PhD. Dissertation

GALOISIAN APPROACH TO SUPERSYMMETRIC QUANTUM MECHANICS

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with our supervision and co-direction.

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Juan J. Morales-Ruiz   Jacques-Arthur Weil

Todo lo puedo en Cristo
que me fortalece.

Filipenses 4, 13.

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Contents

Introduction

Historical Outline

This historical outline begins with two mathematicians: Gastón Darboux and Emile Picard. Darboux published in 1882 the paper [28] in where he presents a proposition in a general way, which in particular case the history proved to be a notable theorem today known as *Darboux transformation.* Darboux had shown that whenever one knows to integrate the equation

$$\frac{d^2y}{dx^2} = (f(x) + m)y$$

for all the values of the constant m, one can obtain an infinite set of equations, displaying the variable parameter in the same way, which are integrable for any value of the parameter. This proposition is also to be found in his book [29, p. 210]. One year after, in 1883, Picard published the paper [66] in which gave the starting point to a *Galois theory for linear differential equations.* Although the analogies between the linear differential equations and the algebraic equations for a long time were announced and continued in different directions, Picard developed an analogue theory to the Galois theory for algebraic equations, arriving to a proposition which seems to correspond to the fundamental Galois theorem, in where he introduces the concept of *group of linear transformations corresponding to the linear differential equation*, which today is known as *Differential Galois Group* (the group of differential automorphism leaving fixed the elements of the field base). Another contribution of Picard to this Galois theory was the paper [65] in 1887 and five years after, in 1892, Ernest Vessiot, doctoral former student of Picard published his thesis [98] giving consolidation to the new Galois theory for linear differential equations, which today is called *Picard-Vessiot theory.* Two years later, in 1894, Picard published the paper [67], summarizing the results presented in [65, 66, 98] which also can be found in his book [68, §7].

Curiously, Picard-Vessiot theory and Darboux transformation were forgotten during decades. The Picard-Vessiot theory was recovered by Joseph Fels Ritt (in 1950, see [75]), Irving Kaplansky (in 1957, see [46]), and fundamentally by Ellis

Kolchin (in 1973, see [48] and references therein). Kolchin wrote the *Differential Galois Theory* in a modern language (algebraic group theory).

Darboux transformation was recovered as an exercise in 1926 by Ince (see exercises 5, 6 and 7 [42, p. 132]) follow closely the formulation of Darboux given in [28, 29].

In 1930, P. Dirac publishes *The Principles of Quantum Mechanics*, in where he gave a mathematically rigorous formulation of quantum mechanics.

In 1938, J. Delsarte wrote the paper [30], in which he introduced the notion of transformation (transmutation) operator, today know as *intertwining operator* which is closely related with Darboux transformation and ladder operators.

In 1941, E. Schrödinger published the paper [79] in which he factorized in several ways the hypergeometric equation. This was a byproduct of his *factorization method* originating an approach that can be traced back to Dirac's raising and lowering operators for the harmonic oscillator.

Ten years later, in 1951, another factorization method was presented. L. Infeld and T. E. Hull published the paper [43] in where they gave the classification of their factorizations of linear second order differential equations for eigenvalue problems of wave mechanics.

In 1955, M.M. Crum inspired in the Liouville's work about Sturm-Liouville systems (see [54, 55]), published the paper [27] giving one kind of iterative generalization of Darboux transformation. Crum surprisingly did not mention to Darboux.

In 1971, G.A. Natanzon published the paper [62], in which he studies a general form of the transformation that converts the hypergeometric equation to the Schrödinger equation writing down the most general *solvable potential*, potential for which the Schrödinger equation can be reduced to hypergeometric or confluent hypergeometric form, concept introduced by himself.

Almost one hundred years later of the Darboux's proposition, in 1981, Edward Witten in his renowned paper [102] gave birth to the *Supersymmetric Quantum Mechanics*, discussing general conditions for dynamical supersymmetry breaking.

Since the work of Witten, thousands of papers, about supersymmetric quantum mechanics, has been written. We mention here some relevant papers.

In 1983, L. É. Gendenshtein published the paper [36] in where the *Shape invariance* condition, i.e. preserving the shape under Darboux transformation, was presented and used to find the complete spectra for a broad class of problems including all known exactly solvable problems of quantum mechanics (bound state and reflectionless potentials). Today this kind of exactly solvable potentials satisfying the shape invariance condition are called *Shape invariant potentials.*

In 1986, A. Turbiner in [89] introduces the concept of *quasi-exactly solvable potentials*, giving an example that is well known as *Turbiner's potential.*

In 1991, V.B. Matveev and M. Salle published the book [59] in where they focused on Darboux transformations and their relation with solitons. Matveev and Salle interpreted the Darboux transformation as Darboux covariance of a Sturm-Liouville problem and also proved that Witten's supersymetric quantum mechanics is equivalent to a single Darboux transformation.

In 1996, C. Bender and G. Dunne studied the *sextic anharmonic oscillator* in [11], which is a quasi-exactly solvable model derived from the Turbiner's potentials. They found that the a portion of the spectrum correspond to the roots of polynomials in the energy. These polynomials are orthogonal and are called *Bender-Dunne polynomials.*

Relationships between the spectral theory and differential Galois theory have been studied by V. Spiridonov [86], F. Beukers [16] and Braverman et. al. [19]. As far as we know, Spiridonov was the first author that considered the useful of the Picard-Vessiot theory in the context of the quantum mechanics. This thesis agree with his point of view.

Structure of the Thesis

This thesis is divided in two parts:

Chapter 1. Theoretical Background. In this part there are not original results. Summaries of Picard-Vessiot theory and supersymmetric quantum mechanics is presented here necessary to understand the chapter 2.

Chapter 2. Differential Galois Theory Approach to Supersymmetric Quantum Mechanics. This part contain the original results of this thesis which were developed using the chapter 1. Up to specific cases, theorems, propositions, corollaries and lemmas given in this chapter are original results for this thesis. Two different Galoisian approaches are studied here, which depends on the differential field. The first one is $\mathbb{C}(x)$ and the second one is $K = \mathbb{C}(z(x), \partial_z(x))$, where $z = z(x)$ is a *Hamiltonian change of variable.* This concept allows to introduce an useful derivation $\widehat{\partial}_z$ which is an important tool to transforms differential equations with non rational coefficients into differential equations with rational coefficients to apply the results given in the case of $\mathbb{C}(x)$.

Chapter 1

Theoretical Background

In this chapter we set the main theoretical background needed to understand the results of this thesis. We start setting conventions and notations that will be used along this work.

- The sets $\mathbb{Z}_+$, $\mathbb{Z}_-$, $\mathbb{Z}_+^*$ and $\mathbb{Z}_-^*$ are defined as

  $$\mathbb{Z}_+ = \{n \in \mathbb{Z} : \quad n \geq 0\}, \quad \mathbb{Z}_- = \{n \in \mathbb{Z} : \quad n \leq 0\}, \quad \mathbb{Z}_+^* = \mathbb{Z}^+, \quad \mathbb{Z}_-^* = \mathbb{Z}^-.$$

- The cardinality of the set A will be denoted by $\mathrm{Card}(A)$.
- The determinant of the matrix A will be denoted by $\det A$.
- The set of matrices $n \times n$ with entries in $\mathbb{C}$ and determinant non-null will be denoted by $\mathrm{GL}(n, \mathbb{C})$.
- The derivation $d/d\xi$ will be denoted by ∂_ξ. For example, the derivations $' = d/dx$ and $\dot{} = d/dt$ are denoted by ∂_x and ∂_t respectively.

1.1 Picard-Vessiot theory

The Picard-Vessiot theory is the Galois theory of linear differential equations. In the classical Galois theory, the main object is a group of permutations of the roots, while in the Picard-Vessiot theory it is a linear algebraic group. For polynomial equations we want a solution in terms of radicals. From classical Galois theory it is well if the Galois group is a solvable group.

An analogous situation holds for linear homogeneous differential equations (see [15, 26, 60, 92]). The following definition is true in general dimension, but for simplicity we are restricting to matrices 2×2.

1.1.1 Definitions and Known Results

Definition 1.1.1. An algebraic group of matrices 2×2 is a subgroup $G \subset \mathrm{GL}(2,\mathbb{C})$, defined by algebraic equations in its matrix elements and in the inverse of its determinant. That is, for $A \in \mathrm{GL}(2,\mathbb{C})$ given by

$$A = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad \det A = x_{11}x_{22} - x_{21}x_{22}$$

there exists a set of polynomials

$$\{P_i(x_{11}, x_{12}, x_{21}, x_{22}, 1/\det A)\}_{i\in I},$$

such that

$$A \in G \quad \Leftrightarrow \quad \forall i \in I, P_i(x_{11}, x_{12}, x_{21}, x_{22}, 1/\det A) = 0.$$

In this case we say that G is an algebraic manifold endowed with a group structure.

Examples (Known algebraic groups). The following algebraic groups should be kept in mind throughout this work.

- Special linear group group:
  $\mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad a,b,c,d \in \mathbb{C} \right\}$
- Borel group: $\mathbb{B} = \mathbb{C}^* \ltimes \mathbb{C} = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}, \quad c \in \mathbb{C}^*, \quad d \in \mathbb{C} \right\}$
- Multiplicative group: $\mathbb{G}_m = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c \in \mathbb{C}^* \right\}$
- Additive group: $\mathbb{G}_a = \left\{ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad d \in \mathbb{C} \right\}$
- Infinite dihedral group (also called meta-abelian group):
  $\mathbb{D}_\infty = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & d \\ -d^{-1} & 0 \end{pmatrix}, \quad d \in \mathbb{C}^* \right\}$
- $n-$quasi-roots: $\mathbb{G}^{\{n\}} = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}, \quad c^n = 1, \quad d \in \mathbb{C} \right\}$
- $n-$roots: $\mathbb{G}^{[n]} = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c^n = 1 \right\}$
- Identity group: $e = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

- The tetrahedral group $A_4^{\mathrm{SL}_2}$ of order 24 is generated by matrices
$$M_1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad M_2 = \frac{1}{3}(2\xi - 1) \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix},$$
where ξ denotes a primitive sixth root of unity, that is $\xi^2 - \xi + 1 = 0$.
- The octahedral group $S_4^{\mathrm{SL}_2}$ of order 48 is generated by matrices
$$M_1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad M_2 = \frac{1}{2}\xi(\xi^2 + 1) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$
where ξ denotes a primitive eighth root of unity, that is $\xi^4 + 1 = 0$.
- The icosahedral group $A_5^{\mathrm{SL}_2}$ of order 120 is generated by matrices
$$M_1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \quad \text{and} \quad M_2 = \frac{1}{5} \begin{pmatrix} \phi & \psi \\ \psi & -\phi \end{pmatrix},$$
where ξ denotes a primitive tenth root of unity, that is $\xi^4 - \xi^3 + \xi^2 - \xi + 1 = 0$, $\phi = \xi^3 - \xi^2 + 4\xi - 2$ and $\psi = \xi^3 + 3\xi^2 - 2\xi + 1$.

Recall that a group G is called solvable if and only if there exists a chain of normal subgroups
$$e = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$$
such that the quotient G_i/G_j is abelian for all $n \geq i \geq j \geq 0$. Also recall that an algebraic group G has a unique connected normal algebraic subgroup G^0 of finite index. This means that the identity connected component G^0 is the largest connected algebraic subgroup of G containing the identity and for instance if $G = G^0$ we say that G is a *connected group*.

Furthermore if G^0 satisfy some property, then we say that G virtually satisfy such property. In this way, virtually solvability of G means solvability of G^0 and virtually abelianity of G means abelianity of G^0 (see [101]).

Theorem 1.1.2 (Lie-Kolchin)**.** *Let $G \subseteq \mathrm{GL}(2, \mathbb{C})$ be a virtually solvable group. Then G^0 is triangularizable, that is conjugate to a subgroup of upper triangular matrices.*

Definition 1.1.3. Let $G \subseteq \mathrm{GL}(2, \mathbb{C})$ be a group acting on a vector space V. We say that (the action of) G is either:

1. *Reducible* if there exists a non-trivial subspace $W \subset V$ such that $G(W) \subset W$. We say that G is *irreducible* if G is not reducible.
2. *Imprimitive* if G is irreducible and there exists subspaces V_i such that $V = V_1 \otimes \cdots \otimes V_m$, where G permutes transitively the V_i, i.e $\forall i = 1, \ldots, m$, $\forall g \in G$, $\exists j \in \{1, \ldots, m\}$ such that $g(V_i) = V_j$. We say that $V_1, \ldots, V_m$ form a system of *imprimitivity* for G.
3. *Primitive* if G is irreducible and not imprimitive.

Examples. Any subgroup of the Borel group is reducible, the infinite dihedral group is imprimitive and the groups A^{SL_2}, $S_4^{\mathrm{SL}_2}$, $A_5^{\mathrm{SL}_2}$, $\mathrm{SL}(2,\mathbb{C})$ are primitives (see [92, 101]).

Definition 1.1.4 (Differential Fields)**.** Let K (depending on x) be a commutative field of characteristic zero, ∂_x a derivation, that is a map $\partial_x : K \to K$ satisfying $\partial_x(a+b) = \partial_x a + \partial_x b$ and $\partial_x(ab) = \partial_x a \cdot b + a \cdot \partial_x b$ for all $a, b \in K$. By $\mathcal{C}$ we denote the field of constants of K

$$\mathcal{C} = \{c \in K | \partial_x c = 0\}$$

which is also of characteristic zero and will be assumed algebraically closed. In this terms, we say that K is a *differential field* with the derivation ∂.

Along this work, up to specific considerations, we consider as differential field the smallest differential containing the coefficients. Furthermore, up to special considerations, we analyze second order linear homogeneous differential equations, that is, equations in the form

$$\mathcal{L} := \partial_x^2 y + a\partial_x y + by = 0, \quad a, b \in K. \tag{1.1}$$

Definition 1.1.5 (Picard-Vessiot Extension)**.** Suppose that y_1, y_2 is a basis of solutions of $\mathcal{L}$ given in the equation (1.1), i.e., y_1, y_2 are linearly independent over K and every solution is a linear combination over $\mathcal{C}$ of these two. Let $L = K\langle y_1, y_2\rangle = K(y_1, y_2, \partial_x y_1, \partial_x y_2)$ the differential extension of K such that $\mathcal{C}$ is the field of constants for K and L. In this terms, we say that L, the smallest differential field containing to K and $\{y_1, y_2\}$, is the *Picard-Vessiot extension* of K for $\mathcal{L}$.

Definition 1.1.6 (Differential Galois Groups)**.** Let assume K, L and $\mathcal{L}$ as in previous definition. The group of all differential automorphisms (automorphisms that commutes with derivation) of L over K is called the *differential Galois group* of L over K and is denoted by $\mathrm{DGal}(L/K)$ or also by DGal_K. This means that for $\sigma \in \mathrm{DGal}(L/K)$, $\sigma(\partial_x a) = \partial_x(\sigma(a))$ for all $a \in$ Ł and $\forall a \in K$, $\sigma(a) = a$. We denote by $\mathrm{DGal}_K(\mathcal{L})$ the differential Galois group $\mathrm{DGal}(L/K)$ of the differential equation $\mathcal{L}$.

Assume that $\{y_1, y_2\}$ is a fundamental system of solutions (basis of solutions) of $\mathcal{L}$. If $\sigma \in \mathrm{DGal}_K(\mathcal{L})$ then $\{\sigma y_1, \sigma y_2\}$ is another fundamental system of $\mathcal{L}$. Hence there exists a matrix

$$A_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}),$$

such that

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sigma(y_1) \\ \sigma(y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} A_\sigma,$$

in a natural way, we can extend to systems:

$$\sigma\begin{pmatrix} y_1 & y_2 \\ \partial_x y_1 & \partial_x y_2 \end{pmatrix} = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \\ \sigma(\partial_x y_1) & \sigma(\partial_x y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ \partial_x y_1 & \partial_x y_2 \end{pmatrix} A_\sigma.$$

This defines a faithful representation $\mathrm{DGal}(L/K) \to \mathrm{GL}(2,\mathbb{C})$ and it is possible to consider $\mathrm{DGal}(L/K)$ as a subgroup of $\mathrm{GL}(2,\mathbb{C})$. It depends on the choice of the fundamental system $\{y_1, y_2\}$, but only up to conjugacy.

One of the fundamental results of the Picard-Vessiot theory is the following theorem (see [46, 48]).

Theorem 1.1.7. *The differential Galois group* $\mathrm{DGal}(L/K)$ *is an algebraic subgroup of* $\mathrm{GL}(2,\mathbb{C})$.

Examples. Consider the following differential equations:

- $\mathcal{L} := \partial_x^2 y = 0$, the basis of solutions is given by $y_1 = 1$, $y_2 = x$. If we set as differential field $K = \mathbb{C}(x)$, we can see that $\sigma(1) = 1$, $\sigma(x) = x$, then the Picard-Vessiot extension $L = K$ and for instance $\mathrm{DGal}_K(\mathcal{L}) = e$:

$$\sigma\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

  Now, if we set $K = \mathbb{C}$, then $L = K\langle x\rangle$, $\partial_x x \in \mathbb{C}$, $\partial_x(\sigma(x)) = \sigma(\partial_x x) = \sigma(1) = 1 = \partial_x x$, so $\sigma(x) = x + d$, $d \in \mathbb{C}$ and for instance $\mathrm{DGal}_K(\mathcal{L}) = \mathbb{G}_a$:

$$\sigma\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y_1 \\ dy_1 + y_2 \end{pmatrix}.$$

- $\mathcal{L} := \partial_x^2 y = \kappa y$, $\kappa \in \mathbb{C}^*$, the basis of solutions is given by $y_1 = e^{\sqrt{\kappa}x}$, $y_2 = e^{-\sqrt{\kappa}x}$, with $\kappa \neq 0$. If we set as differential field $K = \mathbb{C}(x)$, we can see that $L = K\langle e^{\sqrt{k}x}\rangle = K(e^{\sqrt{k}x})$,

$$\sigma\left(\frac{\partial_x y_1}{y_1}\right) = \frac{\partial_x(\sigma(y_1))}{\sigma(y_1)} = \frac{\partial_x y_1}{y_1}, \quad \sigma\left(\frac{\partial_x y_2}{y_2}\right) = \frac{\partial_x(\sigma(y_2))}{\sigma(y_2)} = \frac{\partial_x y_2}{y_2},$$

  $\sigma(y_1y_2) = \sigma(y_1)\sigma(y_2) = y_1y_2 = 1$, $\sigma(y_1) = cy_1$, $\sigma(y_2) = dy_2$, $c, d \in \mathbb{C}$, but $cd = 1$ and for instance $\mathrm{DGal}_K(\mathcal{L}) = \mathbb{G}_m$:

$$\sigma\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} cy_1 \\ c^{-1}y_2 \end{pmatrix},$$

  Now, if we set $K = \mathbb{C}$, we obtain the same result.

- $\mathcal{L} := \partial_x^2 y + \frac{n-1}{nx}\partial_x y = 0$, the basis of solutions is given by $y_1 = z$, where $z^n = x$, $y_2 = 1$. If we set $K = \mathbb{C}(x)$, then $L = K\left\langle x^{\frac{1}{n}}\right\rangle$, $y_1^n = x \in \mathbb{C}(x)$,

$\sigma^n(y_1) = \sigma(y_1^n) = x$, $\sigma(y_1) = cy_1$, so that $c^n = 1$ and for instance $\mathrm{DGal}_K(\mathcal{L})$ is given by:

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} cy_1 \\ y_2 \end{pmatrix}, \quad c^n = 1.$$

- $\mathcal{L} := \partial_x^2 y + \frac{n^2-1}{4n^2x^2} y = 0$, $n \in \mathbb{Z}$, the basis of solutions is given by $y_1 = x^{\frac{n+1}{2n}}$, $y_2 = x^{\frac{n-1}{2n}}$. If we set $K = \mathbb{C}(x)$ and n even, then $L = K\left\langle x^{\frac{1}{2n}} \right\rangle$,

$$\sigma(y_1) = cy_1, \quad \sigma^{2n}(y_1) = c^{2n} y_1^{2n} = \sigma(y_1^{2n}) = y_1^{2n}, \quad c^{2n} = 1,$$

$$\sigma(y_2) = dy_2, \quad \sigma^{2n}(y_2) = d^{2n} y_2^{2n} = \sigma(y_2^{2n}) = y_2^{2n}, \quad d^{2n} = 1,$$

  $\sigma(y_1y_2) = y_1y_2 = \sigma(y_1)\sigma(y_2) = cdy_1y_2$ so that $cd = 1$ and for instance $\mathrm{DGal}_K(\mathcal{L}) = \mathbb{G}^{[2n]}$:

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} cy_1 \\ c^{-1}y_2 \end{pmatrix}, \quad c^{2n} = 1, \quad n > 1.$$

  Now, if we consider n odd, then $L = K\left\langle x^{\frac{1}{n}} \right\rangle$, and $\mathrm{DGal}_K(\mathcal{L}) = \mathbb{G}^{[2n]}$.

- Cauchy-Euler equation

$$\mathcal{L} := \partial_x^2 y = \frac{m(m+1)}{x^2} y, \quad m \in \mathbb{C},$$

  the basis of solutions is $y_1 = x^{m+1}$, $y_2 = x^{-m}$. Setting $K = \mathbb{C}(x)$, we have the following possible cases:

  - for $m \in \mathbb{Z}$, $L = K$ and $\mathrm{DGal}_K(\mathcal{L}) = e$,
  - for $m \in \mathbb{Q} \setminus \mathbb{Z}$, $L = K(x^m)$ and $\mathrm{DGal}_K(\mathcal{L}) = \mathbb{G}^{[d]}$, where $m = n/d$,
  - for $m \in \mathbb{C} \setminus \mathbb{Q}$, $L = K(x^m)$ and $\mathrm{DGal}_K(\mathcal{L}) = \mathbb{G}_m$.

Definition 1.1.8 (Integrability)**.** Let consider the linear differential equation $\mathcal{L}$ such as in equation (1.1). We say that $\mathcal{L}$ is *integrable* if the Picard-Vessiot extension $L \supset K$ is obtained as a tower of differential fields $K = L_0 \subset L_1 \subset \cdots \subset L_m = L$ such that $L_i = L_{i-1}(\eta)$ for $i = 1, \ldots, m$, where either

1. η is *algebraic* over L_{i-1}, that is η satisfies a polynomial equation with coefficients in L_{i-1}.
2. η is *primitive* over L_{i-1}, that is $\partial_x \eta \in L_{i-1}$.
3. η is *exponential* over L_{i-1}, that is $\partial_x \eta / \eta \in L_{i-1}$.

We recall that the differential field of coefficients has been fixed before, i.e., the smallest differential field containing the coefficients.

We remark that the usual terminology in differential algebra for integrable equations is that the corresponding Picard-Vessiot extensions are called *Liouvillian.*

Theorem 1.1.9 (Kolchin)**.** *The equation* $\mathcal{L}$ *given in* (1.1) *is integrable if and only if* $\mathrm{DGal}_K(\mathcal{L})$ *is virtually solvable.*

Let consider the differential equation

$$\mathcal{L} := \partial_x^2\zeta = r\zeta, \quad r \in K. \tag{1.2}$$

We recall that the equation (1.2) can be obtained from the equation (1.1) through the change of variable

$$y = e^{-\frac{1}{2}\int a}\zeta, \quad r = \frac{a^2}{4} + \frac{\partial_x a}{2} - b \tag{1.3}$$

and the equation (1.2) is called the *reduced form* of the equation (1.1).

On the other hand, introducing the change of variable $v = \partial_x\zeta/\zeta$ we get the associated Riccati equation to the equation (1.2)

$$\partial_x v = r - v^2, \quad v = \frac{\partial_x\zeta}{\zeta}, \tag{1.4}$$

where r is obtained by equation (1.3).

Theorem 1.1.10 (Singer 1981, [81])**.** *The Riccati equation* (1.4) *has one algebraic solution over the differential field* K *if and only if the differential equation* (1.2) *is integrable.*

For $\mathcal{L}$ given by the equation (1.2), it is very well known (see [46, 48, 92]) that $\mathrm{DGal}_K(\mathcal{L})$ is an algebraic subgroup of $\mathrm{SL}(2,\mathbb{C})$. The well known classification of subgroups of $\mathrm{SL}(2,\mathbb{C})$ (see [46, p.31], [49, p.7,27]) is the following.

Theorem 1.1.11. *Let* G *be an algebraic subgroup of* $\mathrm{SL}(2,\mathbb{C})$*. Then, up to conjugation, one of the following cases occurs.*

1. $G \subseteq \mathbb{B}$ *and then* G *is reducible and triangularizable.*
2. $G \nsubseteq \mathbb{B}$, $G \subseteq \mathbb{D}_\infty$ *and then* G *is imprimitive.*
3. $G \in \{A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$ *and then* G *is primitive (finite)*
4. $G = \mathrm{SL}(2,\mathbb{C})$ *and then* G *is primitive (infinite).*

Definition 1.1.12. Let consider the differential equation $\mathcal{L}$ given by the equation (1.2). Let $\{\zeta_1,\zeta_2\}$ be a fundamental system of $\mathcal{L}$. Let $f = f(Y_1,Y_2) \in \mathcal{C}[Y_1,Y_2]$ be a homogeneous polynomial, we say that:

1. The polynomial f is *invariant* with respect to $\mathcal{L}$ if its evaluation on a $\mathcal{C}$-basis $\{\zeta_1, \zeta_2\}$ of solutions is invariant under the action of $\mathrm{DGal}_K(\mathcal{L})$, that is, for every $\sigma \in \mathrm{DGal}_K(\mathcal{L})$, $\sigma h(x) = h(x)$, where $h(x) = f(\zeta_1(x), \zeta_2(x)) \in K$. The function $h(x)$ is called the *value* of the invariant polynomial f.

2. The polynomial f is a *semi-invariant* with respect to $\mathcal{L}$ if the logarithmic derivative $\frac{\partial_x h}{h}$ of its evaluation $h(x) = f(\zeta_1(x), \zeta_2(x))$ on any $\mathcal{C}$-basis $\{\zeta_1, \zeta_2\}$ is an element of K, that is, for every $\sigma \in \mathrm{DGal}_K(\mathcal{L})$, $\sigma\theta = \theta$, where $\theta = \partial_x h(x)/h(x) \in K$.

Theorem 1.1.13 (Kovacic, [49]). *Let $\{\zeta_1, \zeta_2\}$ be a fundamental system of solutions of $\mathcal{L}$ given by the differential equation* (1.2)*. Then, for some $i \in \{1, 2\}$ and for every $\sigma \in \mathrm{DGal}_K(\mathcal{L})$, exclusively one of the following cases holds.*

1. $\mathrm{DGal}_K(\mathcal{L})$ *is reducible and then $f = \zeta_i$ is semi-invariant with respect to $\mathcal{L}$, i.e $\partial_x(\ln \zeta_i) \in K$.*

2. $\mathrm{DGal}_K(\mathcal{L})$ *is imprimitive and then $f_1 = \zeta_1\zeta_2$ is semi-invariant with respect to $\mathcal{L}$, $f_2 = (\zeta_1\zeta_2)^2$ is invariant with respect to $\mathcal{L}$, i.e $[K\langle\partial_x(\ln \zeta_i)\rangle : K] = 2$.*

3. $\mathrm{DGal}_K(\mathcal{L})$ *is finite primitive and then the invariants with respect to $\mathcal{L}$ is either $f_1 = (\zeta_1^4 + 8\zeta_1\zeta_2^3)^3$, or $f_2 = (\zeta_1^5\zeta_2 - \zeta_1\zeta_2^5)^2$ or $f_3 = \zeta_1^{11}\zeta_2 - 11\zeta_1^6\zeta_2^6 - \zeta_1\zeta_2^{11}$, i.e $[K\langle\partial_x(\ln \zeta_i)\rangle : K] = 4, 6, 12$.*

4. $\mathrm{DGal}_K(\mathcal{L})$ *is infinite primitive, i.e there are no non-trivial semi-invariants.*

Statements and proofs of the theorems 1.1.9, 1.1.10 and 1.1.11 can be found in [92].

1.1.2 Kovacic's Algorithm

Considering $K = \mathbb{C}(x)$, $\mathcal{C} = \mathbb{C}$ in the theorems 1.1.9, 1.1.11 and 1.1.13, Kovacic in 1986 ([49]) introduced an algorithm to solve the differential equation (1.2) and show that (1.2) is integrable if and only if the solution of the Riccati equation (1.4) is a rational function (case 1), is a root of polynomial of degree two (case 2) or is a root of polynomial of degree 4, 6, or 12 (case 3). For more details see reference [49]. Improvements for this algorithm are given in references [33, 88]. Here, we follow the original version given by Kovacic in reference [49] with an adapted version given in reference [5].

Each case in Kovacic's algorithm is related with each one of the algebraic subgroups of $\mathrm{SL}(2, \mathbb{C})$ and the associated Riccati equation

$$\partial_x v = r - v^2 = \left(\sqrt{r} - v\right)\left(\sqrt{r} + v\right), \quad v = \frac{\partial_x \zeta}{\zeta}.$$

According to Theorem 1.1.11, there are four cases in Kovacic's algorithm. Only for cases 1, 2 and 3 we can solve the differential equation, but for the case 4 the differential equation is not integrable. It is possible that Kovacic's algorithm can provide us only one solution (ζ_1), so that we can obtain the second solution (ζ_2) through

$$\zeta_2 = \zeta_1 \int \frac{dx}{\zeta_1^2}. \tag{1.5}$$

Notations. For the differential equation given by

$$\partial_x^2 \zeta = r\zeta, \qquad r = \frac{s}{t}, \quad s, t \in \mathbb{C}[x],$$

we use the following notations.

1. Denote by Γ' be the set of (finite) poles of r, $\Gamma' = \{c \in \mathbb{C} : t(c) = 0\}$.
2. Denote by $\Gamma = \Gamma' \cup \{\infty\}$.
3. By the order of r at $c \in \Gamma'$, $\circ(r_c)$, we mean the multiplicity of c as a pole of r.
4. By the order of r at ∞, $\circ(r_\infty)$, we mean the order of ∞ as a zero of r. That is $\circ(r_\infty) = \deg(t) - \deg(s)$.

The four cases

Case 1. In this case $[\sqrt{r}]_c$ and $[\sqrt{r}]_\infty$ means the Laurent series of $\sqrt{r}$ at c and the Laurent series of $\sqrt{r}$ at ∞ respectively. Furthermore, we define $\varepsilon(p)$ as follows: if $p \in \Gamma$, then $\varepsilon(p) \in \{+, -\}$. Finally, the complex numbers $\alpha_c^+, \alpha_c^-, \alpha_\infty^+, \alpha_\infty^-$ will be defined in the first step. If the differential equation has not poles it only can fall in this case.

Step 1. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

(c_0) If $\circ(r_c) = 0$, then

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = 0.$$

(c_1) If $\circ(r_c) = 1$, then

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = 1.$$

(c_2) If $\circ(r_c) = 2$, and

$$r = \cdots + b(x-c)^{-2} + \cdots, \quad \text{then}$$

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(c_3) If $\circ(r_c) = 2v \geq 4$, and

$$r = (a\,(x-c)^{-v} + ... + d\,(x-c)^{-2})^2 + b(x-c)^{-(v+1)} + \cdots, \quad \text{then}$$

$$\left[\sqrt{r}\right]_c = a\,(x-c)^{-v} + ... + d\,(x-c)^{-2}, \quad \alpha_c^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a} + v\right).$$

(∞_1) If $\circ(r_\infty) > 2$, then

$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^{+} = 0, \quad \alpha_\infty^{-} = 1.$$

(∞_2) If $\circ(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$, then

$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^{\pm} = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(∞_3) If $\circ(r_\infty) = -2v \leq 0$, and

$$r = (ax^v + ... + d)^2 + bx^{v-1} + \cdots, \quad \text{then}$$

$$\left[\sqrt{r}\right]_\infty = ax^v + ... + d, \quad \textit{and} \quad \alpha_\infty^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a} - v\right).$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ n \in \mathbb{Z}_+ : n = \alpha_\infty^{\varepsilon(\infty)} - \sum_{c\in\Gamma'} \alpha_c^{\varepsilon(c)}, \forall\,(\varepsilon\,(p))_{p\in\Gamma} \right\}.$$

If $D = \emptyset$, then we should start with the case 2. Now, if $\mathrm{Card}(D) > 0$, then for each $n \in D$ we search $\omega \in \mathbb{C}(x)$ such that

$$\omega = \varepsilon\,(\infty)\left[\sqrt{r}\right]_\infty + \sum_{c\in\Gamma'}\left(\varepsilon\,(c)\left[\sqrt{r}\right]_c + \alpha_c^{\varepsilon(c)}(x-c)^{-1}\right).$$

Step 3. For each $n \in D$, search for a monic polynomial P_n of degree n with

$$\partial_x^2 P_n + 2\omega\partial_x P_n + (\partial_x\omega + \omega^2 - r)P_n = 0. \tag{1.6}$$

If success is achieved then $\zeta_1 = P_n e^{\int \omega}$ is a solution of the differential equation. Else, Case 1 cannot hold.

Case 2. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Gamma'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Gamma'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1) If $\circ(r_c) = 1$, then $E_c = \{4\}$

(c_2) If $\circ(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{2 + k\sqrt{1+4b} : k = 0, \pm 2\right\}.$$

(c_3) If $\circ(r_c) = v > 2$, then $E_c = \{v\}$

(∞_1) If $\circ(r_\infty) > 2$, then $E_\infty = \{0, 2, 4\}$

(∞_2) If $\circ(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$, then

$$E_\infty = \left\{2 + k\sqrt{1+4b} : k = 0, \pm 2\right\}.$$

(∞_3) If $\circ(r_\infty) = v < 2$, then $E_\infty = \{v\}$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ n \in \mathbb{Z}_+ : \quad n = \frac{1}{2}\left(e_\infty - \sum_{c \in \Gamma'} e_c\right), \forall e_p \in E_p, \quad p \in \Gamma \right\}.$$

If $D = \emptyset$, then we should start the case 3. Now, if $\mathrm{Card}(D) > 0$, then for each $n \in D$ we search a rational function θ defined by

$$\theta = \frac{1}{2}\sum_{c \in \Gamma'} \frac{e_c}{x-c}.$$

Step 3. For each $n \in D$, search a monic polynomial P_n of degree n, such that

$$\partial_x^3 P_n + 3\theta\partial_x^2 P_n + (3\partial_x\theta + 3\theta^2 - 4r)\partial_x P_n + \left(\partial_x 2\theta + 3\theta\partial_x\theta + \theta^3 - 4r\theta - 2\partial_x r\right) P_n = 0. \tag{1.7}$$

If P_n does not exist, then Case 2 cannot hold. If such a polynomial is found, set $\phi = \theta + \partial_x P_n / P_n$ and let ω be a solution of

$$\omega^2 + \phi\omega + \frac{1}{2}\left(\partial_x\phi + \phi^2 - 2r\right) = 0.$$

Then $\zeta_1 = e^{\int \omega}$ is a solution of the differential equation.

Case 3. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Gamma'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Gamma'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1) If $\circ\left(r_c\right) = 1$, then $E_c = \{12\}$

(c_2) If $\circ\left(r_c\right) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{6 + k\sqrt{1+4b} : \quad k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}.$$

(∞) If $\circ\left(r_\infty\right) = v \geq 2$, and $r = \cdots + bx^2 + \cdots$, then

$$E_\infty = \left\{6 + \frac{12k}{m}\sqrt{1+4b} : \; k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}, \; m \in \{4, 6, 12\}.$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{n \in \mathbb{Z}_+ : \quad n = \frac{m}{12}\left(e_\infty - \sum_{c \in \Gamma'} e_c\right), \forall e_p \in E_p, \quad p \in \Gamma\right\}.$$

In this case we start with $m = 4$ to obtain the solution, afterwards $m = 6$ and finally $m = 12$. If $D = \emptyset$, then the differential equation is not integrable because it falls in the case 4. Now, if $\mathrm{Card}(D) > 0$, then for each $n \in D$ with its respective m, search a rational function

$$\theta = \frac{m}{12}\sum_{c \in \Gamma'} \frac{e_c}{x-c}$$

and a polynomial S defined as

$$S = \prod_{c \in \Gamma'} (x-c).$$

Step 3. Search for each $n \in D$, with its respective m, a monic polynomial $P_n = P$ of degree n, such that its coefficients can be determined recursively by

$$P_{-1} = 0, \quad P_m = -P,$$

$$P_{i-1} = -S\partial_x P_i - \left(\left(m-i\right)\partial_x S - S\theta\right) P_i - \left(m-i\right)\left(i+1\right) S^2 r P_{i+1},$$

where $i \in \{0, 1 \ldots, m-1, m\}$. If P does not exist, then the differential equation is not integrable because it falls in Case 4. Now, if P exists search ω such that

$$\sum_{i=0}^{m} \frac{S^i P}{(m-i)!}\omega^i = 0,$$

then a solution of the differential equation is given by

$$\zeta = e^{\int \omega},$$

where ω is solution of the previous polynomial of degree m.

Some remarks on Kovacic's algorithm (see [5])

Here we assume that the differential equation falls only in one of the four cases.

Remark 1.1.14. **(Case 1).** If the differential equation falls in case 1, then its Galois group is given by one of the following groups:

I1 e when the algorithm provides two rational solutions or only one rational solution and the second solution obtained by (1.5) has not logarithmic term.

I2 $\mathbb{G}^{[n]}$ when the algorithm provides two algebraic solutions ζ_1, ζ_2 such that $\zeta_1^n, \zeta_2^n \in \mathbb{C}(x)$ and $\zeta_1^{n-1}, \zeta_2^{n-1} \notin \mathbb{C}(x)$.

I3 $\mathbb{G}^{\{n\}}$ when the algorithm provides only one algebraic solution ζ such that $\zeta^n \in \mathbb{C}(x)$ and $\zeta^{n-1} \notin \mathbb{C}(x)$.

I4 $\mathbb{G}_m$ when the algorithm provides two non-algebraic solutions.

I5 $\mathbb{G}_a$ when the algorithm provides one rational solution and the second solution is not algebraic.

I6 $\mathbb{B}$ when the algorithm only provides one solution ζ such that ζ and its square are not rational functions.

Kovacic's Algorithm in Maple

In order to analyze second order linear differential equations with rational coefficients, generally without parameters, a standard procedure is using MAPLE, and especially commands `dsolve` and `kovacicsols`. Whenever the command `kovacicsols` yields an output "`[ ]`", it means that the second order linear differential equation being considered is not integrable, and thus its Galois group is non-virtually solvable.

In some cases, moreover, `dsolve` makes it possible to obtain the solutions in terms of special functions such as *Airy functions*, *Bessel functions* and **hypergeometric functions**, among others (see [1]).

There is a number of second order linear equations whose coefficients are not rational, and whose solutions MAPLE can find with the command `dsolve` but the presentation of the solutions is very complicated, furthermore the command `kovacicsols` does not work with such coefficients. These problems, in some cases, can be solved by our *algebrization method* (see section 2.3.1 and see also [5]).

Beyond Kovacic's Algorithm

According to the work of Michael Singer [81] we can have another perspective of the Kovacic's algorithm by means of $\mathcal{L}^{\circledS m}$, the m-th **symmetric power** of a linear differential equation $\mathcal{L}$ (see Michael Singer and Felix Ulmer, [84]).

Theorem 1.1.15. *Let $\mathcal{L}$ be a linear homogeneous differential equation of arbitrary order n. For any $m \geq 1$ there is another linear homogeneous differential equation, denoted by $\mathcal{L}^{\circledS m}$, with the following property. If $\zeta_1, \ldots, \zeta_n$ are any solutions of $\mathcal{L}$ then any homogeneous polynomial in $\zeta_1, \ldots, \zeta_n$ of degree m is a solution of $\mathcal{L}^{\circledS m}$.*

The Kovacic algorithm then can be stated as follows (see also [33, 88]).

Algorithm. Consider $\mathcal{L} = \partial_x^2 \zeta - r\zeta$.

Step 1. Check if $\mathcal{L}$ is integrable, where for a solution ζ, there exists u such that $u = \partial_x \zeta / \zeta \in \mathbb{C}(x)$. If so then $\zeta = e^{\int u}$.

Step 2. Check if $\mathcal{L}^{\circledS 2}$ is integrable, where for a solution ζ there exists u such that $u = \partial_x \zeta / \zeta \in \mathbb{C}(x)$. If so, let v be a root of

$$v^2 + uv + \left(\frac{1}{2}\partial_x u + \frac{1}{2}u^2 - u\right) = 0.$$

Then $\zeta = e^{\int v}$.

Step 3. Check if $\mathcal{L}^{\circledS 4}$, $\mathcal{L}^{\circledS 6}$, or $\mathcal{L}^{\circledS 12}$ is integrable, where for a solution ζ there exists u such that $u = \partial_x \zeta / \zeta \in \mathbb{C}(x)$. If so then there is a polynomial of degree 4, 6 or 12 (respectively) such that if v is a solution of it then $\zeta = e^{\int v}$.

This algorithm can be generalized using the results of Michael Singer in [81]. The trick is to find the correct numbers (like $2, 4, 6, 12$ of the Kovacic's algorithm).

Theorem 1.1.16. *Suppose a linear homogeneous differential equation of order n is integrable. Then it has a solution of the form*

$$\zeta = e^{\int v}$$

where ω is algebraic over $\mathbb{C}(x)$. The degree of v is bounded by $I(n)$, which is defined inductively by

$$\begin{aligned}
&I(0) = 1 \\
&I(n) = \max\{J(n),\ n!I(n-1)\} \\
&J(n) = \left(\sqrt{8n}+1\right)^{2n^2} - \left(\sqrt{8n}-1\right)^{2n^2}.
\end{aligned}$$

Theorem 1.1.17. *If a linear differential equation $\mathcal{L}$ has a solution of the form $\zeta = e^{\int v}$ where v is algebraic over $\mathbb{C}(x)$ of degree m, then $\mathcal{L}^{\circledS m}$ has a solution ζ with $v = \partial_x \zeta / \zeta \in \mathbb{C}(x)$. v is a solution of a generalized Riccati equation.*

This algorithm is not considered implementable; the numbers $I(n)$ are simply much too large. For example $I(2) = 384064$, so this algorithm would require checking if $\mathcal{L}^{\circledS m}$ has a solution ζ with $\partial_x \zeta / \zeta \in \mathbb{C}(x)$ for $m = 1, 2, \ldots, 384064$.

However, the implementation of this algorithm do exist for order 3, due to Michael Singer and Felix Ulmer [85], and order 4, due to Sabrina Hessinger [39].

1.1.3 Eigenrings

We consider two different formalisms for Eigenrings, the matrix and operators formalism. We start with the Matrix formalism of Eigenrings following M. Barkatou in [7], but restricting again to 2×2 matrices.

Let K be a differential field and let A be a matrix in $\mathrm{GL}(2, K)$ such that,

$$\partial_x \mathbf{X} = -A\mathbf{X}. \tag{1.8}$$

Consider a matrix equation (1.8) and let $P \in \mathrm{GL}(2, K)$. The substitution $\mathbf{X} = P\mathbf{Y}$ leads to the matrix equation

$$\partial_x \mathbf{Y} = -B\mathbf{Y}, \quad B = P^{-1}(\partial_x P + AP). \tag{1.9}$$

Definition 1.1.18. The matrices A and B are *equivalents* over K, denoted by $A \sim B$, when there exists a matrix $P \in \mathrm{GL}(2, K)$ satisfying the equation (1.9). The systems (1.8) and (1.9) are equivalents, denoted by $[A] \sim [B]$, when A and B are equivalents.

By the equation (1.9) we have $PB = \partial_x P + AP$. In general, assuming $PB = PA$, where P is a 2×2 matrix, i.e, P is not necessarily in $\mathrm{GL}(2, K)$, we obtain $PA = \partial_x P + AP$, which lead us to the following definition.

Definition 1.1.19. The Eigenring of the system $[A]$, denoted by $\mathcal{E}(A)$, is the set of 2×2 matrices P in K satisfying

$$\partial_x P = PA - AP. \tag{1.10}$$

The equation (1.10) can be viewed as a system of 4 first-order linear differential equations over K. Thus, $\mathcal{E}(A)$ is a $\mathcal{C}-$vector space of finite dimension ≤ 4. Owing to the product of two elements of $\mathcal{E}(A)$ is also an element of $\mathcal{E}(A)$ and the identity matrix I_2 belongs to $\mathcal{E}(A)$, we have that $\mathcal{E}(A)$ is an algebra over $\mathcal{C}$, i.e., $\mathcal{E}(A)$ is a $\mathcal{C}-$algebra. As a consequence, we have the following results that can be found in [7].

Proposition 1.1.20. *Any element P of $\mathcal{E}(A)$ has:*

- *a minimal polynomial with coefficients in $\mathcal{C}$ and*
- *all its eigenvalues in $\mathcal{C}$.*

Proposition 1.1.21. *If two systems $[A]$ and $[B]$ are equivalent, their eigenrings $\mathcal{E}(A)$ and $\mathcal{E}(B)$ are isomorphic as $\mathcal{C}-$algebras. In particular, one has* $\dim_{\mathcal{C}} \mathcal{E}(A) = \dim_{\mathcal{C}} \mathcal{E}(B)$.

Definition 1.1.22. The system $[A]$ is called *reducible* when $A \sim B$, being B given by

$$B = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix}.$$

When $[A]$ is reducible and $b_{21} = 0$, the system $[A]$ is called *decomposable* or *completely reducible*. The system $[A]$ is called *irreducible* or *indecomposable* when $[A]$ is not reducible.

Let assume that the eigenring $\mathcal{E}(A)$ is known.

Theorem 1.1.23. *If $\mathcal{E}(A)$ is not a division ring then $[A]$ is reducible and the reduction can be carried out by a matrix $P \in \mathrm{GL}(2, K)$ that can be computed explicitly.*

Let us note that the condition $\mathcal{E}(A)$ *is not a division ring* implies $\dim_{\mathcal{C}} \mathcal{E}(A) > 1$. Indeed, if $P \in \mathcal{E}(A)\setminus\{0\}$ is not invertible, then the family $\{I, P\}$ is linearly independent (over $\mathcal{C}$) and hence $\dim_{\mathcal{C}} \mathcal{E}(A) > 1$. In our case the converse is true, due to the field of constants $\mathcal{C}$ is algebraically closed. Indeed, suppose that $\dim_{\mathcal{C}} \mathcal{E}(A) > 1$ then there exists $P \in \mathcal{E}(A)$ such that the family $\{I, P\}$ be linearly independent. Since $\mathcal{C}$ is algebraically closed, there exists $\lambda \in \mathcal{C}$ such that $\det(P - \lambda I) = 0$. Hence $\mathcal{E}(A)$ contains an element, namely $P - \lambda I$, which is non-zero and non invertible.

The computation of eigenrings of the system $[A]$ is implemented in **ISOLDE** (Integration of Systems of Ordinary Linear Differential Equations). The function is `eigenring`, the calling sequence is `eigenring(A, x)` being the parameters: A - a square rational function matrix with coefficients in an algebraic extension of the rational numbers and x - the independent variable (a name). ISOLDE was written in Maple V and it is available at `http://isolde.sourceforge.net/`.

In operators formalism we restrict ourselves to second order differential operators and we follows the works of Singer, Barkatou and Van Hoeij (see [83, 7, 93, 94, 95]). A differential equation $\mathcal{L} := \partial_x^2 y + a\partial_x y + by = 0$ with $a, b \in K$ corresponds to a differential operator $f = \partial_x^2 + a\partial_x + b$ acting on y. The differential operator f is an element of the non-commutative ring $K[\partial_x]$.

The factorization of operators is very important to solve differential equations, that is, a factorization $f = \mathfrak{L}\mathfrak{R}$ where $\mathfrak{L}, \mathfrak{R} \in K[\partial_x]$ is useful for computing solutions of f because solutions of the right-hand factor $\mathfrak{R}$ are solutions of f as well.

Definition 1.1.24. Let $\mathfrak{L}$ be a second order differential operator, i.e $\mathcal{L} := \mathfrak{L}(y) = 0$. Denote $V(\mathfrak{L})$ as the solution space of $\mathcal{L}$. The *Eigenring* of $\mathfrak{L}$, denoted by $\mathcal{E}(\mathfrak{L})$, is a basis (a vector space) of the set of all operators $\mathfrak{R}$ for which $\mathfrak{R}(V(\mathfrak{L}))$ is a subset of $V(\mathfrak{L})$, that is $\mathfrak{L}\mathfrak{R} = \mathfrak{S}\mathfrak{L}$, where $\mathfrak{S}$ is also an operator.

As consequence of the previous definition, $\mathfrak{R}$ is an endomorphism of the solution space $V(\mathfrak{L})$. The characteristic polynomial of this map can be computed with the classical methods in linear algebra. For endomorphisms $\mathfrak{R}$, the product of $\mathfrak{L}$ and $\mathfrak{R}$ is divisible on the right by $\mathfrak{L}$. This means that if $\mathfrak{L}(y) = 0$, then $\mathfrak{L}(\mathfrak{R}(y)) = 0$, so that $\mathfrak{R}$ map $V \to V$.

For the general case of operators

$$\mathfrak{L} = \sum_{k=0}^{n} b_k \partial_x^k, \quad \mathfrak{R} = \sum_{i=0}^{m} a_i \partial_x^i, \quad a_i, b_i \in K, \quad n > m, \quad \mathcal{L} := \mathfrak{L}(y) = 0$$

and $G = \mathrm{DGal}_K(\mathcal{L})$, we can see $\mathfrak{R}$ as a G-map. Now, denoting by P the characteristic polynomial of $\mathfrak{R}$, always there exists polynomials P_1, P_2 with $\gcd(P_1, P_2) = 1$ such that $P = P_1P_2$. This means that we can think in $\mathfrak{R}$ as a linear map $V \to V$ and choosing one basis of $V \to \mathfrak{R}$ we can see, by elementary linear algebra, that $\mathfrak{R}$ has a matrix $M_{\mathfrak{R}}$. By Cayley-Hamilton theorem we have that $P(M_{\mathfrak{R}}) = 0$ and by kernel theorem we have that $V = \ker(P_1(M_{\mathfrak{R}})) \oplus \ker(P_2(M_{\mathfrak{R}}))$, in where $\ker(P_1(M_{\mathfrak{R}}))$ and $\ker(P_2(M_{\mathfrak{R}}))$ are invariants under G.

Assuming λ eigenvalue of $\mathfrak{R}$, there exists a non-trivial eigenspace $V_\lambda \subseteq V$, which means that $\mathfrak{L}$ and $\mathfrak{R} - \lambda$ has common solutions and therefore right $-$ $\gcd(\mathfrak{L}, \mathfrak{R} - \lambda)$ is non-trivial, this means that right $- \gcd(\mathfrak{L}, \mathfrak{R} - \lambda)$ it is a factor of $\mathfrak{L}$.

Returning to the second order operators, we establish the relationship between the Eigenring of the system $[A]$ and the Eigenring of the operator $\mathfrak{L}$. We start recalling that $\mathcal{L}$ given by $\partial_x^2 y + a\partial_x y + by = 0$, $a, b \in K$, can be written as the system

$$\partial_x \begin{pmatrix} y \\ \partial_x y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y \\ \partial_x y \end{pmatrix},$$

and the system of linear differential equations

$$\partial_x \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad a, b, c, d \in K,$$

by means of an elimination process, is equivalent to the second-order equation

$$\partial_x^2 y - \left(a + d + \frac{\partial_x b}{b}\right) \partial_x y - \left(\partial_x a + bc - ad - a\frac{\partial_x b}{b}\right) y = 0. \qquad (1.11)$$

In this way, we can go from operators to systems and reciprocally computing the Eigenrings in both formalism. In particular, we emphasize in the operator $\mathfrak{L} = \partial_x^2 + p\partial_x + q$, which is equivalent to the system $[A]$, where A is given by

$$A = \begin{pmatrix} 0 & -1 \\ q & p \end{pmatrix} \quad p, q \in K.$$

Lemma 1.1.25. *Let consider $\mathfrak{L}$, A and P as follows:*

$$\mathfrak{L} = \partial_x^2 + p\partial_x + q, \quad A = \begin{pmatrix} 0 & -1 \\ q & p \end{pmatrix}, \quad P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d, p, q \in K.$$

The following statements holds

1. *If $P \in \mathcal{E}(A)$, then $\mathfrak{R} = a + b\partial_x \in \mathcal{E}(\mathfrak{L})$.*
2. *If $\mathfrak{R} = a + b\partial_x \in \mathcal{E}(\mathfrak{L})$, then $P \in \mathcal{E}(A)$, where P is given by*

$$P = \begin{pmatrix} a & b \\ \partial_x a - bq & a + \partial_x b - bp \end{pmatrix}.$$

3. $1 \leq \dim_{\mathcal{C}} \mathcal{E}(\mathfrak{L}) \leq 4$.
4. $P \in \mathrm{GL}(2, K) \Leftrightarrow \frac{\partial_x a}{a} - \frac{a}{b} + p \neq \frac{\partial_x b}{b} - \frac{b}{a} q$.

Remark 1.1.26. Let $\mathfrak{L}$ be the differential operator $\partial_x^2 + b$, where $b \in K$, $\mathcal{L} := \mathfrak{L}(y) = 0$. The dimension of the eigenring of $\mathfrak{L}$ is related with:

- the number of solutions over K of the differential equation $\mathcal{L}$ and its second symmetric power $\mathcal{L}^{\circledS 2}$ and
- the type of differential Galois group (see [7, 83, 93, 94, 95]).

The previous remark is detailed in the following lemma.

Lemma 1.1.27. *Let assume $\mathfrak{L} = \partial_x^2 + b$, where $b \in K$, $\mathcal{L} := \mathfrak{L}(y) = 0$. The following statements holds.*

1. *If $\dim_{\mathcal{C}} \mathcal{E}(\mathfrak{L}) = 1$, then either differential Galois group is irreducible ($\mathbb{D}_\infty$, primitive or $\mathrm{SL}(2,\mathbb{C})$) or indecomposable ($G \subseteq \mathbb{B}$, $G \notin \{e, \mathbb{G}_m, \mathbb{G}_a, \mathbb{G}^{\{n\}}, \mathbb{G}^{[n]}\}$).*
2. *If $\dim_{\mathcal{C}} \mathcal{E}(\mathfrak{L}) = 2$, then either, the differential Galois group is the additive group or is contained in the multiplicative group, but never will be the identity group. In this case we can have two solutions but not over the differential field K.*
3. *If $\dim_{\mathcal{C}} \mathcal{E}(\mathfrak{L}) = 4$, then the differential Galois group is the identity group. In this case we have 2 independent solutions ζ_1 and ζ_2 in which ζ_1^2, ζ_2^2 and $\zeta_1\zeta_2$ are elements of the differential field K, i.e. the solutions of $\mathcal{L}^{\circledS 2}$ belongs to K.*

The eigenring for a differential operator $\mathfrak{L}$ has been implemented in Maple. The function is `eigenring`, the calling sequences are `eigenring(L,domain)` and `endomorphism-charpoly(L, R, domain)`, where $\mathfrak{L}$ is a differential operator, $\mathfrak{R}$ is the differential operator in the output of eigenring. The argument `domain` describes the differential algebra. If this argument is the list `[Dx,x]` then the differential operators are notated with the symbols `Dx` and `x`, where `Dx` is the operator ∂_x.

Example. Let consider $K = \mathbb{C}(x)$, $\mathfrak{L} = \partial_x^2 - \frac{6}{x^2}$ and $[A]$ where A is given by

$$A = \begin{pmatrix} 0 & -1 \\ -6x^{-2} & 0 \end{pmatrix}.$$

The Eigenring of $[A]$ and the Eigenring of $\mathfrak{L}$ are given by

$$\mathcal{E}(A) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & x \\ 6x^{-1} & 0 \end{pmatrix}, \begin{pmatrix} -3x^5 & x^6 \\ -9x^4 & 3x^5 \end{pmatrix}, \begin{pmatrix} 2x^{-5} & x^{-4} \\ -4x^{-6} & -2x^{-5} \end{pmatrix} \right\},$$

$$\mathcal{E}(\mathfrak{L}) = \left\{ 1, x\partial_x - 1, x^6\partial_x - 3x^5, \frac{\partial_x}{x^4} + \frac{2}{x^5} \right\}.$$

1.1.4 Riemann's Equation

The Riemann's equation is an important differential equation which has been studied for a long time, since Gauss, Riemann, Schwartz, etc., see for example [45, 69]. We are interested in the relationship with the Picard-Vessiot theory. Thus, we follows the works of Kimura [47], Martinet & Ramis [57] and Duval & Loday-Richaud [33].

Definition 1.1.28. The *Riemann's equation* is an homogeneous ordinary linear differential equation of the second order over the Riemann's sphere with at most three singularities which are of the regular type. Assuming a, b and c as regular singularities, the Riemann's equation may be written in the form

$$\begin{aligned} &\partial_x^2 y + \left(\frac{1-\rho-\rho'}{x-a} + \frac{1-\sigma-\sigma'}{x-b} + \frac{1-\tau-\tau'}{x-c} \right) \partial_x y \\ &+ \left(\frac{\rho\rho'(a-b)(a-c)}{(x-a)^2(x-b)(x-c)} + \frac{\sigma\sigma'(b-a)(b-c)}{(x-b)^2(x-a)(x-c)} + \frac{\tau\tau'(c-a)(c-b)}{(x-c)^2(x-a)(x-b)} \right) y = 0, \end{aligned} \tag{1.12}$$

where (ρ, ρ'), (σ, σ') and (τ, τ') are the exponents at the singular points a, b, c respectively and must satisfy the Fuchs relation $\rho + \rho' + \sigma + \sigma' + \tau + \tau' = 1$. The quantities $\rho' - \rho$, $\sigma' - \sigma$ and $\tau' - \tau$ are called the *exponent differences* of the Riemann's equation (1.12) at a, b and c respectively and are denoted by $\widetilde{\lambda}$, $\widetilde{\mu}$ and $\widetilde{\nu}$ as follows:

$$\widetilde{\lambda} = \rho' - \rho, \quad \widetilde{\mu} = \sigma' - \sigma, \quad \widetilde{\nu} = \tau' - \tau.$$

The complete set of solutions of the Riemann's equation (1.12) is denoted by the symbol

$$y = P \left\{ \begin{matrix} a & b & c & \\ \rho & \sigma & \tau & x \\ \rho' & \sigma' & \tau' & \end{matrix} \right\}$$

and is called *Riemann's P-function.*

Now, we will briefly describe here the theorem of Kimura that gives necessary and sufficient conditions for the integrability Riemann's differential equation.

Theorem 1.1.29 (Kimura, [47])**.** *The Riemann's differential equation* (1.12) *is integrable if and only if, either* *(i) At least one of the four numbers* $\widetilde{\lambda}+\widetilde{\mu}+\widetilde{\nu}$, $-\widetilde{\lambda}+\widetilde{\mu}+\widetilde{\nu}$, $\widetilde{\lambda}-\widetilde{\mu}+\widetilde{\nu}$, $\widetilde{\lambda}+\widetilde{\mu}-\widetilde{\nu}$ *is an odd integer, or*

(ii) The numbers $\widetilde{\lambda}$ *or* $-\widetilde{\lambda}$, $\widetilde{\mu}$ *or* $-\widetilde{\mu}$ *and* $\widetilde{\nu}$ *or* $-\widetilde{\nu}$ *belong (in an arbitrary order) to some of the following fifteen families*

1	$1/2+l$	$1/2+m$	*arbitrary complex number*	
2	$1/2+l$	$1/3+m$	$1/3+q$	
3	$2/3+l$	$1/3+m$	$1/3+q$	$l+m+q$ *even*
4	$1/2+l$	$1/3+m$	$1/4+q$	
5	$2/3+l$	$1/4+m$	$1/4+q$	$l+m+q$ *even*
6	$1/2+l$	$1/3+m$	$1/5+q$	
7	$2/5+l$	$1/3+m$	$1/3+q$	$l+m+q$ *even*
8	$2/3+l$	$1/5+m$	$1/5+q$	$l+m+q$ *even*
9	$1/2+l$	$2/5+m$	$1/5+q$	$l+m+q$ *even*
10	$3/5+l$	$1/3+m$	$1/5+q$	$l+m+q$ *even*
11	$2/5+l$	$2/5+m$	$2/5+q$	$l+m+q$ *even*
12	$2/3+l$	$1/3+m$	$1/5+q$	$l+m+q$ *even*
13	$4/5+l$	$1/5+m$	$1/5+q$	$l+m+q$ *even*
14	$1/2+l$	$2/5+m$	$1/3+q$	$l+m+q$ *even*
15	$3/5+l$	$2/5+m$	$1/3+q$	$l+m+q$ *even*

Here n, m, q *are integers.*

Using the Möebius transformation [40], also known as homographic substitution, in the Riemann's equation (1.12), we can maps $x = a, b, c$ to $x' = a', b', c'$, respectively:

$$x' = \frac{px+q}{rx+s}.$$

In particular, we can place the singularities at $x = 0, 1, \infty$ to obtain the following Riemann's equation:

$$\partial_x^2 y + \left(\frac{1-\rho-\rho'}{x} + \frac{1-\sigma-\sigma'}{x-1}\right)\partial_x y + \left(\frac{\rho\rho'}{x^2} + \frac{\sigma\sigma'}{(x-1)^2} + \frac{\tau\tau'-\rho\rho'\sigma\sigma'}{x(x-1)}\right) y = 0, \tag{1.13}$$

where the set of solutions is

$$y = P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ \rho & \sigma & \tau & x \\ \rho' & \sigma' & \tau' & \end{array}\right\}.$$

Sometimes it is very useful maps $x = 0, 1, \infty$ to $x' = -1, 1, \infty$ in the Riemann's equation (1.13), for example, setting $\rho = 0$, we can state the substitution:

$$P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & \sigma & \tau & x \\ \frac{1}{2} & \sigma' & \tau' & \end{array}\right\} = P\left\{\begin{array}{cccc} -1 & 1 & \infty & \\ \sigma & \sigma & 2\tau & \sqrt{x} \\ \sigma' & \sigma' & 2\tau' & \end{array}\right\}.$$

We can transforms the equation (1.13) to the *Gauss Hypergeometric equation* as follows:

$$P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ \rho & \sigma & \tau & x \\ \rho' & \sigma' & \tau' & \end{array}\right\} = x^{\rho}(x-1)^{\sigma} P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & \kappa & x \\ 1-\gamma & \gamma-\kappa-\beta & \beta & \end{array}\right\},$$

where $\kappa = \rho + \sigma + \tau$, $\beta = \rho + \sigma + \tau'$ and $\gamma = 1 + \rho - \rho'$. Then

$$y = P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & \kappa & x \\ 1-\gamma & \gamma-\kappa-\beta & \beta & \end{array}\right\},$$

is the set of solutions of the Gauss Hypergeometric differential equation[1]

$$\partial_x^2 y + \frac{(\gamma - (\kappa + \beta + 1)x)}{x(1-x)}\partial_x y - \frac{\kappa\beta}{x(1-x)} y = 0, \tag{1.14}$$

where the Fuchs relation is trivially satisfied and the exponent differences are given by

$$\widetilde{\lambda} = 1 - \gamma, \quad \widetilde{\mu} = 1 - \gamma - \beta, \quad \widetilde{\nu} = \beta - \kappa.$$

We remark that the Galoisian structure of the Riemann's equation do not change with the Möebius transformation and that the Galoisian structure of the Riemann's equation does not change with the Möebius transformation.

The *confluent Hypergeometric equation* is a degenerate form of the Hypergeometric differential equation where two of the three regular singularities merge into an irregular singularity. For example, making "1 tend to ∞" in a suitable way, the Hypergeometric equation (1.14) has two classical forms:

- *Kummer's* form
$$\partial_x^2 y + \frac{c - x}{x}\partial_x y - \frac{a}{x} y = 0 \tag{1.15}$$

- *Whittaker's* form
$$\partial_x^2 y = \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{4\mu^2 - 1}{4x^2}\right) y \tag{1.16}$$

[1]In general, for the Hypergeometric differential equation, is used α instead of κ, but we want to avoid further confusions.

where the parameters of the two equations are linked by $\kappa = \frac{c}{2} - a$ and $\mu = \frac{c}{2} - \frac{1}{2}$. Furthermore, using the expression (1.3), we can see that the Whittaker's equation is the reduced form of the Kummer's equation. The Galoisian structure of these equations has been deeply studied in [57, 33].

Theorem 1.1.30 (Martinet & Ramis, [57])**.** *The Whittaker's differential equation* (1.16) *is integrable if and only if either,* $\kappa + \mu \in \frac{1}{2} + \mathbb{N}$*, or* $\kappa - \mu \in \frac{1}{2} + \mathbb{N}$*, or* $-\kappa + \mu \in \frac{1}{2} + \mathbb{N}$*, or* $-\kappa - \mu \in \frac{1}{2} + \mathbb{N}$*.*

The *Bessel's equation* is a particular case of the confluent Hypergeometric equation and is given by

$$\partial_x^2 y + \frac{1}{x}\partial_x y + \frac{x^2 - n^2}{x^2} y = 0. \tag{1.17}$$

Under a suitable transformation, the reduced form of the Bessel's equation is a particular case of the Whittaker's equation. Thus, we can obtain the following well known result, see [48, p. 417] and see also [49, 60].

Corollary 1.1.31. *The Bessel's differential equation* (1.17) *is integrable if and only if* $n \in \frac{1}{2} + \mathbb{Z}$*.*

We point out that the integrability of Bessel's equation for half integer of the parameter was known by Daniel Bernoulli [99]. By double confluence of the Hypergeometric equation (1.14), that is making "0 and 1 tend to ∞" in a suitable way, one gets the *parabolic cylinder equation* (also known as *Weber's* equation):

$$\partial_x^2 y = \left(\frac{1}{4}x^2 - \frac{1}{2} - n\right) y, \tag{1.18}$$

which is integrable if and only if $n \in \mathbb{Z}$, see [49, 33]. Setting $n = \frac{b^2 - c}{2a} - \frac{1}{2}$ and making the change $x \mapsto \sqrt{\frac{2}{a}}(ax + b)$, one can gets the Rehm's form of the Weber's equation:

$$\partial_x^2 y = \left(ax^2 + 2bx + c\right) y, \quad a \neq 0, \tag{1.19}$$

so that $\frac{b^2 - c}{a}$ is an odd integer.

The Hypergeometric equation, including confluences, is a particular case of the differential equation

$$\partial_x^2 y + \frac{L}{Q}\partial_x y + \frac{\lambda}{Q} y, \quad \lambda \in \mathbb{C}, \quad L = a_0 + a_1 x, \quad Q = b_0 + b_1 x + b_2 x^2. \tag{1.20}$$

We recall that the *classical orthogonal polynomials* and *Bessel polynomials* are solutions of the equation (1.20), see [23, 44, 63]:

- Hermite, denoted by H_n,
- Chebyshev of first kind, denoted by T_n,

- Chebyshev of second kind, denoted by U_n,
- Legendre, denoted by P_n,
- Laguerre, denoted by L_n,
- associated Laguerre, denoted by $L_n^{(m)}$,
- Gegenbauer, denoted by $C_n^{(m)}$
- Jacobi polynomials, denoted by $\mathcal{P}_n^{(m,\nu)}$ and
- Bessel polynomials, denoted by B_n.

In the following table we give Q, L and λ corresponding to the equation (1.20) for classical orthogonal polynomials and Bessel polynomials.

Polynomial	$\boldsymbol{Q}$	$\boldsymbol{L}$	$\boldsymbol{\lambda}$
H_n	1	$-2x$	$2n$
T_n	$1-x^2$	$-x$	n^2
U_n	$1-x^2$	$-3x$	$n(n+2)$
P_n	$1-x^2$	$-2x$	$n(n+1)$
L_n	x	$1-x$	n
$L_n^{(m)}$	x	$m+1-x$	n
$C_n^{(m)}$	$1-x^2$	$-(2m+1)x$	$n(n+2m)$
$\mathcal{P}_n^{(m,\nu)}$	$1-x^2$	$\nu-m-(m+\nu+2)x$	$n(n+1+m+\nu)$
B_n	x^2	$2(x+1)$	$-n(n+1)$

The *associated Legendre polynomials*, denoted by $P_n^{(m)}$, does not appear in the previous table. They are solutions of the differential equation

$$\partial_x^2 y - \frac{2x}{1-x^2}\partial_x y + \left(\frac{n(n+1) - \frac{m^2}{1-x^2}}{1-x^2}\right) y = 0. \tag{1.21}$$

This equation can be transformed into a Riemann's differential equation through the change $x \mapsto \frac{1}{1-x^2}$. Thus, the complete set of solutions of the equation (1.21) is given by

$$P\left\{\begin{array}{cccc} 0 & 1 & \infty & \\ -\frac{1}{2}n & 0 & \frac{1}{2}m & \frac{1}{1-x^2} \\ \frac{1}{2}+\frac{1}{2}n & \frac{1}{2} & -\frac{1}{2}m & \end{array}\right\},$$

the exponent differences are $\widetilde{\lambda} = \frac{1}{2}$, $\widetilde{\mu} = \frac{1}{2}$ and $\widetilde{\nu} = 0$. By Kimura's theorem this equation is integrable.

Finally, we remark that integrability conditions and solutions of differential equations with solutions orthogonal polynomials, including Bessel polynomials, can be obtained applying Kovacic's algorithm. In the same way, we can apply Kovacic's algorithm to obtain the same results given by Kimura [47] and Martinet & Ramis [57]. Also we recall that Duval & Loday-Richaud applied Kovacic's algorithm to some families of special functions [33].

1.2 Supersymmetric Quantum Mechanics

In this section we establish the basic information on Supersymmetric Quantum Mechanics. We only consider the case of non-relativistic quantum mechanics.

1.2.1 The Schrödinger Equation

In classical mechanics the Hamiltonian corresponding to the energy (kinetic plus potential) is given by

$$H = \frac{\|\overrightarrow{p}\|^2}{2m} + U(\overrightarrow{x}), \quad \overrightarrow{p} = (p_1, \dots, p_n), \quad \overrightarrow{x} = (x_1, \dots, x_n),$$

while in quantum mechanics the momentum $\overrightarrow{p}$ is given by $\overrightarrow{p} = -\imath\hbar\nabla$, the Hamiltonian operator is the Schrödinger (non-relativistic, stationary) operator which is given by

$$H = -\frac{\hbar^2}{2m}\nabla^2 + U(\overrightarrow{x})$$

and the Schrödinger equation is $H\Psi = E\Psi$, where $\overrightarrow{x}$ is the *coordinate* , the eigenfunction Ψ is the *wave function*, the eigenvalue E is the *energy level*, $V(\overrightarrow{x})$ is the *potential or potential energy* and the solutions of the Schrödinger equation are the *states* of the particle. Furthermore, is known that $H^\dagger = H$, i.e., the Schrödinger operator is a self-adjoint operator in a Hilbert space (which in this thesis is complex and separable). Thus, H has a purely real *spectrum* $\mathrm{spec}(H)$ and its spectrum $\mathrm{spec}(H)$ is the disjoint union of the *point spectrum* $\mathrm{spec}_p(H)$ and the continuous spectrum $\mathrm{spec}_c(H)$, i.e., $\mathrm{spec}(H) = \mathrm{spec}_p(H) \cup \mathrm{spec}_c(H)$ with $\mathrm{spec}_p(H) \cap \mathrm{spec}_c(H) = \emptyset$. See for example [6, 71, 87].

Along this memory we only consider the one-dimensional Schrödinger equation written as follows:

$$H\Psi = E\Psi, \quad H = -\partial_z^2 + V(z), \tag{1.22}$$

where $z = x$ (cartesian coordinate) or $z = r$ (radial coordinate) and $\hbar = 2m = 1$. We denote by Ψ_n the wave function for $E = E_n$. The potentials should satisfy some conditions depending of the physic situation such as barrier, scattering, etc., see [24, 35, 53, 59, 78].

Definition 1.2.1 (Bound States)**.** The solution Ψ_n is called a *bound state* when E belongs to the point spectrum of H and its norm is finite, i.e.,

$$E_n \in \operatorname{spec}_p(H), \quad \int |\Psi_n(x)|^2 dx < \infty, \quad n \in \mathbb{Z}_+. \tag{1.23}$$

An interesting property of bound states is given by the Sturm's theorem, see [6, 87].

Theorem 1.2.2 (Sturm's Theorem)**.** *If $\Psi_0, \Psi_1, \ldots, \Psi_n, \ldots$ are the wave functions of the bound states with energies $E_0 < E_1 < \cdots < E_n < \ldots$, then Ψ_n has n nodes (zeros). Furthermore, between two consecutive nodes of Ψ_n, there is a node of Ψ_{n-1}, and moreover Ψ_{n+r} has at least one zero for all $r \geq 1$.*

Definition 1.2.3 (Ground State and Excited States)**.** Let assume $\Psi_0, \Psi_1, \ldots, \Psi_n, \ldots$ as in the Sturm's theorem. The state Ψ_0, which is state with minimum energy is called the *ground state* and the states $\Psi_1, \ldots, \Psi_n, \ldots$ are called the *excited states.*

Definition 1.2.4 (Scattering States)**.** The solution Ψ corresponding to the level energy E is called a *scattering state* when E belongs to the continuous spectrum of H and its norm is infinite.

The wave function belonging to the continuous spectrum have two typical boundary conditions: the first ones, barrier potentials and the second one periodic boundary conditions. The *transmission and reflection coefficients* are related with the barrier potentials, [35].

When the particle moves in one dimension, we use the classical one dimensional Schrödinger equation with cartesian coordinate x.

Example (The Harmonic Oscillator). We consider the Hamiltonian operator H given by (1.22). For $\omega = 1$, the one-dimensional harmonic oscillator potential is $V = \frac{1}{4}x^2$. In terms of raising (creator) and lowering (annihilator) operators $a^\dagger$ and a given by

$$a^\dagger = -\partial_x + \frac{1}{2}x, \quad a = \partial_x + \frac{1}{2}x$$

we can write

$$H = a^\dagger a + \frac{1}{2}, \quad [a, a^\dagger] = 1, \quad [a^\dagger, H] = -a^\dagger, \quad [a, H] = a.$$

Since the operator $aa^\dagger$ in H is positive semi-definite, all eigenvalues $E_n \geq \frac{1}{2}$. Therefore, the successive lowering of eigenstates by the operator a must eventually stop at the ground state wave function Ψ_0 by requiring $a\Psi_0 = 0$, which has as solution

$$\Psi_0 = e^{-\frac{1}{4}x^2}.$$

Operating with $a^\dagger$ yields $a^\dagger a \Psi_0 = \left(H - \frac{1}{2}\right)\Psi_0$, which corresponds to a ground state energy $E_0 = \frac{1}{2}$. All higher eigenstates are generates by repeated applications of $a^\dagger$. The complete energy spectrum is $E_n = n + \frac{1}{2}$, $n \in \mathbb{Z}_+$, which corresponds to the eigenfunctions

$$\Psi_n = (a^\dagger)^n \Psi_0 = H_n \Psi_0,$$

which clearly are bound states.

Also we can consider the particle moving in three dimensions, this means that $\overrightarrow{p} = (p_x, p_y, p_z)$, where $p_x = -\imath\partial_x$, $p_y = -\imath\partial_y$, $p_z = -\imath\partial_z$ and $p = \|\overrightarrow{p}\|$. The *angular momentum operator* is given by $\overrightarrow{L} = (L_x, L_y, L_z)$ where $L_x = yp_z - zp_y$, $L_y = zp_x - xp_z$, $L_z = xp_y - yp_x$ and $L = \|\overrightarrow{L}\|$. The square of the angular momentum operator $\|\overrightarrow{L}\|^2 = L^2 = L_x^2 + L_y^2 + L_z^2$ commutes with all components of the angular momentum operator.

In spherical coordinates $x = r\sin\nu\cos\varphi$, $y = r\sin\nu\sin\varphi$, $z = r\cos\nu$, L^2 is given by

$$L^2 = -\triangle_{\nu,\varphi}, \quad \triangle_{\nu,\varphi} = \frac{1}{\sin\nu}\partial_\nu\left(\sin\nu\partial_\nu\right) + \frac{1}{\sin^2\nu}\partial_\varphi^2,$$

where we denote by $\triangle_{\nu,\varphi}$ that part of the Laplacian acting on the variables ν and φ only. The kinetic energy given by $p^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ reads in polar coordinates as

$$p^2 = p_r^2 + \frac{1}{r^2}L^2, \quad p_r^2 = -\frac{1}{r^2}\partial_r\left(r^2\partial_r\right) = -\left(\partial_r^2 + \frac{2}{r}\partial_r\right).$$

Now, for *central potentials*, where the potential $U(\overrightarrow{r})$ is spherically symmetric, i.e., $U(\overrightarrow{r}) = U(r)$, we can reduce the Schrödinger equation to an one dimensional problem, the so-called *radial equation.*

We start writing the eigenfunctions and eigenvalues of the operator L^2:

$$L^2 Y_{\ell,m}(\nu,\varphi) = \ell(\ell+1)Y_{\ell,m}(\nu,\varphi),$$

the eigenfunctions $Y(\nu,\varphi)$ are the *spherics harmonics* which are related with the associated Legendre Polynomials

$$Y_{\ell,m}(\nu,\varphi) = P_\ell^m(\cos\nu)e^{\imath m\varphi}.$$

Assuming Φ as eigenfunctions of $p^2 + U(r)$ satisfying $\Phi = R_\ell(r)Y_{\ell,m}(\nu,\varphi)$, i.e., the *partial wave function decomposition* see ([35, 53, 78]), we have

$$\left(p_r^2 + \frac{1}{r^2}L^2 + U(r) - E\right)R_\ell(r)Y_{\ell,m}(\nu,\varphi) = 0,$$

so that we obtain

$$\left(p_r^2 + U(r) - E\right)R_\ell(r) + \frac{R_\ell(r)}{r^2 Y_{\ell,m}(\nu,\varphi)}L^2 Y_{\ell,m}(\nu,\varphi) = 0$$

and owing to $L^2 Y_{\ell,m}(\nu,\varphi) = \ell(\ell+1)Y_{\ell,m}(\nu,\varphi)$ we have the radial equation

$$\left(p_r^2 + \frac{\ell(\ell+1)}{r^2} + U(r)\right)R_\ell(r) = ER_\ell(r).$$

Applying the expression (1.3), the radial equation can be reduced to the Schrödinger equation (1.22) as follows:

$$H\Psi = E\Psi, \quad H = -\partial_r^2 + V(r), \quad V(r) = \frac{\ell(\ell+1)}{r^2} + U(r), \quad \Psi = rR_\ell(r).$$

The equation for the angular part is always solved through spherics harmonics, while for the radial part, the analysis depends on the spherically symmetric potential $U(r)$. One example of the radial equation is the Coulomb potential. The complete set of physical and mathematical conditions for the potentials, spectrum and wave functions in one or three dimensions can be found in any book of quantum mechanics, including bound states and scattering cases, see for example [35, 53, 78].

1.2.2 Darboux Transformation

The following theorem is the most general case for Darboux transformation in second order linear differential equations, which is taken faithfully from [28].

Theorem 1.2.5 (Darboux, [28]). *Suppose that we know how to integrate, for any value of the constant m, the following equation*

$$\partial_x^2 y + P\partial_x y + (Q - mR)y = 0. \tag{1.24}$$

If θ is an integral of the equation

$$\partial_x^2 \theta + P\partial_x \theta + Q\theta = 0,$$

then the function

$$u = \frac{\partial_x y - \frac{\partial_x \theta}{\theta} y}{\sqrt{R}}, \tag{1.25}$$

will be an integral of the equation

$$\partial_x^2 u + P\partial_x u + \left(\theta\sqrt{R}\partial_x\left(\frac{P}{\theta\sqrt{R}}\right) - \theta\sqrt{R}\partial_x^2\left(\frac{1}{\theta\sqrt{R}}\right) - mR\right)u = 0, \tag{1.26}$$

for $m \neq 0$.

Darboux in [28, 29] presented the particular for $R = 1$ and $P = 0$, which today is known as *Darboux transformation*, but really is a corollary of the *general Darboux transformation* given in the theorem 1.2.5.

Corollary 1.2.6 (Darboux, [28, 29]). *Suppose that we know integrate*

$$\partial_x^2 y = (f(x) + m)y \tag{1.27}$$

for any value of m. If θ satisfies the equation $\partial_x^2\theta = (f(x) + m_1)\theta$, the function

$$u = \partial_x y - \frac{\partial_x \theta}{\theta} y$$

will be an integral of the equation

$$\partial_x^2 u = \left(\theta \partial_x^2 \left(\frac{1}{\theta}\right) - m_1 + m\right) u, \tag{1.28}$$

for $m \neq m_1$. Furthermore,

$$\theta \partial_x^2 \left(\frac{1}{\theta}\right) - m_1 = f(x) - 2\partial_x \left(\frac{\partial_x \theta}{\theta}\right) = 2\left(\frac{\partial_x \theta}{\theta}\right)^2 - f(x) - 2m_1.$$

Remark 1.2.7. In practice, we need two values of m to apply the Darboux's results.

Example. Let consider the equation $\partial_x^2 y = my$. Employing the solution $\theta = x$, we shall get

$$\partial_x^2 y = \left(\frac{1 \cdot 2}{x^2} + m\right) y.$$

Applying the same method to the latter equation, but taking now $\theta = x^2$, we shall have

$$\partial_x^2 y = \left(\frac{2 \cdot 3}{x^2} + m\right) y$$

and so on. The cases $m_1 = 0$ and $m_1 = -1$ also can be found as exercises in the Ince's book [42, p. 132].

We can see that the equation (1.27) is equivalent to the Schrödinger equation (1.22). Thus, we can apply the Darboux transformation in where $m = -E$ and $m_1 = -E_0$.

The following definition corresponds with Delsarte's transformation operators (isomorphisms of transmutations), which today are called *intertwiner operators*, see [30].

Definition 1.2.8. Two operators $\mathfrak{L}_0$ and $\mathfrak{L}_1$ are said to be *intertwined* by an operator T if

$$\mathfrak{L}_1 T = T \mathfrak{L}_0. \tag{1.29}$$

We can relate the intertwiner operators with the Darboux transformation of the equation (1.22), where $\mathfrak{L}_1$ and $\mathfrak{L}_0$ are Schrödinger operators and T can be either $\mp\partial_x + \frac{\partial_x \Psi_0}{\Psi_0}$.

Crum, inspired in the works of Liouville [54, 55] obtained one kind of iterative generalization of Darboux's result emphasizing in the Sturm-Liouville systems, i.e., he proved that the Sturm-Liouville conditions are preserved under Darboux transformations, see [27]. The Crum's result is presented in the following theorem, defining the Wronskian determinant W of k functions $f_1, f_2, \ldots, f_k$ by

$$W(f_1, \ldots, f_k) = \det A, \quad A_{ij} = \partial_x^{i-1} f_j, \quad i, j = 1, 2, \ldots, k.$$

Theorem 1.2.9 (Crum, [27]). *Let $\Psi_1, \Psi_2, \ldots \Psi_n$ be solutions of the Schrödinger equation* (1.22) *for fixed, arbitrary energy levels $E = E_1, E_2, \ldots, E_n$, respectively. Then, we obtain the Schrödinger equation*

$$H^{[n]}\Psi[n] = E\Psi[n], \quad E \neq E_i, 1 \leq i \leq n, \quad H^{[n]} = -\partial_x^2 + V[n],$$

where

$$\Psi[n] = \frac{W(\Psi_1, \ldots, \Psi_n, \Psi)}{W(\Psi_1, \ldots, \Psi_n)}, \quad V[n] = V - 2\partial_x^2 \ln W(\Psi_1, \ldots, \Psi_n).$$

Darboux transformation coincides with Crum's result in the case $n = 1$ and the iterations of Darboux transformation coincides with Crum iteration. Both formalisms allow us obtain new families of Schrödinger equations preserving the spectrum and the Sturm-Liouville conditions, see [59, 64]. Furthermore, there are extensions of Crum's iteration connecting the Sturm-Liouville theory with orthogonal polynomial theory [51].

Schrödinger in [79] factorized the Hypergeometric equation (1.14). He started making the change $2x - 1 = \cos\theta$, after, using the expression (1.3), he reduced the Hypergeometric equation to establish the conditions of factorization. In this way, setting $\kappa\beta = E$, we obtain families of Schrödinger equations (1.22). This result was used by Natanzon in [62] to obtain the well known *Natanzon's potentials*, i.e, potentials which can be obtained by transformations of the Hypergeometric equation and its confluences, see [24, 25]. In particular, the *Ginocchio potentials* are obtained through the Gegenbauer polynomials.

Witten in [102, §6] presented some models in where dynamical breaking of supersymmetry is plausible. The first model is not a field theory model at all but a model in potential theory-supersymmetric quantum mechanics.

Definition 1.2.10. A *supersymmetric quantum mechanical system* is one in which there are operators Q_i, that commute with the hamiltonian $\mathcal{H}$,

$$[Q_i, \mathcal{H}] = 0, \quad i = 1, \ldots, n \tag{1.30}$$

and satisfy the algebra

$$\{Q_i, Q_j\} = \delta_{ij}\mathcal{H}, \quad \mathcal{H} = 2Q_i^2. \tag{1.31}$$

The case $n = 2$, which involves a spin one half particle moving on the line, is the simplest example of a supersymmetric quantum mechanical system. This case is the main object of this thesis. The wave function of $\mathcal{H}\Phi = E\Phi$ is therefore a two-component *Pauli spinor*,

$$\Phi(x) = \begin{pmatrix} \Psi_+(x) \\ \Psi_-(x) \end{pmatrix}.$$

The *supercharges* Q_i are defined as

$$Q_\pm = \frac{\sigma_1 p \pm \sigma_2 W(x)}{2}, \quad Q_+ = Q_1, \, Q_- = Q_2, \quad p = -i\partial_x, \tag{1.32}$$

where the *superpotential* W is an arbitrary function of x and σ_i are the usual Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using the expressions (1.30), (1.31) and (1.32) we obtain $\mathcal{H}$:

$$\mathcal{H} = 2Q_-^2 = 2Q_+^2 = \frac{I_2 p^2 + I_2 W^2(x) + \sigma_3 \partial_x W(x)}{2}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.33}$$

The *supersymmetric partner Hamiltonians* $H_\pm$ are given by

$$H_\pm = -\frac{1}{2}\partial_x^2 + V_\pm, \quad V_\pm = \left(\frac{W}{\sqrt{2}}\right)^2 \pm \frac{1}{\sqrt{2}}\partial_x \left(\frac{W}{\sqrt{2}}\right).$$

The potentials $V_\pm$ are called *supersymmetric partner potentials* and are linked with the superpotential W through a Riccati equation. So that the equation (1.33) can be written as

$$\mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix},$$

which lead us to the Schrödinger equations $H_+\Psi_+ = E\Psi_+$ and $H_-\Psi_- = E\Psi_-$, and for instance, to solve $\mathcal{H}\Phi = E\Phi$ is equivalent to solve simultaneously $H_+\Psi_+ = E\Psi_+$ and $H_-\Psi_- = E\Psi_-$.

We study the equation $Q_i\Phi = 0$ which must be satisfied by a supersymmetric state. Actually, because of the general relation $Q_1^2 = Q_2^2 = \frac{1}{2}\mathcal{H}$ (or the fact that in this particular model $Q_2 = -i\sigma_3 Q_1$, as can easily be checked), it is enough to satisfy $Q_1\Phi = 0$, or $\sigma_1 p\Phi = -\sigma_2 W\Phi$. Multiplying by σ_1 and using the facts that $p = -i\partial_x$, $\sigma_1\sigma_2 = i\sigma_3$, this equation becomes

$$\partial_x \Phi = W(x)\sigma_3 \Phi(x), \tag{1.34}$$

and the solution is

$$\Phi(x) = e^{\int W(x)\sigma_3 dx}. \tag{1.35}$$

The most important generalization of the above model would be, of course, their extension to four dimensions. The existence of a workable mechanism for dynamical supersymmetry breaking in four dimensions is an open question.

V.B. Matveev and M. Salle in [59] interpret the Darboux Theorem as Darboux covariance of a Sturm-Liouville problem and proved the following result, see also [76, §5-6].

Theorem 1.2.11 (Matveev & Salle, [59]). *Supersymetric Quantum Mechanics is equivalent to a single Darboux transformation*

According to Natanzon [62], by *solvable potential*, also known as *exactly solvable potentials*, one means here a potential for which the Schrödinger equation can be reduced to hypergeometric or confluent hypergeometric form. The following are examples of solvable potentials, .

Potential V(x)	**Name**
$0, \quad x \in [0, L]; \quad \infty, \quad x \notin [0, L]$	*Infinite square well*
$\frac{\ell(\ell+1)}{r^2}, \quad r \in [0, L]; \quad \infty, \quad r \notin [0, L]$	*Radial infinite square well*
$\frac{\mu}{e^{\kappa r}-1},$	*Hulthén*
$\frac{\mu}{e^{\kappa r}-1} + \frac{2\mu^2}{(e^{\mu r}-1)^2},$	*Generalized Hulthén*
$\frac{1}{4}\omega^2 x^2 + g_a \frac{x^2-a^2}{(x^2+a^2)^2}, \quad g_a > 0$	− − − − − − [22]

(1.36)

We remark that our definition of integrability, definition 1.1.8, is different of the concept of solvability given by Natanzon. In the next chapter we come back on this problem.

According Dutt et al. [32] by *conditionally solvable potential* we means potentials for which the entire bound state spectrum can be analytically obtained, provided the parameters in the potential satisfy a specific relation. The following potentials are two examples of conditionally solvable potentials

$$V(x) = \frac{A}{1+e^{-2x}} - \frac{B}{\sqrt{1+e^{-2x}}} - \frac{3}{4(1+e^{-2x})^2} \quad \text{and}$$

$$V(x) = \frac{A}{1+e^{-2x}} - \frac{Be^{-x}}{\sqrt{e^{-2x}+1}} - \frac{3}{4(1+e^{-2x})^2}. \tag{1.37}$$

As a generalization of the method to solve the harmonic oscillator [36, 31] , the ladder (raising and lowering) operators are defined as

$$A^+ = -\partial_x - \frac{\partial_x \Psi_0}{\Psi_0}, \quad A = \partial_x - \frac{\partial_x \Psi_0}{\Psi_0},$$

which are very closed with the supercharges $Q_\pm$ in the Witten's formalism. Thus,

$$A\Psi_0 = 0, \quad A^+ A = H_-, \quad AA^+ = H_+ = -\partial_x^2 + V_+(x), \quad \text{where}$$

$$V_+(x) = V_-(x) - 2\partial_x \left(\frac{\partial_x \Psi_0}{\Psi_0} \right) = -V_-(x) + 2 \left(\frac{\partial_x \Psi_0}{\Psi_0} \right)^2.$$

The supersymmetric partner potentials V_+ and V_- have the same energy levels, except for $E_0^{(-)} = 0$. In terms of the superpotential $W(x)$, the operators A and A^+ are given by

$$A^+ = -\partial_x + W(x), \quad A = \partial_x + W(x).$$

Also, the supersymmetric partner potentials $V_{\pm}(x)$ and the superpotential $W(x)$ satisfies:

$$\frac{V_+(x) + V_-(x)}{2} = W^2(x), \quad [A, A^+] = 2\partial_x W(x).$$

Let $\Psi_n^{(-)}$ and $\Psi_n^{(+)}$ denote the eigenfunctions of the supersymmetric Hamiltonians H_- and H_+ respectively, with eigenvalues $E_n^{(-)}$ and $E_n^{(+)}$. The integer $n = 0, 1, 2, \dots,$ denotes the number of nodes in the wave function.

Theorem 1.2.12 (Dutt et al., [31])**.** *If $\Psi_n^{(-)}$ is any eigenfunction of H_- with eigenvalue $E_n^{(-)}$, then $A\Psi_n^{(-)}$ is an eigenfunction of H_+ with the same eigenvalue. Furthermore*

$$E_n^{(+)} = E_{n+1}^{(-)}, \quad \Psi_n^{(+)} = \frac{A}{\sqrt{E_{n+1}^{(-)}}} \Psi_{n+1}^{(-)}.$$

We can see in Theorem 1.2.12 that the potentials V_+ and V_- have the same spectrum, except that the ground state energy $E_0 = 0$ of V_- has not corresponding level for V_+. Furthermore, if the eigenfunction $\Psi_{n+1}^{(-)}$ of H_- is normalized, then the wave function $\Psi_n^{(+)}$ of H_+ is also normalized.

We can note that the operator A not only converts an eigenfunction of H_- into an eigenfunction of H_+ with the same energy, but it also destroys a node since $\Psi_{n+1}^{(-)}$ has $n+1$ nodes, whereas $\Psi_n^{(+)}$ has n. The operator A^+ creates a node and converts an eigenfunction of H_+ into an eigenfunction of H_- with the same energy. In summary, the operators A and A^+ connect states of the same energy for two different (supersymmetric partner) potentials.

According to Gendenstein [36], the *shape invariance* is a property of some classes of potentials with respect to their parameter(s), say a, and reads

$$V_{n+1}(x, a_n) = V_n(x, a_{n+1}) + R(a_n), \quad V_- = V_0, V_+ = V_1,$$

where R is a remainder. In this way we say that $V_{\pm}$ are *shape invariant potentials*. This property assures a fully algebraic scheme for the spectrum and wave functions. Fixing $E_0 = 0$, the excited spectrum is given by

$$E_n = \sum_{k=2}^{n+1} R(a_k)$$

and the wave functions are obtained from

$$\Psi_n(x, a_1) = \prod_{k=1}^{n} A^+(x, a_k)\Psi_0(x, a_{n+1}).$$

Following [24, 31] we present the list of shape invariant potentials given in expression (1.38).

Potential V	**Name**
$\frac{1}{4}\omega^2 x^2 - \frac{\omega}{2}$	*Harmonic Oscillator*
$\frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+1)}{r^2} - \left(\ell + \frac{3}{2}\right)\omega$	*3D Harmonic Oscillator*
$-\frac{e^2}{r} + \frac{\ell(\ell+1)}{r^2} + \frac{e^4}{8(\ell+1)^2}$	*Coulomb*
$A^2 + B^2 e^{-2ax} - 2B\left(A + \frac{a}{2}\right)e^{-ax}$	*Morse*
$A^2 + \frac{B^2}{A^2} - 2B\coth ar + A\frac{A-a}{\sinh^2 ar}$	*Eckart*
$A^2 + \frac{B^2}{A^2} + 2B\tanh ax - A\frac{A+a}{\cosh^2 ax}$	*Rosen-Morse Hyp.*
$-A^2 + \frac{B^2}{A^2} + 2B\cot ax + A\frac{A+a}{\sin^2 ar}$	*Rosen-Morse Trig.*
$A^2 + \frac{B^2-A^2-Aa}{\cosh^2 ax} + \frac{B(2A+a)\sinh ax}{\cosh^2 ax}$	*Scarf Hyp. I*
$A^2 + \frac{B^2+A^2+Aa}{\sinh^2 ar} - \frac{B(2A+a)\cosh ar}{\sinh^2 ar}$	*Scarf Hyp. II*
$-A^2 + \frac{B^2+A^2-Aa}{\cos^2 ax} - \frac{B(2A-a)\sin ax}{\cos^2 ax}$	*Scarf Trig. I*
$-A^2 + \frac{B^2+A^2-Aa}{\sin^2 ax} - \frac{B(2A-a)\cos ax}{\sin^2 ax}$	*Scarf Trig. II*
$-(A+B)^2 + \frac{A(A-a)}{\cos^2 ax} + \frac{B(B-a)}{\sin^2 ax}$	*Pöschl-Teller 1*
$(A-B)^2 - \frac{A(A+a)}{\cosh^2 ar} + \frac{B(B-a)}{\sinh^2 ar}$	*Pöschl-Teller 2*

(1.38)

Now we explain how to obtain the eigenstates and eigenvalues of shape invariant potentials. We start constructing a series of Hamiltonians $H^{(s)}$, $s \in \mathbb{Z}_+$, where $H^{(0)} = H_-$, $H^{(1)} = H_+$.

$$H^{(s)} = -\partial_x^2 + V_-(x; a_s) + \sum_{k=1}^{s} R(a_k), \quad a_s = f^s(a_0).$$

Now, we compare the spectrum of $H^{(s)}$ with the spectrum of $H^{(s+1)}$, so that

$$H^{(s+1)} = -\partial_x^2 + V_-(x;a_{s+1}) + \sum_{k=1}^{s+1} R(a_k) = -\partial_x^2 + V_+(x;a_s) + \sum_{k=1}^{s} R(a_k).$$

We see that $H^{(s)}$ and $H^{(s+1)}$ are supersymmetric partner Hamiltonians and hence have identical bound state energy spectra except for the lowest level of $H^{(s)}$ whose energy is

$$E_0^{(s)} = \sum_{k=1}^{s} R(a_k), \quad E_0^{(0)} = 0.$$

On going back from $H^{(s)}$ to $H^{(s-1)}$, we would eventually reach $H^{(1)} = H_+$ and $H^{(0)} = H_-$, whose ground state energy is zero and its nth energy level being coincident with the ground state of Hamiltonian $H^{(n)}$, $n \in \mathbb{Z}_+$. Hence, the complete energy spectrum of H_- is given by

$$E_n^{(-)} = \sum_{k=1}^{n} R(a_k), \quad E_0^{(-)} = 0.$$

On the other hand, for any shape invariant potential $V_-(x;a_0)$, the bound state wave function $\Psi_n^{(-)}(x;a_0)$ can be easily constructed from the ground state wave function $\Psi_0^{(-)}(x;a_0)$. This is possible since the operators A and A^+ link up the eigenfunctions of the same energy for supersymetric partner Hamiltonians H_+ and H_-.

Let us start from the Hamiltonian $H^{(s)}$ given above. Its ground state eigenfunction is given by $\Psi_0^{(-)}(x;a_s)$. On going from $H^{(s)}$ to $H^{(s-1)}$ to $H^{(1)} = H_+$ and $H^{(0)} = H_-$ we then find that the nth state unnormalized energy eigenfunction $\Psi_n^{(-)}(x;a_0)$ for the original Hamiltonian $H_-(x;a_0)$ is given by

$$\Psi_n^{(-)}(x;a_0) = A^+(x;a_0)A^+(x;a_1)\dots A^+(x;a_{n-1})\Psi_0^{(-)}(x;a_n).$$

In practice, if one wants explicit expressions for the wave functions, it is simpler to use the result

$$\Psi_{n+1}^{(-)}(x;a_0) = A^+(x;a_0)\Psi_n^{(-)}(x;a_1) = \frac{1}{\Psi_0}\partial_x\left(\Psi_0\Psi_n^{(-)}(x;a_1)\right).$$

We can also notes that for shape invariant potentials

$$\Psi_n^{(+)}(x;a_0) = \Psi_n^{(-)}(x;a_1).$$

Repeated application of these previous equations for $n \in \mathbb{Z}_+$ gives all the eigenfunctions. The procedure for successively obtaining higher energy eigenfunctions stop if any wave function is not normalizable. Of course, this corresponds to the case, where a potential can only hold a finite number of bound states.

Chapter 2

Differential Galois Theory Approach to Supersymmetric Quantum Mechanics

In this chapter we present our original results of this thesis, which corresponds to the Galoisian approach to Supersymmetric quantum mechanics. We start rewriting in a Galoisian context some points of the section 1.2, chapter 1. The results presented here are also true for any differential field K with field of constants $\mathcal{C}$ in agreement with the definition 1.1.4. We are emphasize in the differential fields $K = \mathbb{C}(x)$ and $\widehat{K} = \mathbb{C}(z(x), \partial_x z(x))$, where in both cases $\mathcal{C} = \mathbb{C}$.

2.1 Preliminaries

The main object of our Galoisian analysis is the Schrödinger equation (1.22), which now is written as

$$\mathcal{L}_\lambda := H\Psi = \lambda\Psi, \quad H = -\partial_x^2 + V(x), \quad V \in K, \tag{2.1}$$

being K a differential field (with $\mathbb{C}$ as field of constants). We are interested in the integrability of the equation (2.1) in agreement with the definition 1.1.8.

We introduce the following notations.

- Denote by $\Lambda \subseteq \mathbb{C}$ the set of eigenvalues λ such that the equation (2.1) is integrable according with the definition 1.1.8.
- Assume $\Lambda \subseteq \mathbb{R}$, we denote by Λ_+ the set $\{\lambda \in \Lambda : \lambda \geq 0\}$ and by Λ_- the set $\{\lambda \in \Lambda : \lambda \leq 0\}$.

We remark that Λ can be $\emptyset$, i.e., $\mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathrm{SL}(2,\mathbb{C})\ \forall\lambda \in \mathbb{C}$. On the other hand, by theorem 1.1.9, if $\lambda_0 \in \Lambda$ then $(\mathrm{DGal}_K(\mathcal{L}_{\lambda_0}))^0 \subseteq \mathbb{B}$.

Definition 2.1.1 (Exactly Solvable and Quasi-Exactly Solvable Potentials)**.** We say that the potential $V(x) \in K$ is:

- an *exactly solvable potential* when Λ is an infinite set, or
- a *quasi-exactly solvable potential* when Λ is a non-empty finite set.
- a *non-solvable potential* when $\Lambda = \emptyset$.

Examples. Let assume $K = \mathbb{C}(x)$.

1. If $V(x) = x$, then $\Lambda = \emptyset$, $V(x)$ is non-solvable, see [46, 49].
2. If $V(x) = 0$, then $\Lambda = \mathbb{C}$, i.e., $V(x)$ is exactly solvable. Furthermore,
$$\mathrm{DGal}_K(\mathcal{L}_0) = e, \quad \mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathbb{G}_m, \quad \lambda \neq 0.$$
3. If $V(x) = \frac{x^2}{4} + \frac{1}{2}$, then $\Lambda = \{n : n \in \mathbb{Z}\}$, $V(x)$ is exactly solvable. This example corresponds to the Weber's equation, see subsection 1.1.4.

Remark 2.1.2. In particular, we are interested in the spectrum of the exactly and quasi-exactly solvable potentials, that is, $\mathrm{spec}(H) \cap \Lambda \neq \emptyset$. For example, the potential $V(x) = |x|$ has point spectrum (see [6]) although $V(x)$ is non-solvable. Thus, when $\mathrm{spec}(H) \cap \Lambda$ is an infinite set, our definition is in agreement with the usual physical terminology: these potentials are called *solvable potentials*, see Natanzon [62]. In analogous way, when $\mathrm{spec}(H) \cap \Lambda$ is a finite set, our definition coincides with the usual definition in physics of *quasi-exactly solvable potentials* (Turbiner [89], Bender & Dunne [11], Bender & Boettcher [10], Saad et al. [77], Gibbons & Vesselov [37]).

Definition 2.1.3. Let be $\mathcal{L}$, $\widetilde{\mathcal{L}}$, pairs of linear differential equations defined over differential fields K and $\widetilde{K}$ respectively, with Picard-Vessiot extensions L and $\widetilde{L}$. Let φ be the transformation such that $\mathcal{L} \mapsto \widetilde{\mathcal{L}}$, $K \mapsto \widetilde{K}$ and $L \mapsto \widetilde{L}$, we say that:

1. φ is an **iso-Galoisian transformation** if and only if
$$\mathrm{DGal}(L/K)(\mathcal{L}) = \mathrm{DGal}(\widetilde{L}/\widetilde{K})(\widetilde{\mathcal{L}}).$$
If $\widetilde{L} = L$ and $\widetilde{K} = K$, we say that φ is an **strong iso-Galoisian transformation**.
2. φ is an **virtually iso-Galoisian transformation** if and only if
$$(\mathrm{DGal}(L/K)(\mathcal{L}))^0 = (\mathrm{DGal}(\widetilde{L}/\widetilde{K})(\widetilde{\mathcal{L}}))^0.$$
If $\widetilde{L} = L$ and $\widetilde{K} = K$, we say that φ is an **virtually strong iso-Galoisian transformation**.

Remark 2.1.4. Let $\mathcal{L}, \widetilde{\mathcal{L}}, K$ and $\widetilde{K}$ be as in definition 2.1.3, then the following statements holds.

- If φ is either, an iso-galoisian transformation or an virtually iso-galoisian transformation, then

$$(\mathrm{DGal}(\widetilde{L}/\widetilde{K})(\mathcal{L}))^0 = (\mathrm{DGal}(\widetilde{L}/\widetilde{K})(\widetilde{\mathcal{L}}))^0.$$

- If φ is an iso-Galoisian transformation, then $\mathcal{E}(\mathcal{L}) \simeq \mathcal{E}(\widetilde{\mathcal{L}})$.

Proposition 2.1.5. *Let $\mathcal{L}$ be the differential equation* (1.1) *and $\widetilde{\mathcal{L}}$ the differential equation* (1.2)*. Let $\kappa \in \mathbb{C}$, $a = 2\kappa\partial_x(\ln f)$, $f \in K$ and φ be the transformation such that $\mathcal{L} \mapsto \widetilde{\mathcal{L}}$ through equation* (1.3)*. The following statements holds:*

1. *φ is an strong isogaloisian transformation for $\kappa \in \mathbb{Z}$.*
2. *φ is a virtually strong isogaloisian transformation for $\kappa \in \mathbb{Q} \setminus \mathbb{Z}$.*
3. *If $\mathcal{L}$ has not algebraic solutions over K and $\kappa \in \mathbb{C} \setminus \mathbb{Q}$ (or there is not exists $f \in K$ such that $a = 2\kappa\partial_x(\ln f)$), then φ is an strong isogaloisian transformation.*

Proof. Assume that $\mathcal{B} = \{y_1, y_2\}$ is the basis of solutions and L is the Picard-Vessiot extension of $\mathcal{L}$, $\mathcal{B}' = \{\zeta_1, \zeta_2\}$ is the basis of solutions and $\widetilde{L}$ is the Picard-Vessiot extension of $\widetilde{\mathcal{L}}$. By equation (1.1.9) $K = \widetilde{K}$, but the relationship between L and $\widetilde{L}$ depends on a:

1. If $\kappa = n \in \mathbb{Z}$, then $\mathcal{B}' = \{f^n y_1, f^n y_2\}$ which means that $L = \widetilde{L}$ and φ is strong isogaloisian.
2. If $\kappa = \frac{n}{m}$, with $\gcd(n, m) = 1$, $\frac{n}{m} \notin \mathbb{Z}$, then $\mathcal{B}' = \{f^{\frac{n}{m}} y_1, f^{\frac{n}{m}} y_2\}$ which means that $\widetilde{L}$ is either an algebraic extension of degree at most m of L, and φ is virtually strong isogaloisian, or $L = \widetilde{L}$ when $f^{\frac{n}{m}} \in K$ which means that φ is strong isogaloisian.
3. If $\mathcal{L}$ has not algebraic solutions and either $\kappa \notin \mathbb{Q}$ or there is not exists $f \in K$ such that $a = 2\kappa\partial_x(\ln f)$, then $\widetilde{L}$ is an transcendental extension of L. This means that the Galois group acts in the same way in both basis of solutions, so that φ is strong isogaloisian.

□

Remark 2.1.6. The transformation φ in proposition 2.1.5 is not injective, there are a lot of differential equations $\mathcal{L}$ that are transformed in the same differential equation $\widetilde{\mathcal{L}}$.

As immediate consequence of the previous proposition we have the following corollary.

Corollary 2.1.7 (Sturm-Liouville)**.** *Let $\mathcal{L}$ be the differential equation*

$$\partial_x (a\partial_x y) = (\lambda b - \mu)y, \quad a, b \in K, \quad \lambda, \mu \in \mathbb{C}$$

in where L, $\widetilde{L}$, $\widetilde{\mathcal{L}}$ and Φ are given as in proposition 2.1.5. Then either $\widetilde{L}$ is a quadratic extension of L which means that φ is virtually strong isogaloisian or $\widetilde{L} = L$ when $a^{\frac{1}{2}} \in K$ which means that φ is strong isogaloisian.

2.2 Supersymmetric Quantum Mechanics with Rational Potentials

Along this section we consider as differential field $K = \mathbb{C}(x)$.

2.2.1 Polynomial Potentials

We start considering the Schrödinger equation (2.1) with polynomial potentials, i.e., $V \in \mathbb{C}[x]$, see [18, 91]. For simplicity and without lost of generality, we consider monic polynomials due to the reduced second order linear differential equation with polynomial coefficient $c_n x^n + \ldots + c_1 x + c_0$ can be transformed into the reduced second order linear differential equation with polynomial coefficient $x^n + \ldots + q_1 x + q_0$ through the change of variable $x \mapsto \sqrt[n+2]{c_n}x$.

When V is a polynomial of odd degree, is well known that the Galois group of the Schrödinger equation (2.1) is $\mathrm{SL}(2, \mathbb{C})$, see [49].

We present here the complete result for the Schrödinger equation (2.1) with non-constant polynomial potential (Theorem 2.2.2), see also [5, §2]. The following lemma is useful for our purposes.

Lemma 2.2.1 (Completing Squares, [5])**.** *Every monic polynomial of even degree can be written in one only way completing squares, that is,*

$$Q_{2n}(x) = x^{2n} + \sum_{k=0}^{2n-1} q_k x^k = \left(x^n + \sum_{k=0}^{n-1} a_k x^k\right)^2 + \sum_{k=0}^{n-1} b_k x^k, \tag{2.2}$$

where

$$a_{n-1} = \frac{q_{2n-1}}{2}, \quad a_{n-2} = \frac{q_{2n-2} - a_{n-1}^2}{2}, \quad a_{n-3} = \frac{q_{2n-3} - 2a_{n-1}a_{n-2}}{2}, \cdots,$$

$$a_0 = \frac{q_n - 2a_1 a_{n-1} - 2a_2 a_{n-2} - \cdots}{2}, \quad b_0 = q_0 - a_0^2, \quad b_1 = q_1 - 2a_0 a_1, \quad \cdots,$$

$$b_{n-1} = q_{n-1} - 2a_0 a_{n-1} - 2a_1 a_{n-2} - \cdots.$$

We remark that $V(x)$ as in the equation (2.2) can be written in terms of the superpotential $W(x)$, i.e., $V(x) = W^2(x) - \partial_x W(x)$, when

$$nx^{n-1} + \sum_{k=1}^{n-1} k a_k x^{k-1} = -\sum_{k=0}^{n-1} b_k x^k$$

and $W(x)$ is given by

$$x^n + \sum_{k=0}^{n-1} a_k x^k.$$

The following theorem also can be found in [5, §2] and see also [4]. Here we present a quantum mechanics adapted version.

Theorem 2.2.2 (Polynomial potentials and Galois groups, [5])**.** *Let us consider the Schrödinger equation* (2.1)*, with* $V(x) \in \mathbb{C}[x]$ *a polynomial of degree* $k > 0$*. Then, its differential Galois group* $\mathrm{DGal}_K(\mathcal{L}_\lambda)$ *falls in one of the following cases:*

1. $\mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathrm{SL}(2, \mathbb{C})$,
2. $\mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathbb{B}$,

and the Eigenring of $H-\lambda$ *is trivial, i.e.,* $\mathcal{E}(H-\lambda) = \{1\}$*. Furthermore,* $\mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathbb{B}$ *if and only if the following conditions hold:*

1. $V(x) - \lambda$ *is a polynomial of degree* $k = 2n$ *writing in the form of equation* (2.2)*.*
2. $\pm b_{n-1} - n$ *is a positive even number* $2m$*,* $m \in \mathbb{Z}_+$*.*
3. *There exist a monic polynomial* P_m *of degree* m*, satisfying*

$$\partial_x^2 P_m + 2\left(x^n + \sum_{k=0}^{n-1} a_k x^k\right)\partial_x P_m + \left(nx^{n-1} + \sum_{k=0}^{n-2}(k+1)a_{k+1}x^k - \sum_{k=0}^{n-1} b_k x^k\right) P_m = 0,$$

or

$$\partial_x^2 P_m - 2\left(x^n + \sum_{k=0}^{n-1} a_k x^k\right)\partial_x P_m - \left(nx^{n-1} + \sum_{k=0}^{n-2}(k+1)a_{k+1}x^k + \sum_{k=0}^{n-1} b_k x^k\right) P_m = 0.$$

In such cases, the only possibilities for eigenfunctions with rational superpotentials are given by

$$\Psi_\lambda = P_m e^{f(x)}, \quad \text{or} \quad \Psi_\lambda = P_m e^{-f(x)}, \quad \text{where } f(x) = \frac{x^{n+1}}{n+1} + \sum_{k=0}^{n-1} \frac{a_k x^{k+1}}{k+1}.$$

An easy consequence of the above theorem is the following.

Corollary 2.2.3. *Assume that* $V(x)$ *is an exactly solvable polynomial potential. Then* $V(x)$ *is of degree* 2*.*

Remark 2.2.4. Given a polynomial potential $V(x)$ such that $\mathrm{spec}_p(H)\cap\Lambda\neq\emptyset$, we can obtain bound states and normalized wave functions if and only if the potential $V(x)$ is a polynomial of degree $4n+2$. Furthermore, b_{2n} must be odd integer is one integrability condition of $H\Psi=\lambda\Psi$ for $\lambda\in\Lambda$. In particular, if the potential

$$V(x)=x^{4n+2n}+\mu x^{2n},\quad n>0$$

is a quasi-exactly solvable, then μ is an odd integer. For this kind of potentials, we obtain bound states only when μ is a negative odd integer.

On another hand, the non-constant polynomial potentials $V(x)$ of degree $4n$ are associated to non-hermitian Hamiltonians and $\mathcal{PT}$ invariance which are not considered here, see [10]. Furthermore, b_{2n-1} must be even integer is one integrability condition of $H\Psi=\lambda\Psi$ for $\lambda\in\Lambda$. In particular, if the Schrödinger equation

$$H\Psi=\lambda\Psi,\quad V(x)=x^{4n}+\mu x^{2n-1},\quad \lambda\in\Lambda$$

is integrable, then μ is an even integer.

We present the following examples to illustrate the previous theorem and remark.

Weber's Equation and Harmonic Oscillator. The Schrödinger equation with potential $V(x)=x^2+q_1x+q_0$ corresponds to the Rehm's form of the Weber's equation (1.19), which has been studied in section 1.1.4. By lemma 2.2.1 we have

$$V(x)-\lambda=(x+a_0)^2+b_0,\quad a_0=q_1/2,\quad b_0=q_0-q_1^2/4-\lambda.$$

So that we obtain $\pm b_0-1=2m$, where $m\in\mathbb{Z}_+$. If b_0 is an odd integer, then

$$\mathrm{DGal}_K(\mathcal{L}_\lambda)=\mathbb{B},\quad \mathcal{E}(H-\lambda)=\{1\},\quad \lambda\in\Lambda=\{\pm(2m+1)+q_0-q_1^2/4: m\in\mathbb{Z}_+\}$$

and the set of eigenfunctions is either

$$\Psi_\lambda=P_me^{\frac{1}{2}(x^2+q_1x)},\ \textit{or,}\ \Psi_\lambda=P_me^{-\frac{1}{2}(x^2+q_1x)}.$$

In the second case we have bound states and $\mathrm{spec}_p(H)\cap\Lambda=\mathrm{spec}_p(H)=\{E_m=2m+1+q_0-q_1^2/4: m\in\mathbb{Z}_+\}$, which is infinite. The polynomials P_m are related with the Hermite polynomials H_m, [23, 44, 63].

In particular we have the harmonic oscillator potential, which is given in the list (1.38) and where $H\Psi=E\Psi$. Through the change of independent variable $x\mapsto\sqrt{\frac{2}{\omega}}x$ we obtain $V(x)=x^2-1$ and $\lambda=\frac{2}{\omega}E$, that is, $q_1=0$ and $q_0=-1$. In this way $\Lambda=\{\pm(2m-1)-1: m\in\mathbb{Z}_+\}$ and the set of eigenfunctions is either

$$\Psi_\lambda=P_me^{\frac{1}{2}x^2},\ \textit{or,}\ \Psi_\lambda=P_me^{-\frac{1}{2}x^2},$$

where as below, $\mathrm{DGal}_K(\mathcal{L}_\lambda)=\mathbb{B}$ and $\mathcal{E}(H-\lambda)=\{1\}$ for all $\lambda\in\Lambda$. In the second kind of eigenfunctions we have bound states, $\mathrm{spec}_p(H)\cap\Lambda=\mathrm{spec}_p(H)=\Lambda_+=$

$\{2m : m \in \mathbb{Z}_+\}$ and $P_m = H_m$. The wave functions of $H\Psi = E\Psi$ for the harmonic oscillator potential are given by

$$\Psi_m = H_m\left(\sqrt{\frac{2}{\omega}}x\right)\Psi_0, \quad \Psi_0 = e^{-\frac{\omega}{4}x^2}, \quad E = E_m = m\omega.$$

Quartic and Sextic Anharmonic Oscillator. The Schrödinger equation with potential $V(x) = x^4 + q_3x^3 + q_2x^2 + q_1x + q_0$ can be obtained through transformations of confluent Heun's equation, which is not considered here. By lemma 2.2.1 we have

$$V(x) - \lambda = (x^2 + a_1x + a_0)^2 + b_1x + b_0,$$

where $a_1 = q_3/2$, $a_0 = q_2/2 - a_1^2/2$, $b_1 = q_1 - 2a_0a_1$ and $b_0 = q_0 - a_0^2 - \lambda$. So that we obtain $\pm b_1 - 2 = 2m$, where $m \in \mathbb{Z}_+$. We assume that $\Lambda \neq \emptyset$, then b_1 is an even integer, P_m satisfy the relation (1.6) and $\mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathbb{B}$ for all $\lambda \in \Lambda$. The set of eigenfunctions is either

$$\Psi_\lambda = P_m e^{\frac{x^3}{3}+\frac{a_1x^2}{2}+a_0x}, \ \textit{or, } \Psi_\lambda = P_m e^{-\left(\frac{x^3}{3}+\frac{a_1x^2}{2}+a_0x\right)},$$

where λ and m are related, which means that Λ is finite, i.e., the potential is quasi-exactly solvable. In particular for $q_3 = 2\imath l$, $q_2 = l^2 - 2k$, $q_1 = 2\imath(lk - J)$ and $q_0 = 0$, we have the quartic anharmonic oscillator potential, which can be found in [10].

Now, considering the potentials $V(x,\mu) = x^4 + 4x^3 + 2x^2 - \mu x$, again by lemma 2.2.1 we have that

$$V(x,\mu) - \lambda = (x^2 + 2x - 1)^2 + (4-\mu)x - 1 - \lambda,$$

so that $\pm(4-\mu) - 2 = 2n$, where $n \in \mathbb{Z}_+$ and in consequence $\mu \in 2\mathbb{Z}$. Such μ can be either $\mu = 2 - 2n$ or $\mu = 2n + 6$, where $n \in \mathbb{Z}_+$. By theorem 2.2.2, there exists a monic polynomial P_n satisfying respectively

$$\partial_x^2 P_n + (2x^2 + 4x - 2)\partial_x P_n + ((\mu - 2)x + 3 + \lambda)P_n = 0, \quad \mu = 2 - 2n, \quad \text{or}$$

$$\partial_x^2 P_n - (2x^2 + 4x - 2)\partial_x P_n + ((\mu - 6)x - 1 + \lambda)P_n = 0, \quad \mu = 2n + 6$$

for $\Lambda \neq \emptyset$. This algebraic relation between the coefficients of P_n, μ and λ give us the set Λ in the following way:

1. Write $P_n = x^n + c_{n-1}x^{n-1} + \ldots + c_0$, where c_i are unknown.
2. Pick μ and replace P_n in the algebraic relation (1.6) to obtain a polynomial of degree n with $n+1$ undetermined coefficients involving $c_0, \ldots c_{n-1}$ and λ. Each of such coefficients must be zero.

3. The term $n+1$ is linear in λ and c_{n-1}, thus we write c_{n-1} in terms of λ. After of the elimination of the term $n+1$, we replace c_{n-1} in the term n to obtain a quadratic polynomial in λ and so on until arrive to the constant term which is a polynomial of degree $n+1$ in λ $(Q_{n+1}(\lambda))$. In this way, $\Lambda = \{\lambda : Q_{n+1}(\lambda) = 0\}$ and $c_0, \dots, c_{n-1}$ are determined for each value of λ.

For $\mu = 2n+6$, we have:

$$\begin{array}{llll} n=0, & V(x,6), & P_0 = 1, & \Lambda = \{1\} \\ n=1, & V(x,8), & P_1 = x+1\mp\sqrt{2}, & \Lambda = \{3\pm 2\sqrt{2}\} \\ \vdots & & & \end{array}$$

and the set of eigenfunctions is

$$\Psi_{\lambda,\mu} = P_n e^{-\frac{1}{3}x^3 - x^2 + x}.$$

In the same way, we can obtain Λ, P_n and $\Psi_{\lambda,\mu}$ for $\mu = 2-2n$. However, we have not bound states, $\mathrm{spec}_p(H)\cap\Lambda = \emptyset$, $\mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathbb{B}$ and $\mathcal{E}(H-\lambda) = \{1\}$ for all $\lambda \in \Lambda$.

The well known *sextic anharmonic oscillator* $x^6 + q_5x^5 + \cdots + q_1x + q_0$ can be treated in a similar way, obtaining bound states wave functions and the Bender-Dunne orthogonal polynomials, which corresponds to $Q_{n+1}(\lambda)$, i.e., we can have the same results of [11, 37, 77]. The Schrödinger equation with this potential, under suitable transformations, also falls in a confluent Heun's equation.

2.2.2 Rational Potentials and Kovacic's Algorithm

In this subsection we apply Kovacic's algorithm to solve the Schrödinger equation with rational potentials listed in subsection 1.2.2.

Three dimensional harmonic oscillator potential:

$$V(r) = \frac{1}{4}\omega^2 r^2 + \frac{l(l+1)}{r^2} - \left(l + \frac{3}{2}\right)\omega, \quad \ell \in \mathbb{Z},$$

we can see that the equation (1.22) for this case is

$$\partial_r^2\Psi = \left(\left(\frac{1}{2}\omega r\right)^2 + \frac{\ell(\ell+1)}{r^2} - \left(\ell + \frac{3}{2}\right)\omega - E\right)\Psi.$$

By the change $r \mapsto \left(\sqrt{\frac{2}{\omega}}\right) r$ we obtain the Schrödinger equation

$$\partial_r^2\Psi = \left(r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3) - \lambda\right)\Psi, \quad \lambda = \frac{2}{\hbar\omega}E.$$

In order to apply Kovacic algorithm, we denote:

$$R = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3) - \lambda$$

we can see that this equation could fall in case one, in case two or in case four (of Kovacic's algorithm). We start discarding the case two because by step one (of the Kovacic's algorithm) we should have conditions c_2 and ∞_3, in this way we should have $E_c = \{2, 4+4\ell, -4\ell\}$ and $E_\infty = \{-2\}$, and by step two, we should have that $n = -4 \notin \mathbb{Z}_+$, so that $D = \emptyset$, that is, this Schrödinger equation never falls in case two. Now, we only works with case one, by step one, conditions c_2 and ∞_3 are satisfied, so that

$$\left[\sqrt{R}\right]_c = 0, \quad \alpha_c^{\pm} = \frac{1 \pm (2\ell+1)}{2}, \quad \left[\sqrt{R}\right]_\infty = r, \quad \alpha_\infty^{\pm} = \frac{\mp(\lambda + 2\ell + 3) - 1}{2}.$$

By step two we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\begin{array}{lll}
\Lambda_{++}) & n = \alpha_\infty^+ - \alpha_0^+ = -\frac{1}{2}(4\ell + 6 + \lambda), & \lambda = -2n - 4\ell - 6, \\[2mm]
\Lambda_{+-}) & n = \alpha_\infty^+ - \alpha_0^- = -\frac{1}{2}(4 + \lambda), & \lambda = -2n - 4, \\[2mm]
\Lambda_{-+}) & n = \alpha_\infty^- - \alpha_0^+ = \frac{\lambda}{2}, & \lambda = 2n, \\[2mm]
\Lambda_{--}) & n = \alpha_\infty^- - \alpha_0^- = \frac{1}{2}(4\ell + 2 + \lambda), & \lambda = 2n - 4\ell - 2,
\end{array}$$

where $\Lambda_{++} \cup \Lambda_{+-} \cup \Lambda_{-+} \cup \Lambda_{--} = \Lambda$, which means that $\lambda = 2m$, $m \in \mathbb{Z}$. Now, for $\lambda \in \Lambda$, the rational function ω is given by:

$$\begin{array}{lll}
\Lambda_{++}) & \omega = r + \frac{\ell+1}{r}, & R_n = r^2 + \frac{\ell(\ell+1)}{r^2} + (2\ell+3) + 2n, \\[2mm]
\Lambda_{+-}) & \omega = r - \frac{\ell}{r}, & R_n = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell-1) + 2n, \\[2mm]
\Lambda_{-+}) & \omega = -r + \frac{\ell+1}{r}, & R_n = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3) - 2n, \\[2mm]
\Lambda_{--}) & \omega = -r - \frac{\ell}{r}, & R_n = r^2 + \frac{\ell(\ell+1)}{r^2} + (2\ell-1) - 2n,
\end{array}$$

where R_n is the coefficient of the differential equation $\widetilde{\mathcal{L}}_n := \partial_r^2\Psi = R_n\Psi$, which is integrable for every n and for every $\lambda \in \Lambda$ we can see that $\mathrm{DGal}_K(\widetilde{\mathcal{L}}_n) = \mathrm{DGal}_K(\mathcal{L}_\lambda)$, where $\mathcal{L}_\lambda := H\Psi = \lambda\Psi$.

By step three, there exists a polynomial of degree n satisfying the relation

(1.6):

$$\begin{array}{llcll}
\Lambda_{++}) & \partial_r^2 P_n + 2\left(r + \frac{\ell+1}{r}\right)\partial_r P_n - 2nP_n & = & 0, & \lambda \in \Lambda_-,\\
\Lambda_{+-}) & \partial_r^2 P_n + 2\left(r - \frac{\ell}{r}\right)\partial_r P_n - 2nP_n & = & 0, & \lambda \in \Lambda_-,\\
\Lambda_{-+}) & \partial_r^2 P_n + 2\left(-r + \frac{\ell+1}{r}\right)\partial_r P_n + 2nP_n & = & 0, & \lambda \in \Lambda_+,\\
\Lambda_{--}) & \partial_r^2 P_n + 2\left(-r - \frac{\ell}{r}\right)\partial_r P_n + 2nP_n & = & 0, & \lambda \in \Lambda.
\end{array}$$

These polynomials exists for all $\lambda \in \Lambda$ when their degrees are $n \in 2\mathbb{Z}$, while for $n \in 2\mathbb{Z}+1$, they exists only for $\lambda \in \Lambda_{-+}$) and Λ_{--}) with special conditions. In this way, we have obtained the set $\Lambda = 2\mathbb{Z}$, where $\Lambda_{++} = 4\mathbb{Z}_-$, $\Lambda_{+-} = 2\mathbb{Z}_-$, $\Lambda_{-+} = 4\mathbb{Z}_+$, $\Lambda_{--} = 2\mathbb{Z}$.

The possibilities for eigenfunctions, considering only $\lambda \in 4\mathbb{Z}$, are given by

$$\begin{array}{llcll}
\Lambda_{++}) & \Psi_n(r) & = & r^{\ell+1}P_{2n}(r)e^{\frac{r^2}{2}}, & \lambda \in \Lambda_-,\\
\Lambda_{+-}) & \Psi_n(r) & = & r^{-\ell}P_{2n}(r)e^{\frac{r^2}{2}}, & \lambda \in \Lambda_-,\\
\Lambda_{-+}) & \Psi_n(r) & = & r^{\ell+1}P_{2n}(r)e^{\frac{-r^2}{2}}, & \lambda \in \Lambda_+,\\
\Lambda_{--}) & \Psi_n(r) & = & r^{-\ell}P_{2n}(r)e^{\frac{-r^2}{2}}, & \lambda \in \Lambda.
\end{array}$$

To obtain the spectrum, we look Ψ_n satisfying the conditions (1.23) which is in only true for $\lambda \in \Lambda_{-+}$. With the change $r \mapsto \sqrt{\frac{\omega}{2}}r$, the spectrum and ground state of the Schrödinger equation with the 3D-harmonic oscillator potential are respectively $\mathrm{spec}_p(H) = \{E_n : n \in \mathbb{Z}_+\}$, where $E_n = 2n\omega$, and

$$\Psi_0 = \left(\sqrt{\frac{\omega}{2}}r\right)^{\ell+1} e^{-\frac{\omega}{4}r^2}.$$

The rest of bound state wave functions are obtained as $\Psi_n = P_{2n}\Psi_0$. Now, we can see that $\mathrm{DGal}_K(\mathcal{L}_0) = \mathbb{B}$ and $\mathcal{E}(H) = \{1\}$. Since $\Psi_n = P_{2n}\Psi_0$, for all $\lambda \in \Lambda$ we have that $\mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathbb{B}$ and $\mathcal{E}(H-\lambda) = \{1\}$. In particular, $\mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathbb{B}$ and $\mathcal{E}(H-\lambda) = \{1\}$ for all $\lambda \in \mathrm{spec}_p(H)$, where $\lambda = \frac{2}{\omega}E$.

We remark that the Schrödinger equation with the 3D-harmonic oscillator potential, through the changes $r \mapsto \frac{1}{2}\omega r^2$ and $\Psi \mapsto \sqrt{r}\Psi$, fall in a Whittaker differential equation (1.16) in where the parameters are given by

$$\kappa = \frac{(2\ell+3)\omega + 2E}{4\omega}, \quad \mu = \frac{1}{2}\ell + \frac{1}{4}.$$

Applying the theorem 1.1.30, we can see that for integrability, $\pm\kappa \pm \mu$ must be a half integer. These conditions coincides with our four sets $\Lambda_{\pm\pm}$.

Coulomb potential:

$$V(r) = -\frac{\mathrm{e}^2}{r} + \frac{\ell(\ell+1)}{r^2} + \frac{\mathrm{e}^4}{4(\ell+1)^2}, \quad \ell \in \mathbb{Z}.$$

we can see that the equation (1.22) for this case is

$$\partial_r^2 \Psi = \left(\frac{\ell(\ell+1)}{r^2} - \frac{\mathrm{e}^2}{r} + \frac{\mathrm{e}^4}{4(\ell+1)^2} - E \right) \Psi.$$

By the change $r \mapsto \frac{2(\ell+1)}{\mathrm{e}^2} r$ we obtain the Schrödinger equation

$$\partial_r^2 \Psi = \left(\frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{\xi} + 1 - \lambda \right) \Psi, \quad \lambda = \frac{4(\ell+1)^2}{\mathrm{e}^4} E.$$

In order to apply Kovacic algorithm, we denote

$$R = \frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} + 1 - \lambda.$$

Firstly we analyze the case for $\lambda = 1$: we can see that this equation only could fall in case two or in case four. We start discarding the case two because by step one we should have conditions c_2 and ∞_3, in this way we should have $E_c = \{2, 4+4\ell, -4\ell\}$ and $E_\infty = \{1\}$, and by step two, we should have that $n \notin \mathbb{Z}$, so that $n \notin \mathbb{Z}_+$ and $D = \emptyset$, that is, the Galois group of this Schrödinger equation for $\lambda = 1$ is $\mathrm{SL}(2, \mathbb{C})$.

Now, we analyze the case for $\lambda \neq 1$: we can see that this equation could fall in case one, in case two or in case four. We start discarding the case two because by step one we should have conditions c_2 and ∞_3, in this way we should have $E_c = \{2, 4+4\ell, -4\ell\}$ and $E_\infty = \{0\}$, and by step two, we should have that $n = 2\ell \in \mathbb{Z}_+$, so that $D = \{2\ell\}$ and the rational function θ is $\theta = \frac{-2\ell}{x}$, but we discard this case because only could exists one polynomial of degree 2ℓ for a fixed ℓ, and for instance, only could exist one eigenstate and one eigenfunction for the Schrödinger equation.

Now, we only works with case one, by step one, conditions c_2 and ∞_3 are satisfied, so that

$$\left[\sqrt{R}\right]_c = 0, \quad \alpha_c^{\pm} = \frac{1 \pm (2\ell+1)}{2}, \quad \left[\sqrt{R}\right]_\infty = \sqrt{1-\lambda}, \quad \alpha_\infty^{\pm} = \mp \frac{\ell+1}{\sqrt{1-\lambda}}.$$

By step two we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\Lambda_{++}) \qquad n = \alpha_\infty^+ - \alpha_0^+ = -(\ell+1)\left(1 + \tfrac{1}{\sqrt{1-\lambda}}\right), \quad \lambda = 1 - \left(\tfrac{\ell+1}{\ell+1+n}\right)^2,$$

$$\Lambda_{+-}) \qquad n = \alpha_\infty^+ - \alpha_0^- = -\tfrac{\ell+1}{\sqrt{1-\lambda}} + \ell, \qquad \lambda = 1 - \left(\tfrac{\ell+1}{\ell-n}\right)^2,$$

$$\Lambda_{-+}) \qquad n = \alpha_\infty^- - \alpha_0^+ = (\ell+1)\left(\tfrac{1}{\sqrt{1-\lambda}} - 1\right), \quad \lambda = 1 - \left(\tfrac{\ell+1}{\ell+1+n}\right)^2,$$

$$\Lambda_{--}) \qquad n = \alpha_\infty^- - \alpha_0^- = \tfrac{\ell+1}{\sqrt{1-\lambda}} + \ell, \qquad \lambda = 1 - \left(\tfrac{\ell+1}{\ell-n}\right)^2.$$

We can see that $\lambda \in \Lambda_-$ when $\lambda \leq 0$, while $\lambda \in \Lambda_+$ when $0 \leq \lambda < 1$. Furthermore:

$$\Lambda_{++}) \qquad \ell \leq -1, \qquad \lambda \in \begin{cases} \Lambda_-, & \ell \leq \frac{-n-2}{2} \\ \Lambda_+, & \frac{-n-2}{2} \leq \ell \leq -1 \end{cases}$$

$$\Lambda_{+-}) \qquad \ell > 0, \qquad \lambda \in \begin{cases} \Lambda_-, & \ell \geq \frac{n-1}{2} \\ \Lambda_+, & 0 \leq \ell \leq \frac{n-1}{2} \end{cases}$$

$$\Lambda_{-+}) \qquad \ell \in \mathbb{Z}, \qquad \lambda \in \begin{cases} \Lambda_-, & \ell \leq \frac{-n-2}{2} \\ \Lambda_+, & \ell \geq -1 \end{cases}$$

$$\Lambda_{--}) \qquad \ell > 0, \qquad \lambda \in \begin{cases} \Lambda_-, & \ell \geq \frac{n-1}{2} \\ \Lambda_+, & 0 \leq \ell \leq \frac{n-1}{2} \end{cases}$$

In this way we can obtain the possible set $\Lambda = \Lambda_{++} \cup \Lambda_{+-} \cup \Lambda_{-+} \cup \Lambda_{--}$, that is

$$\Lambda = \left\{1 - \left(\frac{\ell+1}{\ell+1+n}\right)^2 : n \in \mathbb{Z}_+\right\} \cup \left\{1 - \left(\frac{\ell+1}{\ell-n}\right)^2 : n \in \mathbb{Z}_+\right\}, \tag{2.3}$$

Now, for $\lambda \in \Lambda$, the rational function ω is given by:

$$\Lambda_{++}) \qquad \omega = \tfrac{\ell+1}{\ell+1+n} + \tfrac{\ell+1}{r}, \qquad \lambda \in \Lambda_{++}, \quad R_n = \tfrac{\ell(\ell+1)}{r^2} - \tfrac{2(\ell+1)}{r} + \left(\tfrac{\ell+1}{\ell+1+n}\right)^2,$$

$$\Lambda_{+-}) \qquad \omega = \tfrac{\ell+1}{\ell-n} - \tfrac{\ell}{r}, \qquad \lambda \in \Lambda_{+-}, \quad R_n = \tfrac{\ell(\ell+1)}{r^2} - \tfrac{2(\ell+1)}{r} + \left(\tfrac{\ell+1}{\ell-n}\right)^2,$$

$$\Lambda_{-+}) \qquad \omega = -\tfrac{\ell+1}{\ell+1+n} + \tfrac{\ell+1}{r}, \qquad \lambda \in \Lambda_{-+}, \quad R_n = \tfrac{\ell(\ell+1)}{r^2} - \tfrac{2(\ell+1)}{r} + \left(\tfrac{\ell+1}{\ell+1+n}\right)^2,$$

$$\Lambda_{--}) \qquad \omega = -\tfrac{\ell+1}{\ell-n} - \tfrac{\ell}{r}, \qquad \lambda \in \Lambda_{--}, \quad R_n = \tfrac{\ell(\ell+1)}{r^2} - \tfrac{2(\ell+1)}{r} + \left(\tfrac{\ell+1}{\ell-n}\right)^2,$$

where R_n is the coefficient of the differential equation $\partial_r^2\Psi = R_n\Psi$, which is integrable for every n.

By step three, there exists a polynomial of degree n satisfying the relation (1.6),

$$\begin{array}{llcl}
\Lambda_{++}) & \partial_r^2 P_n + 2\left(\frac{\ell+1}{\ell+1+n} + \frac{\ell+1}{r}\right)\partial_r P_n + \frac{2(\ell+1)}{r}\left(1 + \frac{\ell+1}{\ell+1+n}\right)P_n & = & 0,\\
\Lambda_{+-}) & \partial_r^2 P_n + 2\left(\frac{\ell+1}{\ell-n} - \frac{\ell}{r}\right)\partial_r P_n + \frac{2(\ell+1)}{r}\left(1 - \frac{\ell+1}{\ell-n}\right)P_n & = & 0,\\
\Lambda_{-+}) & \partial_r^2 P_n + 2\left(-\frac{\ell+1}{\ell+1+n} + \frac{\ell+1}{r}\right)\partial_r P_n + \frac{2(\ell+1)}{r}\left(1 - \frac{\ell+1}{\ell+1+n}\right)P_n & = & 0,\\
\Lambda_{--}) & \partial_r^2 P_n + 2\left(-\frac{\ell+1}{\ell-n} - \frac{\ell}{r}\right)\partial_r P_n + \frac{2(\ell+1)}{r}\left(1 + \frac{\ell+1}{\ell-n}\right)P_n & = & 0.
\end{array}$$

These polynomials exists for every $\lambda\in\Lambda$ when $n\in\mathbb{Z}$, but $P_0 = 1$ is satisfied only for $\lambda\in\Lambda_{-+}$. In this way, we have obtained the set Λ given by the equation (2.3).

The possibilities for eigenfunctions are given by

$$\begin{array}{lllll}
\Lambda_{++}) & \Psi_n(r) = r^{\ell+1}P_n(r)f_n(r)e^{r}, & f_n(r) = e^{\frac{-nr}{\ell+1+n}}, & \lambda\in\left\{\begin{array}{ll}\Lambda_-, & \ell\leq\frac{-n-2}{2}\\ \Lambda_+, & \frac{-n-2}{2}\leq\ell\leq -1\end{array}\right.\\
\Lambda_{+-}) & \Psi_n(r) = r^{-\ell}P_n(r)f_n(r)e^{r}, & f_n(r) = e^{\frac{n+1}{\ell-n}r}, & \lambda\in\left\{\begin{array}{ll}\Lambda_-, & \ell\geq\frac{n-1}{2}\\ \Lambda_+, & 0\leq\ell\leq\frac{n-1}{2}\end{array}\right.\\
\Lambda_{-+}) & \Psi_n(r) = r^{\ell+1}P_n(r)f_n(r)e^{-r}, & f_n(r) = e^{\frac{nr}{\ell+1+n}}, & \lambda\in\left\{\begin{array}{ll}\Lambda_-, & \ell\leq\frac{-n-2}{2}\\ \Lambda_+, & \ell\geq -1\end{array}\right.\\
\Lambda_{--}) & \Psi_n(r) = r^{-\ell}P_n(r)f_n(r)e^{-r}, & f_n(r) = e^{\frac{n+1}{n-\ell}r}, & \lambda\in\left\{\begin{array}{ll}\Lambda_-, & \ell\geq\frac{n-1}{2}\\ \Lambda_+, & 0\leq\ell\leq\frac{n-1}{2}\end{array}\right.
\end{array}$$

but Ψ_n should satisfy the conditions (1.23) which is in only true for $\Lambda_{-+}\cap\Lambda_+$, so that we choose $\Lambda_{-+}\cap\Lambda_+ = \mathrm{spec}_p(H)$, that is

$$\mathrm{spec}_p(H) = \left\{1 - \left(\frac{\ell+1}{\ell+n+1}\right)^2 : \quad n\in\mathbb{Z}_+, \quad \ell\geq -1\right\}.$$

By the change $r\mapsto\frac{\mathrm{e}^2}{2(\ell+1)}r$, the spectrum and ground state of the Schrödinger equation with Coulomb potential are respectively

$$\mathrm{spec}_p(H) = \{E_n : n\in\mathbb{Z}_+\}, \quad E_n = \frac{\mathrm{e}^4}{4}\left(\frac{1}{(\ell+1)^2} - \frac{1}{(\ell+1+n)^2}\right)$$

and

$$\Psi_0 = \left(\frac{\mathrm{e}^2}{2(\ell+1)}r\right)^{\ell+1} e^{-\frac{\mathrm{e}^2}{2(\ell+1)}r}.$$

the rest of eigenstates $\Psi_n = P_n f_n \Psi_0$, where

$$f_n(r) = e^{\frac{ne^2 r}{2(\ell+1+n)(\ell+1)}}.$$

Now, we can see that $\mathrm{DGal}_K(\mathcal{L}_0) = \mathbb{B}$ and $\mathcal{E}(H) = \{1\}$. Since $\Psi_n = P_{2n} f_n \Psi_0$, for all $\lambda \in \Lambda$ we have that $\mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathbb{B}$ and $\mathcal{E}(H-\lambda) = \{1\}$. In particular, $\mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathbb{B}$ and $\mathcal{E}(H-\lambda) = \{1\}$ for all $\lambda \in \mathrm{spec}_p(H)$, where $E = \frac{\mathrm{e}^4}{4(\ell+1)^2}\lambda$.

We remark that, as in the three dimensional harmonic oscillator, the Schrödinger equation with the Coulomb potential, through the change

$$r \mapsto \frac{\sqrt{-4\,(\ell+1)^2 E + \mathrm{e}^4}}{\ell+1} r,$$

falls in a Whittaker differential equation (1.16) in where the parameters are given by

$$\kappa = \frac{\mathrm{e}^2\,(\ell+1)}{\sqrt{-4\,(\ell+1)^2 E + \mathrm{e}^4}}, \quad \mu = \ell + \frac{1}{2}.$$

Applying the theorem 1.1.30, we can impose $\pm\kappa \pm \mu$ half integer, to coincides with our four sets $\Lambda_{\pm\pm}$.

Remark 2.2.5. By direct application of Kovacic's Algorithm we have:

- The Schrödinger equation (2.1) with potential

  $$V(x) = ax^2 + \frac{b}{x^2}$$

  is integrable for $\lambda \in \Lambda$ when

  - $a = 0, \quad b = \mu(\mu+1), \quad \mu \in \mathbb{C}, \quad \Lambda = \mathbb{C}$,
  - $a = 1, \quad b = 0, \quad \lambda \in \Lambda = 2\mathbb{Z}+1$,
  - $a = 1, \quad b = \ell(\ell+1), \quad \ell \in \mathbb{Z}^*, \Lambda = 2\mathbb{Z}+1$.

- The only rational potentials, up to transformations, in which the levels of energy are in the same distance belongs to the family of potentials given by

  $$V(x) = \sum_{k=-\infty}^{2} a_k x^k, \quad a_2 \neq 0.$$

  In particular, the set Λ for the harmonic oscillator ($a = 1$, $b = 0$) and $3D$ harmonic oscillator ($a = 1$, $b = \ell(\ell+1)$) satisfies this.

Proposition 2.2.6. *Let $\mathcal{L}_\lambda$ be the Schrödinger equation* (2.1) *with $K=\mathbb{C}(x)$. If $\mathrm{DGal}_K(\mathcal{L}_0)$ is finite primitive, then $\mathrm{DGal}_K(\mathcal{L}_\lambda)$ is not finite primitive for all $\lambda\in\Lambda\setminus\{0\}$.*

Proof. Pick $V\in\mathbb{C}(x)$ such that $\mathcal{L}_0$ falls in case 3 of Kovacic algorithm, then $\circ u_\infty\geq 2$. Let assume $t,s\in\mathbb{C}[x]$ such that $V=\frac{s}{t}$, then $\deg(t)\geq\deg(s)+2$ and $V-\lambda=\frac{s-\lambda t}{t}$. Now, for $\lambda\neq 0$ we have that $\deg(s-\lambda t)=\deg(t)$ and therefore $\circ(V-\lambda)_\infty=0$. So that for $\lambda\neq 0$, the equation $\mathcal{L}_\lambda$ does not falls in case 3 of Kovacic algorithm and therefore $\mathrm{DGal}_K(\mathcal{L}_\lambda)$ is not finite primitive. □

Corollary 2.2.7. *Let $\mathcal{L}_\lambda$ be the Schrödinger equation* (2.1) *with $K=\mathbb{C}(x)$. If $\mathrm{Card}(\Lambda)>1$, then there is either zero or one value of λ such that $\mathrm{DGal}_K(\mathcal{L}_\lambda)$ is a finite primitive group.*

It seems that the study of the Galois groups of the Schrödinger equation with the Coulomb potential has been analyzed by Jean-Pierre Ramis using his summability theory since in the eighties of the past century, see [72].

2.2.3 Darboux Transformations

Here present a Galoisian approach to Darboux transformation, Crum iteration and shape invariant potentials. We denote by $W(y_1,\ldots,y_n)$ the Wronskian

$$W(y_1,\ldots,y_n)=\begin{vmatrix} y_1 & \cdots & y_n\\ \vdots & & \vdots\\ \partial_x^{n-1}y_1 & \cdots & \partial_x^{n-1}y_n\end{vmatrix},$$

DT the Darboux transformation, DT_n the n iteration of DT and CI_n the Crum iteration. Also we use the notation of subsection 1.2. We recall that $K=\mathbb{C}(x)$ and for the rest of differential fields as usually along this memory, we mean the smallest differential containing the coefficients of the linear differential equations.

Theorem 2.2.8 (Galoisian version of DT)**.** *Let assume $H_\pm=\partial_x^2+V_\pm(x)$ and $\Lambda\neq\emptyset$. Let $\mathcal{L}_\lambda$ given by the Schrödinger equation $H_-\Psi^{(-)}=\lambda\Psi^{(-)}$ with $V_-(x)\in K$ and $\widetilde{\mathcal{L}}_\lambda$ given by the Schrödinger equation $H_+\Psi^{(+)}=\lambda\Psi^{(+)}$ with $V_+(x)\in\widetilde{K}$. Let* DT *be the transformation such that $\mathcal{L}\mapsto\widetilde{\mathcal{L}}$, $V_-\mapsto V_+$, $\Psi^{(-)}\mapsto\Psi^{(+)}$. Then the following statements holds:*

i) $\mathrm{DT}(V_-)=V_+=\Psi^{(-)}_{\lambda_1}\partial_x^2\left(\frac{1}{\Psi^{(-)}_{\lambda_1}}\right)+\lambda_1=V_--2\partial_x^2(\ln\Psi^{(-)}_{\lambda_1})$,

$\mathrm{DT}(\Psi^{(-)}_{\lambda_1})=\Psi^{(+)}_{\lambda_1}=\frac{1}{\Psi^{(-)}_{\lambda_1}}$, *where $\Psi^{(-)}_{\lambda_1}$ is a particular solution of $\mathcal{L}_{\lambda_1}$, $\lambda_1\in\Lambda$.*

ii) $\mathrm{DT}(\Psi^{(-)}_{\lambda})=\Psi^{(+)}_{\lambda}=\partial_x\Psi^{(-)}_{\lambda}-\partial_x(\ln\Psi^{(-)}_{\lambda_1})\Psi^{(-)}_{\lambda}=\frac{W(\Psi^{(-)}_{\lambda_1},\Psi^{(-)}_{\lambda})}{W(\Psi^{(-)}_{\lambda_1})}$, *$\lambda\neq\lambda_1$, where $\Psi^{(-)}_{\lambda}$ is the general solution of $\mathcal{L}_\lambda$ for $\lambda\in\Lambda\setminus\{\lambda_1\}$ and $\Psi^{(+)}_{\lambda}$ is the general solution of $\widetilde{\mathcal{L}}_\lambda$ also for $\lambda\in\Lambda\setminus\{\lambda_1\}$.*

In agreement with the previous theorem we obtain the following results.

Proposition 2.2.9. DT *is isogaloisian and virtually strong isogaloisian. Furthermore, if* $\partial_x(\ln \Psi_{\lambda_1}^{(-)}) \in K$, *then* DT *is strong isogaloisian.*

Proof. Let K, L be the differential field and the Picard-Vessiot extension of the equation $\mathcal{L}_\lambda$. Let $\widetilde{K}$, $\widetilde{L}$ the differential field and the Picard-Vessiot extension of the equation $\widetilde{\mathcal{L}}_\lambda$. Due to $\mathrm{DT}(V_-) = V_+ = 2W^2 - V_- - 2\lambda_1$, where $W = -\partial_x(\ln \Psi_{\lambda_1}^{(-)})$, we have $\widetilde{K} = K\left\langle \partial_x(\ln \Psi_{\lambda_1}^{(-)})\right\rangle$. By theorem 1.1.10 we have that the Riccati equation $\partial_x W = V_- - W^2$ has one algebraic solution, in this case $W = -\partial_x(\ln \Psi_{\lambda_1}^{(-)})$. Let $\langle \Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)}\rangle$ be the basis of solutions for equation $\mathcal{L}_\lambda$ and $\langle \Psi_{(1,\lambda)}^{(+)}, \Psi_{(2,\lambda)}^{(+)}\rangle$ the basis of solutions for equation $\widetilde{\mathcal{L}}_\lambda$. Since the differential field for equation $\widetilde{\mathcal{L}}_\lambda$ is $\widetilde{K} = K\langle \partial_x(\ln \Psi_{\lambda_1}^{(-)})\rangle$, we have that $L = K\langle \Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)}\rangle$ and

$$\begin{aligned}\widetilde{L} &= \widetilde{K}\langle \Psi_{(1,\lambda)}^{(+)}, \Psi_{(2,\lambda)}^{(+)}\rangle = K\langle \Psi_{(1,\lambda)}^{(+)}, \Psi_{(2,\lambda)}^{(+)}, \partial_x(\ln \Psi_{\lambda_1}^{(-)})\rangle \\ &= K\langle \Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)}, \partial_x(\ln \Psi_{\lambda_1}^{(-)})\rangle = \widetilde{K}\langle \Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)}\rangle,\end{aligned}$$

for $\lambda = \lambda_1$ and for $\lambda \neq \lambda_1$. Since $\partial_x(\ln \Psi_{\lambda_1}^{(-)})$ is algebraic over K, then

$$(\mathrm{DGal}(L/K)(\mathcal{L}_\lambda))^0 = (\mathrm{DGal}(\widetilde{L}/K)(\widetilde{\mathcal{L}}_\lambda))^0, \quad \mathrm{DGal}(L/K)(\mathcal{L}_\lambda) = \mathrm{DGal}(\widetilde{L}/\widetilde{K})(\widetilde{\mathcal{L}}_\lambda),$$

which means that DT is an virtually strong and isogalosian transformation.
In the case $\partial_x(\ln \Psi_{\lambda_1}^{(-)}) \in K$, then $\widetilde{K} = K$ and $\widetilde{L} = L$, which means that DT is an strong isogalosian transformation. □

Proposition 2.2.10. *Let consider* $\mathfrak{L}_\lambda := H_- - \lambda$ *and* $\widetilde{\mathfrak{L}}_\lambda := H_+ - \lambda$ *such that* $\mathrm{DT}(H_- - \lambda) = H_+ - \lambda$, *then the eigenrings of* $\mathfrak{L}_\lambda$ *and* $\widetilde{\mathfrak{L}}_\lambda$ *are isomorphic.*

Proof. Let assume $\mathcal{E}(\mathfrak{L}_\lambda)$ and $\mathcal{E}(\widetilde{\mathfrak{L}}_\lambda)$ the eigenrings of $\mathfrak{L}_\lambda$ and $\widetilde{\mathfrak{L}}_\lambda$ respectively. By proposition 2.2.9 the connected identity component of the Galois group is preserved by Darboux transformation and for instance the eigenrings is preserved by Darboux transformation. Now, suppose that $T \in \mathcal{E}(\mathfrak{L}_\lambda)$, $\mathrm{Sol}(\mathfrak{L}_\lambda)$ and $\mathrm{Sol}(\widetilde{\mathfrak{L}}_\lambda)$ the solutions space for $\mathfrak{L}_\lambda \Psi^{(-)} = 0$ and $\widetilde{\mathfrak{L}}_\lambda \Psi^{(-)} = 0$ respectively. To transform $\mathcal{E}(\mathfrak{L}_\lambda)$ into $\mathcal{E}(\widetilde{\mathfrak{L}}_\lambda)$ we follows the diagram:

$$\begin{tikzcd} \mathrm{Sol}(\mathfrak{L}_\lambda) \arrow[r, "T"] & \mathrm{Sol}(\mathfrak{L}_\lambda) \arrow[d, "A"] \\ \mathrm{Sol}(\widetilde{\mathfrak{L}}_\lambda) \arrow[u, "A^\dagger"] \arrow[r, "\widetilde{T}"'] & \mathrm{Sol}(\widetilde{\mathfrak{L}}_\lambda) \end{tikzcd} \qquad \Rightarrow \widetilde{T} = ATA^\dagger \mathrm{mod}(\widetilde{\mathfrak{L}}_\lambda) \in \mathcal{E}(\widetilde{\mathfrak{L}}_\lambda),$$

where $A^\dagger$ and A are the raising and lowering operators. □

Example. Let assume the Schrödinger equation $\mathcal{L}_\lambda$ with potential $V = V_- = 0$, which means that $\Lambda = \mathbb{C}$. If we choose $\lambda_1 = 0$ and as particular solution $\Psi_0^{(-)} = x$, then for $\lambda \neq 0$ the general solution is given by

$$\Psi_\lambda^{(-)} = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}.$$

Applying the Darboux transformation DT we have $\mathrm{DT}(\mathcal{L}_\lambda) = \widetilde{\mathcal{L}_\lambda}$, where

$$\mathrm{DT}(V_-) = V_+ = \frac{2}{x^2}$$

and for $\lambda \neq 0$

$$\mathrm{DT}(\Psi_\lambda^{(-)}) = \Psi_\lambda^{(+)} = \frac{c_1(\sqrt{-\lambda}x - 1)e^{\sqrt{-\lambda}x}}{x} - \frac{c_2(\sqrt{-\lambda}x + 1)e^{-\sqrt{-\lambda}x}}{x}.$$

We can see that $\widetilde{K} = K = \mathbb{C}(x)$ for all $\lambda \in \Lambda$, the Picard-Vessiot extensions can be either $L = \widetilde{L} = \mathbb{C}(x)$ for $\lambda = 0$ or $L = \widetilde{L} = \mathbb{C}(x, e^{\sqrt{\lambda}x})$ for $\lambda \in \mathbb{C}^*$. In this way we have that $\mathrm{DGal}(L/K)(\mathcal{L}_0) = \mathrm{DGal}(L/K)(\widetilde{\mathcal{L}}_0) = e$, and for $\lambda \neq 0$ we have $\mathrm{DGal}(L/K)(\mathcal{L}_\lambda) = \mathrm{DGal}(L/K)(\widetilde{\mathcal{L}}_\lambda) = \mathbb{G}_m$. The eigenrings are given by

$$\mathcal{E}(\mathfrak{L}_0) = \{1, x\partial_x, x\partial_x - 1, x^2\partial_x - x\},\ \mathcal{E}(\widetilde{\mathfrak{L}}_0) = \left\{1, x\partial_x - 1, x^4\partial_x - 2x^3, \frac{\partial_x}{x^2} + \frac{1}{x^3}\right\}$$

and for $\lambda \neq 0$

$$\mathcal{E}(\mathfrak{L}_\lambda) = \{1, \partial_x\},\ \mathcal{E}(\widetilde{\mathfrak{L}}_\lambda) = \left\{1, -\left(\lambda + \frac{1}{x^2}\right)\partial_x - \frac{1}{x^3}\right\},$$

where $\mathcal{L}_\lambda := \mathfrak{L}_\lambda \Psi^{(-)} = 0$ and $\widetilde{\mathcal{L}}_\lambda := \widetilde{\mathfrak{L}}_\lambda \Psi^{(+)} = 0$.

Applying iteratively the Darboux transformation, theorem 2.2.8, and assuming that the propositions 2.2.9 and 2.2.10 holds, we have the following results.

Proposition 2.2.11 (Galoisian version of DT_n)**.** *Let be $\Lambda \neq \emptyset$, $\mathcal{L}_\lambda^{(n)}$ given by $H^{(n)}\Psi^{(n)} = \lambda\Psi^{(n)}$, $V_n \in K_n$, $K_0 = K$, $V_0 = V_-$, $H^{(0)} = H_-$, $\Psi^{(0)} = \Psi^{(-)}$. Let $\mathcal{L}_\lambda^{(n+1)}$ given by $H^{(n+1)}\Psi^{(n+1)} = \lambda\Psi^{(n+1)}$, $V_{n+1} \in K_{n+1}$. Let DT_n such that $\mathcal{L}_\lambda^{(n)} \mapsto \mathcal{L}_\lambda^{(n+1)}$, $V_n \mapsto V_{n+1}$, $\Psi_\lambda^{(n)} \mapsto \Psi_\lambda^{(n+1)}$. Then the following statements holds:*

i) $\mathrm{DT}_n(V_-) = \mathrm{DT}(V_n) = V_{n+1} = V_n - 2\partial_x^2\left(\ln \Psi_\lambda^{(n)}\right) = V_- - 2\sum_{k=0}^{n} \partial_x^2\left(\ln \Psi_{\lambda_k}^{(k)}\right)$, *where $\Psi_{\lambda_k}^{(k)}$ is a particular solution for $\lambda = \lambda_k$, $k = 0, \ldots, n$. In particular, if $\lambda_n = \lambda_0$ and $\Lambda = \mathbb{C}$, then there exists $\Psi_{\lambda_n}^{(n)}$ such that $V_n \neq V_{n-2}$, with $n \geq 2$.*

ii) $\mathrm{DT}(\Psi_\lambda^{(n)}) = \mathrm{DT}_n(\Psi_\lambda^{(-)}) = \Psi_\lambda^{(n+1)} = \partial_x\Psi_\lambda^{(n)} - \Psi_\lambda^{(n)}\frac{\partial_x\Psi_{\lambda_n}^{(n)}}{\Psi_{\lambda_n}^{(n)}} = \frac{W(\Psi_{\lambda_n}^{(n)}, \Psi_\lambda^{(n)})}{W(\Psi_{\lambda_n}^{(n)})}$ *where $\Psi_\lambda^{(n)}$ is a general solution for $\lambda \in \Lambda \setminus \{\lambda_n\}$ of $\mathcal{L}_\lambda^{(n)}$.*

iii) $K_{n+1} = K_n \left\langle \partial_x(\ln \Psi^{(n)}_{\lambda_n})\right\rangle$.

iv) DT_n *is isogaloisian and virtually strongly isogaloisian. Furthermore, if* $\partial_x(\ln \Psi^{(n)}_{\lambda_n}) \in K_n$ *then* DT_n *is strongly isogaloisian.*

v) *The eigenrings of* $H^{(n)} - \lambda$ *and* $H^{(n+1)} - \lambda$ *are isomorphic.*

Proposition 2.2.12 (Galoisian version of CI_n)**.** *Let consider* $\mathcal{L}_\lambda$ *given by* $H\Psi = \lambda\Psi$, $H = -\partial_x^2 + V$, $V \in K$, *such that* $\mathrm{Card}(\Lambda) > n$ *for a fixed* $n \in \mathbb{Z}_+$. *Let* $\mathcal{L}^{(n)}_\lambda$ *be given by* $H^{(n)}\Psi^{(n)} = \lambda\Psi^{(n)}$, *where* $H^{(n)} = \partial_x^2 + V_n$, $V_n \in K_n$. *Let* CI_n *be the transformation such that* $\mathcal{L}_\lambda \mapsto \mathcal{L}^{(n)}_\lambda$, $V \mapsto V_n$, $(\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_n}, \Psi_\lambda) \mapsto \Psi^{(n)}_\lambda$, *where for* $k = 1, \ldots, n$ *and the equation* $\mathcal{L}_\lambda$, *the function* Ψ_λ *is the general solution for* $\lambda \neq \lambda_k$ *and* Ψ_{λ_k} *is a particular solution for* $\lambda = \lambda_k$. *Then the following statements holds:*

i) $\mathrm{CI}_n(\mathcal{L}_\lambda) = \mathcal{L}^{(n)}_\lambda$ *where* $\mathrm{CI}_n(V) = V_n = V - 2\partial_x^2\left(\ln W(\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_n})\right)$ *and*

$$\mathrm{CI}_n(\Psi_\lambda) = \Psi^{(n)}_\lambda = \frac{W(\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_n}, \Psi_\lambda)}{W(\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_n})},$$

being $\Psi^{(n)}_\lambda$ *the general solution of* $\mathcal{L}^{(n)}_\lambda$.

iii) $K_n = K\langle \partial_x\left(\ln W(\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_n})\right)\rangle$.

iv) CI_n *is isogaloisian and virtually strongly isogaloisian. Furthermore, if*

$$\partial_x\left(\ln W(\Psi_{\lambda_1}, \ldots, \Psi_{\lambda_n})\right) \in K_n,$$

then CI_n *is strongly isogaloisian.*

v) *The eigenrings of* $\mathcal{L}_\lambda$ *and* $\mathcal{L}^{(n)}_\lambda$ *are isomorphic.*

Examples. Starting with $V = 0$, the following potentials can be obtained using Darboux iteration DT_n (see [14, 17]).

$$I)\ V_n = \frac{n(n-1)b^2}{(bx+c)^2}, \quad II)\ V_n = \frac{m^2 n(n-1)(b^2 - a^2)}{(a\cosh(mx) + b\sinh(mx))^2},$$

$$III)\ V_n = \frac{-4abm^2 n(n-1)}{(ae^{mx} + be^{-mx})^2}, \quad IV)\ V_n = \frac{m^2 n(n-1)(b^2 + a^2)}{(a\cos(mx) + b\sin(mx))^2}.$$

In particular for the rational potential given in I), we have $K = K_n = \mathbb{C}(x)$ and for $\lambda_n = \lambda = 0$, we have

$$\Psi^{(n)}_0 = \frac{c_1}{(bx+c)^n} + c_2(bx+c)^{n+1}, \text{ so that } \mathrm{DGal}(L/K)(\mathcal{L}_0) = \mathrm{DGal}(L/K)(\mathcal{L}^{(n)}_0) = e,$$

$$\mathcal{E}(H^{(n)}) = \left\{1, x\partial_x - 1, x^{2n+2}\partial_x - (n+1)x^{2n+1}, \frac{\partial_x}{x^{2n}} + \frac{n}{x^{2n+1}}\right\},$$

whilst for $\lambda \neq 0$ and $\lambda_n = 0$, the general solution $\Psi_\lambda^{(n)}$ is given by

$$\Psi_\lambda^{(n)}(x) = c_1 f_n(x,\lambda) h_n(\sin(\sqrt{\lambda}x) + c_2 g_n(x,\lambda) j_n(\cos(\sqrt{\lambda}x),$$

where $f_n, g_n, h_n, j_n \in \mathbb{C}(x)$, so that

$$\mathrm{DGal}(L/K)(\mathcal{L}_\lambda) = \mathrm{DGal}(L/K)(\mathcal{L}_\lambda^{(n)}) = \mathbb{G}_m,$$

and

$$\dim_{\mathbb{C}} \mathcal{E}(H-\lambda) = \dim_{\mathbb{C}} \mathcal{E}(H^{(n)}-\lambda) = 2.$$

To illustrate the Crum iteration with rational potentials, we consider $V = \frac{2}{x^2}$. The general solution of $\mathcal{L}_\lambda := H\Psi = \lambda\Psi$ is

$$\frac{c_1 e^{kx}(kx-1)}{x} + \frac{c_2 e^{-kx}(kx+1)}{x}, \quad \lambda = -k^2,$$

the eigenfunctions for $\lambda_1 = -1$, and $\lambda_2 = -4$, are respectively given by

$$\Psi_{-1} = \frac{e^{-x}(x+1)}{x}, \quad \Psi_{-4} = \frac{e^{-2x}(2x+1)}{2x}.$$

Thus, we obtain

$$\mathrm{CI}_2(V) = V_2 = \frac{8}{(2x+3)^2}$$

and the general solution of $\mathcal{L}_\lambda^{(2)} := H^{(2)}\Psi^{(2)} = \lambda\Psi^{(2)}$ is

$$\mathrm{CI}_2(\Psi_\lambda) = \Psi_\lambda^{(2)} = \frac{c_1\left(k\left(2\,x+3\right)-2\right)e^{kx}}{2\,x+3} + \frac{c_2\left(2+k\left(2\,x+3\right)\right)e^{-kx}}{4\,x+6}, \quad \lambda = -k^2.$$

The differential Galois groups and eigenrings are given by:

$$\mathrm{DGal}_K(\mathcal{L}_0) = \mathrm{DGal}_K(\mathcal{L}_0^{(2)} = e, \quad \dim_{\mathbb{C}} \mathcal{E}(\mathcal{L}_0) = \dim_{\mathbb{C}} \mathcal{E}(\mathcal{L}_0^{(2)}) = 4,$$

and for $\lambda \neq 0$

$$\mathrm{DGal}_K(\mathcal{L}_\lambda) = \mathrm{DGal}_K(\mathcal{L}_\lambda^{(2)} = \mathbb{G}_m, \quad \dim_{\mathbb{C}} \mathcal{E}(\mathcal{L}_\lambda) = \dim_{\mathbb{C}} \mathcal{E}(\mathcal{L}_\lambda^{(2)}) = 2.$$

Proposition 2.2.13. *The supersymmetric partner potentials $V_\pm$ are rational functions if and only if the superpotential W is a rational function.*

Proof. The supersymmetric partner potentials $V_\pm$ are written as $V_\pm = W^2 \pm \partial_x W$. We start considering the superpotential $W \in \mathbb{C}(x)$, so trivially we have that $V_\pm \in \mathbb{C}(x)$. Now assuming that $V_\pm \in \mathbb{C}(x)$ we have that $\partial_x W \in \mathbb{C}(x)$ and $W^2 \in \mathbb{C}(x)$, which implies that $W\partial_x W \in \mathbb{C}(x)$ and therefore $W \in \mathbb{C}(x)$.

□

Corollary 2.2.14. *The superpotential $W \in \mathbb{C}(x)$ if and only if* DT *is strong isogaloisian.*

The following definition is a partial Galoisian adaptation of the original definition given in [36] ($K = \mathbb{C}(x)$). The complete Galoisian adaptation is given when K is any differential field.

Definition 2.2.15 (Rational Shape Invariant Potentials)**.** Let assume $V_\pm(x;\mu) \in \mathbb{C}(x;\mu)$, where μ is a family of parameters. The potential $V = V_- \in \mathbb{C}(x)$ is said to be rational shape invariant potential with respect to μ and $E = E_n$ being $n \in \mathbb{Z}_+$, if there exists f such that

$$V_+(x;a_0) = V_-(x;a_1) + R(a_1), \quad a_1 = f(a_0), \quad E_n = \sum_{k=2}^{n+1} R(a_k), \quad E_0 = 0.$$

Remark 2.2.16. We propose the following steps to check whether $V \in \mathbb{C}(x)$ is shape invariant.

Step 1. Introduce parameters in $W(x)$ to obtain $W(x;\mu)$, write $V_\pm(x;\mu) = W^2(x;\mu) \pm \partial_x W(x;\mu)$, and replace μ by a_0 and a_1.

Step 2. Obtain polynomials $\mathcal{P} \in \mathbb{C}[x;a_0,a_1]$ and $\mathcal{Q} \in \mathbb{C}[x;a_0,a_1]$ such that

$$\partial_x(V_+(x;a_0) - V_-(x;a_1)) = \frac{\mathcal{P}(x;a_0,a_1)}{\mathcal{Q}(x;a_0,a_1)}.$$

Step 3. Set $\mathcal{P}(x;a_0,a_1) \equiv 0$, as polynomial in x, to obtain a_1 in function of a_0, i.e., $a_1 = f(a_0)$. Also obtain $R(a_1) = V_+(x;a_0) - V_-(x;a_1)$ and verify that exists $k \in \mathbb{Z}^+$ such that $R(a_1) + \cdots + R(a_k) \neq 0$.

Example. Let consider the superpotential of the three dimensional harmonic oscillator $W(r;\ell) = r - \frac{\ell+1}{r}$. By step 1, the supersymmetric partner potentials are

$$V_-(r;\ell) = r^2 + \frac{\ell(\ell+1)}{r^2} - 2\ell - 3, \quad V_+(r;\ell) = r^2 + \frac{(\ell+1)(\ell+2)}{r^2} - 2\ell - 1.$$

By step 2, we have $\partial_r(V_+(r;a_0) - V_-(r;a_1)) = -2\frac{a_0^2+3a_0-a_1^2-a_1+2}{r^3}$. By step 3, $(a_0+1)(a_0+2) = a_1(a_1+1)$, so that $a_1 = f(a_0) = a_0+1$, $a_n = f(a_{n-1}) = a_0+n$, $R(a_1) = 2$. Thus, we obtain the energy levels $E_n = 2n$ and the wave functions $\Psi_n^{(-)}(r;\ell) = A^\dagger(r;\ell)\cdots A^\dagger(r;\ell+n-1)\Psi_0^{(-)}(r;\ell+n)$ (see [31]).

By theorem 2.2.8 and propositions 2.2.9, 2.2.10 and 2.2.13 we have the following result.

Theorem 2.2.17. *Let consider $\mathcal{L}_n := H\Psi^{(-)} = E_n\Psi^{(-)}$ where $n \in \mathbb{Z}_+$ and $V = V_- \in \mathbb{C}(x)$ is a shape invariant potential. Then*

$$\mathrm{DGal}_K(\mathcal{L}_{n+1}) = \mathrm{DGal}_K(\mathcal{L}_n), \quad \mathcal{E}(H - E_{n+1}) \simeq \mathcal{E}(H - E_n), \quad n > 0.$$

Remark 2.2.18. The differential automorphisms σ commutes with the raising and lowering operators A and $A^\dagger$ due to $W \in \mathbb{C}(x)$. Furthermore the wave functions $\Psi_n^{(-)}$ can be written as $\Psi_n^{(-)} = P_n f_n \Psi_0^{(-)}$, where P_n is a polynomial of degree n in x and f_n is a sequence of functions being $f_0(x) = 1$ as we seen in the previous examples.

2.3 The Role of the Algebrization in Supersymmetric Quantum Mechanics

In supersymmetric quantum mechanics there exists potentials which are not rational functions and, for this reason, it is difficult to apply our Galoisian approach such as in section 3.2. In this section we give a solution to this problem presenting some results concerning to differential equations with non-rational coefficients. For these differential equations it is useful, whether is possible, to replace it by a new differential equation over the Riemann sphere $\mathbb{P}^1$ (that is, with rational coefficients). To do this, we can use a change of variables. The equation over $\mathbb{P}^1$ is called the *algebraic form* or *algebrization* of the original equation.

This algebraic form dates back to the 19th century (Liouville, Darboux), but the problem of obtaining the algebraic form (if it exists) of a given differential equation is in general not an easy task. Here we develop a new method using the concept of *Hamiltonian change of variables*. This change of variables allow us to compute the algebraic form of a large number of differential equations of different types. In particular, for second order linear differential equations we can apply *Kovacic's algorithm* over the algebraic form to solve the original equation.

The following definition can be found in [12, 96, 97].

Definition 2.3.1 (Pullbacks of differential equations). Let $\mathfrak{L}_1 \in K_1[\partial_z]$ and $\mathfrak{L}_2 \in K_2[\partial_x]$ be differential operators, the expression $\mathfrak{L}_2 \otimes (\partial_x + v)$ refers to the operator whose solutions are the solutions of $\mathfrak{L}_2$ multiplied by the solution $e^{-\int v dx}$ of $\partial_x + v$.

- $\mathfrak{L}_2$ is a *proper pullback* of $\mathfrak{L}_1$ by means of $f \in K_2$ if the change of variable $z = f(x)$ changes $\mathfrak{L}_1$ into $\mathfrak{L}_2$.
- $\mathfrak{L}_2$ is a *pullback* (also known as weak pullback) of $\mathfrak{L}_1$ by means of $f \in K_2$ if there exists $v \in K_2$ such that $\mathfrak{L}_2 \otimes (\partial_x + v)$ is a proper pullback of $\mathfrak{L}_1$ by means of f.

In the case of compact Riemann surfaces, the geometric mechanism behind the algebrization is a ramified covering of compact Riemann surfaces, see [61, 60].

2.3.1 Second Order Linear Differential Equations

Some results presented in this subsection can be found in [5, §2].

Proposition 2.3.2 (Change of the independent variable, [5]). *Let us consider the following equation, with coefficients in $\mathbb{C}(x)$:*

$$\mathcal{L}_x := \partial_x^2 y + a(x)y\partial_x y + b(x)y = 0, \tag{2.4}$$

and $\mathbb{C}(x) \hookrightarrow L$ the corresponding Picard-Vessiot extension. Let (K,δ) be a differential field with $\mathbb{C}$ as field of constants. Let $\theta \in K$ be a non-constant element. Then, by the change of variable $x = \theta(t)$, the equation (2.4) is transformed in

$$\mathcal{L}_t := \partial_t^2 r + \left(a(\theta)\partial_t\theta - \frac{\partial_t^2\theta}{\partial_t\theta}\right)\partial_t r + b(\theta)(\partial_t\theta)^2 r = 0, \quad \partial_t = \delta, \quad r = y\circ\theta. \tag{2.5}$$

Let $K_0 \subset K$ be the smallest differential field containing θ and $\mathbb{C}$. Then the equation (2.5) is a differential equation with coefficients in K_0. Let $K_0 \hookrightarrow L_0$ be the corresponding Picard-Vessiot extension. Assume that

$$\mathbb{C}(x) \to K_0, \quad x \mapsto \theta$$

is an algebraic extension, then

$$\mathrm{DGal}(L_0/K_0)^0 = \mathrm{DGal}(L/\mathbb{C}(x))^0.$$

Proposition 2.3.3. *Assume $\mathcal{L}_t$ and $\mathcal{L}_x$ as in proposition 2.3.2. Let φ be the transformation given by*

$$\varphi : \begin{array}{l} x \mapsto \theta(t) \\ \partial_x \mapsto \frac{1}{\partial_t\theta}\delta. \end{array}$$

Then $\mathrm{DGal}(L_0/K_0)(\mathcal{L}_t) \simeq \mathrm{DGal}(L/K_0\cap L)(\mathcal{L}_x) \subset \mathrm{DGal}(L/\mathbb{C}(x))(\mathcal{L}_x)$. Furthermore, if $K_0 \cap L$ is algebraic over $\mathbb{C}(x)$, then $(\mathrm{DGal}(L_0/K_0)(\mathcal{L}_t))^0 \simeq (\mathrm{DGal}(L/\mathbb{C}(x))(\mathcal{L}_x))^0$.

Proof. By Proposition 2.3.2, the transformation φ lead us to

$$\mathbb{C}(x) \simeq \varphi(\mathbb{C}(x)) \hookrightarrow K_0,$$

that is, we identify $\mathbb{C}(x)$ with $\varphi(\mathbb{C}(x))$, and so that we can view $\mathbb{C}(x)$ as a subfield of K_0 and then by the Kaplansky's diagram (see [46, 101]),

$$\begin{tikzcd}
& L_0 \arrow[dl, no head] \arrow[dr, no head] \arrow[dr, dotted, bend left, "\mathrm{DGal}(L_0/K_0)(\mathcal{L}_t)"] & \\
L \arrow[dr, no head] \arrow[dr, dotted, bend right, "\mathrm{DGal}(L/(L\bigcap K_0))(\mathcal{L}_x)"'] & & K_0 \arrow[dl, no head] \\
& L\bigcap K_0 \arrow[d, no head] & \\
& \mathbb{C}(x) &
\end{tikzcd}$$

so that we have

$$\mathrm{DGal}(L_0/K_0)(\mathcal{L}_t) \simeq \mathrm{DGal}(L/K_0 \cap L)(\mathcal{L}_x) \subset \mathrm{DGal}(L/\mathbb{C}(x))(\mathcal{L}_x)$$

and if $K_0 \cap L$ is algebraic over $\mathbb{C}(x)$, then

$$(\mathrm{DGal}(L_0/K_0)(\mathcal{L}_t))^0 \simeq (\mathrm{DGal}(L/\mathbb{C}(x))(\mathcal{L}_x))^0.$$

□

Along the rest of this section, for suitability, we write $x = x(t)$ instead of θ.

Remark 2.3.4 (Hard Algebrization). The proper pullback from the equation (2.5) to the equation (2.4) is an algebrization process. Therefore, we can try algebrize any second order linear differential equations with non-rational coefficients (proper pullback) whether we can put it in the form of equation (2.5). To do this, we use the following steps.

Step 1. Find $(\partial_t x)^2$ in the coefficient of y to obtain $\partial_t x$ and x.

Step 2. Obtain $b(x)$ in the coefficient of y dividing by $(\partial_t x)^2$ and check whether $b \in \mathbb{C}(x)$.

Step 3. Obtain $a(x)$ in the coefficient of $\partial_t y$ adding $(\partial_t^2 x)/\partial_t x$ and dividing by $\partial_t x$. After, check whether $a \in \mathbb{C}(x)$.

To illustrate this method, we present the following example.

Example. In [82, p. 256], Singer presents the second order linear differential equation

$$\partial_t^2 r - \frac{1}{t(\ln t + 1)}\partial_t r - (\ln t + 1)^2 r = 0.$$

To algebrize this differential equation we choose $(\partial_t x)^2 = (\ln t + 1)^2$, so that $\partial_t x = \ln t + 1$ and for instance

$$x = \int (\ln t + 1)dt = t \ln t, \quad b(x) = -1.$$

Now we find $a(x)$ in the expression

$$a(x)(\ln t + 1) - \frac{1}{t(\ln t + 1)} = -\frac{1}{t(\ln t + 1)},$$

obtaining $a(x) = 0$. So that the new differential equation is given by $\partial_x^2 y - y = 0$, in which $y(x(t)) = r(t)$ and the basis of solutions is given by $\langle e^x, e^{-x}\rangle$. The basis of solutions of the first differential equation is given by $\langle e^{t\ln t}, e^{-t\ln t}\rangle$.

In general, this method is not clear because the quest of $x = x(t)$ in $b(x)(\partial_t x)^2$ can be purely a lottery, or simply there is not exist x such that $a(x), b(x) \in \mathbb{C}(x)$.

For example, the equations presented by Singer in [82, p. 257, 261, 270] and given by

$$\partial_t^2 r + \frac{\mp 2t\ln^2 t \mp 2t\ln t - 1}{t\ln t + t}\partial_t r + \frac{-2t\ln^2 t - 3t\ln t - t \mp 1}{t\ln t + t} r = 0,$$

$$\partial_t^2 r + \frac{4t\ln t + 2t}{4t^2\ln t}\partial_t r - \frac{1}{4t^2\ln t} r = 0, \quad (t^2\ln^2 t)\partial_t^2 r + (t\ln^2 t - 3t\ln t)\partial_t r + 3r = 0,$$

cannot be algebrized easily with this method although corresponds to pullbacks (not proper pullback) of differential equations with constant coefficients.

In [21], Bronstein and Fredet developed and implemented an algorithm to solve differential equation over $\mathbb{C}(t, e^{\int f})$ without algebrizing the equation, see also [34]. As an application of Proposition 2.3.2 we have the following result[1].

Proposition 2.3.5 (Linear differential equation over $\mathbb{C}(t, e^{\int f})$, [5])**.** *Let $f \in \mathbb{C}(t)$ be a rational function. Then, the differential equation*

$$\partial_t^2 r - \left(f + \frac{\partial_t f}{f} - f e^{\int f} a\left(e^{\int f}\right)\right)\partial_t r + \left(f\left(e^{\int f}\right)\right)^2 b\left(e^{\int f}\right) r = 0, \tag{2.6}$$

is algebrizable by the change $x = e^{\int f}$ and its algebraic form is given by

$$\partial_x^2 y + a(x)\partial_x y + b(x) y = 0, \quad r(t) = y(x(t)).$$

Proof. Let assume that $r(t) = y(x(t))$, and $x = x(t) = e^{\int f dt}$. We can see that

$$\partial_t x = f x, \quad \partial_x y = \frac{\partial_t r}{f x}, \quad \partial_x^2 y = \frac{1}{f x}\partial_t\left(\frac{\partial_t r}{f x}\right) = \frac{1}{\left(f e^{\int f}\right)^2}\left(\partial_t^2 r - f + \left(\frac{\partial_t f}{f}\right)\right)\partial_t r,$$

replacing in $\partial_x^2 y + a(x)\partial_x y + b(x) y = 0$ we obtain the equation (2.6). □

Example. The differential equation

$$\partial_t^2 r - \left(t + \frac{1}{t} - 2t e^{t^2}\right)\partial_t r + \lambda\left(t^2 e^{t^2}\right) r = 0,$$

is algebrizable by the change $x = e^{\frac{t^2}{2}}$ and its algebraic form is given by

$$\partial_x^2 y + 2x\partial_x y + \lambda y = 0.$$

Remark 2.3.6. In this corollary, corollary 2.3.5, we have the following cases.

1. $f = n\frac{\partial_x h}{h}$, for a rational function h, $n \in \mathbb{Z}_+$, we have the trivial case, both equations are over the Riemann sphere and they have the same differential field, so that does not need to be algebrized.

[1] This result is given in [5, §2], but we include here the proof for completeness.

2. $f = \frac{1}{n}\frac{\partial_x h}{h}$, for a rational function h, $n \in \mathbb{Z}^+$, (2.6) is defined over an algebraic extension of $\mathbb{C}(t)$ and so that this equation is not necessarily over the Riemann sphere.
3. $f \neq q\frac{\partial_x h}{h}$, for any rational function h, $q \in \mathbb{Q}$, (2.6) is defined over a transcendental extension of $\mathbb{C}(t)$ and so that this equation is not over the Riemann sphere.

To algebrize second order linear differential equations it is easier when the term in $\partial_t r$ is absent, that is, in the form of the equation (1.2) and the change of variable is *Hamiltonian*.

Definition 2.3.7 (Hamiltonian change of variable, [5])**.** A change of variable $x = x(t)$ is called *Hamiltonian* if and only if $(x(t), \partial_t x(t))$ is a solution curve of the autonomous classical one degree of freedom Hamiltonian system

$$H = H(x,p) = \frac{p^2}{2} + V(x),$$

for some $V \in K$.

Remark 2.3.8. Assume that we algebrize equation (2.5) through a Hamiltonian change of variables, $x = x(t)$, i.e., $V \in \mathbb{C}(x)$. Then, $K_0 = \mathbb{C}(x, \partial_t x, \ldots)$, but, we have the algebraic relation,

$$(\partial_t x)^2 = 2h - 2V(x), \quad h = H(x, \partial_t x) \in \mathbb{C},$$

so that $K_0 = \mathbb{C}(x, \partial_t x)$ is an algebraic extension of $\mathbb{C}(x)$. We can apply Proposition 2.3.2, and then the identity component of the Galois group is conserved. On the other hand, we say that a change of variable $x = x(t)$ is Hamiltonian if and only if there exists α such that $\partial_t x^2 = \alpha(x)$. The *Hamiltonian algebrization* is the algebrization process which has been done using a Hamiltonian change of variable.

The following result, which can be found in [5, §2] is an example of Hamiltonian algebrization and correspond to the case of reduced second order linear differential equations.

Proposition 2.3.9 (Hamiltonian Algebrization, [5])**.** *The differential equation*

$$\partial_t^2 r = q(t) r$$

is algebrizable through a Hamiltonian change of variable $x = x(t)$ if and only if there exist f, α such that

$$\frac{\partial_x \alpha}{\alpha}, \quad \frac{f}{\alpha} \in \mathbb{C}(x), \text{ where } f(x(t)) = q(t), \quad \alpha(x) = 2(H - V(x)) = \partial_t x^2.$$

Furthermore, the algebraic form of the equation $\partial_t^2 r = q(t) r$ is

$$\partial_x^2 y + \frac{1}{2}\frac{\partial_x \alpha}{\alpha}\partial_x y - \frac{f}{\alpha} y = 0, \quad r(t) = y(x(t)). \tag{2.7}$$

Remark 2.3.10 (Using the Algebrization Method). The goal is to algebrize the differential equation $\partial_t^2 r = q(t)r$, so that we propose the following steps.

Step 1 Find a *Hamiltonian change of variable* $x = x(t)$ and two functions f and α such that $q(t) = f(x(t))$ and $(\partial_t x(t))^2 = \alpha(x(t))$.

Step 2 Verify whether or not $f(x)/\alpha(x) \in \mathbb{C}(x)$ and $\partial_x\alpha(x)/\alpha(x) \in \mathbb{C}(x)$ to see if the equation $\partial_t^2 r = q(t)r$ is algebrizable.

Step 3 If the equation $\partial_t^2 r = q(t)r$ is algebrizable, its algebrization is

$$\partial_x^2 y + \frac{1}{2}\frac{\partial_x \alpha}{\alpha}\partial_x y - \frac{f}{\alpha}y = 0, \quad y(x(t)) = r(t).$$

When we have algebrized the differential equation $\partial_t^2 r = q(t)r$, we study its integrability, eigenrings and its Galois group.

Examples. Let consider the following examples.

- Given the differential equation $\partial_t^2 r = f(\tan t)r$ with $f \in \mathbb{C}(\tan t)$, we can choose $x = x(t) = \tan t$ to obtain $\alpha(x) = (1+x^2)^2$, so that $x = x(t)$ is a Hamiltonian change of variable. We can see that $\frac{\partial_x \alpha}{\alpha}, \frac{f}{\alpha} \in \mathbb{C}(x)$ and the algebraic form of the differential equation $\partial_t^2 r = f(\tan t)r$ with this Hamiltonian change of variable is

$$\partial_x^2 y + \frac{2x}{1+x^2}\partial_x y - \frac{f(x)}{(1+x^2)^2}y = 0, \quad y(\tan t) = r(t).$$

- Given the differential equation

$$\partial_t^2 r = \frac{\sqrt{1+t^2}+t^2}{1+t^2}r,$$

we can choose $x = x(t) = \sqrt{1+t^2}$ to obtain

$$f(x) = \frac{x^2+x-1}{x^2}, \quad \alpha(x) = \frac{x^2-1}{x^2},$$

so that $x = x(t)$ is a Hamiltonian change of variable. We can see that $\frac{\partial_x \alpha}{\alpha}, \frac{f}{\alpha} \in \mathbb{C}(x)$ and the algebraic form for this case is

$$\partial_x^2 y + \frac{1}{x(x^2-1)}\partial_x y - \frac{x^2+x-1}{x^2-1}y = 0, \quad y(\sqrt{1+t^2}) = r(t).$$

We remark that the method of Hamiltonian algebrization is not an algorithm, because the problem is to obtain a suitable Hamiltonian H satisfying the definition 2.3.7. We present now an example of algorithm for Hamiltonian algebrization algorithm[2].

[2] Proposition 2.3.11 is a slight improvement of a similar result given in [5, §2]. Furthermore, we include the proof here for completeness.

Proposition 2.3.11 (Hamiltonian Algebrization Algorithm, [5]). *Let us consider $q(t) = g(x_1, \cdots, x_n)$, where $x_i = e^{\lambda_i t}$, $\lambda_i \in \mathbb{C}^*$. The equation $\partial_t^2 r = q(t)r$ is algebrizable if and only if.*

$$\frac{\lambda_i}{\lambda_j} \in \mathbb{Q}^*, \quad 1 \leq i \leq n, 1 \leq j \leq n, \quad g \in \mathbb{C}(x).$$

Furthermore, $\lambda_i = c_i \lambda$, where $\lambda \in \mathbb{C}^*$ and $c_i \in \mathbb{Q}^*$ and for the Hamiltonian change of variable

$$x = e^{\frac{\lambda t}{q}}, \text{ where } c_i = \frac{p_i}{q_i}, \, p_i, q_i \in \mathbb{Z}^*, \, \gcd(p_i, q_i) = 1 \text{ and } q = \operatorname{lcm}(q_1, \cdots, q_n),$$

the algebrization of the differential equation $\partial_t^2 r = q(t)r$ is

$$\partial_x^2 y + \frac{1}{x}\partial_x y - q^2 \frac{g(x^{m_1}, \ldots, x^{m_n})}{\lambda^2 x^2} y = 0, \quad m_i = \frac{qp_i}{q_i}.$$

Proof. Assuming $\lambda_i/\lambda_j = c_{ij} \in \mathbb{Q}^*$ we can see that there exists $\lambda \in \mathbb{C}^*$ and $c_i \in \mathbb{Q}^*$ such that $\lambda_i = \lambda c_i$, so that

$$e^{\lambda_i t} = e^{c_i \lambda t} = e^{\frac{p_i}{q_i}\lambda t} = \left(e^{\frac{\lambda}{q}t}\right)^{\frac{qp_i}{q_i}}, \, p_i, q_i \in \mathbb{Z}^*, \, \gcd(p_i, q_i) = 1, \, \operatorname{lcm}(q_1, \ldots, q_n) = q.$$

Now, setting $x = x(t) = e^{\frac{\lambda}{q}t}$ we can see that

$$f(x) = g(x^{m_1}, \ldots, x^{m_n}), \quad m_i = \frac{qp_i}{q_i}, \quad \alpha = \frac{\lambda^2 x^2}{q^2}.$$

Due to $q|q_i$, we have that $m_i \in \mathbb{Z}$, so that

$$\frac{\partial_x \alpha}{\alpha}, \quad \frac{f}{\alpha} \in \mathbb{C}(x)$$

and the algebraic form is given by

$$\partial_x^2 y + \frac{1}{x}\partial_x y - q^2 \frac{g(x^{m_1}, \ldots, x^{m_n})}{\lambda^2 x^2} y = 0.$$

□

Remark 2.3.12. The propositions 2.3.9 and 2.3.11 allow us the algebrization of a large number of second order differential equations, see for example [70]. In particular, under the assumptions of the proposition 2.3.11, we can algebrize automatically differential equations with trigonometrical or hyperbolic coefficients.

Examples. Let consider the following examples.

- Given the differential equation

$$\partial_t^2 r = \frac{e^{\frac{1}{2}t} + 3e^{-\frac{2}{3}t} - 2e^{\frac{5}{4}t}}{e^t + e^{-\frac{3}{2}}} r,\ \lambda_1 = \frac{1}{2},\ \lambda_2 = -\frac{2}{3},\ \lambda_3 = \frac{5}{4},\ \lambda_4 = 1,\ \lambda_5 = -\frac{3}{2},$$

we obtain we see that $\lambda_i/\lambda_j \in \mathbb{Q}$, $\lambda = 1$, $q = \mathrm{lcm}(1,2,3,4) = 12$ and the Hamiltonian change of variable for this case is $x = x(t) = e^{\frac{1}{12}t}$. We can see that

$$\alpha(x) = \frac{1}{144}x^2, \quad f(x) = \frac{x^6 + 3x^{-8} - 2x^{15}}{x^{12} + x^{-18}}, \quad \frac{\partial_x \alpha}{\alpha}, \frac{f}{\alpha} \in \mathbb{C}(x)$$

and the algebraic form is given by

$$\partial_x^2 y + \frac{1}{x}\partial_x y - 144\frac{x^6 + 3x^{-8} - 2x^{15}}{x^{14} + x^{-16}} y = 0, \quad y(e^{\frac{1}{12}t}) = r(t).$$

- Given the differential equation

$$\partial_t^2 r = (e^{2\sqrt{2}t} + e^{-\sqrt{2}t} - e^{3t})r,\ \lambda_1 = 2\sqrt{2},\ \lambda_2 = -\sqrt{2},\ \lambda_3 = 3,$$

we see that $\lambda_1/\lambda_2 \in \mathbb{Q}$, but $\lambda_1/\lambda_3 \notin \mathbb{Q}$, so that this differential equation cannot be algebrized.

We remark that it is possible to use the algebrization method to transform differential equations, although either the starting equation has rational coefficients or the transformed equation has not rational coefficients.

Examples. As illustration we present the following examples.

- Let consider the following differential equation

$$\partial_t^2 = \frac{t^4 + 3t^2 - 5}{t^2 + 1} y = 0,$$

we can choose $x = x(t) = t^2$ so that $\alpha = 4x$ and the new differential equation is

$$\partial_x^2 + \frac{1}{2x}\partial_x - \frac{x^2 + 2x - 5}{4x(x+1)} y = 0$$

- Let consider the Mathieu's differential equation $\partial_t^2 y = (a + b\cos(t))y$, we can choose $x(t) = \ln(\cos(t))$ so that $\alpha = e^{-2x} - 1$ and the new differential equation is

$$\partial_x^2 - \frac{1}{1 - e^{2x}}\partial_x y - \frac{ae^{2x} + be^{3x}}{1 - e^{2x}} y = 0.$$

Recently, the Hamiltonian algebrization (propositions 2.3.9 and 2.3.11) has been applied in [2, 3, 5] to obtain non-integrability in the framework of *Morales-Ramis theory* [61, 60].

2.3.2 The Operator $\widehat{\partial}_z$ and the Hamiltonian Algebrization

The generalization of proposition 2.3.2 to higher order linear differential equations is difficult. But, it is possible to obtain generalizations of proposition 2.3.9 by means of Hamiltonian change of variable. We recall that $z = z(x)$ is a Hamiltonian change of variable if there exists α such that $(\partial_x z)^2 = \alpha(z)$. More specifically, if $z = z(x)$ is a Hamiltonian change of variable, we can write $\partial_x z = \sqrt{\alpha}$, which lead us to the following notation: $\widehat{\partial}_z = \sqrt{\alpha}\partial_z$.

We can see that $\widehat{\partial}_z$ is a *derivation* because satisfy $\widehat{\partial}_z(f+g) = \widehat{\partial}_z f + \widehat{\partial}_z g$ and the Leibnitz rules

$$\widehat{\partial}_z(f \cdot g) = \widehat{\partial}_z f \cdot g + f \cdot \widehat{\partial}_z g, \quad \widehat{\partial}_z \left(\frac{f}{g}\right) = \frac{\widehat{\partial}_z f \cdot g - f \cdot \widehat{\partial}_z g}{g^2}.$$

We can notice that the chain rule is given by $\widehat{\partial}_z(f \circ g) = \partial_g f \circ g \widehat{\partial}_z(g) \neq \widehat{\partial}_g f \circ g \widehat{\partial}_z(g)$. The iteration of $\widehat{\partial}_z$ is given by

$$\widehat{\partial}_z^0 = 1, \quad \widehat{\partial}_z = \sqrt{\alpha}\partial_z, \quad \widehat{\partial}_z^n = \sqrt{\alpha}\partial_z \widehat{\partial}_z^{n-1} = \underbrace{\sqrt{\alpha}\partial_z \left(\dots\left(\sqrt{\alpha}\partial_z\right)\right)}_{n \text{ times } \sqrt{\alpha}\partial_z}.$$

We say that a Hamiltonian change of variable is rational when the potential $V \in \mathbb{C}(x)$ and for instance $\alpha \in \mathbb{C}(x)$. Along the rest of this memory, we understand $\widehat{\partial}_z = \sqrt{\alpha}\partial_z$ where $z = z(x)$ is a Hamiltonian change of variable and $\partial_x z = \sqrt{\alpha}$. In particular, $\widehat{\partial}_z = \partial_z = \partial_x$ if and only if $\sqrt{\alpha} = 1$, i.e., $z = x$.

Theorem 2.3.13. *Let consider the systems of linear differential equations* $[A]$ *and* $\widehat{A}$ *given respectively by*

$$\partial_x \mathbf{Y} = -A\mathbf{Y}, \quad \widehat{\partial}_z \widehat{\mathbf{Y}} = -\widehat{A}\widehat{\mathbf{Y}}, \quad A = [a_{ij}], \quad \widehat{A} = [\widehat{a}_{ij}], \quad \mathbf{Y} = [y_{i1}], \quad \widehat{\mathbf{Y}} = [\widehat{y}_{i1}],$$

where $1 \leq i \leq n$, $1 \leq j \leq n$, $a_{ij}(x) = \widehat{a}_{ij}(z(x))$ *and* $y_{i1}(x) = y_{i1}(z(x))$. *Let* φ *be the transformation given by*

$$\varphi : \begin{array}{l} x \mapsto z \\ a_{ij} \mapsto \widehat{a}_{ij} \\ y_{i1}(x) \mapsto \widehat{y}_{i1}(z(x)) \\ \partial_x \mapsto \widehat{\partial}_z \end{array}, \qquad \begin{array}{l} a_{ij} \in K = \mathbb{C}(z(x), \partial_x(z(x))), \\ \widehat{a}_{ij} \in \mathbb{C}(z) \subseteq \widehat{K} = \mathbb{C}(z, \sqrt{\alpha}). \end{array}$$

Then the following statements hold.

- $K \simeq \widehat{K}, \quad (K, \partial_x) \simeq (\widehat{K}, \widehat{\partial}_z)$.
- $\mathrm{DGal}_K([A]) \simeq \mathrm{DGal}_{\widehat{K}}([\widehat{A}]) \subset \mathrm{DGal}_{\mathbb{C}(z)}([\widehat{A}]$.
- $(\mathrm{DGal}_K([A]))^0 \simeq (\mathrm{DGal}_{\mathbb{C}(z)}(\widehat{A}))^0$.
- $\mathcal{E}([A]) \simeq \mathcal{E}([\widehat{A}])$.

Proof. Due to $z = z(x)$ is a rational Hamiltonian change of variable, the transformation φ lead us to

$$\mathbb{C}(z) \simeq \varphi(\mathbb{C}(z)) \hookrightarrow K, \quad K \simeq \widehat{K}, \quad \mathbb{C}(z) \hookrightarrow \widehat{K}, \quad (K, \partial_x) \simeq (\widehat{K}, \widehat{\partial}_z)$$

that is, we identify $\mathbb{C}(z)$ with $\varphi(\mathbb{C}(z))$, and so that we can view $\mathbb{C}(z)$ as a subfield of K and then, by the Kaplansky's diagram (see [46, 101]),

$$\begin{tikzcd}
 & L \arrow[dl, dash] \arrow[dr, dash] \arrow[dr, dotted, bend left, "\mathrm{DGal}_K([A])"] & \\
\widehat{L} \arrow[dr, dash] \arrow[dr, dotted, bend right, "\mathrm{DGal}_{\widehat{K}}([\widehat{A}])"'] & & K \arrow[dl, dash] \\
 & \widehat{K} \arrow[d, dash] & \\
 & \mathbb{C}(z) &
\end{tikzcd}$$

so that we have $\mathrm{DGal}_K([A]) \simeq \mathrm{DGal}_{\widehat{K}}([\widehat{A}]) \subset \mathrm{DGal}_{\mathbb{C}(z)}([\widehat{A}]$, $(\mathrm{DGal}_K([A]))^0 \simeq (\mathrm{DGal}_{\mathbb{C}(z)}(\widehat{A}))^0$, and $\mathcal{E}([A]) \simeq \mathcal{E}([\widehat{A}])$. $\square$

We remark that the transformation φ, given in theorem 2.3.13, is virtually strong isogaloisian when $\sqrt{\alpha} \notin \mathbb{C}(z)$ and for $\sqrt{\alpha} \in \mathbb{C}(z)$, φ is strong isogaloisian. Furthermore, by cyclic vector method (see [92]), we can write the systems $[A]$ and $[\widehat{A}]$ in terms of the differential equations $\mathcal{L}$ and $\widehat{\mathcal{L}}$. Thus, $\widehat{\mathcal{L}}$ is the proper pullback of $\mathcal{L}$.

Example. Let consider the system

$$[A] := \begin{array}{l} \partial_x \gamma_1 = -\frac{2\sqrt{2}}{e^x + e^{-x}} \gamma_3, \\[2ex] \partial_x \gamma_2 = \frac{e^x - e^{-x}}{e^x + e^{-x}} \gamma_3, \\[2ex] \partial_x \gamma_3 = \frac{2\sqrt{2}}{e^x + e^{-x}} \gamma_1 - \frac{e^x - e^{-x}}{e^x + e^{-x}} \gamma_2, \end{array}$$

which through the Hamiltonian change of variable $z = e^x$, and for instance $\sqrt{\alpha} = z$, it is transformed in the system

$$\widehat{[A]} := \begin{array}{l} \partial_z \widehat{\gamma}_1 = -\frac{2\sqrt{2}}{z^2+1} \widehat{\gamma}_3, \\[2ex] \partial_z \widehat{\gamma}_2 = \frac{z^2-1}{z(z^2+1)} \widehat{\gamma}_3, \\[2ex] \partial_z \widehat{\gamma}_3 = \frac{2\sqrt{2}}{z^2+1} \widehat{\gamma}_1 - \frac{z^2-1}{x(x^2+1)} \widehat{\gamma}_2. \end{array}$$

One solution of the system $\widehat{[A]}$ is given by

$$\frac{1}{z^2+1}\begin{pmatrix} \frac{\sqrt{2}}{2}(1-z^2) \\ z \\ -z \end{pmatrix},$$

and for instance,

$$\frac{1}{e^{2x}+1}\begin{pmatrix} \frac{\sqrt{2}}{2}(1-e^{2x}) \\ e^x \\ -e^x \end{pmatrix}$$

is the corresponding solution for the system $[A]$.

Remark 2.3.14. The algebrization given in proposition 2.3.9 is an example of how the introduction of the new derivative $\widehat{\partial}_z$ simplifies the proofs and computations. Such proposition is naturally extended to $\partial_x^2 y + a\partial_x y + by = 0$, using φ to obtain $\widehat{\partial}_z^2\widehat{y} + \widehat{a}\widehat{\partial}_z\widehat{y} + \widehat{b}\widehat{y} = 0$, which is equivalent to

$$\alpha\partial_z^2\widehat{y} + \left(\frac{\partial_x\alpha}{2} + \sqrt{\alpha}\widehat{a}\right)\partial_z\widehat{y} + \widehat{b}\widehat{y} = 0, \tag{2.8}$$

where $y(x) = \widehat{y}(z(x))$, $\widehat{a}(z(x)) = a(x)$ and $\widehat{b}(z(x)) = b(x)$.

In general, for $y(x) = \widehat{y}(z(x))$, the equation $F(\partial_x^n y, \ldots, y, x) = 0$ with coefficients given by $a_{i_k}(x)$ is transformed in the equation $\widehat{F}(\widehat{\partial}_z^n\widehat{y}, \ldots, \widehat{y}, z) = 0$ with coefficients given by $\widehat{a}_{i_k}(z)$, where $a_{i_k}(x) = \widehat{a}_{i_k}(z(x))$. In particular, for $\sqrt{\alpha}, \widehat{a}_{i_k} \in \mathbb{C}(z)$, the equation $\widehat{F}(\widehat{\partial}_z^n\widehat{y}, \ldots, \widehat{y}, z) = 0$ is the Hamiltonian algebrization of $F(\partial_x^n y, \ldots, y, x) = 0$. Now, if each derivation ∂_x has order even, then α and $\widehat{a}_{i_k}$ can be rational functions to algebrize the equation $F(\partial_x^n y, \ldots, y, x) = 0$, where $a_{i_k} \in \mathbb{C}(z(x), \partial_x z(x))$. for example, that happens for linear differential equations given by

$$\partial_x^{2n} y + a_{n-1}(x)\partial_x^{2n-2} y + \ldots + a_2(x)\partial_x^4 y + a_1(x)\partial_x^2 y + a_0(x)y = 0.$$

Finally, the algebrization algorithm given in proposition 2.3.11) can be is naturally extended to any differential equation $F(\partial_x^n y, \partial_x^{n-1} y, \ldots, \partial_x y, y, e^{\lambda t}) = 0$, that by means of the change of variable $z = e^{\lambda x}$ is transformed into $\widehat{F}(\widehat{\partial}_z^n\widehat{y}, \widehat{\partial}_z^{n-1}\widehat{y}, \ldots, \widehat{\partial}_z\widehat{y}, y, z) = 0$. Particularly, we consider the algebrization of Riccati equations, higher order linear differential equations and systems.

Example. The following corresponds to some examples of algebrizations for differential equations given in [82, p. 258, 266].

1. The equation $\mathcal{L} := \partial_x^2 y + (-2e^x - 1)\partial_x y + e^{2x} y = 0$ with the Hamiltonian change of variable $z = e^x$, $\sqrt{\alpha} = z$, $\widehat{a} = -2z - 1$ and $\widehat{b} = z^2$ is transformed in the equation $\widehat{\mathcal{L}} := \partial_z^2\widehat{y} - 2\partial_z\widehat{y} + \widehat{y} = 0$ which can be easily solved. The basis of solutions for $\mathcal{L}$ and $\widehat{\mathcal{L}}$ are given by $\langle e^z, ze^z\rangle$ and $\langle e^{e^x}, e^x e^{e^x}\rangle$ respectively. Furthermore $K = \mathbb{C}(e^x)$, $\widehat{K} = \mathbb{C}(z)$, $\mathrm{DGal}_K(\mathcal{L}) = \mathrm{DGal}_{\widehat{K}}(\widehat{\mathcal{L}})$.

2. The differential equation

$$\mathcal{L} := \partial_x^2 y + \frac{-24e^x - 25}{4e^x + 5}\partial_x y + \frac{20e^x}{4e^x + 5} y = 0$$

with the Hamiltonian change of variable $z = e^x$, $\sqrt{\alpha} = z$,

$$\widehat{a} = \frac{-24z - 25}{4z + 5} \text{ and } \widehat{b} = \frac{20z}{4z + 5}$$

is transformed in the equation

$$\widehat{\mathcal{L}} := \partial_z^2 \widehat{y} + \frac{-20(z+1)}{x(4z+5)}\partial_z \widehat{y} + \frac{20}{z(4z+5)}\widehat{y} = 0,$$

which can be solved with Kovacic algorithm. The basis of solutions for $\widehat{\mathcal{L}}$ is $\langle z+1, z^5 \rangle$, so that the basis for $\mathcal{L}$ is $\langle e^x + 1, e^{5x} \rangle$. Furthermore $K = \mathbb{C}(e^x)$, $\widehat{K} = \mathbb{C}(z)$ and $\mathrm{DGal}_K(\mathcal{L}) = \mathrm{DGal}_{\widehat{K}}(\widehat{\mathcal{L}}) = e$.

Remark 2.3.15 (Algebrization of the Riccati equation). The Riccati equation

$$\partial_x v = a(x) + b(x)v + c(x)v^2 \tag{2.9}$$

through the Hamiltonian change of variable $z = z(x)$, becomes in the Riccati equation

$$\partial_z \widehat{v} = \frac{1}{\sqrt{\alpha}}(\widehat{a}(z) + \widehat{b}(z)\widehat{v} + \widehat{c}(z)\widehat{v}^2), \tag{2.10}$$

where $v(x) = \widehat{v}(z(x))$, $\widehat{a}(z(x)) = a(x)$, $\widehat{b}(z(x)) = b(x)$, $\widehat{c}(z(x)) = c(x)$ and $\sqrt{\alpha(z(x))} = \partial_x z(x)$. Furthermore, if $\sqrt{\alpha}$, $\widehat{a}$, $\widehat{b}$, $\widehat{c} \in \mathbb{C}(x)$, the equation (2.10) is the algebrization of the equation (2.9).

Example. Let consider the Riccati differential equation

$$\mathcal{L} := \partial_x v = \left(\tanh x - \frac{1}{\tanh x}\right) v + \left(3\tanh x - 3\tanh^3 x\right) v^2,$$

which through the Hamiltonian change of variable $z = \tanh x$, for instance $\sqrt{\alpha} = 1 - z^2$, is transformed into the Riccati differential equation

$$\widehat{\mathcal{L}} := \partial_z v = -\frac{1}{z} v + 3zv^2.$$

One solution for the equation $\widehat{\mathcal{L}}$ is

$$-\frac{1}{z(3z-c)}, \text{ being } c \text{ a constant },$$

so that the corresponding solution for equation $\mathcal{L}$ is

$$-\frac{1}{\tanh x(3\tanh x - c)}.$$

The following result is the algebrized version of Eigenrings.

Proposition 2.3.16. *Let consider the differential fields K, $\widehat{K}$ and let consider the systems $[A]$ and $[\widehat{A}]$ given by*

$$\partial_x \mathbf{X} = -A\mathbf{X},\ \widehat{\partial}_z \widehat{\mathbf{X}} = -\widehat{A}\widehat{\mathbf{X}},\ \widehat{\partial}_z = \sqrt{\alpha}\partial_z,\ A = [a_{ij}],\ \widehat{A} = [\widehat{a}_{ij}],\ a_{ij} \in K,\ \widehat{a}_{ij} \in \widehat{K},$$

where $z = z(x)$, $\mathbf{X}(x) = \widehat{\mathbf{X}}(z(x))$, $\widehat{a}_{ij}(z(x)) = a_{ij}(x)$, then $\mathcal{E}(A) \simeq \mathcal{E}(\widehat{A})$. In particular, if we consider the linear differential equations

$$\mathcal{L} := \partial_x^n y + \sum_{k=0}^{n-1} a_k \partial_x^k y = 0 \quad \textit{and} \quad \widehat{\mathcal{L}} := \widehat{\partial}_z^n \widehat{y} + \sum_{k=0}^{n-1} \widehat{a}_k \widehat{\partial}_z^k \widehat{y} = 0,$$

where $z = z(x)$, $y(x) = \widehat{z}((x))$, $\widehat{a}_k(z(x)) = a_k(x)$, $a_k \in K$, $\widehat{a}_k \in \widehat{K}$, then $\mathcal{E}(\mathfrak{L}) \simeq \mathcal{E}(\widehat{\mathfrak{L}})$, where $\mathcal{L} := \mathfrak{L}y = 0$ and $\widehat{\mathcal{L}} := \widehat{\mathfrak{L}}\widehat{y} = 0$. Furthermore, assuming

$$T = \begin{pmatrix} p_{11} & \dots & p_{1n} \\ \vdots & & \\ p_{n1} & \dots & p_{nn} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & & & \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{pmatrix},$$

then

$$\mathcal{E}(\mathfrak{L}) = \left\{ \sum_{k=1}^{n} p_{1k} \partial_x^{k-1} : \partial_x T = TA - AT,\ p_{ik} \in K \right\},$$

if and only if

$$\mathcal{E}(\widehat{\mathfrak{L}}) = \left\{ \sum_{k=1}^{n} \widehat{p}_{1k} \widehat{\partial}_z^{k-1} : \widehat{\partial}_z \widehat{T} = \widehat{T}\widehat{A} - \widehat{A}\widehat{T},\ \widehat{p}_{ik} \in \widehat{K} \right\}.$$

Examples. We consider two different examples to illustrate the previous proposition.

- Let consider the differential equation $\mathcal{L}_1 := \partial_x^2 y - (1 + \cos x - \cos^2 x)y = 0$. By means of the Hamiltonian change of variable $z = z(x) = \cos x$, with $\sqrt{\alpha} = -\sqrt{1-z^2}$, $\mathcal{L}_1$ is transformed into the differential equation

$$\widehat{\mathcal{L}}_1 := \partial_z^2 \widehat{y} - \frac{z}{1-z^2}\partial_z \widehat{y} - \frac{1+z-z^2}{1-z^2}\widehat{y} = 0.$$

  Now, computing the eigenring of $\widehat{\mathfrak{L}_1}$ we have that $\mathcal{E}(\widehat{\mathfrak{L}_1}) = \{1\}$, therefore the eigenring of $\mathfrak{L}_1$ is given $\mathcal{E}(\mathfrak{L}_1) = \{1\}$.

- Now we consider the differential equation $\mathcal{L}_2 := \partial_x^2 y = \left(e^{2x} + \frac{9}{4}\right) y$. By means of the Hamiltonian change of variable $z = e^x$, with $\sqrt{\alpha} = x$, $\mathcal{L}_2$ is transformed into the differential equation

$$\widehat{\mathcal{L}}_2 := \partial_z^2 \widehat{y} + \frac{1}{z}\partial_z \widehat{y} - \left(1 + \frac{9}{4x^2}\right)\widehat{y} = 0.$$

Now, computing the eigenring of $\widehat{\mathfrak{L}_2}$ we have that

$$\mathcal{E}(\widehat{\mathfrak{L}_2}) = \left\{1, -2\left(\frac{z^2-1}{z^2}\right)\partial_z - \frac{z^2-3}{z^3}\right\} = \left\{1, -2\left(\frac{z^2-1}{z^3}\right)\widehat{\partial}_z - \frac{z^2-3}{z^3}\right\},$$

therefore the eigenring of $\mathfrak{L}_2$ is given by

$$\mathcal{E}(\mathfrak{L}_2) = \left\{1, -2\left(\frac{e^{2x}-1}{e^{3x}}\right)\partial_x - \frac{e^{2x}-3}{e^{3x}}\right\}.$$

The same result is obtained via matrix formalism, where

$$A = \begin{pmatrix} 0 & 1 \\ e^{2x}+\frac{9}{4} & 0 \end{pmatrix}, \widehat{A} = \begin{pmatrix} 0 & 1 \\ z^2+\frac{9}{4} & 0 \end{pmatrix}, \partial_x T = TA - AT, \widehat{\partial}_z\widehat{T} = \widehat{T}\widehat{A} - \widehat{A}\widehat{T},$$

with $T \in Mat(2, \mathbb{C}(e^x))$ and $\widehat{T} \in Mat(2, \mathbb{C}(z))$.

2.3.3 Applications in Supersymmetric Quantum Mechanics

In this subsection we apply the derivation $\widehat{\partial}_z$ to the Schrödinger equation $H\Psi = \lambda\Psi$, where $H = -\partial_x^2 + V(x)$, $V \in K$. Assume that $z = z(x)$ is a rational Hamiltonian change of variable for $H\Psi = \lambda\Psi$, then $K = \mathbb{C}(z(x), \partial_x z(x))$. Thus, the *algebrized Schrödinger equation* is written as

$$\widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}, \quad \widehat{H} = -\widehat{\partial}_z^2 + \widehat{V}(z), \quad \widehat{\partial}_z^2 = \alpha\partial_z^2 + \frac{1}{2}\partial_z\alpha\partial_z, \quad \widehat{K} = \mathbb{C}(z, \sqrt{\alpha}). \quad (2.11)$$

The *reduced algebrized Schrödinger equation*, obtained through the equation (1.3), is given by

$$\widehat{\mathbf{H}}\Phi = \lambda\Phi, \quad \widehat{\mathbf{H}} = \alpha(z)\left(-\partial_z^2 + \widehat{\mathbf{V}}(z)\right), \quad \begin{array}{l} \widehat{\mathbf{V}}(z) = \mathcal{V} + \frac{\widehat{V}(z)}{\alpha}, \\ \mathcal{V} = \partial_z\mathcal{W} + \mathcal{W}^2, \\ \mathcal{W} = \frac{1}{4}\frac{\partial_z\alpha(z)}{\alpha(z)}. \end{array} \quad (2.12)$$

The eigenfunctions Ψ, $\widehat{\Psi}$ and Φ corresponding to the operators H, $\widehat{H}$ and $\widehat{\mathbf{H}}$ are related respectively as

$$\Phi(z(x)) = \sqrt[4]{\alpha}\widehat{\Psi}(z(x)) = \sqrt[4]{\alpha}\Psi(x).$$

In order to apply the Kovacic's algorithm we only consider the algebrized operator $\widehat{\mathbf{H}} = \alpha(z)$H, whilst the eigenrings will be computed on $\widehat{H} - \lambda$. Also it is possible to apply the version of Kovacic's algorithm given in reference [88] to the algebraized operator $\widehat{H}$. The following results are obtained by applying Kovacic's algorithm to the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$.

Proposition 2.3.17. *Assume the differential equation $\mathcal{L}_\lambda$ given by the reduced algebrized Schrödinger equation* (2.12) *with $\widehat{K} = \mathbb{C}(z)$. If $\alpha, \widehat{u} \in \mathbb{C}[z]$ with $\deg \alpha \leq 2 + \deg \widehat{u}$, then $\mathrm{DGal}_{\widehat{K}}(\mathcal{L})$ is a not finite primitive for every $\lambda \in \Lambda$*

Proposition 2.3.18. *Let consider the differential equation $\mathcal{L}_\lambda$ given by the reduced algebrized Schrödinger equation* (2.12) *with $\widehat{K} = \mathbb{C}(z)$. If $\alpha \in \mathbb{C}[z]$ with $\deg \alpha \leq 4$ and $\circ(\widehat{u})_\infty \leq -2$, then $\mathrm{DGal}_{\widehat{K}}(\mathcal{L})$ is a not finite primitive for every $\lambda \in \Lambda$*

Remark 2.3.19. In a natural way, we obtain the algebrized versions of Darboux transformation, i.e., the *algebrized Darboux transformation*, denoted by $\widehat{\mathrm{DT}}$. By $\widehat{\mathrm{DT}}_n$ we denote the n iteration of $\widehat{\mathrm{DT}}$, and by $\widehat{\mathrm{CI}}_n$ we denote the *algebrized Crum iteration*, where the *algebrized wronskian* is given by

$$\widehat{W}(\widehat{y}_1, \dots, \widehat{y}_n) = \begin{vmatrix} \widehat{y}_1 & \cdots & \widehat{y}_n \\ \vdots & & \vdots \\ \widehat{\partial}_z^{n-1}\widehat{y}_1 & \cdots & \widehat{\partial}_z^{n-1}\widehat{y}_n \end{vmatrix}.$$

In the same way, we define *algebrized shape invariant potentials*, *algebrized superpotential* $\widehat{W}$, *algebrized supersymmetric Hamiltonians* $\widehat{H}_\pm$, algebrized supersymmetric partner potentials $\widehat{V}_\pm$, *algebrized ground state* $\widehat{\Psi_0}^{(-)} = e^{-\int \frac{\widehat{W}}{\sqrt{\alpha}} dz}$, *algebrized wave functions* $\widehat{\Psi_\lambda}^{(-)}$, *algebrized raising and lowering operators* $\widehat{A}$ and $\widehat{A}^\dagger$. Thus, we can rewrite entirely the section 2.2 using the derivation $\widehat{\partial}_z$.

The following theorem show us the relationship between the algebrization and Darboux transformation.

Theorem 2.3.20. *Given the Schrödinger equation $\mathcal{L}_\lambda := H_-\Psi^{(-)} = \lambda\Psi^{(-)}$, the relationship between the algebrization φ and Darboux transformations* DT, $\widehat{\mathrm{DT}}$ *with respect to $\mathcal{L}_\lambda$ is given by $\widehat{\mathrm{DT}}\varphi = \varphi\mathrm{DT}$, that is $\widehat{\mathrm{DT}}\varphi(\mathcal{L}) = \varphi\mathrm{DT}(\mathcal{L})$. In other words, the Darboux transformations* DT *and* $\widehat{\mathrm{DT}}$ *are* intertwined *by the algebrization φ.*

Proof. Let assume the equations $\mathcal{L}_\lambda := H_-\Psi^{(-)} = \lambda\Psi^{(-)}$, $\widehat{\mathcal{L}}_\lambda := \widehat{H}_-\widehat{\Psi}^{(-)} = \lambda\widehat{\Psi}^{(-)}$, $\widetilde{\mathcal{L}}_\lambda := H_+\Psi^{(+)} = \lambda\Psi^{(+)}$ and $\widetilde{\widehat{\mathcal{L}}} := \widehat{H}_+\widehat{\Psi}^{(+)} = \lambda\widehat{\Psi}^{(+)}$, where the Darboux transformations DT and $\widehat{\mathrm{DT}}$ are given by $\mathrm{DT}(\mathcal{L}) = \widetilde{\mathcal{L}}$, $\widehat{\mathrm{DT}}(\widehat{\mathcal{L}}) = \widetilde{\widehat{\mathcal{L}}}$,

$$\mathrm{DT}: \begin{array}{c} V_- \mapsto V_+ \\ \\ \Psi_\lambda^{(-)} \mapsto \Psi_\lambda^{(+)} \end{array}, \qquad \widehat{\mathrm{DT}}: \begin{array}{c} \widehat{V}_- \mapsto \widehat{V}_+ \\ \\ \widehat{\Psi}_\lambda^{(-)} \mapsto \widehat{\Psi}_\lambda^{(+)}, \end{array}$$

and $\varphi(\mathcal{L}_\lambda) = \widehat{\mathcal{L}_\lambda}$, where the algebrization φ is given as in theorem 2.3.13. Then the following diagram commutes

$$\begin{array}{ccc} \mathcal{L}_\lambda & \xrightarrow{\mathrm{DT}} & \widetilde{\mathcal{L}}_\lambda \\ \varphi\downarrow & & \downarrow\varphi \\ \widehat{\mathcal{L}}_\lambda & \xrightarrow[\widehat{\mathrm{DT}}]{} & \widetilde{\widehat{\mathcal{L}}}_\lambda \end{array} \qquad \Rightarrow \widehat{\mathrm{DT}}\varphi(\mathcal{L}) = \varphi\mathrm{DT}(\mathcal{L}) \Leftrightarrow \widehat{\widetilde{\mathcal{L}}} = \widetilde{\widehat{\mathcal{L}}}.$$

□

To illustrate $\widehat{\mathrm{DT}}$ we present the following example.

Example. Let consider the algebrized Schrödinger equation $\widehat{H}\widehat{\Psi}^{(-)} = \lambda\widehat{\Psi}^{(-)}$ with:

- $\sqrt{\alpha(z)} = \sqrt{z^2-1}$ and $\widehat{V}_-(z) = \frac{z}{z-1}$. Taking $\lambda_1 = 1$ and $\widehat{\Psi}_1^{(-)} = \sqrt{\frac{z+1}{z-1}}$, we have that $\widehat{\mathrm{DT}}(\widehat{V}_-) = \widehat{V}_+(z) = \frac{z}{z+1}$ and

$$\widehat{\mathrm{DT}}(\widehat{\Psi}_\lambda^{(-)}) = \widehat{\Psi}_\lambda^{(+)} = \sqrt{z^2-1}\partial_z\widehat{\Psi}_\lambda^{(-)} + \frac{1}{\sqrt{z^2-1}}\widehat{\Psi}_\lambda^{(-)},$$

  where $\widehat{\Psi}_\lambda^{(-)}$ is the general solution of $\widehat{H}_-\widehat{\Psi}^{(-)} = \lambda\widehat{\Psi}^{(-)}$ for $\lambda \neq 1$.

  The original potential corresponding to this example is given by $V_-(x) = \frac{\cosh x}{\cosh x-1}$ and for $\lambda_1 = 1$ the particular solution $\Psi_1^{(-)}$ is given by $\frac{\sinh x}{\cosh x-1}$. Applying DT we have that $\mathrm{DT}(V_-) = V_+(x) = \frac{\cosh x}{\cosh x+1}$ and $\mathrm{DT}(\Psi_\lambda^{(-)}) = \Psi_\lambda^{(+)} = \partial_x\Psi_\lambda^{(-)} + \frac{1}{\sinh x}\Psi_\lambda^{(-)}$.

- $\sqrt{\alpha} = -z$, $\widehat{V}_-(z) = z^2 - z$. Taking $\lambda_1 = 0$ and $\widehat{\Psi}_0^{(-)} = e^{-z}$ we have that $\widehat{\mathrm{DT}}(\widehat{V}_-) = V_+ = z^2 + z$ and $\widehat{\mathrm{DT}}(\widehat{\Psi}_\lambda^{(-)}) = \widehat{\Psi}_\lambda^{(+)} = -z\partial_z\widehat{\Psi}_\lambda^{(-)} - z\widehat{\Psi}_\lambda^{(-)}$, where $\Psi_\lambda^{(-)}$ is the general solution of $\widehat{H}_-\widehat{\Psi}^{(-)} = \lambda\widehat{\Psi}^{(-)}$ for $\lambda \neq 0$. This example corresponds to the Morse potential $V_-(x) = e^{-2x} - e^{-x}$, introduced in the list (1.38).

To illustrate $\widehat{\mathrm{CI}}_n$ we present the following example, which is related with the Chebyshev polynomials.

Example. Now, considering $\sqrt{\alpha} = -\sqrt{1-z^2}$, $V = 0$ with eigenvalues and eigenfunctions $\lambda_1 = 1$, $\lambda_2 = 4$, $\widehat{\Psi}_1 = z$, $\widehat{\Psi}_4 = 2z^2 - 1$, $\widehat{\Psi}_{n^2} = T_n(z)$, where $T_n(z)$ is the Chebyshev polynomial of first kind of degree n. The algebrized Wronskian for $n = 2$ is

$$\widehat{W}(z, 2z^2-1) = -\sqrt{1-z^2}(2z^2+1),$$

and by algebrized Crum iteration we obtain the potential

$$\widehat{\mathrm{CI}}_2(\widehat{V}) = \widehat{V}_2 = ((2z^2-1)\partial_z^2 + z\partial_z)\ln\widehat{W}(z, 2z^2-1)$$

and the algebrized wave functions

$$\widehat{\mathrm{CI}}_2(\widehat{\Psi}_\lambda) = \widehat{\Psi}_\lambda^{(2)} = \frac{\widehat{W}(z, 2z^2-1, T_n)}{\widehat{W}(z, 2z^2-1)}.$$

To illustrate the algebrized shape invariant potentials and the operators $\widehat{A}$ and $\widehat{A}^\dagger$, we present the following example. In a natural way we introduce the concept of algebrized shape invariant potentials $\widehat{V}_{n+1}(z, a_n) = \widehat{V}_n(z, a_{n+1}) + R(a_n)$, where the energy levels for $n > 0$ are given by $E_n = R(a_1) + \cdots R(a_n)$ and the algebrized eigenfunctions are given by $\widehat{\Psi}_n(a_1) = \widehat{A}^\dagger(z, a_1) \cdots \widehat{A}^\dagger(z, a_n)\Psi_0(z, a_n)$.

Example. Let assume $\sqrt{\alpha} = 1 - z^2$ and the algebrized super potential $\widehat{W}(z) = z$. Following the method proposed in remark 2.2.16, step 1, we introduce $\mu \in \mathbb{C}$ to obtain $\widehat{W}(z;\mu) = \mu z$, and

$$\widehat{V}_\pm(z;\mu) = \widehat{W}^2(z;\mu) \pm \widehat{\partial}_z \widehat{W}(z;\mu) = \mu(\mu \mp 1)z^2 \pm \mu,$$

thus, $\widehat{V}_+(z;a_0) = a_0(a_0-1)z^2 + a_0$ and $\widehat{V}_-(z;a_1) = a_1(a_1+1)z^2 - a_1$. By step 2,

$$\widehat{\partial}_z(V_+(z;a_0) - V_-(z;a_1)) = 2z(1-z^2)(a_0(a_0-1) - a_1(a_1+1)).$$

By step 3, we obtain

$$a_1(a_1+1) = a_0(a_0-1), \quad a_1^2 - a_0^2 = -(a_1 + a_0),$$

and assuming $a_1 \neq \pm a_0$ we have $a_1 = f(a_0) = a_0 - 1$ and $R(a_1) = 2a_0 + 1 = (a_0+1)^2 - a_0^2 = a_1^2 - a_0^2$. This means that the potentials $\widehat{V}_\pm$ are algebrized shape invariant potentials where $E = E_n$ is easily obtained,

$$E_n = \sum_{k=1}^{n} R(a_k) = \sum_{k=1}^{n}\left(a_k^2 - a_{k-1}^2\right) = a_n^2 - a_0^2, \quad a_n = f^n(a_0) = a_0 + n.$$

Now, the algebrized ground state wave function of $\widehat{V}_-(z, a_0)$ is

$$\widehat{\Psi}_0 = e^{\int \frac{a_0 z}{1-z^2}dz} = \frac{1}{\left(\sqrt{1-z^2}\right)^{a_0}}.$$

Finally, we can obtain the rest of eigenfunctions using the algebrized raising operator:

$$\widehat{\Psi}_n(z, a_0) = \widehat{A}^\dagger(z, a_0)\widehat{A}^\dagger(z, a_1)\cdots\widehat{A}^\dagger(z, a_{n-1})\widehat{\Psi}_0(z, a_n).$$

This example corresponds to Pöschl-Teller potential introduced in the list (1.38).

Now to illustrate the power of Kovacic's algorithm with the derivation $\widehat{\partial}_z$, we study some Schrödinger equations for non-rational shape invariant potentials given in list (1.38). We work with specific values of these potentials, although we can apply our machinery (algebrization method and Kovacic's algorithm) using all the parameters of such potentials.

Morse potential: $V(x) = e^{-2x} - e^{-x}$.

The Schrödinger equation $H\Psi = \lambda\Psi$ is

$$\partial_x^2\Psi = \left(e^{-2x} - e^{-x} - \lambda\right)\Psi.$$

By the Hamiltonian change of variable $z = z(x) = e^{-x}$, we obtain

$$\alpha(z) = z^2, \quad \widehat{V}(z) = z^2 - z, \quad \widehat{\mathbf{V}}(z) = \frac{z^2 - z - \frac{1}{4}}{z^2}.$$

Thus, $\widehat{K} = \mathbb{C}(z)$ and $K = \mathbb{C}(e^x)$. In this way, the algebrized Schrödinger equation $\widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}$ is

$$z^2\partial_z^2\widehat{\Psi} + z\partial_z\widehat{\Psi} - (z^2 - z - \lambda)\widehat{\Psi} = 0$$

and the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\widehat{\Phi} = \lambda\widehat{\Phi}$ is

$$\partial_z^2\Phi = r\Phi, \quad r = \frac{z^2 - z - \frac{1}{4} - \lambda}{z^2}$$

This equation only could fall in case one, in case two or in case four (of Kovacic's algorithm). We start analyzing the case one: by conditions c_2 and ∞_3 we have that

$$\left[\sqrt{r}\right]_0 = 0, \quad \alpha_0^{\pm} = \frac{1 \pm 2\sqrt{-\lambda}}{2}, \quad \left[\sqrt{r}\right]_\infty = 1, \quad \text{and} \quad \alpha_\infty^{\pm} = \mp\frac{1}{2}.$$

By step two we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\begin{array}{lll}
\Lambda_{++}) & n = \alpha_\infty^+ - \alpha_0^+ = -1 - \sqrt{-\lambda}, & \lambda = -\left(n+1\right)^2, \\
\Lambda_{+-}) & n = \alpha_\infty^+ - \alpha_0^- = -1 + \sqrt{-\lambda}, & \lambda = -\left(n+1\right)^2, \\
\Lambda_{-+}) & n = \alpha_\infty^- - \alpha_0^+ = -\sqrt{-\lambda}, & \lambda = -n^2, \\
\Lambda_{--}) & n = \alpha_\infty^- - \alpha_0^- = \sqrt{-\lambda}, & \lambda = -n^2.
\end{array}$$

We can see that $\lambda \in \Lambda_- = \{-n^2 : n \in \mathbb{Z}_+\}$. Now, for $\lambda \in \Lambda$, the rational function

ω is given by:

$$\begin{array}{llll}\Lambda_{++}) & \omega = 1+\frac{3+2n}{2z}, & \lambda\in\Lambda_{++}, & r_n = \frac{4n^2+8n+3}{4z^2}+\frac{2n+3}{z}+1,\\ \Lambda_{+-}) & \omega = 1-\frac{1+2n}{2z}, & \lambda\in\Lambda_{+-}, & r_n = \frac{4n^2+8n+3}{4z^2}-\frac{2n+1}{z}+1,\\ \Lambda_{-+}) & \omega = -1+\frac{1+2n}{2z}, & \lambda\in\Lambda_{-+}, & r_n = \frac{4n^2-1}{4z^2}-\frac{2n+1}{z}+1,\\ \Lambda_{--}) & \omega = -1+\frac{1-2n}{2z}, & \lambda\in\Lambda_{--}, & r_n = \frac{4n^2-1}{4z^2}+\frac{2n-1}{z}+1,\end{array}$$

where r_n is the coefficient of the differential equation $\partial_z^2\Phi = r_n\Phi$.

By step three, there exists a polynomial of degree n satisfying the relation (1.6),

$$\begin{array}{llcl}\Lambda_{++}) & \partial_z^2\widehat{P}_n+2\left(1+\frac{3+2n}{2z}\right)\partial_z\widehat{P}_n+\frac{2(n+2)}{z}\widehat{P}_n & = & 0,\\ \Lambda_{+-}) & \partial_z^2\widehat{P}_n+2\left(1-\frac{1+2n}{2z}\right)\partial_z\widehat{P}_n+\frac{2(-n)}{z}\widehat{P}_n & = & 0,\\ \Lambda_{-+}) & \partial_z^2\widehat{P}_n+2\left(-1+\frac{1+2n}{2z}\right)\partial_z\widehat{P}_n+\frac{2(-n)}{z}\widehat{P}_n & = & 0,\\ \Lambda_{--}) & \partial_z^2\widehat{P}_n+2\left(-1+\frac{1-2n}{2z}\right)\partial_z\widehat{P}_n+\frac{2n}{z}\widehat{P}_n & = & 0.\end{array}$$

These polynomials only exists for $n=\lambda=0$, with $\lambda\in\Lambda_{-+}\cup\Lambda_{--}$. So that the solutions of $H\Psi=0$, $\widehat{H}\widehat{\Psi}=0$ and $\widehat{\mathbf{H}}\Phi=0$ are given by

$$\Phi_0=\sqrt{z}e^{-z},\quad \widehat{\Psi}_0=e^{-z},\quad \Psi=e^{-e^{-x}}.$$

The wave function Ψ_0 satisfy the conditions (1.23), which means that is ground state (see [31]) and $0\in\mathrm{spec}_p(H)$. Furthermore, we have

$$\mathrm{DGal}_K(H\Psi=0)=\mathrm{DGal}_{\widehat{K}}(\widehat{H}\widehat{\Psi}=0)=\mathrm{DGal}_{\mathbb{C}(z)}(\widehat{\mathbf{H}}\Phi=0)=\mathbb{B},$$

$$\mathcal{E}(H)=\mathcal{E}(\widehat{H})=\mathcal{E}(\widehat{\mathbf{H}})=1.$$

We follow with the case two. The conditions c_2 and ∞_3 are satisfied, in this way we have

$$E_c=\left\{2,2+4\sqrt{-\lambda},2-4\sqrt{-\lambda}\right\}\quad\text{and}\quad E_\infty=\{0\},$$

and by step two, we have that $2\pm\sqrt{-\lambda}=m\in\mathbb{Z}_+$, so that $\lambda=-\left(\frac{m+1}{2}\right)^2$ and the rational function θ has the following possibilities

$$\theta_+=\frac{2+m}{z},\quad \theta_-=-\frac{m}{z}.$$

By step three, there exist a monic polynomial of degree m satisfying the recurrence relation (1.7):

$$\begin{array}{lllll} \theta_+) & \partial_z^3\widehat{P}_m+\frac{3m+6}{z}\partial_z^2\widehat{P}_m-\frac{4z^2-4z-2m^2-7m-6}{z^2}\partial_z\widehat{P}_m-\frac{4mz+8z-4m-6}{z^2}\widehat{P}_m & = & 0, \\ \theta_-) & \partial_z^3\widehat{P}_m-\frac{3m}{z}\partial_z^2\widehat{P}_m-\frac{4z^2-4z-2m^2-m}{z^2}\partial_z\widehat{P}_m+\frac{4mz-4m-2}{z^2}\widehat{P}_m & = & 0. \end{array}$$

We can see that for $m=1$ the polynomial exists only for the case θ_-, being $\widehat{P}_1=z-1/2$. In general, these polynomials could exist only for the case θ_- with $m=2n-1$, $n\geq 1$, that is $\lambda\in\{-n^2:n\geq 1\}$.

For instance, by case one and case two, we obtain $\Lambda=\{-n^2:n\geq 0\}=\mathrm{spec}_p(H)$. Now, the rational function ϕ and the quadratic expression for ω are

$$\phi=-\frac{2n-1}{z}+\frac{\partial_z\widehat{P}_{2n-1}}{\widehat{P}_{2n-1}},\quad \omega^2+M\omega+N=0,\quad \omega=\frac{-M\pm\sqrt{M^2-4N}}{2},$$

where the coefficients M and N are given by

$$M=\frac{2n-1}{z}-\frac{\partial_z\widehat{P}_{2n-1}}{\widehat{P}_{2n-1}},\quad N=\frac{n^2-n+\frac14}{z^2}-\frac{(2n-1)\frac{\partial_z\widehat{P}_{2n-1}}{\widehat{P}_{2n-1}}-2}{z}+\frac{\partial_z^2\widehat{P}_{2n-1}}{\widehat{P}_{2n-1}}-2.$$

Now, $\triangle=M^2-4N\neq 0$, which means that $\widehat{\mathbf{H}}\Phi=-n^2\Phi$ with $n\in\mathbb{Z}^+$ has two solutions given by Kovacic's algorithm:

$$\Phi_{1,n}=\frac{\sqrt{z}\widehat{P}_n e^{-z}}{z^n},\quad \Phi_{2,n}=\frac{\sqrt{z}\widehat{P}_{n-1}e^{z}}{z^n}.$$

The solutions of $\widehat{H}\widehat{\Psi}=-n^2\widehat{\Psi}$ are given by

$$\widehat{\Psi}_{1,n}=\frac{\widehat{P}_n e^{-z}}{z^n},\quad \widehat{\Psi}_{2,n}=\frac{\widehat{P}_{n-1}e^{z}}{z^n},$$

and therefore, the solutions of the Schrödinger equation $H\Psi=-n^2\Psi$ are

$$\Psi_{1,n}=P_n e^{-e^{-x}}e^{nx},\quad \Psi_{2,n}=P_{n-1}e^{e^{-x}}e^{nx},\quad P_n=\widehat{P}_n\circ z.$$

The wave functions $\Psi_{1,n}=\Psi_n$ satisfies the conditions of bound state, and for $n=0$, this solution coincides with the ground state presented above. Therefore we have

$$\Phi_n=\Phi_0\widehat{f}_n\widehat{P}_n,\quad \widehat{\Psi}_n=\widehat{\Psi}_0\widehat{f}_n\widehat{P}_n,\quad \widehat{f}_n(z)=\frac{1}{z^n}.$$

Thus, the bound states wave functions are obtained as

$$\Psi_n=\Psi_0 f_n P_n,\quad f_n(x)=\widehat{f}_n(e^{-x})=e^{nx}.$$

The Eigenrings and differential Galois groups for $n > 0$ satisfies

$$\mathrm{DGal}_K((H+n^2)\Psi = 0) = \mathrm{DGal}_{\widehat{K}}((\widehat{H}+n^2)\widehat{\Psi} = 0) = \mathrm{DGal}_{\mathbb{C}(z)}((\widehat{\mathbf{H}}+n^2)\Phi = 0) = \mathbb{G}_m,$$

$$\dim_{\mathbb{C}} \mathcal{E}(\widehat{\mathbf{H}} + n^2) = \dim_{\mathbb{C}} \mathcal{E}(\widehat{H} + n^2) = \dim_{\mathbb{C}} \mathcal{E}(H + n^2) = 2.$$

We remark that the Schrödinger equation with Morse potential, under suitable changes of variables [53], falls in a Bessel's differential equation. Thus we can obtain its integrability by means of corollary 1.1.31.

It is known that Eckart, Rosen-Morse, Scarf and Pöschl-Teller potentials, under suitable transformations, fall in an Hypergeometric equation which allows apply the theorem 1.1.29. These potentials are inter-related by point canonical coordinate transformations (see [24, p. 314]), so that $\Lambda = \mathbb{C}$ due to Pöschl-Teller potential is obtained by means of Darboux transformations of $V = 0$ ([59, 76]). We consider some particular cases of Eckart, Scarf and Poschl-Teller potentials applying only the case 1 of Kovacic's algorithm. The case 1 allow us to obtain the enumerable set $\Lambda_n \subset \Lambda$, which include the classical results obtained by means of supersymmetric quantum mechanics. Cases 2 and 3 of Kovacic algorithm also can be applied, but are not considered here.

Eckart potential: $V(x) = 4\coth(x) + 5$, $x > 0$.

The Schrödinger equation $H\Psi = \lambda\Psi$ is

$$\partial_x^2\Psi = (4\coth(x) + 5 - \lambda)\,\Psi.$$

By the Hamiltonian change of variable $z = z(x) = \coth(x)$, we obtain

$$\alpha(z) = (1 - z^2)^2, \quad \widehat{V}(z) = 4z + 5, \quad \widehat{\mathbf{V}}(z) = \frac{4}{(z+1)(z-1)^2}.$$

Thus, $\widehat{K} = \mathbb{C}(z)$ and $K = \mathbb{C}(\coth(x))$. In this way, the algebrized Schrödinger equation $\widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}$ is

$$(1 - z^2)^2\partial_z^2\widehat{\Psi} - 2z(1 - z^2)\partial_z\widehat{\Psi} - (4z + 5 - \lambda)\widehat{\Psi} = 0$$

and the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\widehat{\Phi} = \lambda\widehat{\Phi}$ is

$$\partial_z^2\Phi = r\Phi, \quad r = \frac{4z + 4 - \lambda}{(z-1)^2(z+1)^2} = \frac{2 - \frac{\lambda}{4}}{(z-1)^2} + \frac{\frac{\lambda}{4} - 1}{(z-1)} + \frac{-\frac{\lambda}{4}}{(z+1)^2} + \frac{1 - \frac{\lambda}{4}}{(z+1)}$$

We can see that this equation could fall in any case of Kovacic's algorithm. Considering $\lambda = 0$, the conditions $\{c_1, c_2, \infty_1\}$ of case 1 are satisfied, obtaining

$$[\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_\infty = \alpha_\infty^+ = 0, \quad \alpha_{-1}^\pm = \alpha_\infty^- = 1, \quad \alpha_1^+ = 2, \quad \alpha_1^- = -1.$$

By step two, the elements of D are 0 and 1. The rational function ω for $n=0$ and for $n=1$ must be

$$\omega = \frac{1}{z+1} + \frac{-1}{z-1}.$$

By step three we search the monic polynomial of degree n satisfying the relation (1.6). Starting with $n=0$ the only one possibility is $\widehat{P}_0(z)=1$, which effectively satisfy the relation (1.6), while $\widehat{P}_1(z)=z+a_0$ does not exists. In this way we have obtained one solution using Kovacic algorithm:

$$\Phi_0 = \frac{z+1}{z-1}, \quad \widehat{\Psi}_0 = \sqrt{\frac{z+1}{(z-1)^3}},$$

this means that $0 \in \Lambda_n$. We can obtain the second solution using the first solution:

$$\Phi_{0,2} = \frac{z^2+z-4-4\,\ln(z+1)\,z-4\,\ln(z+1)}{z-1}, \quad \widehat{\Psi}_{0,2} = \frac{\Phi_{0,2}}{\sqrt{z^2-1}}.$$

Furthermore the differential Galois groups and Eigenrings for $\lambda=0$ are

$$\mathrm{DGal}_{\mathbb{C}(z)}(\widehat{\mathbf{H}}\Phi=0)=\mathbb{G}_a, \quad \mathrm{DGal}_K(H\Psi=0)=\mathrm{DGal}_{\widehat{K}}(\widehat{H}\widehat{\Psi}=0)=\mathbb{G}^{\{2\}},$$

$$\mathcal{E}(\widehat{\mathbf{H}}) = \left\{1, \frac{(z+1)^2}{(1-z)^2}\partial_z + \frac{2(z+1)}{(1-z)^3}\right\},$$

$$\mathcal{E}(\widehat{H}) = \left\{1, \frac{(z+1)^2}{(1-z)^2}\partial_z - \frac{z^2+3z+2}{(1-z)^3}\right\},$$

$$\mathcal{E}(H) = \left\{1, \frac{(\coth(x)+1)^2}{(1-\coth(x))^2(1-\coth^2(x))}\partial_x - \frac{\coth^2(x)+3\coth(x)+2}{(1-\coth(x))^3}\right\}.$$

Now, for $\lambda \neq 0$, the conditions $\{c_2, \infty_1\}$ of case 1 are satisfied:

$$[\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_\infty = \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1,$$

$$\alpha_{-1}^{\pm} = \tfrac{1\pm\sqrt{1-\lambda}}{2}, \quad \alpha_1^{\pm} = \tfrac{1\pm\sqrt{9-\lambda}}{2}.$$

By step two we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$\Lambda_{++-})$ $\quad n = \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^- = -1 - \tfrac{\sqrt{1-\lambda}-\sqrt{9-\lambda}}{2}, \quad \lambda = 4 - \tfrac{4}{(n+1)^2} - n^2 - 2n,$

$\Lambda_{+--})$ $\quad n = \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^- = -1 + \tfrac{\sqrt{1-\lambda}+\sqrt{9-\lambda}}{2}, \quad \lambda = 4 - \tfrac{4}{(n+1)^2} - n^2 - 2n,$

$\Lambda_{-+-})$ $\quad n = \alpha_\infty^- - \alpha_{-1}^+ - \alpha_1^- = \tfrac{\sqrt{1-\lambda}+\sqrt{9-\lambda}}{2}, \quad \lambda = 5 - \tfrac{4}{n^2} - n^2,$

$\Lambda_{---})$ $\quad n = \alpha_\infty^- - \alpha_{-1}^- - \alpha_1^- = \tfrac{\sqrt{1-\lambda}+\sqrt{9-\lambda}}{2}, \quad \lambda = 5 - \tfrac{4}{n^2} - n^2.$

Therefore, we have that

$$\Lambda_n \subseteq \left\{4-\frac{4}{(n+1)^2}-n^2-2n : n\in\mathbb{Z}_+\right\} \cup \left\{5-\frac{4}{n^2}-n^2 : n\in\mathbb{Z}_+\right\}.$$

Now, for $\lambda\in\Lambda$, the rational function ω is given by:

$$\Lambda_{++-})\qquad \omega=\tfrac{z(n-1)-n^2-2n-1}{(n+1)(z+1)(z-1)},\quad r_n=\tfrac{-2z^2(n-1)+4z(n+1)^2+(n+1)(n^3+3n^2+2n+2)}{(n+1)^2(z+1)^2(z-1)^2},$$

$$\Lambda_{+--})\qquad \omega=\tfrac{nz(n+1)+2}{(n+1)(z+1)(1-z)},\quad r_n=\tfrac{nz^2(n+1)^3+4z(n+1)^2+n^3+2n^2+n+4}{(n+1)^2(z+1)^2(z-1)^2},$$

$$\Lambda_{-+-})\qquad \omega=\tfrac{z(n-2)-n^2}{n(z+1)(z-1)},\quad r_n=\tfrac{-2z^2(n-2)+4n^2z+n(n^3-n+2)}{n^2(z+1)^2(z-1)^2},$$

$$\Lambda_{---})\qquad \omega=\tfrac{nz(n-1)+2}{n(z+1)(1-z)},\quad r_n=\tfrac{n^3z^2(n-1)+4n^2z+n^3-n^2+4}{n^2(z+1)^2(z-1)^2},$$

where r_n is the coefficient of the differential equation $\partial_z^2\Phi=r_n\Phi$.

By step three, there exists a monic polynomial of degree n satisfying the relation (1.6),

$$\Lambda_{++-})\qquad \partial_z^2\widehat{P}_n+2\left(\tfrac{z(n-1)-n^2-2n-1}{(n+1)(z+1)(z-1)}\right)\partial_z\widehat{P}_n+\tfrac{2(1-n)}{((n+1)^2(z+1)(z-1)}\widehat{P}_n = 0,$$

$$\Lambda_{+--})\qquad \partial_z^2\widehat{P}_n+2\left(\tfrac{nz(n+1)+2}{(n+1)(z+1)(1-z)}\right)\partial_z\widehat{P}_n+\tfrac{n(n+1)}{(z+1)(z-1)}\widehat{P}_n = 0,$$

$$\Lambda_{-+-})\qquad \partial_z^2\widehat{P}_n+2\left(\tfrac{z(n-2)-n^2}{n(z+1)(z-1)}\right)\partial_z\widehat{P}_n+\tfrac{2(2-n)}{n^2(z+1)(z-1)}\widehat{P}_n = 0,$$

$$\Lambda_{---})\qquad \partial_z^2\widehat{P}_n+2\left(\tfrac{nz(n-1)+2}{n(z+1)(1-z)}\right)\partial_z\widehat{P}_n+\tfrac{n(n-1)}{(z+1)(z-1)}\widehat{P}_n = 0.$$

The only one case in which there exist the polynomial $\widehat{P}_n$ of degree n is for $\Lambda_{+--})$. The solutions of the equation $\widehat{\mathbf{H}}\Phi=\lambda\Phi$, with $\lambda\neq 0$, are:

$$\Lambda_{++-})\qquad \Phi_n=\widehat{P}_n\widehat{f}_n\Phi_0,\quad \Phi_0=\tfrac{1}{z-1}\quad \widehat{f}_n=(z-1)^{\frac{n(1-n)}{2(n+1)}}(z+1)^{\frac{n(n+3)}{2(n+1)}},$$

$$\Lambda_{+--})\qquad \Phi_n=\widehat{P}_n\widehat{f}_n\Phi_0,\quad \Phi_0=\tfrac{z+1}{z-1}\quad f_n=(z-1)^{\frac{n(1-n)}{2(n+1)}}(z+1)^{\frac{-n(n+3)}{2(n+1)}},$$

$$\Lambda_{-+-})\qquad \Phi_n=\widehat{P}_n\widehat{f}_n\Phi_1,\quad \Phi_1=\tfrac{1}{z-1}\quad \widehat{f}_n=(z+1)^{\frac{n^2+n-2}{2n}}(z-1)^{\frac{-n^2+3n-2}{2n}},$$

$$\Lambda_{---})\qquad \Phi_n=\widehat{P}_n\widehat{f}_n\Phi_1,\quad \Phi_1=\tfrac{z+1}{z-1}\quad \widehat{f}_n=(z+1)^{\frac{-n^2-n+2}{2n}}(z-1)^{\frac{-n^2+3n-2}{2n}}.$$

In any case $\widehat{\Psi}_n=\frac{\Phi_n}{1-z^2}$, but the case $\Lambda_{+--})$ includes the classical results obtained by means of supersymmetric quantum mechanics. Thus, replacing z by $\coth(x)$ we obtain the eigenstates Ψ_n. The Eigenrings and differential Galois groups for $n > 0$ satisfies

$$\mathrm{DGal}_K((H-\lambda)\Psi = 0) \subseteq \mathbb{G}^{\{2m\}}, \quad \mathrm{DGal}_{\widehat{K}}((\widehat{H}-\lambda)\widehat{\Psi} = 0) \subseteq \mathbb{G}^{\{2m\}},$$

$$\mathrm{DGal}_{\mathbb{C}(z)}((\widehat{\mathbf{H}}+n^2)\Phi = 0) = \mathbb{G}_m,$$

$$\mathcal{E}(\widehat{\mathbf{H}}+n^2) = \mathcal{E}(\widehat{H}+n^2) = \mathcal{E}(H+n^2) = \{1\}.$$

Scarf potential: $V(x) = \frac{\sinh^2 x - 3\sinh x}{\cosh^2 x}$.

The Schrödinger equation $H\Psi = E\Psi$ is

$$\partial_x^2\Psi = \left(\frac{\sinh^2 x - 3\sinh x}{\cosh^2 x} - E\right)\Psi.$$

By the Hamiltonian change of variable $z = z(x) = \sinh(x)$, we obtain

$$\alpha(z) = 1+z^2, \quad \widehat{V}(z) = \frac{z^2-3z}{1+z^2}.$$

Thus, $\widehat{K} = \mathbb{C}(z, \sqrt{1+z^2})$ and $K = \mathbb{C}(\sinh(x), \cosh(x))$. In this way, the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$ is

$$\partial_z^2 = \left(\frac{\lambda z^2 - 12z + \lambda - 1}{4(z^2+1)^2}\right)\Phi, \quad \lambda = 3-4E.$$

Applying Kovacic's algorithm for this equation with $\lambda = 0$, we see that does not falls in case 1. We consider only $\lambda \neq 0$. By conditions $\{c_2, \infty_2\}$ of case 1 we have that

$$[\sqrt{r}]_{-\mathrm{i}} = [\sqrt{r}]_{\mathrm{i}} = [\sqrt{r}]_\infty = 0, \quad \alpha_\infty^\pm = \tfrac{1\pm\sqrt{1+\lambda}}{2},$$

$$\alpha_{-\mathrm{i}}^+ = \tfrac{5}{4} - \tfrac{\mathrm{i}}{2}, \quad \alpha_{-\mathrm{i}}^- = -\tfrac{1}{4} + \tfrac{\mathrm{i}}{2} \quad \alpha_{\mathrm{i}}^+ = \tfrac{5}{4} + \tfrac{\mathrm{i}}{2}, \quad \alpha_{\mathrm{i}}^- = -\tfrac{1}{4} - \tfrac{\mathrm{i}}{2}.$$

By step two we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\Lambda_{+++}) \quad n = \alpha_\infty^+ - \alpha_{-\mathrm{i}}^+ - \alpha_{\mathrm{i}}^+ = \tfrac{\sqrt{\lambda+1}-4}{2}, \quad \lambda = 4n^2+16n+15,$$

$$\Lambda_{+--}) \quad n = \alpha_\infty^+ - \alpha_{-\mathrm{i}}^- - \alpha_{\mathrm{i}}^- = \tfrac{\sqrt{\lambda+1}+2}{2}, \quad \lambda = 4n^2-8n+3,$$

obtaining in this way

$$\Lambda_n \subseteq \left\{4n^2+16n+15 : n \in \mathbb{Z}_+\right\} \cup \left\{4n^2-8n+3 : n \in \mathbb{Z}_+\right\}.$$

Now, the rational function ω is given by:

$$\Lambda_{+++}) \qquad \omega = \frac{5z-2}{2(z^2+1)}, \qquad \Lambda_{+--}) \quad \frac{2-z}{2(z^2+1)}.$$

By step three, there exists $\widehat{P}_0 = 1$ and a polynomial of degree $n \geq 1$ should satisfy either of the relation (1.6),

$$\begin{array}{llll} \Lambda_{+++}) & \partial_z^2 \widehat{P}_n + \frac{5z-2}{z^2+1}\partial_z \widehat{P}_n - \frac{nz^2(n+4)+3z+n^2+4n-3}{(z^2+1)^2}\widehat{P}_n & = & 0, \\ \Lambda_{+--}) & \partial_z^2 \widehat{P}_n + \frac{2-z}{z^2+1}\partial_z \widehat{P}_n - \frac{nz^2(n-2)+3z+n^2-2n-3}{(z^2+1)^2}\widehat{P}_n & = & 0. \end{array}$$

In both cases there exists the polynomial $\widehat{P}_n$ of degree $n \geq 1$. The basis of solutions $\{\Phi_{1,n}, \Phi_{2,n}\}$ of the equation reduced algebrized Schrödinger are:

$$\begin{array}{llll} \Lambda_{+++}) & \Phi_{1,n} = \widehat{P}_n \widehat{f}_n \Phi_{1,0}, & \Phi_{1,0} = (1+z^2)^{\frac{5}{4}} e^{-\arctan z}, & \widehat{f}_n = 1, \\ \Lambda_{+--}) & \Phi_{1,n} = \widehat{P}_n \widehat{f}_n \Phi_{1,0}, & \Phi_{1,0} = \frac{1}{\sqrt[4]{1+z^2}} e^{\arctan z}, & f_n = 1. \end{array}$$

The second solutions for these cases with associated polynomials $\widehat{P}_n$ of degree n are given by

$$\begin{array}{llll} \Lambda_{+++}) & \Phi_{2,n} = \widetilde{P}_n f_n \Phi_0, & \Phi_0 = \frac{22+21x+12x^2+6x^3}{\sqrt[4]{1+z^2}} e^{-\arctan z}, & f_n = 1, \\ \Lambda_{+--}) & \Phi_{2,n} = \widetilde{P}_n f_n \Phi_0, & \Phi_0 = \frac{1}{\sqrt[4]{1+z^2}} e^{\arctan z} \int \frac{1}{\sqrt{1+z^2}} e^{-2\arctan z} dz, & f_n = 1. \end{array}$$

In both cases $\widehat{\Psi} = \frac{\Phi}{\sqrt[4]{1+z^2}}$, but the classical case (see references [24, 31]) is $\Lambda_{+--})$, so that replacing z by $\sinh x$ and λ by $3-4E$ we obtain the eigenstates Ψ_n.

The Eigenrings and differential Galois groups are

$$\mathcal{E}(H-\lambda) = \mathcal{E}(\widehat{H}-\lambda) = \mathcal{E}(\widehat{\mathbf{H}}-\lambda) = \{1\},$$

$$\mathrm{DGal}_K((H-\lambda)\Psi = 0) = \mathrm{DGal}_{\widehat{K}}((\widehat{H}-\lambda)\widehat{\Psi} = 0) = \mathrm{DGal}_{\mathbb{C}(x)}((\widehat{\mathbf{H}}-\lambda)\Phi = 0) = \mathbb{B}.$$

Pöschl-Teller potential: $V(\mathrm{r}) = \frac{\cosh^4(x)-\cosh^2(x)+2}{\sinh^2(x)\cosh^2(x)}$, $x > 0$. The reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = E\Phi$ is

$$\partial_z^2 \Phi = \left(\frac{\lambda z^4 - (\lambda+3)z^2 + 8}{4z^2(z^2-1)^2}\right)\Phi, \quad \lambda = 3-4E.$$

Considering $\lambda = 0$ and starting with the conditions $\{c_2, \infty_1\}$ of case 1, we obtain

$$[\sqrt{r}]_0 = [\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_\infty = \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1,$$

$$\alpha_{-1}^+ = \alpha_1^+ = \tfrac{5}{4}, \quad \alpha_{-1}^- = \alpha_1^- = -\tfrac{1}{4}, \quad \alpha_0^+ = 2, \quad \alpha_0^- = -1.$$

By step two, the elements of D are 0 and 1. The rational function ω has the following possibilities for $n = 0$ and for $n = 1$:

$$\Lambda_{++--}) \quad n = 0, \quad \omega = \tfrac{5/4}{z+1} + \tfrac{-1/4}{z-1} + \tfrac{-1}{z},$$

$$\Lambda_{+-+-}) \quad n = 0, \quad \omega = \tfrac{-1/4}{z+1} + \tfrac{5/4}{z-1} + \tfrac{-1}{z},$$

$$\Lambda_{-+--}) \quad n = 1, \quad \omega = \tfrac{5/4}{z+1} + \tfrac{-1/4}{z-1} + \tfrac{-1}{z},$$

$$\Lambda_{--+-}) \quad n = 1, \quad \omega = \tfrac{-1/4}{z+1} + \tfrac{5/4}{z-1} + \tfrac{-1}{z}.$$

By step three we search the monic polynomial of degree n satisfying the relation (1.6). Starting with $n = 0$ the only one possibility for $\Lambda_{++--})$ and $\Lambda_{+-+-})$ is $\widehat{P}_0(z) = 1$, which does not satisfy the relation (1.6) in both cases, while $\widehat{P}_1(z) = z + a_0$ effectively does exists, in where $a_0 = -\frac{2}{3}$ for $\Lambda_{-+--})$ and $a_0 = \frac{2}{3}$ for $\Lambda_{--+-})$. In this way we have obtained two solutions $(\Phi_{1,0}, \Phi_{2,0})$ using Kovacic's algorithm:

$$\Phi_{1,0} = \left(1 - \tfrac{2}{3z}\right) \sqrt[4]{\tfrac{(z+1)^5}{z-1}}, \quad \widehat{\Psi}_{1,0} = \left(1 - \tfrac{2}{3z}\right) \tfrac{z+1}{\sqrt{z-1}},$$

$$\Phi_{2,0} = \left(1 + \tfrac{2}{3z}\right) \sqrt[4]{\tfrac{(z-1)^5}{z+1}}, \quad \widehat{\Psi}_{2,0} = \left(1 + \tfrac{2}{3z}\right) \tfrac{z-1}{\sqrt{z+1}},$$

this means that $0 \in \Lambda_n$. Furthermore,

$$\mathrm{DGal}_{\mathbb{C}(x)}(\widehat{\mathbf{H}}\Phi = 0) = \mathbb{G}^{[4]}, \quad \mathrm{DGal}_{\widehat{K}}(\widehat{H}\widehat{\Psi} = 0) = \mathrm{DGal}_K(H\Psi = 0) = e,$$

$$\dim_{\mathbb{C}} \mathcal{E}(\widehat{\mathbf{H}}) = 2, \quad \dim_{\mathbb{C}} \mathcal{E}(\widehat{H}) = \dim_{\mathbb{C}} \mathcal{E}(H) = 4.$$

Now, for $\lambda \neq 0$ we see that conditions $\{c_2, \infty_1\}$ of case 1 lead us to

$$[\sqrt{r}]_0 = [\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_\infty = 0, \qquad \alpha_\infty^\pm = \tfrac{1 \pm \sqrt{1+\lambda}}{2},$$

$$\alpha_{-1}^+ = \alpha_1^+ = \tfrac{5}{4}, \quad \alpha_{-1}^- = \alpha_1^- = -\tfrac{1}{4}, \quad \alpha_0^+ = 2, \quad \alpha_0^- = -1.$$

By step two we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\Lambda_{++++}) \quad n = \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^+ - \alpha_0^+ = \tfrac{\sqrt{\lambda+1}-8}{2}, \quad \lambda = 4n^2 + 32n + 63,$$

$$\Lambda_{+++-}) \quad n = \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^+ - \alpha_0^- = \tfrac{\sqrt{\lambda+1}-2}{2}, \quad \lambda = 4n^2 + 8n + 3,$$

$$\Lambda_{++-+}) \quad n = \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^- - \alpha_0^+ = \tfrac{\sqrt{\lambda+1}-5}{2}, \quad \lambda = 4n^2 + 20n + 24,$$

$$\Lambda_{++--}) \quad n = \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^- - \alpha_0^- = \tfrac{\sqrt{\lambda+1}+1}{2}, \quad \lambda = 4n^2 - 4n,$$

$$\Lambda_{+-++}) \quad n = \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^+ - \alpha_0^+ = \tfrac{\sqrt{\lambda+1}-5}{2}, \quad \lambda = 4n^2 + 20n + 24,$$

$$\Lambda_{+-+-}) \quad n = \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^+ - \alpha_0^- = \tfrac{\sqrt{\lambda+1}+1}{2}, \quad \lambda = 4n^2 - 4n,$$

$$\Lambda_{+--+}) \quad n = \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^- - \alpha_0^+ = \tfrac{\sqrt{\lambda+1}-2}{2}, \quad \lambda = 4n^2 + 8n + 3,$$

$$\Lambda_{+---}) \quad n = \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^- - \alpha_0^- = \tfrac{\sqrt{\lambda+1}+4}{2}, \quad \lambda = 4n^2 - 16n + 15,$$

obtaining $\Lambda_n \subseteq \Lambda_a \cup \Lambda_b \cup \Lambda_c \cup \Lambda_d \cup \Lambda_e$, where

$$\begin{aligned}
&\Lambda_a = \left\{4n^2 + 32n + 63 : n \in \mathbb{Z}_+\right\}, \quad \Lambda_b = \left\{4n^2 + 8n + 3 : n \in \mathbb{Z}_+\right\}\\
&\Lambda_c = \left\{4n^2 + 20n + 24 : n \in \mathbb{Z}_+\right\}, \quad \Lambda_d = \left\{4n^2 - 4n : n \in \mathbb{Z}_+\right\},\\
&\Lambda_e = \left\{4n^2 - 16n + 15 : n \in \mathbb{Z}_+\right\}.
\end{aligned}$$

Now, the rational function ω is given by:

$$\begin{array}{ll}
\Lambda_{++++}) \quad \omega = \frac{5/4}{z+1} + \frac{5/4}{z-1} + \frac{2}{z}, & \Lambda_{+++-}) \quad \omega = \frac{5/4}{z+1} + \frac{5/4}{z-1} + \frac{-1}{z},\\
\Lambda_{++-+}) \quad \omega = \frac{5/4}{z+1} + \frac{-1/4}{z-1} + \frac{2}{z}, & \Lambda_{++--}) \quad \omega = \frac{5/4}{z+1} + \frac{-1/4}{z-1} + \frac{-1}{z},\\
\Lambda_{+-++}) \quad \omega = \frac{-1/4}{z+1} + \frac{5/4}{z-1} + \frac{2}{z}, & \Lambda_{+-+-}) \quad \omega = \frac{-1/4}{z+1} + \frac{5/4}{z-1} + \frac{-1}{z},\\
\Lambda_{+--+}) \quad \omega = \frac{-1/4}{z+1} + \frac{-1/4}{z-1} + \frac{2}{z}, & \Lambda_{+---}) \quad \omega = \frac{-1/4}{z+1} + \frac{-1/4}{z-1} + \frac{-1}{z}.
\end{array}$$

By step three, there exists a monic polynomial of degree n satisfying the relation (1.6),

$$\Lambda_{++++})\quad \partial_z^2\widehat{P}_n+\tfrac{(3z+2)(3z-2)}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{n(n+8)}{(z+1)(1-z)}P_n=0,$$

$$\Lambda_{+++-})\quad \partial_z^2\widehat{P}_n+\tfrac{3z^2+2}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{n(n+2)}{(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{++-+})\quad \partial_z^2\widehat{P}_n+\tfrac{6z^2-3x-4}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{nz(n+5)+6}{z(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{++--})\quad \partial_z^2\widehat{P}_n+\tfrac{3z-2}{z(z+1)(1-z)}\partial_z\widehat{P}_n+\tfrac{nz(n-1)-12}{4z(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{+-++})\quad \partial_z^2\widehat{P}_n+\tfrac{6z^2+3z-4}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{z(n^2+5)-6}{z(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{+-+-})\quad \partial_z^2\widehat{P}_n+\tfrac{3z+2}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{nz(n-1)+3}{z(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{+--+})\quad \partial_z^2\widehat{P}_n+\tfrac{3z^2-4}{z(z+1)(z-1)}\partial_z\widehat{P}_n+\tfrac{n(n+2)}{(z+1)(1-z)}\widehat{P}_n=0,$$

$$\Lambda_{+---})\quad \partial_z^2\widehat{P}_n+\tfrac{3z^2-2}{z(z+1)(1-z)}\partial_z\widehat{P}_n+\tfrac{n(4-n)}{(z+1)(z-1)}P_n=0.$$

The polynomial P_n of degree n exists for $\lambda_n\in\Lambda_b$ with n even, that is, $\Lambda_n=\{n\in\mathbb{Z}:16n^2+16n+3\}$, for $\Lambda_{++-+})$ and $\Lambda_{+--+})$. Therefore $E=E_n=\{n\in\mathbb{Z}:-4n^2-4n\}$.
The corresponding solutions for Λ_n are

$$\Lambda_{+++-})\qquad \Phi_{1,n}=\widehat{P}_{2n}\widehat{f}_n\Phi_{1,0},\quad \Phi_{1,0}=\frac{\sqrt[4]{(z^2-1)^5}}{z}\qquad \widehat{f}_n=1,\quad \widehat{\Psi}_{1,0}=z-\tfrac{1}{z},$$

$$\Lambda_{+--+})\qquad \widehat{\Phi}_{2,n}=\widetilde{P}_{2n}\widehat{f}_n\widehat{\Phi}_{2,0},\quad \widehat{\Phi}_{2,0}=\tfrac{z^2}{\sqrt[4]{z^2-1}}\qquad \widehat{f}_n=1\quad \widehat{\Psi}_{2,0}=\tfrac{z^2}{\sqrt{z^2-1}}.$$

These two solutions are equivalent to the same solution of the original Schrödinger equation and corresponds to the well known supersymmetric quantum mechanics approach to this Pöschl-Teller potential, [24, 25].

Searching New Potentials in Parameterized Differential Equations.

The main object to search new potentials using $\widehat{\partial}_z$ is the family of differential equations presented by Darboux in [28], see section 1.2.2 and equation (1.24),

$$\partial_z^2 y+P\partial_z y+(Q-\lambda R)y=0.$$

We recall that the differential equations presented in section 1.1.4 corresponds to this kind. After, we reduce the previous equation to put it in the form of the reduced algebrized Schrödinger equation $\alpha\mathrm{H}\Phi=\lambda\Phi$, checking that $\mathrm{Card}(\Lambda)>1$, to obtain the Schrödinger equation $H\Psi=\lambda\Psi$ (starting with the potential $\widehat{\mathbf{V}}$ and arriving to the potential V). To do this, we propose the following heuristic:

1. Reduce a differential equation of the form (1.24) and put it in the form $\alpha \mathrm{H}\Phi = \lambda\Phi$, checking that $\mathrm{Card}(\Lambda) > 1$ and to avoid triviality, α must be a non-constant function.
2. Write $\mathcal{W} = \frac{1}{4}\partial_z(\ln\alpha)$ and obtain $\widehat{V}(z) = \alpha(\widehat{\mathbf{V}} - \partial_z\mathcal{W} - \mathcal{W}^2)$.
3. Solve the differential equation $(\partial_x z)^2 = \alpha$, write $z = z(x)$, $V(x) = \widehat{V}(z(x))$.

To illustrate this method, we present the following examples.

Bessel Potentials

- (From Darboux transformations over $V = 0$) In the differential equation
$$\partial_z^2\Phi = \left(\frac{n(n+1)}{z^2} + \mu\right)\Phi, \quad \mu \in \mathbb{C},$$
we see that $\lambda = -n(n+1)$ and $\alpha = z^2$. Applying the method, we obtain $\widehat{\mathbf{V}} = \mu$ we obtain $\widehat{V}(z) = \mu z^2 + \frac{1}{4}$ and $z = z(x) = e^{\pm x}$. Thus, we have obtained the potentials $V(x) = \widehat{V}(z(x)) = \mu e^{\pm 2x} + \frac{1}{4}$ (compare with [35, §6.9]).
- (From Bessel differential equation) The equation
$$\partial_z^2 y + \frac{1}{z}\partial_z y + \frac{z^2 - n^2}{z^2} y = 0, \quad n \in \frac{1}{2} + \mathbb{Z},$$
is transformed to the reduced equation
$$\partial_z^2\Phi = \left(\frac{n^2}{z^2} - \frac{4z^2+1}{4z^2}\right)\Psi.$$
We can see that $\lambda = -n^2$, $\alpha = z^2$, obtaining $\widehat{\mathbf{V}} = -z^2 - \frac{1}{4}$, $\widehat{V} = -z^4 - \frac{1}{4}z^2 + \frac{1}{4}$ and $z = z(x) = e^{\pm x}$. Thus, so that we have obtained the potential $V(x) = \widehat{V}(z(x)) = -e^{\pm 4x} - \frac{1}{4}e^{\pm 2x} + \frac{1}{4}$ (compare with [35, §6.9]).

We remark that the previous examples give us potentials related with the Morse potential, due to their solutions are given in term of Bessel functions.

We can apply this method to equations such as Whittaker, Hypergeometric and in particular, differential equations involving orthogonal polynomials (compare with [24, §5]).

Final Remark

The aim of this work is, in contemporary terms, a formalization of original ideas and intuitions given by G. Darboux, E. Witten and L. É. Gendenshtein in the context of the Galois theory of linear differential equations. We found the following facts:

- We constructed integrable Schrödinger equations through the superpotential which is an algebraic solution of Riccati equation associated with the potential, defined over a differential field.
- Darboux transformation was interpreted as an isogaloisian transformation, allowing to obtain isomorphisms between their eigenrings.
- We introduced in a general way, the Hamiltonian algebrization method, allowing to apply algorithmic tools such as Kovacic's algorithm to obtain the solutions, differential Galois groups and Eigenrings. Also, we applied this algebrization procedure in Supersymmetric quantum mechanics.

As a conclusion, as happen in other areas of the field of differential equations, in view of the many families of examples studied along this thesis, we can conclude that the *differential Galois theory is the natural framework where should be studied the supersymmetric quantum mechanics.*

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