,\ldots,z_p)\,[b_1/\mathcal{U},\ldots,b_p/\mathcal{U}].$$

This completes the proof of (4.3).

Apply now part (i) of Diagrams' Lemma to (4.3). We have that the mapping $a^* \longmapsto (b_a/\mathcal{U})^*$ defines a strong embedding from $\mathfrak{A}^*$ into $\mathfrak{C}^*$. Moreover, h is surjective, so that once more the Homomorphism Theorem gives $\mathfrak{A}^* \cong \mathfrak{C}^*$. As a result, $\mathfrak{A} \in ERS\,\overline{P}_u(\mathsf{K}') = \mathsf{K}$ and the theorem is proved. ∎

Let K be any class of $\mathcal{L}$-structures. The *full universal class* generated by K, or simply the *universal class* generated by K, is defined as $K^U = Mod\,Un\,K$, whereas the *reduced universal class* generated by K is taken to be $(K^U)^* = L(Mod\,Un\,K)$. Once more, $(K^U)^*$ need not be even an elementary class. The next result looks like Corollary 4.2.2.

COROLLARY 4.3.2. *If* K *is any class of* $\mathcal{L}$*-structures, the following holds.*

(i) $K^U = ERS\overline{P}_u(K)$.

(ii) $(K^U)^* = S^*\overline{P}_u^*(K^*)$.

Proof. We just repeat the argument for the proof of Corollary 4.2.2. ■

COROLLARY 4.3.3. *A class* K *of reduced* $\mathcal{L}$*-structures is a reduced universal class (i.e.,* $K = (K^U)^*$*) iff it is closed under reduced substructures and reduced ultraproducts modulo ultrafilters over nonempty sets.* ■

A reduced universal class is not in general closed under the operators S and $\overline{P}_u$. For ultraproducts the example given in the preceding Section keeps on working now. For substructures we can find simple counterexamples. For instance, consider the language of groups together with a unary relation symbol, $\mathcal{L} = \{\cdot, e, r\}$. The whole class $Str\,\mathcal{L}$ is universal (the sentence $\forall x(rx \rightarrow rx)$ provides an axiomatization). Let $\mathbf{A}$ be a simple group and $\mathbf{B}$ a nonsimple subgroup of $\mathbf{A}$. Then, if N is the universe of a normal subgroup of $\mathbf{B}$, $\mathfrak{A} = \langle \mathbf{A}, N\rangle \in Str^*\mathcal{L}$, $\mathfrak{B} = \langle \mathbf{B}, N\rangle \subseteq \mathfrak{A}$ and $\mathfrak{B}$ is not reduced.

4.4. Quasivarieties

We say that a class K of $\mathcal{L}$-structures is a *quasivariety* if there exists some set Γ of implicative $\mathcal{L}$-formulas such that $K = Mod\,\Gamma$, or equivalently, if $K = Mod\,Imp\,K$[15]. Our purpose now is to provide some algebraic characterizations of quasivarieties that hold for languages with as well as without equality. The results we are going to establish generalize the classical theorem of Mal'cev [87] and some more recent theorems due to Czelakowski [29,37]. The technique of the proof given here differs from the one used by Czelakowski, but we shall see in Section 8.2 below that his proof can also be extended.

THEOREM 4.4.1. *For any class* K *of* $\mathcal{L}$*-structures, the following statements are equivalent:*

(i) K *is a quasivariety.*

[15] The common expression among Western model-theorists to refer to quasivarieties as defined here is "strict universal Horn class". We have chosen the former terminology, which goes back to Mal'cev and is also pretty usual among Eastern model-theorists. The choice purports to show the algebraic spirit that the model theory we try to develop (mainly for UHL) has. Also, it has been picked out for it is consistent with the more convenient terms of *variety* and *relative subvariety* used later on. To avoid any possible confusion with the usual meaning of the term "quasivariety" in the West (a class of algebras defined by a set of quasi-identities) we shall always speak of "quasivariety of $\mathcal{L}$-structures" as opposed to "quasivariety of $\mathcal{L}$-algebras".

(ii) K *is closed under* E, R, S *and* P_f.
(iii) K$= ERSP_f$ (K$'$), *for some* K$'$.

Proof. As for universal classes, (i) implies (ii) is easily checked using Theorem 1.3.1 instead of Los Theorem. Likewise, (ii) implies (iii) is clear. Let us prove the implication from (iii) to (i). For this, we shall see that (iii) entails K is axiomatizable by $Imp\,\mathsf{K}'$. Certainly $\mathsf{K} \subseteq Mod\,Imp\,\mathsf{K}'$ (observe that K must contain the trivial, one-element structure). Suppose $\mathfrak{A} \in Mod\,Imp\,\mathsf{K}'$. Let $\Delta = Sb_\omega(D\mathfrak{A})$. If we are given $\Phi \in \Delta$, $\Phi := \Phi(c_{a_1}, \ldots, c_{a_k})$, then

$$\mathfrak{A} \models \exists x_1 \ldots \exists x_k \bigwedge \Phi(x_1, \ldots, x_k).$$

We want to show that some member of $P(\mathsf{K}')$ satisfies this sentence as well. For this purpose it suffices to prove

$$\forall x_1 \ldots \forall x_k \neg \bigwedge \Phi(x_1, \ldots, x_k) \notin Th\,P(\mathsf{K}'). \tag{4.6}$$

We distinguish three cases. If none of the elements of Φ is a negated atomic $\mathcal{L}$-formula then (4.6) holds, for $P(\mathsf{K}')$ contains the trivial, one-element structure which does not satisfy the negation of any atomic $\mathcal{L}$-formula. If exactly one element of Φ is negated atomic then the universal sentence above is logically equivalent to the universal closure of some implicative $\mathcal{L}$-formula which is not true in $\mathfrak{A}$ and, consequently, since $\mathfrak{A} \in Mod\,Imp\,\mathsf{K}'$, in K'. The last case is the most difficult to argue. Let $\Phi := \{\varphi_1, \ldots, \varphi_q\}$ and let us suppose at least two elements of Φ are negated atomic formulas, say φ_i for $1 \leq i \leq p$, where $2 \leq p \leq q$. Then one can reason as above that

$$\forall x_1 \ldots \forall x_k\, (\neg\varphi_i(x_1, \ldots, x_k) \vee \neg\varphi_{p+1}(x_1, \ldots, x_k) \vee \ldots \vee \neg\varphi_q(x_1, \ldots, x_k)) \notin Th\,\mathsf{K}',$$

for $1 \leq i \leq p$. Consequently, for some $\mathfrak{B}_i \in \mathsf{K}'$ and some $b_{i1}, \ldots, b_{ik} \in B_i$, $1 \leq i \leq p$,

$$\mathfrak{B}_i \models \varphi_i(x_1, \ldots, x_k) \wedge \varphi_{p+1}(x_1, \ldots, x_k) \wedge \ldots \wedge \varphi_q(x_1, \ldots, x_k))\,[b_{i1}, \ldots, b_{ik}].$$

Define

$$\mathbf{b}_j := (b_{1j}, \ldots, b_{pj}) \in \textstyle\prod_{1 \leq i \leq p} B_i, \quad 1 \leq j \leq k.$$

Then Theorem 1.3.1 implies

$$\textstyle\prod_{1 \leq i \leq p} \mathfrak{B}_i \models \bigwedge \Phi(x_1, \ldots, x_k)\,[\mathbf{b}_1, \ldots, \mathbf{b}_k],$$

and hence, since $\prod_{1 \leq i \leq p} \mathfrak{B}_i \in P(\mathsf{K}')$, (4.6) is proved.

Now, for each $\Phi \in \Delta$, consider $\mathfrak{B}_\Phi \in P(\mathsf{K}')$ and $\{b_{a,\Phi} : a \in A\} \subseteq B_\Phi$ such that

$$(\mathfrak{B}_\Phi, b_{a,\Phi})_{a \in A} \models \bigwedge \Phi(c_{a_1}, \ldots, c_{a_k}).$$

We can now proceed as in the proof of 4.3.1 to obtain an $\mathcal{L}_A$-structure $(\mathfrak{C}, b_a/\mathcal{U})_{a \in A}$ of $SP_uP(\mathsf{K}')$ such that $(\mathfrak{C}, b_a/\mathcal{U})_{a \in A} \in Mod\,D_l\mathfrak{A}$. So, a new application of tha Diagrams' Lemma gives $\mathfrak{A} \in ERSP_uP(\mathsf{K}')$. But

$$\begin{aligned} SP_uP &\leq SP_fP_f, && \text{by definition,} \\ &= SP_f, && \text{by Lemma 4.1.1.} \end{aligned}$$

Hence, $\mathfrak{A} \in ERSP_uP(\mathsf{K}') \subseteq ERSP_f(\mathsf{K}')$ and the assumption (iii) says that $\mathfrak{A} \in \mathsf{K}$. This finishes the proof of the theorem. ∎

Given any class K of $\mathcal{L}$-structures, we define the *full quasivariety* generated by K, or simply the *quasivariety* generated by K, as $\mathsf{K}^Q = Mod\,Imp\,\mathsf{K}$, and the *reduced quasivariety* generated by K as $(\mathsf{K}^Q)^* = Mod^*Imp\,\mathsf{K}$. The next result includes a generalization of [12, Thm. 6.2].

COROLLARY 4.4.2. *The following is true for any class* K *of* $\mathcal{L}$*-structures.*
(i) $\mathsf{K}^Q = ERSP_f(\mathsf{K})$.
(ii) $(\mathsf{K}^Q)^* = S^*P_f^*(\mathsf{K}^*)$. ∎

COROLLARY 4.4.3. *A class* K *of reduced* $\mathcal{L}$*-structures is a reduced quasivariety (i.e.,* $\mathsf{K} = (\mathsf{K}^Q)^*$*) iff it is closed under both reduced substructures and reduced filtered products.* ∎

We sometimes use the notation Q to abbreviate the composed operator $ERSP_f$, so that we have just proved that $\mathsf{K}^Q = Q(\mathsf{K})$ and $Q^* = S^*P_f^*L$. The next lemmas can be used to derive some other useful descriptions of these operators Q and Q^* for generating quasivarieties.

LEMMA 4.4.4. *(Grätzer and Lasker* [58, Lemma 2]*)* $SP_f = SPP_u$. ∎

LEMMA 4.4.5. *(Czelakowski* [37]*)* $SP_f = P_{fs} = P_{sd}SP_u$.

Proof. Let us prove first the equality $SP_f = P_{fs}$. The inclusion $P_{fs} \leq SP_f$ is obvious: by definition, a filtered subdirect product of a system of structures is always isomorphic to a substructure of a filtered product of the system. Also, $P_f \leq P_{fs}$. So let us see that $S \leq P_{fs}$. Take an arbitrary class K of $\mathcal{L}$-structures, and suppose $\mathfrak{A} \subseteq \mathfrak{B} \in \mathsf{K}$. Define

$$C := \{\mathbf{b} \in B^\omega : b_i = a \text{ if } i \geq m, \text{ for some } a \in A \text{ and } m \in \omega\}. \tag{4.7}$$

Note that, for every $\mathbf{b} \in C$, the element a in (4.7) is unique; let us denote it by $a(\mathbf{b})$. Also, C is the universe of a subalgebra of the direct power $\mathbf{B}^\omega$; rather, it is the universe of a subdirect power of $\mathbf{B}^\omega$, for the projection of C into each component is surjective. So, let $\mathfrak{C} := \mathfrak{B}^\omega \restriction C$. If

$$\mathcal{F} := \{X \in Sb(\omega) : \overline{X} \text{ is finite}\},$$

we claim that the mapping h from $C/\mathcal{F}$ into A given by $\mathbf{b}/\mathcal{F} \longmapsto a(\mathbf{b})$ defines an isomorphism between the filtered subdirect power $\mathfrak{C}/\mathcal{F}$ and the substructure $\mathfrak{A}$. Indeed, if $\mathbf{b}, \mathbf{b}' \in C$, then

$$\begin{aligned} \mathbf{b}/\mathcal{F} = \mathbf{b}'/\mathcal{F} \ &\text{iff there exists } m \in \omega \text{ such that } b_i = b'_i \text{ for all } i \geq m \\ &\text{iff } a(\mathbf{b}) = a(\mathbf{b}'). \end{aligned}$$

Thus h is well defined and bijective. Now chose elements $\mathbf{b}_1, \ldots, \mathbf{b}_n \in C$ and let f and r a function and a relation symbol, repectively, of arity n. Since $a(f^{\mathbf{C}}\mathbf{b}_1 \ldots \mathbf{b}_n) =$

$f^{\mathbf{B}}b_{1m}\dots b_{nm}$ for some $m \in \omega$, we have $h(\mathrm{b}_i/\mathcal{F}) = a(\mathrm{b}) = b_{im}$ for all $1 \leq i \leq n$, and consequently

$$hf^{\mathbf{C}/\mathcal{F}}\mathrm{b}_1/\mathcal{F}\dots\mathrm{b}_n/\mathcal{F} = a(f^{\mathbf{C}}\mathrm{b}_1\dots\mathrm{b}_n) = f^{\mathbf{A}}a(\mathrm{b}_1/\mathcal{F})\dots a(\mathrm{b}_n/\mathcal{F}).$$

Moreover, by the definition of filtered subdirect product,

$$\begin{aligned}\langle \mathrm{b}_1/\mathcal{F},\dots,\mathrm{b}_n/\mathcal{F}\rangle \in r^{\mathfrak{C}/\mathcal{F}} \text{ iff } & \{i\in\omega : \langle b_{1i},\dots,b_{ni}\rangle \in r^{\mathfrak{B}}\} \in \mathcal{F}\\ \text{iff } & \text{there exists } m\in\omega \text{ such that } \langle b_{1i},\dots,b_{ni}\rangle \in r^{\mathfrak{B}} \text{ for all } i\geq m\\ \text{iff } & \langle a(\mathrm{b}_1),\dots,a(\mathrm{b}_n)\rangle \in r^{\mathfrak{A}}.\end{aligned}$$

So h is the desired isomorphism. From the claim we conclude that $\mathfrak{A} \in P_{fs}(\mathfrak{B})$ and hence $S(\mathsf{K}) \subseteq P_{fs}(\mathsf{K})$. This completes the proof of the equality $SP_f = P_{fs}$.

To see $P_{fs} = P_{sd}SP_u$, we first notice that $P_{sd}SP_u \leq P_{fs}$, for P_{fs} is idempotent by Lemma 4.1.1 and each one of the operators P_{sd}, S and P_u is less than P_{fs}. For the reverse inclusion, let $\{\mathfrak{A}_i : i \in I\}$ be a system of structures and let $\mathcal{F}$ be a proper filter on I. Clearly $\mathcal{F}$ may be expressed as the intersection of some family of ultrafilters on I; for simplicity suppose $\{\mathcal{U}_j : j \in J\}$ is such a family, i.e., $\mathcal{F} = \bigcap_{j\in J}\mathcal{U}_j$, where $\mathcal{U}_j$ is an ultrafilter of $Sb(I)$. Then the congruence $\Theta_{\mathcal{F}}$ is the intersection of the family $\{\Theta_{\mathcal{U}_j} : j \in J\}$, and so the filtered product $\prod_{i\in I}\mathfrak{A}_i/\mathcal{F}$ is subdirectly embeddable in $\prod_{j\in J}(\prod_{i\in I}\mathfrak{A}_i/\mathcal{U}_j)$ (this follows from a property proved below in Chapter 6, Proposition 6.1.1). Let us say h the subdirect embedding. Then, if $\mathfrak{A}/\mathcal{F}$ is a filtered subdirect product of the system $\{\mathfrak{A}_i : i \in I\}$, the image $h(\mathfrak{A}/\mathcal{F})$ can be easily proved to be isomorphic to a subdirect product of the structures $\mathfrak{A}/\mathcal{U}_j$, $j \in J$. In conclusion, $P_{fs} \leq P_{sd}SP_u$, which finishes the proof of the second equality and the Lemma. ■

COROLLARY 4.4.6. *The following equalities hold.*
(i) $Q = ERSPP_u = ERP_{fs} = ERP_{sd}SP_u$.
(ii) $Q^* = S^*P^*P_u^* = P_{fs}^* = P_{sd}^*S^*P_u^*$.

Proof. Part (i) follows directly from 4.4.4 and 4.4.5. To obtain (ii) we can apply Proposition 4.1.5(i) and the preceding lemmas. ■

The examples provided in the previous sections show that reduced quasivarieties are not in general closed under the operators S and P_u. An easy counterexample borrowed from [12] proves that they are neither closed under P and hence P_f. Indeed, if $\mathcal{L}$ consists of one relation symbol, of arity 1, and no function symbol, then the reduced $\mathcal{L}$-structures are of the form $\mathfrak{A} = \langle\{a,b\},\{a\}\rangle$ for distinct elements a, b. So, $\mathfrak{A}^2$ is not reduced, since $|A^2| = 4$.

4.5. Varieties and Relative Subvarieties

Let K be any class of $\mathcal{L}$-structures. We say K is a *variety* if $\mathsf{K} = Mod\,\Gamma$ for some set Γ of atomic formulas over $\mathcal{L}$; equivalently, if $\mathsf{K} = Mod\,Atm\,\mathsf{K}$. Next result provides a generalization of Birkhoff's Variety Theorem to general first-order languages, with or without equality. The proof is entirely of the same nature as the proof of the previous theorems in the Chapter, and so runs far from Birkhoff's original proof.

THEOREM 4.5.1. *For any class* K *of* $\mathcal{L}$*-structures, the following statements are equivalent.*

(i) K *is a variety.*

(ii) K *is closed under* H, E, S *and* P.

(iii) $\mathsf{K} = HESP(\mathsf{K}')$, *for some* K'.

Proof. (i) implies (ii) and (ii) implies (iii) are clear. Let us show (iii) implies (i) by proving that K is axiomatizable by $Atm\,\mathsf{K}'$. Once more the inclusion $\mathsf{K} \subseteq Mod\,Atm\,\mathsf{K}'$ is easy to check. Assume $\mathfrak{A} \in Mod\,Atm\,\mathsf{K}'$ and let Δ be the set $D^{-}\mathfrak{A}$ of negated atomic $\mathcal{L}_A$-sentences which are satisfied by $(\mathfrak{A}, a)_{a\in A}$. Let $\varphi := \varphi(c_{a_1}, \ldots, c_{a_k}) \in \Delta$. We claim there exist $\mathfrak{B}_\varphi \in \mathsf{K}'$ and $\{b_{a,\varphi} : a \in A\} \subseteq B_\varphi$ such that

$$(\mathfrak{B}_\varphi, b_{a,\varphi})_{a\in A} \models \varphi(c_{a_1}, \ldots, c_{a_k}).$$

Otherwise, the sentence $\forall x_1 \ldots \forall x_k\, \neg\varphi(x_1, \ldots, x_k)$ is logically equivalent to the universal closure of some member of $Atm\,\mathsf{K}'$, and hence $\mathfrak{A} \models \forall x_1 \ldots \forall x_k\, \neg\varphi(x_1, \ldots, x_k)$. But this contradicts the assumption $\varphi \in \Delta$. So let

$$\begin{aligned} &\mathfrak{B} := \textstyle\prod_{\varphi\in\Delta} \mathfrak{B}_\varphi, \\ &b_a := (b_{a,\varphi} : \varphi \in \Delta) \in \textstyle\prod_{\varphi\in\Delta} B_\varphi, \ \text{ for each } a \in A, \\ &\mathfrak{C} := \mathfrak{B} \restriction \{b_a : a \in A\}. \end{aligned}$$

Obviously $\mathfrak{C} \in SP(\mathsf{K}')$. Moreover, by 1.3.1, we have $\mathfrak{B} \models \Delta$ and thus $\mathfrak{C} \models \Delta$.

Consider the absolutely free $\mathcal{L}$-algebra $\mathbf{Te}_{\mathcal{L},|A|}$ over $|A|$-variables $\{x_a : a \in A\}$, and define $h : Te_{\mathcal{L},|A|} \to C$ by $x_a \longmapsto b_a$. Let $\mathfrak{F} = \langle Te_{\mathcal{L},|A|}, h^{-1}R_{\mathfrak{C}}\rangle$ be the inverse image of $\mathfrak{C}$ under h, so that we have $h : \mathfrak{F} \twoheadrightarrow_s \mathfrak{C}$. We want the mapping $x_a \longmapsto a$ to be a surjective homomorphism from $\mathfrak{F}$ onto $\mathfrak{A}$. Clearly h is an algebra homomorphism. Also, since $(\mathfrak{C}, b_a)_{a\in A}$ is a model of $\Delta = D^{-}\mathfrak{A}$, the following is true for any atomic $\mathcal{L}$-formula $rt_1 \ldots t_n$, where $t_1, \ldots, t_n$ are terms in k variables:

$$\begin{aligned} &\langle t_1(x_{a_1}, \ldots, x_{a_k}), \ldots, t_n(x_{a_1}, \ldots, x_{a_k})\rangle \in r^{\mathfrak{F}} \\ &\qquad \text{iff } \langle t_1^{\mathbf{C}}(b_{a_1}, \ldots, b_{a_k}), \ldots, t_n^{\mathbf{C}}(b_{a_1}, \ldots, b_{a_k})\rangle \in r^{\mathfrak{C}} \\ &\qquad \text{implies } \langle t_1^{\mathbf{A}}(a_1, \ldots, a_k), \ldots, t_n^{\mathbf{A}}(a_1, \ldots, a_k)\rangle \in r^{\mathfrak{A}}. \\ &\qquad \text{iff } \langle ht_1(x_{a_1}, \ldots, x_{a_k}), \ldots, ht_n(x_{a_1}, \ldots, x_{a_k})\rangle \in r^{\mathfrak{A}}. \end{aligned}$$

Therefore, $h : \mathfrak{F} \twoheadrightarrow \mathfrak{A}$. From here we conclude that $\mathfrak{A} \in H(\mathfrak{F})$ and $\mathfrak{F} \in E(\mathfrak{C})$. As a result, $\mathfrak{A} \in HESP(\mathsf{K}')$. ∎

Remark. Notice that this result specializes to Birkhoff's Variety Theorem, for reductive homomorphisms are just isomorphisms when $\mathcal{L}$ has equality. In fact, the preceding proof simplifies in this case and provides a proof of a general form of Birkhoff's Variety Theorem strictly based on model-theoretic techniques. The simplification goes as follows. If $\approx$ is a symbol of $\mathcal{L}$, then the $\mathcal{L}_A$-sentence $\neg c_a \approx c_{a'}$

belongs to Δ, for each $a, a' \in A$ such that $a \neq a'$; therefore, $b_a \neq b_{a'}$ must hold. Also, if $\langle a_1, \ldots, a_n \rangle$, $\langle a'_1, \ldots, a'_n \rangle \in A^n$, then $f^{\mathbf{A}} a_1 \ldots a_n \neq f^{\mathbf{A}} a'_1 \ldots a'_n$ implies $\neg f c_{a_1} \ldots c_{a_n} \approx f c_{a'_1} \ldots c_{a'_n} \in \Delta$, and hence $f^{\mathbf{C}} b_{a_1} \ldots b_{a_n} \neq f^{\mathbf{C}} b_{a'_1} \ldots b_{a'_n}$. In general, we can iterate this argument and prove that we can construct directly a surjective homomorphism from $\mathfrak{C}$ onto $\mathfrak{A}$ such that $b_a \longmapsto a$, and hence we obtain $\mathfrak{A} \in H(\mathfrak{C}) \subseteq HSP(\mathsf{K}')$. ⊣

We define the *full variety* generated by a class K, or simply the *variety* generated by K, as $\mathsf{K}^V = Mod\, Atm\, \mathsf{K}$, and the *reduced variety* generated by K as $(\mathsf{K}^V)^* = Mod^*\, Atm\, \mathsf{K}$. Then we have:

COROLLARY 4.5.2. *The following is true for any class* K *of* $\mathcal{L}$*-structures.*

(i) $\mathsf{K}^V = HESP(\mathsf{K})$.

(ii) $(\mathsf{K}^V)^* = F^*ESP(\mathsf{K})$.

Proof. It follows directly from Lemmas 4.1.3(ii) and 4.1.4(ii). ∎

As for quasivarieties, we introduce the notation V to express the composed operator $HESP$, so that we have proved the equality $\mathsf{K}^V = V(\mathsf{K})$. In general, however, the operators E and F do not commute, nor F^* coincides with F^*L as it occurs for the remaining operators (cf. Lemma 4.1.5 above). There are easy counterexamples of that. For instance, let

$$\mathfrak{A} := \langle \mathbb{N} \times \mathbb{N}, +, (0,0), \sim \rangle, \quad \mathfrak{B} := \langle \mathbb{N} \times \mathbb{N}, +, (0,0), \sim' \rangle,$$

where $\sim$ is the binary relation on $\mathbb{N}\times\mathbb{N}$ given by

$$(a,b) \sim (a',b') \text{ iff } a + b' = a' + b,$$

and $\sim'$ is the relation that results from $\sim$ by joining the set $\{(0,1),(1,0)\}^2$. Then it is easy to check that the Leibniz congruence on $\mathfrak{A}$ is the relation $\sim$ (recall the construction of the integers by the symmetrization process). Also, $\Omega\mathfrak{B}$ coincides with the set of all pairs $\langle (a,b),(a',b') \rangle$ of $\sim$ that satisfy the following additional condition:

$$(a,b) \in \{(0,0),(1,0),(0,1)\} \text{ or } (a',b') \in \{(0,0),(1,0),(0,1)\}$$
$$\text{implies } (a,b) = (a',b'). \tag{4.8}$$

Indeed, denote by θ such set of pairs. Clearly θ is an equivalence relation and $\theta \subseteq \sim$. So we have that θ is compatible with $\sim'$. It remains to show that θ is also compatible with the addition. For this purpose, assume $(a,b)\theta(a',b')$ and $(c,d)\theta(c',d')$. We distinguish three cases. If none of the pairs (a,b), (c,d) belongs to $\{(0,0),(1,0),(0,1)\}$, then we actually have that $(a,b) \sim (a',b')$ and 8c, d) $\sim (c',d')$ and hence

$$(a+c, b+d)\,\theta\,(a'+c', b'+d'). \tag{4.9}$$

If (a,b) is one of the pairs $\{(0,0),(1,0),(0,1)\}$, then (4.8) says that $(a,b) = (a',b')$ and consequently (4.9) also holds. Finally, if (a,b), (c,d) are both members of the set $\{(0,0),(1,0),(0,1)\}$, we reason as before and obtain the same conclusion. All this proves our claim and therefore we have that $\mathfrak{B} \in FE(\mathfrak{A}^*)$ but $\mathfrak{B} \notin EF(\mathfrak{A}^*)$, for no quotient of $\mathfrak{B}$ can have as underlying algebra the additive group of integers. A similar counterexample can be found that proves $F^* \neq F^*L$.

The previous remark says that the variety generated by a class K cannot be obtained by adding ER to the classical operator HSP that generates varieties in presence of the equality symbol; so far, this had constituted the only necessary modification with respect to the model theory developed by Mal'cev. Also, since $F^* \neq F^*L$, $F^*S^*P^*(\mathsf{K}^*)$ does not necessarily coincide with $(\mathsf{K}^V)^*$. In view of that, two interesting issues arise naturally: to determine sufficient conditions for the class K to satisfy the equalities $\mathsf{K}^V = ERHSP(\mathsf{K})$ and $(\mathsf{K}^V)^* = F^*S^*P^*(\mathsf{K}^*)$. An answer to these problems is given in the next Chapter, Corollary 5.3.10 and Theorem 5.3.11; see also Chapter 7, Corollaries 7.2.3 and 7.2.6. Meanwhile, notice that none of the inequalities $F^*S^*P^* \leq F^*ESP$ and $F^*ESP \leq F^*S^*P^*$ seem to hold in general.

There are absolutely no examples of interesting classes of structures defined without equality and closed under H, E, S and P; even the many well-known purely algebraic varieties (such as groups, rings, lattices and so on) are not closed under homomorphic images when they are defined using a language without equality, for in this case they become quasivarieties of the form K_{Q} described in Section 3.3. We close this section by introducing what seems to be the natural counterpart of the concept of variety when we deal with such a kind of languages. This concept is central to the purpose of generalizing the theory of varieties to arbitrary structures. Let K be any class of $\mathcal{L}$-structures. A subclass V of K is called a *relative subvariety of* K if there exists a set of atomic $\mathcal{L}$-formulas Σ such that $\mathfrak{A} \in \mathsf{V}$ iff $\mathfrak{A} \in \mathsf{K}$ and $\mathfrak{A} \in Mod\Sigma$. In this case we say that the reduced class V^* is a *reduced relative subvariety of* K^*. As it was first noted by Blok and Pigozzi [11,12], this latter notion specializes to varieties in the usual universal algebraic sense when we take K to be the whole class K_{co}.

The following is an easy consequence from the preceding results; H_{Q} is the operator that gives all the homomorphic images belonging to the class Q and F_{Q} denotes the operator that provides filter extensions that are members of Q.

COROLLARY 4.5.3. *Let* Q *be a quasivariety of* $\mathcal{L}$*-structures and* K *a subclass of* Q*. The relative subvariety of* Q *generated by* K *is* $\mathsf{K}^V \cap \mathsf{Q} = H_{\mathsf{Q}}ESP(\mathsf{K})$*. Similarly, the reduced relative subvariety is* $\mathsf{K}^V \cap \mathsf{Q}^* = F^*_{\mathsf{Q}}ESP(\mathsf{K})$. ∎

5. The Leibniz Operator and some Well-behaved Classes

As is well known, most of the results in universal algebra involve, in one way or another, lattices of congruences; let us mention, for instance, the profound influence that congruence identities have on the structure of varieties. For such a reason, the concept of congruence is central to the development of a model theory that tries to generalize as much as possible of universal algebra. But this concept splits into two different notions when dealing with arbitrary structures. The first one of this notions is the straightforward extension that we obtain when the compatibility with relations is required; it turns out to be the notion of congruence on a structure studied in detail all over Chapter 2. The motivation of the second extension is based upon the semantics of the theory Γ_{co} defined in Section 3.3. Indeed, we have that the relational part of the members of K_{co} with underlying algebra $\mathbf{A}$ are just the congruences on $\mathbf{A}$. So, in some sense, it is reasonable to think of the relational part of structures as another generalization of the concept of congruence when passing from algebras to arbitrary structures[16].

Such a splitting of the concept of congruence causes that the generalization of universal algebraic results could take place into two different directions. Thus, we have already seen in Section 2.2 that the Isomorphisms Theorems of universal algebra (we include here the classical Homomorphism Theorem and Correspondence Theorem) have an easy counterpart when we replace the notion of congruence on an algebra by that of congruence on a structure; for instance, we already know the close connection that exists between congruences on two structures $\mathfrak{A}$ and $\mathfrak{B}$ when $\mathfrak{B}$ is a reduction of $\mathfrak{A}$ (see Corollary 2.2.7). Now a similar problem emerges for the second extension of the concept of congruence, and this problem turns out to be in the very base of the solution to the main issue we posed at the end of Chapter 3.

Indeed, one might expect the close link between properties of congruence lattices and properties of classes of algebras to carry over to a similar link between properties of the posets formed of the relational part of the structures of a class (on a given underlying algebra) and properties of this class. But, as it has already been observed in a restricted context (see, e.g., [7, p. 338]), the latter link does not exist without some restrictions.

[16] Later it will become suitable to combine these two extensions of the concept of congruence into what we call *congruence-filter pair* on a structure (see Section 6.1 below).

Roughly speaking, the approach that guides this work and that fortunately works out beautifully is the following: the algebraic character of a class K and its reduced class K^* relies on the properties the posets of predicates (or relations) have in comparison with the ones satisfied by lattices of congruences. So the main purpose of this Chapter is to introduce what we call *Leibniz operator*, a mapping that establishes a correspondence between the relational part of a structure and a congruence on the corresponding underlying algebra (it is going to be the Leibniz equality!). Then we use the properties of this operator as the primary criterion to describe a hierarchy of special classes for which we shall be able to derive here and subsequent chapters more general forms of some classical results of universal algebra, results that do not hold any longer for more arbitrary classes. On the top of this hierarchy we shall find those classes for which the Leibniz operator is an isomorphism in a sense we make precise below (Definition 5.4.7); they constitute an important category of classes to which we come back in Chapter 10 for their relevance in algebraic logic. The starting point of the Chapter is to describe those full classes that are better amenable of universal algebraic methods, and which turn out to be the quasivarieties of $\mathcal{L}$-structures (Theorem 5.1.1 below).

5.1. Lattices of Relative Filter Extensions

Let K be a full class of $\mathcal{L}$-structures, and consider any $\mathcal{L}$-algebra **A**. We define the set of *K-structures on* **A**, denoted by $K_{\mathbf{A}}$, as the set of elements of K whose underlying algebra is **A**; when **A** is the term algebra $\mathbf{Te}_{\mathcal{L},\alpha}$, we talk about the *term-structures of* K (with α generators). In general, the set of K-structures on **A** is a partially ordered set with respect to the filter extension relation $\preccurlyeq$. If, in addition, it is an algebraic closure system[17] for all **A**, then we say that K satisfies the *filter-lattice condition* (FL condition for short). In this case, $\mathbb{K}_{\mathbf{A}} = \langle K_{\mathbf{A}}, \cap, \vee \rangle$ is an algebraic complete lattice, where

$$\bigvee_{i \in I} \mathfrak{A}_i = \bigcap \{\mathfrak{A} \in K_{\mathbf{A}} : \mathfrak{A}_i \preccurlyeq \mathfrak{A} \text{ for each } i \in I\}.$$

The relevance of such a condition rests on the fact that most of the known properties of algebras and varieties or quasivarieties of algebras strongly depends on this property of sets of congruences. So, the problem of characterizing the full classes that satisfy the filter-lattice condition calls for an answer. The next theorem says that these classes are exactly the quasivarieties. A proof very different in nature to the one provided here was pointed out to the author by Czelakowski [37].

THEOREM 5.1.1. *Let* K *be any full class of $\mathcal{L}$-structures. Then* K *satisfies the filetr-lattice condition iff* K *is a quasivariety.*

[17] We use indistinctly the terms *inductive closure system* and *algebraic closure system* to mean a nonempty system of subsets of a set closed under arbitrary intersections and unions of directed families. It was proved by Schmidt (see [26, Thm. III.1.1]) that they are exactly those nonempty systems closed under arbitrary intersections and such that each member can be expressed as the union of all its finite subsets.

Proof. The backward implication is easy to check: it suffices to show that implicative formulas are preserved under arbitrary intersections and unions of well ordered $\preccurlyeq$-chains (recall that closure under unions of directed families in a poset is equivalent to closure under unions of chains [26, Prop. I.5.9]. For this, consider any implicative $\mathcal{L}$-formula $\sigma := \bigwedge\Phi \rightarrow \varphi$ and let $\mathbf{A}$ be a fixed $\mathcal{L}$-algebra. The intersection of the empty family of $\mathbf{A}$-structures is the structure with all relations holding, i.e., where the interpretation of any relation symbol r is $A^{\rho(r)}$. So σ is true in $\bigcap\emptyset$. Let now $\mathfrak{A}_i$, $i \in I$, be a nonempty system of $\mathbf{A}$-structures such that $\mathfrak{A}_i \vDash \sigma$, for $i \in I$. Given an assignment $g : \mathbf{Te}_{\mathcal{L}} \rightarrow \mathbf{A}$, we have

$$\begin{aligned} \textstyle\bigcap_{i\in I} \mathfrak{A}_i \vDash \bigwedge\Phi\ [g] \quad &\text{iff} \quad \mathfrak{A}_i \vDash \varphi_j\ [g],\ 1 \leq j \leq m,\ i \in I \\ &\text{implies} \quad \mathfrak{A}_i \vDash \varphi\ [g],\ i \in I \\ &\text{iff} \quad \textstyle\bigcap_{i\in I} \mathfrak{A}_i \vDash \varphi\ [g]. \end{aligned}$$

Hence, $\bigcap_{i\in I} \mathfrak{A}_i \vDash \sigma$.

Assume finally that α is an ordinal and $\mathfrak{A}_\lambda$, $\lambda < \alpha$, are such that $\mathfrak{A}_\lambda \preccurlyeq \mathfrak{A}_\mu$ whenever $\lambda, \mu < \alpha$ and $\lambda \leq \mu$. Also, suppose $\mathfrak{A}_\lambda \vDash \sigma$ for all $\lambda < \alpha$. Let $\mathfrak{A} = \bigcup_{\lambda<\alpha} \mathfrak{A}_\lambda$. Then, since $r^{\mathfrak{A}} = \bigcup_{\lambda<\alpha} r^{\mathfrak{A}_\lambda}$ for all $r \in R$, $\mathfrak{A} \vDash \bigwedge\Phi\ [g]$ implies $\mathfrak{A}_{\lambda_0} \vDash \bigwedge\Phi\ [h]$ for some $\lambda_0 < \alpha$. Thus, the assumption says $\mathfrak{A}_{\lambda_0} \vDash \varphi\ [g]$ and finally $\mathfrak{A} \vDash \varphi\ [g]$. Hence again we conclude $\mathfrak{A} \vDash \sigma$.

For the converse, consider any member $\mathfrak{A}$ from K^Q and let $\alpha := \max\{\omega, |A|\}$. Take h to be an algebra homomorphism from $\mathbf{Te}_{\mathcal{L},\alpha}$ onto $\mathbf{A}$. Since K satisfies the filter-lattice condition, the structure $\mathfrak{F}_\alpha := \bigcap \mathsf{K}_{\mathbf{Te}_{\mathcal{L},\alpha}}$ belongs to K; in particular, $h^{-1}\mathfrak{A}$ is a filter extension of $\mathfrak{F}_\alpha$. Also, $h^{-1}\mathfrak{A} \in E(\mathfrak{A}) \subseteq \mathsf{K}^Q$ and $\mathfrak{A} \in R(h^{-1}\mathfrak{A})$. So the forward implication will be proved if we show that every filter extension of $\mathfrak{F}_\alpha$ that belongs to K^Q is also a member of K, for K is closed under R by hypothesis.

To this goal, let $\mathfrak{F}$ be an arbitrary filter extension of $\mathfrak{F}_\alpha$. We write

$$Fg_{\mathsf{K}}\mathfrak{F} := \bigcap \{\mathfrak{A} \in \mathsf{K} : \mathfrak{F} \preccurlyeq \mathfrak{A}\}.$$

Notice right off that $Fg_{\mathsf{K}}\mathfrak{F} \in \mathsf{K}$, for K satisfies the filter-lattice condition. We are going to see that $\mathfrak{F} \in \mathsf{K}^Q$ implies $Fg_{\mathsf{K}}\mathfrak{F} = \mathfrak{F}$; this will prove $\mathfrak{F} \in \mathsf{K}$ and thus the theorem. Since K satisfies the filter-lattice condition,

$$Fg_{\mathsf{K}}\mathfrak{F} = \bigcup\{Fg_{\mathsf{K}}\mathfrak{A} : \mathfrak{A} \in \mathsf{K},\ \mathfrak{A} \preccurlyeq \mathfrak{F} \text{ and } \bigcup R_{\mathfrak{A}} \text{ is finite}\}.$$

Therefore, let us prove $Fg_{\mathsf{K}}\mathfrak{A} \preccurlyeq \mathfrak{F}$ for all $\mathfrak{A}$ satisfying the preceding three conditions.

We may assume without loss of generality that $\alpha = \omega$; if it did not, we use the same argument and apply the equality $Mod\,Imp_\alpha\mathsf{K} = Mod\,Imp\,\mathsf{K}$, which holds whenever $\alpha \geq \omega$. Define the set

$$\Phi[\mathfrak{A}] := \{\psi \in Atm\,\mathcal{L} : \mathfrak{A} \vDash \psi\ [id]\},$$

where id denotes the identity function on $Te_{\mathcal{L}}$, and take an arbitrary atomic $\mathcal{L}$-formula φ. (Observe that the members of $\Phi[\mathfrak{A}]$ are exactly the atomic $\mathcal{L}$-formulas $rt_1 \ldots t_n$ such that $\langle t_1, \ldots, t_n\rangle \in r^{\mathfrak{A}}$; so, in particular, $\Phi[\mathfrak{A}]$ is a finite set, for $\bigcup R_{\mathfrak{A}}$ is finite by assumption.). We claim:

$$Fg_{\mathsf{K}}\mathfrak{A} \vDash \varphi\ [id] \text{ implies } \bigwedge\Phi[\mathfrak{A}] \rightarrow \varphi \in Imp\,\mathsf{K}.$$

If this is true, then $\mathfrak{F} \models \varphi\,[id]$, because we are assuming that $\mathfrak{F}$ belongs to $Mod\,Imp\,\mathsf{K}$ and $\mathfrak{A} \preccurlyeq \mathfrak{F}$. Therefore, $Fg_{\mathsf{K}}\mathfrak{A} \preccurlyeq \mathfrak{F}$ and the assertion will be proved.

It only remains to show the claim. The proof runs as follows. Let $\mathfrak{B}$ be any element of K and $h : \mathbf{Te}_{\mathcal{L}} \to \mathbf{Te}_{\mathcal{L}}$ any homomorphism, and suppose $\mathfrak{B} \models \bigwedge \Phi[\mathfrak{A}]\,[h]$. This assumption is clearly equivalent to $\mathfrak{A} \preccurlyeq h^{-1}\mathfrak{B}$ by the previous observation. Moreover, since K is a full class, we must have $h^{-1}\mathfrak{B} \in \mathsf{K}$. So, $Fg_{\mathsf{K}}\mathfrak{A} \preccurlyeq h^{-1}\mathfrak{B}$. We now apply the hypothesis $Fg_{\mathsf{K}}\mathfrak{A} \models \varphi\,[id]$ and conclude $h^{-1}\mathfrak{B} \models \varphi\,[id]$, i.e., $\mathfrak{B} \models \varphi\,[h]$. That is what we wanted. ■

According to the preceding theorem, we shall be forced to restrict our attention to quasivarieties of $\mathcal{L}$-structures whenever we want a class to exhibit certain algebraic properties, namely those that entirely depend on the filter-lattice condition. This explains why the rest of the work will often be centered on the development of a model theory for the strict universal Horn fragment of first-order logic without equality. The next definition is a central one to this purpose.

Let $\mathfrak{A}$ any $\mathcal{L}$-structure. We say $\mathfrak{B}$ is a *filter extension of* $\mathfrak{A}$ *relative to* K, or simply a K*-filter extension of* $\mathfrak{A}$, if $\mathfrak{B} \in \mathsf{K}$ and $\mathfrak{A} \preccurlyeq \mathfrak{B}$. The set of all such relative filter extensions is denoted by $Fe_{\mathsf{K}}\mathfrak{A}$. If K is a quasivariety then it coincides with the principal sublattice of $\mathbb{K}_{\mathbf{A}}$ generated by $\mathfrak{A}$, and thus forms an algebraic complete lattice $\mathbf{Fe}_{\mathsf{K}}\mathfrak{A} = \langle Fe_{\mathsf{K}}\mathfrak{A}, \cap, \vee\rangle$. Note that $Fe_{\mathsf{K}}(\bigcap \mathsf{K}_{\mathbf{A}}) = \mathsf{K}_{\mathbf{A}}$. For convenience, we standarize a notation introduced in the proof of the last theorem; we write

$$Fg_{\mathsf{K}}\mathfrak{A} := \bigcap\{\mathfrak{B} \in \mathsf{K} : \mathfrak{A} \preccurlyeq \mathfrak{B}\}.$$

The structure $Fg_{\mathsf{K}}\mathfrak{A}$ is called the K*-filter extension generated by* $\mathfrak{A}$. Obviously, if $\mathfrak{A} \in \mathsf{K}$ then $Fg_{\mathsf{K}}\mathfrak{A} = \mathfrak{A}$.

Examples. If $\mathsf{K} = \mathsf{K}_{\mathrm{co}}$ and $\langle \mathbf{A}, \theta\rangle$ is a member of K, then $\mathbf{Fe}_{\mathsf{K}}\langle \mathbf{A}, \theta\rangle \cong \mathbf{CoA}/\theta$ by the Correspondence Theorem of universal algebra. More generally, let $\mathcal{Q}$ be any class of $\mathcal{L}$-algebras and $\mathbf{A}$ any $\mathcal{L}$-algebra. Define the set of $\mathcal{Q}$*-congruences on* $\mathbf{A}$ as

$$Co_{\mathcal{Q}}\mathbf{A} = \{\phi \in Co\,\mathbf{A} : \mathbf{A}/\phi \in \mathcal{Q}\}.$$

Then $\mathbf{Fe}_{\mathsf{K}_{\mathcal{Q}}}\langle \mathbf{A}, \theta\rangle \cong Co_{\mathcal{Q}}\mathbf{A}/\theta$ for all $\langle \mathbf{A}, \theta\rangle \in \mathsf{K}_{\mathcal{Q}}$, whenever $\mathcal{Q}$ is a quasivariety.

Another example of this kind will be provided in Section 5.3, using a more general form of the Correspondence Theorem. ⊣

In the last part of this Section we look at the connection between the lattice of relative filter extensions of a structure and that of its homomorphic images. For this, given a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ between two structures $\mathfrak{A}, \mathfrak{B}$ of a quasivariety K, let $h_{\mathsf{K}} : Fe_{\mathsf{K}}\mathfrak{A} \to Fe_{\mathsf{K}}\mathfrak{B}$ be the mapping defined by setting

$$h_{\mathsf{K}}\mathfrak{A}' := Fg_{\mathsf{K}}\langle \mathbf{B}, hR_{\mathfrak{A}'} \cup R_{\mathfrak{B}}\rangle,$$

for all $\mathfrak{A}' \in Fe_{\mathsf{K}}\mathfrak{A}$. Then the connection is summarized in the next lemma. Something more can be said by imposing some restrictions on the class of structures; see, e.g., Theorem 5.3.8 below.

LEMMA 5.1.2. *Let* $\mathbf{K}$ *be a quasivariety of* $\mathcal{L}$*-structures. Let* $\mathfrak{A}, \mathfrak{B}$ *be elements of* $\mathbf{K}$ *and* $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$. *The following statements hold.*

(i) $h^{-1}\mathfrak{B}' \in Fe_{\mathbf{K}}\mathfrak{A}$, *for all* $\mathfrak{B}' \in Fe_{\mathbf{K}}\mathfrak{B}$.

(ii) *If* $\mathfrak{B}' \in Fe_{\mathbf{K}}\mathfrak{B}$ *then* $h_{\mathbf{K}}h^{-1}\mathfrak{B}' = hh^{-1}\mathfrak{B}' = \mathfrak{B}'$.

(iii) *If* $\mathfrak{A}' \in Fe_{\mathbf{K}}\mathfrak{A}$ *then* $h^{-1}h_{\mathbf{K}}\mathfrak{A}' = h^{-1}h\mathfrak{A}' = \mathfrak{A}'$ *iff* $h^{-1}\mathfrak{B} \preccurlyeq \mathfrak{A}'$ *and* $Ker\, h \in Co\,\mathfrak{A}'$. *In particular,* $h^{-1}h_{\mathbf{K}}\mathfrak{A} = h^{-1}h\mathfrak{A} = \mathfrak{A}$ *iff* h *is a reductive homomorphism.*

(iv) $h^{-1}(\mathfrak{B}' \cap \mathfrak{B}'') = h^{-1}\mathfrak{B}' \cap h^{-1}\mathfrak{B}''$, *for all* $\mathfrak{B}', \mathfrak{B}'' \in Fe_{\mathbf{K}}\mathfrak{B}$.

(v) $h_{\mathbf{K}}Fg_{\mathbf{K}}\mathfrak{C} = h_{\mathbf{K}}\mathfrak{C}$, *for all* $\mathfrak{C} \in Fe\,\mathfrak{A}$.

Proof. (i) It is a consequence of the fact that h is a reductive homomorphism from $h^{-1}\mathfrak{B}'$ onto $\mathfrak{B}'$ and the assumption that $\mathbf{K}$ is a full class.

(ii) The equality $hh^{-1}\mathfrak{B}' = \mathfrak{B}'$ follows directly from the surjectivity of h, and implies that $Fg_{\mathbf{K}}(hh^{-1}\mathfrak{B}') = \mathfrak{B}'$. So, (ii) holds.

(iii) Since h is surjective, $h^{-1}h_{\mathbf{K}}\mathfrak{A}' = h^{-1}h\mathfrak{A}'$ implies $h_{\mathbf{K}}\mathfrak{A}' = h\mathfrak{A}'$, so that $\mathfrak{B} \preccurlyeq h\mathfrak{A}'$. Hence, the equality $h^{-1}h\mathfrak{A}' = \mathfrak{A}'$ gives $h^{-1}\mathfrak{B}' \preccurlyeq \mathfrak{A}'$. On the other hand, $h^{-1}h\mathfrak{A}' = \mathfrak{A}'$ entails h is a reductive homomorphism from $\mathfrak{A}'$ onto $h\mathfrak{A}'$, so by Lemma 2.1.5, $Ker\, h \in Co\,\mathfrak{A}'$. This proves one implication. For the converse, we know that $Ker\, h \in Co\,\mathfrak{A}'$ implies $h^{-1}h\mathfrak{A}' = \mathfrak{A}'$, and consequently $h : \mathfrak{A}' \twoheadrightarrow_s h\mathfrak{A}'$. Therefore, $h\mathfrak{A}' \in \mathbf{K}_{\mathbf{B}}$, for $\mathbf{K}$ is a full class and $\mathfrak{A}' \in \mathbf{K}_{\mathbf{A}}$. On the other hand, $h^{-1}\mathfrak{B} \preccurlyeq \mathfrak{A}'$, and so $\mathfrak{B} \preccurlyeq h\mathfrak{A}'$. Thus, $h\mathfrak{A}' \in Fe_{\mathbf{K}}\mathfrak{B}$ and finally $h\mathfrak{A}' = h_{\mathbf{K}}\mathfrak{A}'$. The case $\mathfrak{A}' = \mathfrak{A}$ holds as a consequence of 2.1.5.

(iv) It is obvious.

(v) The inclusion $h_{\mathbf{K}}\mathfrak{C} \preccurlyeq h_{\mathbf{K}}Fg_{\mathbf{K}}\mathfrak{C}$ is clear. To see the opposite inclusion assume $\mathfrak{B}'$ is any $\mathbf{K}$-filter extension of $\mathfrak{B}$ that includes $h\mathfrak{C}$. Then $\mathfrak{C} \preccurlyeq h^{-1}\mathfrak{B}'$. Also, since $h^{-1}\mathfrak{B}' \in Fe_{\mathbf{K}}\mathfrak{A}$, we have $Fg_{\mathbf{K}}\mathfrak{C} \preccurlyeq h^{-1}\mathfrak{B}'$ and hence, by (ii),

$$h_{\mathbf{K}}Fg_{\mathbf{K}}\mathfrak{C} \preccurlyeq Fg_{\mathbf{K}}(hh^{-1}\mathfrak{B}') \vee \mathfrak{B} = \mathfrak{B}'.$$

Thus, $h_{\mathbf{K}}Fg_{\mathbf{K}}\mathfrak{C} \preccurlyeq h_{\mathbf{K}}\mathfrak{C}$ also holds. ∎

5.2. Leibniz Operator and Relative Congruences

Following the terminology introduced by Blok and Pigozzi [8, p.10], we call *Leibniz operator* the mapping $\Omega : \mathfrak{A} \longmapsto \Omega\mathfrak{A}$ defined on the whole class of $\mathcal{L}$-structures. Given a quasivariety $\mathbf{K}$, this mapping can be restricted to $\mathbf{K}_{\mathbf{A}}$, for each $\mathcal{L}$-algebra $\mathbf{A}$, and then gives rise to a mapping between two algebraic complete lattices, namely $\mathbb{K}_{\mathbf{A}}$ and $\mathbf{Co\,A}$. An element of the image of $\mathbf{K}_{\mathbf{A}}$ under Ω is called a *congruence on* $\mathbf{A}$ *relative to* $\mathbf{K}$, or simply a $\mathbf{K}$*-congruence on* $\mathbf{A}$; we write $Co_{\mathbf{K}}\mathbf{A}$ to denote the set of all such congruences, i.e.,

$$Co_{\mathbf{K}}\mathbf{A} := \{\theta \in Co\,\mathbf{A} : \theta = \Omega\mathfrak{A} \text{ for some } \mathfrak{A} \in \mathbf{K}_{\mathbf{A}}\}.$$

Examples. We already know that Ω restricted to $\mathbf{K}_{\mathrm{co}}$ is the projection onto the second component. So we trivially have that $Co_{\mathbf{K}_{\mathbf{Q}}}\mathbf{A} = Co_{\mathbf{Q}}\mathbf{A}$, for each $\mathcal{L}$-algebra

A and each quasivariety $\mathcal{Q}$ of algebras of type $\mathcal{L}$. Likewise, since $\mathsf{K}_{co} \subseteq \mathsf{K}_{qo}$, $Co_{\mathsf{K}_{qo,\mathcal{Q}}}\mathbf{A}$ and $Co_{\mathsf{K}_{po,\mathcal{Q}}}\mathbf{A}$ also coincide with $Co_{\mathcal{Q}}\mathbf{A}$. These three equalities are in fact trivial cases of a general result proved in Chapter 9, Proposition 9.1.1. ⊣

Remark. Observe that, although all the preceding three classes determine the same set of K-congruences, the Ω operator restricted to $(\mathsf{K}_{\mathcal{Q}})_{\mathbf{A}}$ is essentially the identity function and thus an isomorphism onto $Co_{\mathcal{Q}}\mathbf{A}$, whereas it is not one-one, nor even a lattice homomorphism, when the domain is one of the broader classes $(\mathsf{K}_{qo,\mathcal{Q}})_{\mathbf{A}}$ or $(\mathsf{K}_{po,\mathcal{Q}})_{\mathbf{A}}$. This is the ultimate reason for $\mathsf{K}_{\mathcal{Q}}$ to exhibit a better algebraic character than $\mathsf{K}_{qo,\mathcal{Q}}$ or $\mathsf{K}_{po,\mathcal{Q}}$. ⊣

The importance of the Leibniz operator just rests on the connection between its properties when restricted to a given class K and the fulfilment of some properties by K itself and the associated reduced class K^*. Using the notion introduced above, the idea turns out to be quite simple: the more assumptions on Ω to guarantee that the relational part of members of K can be replaced by the K-congruences with "no loss of information", the nicer algebraic character of K and K^*. The same idea can still be expressed in other words by noting that, in essence, congruences are weaker forms of equality; under this view, the restrictions on Ω are better thought of as restrictions to ensure that the set of all predicates in the members of K is "close" to an equality predicate.

At this point, a key issue that arises naturally is to find out the properties that must be assumed on Ω. For the case we are interested in, i.e., when K is a quasivariety, the properties that seem to be of interest include the ones typical of mappings between two algebraic complete lattices (e.g. to be a meet or join homomorphism, to be monotone or injective, and so on), for Ω is just of this type when it is restricted to the posets of K-structures on the $\mathcal{L}$-algebras. Apparently, however, on the base of some special cases investigated in detail in the context of algebraic logic (see, e.g., [12]), few of this properties seem to be enough to reflect the algebraic character of K and K^*. But this is still an obscure point that asks for a systematic investigation.

On the other hand, the behaviour of the Ω operator with respect to the different algebraic constructions described in Chapter 1 also seems to be relevant. For instance, the property used in Definition 5.4.1 below to distinguish a special kind of quasivarieties turns out to be of this sort. In fact, all of them are closely connected, and an open problem is to express them in terms of properties of a purely syntactical nature that describe the explicit connection between the Leibniz equality predicate and the predicates of the language. We come back to this point later in Chapter 7, after Theorem 7.1.4.

The next result concerns the conditions under which the poset $Co_{\mathsf{K}}\mathbf{A}$ can be endowed with a structure of complete lattice for every $\mathcal{L}$-algebra A.

PROPOSITION 5.2.1. *Let* K *be a quasivariety of* $\mathcal{L}$*-structures and* A *an* $\mathcal{L}$*-algebra. If* Ω *is a complete meet-homomorphism between the lattices* $\mathbb{K}_{\mathbf{A}}$ *and* Co A, *then* $\mathbf{Co}_{\mathsf{K}}\mathbf{A} = \langle Co_{\mathsf{K}}\mathbf{A}, \cap, \vee_{\mathsf{K}} \rangle$ *is a meet-complete subsemilattice of* Co A, *where*

$$\theta \vee_{\mathsf{K}} \phi := \bigcap\{\Theta \in Co_{\mathsf{K}}\mathbf{A} : \theta, \phi \subseteq \Theta\}. \blacksquare$$

Notice that, as the previous result shows, the lattice structure of $\mathbf{Co_K A}$ is the one inherited from $\mathbf{K_A}$ by Ω; in general, $\mathbf{Co_K A}$ is not a sublattice of $\mathbf{Co\,A}$, for $\theta \vee_K \phi$ may strictly include every congruence on $\mathbf{A}$ that contains θ and ϕ. Even more, the mapping $\Omega : \mathbb{K}_\mathbf{A} \longrightarrow \mathbf{Co_K A}$ rarely is a join-homomorphism, though the assumption Ω is a complete meet-homomorphism is enough to prove the inclusion $\Omega\mathfrak{A} \vee_K \Omega\mathfrak{B} \subseteq \Omega(\mathfrak{A} \vee \mathfrak{B})$. We shall delay however further discussion on the lattice structure of $\mathbf{Co_K A}$ until Chapter 9, for it is right there where the problem of finding out similar conditions to ensure $Co_K\mathbf{A}$ has a structure of algebraic complete lattice is specially meaningful.

5.3. Protoalgebraic Classes

Let $\mathfrak{A}, \mathfrak{B}$ two $\mathcal{L}$-structures such that $\mathfrak{A} \preccurlyeq \mathfrak{B}$. Clearly, for any atomic $\mathcal{L}$-formula φ and any assignment $g : \mathbf{Te}_\mathcal{L} \longrightarrow \mathbf{A}$, the condition $\mathfrak{A} \models \varphi\,[g]$ entails $\mathfrak{B} \models \varphi\,[g]$. Nevertheless, this does not mean that two elements a, b of the common universe of $\mathfrak{A}$ and $\mathfrak{B}$ staisfy exactly the same first-order properties in the model $\mathfrak{B}$ whenever they do in the model $\mathfrak{A}$, i.e., $a \equiv b\ (\Omega\mathfrak{A})$ does not imply $a \equiv b\ (\Omega\mathfrak{B})$. In spite of this fact, it seems up to some point reasonable to assume that this is the case; i.e., the more positive information we gain about the common univers in passing from $\mathfrak{A}$ to $\mathfrak{B}$, the more denotations of its elements can be identified. In other words, this information can never be used to distinguish two elements that were formerly identified. This assumption is in the origin of one of the main notions in the paper, a notion due to Blok and Pigozzi [7][18].

DEFINITION 5.3.1. *A full class* K *of* $\mathcal{L}$*-structures is said to be protoalgebraic if* Ω *is* $\preccurlyeq$*-monotone in* K, *i.e., for each* $\mathcal{L}$*-algebra* $\mathbf{A}$, *and each* $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}_\mathbf{A}$, $\mathfrak{A} \preccurlyeq \mathfrak{B}$ *implies* $\Omega\mathfrak{A} \subseteq \Omega\mathfrak{B}$.

Let us notice that in the previous definition we do not assume anything on the class K except being full, and even this restriction is superflous. But we shall see that the best properties of protoalgebraic classes hold when K is a quasivariety.

PROPOSITION 5.3.2. *For every first-order language* $\mathcal{L}$ *and every quasivariety* $\mathcal{Q}$ *of* $\mathcal{L}$*-algebras, the classes* K_{eq}, $\mathsf{K}_{qo,\mathcal{Q}}$, $\mathsf{K}_{po,\mathcal{Q}}$ *and* $\mathsf{K}_\mathcal{Q}$ *are protoalgebraic.*

Proof. We saw in Section 2.1 that Ω is the function $\langle \mathbf{A}, \theta \rangle \longmapsto \bigvee\{\phi \in Co\,\mathbf{A} : \phi \subseteq \theta\}$ on K_{eq}, the function $\langle \mathbf{A}, \theta \rangle \longmapsto \theta \cap \theta^{-1}$ on K_{qo} and the projection onto the second component on K_{co}. Thus, the above four classes are trivially protoalgebraic. ∎

In contrast to what happens for the preceding classes, neither K_{to} nor $\mathsf{K}_{to,\mathcal{Q}}$ are in general protoalgebraic. For instance, an easy counterexemple for K_{to} is the following. Consider the language $\mathcal{L}$ with no function symbols and just one relation

[18] Actually, Blok and Pigozzi consider sentential logics whose class of matrix models satisfy the above monotonicity of the Leibniz operator. In [12], they prove that such logics are exactly those that Czelakowski called *non-pathological logics* in earlier papers [33,36].

symbol r, of arity 2, and let $\mathfrak{A}, \mathfrak{B}$ the following $\mathcal{L}$-structures:

$$\mathfrak{A} := \langle \{0,1,2\}, r^{\mathfrak{A}} \rangle, \quad r^{\mathfrak{A}} := \Delta_{\{0,1,2\}} \cup \{(0,1),(1,0)\};$$
$$\mathfrak{B} := \langle \{0,1,2\}, r^{\mathfrak{B}} \rangle, \quad r^{\mathfrak{B}} := \Delta_{\{0,1,2\}} \cup \{(0,1),(1,0),(0,2),(2,0)\}.$$

Then $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}_{\mathrm{to}}$ and $\mathfrak{A} \preccurlyeq \mathfrak{B}$. On the other hand, we easily have that $\Omega\mathfrak{A} = r^{\mathfrak{A}}$ and $\Omega\mathfrak{B} = \Delta_{\{0,1,2\}} \cup \{(1,2),(2,1)\}$. Hence, $\Omega\mathfrak{A} \not\subseteq \Omega\mathfrak{B}$.

There are several alternative characterizations of protoalgebraicity that provide insight into the various aspects of the notion. It is important however to realize that some of these characterizations hold in general, whereas some others only hold for classes satisfying more than the condition of fullness. The first two are just useful reformulations of the definition.

PROPOSITION 5.3.3. *A full class* K *is protoalgebraic iff for any* $\mathcal{L}$*-algebra* $\mathbf{A}$ *and any* $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}_{\mathbf{A}}$ *such that* $\mathfrak{A} \preccurlyeq \mathfrak{B}$, *we have* $Co\,\mathfrak{A} \subseteq Co\,\mathfrak{B}$, *i.e., if* $\theta \in Co\,\mathbf{A}$ *is compatible with the relations on* $\mathfrak{A}$, *it is also compatible with the relations on any other* K*-filter extension of* $\mathfrak{A}$.

Proof. It follows directly from 2.1.1. ∎

PROPOSITION 5.3.4. *Let* K *be a full class of* $\mathcal{L}$*-structures satisfying the following property (see Corollary 6.1.2 below for an equivalent formulation of this condition):*

(5.1) $\mathsf{K}_{\mathbf{A}}$ *is a closure system for all* $\mathcal{L}$*-algebra* $\mathbf{A}$.

Then K *is protoalgebraic iff* Ω *is meet-continuous in* K, *i.e., for any* $\mathcal{L}$*-algebra* $\mathbf{A}$ *and any set* $\{\mathfrak{A}_i : i \in I\}$ *of* K*-structures on* $\mathbf{A}$, *the equality* $\Omega(\bigcap_{i\in I} \mathfrak{A}_i) = \bigcap_{i \in I} \Omega\mathfrak{A}_i$ *holds.*

Proof. Suppose K is protoalgebraic and let $\mathbf{A}$ be any $\mathcal{L}$-algebra. Let $\mathfrak{A}_i$, $i \in I$, be a family of K-structures on $\mathbf{A}$. We must show that $\Omega\mathfrak{A} = \bigcap_{i\in I} \Omega\mathfrak{A}_i$, where $\mathfrak{A} = \bigcap_{i\in I} \mathfrak{A}_i$. The inclusion from left to right is a direct consequence of Ω being $\preccurlyeq$-monotone. To see the opposite inclusion, we are going to show that $\bigcap_{i\in I} \Omega\mathfrak{A}_i$ is a congruence on $\mathfrak{A}$. Indeed, let r be any relation symbol and let $\mathbf{a}, \mathbf{b} \in A^{\rho(r)}$ be such that $\mathbf{a} \in r^{\mathfrak{A}}$ and $\mathbf{a} \equiv \mathbf{b}\ (\bigcap \Omega\mathfrak{A}_i)$. For all i, we have $\mathbf{a} \equiv \mathbf{b}\ (\Omega\mathfrak{A}_i)$ and hence $\mathbf{b} \in r_i^{\mathfrak{A}}$. Therefore, $\mathbf{b} \in r^{\mathfrak{A}}$. This proves the forward implication.

The backward implication is easier: if $\mathfrak{A} \preccurlyeq \mathfrak{B}$, then $\mathfrak{A} = \mathfrak{A} \cap \mathfrak{B}$ and consequently $\Omega\mathfrak{A} = \Omega\mathfrak{A} \cap \Omega\mathfrak{B}$. ∎

A third characterization that holds for classes satisfying condition (5.1) is the following. Let K be a full class of $\mathcal{L}$-structures such that (5.1) is true for K. Given any relation symbol r of $\mathcal{L}$ and a tuple $\mathbf{a} \in A^{\rho(r)}$, set

$$Fg_{\mathsf{K}}^{\mathfrak{A}}[r; \mathbf{a}] := \bigcap \{\mathfrak{B} \in Fe_{\mathsf{K}}\mathfrak{A} : \mathbf{a} \in r^{\mathfrak{B}}\}.$$

Then we can prove the following.

PROPOSITION 5.3.5. *Let* K *be any full class of* $\mathcal{L}$*-structures satisfying (5.1). Then* K *is protoalgebraic iff for all members* $\mathfrak{A}$ *of* K, *all relation symbols* $r \in R$ *and all tuples* $\mathbf{a}, \mathbf{b} \in A^{\rho(r)}$, *the condition* $\mathbf{a} \equiv \mathbf{b}\ (\Omega\mathfrak{A})$ *implies* $Fg_{\mathsf{K}}^{\mathfrak{A}}[r; \mathbf{a}] = Fg_{\mathsf{K}}^{\mathfrak{A}}[r; \mathbf{b}]$.

Proof. Assume K is protoalgebraic. Fix a member $\mathfrak{A}$ of K and a relation symbol r. Let $n = \rho(r)$ and let $\mathbf{a} = \langle a_1, \dots, a_n \rangle, \mathbf{b} = \langle b_1, \dots, b_n \rangle$ be two arbitrary elements of A^n such that $\mathbf{a} \equiv \mathbf{b}$ $(\Omega\mathfrak{A})$. For all $i \leq n$, define $\mathbf{b}_i = \langle b_1, \dots, b_i, a_{i+1}, \dots, a_n \rangle$; in particular, we have $\mathbf{a} = \mathbf{b}_0$ and $\mathbf{b} = \mathbf{b}_n$. We claim that

$$Fg^{\mathfrak{A}}_{\mathsf{K}}\,[r; \mathbf{b}_{i-1}] = Fg^{\mathfrak{A}}_{\mathsf{K}}\,[r; \mathbf{b}_i], \text{ for all } i > 0. \tag{5.2}$$

If this is true, then the forward implication follows trivially. So let us prove (5.2). To this goal, we use 2.1.2 and the hypothesis that K is protoalgebraic. Then we obatin the following chain of implications:

$$\begin{aligned} a_i \equiv b_i\ (\Omega\mathfrak{A}) \text{ implies } & a_i \equiv b_i\ (\Omega\mathfrak{B}) \text{ for all } \mathfrak{B} \in Fe_{\mathsf{K}}\mathfrak{A} \\ \text{implies } & \mathfrak{B} \models \forall z_1 \dots \forall z_{n-1}(rz_1 \dots z_{i-1}xz_i \dots z_{n-1} \\ & \quad \leftrightarrow rz_1 \dots z_{i-1}yz_i \dots z_{n-1})\,[a_i, b_i], \text{ for all } \mathfrak{B} \in Fe_{\mathsf{K}}\mathfrak{A} \\ \text{implies } & \mathbf{b}_{i-1} \in r^{\mathfrak{B}} \text{ iff } \mathbf{b}_i \in r^{\mathfrak{B}}, \text{ for all } \mathfrak{B} \in Fe_{\mathsf{K}}\mathfrak{A} \\ \text{implies } & Fg^{\mathfrak{A}}_{\mathsf{K}}\,[r; \mathbf{b}_{i-1}] = Fg^{\mathfrak{A}}_{\mathsf{K}}\,[r; \mathbf{b}_i]. \end{aligned}$$

To see the converse, let $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ be such that $\mathfrak{A} \preccurlyeq \mathfrak{B}$. We are going to show $\Omega\mathfrak{A}$ is a congruence on $\mathfrak{B}$. Consider any relation symbol r and let $\mathbf{a}, \mathbf{b}$ be two tuples of length $\rho(r)$ satisfying the conditions $\mathbf{a} \in r^{\mathfrak{B}}$ and $\mathbf{a} \equiv \mathbf{b}$ $(\Omega\mathfrak{A})$. Then we have that $Fg^{\mathfrak{A}}_{\mathsf{K}}\,[r; \mathbf{b}] = Fg^{\mathfrak{A}}_{\mathsf{K}}\,[r; \mathbf{a}] \preccurlyeq \mathfrak{B}$. As a result, $\mathbf{b} \in r^{\mathfrak{B}}$. This finishes the proof of the proposition. ∎

From Proposition 5.3.3 and Theorem 2.2.6 we easily obtain that protoalgebraic classes satisfy a generalized form of the Second Isomorphism Theorem of universal algebra.

COROLLARY 5.3.6. *For every protoalgebraic class* K *of* $\mathcal{L}$*-structures, if* $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ *are such that* $\mathfrak{A} \preccurlyeq \mathfrak{B}$ *then* $\Omega\mathfrak{A} \subseteq \Omega\mathfrak{B}$ *and* $(\mathfrak{B}/\Omega\mathfrak{A})/(\Omega\mathfrak{B}/\Omega\mathfrak{A}) \cong \mathfrak{B}^*$. ∎

Related to the above corollary, we have the following definition, which isolates the property that is mainly responsible for the distinctive algebraic character of protoalgebraic quasivarieties; it consists precisely in satisfying a kind of "filter version" of the Correspondence Theorem of universal algebra.

DEFINITION 5.3.7. *A quasivariety* K *of* $\mathcal{L}$*-structures is said to have the filter correspondence property (FCP for short) if for any* $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ *and any reductive homomorphism* $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, *the mapping* $\mathfrak{B}' \longmapsto h^{-1}\mathfrak{B}'$ *defines an isomorphism between* $\mathbf{Fe}_{\mathsf{K}}\mathfrak{B}$ *and* $\mathbf{Fe}_{\mathsf{K}}\mathfrak{A}$ *with inverse* $\mathfrak{A}' \longmapsto h\mathfrak{A}'$.

The next theorem contains some different characterizations, very close in spirit, of protoalgebraic quasivarieties. Similar results are included in [7,12].

THEOREM 5.3.8. *Let* K *be a quasivariety of* $\mathcal{L}$*-structures. Then the following statements are equivalent.*

(i) K *is protoalgebraic.*
(ii) For all $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ *and all* $h : \mathfrak{A} \to_s \mathfrak{B}$, $h^{-1}h\mathfrak{A}' = \mathfrak{A}'$ *whenever* $\mathfrak{A}' \in Fe_{\mathsf{K}}\mathfrak{A}$.
(iii) K *has the FCP.*

(iv) *For all* $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ *and all* $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, $h^{-1}h_{\mathsf{K}}\mathfrak{A}' = \mathfrak{A}' \vee h^{-1}\mathfrak{B}$ *whenever* $\mathfrak{A}' \in Fe_{\mathsf{K}}\mathfrak{A}$.

Proof. Assume K is protoalgebraic, and let $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ and $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$. Consider any $\mathfrak{A}' \in Fe_{\mathsf{K}}\mathfrak{A}$. By Lemma 2.1.5, $Ker\, h \in Co\,\mathfrak{A}$, so 5.3.3 gives $Ker\, h \in Co\,\mathfrak{A}'$. Moreover, since h is strong, $h^{-1}\mathfrak{B} = \mathfrak{A} \preccurlyeq \mathfrak{A}'$. Hence, from Lemma 5.1.2(iii) we conclude that $h^{-1}h\mathfrak{A}' = \mathfrak{A}'$ and the implication from (i) to (ii) is proved.

Suppose now (ii) and let us show (iii). Consider $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$, for $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$. Clearly, the mapping $\mathfrak{B}' \longmapsto h^{-1}\mathfrak{B}'$ from $Fe_{\mathsf{K}}\mathfrak{B}$ into $Fe_{\mathsf{K}}\mathfrak{A}$ is well defined and order-preserving. If $\mathfrak{A}' \in Fe_{\mathsf{K}}\mathfrak{A}$ then (ii) says $h^{-1}h\mathfrak{A}' = \mathfrak{A}'$, so that $h\mathfrak{A}' \in R(\mathfrak{A}') \subseteq \mathsf{K}$. Thus, $\mathfrak{A}' \longmapsto h\mathfrak{A}'$ is also well defined and order-preserving, and consequently the class K has the FCP.

Assume (iii) and let $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ for $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$. Let $\mathfrak{A}' \in Fe_{\mathsf{K}}\mathfrak{A}$. The inclusion $\mathfrak{A}' \vee h^{-1}\mathfrak{B} \preccurlyeq h^{-1}h_{\mathsf{K}}\mathfrak{A}'$ is clear, for $\mathfrak{A}' \preccurlyeq h^{-1}h_{\mathsf{K}}\mathfrak{A}'$ and $\mathfrak{B} \preccurlyeq h_{\mathsf{K}}\mathfrak{A}'$. To show the reverse, we use that h is a reductive homomorphism from $h^{-1}\mathfrak{B}$ onto $\mathfrak{B}$. Then, $\mathfrak{A}' \vee h^{-1}\mathfrak{B} \in Fe_{\mathsf{K}}h^{-1}\mathfrak{B}$, and so the FCP gives that $h(\mathfrak{A}' \vee h^{-1}\mathfrak{B})$ is a K-filter extension of $\mathfrak{B}$ such that $h^{-1}h(\mathfrak{A}' \vee h^{-1}\mathfrak{B}) = \mathfrak{A}' \vee h^{-1}\mathfrak{B}$. Therefore, since $h\mathfrak{A}' \preccurlyeq h(\mathfrak{A}' \vee h^{-1}\mathfrak{B})$, we conclude $h^{-1}h_{\mathsf{K}}\mathfrak{A}' \preccurlyeq h^{-1}h(\mathfrak{A}' \vee h^{-1}\mathfrak{B}) = \mathfrak{A}' \vee h^{-1}\mathfrak{B}$. So, the implication from (iii) to (iv) is proved.

Let us finally see that (iv) entails (i). For this, assume $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ are such that $\mathfrak{A} \preccurlyeq \mathfrak{B}$. Consider the natural projection $\pi : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}^*$. Since $\mathfrak{B} \in Fe_{\mathsf{K}}\mathfrak{A}$, (iv) implies that $\pi^{-1}\pi_{\mathsf{K}}\mathfrak{B} = \mathfrak{B} \vee \pi^{-1}\mathfrak{A}^*$. Hence, as π is a reductive homomorphism, $\mathfrak{B} \preccurlyeq \pi^{-1}\pi\mathfrak{B} \preccurlyeq \pi^{-1}\pi_{\mathsf{K}}\mathfrak{B} = \mathfrak{B}$, and consequently, $\pi^{-1}\pi_{\mathsf{K}}\mathfrak{B} = \pi^{-1}\pi\mathfrak{B} = \mathfrak{B}$. We apply 5.1.2(iii) and obtain $Ker\,\pi = \Omega\mathfrak{A} \in Co\,\mathfrak{B}$. As a result, $\Omega\mathfrak{A} \subseteq \Omega\mathfrak{B}$. ∎

Examples. The Correspondence Theorem of universal algebra is a particular case of the FCP obtained when K is taken to be the quasivariety K_{co}; in this case, given $\langle \mathbf{A}, \theta\rangle \in \mathsf{K}_{co}$, we have that $\mathbf{Fe}_{\mathsf{K}}\langle \mathbf{A}, \theta\rangle \cong \mathbf{Fe}_{\mathsf{K}}\langle \mathbf{A}, \theta\rangle^*$. But, since the Leibniz quotient of $\langle \mathbf{A}, \theta\rangle$ is $\langle \mathbf{A}/\theta, \Delta_{A/\theta}\rangle$, this means that $[\theta, \nabla_A] \cong Co\ \mathbf{A}/\theta$, where the isomorphism is given by the mapping $\phi \longmapsto \phi/\theta$.

A similar correspondence theorem can be obtained when we apply the preceding theorem to the protoalgebraic quasivariety K_{qo}. For every $\mathcal{L}$-algebra $\mathbf{A}$, define the set

$$Qo\ \mathbf{A} = \{\phi \subseteq A^2 : \phi \text{ is a quasi-order on } \mathbf{A}\}.$$

Then, if $\mathsf{K} = \mathsf{K}_{qo}$ and $\langle \mathbf{A}, \theta\rangle$ is a member of K, the lattice $\mathbf{Fe}_{\mathsf{K}}\langle \mathbf{A}, \theta\rangle$ is isomorphic to the sublattice $[\theta, \nabla_A]$ of $\mathbf{Qo\,A}$, whereas $\mathbf{Fe}_{\mathsf{K}}\langle \mathbf{A}, \theta\rangle^* \cong \mathbf{Qo\,A}/\theta \cap \theta^{-1}$. Thus the FCP says that

$$[\theta, \nabla_A] \cong \mathbf{Qo\,A}/\theta \cap \theta^{-1}$$

by the mapping $\phi \longmapsto \phi/\theta \cap \theta^{-1}$. ⊣

Certainly, the last characterization contributes significantly to get some idea about the nice properties that can be expected of protoalgebraic classes (quasivarieties) of structures. We are going to end this Section with some results which illustrate this point successfully. The first one expresses basically that, if K is a protoalgebraic class, then K^* is a full reflective subcategory of K (this latter category with all surjective homomorphisms as arrows)[19]. An earlier version of this

[19] For the notion of *reflective subcategory* see, e.g., [80, p.89].

theorem can be found in [12, Thm. 8.2].

THEOREM 5.3.9. *Let* K *be any protoalgebraic class of* $\mathcal{L}$*-structures, and let* $\mathfrak{A}, \mathfrak{B}$ *be members of* K. *Then every surjective homomorphism* $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ *induces a surjective homomorphism* $h^* : \mathfrak{A}^* \twoheadrightarrow \mathfrak{B}^*$ *of the respective Leibniz quotients, defined by* $a/\Omega\mathfrak{A} \longmapsto ha/\Omega\mathfrak{B}$.

Proof. Certainly $\mathfrak{A} \preccurlyeq h^{-1}\mathfrak{B}$ and h maps strong homomorphically $h^{-1}\mathfrak{B}$ onto $\mathfrak{B}$. Thus, by protoalgebraicity and 2.1.8(i), we have $\Omega\mathfrak{A} \subseteq \Omega(h^{-1}\mathfrak{B}) = h^{-1}\Omega\mathfrak{B}$. So h^* is well defined and thus is an algebra homomorphism from $\mathbf{A}^*$ onto $\mathbf{B}^*$. Also, it is a homomorphism of $\mathfrak{A}^*$ onto $\mathfrak{B}^*$ since $hr^{\mathfrak{A}^*} = (hr^{\mathfrak{A}})/\Omega\mathfrak{B} \subseteq r^{\mathfrak{B}}/\Omega\mathfrak{B} = r^{\mathfrak{B}^*}$, for all relation symbol r. ∎

The second one concerns the commutativity of the operators E and F and allows us to get a description, which holds in some cases, of the relative subvariety generated by a class.

THEOREM 5.3.10. *Let* Q *be a protoalgebraic class of* $\mathcal{L}$*-structures, and let* K *be any subclass of* Q. *Then* $EF_{\mathsf{Q}}(\mathsf{K}) = F_{\mathsf{Q}}E(\mathsf{K})$.

Proof. The inclusion $EF_{\mathsf{Q}}(\mathsf{K}) \subseteq F_{\mathsf{Q}}E(\mathsf{K})$ follows from 4.1.3(i). So, assume $\mathfrak{A} \in F_{\mathsf{Q}}E(\mathsf{K})$. Let $h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{C}$ with $\mathfrak{C} \in \mathsf{K}$ and $\mathfrak{A} \in Fe_{\mathsf{Q}}\mathfrak{B}$. By 2.1.5, we know that $Ker\, h \in Co\,\mathfrak{B}$. Hence, as Q is protoalgebraic, $Ker\, h \in Co\,\mathfrak{A}$. Thus we just need to apply the Homomorphism Theorem: $\mathfrak{C} \cong \mathfrak{B}/Ker\, h \preccurlyeq \mathfrak{A}/Ker\, h$. ∎

COROLLARY 5.3.11. *Let* Q *be a protoalgebraic quasivariety of* $\mathcal{L}$*-structures and* K *any subclass of* Q. *Then* $\mathsf{K}^V \cap \mathsf{Q} = ERF_{\mathsf{Q}}SP(\mathsf{K})$.

In particular, if Q *is itself a variety, then* $\mathsf{K}^V = ERFSP(\mathsf{K})$. ∎

The last corollary solves an open problem suggested in the preceding Chapter. Remember that in Section 4.5 we proved a generalized form of Birkhoff's Variety Theorem to describe the variety generated by a given class. From this result we derived immediately a characterization of the reduced variety generated by a class as the one that results by applying the operator F^*ESP. The question was to find out sufficient conditions under which this operator is not distinct but coincides with $F^*S^*P^*$. Now we are going to show that protoalgebraicity is enough; in other words, the assumption of protoalgebraicity guarantees a good behaviour of the operator F when passing from full to reduced semantics, as Proposition 4.1.5 says that happens with the remaining operators. A special case of this result was proved in [12, Thm. 11.1] following a different argument. Chapter 7 below contains improved forms of the Theorem (Corollaries 7.2.3 and 7.2.6).

THEOREM 5.3.12. *Let* Q *be a protoalgebraic quasivariety of* $\mathcal{L}$*-structures, and let* K *be any subclass of* Q. *Then the reduced relative subvariety of* Q^* *generated by* K^* *is* $\mathsf{K}^V \cap \mathsf{Q}^* = F^*_{\mathsf{Q}}S^*P^*(\mathsf{K}^*)$.

In particular, if Q *is itself a variety, then* $(\mathsf{K}^V)^* = F^*S^*P^*(\mathsf{K}^*)$.

Proof. By Corollary 4.5.3, we know that $\mathsf{K}^V \cap \mathsf{Q}^* = LF_{\mathsf{Q}}ESP(\mathsf{K})$. So we must prove

$$LF_{\mathrm{Q}}ESP(\mathsf{K}) = F^*_{\mathrm{Q}}S^*P^*(\mathsf{K}^*).$$

It is easy to check that if φ is any atomic $\mathcal{L}$-formulauniversally satisfied by all the members of K, then φ is also satisfied by any structure in $F^*_{\mathrm{Q}}(\mathsf{K})$, $S^*(\mathsf{K})$ and $P^*(\mathsf{K})$. Thus, using 4.1.5(i), $F^*_{\mathrm{Q}}S^*P^*(\mathsf{K}^*) \subseteq F^*_{\mathrm{Q}}S^*P^*(\mathsf{K}) \subseteq LF_{\mathrm{Q}}ESP(\mathsf{K})$. Let us see the opposite inclusion. By the preceding theorem, the operators E and F_{Q} commute, and hence $LF_{\mathrm{Q}}ESP(\mathsf{K}) = F^*_{\mathrm{Q}}SP(\mathsf{K})$. We claim that

$$F^*_{\mathrm{Q}}(\mathfrak{A}) \subseteq F^*_{\mathrm{Q}}L(\mathfrak{A})$$

holds for all $\mathfrak{A} \in \mathrm{Q}$. The proof is as follows. Let $\mathfrak{A}$ be an element of Q, and let $\mathfrak{B}$ be any Q-filter extension of $\mathfrak{A}$. Since Q is protoalgebraic, $\Omega\mathfrak{A} \in Co\,\mathfrak{B}$. Hence, $\mathfrak{A}^* \preccurlyeq \mathfrak{B}/\Omega\mathfrak{A}$. Moreover, by 2.3.1, $(\mathfrak{B}/\Omega\mathfrak{A})^* \cong \mathfrak{B}^*$. Thus, $\mathfrak{B}^* \in F^*_{\mathrm{Q}}L(\mathfrak{A})$, as required.

Now, using the claim, the desired inclusion follows immediately from Lemma 4.1.5(i). ■

The preceding theorem can be applied, for instance, to the quasivariety K_{qo}. Recall from Section 3.3 that, for any variety of $\mathcal{L}$-algebras $\mathcal{V}$, the reduced relative subvariety $\mathsf{K}^*_{\mathrm{po},\mathcal{V}}$ of $\mathsf{K}^*_{\mathrm{qo}}$ is the class of all ordered $\mathcal{V}$-algebras, so that in this case Theorem 5.3.12 amounts to a result of Bloom [14, Thm. 2.6] recently called Bloom's Order Variety Theorem [121, p. 271].

There are still other alternative characterizations of classes which are protoalgebraic. We shall give two more characterizations in subsequent chapters.

5.4. Some Other Types of Classes

Another important property of the Leibniz operator that seems to provide a nice algebraic character of classes of structures, specially when passing to reduced semantics, concerns the connection between the Leibniz congruences of a structure and its substructures. Let $\mathfrak{A}, \mathfrak{B}$ be two $\mathcal{L}$-structures such that $\mathfrak{A} \subseteq \mathfrak{B}$. We are interested in those classes for which the expansion of the universe that occurs in passing from $\mathfrak{A}$ to $\mathfrak{B}$ does not result into the distinction of two elements of the original universe A that were formerly identified. This carries us to introduce a second type of monotonicity of the Ω operator.

DEFINITION 5.4.1. *A full class* K *of* $\mathcal{L}$*-structures is said to be* ***semialgebraic*** *if it is protoalgebraic and, in addition,* Ω *is* ***$\subseteq$-monotone*** *in* K, *i.e, for all* $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$, $\mathfrak{A} \subseteq \mathfrak{B}$ *implies* $\Omega\mathfrak{A} \subseteq \Omega\mathfrak{B}$.

PROPOSITION 5.4.2. *For every first-order language* $\mathcal{L}$ *and every quasivariety* $\mathcal{Q}$ *of* $\mathcal{L}$*-algebras, the classes* $\mathsf{K}_{\mathrm{qo},\mathcal{Q}}$, $\mathsf{K}_{\mathrm{po},\,\mathcal{Q}}$ *and* $\mathsf{K}_{\mathcal{Q}}$ *are semialgebraic.*

Proof. Certainly, if $\mathfrak{A} = \langle \mathrm{A}, \theta, \phi \rangle \in \mathsf{K}_{\mathrm{qo},\mathcal{Q}}$ then $\Omega\mathfrak{A} = \theta$ and hence Ω is $\subseteq$-monotone. The remaining cases are also immediate and their proof is omitted. ■

Obviously, if $\mathcal{L}$ has no function symbol, K_{eq} is semialgebraic, for it coincides with K_{co}. But in general, as it occurs with K_{to} but for different reasons, K_{eq} is not $\subseteq$-monotone. For a counterexample, consider once more the language of rings $\mathcal{L} = \{+, \cdot, 0\}$ and let

$$\mathfrak{A} := \langle\{\mathbb{Z}_+, +, \cdot, 0\}, \nabla_{\mathbb{Z}_+}\rangle, \qquad \mathfrak{B} := \langle\{\mathbb{Z}, +, \cdot, 0\}, \nabla_{\mathbb{Z}_+} \cup \nabla_{\mathbb{Z}^*_-}\rangle.$$

($\mathbb{Z}_+$ denotes the set of nonnegative integers and $\mathbb{Z}^*_-$ the set of negative integers). Clearly $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}_{\text{eq}}$ and $\Omega\mathfrak{A} = \nabla_{\mathbb{Z}_+}$. On the other hand, $\Omega\mathfrak{B} = \Delta_{\mathbb{Z}}$, for $\Delta_{\mathbb{Z}}$ is the only congruence on $\mathbf{B}$ contained in $\nabla_{\mathbb{Z}_+} \cup \nabla_{\mathbb{Z}^*_-}$. Hence, $\mathfrak{A} \subseteq \mathfrak{B}$ but $\Omega\mathfrak{A} \nsubseteq \Omega\mathfrak{B}$.

It is still an open problem to obtain a characterization of the $\subseteq$-monotonicity of the Leibniz operator in terms of a property of Ω as a mapping between (algebraic complete) lattices. The next result just includes an easy but interesting reformulation of the definition.

PROPOSITION 5.4.3. *A class* K *of* $\mathcal{L}$*-structures is* $\subseteq$*-monotone iff for all* $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$, $\mathfrak{A} \subseteq \mathfrak{B}$ *implies* $\Omega\mathfrak{A} = \Omega\mathfrak{B} \cap A^2$.

Proof. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be two structures of K. By Lemma 2.1.6, $\Omega\mathfrak{B} \cap A^2 \subseteq \Omega\mathfrak{A}$. Hence, if K is $\subseteq$-monotone, $\Omega\mathfrak{A} = \Omega\mathfrak{B} \cap A^2$. This proves the forward implication. The converse is clear. ∎

Recall from the preceding Section that protoalgebraic classes satisfy a general form of the Second Isomorphism Theorem of universal algebra (Corollary 5.3.6). Now, it is well worth noting that also a general form of the First Isomorphism Theorem holds for semialgebraic classes. So we can expect better algebraic properties of this type of classes, since most of the typical properties that set algebras apart from arbitrary structures derive from the Isomorphisms Theorems.

PROPOSITION 5.4.4. *Let* K *be a semialgebraic universal class of* $\mathcal{L}$*-structures, and assume* $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ *are such that* $\mathfrak{A} \subseteq \mathfrak{B}$. *For every* K*-filter extension* $\mathfrak{B}'$ *of* $\mathfrak{B}$, *if* $\mathfrak{A}' = \mathfrak{B}' \restriction A$ *then*

$$\mathfrak{A}'/\Omega\mathfrak{A}' \cong \widetilde{\mathfrak{A}}/\Omega\mathfrak{B}' \cap \widetilde{A}^2,$$

where $\widetilde{A} = \{a : a \equiv b\ (\Omega\mathfrak{B}') \text{ for some } b \in A\}$ *and* $\widetilde{\mathfrak{A}} = \mathfrak{B}' \restriction \widetilde{A}$.

Proof. The proof is an immediate consequence of 5.4.3 and 2.2.5. Note that the hypothesis of K being universal is required to ensure that $\mathfrak{A}'$ and $\widetilde{\mathfrak{A}}$ are still members of K. ∎

We are not going to prove now other characterizations of semialgebraic classes as we did for protoalgebraic ones in the preceding Section. Let us observe however that Theorem 5.3.9 can be sharpened by assuming semialgebraicity, for we can take K to be the category with all homomorphisms as arrows (not only the surjective ones!) and then K^* is still a full reflective subcategory of K. This easy property, which will be used later on (see the proof of Theorem 8.1.8), can be stated as follows.

THEOREM 5.4.5. *Let* K *be a semialgebraic universal class of* $\mathcal{L}$*-structures, and let* $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$. *Then every homomorphism* $h : \mathfrak{A} \to \mathfrak{B}$ *induces a homomorphism* $h^* : \mathfrak{A}^* \to \mathfrak{B}^*$ *of the respective Leibniz quotients, defined by* $a/\Omega\mathfrak{A} \longmapsto ha/\Omega\mathfrak{B}$.

Proof. Let $\mathfrak{C} = \mathfrak{B} \upharpoonright hA$. The structure $\mathfrak{C}$ is a member of the class K, for this is a universal class and thus closed under substructures. So, using Theorem 5.3.9, we have that $h^* : \mathfrak{A}^* \twoheadrightarrow \mathfrak{C}^*$. Also, since K is $\subseteq$-monotone, $\Omega\mathfrak{C} = \Omega\mathfrak{B} \cap C^2$. Hence, we can apply the sublemma stated in the proof of 4.1.5. We conclude that $\mathfrak{C}^* \rightarrowtail \mathfrak{B}^*$ by the natural embedding $b/\Omega\mathfrak{C} \longmapsto b/\Omega\mathfrak{B}$. In conclusion, by composing appropiate maps, h^* is a well defined homomorphism from $\mathfrak{A}^*$ into $\mathfrak{B}^*$ and thus the theorem is proved. ■

Both kinds of monotonicity of the Leibniz operator considered so far can be apparently strengthened by a third property of Ω whose motivation is not so clear (that this is so follows from Theorem 7.2.3 below). The property is used to distinguish a new category of well-behaved classes, this time quasivarieties.

DEFINITION 5.4.6. *A quasivariety* K *of $\mathcal{L}$-structures is said to be* ***algebraic*** *if Ω is join-continuous in* K, *i.e., if $\Omega(\bigcup_{i\in I}\mathfrak{A}_i) = \bigcup_{i\in I}\Omega\mathfrak{A}_i$ for any $\mathcal{L}$-algebra* **A** *and any $\preccurlyeq$-directed system $\{\mathfrak{A}_i : i \in I\}$ of* K*-structures on* **A**.

A last kind of quasivarieties that can be distinguished for the algebraic character they exhibit is obtained by assuming an additional property on the Leibniz operator, considered as a mapping between algebraic complete lattices. We take account of these quasivarieties again in Chapter 10 with the notion of ***algebraizable deductive system*** [8].

DEFINITION 5.4.7. *A quasivariety* K *of $\mathcal{L}$-structures is said to be* ***purely algebraic*** *if Ω is join-continuous in* K *and, moreover, one-one when restricted to the set $\mathsf{K}_{\mathbf{A}}$, for any $\mathcal{L}$-algebra* **A**

The above distinction of several categories of classes of structures (without equality) on the base of the Leibniz operator can be found explicit for the first time in a paper by Blok and Pigozzi [12] which works out an approach to the study of algebraic semantics for deductive systems. So not only the first examples of protoalgebraic classes were given in the context of general sentential logic, as we already noted in Section 5.3, but also the first examples for the remaining special classes can be found in several papers on algebraic logic. For instance, when the language has just one relation symbol (of arity 1), semialgebraic classes correspond to the model classes of a special type of deductive systems; see [12, Thm. 13.13]. We shall come back to this point in the last Chapter[20].

[20] We could have adopted the common terminology in algebraic logic to refer to the different types of classes. But several reasons leads us not to follow this option, the main one being that such a terminology is inspired in certain syntactical characterizations of these classes that are not known to hold in our broader context. Anyway, although the terminology we have adopted here has no convincing justification, it has the advantage of expressing the fundamental idea that there exists a hierarchy among classes of structures relying on the exhibition or not of some of the nice features of classes of algebras.

6. Subdirect Representation Theory for Structures

The representation of algebras as direct or subdirect products was extensively studied in the 1940's and 50's by many authors; see, e.g., [5, 6, 50, 60]. Quite probably, the main and more influential result on this decomposition problem was the result obtained by Birkhoff [5], which, concerning the class of all algebras of a fixed similarity type, fashioned the familiar subdirect representation theory so useful in universal algebra. Not very later, this result was improved by Mal'cev [86]; he pointed out the interest of having an analogous result for narrower or broader classes, and established a relativized version of Birkhoff's theorem, this time available not only for classes of algebras but for a wide range of classes of structures over arbitrary first-order languages with equality.

The aim of this Chapter is to examine Mal'cev's result when we consider general languages without equality and state appropiate versions for both the full and the reduced semantics. The resulting theorem (Theorem 6.1.8) can be applied in the same way as one uses Birkhoff's Theorem to understand the structure of algebras in a given class, i.e., by identifying the (relatively) subdirectly irreducible members of the class. In this sense, we also investigate the characterization of relatively subdirectly irreducible members of certain classes of structures in the style of Jónsson's Theorem [67] on congruence-distributive varieties.

6.1. Relative Subdirect Representations

Let K be any class of $\mathcal{L}$-structures and consider an arbitrary $\mathcal{L}$-structure $\mathfrak{A}$. A *congruence-filter pair of* $\mathfrak{A}$ *relative to* K, or simply a K-*congruence-filter pair of* $\mathfrak{A}$, is any pair $\langle \mathfrak{B}, \theta \rangle$ such that $\mathfrak{B} \in Fe_{\mathsf{K}}\mathfrak{A}$ and $\theta \in Co\ \mathfrak{B}$[21]. The set of all K-congruence-filter pairs of $\mathfrak{A}$ is denoted by $Cf_{\mathsf{K}}\mathfrak{A}$; by virtue of Theorem 5.1.1, it forms an inductive closure system iff K is a quasivariety. In the case $\mathcal{L}$ has equality, congruence-filter pairs of a structure $\mathfrak{A}$ are of the form $\langle \mathfrak{B}, \Delta_B \rangle$ for $\mathfrak{B} \in Fe_{\mathsf{K}}\mathfrak{A}$, and hence they naturally identify with the K-filter extensions of $\mathfrak{A}$. Also, if $\mathcal{L}$ has no other relation symbol except the equality, the notion of relative filter extension

[21] A non-relativized form of this notion first appeared in Nelson [93, p.34] under the name of *relation kernel*. See also [56], [120].

becomes superflous and the preceding definition of congruence-filter pair can be conveniently replaced by the usual notion of congruence on an algebra.

An $\mathcal{L}$-structure $\mathfrak{A}$ is said to be *subdirectly representable relative to* K if $\mathfrak{A} \in P_{sd}(\mathsf{K})$. One of the main results in the Chapter will state that under certain natural conditions each member of a class K is subdirectly representable relative to K. The following result is fundamental for this purpose; in particular, it shows that the role congruences on algebras play in the corresponding subdirect representation theory is now performed by congruence-filter pairs.

PROPOSITION 6.1.1. *Let* K *be any full class of $\mathcal{L}$-structures and let $\mathfrak{B}_i \in \mathsf{K}$, for $i \in I$. Then $\mathfrak{A}$ is isomorphic to a subdirect product of $\{\mathfrak{B}_i : i \in I\}$ iff there exists a corresponding system $\{\langle \mathfrak{A}_i, \theta_i \rangle : i \in I\}$ of* K*-congruence-filter pairs of $\mathfrak{A}$ such that*
(i) $\bigcap_{i \in I} \langle \mathfrak{A}_i, \theta_i \rangle = \langle \mathfrak{A}, \Delta_A \rangle$;
(ii) $\mathfrak{A}_i/\theta_i \cong \mathfrak{B}_i$ for all $i \in I$.

Proof. Assume $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i \in I} \mathfrak{B}_i$ for $\mathfrak{B}_i \in \mathsf{K}$. Let π_i denote the natural projection of $\prod_{i \in I} \mathfrak{B}_i$ onto $\mathfrak{B}_i$ and let $\mathfrak{A}_i = (\pi_i \circ h)^{-1}\mathfrak{B}_i$. Since π_i is onto, $\mathfrak{A}_i \in E(\mathfrak{B}_i) \subseteq \mathsf{K}$, and thus $\mathfrak{A}_i \in Fe_{\mathsf{K}}\mathfrak{A}$ (we do not assume $\mathfrak{A} \in \mathsf{K}$!). Moreover, given an n-ary relation symbol r and arbitrary elements $a_1, \ldots, a_n \in A$, we have

$$\langle a_1, \ldots, a_n \rangle \in \bigcap_{i \in I} r^{\mathfrak{A}_i} \text{ iff } \langle ha_1, \ldots, ha_n \rangle \in r^{\prod_{i \in I} \mathfrak{B}_i}$$
$$\text{iff } \langle ha_1, \ldots, ha_n \rangle \in r^{h\mathfrak{A}} \text{ iff } \langle a_1, \ldots, a_n \rangle \in r^{\mathfrak{A}},$$

for by 2.1.5 h is strong. Hence $\bigcap_{i \in I} \mathfrak{A}_i = \mathfrak{A}$. Let us take $\theta_i = Ker(\pi_i \circ h)$, for $i \in I$. We know that $\theta_i \in Co\, \mathfrak{A}_i$, so the Homomorphism Theorem says $\mathfrak{A}_i/\theta_i \cong \mathfrak{B}_i$ for all i. Finally $\bigcap_{i \in I} \theta_i = \Delta_A$, for h is an embedding by assumption. So the implication from left to right is proved.

For the converse, consider the mapping $h : A \rightarrowtail_{sd} \prod_{i \in I} A/\theta_i$ defined by $ha = \langle a/\theta_i : i \in I \rangle$, and suppose (i)-(ii) hold. By (i), h is a subdirect embedding of the algebra $\mathbf{A}$ into $\prod_{i \in I} \mathbf{A}/\theta_i$, so it is enough to see that $h\mathfrak{A} \subseteq \prod_{i \in I} \mathfrak{A}_i/\theta_i$. Let r be an n-ary relation symbol. Using (i) and the fact that h is injective, we have $r^{h\mathfrak{A}} = \bigcap_{i \in I} hr^{\mathfrak{A}_i}$. Hence, for all $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \prod_{i \in I} A/\theta_i$, with $\mathbf{a}_j := \langle a_{ij}/\theta_i : i \in I \rangle$ for $1 \leq j \leq n$, we have

$$\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle \in r^{h\mathfrak{A}} \text{ iff } \langle a_{i1}, \ldots, a_{in} \rangle \in r^{\mathfrak{A}_i} \text{ for all } i \in I.$$

But the last condition is equivalent to $\langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle \in r^{\prod_{i \in I} \mathfrak{A}_i/\theta_i}$. So $h\mathfrak{A} \subseteq \prod_{i \in I} \mathfrak{A}_i/\theta_i$ and the proof is finished. ∎

From this proposition we can draw several interesting properties. The first one is a generalization of a result due to Mal'cev [86, Thm. 2].

COROLLARY 6.1.2. *An full class* K *of $\mathcal{L}$-structures is closed under P_{sd} iff $\mathsf{K}_{\mathbf{A}}$ is closed under arbitrary intersections, for all $\mathcal{L}$-algebra* $\mathbf{A}$. ∎

The result analogue to 6.1.1 for reduced semantics is the following.

COROLLARY 6.1.3. *Given a full class* K *of $\mathcal{L}$-structures, a reduced $\mathcal{L}$-structure $\mathfrak{A}$ and a system $\{\mathfrak{B}_i : i \in I\}$ of reduced members of* K*, we have $\mathfrak{A} \rightarrowtail_{sd} \prod_{i \in I} \mathfrak{B}_i$ iff there exist $\mathfrak{A}_i \in Fe_{\mathsf{K}}\mathfrak{A}$, for $i \in I$, such that*

(i) $\bigcap_{i\in I} \mathfrak{A}_i = \mathfrak{A}$;
(ii) $\mathfrak{A}_i^* \cong \mathfrak{B}_i$, *for each* $i \in I$.

Proof. Once more assume $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i\in I} \mathfrak{B}_i$ for $\mathfrak{B}_i$ reduced structures of K, and let $\mathfrak{A}_i = (\pi_i \circ h)^{-1}\mathfrak{B}_i$. By the proof of 6.1.1, the necessity will be proved if we verify that $Ker\,(\pi_i \circ h) = \Omega\mathfrak{A}_i$. But this is an easy consequence of the assumption that $\mathfrak{B}_i$ is reduced and $\pi_i \circ h : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i$ for all i. Indeed, the Homomorphism Theorem says $\mathfrak{A}_i/Ker\,(\pi_i \circ h) \cong \mathfrak{B}_i$. This means that the quotient $\mathfrak{A}_i/Ker\,(\pi_i \circ h)$ is reduced. So, by virtue of the Correspondence Theorem stated in 2.2.7, $Ker\,(\pi_i \circ h) = \Omega\mathfrak{A}_i$, as required.

For the converse, we have that the intersection $\bigcap_{i\in I} \Omega\mathfrak{A}_i$ is a congruence on $\mathfrak{A}$, for $\bigcap_{i\in I} \mathfrak{A}_i = \mathfrak{A}$. Hence, since $\mathfrak{A}$ is reduced, $\bigcap_{i\in I} \Omega\mathfrak{A}_i = \Delta_A$, and thus the implication follows trivially from 6.1.1. ∎

Let us notice that in the proofs of the preceding Corollary and Proposition 6.1.1, the hypothesis of K being a full class is only used to show the forward implications. In fact, the hypothesis can be removed but then we obtain weaker results of the forecoming results; namely, we cannot guarantee that the filter extensions $\mathfrak{A}_i$ of $\mathfrak{A}$ are relative to K. This observation is going to be used in the proofs of Propositions 6.1.4 and 6.1.5 below.

Let K be any class of $\mathcal{L}$-structures (not necessarily full). A nontrivial $\mathfrak{A} \in$ K is *(completely) subdirectly irreducible relative to* K if $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i\in I} \mathfrak{A}_i$ with $\mathfrak{A}_i \in$ K for all $i \in I$ implies $\pi_i \circ h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}_i$ ($\pi_i \circ h : \mathfrak{A} \cong \mathfrak{A}_i$) for some i. The class of all relatively (completely) sudirectly irreducible members of K is denoted by $\mathsf{K}_{\mathrm{RSI}}$ ($\mathsf{K}_{\mathrm{RCSI}}$), and their elements are called RSI (RCSI) for short, omitting any mention to the class K, which is always clear from context; $\mathsf{K}^*_{\mathrm{RSI}}$ and $\mathsf{K}^*_{\mathrm{RCSI}}$ are shortened notations to mean $(\mathsf{K}_{\mathrm{RSI}})^*$ and $(\mathsf{K}_{\mathrm{RCSI}})^*$ respectively. These classes must be distinguished from the one formed of the relatively subdirectly irreducible members of K^*, i.e., $(\mathsf{K}^*)_{\mathrm{RSI}}$. By virtue of the Homomorphism Theorem, the latter class coincides with $(\mathsf{K}^*)_{\mathrm{RCSI}}$, for a reduced structure is RSI iff it is RCSI. This equivalence does not hold any longer for arbitrary non-reduced structures; we give an easy counterexample a few paragraphs below. The following lemma summarizes the relationship between all the forecoming classes without any assumption on K.

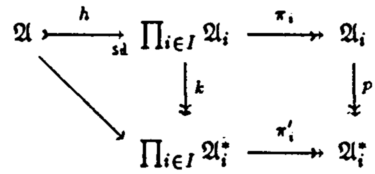
LEMMA 6.1.4. *For any class* K *of* $\mathcal{L}$*-structures, the following holds:* $(\mathsf{K}^*)_{\mathrm{RSI}} = (\mathsf{K}^*)_{\mathrm{RCSI}} \subseteq \mathsf{K}^*_{\mathrm{RCSI}} \subseteq \mathsf{K}^*_{\mathrm{RSI}}$.

Proof. The equality $(\mathsf{K}^*)_{\mathrm{RSI}} = (\mathsf{K}^*)_{\mathrm{RCSI}}$ and the inclusion $\mathsf{K}^*_{\mathrm{RCSI}} \subseteq \mathsf{K}^*_{\mathrm{RSI}}$ are clear. Let us see that the relatively subdirectly irreducible members of K^* are also completely subdirectly irreducible relative to K. For this, take $\mathfrak{A} \in (\mathsf{K}^*)_{\mathrm{RSI}}$ and let $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i\in I} \mathfrak{A}_i$, with $\mathfrak{A}_i \in$ K for all $i \in I$. Consider the mapping k from $\prod_{i\in I} \mathfrak{A}_i$ onto $\prod_{i\in I} \mathfrak{A}_i^*$ given by

$$\langle a_i : i \in I\rangle \longmapsto \langle a_i^* : i \in I\rangle.$$

We claim that $k \circ h$ is a subdirect embedding from $\mathfrak{A}$ into $\prod_{i\in I} \mathfrak{A}_i^*$. Indeed, the compositions of $k \circ h$ with the natural projections are clearly surjective. Moreover, since k and h are strong homomorphisms (the latter for being a subdirect embedding!), $k \circ h$ is strong, and thus $h\mathfrak{A}$ is a substructure of $\prod_{i\in I} \mathfrak{A}_i^*$, by virtue of 1.2.2.

Also, $Ker(k \circ h)$ is a congruence on $\mathfrak{A}$, this time by 2.1.5. So the hypothesis that $\mathfrak{A}$ is reduced entails $Ker(k \circ h) = \Delta_A$. In conclusion, $k \circ h$ is one-one, and the claim is proved. The situation is reflected in the next commutative diagram, where the onto homomorphisms are natural projections.



Apply now that $\mathfrak{A}$ is subdirectly completely irreducible relative to $\mathbf{K}^*$. We obtain that, for some $i \in I$, the composition $\pi_i' \circ k \circ h$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{A}_i^*$. But $\pi_i' \circ k \circ h = p_i \circ \pi_i \circ h$ and hence

$$Ker(\pi_i \circ h) \subseteq Ker(\varphi_i \circ \varphi_i \circ h) = Ker(\pi_i' \circ k \circ h) = \Delta_A.$$

So, $\pi_i \circ h$ is a bijective homomorphism. It remains to see that $\pi_i \circ h$ is strong. But this follows from the next chain of equivalences: for all $r \in R$ and all $\mathbf{a} \in A^{\rho(r)}$,

$$\begin{aligned} \mathbf{a} \in r^{\mathfrak{A}} &\text{ iff } (\pi_i' \circ k \circ h)(\mathbf{a}) \in r^{\mathfrak{A}_i^*} \\ &\text{ iff } (p_i \circ \pi_i \circ h)(\mathbf{a}) \in r^{\mathfrak{A}_i^*} \text{ iff } (\pi_i \circ h)(\mathbf{a}) \in r^{\mathfrak{A}_i}. \ \blacksquare \end{aligned}$$

The next proposition states a characterization of the relatively subdirectly irreducible members of a given class of structures that extends trivially the one obtained by Birkhoff for subdirectly irreducible algebras; in this generalization, the role of congruences is performed by the relational part of structures. It is also an easy consequence of 6.1.1.

PROPOSITION 6.1.5. *Let* $\mathbf{K}$ *be any class of* $\mathcal{L}$*-structures (not necessarily full) and let* $\mathfrak{A} \in \mathbf{K}$ *be nontrivial. Then the following statements are equivalent.*

(i) *$\mathfrak{A}$ is subdirectly irreducible relative to* $\mathbf{K}$.

(ii) *For any* $\mathfrak{A}_i \in Fe_{\mathbf{K}}\mathfrak{A}$, $i \in I$, *the condition* $\bigcap_{i \in I} \mathfrak{A}_i = \mathfrak{A}$ *implies* $\mathfrak{A} = \mathfrak{A}_i$ *for some* $i \in I$.

(iii) *There exist an* $r \in R$ *and an element* $\mathbf{a} \in A^{\rho(r)}$ *such that* $\mathbf{a} \notin r^{\mathfrak{A}}$ *but* $\mathbf{a} \in r^{\mathfrak{B}}$ *for all* $\mathfrak{B} \in Fe_{\mathbf{K}}\mathfrak{A}$.

Proof. Let $\mathfrak{A}_i \in Fe_{\mathbf{K}}\mathfrak{A}$, $i \in I$, be such that $\mathfrak{A} = \bigcap_{i \in I} \mathfrak{A}_i$. As Δ_A is a congruence on $\mathfrak{A}_i$ for all i, the proof of 6.1.1 says $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i \in I} \mathfrak{A}_i$, where $ha = \langle a : i \in I \rangle$. Consequently, if $\mathfrak{A} \in \mathbf{K}_{\mathrm{RSI}}$ then $\pi_i \circ h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}_i$ for some i, and thus $\mathfrak{A} = \mathfrak{A}_i$. This proves the implication from (i) to (ii).

Suppose now that (ii) holds. Let $\mathfrak{A}_0 = \bigcap\{\mathfrak{A} \in Fe_{\mathbf{K}}\mathfrak{A} : \mathfrak{B} \neq \mathfrak{A}\}$. Since $\mathfrak{A}_0$ must be different from $\mathfrak{A}$, there exists an $r \in R$ such that $r^{\mathfrak{A}} \neq r^{\mathfrak{A}_0}$. Thus (iii) follows trivially from the fact that $\mathfrak{A}_0 \preccurlyeq \mathfrak{B}$ for all $\mathfrak{B} \in Fe_{\mathbf{K}}\mathfrak{A} \setminus \{\mathfrak{A}\}$.

Let us prove finally that (iii) entails (i). If $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i \in I} \mathfrak{B}_i$ for $\mathfrak{B}_i \in \mathbf{K}$, by 6.1.1 again we have that $\mathfrak{A} = \bigcap_{i \in I} \mathfrak{A}_i$, where $\mathfrak{A}_i = (\pi_i \circ h)^{-1}\mathfrak{B}_i \in Fe_{\mathbf{K}}\mathfrak{A}$, for all

$i \in I$. Thus (iii) implies $\mathfrak{A} = \mathfrak{A}_i$ for some i, and therefore $\pi_i \circ h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}_i$. So $\mathfrak{A} \in \mathsf{K}_{\mathrm{RSI}}$, as required. ∎

The preceding proposition and the fact that $(\mathsf{K}^*)_{\mathrm{RSI}} = (\mathsf{K}^*)_{\mathrm{RCSI}}$ suggest the notion of relatively subdirectly irreducible structure, and not that of relatively completely subdirectly irreducible structure, is the proper generalization of subdirectly irreducible algebras when we consider general first-order languages without equality. The subdirect representation theorem available for full semantics (see Theorem 6.1.8 below) will convince us of that. Actually, in order to obtain a characterization of RCSI nonreduced structures, it is not enough to think of the relational part of sructures as playing the role of congruences, but we must use the forecoming notion of congruence-filter pair. The following is no more than a direct consequence of Proposition 6.1.1.

COROLLARY 6.1.6. *Let* K *be any class of* $\mathcal{L}$*-structures (not necessarily full) and let* $\mathfrak{A} \in \mathsf{K}$ *be nontrivial. Then* $\mathfrak{A}$ *is subdirectly completely irreducible relative to* K *iff for all* $\langle \mathfrak{A}_i, \theta_i \rangle \in Cf_{\mathsf{K}}\mathfrak{A}$ *such that* $\bigcap_{i\in I} \langle \mathfrak{A}_i, \theta_i \rangle = \langle \mathfrak{A}, \Delta_A \rangle$, *we have* $\langle \mathfrak{A}, \Delta_A \rangle = \langle \mathfrak{A}_i \theta_i \rangle$ *for some* $i \in I$. ∎

As in Proposition 6.1.5, we do not require K to be a full class in the preceding Corollary, so both results may be applied to characterize subdirectly (completely) irreducible structures relative to reduced classes. Also, observe that if we assume $\mathsf{K}_{\mathbf{A}}$ is closed under arbitrary intersections, for all $\mathcal{L}$-algebra $\mathbf{A}$ (for instance, if K is closed under P_{sd}, cf. Corollary 6.1.2), then we have the following equivalences: $\mathfrak{A}$ is RSI iff $Fe_{\mathsf{K}}\mathfrak{A} \setminus \{\mathfrak{A}\}$ has a minimum element, whereas $\mathfrak{A}$ is RCSI iff $Cf_{\mathsf{K}}\mathfrak{A} \setminus \{\mathfrak{A}\}$ has a minimum element.

The following necessary condition for a structure to be completely subdirectly irreducible relative to a full class is interesting, for it involves the lattice of congruences instead of the lattice of congruence-filter pairs and must be added to the necessary condition that follows from 6.1.5.

PROPOSITION 6.1.7. *Given any full class* K *of* $\mathcal{L}$*-structures and a nontrivial* $\mathfrak{A} \in \mathsf{K}$, *if* $\mathfrak{A}$ *is subdirectly completely irreducible relative to* K *then* $Co\,\mathfrak{A} \setminus \{\Delta_A\}$ *has a minimum element, i.e., for all* $\theta_i \in Co\,\mathfrak{A}$, $i \in I$, *the condition* $\bigcap_{i\in I} \theta_i = \Delta_A$ *implies* $\theta_i = \Delta_A$ *for some* $i \in I$.

Proof. Assume $Co\,\mathfrak{A}\setminus\{\Delta_A\}$ has no minimum element. Then $\bigcap Co\,\mathfrak{A}\setminus\{\Delta_A\} = \Delta_A$. So, using 6.1.1, $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{\theta\neq\Delta_A} \mathfrak{A}/\theta$, where $ha = \langle a/\theta : \theta \neq \Delta_A \rangle$. But none of the projections $\pi_\theta : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}/\theta$ is injective if $\theta \neq \Delta_A$. Consequently, as K is full, $\mathfrak{A}/\theta \in \mathsf{K}$ and $\mathfrak{A}$ is not completely subdirectly irreducible relative to K. ∎

We can apply 6.1.7 to give an easy example that shows RSI non-reduced structures are not in general RCSI; once more, the example is one in the type with some operations (maybe none) and just one relation, of arity 2. Concretely, consider the class K_{co} and let $\mathbf{A}$ be a non subdirectly irreducible algebra of the appropriate similarity type. Then $\langle \mathbf{A}, \nabla_A \rangle$ is non-reduced and, by 6.1.5, is subdirectly irreducible relative to K_{co}. On the other hand, $\mathfrak{A} \notin (\mathsf{K}_{\mathrm{co}})_{\mathrm{RCSI}}$, since $Co\,\mathfrak{A} = Co\,\mathbf{A}$ and $\mathbf{A}$ has been taken to be not subdirectly irreducible.

We are ready to establish the Subdirect Representation Theorem for full and reduced semantics. The minimal requirements on the class of structures that are needed in order to have such a subdirect decomposition into relatively subdirectly irreducible members were discovered by Mal'cev [86, Thm. 3].

THEOREM 6.1.8 (Subdirect Representation Theorem). *Let* K *be any full class of* $\mathcal{L}$*-structures satisfying that* $\mathsf{K}_{\mathbf{A}}$ *is closed under the union of* $\preccurlyeq$*-chains, for all* $\mathbf{A}$. *Then the following holds:*

- *(i)* $\mathsf{K} \subseteq P_{sd}(\mathsf{K}_{\mathrm{RSI}})$, *i.e., every structure of* K *is isomorphic to a subdirect product of members of* K *that are subdirectly irreducible relative to* K.
- *(ii)* $\mathsf{K}^* \subseteq P^*_{sd}(\mathsf{K}^*_{\mathrm{RSI}})$, *i.e., every structure of* K^* *is isomorphic to the Leibniz quotient of a subdirect product of reduced members of* K *that are subdirectly irreducible relative to* K.

Proof. (i) Let $\mathfrak{A} \in \mathsf{K}$ and define $\Delta := \{(r, \mathbf{a}) : r \in R, \mathbf{a} \in A^{ar(r)} \text{ and } \mathbf{a} \notin r^{\mathfrak{A}}\}$. For each $(r, \mathbf{a}) \in \Delta$, choose an $\mathfrak{A}(r, \mathbf{a}) \in Fe_{\mathsf{K}}\mathfrak{A}$ that is maximal with respect to the property $\mathbf{a} \notin r^{\mathfrak{A}(r,\mathbf{a})}$; since $\mathsf{K}_{\mathbf{A}}$ is closed under unions of $\preccurlyeq$-chains, for all $\mathbf{A}$, we can apply Zorn's lemma and such a filter extension exists. It is easy to verify that

$$\bigcap\nolimits_{(r,\mathbf{a})\in\Delta} \mathfrak{A}(r, \mathbf{a}) = \mathfrak{A}.$$

So, by 6.1.1, $\mathfrak{A} \rightarrowtail_{sd} \prod_{(r,\mathbf{a})\in\Delta} \mathfrak{A}(r, \mathbf{a})$. Moreover, the definition of $\mathfrak{A}(r, \mathbf{a})$ and 6.1.5 say that $\mathfrak{A}(r, \mathbf{a})$ is subdirectly irreducible relative to K. Therefore $\mathfrak{A} \in P_{sd}(\mathsf{K}_{\mathrm{RSI}})$.

(ii) It follows directly from part (i) and Proposition 4.1.5(ii). ∎

Remark. By the above, the Subdirect Representation Theorem holds whenever K is assumed to be a full class and to satisfy that

(6.1) $\mathsf{K}_{\mathbf{A}}$ is closed under unions of $\preccurlyeq$-chains, for all $\mathbf{A}$.

This is in accordance with Mal'cev's conditions for languages with equality (see the remark following [86, Thm. 3]), for in this case we know that the full classes are the ones closed under isomorphisms. ⊣

Certainly, by virtue of 5.1.1, the Subdirect Representation Theorem stated above holds in particular for arbitrary quasivarieties. We shall see, however, that by imposing some more restrictions on K, part (ii) can be slightly improved and consequently K^* acquires a nicer structure theory (Theorem 6.2.2 below). On the other hand, it is not difficult to find examples of other full classes K that are not quasivarieties but of where the Subdirect Representation Theorem holds, because they still satisfy (6.1). The following result describes a sufficient condition for K to be of this sort.

PROPOSITION 6.1.9. *If* K *is an elementary class of* $\mathcal{L}$*-structures axiomatized by positive and/or universal sentences, then* K *satisfies condition (6.1).*

Proof. Remember that a sentence is called positive if it is of the form

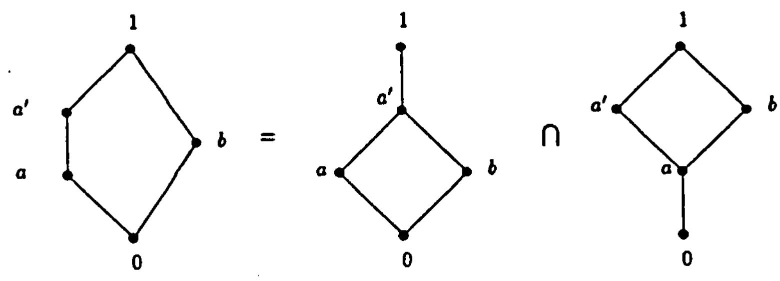
$$Q_1x_1 \ldots Q_kx_k\Phi(x_1, \ldots, x_k),$$

where Q_i are arbitrary quantifiers and $\Phi(x_1, \ldots, x_k)$ is a quantifier-free formula constructed from atomic expressions with the aid of the connectives $\vee$ and $\wedge$ only. Let us show that positive and universal sentences are preserved under unions of $\preccurlyeq$-chains. The statement for positive sentences follows trivially from the next claim: if φ is a positive formula, $\mathfrak{A} \preccurlyeq \mathfrak{B}$, $g : \mathbf{Te}_{\mathcal{L}} \rightarrow \mathbf{A}$ and $\mathfrak{A} \models \varphi[g]$, then $\mathfrak{B} \models \varphi[g]$. The claim is proved by an obvious induction on φ; we take the atomic case and the passage from φ to $\exists x\varphi$ as examples. Suppose first that $\varphi := rt_1 \ldots t_n$ for some $\mathcal{L}$-terms $t_1, \ldots, t_n$. Since $\mathfrak{A} \models \varphi[g]$, we have $\langle ht_1, \ldots, ht_n \rangle \in r^{\mathfrak{A}}$. Hence, $\mathfrak{A} \preccurlyeq \mathfrak{B}$ implies $\langle ht_1, \ldots, ht_n \rangle \in r^{\mathfrak{B}}$ and consequently $\mathfrak{B} \models \varphi[g]$. Now assume the claim for φ, and suppose that $\mathfrak{A} \models \exists x\varphi[g]$. Let b be an element of the common universe of $\mathfrak{A}$ and $\mathfrak{B}$ for which $\mathfrak{A} \models \varphi[g(x/b)]$. By the induction hypothesis $\mathfrak{B} \models \varphi[g(x/b)]$, so that $\mathfrak{B} \models \exists x\varphi[g]$, as desired.

The proof of the statement for universal sentences is in very much the same manner and so it is omitted. ■

Examples. There is a big amount of interesting classes to which the above proposition can be applied to conclude the validity of the Subdirect Representation Theorem for them. Examples of such classes that arise in the algebra context are provided by Mal'cev [86], who was aware of the property contained in 6.1.9: rings without zero divisors, rings embeddable in skewfields, torsion free groups,... We know these cases are not immediately covered by Birkhoff's Theorem because homomorphic images of rings without zero divisors, e.g., may not be without zero divisors. Other exemples of where the Subdirect Representation Theorem holds and which also are not a consequence of Birkhoff's Theorem are the class K_{qo} and, for instance, its subclasses of directed quasi-ordered algebras or totally quasi-ordered algebras, i.e., pairs $\langle \mathbf{A}, \theta \rangle$ such that θ is a quasi-order on $\mathbf{A}$ and satisfies respectively the conditions: (1) for all $a, b \in A$ there exists a $c \in A$ such that $a\theta c$ and $b\theta c$; (2) for all $a, b \in A$, $a\theta b$ or $b\theta a$. ⊣

To close this Section, it is worth noting that when applied to lattices as ordered sets (i.e., members of the class $\mathsf{K}^{*}_{\mathrm{qo}}$ in the case $\mathcal{L}$ has no function symbol), our structure theory need not yield the usual Subdirect Representation Theory for lattices as algebras. For example, the lattice N_5 illustrated bellow is nontrivially subdirectly representable relative to $\mathsf{K}^{*}_{\mathrm{qo}}$, for it decomposes in the following way:



N_5

Besides, it is nontrivially subdirectly irreducible relative to the narrower class of all members of $\mathsf{K}^*_{\mathrm{qo}}$ that are lattices, i.e., that satisfy the next additional axioms:

$$\forall x \forall y \exists z (r(x,z) \wedge r(y,z) \wedge \forall u (r(x,u) \wedge r(y,u) \rightarrow r(z,u))),$$
$$\forall x \forall y \exists z (r(z,x) \wedge r(z,y) \wedge \forall u (r(u,x) \wedge r(u,y) \rightarrow r(u,z))),$$

However, as an algebra it is easy to see that N_5 has no nontrivial subdirect representation. This surprising fact simply shows that the structure theory we describe here strongly depends on the language; namely, there exist two classes, defined over distinct languages, that exhibit the same logical properties but whose structure theories do not coincide (take K to be the subclass of K^* mentioned above and K' the class $\mathsf{K}^*_{\mathcal{V}}$, where $\mathcal{V}$ is the variety of lattices as algebras)[22].

6.2. Structure Theory for Protoalgebraic Classes

A remarkable point in the development of a structure theory for a class K of structures defined without equality is that we can obtain better properties of the reduced class K^* by imposing some restrictions on K of the type considered in Chapter 5, i.e., restrictions on the Leibniz operator. Certainly, the monotonicity of the Ω operator with respect to $\preccurlyeq$ has not been a necessary requirment for the results stated so far. However, as we already suggested, this assumption is needed when we try to obtain a generalization of some other universal algebraic results, as we want now to do. Partly, this is due to the fact that relative subdirect irreducibility is not in general preserved under expansions, though it really is under reductions. In this Section and the following we put our attention basically on quasivarieties of structures. To start with, the next proposition contains a property of full classes that is going to be used later and clarifies the preceding comment.

PROPOSITION 6.2.1. *Let* K *be any full class of* $\mathcal{L}$*-structures. If* $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ *are such that* $\mathfrak{B}$ *is a reduction of* $\mathfrak{A}$*, then* $\mathfrak{A} \in \mathsf{K}_{\mathrm{RSI}}$ *implies* $\mathfrak{B} \in \mathsf{K}_{\mathrm{RSI}}$*. The converse is true if* K *is protoalgebraic.*

Proof. Suppose that $h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{B}$ and let $\{\mathfrak{B}_i : i \in I\}$ be a family of members of $\mathsf{K}_{\mathfrak{B}}$ such that $\mathfrak{B} = \bigcap_{i \in I} \mathfrak{B}_i$. Since K is full, $\mathfrak{A}_i = h^{-1}\mathfrak{B}_i \in Fe_{\mathsf{K}}\mathfrak{A}$. So the strongness of h says that $\mathfrak{A}$ can be expressed as the intersection of some of its K-filter extensions:

$$\mathfrak{A} = h^{-1}\mathfrak{B} = h^{-1}(\textstyle\bigcap_{i \in I} \mathfrak{B}_i) = \bigcap_{i \in I} \mathfrak{A}_i.$$

Therefore, $\mathfrak{A} \in \mathsf{K}_{\mathrm{RSI}}$ entails $\mathfrak{A} = \mathfrak{A}_i$ for some i, and hence $\mathfrak{B} = hh^{-1}\mathfrak{A} = \mathfrak{B}_i$, for h is surjective. Thus the first implication follows from Proposition 6.1.5.

To see the converse, assume $\mathfrak{A} = \bigcap_{i \in I} \mathfrak{A}_i$ for $\mathfrak{A}_i \in Fe_{\mathsf{K}}\mathfrak{A}$. By 5.3.8, we have that $h^{-1}h\mathfrak{A}_i = \mathfrak{A}_i$ for all i (for this, we do not need K to be a quasivariety). Consequently,

$$\mathfrak{A} = \textstyle\bigcap_{i \in I} h^{-1}h\mathfrak{A}_i = h^{-1}(\bigcap_{i \in I} h\mathfrak{A}_i).$$

[22] An interesting structure theory for ordered sets distinct from the one provided here is developed by Duffus and Rival [42]. It differs substantially from ours, for it adds to the idea of subdirect representation the concept of *retraction* as a substitute of the common homomorphisms.

The surjectivity of h entails $\mathfrak{B} = \bigcap_{i\in I} h\mathfrak{A}_i$. Thus, if $\mathfrak{B} \in \mathsf{K}_{\mathrm{RSI}}$ we have $\mathfrak{B} = h\mathfrak{A}_i$ for some i. As a result, $\mathfrak{A} = \mathfrak{A}_i$, and Proposition 6.1.5 finishes the proof. ∎

Theorem 6.1.8(ii) can be sharpened for full classes that are also protoalgebraic quasivarieties, in the sense that $\mathsf{K}^*_{\mathrm{RSI}}$ can be replaced by the narrower subclass $(\mathsf{K}^*)_{\mathrm{RCSI}}$; cf. Lemma 6.1.4.

THEOREM 6.2.2. *If* K *is a protoalgebraic quasivariety of* $\mathcal{L}$*-structures, then* $\mathsf{K}^* \subseteq P_{sd}((\mathsf{K}^*)_{\mathrm{RCSI}})$, *i.e., every structure of* K^* *is isomorphic to a subdirect product of relatively completely subdirectly irreducible members of* K^*.

Proof. Let $\mathfrak{A}$ be an element of K^*. By part (i) of 6.1.8, there exist a set I and an structure $\mathfrak{A}_i \in \mathsf{K}_{\mathrm{RSI}}$ for $i \in I$ such that $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i\in I} \mathfrak{A}_i$ for some h. Thus, reasoning as in the proof of 6.1.4, $\mathfrak{A}$ can be subdirectly embedded into the product $\prod_{i\in I} \mathfrak{A}_i^*$. It suffices to show that $\mathfrak{A}_i$ belongs to $(\mathsf{K}^*)_{\mathrm{RSI}}$. First of all observe that the lattice $\mathbf{Fe}_{\mathsf{K}}\mathfrak{A}_i$ has a smallest proper element, by 6.1.5. Also, as K is protoalgebraic, Theorem 5.3.8 entails that $\mathbf{Fe}_{\mathsf{K}}\mathfrak{A}_i^*$ is isomorphic to $\mathbf{Fe}_{\mathsf{K}}\mathfrak{A}_i$. Hence, $\mathbf{Fe}_{\mathsf{K}}\mathfrak{A}_i^*$ also has a smallest element. So the desired condition follows from 6.1.5. ∎

Given a quasivariety K, a structure $\mathfrak{A} \in \mathsf{K}$ is said to be *(finitely) meet prime* in $Fe_{\mathsf{K}}\mathfrak{B}$ if $\bigcap_{i\in I} \mathfrak{A}_i \preccurlyeq \mathfrak{A}$, where $\{\mathfrak{A}_i : i \in I\}$ is a (finite) system of K-filter extensions of $\mathfrak{B}$, implies $\mathfrak{A}_i \preccurlyeq \mathfrak{A}$ for some $i \in I$. Note that, by Proposition 6.1.5, $\mathfrak{A}$ is (finitely) meet prime in $Fe_{\mathsf{K}}\mathfrak{A}$ iff $\mathfrak{A}$ is (finitely) subdirectly irreducible relative to K. This equivalence, however, does not hold in general. Certainly, if $\mathfrak{A}$ is (finitely) meet prime in $Fe_{\mathsf{K}}\mathfrak{B}$, for some $\mathfrak{B}$ such that $\mathfrak{B} \preccurlyeq \mathfrak{A}$, then it is (finitely) subdirectly irreducible relative to K. But the converse is true only if the lattice $Fe_{\mathsf{K}}\mathfrak{B}$ is distributive; in this case, $\bigcap_{i\in I} \mathfrak{A}_i \preccurlyeq \mathfrak{A}$ implies $\mathfrak{A} = \mathfrak{A} \vee (\bigcap_{i\in I} \mathfrak{A}_i) = \bigcap_{i\in I}(\mathfrak{A} \vee \mathfrak{A}_i)$, and consequently the subdirect irreducibility of $\mathfrak{A}$ relative to K entails $\mathfrak{A} = \mathfrak{A} \vee \mathfrak{A}_i$, i.e., $\mathfrak{A}_i \preccurlyeq \mathfrak{A}$, for some $i \in I$.

LEMMA 6.2.3. *Assume* K *is a quasivariety of* $\mathcal{L}$*-structures. Let* $\mathfrak{C}_i \in \mathsf{K}$ *for* $i \in I$, *and* $\mathfrak{B} \subseteq \prod_{i\in I} \mathfrak{C}_i$. *Then, if* $\mathfrak{A} \in Fe_{\mathsf{K}}\mathfrak{B}$ *is finitely meet prime in* $Fe_{\mathsf{K}}\mathfrak{B}$, *there exists an ultrafilter* $\mathcal{U}$ *on* I *such that* $\mathfrak{C}^{\mathcal{U}} \restriction B \preccurlyeq \mathfrak{A}$, *where* $\mathfrak{C}^{\mathcal{U}} = \prod_{i\in I}^{\mathcal{U}} \mathfrak{C}_i$.

Proof. Suppose $r_0^{\mathfrak{A}} \neq B^n$ for some n-ary relation symbol $r_0 \in R$; otherwise, every ultrafilter $\mathcal{U}$ on I satisfies the desired condition $\mathfrak{C}^{\mathcal{U}} \restriction B \preccurlyeq \mathfrak{A}$. For each $J \subseteq I$, if π_J denotes the natural projection from $\prod_{i\in I} \mathfrak{C}_i$ onto $\prod_{j\in J} \mathfrak{C}_j$, let

$$\begin{aligned} &\mathfrak{C}_J := \pi_J^{-1}(\textstyle\prod_{j\in J} \mathfrak{C}_j), \\ &\mathfrak{B}_J := \mathfrak{C}_J \restriction B, \\ &U := \{J \in Sb(I) : \mathfrak{B}_{\bar{J}} \not\preccurlyeq \mathfrak{A}\}, \end{aligned}$$

where $\bar{J}$ means $I \setminus J$. Note that $\mathfrak{B}_J \in Fe_{\mathsf{K}}\mathfrak{B}$ for all $J \in Sb(I)$. We claim that U is a proper filter. Indeed, we have that $r_0^{\mathfrak{C}_{\bar{I}}} = C^n$ and hence $r_0^{\mathfrak{B}_{\bar{I}}} = B^n \not\subseteq r^{\mathfrak{A}}$. So $I \in U$. Assume now $J \in U$ and $J \subseteq K$. Clearly $\mathfrak{C}_{\bar{J}} \preccurlyeq \mathfrak{C}_{\bar{K}}$, and thus $\mathfrak{B}_{\bar{J}} \preccurlyeq \mathfrak{B}_{\bar{K}}$. Hence $\mathfrak{B}_{\bar{J}} \not\preccurlyeq \mathfrak{A}$ implies $\mathfrak{B}_{\bar{K}} \not\preccurlyeq \mathfrak{A}$, i.e., $K \in U$. Let us see U is closed under finite intersections. Consider $J, K \in U$. Since K is a quasivariety,

$$\mathfrak{B}_{\bar{J}} \in SEP(\mathsf{K}) \subseteq \mathsf{K},$$

for all $J \in Sb(I)$. So, as $\mathfrak{B}_{\bar{J}} \not\preccurlyeq \mathfrak{A}$, $\mathfrak{B}_{\bar{K}} \not\preccurlyeq \mathfrak{A}$ and $\mathfrak{A}$ is finitely meet prime in $Fe_{\mathsf{K}}\mathfrak{B}$, we have $\mathfrak{B}_{\bar{J}} \cap \mathfrak{B}_{\bar{K}} \not\preccurlyeq \mathfrak{A}$. It suffices to verify that $\mathfrak{B}_{\bar{J}} \cap \mathfrak{B}_{\bar{K}} = \mathfrak{B}_{\overline{J \cap K}}$. But this is an easy consequence from the definitions involved: for each $r \in R$, if $\rho(r) = n$,

$$\begin{aligned} r^{\mathfrak{C}_{\bar{J}}} \cap r^{\mathfrak{C}_{\bar{K}}} &= \{\bar{\mathbf{a}} \in C^n : \langle a_{j1}, \ldots, a_{jn} \rangle \in r^{\mathfrak{C}_j} \text{ for all } j \notin J \text{ and} \\ &\qquad \langle a_{k1}, \ldots, a_{kn} \rangle \in r^{\mathfrak{C}_k} \text{ for all } k \notin K\} \\ &= \{\bar{\mathbf{a}} \in C^n : \langle a_{j1}, \ldots, a_{jn} \rangle \in r^{\mathfrak{C}_j} \text{ for all } j \in \bar{J} \cup \bar{K}\} = r^{\mathfrak{C}_{\overline{J \cap K}}}, \end{aligned}$$

where $\bar{\mathbf{a}} = \langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle$. This proves U is a filter. Also, since $\mathfrak{C}_{\bar{\emptyset}} = \prod_{i \in I} \mathfrak{C}_i$, we have $\mathfrak{B}_{\bar{\emptyset}} = \mathfrak{B} \preccurlyeq \mathfrak{A}$ and consequently $\emptyset \notin U$, i.e., U is proper. So the proof of the claim is finished.

Consider now an ultrafilter $\mathcal{U}$ containing U. Given an n-ary relation symbol r and elements $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \prod_{i \in I} C_i$,

$$\begin{aligned} \langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle \in r^{\bigcup_{J \in \mathcal{U}} \mathfrak{C}_J} &\text{ iff there is a } J \in \mathcal{U} \text{ such that } \langle a_{j1}, \ldots, a_{jn} \rangle \in r^{\mathfrak{C}_j} \text{ for } j \in J \\ &\text{ iff } \{i \in I : \langle a_{i1}, \ldots, a_{in} \rangle \in r^{\mathfrak{C}_i}\} \in \mathcal{U} \\ &\text{ iff } \langle \mathbf{a}_1, \ldots, \mathbf{a}_n \rangle \in \textstyle\prod_{i \in I}^{\mathcal{U}} r^{\mathfrak{C}_i}. \end{aligned}$$

Therefore, $\bigcup_{J \in \mathcal{U}} \mathfrak{C}_J = \prod_{i \in I}^{\mathcal{U}} \mathfrak{C}_i$. On the other hand, if $J \in \mathcal{U}$ then $\bar{J} \notin \mathcal{U}$ and hence $\bar{J} \notin U$, i.e., $\mathfrak{B}_J \preccurlyeq \mathfrak{A}$. As a result, $\mathfrak{C}^{\mathcal{U}} \restriction B = \bigcup_{J \in \mathcal{U}} \mathfrak{B}_J \preccurlyeq \mathfrak{A}$. ∎

The following theorem can be viewed as an analogue for protoalgebraic quasivarieties of the well known result of Jónsson [3,67] characterizing the finitely subdirectly irreducible algebras in the variety generated by a set K of algebras, provided the variety is congruence-distributive. Of course, in the present case the theorem splits into two parts, one concerning the full semantics and the other the reduced semantics. If K is any class of $\mathcal{L}$-structures, K_{RFSI} means the class of all *finitely subdirectly irreducible structures relative to* K, i.e., those members $\mathfrak{A}$ of K for which $\mathfrak{A} = \bigcap_{i \leq n} \mathfrak{A}_i$ with $\mathfrak{A}_i \in \mathsf{K}$ implies $\mathfrak{A} = \mathfrak{A}_i$ for some $i \leq n$ (or equivalently, $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i \leq n} \mathfrak{A}_i$ implies $\pi_i \circ h : \mathfrak{A} \twoheadrightarrow_s \mathfrak{A}_i$ for some i). Note that $\mathsf{K}_{RSI} \subseteq \mathsf{K}_{RFSI}$ for each K.

THEOREM 6.2.4. *Let* Q *be a protoalgebraic quasivariety of* $\mathcal{L}$*-structures. The following holds for any subclass* $\mathsf{K} \subseteq \mathsf{Q}$.

(i) $(\mathsf{K}^{\mathsf{Q}})_{\mathrm{RFSI}} \subseteq ERSP_u(\mathsf{K})$.

(ii) $(\mathsf{K}^{\mathsf{Q}})^*_{\mathrm{RFSI}} \subseteq S^* P_u^*(\mathsf{K}^*)$.

Proof. (i) Let $\mathfrak{A}$ be finitely subdirectly irreducible relative to K^{Q}. By 4.4.2 and 4.4.4, $\mathsf{K}^{\mathsf{Q}} = ERSPP_u(\mathsf{K}) \subseteq \mathsf{Q}$, so there exist a $\mathfrak{C}_i \in P_u(\mathsf{K})$, with $i \in I$, a substructure $\mathfrak{B} \subseteq \prod_{i \in I} \mathfrak{C}_i$ and homomorphisms $h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{D}$, $g : \mathfrak{A} \twoheadrightarrow_s \mathfrak{D}$. Now $\mathfrak{A}$ is finitely subdirectly irreducible relative to K^{Q} and therefore $\mathfrak{B}$ is also finitely subdirectly irreducible relative to K^{Q}, by 6.2.1. Hence, as $\mathfrak{B}$ is the smallest K^{Q}-filter extension of $\mathfrak{B}$, $\mathfrak{B}$ is finitely meet prime in $Fe_{\mathsf{K}^{\mathsf{Q}}}\mathfrak{B}$. So we can apply Lemma 6.2.3 by taking $\mathfrak{A}$ as $\mathfrak{B}$; we conclude that there is an ultrafilter $\mathcal{U}$ on I such that $\mathfrak{C}^{\mathcal{U}} \restriction B \preccurlyeq \mathfrak{B}$. Thus, $\mathfrak{B} = \mathfrak{C}^{\mathcal{U}} \restriction B$. Also we know that, by definition of $\mathfrak{C}^{\mathcal{U}}$, the natural projection from

$\mathfrak{C}^{\mathcal{U}}$ onto the ultraproduct $\prod_{i\in I}\mathfrak{C}_i/\mathcal{U}$ is a reductive homomorphism. As a result, 4.1.1 and 4.1.2(i) say that

$$\mathfrak{B} \in SEP_uP_u(\mathsf{K}) = SEP_u(\mathsf{K}) \subseteq ESP_u(\mathsf{K}).$$

Finally, $\mathfrak{A} \in ER(\mathfrak{B}) \subseteq ERESP_u(\mathsf{K}) = ERSP_u(\mathsf{K})$, as desired.

(ii) We apply part (i) and repeat the proof of 4.2.2(ii). ∎

The special case of this theorem for quasivarieties of algebras was first proved by Dziobiak [44], whereas earlier versions for classes of structures other than algebras can be found in [12, Thm. 9.6] and [34, Thm. III.8].

Notice that the assumption of filter-distributivity is not required in the previous result. The aim in the next Section is precisely to give another analogue of Jónsson's Theorem, which does require filter-distributivity and is much closer in spirit to the original result.

6.3. Relative Filter-Distributivity and Generalized Jónsson's Theorem

A quasivariety K of $\mathcal{L}$-structures is said to be *relative filter-distributive* (RFD for short) if $\mathbf{Fe}_{\mathsf{K}}\mathfrak{A}$ is a distributive lattice for all $\mathfrak{A} \in \mathsf{K}$; equivalently, $\mathbb{K}_{\mathbf{A}}$ is distributive for all $\mathcal{L}$-algebra $\mathbf{A}$. Actually, the assumption of K being a quasivariety entails a stronger condition on the lattices of relative filter extensions: if $\mathfrak{B} \in \mathbf{Fe}_{\mathsf{K}}\mathfrak{A}$ and $\{\mathfrak{A}_i : i \in I\} \in \mathbf{Fe}_{\mathsf{K}}\mathfrak{A}$, then

$$\mathfrak{B} \cap \left(\bigvee_{i\in I}\mathfrak{A}_i\right) = \bigvee_{i\in I}(\mathfrak{B} \cap \mathfrak{A}_i).$$

Relative filter-distributivity in the sense we have just defined was first considered by Dzik and Suszko [43], and it seems to be the most fruitfull generalization to classes of structures of the concept of congruence-distributivity in universal algebra (take K to be any subclass of K_{co} and observe that RFD amounts to such concept!). There is, however, another feasible generalization, namely, the one we obtain by looking at the relative congruences on a structure defined in Section 5.2 as another extension of the notion of congruence on an algebra. But we shall not enter into this subject until Chapter 9 below.

The following is an easy but interesting characterization of RFD protoalgebraic quasivarieties inspired in a result of Blok and Pigozzi [7].

THEOREM 6.3.1. *Let K be a protoalgebraic quasivariety of $\mathcal{L}$-structures. Then the following statements are equivalent.*

(i) K is RFD.

(ii) For all $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ and all $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, the mapping $h_{\mathsf{K}} : \mathbf{Fe}_{\mathsf{K}}\mathfrak{A} \to \mathbf{Fe}_{\mathsf{K}}\mathfrak{B}$ is a surjective lattice homomorphism.

(iii) For all $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ and all $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, the mapping $h_{\mathsf{K}} : \mathbf{Fe}_{\mathsf{K}}\mathfrak{A} \to \mathbf{Fe}_{\mathsf{K}}\mathfrak{B}$ is a meet-homomorphism.

Proof. Assume that K is RFD, and let $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ for $\mathfrak{A}, \mathfrak{B} \in$ K. The mapping $h_{\mathsf{K}} : Fe_{\mathsf{K}}\mathfrak{A} \rightarrow Fe_{\mathsf{K}}\mathfrak{B}$ is clearly surjective, for if $\mathfrak{B}' \in Fe_{\mathsf{K}}\mathfrak{B}$ then $hh^{-1}\mathfrak{B}' = \mathfrak{B}'$. Consider two K-filter extensions $\mathfrak{A}_0, \mathfrak{A}_1$ of $\mathfrak{A}$. By Theorem 5.3.8, we have

$$\begin{aligned} h^{-1}h_{\mathsf{K}}(\mathfrak{A}_0 \cap \mathfrak{A}_1) &= (\mathfrak{A}_0 \cap \mathfrak{A}_1) \vee h^{-1}\mathfrak{B} = (\mathfrak{A}_0 \vee h^{-1}\mathfrak{B}) \cap (\mathfrak{A}_1 \vee h^{-1}\mathfrak{B}) \\ &= h^{-1}h_{\mathsf{K}}\mathfrak{A}_0 \cap h^{-1}h_{\mathsf{K}}\mathfrak{A}_1 = h^{-1}(h_{\mathsf{K}}\mathfrak{A}_0 \cap h_{\mathsf{K}}\mathfrak{A}_1). \end{aligned}$$

Thus, applying h_{K} to both sides of the equality and using 5.1.2(ii), we get $h_{\mathsf{K}}(\mathfrak{A}_0 \cap \mathfrak{A}_1) = h_{\mathsf{K}}\mathfrak{A}_0 \cap h_{\mathsf{K}}\mathfrak{A}_1$, which says that h_{K} is a meet-homomorphism. Also,

$$\begin{aligned} h^{-1}h_{\mathsf{K}}(\mathfrak{A}_0 \vee \mathfrak{A}_1) &= (\mathfrak{A}_0 \vee h^{-1}\mathfrak{B}) \vee (\mathfrak{A}_1 \vee h^{-1}\mathfrak{B}) \\ &= h^{-1}h_{\mathsf{K}}\mathfrak{A}_0 \vee h^{-1}h_{\mathsf{K}}\mathfrak{A}_1 \preccurlyeq h^{-1}(h_{\mathsf{K}}\mathfrak{A}_0 \vee h_{\mathsf{K}}\mathfrak{A}_1). \end{aligned}$$

So we repeat the preceding argument and obtain $h_{\mathsf{K}}(\mathfrak{A}_0 \vee \mathfrak{A}_1) \preccurlyeq h_{\mathsf{K}}\mathfrak{A}_0 \vee h_{\mathsf{K}}\mathfrak{A}_1$. Since the reverse inclusion holds trivially, we conclude that h_{K} is a join-homomorphism. Hence we have proved that (i) entails (ii).

The implication from (ii) to (iii) is trivial. Let us see that (iii) implies (i). For this, take an element $\mathfrak{A} \in$ K and let $\mathfrak{A}_0, \mathfrak{A}_1$ and $\mathfrak{B}$ be K-filter extensions of $\mathfrak{A}$. We want to show the equality $(\mathfrak{A}_0 \cap \mathfrak{A}_1) \vee \mathfrak{B} = (\mathfrak{A}_0 \vee \mathfrak{B}) \cap (\mathfrak{A}_1 \vee \mathfrak{B})$. Obviously, the identity function on A, call it h, is a surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. Moreover, by 5.3.8, we have $h^{-1}h_{\mathsf{K}}\mathfrak{A}_0 = \mathfrak{A}_0 \vee h^{-1}\mathfrak{B} = \mathfrak{A}_0 \vee \mathfrak{B}$, and similarly, $h^{-1}h_{\mathsf{K}}\mathfrak{A}_1 = \mathfrak{A}_0 \vee \mathfrak{B}$ and $h^{-1}h_{\mathsf{K}}(\mathfrak{A}_0 \cap \mathfrak{A}_1) = (\mathfrak{A}_0 \cap \mathfrak{A}_1) \vee \mathfrak{B}$. So, since by assumption

$$h^{-1}h_{\mathsf{K}}(\mathfrak{A}_0 \cap \mathfrak{A}_1) = h^{-1}(h_{\mathsf{K}}\mathfrak{A}_0 \cap h_{\mathsf{K}}\mathfrak{A}_1) = h^{-1}h_{\mathsf{K}}\mathfrak{A}_0 \cap h^{-1}h_{\mathsf{K}}\mathfrak{A}_1,$$

we conclude $(\mathfrak{A}_0 \cap \mathfrak{A}_1) \vee \mathfrak{B} = (\mathfrak{A}_0 \vee \mathfrak{B}) \cap (\mathfrak{A}_1 \vee \mathfrak{B})$. ∎

The next result is the natural generalization of Jónsson's Theorem. Earlier generalizations in more restricted contexts can be found in [7], [12], [31].

THEOREM 6.3.2. *Let* Q *be a protoalgebraic quasivariety and let* $\mathsf{K} \subseteq \mathsf{Q}$. *Let* V *denote the relative subvariety of* Q *generated by* K. *If* V *is RFD, then we have:*

(i) $\mathsf{V}_{RFSI} \subseteq ERF_{\mathsf{Q}}SP_u(\mathsf{K})$;

(ii) $\mathsf{V}^*_{RFSI} \subseteq F^*_{\mathsf{Q}}S^*P^*_u(\mathsf{K}^*)$.

Proof. (i) Suppose $\mathfrak{A} \in \mathsf{V}_{RFSI}$. By Corollary 5.3.11, $\mathsf{V} = ERF_{\mathsf{Q}}SP(\mathsf{K})$, and so there exists a $\mathfrak{C}_i \in \mathsf{K}$, for $i \in I$, a substructure $\mathfrak{C} \subseteq \prod_{i \in I}\mathfrak{C}_i$, a Q-filter extension $\mathfrak{B}$ of $\mathfrak{C}$ and homomorphisms $h : \mathfrak{B} \twoheadrightarrow_s \mathfrak{D}$, $g : \mathfrak{A} \twoheadrightarrow_s \mathfrak{D}$. Since $\mathfrak{A}$ is finitely subdirectly irreducible relative to V, 6.1.1 gives that $\mathfrak{B}$ also belongs to V_{RFSI}. Hence, $\mathfrak{B}$ is finitely meet prime in $Fe_{\mathsf{V}}\mathfrak{C}$, for the lattice $\mathrm{Fe}_{\mathsf{V}}\mathfrak{C}$ is distributive by hypothesis. We use now Lemma 6.2.3. There is an ultrafilter $\mathcal{U}$ on I such that $\mathfrak{C}^{\mathcal{U}} \restriction C \preccurlyeq \mathfrak{B}$. Thus, reasoning as in the proof of 6.2.4(i),

$$\begin{aligned} \mathfrak{A} \in ER(\mathfrak{B}) &\subseteq ERF_{\mathsf{Q}}SEP_u(\mathsf{K}) \\ &\subseteq ERF_{\mathsf{Q}}ESP_u(\mathsf{K}) \subseteq ERF_{\mathsf{Q}}SP_u(\mathsf{K}). \end{aligned}$$

We have applied Theorem 5.3.10 in the last inclusion.

(ii) Using part (i), the proof runs as in 5.3.12. ■

Remark. The assumption that V is RFD in the above theorem can be replaced by the condition that the whole class Q is RFD. Actually, for each $\mathfrak{A} \in \mathsf{V}$, we have that $Fe_{\mathsf{V}}\mathfrak{A} = Fe_{\mathsf{Q}}\mathfrak{A}$, for V is closed under F_{Q}. Hence, V is RFD whenever Q is. A trivial consequence of this property, it follows trivially that $\mathsf{V}_{\mathrm{RFSI}} = \mathsf{V} \cap \mathsf{Q}_{\mathrm{RFSI}}$. ⊣

COROLLARY 6.3.3. *Under the assumptions of the preceding theorem, we have:*

(i) $\mathsf{V} = EP_{sd}H_{\mathsf{Q}}SP_{u}(\mathsf{K})$;

(ii) $\mathsf{V}^{*} = P^{*}_{sd}F^{*}_{\mathsf{Q}}S^{*}P^{*}_{u}(\mathsf{K}^{*})$. ■

7. Reduced Quasivarieties

As we already noticed in Section 4.4, reduced quasivarieties of $\mathcal{L}$-structures are not in general elementary over the language $\mathcal{L}$. Even, if $\mathcal{L}_{\approx}$ denotes the language that results from $\mathcal{L}$ by adding the equality symbol $\approx$ (not necessarily included in $\mathcal{L}$), they also may not be elementary over $\mathcal{L}_{\approx}$ for they are not closed under ultraproducts. The same is true for any other type of reduced model class, but our interest is mainly centered on quasivarieties for the reasons adduced in Chapter 5.

In this Chapter we propose to find out the relation between the properties of the Ω operator when restricted to a quasivariety K and the closure of the corresponding reduced class K^* under certain algebraic constructions. Concretely, we shall provide new characterizations, purely algebraic in nature, of the classes defined in Sections 5.3 and 5.4, and conclude stronger forms of previous results that hold for such classes by replacing operators of the sort $\mathcal{O}^*$ by $\mathcal{O}$. Firstly, it becomes appropiate to introduce a new model-theoretic notion that leads to consider another type of full model classes (and so quasivarieties); it is the concept of *elementarily reducible class*, of which we investigate some useful aspects.

7.1. Elementarily Reducible Classes

Let K be a full class of $\mathcal{L}$-structures. We say that K is *elementarily reducible* if K^* is an elementary class over the language with equality $\mathcal{L}_{\approx}$. By a classical result of Los, Suszko and Chang [24, 75], the assumption of K being elementarily reducible entails in fact a stronger condition. Namely, we have the following proposition.

PROPOSITION 7.1.1. *If K is an elementarily reducible class of $\mathcal{L}$-structures, then K^* is a universal-existential class over $\mathcal{L}_{\approx}$, i.e., $\mathsf{K}^* = Mod\,\Gamma$ for some set Γ of universal-existential sentences over $\mathcal{L}_{\approx}$.*

Proof. Using [25, Thm. 5.2.6], it suffices to see that K^* is closed under unions of $\subseteq$-directed systems. So let $\{\mathfrak{A}_i : i \in I\}$ be a family of members of K^* such that $\langle I, \leq\rangle$ is a directed poset and, for all $i, j \in I$, $i \leq j$ iff $\mathfrak{A}_i \subseteq \mathfrak{A}_j$. Define the union of the system $\{\mathfrak{A}_i : i \in I\}$ as usual, and denote it by $\mathfrak{A}$. We claim that $\mathfrak{A}$ still belongs to K and it is reduced. To see this, consider a finite set $\Phi := \Phi(x_1, \dots, x_n)$

of atomic $\mathcal{L}$-formulas and a single atomic $\mathcal{L}$-formula $\varphi := \varphi(x_1, \dots, x_n)$ such that $\mathfrak{A}_i$ is a model of the implicative formula

$$\bigwedge \Phi(x_1, \dots, x_n) \to \varphi(x_1, \dots, x_n),$$

for all $i \in I$. Take elements $a_1, \dots, a_n \in A$. If $\mathfrak{A} \models \bigwedge \Phi(x_1, \dots, x_n)\ [a_1, \dots, a_n]$, then there exists $i_0 \in I$ such that

$$a_1, \dots, a_n \in A_i \text{ and } \mathfrak{A}_i \models \bigwedge \Phi(x_1, \dots, x_n)\ [a_1, \dots, a_n],$$

for all $i \geq i_0$. Therefore, $\mathfrak{A}_i \models \varphi(x_1, \dots, x_n)\ [a_1, \dots, a_n]$ whenever $i \geq i_0$, and from here we conclude that $\mathfrak{A} \models \varphi(x_1, \dots, x_n)\ [a_1, \dots, a_n]$. This shows the first part of the claim, i.e., that $\mathfrak{A} \in \mathsf{K}$.

Suppose now a, b are distinct elements of A. We have that $a, b \in A_{i_0}$ for some $i_0 \in I$. Hence, as $\mathfrak{A}_{i_0}$ is reduced by hypothesis, $\langle a, b\rangle \notin \Omega\mathfrak{A}_{i_0}$. By Theorem 2.1.2, this implies that there is some Leibniz $\mathcal{L}$-formula, let us say $\psi(x, y)$, such that $\mathfrak{A}_{i_0}$ is not a model of $\psi(x, y)$ when x and y are interpreted as a and b respectively. So we just need to apply that $\mathfrak{A}_{i_0} \subseteq \mathfrak{A}$ and obtain $\mathfrak{A} \not\models \psi(x, y)\ [a, b]$. Once more 2.1.2 says $\langle a, b\rangle \notin \Omega\mathfrak{A}$, and consequently $\Omega\mathfrak{A}$ must be equal to Δ_A. Thus the claim is proved. ■

Recall from classical model theory that a class of structures over a language with equality is elementary iff it is closed under ultraproducts of nonempty families, elementary substructures and isomorphisms (this also turns out to be a consequence of Theorem 4.2.1 above). So, using that reduced elementary classes are in general closed under S_e (cf. Corollary 4.2.3), the previous result can be sharpenend as follows.

PROPOSITION 7.1.2. *Let* K *be an elementary class of* $\mathcal{L}$*-structures. Then* K *is elementarily reducible iff* K^* *is closed under* $\overline{P}_u$.

Proof. The forward implication is well-known. To see the converse, we use Lemma 4.1.5(ii); we have that $S_e(\mathsf{K}^*) = S_e L(\mathsf{K}) = LS_e(\mathsf{K}) = \mathsf{K}^*$. Therefore, K^* is closed under S_e and Theorem 4.2.1 completes the proof. ■

COROLLARY 7.1.3. *A quasivariety* K *of* $\mathcal{L}$*-structures is elementarily reducible iff* K^* *is closed under* P_u.

Proof. It suffices to observe that K^* contains the trivial structure, for K does. ■

A question concerning elementary classes of $\mathcal{L}$-structures arises naturally right now. It consists in finding out the condition that the Leibniz operator rectricted to such a kind of class must satisfy for this to be elementarily reducible. In a sense this is an open problem, for apparently such condition cannot be expressed in terms Ω when this is understood as a mapping between two posets, even in the case that the class is a quasivariety and so the mapping is between two algebraic complete lattices. On the contrary, a sufficient condition of purely syntactical nature is known; it refers to the formal representability of the Leibniz equality by means of some $\mathcal{L}$-formula. To be precise, let K be any full class of $\mathcal{L}$-structures, as usual.

We say that the Leibniz equality is *(uniformly elementarily) definible in* K if there exists some $\mathcal{L}$-formula in two free variables, $\alpha(x,y)$, such that, for all $\mathfrak{A} \in \mathsf{K}$,

$$\Omega\mathfrak{A} = \{\langle a,b\rangle \in A^2 : \mathfrak{A} \models \alpha(x,y)\,[a,b]\}.$$

Then we have the next theorem, a special case of which has been recently obtained by Rautenberg [100, Prop. 2].

THEOREM 7.1.4. *Let* K *be an elementary class of* $\mathcal{L}$*-structures. The following statements are equivalent.*

(i) K *is elementarily reducible.*

(ii) There exists a finite set $\psi_1, \dots, \psi_m$ *of Leibniz* $\mathcal{L}$*-formulas such that, for all* $frakA \in \mathsf{K}$, $\Omega\mathfrak{A} = \{\langle a,b\rangle \in A^2 : \mathfrak{A} \models \bigwedge_{i=1}^{m} \psi_i(x,y)\,[a,b]\}$.

(iii) The Leibniz equality is definible in K.

Proof. To show the implication from (i) to (ii), assume (ii) does not hold; we shall see that (i) is also false. Let $\{\psi_k : k \in \omega\}$ be an enumeration of the Leibniz $\mathcal{L}$-formulas, and set $\Psi_k := \{\psi_i : i \leq k\}$. By 2.1.2, $\Omega\mathfrak{A} \subseteq \{\langle a,b\rangle \in A^2 : \mathfrak{A} \models \Psi_k(x,y)\,[a,b]\}$ for each $\mathfrak{A} \in \mathsf{K}$. Thus, for every $k \in \omega$, there exist an $\mathfrak{A}_k \in \mathsf{K}$ and elements $a_k, b_k \in A_k$ such that $\mathfrak{A}_k \models \Psi_k(x,y)\,[a_k,b_k]$ and $\langle a_k,b_k\rangle \notin \Omega\mathfrak{A}_k$. Using Proposition 3.1.1, this condition is equivalent to

$$\mathfrak{A}_k^* \models \Psi_k(x,y)\,[a_k^*,b_k^*] \text{ and } a_k^* \neq b_k^*. \tag{7.1}$$

So define

$$\begin{aligned} &\mathcal{U} := \{X \in Sb(\omega) : \overline{X} \text{ is finite }\},\\ &\mathfrak{A} := \textstyle\prod_{k\in\omega} \mathfrak{A}_k^*/\mathcal{U},\\ &\mathbf{a}^*/\mathcal{U} := \langle a_k^* : k \in \omega\rangle, \quad \mathbf{b}^*/\mathcal{U} := \langle b_k^* : k \in \omega\rangle. \end{aligned}$$

Clearly $\mathfrak{A} \in P_u(\mathsf{K}^*)$. Also, $\mathbf{a}^*/\mathcal{U} \neq \mathbf{b}^*/\mathcal{U}$, for $\{k \in \omega : a_k^* = b_k^*\} = \emptyset \notin \mathcal{U}$. It suffices to show that

$$\mathfrak{A} \models \Psi_m(x,y)\,[\mathbf{a}^*/\mathcal{U}, \mathbf{b}^*/\mathcal{U}], \quad \text{for each } m \in \omega. \tag{7.2}$$

If so, Theorem 2.1.2 implies $\langle \mathbf{a}^*/\mathcal{U}, \mathbf{b}^*/\mathcal{U}\rangle \in \Omega\mathfrak{A}$, and hence K^* is not closed under ultraproducts; thus, the negation of (i) follows from the preceding proposition.

Let us verify (7.2). Fix $m \in \omega$. By (7.1), the infinite set $\{m, m+1, \dots\}$ is included in $\{k \in \omega : \mathfrak{A}_k^* \models \Psi_m(x,y)\,[a_k^*,b_k^*]\}$. Therefore, this second set belongs to $\mathcal{U}$. So we just need to apply Los Theorem to conclude the desired condition.

The implication from (ii) to (iii) is trivial. Assume finally that (iii) holds. Let $\alpha(x,y)$ be an $\mathcal{L}$-formula that defines the Leibniz equality in K. Since K is a elementary by hypothesis, K^* is the class of models of the set of $\mathcal{L}$-formulas $Th\,\mathsf{K}$ and the single $\mathcal{L}_{\approx}$-sentence $\forall x \forall y (\alpha(x,y) \rightarrow x \approx y)$. Hence, K^* is elementary over the language $\mathcal{L}_{\approx}$ and (i) holds. ∎

The above theorem turns out to be a very interesting one; it suggests the possibility of translating the various properties of the Leibniz operator (when restricted to a given elementary class) into the uniform definibility of the Leibniz equality in the class by means of some set of $\mathcal{L}$-formulas. We do not deal with this problem here but invite the reader to look at the paper by Blok and Pigozzi [12, §13], which contains some results in this direction.

7.2. Characterizing some Reduced Quasivarieties

It has already been established in Chapter 5 that the various properties of the Leibniz operator when restricted to quasivarieties of structures derives into a good behaviour of the corresponding reduced classes. In essence, the nature of these properties concerned the handling of the operator Ω when dealing with certain methods of construction of new structures from old ones, namely, filter extensions, substructures and unions of directed systems of filter extensions. So, as there exists a close connection between these methods and the distinct product constructions (cf. e.g. the proofs of 4.4.5, 5.2.3 and 6.1.2), one can also expect to handle easily the Leibniz operator when dealing with them. This idea suggests the feasibility of attaining purely algebraic characterizations for the distinct types of classes introduced some pages back in Sections 5.3 and 5.4. The next theorems summarize such characterizations.

THEOREM 7.2.1. *Let* K *be a full class of $\mathcal{L}$-structures closed under subdirect products. Then* K *is protoalgebraic iff* K^* *is closed under* P_{sd}.

Proof. Consider the forward direction. Suppose $h : \mathfrak{A} \rightarrowtail_{sd} \prod_{i\in I} \mathfrak{B}_i$ with $\mathfrak{B}_i \in \mathsf{K}^*$. The hypothesis that K is a full class closed under P_{sd} says that $\mathfrak{A} \in \mathsf{K}$, so it is enough to see $\mathfrak{A}$ is reduced. And this is easy to check. By the proof of 6.1.1, $\mathfrak{A} = \bigcap_{i\in I} \mathfrak{A}_i$ and $\bigcap_{i\in I} Ker(\pi_i \circ h) = \Delta_A$, where $\mathfrak{A}_i = (\pi_i \circ h)^{-1}\mathfrak{B}_i$ for each i. Also, since $\mathfrak{B}_i$ is reduced and $\pi_i \circ h : \mathfrak{A}_i \twoheadrightarrow_s \mathfrak{B}_i$ by definition, $Ker(\pi_i \circ h) = \Omega\mathfrak{A}_i$. So, the monotonicity of Ω implies

$$\Omega\mathfrak{A} \subseteq \textstyle\bigcap_{i\in I} \Omega\mathfrak{A}_i = \Delta_A$$

and thus $\mathfrak{A}$ is reduced.

To show the backward direction, consider $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ such that $\mathfrak{A} \preccurlyeq \mathfrak{B}$. Construct a map $h : A \rightarrow A/\Omega\mathfrak{A} \times A/\Omega\mathfrak{B}$ by setting $ha = (a/\Omega\mathfrak{A}, a/\Omega\mathfrak{B})$. Since $\mathfrak{B}$ is a filter extension of $\mathfrak{A}$, h induces an algebra homomorphism from $\mathbf{A}$ into $\mathbf{A}/\Omega\mathfrak{A} \times \mathbf{A}/\Omega\mathfrak{B}$ that satisfies the strongness condition:

$$\begin{aligned}\langle a_1,\dots,a_n\rangle \in r^{\mathfrak{A}} \text{ iff } &\langle a_1/\Omega\mathfrak{A},\dots,a_n/\Omega\mathfrak{A}\rangle \in r^{\mathfrak{A}^*} \text{ and } \langle a_1/\Omega\mathfrak{B},\dots,a_n/\Omega\mathfrak{B}\rangle \in r^{\mathfrak{B}^*}\\ \text{iff } &\langle ha_1,\dots,ha_n\rangle \in r^{\mathfrak{A}^*\times\mathfrak{B}^*}.\end{aligned}$$

Moreover, $Ker\, h = \Omega\mathfrak{A} \cap \Omega\mathfrak{B}$, and hence $\mathfrak{A}/\Omega\mathfrak{A} \cap \Omega\mathfrak{B} \rightarrowtail_{sd} \mathfrak{A}^* \times \mathfrak{B}^*$ by the Homomorphism Theorem. So, since K^* is closed under P_{sd}, we conclude that $\mathfrak{A}/\Omega\mathfrak{A}\cap\Omega\mathfrak{B}$ must belong to K^* and consequently must be reduced. Therefore $\Omega\mathfrak{A} \cap \Omega\mathfrak{B}$ is the largest congruence on $\mathfrak{A}$, i.e., $\Omega\mathfrak{A} \cap \Omega\mathfrak{B} = \Omega\mathfrak{A}$, and finally $\Omega\mathfrak{A} \subseteq \Omega\mathfrak{B}$. This proves K is protoalgebraic. ∎

COROLLARY 7.2.2. *Let* K *be any quasivariety of* $\mathcal{L}$*-structures. Then the following statements are equivalent.*

(i) K *is protoalgebraic.*
(ii) K^* *is closed under* P_{sd}.
(iii) $\mathsf{K}^* = P_{sd}S^*P_u^*(\mathsf{K}^*)$.

Proof. We apply Lemma 4.4.5 and the above theorem. ∎

The following is a trivial consequence of the preceding Corollary and Theorem 5.3.12. Notice however that it can also be derived without using the Corollary; for this, we just need to apply 5.3.12 and Lemma 4.1.5(i).

COROLLARY 7.2.3. *Let* Q *be any protoalgebraic quasivariety of* $\mathcal{L}$*-structures, and let* K *be a subclass of* Q. *Then the reduced relative subvariety of* Q^* *generated by* K^* *is* $\mathsf{K}^V \cap \mathsf{Q}^* = F_{\mathsf{Q}}^* S^* P(\mathsf{K}^*)$.∎

THEOREM 7.2.4. *Let* K *be a full class of* $\mathcal{L}$*-structures closed under substructures and direct products. Then* K *is semialgebraic iff* K^* *is closed under* S *and* P.

Proof. By 7.2.1, if K is semialgebraic then K^* is closed under P. So it is enough to prove that K^* is closed under S^*. Let $\mathfrak{B} \in \mathsf{K}^*$ and $\mathfrak{A} \subseteq \mathfrak{B}$. By assumption, K is full and closed under substructures, so that $\mathfrak{A}, \mathfrak{B}$ are both members of K. Thus the monotonicity of Ω with respect to $\subseteq$ entails that $\Omega\mathfrak{A} \subseteq \Omega\mathfrak{B} = \Delta_B$. Consequently, $\Omega\mathfrak{A} = \Delta_A$ and $\mathfrak{A}$ is reduced. This proves the implication from left to right.

Assume now K^* is closed under S and P. In particular, K^* is closed under P_{sd}. So, 7.2.1 again (this time the backward implication) implies K^* is protoalgebraic. Let us show that Ω is $\subseteq$-monotone. Take $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$. The natural projection $\pi_{\mathfrak{B}} : \mathfrak{B} \twoheadrightarrow_s \mathfrak{B}^*$ gives rise to a strong homomorphism from $\mathfrak{A}$ into $\mathfrak{B}^*$. Let us call h this homomorphism; it sends an element $a \in A$ to the equivalence class $a/\Omega\mathfrak{B}$. By 1.2.1, $h\mathfrak{A} \subseteq \mathfrak{B}^*$ and hence $h\mathfrak{A}$ is reduced, for K^* is closed under S. So, using 2.3.1, $h\mathfrak{A}$ must be isomorphic to $\mathfrak{A}^*$ by the mapping

$$a/\Omega\mathfrak{B} \longmapsto a/\Omega\mathfrak{A}.$$

On the other hand, the Homomorphism Theorem stated in 2.2.3 says that the correspondence $a/Ker\,h \longmapsto a/\Omega\mathfrak{B}$ establishes an isomorphism between $\mathfrak{A}/Ker\,h$ and $h\mathfrak{A}$. So we actually have that $\mathfrak{A}/Ker\,h \cong \mathfrak{A}^*$ by the mapping $a/Ker\,h \longmapsto a/\Omega\mathfrak{A}$, and therefore $Ker\,h = \Omega\mathfrak{A}$. Hence, since $Ker\,h \subseteq \Omega\mathfrak{B}$ by the definition of h, the desired inclusion $\Omega\mathfrak{A} \subseteq \Omega\mathfrak{B}$ holds. ∎

COROLLARY 7.2.5. *Let* K *be any quasivariety of* $\mathcal{L}$*-structures. Then the following statements are equivalent.*

(i) K *is semialgebraic.*
(ii) K^* *is closed under* S *and* P.
(iii) $\mathsf{K}^* = SPP_u^*(\mathsf{K}^*)$.

Proof. We use again Lemma 4.4.5 and the preceding theorem. ∎

In contrast to what happened before, now it seems that the next result cannot be derived without using the above Corollary, for in general the reduction operator L does not commute with F in the sense of Lemma 4.1.5(i).

COROLLARY 7.2.6. *Let* Q *be any semialgebraic quasivariety of* $\mathcal{L}$*-structures, and let* K *be a subclass of* Q*. Then the reduced relative subvariety of* Q^* *generated by* K^* *can be expressed as* $\mathsf{K}^V \cap \mathsf{Q}^* = F^*_{\mathsf{Q}}SP(\mathsf{K}^*)$. ∎

THEOREM 7.2.7. *Assume* K *is a quasivariety of* $\mathcal{L}$*-structures. Then* K *is algebraic iff* K^* *is closed under* S, P *and* P_u.

Proof. Suppose first that K is an algebraic quasivariety and let us prove K^* is closed under S, P and P_u. By 4.4.4 and 4.4.5, it is enough to see that K^* is closed under filtered subdirect products. Consider first the case of filtered subdirect products modulo principal filters. Let $\{\mathfrak{A}_i : i \in I\}$ be a family of members of K^* and let $\mathcal{F}$ be the principal filter of I generated by $X \in Sb(I)$. We just need to verify that if $\mathfrak{A} \subseteq_{sd} \prod_{i\in I} \mathfrak{A}_i$, then $\mathfrak{A}/\mathcal{F}$ is reduced, for clearly it belongs to K. Take elements $\mathbf{a}/\mathcal{F}, \mathbf{b}/\mathcal{F} \in A/\mathcal{F}$ and let $\varphi := \varphi(x, z_1, \dots, z_k)$ be any atomic $\mathcal{L}$-formula. Since $\mathfrak{A}/\mathcal{F}$ is isomorphic to a substructure of the filtered product $\prod_{i\in I} \mathfrak{A}_i/\mathcal{F}$,

$$\mathfrak{A}/\mathcal{F} \models \varphi(x, z_1, \dots, z_k)\, [g] \text{ iff } \textstyle\prod_{i\in I} \mathfrak{A}_i/\mathcal{F} \models \varphi(x, z_1, \dots, z_k)\, [g],$$

for each assignment $g : \mathbf{Te}_{\mathcal{L}} \to \mathbf{A}/\mathcal{F}$. Therefore, using 2.1.2 and 1.3.1, the condition $\langle \mathbf{a}/\mathcal{F}, \mathbf{b}/\mathcal{F} \rangle \in \Omega(\mathfrak{A}/\mathcal{F})$ is equivalent to the next one: for all atomic $\mathcal{L}$-formula $\varphi(x, z_1, \dots, z_k)$ and all $\mathbf{c}_1, \dots, \mathbf{c}_k \in \mathbf{A}$,

$$\begin{aligned} &X \subseteq \{i \in I : \mathfrak{A}_i \models \varphi(x, z_1, \dots, z_k)\, [\mathbf{a}/\mathcal{F}), \mathbf{c}_1/\mathcal{F}, \dots, \mathbf{c}_k/\mathcal{F}] \text{ iff} \\ &X \subseteq \{i \in I : \mathfrak{A}_i \models \varphi(x, z_1, \dots, z_k)\, [\mathbf{b}/\mathcal{F}), \mathbf{c}_1/\mathcal{F}, \dots, \mathbf{c}_k/\mathcal{F}]. \end{aligned} \tag{7.3}$$

We now apply the definition of direct product; (7.3) can be expressed as

$$\textstyle\prod_{i\in X} \mathfrak{A}_i \models \forall z_1 \dots \forall z_k (\varphi(x, z_1, \dots, z_k) \leftrightarrow \varphi(y, z_1, \dots, z_k))\, [\mathbf{a}{\restriction}X, \mathbf{b}{\restriction}X],$$

for all $\varphi \in Atm\ \mathcal{L}$. Hence, a new application of 2.1.2 leads us to conclude that $\langle \mathbf{a}/\mathcal{F}, \mathbf{b}/\mathcal{F} \rangle \in \Omega(\mathfrak{A}/\mathcal{F})$ iff $\langle \mathbf{a}{\restriction}X, \mathbf{b}{\restriction}X \rangle \in \Omega(\prod_{i\in X} \mathfrak{A}_i)$. Since Ω is $\preccurlyeq$-monotone, Theorem 7.2.1 says that K^* is closed under P and consequently $\Omega(\prod_{i\in X} \mathfrak{A}_i) = \Delta_{\prod_X A_i}$. So $X \subseteq \{i \in I : a_i = b_i\} \in \mathcal{F}$, which implies that $\mathbf{a}/\mathcal{F} = \mathbf{b}/\mathcal{F}$. This proves $\mathfrak{A}/\mathcal{F}$ is reduced and thus a member of K^*.

Consider now the general case, i.e., let $\mathcal{F}$ be an arbitrary (proper) filter on I. Keeping the notation introduced in the proof of Lemma 5.2.3, define the following K-filter extensions of $\mathfrak{A}$:

$$\begin{aligned} \mathfrak{A}^{\mathcal{F}} &:= \pi_{\mathcal{F}}^{-1}(\mathfrak{A}/\mathcal{F}), \\ \mathfrak{A}^{\mathcal{F}_X} &:= \pi_{\mathcal{F}_X}^{-1}(\mathfrak{A}/\mathcal{F}_X), \quad \text{for all } X \in \mathcal{F}. \end{aligned}$$

Clearly the family $\{\mathfrak{A}^{\mathcal{F}_X} : X \in \mathcal{F}\}$ is a directed system of members of $\mathsf{K}_{\mathbf{A}}$: for all $X, Y \in \mathcal{F}$, the structure $\mathfrak{A}^{\mathcal{F}_{X\cap Y}}$ is a filter extension of both $\mathfrak{A}^{\mathcal{F}_X}$ and $\mathfrak{A}^{\mathcal{F}_Y}$. Also, if $\mathbf{a}_1, \dots, \mathbf{a}_n \in A$ and r is an n-ary relation symbol of $\mathcal{L}$, we have

$$\begin{aligned} \langle \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \in r^{\mathfrak{A}^{\mathcal{F}}} &\text{ iff } \{i \in I : \langle a_{1i}, \dots, a_{ni} \rangle \in r^{\mathfrak{A}_i}\} \in \mathcal{F} \\ &\text{ iff } \{i \in I : \langle a_{1i}, \dots, a_{ni} \rangle \in r^{\mathfrak{A}_i}\} \in \mathcal{F}_X \text{ for some } X \in \mathcal{F}. \end{aligned}$$

Thus, $\mathfrak{A}^{\mathcal{F}} = \bigcup_{X\in\mathcal{F}} \mathfrak{A}^{\mathcal{F}_X}$. So, applying Theorem 2.1.8(i) and the join-continuity of Ω on $\mathbf{K}$, we can argue as follows to conclude that $\mathfrak{A}/\mathcal{F}$ is also reduced in the general case:

$$\begin{aligned}\langle \mathbf{a}/\mathcal{F}, \mathbf{b}/\mathcal{F}\rangle \in \Omega(\mathfrak{A}/\mathcal{F}) &\text{ iff } \langle \mathbf{a},\mathbf{b}\rangle \in \Omega(\mathfrak{A}^{\mathcal{F}}) = \textstyle\bigcup_{X\in\mathcal{F}} \Omega\mathfrak{A}^{\mathcal{F}_X}\\ &\text{ iff } \langle \mathbf{a},\mathbf{b}\rangle \in \Omega\mathfrak{A}^{\mathcal{F}_X} \text{ for some } X\in\mathcal{F}\\ &\text{ iff } \langle \mathbf{a}/\mathcal{F}_X, \mathbf{b}/\mathcal{F}_X\rangle \in \Omega(\mathfrak{A}/\mathcal{F}_X) \text{ for some } X\in\mathcal{F}\\ &\text{ iff } \mathbf{a}/\mathcal{F}_X = \mathbf{b}/\mathcal{F}_X \text{ for some } X\in\mathcal{F}\\ &\text{ iff } \{i\in I : a_i = b_i\}\in\mathcal{F}\\ &\text{ iff } \mathbf{a}/\mathcal{F} = \mathbf{b}/\mathcal{F}.\end{aligned}$$

Suppose now $\mathbf{K}^*$ is closed under S, P and P_u or, equivalently, under S and P_f, and let us show that the Leibniz operator is join-continuous on $\mathbf{K}$. Choose any $\mathcal{L}$-algebra $\mathbf{A}$ and consider an arbitrary directed system $\{\mathfrak{A}_i : i\in I\}$ of $\mathbf{A}$-structures from $\mathbf{K}$. Denote by $\mathfrak{A}$ their union. Let $\mathcal{F}$ be the filter on I generated by the family $\{[i) : i\in I\}$ of subsets of I, where $[i) = \{j\in I : i\le j\}$. Then we define the mapping h from A into $\prod_{i\in I} A_i^*/\mathcal{F}$ by setting $ha = \langle a/\Omega\mathfrak{A}_i : i\in I\rangle/\mathcal{F}$. We claim that so defined h is a strong homomorphism from $\mathfrak{A}$ into $\prod_{i\in I}\mathfrak{A}_i^*/\mathcal{F}$. Indeed, we omit the proof that h is an algebra homomorphism for it is a straightforward consequence from the definitions involved. To verify the strongness condition, let r be an n-ary relation symbol of $\mathcal{L}$ and let $a_1,\dots,a_n$ be arbitrary elements of A. The assumption that $\{\mathfrak{A}_i : i\in I\}$ is a directed system and the definition of $\mathcal{F}$ entail the following equivalences:

$$\begin{aligned}\langle a_1,\dots,a_n\rangle\in r^{\mathfrak{A}} &\text{ iff } \langle a_1,\dots,a_n\rangle \in r^{\mathfrak{A}_i} \text{ for some } i\in I\\ &\text{ iff } \langle a_1/\Omega\mathfrak{A}_j,\dots,a_n/\Omega\mathfrak{A}_j\rangle \in r^{\mathfrak{A}_j^*} \text{ for all } j\in[i) \text{ and some } i\in I\\ &\text{ iff } \{j\in I : \langle a_1/\Omega\mathfrak{A}_j,\dots,a_n/\Omega\mathfrak{A}_j\rangle\in r^{\mathfrak{A}_j^*}\}\in\mathcal{F}.\end{aligned}$$

Therefore $h : \mathfrak{A} \to_s \prod_{i\in I}\mathfrak{A}_i^*/\mathcal{F}$. We now apply the Homomorphism Theorem stated in 2.2.3 and conclude that $\mathfrak{A}/Ker\ h$ belongs to the class $SP_f(\mathbf{K}^*)$, which by hypothesis coincides with $\mathbf{K}^*$. Hence we have proved that $\mathfrak{A}/Ker\ h$ is a reduced structure. So, to show that $\Omega\mathfrak{A} = \bigcup_{i\in I}\Omega\mathfrak{A}_i$ we just need to verify the condition $Ker\ h = \bigcup_{i\in I}\Omega\mathfrak{A}_i$. For this, we use again the fact that $\{\mathfrak{A}_i : i\in I\}$ is a directed system and the definition of $\mathcal{F}$. We have:

$$\begin{aligned}ha = hb &\text{ iff } \langle a/\Omega\mathfrak{A}_i : i\in I\rangle/\mathcal{F} = \langle b/\Omega\mathfrak{A}_i : i\in I\rangle/\mathcal{F}\\ &\text{ iff } \{i\in I : a/\Omega\mathfrak{A}_i = b/\Omega\mathfrak{A}_i\}\in\mathcal{F}\\ &\text{ iff } \textstyle\bigcap_{j=1}^{k}[i_j) \subseteq \{i\in I : a/\Omega\mathfrak{A}_i = b/\Omega\mathfrak{A}_i\} \text{ for some } i_1,\dots,i_k\in I\\ &\text{ iff } a/\Omega\mathfrak{A}_j = b/\Omega\mathfrak{A}_j \text{ for all } j\ge i \text{ and some } i\in I\\ &\text{ iff } \langle a,b\rangle \in \textstyle\bigcup_{i\in I}\Omega\mathfrak{A}_i. \blacksquare\end{aligned}$$

A special case of the above result can be found in [12]; however the proof there takes quite a different path, since it rests on the effective use of syntactical characterizations of the properties of the Leibniz operator that hardly extend to the general case.

COROLLARY 7.2.8. *A quasivariety* K *of* $\mathcal{L}$*-structures is algebraic iff it is an elementarily reducible semialgebraic class.*

Proof. It is a direct consequence of Proposition 7.1.2 and Corollary 7.2.4. ∎

COROLLARY 7.2.9. *A quasivariety* K *of* $\mathcal{L}$*-structures is algebraic iff* K^* *is an* $\mathcal{L}_{\approx}$*-quasivariety, i.e.,* $\mathsf{K}^* = SPP_u(\mathsf{K}^*)$. ∎

8. Free Structures

In this Chapter we deal with the existence of free members in classes of structures, a problem that attracted the attention of several authors during the last decade for its applications to computer science [81, 82], and more recently for its interest in algebraic logic. The origin of the problem can be traced back to the 1930's in the context of universal algebra when Birkhoff [4] stated a sufficient condition for the existence of free members in a class of abstract algebras (see also [6, Thm. 13']); Birkhoff's result was improved a few years later by Mal'cev [85, Thm. 1], who provided a sufficient as well as necessary condition. The investigation of the same problem for classes of structures over arbitrary first-order languages with equality was solved at the end of the 1960's [109].

Now, in the context of first-order logic without equality, we are obliged to investigate what happens for both full and reduced classes. We go further specially in the context of reduced semantics obtaining a characterization of protoalgebraic and semialgebraic quasivarieties in terms of the existence of some sort of free structures (Theorems 8.1.7 and 8.1.8 below). Also, we examine the nature of the term-structures of the quasivariety generated by a class and use them to supply new proofs for Theorems 4.4.1 and 4.5.1. A final Section is devoted to study the correspondence of quasivarieties with some specific types of lattice structures; such a correspondence will generalize Neumann's result [94] relating varieties of algebras with fully invariant congruences and offers the possibility of turning the logical methods used in the theory of varieties and quasivarieties into purely algebraic ones.

8.1. Free Structures in Full and Reduced Classes

Let K be a class of $\mathcal{L}$-structures, $\mathfrak{A}$ a single $\mathcal{L}$-structure and X a subset of elements of the universe of $\mathfrak{A}$, which can be empty iff $\mathcal{L}$ contains some constant symbol. We say $\mathfrak{A}$ is *freely generated over* K *by* X if, for any $\mathfrak{B} \in \mathsf{K}$ and any $g : X \to B$, there exists a unique homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ such that $h \restriction X = g$. $\mathfrak{A}$ is *free in* K *with* α *generators* if $\mathfrak{A} \in \mathsf{K}$ and $\mathfrak{A}$ is freely generated over K by some set of generators of cardinal α. Clearly any two structures freely generated over K by some sets of generators of the same cardinality are isomorphic. The first purpose of this Section is to characterize those full model classes which contain free structures.

Two basic notions for solving this problem turns out to be that of Herbrand structure of a class over some set of variables, and that of relative ξ-subvariety of a class, where ξ is an arbitrary cardinal. Let us start with the notion of Herbrand structure. For this, fix a cardinal $\alpha \geq 0$ and an arbitrary class K of type $\mathcal{L}$. We define the *α-Herbrand structure of* K as the structure

$$\mathcal{H}_\alpha \mathsf{K} = \langle \mathbf{Te}_{\mathcal{L},\alpha}, R_{\mathsf{K},\alpha}\rangle,$$

where $R_{\mathsf{K},\alpha} = \{r^{\mathsf{K},\alpha} : r \in R\}$ is given by setting

$$r^{\mathsf{K},\alpha} = \{\langle t_1, \ldots, t_n\rangle \in Te^n_{\mathcal{L},\alpha} : \mathsf{K} \vDash rt_1 \ldots t_n\}, \tag{8.1}$$

for each $r \in R$. (As before, the case $\alpha = 0$ is allowed iff the set of constant symbols of $\mathcal{L}$ is nonempty).[23]

Remark. If $\mathcal{L}$ has equality, then $\mathcal{H}_\alpha \mathsf{K}$ as defined above may not be directly an $\mathcal{L}$-structure, because the interpretation of two distinct closed terms may coincide in every element of K. So in this case the definition of Herbrand structure must be modified as follows. Let $\approx$ denote the equality symbol in $\mathcal{L}$, and assume $\approx^{\mathsf{K},\alpha}$ is given by (8.1). It is easy to see that $\approx^{\mathsf{K},\alpha}$ is a congruence on $\langle \mathbf{Te}_{\mathcal{L},\alpha}, R_{\mathsf{K},\alpha}\rangle$; in fact, it is the Leibniz congruence. Thus, the α-Herbrand structure of K can now be defined as

$$\mathcal{H}_\alpha \mathsf{K} = \langle \mathbf{Te}_{\mathcal{L},\alpha}/ \approx^{\mathsf{K},\alpha}, R_{\mathsf{K},\alpha}/ \approx^{\mathsf{K},\alpha}\rangle.$$

We must notice right now that with this new notion all the subsequents results remain true with no changes. ⊣

Herbrand structures have several important properties whose proof is rather immediate. The first one is just a useful reformulation of their definition, which says that every α-Herbrand structure of K can be obtained from $\mathcal{H}_\omega \mathsf{K}$.

LEMMA 8.1.1. *Let* K *be any class of $\mathcal{L}$-structures. For any cardinal α and any n-ary relation symbol $r \in R$,*

$$r^{\mathsf{K},\alpha} = \{\langle t_1, \ldots, t_n\rangle \in Te^n_{\mathcal{L},\alpha} : \text{there exist } \varphi \in Atm\,\mathcal{L} \text{ and } \sigma : \mathbf{Te}_{\mathcal{L}} \to \mathbf{Te}_{\mathcal{L},\alpha} \text{ such that } \mathsf{K} \vDash \varphi \text{ and } \sigma\varphi = rt_1 \ldots t_n\}.$$

Proof. The inclusion from right to left is clear. Suppose $t_1, \ldots, t_n \in Te_{\mathcal{L},\alpha}$ are such that $\mathsf{K} \vDash rt_1 \ldots t_n$. Let $x_{\lambda_1}, \ldots, x_{\lambda_k}$ be the distinct variables occurring among the terms t_i, $1 \leq i \leq n$. If $\varphi := rt_1 \ldots t_n$, let $\tau_\varphi : \mathbf{Te}_{\mathcal{L},\alpha} \to \mathbf{Te}_{\mathcal{L}}$ be such that

$$\tau_\varphi x_{\lambda_i} = x_i, \quad 1 \leq i \leq n.$$

[23]The reason of naming such structures in this way is that the terminology of *Herbrand structure* is presently quite extended among logicians and computer scientists to refer to our 0-Herbrand structures, as Herbrand [64] used a closely related concept in the proof of his famous theorem (see, e.g., [63, 71]). Nevertheless, to be historically accurate, we should call them *Lindenbaum structures*, because it was Lindenbaum who considered for the first time, during the 20's, models built up out of the same language. On the other hand, according to Tarski's contribution, the term *Lindenbaum-Tarski matrix* used by Blok and Pigozzi [12, p.32] would appear to be more adequate to refer to the Leibniz quotient of our Herbrand structure.

Since τ_φ is locally one-one, there exists by the Axiom of Choice a homomorphism $\sigma_\varphi : \mathbf{Te}_\mathcal{L} \to \mathbf{Te}_{\mathcal{L},\alpha}$ satisfying $(\sigma_\varphi \circ \tau_\varphi)(t_i) = t_i$, for all i. So, for the reverse inclusion it suffices to show that $\mathsf{K} \vDash \tau_\varphi \varphi$. But this is a clear consequence from the following equivalence: for any member $\mathfrak{A}$ of K and any assignment $g : \mathbf{Te}_\mathcal{L} \to \mathbf{A}$, $\mathfrak{A} \vDash \tau_\varphi \varphi\ [g]$ iff $\mathfrak{A} \vDash \varphi\ [g \circ \tau_\varphi]$. ■

The second property says that, for quasivarieties, Herbrand structures are minimal elements in the set of all term-structures of the class.

PROPOSITION 8.1.2. *Let* K *be a quasivariety of* $\mathcal{L}$*-structures. Then, for all* $\alpha \geq 0$, $\mathcal{H}_\alpha \mathsf{K} = \bigcap \mathsf{K}_{\mathbf{Te}_{\mathcal{L},\alpha}}$, *i.e.,* $\mathcal{H}_\alpha \mathsf{K}$ *is the minimal structure of* K *with underlying algebra* $\mathbf{Te}_{\mathcal{L},\alpha}$. ■

The third property extends the well known peculiarity of freely generated algebras pointed out by Birkhoff, according to which the set of atomic formulas (over a fixed set of variables) satisfied by a class of algebras coincide with the one satisfied by its corresponding free algebra. In the terminology of Makowsky [82, p. 274], we should say that the Herbrand structures are *generic* relatively to atomic formulas. This third property has an interesting consequence that we shall analyze in detail in Section 8.3 below.

LEMMA 8.1.3. *Let* K *be any class of* $\mathcal{L}$*-structures. For all cardinals* $\beta \geq \alpha > 0$, *if* φ *is an atomic* $\mathcal{L}$*-formula over* α *variables, then* $\mathsf{K} \vDash \varphi$ *iff* $\mathcal{H}_\beta \mathsf{K} \vDash \varphi$.

Proof. The "only if" part follows trivially from the definitions involved. For the converse, it suffices to observe that, if $\varphi := rt_1 \dots t_n$, where $t_1, \dots, t_n \in Te_{\mathcal{L},\alpha}$, then $\mathcal{H}_\beta \mathsf{K} \vDash \varphi$ entails $\langle t_1, \dots, t_n \rangle \in r^{\mathsf{K},\beta}$. For this, we use that $\beta \geq \alpha$. ■

A fourth property can be derived from 8.1.3 and provides a sufficient condition for a class K to contain almost all of its Herbrand structures. This property is going to be quite a lot improved in the next Section, Theorem 8.2.2.

PROPOSITION 8.1.4. *If* K *is a quasivariety of* $\mathcal{L}$*-structures, then* $\mathcal{H}_\alpha \mathsf{K} \in \mathsf{K}$, *for every cardinal* α.

Proof. Let us sketch the proof for $\alpha = \omega$; the general case entails only some additional technical difficulties. Let $\varphi_1 \wedge \dots \wedge \varphi_m \to \varphi$ be any implicative $\mathcal{L}$-formula satisfied by K and let $\sigma : \mathbf{Te}_\mathcal{L} \to \mathbf{Te}_\mathcal{L}$. If $\mathcal{H}_\omega \mathsf{K} \vDash \varphi_i\ [\sigma]$, for $1 \leq i \leq m$, then $\sigma\varphi_1, \dots, \sigma\varphi_m \in Atm\, \mathsf{K}$; hence, $\sigma\varphi \in Atm\, \mathsf{K}$, which by 8.1.3 equals to $Atm\, \mathcal{H}_\omega \mathsf{K}$. But this implies that $\mathcal{H}_\omega \mathsf{K} \vDash \varphi\ [\sigma]$. Thus, since σ was arbitrary, $\mathcal{H}_\omega \mathsf{K} \in Mod\, Imp\, \mathsf{K} = \mathsf{K}$, for by hypothesis K is a quasivariety. ■

Finally, the last property follows immediately from the definition and is mainly responsible for the usefulness of Herbrand structures.

PROPOSITION 8.1.5. *For each class* K *of* $\mathcal{L}$*-structures and each* $\mathfrak{A} \in \mathsf{K}$, *there exists a cardinal* α *such that* $\mathfrak{A} \in H(\mathcal{H}_\alpha \mathsf{K})$.

Proof. It suffices to take $\alpha = |A|$ and consider any bijection between X_α and A. ■

The other basic notion we mentioned a few paragraphs back is introduced to avoid the use of a more general concept of free structure that would correspond to what Mal'cev called *algebras with defining relations* [85]. Take an arbitrary cardinal ξ and let $\mathcal{L}_\xi$ mean the language obtained from $\mathcal{L}$ by joining ξ new constant symbols $\{c_\lambda : \lambda < \xi\}$. Then a class V of $\mathcal{L}_\xi$-structures is said to be a *relative ξ-subvariety of* K, or simply a *relative subvariety of* K when $\xi = 0$, if there exists a set of atomic $\mathcal{L}_\xi$-formulas Σ such that $\mathfrak{A} \in \mathsf{V}$ iff $\mathfrak{A}$ is a model of Σ and the $\mathcal{L}$-reduct of $\mathfrak{A}$ belongs to K.

Earlier versions of the next theorem can be found in several papers; e.g. [53], [81], [109]. Our proof closely resembles the one given in [109], as it is also based in the model-theoretic method of diagrams initiated by Henkin and Robinson in the 50's; for a proof that takes a different path, see [53].

THEOREM 8.1.6. *Assume* K *is an elementary class of* $\mathcal{L}$*-structures. Then the following statements are equivalent.*

(i) K *is a quasivariety.*

(ii) Every relative ξ*-subvariety of* K *has a free structure with* α *generators, for all cardinals* ξ, α *such that* $\xi + \alpha > 0$*. Concretely,* $\mathcal{H}_\alpha \mathsf{V}$ *is the free structure in* V *with* α *generators, for each relative* ξ*-subvariety* V *of* K.

Proof. Assume (i) and fix $\xi, \alpha \geq 0$ such that $\xi + \alpha > 0$; this condition is added to ensure that it has sense to consider α-Herbrand structures on the language $\mathcal{L}_\xi$. Let V be a relative ξ-subvariety of K and denote by Σ a set of atomic $\mathcal{L}_\xi$-formulas that defines V. We are going to show the Herbrand structure $\mathcal{H}_\alpha \mathsf{V}$ is free in V with α generators $X_\alpha = \{x_\lambda : \lambda < \alpha\}$. Indeed, since K is a quasivariety, V coincides with the class of models of $\Sigma \cup Imp\,\mathsf{K}$. Consequently, Proposition 8.1.4 ensures that $\mathcal{H}_\alpha \mathsf{V}$ belongs to V. Also, given a structure $\mathfrak{B} \in \mathsf{V}$ and $g : X_\alpha \to B$, g extends uniquely to an algebra homomorphism $h : \mathbf{Te}_{\mathcal{L}_\xi,\alpha} \to \mathbf{B}$. So, since trivially h preserves relations, $\mathcal{H}_\alpha \mathsf{V}$ is free in V with set of generators X_α. This proves (i) implies (ii).

Assume now (ii) and let us see that $Mod\,Imp\,\mathsf{K} \subseteq \mathsf{K}$; the opposite inclusion is clear. Let $\mathfrak{A}$ be an $\mathcal{L}$-structure such that $\mathfrak{A} \models Imp\,\mathsf{K}$. Take Σ to be the positive diagram of $\mathfrak{A}$, i.e., the set of atomic $\mathcal{L}_{|A|}$-sentences that hold in $(\mathfrak{A}, a)_{a \in A}$, and denote the relative $|A|$-subvariety of K determined by Σ by V. Then, using (ii), $\mathcal{H}_0 \mathsf{K}$ is a member of V and hence $\mathcal{H}_0 \mathsf{V} \restriction \mathcal{L} \in \mathsf{K}$. We claim that $\mathfrak{A}$ is a reduction of $\mathcal{H}_0 \mathsf{V} \restriction \mathcal{L}$. To establish a precise reductive homomorphism, consider the algebra homomorphism h which sends the constant c_a to a, for all $a \in A$, and each one of the original constants of $\mathcal{L}$ to their interpretations in $\mathfrak{A}$. We must show that for all n-ary $r \in R$ and all terms $t_1, \ldots, t_n \in Te_{\mathcal{L}_{|A|},0}$,

$$(8.2) \qquad \langle t_1, \ldots, t_n \rangle \in r^{\mathsf{V},0} \quad \text{iff} \quad \langle ht_1, \ldots, ht_n \rangle \in r^{\mathfrak{A}}.$$

Notice right off that $\langle ht_1, \ldots, ht_n \rangle \in r^{\mathfrak{A}}$ implies $rt_1 \ldots t_n \in \Sigma$, and consequently the backward direction of (8.2) is obvious. So assume the left-hand side condition. Since K is elementary by hypothesis, we have that $\mathsf{V} = Mod(\Sigma \cup Th\,\mathsf{K})$, and consequently

$$\Sigma \cup Th\,\mathsf{K} \models rt_1 \ldots t_n.$$

Thus, by the Compactness Theorem, there exists a finite subset $\{\sigma_1, \ldots, \sigma_k\}$ of Σ such that

$$Th\,\mathsf{K} \models \sigma_1 \wedge \ldots \wedge \sigma_k \rightarrow rt_1 \ldots t_n.$$

We now use a well known property of first-order logic and obtain that the formula that results by simultaneously substituting the distinct constants c_a of $\sigma_1 \wedge \ldots \wedge \sigma_k \rightarrow rt_1 \ldots t_n$ by distinct variables is included in $Imp\,\mathsf{K}$. Therefore, as $\mathfrak{A} \models Imp\,\mathsf{K}$, we conclude $\langle ht_1, \ldots, ht_n\rangle \in r^{\mathfrak{A}}$.

This finishes the proof of the claim, so it suffices to apply once more that K is elementary; using 4.2.1, we get $\mathfrak{A} \in R(\mathcal{H}_0\mathsf{V}{\restriction}\mathcal{L}) \subseteq R(\mathsf{K}) = \mathsf{K}$. ■

Remark. The difference between the above proof and the one for the case $\mathcal{L}$ has equality rests on the fact that the mapping defined from $\mathcal{H}_0\mathsf{V} \restriction \mathcal{L}$ into $\mathfrak{A}$ is directly one-one and thus an isomorphism in this case (see the remark following the definition of Herbrand structure). ⊣

Observe that (ii) can be replaced in the previous theorem by the following weaker condition: "every relative ξ-subvariety of K has a free structure over 0 generators, for all cardinals $\xi > 0$". This follows from the proof of (ii) implies (i). On the contrary, (ii) cannot be replaced by the statement "every relative subvariety of K has a free structure with α generators, for all cardinals α". The next one is an easy counterexample that shows the equivalence would not hold any longer in this case. Consider the language $\mathcal{L}$ with exactly one function symbol 0 and one relation symbol $\leq$, of arities 0 and 2 respectively, together with the equality symbol $\approx$. Let K be the class of $\mathcal{L}$-structures axiomatized by the set of formulas

$$\begin{aligned}
&0 \leq x\\
&x \leq x\\
&x \leq y \wedge y \leq x \rightarrow x \approx y\\
&x \leq y \wedge y \leq z \rightarrow x \leq z\\
&x \leq y \vee y \leq x
\end{aligned}$$

(the members of K are just totally ordered sets with a minimum element). The only relative subvarieties of K are K itself and the class that in addition satisfies the axiom $x \leq y$ and whose members are structures with all relations holding. Thus the weaker condition above is clearly satisfied in this case; the free structures are respectively $\langle \mathbf{Te}_{\mathcal{L},\alpha}, \Delta\rangle$ and $\langle \mathbf{Te}_{\mathcal{L},\alpha}, \nabla\rangle$. But obviously K is not a quasivariety.

Let us turn now our attention into the existence of free structures in reduced classes. To start with, we introduce two distinct notions of freeness. Let K be a class of $\mathcal{L}$-structures, as usual. Then we say that a reduced $\mathcal{L}$-structure $\mathfrak{A}$ is *(weakly) freely generated over* K^* by a subset X of A if, for all $\mathfrak{B} \in \mathsf{K}^*$ and all $g : X \rightarrow B$ (such that gX generates $\mathbf{B}$), there exists a unique (surjective) homomorphism $h : \mathfrak{A} \rightarrow \mathfrak{B}$ such that $h \restriction X = g$. $\mathfrak{A}$ is *(weakly) free in* K^* with α generators if $\mathfrak{A} \in \mathsf{K}^*$ and $\mathfrak{A}$ is (weakly) freely generated over K^* by some set of generators of cardinality α. Once more, any two structures (weakly) freely generated over K^* by some sets of generators of the same cardinality are isomorphic.

Notice that, in general, an arbitrary nontrivial reduced quasivariety need not have free structures, nor even weakly free structures, over α generators, for all cardinals α. The point is that, if $\mathfrak{F}$ is free in a class K with α generators, let us say $X = \{x_\lambda : \lambda < \alpha\}$, then the set $X^* = \{x^*_\lambda : \lambda < \alpha\}$ in the quotient $\mathfrak{F}^*$ may not be of cardinal α, or even $\mathfrak{F}^*$ may not be freely generated. A simple example of the first situation is provided by the whole class of structures over the language with no function symbols and just one relation symbol, of arity 1. Indeed, the nontrivial reduced members of such class are of the form $\langle\{a, b\}, \{a\}\rangle$, for distinct elements a, b (if $\mathfrak{A} = \langle A, D\rangle$ is a member of this class, then $\Omega\mathfrak{A}$ is the equivalence relation whose equivalence classes are D and $A \setminus D$!). Even when we restrict our attention to protoalgebraic quasivarieties, the admission of free structures with an arbitrary number of generators is not inherited, strictly speaking, in passing to reduced semantics. Nevertheless, protoalgebraicity is enough to ensure that the reduced class of a quasivariety admit at least free structures in the weak sense for almost all cardinals α. Concretely, let us denote by $\mathcal{H}^*_\alpha$K the Leibniz quotient of $\mathcal{H}_\alpha$K, for any class K, and call V a *reduced relative ξ-subvariety of* K* if V is obtained from a relative ξ-subvariety of K by applying the reduction operator. Then we have the next theorem.

THEOREM 8.1.7. *Let* K *be a quasivariety of* $\mathcal{L}$*-structures. The following statements are equivalent.*

(i) K *is protoalgebraic.*

(ii) Every nontrivial reduced relative ξ-subvariety of K* *has a weakly free structure with α generators, for all cardinals ξ, α such that $\xi + \alpha > 0$ and, either α is infinite or $\alpha \leq 1$. Concretely,* $\mathcal{H}^*_\alpha$V *is the weakly free structure in* V* *with α generators, for each relative ξ-subvariety* V *of* K *such that* V* *is nontrivial.*

Proof. Consider the implication from (i) to (ii). Let ξ, α be two arbitrary cardinals satisfying the conditions required in (ii), and let V be a relative ξ-subvariety of K determined by some set Σ of atomic $\mathcal{L}_\xi$-formulas such that V* is nontrivial (recall that $\mathfrak{A} \in$ V* iff $\mathfrak{A} \in Mod\,\Sigma$ and $\mathfrak{A}\restriction\mathcal{L} \in$ K*). If K is protoalgebraic so is V; to get this conclusion, we just need to apply 2.1.4 and obtain that

$$\Omega\mathfrak{A} = \Omega(\mathfrak{A}\restriction\mathcal{L}) \subseteq \Omega(\mathfrak{B}\restriction\mathcal{L}) = \Omega\mathfrak{B}$$

for all $\mathfrak{A}, \mathfrak{B} \in$ V such that $\mathfrak{A} \preccurlyeq \mathfrak{B}$. On the other hand, by Proposition 8.1.4, $\mathcal{H}_\alpha$V is contained in V and hence its Leibniz quotient belongs to V*. Let us show that this reduced Herbrand structure is weakly freely generated over V* by the set $X^*_\alpha = \{x^*_\lambda : \lambda < \alpha\}$. For this, consider a structure $\mathfrak{B} \in$ V* and a map $g^* : X^*_\alpha \to B$ such that $g^*X^*_\alpha$ generates B. Define $g : X_\alpha \to B$ by setting $gx_\lambda = g^*x^*_\lambda$ for each $\lambda < \alpha$. The proof of Theorem 8.1.6 says in this case that g extends to a surjective homomorphism h from $\mathcal{H}_\alpha$V onto $\mathfrak{B}$. So, by 5.3.9, we have $h^* : \mathcal{H}^*_\alpha\text{V} \twoheadrightarrow \mathfrak{B}^* \cong \mathfrak{B}$, as required.

It remains to see that X^*_α is still a set of cardinal α. If $\alpha \leq 1$ the statement is trivial. So let us consider the case $\alpha \geq \omega$. Take two distinct elements $\lambda, \mu < \alpha$ and suppose $x^*_\lambda = x^*_\mu$; we shall derive a contradiction. To simplify the notation, put $u := x_\lambda$ and $v := x_\mu$, so that our assumption is $u \equiv v$ $(\Omega\mathcal{H}_\alpha\text{V})$. We claim that

$u \equiv v$ $(\Omega\mathcal{H}_\beta\mathsf{V})$ for all $\beta > \alpha$. Indeed, consider an arbitrary, fixed atomic $\mathcal{L}$-formula $\varphi := \varphi(x, z_1, \ldots, z_k)$. Using 2.1.2, it suffices to show that the following equivalence holds for all $t_1, \ldots, t_k \in Te_{\mathcal{L}_\ell,\beta}$:

$$\mathcal{H}_\beta\mathsf{K} \vDash \varphi(x, z_1, \ldots, z_k)\,[u, t_1, \ldots, t_k] \text{ iff } \tag{8.3}$$
$$\mathcal{H}_\beta\mathsf{K} \vDash \varphi(x, z_1, \ldots, z_k)\,[v, t_1, \ldots, t_k].$$

To this end, we reason as follows. Since $Te_{\mathcal{L}_\ell,\beta}$ is obtained by adding to the generating set of $Te_{\mathcal{L}_\ell,\alpha}$ as many variables as necessary, we can assume that the free variables of the terms t_i are among $x_{\lambda_1}, \ldots, x_{\lambda_p}, x_{\lambda_{p+1}}, \ldots, x_{\lambda_{p+q}}$, where $x_{\lambda_i} \in X_\alpha$ whenever $1 \leq i \leq p$, and $x_{\lambda_{p+j}} \in X_\beta \setminus X_\alpha$ whenever $1 \leq j \leq q$. Choose q new variables $x_{\mu_1}, \ldots, x_{\mu_q}$ of X_α distinct from $u, v, x_{\lambda_1}, \ldots, x_{\lambda_p}$; we can do that for $\alpha \geq \omega$ by hypothesis. Then the new terms

$$t_i' := t_i(x_{\mu_1}/x_{\lambda_{p+1}}, \ldots, x_{\mu_q}/x_{\lambda_{p+q}}), \quad 1 \leq i \leq k,$$

are elements of $Te_{\mathcal{L}_\ell,\alpha}$. So the definition of Herbrand structure and the assumption that $u \equiv v$ $(\Omega\mathcal{H}_\alpha\mathsf{V})$ entail the following chain of equivalences, which includes the required condition (8.3):

$$\begin{aligned}
\mathcal{H}_\beta\mathsf{V} \vDash \varphi(x, z_1, \ldots, z_k)\,[u, t_1, \ldots, t_k] &\text{ iff } \mathcal{H}_\beta\mathsf{V} \vDash \varphi(x, z_1, \ldots, z_k)\,[u, t'_1, \ldots, t'_k]\\
&\text{ iff } \mathcal{H}_\alpha\mathsf{V} \vDash \varphi(x, z_1, \ldots, z_k)\,[u, t'_1, \ldots, t'_k]\\
&\text{ iff } \mathcal{H}_\alpha\mathsf{V} \vDash \varphi(x, z_1, \ldots, z_k)\,[v, t'_1, \ldots, t'_k]\\
&\text{ iff } \mathcal{H}_\beta\mathsf{V} \vDash \varphi(x, z_1, \ldots, z_k)\,[v, t'_1, \ldots, t'_k]\\
&\text{ iff } \mathcal{H}_\beta\mathsf{V} \vDash \varphi(x, z_1, \ldots, z_k)\,[v, t_1, \ldots, t_k].
\end{aligned}$$

Once the claim is proved, let $\mathfrak{A} \in \mathsf{V}$ and let a, b be two arbitrary elements of A. We are going to see that $a \equiv b$ $(\Omega\mathfrak{A})$. For this, let us distinguish two cases. If $|A| \leq \alpha$, there exists a surjective homomorphism from $\mathcal{H}_\alpha\mathsf{V}$ onto $\mathfrak{A}$ such that $u \longmapsto a$ and $v \longmapsto b$, for we already know $\mathcal{H}_\alpha\mathsf{V}$ is freely generated over V by X_α. Let us call h this homomorphism. Then, since V is protoalgebraic, we can apply Theorem 5.3.9 and obtain that h^* is a well defined homomorphism from $\mathcal{H}_\alpha^*\mathsf{V}$ onto $\mathfrak{A}^*$. So, as $u \equiv v$ $(\Omega\mathcal{H}_\alpha\mathsf{V})$ by hypothesis, we conclude that $a \equiv b$ $(\Omega\mathfrak{A})$. For the second case, $|A| > \alpha$, we reason in very much the same manner; this time we use the claim proved previously and the fact that there exists a surjective homomorphism from $\mathcal{H}_{|A|}\mathsf{V}$ onto $\mathfrak{A}$ sending $u \longmapsto a$ and $v \longmapsto b$. Therefore, the condition $a \equiv b$ $(\Omega\mathfrak{A})$ always holds. In conclusion, we have proved the following:

$$u \equiv v\ (\Omega\mathcal{H}_\alpha\mathsf{V}) \text{ implies } \Omega\mathfrak{A} = \nabla_A \text{ for all } \mathfrak{A} \in \mathsf{V}.$$

But this contradicts the assumption that V^* is a nontrivial reduced class. Thus, $u \not\equiv v$ $(\Omega\mathcal{H}_\alpha\mathsf{V})$, and hence we have $|X_\alpha^*| = \alpha$, as desired.

To see that (ii) implies (i), let $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ be such that $\mathfrak{A} \preccurlyeq \mathfrak{B}$. If $\Omega\mathfrak{B} = \nabla_A$, then the inclusion $\Omega\mathfrak{A} \subseteq \Omega\mathfrak{B}$ holds trivially. So suppose that $\Omega\mathfrak{B} \neq \nabla_A$. Consider the relative $|A|$-subvariety V of K determined by the positive diagram of $\mathfrak{A}$. Since $\mathfrak{B}$ is a filter extension of $\mathfrak{A}$, we have $(\mathfrak{B}, a)_{a \in A} \in \mathsf{V}$ and consequently V^* is nontrivial.

Thus, using (ii), $\mathcal{H}_0^*\mathsf{V}$ is weakly free in V^*. Consider the only homomorphism that applies $\mathcal{H}_0^*\mathsf{V}$ onto $(\mathfrak{B}^*, a/\Omega\mathfrak{B})_{a\in A}$; it must send c_a^* to $a/\Omega\mathfrak{B}$, so that $c_a \equiv c_b\ (\Omega\mathcal{H}_0\mathsf{V})$ implies $a \equiv b\ (\Omega\mathfrak{B})$. On the other hand, the unique homomorphism from $\mathcal{H}_0\mathsf{V}$ into the expanded structure $(\mathfrak{A}, a)_{a\in A}$ is clearly a reductive one and such that $c_a \longmapsto a$. Hence, if h denotes this homomorphism, Theorem 2.1.8 and Proposition 2.1.4 say $\Omega\mathcal{H}_0\mathsf{V} = h^{-1}\Omega\mathfrak{A}$. In conclusion, we have the following implications:

$$a \equiv b\ (\Omega\mathfrak{A}) \text{ implies } c_a \equiv c_b\ (\Omega\mathcal{H}_0\mathsf{V}) \text{ implies } a \equiv b\ (\Omega\mathfrak{B}).$$

So, $\Omega\mathfrak{A} \subseteq \Omega\mathfrak{B}$ and K is protoalgebraic. ■

Once more, observe that the condition that every nontrivial reduced relative ξ-subvariety of K^* has a weakly free structure with 0 generators, for all $\xi > 0$, suffices to guarantee the protoalgebraicity of K.

For languages with just one relation symbol, it has been proved in [12, Thm. 10.1] that reduced protoalgebraic quasivarieties do admit weakly free structures with an arbitrary number of generators; the proof relies on the effective use of a syntactic characterization of protoalgebraicity that is not known to have a simple extension for more general languages. The necessary and sufficient requirement for any quasivariety to keep admitting free structures (this time with any number of generators!) is semialgebraicity. Concretely, we have the next result.

THEOREM 8.1.8. *For any quasivariety* K *of* $\mathcal{L}$*-structures, the following statements are equivalent.*

(i) K *is semialgebraic.*

(ii) Every nontrivial reduced relative ξ*-subvariety of* K^* *has a free structure with* α *generators, for all cardinals* ξ, α *such that* $\xi + \alpha > 0$. *Concretely,* $\mathcal{H}_\alpha^*\mathsf{V}$ *is the free structure in* V^* *with* α *generators, for each relative* ξ*-subvariety* V *of* K *such that* V^* *is nontrivial.*

Proof. The argument that proves the first part of (i) implies (ii) is the same as in 8.1.7, except this time we use Theorem 5.4.5 instead of 5.3.9. Moreover, to see that $|X_\alpha^*| = \alpha$ for any cardinal α, infinite or not, we reason as follows. If $\beta > \alpha$ then $\mathcal{H}_\alpha\mathsf{V} \subseteq \mathcal{H}_\beta\mathsf{V}$ and hence, by semialgebraicity, $\Omega\mathcal{H}_\alpha\mathsf{V} \subseteq \Omega\mathcal{H}_\beta\mathsf{V}$. So this time the implication

$$u \equiv v\ (\Omega\mathcal{H}_\alpha\mathsf{V}) \text{ implies } u \equiv v\ (\Omega\mathcal{H}_\beta\mathsf{V}), \quad \text{for all } \beta > \alpha,$$

holds no matter the value of α is. Therefore, we can repeat the argument of 8.1.7 to obtain again that $u \equiv v\ (\Omega\mathcal{H}_\alpha\mathsf{V})$ implies V^* is trivial.

For the reverse implication, it suffices to see that the Leibniz operator restricted to K is $\subseteq$-monotone, by virtue of 8.1.7. So take $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$. As before, we obtain that the Leibniz quotient of the Herbrand structure of V, where V is the relative $|A|$-subvariety of K determined by the positive diagram of $\mathfrak{A}$, applies into $(\mathfrak{B}^*, a/\Omega\mathfrak{B})_{a\in A}$, maybe not surjectively. Therefore, followwing the steps of the proof of 8.1.7, we also conclude that $\Omega\mathfrak{A} \subseteq \Omega\mathfrak{B}$. ■

Remark. We can also prove the last theorem using 7.2.3 and Fujiwara's Theorem [53, Thm. 2]. However, anything similar happens in the case of 8.1.7, for we already know that protoalgebraic reduced quasivarieties are not in general closed under substructures. ⊣

The preceding result can be applied, for instance, to the class K_{qo} of quasi-ordered algebras of a given similarity type. In this case, Theorem 8.1.8 specializes to a well known result of Bloom [14, Thm. 2.2].

8.2. Further Properties of Herbrand Structures

So far we know that, given any class K of $\mathcal{L}$-structures, its Herbrand structures belong to the quasivariety generated by K, which can be expressed as $ERSPP_u(\mathsf{K})$; this is just a consequence of Proposition 8.1.4. The next result sharpens this by showing that, in fact, all the Herbrand structures of K are in the smaller class $EP_{sd}S(\mathsf{K})$; furthermore, we shall see that their filter extensions relative to K^Q are included in $EP_{sd}SP_u(\mathsf{K})$. Of course, the proof now runs quite differently. To start with we need a lemma.

LEMMA 8.2.1. *Let* K *be any class of* $\mathcal{L}$*-structures closed under ultraproducts. Let* $\Phi \cup \{\psi\} \subseteq Atm\,\mathcal{L}$ *be such that, for all* $\mathfrak{A} \in \mathsf{K}$ *and all assignment* $g : \mathbf{Te}_{\mathcal{L}} \to \mathbf{A}$, *if* $\mathfrak{A} \models \Phi\ [g]$ *then* $\mathfrak{A} \models \psi\ [g]$. *Then there exists a finite subset* $\{\varphi_1, \dots, \varphi_m\}$ *of* Φ *such that* $\varphi_1 \wedge \cdots \wedge \varphi_m \to \varphi \in Imp\,\mathsf{K}$.

Proof. Suppose $\varphi_1 \wedge \cdots \wedge \varphi_m \to \psi \notin Imp\,\mathsf{K}$ for all $\{\varphi_1, \dots, \varphi_m\} \subseteq \Phi$ and let us get a contradiction. For each $\Psi \in Sb_\omega(\Phi)$, there exist an $\mathfrak{A}_\Psi \in \mathsf{K}$ and an algebra homomorphism $g_\Psi : \mathbf{Te}_{\mathcal{L}} \to \mathbf{A}_\Psi$ such that

$$\mathfrak{A}_\Psi \models \bigwedge \Psi \wedge \neg\psi\ [g_\Psi].$$

Consider an ultrafilter $\mathcal{U}$ of $Sb_\omega(\Phi)$ that includes all the sets $U_\Psi = \{\Xi \in Sb_\omega(\Phi) : \Psi \subseteq \Xi\}$, for $\Psi \in Sb_\omega(\Phi)$, and let

$$\mathfrak{A} := \prod_{\Psi \in Sb_\omega(\Phi)} \mathfrak{A}_\Psi,$$
$$g : \mathbf{Te}_{\mathcal{L}} \to \mathbf{A},\quad gx_i = \langle g_\Psi x_i : \Psi \in Sb_\omega(\Phi)\rangle,\quad \text{for } i < \omega.$$

Since $U_{\{\varphi\}} \subseteq \{\Psi \in Sb_\omega(\Phi) : \mathfrak{A}_\Psi \models \varphi \wedge \neg\psi\ [g_\Psi]\}$ and $U_{\{\varphi\}} \in \mathcal{U}$, the set $\{\Psi \in Sb_\omega(\Phi) : \mathfrak{A}_\Psi \models \varphi \wedge \neg\psi\ [g_\Psi]\}$ belongs to $\mathcal{U}$, for all $\varphi \in \Phi$. Thus, Loš Theorem says

$$\mathfrak{A}/\mathcal{U} \models \varphi \wedge \neg\psi\ [\pi_{\mathcal{U}} \circ g],$$

for each $\varphi \in \Phi$, i.e., $\mathfrak{A}/\mathcal{U} \models \Phi\ [\pi_{\mathcal{U}} \circ g]$ whereas $\mathfrak{A}/\mathcal{U} \not\models \psi\ [\pi_{\mathcal{U}} \circ g]$. But this contradicts the assumption K is closed under ultraproducts. ∎

The following result can be viewed as a characterization, in the style of Jónsson's Theorem on finitely subdirectly irreducible algebras, of the term-structures of the quasivariety generated by a given class.

THEOREM 8.2.2. *The following holds for any quasivariety* K *of* $\mathcal{L}$*-structures.*

(i) $\mathcal{H}_\alpha \mathsf{K} \in EP_s{}_dS(\mathsf{K})$, *for every cardinal* α.
(ii) *For every cardinal* α, *if* $\mathfrak{F}$ *is a filter extension of* $\mathcal{H}_\alpha \mathsf{K}$ *relative to* K^Q, *then* $\mathfrak{F} \in EP_s{}_dSP_u(\mathsf{K})$.

Proof. (i) Fix a cardinal α and define

$$\Delta_\alpha := \{rt_1 \dots t_n \in Atm_\alpha \mathcal{L} : \langle t_1, \dots, t_n\rangle \notin r^{\mathsf{K},\alpha}\}.$$

By definition, $\varphi \in \Delta_\alpha$ implies there exist $\mathfrak{A}_\varphi \in \mathsf{K}$ and $h_\varphi : \mathbf{Te}_{\mathcal{L},\alpha} \to \mathbf{A}_\varphi$ such that $\mathfrak{A}_\varphi \not\Vdash \varphi\, [h_\varphi]$. Let

$$\mathfrak{I}_\varphi := \langle \mathbf{Te}_{\mathcal{L},\alpha}, h_\varphi^{-1} R_{\mathfrak{A}_\varphi}\rangle.$$

Clearly $h_\varphi : \mathfrak{I}_\varphi \to_s \mathfrak{A}_\varphi$. So, by 1.2.2 and the Homomorphism Theorem, we have that $\mathfrak{I}_\varphi \in ES(\mathsf{K})$. Also $Ker\, h_\varphi \in Co\,\mathfrak{I}_\varphi$, by virtue of 2.1.5. We are going to show that a quotient of $\mathcal{H}_\alpha \mathsf{K}$ is isomorphic to a subdirect product of $\{\mathfrak{I}_\varphi : \varphi \in \Delta\}$; if this is true, then we just need to apply 4.1.2(i) to obtain that $\mathcal{H}_\alpha \mathsf{K} \in EP_s{}_dES(\mathsf{K}) \subseteq EP_s{}_dS(\mathsf{K})$, and thus part (i) of the theorem will be proved.

Take an n-ary $r \in R$ and let $t_1, \dots, t_n \in Te_{\mathcal{L},\alpha}$. Since $\mathfrak{A}_\varphi$ is a member of K for all $\varphi \in \Delta_\alpha$, the condition $\langle t_1, \dots, t_n\rangle \in r^{\mathsf{K},\alpha}$ implies that $\mathfrak{A}_\varphi \vDash rt_1 \dots t_n\, [h_\varphi]$ for all $\varphi \in \Delta_\alpha$; so, $\langle t_1, \dots, t_n\rangle \in r^{\mathfrak{I}_\varphi}$ for all $\varphi \in \Delta_\alpha$. Conversely, if $\langle t_1, \dots, t_n\rangle \notin r^{\mathsf{K},\alpha}$ then $\varphi := rt_1, \dots, t_n \in \Delta_\alpha$, so that $\langle h_\varphi t_1, \dots, h_\varphi t_n\rangle \notin r^{\mathfrak{A}_\varphi}$; thus $\langle t_1, \dots, t_n\rangle \notin r^{\mathfrak{I}_\varphi}$. In conclusion, we have that $\mathcal{H}_\alpha \mathsf{K} = \bigcap_{\varphi \in \Delta} \mathfrak{I}_\varphi$. We now apply Proposition 6.1.1. Define

$$\theta_\alpha(\mathsf{K}) := \bigcap_{\varphi \in \Delta_\alpha} Ker\, h_\varphi.$$

Then 6.1.1 says that the quotient $\mathcal{H}_\alpha \mathsf{K}/\theta_\alpha(\mathsf{K})$ is isomorphic to a subdirect product of $\{\mathfrak{I}_\varphi : \varphi \in \Delta\}$, as required.

(ii) The idea for the proof is quite the same as in part (i). Now, given a filter extension $\mathfrak{F}$ of $\mathcal{H}_\alpha \mathsf{K}$ relative to K^Q, we must consider the index set

$$\Delta_{\alpha,\mathfrak{F}} := \{rt_1 \dots t_n \in Atm_\alpha \mathcal{L} : \langle t_1, \dots, t_n\rangle \notin r^{\mathfrak{F}}\}.$$

If $\Delta_{\alpha,\mathfrak{F}} = \emptyset$ then $\mathfrak{F}$ is the structure on $\mathbf{Te}_{\mathcal{L},\alpha}$ with all relations holding, so that $\Omega\mathfrak{F} = \nabla_{Te_{\mathcal{L},\alpha}}$ and hence $\mathfrak{F}$ is an expansion of the empty direct product. If $\Delta_{\alpha,\mathfrak{F}} \neq \emptyset$, we claim that, for all $\varphi \in \Delta_{\alpha,\mathfrak{F}}$, there is an $\mathfrak{A}_\varphi \in P_u(\mathsf{K})$ and a homomorphism $h_\varphi : \mathbf{Te}_{\mathcal{L},\alpha} \to \mathbf{A}_\varphi$ such that

$$\mathfrak{F} \preccurlyeq h_\varphi^{-1}\mathfrak{A}_\varphi \text{ and } \mathfrak{A}_\varphi \nvDash \varphi\, [h_\varphi].$$

The proof of the claim is based on Lemma 8.2.1. Take $\varphi \in \Delta_{\alpha,\mathfrak{F}}$. Certainly, if Δ_α denotes the same set as in (i), then $\Delta_{\alpha,\mathfrak{F}} \subseteq \Delta_\alpha$. Thus there exist $\mathfrak{A} \in \mathsf{K}$ and $h : \mathbf{Te}_{\mathcal{L},\alpha} \to \mathbf{A}$ satisfying $\mathfrak{A} \nvDash \varphi\, [h]$. If, in addition, $\mathfrak{A}$ and h are such that $\mathfrak{F} \preccurlyeq h^{-1}\mathfrak{A}$ the claim is proved. So, suppose otherwise. We shall have that, for all $\mathfrak{A} \in \mathsf{K}$ and all $h : \mathbf{Te}_{\mathcal{L},\alpha} \to \mathbf{A}$, if $\mathfrak{F} \preccurlyeq h^{-1}\mathfrak{A}$ then $\mathfrak{A} \vDash \varphi\, [h]$. Let us see we can find a structure in $P_u(\mathsf{K})$ satisfying the condition asserted in the claim. For this, we proceed again by "reductio ad absurdum". Assume $\mathfrak{F} \preccurlyeq h^{-1}\mathfrak{A}$ implies $\mathfrak{A} \vDash \varphi\, [h]$, not only for all $\mathfrak{A} \in \mathsf{K}$ but also for any other element of $P_u(\mathsf{K})$ and any homomorphism h. Observe that, given any $\mathcal{L}$-structure $\mathfrak{A}$, if $\mathfrak{A} \models \psi\,[h]$ for all $\psi \notin \Delta_{\alpha,\mathfrak{F}}$ then $\mathfrak{F} \preccurlyeq h^{-1}\mathfrak{A}$. So, our assumption says that, for every $\mathfrak{A} \in P_u(\mathsf{K})$ and every $h : \mathbf{Te}_{\mathcal{L},\alpha} \to \mathbf{A}$,

$$\mathfrak{A} \models \psi\,[h] \text{ for all } \psi \notin \Delta_{\alpha,\mathfrak{F}} \text{ implies } \mathfrak{A} \models \varphi\,[h].$$

We can now use the lemma, since 4.1.1 ensures $P_u(\mathsf{K})$ is closed under unltraproducts. We conclude that there exists a finite set $\{\psi_1, \dots, \psi_m\}$ of atomic $\mathcal{L}$-formulas over α variables, not in $\Delta_{\alpha,\mathfrak{F}}$, such that

$$\psi_1 \wedge \cdots \wedge \psi_m \to \varphi \in Imp\, P_u(\mathsf{K}) = Imp\, \mathsf{K}.$$

But $\mathfrak{F} \models Imp\,\mathsf{K}$, since by assumption $\mathfrak{F}$ is included in K^Q. Therefore, $\varphi \notin \Delta_{\alpha,\mathfrak{F}}$, and this contradicts that φ had been choosen from $\Delta_{\alpha,\mathfrak{F}}$. So the claim is proved.

For the rest of the proof we just repeat the argument of part (i) and obtain the desired condition, i.e., $\mathfrak{F} \in EP_{sd}SP_u(\mathsf{K})$. ■

Remark. The proof of 8.2.2(i) can be technically simplified by considering, instead of the set $\{h_\varphi : \varphi \in \Delta_\alpha\}$, the whole class of homomorphisms with domain $\mathbf{Te}_{\mathcal{L},\alpha}$ and range the algebra reduct of some member of K; concretely, we have that

$$\mathcal{H}_\alpha\mathsf{K} = \bigcap\{h^{-1}\mathfrak{A} : \mathfrak{A} \in \mathsf{K} \text{ and } h : \mathbf{Te}_{\mathcal{L},\alpha} \to \mathbf{A}\},$$
$$\theta_\alpha(\mathsf{K}) = \bigcap\{Ker\, h : h : \mathbf{Te}_{\mathcal{L},\alpha} \to \mathbf{A} \text{ for some } \mathfrak{A} \in \mathsf{K}\}.$$

The same is true for 8.2.2(ii); in this case, for example, we have

$$\mathfrak{F} = \bigcap\{h^{-1}\mathfrak{A} : \mathfrak{A} \in P_u(\mathsf{K}),\ h : \mathbf{Te}_{\mathcal{L},\alpha} \to \mathbf{A} \text{ and } \mathfrak{F} \preccurlyeq h^{-1}\mathfrak{A}\}.$$

Nevertheless, we believe the given proofs eventually provide some additional insight into the concepts we deal with, and for this reason we have chosen them.

On the other hand, it is clear that in our proof we are actually constructing, by a method different from the one discovered by Birkhoff [4], a structure which is in general distinct from $\mathcal{H}_\alpha\mathsf{K}$ but which is also freely generated over K by α generators; it is the structure $\mathcal{H}_\alpha\mathsf{K}/\theta_\alpha(\mathsf{K})$. This does not contradict the uniqueness of free structures, because none of $\mathcal{H}_\alpha\mathsf{K}$ and $\mathcal{H}_\alpha\mathsf{K}/\theta_\alpha(\mathsf{K})$ must necessarily belong to K. In fact, if $\mathcal{H}_\alpha\mathsf{K} \in \mathsf{K}$ then both structures coincide, since the identity function is included in the class of homomorphism from $\mathbf{Te}_{\mathcal{L},\alpha}$ into the algebra reduct of a member of K (i.e., $\mathfrak{A}_\varphi$ can be chosen to be $\mathcal{H}_\alpha\mathsf{K}$ for all $\varphi \in \Delta_\alpha$). ⊣

It is worth noting that the preceding proof can be slightly modified to obtain that the Herbrand structures with a sufficiently large number of generators are in fact in the smaller class $EP_{sd}(\mathsf{K})$. Rather, we have the following.

THEOREM 8.2.3. *Let* K *be an arbitrary class of* $\mathcal{L}$*-structures. There exists a cardinal* α_0 *such that* $\mathcal{H}_\alpha\mathsf{K} \in EP_{sd}(\mathsf{K})$*, for all* $\alpha \geq \alpha_0$.

Proof. Once more, define the set

$$\Delta_\omega := \{rt_1 \dots t_n \in Atm\,\mathcal{L} : \langle t_1, \dots, t_n\rangle \notin r^{\mathsf{K},\omega}\}.$$

For each $\varphi \in \Delta_\omega$ there are $\mathfrak{A}_\varphi \in \mathsf{K}$ and $h_\varphi : \mathbf{Te}_{\mathcal{L}} \to \mathbf{A}_\varphi$ such that $\mathfrak{A}_\varphi \nvDash \varphi\,[h_\varphi]$. Set

$$\alpha_0 := \omega + \sup\{|A_\varphi| : \varphi \in \Delta_\omega\},$$

and take $\alpha \geq \alpha_0$. We are going to show that $\mathcal{H}_\alpha \mathsf{K} \in EP_{sd}(\mathsf{K})$. Let Δ_α be defined as in the proof of 8.2.2(i). For each $\psi \in \Delta_\alpha$, $\psi := \psi(x_{\lambda_1}, \ldots, x_{\lambda_k})$, let $\tau_\psi : \mathbf{Te}_{\mathcal{L},\alpha} \to \mathbf{Te}_{\mathcal{L}}$ be such that $\tau_\psi x_{\lambda_i} = x_i$, where $1 \leq i \leq k$. By Proposition 8.1.1, there are $\mathfrak{A}_\psi \in \mathsf{K}$ and $g_\psi : \mathbf{Te}_{\mathcal{L}} \to \mathbf{A}_\psi$ satisfying

$$\mathfrak{A}_\psi \nvDash \tau_\psi \psi\,[g_\psi].$$

We may assume that the structure $\mathfrak{A}_\psi$ is one from the set $\{\mathfrak{A}_\varphi : \varphi \in \Delta_\omega\}$, because in fact $\tau_\psi \psi \in \Delta_\omega$. So, consider a surjective algebra homomorphism h_ψ from $\mathbf{Te}_{\mathcal{L},\alpha}$ onto $\mathbf{A}_\psi$ such that $h_\psi x_{\lambda_i} = g_\psi x_i$, whenever $1 \leq i \leq k$. For such a homomorphism, we have $\mathfrak{A}_\psi \nvDash \psi\,[h_\psi]$, and hence we can follow the preceding argument to obtain $\mathcal{H}_\alpha \mathsf{K} \in EP_{sd}(\mathsf{K})$ (now using that the inverse image of $\mathfrak{A}_\psi$ under h_ψ belongs directly to $E(\mathsf{K})$, for all ψ in Δ_α). ∎

We can infer some interesting consequences from the above theorem. The first one concerns the Leibniz quotient of Herbrand structures and part of it can be found in [12], proved under the assumption that the class K generates a protoalgebraic quasivariety.

COROLLARY 8.2.4. *The following holds for any class* K *of* $\mathcal{L}$*-structures.*

(i) $\mathcal{H}^*_\alpha \mathsf{K} \in P^*_{sd} S^*(\mathsf{K}^*)$, *for every cardinal* α.

(ii) *There exists a cardinal* α_0 *such that* $\mathcal{H}^*_\alpha \mathsf{K} \in P^*_{sd}(\mathsf{K}^*)$, *for all* $\alpha \geq \alpha_0$.

Proof. We apply Theorems 8.2.2 and 8.2.3, and Lemma 4.1.5(ii). ∎

The second one is another characterization of the variety (relative subvariety) generated by a given class of structures (included in a quasivariety), different from the one given in Chapter 4. It generalizes to arbitrary first-order languages, with or without equality, a result of Kogalovskii; see [19], [69].

COROLLARY 8.2.5. *(i) For any class* K *of* $\mathcal{L}$*-structures,* $\mathsf{K}^V = HEP_{sd}(\mathsf{K})$.

(ii) *If* Q *is a quasivariety and* K *is any subclass contained in* Q, *then* $\mathsf{K}^V \cap \mathsf{Q} = H_{\mathsf{Q}} EP_{sd}(\mathsf{K})$.

Proof. (i) Let $\mathfrak{A}$ be a member of K^V. Proposition 8.1.3 (see also Lemma 8.3.2(ii) below) implies $\mathcal{H}_\alpha \mathsf{K} = \mathcal{H}_\alpha \mathsf{K}^V$; so there exists $\alpha \geq 0$ such that $\mathfrak{A} \in H(\mathcal{H}_\alpha \mathsf{K})$, by virtue of 8.1.5. Also, 8.2.3 says that for some α_0, $\mathcal{H}_\alpha \mathsf{K} \in EP_{sd}(\mathsf{K})$ whenever $\alpha \geq \alpha_0$, so that we actually have $\mathfrak{A} \in HEP_{sd}(\mathsf{K})$. Therefore the inclusion $\mathsf{K}^V \subseteq HEP_{sd}(\mathsf{K})$ is proved. The reverse inclusion follows immediately from the characterization of varieties stated in 4.5.1.

(ii) It is a trivial consequence of part (i). ∎

In addition to the above consequences, free structures have some other worth-noting applications. For instance, we can use them to get a proof of 4.5.1 closer in spirit to the original proof of Birkhoff's Theorem that characterizes varieties of algebras as classes closed under H, S and P. This new proof uses 8.2.2(i) to see the hard implication, i.e., that any class of structures closed under H, E, S and P is a variety. The idea of the proof is as follows. If $\mathfrak{A} \in Mod\, Atm\, \mathsf{K}$ then any atomic $\mathcal{L}$-formula satisfied by $\mathcal{H}_{|A|}\mathsf{K}$ is also satisfied by $\mathfrak{A}$, and thus $\mathfrak{A} \in H(\mathcal{H}_{|A|}\mathsf{K})$; hence, if K is closed under H, E, S and P, the condition $\mathfrak{A} \in \mathsf{K}$ must hold.

Also, using the characterization of term-structures stated in Theorem 8.2.2(ii), we can provide another proof that $ERSPP_u(\mathsf{K})$ is the least quasivariety containing K, and in this case we run close to an argument of Czelakowski [29, Thm. 5.1], successfully generalized in [38]. Again, we summarize the main steps for proving the difficult implication. Suppose $\mathfrak{A} \in Mod\, Imp\, \mathsf{K}$, and let $\alpha = \max\{|A|, \omega\}$. Clearly, there exists a surjective homomorphism h from $\mathcal{H}_\alpha \mathsf{K}$ onto $\mathfrak{A}$, since $Atm\, \mathsf{K} = Atm\, \mathcal{H}_\alpha \mathsf{K}$ by 8.1.3. Let $\mathfrak{F}$ be the inverse image of $\mathfrak{A}$ under this homomorphism h. Then $\mathfrak{F} \in EP_{sd}SP_u(\mathsf{K})$ and consequently $\mathfrak{A} \in REP_{sd}SP_u(\mathsf{K}) \subseteq ERSPP_u(\mathsf{K})$. So, $\mathsf{K}^Q \subseteq ERSPP_u(\mathsf{K})$. The last argument also sketches an alternative proof of a further result, namely, that $\mathsf{K}^Q = ERP_{sd}SP_u(\mathsf{K})$; cf. Corollary 4.4.6 above.

8.3. Fully Invariance and (Quasi)Varieties

A well known result in universal algebra, due to Neumann [94], says that those algebras which are free in some class are the ones that can be obtained by factorizing the absolutely free algebra by a fully invariant congruence. And from this property it is easy to conclude that the lattice of varieties of algebras of a given similarity type is isomorphic to the lattice of fully invariant congruences of the absolutely free algebra. More recently, Hoehnke [65] have also proved that an analoguous correspondence can be established between lattices of quasivarieties of algebras and certain systems of congruences on the absolutely free algebra. In this Section we propose to extend these results to structures over arbitrary first-order languages, with or without equality. For this purpose, the natural generalization of the notion of congruence on an algebra is the relational part of structures. We introduce the following definition: an $\mathcal{L}$-structure $\mathfrak{A}$ is called *fully invariant* iff every algebra homomorphism h from $\mathbf{A}$ into itself is also a homomorphism from $\mathfrak{A}$ into $\mathfrak{A}$, i.e., h satisfies that $hR_\mathfrak{A} \subseteq R_\mathfrak{A}$. Then we have the next result.

PROPOSITION 8.3.1. *For any class K of $\mathcal{L}$-structures, the structure $\mathcal{H}_\omega \mathsf{K}$ is fully invariant. Conversely, if $\mathfrak{A}$ is a fully invariant structure with underlying algebra $\mathbf{Te}_\mathcal{L}$, then $\mathfrak{A} = \mathcal{H}_\omega V(\mathfrak{A})$.*

Proof. It is a straightforward excercise to verify that $\mathcal{H}_\omega \mathsf{K}$ is always fully invariant, so the proof of the forward implication is omitted. Assume conversely that $\mathfrak{A} = \langle \mathbf{Te}_\mathcal{L}, R_\mathfrak{A}\rangle$ is fully invariant. Consider an n-ary relation symbol $r \in R$ and terms $t_1, \ldots, t_n \in Te_\mathcal{L}$ such that $\langle t_1, \ldots, t_n\rangle \in r^\mathfrak{A}$. Since $\mathfrak{A}$ is fully invariant, for all $h : \mathbf{Te}_\mathcal{L} \to \mathbf{Te}_\mathcal{L}$ we have $\langle ht_1, \ldots, ht_n\rangle \in r^\mathfrak{A}$, and hence $\mathfrak{A} \models rt_1 \ldots t_n$. Thus $V(\mathfrak{A}) \models rt_1 \ldots t_n$ and finally $\langle t_1, \ldots, t_n\rangle \in r^{V(\mathfrak{A}),\omega}$. This proves the inclusion $R_\mathfrak{A} \subseteq R_{V(\mathfrak{A}),\omega}$. The argument for the reverse inclusion is easier and it is also omitted. In conclusion, $R_\mathfrak{A} = R_{V(\mathfrak{A}),\omega}$ and consequently $\mathfrak{A} = \mathcal{H}_\omega V(\mathfrak{A})$. ■

It is an easy matter to check that the set of fully invariant structures on a given underlying algebra is closed under arbitrary intersections; thus, the preceding proposition says that the set of ω-Herbrand structures forms a complete lattice. We are going to show this lattice is actually isomorphic to the lattice of varieties of structures of type $\mathcal{L}$. To this goal, the next lemma, a consequence of 8.1.3, establishes the existence of a one-one correspondence between both lattices.

LEMMA 8.3.2. *The following properties hold for any class* K *of* $\mathcal{L}$*-structures and any cardinal* α.

(i) $V(\mathcal{H}_\alpha \mathsf{K}) = V(\mathsf{K})$.

(ii) $\mathcal{H}_\alpha \mathsf{K} = \mathcal{H}_\alpha \mathsf{K}'$ *whenever* $\mathsf{K} \subseteq \mathsf{K}' \subseteq \mathsf{K}^V$.

Proof. Certainly (i) is a direct consequence of 8.1.3. To see (ii) we must use that $Atm\, \mathsf{K}^V = Atm\, \mathsf{K}$, equality that is immediate from the definition of K^V. ∎

The desired generalization of Neumann's result is the following. A similar isomorphism was already pointed out by Mal'cev [87], who calls ***totally characteristic*** the structures that we name fully invariant.

THEOREM 8.3.3. *Let* $\mathbf{V}(\mathcal{L}) = \langle \mathcal{V}(\mathcal{L}), \subseteq \rangle$ *denote the lattice of varieties of* $\mathcal{L}$*-structures, and* $\mathbf{H}(\mathcal{L}) = \langle \mathcal{H}(\mathcal{L}), \preccurlyeq \rangle$ *the lattice of* ω*-Herbrand structures of type* $\mathcal{L}$*. Then* $\mathbf{V}(\mathcal{L})$ *and* $\mathbf{H}(\mathcal{L})$ *are dually isomorphic by the mapping* $\mathsf{V} \longmapsto \mathcal{H}_\omega \mathsf{V}$.

Proof. Indeed, the previous lemma guarantees that the mapping $\mathsf{V} \longmapsto \mathcal{H}_\omega \mathsf{V}$ is a bijection. Thus the theorem follows from the next easy equivalence: $\mathsf{V} \subseteq \mathsf{V}'$ iff $Atm\, \mathsf{V} \supseteq Atm\, \mathsf{V}'$. ∎

The problem of establishing a similar correspondence between quasivarieties of structures of type $\mathcal{L}$ and some other kind of objects related to Herbrand structures seems to be a little bit more intricate. For instance, notice that 8.1.3 does not hold any longer for quasivarieties, since $Q(\mathcal{H}_\omega \mathsf{K})$ may be strictly included in $Q(\mathsf{K})$; in other words, there can be an implicative formula which is true in $\mathcal{H}_\omega \mathsf{K}$ but not in K. However, we do have the next equivalence.

PROPOSITION 8.3.4. *Let* K *be any class of* $\mathcal{L}$*-structures and let* α, β *be two arbitrary cardinals such that* $\beta \geq \alpha > 0$*. Then, if* $\varphi \in Imp_\alpha \mathcal{L}$*, we have that* $\mathsf{K} \models \varphi$ *iff* $\mathfrak{F} \models \varphi$ *for every* K^Q*-filter extension of* $\mathcal{H}_\beta \mathsf{K}$.

Proof. Fix an arbitrary implicative $\mathcal{L}$-formula φ over α variables. Clearly, if $\mathsf{K} \models \varphi$ then $\mathsf{K}^Q \models \varphi$ and hence $\mathfrak{F} \models \varphi$ for all $\mathfrak{F} \in Fe_{\mathsf{K}^Q} \mathcal{H}_\beta \mathsf{K}$. So the forward implication holds. Assume conversely that $\mathfrak{F} \models \varphi$ for every K^Q-filter extension $\mathfrak{F}$ of $\mathcal{H}_\beta \mathsf{K}$, and let $\mathfrak{A} \in \mathsf{K}$ and $g : \mathbf{Te}_{\mathcal{L},\alpha} \to \mathbf{A}$. We must show $\mathfrak{A} \models \varphi\, [g]$. Indeed, consider a homomorphism h from $\mathbf{Te}_{\mathcal{L},\beta}$ into $\mathbf{A}$ such that $h \restriction Te_{\mathcal{L},\alpha} = g$, and define $\mathfrak{F} = h^{-1}\mathfrak{A}$. Then $\mathfrak{F}$ applies strong homomorphically into $\mathfrak{A}$, so once more we use 1.2.2 and the Homomorphism Theorem to conclude that $\mathfrak{F} \in ES(\mathfrak{A})$. But this says $\mathfrak{F} \in \mathsf{K}^Q$, since obviously $ES(\mathfrak{A}) \subseteq \mathsf{K}^Q$. So, by virtue of 8.1.2, $\mathfrak{F}$ is a K^Q-filter extension of the Herbrand structure $\mathcal{H}_\beta \mathsf{K}$. We can now apply the hypothesis and obtain $\mathfrak{F} \models \varphi$. In particular, this means that, if k is the canonical embedding from $\mathbf{Te}_{\mathcal{L},\alpha}$ into $\mathbf{Te}_{\mathcal{L},\beta}$, then we have $\mathfrak{F} \models \varphi\, [k]$ (recall that $\beta \geq \alpha$ by assumption). But using the definition of $\mathfrak{F}$, the equivalence

$$\mathfrak{F} \vDash \psi\ [k] \text{ iff } \mathfrak{A} \vDash \psi\ [g]$$

holds whenever $\psi \in Atm\,\mathcal{L}$. Hence, the condition $\mathfrak{F} \vDash \varphi\ [k]$ implies $\mathfrak{A} \vDash \varphi\ [g]$, as required. ■

The preceding result suggests the right way to obtain the desired correspondence. Indeed, it will turn out that a special sort of closure systems determined by the Herbrand structures are the algebraic counterpart of quasivarieties of structures. With this goal, let us follow Hoehnke [65] in defining a *fully invariant system* on an algebra **A** as any set S of structures with underlying algebra **A** satisfying the next two conditions: (i) S is an algebraic closure system; (ii) for all $\mathfrak{A} \in$ S and all $h : \mathbf{A} \to \mathbf{A}$, $h^{-1}\mathfrak{A} \in$ S. Then we have the following result, which looks like 8.3.1.

THEOREM 8.3.5. *For any quasivariety* K *of* $\mathcal{L}$*-structures, the set* $Fe_{\mathsf{K}}\mathcal{H}_\omega\mathsf{K}$ *is a fully invariant system. Conversely, if* S *is a fully invariant system on* $\mathbf{Te}_\mathcal{L}$*, then* $\mathsf{S} = Fe_{\mathsf{S}^Q}\mathcal{H}_\omega\mathsf{S}^Q$.

Proof. By 5.1.1, $Fe_{\mathsf{K}}\mathcal{H}_\omega\mathsf{K}$ is an algebraic closure system for any quasivariety K. Moreover, if $\mathfrak{F}$ is a K-filter extension of $\mathcal{H}_\omega\mathsf{K}$ and $h : \mathbf{Te}_\mathcal{L} \to \mathbf{Te}_\mathcal{L}$ is an algebra homomorphism, then we have

$$\mathcal{H}_\omega\mathsf{K} \preccurlyeq h^{-1}\mathfrak{F} \in ES(\mathfrak{F}),$$

and consequently $h^{-1}\mathfrak{F} \in Fe_{\mathsf{K}}\mathcal{H}_\omega\mathsf{K}$. So the first implication holds.

Suppose now S is a fully invariant system on the algebra of $\mathcal{L}$-terms. Let $\mathfrak{F}_0$ be the least element of S, i.e., $\mathfrak{F}_0 = \bigcap \mathsf{S} \in \mathsf{S}$. For each algebra homomorphism $h : \mathbf{Te}_\mathcal{L} \to \mathbf{Te}_\mathcal{L}$, we have $h^{-1}\mathfrak{F}_0 \in \mathsf{S}$ and hence $\mathfrak{F}_0 \preccurlyeq h^{-1}\mathfrak{F}_0$. Therefore $\mathfrak{F}_0$ is a fully invariant structure on $\mathbf{Te}_\mathcal{L}$. Using 8.3.1, we obtain that $\mathfrak{F}_0 = \mathcal{H}_\omega V(\mathfrak{F}_0)$ or, which is equivalent by 8.3.2(ii), $\mathfrak{F}_0 = \mathcal{H}_\omega Q(\mathfrak{F}_0)$. Also, $\mathfrak{F}_0 \vDash \varphi$ iff $\mathsf{S} \vDash \varphi$ for any atomic formula φ. So we apply once more 8.3.2 to get that $Q(\mathfrak{F}_0)$ and S^Q have the same ω-Herbrand structure. All this says the converse will be proved if we show that $\mathsf{S} = Fe_{\mathsf{S}^Q}\mathfrak{F}_0$. Indeed, the inclusion from left to right is trivial. For the reverse, let $\mathfrak{A}$ be an element of $Fe_{\mathsf{S}^Q}\mathfrak{F}_0$. We extend the notation introduced in Chapter 5 and write $Fg_{\mathsf{S}}\mathfrak{A}$ to mean the least element of S that is a filter extension of $\mathfrak{A}$, i.e.,

$$Fg_{\mathsf{S}}\mathfrak{A} := \bigcap\{\mathfrak{B} \in \mathsf{S} : \mathfrak{A} \preccurlyeq \mathfrak{B}\}.$$

Then $Fg_{\mathsf{S}}\mathfrak{A} \in \mathsf{S}$, for S is a closure system by hypothesis. So, by repeating the argument that proves the "only-if" part of 5.1.1, we obtain that $Fg_{\mathsf{S}}\mathfrak{A} = \mathfrak{A}$. This completes the proof. ■

Again, it is easy to see that the set of fully invariant systems on a given algebra is closed under arbitrary intersections. We use this fact in the next theorem, whose universal algebraic version is the main result in Hoehnke [65, Thm. 2.3].

THEOREM 8.3.6. *Let* $\mathbf{\Omega}(\mathcal{L}) = \langle \Omega(\mathcal{L}), \subseteq \rangle$ *denote the lattice of quasivarieties of* $\mathcal{L}$*-structures, and let* $\mathbf{\mathcal{FH}}(\mathcal{L}) = \langle \mathcal{FH}(\mathcal{L}), \subseteq \rangle$ *the lattice associated to the set of all lattices of the form* $Fe_{\mathsf{K}}\mathcal{H}_\omega\mathsf{K}$*, for* K *a quasivariety of type* $\mathcal{L}$*. Then* $\mathbf{\Omega}(\mathcal{L})$ *and* $\mathbf{\mathcal{FH}}(\mathcal{L})$ *are isomorphic by the map* $\mathsf{K} \longmapsto Fe_{\mathsf{K}}\mathcal{H}_\omega\mathsf{K}$.

Proof. Clearly, the mapping is a bijection, whose inverse is $Fe_{\mathsf{K}}\mathcal{H}_\omega\mathsf{K} \longmapsto \mathsf{K}$. Also, if K and K' are quasivarieties of the same type, $\mathsf{K} \subseteq \mathsf{K}'$ implies $\mathcal{H}_\omega\mathsf{K}' \preccurlyeq \mathcal{H}_\omega\mathsf{K}$ and consequently $Fe_{\mathsf{K}}\mathcal{H}_\omega\mathsf{K} \subseteq Fe_{\mathsf{K}'}\mathcal{H}_\omega\mathsf{K}'$. So, the map $\mathsf{K} \longmapsto Fe_{\mathsf{K}}\mathcal{H}_\omega\mathsf{K}$ is order-preserving. Finally, $Fe_{\mathsf{K}}\mathcal{H}_\omega\mathsf{K} \subseteq Fe_{\mathsf{K}'}\mathcal{H}_\omega\mathsf{K}'$ entails that any K-filter extension of $\mathcal{H}_\omega\mathsf{K}$ is a member of K' and thus a model of $Imp\,\mathsf{K}'$. Hence, by Proposition 8.3.4, $\mathsf{K} \models Imp\,\mathsf{K}'$, and from here it immediately follows that $\mathsf{K} \subseteq \mathsf{K}'$. Therefore, the function $Fe_{\mathsf{K}}\mathcal{H}_\omega\mathsf{K} \longmapsto \mathsf{K}$ is an order-preserving inverse of the preceding map. In conclusion, $\mathfrak{Q}(\mathcal{L})$ and $\mathfrak{FH}(\mathcal{L})$ are isomorphic lattices. ∎

We close this section noting that the varieties of $\mathfrak{Q}(\mathcal{L})$ correspond exactly to the elements $\mathbf{Fe}_{\mathsf{K}}\mathcal{H}_\omega\mathsf{K}$ of $\mathfrak{FH}(\mathcal{L})$ that are principal ideals in $\mathbf{Fe}\,\mathcal{H}_\omega\mathsf{K}$; see [65, Thm. 5.2].

9. Some Mal'cev-Type Theorems

It has already been established in earlier chapters that in passing from the study of classes of algebras to classes of structures, the very central notion of congruence on an algebra need to be replaced, depending on the purpose, by that of congruence on a structure or by that of filter extension (or even by the concept of congruence-filter pair, as it happens when we want to develop a Subdirect Representation Theory). Besides, as our real interest is in languages without equality, the usual notion of congruence on an algebra is still susceptible of another generalization, viz. the concept of relative congruence introduced in Section 5.2 by means of the Leibniz operator. Our opinion is that it may be of interest to know if some of the Mal'cev conditions proved for (relative) congruence identities also generalize in some of the forecoming senses, and if so, how the extensions look like. In view of the Generalized Jónsson's Theorem (Theorem 6.3.2), this is specially true in the case of identities concerning lattices of relative filter extensions; one really would like to have something like a Mal'cev condition for relative filter distributivity of a quasivariety of structures.

In this Chapter we provide an answer to a very few questions that are inside the scope of this general problem, and that they concern the characterization of some properties of lattices of relative congruences.

9.1. Relatively Congruence Permutable Classes

Let K be any class of $\mathcal{L}$-structures. An $\mathcal{L}$-algebra **A** is said to be *congruence permutable relative to* K if every pair of K-congruences on **A** permute, i.e., $\theta{\cdot}\phi = \phi{\cdot}\theta$ for all $\theta, \phi \in Co_{\mathbf{K}}\mathbf{A}$. The class K is called *relatively congruence permutable* (RCP for short) if every $\mathcal{L}$-algebra **A** is congruence permutable relative to K. The first result provides an interesting reformulation of the concept of relative congruence introduced in Section 5.2 which is going to play quite an important role in the present Chapter. Let

$$Alg(\mathbf{K}^*) = \{\mathbf{A} : \Delta_A \in Co_{\mathbf{K}}\mathbf{A}\},$$

i.e., $Alg(\mathbf{K}^*)$ is the whole class of $\mathcal{L}$-algebras **A** such that $\mathbf{K}_{\mathbf{A}}$ contains some reduced structure. Then we have the following.

PROPOSITION 9.1.1. *For any full class* K *of* $\mathcal{L}$*-structures and any* $\mathcal{L}$*-algebra* **A**, $Co_{\mathbf{K}}\mathbf{A} = Co_{Alg(\mathbf{K}^*)}\mathbf{A}$.

Proof. By definition, if θ is a K-congruence on **A**, there exists a structure $\mathfrak{A} \in \mathbf{K}_{\mathbf{A}}$ such that $\theta = \Omega\mathfrak{A}$. Hence, $\mathfrak{A}/\theta \in \mathbf{K}^*$ and consequently $\mathbf{A}/\theta$ belongs to $Alg(\mathbf{K}^*)$.

Conversely, suppose $\theta \in Co_{Alg(\mathbf{K}^*)}\mathbf{A}$. Then we have that $\Omega\mathfrak{A} = \Delta_{A/\theta}$ for some member $\mathfrak{A}$ of K with underlying algebra $\mathbf{A}/\theta$. Define $\mathfrak{B} := \pi_\theta^{-1}\mathfrak{A}$, where π_θ denotes the natural projection from the algebra **A** onto the quotient $\mathbf{A}/\theta$. Then $\mathfrak{B}$ is an element of K, for this is a full class by hypothesis. Moreover, by Theorem 2.1.8, $\Omega\mathfrak{B} = \pi_\theta^{-1}\Omega\mathfrak{A} = \pi_\theta^{-1}\Delta_{A/\theta} = Ker\,\pi_\theta^{-1} = \theta$. In conclusion, $\theta \in Co_{\mathbf{K}}\mathbf{A}$. ∎

An obvious consequence from the above proposition is that relative congruence permutability has an easy universal algebraic interpretation as follows.

COROLLARY 9.1.2. *Let* K *be any full class of* $\mathcal{L}$*-structures. Then the following statements are equivalent.*

(i) K *is RCP.*

(ii) *Every pair of* $Alg(\mathbf{K}^*)$*-congruences on* **A** *permute, for all* $\mathcal{L}$*-algebra* **A**.

If, in addition, K *is protoalgebraic and satisfies condition (5.1), then both properties above are equivalent to the following one:*

(iii) *Every pair of* $Alg(\mathbf{K}^*)$*-congruences on* **A** *permute, for all* $\mathbf{A} \in Alg(\mathbf{K}^*)$.

Proof. The equivalence between (i) and (ii) and the implication form (ii) to (iii) are clear. So let us prove that (iii) implies (ii). For this, consider any $\mathcal{L}$-algebra **A**. We must show that every pair of $Alg(\mathbf{K}^*)$-congruences on **A** permute. Let

$$\theta_{\mathbf{K},\mathbf{A}} := \bigcap Co_{\mathbf{K}}\mathbf{A}.$$

Since K is protoalgebraic and satisfies condition (5.1) by hypothesis, $\theta_{\mathbf{K},\mathbf{A}}$ is also a K-congruence on **A** by virtue of 5.3.4; actually, we have that $\theta_{\mathbf{K},\mathbf{A}} = \Omega(\bigcap \mathbf{K}_{\mathbf{A}})$. Hence the quotient algebra $\mathbf{A}/\theta_{\mathbf{K},\mathbf{A}}$ is a member of $Alg(\mathbf{K}^*)$. Moreover, the Correspondence Theorem entails that

$$\mathbf{Co}_{Alg(\mathbf{K}^*)}\mathbf{A} \cong \mathbf{Co}_{Alg(\mathbf{K}^*)}\mathbf{A}/\theta_{\mathbf{K},\mathbf{A}} \tag{9.1}$$

by the map $\phi \longmapsto \phi/\theta_{\mathbf{K},\mathbf{A}}$. So, since $(\theta/\Theta)\cdot(\phi/\Theta) = \theta\cdot\phi/\Theta$ for all $\theta,\phi,\Theta \in Co\,\mathbf{A}$ such that $\Theta \subseteq \theta,\phi$, we apply (iii) and conclude the desired conditon. ∎

COROLLARY 9.1.3. *For any quasivariety* $\mathcal{Q}$ *of* $\mathcal{L}$*-algebras, the class* $\mathbf{K}_{\mathcal{Q}}$ *is RCP iff every pair of* $\mathcal{Q}$*-congruences on* **A** *permute, for all* $\mathbf{A} \in \mathcal{Q}$. ∎

The following is the main result of the Section; it states a Mal'cev-like condition for relative congruence permutability in semialgebraic quasivarieties of structures.

THEOREM 9.1.4. *Assume* $\mathcal{L}$ *contains some function symbol and let* K *be a semialgebraic quasivariety of* $\mathcal{L}$*-structures. Then* K *is RCP iff there exists a ternary* $\mathcal{L}$*-term* $t(x,y,z)$ *such that for all* $\mathfrak{A} \in \mathbf{K}$ *and all* $a,b \in A$,

$$t^{\mathbf{A}}(a,b,b) \equiv a\ (\Omega\mathfrak{A}) \text{ and } t^{\mathbf{A}}(a,a,b) \equiv b\ (\Omega\mathfrak{A}).$$

Proof. Consider first the backward direction. Let $\mathbf{A}$ be any $\mathcal{L}$-algebra and let θ, ϕ be two K-congruences on $\mathbf{A}$. Take two arbitrary elements $a, b \in A$ such that $\langle a, b\rangle \in \theta \cdot \phi$. By definition of the product $\cdot$, there exists a $c \in A$ satisfying $a\theta c$ and $c\phi b$. Hence, by (ii), we have

$$a\,\phi\, t^{\mathbf{A}}(a,c,c)\,\phi\, t^{\mathbf{A}}(a,c,b)\,\theta\, t^{\mathbf{A}}(a,b,b)\,\theta\, b,$$

and so $\langle a, b\rangle \in \phi \cdot \theta$. This proves the inclusion $\theta \cdot \phi \subseteq \phi \cdot \theta$, and so the required condition.

To show the converse, let $\mathbf{Te}_{\mathcal{L},3}$ denote the algebra of $\mathcal{L}$-terms over the three variables x, y and z, and define the class

$$H_{x,y} := \{\langle h, \mathfrak{A}\rangle : \mathfrak{A} \in \mathsf{K},\ h : \mathbf{Te}_{\mathcal{L},3} \to \mathbf{A} \text{ and } hx \equiv hy\ (\Omega\mathfrak{A})\}.$$

For every pair $\langle h, \mathfrak{A}\rangle$ of $H_{x,y}$, the structure $\mathfrak{I}_{h,\mathfrak{A}} = h^{-1}\mathfrak{A}$ belongs to $ES(\mathsf{K})$, which is included in K for this class is a quasivariety by hypothesis. On the other hand, Proposition 6.1.1 says that the structure

$$\mathfrak{I}_{x,y} := \bigcap\nolimits_{\langle h,\mathfrak{A}\rangle \in H_{x,y}} \mathfrak{I}_{h,\mathfrak{A}}$$

is isomorphic to a subdirect product of the system $\{\mathfrak{I}_{h,\mathfrak{A}} : \langle h, \mathfrak{A}\rangle \in H_{x,y}\}$ (observe this system is in fact a set, though $H_{x,y}$ may be a proper class). Consequently, once more the assumption K is a quasivariety entails that $\mathfrak{I}_{x,y} \in \mathsf{K}$. We can define exactly in the same way the class $H_{y,z}$ and the structure $\mathfrak{I}_{y,z}$, this time y and z playing the role of x and y respectively, and again we can prove that $\mathfrak{I}_{y,z} \in \mathsf{K}$. Hence, using (i), we obtain

$$\Omega\mathfrak{I}_{x,y} \cdot \Omega\mathfrak{I}_{y,z} = \Omega\mathfrak{I}_{y,z} \cdot \Omega\mathfrak{I}_{x,y}. \tag{9.2}$$

Now let us see that $x \equiv y\ (\Omega\mathfrak{I}_{x,y})$. Indeed, for each $\langle h, \mathfrak{A}\rangle \in H_{x,y}$, the relation $h^{-1}\Omega\mathfrak{A}$ is a congruence on $\mathfrak{I}_{h,\mathfrak{A}}$, by Lemma 2.1.7. Moreover, $x \equiv y\ (h^{-1}\Omega\mathfrak{A})$. Therefore, we have

$$\bigcap\nolimits_{\langle h,\mathfrak{A}\rangle \in H_{x,y}} h^{-1}\Omega\mathfrak{A} \in Co\,\mathfrak{I}_{x,y} \text{ and } x \equiv y\ (\bigcap\nolimits_{\langle h,\mathfrak{A}\rangle \in H_{x,y}} h^{-1}\Omega\mathfrak{A}).$$

So we just need to apply the definition of Leibniz congruence; $\bigcap_{\langle h,\mathfrak{A}\rangle \in H_{x,y}} h^{-1}\Omega\mathfrak{A}$ must be included in $\Omega\mathfrak{I}_{x,y}$ and consequently $x \equiv y\ (\Omega\mathfrak{I}_{x,y})$. The same argument proves that $y \equiv z\ (\Omega\mathfrak{I}_{y,z})$. Therefore, using (9.2), we obtain $\langle x, z\rangle \in \Omega\mathfrak{I}_{y,z} \cdot \Omega\mathfrak{I}_{x,y}$. This means there exists an element $t(x,y,z) \in Te_{\mathcal{L},3}$ such that

$$x \equiv t(x,y,z)\ (\Omega\mathfrak{I}_{y,z}), \qquad z \equiv t(x,y,z)\ (\Omega\mathfrak{I}_{x,y}). \tag{9.3}$$

It suffices to show that $t(x,y,z)$ satisfies (ii). To this end, consider an $\mathfrak{A} \in \mathsf{K}$ and choose arbitrary elements $a, b \in A$. Let h be an algebra homomorphism from $\mathbf{Te}_{\mathcal{L},3}$ into $\mathbf{A}$ such that $hx = a$ and $hy = hz = b$. Clearly $\langle h, \mathfrak{A}\rangle \in H_{y,z}$, and hence $\mathfrak{I}_{y,z} \preccurlyeq \mathfrak{I}_{h,\mathfrak{A}}$. We now apply that K is semialgebraic. Since $h : \mathfrak{I}_{h,\mathfrak{A}} \twoheadrightarrow_s h\mathfrak{I}_{h,\mathfrak{A}}$ and $h\mathfrak{I}_{h,\mathfrak{A}} \subseteq \mathfrak{A}$, we have

$$\begin{aligned} \Omega\mathfrak{I}_{y,z} &\subseteq \Omega\mathfrak{I}_{h,\mathfrak{A}}, && \text{by } \preccurlyeq\text{-monotonicity of } \Omega,\\ &= h^{-1}\Omega h\mathfrak{I}_{h,\mathfrak{A}}, && \text{by Theorem 2.1.8(i)},\\ &\subseteq h^{-1}\Omega\mathfrak{A}, && \text{by } \subseteq\text{-monotonicity of } \Omega. \end{aligned}$$

Consequently, the first part of (9.3) implies $a \equiv t^{\mathbf{A}}(a,b,b)\ (\Omega\mathfrak{A})$.

The conclusion $b \equiv t^{\mathbf{A}}(a,a,b)\ (\Omega\mathfrak{A})$ is proven similarly. ∎

Observe that, if we use as usual the symbol $\approx$ to formally represent the common identity relation, then condition (ii) in Theorem 9.1.4 can be expressed as follows:

$$Alg(\mathsf{K}^*) \models t(x,y,y) \approx x \wedge t(x,x,y) \approx y.$$

Hence, by Corollary 9.1.2, the equivalence of (i) with (ii) in the preceding theorem amounts to Mal'cev's Theorem on congruence permutability whenever $Alg(\mathsf{K}^*)$ is a variety (e.g., if $\mathsf{K} = \mathsf{K}_{\mathcal{V}}$ for some variety $\mathcal{V}$ of $\mathcal{L}$-algebras). But in general this is not the case; $Alg(\mathsf{K}^*)$ may not be a variety nor even a quasivariety.

Using the previous theorem, we can still prove a further characterization of relative congruence permutability that sharpens the universal algebraic reformulation stated in Corollary 9.1.2.

COROLLARY 9.1.5. *Assume $\mathcal{L}$ contains some function symbol and let K be an arbitrary semialgebraic quasivariety of $\mathcal{L}$-structures. Then the following statements are equivalent.*

- *(i) K is RCP.*
- *(ii) $Alg(\mathsf{K}^*)$ is congruence permutable in the usual universal algebraic sense, i.e., every pair of congruences on $\mathbf{A}$ permute, for all $\mathbf{A} \in Alg(\mathsf{K}^*)$.*
- *(iii) Every pair of congruences on $\mathbf{A}$ containing $\theta_{\mathsf{K},\mathbf{A}}$ permute, for all $\mathcal{L}$-algebra $\mathbf{A}$.*

Proof. Assume K is RCP and let us prove (ii). Take $\mathbf{A} \in Alg(\mathsf{K}^*)$ and consider two arbitrary congruences θ, ϕ on $\mathbf{A}$. The definition of $Alg(\mathsf{K}^*)$ says that there exists an $\mathfrak{A} \in \mathsf{K}_{\mathbf{A}}$ such that $\Omega\mathfrak{A} = \Delta_A$. So, by Theorem 9.1.4, $a\theta c\phi b$ implies

$$a = t^{\mathbf{A}}(a,c,c)\,\phi\,t^{\mathbf{A}}(a,c,b)\,\theta\,t^{\mathbf{A}}(a,a,b) = b,$$

for all $a,b,c \in A$. As a result, $\theta \cdot \phi \subseteq \phi \cdot \theta$ and consequently θ and ϕ permute, as required.

Suppose now that (ii) holds and fix any $\mathcal{L}$-algebra $\mathbf{A}$. Define $\mathfrak{A} = \bigcap \mathsf{K}_{\mathbf{A}}$. Since K is a protoalgebraic quasivariety, we have $\mathfrak{A} \in \mathsf{K}_{\mathbf{A}}$ and $\Omega\mathfrak{A} = \theta_{\mathsf{K},\mathbf{A}}$, and hence $\mathbf{A}/\theta_{\mathsf{K},\mathbf{A}} \in Alg(\mathsf{K}^*)$. So, every pair of $Co\,\mathbf{A}/\theta_{\mathsf{K},\mathbf{A}}$ permute by hypothesis. The rest of the proof runs as in 9.1.2, using the Correspondence Property.

Finally, the implication from (iii) to (i) is clear, for $\theta_{\mathsf{K},\mathbf{A}}$ is contained in every K-congruences of $\mathbf{A}$, for all $\mathcal{L}$-algebra $\mathbf{A}$. ∎

COROLLARY 9.1.6. *Assume $\mathcal{L}$ contains some function symbol. Then the following are equivalent for each quasivariety $\mathcal{Q}$ of $\mathcal{L}$-algebras.*

- *(i) Every pair of $\mathcal{Q}$-congruences on $\mathbf{A}$ permute, for all $\mathbf{A} \in \mathcal{Q}$.*
- *(ii) $\mathcal{Q}$ is congruence permutable in the usual universal algebraic sense.*
- *(iii) There exists a ternary $\mathcal{L}$-term $t(x,y,z)$ such that*

$$\mathcal{Q} \models t(x,y,y) \approx x \wedge t(x,x,y) \approx y.$$

Proof. It follows immediately from 9.1.3, 9.1.4 and 9.1.5. ■

It is natural to ask about the consequences and the interest of taking as notion of congruence permutability of a class the following: K is *congruence permutable* iff every pair of congruences on $\mathfrak{A}$ permute, for all $\mathfrak{A} \in$ K. The fact is that it becomes hard to find in this case a congruence permutable class; actually, we can argue there is no way in our context of proving a Mal'cev-like result characterizing congruence permutability in this sense and that specializes to Mal'cev's Theorem. Indeed, if $\mathcal{Q}$ is any class of $\mathcal{L}$-algebras, then we have $\mathsf{K}_{\mathcal{Q}}$ is congruence permutable (in the above sense) iff for each $\mathcal{L}$-algebra **A** and each $\mathcal{Q}$-congruence θ on **A**, every pair of $[\Delta_A, \theta]$ permute, and this condition is far from the common notion of congruence permutability of $\mathcal{Q}$.

In spite of this, a proper generalization of Mal'cev's Theorem that points towards this direction is proved by Weaver [120], certainly in a different context[24]. But it is not clear Weaver's result really strengthens Mal'cev's one, for no example of a class of structures satisfying the stated notion of congruence permutability is provided in his paper, aside from the cases already covered by Mal'cev's Theorem. Something similar occurs with other properties of lattices of congruences, like distributivity, arithmeticity and so on.

9.2. Study of Other Mal'cev Conditions

In light of the results of the preceding Section, we can think of two feasible extensions of the usual concept of congruence identity in universal algebra, both refering to our notion of relative congruence. For the sake of simplicity, let us center our attention on the property of congruence distributivity; the situation is very much the same for some other properties (none of them the one considered previously!). Let K be any class of $\mathcal{L}$-structures. Then K is said to be *congruence distributive* (CD for short) if the class $Alg(\mathsf{K}^*)$ is congruence distributive in the usual sense, i.e., if Co **A** is a distributive lattice for all $\mathbf{A} \in Alg(\mathsf{K}^*)$. If, moreover, K is protoalgebraic and satisfies condition (5.1), K is called *relatively congruence distributive* (RCD for short) if $\mathrm{Co}_{\mathsf{K}}\mathbf{A}$ is a distributive lattice for all $\mathcal{L}$-algebra **A**. (Recall that assuming protoalgebraicity and (5.1) on K is enough to guarantee that $Co_{\mathsf{K}}\mathbf{A}$ has a lattice structure, inherited by that of $\mathsf{K}_{\mathbf{A}}$ through the Leibniz operator).

The first two results are easy reformulations of CD and RCD, respectively; compare them with the contents of Corollaries 9.1.2 and 9.1.5 above.

PROPOSITION 9.2.1. *Let* K *be any class of* $\mathcal{L}$*-structures. Then* K *is CD in the above sense iff the sublattice* $[\theta_{\mathsf{K},\mathbf{A}}, \nabla_A]$ *of* Co **A** *is distributive for each* $\mathcal{L}$*-algebra* **A**.

[24] The main difference of Weaver's context compared with ours relies on the fact that he assumes the existence of an equality symbol $\approx$ in the language and consider as a generalization of varieties those classes of structures axiomatized by implications of the form $\bigwedge\Phi \longrightarrow \varphi$, where Φ is an arbitrary set of atomic formulas (maybe infinite), none of them of the form $s \approx t$, and φ is any atomic formula.

Proof. Clearly, if $\mathbf{A} \in Alg(\mathsf{K}^*)$ then $\theta_{\mathsf{K},\mathbf{A}} = \Delta_A$; so the backward implication is trivial. To see the reverse implication, assume K is CD. We already know that for each $\mathcal{L}$-algebra **A**, the quotient $\mathbf{A}/\theta_{\mathsf{K},\mathbf{A}}$ belongs to $Alg(\mathsf{K}^*)$. Hence $\mathbf{Co}\,\mathbf{A}/\theta_{\mathsf{K},\mathbf{A}}$ is a distributive lattice by hypothesis. Thus it suffices to apply the Correspondence Theorem of universal algebra and we complete the proof. ∎

PROPOSITION 9.2.2. *Let* K *be any protoalgebraic class of $\mathcal{L}$-structures satisfying condition (5.1). Then* K *is RCD iff $\mathbf{Co}_{Alg(\mathsf{K}^*)}\mathbf{A}$ is distributive for all $\mathbf{A} \in Alg(\mathsf{K}^*)$.*

Proof. The implication from left to right is trivial. For the converse, we apply (9.1). Then $\mathbf{Co}_{Alg(\mathsf{K}^*)}\mathbf{A}$ is distributive iff $\mathbf{Co}_{Alg(\mathsf{K}^*)}\mathbf{A}/\theta_{\mathsf{K},\mathbf{A}}$ is. So the backward implication also holds. ∎

In contrast to what happened for congruence permutability, this time CD and RCD are not equivalent assumptions on a class K, except of course in the trivial case that $Alg(\mathsf{K}^*)$ is a variety; clearly, in this case the lattice $\mathbf{Co}_{Alg(\mathsf{K}^*)}\mathbf{A}$ coincides with the lattice of all congruence relations on **A**. The point is that, as we already noticed in Section 5.2, $\mathbf{Co}_{\mathsf{K}}\mathbf{A}$ is not in general a sublattice of $\mathbf{Co}\,\mathbf{A}$. Besides, it is still an open problem to obtain a manageable description of the lattice operation $\vee_{\mathsf{K}}$, which eventually could be helpful to get a Mal'cev-like condition for relative congruence distributivity. Recently, it has been proved such a kind of result for quasivarieties of algebras; in our context, this result says the following.

THEOREM 9.2.3. *(Dziobiak [45], Nurakunov [95]) Let Q be any quasivariety of $\mathcal{L}$-algebras. Then the following statements are equivalent.*

(i) K_Q is RCD.

(ii) $(\mathsf{K}_Q^*)_{\mathrm{RFSI}} = (\mathsf{K}_Q^*)_{\mathrm{FSI}}$ and there exists a finite nonempty sequence

$$\langle t_i(x,y,z), u_i(x,y,z), v_i(x,y,z)\rangle, \quad i \leq n,$$

of triples of ternary $\mathcal{L}$-terms such that the next conditions hold for all $\mathfrak{A} \in \mathsf{K}_Q$ and all $a,b,c \in A$:

$$t_i^{\mathbf{A}}(a,a,b) \equiv u_i^{\mathbf{A}}(a,a,b)\ (\Omega\mathfrak{A}), \quad i \leq n;$$
$$u_i^{\mathbf{A}}(a,b,b) \equiv v_i^{\mathbf{A}}(a,b,b)\ (\Omega\mathfrak{A}), \quad i \leq n;$$
$$t_i^{\mathbf{A}}(a,b,a) \equiv u_i^{\mathbf{A}}(a,b,a) \equiv v_i^{\mathbf{A}}(a,b,a)\ (\Omega\mathfrak{A}), \quad i \leq n;$$
$$t_i^{\mathbf{A}}(a,b,c) \equiv v_i^{\mathbf{A}}(a,b,v)\ (\Omega\mathfrak{A}), \text{ for all } i \leq n \text{ implies } a \equiv c\ (\Omega\mathfrak{A}). \blacksquare$$

The proof of the preceding theorem largely rests on the fact that $\mathbf{Co}_{\mathsf{K}_Q}\mathbf{A}$ (or equivalently $\mathbf{Co}_Q\mathbf{A}$) is an algebraic complete lattice. We are going to see, however, that this is in general quite a strong assumption. For this, we first need a purely universal algebraic lemma. The proof of the necessity was suggested to the author by Czelakowski; for the sufficiency, see [105].

LEMMA 9.2.4. *Let Q be any class of $\mathcal{L}$-algebras closed under isomorphisms. Then the set $Co_Q\mathbf{A}$ is an inductive closure system iff Q is a quasivariety.*

Proof. Suppose $Co_Q\mathbf{A}$ is an inductive closure system and let us see that Q is a quasivariety. Using 4.4.1 and 4.4.5, it suffices to show that Q is closed under P_{fs}. So, let $\mathbf{A}_i \in Q$, $i \in I$, and assume $\mathbf{A} \subseteq_{sd} \prod_{i\in I} \mathbf{A}_i$. If π_i denotes the natural projection from $\mathbf{A}$ onto $\mathbf{A}_i$, then $\mathbf{A}/Ker\ \pi_i \cong \mathbf{A}_i$, so that $Ker\ \pi_i \in Co_Q\mathbf{A}$ for all $i \in I$. Moreover, $\bigcap_{i\in I} Ker\ \pi_i = \Delta_A$ for $\mathbf{A}$ is by hypothesis a subdirect product of the system $\{\mathbf{A}_i : i \in I\}$. Hence, since $Co_Q\mathbf{A}$ is a closure system, $\Delta_A \in Co_Q\mathbf{A}$, i.e., $\mathbf{A} \in Q$. This proves that Q is closed under P_s.

Take now $X \in Sb(I)$ and let $\mathcal{F}_X$ be the principal filter on I generated by X. Define the set

$$\mathbf{A}_X := \textstyle\prod_{i\in X} \mathbf{A}_i \restriction A.$$

It follows immediately from $\mathbf{A} \subseteq_{sd} \prod_{i\in I} \mathbf{A}_i$ that $\mathbf{A}\restriction X \subseteq_{sd} \prod_{i\in X} \mathbf{A}_i$. Also, $\mathbf{A}/\mathcal{F}_X$ is isomorphic to $\mathbf{A}_X$. To see this, consider the surjective mapping h from $\mathbf{A}/\Theta_{\mathcal{F}_X,A}$ onto A_X defined by setting $h(\mathbf{a}/\mathcal{F}_X) = \mathbf{a}\restriction X$. So given h is clearly well defined and one-one, for if $\mathbf{a}/\mathcal{F}_X, \mathbf{b}/\mathcal{F}_X \in A/\Theta_{\mathcal{F}_X,A}$,

$$\begin{aligned} \mathbf{a}/\mathcal{F}_X = \mathbf{b}/\mathcal{F}_X &\text{ iff } \{i \in I : a_i = b_i\} \supseteq X \\ &\text{ iff } \mathbf{a}\restriction X = \mathbf{b}\restriction X. \end{aligned}$$

Thus, since

$$\begin{aligned} h(f^{\mathbf{A}/\mathcal{F}_X}\mathbf{a}_1/\mathcal{F}_X \ldots \mathbf{a}_n/\mathcal{F}_X) &= h(f^{\mathbf{A}}\mathbf{a}_1 \ldots \mathbf{a}_n/\mathcal{F}_X) \\ &= f^{\mathbf{A}}\mathbf{a}_1 \ldots \mathbf{a}_n \restriction X = f^{\mathbf{A}\restriction X} h(\mathbf{a}_1/\mathcal{F}_X)\ldots h(\mathbf{a}_n/\mathcal{F}_X), \end{aligned}$$

h is the desired isomorphism. This proves Q is closed under filtered subdirect products modulo principal filters. The closure of Q under arbitrary filtered subdirect products follows from the fact that $\mathcal{F} = \bigcup_{X\in\mathcal{F}} \mathcal{F}_X$. The set of congruences $\{\Theta_{\mathcal{F}_X,A} : X \in \mathcal{F}\}$ is a directed system of $Co_Q\mathbf{A}$ whose union is $\Theta_{\mathcal{F},A}$. Therefore, since $Co_Q\mathbf{A}$ is an inductive closure system by hypothesis, $\Theta_{\mathcal{F},A} \in Co_Q\mathbf{A}$, i.e., $\mathbf{A}/\mathcal{F} \in Q$. This completes the proof of the necessity.

For the converse, suppose Q is a quasivariety and let $\{\theta_i : i \in I\}$ any family of Q-congruences on $\mathbf{A}$. The quotient algebra $\mathbf{A}/\bigcap_{i\in I}\theta_i$ is subdirectly embeddable into the direct product $\prod_{i\in I} \mathbf{A}/\theta_i$. Hence, as by assumption $\mathbf{A}/\theta_i \in Q$ for all $i \in I$, we have $\mathbf{A}/\bigcap_{i\in I}\theta_i \in P_{sd}(Q) = Q$. This shows that $Co_Q\mathbf{A}$ is closed under arbitrary intersections. Assume now $\{\theta_i : i \in I\}$ is a nonempty chain of Q-congruences on $\mathbf{A}$. The family $\{[i) : i \in I\}$ of subsets of I ($[i) = \{j \in I : i \leq j\}$) satisfies the finite intersection property and thus it is included in some ultrafilter of $Sb(I)$. Denote it by $\mathcal{U}$ and let $\mathbf{B} := \prod_{i\in I} \mathbf{A}/\theta_i$. It is an easy matter to check that the kernel of the function $h : a \longmapsto \langle a/\theta_i : i \in I\rangle$ composed with the natural projection $\pi_{\mathcal{U}}$ from $\mathbf{B}$ onto $\mathbf{B}/\mathcal{U}$ coincides with the union $\bigcup_{i\in I}\theta_i$; actually, using that $\{\theta_i : i \in I\}$ is a chain, we have the following equivalences:

$$\begin{aligned} \langle a,b\rangle \in Ker\ (\pi_{\mathcal{U}} \circ h) &\text{ iff } \{i \in I : a/\theta_i = b/\theta_i\} \in \mathcal{U} \\ &\text{ iff } a/\theta_i = b/\theta_i \text{ for some } i \in I \\ &\text{ iff } \langle a,b\rangle \in \textstyle\bigcup_{i\in I}\theta_i. \end{aligned}$$

In conclusion, the quotient $\mathbf{A}/\bigcup_{i\in I}\theta_i$ is isomorphic to a subalgebra of $\mathbf{B}/\mathcal{U}$, and thus, as Q is a quasivariety, $\mathbf{A}/\bigcup_{i\in I}\theta_i \in Q$. This means $\bigcup_{i\in I}\theta_i \in Co_Q\mathbf{A}$ and finishes the proof of the lemma. ∎

THEOREM 9.2.5. *Let* K *be any full class of* $\mathcal{L}$*-structures and* **A** *any* $\mathcal{L}$*-algebra. The set* $Co_{\mathsf{K}}\mathbf{A}$ *is an inductive closure system iff the class of* $\mathcal{L}$*-algebras* $Alg(\mathsf{K}^*)$ *is a quasivariety.*

Proof. It follows trivially from Proposition 9.1.1 and the previous lemma. ∎

It is still an open problem to find out a necessary and sufficient condition, expressed in terms of the Leibniz operator or by other means, for the class $Alg(\mathsf{K}^*)$ to be a quasivariety. But in any case the above theorem has a negative conclusion: it seems hard to strengthen the content of Theorem 9.2.3 by considering more arbitrary quasivarieties of $\mathcal{L}$-structures.

10. Connections with Algebraic Logic

It is generally understood that three distinct traditions can be traced back in the history of algebraic logic[25]; see, e.g., [9] or [91]. The first one of these traditions originated with the work of Boole and De Morgan [16,39], and it is characterized by the fact that algebra is viewed as the embodiment of logic rather than merely a representation of it. At the present time this tradition is partly overshadowed by other algebraic approaches to logic, mainly because of the huge influence of the so-called *logicist method* of Frege, Whitehead and Russell. It remains alive however in the modern theory of relation algebras.

The second tradition largely started with Tarski [114]. He established the bases of a method for the algebraization of various *logical* or *deductive systems* that arised from the formalism of the preceding three authors[26]. The main fact on which this second tradition rests on is that the whole deductive apparatus of a logical system, as well as many of its higher-order metalogical properties (e.g., deduction and interpolation theorems), can be interpreted algebraically in many cases. So, according to this tradition, algebra plays an auxiliary role as just a useful way of representing logic. The very precise investigation of the connection between deductive systems and classes of algebras is the main purpose of this algebraic approach to logic and it has been carried on by several authors under different views; e.g., [1], [2], [15], [17], [47], [98], [99]. A culminating point in this investigation was the abstract analysis of the notion of algebraization worked out by Blok and Pigozzi [8]. They developed a *theory of algebraizable logics* that have deeply influenced most of the subsequent contributions to the subject.

The third tradition concerns the investigation of algebraic semantics of deductive systems on a more general level; instead of looking for a class of algebras whose quasi-equational theory fairly describes the entire deductive apparatus of the sys-

[25] The phrase "algebraic logic" originated with Halmos [59]. He intended "algebraic logic" to refer specifically to the algebraization of first-order prediacte logic, but it has come to mean the whole body of work in logic in which algebraic methods are dominant.

[26] The definition of *deductive system* was carried out in several stages by Tarski during the 1930's [111, 112, 114, 116]; nowadays, it usually means a particular purely algebraic language together with a consequence relation attached to it (see, e.g., [8, p. 5]). This notion is general enough to include not only all the classical and non-classical systems of sentetial logics but also of first-order logics. For instance, a formalization of classical first-order logic as a deductive system in the above sense is provided in [8, Appendix C].

tem, as it happens when it is algebraizable in the sense of Blok and Pigozzi [8], now the "algebraic" properties of its class of *matrix models* are examined[27]. So this time the role algebra plays in logic need to be understood in a more abstract sense. The roots of this last tradition can be found not only in the development of matrix semantics for propositional logic by Lukasiewicz and Tarski and their collaborators in the 1920's, but also in the model theory worked out by Birkhoff and, a bit later, by Mal'cev for equational logic and first-order logic with equality, respectively. Roughly speaking, it relies on the fact that by imposing some restrictions on a deductive system, much weaker than the system to be algebraizable in the sense of [8], the matrix semantics begins to show many of the characteristics of a purely algebraic semantics, and thus a good part of the theory of varieties and quasivarieties carries over to classes of matrices[28].

The whole of our work can be seen as a contribution to this third tradition in algebraic logic, which can be traced back to model theory and that now we intend to develop into a new trend in this area. The primary motivation of our shift from the investigation of matrix semantics of deductive systems to the study of classes of arbitrary first-order structures defined without equality was the emergence of some more general kinds of logical systems, viz. the *k-dimensional deductive systems* of [11] and the *Gentzen systems* of [101], and the much attention they have attracted recently. Since Bloom [13], it is well known that any deductive system in the sense defined above can be formalized as an (elementary) strict universal Horn theory without equality and with a single, unary relation symbol. A similar interpretation, however, can be carried out for a rather general sort of logical systems that include the preceding ones and even some other consequences naturally associated to classes of algebras, like the *quasiequational consequences* [46]. Although the obtained theories need to be defined this time over languages with maybe more than just one relation symbol, the truth of the matter is that the strict universal Horn fragment of first-order logic is still enough to get such an elementary characterization for this broader class of logical systems.

This is a fundamental fact that, together with Theorem 5.1.1 of Chapter 5, brings us to what in our opinion is the core of the connection between the work developed previously and algebraic logic: strict universal Horn theories (without equality) seem to be the elementary notion that better retains the main features of Tarski's deductive systems. Just because of this conviction, a great deal of the theory of deductive systems, including their matrix semantics, is extended in the previous chapters to the formalism of UHL without equality. The ultimate purpose is to lay the foundations for a theory general enough to encompass the investigation of algebraic semantics of particular types of logical systems.

Many examples of the well-behaved classes introduced in Chapter 5 are furnished in the literature on algebraic logic, a small part of which have been cited before and can be found in the bibliography. In fact, we already pointed out that the concept of *Leibniz operator* and the hierarchy

[27] See, e.g., [8, p.9] for a definition of *matrix model* of a deductive system.

[28] We mentioned this point in the Introduction. So see there in for some references.

Purely Algebraic Quasivarieties

|

Algebraic Quasivarieties

|

Semialgebraic Quasivarieties

|

Protoalgebraic Quasivarieties

|

Quasivarieties

comes from this research area. Thus, for instance, the *protoalgebraic deductive systems* introduced in [7] are exactly those systems whose class of matrix models form a protoalgebraic quasivariety in our sense. To better illustrate this influence, it is appropriate to mention some examples, borrowed from sentential logic, of well-behaved quasivarieties of structures. For instance, the classes of matrix models of the following sentential logics are of the type indicated below:

1. Purely algebraic quasivarieties: classical logic, intuitionistic logic, normal modal logics, the relevance logics **R** and **RM**, many-valued logics of Lukasiewicz; see [8], [98].
2. Algebraic quasivarieties that are not purely algebraic ones: quasi-normal modal logics $\mathbf{S4}_{MP}$ and $\mathbf{S5}_{MP}$ [66].
3. Semialgebraic quasivarieties but not algebraic ones: quasi-normal modal logics $\mathbf{K}_{MP}$ and $\mathbf{T}_{MP}$ [66].
4. Protoalgebraic quasivarieties that are not semialgebraic: the sentential logics defined by the Gentzen calculi $\mathcal{G}_1$ and $\mathcal{G}^1$ of [48] .
5. Quasivarieties that are not even protoalgebraic: the $\{\vee, \wedge\}$-fragment of classical logic and the $\{\vee, \wedge, \neg\}$-fragment of intuitionistic logic are not protoalgebraic; see [8], [49], [101].

The property of a quasivariety of structures to be purely algebraic deserves a special consideration for it is closely related to the notion of *algebraizable deductive system* introduced by Blok and Pigozzi. In [8] they showed that the algebraizable deductive systems are just those systems whose class of matrix models form a purely algebraic quasivariety. Their result has been recently extended to general strict universal Horn theories in [41], and by virtue of this generalization we can find some other examples of purely algebraic quasivarieties that are not matrix semantics but come from particular Gentzen systems; see [101]. A remarkable fact is that purely algebraic quasivarieties of $\mathcal{L}$-structures are, roughly speaking, the quasivarieties elementary definitionally equivalent to some quasivariety of $\mathcal{L}$-algebras, where the possible definitions are required to be of a very special form; see [8, Appendix A]. That is way they exhibit so nice algebraic properties.

Bibliography

[1] H. Andréka, G. Gegerly and I. Németi, *On universal algebraic constructions of logics*, Studia Logica **36** (1977), 9–47.

[2] H. Andréka, I. Németi and I. Sain, *Abstract model theoretic approach to algebraic logic*, Preprint, 1988.

[3] J. van Benthem, *A note on Jónsson's Theorem*, Algebra Universalis **25** (1988), 391–393.

[4] G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. **31** (1935), 433–454.

[5] ———, *Subdirect unions in universal algebra*, Bull. Amer. Math. Soc. **50** (1944), 764–768.

[6] ———, "Lattice Theory", Colloquim Publications, Vol. 25, Amer. Math. Soc., Providence, 1948.

[7] W. Blok and D. Pigozzi, *Protoalgebraic logics*, Studia Logica **45** (1986), 337–369.

[8] ———, *Algebraizable logics*, Memoirs of the AMS **396** (1989), 78pp.

[9] ———, *Introduction* to the special issue on Algebraic Logic, Studia Logica **50** (1991) 365–374.

[10] ———, *Local Deduction Theorems in Algebraic Logic*, in H. Andréka, J.D. Monk, I. Németi (eds.), "Algebraic Logic", Colloq. Math. Soc. J. Bolyai 54, North-Holland, Amsterdam, 1991, pp. 75–109.

[11] ———, *The Deduction Theorem in Algebraic Logic*, Preprint, 1991.

[12] ———, *Algebraic semantics for Universal Horn Logic without equality*, in A. Romanowska, J.D. Smith (eds.), "Universal Algebra and Quasigroup Theory", Research and Exposition in Mathematics, Vol. 19, Heldermann Verlag, Berlin, 1992, pp. 1–56.

[13] S.L. Bloom, *Some theorems on structural consequence relations*, Studia Logica **34** (1975), 1–9.

[14] ———, *Varieties of ordered algeberas*, Jour. of Comp. Sys. Sci. **13** (1976), 200–

212.

[15] S.L. Bloom and D.J. Brown, *Classical abstract logics,* Dissertationes Math. **102** (1973), 43-51.

[16] G. Boole, "The metamathematical analysis of logic, being an essay toward a calculus of deductive reasoning", London and Cambridge, 1847.

[17] D.J. Brown and R. Suzko, *Abstract logics,* Dissertationes Math. **102** (1973), 9-41.

[18] S. Burris, *Subdirect representation in axiomatic classes,* Colloq. Math. **34** (1976), 191-197.

[19] ———, *Free algebras as subdirect products,* Algebra Universalis **11** (1980), 133-134.

[20] S. Burris and H.P. Sankappanavar, "A course in Universal Algebra", Graduate Texts in Mathematics, Vol. 78, Springer-Verlag, New York, 1981.

[21] I. Chajda, *Lattices of compatible relations,* Archiv. Math. (Brno) **13** (1977), 89-96.

[22] ———, *Distributivity and modularity of lattices of tolerance relations,* Algebra Universalis **12** (1981), 247-255.

[23] ———, *Tolerances in congruence permutable algebras,* Czechoslovak Math. Jour. **38** (1988), 218-225.

[24] C.C. Chang, *On unions of chains of models,* Proc. Amer. Math. Soc. **10** (1959), 120-127.

[25] C.C. Chang and H.J. Keisler, "Model Theory" (3rd edition), Studies in Logic and the Foundations of Mathemantics, Vol. 73, North-Holland, Amsterdam, 1990.

[26] P.C. Cohn, "Universal Algebra", Mathematics and Its Applications, Vol. 6, Reidel Publishing, Dordrecht, 1981.

[27] G. Czédli and A. Lenkehegyi, *On classes of ordered algebras and quasiorder distributivity,* Acta Sci. Math. **46** (1983), 41-54.

[28] J. Czelakowski, "Model-theoretic methods in methodology of propositional calculii", The Inst. of Phil. and Sociol. of the Polish Acad. Sci., Warszawa, 1980.

[29] ———, *Reduced products of logical matrices,* Studia Logica **39** (1980), 19-43.

[30] ———, *Equivalential logics, I-II,* Studia Logica **40** (1981), 227-236, 335-372.

[31] ———, *Matrices, primitive satisfaction and finitely based logics,* Studia Logica **42** (1983), 89-104.

[32] ———, *Some theorems on structural entailment relations,* Studia Logica **42** (1983), 417-430.

[33] ———, *Filter distributive logics,* Studia Logica **43** (1984), 353-377.

[34] ———, *Remarks on finitely based logics,* in "Logic Colloquim, Aachen '83", Vol. 1, Models and Sets. Lecture Notes in Math., No. 1103, Springer-Verlag, Berlin, 1984, pp. 147-168.

[35] ———, *Algebraic aspects of deduction theorems,* Studia Logica **44** (1985), 369-387.

[36] ——, *Local deduction theorems*, Studia Logica **45** (1986), 377–391.

[37] ——, *Personal communication*, 1993.

[38] J. Czelakowski and W. Dziobiak, *Another proof that ISP_rK is the least quasivariety containing K*, Studia Logica **39** (1980), 343–345.

[39] A. De Morgan, *On the syllogism: IV, and on the logic of relations*, Trans. Camb. Phil. Soc. **10** (1864), 331–358. Reprinted in [40].

[40] ——, "On the syllogism, and other logical writings", P. Heath (ed.), Yale University Press, New Haven, 1960.

[41] P. Dellunde and R. Jansana, *On structural equivalence of theories*, Manuscript, 1993.

[42] D. Duffus and I. Rival, *A structure theory for ordered sets*, Discrete Math. **35** (1981), 53–118.

[43] W. Dzik and R. Suszko, *On distributivity of closure systems*, Polish Acad. Sci. Inst. Philos. Sociol. Bull. Sect. Logic **6** (1977), 64–66.

[44] W. Dziobiak, *Finitely generated congruence distributive quasivarieties of algebras*, Fund. Math. **133** (1989), 47-57.

[45] ——, *Relatively congruence distributivity within quasivarieties of nearly associative Φ-algebras*, Fund. Math. **135** (1990), 77–95.

[46] R. Elgueta, *A general notion of deductive system in elementary terms*, Manuscript, 1992.

[47] W. Felscher and J. Schülte-Mönting, *Algebraic and deductive consequence operations*, in P. Burmeister et al. (eds.), "Universal Algebra and its links with Logic, Algebra, Combinatorics and Computer Science", Research and Exposition in Mathematics, Vol. 4, Heldermann Verlag, Berlin, 1984, pp. 41–66.

[48] J.M. Font and J.L. García Lapresta, *Logics and algebras motivated by cardinality restrictions in the Deduction Theorem*, Manuscript, 1991.

[49] J.M. Font and V. Verdú, *Algebraic logic for classical conjunction and disjunction*, Studia Logica **50** (1991), 391–419.

[50] L. Fuchs, *On subdirect unions I*, Acta Math. Acad. Sci. Hungarica **3** (1952), 103–119.

[51] ——, *On partially ordered algebras I*, Colloq. Math. **14** (1966), 115–130.

[52] ——, "Partially ordered algebraic systems", Pergamon Press, Oxford, 1969.

[53] L. Fujiwara, *On the construction of the least universal Horn class containing a given class*, Osaka Jour. Math. **8** (1971), 425-436.

[54] K. Gödel, *Die Vollständigkeit der Axiome des logischen Funcktionenkalküls*, Monat. Math. Phys. **37** (1930), 349-360. English translation in [55]: *The completeness of the axioms of the functional calculus of logic*.

[55] ——, "Collected Works, Vol. I. Publications 1929-1936", S. Feferman et al. (eds.),

Oxford University Press, New York, 1986.

[56] V.A. Gorbunov, *The cardinality of sudirectly irreducible systems in quasivarieties,* Algebra and Logic **25** (1986), 1–34.

[57] G. Grätzer, "Universal Algebra" (2nd edition), Springer-Verlag, New York, 1979.

[58] G. Grätzer and H. Lasker, *A note on the implicational class generated by a class of structures,* Canadian Math. Bull. **16** (1973), 603–605.

[59] P. Halmos, *The basic concepts of algebraic logic,* Amer. Math. Monthly **56** (1956), 363–387.

[60] J. Hashimoto, *Direct, subdirect decompositions and congruence relations,* Osaka Math. Jour. **9** (1957), 87–112.

[61] L. Henkin, *The completeness of the first-order functional calculus,* Jour. Symbolic Logic **14** (1949), 159–166.

[62] ——, *Some interconnections between modern algebra and mathematical logic,* Trans. Amer. Math. Soc. **74** (1953), 410–427.

[63] W. Hodges, *Logical Features of Horn Clauses,* in D.M. Gabbay et al. (eds.), "Handbook of Logic in Artificial Intelligence and Logic Programming", Vol. I, Clarendon Press, Oxford, 1993, pp. 449-503.

[64] J. Herbrand, *Investigations into Proof Theory,* in W. Goldfarb (ed.), "Jacques Herbrand. Logical Writings", Reidel Publishing, Dordrecht, 1971, pp. 44–202.

[65] H.J. Hoehnke, *Fully invariant algebraic closure systems of congruences and quasivarieties of algebras,* in L. Szabo and A. Szendrei (eds.), "Lectures in Universal Algebra, Szeged '83", Colloq. Math. Soc. J. Bolyai **43**, North-Holland, Amsterdam, 1986, pp. 189–207.

[66] R. Jansana, *Abstract modal logics,* Preprint, 1991.

[67] B. Jónsson, *Algebras whose congruence lattices are distributive,* Math. Scand. **21** (1967), 110–121.

[68] D. Kelly, *Complete rules of inference for universal sentences,* Stud. Sci. Math. Hungarica **19** (1984), 347–361.

[69] S.R. Kogalovskii, *On a theorem of Birkhoff,* Uspehi Mat. Nauk. **20** (1965), 206–207.

[70] A. Kolibiar, *Congruence relations and direct decompositions of ordered sets,* Acta Sci. Math. **51** (1987), 129–135.

[71] J.W. Lloyd, "Foundations of Logic Programming", Springer-Verlag, Berlin, 1984.

[72] J. Los, *O matrycach logicznych,* Travaux Soc. Sci. Lettres Wroclaw, Ser. B **19**, 1949.

[73] ——, *Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres,* in "Mathematical Interpretation of Formal Systems", Studies in Logic and the Foundations of Mathematics, Vol. 10, North-Holland, Amsterdam, 1955, pp. 98–113.

[74] J. Łos and R. Suszko, *Remarks on sentential logics,* Indag. Math. **20** (1958), 177–183.

[75] ———, *On the extending of models IV: Infinite sums of models,* Fund. Math. **44** (1957), 52–60.

[76] ———, *On the extending of models V: Embedding theorems for relational structures,* Fund. Math. **48** (1960), 113–121.

[77] L. Löwenheim, *Über Möglichkeiten im Relativkakül,* Math. Ann. **76** (1915), 447–470.

[78] J. Lukasiewicz and A. Tarski, *Untersuchungen über den Aussagenkalkül,* Compt. Rend. Soc. Sci. Lettres de Varsovie, Classe III **23** (1930), 30–50. Revised english translation in [?]: *Investigations into the sentential calculus.*

[79] R. Lyndon, *Properties preserved in subdirect products,* Pacific Jour. Math. **9** (1959), 155–166

[80] S. MacLane, "Categories for the working mathematicien", Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York, 1971.

[81] B. Mahr and J.A. Makowsky, *Characterizing specification languages which admit initial semantics,* Theor. Comp. Sci. **31** (1984), 49–59.

[82] J.A. Makowsky, *Why Horn formulas matter in computer science: initial structures and generic exemples,* Jour. of Comp. Sys. Sci. **34** (1987), 266–292.

[83] A.I. Mal'cev, *Untersuchungen aus dem Gebiete der mathematischen Logik,* Mat. Sbornik **1(43)** (1936), 323–336. English translation in [88]: *Investigations in the realm of mathematical logic.*

[84] ———, *K obscei teorii algebraice skih sistem (On the general theory of algebraic systems),* Mat. Sbornik **35(71)** (1954), 3–20.

[85] ———, *Kvaziprimitivnye klassy abstraktnyh algebr,* Dokl. Akad. Nauk SSSR **108** (1956), 187–189. English translation in [88]: *Quasiprimitive classes of abstract algebras.*

[86] ———, *Podprjamye proizvedenija modelei,* Dokl. Akad. Nauk SSSR **109** (1956), 264–266. English translation in [88]: *Subdirect products of models.*

[87] ———, *Neskol'ko zamecanii Kvazimnogoobrazijah algebraiceskih sistem,* Algebra i Logica **5** (1966), 3–9. English translation in [88]: *A few remarks on quasivarieties of algebraic systems.*

[88] ———, "The metamathematics of algebraic systems", Collected Papers: 1936–1937, Studies in Logic and the Foundations of Mathematics, Vol. 66, North- Holland, Amsterdam, 1971.

[89] ———, "Algebraic Systems", Die Grundlehren der Mathematischen Wissenschaften, Band 152, Springer-Verlag, Berlin, 1973.

[90] J. Malinowski, *Modal equivalential logics,* Jour. of Non-Classical Logics **3** (1986), 13–34.

[91] C. Mangione and S. Bozzi, "Storia della Logica. Da Boole ai nostri giorni", Garzanti, 1993.

[92] G.F. McNulty, *Fragments of first-orden logic I: Universal Horn Logic,* Jour. Sym. Logic **42** (1977), 221–237.

[93] E. Nelson, *On the adjointness between operations and relations and its impact on atomic compactness,* Colloq. Math. **33** (1975), 33–40.

[94] B.H. Neumann, "Universal Algebra", Lecture Notes, Courant Institute of Math. Sci., New York University, 1962.

[95] A.M. Nurakunov, *Characterization of relatively distributed quasivarieties of algebras,* Algebra and Logic **29** (1990), 451– 458.

[96] H.E. Pickett, *Subdirect representations of relational systems,* Fund. Math. **56** (1964), 223–240.

[97] T. Prucnal and A. Wronski, *An algebraic characterization of the notion of structural completeness,* Polish Acad. Sci. Inst. Philos. Sociol. Bull. Sect. Logic **3** (1974), 30–33.

[98] H. Rasiwoa, "An algebraic approach to non-classical logics", North-Holland, Amsterdam, 1974.

[99] H. Rasiwoa and R. Sikorski, "The mathematics of metamathematics", PWN, Warsaw, 1963.

[100] W. Rautenberg, *On reduced matrices,* Studia Logica **52** (1993), 63–72.

[101] J. Rebagliato and V. Verdú, *On the algebraization of some Gentzen systems,* Fundamenta Informaticae **18** (1993), 319–338.

[102] A. Robinson, "On the metamathematics of Algebra", Studies in Logic and the Foundations of Mathematics, Vol. 44, North-Holland, Amsterdam, 1951.

[103] ———, "Complete Theories", Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1956.

[104] I.G. Rosenberg and D. Schweigert, *Compatible orderings and tolerances on lattices,* Ann. Discrete Math. **23** (1984), 119–150.

[105] I.G. Rosenberg and T. Sturm, *Congruence relations on finite models,* Czech. Math. Jour. **42 (117)** (1992), 461–470.

[106] D.J. Shoesmith and T.J. Smiley, "Multiple-conclusion logic", Cambridge Univ. Press, Cambridge, 1978.

[107] T. Skolem, *Logish-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischen Sätze nebst einem Theorem über dichte Mengen,* Skrifter utgitt av Videnskapsselskapet i Krintiania, I, Mat. Naturv. Kl. **4** (1920), 36pp.

[108] S.J. Surma, *On the origin and the subsequent application of the concept of Lindenbaum Algebra,* in "Proc. 6th Int. Congress of Logic, Methodology and Philosophy of Science, Hannover 79", Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1982, pp. 719–734.

[109] H. Tabata, *Free structures and Horn sentences,* Math. Japonicae **14** (1969), 101–104.

[110] ———, *A generalized free structure and several properties of universal Horn classes,* Math. Japonicae **16** (1971), 91–102.

[111] A. Tarski, *Über einige fundamentalen Begriffe der Metamathematik,* Compt. Rend. Soc. Sci. Lettres de Varsovie, Classe III **23** (1930), 22–29. Revised english translation in [118]: *On some fundamental concepts of metamathematics.*

[112] ———, *Fundamentale Begriffe der Methodologie der deductiven Wissenschaften, I,* Monat. Math. Phys. **37** (1930), 361–404. Revised english translation in [118]: *Fundamental concepts of the methodology of the deductive sciences.*

[113] ———, *Sur les ensembles définissables de nombres réels I,* Fund. Math. **17** (1931), 210–239. Revised english translation in [118]: *On definible sets of real numbers.*

[114] ———, *Grundzüge des Systemenkalküls. Erster Teil,* Fund. Math. **25** (1935), 503–526. Revised english translation in [118]: *Foundations of the calculus of systems.*

[115] ———, *Grundzüge des Systemenkalküls. Zweiter Teil,* Fund. Math. **26** (1936), 283–301. Revised english translation in [118]: *Foundations of the calculus of systems.*

[116] ———, *Some notions and methods on the borderline of algebra and metamathematics,* in "Proc. Inter. Congress of Mathematics, Cambridge 1950", Amer. Math. Soc., Providence, 1952, pp. 705–720.

[117] ———, *Contributions to the Theory of Models I-II-III,* Indag. Math. **16** (1954), 572–588; **17** (1955), 56–64.

[118] ———, "Logics, semantics and metamathematics" (2nd edition), J. Corcoran (ed.), Hackett Publishing, 1983.

[119] A. Torrens, *Model Theory for Sequential Deductive Systems,* Manuscript, 1992.

[120] N. Weaver, *Generalized varieties,* Algebra Universalis **30** (1993), 27–52.

[121] W. Wechler, "Universal Algebra for Computer Scientists", E.A.T.C.S. Monographs, Vol. 25, Springer-Verlag, New York, 1992.

[122] R. Wójcicki, *Matrix approach in the methodology of sentential calculi,* Studia Logica **68** (1973), 269–279.

[123] ———, "Lectures on propositional calculi", The Publishing House of the Polish Academy of Sciences, Polish Acad. Sci. Inst. Philosophy and Sociology, Ossolineum, Warsaw, 1984.

[124] ———, "Theory of logical calculi. Basic theory of consequence operations", Synthese Library, Vol. 199, Reidel Publishing, Dordrecht, 1988.

[125] B. Zelinka, *Tolerances in algebraic structures,* Czech. Math. Jour. **20** (1970), 179–183.

[126] G. Zubieta, *Clases aritméticas definidas sin igualdad,* Bol. Soc. Mat. Mexicana **2** (1957), 45–53.

[127] J. Zygmunt, *A note on direct products and ultraproducts of logical matrices,* Studia Logica **33** (1974), 349–357.

DEPARTAMENT DE MATEMÀTICA APLICADA II
UNIVERSITAT POLITÈCNICA DE CATALUNYA
PAU GARGALLO 5
08071 BARCELONA, SPAIN

E-mail: **elgueta@ma2.upc.es**

Special Notation

Numbers refer to the page on which the notation is defined or first used.

Index

*See the footnote(s) at the bottom of the quoted page.