Elliptic and parabolic PDEs:
regularity for nonlocal diffusion equations and two isoperimetric problems

by

Joaquim Serra

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Summary

This thesis is divided into two parts.

The *first part* is mainly concerned with regularity issues for *integro-differential* or *nonlocal* equations. The elliptic integro-differential operators L of the form

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) \frac{b(y)}{|y|^{n+2s}}\, dy \quad \text{with } 0 < \lambda \leq b(y) \leq \Lambda \tag{0.1}$$

are infinitesimal generators of *Lévy processes*. Thus, in the same way that densities of particles with Brownian motion solve second order elliptic or parabolic equations, the equations $Lu = 0$ or $u_t = Lu$ are satisfied by densities of particles with Lévy motion.

When $b(y)$ is constant, the operator L is the fractional Laplacian $-(-\Delta)^s$, which can be also defined via Fourier transform as $\mathcal{F}\big((-\Delta)^s u\big) = |\xi|^{2s} \mathcal{F}(u)$.

The well-posed Dirichlet problem for these operators is a problem with complement data:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{0.2}$$

There are many classical regularity results for $(-\Delta)^s$ —whose "inverse" is the Riesz potential. For instance, the explicit Poisson kernel for a ball is an "old" result, as well as the L^p solvability of the equation $(-\Delta)^s u = f$ in the whole $\mathbb{R}^n$. However, very little was known on *boundary regularity* for problems of the type (0.2). A main topic of this thesis is the study of this boundary regularity, which is qualitatively different from that for second order equations.

Our first result in this direction is for problem (0.2) with $L = -(-\Delta)^s$. In this case we prove that solutions u are C^s up to the boundary and, more importantly, that the quotient $u/d^s \in C^\alpha(\overline{\Omega})$, for some small $\alpha > 0$, where d is the distance to the boundary $\partial\Omega$. Note that the solution to $(-\Delta)^s u = 1$ in B_1, $u \equiv 0$ outside B_1 is given by the explicit expression $u(x) = c(1 - |x|^2)_+^s$, where c is some positive constant. Hence, the regularity $u \in C^s$ cannot be improved. Instead, finer boundary regularity for these fractional order equations means higher order Hölder regularity of u/d^s.

The previous estimates for $(-\Delta)^s$ are crucial to establish the Pohozaev identity for the fractional Laplacian, a main result of this thesis. This new identity applies

to solutions of $(-\Delta)^s u = f(u, x)$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, and reads as follows

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u \; dx = \frac{2s - n}{2} \int_\Omega u(-\Delta)^s u \; dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{d^s}\right)^2 (x \cdot \nu) d\sigma.$$

This identity does not follow from some "vector calculus identity" and the divergence theorem, as it is the case in the classical Pohozaev identity. Instead, its proof is more delicate mainly due to the more intricate boundary behavior of solutions.

Our methods to prove Hölder regularity for u/d^s are based only on the maximum principle, the Harnack inequality, and on suitable barriers (we develop a nonlocal version of the Krylov method for second order elliptic equations with bounded measurable coefficients). This allows us to obtain results also for fully nonlinear integro-differential equations, arising in stochastic differential games. Our results apply to fully nonlinear equations with respect to the class $\mathcal{L}_*$, containing all operators L of the form (0.1) which are *scale invariant*, i.e. $b(y) = a(y/|y|)$. This class is comprised by infinitesimal generators of $2s$-stable Lévy processes. We prove that solutions u to (0.2) where L is replaced by a fully nonlinear operator (such as $\inf_\beta \sup_\alpha L_{\alpha\beta}$ with $L_{\alpha\beta} \in \mathcal{L}_*$), satisfy $u/d^s \in C^\beta(\overline{\Omega})$ for all $\beta \in (0, s)$. In addition, we show counterexamples to this boundary regularity in the larger ellipticity class $\mathcal{L}_0$ of Caffarelli and Silvestre, which contains *all* operators of the form (0.1).

We also obtain new interior regularity results for fully nonlinear nonlocal parabolic equations with *rough kernels* in the class $\mathcal{L}_0$. We prove that solutions to these equations with merely bounded complement data are C^β in space and $C^{\frac{\beta}{2s}}$ in time for all $\beta < \{2s, 1 + \alpha\}$, where $\alpha > 0$. These results are nearly optimal. For it, we develop a new regularity method for nonlocal equations based on a Liouville theorem and a blow up and compactness argument. It allows to avoid a recurrent difficulty in integro-differential equations when trying to iterate nonlocal estimates We believe that this method, which is very flexible and solves the previous difficulty, is a main contribution of the thesis. For instance its use has been essential to prove our boundary regularity results for fully nonlinear equations, described above.

In this first part of the thesis we have also studied semilinear equations with nonlocal diffusion operators. On the one hand, by finding an extension problem for a sum of fractional Laplacians, we are able to prove 1-D symmetry of *phase transitions* in dimension $n = 2$ for equations of the type $\sum_i (-\Delta)^{s_i} u + W'(u) = 0$ in $\mathbb{R}^n$, where W is a double well potential and $s_i \in (0, 1)$. We also obtain symmetry in dimension $n = 3$ provided $\min s_i \geq 1/2$. One the other hand, we study the nonlocal version of the *extremal solution* problem $(-\Delta)^s u = \lambda f(u)$ in Ω. For this problem we obtain some initial results on boundedness of the extremal solution extending well-known and important ones for $s = 1$. In addition, and as an application of our Pohozaev identity, we prove that the extremal solution belongs to H^s.

In the *second part* we give two instances of interaction between isoperimetry and Partial Differential Equations. In the first one we use the Alexandrov-Bakelman-Pucci method for elliptic PDE to obtain new sharp isoperimetric inequalities in

cones with *densities*. We show that given a convex cone $\Sigma \subset \mathbb{R}^n$ and a weight $w \in C(\overline{\Sigma})$ which is homogeneous of degree $\alpha > 0$ and such that $w^{1/\alpha}$ is concave in Σ, the isoperimteric quotient

$$\frac{\left(\int_{\partial\Omega\cap\Sigma} w \, d\sigma\right)^{1/(n+\alpha-1)}}{\left(\int_{\Omega\cap\Sigma} w \, dx\right)^{1/(n+\alpha)}}$$

is minimized by balls centered at the origen. We also obtain an anisotropic version of this result. This is done by generalizing a proof of the classical isoperimetric inequality due to X. Cabré. Our new results contain as particular cases the classical Wulff inequality and the isoperimetric inequality in cones of Lions and Pacella.

In the second instance we use the isoperimetric inequality and the classical Pohozaev identity to establish a radial symmetry result for second order reaction-diffusion equations. The novelty here is to include discontinuous nonlinearities. For this, we extend a two-dimensional argument of P.-L. Lions from 1981 to obtain now results in higher dimensions.

The thesis is made up of the following articles:

A. X. Ros-Oton, J. Serra, *The Pohozaev identity for the fractional Laplacian*, to appear in Arch. Rational Mech. Anal.

B. X. Ros-Oton, J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. 101 (2014), 275-302.

C. X. Ros-Oton, J. Serra, *Nonexistence results for nonlocal equations with critical and supercritical nonlinearities*, preprint ArXiv (2013).

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I. J. Serra, *Radial symmetry for diffusion equations with discontinuous nonlinearities*, J. Differential Equations 254 (2013), 1893-1902.

Contents

Introduction

The contents of this thesis are divided into two parts. The first, and main one, concerns *nonlocal* —or fractional— *diffusion equations*, an extension of standard Brownian diffusion and of the heat equation. The second part includes two instances of interaction between *isoperimetry* and Partial Differential Equations.

In the first part we study fractional semilinear problems, as well as nonlocal fully nonlinear diffusion equations —both in their parabolic and elliptic versions. We obtain results on interior and *boundary regularity* for these equations. The boundary regularity results for equations of fractional order are a main novelty of the thesis. They are qualitatively different from their classical versions for second order elliptic equations. In addition, they play an important role in the *Pohozaev identity* for the fractional Laplacian, which is another central result of the thesis. In this first part we also study a nonlocal phase transition problem, as well as the fractional version the semilinear extremal solution problem.

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In the second part we include two instances of interaction between isoperimetric inequalities and elliptic PDE. In the first one we use the Alexandrov-Bakelman-Pucci method for elliptic PDE to obtain new sharp isoperimetric inequalities in cones with *densities*. This is done by generalizing a proof of the classical isoperimetric inequality due to X. Cabré. Our new result contains as particular cases the classical

Wulff inequality and the isoperimetric inequality in cones of Lions and Pacella. In the second instance the interaction goes on the opposite direction: we use the isoperimetric inequality and the classical Pohozaev identity to establish a radial symmetry result for second order reaction-diffusion equations. The novelty here is to include discontinuous nonlinearities. For this, we extend a two-dimensional argument of P.-L. Lions from 1981 to obtain now results in higher dimensions.

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The introduction is divided into two sections, corresponding to the division explained above.

After the Introduction, we include all articles. Each of them has its own Bibliography.

1. Nonlocal diffusions

1.1. From Brownian to Lévy models. In mathematics, and more particularly in PDE, the (main) equation to model diffusion is the heat equation

$$u_t - \Delta u = 0. \tag{1.1}$$

It is a well-known fact that the probability distribution function (which depends on time) for a Brownian motion satisfies (1.1). By this reason, a wide variety of physical phenomena are modeled by the heat equation. A short list of typical examples includes: diffusion of a pollutant in the air, bacterial diffusion, disease propagation, error propagation in numerical analysis, or stock market prices.

Let us develop in more detail the example of the stock prices. This example turns out to be very appropriate to motivate the results of this thesis, and it will appear again in this introduction. Let

$$\boldsymbol{X}(t) = \big(X_1(t), \ldots, X_n(t)\big)$$

be a vector with the share prices of n different corporations at time t. The Brownian model for share prices, also known as Merton-Samuelson model, describes the fluctuation of the logarithmic prices as a n-dimensional Brownian motion. Namely, letting

$$Y_i = \log(X_i),$$

the model states (in the language of stochastic differential equations) that

$$dY_i = \mu_i \, dt + \sum_{k=1}^{d} \sigma_{ik} \, dW_k. \tag{1.2}$$

Here, μ_i are the drift coefficients, σ_{ik} are the (joint) volatilities, and W_k are independent Wiener processes, also called Brownian processes. Actually, (1.2) corresponds to the simplest version of the model, in which drifts and volatilities are taken constant in time.

For all modeling purposes, after discretizing time by choosing a small time step $\tau > 0$, the previous stochastic differential equation can be understood as the following recursive relation

$$Y_i(t+\tau) - Y_i(t) = \mu_i \tau + \sum_{k=1}^{d} \sigma_{ik} \sqrt{\tau} \, \xi_k(t), \tag{1.3}$$

where $t \in \tau\mathbb{Z}$ and $\xi_k(t)$ are *noise variables* that have Gaussian distribution $N(0,1)$, with zero mean and unit variance. The d random variables $\xi_k(t)$ are assumed to be independent of each other and also independent of their past values $\{\xi_k(s), \ s < t\}$.

It is of course important the square root $\sqrt{\tau}$, appearing in (1.3) in front of the noise variables $\xi_j(t)$. If $\sqrt{\tau}$ was replaced by a smaller factor (say τ or $\tau^{2/3}$), then in the limit $\tau \searrow 0$ we would obtain a deterministic model; the stochastic part would be *killed* by the too small factor. This is clearly related to the Central Limit Theorem:

the sum of a large number N of independent random variables divided by $\sqrt{N}$ (and not N nor $N^{2/3}$) converges to a Gaussian law $N(0,1)$.

If $u(x,t)$ is the probability density function of $\boldsymbol{Y}(t)$ then, when $\mu_i = 0$, we have

$$u_t - \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u = 0 \quad \text{in } \mathbb{R}^n,$$

for $a_{ij} = \sum_k \frac{1}{2}\sigma_{ik}\sigma_{jk}$. This equation gives the (a priori unclear) link between the fluctuation of market prices and heat conduction. This connection also makes the model (1.2) to enjoy nice mathematical properties.

Although the model (1.2) is quite simple (contains few parameters), it is known to be quite accurate. Other than the drift parameters μ_i, which quantify long-term trends of prices, this model depends only on a matrix of parameters σ_{ij}, the volatilizes. Until the last decades, simplicity was a very important advantage of this (and every) model. Indeed, the values of the parameters need to be calibrated by *fitting* the model to the existing market data —for instance by a maximum likelihood criterion. A small number of parameters is crucial when data is (or was) scarce since, otherwise, the model will easily *overfit*[1].

However, the available amount of market data increases extremely fast, and so does computational power. For instance, it is nowadays easy to download gigabytes of historical market data in few minutes and to fit quite involved models to it using a regular laptop. Thus, it seems no longer necessary in applications to consider so simple models, but rather it may be an oversimplification in some situations. Still, the mathematical simplicity of a model will always be a virtue and allows to understand the crucial issues of a problem.

From the more theoretical point of view, there are some well-known inconsistencies of the model (1.2) like the *implied volatility smile*. This is a famous *paradox* to the Black-Scholes theory for option pricing, and it is usually attributed to the inaccuracy of (1.2) —a main assumption of the Black-Scholes theory. At the same time, a reasonable criticism to the Brownian model is that, according to it, stock market prices should be scale invariant (since Brownian motion is). But they are not Ideed, it is known that the behavior of stock market prices is typically different in middle and large times scales —with the Brownian model being more accurate in larger time scales, due to the Central Limit Theorem.

It is thus not surprising that, since the seventies, many authors have considered more general models in which (1.2) is replaced by the natural assumption that $\boldsymbol{Y}(t)$ is a *Lévy process* —see for instance [**91, 48, 88, 96**] and references therein. As explained in the next section, Lévy processes are Stochastic processes with no memory and stationary increments. These properties make Lévy processes the rational model of noise or random perturbations. Brownian motion is a distinguished particular case in this large class of processes. From the mathematical point of view,

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[1]Like if we use a polynomial of degree 99 to fit a cloud of 100 points which are rather aligned: it fits *exactly* the data but it is completely useless as a model.

Brownian motion is the only Lévy process with continuous sample paths. However, in real world data, time is always discrete —as in (1.3)— and hence continuity of sample paths is only an ideal property which is impossible to see or test. What we do see is whether time increments are stationary and also the absence of memory, and these two properties are shared by *all* Lévy processes.

Although this discussion was focussed in mathematical models of the financial market, it is clear that the same issues (need of more accuracy, known inconsistencies, increasing amount of data, discrete time models) apply to many diffusive models in science where the Brownian diffusion has also been replaced by a Lévy one. For instance, this seems specifically relevant in models of population dynamics in biology and social sciences [**79, 100, 123**].

In the next section we introduce Lévy processes and some of their main properties.

1.2. Lévy processes. A n-dimensional Lévy process is a stochastic process taking values in $\mathbb{R}^n$ and that has stationary and independent increments. It is also assumed to satisfy $\boldsymbol{Y}(0) = 0$ and to be continuous in probability, i.e., given $t > 0$, the probability that $\big|\boldsymbol{Y}(t+h) - \boldsymbol{Y}(t)\big| > \epsilon$ tends to zero as $h \searrow 0$ for all $\epsilon > 0$ and for a.e. t. This does *not* mean that the sample paths of a Lévy process are continuous, and in fact they are not —with the sole exception of Brownian motion. That $\boldsymbol{Y}(t)$ has independent, stationary increments means that the law of $\boldsymbol{Y}(t + h) - \boldsymbol{Y}(t)$ depends only on h (not on t), and that $\boldsymbol{Y}(t + h) - \boldsymbol{Y}(t)$ is independent of the past $\{\boldsymbol{Y}(s), \ s < t\}$.

While (1.2) depends only of a finite number of real parameters (μ_i and σ_{ij}), a Lévy process depends on the choice of a measure μ in $\mathbb{R}^n \setminus \{0\}$ satisfying the condition

$$\int_{\mathbb{R}^n \setminus \{0\}} 1 \wedge |y|^2 \, d\mu(y) < \infty, \tag{1.4}$$

where $\wedge$ denotes the minimum. Measures μ satisfying (1.4) are called *Lévy measures*.

If $Y(t)$ is a Lévy process, we define its *infinitesimal generator* L as the linear translation invariant operator that acts on functions $u \in C_c^\infty(\mathbb{R}^n)$ as follows:

$$Lu(x) = \lim_{t \searrow 0} \frac{\mathbb{E}\, u\big(x + \boldsymbol{Y}(t)\big) - u(x)}{t}. \tag{1.5}$$

As a consequence of the Lévy-Khintchine representation formula [**8**], the infinitesimal generators of Lévy processes are exactly the operators of the form

$$Lu(x) = a_{ij}\partial_{ij}u(x) + b_i\partial_i u(x) + \int_{\mathbb{R}^n \setminus \{0\}} \big\{u(x+y) - u(x) - \nabla u(x) \cdot y \chi_{B_1}(x)\big\}\mu(dy), \tag{1.6}$$

where μ is a Lévy measure. Note that $Lu(x)$ is meaningful whenever u is bounded in $\mathbb{R}^n$ and is C^2 in a neighborhood of x. Under these conditions the integral in (1.6) is well defined and finite, since μ satisfies (1.4) and the integrand inside the brackets is $O(|y|^2)$ at $y = 0$.

General references for Lévy processes are [**2, 8**].

The adjoint operator L^{adj} of the infinitesimal generator L carries all the information on the law of $\boldsymbol{Y}(t)$. Namely, if $p(x,t)$ denotes the probability distribution of a n-dimensional Lévy process $\boldsymbol{Y}(t)$, defined by

$$\int_A p(x,t)\,dx = \mathbb{P}(\boldsymbol{Y}(t) \in A) \quad \text{for all } A,$$

then $p(x,t)$ solves the evolution equation

$$p_t = L^{\mathrm{adj}}p \qquad \text{in } \mathbb{R}^n. \tag{1.7}$$

That the adjoint operator L^{adj} (and not L) must appear in (1.7) also happens for second order operators $a_{ij}(x)\partial_{ij}$ and $\partial_{ij}\big(a_{ij}(x)\,\cdot\,\big)$, and their relation with Markov processes. Namely, if a n-dimensional Markov process $\boldsymbol{Z}(t)$ solves

$$\begin{cases} dZ_i(t) = \sum_k \sigma_{ik}\big(\boldsymbol{Z}(t)\big)dW_k(t), \\ \boldsymbol{Z}(0) = x_0. \end{cases}$$

then the probability *density* function $p(x,t)$ of $\boldsymbol{Z}(t)$ satisfies the Fokker-Planck equation

$$p_t = \partial_{ij}\big(a_{ij}(x)p\big),$$

where $a_{ij}(x) = \frac{1}{2}\sum_k \sigma_{ik}(x)\sigma_{jk}(x)$. Instead, the operator $a_{ij}(x)\partial_{ij}$ appears in the Kolomogorov equation, which is satisfied by *expectation* type quantities like $u(x_0,t) = \mathbb{E}\,\varphi\big(\boldsymbol{Z}_{x_0}(t)\big)$, where $\varphi \in C_c^\infty(\mathbb{R}^n)$. Similarly, expectation type quantities for Lévy processes are often described by equations involving L instead of L^{adj}, as seen in the following example.

EXAMPLE 1.1. Let $\boldsymbol{Y}(t)$ be a Lévy process, Ω be a bounded domain and let $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \Omega)$. Given $x_0 \in \Omega$ let

$$u(x_0) = \mathbb{E}\varphi\big(x_0 + \boldsymbol{Y}(\mathcal{T}_{x_0})\big),$$

where $\mathcal{T}_{x_0}$ is the first time at with the process $x_0 + \boldsymbol{Y}$ leaves Ω, that is

$$\mathcal{T}_{x_0} = \inf\big\{t > 0 : \big(x_0 + \boldsymbol{Y}(t)\big) \notin \Omega\big\}.$$

Note that $\mathcal{T}_{x_0}$ is a random variable (a so called *stopping time*). Note also that

$$u(x_0) = \mathbb{E}\,u\big(x_0 + \boldsymbol{Y}(t \wedge \mathcal{T}_{x_0})\big) \quad \text{for all } t > 0.$$

Thus, (at least formally) we have

$$0 = \lim_{t \searrow 0} \frac{\mathbb{E}\,u\big(x_0 + \boldsymbol{Y}(t \wedge \mathcal{T}_{x_0})\big) - u(x_0)}{t} = Lu(x_0) \quad \text{for all } x_0 \in \Omega.$$

It follows that u is a solution of the nonlocal problem

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = \varphi & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where L, as in (1.5) and (1.6), is the infinitesimal generator of $\boldsymbol{Y}(t)$. Although this argument is formal, it can be made rigorous if the previous equation has good regularity estimates.

1.3. Nonlocal elliptic operators. Key differences with the second order case. The infinitesimal generators of Lévy processes are linear elliptic integro-differential operators. They are typically *nonlocal*, meaning that the value of $L\varphi$ at a point $x_0 \in \mathbb{R}^n$, given by (1.6), depends on the values of φ outside a neighborhood of x_0. When $\mu > 0$ in $\mathbb{R}^n \setminus \{0\}$, $L\varphi(x_0)$ depends on the values of φ at *all* points of $\mathbb{R}^n$. This is a clear contrast with local second order operators, which can be evaluated at one point only knowing the values of the function in an arbitrarily small neighborhood.

Let us explain next in what sense L is *elliptic*. Assume that $\varphi \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ has a global minimum at $x_0 \in \mathbb{R}^n$. Then, either using (1.5) or (1.6) we obtain

$$L\varphi(x_0) \geq 0.$$

This property of the operator being nonnegative at points of minimum is the viscosity notion of ellipticity.

However, let us point out two important differences with second order elliptic operators. Again consider $\mu > 0$ in $\mathbb{R}^n \setminus \{0\}$. We then have:

(a) That $\varphi \geq \varphi(x_0)$ in a neighborhood of x_0 is not enough to ensure $L\varphi(x_0) \geq 0$. Instead, we must require $\varphi \geq \varphi(x_0)$ in all of $\mathbb{R}^n$.

(b) If $\varphi \geq \varphi(x_0)$ in all of $\mathbb{R}^n$ then either $L\varphi(x_0) > 0$ or $\varphi \equiv \varphi(x_0)$ in all of $\mathbb{R}^n$.

Both (a) and (b) follow easily from the definition of L in (1.6) when $\mu > 0$. While (a) is a "disadvantage" with respect to second order local operators, (b) is a very favorable counterpart. Note that (b) has the flavor of a strong maximum principle but, in a dramatic contrast with second order operators, the sole information $L\varphi(x_0) = 0$ at the point of minimum x_0 is enough to conclude that φ is constant in all of $\mathbb{R}^n$!

The two key differences (a) and (b) of nonlocal elliptic operators with respect to local ones appear repeatedly in the regularity theory of elliptic and parabolic integro-differential equations. The global nature of the maximum principle in (a) causes difficulties and forces to control the solutions in the whole $\mathbb{R}^n$ —estimates are nonlocal, Harnack inequality is only available for solutions which are nonnegative in the whole $\mathbb{R}^n$, etc. Instead, property (b) makes some things easier than for local equations. A good example of this is the quick proof of Luis Silvestre [**110**] of the Hölder regularity for nonlocal elliptic equations with "bounded measurable coefficients". Another example is the proof of the Bernstein theorem for nonlocal minimal surfaces of Figalli and Valdinoci [**68**].

In the same way that the model for second order elliptic operators is the Laplacian, denoted Δ, the model nonlocal elliptic operator is the fractional Laplacian

$-(-\Delta)^s$, where $s \in (0, 1)$. It is equivalently defined either by the integral

$$- (-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) \frac{1}{|y|^{n+2s}} \, dy \qquad (1.8)$$

or via Fourier transform as

$$\mathcal{F}\big[(-\Delta)^s\big](\xi) = |\xi|^{2s} \mathcal{F}[u](\xi). \qquad (1.9)$$

As a consequence of (1.9) we have $(-\Delta)^s \circ (-\Delta)^t = (-\Delta)^{s+t}$, which motivates the name *fractional Laplacian*.

The fractional Laplacian is the only nonlocal elliptic operator which is translation, rotation, and scale invariant. In this sense, it is similar to the Δ. However, note the following important difference between the cases $s = 1$ and $s < 1$. While every linear translation invariant elliptic operator of order 2 is the Laplacian after some affine change of coordinates, for $s < 1$ there are many more linear translation invariant operators of order $2s$ than just affine transformations of $(-\Delta)^s$ —for instance all the infinitesimal generators of $2s$-stable Lévy processes of the form (1.21). Thus, zoology of operators in the nonlocal setting is richer than in the second order case.

1.4. Nonlinear analysis for nonlocal operators: mathematical background. The study of linear integro-differential elliptic equations was initiated by the Probability community in the 1950's —not surprisingly given the strong probabilistic motivation of these equations. Many authors, like Blumenthal, Getoor, Kac, or more recently Bogdan, Bass, and Kassmann (to name only a few) have made important contributions using mostly probabilistic techniques. In parallel to this, the potential theory for the fractional Laplacian (Riesz potentials) was studied in detail, starting by Riesz [101] himself, and there are even classical references on the topic like the book of Landkov [85]. For instance, the explicit Poisson kernel in a ball for the fractional Dirichlet problem $(-\Delta)u^s = f$ in B_1, $u = g$ in $\mathbb{R}^n \setminus B_1$ is a classical result [81, 74]. These early results for nonlocal equations concerned mainly linear equations.

It has not been until the last decade, coinciding with the irruption of nonlocal equations in the PDE community, that *nonlinear* integro-differential equations have been studied in depth. Some fractional nonlinear problems (with many relations existing among them) that have been studied in the last years include:

- Reactions on the boundary; layer solutions to nonlocal reaction-diffusion equations; De Giorgi type conjecture; nonlocal fractional perimeters; nonlocal minimal surfaces.
- Fractional obstacle problem; one phase problem.
- Fully nonlinear elliptic and parabolic integro-differential equations; perturbative methods for these equations.
- Nonlinear nonlocal elliptic variational equations; De Giorgi-Nash-Moser type theory.

- Nonlinear diffusions; front propagation; fractional versions of the porous media equation.
- Fractional semilinear problems; existence, symmetry, and qualitative properties, uniqueness of ground states.
- Fractional equations and curvatures from conformal geometry.

In the remarkable paper [**3**] from 1991, Amick and Toland found *all* solutions in $\mathbb{R}$ to the Benjamin-Ono equation $(-\Delta)^{1/2}u = -u + u^2$. As already observed by Benjamin, the previous equation is equivalent to

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2_+ \\ \partial_\nu v = -v + v^2 & \text{on } \partial\mathbb{R}^2_+ = \{x_2 = 0\}, \end{cases} \tag{1.10}$$

This fact is crucially used in the analysis of [**3**]. Later, Toland [**118**] classified the solutions to the Peierls-Nabarro equation $(-\Delta)^{1/2}u = \sin(\pi u)$ in $\mathbb{R}$ by unraveling an intrinsic link of this equation with the Benjamin-Ono equation.

For more general nonlinearities, Cabré and Solà-Morales [**27**] studied *boundary reaction* problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ \partial_\nu v = f(v) & \text{on } \partial\mathbb{R}^{n+1}_+ = \{x_{n+1} = 0\}, \end{cases} \tag{1.11}$$

where f is a bistable nonlinearity. In [**27**] some important classical results for interior reactions $-\Delta u = f(u)$ were proved to hold also for (1.11). Among other results, the authors showed a Modica type estimate in dimension $n = 1$ and proved the analogue of the De Giorgi conjecture in dimension $n = 2$.

As in [**3, 118**], problem (1.11) is equivalent to $(-\Delta)^{1/2}u = f(u)$ in $\mathbb{R}^n$, where $u(x) = v(x, 0)$, $x \in \mathbb{R}^n$. Heuristically, the reason why the Dirichlet-to-Neumann operator $T : u \mapsto \frac{\partial}{\partial\nu}v$, where v is the harmonic extension of u in $\mathbb{R}^{n+1}_+$, coincides with the half Laplacian $(-\Delta)^{1/2}$ is the following. If we apply it twice we obtain

$$T^2 u(x) = \partial_{\nu\nu} v(x, 0) = \partial_{x_{n+1}x_{n+1}} v(x.0) = -\sum_{i=1}^{n} \partial_{x_i x_i} v(x, 0) = (-\Delta)|_{\mathbb{R}^n} u(x).$$

The Hölder regularity for integro-differential elliptic equations "with bounded measurable coefficients", proved by Bass and Levin [**5**] and Silvestre [**110**], opened the door to a regularity theory for fully nonlinear nonlocal elliptic equations, although a precise definition of these equations was not given until some year later in [**34**].

After the works [**5, 110**], the *fractional obstacle problem* was addressed. It arose as a generalization of the thin obstacle problem (also known as Signorini problem), although it is also motivated by a pricing model for american options with Lévy behavior of underlying assets. In the paper [**111**], Silvestre proved almost optimal regularity for the solution of the fractional obstacle problem, and established some important guidelines on how to apply PDE methods to integro-differential equations.

Later, Caffarelli and Silvestre [33] introduced the *extension problem* tool. Similarly to what happens for $s = 1/2$ with the nonlinear problem for (1.11), the extension problem transforms an equation involving $(-\Delta)^s$ in $\mathbb{R}^n$ with $s \in (0,1)$ into a PDE in one more dimension. The new PDE involves the singular elliptic differential operator $\mathrm{div}\big((x_{n+1})^{1-2s}\nabla \cdot\big)$ and a Neumann type boundary condition. A remarkable consequence of this new tool is an Almgren type monotonicty formula for solutions to $(-\Delta)^s u = 0$. This was used by the previous two authors and Salsa [37] to prove regularity of the solution and of the free boundary for the fractional obstacle problem.

Also using the extension problem, a *fractional version of the De Giorgi conjecture* was proved in dimension $n = 2$ for all $s \in (0,1)$ and in dimension $n = 3$ for $s \in [1/2, 1)$ by Cabré and Cinti [20, 21]. Related to this (although not using the extension), Savin and Valdinoci [107, 108] proved a Γ-convergence result (in the spirt of the classical one of Modica and Mortola [92]) for the fractional Allen-Cahn equation. They consider the renormalized energy functional for the rescaled equation $(-\Delta)^s u = \varepsilon^{-2s} f(u)$. For $s \in [1/2, 1]$ they obtain the Γ-convergence to the classical perimeter of the renormalized energy functional, as in [92]. Instead, for $s \in (0, 1/2)$ they obtain Γ-convergence to a *fractional perimeter*, as introduced by Caffarelli, Roquejoffre, and Savin [31].

An important result which uses the extension problem in an essential way is *uniqueness of ground states* for $(-\Delta)^s u = -u + u^p$ in dimension 1, proved by Frank and Lenzmann [70]. Recently, incorporating some ideas of Cabré and Sire [28], Frank, Lenzmann, and Silvestre [71] have proved the uniqueness of ground states in every dimension.

In parallel with the analysis of semilinear equations, the theory of *nonlocal fully nonlinear elliptic equations* has been developed during the last years. In the foundational paper [34], Caffarelli and Silvestre gave a definition of fully nonlinear elliptic integro-differential equation based in the motivation from stochastic differential games. They proved the existence of viscosity solutions and an ABP type estimate which served to establish the Harnack inequality for solutions to the linearized equations. Although for equations of order $\sigma \in (0,2)$ these type of results for equations with bounded measurable coefficients had already been proved in [5, 110], an important novelty in [34] is that their estimates are uniform as $\sigma \to 2$. Hence, remarkably, the results in [34] contain the classical theory for second order equations as a limit case. Using this C^α estimate, they also proved a $C^{1,\alpha}$ interior estimate for translation invariant elliptic fully nonlinear equations. The perturbative theory for nonlocal equations was addressed later by the same authors in [35], who also obtained a nonlocal version of the Evans-Krylov theorem for convex equations in the important paper [36]. The adaptation of these elliptic methods to the parabolic fully nonlinear setting has been done by Chang and Dávila [44, 45].

In [**4**], Barles, Chasseigne, and Imbert studied a different class of fully nonlinear integro-differential equations using methods à la Jensen-Ishi-Lions. The ideas in [**4**] are interesting and useful also for the equations considered in [**34**].

The nonlocal variational theory has also been developed. A main step in this direction was the nonlocal analogue to the *De Giorgi-Nash-Moser theory* obtained by Kassmann in [**84**]. Later, motivated by their previous works on the surface quasi-geostrophic equation [**38**] and on the Navier-Stokes equation [**120**], Caffarelli, Chan, and Vasseur established the regularity theory for nonlocal parabolic equations in divergence form [**30**].

Front propagation and nonlinear diffusions have also been studied in the fractional context. The exponential speed of invasions has been proved by Cabré, Coulon, and Roquejoffre in [**22, 23**]. The *porous media equation with fractional pressure* was studied by Caffarelli and Vázquez in [**39, 40**]. See also [**109, 61, 62, 121**].

1.5. Fully nonlinear elliptic and parabolic integro-differential equations: the Stochastic control motivation. Let us consider the following variation of Example 1.1 in section 1.2 in which now a single player controls the law of increments of the process, with the goal of maximizing the expectation of the *payoff* received at the first visited point outside Ω.

EXAMPLE 1.2. Let $\boldsymbol{Y}(\alpha; t)$ be a family of Lévy processes indexed by a controllable parameter α, and let L_α be the infinitesimal generator of $\boldsymbol{Y}(\alpha; \cdot)$. A strategy of the player assigns to each $x \in \Omega$ some control $\alpha[x]$. The random motion, stating at $x_0 \in \Omega$, associated to some strategy $\alpha[\,\cdot\,]$ is (heuristically)

$$\begin{cases} d\boldsymbol{X}(t) = d\boldsymbol{Y}\big(\alpha[\boldsymbol{X}(t)]; t\big) \\ \boldsymbol{X}(0) = x_0. \end{cases}$$

Given a bounded payoff function $\varphi \in C^2(\mathbb{R}^n \setminus \Omega)$, the player wants maximize the expected payoff at the stopping time $\mathcal{T} = \inf\{t > 0 : X(t) \notin \Omega\}$. Let us call

$$u(x_0, t) = \max_{\alpha[\,\cdot\,]} \mathbb{E}\,\varphi(\mathcal{T}).$$

Then, we formally have

$$u(x_0) = \lim_{t \searrow 0} \max_{\alpha} u\big(x_0 + \boldsymbol{Y}(\alpha; t \wedge \mathcal{T})\big).$$

FUrthermore,

$$0 = \lim_{t \searrow 0} \frac{\max_\alpha u\big(x_0 + \boldsymbol{Y}(\alpha; t \wedge \mathcal{T}))\big) - u(x_0)}{t} = \max_\alpha L_\alpha u(x_0).$$

In Example 1.2 we see that the *value* (expected payoff) of a single player game (control problem) formally satisfies the equation

$$\begin{cases} \max_\alpha L_\alpha u &= 0 & \text{in } \Omega \\ u &= \varphi & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.12}$$

In the situation of a zero sum game between two players the equation for the *value* of the game is

$$\begin{cases} \min_\beta \max_\alpha L_{\alpha\beta}u &=& 0 & \text{in } \Omega \\ u &=& \varphi & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \qquad (1.13)$$

where β stands for the possible controls of the second player, whose objective is to minimize the payoff for the first player.

When L_α or L_{α_β} are restricted to be linear second order elliptic operators, the previous equations are the classical Bellman and Isaacs equations.

Example 1.2 motivates the abstract definition in [**34**] of fully nonlinear elliptic operator I, explained below, which is the one that we follow.

The equivalent notion in nonlocal equations to the uniform ellipticity for second order equations is ellipticity with respect to some class. The *ellipticity class* is a given set of linear translation invariant operators, denoted by $\mathcal{L}$, of the form (1.6). The extremal Pucci type operators for a given class $\mathcal{L}$ are defined as

$$M_\mathcal{L}^+ u(x) = \sup_{L \in \mathcal{L}} Lu(x) \quad \text{and} \quad M_\mathcal{L}^- u(x) = \inf_{L \in \mathcal{L}} Lu(x).$$

Then a fully nonlinear operator I is said to be elliptic with respect to $\mathcal{L}$ if the inequalities

$$M_\mathcal{L}(u - v)(x) \leq Iu(x) - Iv(x) \leq M_\mathcal{L}(u - v)(x)$$

hold for every pair of test functions u, v at x —i.e., C^2 functions in a neighborhood of x and bounded in the whole space. It is not difficult to see that this definition coincides with the usual definition of second order uniformly elliptic fully nonlinear operator when $\mathcal{L} = \{a_{ij}\partial_{ij} \text{ with } 0 < \lambda \text{Id} \leq (a_{ij}) \leq \Lambda \text{Id}\}$.

As shown in [**34**], the "convex" operator $Iu = \max_\alpha L_\alpha u$ in (1.12) is elliptic with respect to $\mathcal{L} = \bigcup_\alpha \{L_\alpha\}$ and $Iu = \min_\beta \max_\alpha L_\alpha u$ in (1.13) is elliptic with respect to $\mathcal{L} = \bigcup_{\alpha,\beta} \{L_{\alpha,\beta}\}$.

We say that I is translation invariant when

$$I\big(u(x_0 + \cdot)\big)(x) = (Iu)(x_0 + x).$$

Other examples of fully nonlinear translation invariant elliptic operators are those of the form

$$Iu(x) = \inf_\alpha \sup_\beta \big(L_{\alpha\beta}u + c_{\alpha\beta}\big).$$

As for second order equations, the incremental quotients v of a solution u to a translation invariant elliptic equation $Iu = 0$ satisfy the two inequalities $M_\mathcal{L}^+ v \geq 0$ and $M_\mathcal{L}^- v \leq 0$. This pair of inequalities is what we sometimes refer as elliptic equation with "bounded measurable coefficients", even though in the integro-differential context there are no coefficients but kernels.

The definition we use of viscosity solutions (and inequalities) for elliptic and parabolic equations are the ones in [**34**] and [**44**]. Up to technical details, a viscosity solution of $Iu = 0$ is, as usual, a continuous function such that every time that a

smooth function φ touches it from above (resp. below) at a point x then $I\varphi(x) \geq 0$ (resp. $\leq$).

1.6. Results: Pohozaev identity for the fractional Laplacian. A main result of this Thesis is the Pohozaev identity for the fractional Laplacian. For second order equations, the Pohozaev identitity applies to solutions of

$$-\Delta u = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

The original identity was obtained by Pohozaev [**98**], who used it to prove the nonexistence of solutions for critical and supercritical nonlinearities f.

There is a collection of identities of Pohozhaev type which have been widely used in the analysis of elliptic PDE. These identities are usually related some to divergence free quantity for the corresponding homogeneous equation. Divergence free quantities are to PDE what conserved quantities are to ODE. They can be found exploiting symmetries of the problem, i.e., using the PDE version of the Noether's theorem; see [**63**]. In the original Pohozaev identity (as well as in ours) the underlying symmetry is the scale invariance of the (fractional) Laplacian. Pohozaev identities are used in a several different contexts: monotonicity formulas, unique continuation properties, concentration-compactness results, energy estimates for ground states in $\mathbb{R}^n$, radial symmetry of solutions, controllability of wave equations, etc.

We have obtained the following fractional version of the Pohozaev identity. It applies to functions that satisfy an equation of the type $(-\Delta)^s u = f(x, u)$, and $u = 0$ in $\mathbb{R}^n \setminus \Omega$. These functions u are only $C^s(\overline{\Omega})$. However, the quotient u/d^s, where d is the distance to the boundary, belongs to $C^\alpha(\overline{\Omega})$, in the sense that the function u/d^s, defined in Ω admits a continuous extension to $\overline{\Omega}$. In the last term of our identity, the quantity $u/d^s|_{\partial\Omega}$ is understood as the limit from inside Ω of the function u/d^s.

THEOREM 1.3 ([**A**]). *Let Ω be a bounded and $C^{1,1}$ domain. Assume that u is a bounded $H^s(\mathbb{R}^n)$ solution of a semilinear equation of the type $(-\Delta)^s u = f(x, u)$, with f Lipschitz, and $u = 0$ in $\mathbb{R}^n \setminus \Omega$. Then $u/d^s \in C^\alpha(\overline{\Omega})$ and the following identity holds*

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u \, dx = \frac{2s - n}{2} \int_\Omega u(-\Delta)^s u \, dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{d^s}\right)^2 (x \cdot \nu) d\sigma,$$

where $d = \text{dist}(\cdot, \partial\Omega)$, ν is the unit outward normal to $\partial\Omega$ at x, and Γ is the Gamma function.

Setting $s = 1$ we retrieve the original identity of Pohozaev.

As a corollary of Theorem 1.3, using that the origin can be arbitrarily chosen, we obtain a new identity for the fractional Laplacian with the flavor of an integration by parts formula.

COROLLARY 1.4 ([**A**]). *Let Ω be a bounded and $C^{1,1}$ domain, and u and v be functions satisfying the hypotheses in Theorem 1.3. Then, the following identity*

holds

$$\int_\Omega (-\Delta)^s u \; v_{x_i} \, dx = -\int_\Omega u_{x_i}(-\Delta)^s v \, dx + \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{d^s} \frac{v}{d^s} \nu_i \, d\sigma$$

for $i = 1, ..., n$, where $d = \mathrm{dist}(\cdot, \partial\Omega)$, ν is the unit outward normal to $\partial\Omega$ at x, and Γ is the Gamma function.

The proof of Theorem 1.3 is completely different when $s < 1$ from that of the classical case $s = 1$. For $s = 1$ it follows from the identity

$$\mathrm{div}\left\{\left(2\nabla u \cdot x + (n-2)u\right)\nabla u - |\nabla u|^2 x\right\} = \left(2\nabla u \cdot x + (n-2)u\right)\Delta u.$$

by integrating it over Ω, using the divergence theorem, and that $u = 0$ on $\partial\Omega$.

Instead, when $s < 1$ the proof of our identity in bounded domains is more delicate mainly due to the non-regular behavior of the solutions near the boundary (recall that u behaves like d^s). Actually, the mere existence of such an identity was unexpected when we announced it. The factor $\Gamma(1+s)^2$ and the nature of the boundary term in the right hand side of our identity suggest that the proof needs to be more involved than for the second order case.

To prove Theorem 1.3 we first assume the domain Ω to be star-shaped with respect to the origin. The result for general domains follows from the star-shaped case, using a trick which involves a bilinear version of our Pohozaev identity and a partition of unity.

For star-shaped domains, a key idea of the proof is the following computation. First, we write the left hand side of the identity as

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u \, dx = \left.\frac{d}{d\lambda}\right|_{\lambda=1^+} \int_\Omega u_\lambda (-\Delta)^s u \, dx,$$

where

$$u_\lambda(x) = u(\lambda x).$$

Note that $u_\lambda \equiv 0$ in $\mathbb{R}^n \backslash \Omega$, since Ω is star-shaped and we take $\lambda > 1$ in the above derivative. As a consequence, we may integrate by parts and make the change of variables $y = \sqrt{\lambda}x$, to obtain

$$\int_\Omega u_\lambda(-\Delta)^s u \, dx = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u_\lambda (-\Delta)^{s/2} u \, dx = \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \, dy,$$

where

$$w(x) = (-\Delta)^{s/2} u(x).$$

Thus,

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u \, dx = \frac{d}{d\lambda}\bigg|_{\lambda=1^+} \left\{ \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \, dy \right\}$$

$$= \frac{2s-n}{2} \int_{\mathbb{R}^n} w^2 dx + \frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_{\sqrt{\lambda}} \qquad (1.14)$$

$$= \frac{2s-n}{2} \int_{\mathbb{R}^n} u(-\Delta)^s u \, dx + \frac{1}{2}\frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda,$$

where

$$I_\lambda = \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} dy.$$

Therefore, the identity of Theorem 1.3 is equivalent to the following equality

$$-\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} \, dy = \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{d^s}\right)^2 (x \cdot \nu)d\sigma. \qquad (1.15)$$

The quantity $\frac{d}{d\lambda}|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}$ vanishes for any $C^1(\mathbb{R}^n)$ function w, as can be seen by differentiating under the integral sign. Instead, we are able to prove that the function $w = (-\Delta)^{s/2}u$ has a singularity along $\partial\Omega$, and that (1.15) holds.

To prove this, it turns to be to very useful to define the following operator $\mathfrak{I}$ (a kind of quadratic from)

$$\mathfrak{I}(\varphi) = -\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} \varphi(\lambda x)\varphi(\lambda^{-1}x) \, dx,$$

and to understand how it acts on certain singular functions φ. The following properties of $\mathfrak{I}$ make it useful:

(1) $\mathfrak{I}(\varphi) \geq 0$ since

$$\int_{\mathbb{R}^n} \varphi(\lambda x)\varphi(\lambda^{-1}x)dx \leq \left(\int_{\mathbb{R}^n} \varphi^2(\lambda x)dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \varphi^2(\lambda^{-1}x)dx\right)^{\frac{1}{2}} = \int_{\mathbb{R}^n} \varphi^2$$

(2) ψ smooth $\Rightarrow$ $\mathfrak{I}(\psi) = 0$
(3) If $\mathfrak{I}(\psi) = 0$ $\Rightarrow$ $\mathfrak{I}(\varphi + \psi) = \mathfrak{I}(\varphi)$

It seems that this operator $\mathfrak{I}$ had not been used before in the literature, although it is a quite natural nonnegative quadratic form. A reason explaining this fact might be that, as said before, $\mathfrak{I}$ vanishes when computed at smooth functions.

Besides a general understanding of the operator $\mathfrak{I}$ it is crucial to our proof to have a good description of the singular behavior of $w = (-\Delta)^{s/2}u$ near the boundary. Namely, we need to show that

$$w(x) = (-\Delta)^{s/2}u(x) = c_1\big\{-\log d(x) + c_2\chi_\Omega(x)\big\}(u/d^s)(x) + h(x) \qquad (1.16)$$

where c_1 and c_2 are constants which depend only on s (and that we compute explicitly) and where $h \in C^\alpha$ is not seen by the operator $\mathfrak{I}$, i.e., it satisfies $\mathfrak{I}(h) = 0$. To prove this "expansion" we need to control how fast high order Hölder norms of

u and u/δ^s blow up near the boundary. This brought us to study the regularity up to the boundary for the fractional Laplacian in [**B**].

With the expansion (1.16) for w and the properties of $\Im$ at hand, we are able to compute $\Im(w)$, showing (1.15) and establishing the identity.

Our scaling argument in the proof of Theorem 1.3 can be used to show nonexistence of bounded solutions to some nonlinear problems involving quite general integro-differential operators. These nonexistence results follow from a general variational inequality in the spirit of Pucci and Serrin [**99**]. Essentially we repeat the scaling augment and, instead of proving an equality like (1.15), we show only an inequality when the domain Ω is star-shaped. Doing this we may consider more general operators like

$$Lu(x) = -a_{ij}\partial_{ij}u + \mathrm{PV}\int_{\mathbb{R}^n} (u(x) - u(x+y))K(y)dy,$$

where K is a symmetric kernel satisfying an appropriate monotonicity property. More precisely, we assume that either $a_{ij} = 0$ and $K(y)|y|^{n+\sigma}$ is nondecreasing along rays from the origin for some $\sigma \in (0,2)$, or that (a_{ij}) is positive definite and $K(y)|y|^{n+2}$ is nondecreasing along rays from the origin. This is the content of the paper [**C**].

1.7. Results: interior regularity for fully nonlinear parabolic equations. In [**34**], Caffarelli and Silvestre introduced the ellipticity class $\mathcal{L}_0 = \mathcal{L}_0(\sigma)$, with order $\sigma \in (0,2)$. The class $\mathcal{L}_0$ contains all linear operators L of the form

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) K(y)\,dy,$$

where the kernels $K(y)$ satisfy the ellipticity bounds

$$0 < \lambda\frac{2-\sigma}{|y|^{n+\sigma}} \leq K(y) \leq \Lambda\frac{2-\sigma}{|y|^{n+\sigma}}.$$

This includes kernels that may be very oscillating and irregular. That is why the words *rough kernels* are sometimes used to refer to $\mathcal{L}_0$. The extremal operators M_σ^+ and M_σ^- for $\mathcal{L}_0$ are

$$M_\sigma^+ u(x) = \sup_{L\in\mathcal{L}_0} Lu(x) \quad\text{and}\quad M_\sigma^- u(x) = \inf_{L\in\mathcal{L}_0} Lu(x).$$

If $u \in L^\infty(\mathbb{R}^n)$ satisfies the two viscosity inequalities $M_\sigma^+ u \geq 0$ and $M_\sigma^- u \leq 0$ in B_1, then u belongs to $C^\alpha(\overline{B_{1/2}})$. More precisely, one has the estimate

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}. \tag{1.17}$$

This estimate, with constants that remain bounded as the $\sigma \nearrow 2$, is one of the main results in [**34**].

For second order equations ($\sigma = 2$) the analogous of (1.17) is the classical estimate of Krylov and Safonov, and differs from (1.17) only from the fact that it has $\|u\|_{L^\infty(B_1)}$ instead of $\|u\|_{L^\infty(\mathbb{R}^n)}$ on the right hand side. This apparently harmless

difference comes from the fact that elliptic equations of order $\sigma < 2$ are nonlocal. By analogy with second order equations, from (1.17) one expects to obtain $C^{1,\alpha}$ interior regularity of solutions to translation invariant elliptic equations $\mathrm{I}u = 0$ in B_1. When $\sigma = 2$, this is done by applying iteratively the estimate (1.17) to incremental quotients of u, improving at each step by α the Hölder exponent in a smaller ball (see [**29**]). However, in the case $\sigma < 2$ the same iteration does not work since, right after the first step, the L^∞ norm of the incremental quotient of u is only bounded in $B_{1/2}$, and not in the whole $\mathbb{R}^n$.

The previous difficulty is strongly related to the fact that the operator will "see" possible distant high frequency oscillations in the exterior Dirichlet datum. In [**34**], this issue is bypassed by restricting the ellipticity class, i.e., introducing a new class $\mathcal{L}_1 \subset \mathcal{L}_0$ of operators with C^1 kernels (away from the origin). The additional regularity of the kernels has the effect of averaging distant high frequency oscillations, balancing out its influence. This is done with an integration by parts argument. Hence, the $C^{1+\alpha}$ estimates in [**34**] are "only" proved for elliptic equations with respect to $\mathcal{L}_1$ (instead of $\mathcal{L}_0$).

Very recently, Kriventsov [**83**] succeeded in proving the same $C^{1+\alpha}$ estimates for elliptic equations of order $\sigma > 1$ with rough kernels, that is, for $\mathcal{L}_0$. The proof in [**83**] is quite involved and combines fine new estimates with a compactness argument. The same methods are used there to obtain other interesting applications, including nearly sharp Schauder type estimates for linear, non translation invariant, nonlocal elliptic equations.

Here, we extend the main result in [**83**] in two ways, providing in addition a new proof of it. First, we pass from elliptic to parabolic equations. Second, we allow also $\sigma \le 1$, proving in this case $C^{\sigma-\epsilon}$ regularity in space and $C^{1-\epsilon}$ in time (for all $\epsilon > 0$) for solutions to nonlocal translation invariant parabolic equations with rough kernels. Our result reads as follows.

THEOREM 1.5 ([**D**]). *Let $\sigma_0 \in (0, 2)$ and $\sigma \in [\sigma_0, 2]$. Let $u \in L^\infty\big(\mathbb{R}^n \times (-1, 0)\big)$ be a viscosity solution of $u_t - \mathrm{I}u = f$ in $B_1 \times (-1, 0]$, where I is a translation invariant elliptic operator with respect to the class $\mathcal{L}_0(\sigma)$ with $\mathrm{I}0 = 0$.*

Then, there is $\alpha > 0$ such that for all $\epsilon > 0$ and letting

$$\beta = \min\{\sigma, 1 + \alpha\} - \epsilon,$$

the following estimate holds

$$\sup_{t \in [-1/2, 0]} \big\| u(\,\cdot\,, t) \big\|_{C^\beta(B_{1/2})} + \sup_{x \in B_{1/2}} \big\| u(x, \,\cdot\,) \big\|_{C^{\beta/\sigma}([-1/2, 0])} \le C C_0,$$

where

$$C_0 = \big\| u \big\|_{L^\infty(\mathbb{R}^n \times (-1, 0))} + \| f \|_{L^\infty(B_1 \times (-1, 0))}.$$

The constants $\alpha > 0$ and C depend only on σ_0, ϵ, ellipticity constants, and dimension.

To prove this result we introduce a new method, different from that in [**83**]. The result is new and provides a nearly optimal estimate which was guessed to be the "right" one by the experts. But more importantly, the method we introduce is very flexible and provides a clean way to overcome a difficulty that is recurrent in nonlocal equations. Thus, as we will see later, the same method is useful in other situations.

Our strategy consists on proving first a Liouville type theorem for global solutions, and deducing later the interior estimates from this Liouville theorem, using a blow up and compactness argument. That a regularity estimate and a Liouville theorem are somehow equivalent is an old principle in PDEs, but here it turns out to be very useful to bypass the difficulty iterating the "nonlocal" estimate (1.17). As said above this method is very flexible and can be useful in different contexts with nonlocal equations. For instance, it can be used to study equations which are nonlocal also in time, and also to analyze boundary regularity for nonlocal equation (as seen in the next section).

To have a local $C^{1+\alpha}$ estimate for solutions that are merely bounded in $\mathbb{R}^n$, it is necessary that the order σ of the equation be greater than one. Indeed, for nonlocal equations of order σ with rough kernels there is no hope to prove a local Hölder estimate of order greater than σ for solutions that are merely bounded in $\mathbb{R}^n$. The reason being that influence of the distant oscillations is too strong. Counterexamples can be constructed even for linear equations. That is why the condition $\sigma > 1$ is necessary for the $C^{1,\alpha}$ estimates of Kriventsov [**83**]. Also, this is why we prove C^β estimates in space only for $\beta < \sigma$.

As explained above, the difficulty of nonlocal equations with rough kernels, with respect to local ones, is that the estimate (1.17) is not immediately useful to prove higher order Hölder regularity for solutions of $\mathrm{I}u = 0$ in B_1. Recall that the classical iteration fails because, after the first step, the L^∞ norm of the incremental quotient of order α is only controlled in $B_{1/2}$, and not in the whole $\mathbb{R}^n$. The idea in our approach is that the iteration does work if one considers a solution in the whole space. If we have a global solution u, then we can apply (1.17) at every scale and deduce that u is C^α in all space. Then, we consider the incremental quotients of order α of u, which we control in the whole $\mathbb{R}^n$, and we prove that u is $C^{2\alpha}$. And so on. When this is done with estimates, taking into account the growth at infinity of the function u and the scaling of the estimates, we obtain a Liouville theorem. Using it, we deduce the higher order interior regularity of solutions in the bounded domain directly, using a blow up and compactness argument. In order to have compactness of sequences of viscosity solutions we only need the C^α estimate (1.17). For the parabolic problem, we actually need to establish a parabolic Liouville type theorem, which is proved by iterating the C^α estimate of Chang and Dávila [**44**] —this is the parabolic version of (1.17).

For translation invariant second order elliptic equations like $F(D^2u) = 0$ in B_1 it would be a unnecessary complication to first prove the Liouville theorem and

then obtain the interior estimate by the blow up and compactness argument in this paper. Indeed, as said above, the iteration already works in the bounded domain B_1. Nevertheless, it is worth noting that equations of the type $F(D^2u, Du, x) = 0$, with continuous dependence on x, become $\tilde{F}(D^2u) = 0$ after blow up at some point. By this reason, one can see that the second order Liouville theorem and the blow up method provide a $C^{1,\alpha}$ bound for solutions to $F(D^2u, Du, x) = 0$ in B_1. However, this approach gives nothing new in the second order case with respect to classical perturbative methods (as in [**29**]).

1.8. Results: boundary regularity for fully nonlinear elliptic integro-differential equations. As explained in Section 1.6, our interest in the boundary regularity for integro-differential equations was initially motivated by our Pohozaev identity for the fractional Laplacian [**A**]. More specifically, in our proof of this identity we crucially need to know a quite precise description of the behavior near the boundary for solutions to

$$\begin{cases} -(-\Delta)^s u = f(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.18}$$

Given that the optimal regularity up to the boundary of solutions u is C^s —for $f \equiv 1$ and $\Omega = B_1$ the problem admits the explicit solution $u(x) = c(1 - |x|^2)_+^s$—, the problem of studying the regularity up to the boundary of u/d^s arises naturally. Here, d is the distance to the boundary. In [**B**] we develop a nonlocal version of the Krylov method for second order equations, and with it we establish

$$u/d^s \in C^\alpha(\overline{\Omega}) \quad \text{for some small } \alpha > 0.$$

The Krylov method for second order equations is used to prove a $C^{2,\alpha}$ estimate on the boundary for fully nonlinear elliptic equations $F(D^2u) = 0$. Since it is conceived for equations with bounded measurable coefficients it uses only barriers, the comparison principle, and the interior Harnack inequality of Krylov and Safonov. Since the Harnack inequality for the fractional Laplacian was known, we need to construct suitable barriers —which are comparable to d^s near the boundary. However, there is a technical issue with the Harnack inequality: it requires (an it is a necessary assumption) that solutions to be nonnegative in the whole $\mathbb{R}^n$. This causes technical complications and forces us to control "errors", since we "would like" to apply the Harnack inequality to functions that are only positive in a ball. As explained in Section 1.3 this a typical issue in nonlocal equations.

The C^α regularity of u/d^s is important in our proof of the Pohozaev identity [**A**]. However, a more precise knowledge of the regularity of u/d^s is needed to complete the proof. Thus, to accomplish this, in [**B**] we derive a (singular) nonlocal equation for the quotient u/d^s in Ω. Using this equation we prove that if $f \in C^\beta$, $\beta > 0$ then

$$u/d^s \in C^{2s+\beta}(\Omega)$$

and the $C^{2s+\beta}$ seminorm of u/d^s in a small ball of radius comparable to d is controlled by $d^{\alpha-2s-\beta}$. These estimates are crucial in the proof of (1.16), which is a main step in the proof of our Pohozaev identity, as explained in Section 1.6.

The closest previous result to our work [**B**] had been obtained by Bogdan, who established the boundary Harnack principle for s-harmonic functions [**9**] —i.e., for solutions to $(-\Delta)^s u = 0$ near some piece of boundary. After our results in [**B**], Grubb [**77**] has showed that, when $f \in C^\beta$ (resp. $f \in L^\infty$), and Ω is smooth, then the solution u to (1.18) satisfies $u/d^s \in C^{\beta+s-\epsilon}(\overline{\Omega})$ (resp. $u/d^s \in C^{s-\epsilon}(\overline{\Omega})$) for all $\epsilon > 0$. This is a remarkable result and represents a great improvement with respect to our result. In addition, the results in [**77**], which use fine (linear) Hörmander theory are not based in the maximum principle and hold also for higher order fractional Laplacians $(-\Delta)^s$ for $s > 1$. These operators do not satisfy the viscosity notion of ellipticity and hence our methods do not work for them. As an important counterpart, our methods based in comparison principle and barriers work also for fully nonlinear equations, as explained below.

In the paper [**E**] we have been able to extend the result of Hölder continuity of u/d^s to solutions of fully nonlinear elliptic integro-differential equations. More importantly, we can lift the Hölder exponent from some small positive α to any number $\beta < s$. Next we explain in more detail these results for fully nonlinear equations, which are a main contribution of this tesis.

Let us recall that, since the foundational paper of Caffarelli and Silvestre [**34**] ellipticity for a nonlinear integro-differential operator is defined relatively to a given set $\mathcal{L}$ of linear translation invariant elliptic operators. This set $\mathcal{L}$ is called the ellipticity class.

The reference ellipticity class from [**34**] is the class $\mathcal{L}_0 = \mathcal{L}_0(s)$, containing all operators L of the form

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) K(y) \, dy \qquad (1.19)$$

with even kernels $K(y)$ bounded between two positive multiples of $(1-s)|y|^{-n-2s}$, which is the kernel of the fractional Laplacian $(-\Delta)^s$.

In the three papers [**34, 35, 36**], Caffarelli and Silvestre studied the interior regularity of solutions u to

$$\begin{cases} Iu = f & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \qquad (1.20)$$

being I a translation invariant fully nonlinear integro-differential operator of order $2s$ (see the definition later on in this Introduction). They proved existence of viscosity solutions, established $C^{1+\alpha}$ interior regularity of solutions [**34**], $C^{2s+\alpha}$ regularity in case of convex equations [**36**], and developed a perturbative theory for non translation invariant equations [**35**]. Thus, the interior regularity for these equations is well understood.

In contrast with the good understanding of interior regularity, there were no previous results on regularity for u/d^s that applied to fully nonlinear nonlocal equations of order $2s$. All that was known is that solutions u to a fully nonlinear equation elliptic with respect to $\mathcal{L}_0$ are C^α up to the boundary (a result for u but not for u/d^s).

Here, we obtain fine boundary regularity for fully nonlinear integro-differential problems of the form (1.20) which are elliptic with respect to a class $\mathcal{L}_* \subset \mathcal{L}_0$ defined as follows. $\mathcal{L}_*$ consists of all linear operators of the form

$$Lu(x) = (1-s) \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) \frac{a(y/|y|)}{|y|^{n+2s}} \, dy, \qquad (1.21)$$

with

$$a \in L^\infty(S^{n-1}) \quad \text{satisfying} \quad \lambda \le a \le \Lambda, \qquad (1.22)$$

where $0 < \lambda \le \Lambda$ are called ellipticity constants. The class $\mathcal{L}_*$ consists of all infinitesimal generators of *stable* Lévy processes belonging to $\mathcal{L}_0$. Our main result establishes that when $f \in L^\infty$, $g \equiv 0$, and Ω is $C^{1,1}$, viscosity solutions u of (1.20) satisfy

$$u/d^s \in C^{s-\epsilon}(\overline{\Omega}) \text{ for all } \epsilon > 0. \qquad (1.23)$$

To state our result "near a piece of boundary" of a $C^{1,1}$ domain it is useful the following:

DEFINITION 1.6. We say that Γ is $C^{1,1}$ surface with radius $\rho_0 > 0$ splitting B_1 into Ω^+ and Ω^- if the following happens.

- The two disjoint domains Ω^+ and Ω^- partition B_1, i.e., $\overline{B_1} = \overline{\Omega^+} \cup \overline{\Omega^-}$.
- The boundary $\Gamma := \partial\Omega^+ \setminus \partial B_1 = \partial\Omega^- \setminus \partial B_1$ is $C^{1,1}$ surface with $0 \in \Gamma$.
- All points on $\Gamma \cap \overline{B_{3/4}}$ can be touched by two balls of radii ρ_0, one contained in Ω^+ and the other contained in Ω^-.

THEOREM 1.7 ([**E**]). *Let Γ be a $C^{1,1}$ surface with radius ρ_0 splitting B_1 into Ω^+ and Ω^-; see Definition 1.6. Let $d(x) = \text{dist}\,(x, \Gamma)$.*

Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. Assume that I is a fully nonlinear and translation invariant operator, elliptic with respect to $\mathcal{L}_*(s)$, with $\mathrm{I}0 = 0$. Let $f \in C(\overline{\Omega^+})$, and $u \in L^\infty(\mathbb{R}^n) \cap C(\overline{\Omega^+})$ be a viscosity solution of

$$\begin{cases} \mathrm{I}u = f & \text{in } \Omega^+ \\ u = 0 & \text{in } \Omega^-. \end{cases}$$

Then, u/d^s belongs to $C^{s-\epsilon}(\overline{\Omega^+ \cap B_{1/2}})$ for all $\epsilon > 0$ with the estimate

$$\left\| u/d^s \right\|_{C^{s-\epsilon}(\Omega^+ \cap B_{1/2})} \le C \big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega^+)} \big),$$

where the constant C depends only on ρ_0, s_0, ϵ, ellipticity constants, and dimension.

In [**E**] we also obtain boundary regularity for problem (1.20) with exterior data $g \in C^2$, and also for non translation invariant operators $\mathfrak{I}(u, x)$.

Theorem 1.7 is, to our knowledge, the first boundary regularity result for fully nonlinear integro-differential equations. For solutions u to elliptic equations with respect to $\mathcal{L}_*$, our result gives a quite accurate description of the boundary behavior. Namely, u/d^s is $C^{s-\epsilon}$ for all $\epsilon > 0$, where d is the distance to the boundary.

We believe the Hölder exponent $s - \epsilon$ in (1.23) to be optimal (or almost optimal) for merely bounded right hand sides f. Moreover, we expect the class $\mathcal{L}_*$ to be the largest scale invariant subclass of $\mathcal{L}_0$ for which this result is true.

For general elliptic equations with respect to $\mathcal{L}_0$, no fine boundary regularity results like (1.23) hold. In fact, the class $\mathcal{L}_0$ is too large for all solutions to be comparable to d^s near the boundary. Indeed, in [**E**] we show that there are powers $0 < \beta_1 < s < \beta_2$ for which the functions $(x_n)_+^{\beta_1}$ and $(x_n)_+^{\beta_2}$ satisfy

$$M_{\mathcal{L}_0}^+ (x_n)_+^{\beta_1} = 0 \quad \text{and} \quad M_{\mathcal{L}_0}^- (x_n)_+^{\beta_2} = 0 \quad \text{in } \{x_n > 0\},$$

where $M_{\mathcal{L}_0}^+$ and $M_{\mathcal{L}_0}^-$ are the extremal operators for the class $\mathcal{L}_0$, Hence, since $(-\Delta)^s (x_n)_+^s = 0$ in $\{x_n > 0\}$, we have at least three functions that solve fully nonlinear elliptic equations with respect to $\mathcal{L}_0$, but which are not even comparable near the boundary $\{x_n = 0\}$. As we show in Section 2, the same happens for the subclasses $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}_0$ which have more regular kernels and were considered in [**34, 35, 36**].

It is important to notice that our result is not only an a priori estimate for classical solutions but also applies to viscosity solutions. For local equations of second order $F(D^2u) = 0$, the boundary regularity for viscosity solutions to fully nonlinear equations has been recently obtained by Silvestre-Sirakov [**112**].

Besides its own interest, the boundary regularity of solutions to integro-differential equations plays an important role in different contexts. For example, it is needed in overdetermined problems arising in shape optimization [**52, 64**] and also in Pohozaev-type or integration by parts identities [**A**]. Moreover, boundary regularity issues appear naturally in free boundary problems [**32, 111**].

Theorem 1.7 follows by combining an estimate *on* the boundary, (1.24) below, with the known interior regularity estimates in [**34, 83**]. The estimate on the boundary reads as follows. If u satisfies the hypotheses of Theorem 1.7, then for all $z \in \Gamma \cap \overline{B_{1/2}}$ there exists $Q(z) \in \mathbb{R}$ for which

$$\left| u(x) - Q(z)\big((x - z) \cdot \nu(z)\big)_+^s \right| \leq C|x - z|^{2s-\epsilon} \quad \text{for all } x \in B_1. \qquad (1.24)$$

Here, $\nu(z)$ is the unit normal vector to Γ at z pointing towards Ω^+.

From this point on, our proof differs substantially from that in second order equations. A main reason for this is not only the nonlocal character of the estimates, but also that tangential and normal derivatives of the solution behave differently on the boundary; recall that the solution is C^s but cannot be Lipschitz up to the boundary.

The estimate on the boundary (1.24) relies heavily on two ingredients, as explained next.

The first ingredient is the following Liouville-type theorem for solutions in a half space.

THEOREM 1.8 ([**E**]). *Let $u \in C(\mathbb{R}^n)$ be a viscosity solution of*

$$\begin{cases} \mathrm{I}u = 0 & in \ \{x_n > 0\} \\ u = 0 & in \ \{x_n < 0\}, \end{cases}$$

where I *is a fully nonlinear and translation invariant operator, elliptic with respect to $\mathcal{L}_*$ and with* I$0 = 0$. *Assume that for some positive $\beta < 2s$, u satisfies the growth control at infinity*

$$\|u\|_{L^\infty(B_R)} \leq CR^\beta \quad for \ all \ R \geq 1. \tag{1.25}$$

Then,

$$u(x) = K(x_n)_+^s$$

for some constant $K \in \mathbb{R}$.

The second ingredient towards (1.24) is a compactness argument, a boundary version of our interior regularity method for parabolic equations with rough kernels (see Section 1.7). With u as in Theorem 1.7, we suppose by contradiction that (1.24) does not hold, and we blow up the fully nonlinear equation at a boundary point (after subtracting appropriate terms to the solution). We then show that the solution converges to an entire solution in $\{x \cdot \nu > 0\}$ for some unit vector ν. Finally, the contradiction is reached by applying the Liouville-type theorem stated above to the entire solution in $\{x \cdot \nu > 0\}$.

These are the main ideas used to prove (1.24). A byproduct of using this blow-up method is that the same proof yields results for non translation invariant equations. Finally, Theorem 1.7 follows by combining (1.24) with the interior regularity estimates in [**34, 83**].

1.9. Results: regularity of the fractional extremal solution. In the paper [**F**] of this thesis we study the extremal solution problem for the fractional Laplacian

$$\begin{cases} (-\Delta)^s u = \lambda f(u) & in \ \Omega \\ u = 0 & in \ \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.26}$$

where λ is a positive parameter and $f : [0, \infty) \longrightarrow \mathbb{R}$ satisfies

$$f \text{ is } C^1 \text{ and nondecreasing, } f(0) > 0, \text{ and } \lim_{t \to +\infty} \frac{f(t)}{t} = +\infty. \tag{1.27}$$

It is well known —see [**11**] or the excellent monograph [**60**] and references therein— that in the classical case $s = 1$ there exists a finite extremal parameter λ^* such that if $0 < \lambda < \lambda^*$ then problem (1.26) admits a minimal classical solution u_λ, while for $\lambda > \lambda^*$ it has no solution, even in the weak sense. Moreover, the family of functions $\{u_\lambda : 0 < \lambda < \lambda^*\}$ is increasing in λ, and its pointwise limit $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a weak solution of problem (1.26) with $\lambda = \lambda^*$. It is called the extremal solution of (1.26).

When $f(u) = e^u$, we have that $u^* \in L^\infty(\Omega)$ if $n \leq 9$ [50], while $u^*(x) = \log \frac{1}{|x|^2}$ if $n \geq 10$ and $\Omega = B_1$ [80]. An analogous result holds for other nonlinearities such as powers $f(u) = (1 + u)^p$ and also for functions f satisfying a limit condition at infinity; see [105]. In the nineties H. Brezis and J.L. Vázquez [11] raised the question of determining the regularity of u^*, depending on the dimension n, for general nonlinearities f satisfying (1.27). The first result in this direction was proved by G. Nedev [95], who obtained that the extremal solution is bounded in dimensions $n \leq 3$ whenever f is convex. Some years later, X. Cabré and A. Capella [19] studied the radial case. They showed that when $\Omega = B_1$ the extremal solution is bounded for all nonlinearities f whenever $n \leq 9$. For general nonlinearities, the best known result at the moment is due to X. Cabré [18], and states that in dimensions $n \leq 4$ then the extremal solution is bounded for any convex domain Ω —later, S. Villegas removed the convexity assumption on Ω.

In [F] we define an appropriate notion of weak solution for problem (1.26) and we prove the existence of a minimal branch of solutions, $\{u_\lambda, \ 0 < \lambda < \lambda^*\}$, with the same properties as in the case $s = 1$. These solutions are proved to be positive, bounded, increasing in λ, and semistable. Recall that a weak solution u of (1.26) is said to be *semistable* if

$$\int_\Omega \lambda f'(u)\eta^2 dx \leq \|\eta\|_{\mathring{H}^s}^2 \tag{1.28}$$

for all $\eta \in H^s(\mathbb{R}^n)$ with $\eta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. When u is an energy solution this is equivalent to saying that the second variation of energy $\mathcal{E}$ at u is nonnegative.

The weak solution u^* for $\lambda = \lambda*$ is called the *extremal solution* of problem (1.26). As explained above, the main question about the extremal solution u^* is to decide whether it is bounded or not. Once the extremal solution is bounded then it is a classical solution, in the sense that it satisfies equation (1.26) pointwise. For example, if $f \in C^\infty$ then u^* bounded yields $u^* \in C^\infty(\Omega) \cap C^s(\overline{\Omega})$.

Our main result, stated next, concerns the regularity of the extremal solution for problem (1.26). To our knowledge this is the first result concerning extremal solutions for (1.26). In particular, the following are new results even for the unit ball $\Omega = B_1$ and for the exponential nonlinearity $f(u) = e^u$.

THEOREM 1.9 ([F]). *Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $s \in (0, 1)$, f be a function satisfying* (1.27)*, and u^* be the extremal solution of* (1.26)*.*

(i) *Assume that f is convex. Then, u^* is bounded whenever $n < 4s$.*
(ii) *Assume that f is C^2 and that the following limit exists:*

$$\tau := \lim_{t \to +\infty} \frac{f(t)f''(t)}{f'(t)^2}. \tag{1.29}$$

   Then, u^* is bounded whenever $n < 10s$.
(iii) *Assume that Ω is convex. Then, u^* belongs to $H^s(\mathbb{R}^n)$ for all $n \geq 1$ and all $s \in (0, 1)$.*

Note that the exponential and power nonlinearities e^u and $(1+u)^p$, with $p > 1$, satisfy the hypothesis in part (ii) whenever $n < 10s$. Also in (ii), the limiting assumption as $s \nearrow 1$ in $n < 10$, which is optimal since the extremal solution may be singular for $s = 1$ and $n = 10$ (as explained before in this introduction).

Note that the results in parts (i) and (ii) of Theorem 1.9 do not provide any estimate when s is small (more precisely, when $s \leq 1/4$ and $s \leq 1/10$, respectively). The boundedness of the extremal solution for small s seems to require different methods from the ones that we present here.

Related problems to (1.26) where studied in [**42, 55**]. The regularity of the extremal solution was for the *spectral* fractional Laplacian A^s in the unit ball B_1 was studied by Capella-Dávila-Dupaigne-Sire in [**42**], who proved boundedness of all extremal solutions in dimensions $n \leq 6$ for all $s \in (0, 1)$. Recall that the spectral fractional Laplacian A^s is defined via the Dirichlet eigenfunctions of the Laplacian or trough the extension problem to a cylinder. Also in this direction, Dávila-Dupaigne-Montenegro [**55**] studied the extremal solution for a boundary reaction problem with mixed Dirichlet and Neumann conditions.

1.10. Results: extension problem for sums of fractional Laplacians and 1-D symmetry of phase transitions.
In the paper [**G**] we study *layer solutions* of phase transition problems with a nonlocal diffusion. The main novelty is that the diffusion operator we consider does not have self-similarity properties, in particular the extension problem of Caffarelli and Silvestre does not apply.

We consider nonlocal Allen-Cahn type equations

$$\sum_{i=1}^{K} \mu_i (-\Delta)^{s_i} u + W'(u) = 0 \quad \text{in } \mathbb{R}^n,\tag{1.30}$$

where $\mu_i > 0$ with $\sum \mu_i = 1$, $0 < s_1 < \cdots < s_K < 1$, and W is an double-well potential with wells of the same height located at ± 1. By definition, a *layer solution* is monotone in the direction x_n with limits ± 1 as $x_n \to \pm\infty$. That is,

$$u_{x_n} \geq 0 \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1},\tag{1.31}$$

Having always (1.30) in mind, we actually allow the more general equation

$$Lu + W'(u) = 0 \quad \text{in } \mathbb{R}^n,\tag{1.32}$$

where for some $s_* \in (0, 1)$ we have

$$Lu = \int_{[s_*, 1)} (-\Delta)^s u \, d\mu(s), \quad \mu \geq 0, \quad \mu\big([s_*, 1)\big) = 1.\tag{1.33}$$

We assume that μ is a probability measure supported in $[s_*, 1)$, i.e.,

$$\mu \geq 0 \quad \text{and} \quad \mu\big([s_*, 1)\big) = \mu(\mathbb{R}) = 1.$$

The operator L is the infinitesimal generator of a Lévy process $\mathbf{Y}(t)$ which is isotropic but not stable. It has different behaviors at large and small time scales. The

interpretation of the probability μ is the following: for a very small time step h, the probability that the increment $\boldsymbol{Y}(t + h) - \boldsymbol{Y}(t)$ coincides with that of a $2s$-stable Lévy process is given by $\mu(ds)$.

We assume that

$$s_* = \max\{s \, : \, \operatorname{support} \mu \subset [s, 1)\} \, . \tag{1.34}$$

and that W satisfies

$$W \in C^3(\mathbb{R}) \, , \quad W(\pm 1) = 0 \quad \text{and} \quad W(t) > 0 \text{ for } t \neq \pm 1 \, . \tag{1.35}$$

In the paper [**G**] we establish an extension problem for the operator L. As a main application we obtain the following 1-D symmetry result for layer solutions to (1.32).

THEOREM 1.10. *Assume that $u \in L^\infty(\mathbb{R}^n)$ is a layer solution of* (1.32), *that is, satisfying* (1.31). *Assume that either $n = 2$ and $s_* > 0$, or that $n = 3$ and $s_* \geq 1/2$, where s_* is given by* (1.34).

Then, u has 1-D symmetry. That is, $u(x) = u_0(a \cdot x)$ where $u_0 : \mathbb{R} \to \mathbb{R}$ is a layer solution in dimension one of $Lu_0 + W'(u_0) = 0$ in $\mathbb{R}$ and $a \in \mathbb{R}^n$ is some unit vector.

The existence of a 1-D solution relies on interior estimates for the operator L in bounded domains. These estimates are not very simple by the following two reasons. First, since the support of μ may arrive all the way to $s = 1$ we can not take advantage of the operator begin nonlocal to show Hölder continuity of solutions in a bounded domain. Second, since the measure μ can be continuous (not discrete) then the operator L may not have a well definite leading order. Therefore, the Hölder regularity in bounded domains for L requires some analysis based on the smoothness and growth of the Fourier multiplier. It will be established in a future work. Here, we use a factorization of the operator trick to deduce estimates in the whole $\mathbb{R}^n$. When L is of the form (1.30) the interior estimates in a bounded domain are very elementary and the existence of a 1-D solution follows from a similar argument as in Palatuci, Savin, and Valdinoci [**97**].

Theorem 1.10 is clearly inspired in a conjecture of De Giorgi for the Allen-Cahn equation: $-\Delta u = u - u^3$ in all $\mathbb{R}^n$. This conjecture states that, if $n \leq 8$, then solutions u which are monotone in one variable have 1-D symmetry. This has been proved in dimensions $n = 2$ by Ghoussoub and Gui [**78**], $n = 3$ by Ambrosio and Cabré [**1**], and $4 \leq n \leq 8$ by Savin [**106**] —under the additional assumption that u is a layer solution, or more generally a minimizer of the energy.

For the related nonlocal equation, $(-\Delta)^s u + W'(u) = 0$ in all $\mathbb{R}^n$, analog results have been found for $n = 2$ and $s = 1/2$ by Cabré and Solà-Morales [**27**], for $n = 2$ and $s \in (0, 1)$, by Cabré and Sire [**28**] and for $n = 3$ and $s \in [1/2, 1)$ by Cabré and Cinti [**20, 21**].

In this paper, we show how several arguments in [**78, 1, 27, 20, 21, 28**] can be adapted to equation (1.32) to obtain 1-D symmetry results. In these papers, symmetry is deduced from a Liouville type theorem. Provided that u satisfies certain

energy estimates, this Liouville type theorem implies that any two directional derivatives of u coincide up to a multiplicative constant. This is equivalent to the 1-D symmetry. All the known symmetry results for fractional equations [**20, 21, 28, 27**] were proven using the extension problem of Caffarelli and Silvestre [**34**], which seems necessary to prove and even to state the Liouville theorem. The main novelty of [**G**] is that we have an non scale invariant operator and the existence of an extension problem is a priori unclear. Here, we show what is the natural extension problem, and how one can prove the symmetry result using it. This new extension problem, consists of a "system" of (possibly infinitely many) singular elliptic PDEs which are coupled by a single Neumann type boundary condition and a common trace constrain. Although this extension is a somehow exotic mathematical object, it turns out to be useful, for instance to prove Theorem 1.10.

This idea could be useful in other contexts where an extension operator is known for a family of operators and one wants to consider also sums (or integrals) of the operators.

Energy estimates for the new extension problem are also an important point in our proof. Here, some ideas from the theory of nonlocal minimal surfaces [**31**] play an important role.

2. Isoperimetric problems

2.1. Isoperimetric inequalities. The classical isoperimetric inequality states that, among all sets of finite perimeter with given volume, balls are the (only) ones with minimal perimeter. Up to the important issue of the uniqueness of minimizer, this can be written as

$$\frac{P(E)}{|E|^{\frac{n-1}{n}}} \geq \frac{P(B_1)}{|B_1|^{\frac{n-1}{n}}}, \tag{2.1}$$

for every measurable set E with finite volume, $|E| < \infty$ —when E is not of finite perimeter the left hand side of the inequality is infinite.

This fact is heuristically known since ancient times, but the history of rigorous proofs is more recent. Isoperimetric inequalities interact with many areas of mathematics, mainly analysis and geometry.

The "first variation of perimeter" approach to the isoperimetric problem has brought many interesting results. Assuming that a piece of the boundary ∂E is a smooth graph $x_n = u(x')$, one can compute the variation of perimeter with respect to perturbations that do not change the volume under the graph. Doing this, one obtains that ∂E must have constant mean curvature:

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u^2|}}\right) = \text{constant.}$$

The theory of constant mean curvature surfaces is very rich from both the geometry and from the PDE sides. For instance, for 2-dimensional surfaces there is representation formula for constant mean curvature surfaces in terms of holomorphic functions in the spirit of the Weierstrass-Enneper formula for minimal surfaces. This is based on the fact that, on constant mean curvature surfaces, the Gauss map in an *harmonic map.* In addition, Alexandrov's proof that spheres are the only compactly embedded surfaces in $\mathbb{R}^3$ with constant mean curvature, inspired the important moving planes technique for PDE.

A short list of uses of isoperimetric inequalities in the analysis of PDE and calculus of variations includes:

- Faber-Krahn inequality (Rayleigh's statement on the fundamental tone of a drum);
- Sobolev inequalities with the best constant (using the rearrangement result of Talenti [**114**]);
- Density estimates (clean ball condition lemma) in the classical theory of minimal surfaces;
- Variational problems involving BV type norms (minimal surfaces, image processing models, etc.).
- Radial symmetry of minimizers for variational PDE (mainly via rearrangements)

Related to this last point, in [**I**] we give another example of sharp insoperimetric inequalities leading to radial symmetry results in PDEs (for all solutions and not only for minimizers).

2.2. The isoperimetric problem in cones and anisotropic perimeters. The classical isoperimetric problem in convex cones was solved by P.-L. Lions and F. Pacella [**87**] in 1990. Their result states that among all sets E with fixed volume inside an open convex cone Σ, the balls centered at the vertex of the cone minimize the perimeter relative to the cone (the part of the boundary of E that lies on the boundary of the cone is not counted).

Throughout the section Σ is an open convex cone in $\mathbb{R}^n$. When E is smooth enough, the relative perimeter is defined as

$$P(E; \Sigma) = \int_{\partial E \cap \Sigma} dS.$$

The isoperimetric inequality in cones of Lions and Pacella reads as follows.

THEOREM 2.1 ([**87**]). *Let Σ be an open convex cone in $\mathbb{R}^n$ with vertex at 0, and $B_1 := B_1(0)$. Then,*

$$\frac{P(E; \Sigma)}{|E \cap \Sigma|^{\frac{n-1}{n}}} \geq \frac{P(B_1; \Sigma)}{|B_1 \cap \Sigma|^{\frac{n-1}{n}}} \tag{2.2}$$

for every measurable set $E \subset \mathbb{R}^n$ with $|E \cap \Sigma| < \infty$.

The proof of Theorem 2.1 given in [**87**] is based on the Brunn-Minkowski inequality.

Theorem 2.1 can be deduced from a degenerate case of the classical Wulff inequality stated in Theorem 2.2 below. This is because the convex set $B_1 \cap \Sigma$ is the Wulff shape (2.4) associated to some appropriate anisotropic perimeter. This idea, which is crucial in our proofs in [**H**], has also been used by Figalli and Indrei [**65**] to prove a quantitative isoperimetric inequality in convex cones. From it, one deduces that balls centered at the origin are the unique minimizers in (2.2) up to translations that leave invariant the cone (if they exist). This had been established in [**87**] in the particular case when $\partial \Sigma \setminus \{0\}$ is smooth (and later in [**102**], which also classified stable hypersurfaces in smooth cones).

Next we recall the notion of anisotropic perimeter. We say that a function H defined in $\mathbb{R}^n$ is a *gauge* when

H is nonnegative, positively homogeneous of degree one, and convex.   (2.3)

Any norm is a gauge, but a gauge may vanish on some unit vectors. We need to allow this case since it will occur in our new proof of Theorem 2.1 —which builds from the cone Σ a gauge that is not a norm.

The anisotropic perimeter associated to the gauge H is defined (when ∂E is smooth) by

$$P_H(E) = \int_{\partial E} H\big(\nu(x)\big) dS,$$

where $\nu(x)$ is the unit outward normal at $x \in \partial E$.

The Wulff shape associated to H is defined as

$$W = \{x \in \mathbb{R}^n \ : \ x \cdot \nu < H(\nu) \ \text{ for all } \nu \in S^{n-1}\}. \tag{2.4}$$

We will always assume that $W \neq \varnothing$. Note that W is an open set with $0 \in \overline{W}$. To visualize W, it is useful to note that it is the intersection of the half-spaces $\{x \cdot \nu < H(\nu)\}$ among all $\nu \in S^{n-1}$. In particular, W is a convex set.

The following is the celebrated Wulff inequality.

THEOREM 2.2 ([**124, 115, 116**]). *Let H be a gauge in $\mathbb{R}^n$ which is positive on S^{n-1}, and let W be its associated Wulff shape. Then, for every measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$, we have*

$$\frac{P_H(E)}{|E|^{\frac{n-1}{n}}} \geq \frac{P_H(W)}{|W|^{\frac{n-1}{n}}}. \tag{2.5}$$

Moreover, equality holds if and only if $E = aW + b$ for some $a > 0$ and $b \in \mathbb{R}^n$ except for a set of measure zero.

This result was first stated without proof by Wulff [**124**] in 1901. His work was followed by Dinghas [**57**], who studied the problem within the class of convex polyhedra. He used the Brunn-Minkowski inequality. Some years later, Taylor [**115, 116**] finally proved Theorem 2.2 among sets of finite perimeter; see [**117, 69, 90**] for more information on this topic.

2.3. Isoperimetric inequalities with densities. The isoperimetric problem with a weight —also called density— is the following. Given a weight w (that is, a positive function w), one wants to characterize minimizers of the weighted perimeter $\int_{\partial E} w$ among those sets E having weighted volume $\int_E w$ equal to a given constant. A set solving the problem, if it exists, is called an isoperimetric set or simply a minimizer. This question, and the associated isoperimetric inequalities with weights, have attracted much attention recently; see for example [**94, 89, 47, 66, 93**].

The solution to the isoperimetric problem in $\mathbb{R}^n$ with a weight w is known only for very few weights, even in the case $n = 2$. For example, in $\mathbb{R}^n$ with the Gaussian weight $w(x) = e^{-|x|^2}$ all the minimizers are half-spaces [**10, 46**], and with $w(x) = e^{|x|^2}$ all the minimizers are balls centered at the origin [**104**]. Instead, mixed Euclidean-Gaussian densities lead to minimizers that have a more intricate structure of revolution [**72**]. The radial homogeneous weight $|x|^\alpha$ has been considered very recently. In the plane ($n = 2$), minimizers for this homogeneous weight depend on the values of α. On the one hand, Carroll-Jacob-Quinn-Walters [**43**] showed that when $\alpha < -2$ all minimizers are $\mathbb{R}^2 \setminus B_r(0)$, $r > 0$, and that when $-2 \leq \alpha < 0$ minimizers do not exist. On the other hand, when $\alpha > 0$ Dahlberg-Dubbs-Newkirk-Tran [**51**] proved that all minimizers are circles passing through the origin (in particular, not centered at the origin). Note that this result shows that even radial homogeneous weights may lead to nonradial minimizers.

Weighted isoperimetric inequalities in cones have also been considered. In these results, the perimeter of E is taken relative to the cone, that is, not counting the part of ∂E that lies on the boundary of the cone. In [**56**] Díaz-Harman-Howe-Thompson consider again the radial homogeneous weight $w(x) = |x|^\alpha$, with $\alpha > 0$, but now in an open convex cone Σ of angle β in the plane $\mathbb{R}^2$. Among other things, they prove that there exists $\beta_0 \in (0, \pi)$ such that for $\beta < \beta_0$ all minimizers are $B_r(0) \cap \Sigma$, $r > 0$, while these circular sets about the origin are not minimizers for $\beta > \beta_0$.

Also related to the weighted isoperimetric problem in cones, the following is a recent result by Brock-Chiaccio-Mercaldo [**14**]. Assume that Σ is any cone in $\mathbb{R}^n$ with vertex at the origin, and consider the isoperimetric problem in Σ with any weight w. Then, for $B_R(0) \cap \Sigma$ to be an isoperimetric set for every $R > 0$ a necessary condition is that w admits the factorization

$$w(x) = A(r)B(\Theta), \tag{2.6}$$

where $r = |x|$ and $\Theta = x/r$. Related to this, a main result of this thesis —Theorem 2.3 below— gives a sufficient condition on $B(\Theta)$ whenever Σ is convex and $A(r) = r^\alpha$, $\alpha \geq 0$, to guarantee that $B_R(0) \cap \Sigma$ are isoperimetric sets.

2.4. Results: sharp isoperimetric inequalities in cones with densities.
The weighted anisotropic perimeter relative to an open cone Σ is defined as follows. We will denote the weight by w. We assume that w is continuous function in $\overline{\Sigma}$, positive and locally Lipschitz in Σ, and homogeneous of degree $\alpha \geq 0$. Given a gauge H in $\mathbb{R}^n$ and a weight w, we define (as in [**6**]) the weighted anisotropic perimeter relative to the cone Σ by

$$P_{w,H}(E; \Sigma) = \int_{\partial E \cap \Sigma} H\big(\nu(x)\big) w(x) dS \tag{2.7}$$

whenever E is smooth enough. We actually consider a more general definition of $P_{w,H}$ that is defined (but possibly infinite) for all measurable sets E. We denote by $w(F)$ the weighted volume of a measurable set F

$$w(F) := \int_F w \, dx.$$

Finally, we denote

$$D = n + \alpha.$$

The following is a main result of the thesis and establishes that the Wulff shapes are the constraint minimizers of anisotropic weighted perimeter in cones for a large class of weights satisfying a concavity condition.

THEOREM 2.3 ([**H**]). *Let H be a gauge in $\mathbb{R}^n$, i.e., a function satisfying (2.3), and W its associated Wulff shape defined by (2.4). Let Σ be an open convex cone in $\mathbb{R}^n$ with vertex at the origin, and such that $W \cap \Sigma \neq \varnothing$. Let w be a continuous function in $\overline{\Sigma}$, positive in Σ, and positively homogeneous of degree $\alpha \geq 0$. Assume in addition that $w^{1/\alpha}$ is concave in Σ in case $\alpha > 0$.*

Then, for each measurable set $E \subset \mathbb{R}^n$ with $w(E \cap \Sigma) < \infty$,

$$\frac{P_{w,H}(E;\Sigma)}{w(E \cap \Sigma)^{\frac{D-1}{D}}} \geq \frac{P_{w,H}(W;\Sigma)}{w(W \cap \Sigma)^{\frac{D-1}{D}}}, \tag{2.8}$$

where $D = n + \alpha$.

Note that the classical inequalities of Lions-Pacella and Wulff follow as the particular cases $w =$ constant of Theorem 2.3. In particular we give new proofs of these inequalities based on the ABP method.

In the isotropic case, making the first variation of weighted perimeter (see [**104**]), one sees that the (generalized) mean curvature of $\partial\Omega$ with the density w is

$$H_w = H_{\mathrm{eucl}} + \frac{1}{n}\frac{\partial_\nu w}{w}, \tag{2.9}$$

where ν is is the unit outward normal to $\partial\Omega$ and H_{eucl} is the Euclidean mean curvature of $\partial\Omega$. It follows that balls centered at the origin intersected with the cone have constant mean curvature whenever the weight is of the form (2.6). However, as we have seen in several examples presented above, it is far from being true that the solution of the isoperimetric problem for all the weights satisfying (2.6) are balls centered at the origin intersected with the cone. Our result provides a large class of nonradial weights for which, remarkably, Euclidean balls centered at the origin (intersected with the cone) solve the isoperimetric problem.

REMARK 2.4. Our key hypothesis that $w^{1/\alpha}$ is a concave function is equivalent to a natural curvature-dimension bound (in fact, to the nonnegativeness of the Bakry-Émery Ricci tensor in dimension $D = n + \alpha$). This was suggested to us by Cédric Villani, and has also been noticed by Cañete and Rosales (see Lemma 3.9 in [**41**]). More precisely, we see the cone $\Sigma \subset \mathbb{R}^n$ as a Riemannian manifold of dimension n equipped with a reference measure $w(x)dx$. We are also given a "dimension" $D = n + \alpha$. Consider the Bakry-Émery Ricci tensor, defined by

$$\mathrm{Ric}_{D,w} = \mathrm{Ric} - \nabla^2 \log w - \frac{1}{D-n}\nabla \log w \otimes \nabla \log w.$$

Now, our assumption $w^{1/\alpha}$ being concave is equivalent to

$$\mathrm{Ric}_{D,w} \geq 0. \tag{2.10}$$

Indeed, since $\mathrm{Ric} \equiv 0$ and $D - n = \alpha$, (2.10) reads as

$$-\nabla^2 \log w^{1/\alpha} - \nabla \log w^{1/\alpha} \otimes \nabla \log w^{1/\alpha} \geq 0,$$

which is the same condition as $w^{1/\alpha}$ being concave. Condition (2.10) is called a curvature-dimension bound; in the terminology of [**122**] we say that $\mathrm{CD}(0,D)$ is satisfied by $\Sigma \subset \mathbb{R}^n$ with the reference measure $w(x)dx$.

In addition, C. Villani pointed out that optimal transport techniques could also lead to weighted isoperimetric inequalities in convex cones.

Due to the homogeneity of w, the exponent $D = n + \alpha$ can be found just by a scaling argument in our inequality (2.8). Note that this exponent D has a dimension flavor if one compares (2.8) with (2.2) or with (2.5). Also, it is the exponent for the volume growth, in the sense that $w(B_r(0) \cap \Sigma) = Cr^D$ for all $r > 0$. The interpretation of D as a dimension is more clear in the following example that motivated our work.

REMARK 2.5. The monomial weights

$$w(x) = x_1^{A_1} \cdots x_n^{A_n} \qquad \text{in} \quad \Sigma = \{x \in \mathbb{R}^n \,:\, x_i > 0 \text{ whenever } A_i > 0\}, \qquad (2.11)$$

where $A_i \geq 0$, $\alpha = A_1 + \cdots + A_n$, and $D = n + A_1 + \cdots + A_n$, are important examples for which (2.8) holds. The isoperimetric inequality —and the corresponding Sobolev inequality— with the above monomial weights were studied by Cabré and Ros-Oton in [25] motivated by the need of these inequalities in the analysis of the extremal solution semilinear problem in domains of double revolution [24].

The proof of Theorem 2.3 consists of applying the ABP method to a linear Neumann problem involving the operator $w^{-1}\text{div}(w\nabla u)$, where w is the weight. When $w \equiv 1$, the idea goes back to 2000 in the works [16, 17] of the first author, where the classical isoperimetric inequality in all of $\mathbb{R}^n$ (here $w \equiv 1$) was proved with a new method. It consisted of solving the problem

$$\begin{cases} \Delta u = b_\Omega & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega \end{cases}$$

for a certain constant b_Ω, to produce a bijective map with the gradient of u, $\nabla u :$ $\Gamma_{u,1} \longrightarrow B_1$, which leads to the isoperimetric inequality. Here $\Gamma_{u,1} \subset \Gamma_u \subset \Omega$ and $\Gamma_{u,1}$ is a certain subset of the lower contact set Γ_u of u. The use of the ABP method is crucial in the proof.

Previously, Trudinger [119] had given a proof of the classical isoperimetric inequality in 1994 using the theory of Monge-Ampère equations and the ABP estimate. His proof consists of applying the ABP estimate to the Monge-Ampère problem

$$\begin{cases} \det D^2 u = \chi_\Omega & \text{in } B_R \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

where χ_Ω is the characteristic function of Ω and $B_R = B_R(0)$, and then letting $R \to \infty$.

Before these two works ([119] and [16]), there was already a proof of the isoperimetric inequality using a certain map (or coupling). This is Gromov's proof, which used the Knothe map; see [122].

After these three proofs, in 2004 Cordero-Erausquin, Nazaret, and Villani [49] used the Brenier map from optimal transportation to give a beautiful proof of the anisotropic isoperimetric inequality; see also [122]. More recently, Figalli-Maggi-Pratelli [67] established a sharp quantitative version of the anisotropic isoperimetric

inequality, using also the Brenier map. In the case of the Lions-Pacella isoperimetric inequality, this has been done by Figalli-Indrei [**65**] very recently. Interestingly, our new proof in [**H**] is also suited for a quantitative version, as we will show in a future work with Cinti and Pratelli.

2.5. Results: radial symmetry for diffusion equations with discontinuous nonlinearities. One of the first results of this thesis was concerned with radial symmetry of solutions to

$$\begin{cases} -\Delta u = f(u) & \text{in } B \subset \mathbb{R}^n\,, \\ u > 0 & \text{in } B\,, \\ u = 0 & \text{on } \partial B\,, \end{cases} \tag{2.12}$$

where B is a ball.

A well-known theorem of Gidas-Ni-Nirenberg [**76**] states that if $f = f_1 + f_2$ with f_1 Lipschitz and f_2 nondecreasing, then a solution $u \in C^2(\overline{B})$ to (2.12) has radial symmetry. Since f_2 might be any nondecreasing function, this result allows f to be discontinuous, but only with increasing jumps. Besides this, the only other general result for f discontinuous is, to our knowledge, the one of P. L. Lions [**86**] in 1981, that establishes radial symmetry of solutions for every locally bounded $f \geq 0$ in dimension $n = 2$.

In [**I**] we establish radial symmetry of solutions to (2.12) in every dimension $n \geq 3$ under the assumption

$$\phi \leq f \leq \frac{2n}{n-2}\,\phi$$

for some nonincreasing function $\phi \geq 0$. In addition, we also obtain results for the p-Laplace equation

$$\begin{cases} -\Delta_p u := -\nabla \cdot (|\nabla u|^{p-2}\nabla u) = f(u) & \text{in } B\,, \\ u \geq 0 & \text{in } B\,, \\ u = 0 & \text{on } \partial B,, \end{cases} \tag{2.13}$$

where $\Omega \subset \mathbb{R}^n$ is a ball. For instance, under the assumption $p \geq n$, we establish radial symmetry of bounded solutions to (2.13) for every $f \geq 0$ locally bounded but possibly discontinuous.

The result we obtain is the following:

THEOREM 2.6 ([**I**]). *Let Ω be a ball in $\mathbb{R}^n$, $n \geq 2$, and let $1 < p < \infty$. Assume that $f \in L^\infty_{loc}([0, +\infty))$ is nonnegative. Let $u \in C^1(B) \cap C^0(\overline{B})$ be a solution of (2.13) in the weak sense. Assume that either*

(a) *$p \geq n$,*

or

(b) *$p < n$ and, for some nonincreasing function $\phi \geq 0$, we have $\phi \leq f \leq \frac{np}{n-p}\phi$.*

Then, u is a radially symmetric and nonincreasing function. Moreover, $\frac{\partial u}{\partial r} < 0$ in $\{0 < u < \max_{\overline{\Omega}} u\}$, a set that will be an annulus or a punctured ball.

This result follows the approach introduced in 1981 by P. L. Lions within the paper [**86**], where the case $p = n = 2$ of Theorem 2.6 is proved (also with the hypothesis $f \geq 0$). In the same direction, Kesavan and Pacella [**82**] established the cases $p = n \geq 2$ of Theorem 2.6. In Lions' method, the isoperimetric inequality and the Pohozaev identity are combined to conclude the symmetry of u.

For some nonlinearities f which change sign, there exist positive solutions of (2.13) in a ball which are not radially symmetric, even with $p = 2$ and f Hölder continuous (see [**13**] for an example).

For $1 < p < \infty$, assuming that f is locally Lipschitz and *positive*, and that $u \in C^1(\overline{B})$ is a positive solution of (2.13) in a ball, Damascelli and Pacella [**53**] $(1 < p < 2)$ and Damascelli and Sciunzi [**54**] $(p > 2)$ succeeded in applying the moving planes method to prove the radial symmetry of u.

Another symmetry result for (2.12) with possibly non-Lipschitz f is due to Dolbeault and Felmer [**58**]. They assume that f is continuous and that, in a neighborhood of each point of its domain, f is either decreasing, or is the sum of a Lipschitz and a nondecreasing functions. If, in addition, $f \geq 0$, solutions $u \in C^1(B) \cap C^0(\overline{B})$ to (2.12) are radially symmetric. A similar result for the p-Laplacian equation (2.13) is found in [**59**]. These works use a local version of the moving planes technique.

Under the weaker assumption that $f \geq 0$ is only continuous, for $1 < p < \infty$, Brock [**12**] proved that $C^1(\overline{B})$ positive solutions of (2.13) are radially symmetric using the so called "continuous Steiner symmetrization".

The radial symmetry results in [**12**] (via continuous symmetrization) and in [**58, 59**] (via local moving planes) follow from more general local symmetry results [**13, 58, 59**] which do not require $f \geq 0$. These describe the only way in which radial symmetry may be broken through the formation of "plateaus" and radially symmetric cores placed arbitrarily on the top of them. The notion of local symmetry, introduced by Brock in [**13**], is strongly related to rearrangements. Nevertheless, in [**58, 59**], local symmetry results are proved using a local version of the moving planes method.

In contrast, our technique leads to symmetry for very general discontinuous nonlinearities. However, it only when Ω is a ball. Instead, the technique used in [**12**], as well as the moving planes method used in [**76, 54, 58, 59**] are still applicable when the domain is not a ball, but is symmetric about some hyperplane and convex in the normal direction to this hyperplane. See [**7**] for an improved version of the moving planes method that allowed to treat domains with corners.

A feature of the original moving planes method in [**76**] and [**54**] is that, in addition to the radial symmetry, leads to $\frac{\partial u}{\partial r} < 0$, for $r = |x| \in (0, R)$, R being the radius of the ball Ω. However, with discontinuous f we cannot expect so much, even with $p = 2$. A simple counterexample is constructed as follows: let v be the solution of

$$\begin{cases} -\Delta_p v = 1 & \text{in } A = \{1/2 < r < 1\}, \\ v = 0 & \text{on } \partial A. \end{cases}$$

Then, v is radial and positive, and thus it attains its maximum on a sphere $\{r = \rho_0\}$, for some $\rho_0 \in (1/2, 1)$. We readily check that $u = v\chi_{\{r > \rho_0\}} + (\max_{\overline{A}} v)\chi_{\{r \leq \rho_0\}}$ is a solution of (2.13) for $\Omega = \{r < 1\}$ and $f = \chi_{[0, \max_{\overline{A}} v)} \geq 0$, and u is constant on the ball $\{r \leq \rho_0\}$.

Related to this, Theorem 2.6 states that u is radial with $\frac{\partial u}{\partial r} < 0$ in the annulus or punctured ball $\{0 < u < \max_{\overline{\Omega}} u\}$. Nevertheless, u might attain its maximum in a concentric ball of positive radius $\{u = \max_{\overline{\Omega}} u\}$, as occurs in the preceding example.

The following three distribution-type functions will play a central role in our proof:

$$I(t) = \int_{\{u > t\}} f(u)\, d\mathcal{H}^n\,, \quad J(t) = \mathcal{H}^n\big(\{u > t\}\big)\,, \quad K = I^\alpha J^\beta\,. \tag{2.14}$$

These functions are defined for $t \in (-\infty, M)$, where $M = \max_\Omega u$. The parameters α, β in (2.14), that are appropriately chosen depending on p and n, are given by

$$\alpha = p' = \frac{p}{p-1}\,, \quad \beta = \frac{p-n}{n(p-1)}\,. \tag{2.15}$$

Lions [86] in the case $p = n = 2$ and Kesavan-Pacella [82] in the cases $p = n \geq 2$ used the distribution type function $K = I^\alpha$ (note that our β is zero in their cases). By considering the function $K = I^\alpha J^\beta$ we are able to treat the cases $p \neq n$.

That the function K in (2.14) be nonincreasing is essential for our argument to work. This is trivially the case when α, β given by (2.15) are nonnegative, and thus this occurs when $p \geq n$. When $\beta < 0$, the bounds (b) on f guarantee that K is nonincreasing.

References for the Introduction

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A. The Pohozaev identity for the fractional Laplacian

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THE POHOZAEV IDENTITY FOR THE FRACTIONAL LAPLACIAN

XAVIER ROS-OTON AND JOAQUIM SERRA

ABSTRACT. In this paper we prove the Pohozaev identity for the semilinear Dirichlet problem $(-\Delta)^s u = f(u)$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \backslash \Omega$. Here, $s \in (0,1)$, $(-\Delta)^s$ is the fractional Laplacian in $\mathbb{R}^n$, and Ω is a bounded $C^{1,1}$ domain.

To establish the identity we use, among other things, that if u is a bounded solution then $u/\delta^s|_\Omega$ is C^α up to the boundary $\partial\Omega$, where $\delta(x) = \text{dist}(x, \partial\Omega)$. In the fractional Pohozaev identity, the function $u/\delta^s|_{\partial\Omega}$ plays the role that $\partial u/\partial\nu$ plays in the classical one. Surprisingly, from a nonlocal problem we obtain an identity with a boundary term (an integral over $\partial\Omega$) which is completely local.

As an application of our identity, we deduce the nonexistence of nontrivial solutions in star-shaped domains for supercritical nonlinearities.

1. INTRODUCTION AND RESULTS

Let $s \in (0,1)$ and consider the fractional elliptic problem

$$\begin{cases} (-\Delta)^s u &=& f(u) & \text{in } \Omega \\ u &=& 0 & \text{in } \mathbb{R}^n \backslash \Omega \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where

$$(-\Delta)^s u(x) = c_{n,s} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \tag{1.2}$$

is the fractional Laplacian. Here, $c_{n,s}$ is a normalization constant given by (A.1).

When $s = 1$, a celebrated result of S. I. Pohozaev states that any solution of (1.1) satisfies an identity, which is known as the Pohozaev identity [16]. This classical result has many consequences, the most immediate one being the nonexistence of nontrivial bounded solutions to (1.1) for supercritical nonlinearities f.

The aim of this paper is to give the fractional version of this identity, that is, to prove the Pohozaev identity for problem (1.1) with $s \in (0,1)$. This is the main result of the paper, and it reads as follows. Here, since the solution u is bounded, the notions of weak and viscosity solutions agree (see Remark 1.5).

Theorem 1.1. *Let Ω be a bounded and $C^{1,1}$ domain, f be a locally Lipschitz function, u be a bounded solution of (1.1), and*

$$\delta(x) = \text{dist}(x, \partial\Omega).$$

Then,

$$u/\delta^s|_\Omega \in C^\alpha(\overline{\Omega}) \qquad \text{for some } \ \alpha \in (0,1),$$

meaning that $u/\delta^s|_\Omega$ has a continuous extension to $\overline{\Omega}$ which is $C^\alpha(\overline{\Omega})$, and the following identity holds

$$(2s - n)\int_\Omega uf(u)dx + 2n\int_\Omega F(u)dx = \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu)d\sigma,$$

where $F(t) = \int_0^t f$, ν is the unit outward normal to $\partial\Omega$ at x, and Γ is the Gamma function.

Note that in the fractional case the function $u/\delta^s|_{\partial\Omega}$ plays the role that $\partial u/\partial\nu$ plays in the classical Pohozaev identity. Moreover, if one sets $s = 1$ in the above identity one recovers the classical one, since $u/\delta|_{\partial\Omega} = \partial u/\partial\nu$ and $\Gamma(2) = 1$.

It is quite surprising that from a nonlocal problem (1.1) we obtain a completely local boundary term in the Pohozaev identity. That is, although the function u has to be defined in all $\mathbb{R}^n$ in order to compute its fractional Laplacian at a given point, knowing u only in a neighborhood of the boundary we can already compute $\int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu)d\sigma$.

Recall that problem (1.1) has an equivalent formulation given by the Caffarelli-Silvestre [9] associated extension problem —a local PDE in $\mathbb{R}^{n+1}_+$. For such extension, some Pohozaev type identities are proved in [4, 5, 6]. However, these identities contain boundary terms on the cylinder $\partial\Omega \times \mathbb{R}^+$ or in a half-sphere $\partial B_R^+ \cap \mathbb{R}^{n+1}_+$, which have no clear interpretation in terms of the original problem in $\mathbb{R}^n$. The proofs of these identities are similar to the one of the classical Pohozaev identity and use PDE tools (differential calculus identities and integration by parts).

Sometimes it may be useful to write the Pohozaev identity as

$$2s[u]^2_{H^s(\mathbb{R}^n)} - 2n\mathcal{E}[u] = \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu)d\sigma,$$

where $\mathcal{E}$ is the energy functional

$$\mathcal{E}[u] = \frac{1}{2}[u]^2_{H^s(\mathbb{R}^n)} - \int_\Omega F(u)dx, \tag{1.3}$$

$F' = f$, and

$$[u]_{H^s(\mathbb{R}^n)} = \||\xi|^s \mathcal{F}[u]\|_{L^2(\mathbb{R}^n)} = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dxdy. \tag{1.4}$$

We have used that if u and v are $H^s(\mathbb{R}^n)$ functions and $u \equiv v \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, then

$$\int_\Omega v(-\Delta)^s u \, dx = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} v \, dx, \tag{1.5}$$

which yields

$$\int_\Omega u f(u) dx = \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 dx = [u]_{H^s(\mathbb{R}^n)}.$$

As a consequence of our Pohozaev identity we obtain nonexistence results for problem (1.1) with supercritical nonlinearities f in star-shaped domains Ω. In Section 2 we will give, however, a short proof of this result using our method to establish the Pohozaev identity. This shorter proof will not require the full strength of the identity.

Corollary 1.2. *Let Ω be a bounded, $C^{1,1}$, and star-shaped domain, and let f be a locally Lipschitz function. If*

$$\frac{n - 2s}{2n} u f(u) \geq \int_0^u f(t) dt \qquad \text{for all } u \in \mathbb{R}, \tag{1.6}$$

then problem (1.1) admits no positive bounded solution. Moreover, if the inequality in (1.6) is strict, then (1.1) admits no nontrivial bounded solution.

For the pure power nonlinearity, the result reads as follows.

Corollary 1.3. *Let Ω be a bounded, $C^{1,1}$, and star-shaped domain. If $p \geq \frac{n+2s}{n-2s}$, then problem*

$$\begin{cases} (-\Delta)^s u = |u|^{p-1} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega \end{cases} \tag{1.7}$$

admits no positive bounded solution. Moreover, if $p > \frac{n+2s}{n-2s}$ then (1.7) admits no nontrivial bounded solution.

The nonexistence of changing-sign solutions to problem (1.7) for the critical power $p = \frac{n+2s}{n-2s}$ remains open.

Recently, M. M. Fall and T. Weth [12] have also proved a nonexistence result for problem (1.1) with the method of moving spheres. In their result no regularity of the domain is required, but they need to assume the solutions to be positive. Our nonexistence result is the first one allowing changing-sign solutions. In addition, their condition on f for the nonexistence —(1.16) in our Remark 1.14— is more restrictive than ours, i.e., (1.6) and, when $f = f(x, u)$, condition (1.15).

The existence of weak solutions $u \in H^s(\mathbb{R}^n)$ to problem (1.1) for subcritical f has been recently proved by R. Servadei and E. Valdinoci [19].

The Pohozaev identity will be a consequence of the following two results. The first one establishes $C^s(\mathbb{R}^n)$ regularity for u, $C^\alpha(\overline{\Omega})$ regularity for $u/\delta^s|_\Omega$, and higher order interior Hölder estimates for u and u/δ^s. It is proved in our paper [18].

Throughout the article, and when no confusion is possible, we will use the notation $C^\beta(U)$ with $\beta > 0$ to refer to the space $C^{k,\beta'}(U)$, where k is the is greatest integer such that $k < \beta$, and $\beta' = \beta - k$. This notation is specially appropriate when we work with $(-\Delta)^s$ in order to avoid the splitting of different cases in the statements

of regularity results. According to this, $[\cdot]_{C^\beta(U)}$ denotes the $C^{k,\beta'}(U)$ seminorm

$$[u]_{C^\beta(U)} = [u]_{C^{k,\beta'}(U)} = \sup_{x,y\in U,\ x\neq y} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\beta'}}.$$

Here, by $f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})$ we mean that f is Lipschitz in every compact subset of $\overline{\Omega} \times \mathbb{R}$.

Theorem 1.4 ([18]). *Let Ω be a bounded and $C^{1,1}$ domain, $f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})$, u be a bounded solution of*

$$\begin{cases} (-\Delta)^s u & = & f(x,u) & in\ \Omega \\ u & = & 0 & in\ \mathbb{R}^n\backslash\Omega, \end{cases} \tag{1.8}$$

and $\delta(x) = \text{dist}(x, \partial\Omega)$. Then,

(a) *$u \in C^s(\mathbb{R}^n)$ and, for every $\beta \in [s, 1+2s)$, u is of class $C^\beta(\Omega)$ and*

$$[u]_{C^\beta(\{x\in\Omega\,:\,\delta(x)\geq\rho\})} \leq C\rho^{s-\beta} \qquad for\ all\ \ \rho \in (0,1).$$

(b) *The function $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, u/δ^s belongs to $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ depending only on Ω, s, f, $\|u\|_{L^\infty(\mathbb{R}^n)}$. In addition, for all $\beta \in [\alpha, s+\alpha]$, it holds the estimate*

$$[u/\delta^s]_{C^\beta(\{x\in\Omega\,:\,\delta(x)\geq\rho\})} \leq C\rho^{\alpha-\beta} \qquad for\ all\ \ \rho \in (0,1).$$

The constant C depends only on Ω, s, f, $\|u\|_{L^\infty(\mathbb{R}^n)}$, and β.

Remark 1.5. For bounded solutions of (1.8), the notions of energy and viscosity solutions coincide (see more details in Remark 2.9 in [18]). Recall that u is an energy (or weak) solution of problem (1.8) if $u \in H^s(\mathbb{R}^n)$, $u \equiv 0$ in $\mathbb{R}^n\backslash\Omega$, and

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2}u(-\Delta)^{s/2}v\,dx = \int_\Omega f(x,u)v\,dx$$

for all $v \in H^s(\mathbb{R}^n)$ such that $v \equiv 0$ in $\mathbb{R}^n \setminus \Omega$.

By Theorem 1.4 (a), any bounded weak solution is continuous up to the boundary and solve equation (1.8) in the classical sense, i.e., in the pointwise sense of (1.2). Therefore, it follows from the definition of viscosity solution (see [8]) that bounded weak solutions are also viscosity solutions.

Reciprocally, by uniqueness of viscosity solutions [8] and existence of weak solution for the linear problem $(-\Delta)^s v = f(x, u(x))$, any viscosity solution u belongs to $H^s(\mathbb{R}^n)$ and it is also a weak solution. See [18] for more details.

The second result towards Theorem 1.1 is the new Pohozaev identity for the fractional Laplacian. The hypotheses of the following proposition are satisfied for any bounded solution u of (1.8) whenever $f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})$, by our results in [18] (see Theorem 1.4 above).

Proposition 1.6. *Let Ω be a bounded and $C^{1,1}$ domain. Assume that u is a $H^s(\mathbb{R}^n)$ function which vanishes in $\mathbb{R}^n \setminus \Omega$, and satisfies*

(a) $u \in C^s(\mathbb{R}^n)$ and, for every $\beta \in [s, 1 + 2s)$, u is of class $C^\beta(\Omega)$ and

$$[u]_{C^\beta(\{x \in \Omega \,:\, \delta(x) \geq \rho\})} \leq C\rho^{s-\beta} \qquad \text{for all} \ \ \rho \in (0, 1).$$

(b) The function $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, there exists $\alpha \in (0, 1)$ such that $u/\delta^s \in C^\alpha(\overline{\Omega})$. In addition, for all $\beta \in [\alpha, s + \alpha]$, it holds the estimate

$$[u/\delta^s]_{C^\beta(\{x \in \Omega \,:\, \delta(x) \geq \rho\})} \leq C\rho^{\alpha-\beta} \qquad \text{for all} \ \ \rho \in (0, 1).$$

(c) $(-\Delta)^s u$ is pointwise bounded in Ω.

Then, the following identity holds

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u \, dx = \frac{2s - n}{2} \int_\Omega u(-\Delta)^s u \, dx - \frac{\Gamma(1 + s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma,$$

where ν is the unit outward normal to $\partial\Omega$ at x, and Γ is the Gamma function.

Remark 1.7. Note that hypothesis (a) ensures that $(-\Delta)^s u$ is defined pointwise in Ω. Note also that hypotheses (a) and (c) ensure that the integrals appearing in the above identity are finite.

Remark 1.8. By Propositions 1.1 and 1.4 in [18], hypothesis (c) guarantees that $u \in C^s(\mathbb{R}^n)$ and $u/\delta^s \in C^\alpha(\overline{\Omega})$, but not the interior estimates in (a) and (b). However, under the stronger assumption $(-\Delta)^s u \in C^\alpha(\overline{\Omega})$ the whole hypothesis (b) is satisfied; see Theorem 1.5 in [18].

As a consequence of Proposition 1.6, we will obtain the Pohozaev identity (Theorem 1.1) and also a new integration by parts formula related to the fractional Laplacian. This integration by parts formula follows from using Proposition 1.6 with two different origins.

Theorem 1.9. *Let Ω be a bounded and $C^{1,1}$ domain, and u and v be functions satisfying the hypotheses in Proposition 1.6. Then, the following identity holds*

$$\int_\Omega (-\Delta)^s u \, v_{x_i} \, dx = - \int_\Omega u_{x_i}(-\Delta)^s v \, dx + \Gamma(1 + s)^2 \int_{\partial\Omega} \frac{u}{\delta^s} \frac{v}{\delta^s} \nu_i \, d\sigma$$

for $i = 1, ..., n$, where ν is the unit outward normal to $\partial\Omega$ at x, and Γ is the Gamma function.

To prove Proposition 1.6 we first assume the domain Ω to be star-shaped with respect to the origin. The result for general domains will follow from the star-shaped case, as seen in Section 5. When the domain is star-shaped, the idea of the proof is the following. First, one writes the left hand side of the identity as

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u \, dx = \frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_\Omega u_\lambda (-\Delta)^s u \, dx,$$

where

$$u_\lambda(x) = u(\lambda x).$$

Note that $u_\lambda \equiv 0$ in $\mathbb{R}^n \backslash \Omega$, since Ω is star-shaped and we take $\lambda > 1$ in the above derivative. As a consequence, we may use (1.5) with $v = u_\lambda$ and make the change of variables $y = \sqrt{\lambda}x$, to obtain

$$\int_\Omega u_\lambda (-\Delta)^s u \ dx = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u_\lambda (-\Delta)^{s/2} u \ dx = \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \ dy,$$

where

$$w(x) = (-\Delta)^{s/2} u(x).$$

Thus,

$$\begin{aligned}
\int_\Omega (x \cdot \nabla u)(-\Delta)^s u \ dx &= \frac{d}{d\lambda}\bigg|_{\lambda=1^+} \left\{ \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \ dy \right\} \\
&= \frac{2s-n}{2} \int_{\mathbb{R}^n} w^2 dx + \frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_{\sqrt{\lambda}} \qquad (1.9) \\
&= \frac{2s-n}{2} \int_{\mathbb{R}^n} u(-\Delta)^s u \ dx + \frac{1}{2}\frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda,
\end{aligned}$$

where

$$I_\lambda = \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} dy.$$

Therefore, Proposition 1.6 is equivalent to the following equality

$$-\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} \ dy = \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma. \qquad (1.10)$$

The quantity $\frac{d}{d\lambda}|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}$ vanishes for any $C^1(\mathbb{R}^n)$ function w, as can be seen by differentiating under the integral sign. Instead, we will prove that the function $w = (-\Delta)^{s/2}u$ has a singularity along $\partial\Omega$, and that (1.10) holds.

Next we give an easy argument to give a direct proof of the nonexistence result for supercritical nonlinearities without using neither equality (1.10) nor the behavior of $(-\Delta)^{s/2}u$; the detailed proof is given in Section 2.

Indeed, in contrast with the delicate equality (1.10), the inequality

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda \leq 0 \qquad (1.11)$$

follows easily from Cauchy-Schwarz. Namely,

$$I_\lambda \leq \|w_\lambda\|_{L^2(\mathbb{R}^n)} \|w_{1/\lambda}\|_{L^2(\mathbb{R}^n)} = \|w\|^2_{L^2(\mathbb{R}^n)} = I_1,$$

and hence (1.11) follows.

With this simple argument, (1.9) leads to

$$-\int_\Omega (x \cdot \nabla u)(-\Delta)^s u \ dx \geq \frac{n-2s}{2} \int_\Omega u(-\Delta)^s u \ dx,$$

which is exactly the inequality used to prove the nonexistence result of Corollary 1.2 for supercritical nonlinearities. Here, one also uses that, when u is a solution of (1.1), then

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u \ dx = \int_\Omega (x \cdot \nabla u)f(u)dx = \int_\Omega x \cdot \nabla F(u)dx = -n\int_\Omega F(u)dx.$$

This argument can be also used to obtain nonexistence results (under some decay assumptions) for weak solutions of (1.1) in the whole $\mathbb{R}^n$; see Remark 2.2.

The identity (1.10) is the difficult part of the proof of Proposition 1.6. To prove it, it will be crucial to know the precise behavior of $(-\Delta)^{s/2}u$ near $\partial\Omega$ —from both inside and outside Ω. This is given by the following result.

Proposition 1.10. *Let Ω be a bounded and $C^{1,1}$ domain, and u be a function such that $u \equiv 0$ in $\mathbb{R}^n \backslash \Omega$ and that u satisfies* (b) *in Proposition 1.6. Then, there exists a $C^\alpha(\mathbb{R}^n)$ extension v of $u/\delta^s|_\Omega$ such that*

$$(-\Delta)^{s/2}u(x) = c_1\left\{\log^- \delta(x) + c_2\chi_\Omega(x)\right\}v(x) + h(x) \quad in \ \ \mathbb{R}^n, \tag{1.12}$$

where h is a $C^\alpha(\mathbb{R}^n)$ function, $\log^- t = \min\{\log t, 0\}$,

$$c_1 = \frac{\Gamma(1+s)\sin\left(\frac{\pi s}{2}\right)}{\pi}, \qquad and \qquad c_2 = \frac{\pi}{\tan\left(\frac{\pi s}{2}\right)}. \tag{1.13}$$

Moreover, if u also satisfies (a) *in Proposition 1.6, then for all $\beta \in (0, 1+s)$*

$$[(-\Delta)^{s/2}u]_{C^\beta(\{x\in\mathbb{R}^n\,:\,\delta(x)\geq\rho\})} \leq C\rho^{-\beta} \qquad for \ all \ \ \rho \in (0,1), \tag{1.14}$$

for some constant C which does not depend on ρ.

The values (1.13) of the constants c_1 and c_2 in (1.12) arise in the expression for the $s/2$ fractional Laplacian, $(-\Delta)^{s/2}$, of the 1D function $(x_n^+)^s$, and they are computed in the Appendix.

Writing the first integral in (1.10) using spherical coordinates, equality (1.10) reduces to a computation in dimension 1, stated in the following proposition. This result will be used with the function φ in its statement being essentially the restriction of $(-\Delta)^{s/2}u$ to any ray through the origin. The constant γ will be chosen to be any value in $(0, s)$.

Proposition 1.11. *Let A and B be real numbers, and*

$$\varphi(t) = A\log^- |t-1| + B\chi_{[0,1]}(t) + h(t),$$

where $\log^- t = \min\{\log t, 0\}$ and h is a function satisfying, for some constants α and γ in $(0, 1)$, and $C_0 > 0$, the following conditions:

(i) $\|h\|_{C^\alpha([0,\infty))} \leq C_0$.

(ii) *For all $\beta \in [\gamma, 1+\gamma]$*

$$\|h\|_{C^\beta((0,1-\rho)\cup(1+\rho,2))} \leq C_0\rho^{-\beta} \qquad for \ all \ \ \rho \in (0,1).$$

(iii) $|h'(t)| \leq C_0 t^{-2-\gamma}$ *and* $|h''(t)| \leq C_0 t^{-3-\gamma}$ *for all* $t > 2$.

Then,

$$-\frac{d}{d\lambda}\bigg|_{\lambda=1^+}\int_0^\infty \varphi\left(\lambda t\right)\varphi\left(\frac{t}{\lambda}\right)dt = A^2\pi^2 + B^2.$$

Moreover, the limit defining this derivative is uniform among functions φ satisfying (i)-(ii)-(iii) with given constants C_0, α, and γ.

From this proposition one obtains that the constant in the right hand side of (1.10), $\Gamma(1+s)^2$, is given by $c_1^2(\pi^2 + c_2^2)$. The constant c_2 comes from an involved expression and it is nontrivial to compute (see Proposition 3.2 in Section 5 and the Appendix). It was a surprise to us that its final value is so simple and, at the same time, that the Pohozaev constant $c_1^2(\pi^2 + c_2^2)$ also simplifies and becomes $\Gamma(1+s)^2$.

Instead of computing explicitly the constants c_1 and c_2, an alternative way to obtain the constant in the Pohozaev identity consists of using an explicit nonlinearity and solution to problem (1.1) in a ball. The one which is known [13, 3] is the solution to problem

$$\begin{cases} (-\Delta)^s u & = \quad 1 \quad \text{in } B_r(x_0) \\ u & = \quad 0 \quad \text{in } \mathbb{R}^n \backslash B_r(x_0). \end{cases}$$

It is given by

$$u(x) = \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\left(r^2 - |x-x_0|^2\right)^s \qquad \text{in } B_r(x_0).$$

From this, it is straightforward to find the constant $\Gamma(1+s)^2$ in the Pohozaev identity; see Remark A.4 in the Appendix.

Using Theorem 1.4 and Proposition 1.6, we can also deduce a Pohozaev identity for problem (1.8), that is, allowing the nonlinearity f to depend also on x. In this case, the Pohozaev identity reads as follows.

Proposition 1.12. *Let Ω be a bounded and $C^{1,1}$ domain, $f \in C^{0,1}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R})$, u be a bounded solution of (1.8), and $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. Then*

$$u/\delta^s|_\Omega \in C^\alpha(\overline{\Omega}) \qquad \text{for some } \alpha \in (0,1),$$

and the following identity holds

$$(2s-n)\int_\Omega uf(x,u)dx + 2n\int_\Omega F(x,u)dx =$$

$$= \Gamma(1+s)^2\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2(x\cdot\nu)d\sigma - 2\int_\Omega x\cdot F_x(x,u)dx,$$

where $F(x,t) = \int_0^t f(x,\tau)d\tau$, ν is the unit outward normal to $\partial\Omega$ at x, and Γ is the Gamma function.

From this, we deduce nonexistence results for problem (1.8) with supercritical nonlinearities f depending also on x. This has been done also in [12] for positive solutions. Our result allows changing sign solutions as well as a slightly larger class of nonlinearities (see Remark 1.14).

Corollary 1.13. *Let Ω be a bounded, $C^{1,1}$, and star-shaped domain, $f \in C_{\text{loc}}^{0,1}(\overline{\Omega} \times \mathbb{R})$, and $F(x,t) = \int_0^t f(x,\tau)d\tau$. If*

$$\frac{n-2s}{2} u f(x,t) \geq nF(x,t) + x \cdot F_x(x,t) \qquad \text{for all} \ \ x \in \Omega \ \ \text{and} \ \ t \in \mathbb{R}, \quad (1.15)$$

then problem (1.8) admits no positive bounded solution. Moreover, if the inequality in (1.15) is strict, then (1.8) admits no nontrivial bounded solution.

Remark 1.14. For locally Lipschitz nonlinearities f, condition (1.15) is more general than the one required in [12] for their nonexistence result. Namely, [12] assumes that for each $x \in \Omega$ and $t \in \mathbb{R}$, the map

$$\lambda \mapsto \lambda^{-\frac{n+2s}{n-2s}} f(\lambda^{-\frac{2}{n-2s}} x, \lambda t) \qquad \text{is nondecreasing for} \ \lambda \in (0,1]. \quad (1.16)$$

Such nonlinearities automatically satisfy (1.15).

However, in [12] they do not need to assume any regularity on f with respect to x.

The paper is organized as follows. In Section 2, using Propositions 1.10 and 1.11 (to be established later), we prove Proposition 1.6 (the Pohozaev identity) for strictly star-shaped domains with respect to the origin. We also establish the nonexistence results for supercritical nonlinearities, and this does not require any result from the rest of the paper. In Section 3 we establish Proposition 1.10, while in Section 4 we prove Proposition 1.11. Section 5 establishes Proposition 1.6 for non-star-shaped domains and all its consequences, which include Theorems 1.1 and 1.9 and the nonexistence results. Finally, in the Appendix we compute the constants c_1 and c_2 appearing in Proposition 1.10.

2. STAR-SHAPED DOMAINS: POHOZAEV IDENTITY AND NONEXISTENCE

In this section we prove Proposition 1.6 for strictly star-shaped domains. We say that Ω is strictly star-shaped if, for some $z_0 \in \mathbb{R}^n$,

$$(x - z_0) \cdot \nu > 0 \qquad \text{for all} \ \ x \in \partial\Omega. \quad (2.1)$$

The result for general $C^{1,1}$ domains will be a consequence of this strictly star-shaped case and will be proved in Section 5.

The proof in this section uses two of our results: Proposition 1.10 on the behavior of $(-\Delta)^{s/2}u$ near $\partial\Omega$ and the one dimensional computation of Proposition 1.11.

The idea of the proof for the fractional Pohozaev identity is to use the integration by parts formula (1.5) with $v = u_\lambda$, where

$$u_\lambda(x) = u(\lambda x), \quad \lambda > 1,$$

and then differentiate the obtained identity (which depends on λ) with respect to λ and evaluate at $\lambda = 1$. However, this apparently simple formal procedure requires a quite involved analysis when it is put into practice. The hypothesis that Ω is star-shaped is crucially used in order that u_λ, $\lambda > 1$, vanishes outside Ω so that (1.5) holds.

Proof of Proposition 1.6 for strictly star-shaped domains. Let us assume first that Ω is strictly star-shaped with respect to the origin, that is, $z_0 = 0$.

Let us prove that

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u\, dx = \frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_\Omega u_\lambda (-\Delta)^s u\, dx, \qquad (2.2)$$

where $\frac{d}{d\lambda}\big|_{\lambda=1^+}$ is the derivative from the right side at $\lambda = 1$. Indeed, let $g = (-\Delta)^s u$. By assumption (a) g is defined pointwise in Ω, and by assumption (c) $g \in L^\infty(\Omega)$. Then, making the change of variables $y = \lambda x$ and using that $\operatorname{supp} u_\lambda = \frac{1}{\lambda}\Omega \subset \Omega$ since $\lambda > 1$, we obtain

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_\Omega u_\lambda(x) g(x) dx = \lim_{\lambda\downarrow 1} \int_\Omega \frac{u(\lambda x) - u(x)}{\lambda - 1} g(x) dx$$

$$= \lim_{\lambda\downarrow 1} \lambda^{-n} \int_{\lambda\Omega} \frac{u(y) - u(y/\lambda)}{\lambda - 1} g(y/\lambda) dy$$

$$= \lim_{\lambda\downarrow 1} \int_\Omega \frac{u(y) - u(y/\lambda)}{\lambda - 1} g(y/\lambda) dy + \lim_{\lambda\downarrow 1} \int_{(\lambda\Omega)\backslash\Omega} \frac{-u(y/\lambda)}{\lambda - 1} g(y/\lambda) dy.$$

By the dominated convergence theorem,

$$\lim_{\lambda\downarrow 1} \int_\Omega \frac{u(y) - u(y/\lambda)}{\lambda - 1} g(y/\lambda)\, dy = \int_\Omega (y \cdot \nabla u) g(y)\, dy,$$

since $g \in L^\infty(\Omega)$, $|\nabla u(\xi)| \leq C\delta(\xi)^{s-1} \leq C\lambda^{1-s}\delta(y)^{s-1}$ for all ξ in the segment joining y and y/λ, and δ^{s-1} is integrable. The gradient bound $|\nabla u(\xi)| \leq C\delta(\xi)^{s-1}$ follows from assumption (a) used with $\beta = 1$. Hence, to prove (2.2) it remains only to show that

$$\lim_{\lambda\downarrow 1} \int_{(\lambda\Omega)\backslash\Omega} \frac{-u(y/\lambda)}{\lambda - 1} g(y/\lambda) dy = 0.$$

Indeed, $|(\lambda\Omega)\backslash\Omega| \leq C(\lambda - 1)$ and —by (a)— $u \in C^s(\mathbb{R}^n)$ and $u \equiv 0$ outside Ω. Hence, $\|u\|_{L^\infty((\lambda\Omega)\backslash\Omega)} \to 0$ as $\lambda \downarrow 1$ and (2.2) follows.

Now, using the integration by parts formula (1.5) with $v = u_\lambda$,

$$\int_\Omega u_\lambda(-\Delta)^s u\, dx = \int_{\mathbb{R}^n} u_\lambda(-\Delta)^s u\, dx$$

$$= \int_{\mathbb{R}^n} (-\Delta)^{s/2} u_\lambda (-\Delta)^{s/2} u\, dx$$

$$= \lambda^s \int_{\mathbb{R}^n} \left((-\Delta)^{s/2} u\right)(\lambda x)(-\Delta)^{s/2} u(x) dx$$

$$= \lambda^s \int_{\mathbb{R}^n} w_\lambda w\, dx,$$

where

$$w(x) = (-\Delta)^{s/2} u(x) \qquad \text{and} \qquad w_\lambda(x) = w(\lambda x).$$

With the change of variables $y = \sqrt{\lambda}x$ this integral becomes

$$\lambda^s \int_{\mathbb{R}^n} w_\lambda w \, dx = \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \, dy,$$

and thus

$$\int_\Omega u_\lambda (-\Delta)^s u \, dx = \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \, dy.$$

Furthermore, this leads to

$$
\begin{aligned}
\int_\Omega (\nabla u \cdot x)(-\Delta)^s u \, dx &= \left. \frac{d}{d\lambda} \right|_{\lambda=1^+} \left\{ \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \, dy \right\} \\
&= \frac{2s-n}{2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \, dx + \left. \frac{d}{d\lambda} \right|_{\lambda=1^+} \int_{\mathbb{R}^n} w_{\sqrt{\lambda}} w_{1/\sqrt{\lambda}} \, dy \\
&= \frac{2s-n}{2} \int_\Omega u(-\Delta)^s u \, dx + \frac{1}{2} \left. \frac{d}{d\lambda} \right|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} \, dy. \quad (2.3)
\end{aligned}
$$

Hence, it remains to prove that

$$- \left. \frac{d}{d\lambda} \right|_{\lambda=1^+} I_\lambda = \Gamma(1+s)^2 \int_{\partial\Omega} \left(\frac{u}{\delta^s} \right)^2 (x \cdot \nu) \, d\sigma, \qquad (2.4)$$

where we have denoted

$$I_\lambda = \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} \, dy. \qquad (2.5)$$

Now, for each $\theta \in S^{n-1}$ there exists a unique $r_\theta > 0$ such that $r_\theta \theta \in \partial\Omega$. Write the integral (2.5) in spherical coordinates and use the change of variables $t = r/r_\theta$:

$$
\begin{aligned}
\left. \frac{d}{d\lambda} \right|_{\lambda=1^+} I_\lambda &= \left. \frac{d}{d\lambda} \right|_{\lambda=1^+} \int_{S^{n-1}} d\theta \int_0^\infty r^{n-1} w(\lambda r \theta) w \left(\frac{r}{\lambda} \theta \right) dr \\
&= \left. \frac{d}{d\lambda} \right|_{\lambda=1^+} \int_{S^{n-1}} r_\theta d\theta \int_0^\infty (r_\theta t)^{n-1} w(\lambda r_\theta t \theta) w \left(\frac{r_\theta t}{\lambda} \theta \right) dt \\
&= \left. \frac{d}{d\lambda} \right|_{\lambda=1^+} \int_{\partial\Omega} (x \cdot \nu) d\sigma(x) \int_0^\infty t^{n-1} w(\lambda t x) w \left(\frac{tx}{\lambda} \right) dt,
\end{aligned}
$$

where we have used that

$$r_\theta^{n-1} \, d\theta = \left(\frac{x}{|x|} \cdot \nu \right) d\sigma = \frac{1}{r_\theta} (x \cdot \nu) \, d\sigma$$

with the change of variables $S^{n-1} \to \partial\Omega$ that maps every point in S^{n-1} to its radial projection on $\partial\Omega$, which is unique because of the strictly star-shapedness of Ω.

Fix $x_0 \in \partial\Omega$ and define

$$\varphi(t) = t^{\frac{n-1}{2}} w(tx_0) = t^{\frac{n-1}{2}} (-\Delta)^{s/2} u(tx_0).$$

By Proposition 1.10,

$$\varphi(t) = c_1 \{ \log^- \delta(tx_0) + c_2 \chi_{[0,1]} \} v(tx_0) + h_0(t)$$

in $[0, \infty)$, where v is a $C^\alpha(\mathbb{R}^n)$ extension of $u/\delta^s|_\Omega$ and h_0 is a $C^\alpha([0, \infty))$ function. Next we will modify this expression in order to apply Proposition 1.11.

Using that Ω is $C^{1,1}$ and strictly star-shaped, it is not difficult to see that $\frac{|r - r_\theta|}{\delta(r\theta)}$ is a Lipschitz function of r in $[0, \infty)$ and bounded below by a positive constant (independently of x_0). Similarly, $\frac{|t-1|}{\delta(tx_0)}$ and $\frac{\min\{|t-1|,1\}}{\min\{\delta(tx_0),1\}}$ are positive and Lipschitz functions of t in $[0, \infty)$. Therefore,

$$\log^- |t - 1| - \log^- \delta(tx_0)$$

is Lipschitz in $[0, \infty)$ as a function of t.

Hence, for $t \in [0, \infty)$,

$$\varphi(t) = c_1\{\log^- |t - 1| + c_2\chi_{[0,1]}\}v(tx_0) + h_1(t),$$

where h_1 is a C^α function in the same interval.

Moreover, note that the difference

$$v(tx_0) - v(x_0)$$

is C^α and vanishes at $t = 1$. Thus,

$$\varphi(t) = c_1\{\log^- |t - 1| + c_2\chi_{[0,1]}(t)\}v(x_0) + h(t)$$

holds in all $[0, \infty)$, where h is C^α in $[0, \infty)$ if we slightly decrease α in order to kill the logarithmic singularity. This is condition (i) of Proposition 1.11.

From the expression

$$h(t) = t^{\frac{n-1}{2}}(-\Delta)^{s/2}u\,(tx_0) - c_1\{\log^- |t - 1| + c_2\chi_{[0,1]}(t)\}v(x_0)$$

and from (1.14) in Proposition 1.10, we obtain that h satisfies condition (ii) of Proposition 1.11 with $\gamma = s/2$.

Moreover, condition (iii) of Proposition 1.11 is also satisfied. Indeed, for $x \in \mathbb{R}^n\backslash(2\Omega)$ we have

$$(-\Delta)^{s/2}u(x) = c_{n,\frac{s}{2}} \int_\Omega \frac{-u(y)}{|x - y|^{n+s}}dy$$

and hence

$$|\partial_i(-\Delta)^{s/2}u(x)| \leq C|x|^{-n-s-1} \quad \text{and} \quad |\partial_{ij}(-\Delta)^{s/2}u(x)| \leq C|x|^{-n-s-2}.$$

This yields $|\varphi'(t)| \leq Ct^{\frac{n-1}{2}-n-s-1} \leq Ct^{-2-\gamma}$ and $|\varphi''(t)| \leq Ct^{\frac{n-1}{2}-n-s-2} \leq Ct^{-3-\gamma}$ for $t > 2$.

Therefore we can apply Proposition 1.11 to obtain

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_0^\infty \varphi(\lambda t)\varphi\left(\frac{t}{\lambda}\right) dt = (v(x_0))^2\, c_1^2(\pi^2 + c_2^2),$$

and thus

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_0^\infty t^{n-1}w(\lambda tx_0)w\left(\frac{tx_0}{\lambda}\right) dt = (v(x_0))^2\, c_1^2(\pi^2 + c_2^2)$$

for each $x_0 \in \partial\Omega$.

Furthermore, by uniform convergence on x_0 of the limit defining this derivative (see Proposition 4.2 in Section 4), this leads to

$$\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda = c_1^2(\pi^2 + c_2^2) \int_{\partial\Omega} (x_0 \cdot \nu) \left(\frac{u}{\delta^s}(x_0)\right)^2 dx_0.$$

Here we have used that, for $x_0 \in \partial\Omega$, $v(x_0)$ is uniquely defined by continuity as

$$\left(\frac{u}{\delta^s}\right)(x_0) = \lim_{x\to x_0,\ x\in\Omega} \frac{u(x)}{\delta^s(x)}.$$

Hence, it only remains to prove that

$$c_1^2(\pi^2 + c_2^2) = \Gamma(1+s)^2.$$

But

$$c_1 = \frac{\Gamma(1+s)\sin\left(\frac{\pi s}{2}\right)}{\pi} \qquad \text{and} \qquad c_2 = \frac{\pi}{\tan\left(\frac{\pi s}{2}\right)},$$

and therefore

$$\begin{aligned}
c_1^2(\pi^2 + c_2^2) &= \frac{\Gamma(1+s)^2 \sin^2\left(\frac{\pi s}{2}\right)}{\pi^2}\left(\pi^2 + \frac{\pi^2}{\tan^2\left(\frac{\pi s}{2}\right)}\right) \\
&= \Gamma(1+s)^2 \sin^2\left(\frac{\pi s}{2}\right)\left(1 + \frac{\cos^2\left(\frac{\pi s}{2}\right)}{\sin^2\left(\frac{\pi s}{2}\right)}\right) \\
&= \Gamma(1+s)^2.
\end{aligned}$$

Assume now that Ω is strictly star-shaped with respect to a point $z_0 \neq 0$. Then, Ω is strictly star-shaped with respect to all points z in a neighborhood of z_0. Then, making a translation and using the formula for strictly star-shaped domains with respect to the origin, we deduce

$$\int_\Omega \{(x-z)\cdot\nabla u\}\,(-\Delta)^s u\,dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u\,dx + \\
- \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 (x-z)\cdot\nu\,d\sigma \tag{2.6}$$

for each z in a neighborhood of z_0. This yields

$$\int_\Omega u_{x_i}(-\Delta)^s u\,dx = -\frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 \nu_i\,d\sigma \tag{2.7}$$

for $i = 1,...,n$. Thus, by adding to (2.6) a linear combination of (2.7), we obtain

$$\int_\Omega (x\cdot\nabla u)(-\Delta)^s u\,dx = \frac{2s-n}{2}\int_\Omega u(-\Delta)^s u\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2 x\cdot\nu\,d\sigma.$$

$\square$

Next we prove the nonexistence results of Corollaries 1.2, 1.3, and 1.13 for su-percritical nonlinearities in star-shaped domains. Recall that star-shaped means $x \cdot \nu \geq 0$ for all $x \in \partial\Omega$. Although these corollaries follow immediately from Proposition 1.12 —as we will see in Section 5—, we give here a short proof of their second part, i.e., nonexistence when the inequality (1.6) or (1.15) is strict. That is, we establish the nonexistence of nontrivial solutions for supercritical nonlinearities (not including the critical case).

Our proof follows the method above towards the Pohozaev identity but does not require the full strength of the identity. In addition, in terms of regularity results for the equation, the proof only needs an easy gradient estimate for solutions u. Namely,

$$|\nabla u| \leq C\delta^{s-1} \text{ in } \Omega,$$

which follows from part (a) of Theorem 1.4, proved in [18].

Proof of Corollaries 1.2, 1.3, and 1.13 for supercritical nonlinearities. We only have to prove Corollary 1.13, since Corollaries 1.2 and 1.3 follow immediately from it by setting $f(x,u) = f(u)$ and $f(x,u) = |u|^{p-1}u$ respectively.

Let us prove that if Ω is star-shaped and u is a bounded solution of (1.8), then

$$\frac{2s-n}{2} \int_{\Omega} uf(x,u)dx + n \int_{\Omega} F(x,u)dx - \int_{\Omega} x \cdot F_x(x,u)dx \geq 0. \qquad (2.8)$$

For this, we follow the beginning of the proof of Proposition 1.6 (given above) to obtain (2.3), i.e., until the identity

$$\int_{\Omega} (\nabla u \cdot x)(-\Delta)^s u\, dx = \frac{2s-n}{2} \int_{\Omega} u(-\Delta)^s u\, dx + \frac{1}{2} \frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda,$$

where

$$I_\lambda = \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}\, dx, \qquad w(x) = (-\Delta)^{s/2} u(x), \qquad \text{and} \qquad w_\lambda(x) = w(\lambda x).$$

This step of the proof only need the star-shapedness of Ω (and not the strictly star-shapedness) and the regularity result $|\nabla u| \leq C\delta^{s-1}$ in Ω, which follows from Theorem 1.4, proved in [18].

Now, since $(-\Delta)^s u = f(x,u)$ in Ω and

$$(\nabla u \cdot x)(-\Delta)^s u = x \cdot \nabla F(x,u) - x \cdot F_x(x,u),$$

by integrating by parts we deduce

$$-n \int_{\Omega} F(x,u)dx - \int_{\Omega} x \cdot F_x(x,u)dx = \frac{2s-n}{2} \int_{\Omega} uf(x,u)dx + \frac{1}{2} \frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda.$$

Therefore, we only need to show that

$$\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda \leq 0. \qquad (2.9)$$

But applying Hölder's inequality, for each $\lambda > 1$ we have

$$I_\lambda \leq \|w_\lambda\|_{L^2(\mathbb{R}^n)}\|w_{1/\lambda}\|_{L^2(\mathbb{R}^n)} = \|w\|^2_{L^2(\mathbb{R}^n)} = I_1,$$

and (2.9) follows. $\qquad\square$

Remark 2.1. For this nonexistence result the regularity of the domain Ω is only used for the estimate $|\nabla u| \leq C\delta^{s-1}$. This estimate only requires Ω to be Lipschitz and satisfy an exterior ball condition; see [18]. In particular, our nonexistence result for supercritical nonlinearities applies to any convex domain, such as a square for instance.

Remark 2.2. When $\Omega = \mathbb{R}^n$ or when Ω is a star-shaped domain with respect to infinity, there are two recent nonexistence results for subcritical nonlinearities. They use the method of moving spheres to prove nonexistence of bounded positive solutions in these domains. The first result is due to A. de Pablo and U. Sánchez [15], and they obtain nonexistence of bounded positive solutions to $(-\Delta)^s u = u^p$ in all of $\mathbb{R}^n$, whenever $s > 1/2$ and $1 < p < \frac{n+2s}{n-2s}$. The second result, by M. Fall and T. Weth [12], gives nonexistence of bounded positive solutions of (1.8) in star-shaped domains with respect to infinity for subcritical nonlinearities.

Our method in the previous proof can also be used to prove nonexistence results for problem (1.7) in star-shaped domains with respect to infinity or in the whole $\mathbb{R}^n$. However, to ensure that the integrals appearing in the proof are well defined, one must assume some decay on u and ∇u. For instance, in the supercritical case $p > \frac{n+2s}{n-2s}$ we obtain that the only solution to $(-\Delta)^s u = u^p$ in all of $\mathbb{R}^n$ decaying as

$$|u| + |x \cdot \nabla u| \leq \frac{C}{1 + |x|^\beta},$$

with $\beta > \frac{n}{p+1}$, is $u \equiv 0$.

In the case of the whole $\mathbb{R}^n$, there is an alternative proof of the nonexistence of solutions which decay fast enough at infinity. It consists of using a Pohozaev identity in all of $\mathbb{R}^n$, that is easily deduced from the pointwise equality

$$(-\Delta)^s(x \cdot \nabla u) = 2s(-\Delta)^s u + x \cdot \nabla(-\Delta)^s u.$$

The classification of solutions in the whole $\mathbb{R}^n$ for the critical exponent $p = \frac{n+2s}{n-2s}$ was obtained by W. Chen, C. Li, and B. Ou in [10]. They are of the form

$$u(x) = c\left(\frac{\mu}{\mu^2 + |x - x_0|^2}\right)^{\frac{n-2s}{2}},$$

where μ is any positive parameter and c is a constant depending on n and s.

3. Behavior of $(-\Delta)^{s/2}u$ near $\partial\Omega$

The aim of this section is to prove Proposition 1.10. We will split this proof into two propositions. The first one is the following, and compares the behavior of $(-\Delta)^{s/2}u$ near $\partial\Omega$ with the one of $(-\Delta)^{s/2}\delta_0^s$, where $\delta_0(x) = \text{dist}(x, \partial\Omega)\chi_\Omega(x)$.

Proposition 3.1. *Let Ω be a bounded and $C^{1,1}$ domain, u be a function satisfying* (b) *in Proposition 1.6. Then, there exists a $C^\alpha(\mathbb{R}^n)$ extension v of $u/\delta^s|_\Omega$ such that*

$$(-\Delta)^{s/2}u(x) = (-\Delta)^{s/2}\delta_0^s(x)v(x) + h(x) \quad in \ \mathbb{R}^n,$$

where $h \in C^\alpha(\mathbb{R}^n)$.

Once we know that the behavior of $(-\Delta)^{s/2}u$ is comparable to the one of $(-\Delta)^{s/2}\delta_0^s$, Proposition 1.10 reduces to the following result, which gives the behavior of $(-\Delta)^s\delta_0^s$ near $\partial\Omega$.

Proposition 3.2. *Let Ω be a bounded and $C^{1,1}$ domain, $\delta(x) = \mathrm{dist}(x, \partial\Omega)$, and $\delta_0 = \delta\chi_\Omega$. Then,*

$$(-\Delta)^{s/2}\delta_0^s(x) = c_1\left\{\log^- \delta(x) + c_2\chi_\Omega(x)\right\} + h(x) \quad in \ \mathbb{R}^n,$$

where c_1 and c_2 are constants, h is a $C^\alpha(\mathbb{R}^n)$ function, and $\log^- t = \min\{\log t, 0\}$. The constants c_1 and c_2 are given by

$$c_1 = c_{1,\frac{s}{2}} \qquad and \qquad c_2 = \int_0^\infty \left\{\frac{1-z^s}{|1-z|^{1+s}} + \frac{1+z^s}{|1+z|^{1+s}}\right\} dz,$$

where $c_{n,s}$ is the constant appearing in the singular integral expression (1.2) *for $(-\Delta)^s$ in dimension n.*

The fact that the constants c_1 and c_2 given by Proposition 3.2 coincide with the ones from Proposition 1.10 is proved in the Appendix.

In the proof of Proposition 3.1 we need to compute $(-\Delta)^{s/2}$ of the product $u = \delta_0^s v$. For it, we will use the following elementary identity, which can be derived from (1.2):

$$(-\Delta)^s(w_1 w_2) = w_1(-\Delta)^s w_2 + w_2(-\Delta)^s w_1 - I_s(w_1, w_2),$$

where

$$I_s(w_1, w_2)(x) = c_{n,s}\mathrm{PV}\int_{\mathbb{R}^n} \frac{\big(w_1(x) - w_1(y)\big)\big(w_2(x) - w_2(y)\big)}{|x-y|^{n+2s}}\, dy. \qquad (3.1)$$

Next lemma will lead to a Hölder bound for $I_s(\delta_0^s, v)$.

Lemma 3.3. *Let Ω be a bounded domain and $\delta_0 = \mathrm{dist}(x, \mathbb{R}^n \setminus \Omega)$. Then, for each $\alpha \in (0, 1)$ the following a priori bound holds*

$$\|I_{s/2}(\delta_0^s, w)\|_{C^{\alpha/2}(\mathbb{R}^n)} \leq C[w]_{C^\alpha(\mathbb{R}^n)}, \qquad (3.2)$$

where the constant C depends only on n, s, and α.

Proof. Let $x_1, x_2 \in \mathbb{R}^n$. Then,

$$|I_{s/2}(\delta_0^s, w)(x_1) - I_{s/2}(\delta_0^s, w)(x_2)| \leq c_{n,\frac{s}{2}}(J_1 + J_2),$$

where

$$J_1 = \int_{\mathbb{R}^n} \frac{\big|w(x_1) - w(x_1+z) - w(x_2) + w(x_2+z)\big|\big|\delta_0^s(x_1) - \delta_0^s(x_1+z)\big|}{|z|^{n+s}}\, dz$$

and

$$J_2 = \int_{\mathbb{R}^n} \frac{\left|w(x_2) - w(x_2 + z)\right| \left|\delta_0^s(x_1) - \delta_0^s(x_1 + z) - \delta_0^s(x_2) + \delta_0^s(x_2 + z)\right|}{|z|^{n+s}} \, dz \, .$$

Let $r = |x_1 - x_2|$. Using that $\|\delta_0^s\|_{C^s(\mathbb{R}^n)} \le 1$ and $\operatorname{supp} \delta_0^s = \overline{\Omega}$,

$$J_1 \le \int_{\mathbb{R}^n} \frac{\left|w(x_1) - w(x_1 + z) - w(x_2) + w(x_2 + z)\right| \min\{|z|^s, (\operatorname{diam}\Omega)^s\}}{|z|^{n+s}} \, dz$$

$$\le C \int_{\mathbb{R}^n} \frac{[w]_{C^\alpha(\mathbb{R}^n)} r^{\alpha/2} |z|^{\alpha/2} \min\{|z|^s, 1\}}{|z|^{n+s}} \, dz$$

$$\le C r^{\alpha/2} [w]_{C^\alpha(\mathbb{R}^n)} \, .$$

Analogously,

$$J_2 \le C r^{\alpha/2} [w]_{C^\alpha(\mathbb{R}^n)} \, .$$

The bound for $\|I_{s/2}(\delta_0^s, w)\|_{L^\infty(\mathbb{R}^n)}$ is obtained with a similar argument, and hence (3.2) follows. $\qquad\square$

Before stating the next result, we need to introduce the following weighted Hölder norms; see Definition 1.3 in [18].

Definition 3.4. Let $\beta > 0$ and $\sigma \ge -\beta$. Let $\beta = k + \beta'$, with k integer and $\beta' \in (0, 1]$. For $w \in C^\beta(\Omega) = C^{k,\beta'}(\Omega)$, define the seminorm

$$[w]_{\beta;\Omega}^{(\sigma)} = \sup_{x,y \in \Omega} \left(\min\{\delta(x), \delta(y)\}^{\beta+\sigma} \frac{|D^k w(x) - D^k w(y)|}{|x - y|^{\beta'}} \right).$$

For $\sigma > -1$, we also define the norm $\|\cdot\|_{\beta;\Omega}^{(\sigma)}$ as follows: in case that $\sigma \ge 0$,

$$\|w\|_{\beta;\Omega}^{(\sigma)} = \sum_{l=0}^{k} \sup_{x \in \Omega} \left(\delta(x)^{l+\sigma} |D^l w(x)| \right) + [w]_{\beta;\Omega}^{(\sigma)} \, ,$$

while for $-1 < \sigma < 0$,

$$\|w\|_{\beta;\Omega}^{(\sigma)} = \|w\|_{C^{-\sigma}(\overline{\Omega})} + \sum_{l=1}^{k} \sup_{x \in \Omega} \left(\delta(x)^{l+\sigma} |D^l w(x)| \right) + [w]_{\beta;\Omega}^{(-\alpha)} \, .$$

The following lemma, proved in [18], will be used in the proof of Proposition 3.1 below —with w replaced by v— and also at the end of this section in the proof of Proposition 1.10 —with w replaced by u.

Lemma 3.5 ([18, Lemma 4.3]). *Let Ω be a bounded domain and α and β be such that $\alpha \le s < \beta$ and $\beta - s$ is not an integer. Let k be an integer such that $\beta = k + \beta'$ with $\beta' \in (0, 1]$. Then,*

$$[(-\Delta)^{s/2} w]_{\beta-s;\Omega}^{(s-\alpha)} \le C \left(\|w\|_{C^\alpha(\mathbb{R}^n)} + \|w\|_{\beta;\Omega}^{(-\alpha)} \right) \tag{3.3}$$

for all w with finite right hand side. The constant C depends only on n, s, α, and β.

Before proving Proposition 3.1, we give an extension lemma —see [11, Theorem 1, Section 3.1] where the case $\alpha = 1$ is proven in full detail.

Lemma 3.6. *Let $\alpha \in (0,1]$ and $V \subset \mathbb{R}^n$ a bounded domain. There exists a (non-linear) map $E : C^{0,\alpha}(\overline{V}) \to C^{0,\alpha}(\mathbb{R}^n)$ satisfying*

$$E(w) \equiv w \quad in \ \overline{V}, \quad [E(w)]_{C^{0,\alpha}(\mathbb{R}^n)} \leq [w]_{C^{0,\alpha}(\overline{V})}, \quad and \quad \|E(w)\|_{L^\infty(\mathbb{R}^n)} \leq \|w\|_{L^\infty(V)}$$

for all $w \in C^{0,\alpha}(\overline{V})$.

Proof. It is immediate to check that

$$E(w)(x) = \min \left\{ \min_{z \in \overline{V}} \left\{ w(z) + [w]_{C^\alpha(\overline{V})} |z - x|^\alpha \right\}, \|w\|_{L^\infty(V)} \right\}$$

satisfies the conditions since, for all x, y, z in $\mathbb{R}^n$,

$$|z - x|^\alpha \leq |z - y|^\alpha + |y - x|^\alpha.$$

$\square$

Now we can give the

Proof of Proposition 3.1. Since $u/\delta^s|_\Omega$ is $C^\alpha(\overline{\Omega})$ —by hypothesis (b)— then by Lemma 3.6 there exists a $C^\alpha(\mathbb{R}^n)$ extension v of $u/\delta^s|_\Omega$.

Then, we have that

$$(-\Delta)^{s/2}u(x) = v(x)(-\Delta)^{s/2}\delta_0^s(x) + \delta_0(x)^s(-\Delta)^{s/2}v(x) - I_{s/2}(v, \delta_0^s),$$

where

$$I_{s/2}(v, \delta_0^s) = c_{n,\frac{s}{2}} \int_{\mathbb{R}^n} \frac{\left(v(x) - v(y)\right)\left(\delta_0^s(x) - \delta_0^s(y)\right)}{|x - y|^{n+s}} \, dy,$$

as defined in (3.1). This equality is valid in all of $\mathbb{R}^n$ because $\delta_0^s \equiv 0$ in $\mathbb{R}^n \backslash \Omega$ and $v \in C^{\alpha+s}$ in Ω —by hypothesis (b). Thus, we only have to see that $\delta_0^s(-\Delta)^{s/2}v$ and $I_{s/2}(v, \delta_0^s)$ are $C^\alpha(\mathbb{R}^n)$ functions.

For the first one we combine assumption (b) with $\beta = s + \alpha < 1$ and Lemma 3.5. We obtain

$$\|(-\Delta)^{s/2}v\|_{\alpha;\Omega}^{(s-\alpha)} \leq C, \tag{3.4}$$

and this yields $\delta_0^s(-\Delta)^{s/2}v \in C^\alpha(\mathbb{R}^n)$. Indeed, let $w = (-\Delta)^{s/2}v$. Then, for all $x, y \in \Omega$ such that $y \in B_R(x)$, with $R = \delta(x)/2$, we have

$$\frac{|\delta^s(x)w(x) - \delta^s(y)w(y)|}{|x - y|^\alpha} \leq \delta(x)^s \frac{|w(x) - w(y)|}{|x - y|^\alpha} + |w(x)| \frac{|\delta^s(x) - \delta^s(y)|}{|x - y|^\alpha}.$$

Now, since

$$|\delta^s(x) - \delta^s(y)| \leq CR^{s-\alpha}|x - y|^\alpha \leq C \min\{\delta(x), \delta(y)\}^{s-\alpha}|x - y|^\alpha,$$

using (3.4) and recalling Definition 3.4 we obtain

$$\frac{|\delta^s(x)w(x) - \delta^s(y)w(y)|}{|x - y|^\alpha} \leq C \quad \text{whenever } y \in B_R(x), \ R = \delta(x)/2.$$

This bound can be extended to all $x, y \in \Omega$, since the domain is regular, by using a dyadic chain of balls; see for instance the proof of Proposition 1.1 in [18].

The second bound, that is,

$$\|I_{s/2}(v, \delta_0^s)\|_{C^\alpha(\mathbb{R}^n)} \leq C,$$

follows from assumption (b) and Lemma 3.3 (taking a smaller α if necessary). $\square$

To prove Proposition 3.2 we need some preliminaries.

Fixed $\rho_0 > 0$, define $\phi \in C^s(\mathbb{R})$ by

$$\phi(x) = x^s \chi_{(0,\rho_0)}(x) + \rho_0^s \chi_{(\rho_0,+\infty)}(x). \tag{3.5}$$

This function ϕ is a truncation of the s-harmonic function x_+^s. We need to introduce ϕ because the growth at infinity of x_+^s prevents us from computing its $(-\Delta)^{s/2}$.

Lemma 3.7. Let $\rho_0 > 0$, and let $\phi : \mathbb{R} \to \mathbb{R}$ be given by (3.5). Then, we have

$$(-\Delta)^{s/2}\phi(x) = c_1\{\log|x| + c_2\chi_{(0,\infty)}(x)\} + h(x)$$

for $x \in (-\rho_0/2, \rho_0/2)$, where $h \in C^s([-\rho_0/2, \rho_0/2])$. The constants c_1 and c_2 are given by

$$c_1 = c_{1,\frac{s}{2}} \qquad \text{and} \qquad c_2 = \int_0^\infty \left\{ \frac{1 - z^s}{|1 - z|^{1+s}} + \frac{1 + z^s}{|1 + z|^{1+s}} \right\} dz,$$

where $c_{n,s}$ is the constant appearing in the singular integral expression (1.2) for $(-\Delta)^s$ in dimension n.

Proof. If $x < \rho_0$,

$$(-\Delta)^{s/2}\phi(x) = c_{1,\frac{s}{2}} \left(\int_{-\infty}^{\rho_0} \frac{x_+^s - y_+^s}{|x - y|^{1+s}} \, dy + \int_{\rho_0}^\infty \frac{x_+^s - \rho_0^s}{|x - y|^{1+s}} \, dy \right).$$

We need to study the first integral:

$$J(x) = \int_{-\infty}^{\rho_0} \frac{x_+^s - y_+^s}{|x - y|^{1+s}} \, dy = \begin{cases} J_1(x) = \displaystyle\int_{-\infty}^{\rho_0/x} \frac{1 - z_+^s}{|1 - z|^{1+s}} \, dz & \text{if } x > 0 \\[20pt] J_2(x) = \displaystyle\int_{-\infty}^{\rho_0/|x|} \frac{-z_+^s}{|1 + z|^{1+s}} \, dz & \text{if } x < 0 \,, \end{cases} \tag{3.6}$$

since

$$(-\Delta)^{s/2}\phi(x) - c_1 J(x) = c_1 \int_{\rho_0}^\infty \frac{x_+^s - \rho_0^s}{|x - y|^{1+s}} \, dy \tag{3.7}$$

belongs to $C^s([-\rho_0/2, \rho_0/2])$ as a function of x.

Using L'Hôpital's rule we find that

$$\lim_{x \downarrow 0} \frac{J_1(x)}{\log|x|} = \lim_{x \uparrow 0} \frac{J_2(x)}{\log|x|} = 1.$$

Moreover,

$$\lim_{x\downarrow 0} x^{1-s}\left(J_1'(x) - \frac{1}{x}\right) = \lim_{x\downarrow 0} x^{1-s}\left(-\frac{\rho_0}{x^2}\frac{1-(\rho_0/x)^s}{((\rho_0/x)-1)^{1+s}} - \frac{1}{x}\right)$$

$$= \rho_0^{-s}\lim_{y\downarrow 0} y^{1-s}\left(\frac{1-y^s}{y(1-y)^{1+s}} - \frac{(1-y)^{1+s}}{y(1-y)^{1+s}}\right)$$

$$= \rho_0^{-s}\lim_{y\downarrow 0}\frac{1-y^s-(1-y)^{1+s}}{y^s} = -\rho_0^{-s}$$

and

$$\lim_{x\uparrow 0}(-x)^{1-s}\left(J_2'(x) - \frac{1}{x}\right) = \lim_{x\uparrow 0}(-x)^{1-s}\left(\frac{\rho_0}{x^2}\frac{-(-\rho_0/x)^s}{(1+(-\rho_0/x))^{1+s}} - \frac{1}{-x}\right)$$

$$= \rho_0^{-s}\lim_{y\downarrow 0} y^{1-s}\left(\frac{-1}{y(1+y)^{1+s}} + \frac{(1+y)^{1+s}}{y(1+y)^{1+s}}\right)$$

$$= \rho_0^{-s}\lim_{y\downarrow 0}\frac{(1+y)^{1+s}-1}{y^s} = 0.$$

Therefore,

$$(J_1(x) - \log|x|)' \leq C|x|^{s-1} \quad \text{in } (0, \rho_0/2]$$

and

$$(J_2(x) - \log|x|)' \leq C|x|^{s-1} \quad \text{in } [-\rho_0/2, 0),$$

and these gradient bounds yield

$$(J_1 - \log|\cdot|) \in C^s([0, \rho_0/2]) \quad \text{and} \quad (J_2 - \log|\cdot|) \in C^s([-\rho_0/2, 0]).$$

However, these two Hölder functions do not have the same value at 0. Indeed,

$$\lim_{x\downarrow 0}\left\{(J_1(x) - \log|x|) - (J_2(-x) - \log|-x|)\right\} = \lim_{x\downarrow 0}\left\{J_1(x) - J_2(-x)\right\}$$

$$= \int_{-\infty}^{\infty}\left\{\frac{1-z_+^s}{|1-z|^{1+s}} + \frac{z_+^s}{|1+z|^{1+s}}\right\} dz$$

$$= \int_0^{\infty}\left\{\frac{1-z^s}{|1-z|^{1+s}} + \frac{1+z^s}{|1+z|^{1+s}}\right\} dz = c_2.$$

Hence, the function $J(x) - \log|x| - c_2\chi_{(0,\infty)}(x)$, where J is defined by (3.6), is $C^s([-\rho_0/2, \rho_0/2])$. Recalling (3.7), we obtain the result. □

Next lemma will be used to prove Proposition 3.2. Before stating it, we need the following

Remark 3.8. From now on in this section, $\rho_0 > 0$ is a small constant depending only on Ω, which we assume to be a bounded $C^{1,1}$ domain. Namely, we assume that that every point on $\partial\Omega$ can be touched from both inside and outside Ω by balls of radius ρ_0. In other words, given $x_0 \in \partial\Omega$, there are balls of radius ρ_0, $B_{\rho_0}(x_1) \subset \Omega$ and $B_{\rho_0}(x_2) \subset \mathbb{R}^n\backslash\Omega$, such that $\overline{B_{\rho_0}(x_1)}\cap\overline{B_{\rho_0}(x_2)} = \{x_0\}$. A useful observation is that all points y in the segment that joins x_1 and x_2 —through x_0— satisfy $\delta(y) = |y - x_0|$.

Lemma 3.9. *Let Ω be a bounded $C^{1,1}$ domain, $\delta(x) = \text{dist}(x, \partial\Omega)$, $\delta_0 = \delta\chi_\Omega$, and ρ_0 be given by Remark 3.8. Fix $x_0 \in \partial\Omega$, and define*

$$\phi_{x_0}(x) = \phi\left(-\nu(x_0) \cdot (x - x_0)\right)$$

and

$$S_{x_0} = \{x_0 + t\nu(x_0), \ t \in (-\rho_0/2, \rho_0/2)\}, \tag{3.8}$$

where ϕ is given by (3.5) and $\nu(x_0)$ is the unit outward normal to $\partial\Omega$ at x_0. Define also $w_{x_0} = \delta_0^s - \phi_{x_0}$.

Then, for all $x \in S_{x_0}$,

$$|(-\Delta)^{s/2}w_{x_0}(x) - (-\Delta)^{s/2}w_{x_0}(x_0)| \leq C|x - x_0|^{s/2},$$

where C depends only on Ω and ρ_0 (and not on x_0).

Proof. We denote $w = w_{x_0}$. Note that, along S_{x_0}, the distance to $\partial\Omega$ agrees with the distance to the tangent plane to $\partial\Omega$ at x_0; see Remark 3.8. That is, denoting $\delta_\pm = (\chi_\Omega - \chi_{\mathbb{R}^n\setminus\Omega})\delta$ and $d(x) = -\nu(x_0) \cdot (x - x_0)$, we have $\delta_\pm(x) = d(x)$ for all $x \in S_{x_0}$. Moreover, the gradients of these two functions also coincide on S_{x_0}, i.e., $\nabla\delta_\pm(x) = -\nu(x_0) = \nabla d(x)$ for all $x \in S_{x_0}$.

Therefore, for all $x \in S_{x_0}$ and $y \in B_{\rho_0/2}(0)$, we have

$$|\delta_\pm(x + y) - d(x + y)| \leq C|y|^2$$

for some C depending only on ρ_0. Thus, for all $x \in S_{x_0}$ and $y \in B_{\rho_0/2}(0)$,

$$|w(x + y)| = |(\delta_\pm(x + y))_+^s - (d(x + y))_+^s| \leq C|y|^{2s}, \tag{3.9}$$

where C is a constant depending on Ω and s.

On the other hand, since $w \in C^s(\mathbb{R}^n)$, then

$$|w(x + y) - w(x_0 + y)| \leq C|x - x_0|^s. \tag{3.10}$$

Finally, let $r < \rho_0/2$ to be chosen later. For each $x \in S_{x_0}$ we have

$$|(-\Delta)^{s/2}w(x) - (-\Delta)^{s/2}w(x_0)| \leq C\int_{\mathbb{R}^n} \frac{|w(x + y) - w(x_0 + y)|}{|y|^{n+s}}\,dy$$

$$\leq C\int_{B_r} \frac{|w(x + y) - w(x_0 + y)|}{|y|^{n+s}}\,dy + C\int_{\mathbb{R}^n\setminus B_r} \frac{|w(x + y) - w(x_0 + y)|}{|y|^{n+s}}\,dy$$

$$\leq C\int_{B_r} \frac{|y|^{2s}}{|y|^{n+s}}\,dy + C\int_{\mathbb{R}^n\setminus B_r} \frac{|x - x_0|^s}{|y|^{n+s}}\,dy$$

$$= C(r^s + |x - x_0|^s r^{-s}),$$

where we have used (3.9) and (3.10). Taking $r = |x - x_0|^{1/2}$ the lemma is proved. $\square$

The following is the last ingredient needed to prove Proposition 3.2.

Claim 3.10. *Let Ω be a bounded $C^{1,1}$ domain, and ρ_0 be given by Remark 3.8. Let w be a function satisfying, for some $K > 0$,*

(i) w is locally Lipschitz in $\{x \in \mathbb{R}^n : 0 < \delta(x) < \rho_0\}$ and

$$|\nabla w(x)| \leq K\delta(x)^{-M} \quad in \quad \{x \in \mathbb{R}^n : 0 < \delta(x) < \rho_0\}$$

for some $M > 0$.

(ii) There exists $\alpha > 0$ such that

$$|w(x) - w(x^*)| \leq K\delta(x)^\alpha \quad in \quad \{x \in \mathbb{R}^n : 0 < \delta(x) < \rho_0\},$$

where x^* is the unique point on $\partial\Omega$ satisfying $\delta(x) = |x - x^*|$.

(iii) For the same α, it holds

$$\|w\|_{C^\alpha(\{\delta \geq \rho_0\})} \leq K.$$

Then, there exists $\gamma > 0$, depending only on α and M, such that

$$\|w\|_{C^\gamma(\mathbb{R}^n)} \leq CK, \tag{3.11}$$

where C depends only on Ω.

Proof. First note that from (ii) and (iii) we deduce that $\|w\|_{L^\infty(\mathbb{R}^n)} \leq CK$. Let $\rho_1 \leq \rho_0$ be a small positive constant to be chosen later. Let $x, y \in \{\delta \leq \rho_0\}$, and $r = |x - y|$.

If $r \geq \rho_1$, then

$$\frac{|w(x) - w(y)|}{|x - y|^\gamma} \leq \frac{2\|w\|_{L^\infty(\mathbb{R}^n)}}{\rho_1^\gamma} \leq CK.$$

If $r < \rho_1$, consider

$$x' = x^* + \rho_0 r^\beta \nu(x^*) \quad and \quad y' = y^* + \rho_0 r^\beta \nu(y^*),$$

where $\beta \in (0, 1)$ is to be determined later. Choose ρ_1 small enough so that the segment joining x' and y' contained in the set $\{\delta > \rho_0 r^\beta / 2\}$. Then, by (i),

$$|w(x') - w(y')| \leq CK(\rho_0 r^\beta / 2)^{-M}|x' - y'| \leq Cr^{1-\beta M}. \tag{3.12}$$

Thus, using (ii) and (3.12),

$$|w(x) - w(y)| \leq |w(x) - w(x^*)| + |w(x^*) - w(x')| +$$
$$+ |w(y) - w(y^*)| + |w(y^*) - w(y')| + |w(x') - w(y')|$$
$$\leq K\delta(x)^\alpha + K\delta(y)^\alpha + 2K(\rho_0 r^\beta)^\alpha + CKr^{1-\beta M}.$$

Taking $\beta < 1/M$ and $\gamma = \min\{\alpha\beta, 1 - \beta M\}$, we find

$$|w(x) - w(y)| \leq CKr^\gamma = CK|x - y|^\gamma.$$

This proves

$$[w]_{C^\gamma(\{\delta \leq \rho_0\})} \leq CK.$$

To obtain the bound (3.11) we combine the previous seminorm estimate with (iii). $\qquad\square$

Finally, we give the proof of Proposition 3.2.

Proof of Proposition 3.2. Let

$$h(x) = (-\Delta)^{s/2}\delta_0^s(x) - c_1\left\{\log^- \delta(x) + c_2\chi_\Omega(x)\right\}.$$

We want to prove that $h \in C^\alpha(\mathbb{R}^n)$ by using Claim 3.10.

On one hand, by Lemma 3.7, for all $x_0 \in \partial\Omega$ and for all $x \in S_{x_0}$, where S_{x_0} is defined by (3.8), we have

$$h(x) = (-\Delta)^{s/2}\delta_0^s(x) - (-\Delta)^{s/2}\phi_{x_0}(x) + \tilde{h}\big(\nu(x_0) \cdot (x - x_0)\big),$$

where $\tilde{h}$ is the $C^s([-\rho_0/2, \rho_0/2])$ function from Lemma 3.7. Hence, using Lemma 3.9, we find

$$|h(x) - h(x_0)| \leq C|x - x_0|^{s/2} \quad \text{for all } x \in S_{x_0}$$

for some constant independent of x_0.

Recall that for all $x \in S_{x_0}$ we have $x^* = x_0$, where x^* is the unique point on $\partial\Omega$ satisfying $\delta(x) = |x - x^*|$. Hence,

$$|h(x) - h(x^*)| \leq C|x - x^*|^{s/2} \quad \text{for all } x \in \{\delta < \rho_0/2\}. \tag{3.13}$$

Moreover,

$$\|h\|_{C^\alpha(\{\delta \geq \rho_0/2\})} \leq C \tag{3.14}$$

for all $\alpha \in (0, 1 - s)$, where C is a constant depending only on α, Ω and ρ_0. This last bound is found using that $\|\delta_0^s\|_{C^{0,1}(\{\delta \geq \rho_0/2\})} \leq C$, which yields

$$\|(-\Delta)^{s/2}\delta_0^s\|_{C^\alpha(\{\delta \geq \rho_0\})} \leq C$$

for $\alpha < 1 - s$.

On the other hand, we claim now that if $x \notin \partial\Omega$ and $\delta(x) < \rho_0/2$, then

$$|\nabla h(x)| \leq |\nabla(-\Delta)^{s/2}\delta_0^s(x)| + c_1|\delta(x)|^{-1} \leq C|\delta(x)|^{-n-s}. \tag{3.15}$$

Indeed, observe that $\delta_0^s \equiv 0$ in $\mathbb{R}^n\backslash\Omega$, $|\nabla\delta_0^s| \leq C\delta_0^{s-1}$ in Ω, and $|D^2\delta_0^s| \leq C\delta_0^{s-2}$ in Ω_{ρ_0}. Then, $r = \delta(x)/2$,

$$\begin{aligned}
|(-\Delta)^{s/2}\nabla\delta_0^s(x)| &\leq C\int_{\mathbb{R}^n} \frac{|\nabla\delta_0^s(x) - \nabla\delta_0^s(x+y)|}{|y|^{n+s}}\, dy \\
&\leq C\int_{B_r} \frac{Cr^{s-2}|y|\, dy}{|y|^{n+s}} + C\int_{\mathbb{R}^n\backslash B_r}\left(\frac{|\nabla\delta_0^s(x)|}{|y|^{n+s}} + \frac{|\nabla\delta_0^s(x+y)|}{r^{n+s}}\right) dy \\
&\leq \frac{C}{r} + \frac{C}{r} + \frac{C}{r^{n+s}}\int_{\mathbb{R}^n}\delta_0^{s-1} \leq \frac{C}{r^{n+s}},
\end{aligned}$$

as claimed.

To conclude the proof, we use bounds (3.13), (3.14), and (3.15) and Claim 3.10. $\square$

To end this section, we give the

Proof of Proposition 1.10. The first part follows from Propositions 3.1 and 3.2. The second part follows from Lemma 3.5 with $\alpha = s$ and $\beta \in (s, 1 + 2s)$. $\square$

$$4. \text{ THE OPERATOR } -\frac{d}{d\lambda}\Big|_{\lambda=1^+} \int_{\mathbb{R}} w_\lambda w_{1/\lambda}$$

The aim of this section is to prove Proposition 1.11. In other words, we want to evaluate the operator

$$\mathfrak{I}(w) = -\frac{d}{d\lambda}\Big|_{\lambda=1^+} \int_0^\infty w\,(\lambda t)\,w\left(\frac{t}{\lambda}\right) dt \tag{4.1}$$

on

$$w(t) = A \log^- |t-1| + B\chi_{[0,1]}(t) + h(t),$$

where $\log^- t = \min\{\log t, 0\}$, A and B are real numbers, and h is a function satisfying, for some constants $\alpha \in (0,1)$, $\gamma \in (0,1)$, and C_0, the following conditions:

(i) $\|h\|_{C^\alpha((0,\infty))} \le C_0$.
(ii) For all $\beta \in [\gamma, 1+\gamma]$,

$$\|h\|_{C^\beta((0,1-\rho)\cup(1+\rho,2))} \le C_0\rho^{-\beta} \qquad \text{for all } \rho \in (0,1).$$

(iii) $|h'(t)| \le Ct^{-2-\gamma}$ and $|h''(t)| \le Ct^{-3-\gamma}$ for all $t > 2$.

We will split the proof of Proposition 1.11 into three parts. The first part is the following, and evaluates the operator $\mathfrak{I}$ on the function

$$w_0(t) = A \log^- |t-1| + B\chi_{[0,1]}(t). \tag{4.2}$$

Lemma 4.1. *Let w_0 and $\mathfrak{I}$ be given by (4.2) and (4.1), respectively. Then,*

$$\mathfrak{I}(w_0) = A^2\pi^2 + B^2.$$

The second result towards Proposition 1.11 is the following.

Lemma 4.2. *Let h be a function satisfying* (i), (ii), *and* (iii) *above, and $\mathfrak{I}$ be given by (4.1). Then,*

$$\mathfrak{I}(h) = 0.$$

Moreover, there exist constants C and $\nu > 1$, depending only on the constants α, γ, and C_0 appearing in (i)-(ii)-(iii), *such that*

$$\left| \int_0^\infty \left\{ h\,(\lambda t)\,h\left(\frac{t}{\lambda}\right) - h(t)^2 \right\} dt \right| \le C|\lambda - 1|^\nu$$

for each $\lambda \in (1, 3/2)$.

Finally, the third one states that $\mathfrak{I}(w_0 + h) = \mathfrak{I}(w_0)$ whenever $\mathfrak{I}(h) = 0$.

Lemma 4.3. *Let w_1 and w_2 be $L^2(\mathbb{R})$ functions. Assume that the derivative at $\lambda = 1^+$ in the expression $\mathfrak{I}(w_1)$ exists, and that*

$$\mathfrak{I}(w_2) = 0.$$

Then,

$$\mathfrak{I}(w_1 + w_2) = \mathfrak{I}(w_1).$$

Let us now give the proofs of Lemmas 4.1, 4.2, and 4.3. We start proving Lemma 4.3. For it, is useful to introduce the bilinear form

$$(w_1, w_2) = -\frac{1}{2} \frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_0^\infty \left\{ w_1(\lambda t) w_2\left(\frac{t}{\lambda}\right) + w_1\left(\frac{t}{\lambda}\right) w_2(\lambda t) \right\} dt,$$

and more generally, the bilinear forms

$$(w_1, w_2)_\lambda = -\frac{1}{2(\lambda-1)} \int_0^\infty \left\{ w_1(\lambda t) w_2\left(\frac{t}{\lambda}\right) + w_1\left(\frac{t}{\lambda}\right) w_2(\lambda t) - 2w_1(t)w_2(t) \right\} dt, \tag{4.3}$$

for $\lambda > 1$.

It is clear that $\lim_{\lambda\downarrow 1}(w_1, w_2)_\lambda = (w_1, w_2)$ whenever the limit exists, and that $(w, w) = \mathfrak{I}(w)$. The following lemma shows that these bilinear forms are positive definite and, thus, they satisfy the Cauchy-Schwarz inequality.

Lemma 4.4. *The following properties hold.*
   (a) $(w_1, w_2)_\lambda$ *is a bilinear map.*
   (b) $(w, w)_\lambda \geq 0$ *for all $w \in L^2(\mathbb{R}_+)$.*
   (c) $|(w_1, w_2)_\lambda|^2 \leq (w_1, w_1)_\lambda (w_2, w_2)_\lambda$.

Proof. Part (a) is immediate. Part (b) follows from the Hölder inequality

$$\|w_\lambda w_{1/\lambda}\|_{L^1} \leq \|w_\lambda\|_{L^2} \|w_{1/\lambda}\|_{L^2} = \|w\|_{L^2}^2,$$

where $w_\lambda(t) = w(\lambda t)$. Part (c) is a consequence of (a) and (b). $\qquad\square$

Now, Lemma 4.3 is an immediate consequence of this Cauchy-Schwarz inequality.

Proof of Lemma 4.3. By Lemma 4.4 (iii) we have

$$0 \leq |(w_1, w_2)_\lambda| \leq \sqrt{(w_1, w_1)_\lambda}\sqrt{(w_2, w_2)_\lambda} \longrightarrow 0.$$

Thus, $(w_1, w_2) = \lim_{\lambda\downarrow 1}(w_1, w_2)_\lambda = 0$ and

$$\mathfrak{I}(w_1 + w_2) = \mathfrak{I}(w_1) + \mathfrak{I}(w_2) + 2(w_1, w_2) = \mathfrak{I}(w_1).$$

$\qquad\square$

Next we prove that $\mathfrak{I}(h) = 0$. For this, we will need a preliminary lemma.

Lemma 4.5. *Let h be a function satisfying* (i), (ii), *and* (iii) *in Propostion 1.11, $\lambda \in (1, 3/2)$, and $\tau \in (0, 1)$ be such that $\tau/2 > \lambda - 1$. Let α, γ, and C_0 be the constants appearing in* (i)-(ii)-(iii). *Then,*

$$\left| h(\lambda t) h\left(\frac{t}{\lambda}\right) - h(t)^2 \right| \leq \begin{cases} C\max\left\{|t - \lambda|^\alpha, |t - 1/\lambda|^\alpha\right\} & t \in (1-\tau, 1+\tau) \\ C(\lambda-1)^{1+\gamma}|t-1|^{-1-\gamma} & t \in (0, 1-\tau) \cup (1+\tau, 2) \\ C(\lambda-1)^2 t^{-1-\gamma} & t \in (2, \infty), \end{cases}$$

where the constant C depends only on C_0.

Proof. Let $t \in (1 - \tau, 1 + \tau)$. Let us denote $\widetilde{h} = h - h(1)$. Then,

$$h\left(\lambda t\right) h\left(\frac{t}{\lambda}\right) - h(t)^2 = \widetilde{h}\left(\lambda t\right) \widetilde{h}\left(\frac{t}{\lambda}\right) - \widetilde{h}(t)^2 + h(1)\left(\widetilde{h}\left(\lambda t\right) + \widetilde{h}\left(\frac{t}{\lambda}\right) - 2\widetilde{h}(t)\right).$$

Therefore, using that $|\widetilde{h}(t)| \leq C_0|t - 1|^\alpha$ and $\|h\|_{L^\infty(\mathbb{R})} \leq C_0$, we obtain

$$
\begin{aligned}
\left| h\left(\lambda t\right) h\left(\frac{t}{\lambda}\right) - h(t)^2 \right| \;\leq\;& C\,|\lambda t - 1|^\alpha \left|\frac{t}{\lambda} - 1\right|^\alpha + C|t - 1|^{2\alpha} + C|\lambda t - 1|^\alpha + \\
& + C\left|\frac{t}{\lambda} - 1\right|^\alpha + C|t - 1|^\alpha \\
\leq\;& C\max\left\{|t - \lambda|^\alpha, \left|t - \frac{1}{\lambda}\right|^\alpha\right\}.
\end{aligned}
$$

Let now $t \in (0, 1 - \tau) \cup (1 + \tau, 2)$ and recall that $\lambda \in (1, 1 + \tau/2)$. Define, for $\mu \in [1, \lambda]$,

$$\psi(\mu) = h\left(\mu t\right) h\left(\frac{t}{\mu}\right) - h(t)^2.$$

By the mean value theorem, $\psi(\lambda) = \psi(1) + \psi'(\mu)(\lambda - 1)$ for some $\mu \in (1, \lambda)$. Moreover, observing that $\psi(1) = \psi'(1) = 0$, we deduce

$$|\psi(\lambda)| \leq (\lambda - 1)|\psi'(\mu) - \psi'(1)|.$$

Next we claim that

$$|\psi'(\mu) - \psi'(1)| \leq C|\mu - 1|^\gamma|t - 1|^{-1-\gamma}. \tag{4.4}$$

This yields the desired bound for $t \in (0, 1 - \tau) \cup (1 + \tau, 2)$.

To prove this claim, note that

$$\psi'(\mu) = th'\left(\mu t\right) h\left(\frac{t}{\mu}\right) - \frac{t}{\mu^2} h\left(\mu t\right) h'\left(\frac{t}{\mu}\right).$$

Thus, using the bounds from (ii) with β replaced by γ, 1, and $1 + \gamma$,

$$
\begin{aligned}
|\psi'(\mu) - \psi'(1)| \leq\;& t|h'(\mu t) - h'(t)|\left|h\left(\frac{t}{\mu}\right)\right| + t\left|h\left(\frac{t}{\mu}\right) - h(t)\right||h'(t)| + \\
& + t\left|h'\left(\frac{t}{\mu}\right) - h'(t)\right|\frac{|h(\mu t)|}{\mu^2} + t\left|\frac{h(\mu t)}{\mu^2} - h(t)\right||h'(t)| \\
\leq\;& C|\mu t - t|^\gamma m^{-1-\gamma} + C\left|\frac{t}{\mu} - t\right|^\gamma m^{-\gamma}|t - 1|^{-1} + \frac{C}{\mu^2}\left|\frac{t}{\mu} - t\right|^\gamma m^{-1-\gamma} + \\
& + \frac{C}{\mu^2}|\mu t - t|^\gamma m^{-\gamma}|t - 1|^{-1} + C(\mu - 1)|t - 1|^{-1} \\
\leq\;& C(\mu - 1)^\gamma m^{-1-\gamma},
\end{aligned}
$$

where $m = \min\{|\mu t - 1|, |t - 1|, |t/\mu - 1|\}$.

Furthermore, since $\mu - 1 < |t - 1|/2$, we have $m \geq \frac{1}{4}|t - 1|$, and hence (4.4) follows.

Finally, if $t \in (2, \infty)$, with a similar argument but using the bound (iii) instead of (ii), we obtain

$$|\psi(\lambda)| \leq C(\lambda - 1)^2 t^{-1-\gamma},$$

and we are done. $\qquad\square$

Let us now give the

Proof of Lemma 4.2. Let us call

$$I_\lambda = \int_0^\infty \left\{ h\left(\lambda t\right) h\left(\frac{t}{\lambda}\right) - h(t)^2 \right\} dx.$$

For each $\lambda \in (1, 3/2)$, take $\tau \in (0, 1)$ such that $\lambda - 1 < \tau/2$ to be chosen later. Then, by Lemma 4.5,

$$
\begin{aligned}
|I_\lambda| \quad \leq \quad & C(\lambda - 1)^{1+\gamma} \int_0^{1-\tau} |t - 1|^{-1-\gamma} dt + C \int_{1-\tau}^1 |t - \lambda|^\alpha \, dt + \\
& + C \int_1^{1+\tau} \left| t - \frac{1}{\lambda} \right|^\alpha dt + C(\lambda - 1)^{1+\gamma} \int_{1+\tau}^2 |t - 1|^{-1-\gamma} dt + \\
& + C(\lambda - 1)^2 \int_2^\infty t^{-1-s} dt \\
\leq \quad & C(\lambda - 1)^{1+\gamma} \tau^{-\gamma} + C \left(\tau + \lambda - 1 \right)^{\alpha+1} + C(\lambda - 1)^{1+\gamma} \tau^{-\gamma} + \\
& + C \left(\tau + 1 - \frac{1}{\lambda} \right)^{\alpha+1} + C(\lambda - 1)^2.
\end{aligned}
$$

Choose now

$$\tau = (\lambda - 1)^\theta,$$

with $\theta < 1$ to be chosen later. Then,

$$\tau + \lambda - 1 \leq 2\tau \qquad \text{and} \qquad \tau + 1 - \frac{1}{\lambda} \leq 2\tau,$$

and hence

$$|I_\lambda| \leq C(\lambda - 1)^{(\alpha+1)\theta} + C(\lambda - 1)^{1+\gamma-\theta\gamma} + C(\lambda - 1)^2.$$

Finally, choose θ such that $(\alpha + 1)\theta > 1$ and $1 + \gamma - \theta\gamma > 1$, that is, satisfying

$$\frac{1}{1+\alpha} < \theta < 1.$$

Then, for $\nu = \min\{(\alpha + 1)\theta, 1 + \gamma - \gamma\theta\} > 1$, it holds

$$\left| \int_0^\infty \left\{ h\left(\lambda t\right) h\left(\frac{t}{\lambda}\right) - h(t)^2 \right\} dt \right| \leq C|\lambda - 1|^\nu,$$

as desired. $\qquad\square$

Next we prove Lemma 4.1.

Proof of Lemma 4.1. Let

$$w_1(t) = \log^- |t - 1| \qquad \text{and} \qquad w_2(t) = \chi_{[0,1]}(t).$$

We will compute first $\Im(w_1)$.

Define

$$\Psi(t) = \int_0^t \frac{\log |r - 1|}{r} dr.$$

It is straightforward to check that, if $\lambda > 1$, the function

$$
\begin{aligned}
\vartheta_\lambda(t) &= \left(t - \frac{1}{\lambda}\right) \log |\lambda t - 1| \log \left|\frac{t}{\lambda} - 1\right| + (\lambda - t) \log \left|\frac{t}{\lambda} - 1\right| \\
&\quad - \frac{\lambda^2 - 1}{\lambda} \log(\lambda^2 - 1) \log \left|\frac{t}{\lambda} - 1\right| - \frac{\lambda^2 - 1}{\lambda} \Psi \left(\frac{\lambda(\lambda - t)}{\lambda^2 - 1}\right) \\
&\quad + 2t - \frac{\lambda t - 1}{\lambda} \log |\lambda t - 1|
\end{aligned}
$$

is a primitive of $\log |\lambda t - 1| \log \left|\frac{t}{\lambda} - 1\right|$. Denoting $I_\lambda = \int_0^\infty w_1(\lambda t) w_1 \left(\frac{t}{\lambda}\right) dt$, we have

$$
\begin{aligned}
I_\lambda - I_1 &= \int_0^{\frac{2}{\lambda}} \log |\lambda t - 1| \log \left|\frac{t}{\lambda} - 1\right| dt - \int_0^2 \log^2 |t - 1| dt \\
&= \vartheta_\lambda \left(\frac{2}{\lambda}\right) - \vartheta_\lambda(0) - 4 \\
&= \left(\frac{\lambda^2 - 1}{\lambda}\right) \left\{ \Psi \left(\frac{\lambda^2}{\lambda^2 - 1}\right) - \Psi \left(\frac{\lambda^2 - 2}{\lambda^2 - 1}\right) \right\} + \left(\lambda - \frac{2}{\lambda}\right) \log \left(\frac{2}{\lambda^2} - 1\right) + \\
&\quad + \left(\lambda - \frac{1}{\lambda}\right) \log(\lambda^2 - 1) \log \left(\frac{2}{\lambda^2} - 1\right) - \frac{4(\lambda - 1)}{\lambda},
\end{aligned}
$$

where we have used that

$$I_1 = \int_0^2 \log^2 |t - 1| dt = 2 \int_0^1 \log^2 t' dt' = 2 \int_0^\infty r^2 e^{-r} dr = 2\Gamma(3) = 4.$$

Therefore, dividing by $\lambda - 1$ and letting $\lambda \downarrow 1$,

$$
\begin{aligned}
\frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda &= 2 \lim_{\lambda \downarrow 1} \int_{\frac{\lambda^2-2}{\lambda^2-1}}^{\frac{\lambda^2}{\lambda^2-1}} \frac{\log |t - 1|}{t} dt + \\
&\quad + \lim_{\lambda \downarrow 1} \left\{ 2\log(\lambda^2 - 1) \log \left(\frac{2}{\lambda^2} - 1\right) - \frac{\log \left(\frac{2}{\lambda^2} - 1\right)}{\lambda - 1} - \frac{4}{\lambda} \right\}.
\end{aligned}
$$

The first term equals to

$$\lim_{M \to +\infty} \int_{-M}^M \frac{2 \log |t - 1|}{t} dt,$$

while the second, using that $\log(1+x) \sim x$ for $x \sim 0$, equals to

$$\lim_{\lambda \downarrow 1} \left\{ 2\log(\lambda^2 - 1)\left(\frac{2}{\lambda^2} - 2\right) - \frac{\frac{2}{\lambda^2} - 2}{\lambda - 1} - \frac{4}{\lambda} \right\} = 0 + 4 - 4 = 0.$$

Hence,

$$
\begin{aligned}
\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda &= \lim_{M\to+\infty} \int_{-M}^{M} \frac{2\log|t-1|}{t}\,dt = \lim_{M\to+\infty} \int_{-M}^{M} \frac{2\log|t|}{t+1}\,dt \\
&= \lim_{M\to+\infty} \left\{ \int_{-M}^{0} \frac{2\log(-t)}{t+1}\,dt + \int_{0}^{M} \frac{2\log t}{t+1}\,dt \right\} \\
&= \lim_{M\to+\infty} \left\{ \int_{0}^{M} \frac{2\log t}{1-t}\,dt + \int_{0}^{M} \frac{2\log t}{t+1}\,dt \right\} = \int_{0}^{+\infty} \frac{4\log t}{1-t^2}\,dt \\
&= \int_{0}^{1} \frac{4\log t}{1-t^2}\,dt + \int_{1}^{+\infty} \frac{-4\log\frac{1}{t}}{\frac{1}{t^2}-1}\frac{dt}{t^2} = 2\int_{0}^{1} \frac{4\log t}{1-t^2}\,dt.
\end{aligned}
$$

Furthermore, using that $\frac{1}{1-t^2} = \sum_{n\geq 0} t^{2n}$ and that

$$\int_{0}^{1} t^n \log t\,dt = -\int_{0}^{1} \frac{t^{n+1}}{n+1}\frac{1}{t}\,dt = -\frac{1}{(n+1)^2},$$

we obtain

$$\int_{0}^{1} \frac{\log t}{1-t^2}\,dt = -\sum_{n\geq 0} \frac{1}{(2n+1)^2} = -\frac{\pi^2}{8},$$

and thus

$$\Im(w_1) = -\frac{d}{d\lambda}\Big|_{\lambda=1^+} I_\lambda = \pi^2.$$

Let us evaluate now $\Im(w_2) = \Im(\chi_{[0,1]})$. We have

$$\int_{0}^{+\infty} \chi_{[0,1]}(\lambda t)\,\chi_{[0,1]}\left(\frac{t}{\lambda}\right)\,dt = \int_{0}^{\frac{1}{\lambda}} dt = \frac{1}{\lambda}.$$

Therefore, differentiating with respect to λ we obtain $\Im(w_2) = 1$.

Let us finally prove that $(w_1, w_2) = 0$, i.e., that

$$\frac{d}{d\lambda}\Big|_{\lambda=1^+} \left\{ \int_{0}^{\lambda} \log|1-\lambda t|dt + \int_{0}^{\frac{1}{\lambda}} \log\left|1 - \frac{t}{\lambda}\right|dt \right\} = 0. \tag{4.5}$$

We have

$$
\begin{aligned}
\int_{0}^{\lambda} \log|1-\lambda t|dt &= \frac{1}{\lambda}\left[(\lambda t - 1)\log|1-\lambda t| - \lambda t\right]_0^{\lambda} \\
&= \left(\lambda - \frac{1}{\lambda}\right)\log(\lambda^2 - 1) - \lambda,
\end{aligned}
$$

and similarly,

$$\int_0^{\frac{1}{\lambda}} \log\left|1 - \frac{t}{\lambda}\right| dt = \left(\frac{1}{\lambda} - \lambda\right) \log\left(1 - \frac{1}{\lambda^2}\right) - \frac{1}{\lambda}.$$

Thus,

$$\left|\int_0^\lambda \log|1 - \lambda t| dt + \int_0^{\frac{1}{\lambda}} \log\left|1 - \frac{t}{\lambda}\right| dt - 2\int_0^1 \log|1 - t| dt\right| =$$

$$= \left|\frac{2(\lambda^2 - 1)}{\lambda} \log\lambda - \frac{(\lambda - 1)^2}{\lambda}\right| \le 4(\lambda - 1)^2.$$

Therefore (4.5) holds, and the proposition is proved. □

Finally, to end this section, we give the:

Proof of Proposition 1.11. Let us write $\varphi = w_0 + h$, where w_0 is given by (4.2). Then, for each $\lambda > 1$ we have

$$(\varphi, \varphi)_\lambda = (w_0, w_0)_\lambda + 2(w_0, h)_\lambda + (h, h)_\lambda,$$

where $(\cdot, \cdot)_\lambda$ is defined by (4.3). Using Lemma 4.4 (c) and Lemma 4.2, we deduce

$$\left|(\varphi, \varphi)_\lambda - A^2\pi^2 - B^2\right| \le \left|(w_0, w_0)_\lambda - A^2\pi^2 - B^2\right| + C|\lambda - 1|^\nu.$$

The constants C and ν depend only on α, γ, and C_0, and by Lemma 4.1 the right hand side goes to 0 as $\lambda \downarrow 1$, since $(w_0, w_0)_\lambda \to \mathfrak{I}(w_0)$ as $\lambda \downarrow 1$. □

5. Proof of the Pohozaev identity in non-star-shaped domains

In this section we prove Proposition 1.6 for general $C^{1,1}$ domains. The key idea is that every $C^{1,1}$ domain is locally star-shaped, in the sense that its intersection with any small ball is star-shaped with respect to some point. To exploit this, we use a partition of unity to split the function u into a set of functions $u_1, ..., u_m$, each one with support in a small ball. However, note that the Pohozaev identity is quadratic in u, and hence we must introduce a bilinear version of this identity, namely

$$\int_\Omega (x \cdot \nabla u_1)(-\Delta)^s u_2\, dx + \int_\Omega (x \cdot \nabla u_2)(-\Delta)^s u_1\, dx = \frac{2s - n}{2}\int_\Omega u_1(-\Delta)^s u_2\, dx +$$

$$+ \frac{2s - n}{2}\int_\Omega u_2(-\Delta)^s u_1\, dx - \Gamma(1 + s)^2\int_{\partial\Omega} \frac{u_1}{\delta^s}\frac{u_2}{\delta^s}(x \cdot \nu)\, d\sigma. \tag{5.1}$$

The following lemma states that this bilinear identity holds whenever the two functions u_1 and u_2 have disjoint compact supports. In this case, the last term in the previous identity equals 0, and since $(-\Delta)^s u_i$ is evaluated only outside the support of u_i, we only need to require $\nabla u_i \in L^1(\mathbb{R}^n)$.

Lemma 5.1. *Let u_1 and u_2 be $W^{1,1}(\mathbb{R}^n)$ functions with disjoint compact supports K_1 and K_2. Then,*

$$\int_{K_1}(x \cdot \nabla u_1)(-\Delta)^s u_2\, dx + \int_{K_2}(x \cdot \nabla u_2)(-\Delta)^s u_1\, dx =$$

$$= \frac{2s-n}{2}\int_{K_1} u_1(-\Delta)^s u_2\, dx + \frac{2s-n}{2}\int_{K_2} u_2(-\Delta)^s u_1\, dx.$$

Proof. We claim that

$$(-\Delta)^s(x \cdot \nabla u_i) = x \cdot \nabla(-\Delta)^s u_i + 2s(-\Delta)^s u_i \qquad \text{in } \mathbb{R}^n \backslash K_i. \qquad (5.2)$$

Indeed, using $u_i \equiv 0$ in $\mathbb{R}^n \setminus K_i$ and the definition of $(-\Delta)^s$ in (1.2), for each $x \in \mathbb{R}^n \backslash K_i$ we have

$$(-\Delta)^s(x \cdot \nabla u_i)(x) = c_{n,s}\int_{K_i} \frac{-y \cdot \nabla u_i(y)}{|x-y|^{n+2s}}dy$$

$$= c_{n,s}\int_{K_i} \frac{(x-y)\cdot\nabla u_i(y)}{|x-y|^{n+2s}}dy + c_{n,s}\int_{K_i}\frac{-x\cdot\nabla u_i(y)}{|x-y|^{n+2s}}dy$$

$$= c_{n,s}\int_{K_i} \text{div}_y\left(\frac{x-y}{|x-y|^{n+2s}}\right)u_i(y)dy + x\cdot(-\Delta)^s\nabla u_i(x)$$

$$= c_{n,s}\int_{K_i} \frac{-2s}{|x-y|^{n+2s}}u_i(y)dy + x\cdot\nabla(-\Delta)^s u_i(x)$$

$$= 2s(-\Delta)^s u_i(x) + x\cdot\nabla(-\Delta)^s u_i(x),$$

as claimed.

We also note that for all functions w_1 and w_2 in $L^1(\mathbb{R}^n)$ with disjoint compact supports W_1 and W_2, it holds the integration by parts formula

$$\int_{W_1} w_1(-\Delta)^s w_2 = \int_{W_1}\int_{W_2}\frac{-w_1(x)w_2(y)}{|x-y|^{n+2s}}dy\, dx = \int_{W_2} w_2(-\Delta)^s w_1. \qquad (5.3)$$

Using that $(-\Delta)^s u_2$ is smooth in K_1 and integrating by parts,

$$\int_{K_1}(x\cdot\nabla u_1)(-\Delta)^s u_2 = -n\int_{K_1} u_1(-\Delta)^s u_2 - \int_{K_1} u_1 x\cdot\nabla(-\Delta)^s u_2.$$

Next we apply the previous claim and also the integration by parts formula (5.3) to $w_1 = u_1$ and $w_2 = x \cdot \nabla u_2$. We obtain

$$\int_{K_1} u_1 x\cdot\nabla(-\Delta)^s u_2 = \int_{K_1} u_1(-\Delta)^s(x\cdot\nabla u_2) - 2s\int_{K_1} u_1(-\Delta)^s u_2$$

$$= \int_{K_2}(-\Delta)^s u_1(x\cdot\nabla u_2) - 2s\int_{K_1} u_1(-\Delta)^s u_2.$$

Hence,

$$\int_{K_1}(x\cdot\nabla u_1)(-\Delta)^s u_2 = -\int_{K_2}(-\Delta)^s u_1(x\cdot\nabla u_2) + (2s-n)\int_{K_1} u_1(-\Delta)^s u_2.$$

Finally, again by the integration by parts formula (5.3) we find

$$\int_{K_1} u_1 (-\Delta)^s u_2 = \frac{1}{2} \int_{K_1} u_1 (-\Delta)^s u_2 + \frac{1}{2} \int_{K_2} u_2 (-\Delta)^s u_1,$$

and the lemma follows. $\square$

The second lemma states that the bilinear identity (5.1) holds whenever the two functions u_1 and u_2 have compact supports in a ball B such that $\Omega \cap B$ is star-shaped with respect to some point z_0 in $\Omega \cap B$.

Lemma 5.2. *Let Ω be a bounded $C^{1,1}$ domain, and let B be a ball in $\mathbb{R}^n$. Assume that there exists $z_0 \in \Omega \cap B$ such that*

$$(x - z_0) \cdot \nu(x) > 0 \qquad \text{for all } x \in \partial\Omega \cap \overline{B}.$$

Let u be a function satisfying the hypothesis of Proposition 1.6, and let $u_1 = u\eta_1$ and $u_2 = u\eta_2$, where $\eta_i \in C_c^\infty(B)$, $i = 1, 2$. Then, the following identity holds

$$\int_B (x \cdot \nabla u_1)(-\Delta)^s u_2 \, dx + \int_B (x \cdot \nabla u_2)(-\Delta)^s u_1 \, dx = \frac{2s - n}{2} \int_B u_1 (-\Delta)^s u_2 \, dx +$$

$$+ \frac{2s - n}{2} \int_B u_2 (-\Delta)^s u_1 \, dx - \Gamma(1 + s)^2 \int_{\partial\Omega \cap B} \frac{u_1}{\delta^s} \frac{u_2}{\delta^s} (x \cdot \nu) \, d\sigma.$$

Proof. We will show that given $\eta \in C_c^\infty(B)$ and letting $\tilde{u} = u\eta$ it holds

$$\int_B (x \cdot \nabla \tilde{u})(-\Delta)^s \tilde{u} \, dx = \frac{2s - n}{2} \int_B \tilde{u} (-\Delta)^s \tilde{u} \, dx - \Gamma(1 + s)^2 \int_{\partial\Omega \cap B} \left(\frac{\tilde{u}}{\delta^s} \right)^2 (x \cdot \nu) d\sigma. \tag{5.4}$$

From this, the lemma follows by applying (5.4) with $\tilde{u}$ replaced by $(\eta_1 + \eta_2)u$ and by $(\eta_1 - \eta_2)u$, and subtracting both identities.

We next prove (5.4). For it, we will apply the result for strictly star-shaped domains, already proven in Section 2. Observe that there is a $C^{1,1}$ domain $\tilde{\Omega}$ satisfying

$$\{\tilde{u} > 0\} \subset \tilde{\Omega} \subset \Omega \cap B \quad \text{and} \quad (x - z_0) \cdot \nu(x) > 0 \quad \text{for all } x \in \partial\tilde{\Omega}.$$

This is because, by the assumptions, $\Omega \cap B$ is a Lipschitz polar graph about the point $z_0 \in \Omega \cap B$ and supp $\tilde{u} \subset B' \subset\subset B$ for some smaller ball B'; see Figure 5.1. Hence, there is room enough to round the corner that $\Omega \cap B$ has on $\partial\Omega \cap \partial B$.

Hence, it only remains to prove that $\tilde{u}$ satisfies the hypotheses of Proposition 1.6. Indeed, since u satisfies (a) and η is $C_c^\infty(B')$ then $\tilde{u}$ satisfies

$$[\tilde{u}]_{C^\beta(\{x \in \tilde{\Omega} : \tilde{\delta}(x) > \rho\})} \leq C\rho^{s - \beta}$$

for all $\beta \in [s, 1 + 2s)$, where $\tilde{\delta}(x) = \text{dist}(x, \partial\tilde{\Omega})$.

On the other hand, since u satisfies (b) and we have $\eta\delta^s/\tilde{\delta}^s$ is Lipschitz in supp $\tilde{u}$ —because $\text{dist}(x, \partial\tilde{\Omega} \setminus \partial\Omega) \geq c > 0$ for all $x \in \text{supp } \tilde{u}$—, then we find

$$[\tilde{u}/\tilde{\delta}^s]_{C^\beta(\{x \in \tilde{\Omega} : \tilde{\delta}(x) > \rho\})} \leq C\rho^{\alpha - \beta}$$

for all $\beta \in [\alpha, s + \alpha]$.
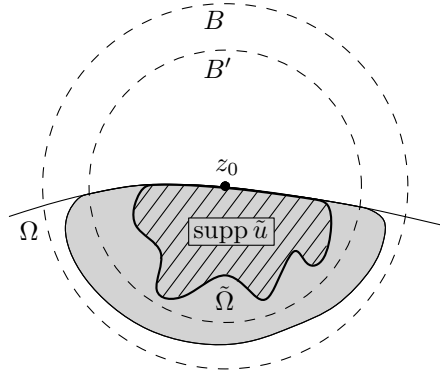
FIGURE 5.1.

Let us see now that $\tilde{u}$ satisfies (c), i.e., that $(-\Delta)^s\tilde{u}$ is bounded. For it, we use

$$(-\Delta)^s(u\eta) = \eta(-\Delta)^s u + u(-\Delta)^s\eta - I_s(u,\eta)$$

where I_s is given by (3.1), i.e.,

$$I_s(u,\eta)(x) = c_{n,s}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\, dy\,.$$

The first term is bounded since $(-\Delta)^s u$ so is by hypothesis. The second term is bounded since $\eta \in C_c^\infty(\mathbb{R}^n)$. The third term is bounded because $u \in C^s(\mathbb{R}^n)$ and $\eta \in \mathrm{Lip}(\mathbb{R}^n)$.

Therefore, $\tilde{u}$ satisfies the hypotheses of Proposition 1.6 with Ω replaced by $\tilde{\Omega}$, and (5.4) follows taking into account that for all $x_0 \in \partial\tilde{\Omega}\cap\mathrm{supp}\,\tilde{u} = \partial\Omega\cap\mathrm{supp}\,\tilde{u}$ we have

$$\lim_{x\to x_0,\, x\in\tilde{\Omega}} \frac{\tilde{u}(x)}{\tilde{\delta}^s(x)} = \lim_{x\to x_0,\, x\in\Omega} \frac{\tilde{u}(x)}{\delta^s(x)}.$$

$\square$

We now give the

Proof of Proposition 1.6. Let $B_1, ..., B_m$ be balls of radius $r > 0$ covering $\overline{\Omega}$. By regularity of the domain, if r is small enough, for each i, j such that $\overline{B_i}\cap\overline{B_j} \neq \varnothing$ there exists a ball B containing $B_i \cup B_j$ and a point $z_0 \in \Omega\cap B$ such that

$$(x - z_0)\cdot\nu(x) > 0 \qquad \text{for all } x \in \partial\Omega\cap B.$$

Let $\{\psi_k\}_{k=1,...,m}$ be a partition of the unity subordinated to $B_1, ..., B_m$, that is, a set of smooth functions $\psi_1, ..., \psi_m$ such that $\psi_1 + \cdots + \psi_m = 1$ in Ω and that ψ_k has compact support in B_k for each $k = 1, ..., m$. Define $u_k = u\psi_k$.

Now, for each $i, j \in \{1, ..., m\}$, if $\overline{B_i} \cap \overline{B_j} = \varnothing$ we use Lemma 5.1, while if $\overline{B_i} \cap \overline{B_j} \neq \varnothing$ we use Lemma 5.2. We obtain

$$\int_\Omega (x \cdot \nabla u_i)(-\Delta)^s u_j \, dx + \int_\Omega (x \cdot \nabla u_j)(-\Delta)^s u_i \, dx = \frac{2s-n}{2} \int_\Omega u_i(-\Delta)^s u_j \, dx +$$

$$+ \frac{2s-n}{2} \int_\Omega u_j(-\Delta)^s u_i \, dx - \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u_i}{\delta^s} \frac{u_j}{\delta^s} (x \cdot \nu) \, d\sigma$$

for each $1 \leq i \leq m$ and $1 \leq j \leq m$. Therefore, adding these identities for $i = 1, ..., m$ and for $j = 1, ..., m$ and taking into account that $u_1 + \cdots + u_m = u$, we find

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u \, dx = \frac{2s-n}{2} \int_\Omega u(-\Delta)^s u \, dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) \, d\sigma,$$

and the proposition is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To end this section we prove Theorem 1.1, Proposition 1.12, Theorem 1.9, and Corollaries 1.2, 1.3, and 1.13.

Proof of Proposition 1.12 and Theorem 1.1. By Theorem 1.4, any solution u to problem (1.8) satisfies the hypothesis of Proposition 1.6. Hence, using this proposition and that $(-\Delta)^s u = f(x, u)$, we obtain

$$\int_\Omega (\nabla u \cdot x) f(x, u) dx = \frac{2s-n}{2} \int_\Omega u f(x, u) dx + \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma.$$

On the other hand, note that $(\nabla u \cdot x) f(x, u) = \nabla (F(x, u)) \cdot x - x \cdot F_x(x, u)$. Then, integrating by parts,

$$\int_\Omega (\nabla u \cdot x) f(x, u) dx = -n \int_\Omega F(x, u) dx - \int_\Omega x \cdot F_x(x, u) dx.$$

If f does not depend on x, then the last term do not appear, as in Theorem 1.1. $\quad\square$

Proof of Theorem 1.9. As shown in the final part of the proof of Proposition 1.6 for strictly star-shaped domains given in Section 2, the freedom for choosing the origin in the identity from this proposition leads to

$$\int_\Omega w_{x_i}(-\Delta)^s w \, dx = \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{w}{\delta^s}\right)^2 \nu_i \, d\sigma$$

for each $i = 1, ..., n$. Then, the theorem follows by using this identity with $w = u + v$ and with $w = u - v$ and subtracting both identities. $\qquad\qquad\qquad\square$

Proof of Corollaries 1.2, 1.3, and 1.13. We only have to prove Corollary 1.13, since Corollaries 1.2 and 1.3 follow immediately from it by setting $f(x, u) = f(u)$ and $f(x, u) = |u|^{p-1} u$ respectively.

By hypothesis (1.15), we have

$$\frac{n-2s}{2} \int_\Omega u f(x, u) dx \geq n \int_\Omega F(x, u) dx + \int_\Omega x \cdot F_x(x, u) dx.$$

This, combined with Proposition 1.12 gives

$$\int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu)d\sigma \le 0.$$

If Ω is star-shaped and inequality in (1.15) is strict, we obtain a contradiction. On the other hand, if inequality in (1.15) is not strict but u is a positive solution of (1.8), then by the Hopf Lemma for the fractional Laplacian (see, for instance, [7] or Lemma 3.2 in [18]) the function u/δ^s is strictly positive in $\overline{\Omega}$, and we also obtain a contradiction. $\qquad\square$

Appendix A. Calculation of the constants c_1 and c_2

In Proposition 3.2 we have obtained the following expressions for the constants c_1 and c_2:

$$c_1 = c_{1,\frac{s}{2}}, \qquad \text{and} \qquad c_2 = \int_0^\infty \left\{ \frac{1-x^s}{|1-x|^{1+s}} + \frac{1+x^s}{|1+x|^{1+s}} \right\} dx,$$

where $c_{n,s}$ is the constant appearing in the singular integral expression for $(-\Delta)^s$ in dimension n.

Here we prove that the values of these constants coincide with the ones given in Proposition 1.10. We start by calculating c_1.

Proposition A.1. *Let $c_{n,s}$ be the normalizing constant of $(-\Delta)^s$ in dimension n. Then,*

$$c_{1,\frac{s}{2}} = \frac{\Gamma(1+s)\sin\left(\frac{\pi s}{2}\right)}{\pi}.$$

Proof. Recall that

$$c_{n,s} = \frac{s2^{2s}\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2}\Gamma(1-s)}. \tag{A.1}$$

Thus,

$$c_{1,\frac{s}{2}} = \frac{s2^{s-1}\Gamma\left(\frac{1+s}{2}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{s}{2}\right)}.$$

Now, using the properties of the Gamma function (see for example [1])

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z) \qquad \text{and} \qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

we obtain

$$c_{1,\frac{s}{2}} = \frac{s2^{s-1}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)} = \frac{s2^{s-1}}{\sqrt{\pi}} \cdot \frac{2^{1-s}\sqrt{\pi}\Gamma(s)}{\pi/\sin\left(\frac{\pi s}{2}\right)} = \frac{s\Gamma(s)\sin\left(\frac{\pi s}{2}\right)}{\pi}.$$

The result follows by using that $z\Gamma(z) = \Gamma(1+z)$. $\qquad\square$

Let us now compute the constant c_2.

Proposition A.2. *Let $0 < s < 1$. Then,*

$$\int_0^\infty \left\{ \frac{1 - x^s}{|1 - x|^{1+s}} + \frac{1 + x^s}{|1 + x|^{1+s}} \right\} dx = \frac{\pi}{\tan\left(\frac{\pi s}{2}\right)}.$$

For it, we will need some properties of the hypergeometric function $_2F_1$, which we prove in the next lemma. Recall that this function is defined as

$$_2F_1(a, b; c; z) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad \text{for } |z| < 1,$$

where $(a)_n = a(a + 1) \cdots (a + n - 1)$, and by analytic continuation in the whole complex plane.

Lemma A.3. *Let $_2F_1(a, b; c; z)$ be the ordinary hypergeometric function, and $s \in \mathbb{R}$. Then,*

(i) *For all $z \in \mathbb{C}$,*

$$\frac{d}{dz} \left\{ \frac{z^{s+1}}{s+1} \, _2F_1(1 + s, 1 + s; 2 + s; z) \right\} = \frac{z^s}{(1 - z)^{1+s}}.$$

(ii) *If $s \in (0, 1)$, then*

$$\lim_{x \to 1} \left\{ \frac{1}{s+1} \, _2F_1(1 + s, 1 + s; 2 + s; x) - \frac{1}{s(1 - x)^s} \right\} = -\frac{\pi}{\sin(\pi s)}.$$

(iii) *If $s \in (0, 1)$, then*

$$\lim_{x \to +\infty} \left\{ \frac{(-x)^{s+1}}{s+1} \, _2F_1(1+s, 1+s; 2+s; x) - \frac{x^{s+1}}{s+1} \, _2F_1(1+s, 1+s; 2+s; -x) \right\} = i\pi,$$

where the limit is taken on the real line.

Proof. (i) Let us prove the equality for $|z| < 1$. In this case,

$$\frac{d}{dz} \left\{ \frac{z^{s+1}}{s+1} \, _2F_1(1 + s, 1 + s; 2 + s; z) \right\} = \frac{d}{dz} \sum_{n \geq 0} \frac{(1 + s)_n^2}{(2 + s)_n} \frac{z^{n+1+s}}{n!(s + 1)} =$$

$$= \sum_{n \geq 0} \frac{(1 + s)_n}{n!} z^{n+s} = z^s \sum_{n \geq 0} \binom{-1 - s}{n} (-z)^n = z^s (1 - z)^{-1-s},$$

where we have used that $(2 + s)_n = \frac{n+1+s}{1+s}(1 + s)_n$ and that $\frac{(a)_n}{n!} = (-1)^n \binom{-a}{n}$. Thus, by analytic continuation the identity holds in $\mathbb{C}$.

(ii) Recall the Euler transformation (see for example [1])

$$_2F_1(a, b; c; x) = (1 - x)^{c-a-b} \, _2F_1(c - a, c - b; c; x), \tag{A.2}$$

and the value at $x = 1$

$$_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \qquad \text{whenever} \qquad a + b < c. \tag{A.3}$$

Hence,

$$\frac{1}{s+1} \, {}_2F_1(1+s, 1+s; 2+s; x) - \frac{1}{s(1-x)^s} = \frac{\frac{1}{s+1} \, {}_2F_1(1, 1; 2+s; x) - \frac{1}{s}}{(1-x)^s},$$

and we can use l'Hôpital's rule,

$$\begin{aligned}
\lim_{x \to 1} \frac{\frac{1}{s+1} \, {}_2F_1(1, 1; 2+s; x) - \frac{1}{s}}{(1-x)^s} &= \lim_{x \to 1} \frac{\frac{1}{s+1} \frac{d}{dx} \, {}_2F_1(1, 1; 2+s; x)}{-s(1-x)^{s-1}} \\
&= -\lim_{x \to 1} \frac{(1-x)^{1-s}}{s(s+1)(s+2)} \, {}_2F_1(2, 2; 3+s; x) \\
&= -\lim_{x \to 1} \frac{1}{s(s+1)(s+2)} \, {}_2F_1(1+s, 1+s; 3+s; x) \\
&= -\frac{1}{s(s+1)(s+2)} \, {}_2F_1(1+s, 1+s; 3+s; 1) \\
&= -\frac{1}{s(s+1)(s+2)} \frac{\Gamma(3+s)\Gamma(1-s)}{\Gamma(2)\Gamma(2)} \\
&= -\Gamma(s)\Gamma(1-s) \\
&= -\frac{\pi}{\sin(\pi s)}.
\end{aligned}$$

We have used that

$$\frac{d}{dx} \, {}_2F_1(1, 1; 2+s; x) = \frac{1}{s+2} \, {}_2F_1(2, 2; 3+s; x),$$

the Euler transformation (A.2), and the properties of the Γ function

$$x\Gamma(x) = \Gamma(x+1), \qquad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}.$$

(iii) In [2] it is proved that

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \, {}_2F_1(a, b; a+b; x) = \log\frac{1}{1-x} + R + o(1) \qquad \text{for } x \sim 1, \qquad (A.4)$$

where

$$R = -\psi(a) - \psi(b) - \gamma,$$

ψ is the digamma function, and γ is the Euler-Mascheroni constant. Using the Pfaff transformation [1]

$$_2F_1(a, b; c; x) = (1-x)^{-a} \, {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right)$$

and (A.4), we obtain

$$\frac{(1-x)^{1+s}}{1+s}\,{}_2F_1(1+s,1+s;2+s;x) \;=\; \frac{1}{1+s}\,{}_2F_1\left(1+s,1;2+s;\frac{x}{x-1}\right)$$

$$= \; \log\frac{1}{1-x} + R + o(1) \quad \text{for} \ \ x\sim\infty.$$

Thus, it also holds

$$\frac{(-x)^{1+s}}{1+s}\,{}_2F_1(1+s,1+s;2+s;x) = \log\frac{1}{1-x} + R + o(1) \quad \text{for} \ \ x\sim\infty,$$

and therefore the limit to be computed is now

$$\lim_{x\to+\infty}\left\{\left(\log\frac{1}{1-x}+R\right)-\left(\log\frac{1}{1+x}+R\right)\right\} = i\pi.$$

$\square$

Next we give the:

Proof of Proposition A.2. Let us compute separately the integrals

$$I_1 = \int_0^1 \left\{\frac{1-x^s}{|1-x|^{1+s}} + \frac{1+x^s}{|1+x|^{1+s}}\right\} dx$$

and

$$I_2 = \int_1^\infty \left\{\frac{1-x^s}{|1-x|^{1+s}} + \frac{1+x^s}{|1+x|^{1+s}}\right\} dx.$$

By Lemma A.3 (i), we have that

$$\int \left\{\frac{1-x^s}{(1-x)^{1+s}} + \frac{1+x^s}{(1+x)^{1+s}}\right\} dx = \frac{1}{s}(1-x)^{-s} - \frac{x^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;x)$$

$$-\frac{1}{s}(1+x)^{-s} + \frac{x^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;-x).$$

Hence, using A.3 (ii),

$$I_1 = \frac{\pi}{\sin(\pi s)} - \frac{1}{s2^s} + \frac{1}{s+1}\,{}_2F_1(1+s,1+s;2+s;-1).$$

Let us evaluate now I_2. As before, by Lemma A.3 (i),

$$\int \left\{\frac{1-x^s}{(x-1)^{1+s}} + \frac{1+x^s}{(x+1)^{1+s}}\right\} dx = \frac{1}{s}(x-1)^{-s} + (-1)^s\frac{x^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;x)$$

$$-\frac{1}{s}(1+x)^{-s} + \frac{x^{s+1}}{s+1}\,{}_2F_1(1+s,1+s;2+s;-x).$$

Hence, using A.3 (ii) and (iii),

$$
\begin{aligned}
I_2 &= -i\pi + (-1)^s \frac{\pi}{\sin(\pi s)} + \frac{1}{s2^s} - \frac{1}{s+1} \, {}_2F_1(1+s, 1+s; 2+s; -1) \\
&= -i\pi + \cos(\pi s)\frac{\pi}{\sin(\pi s)} + i\sin(\pi s)\frac{\pi}{\sin(\pi s)} + \\
&\qquad + \frac{1}{s2^s} - \frac{1}{s+1} \, {}_2F_1(1+s, 1+s; 2+s; -1) \\
&= \frac{\pi}{\tan(\pi s)} + \frac{1}{s2^s} - \frac{1}{s+1} \, {}_2F_1(1+s, 1+s; 2+s; -1).
\end{aligned}
$$

Finally, adding up the expressions for I_1 and I_2, we obtain

$$
\begin{aligned}
\int_0^\infty \left\{ \frac{1-x^s}{|1-x|^{1+s}} + \frac{1+x^s}{|1+x|^{1+s}} \right\} dx &= \frac{\pi}{\sin(\pi s)} + \frac{\pi}{\tan(\pi s)} = \pi \cdot \frac{1+\cos(\pi s)}{\sin(\pi s)} \\
&= \pi \cdot \frac{2\cos^2\left(\frac{\pi s}{2}\right)}{2\sin\left(\frac{\pi s}{2}\right)\cos\left(\frac{\pi s}{2}\right)} = \frac{\pi}{\tan\left(\frac{\pi s}{2}\right)},
\end{aligned}
$$

as desired. $\qquad\square$

Remark A.4. It follows from Proposition 1.11 that the constant appearing in (1.10) (and thus in the Pohozaev identity), $\Gamma(1+s)^2$, is given by

$$
c_3 = c_1^2(\pi^2 + c_2^2).
$$

We have obtained the value of c_3 by computing explicitly c_1 and c_2. However, an alternative way to obtain c_3 is to exhibit an explicit solution of (1.1) for some nonlinearity f and apply the Pohozaev identity to this solution. For example, when $\Omega = B_1(0)$, the solution of

$$
\begin{cases}
(-\Delta)^s u = 1 & \text{in } B_1(0) \\
u = 0 & \text{in } \mathbb{R}^n \backslash B_1(0)
\end{cases}
$$

can be computed explicitly [13, 3]:

$$
u(x) = \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\left(1 - |x|^2\right)^s. \tag{A.5}
$$

Thus, from the identity

$$
(2s - n)\int_{B_1(0)} u\,dx + 2n\int_{B_1(0)} u\,dx = c_3\int_{\partial B_1(0)}\left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu)d\sigma \tag{A.6}
$$

we can obtain the constant c_3, as follows.

On the one hand,

$$
\begin{aligned}
\int_{B_1(0)} u \, dx &= \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)} \int_{B_1(0)} \left(1 - |x|^2\right)^s dx \\
&= \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)} |S^{n-1}| \int_0^1 r^{n-1}(1-r^2)^s dr \\
&= \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)} |S^{n-1}| \frac{1}{2} \int_0^1 r^{n/2-1}(1-r)^s dr \\
&= \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)} |S^{n-1}| \frac{1}{2} \frac{\Gamma(n/2)\Gamma(1+s)}{\Gamma(n/2+1+s)},
\end{aligned}
$$

where we have used the definition of the Beta function

$$
B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt
$$

and the identity

$$
B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.
$$

On the other hand,

$$
\int_{\partial B_1(0)} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu) d\sigma = \left(\frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\right)^2 |S^{n-1}| 2^{2s}.
$$

Thus, (A.6) is equivalent to

$$
(n+2s)\frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)} \frac{1}{2} \frac{\Gamma(n/2)\Gamma(1+s)}{\Gamma(n/2+1+s)} = c_3 \left(\frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\right)^2 2^{2s}.
$$

Hence, after some simplifications,

$$
c_3 = \frac{\Gamma(1+s)^2}{\Gamma(n/2+1+s)} \frac{n+2s}{2} \Gamma\left(\frac{n+2s}{2}\right),
$$

and using that

$$
z\Gamma(z) = \Gamma(1+z)
$$

one finally obtains

$$
c_3 = \Gamma(1+s)^2,
$$

as before.

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Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain

E-mail address: xavier.ros.oton@upc.edu

Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain

E-mail address: joaquim.serra@upc.edu

B. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary

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Collaboration with X. Ros-Oton

THE DIRICHLET PROBLEM FOR THE FRACTIONAL LAPLACIAN: REGULARITY UP TO THE BOUNDARY

XAVIER ROS-OTON AND JOAQUIM SERRA

ABSTRACT. We study the regularity up to the boundary of solutions to the Dirichlet problem for the fractional Laplacian. We prove that if u is a solution of $(-\Delta)^s u = g$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \backslash \Omega$, for some $s \in (0,1)$ and $g \in L^\infty(\Omega)$, then u is $C^s(\mathbb{R}^n)$ and $u/\delta^s|_\Omega$ is C^α up to the boundary $\partial\Omega$ for some $\alpha \in (0,1)$, where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. For this, we develop a fractional analog of the Krylov boundary Harnack method.

Moreover, under further regularity assumptions on g we obtain higher order Hölder estimates for u and u/δ^s. Namely, the C^β norms of u and u/δ^s in the sets $\{x \in \Omega : \delta(x) \geq \rho\}$ are controlled by $C\rho^{s-\beta}$ and $C\rho^{\alpha-\beta}$, respectively.

These regularity results are crucial tools in our proof of the Pohozaev identity for the fractional Laplacian [19, 20].

1. INTRODUCTION AND RESULTS

Let $s \in (0,1)$ and $g \in L^\infty(\Omega)$, and consider the fractional elliptic problem

$$\begin{cases} (-\Delta)^s u & = & g & \text{in } \Omega \\ u & = & 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where

$$(-\Delta)^s u(x) = c_{n,s} \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \tag{1.2}$$

and $c_{n,s}$ is a normalization constant.

Problem (1.1) is the Dirichlet problem for the fractional Laplacian. There are classical results in the literature dealing with the interior regularity of s-harmonic functions, or more generally for equations of the type (1.1). However, there are few results on regularity up to the boundary. This is the topic of study of the paper.

Our main result establishes the Hölder regularity up to the boundary $\partial\Omega$ of the function $u/\delta^s|_\Omega$, where

$$\delta(x) = \mathrm{dist}(x, \partial\Omega).$$

For this, we develop an analog of the Krylov [17] boundary Harnack method for problem (1.1). As in Krylov's work, our proof applies also to operators with "bounded

---

measurable coefficients", more precisely those of the type (1.5). This will be treated in a future work [21]. In this paper we only consider the constant coefficient operator $(-\Delta)^s$, since in this case we can establish more precise regularity results. Most of them will be needed in our subsequent work [20], where we find and prove the Pohozaev identity for the fractional Laplacian, announced in [19]. For (1.1), in addition to the Hölder regularity up to the boundary for u/δ^s, we prove that any solution u is $C^s(\mathbb{R}^n)$. Moreover, when g is not only bounded but Hölder continuous, we obtain better interior Hölder estimates for u and u/δ^s.

The Dirichlet problem for the fractional Laplacian (1.1) has been studied from the point of view of probability, potential theory, and PDEs. The closest result to the one in our paper is that of Bogdan [2], establishing a boundary Harnack inequality for nonnegative s-harmonic functions. It will be described in more detail later on in the Introduction (in relation with Theorem 1.2). Related regularity results up to the boundary have been proved in [16] and [7]. In [16] it is proved that u/δ^s has a limit at every boundary point when u solves the homogeneous fractional heat equation. The same is proven in [7] for a free boundary problem for the fractional Laplacian.

Some other results dealing with various aspects concerning the Dirichlet problem are the following: estimates for the heat kernel (of the parabolic version of this problem) and for the Green function, e.g., [3, 10]; an explicit expression of the Poisson kernel for a ball [18]; and the explicit solution to problem (1.1) in a ball for $g \equiv 1$ [13]. In addition, the interior regularity theory for viscosity solutions to nonlocal equations with "bounded measurable coefficients" is developed in [9].

The first result of this paper gives the optimal Hölder regularity for a solution u of (1.1). The proof, which is given in Section 2, is based on two ingredients: a suitable upper barrier, and the interior regularity results for the fractional Laplacian. Given $g \in L^\infty(\Omega)$, we say that u is a solution of (1.1) when $u \in H^s(\mathbb{R}^n)$ is a weak solution (see Definition 2.1). When g is continuous, the notions of weak solution and of viscosity solution agree; see Remark 2.11.

We recall that a domain Ω satisfies the exterior ball condition if there exists a positive radius ρ_0 such that all the points on $\partial\Omega$ can be touched by some exterior ball of radius ρ_0.

Proposition 1.1. *Let Ω be a bounded Lipschitz domain satisfying the exterior ball condition, $g \in L^\infty(\Omega)$, and u be a solution of* (1.1). *Then, $u \in C^s(\mathbb{R}^n)$ and*

$$\|u\|_{C^s(\mathbb{R}^n)} \leq C\|g\|_{L^\infty(\Omega)},$$

where C is a constant depending only on Ω and s.

This C^s regularity is optimal, in the sense that a solution to problem (1.1) is not in general C^α for any $\alpha > s$. This can be seen by looking at the problem

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } B_r(x_0) \\ u = 0 & \text{in } \mathbb{R}^n \backslash B_r(x_0), \end{cases} \tag{1.3}$$

for which its solution is explicit. For any $r > 0$ and $x_0 \in \mathbb{R}^n$, it is given by [13, 3]

$$u(x) = \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\left(r^2 - |x - x_0|^2\right)^s \quad \text{in} \ \ B_r(x_0). \tag{1.4}$$

It is clear that this solution is C^s up to the boundary but it is not C^α for any $\alpha > s$.

Since solutions u of (1.1) are C^s up to the boundary, and not better, it is of importance to study the regularity of u/δ^s up to $\partial\Omega$. For instance, our recent proof [20, 19] of the Pohozaev identity for the fractional Laplacian uses in a crucial way that u/δ^s is Hölder continuous up to $\partial\Omega$. This is the main result of the present paper and it is stated next.

For local equations of second order with bounded measurable coefficients and in non-divergence form, the analog result is given by a theorem of N. Krylov [17], which states that u/δ is C^α up to the boundary for some $\alpha \in (0,1)$. This result is the key ingredient in the proof of the $C^{2,\alpha}$ boundary regularity of solutions to fully nonlinear elliptic equations $F(D^2u) = 0$ —see [15, 6].

For our nonlocal equation (1.1), the corresponding result is the following.

Theorem 1.2. *Let Ω be a bounded $C^{1,1}$ domain, $g \in L^\infty(\Omega)$, u be a solution of (1.1), and $\delta(x) = \text{dist}(x, \partial\Omega)$. Then, $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, we have $u/\delta^s \in C^\alpha(\overline{\Omega})$ and*

$$\|u/\delta^s\|_{C^\alpha(\overline{\Omega})} \le C\|g\|_{L^\infty(\Omega)}$$

for some $\alpha > 0$ satisfying $\alpha < \min\{s, 1 - s\}$. The constants α and C depend only on Ω and s.

To prove this result we use the method of Krylov (see [15]). It consists of trapping the solution between two multiples of δ^s in order to control the oscillation of the quotient u/δ^s near the boundary. For this, we need to prove, among other things, that $(-\Delta)^s\delta_0^s$ is bounded in Ω, where $\delta_0(x) = \text{dist}(x, \mathbb{R}^n \setminus \Omega)$ is the distance function in Ω extended by zero outside. This will be guaranteed by the assumption that Ω is $C^{1,1}$.

To our knowledge, the only previous results dealing with the regularity up to the boundary for solutions to (1.1) or its parabolic version were the ones by K. Bogdan [2] and S. Kim and K. Lee [16]. The first one [2] is the boundary Harnack principle for nonnegative s-harmonic functions, which reads as follows: assume that u and v are two nonnegative functions in a Lipschitz domain Ω, which satisfy $(-\Delta)^su \equiv 0$ and $(-\Delta)^sv \equiv 0$ in $\Omega \cap B_r(x_0)$ for some ball $B_r(x_0)$ centered at $x_0 \in \partial\Omega$. Assume also that $u \equiv v \equiv 0$ in $B_r(x_0) \setminus \Omega$. Then, the quotient u/v is $C^\alpha(\overline{B_{r/2}(x_0)})$ for some $\alpha \in (0,1)$. In [4] the same result is proven in open domains Ω, without any regularity assumption.

While the result in [4] assumes no regularity on the domain, we need to assume Ω to be $C^{1,1}$. This assumption is needed to compare the solutions with the function δ^s. As a counterpart, we allow nonzero right hand sides $g \in L^\infty(\Omega)$ and also changing-sign solutions. In $C^{1,1}$ domains, our results in Section 3 (which are local near any

boundary point) extend Bogdan's result. For instance, assume that u and v satisfy $(-\Delta)^s u = g$ and $(-\Delta)^s v = h$ in Ω, $u \equiv v \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, and that h is positive in Ω. Then, by Theorem 1.2 we have that u/δ^s and v/δ^s are $C^\alpha(\overline{\Omega})$ functions. In addition, by the Hopf lemma for the fractional Laplacian we find that $v/\delta^s \geq c > 0$ in Ω. Hence, we obtain that the quotient u/v is C^α up to the boundary, as in Bogdan's result for s-harmonic functions.

As in Krylov's result, our method can be adapted to the case of nonlocal elliptic equations with "bounded measurable coefficients". Namely, in another paper [21] we will prove the boundary Harnack principle for solutions to $\mathcal{L}u = g$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, where $g \in L^\infty(\Omega)$,

$$\mathcal{L}u(x) = \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y^T A(x) y|^{\frac{n+2s}{2}}} dy, \tag{1.5}$$

and $A(x)$ is a symmetric matrix, measurable in x, and with $0 < \lambda \mathrm{Id} \leq A(x) \leq \Lambda \mathrm{Id}$.

A second result (for the parabolic problem) related to ours is contained in [16]. The authors show that any solution of $\partial_t u + (-\Delta)^s u = 0$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, satisfies the following property: for any $t > 0$ the function u/δ^s is continuous up to the boundary $\partial \Omega$.

Our results were motivated by the study of nonlocal semilinear problems $(-\Delta)^s u = f(u)$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, more specifically, by the Pohozaev identity that we establish in [20]. Its proof requires the precise regularity theory up to the boundary developed in the present paper (see Corollary 1.6 below). Other works treating the fractional Dirichlet semilinear problem, which deal mainly with existence of solutions and symmetry properties, are [22, 23, 12, 1].

In the semilinear case, $g = f(u)$ and therefore g automatically becomes more regular than just bounded. When g has better regularity, the next two results improve the preceding ones. The proofs of these results require the use of the following weighted Hölder norms, a slight modification of the ones in Gilbarg-Trudinger [14, Section 6.1].

Throughout the paper, and when no confusion is possible, we use the notation $C^\beta(U)$ with $\beta > 0$ to refer to the space $C^{k,\beta'}(U)$, where k is the is greatest integer such that $k < \beta$ and where $\beta' = \beta - k$. This notation is specially appropriate when we work with $(-\Delta)^s$ in order to avoid the splitting of different cases in the statements of regularity results. According to this, $[\,\cdot\,]_{C^\beta(U)}$ denotes the $C^{k,\beta'}(U)$ seminorm

$$[u]_{C^\beta(U)} = [u]_{C^{k,\beta'}(U)} = \sup_{x,y \in U, \ x \neq y} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\beta'}}.$$

Moreover, given an open set $U \subset \mathbb{R}^n$ with $\partial U \neq \varnothing$, we will also denote

$$d_x = \mathrm{dist}(x, \partial U) \qquad \text{and} \qquad d_{x,y} = \min\{d_x, d_y\}.$$

Definition 1.3. Let $\beta > 0$ and $\sigma \geq -\beta$. Let $\beta = k + \beta'$, with k integer and $\beta' \in (0, 1]$. For $w \in C^\beta(U) = C^{k,\beta'}(U)$, define the seminorm

$$[w]_{\beta;U}^{(\sigma)} = \sup_{x,y \in U} \left(d_{x,y}^{\beta+\sigma} \frac{|D^k w(x) - D^k w(y)|}{|x - y|^{\beta'}} \right).$$

For $\sigma > -1$, we also define the norm $\| \cdot \|_{\beta;U}^{(\sigma)}$ as follows: in case that $\sigma \geq 0$,

$$\|w\|_{\beta;U}^{(\sigma)} = \sum_{l=0}^{k} \sup_{x \in U} \left(d_x^{l+\sigma} |D^l w(x)| \right) + [w]_{\beta;U}^{(\sigma)},$$

while for $-1 < \sigma < 0$,

$$\|w\|_{\beta;U}^{(\sigma)} = \|w\|_{C^{-\sigma}(\overline{U})} + \sum_{l=1}^{k} \sup_{x \in U} \left(d_x^{l+\sigma} |D^l w(x)| \right) + [w]_{\beta;U}^{(\sigma)}.$$

Note that σ is the rescale order of the seminorm $[\,\cdot\,]_{\beta;U}^{(\sigma)}$, in the sense that $[w(\lambda\cdot)]_{\beta;U/\lambda}^{(\sigma)} = \lambda^\sigma [w]_{\beta;U}^{(\sigma)}$.

When g is Hölder continuous, the next result provides optimal estimates for higher order Hölder norms of u up to the boundary.

Proposition 1.4. *Let Ω be a bounded domain, and $\beta > 0$ be such that neither β nor $\beta + 2s$ is an integer. Let $g \in C^\beta(\Omega)$ be such that $\|g\|_{\beta;\Omega}^{(s)} < \infty$, and $u \in C^s(\mathbb{R}^n)$ be a solution of (1.1). Then, $u \in C^{\beta+2s}(\Omega)$ and*

$$\|u\|_{\beta+2s;\Omega}^{(-s)} \leq C \big(\|u\|_{C^s(\mathbb{R}^n)} + \|g\|_{\beta;\Omega}^{(s)} \big),$$

where C is a constant depending only on Ω, s, and β.

Next, the Hölder regularity up to the boundary of u/δ^s in Theorem 1.2 can be improved when g is Hölder continuous. This is stated in the following theorem, whose proof uses a nonlocal equation satisfied by the quotient u/δ^s in Ω —see (4.2)— and the fact that this quotient is $C^\alpha(\overline{\Omega})$.

Theorem 1.5. *Let Ω be a bounded $C^{1,1}$ domain, and let $\alpha \in (0, 1)$ be given by Theorem 1.2. Let $g \in L^\infty(\Omega)$ be such that $\|g\|_{\alpha;\Omega}^{(s-\alpha)} < \infty$, and u be a solution of (1.1). Then, $u/\delta^s \in C^\alpha(\overline{\Omega}) \cap C^\gamma(\Omega)$ and*

$$\|u/\delta^s\|_{\gamma;\Omega}^{(-\alpha)} \leq C \big(\|g\|_{L^\infty(\Omega)} + \|g\|_{\alpha;\Omega}^{(s-\alpha)} \big),$$

where $\gamma = \min\{1, \alpha + 2s\}$ and C is a constant depending only on Ω and s.

Finally, we apply the previous results to the semilinear problem

$$\begin{cases} (-\Delta)^s u &= f(x, u) & \text{in } \Omega \\ u &= 0 & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.6}$$

where Ω is a bounded $C^{1,1}$ domain and f is a Lipschitz nonlinearity.

In the following result, the meaning of "bounded solution" is that of "bounded weak solution" (see definition 2.1) or that of "viscosity solution". By Remark 2.11, these two notions coincide. Also, by $f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})$ we mean that f is Lipschitz in every compact subset of $\overline{\Omega} \times \mathbb{R}$.

Corollary 1.6. *Let Ω be a bounded and $C^{1,1}$ domain, $f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})$, u be a bounded solution of (1.6), and $\delta(x) = \text{dist}(x, \partial\Omega)$. Then,*

(a) *$u \in C^s(\mathbb{R}^n)$ and, for every $\beta \in [s, 1 + 2s)$, u is of class $C^\beta(\Omega)$ and*

$$[u]_{C^\beta(\{x \in \Omega \,:\, \delta(x) \geq \rho\})} \leq C\rho^{s-\beta} \qquad \text{for all} \ \ \rho \in (0, 1).$$

(b) *The function $u/\delta^s|_\Omega$ can be continuously extended to $\overline{\Omega}$. Moreover, there exists $\alpha \in (0, 1)$ such that $u/\delta^s \in C^\alpha(\overline{\Omega})$. In addition, for all $\beta \in [\alpha, s + \alpha]$, it holds the estimate*

$$[u/\delta^s]_{C^\beta(\{x \in \Omega \,:\, \delta(x) \geq \rho\})} \leq C\rho^{\alpha-\beta} \qquad \text{for all} \ \ \rho \in (0, 1).$$

The constants α and C depend only on Ω, s, f, $\|u\|_{L^\infty(\mathbb{R}^n)}$, and β.

The paper is organized as follows. In Section 2 we prove Propositions 1.1 and 1.4. In Section 3 we prove Theorem 1.2 using the Krylov method. In Section 4 we prove Theorem 1.5 and Corollary 1.6. Finally, the Appendix deals with some basic tools and barriers which are used throughout the paper.

2. Optimal Hölder regularity for u

In this section we prove that, assuming Ω to be a bounded Lipschitz domain satisfying the exterior ball condition, every solution u of (1.1) belongs to $C^s(\mathbb{R}^n)$. For this, we first establish that u is C^β in Ω, for all $\beta \in (0, 2s)$, and sharp bounds for the corresponding seminorms near $\partial\Omega$. These bounds yield $u \in C^s(\mathbb{R}^n)$ as a corollary. First, we make precise the notion of weak solution to problem (1.1).

Definition 2.1. We say that u is a weak solution of (1.1) if $u \in H^s(\mathbb{R}^n)$, $u \equiv 0$ (a.e.) in $\mathbb{R}^n \setminus \Omega$, and

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} v \, dx = \int_\Omega gv \, dx$$

for all $v \in H^s(\mathbb{R}^n)$ such that $v \equiv 0$ in $\mathbb{R}^n \setminus \Omega$.

We recall first some well known interior regularity results for linear equations involving the operator $(-\Delta)^s$, defined by (1.2). The first one states that $w \in C^{\beta+2s}(\overline{B_{1/2}})$ whenever $w \in C^\beta(\mathbb{R}^n)$ and $(-\Delta)^s w \in C^\beta(\overline{B_1})$. Recall that, throughout this section and in all the paper, we denote by C^β, with $\beta > 0$, the space $C^{k,\beta'}$, where k is an integer, $\beta' \in (0, 1]$, and $\beta = k + \beta'$.

Proposition 2.2. *Assume that $w \in C^\infty(\mathbb{R}^n)$ solves $(-\Delta)^s w = h$ in B_1 and that neither β nor $\beta + 2s$ is an integer. Then,*

$$\|w\|_{C^{\beta+2s}(\overline{B_{1/2}})} \leq C\big(\|w\|_{C^\beta(\mathbb{R}^n)} + \|h\|_{C^\beta(\overline{B_1})}\big),$$

where C is a constant depending only on n, s, and β.

Proof. Follow the proof of Proposition 2.1.8 in [24], where the same result is proved with B_1 and $B_{1/2}$ replaced by the whole $\mathbb{R}^n$. □

The second result states that $w \in C^\beta(\overline{B_{1/2}})$ for each $\beta \in (0, 2s)$ whenever $w \in L^\infty(\mathbb{R}^n)$ and $(-\Delta)^s w \in L^\infty(B_1)$.

Proposition 2.3. *Assume that $w \in C^\infty(\mathbb{R}^n)$ solves $(-\Delta)^s w = h$ in B_1. Then, for every $\beta \in (0, 2s)$,*

$$\|w\|_{C^\beta(\overline{B_{1/2}})} \le C\big(\|w\|_{L^\infty(\mathbb{R}^n)} + \|h\|_{L^\infty(B_1)}\big),$$

where C is a constant depending only on n, s, and β.

Proof. Follow the proof of Proposition 2.1.9 in [24], where the same result is proved in the whole $\mathbb{R}^n$. □

The third result is the analog of the first, with the difference that it does not need to assume $w \in C^\beta(\mathbb{R}^n)$, but only $w \in C^\beta(\overline{B_2})$ and $(1 + |x|)^{-n-2s} w(x) \in L^1(\mathbb{R}^n)$.

Corollary 2.4. *Assume that $w \in C^\infty(\mathbb{R}^n)$ is a solution of $(-\Delta)^s w = h$ in B_2, and that neither β nor $\beta + 2s$ is an integer. Then,*

$$\|w\|_{C^{\beta+2s}(\overline{B_{1/2}})} \le C\left(\|(1+|x|)^{-n-2s} w(x)\|_{L^1(\mathbb{R}^n)} + \|w\|_{C^\beta(\overline{B_2})} + \|h\|_{C^\beta(\overline{B_2})}\right)$$

where the constant C depends only on n, s, and β.

Proof. Let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $\eta \equiv 0$ outside B_2 and $\eta \equiv 1$ in $B_{3/2}$. Then $\tilde{w} := w\eta \in C^\infty(\mathbb{R}^n)$ and $(-\Delta)^s \tilde{w} = \tilde{h} := h - (-\Delta)^s\big(w(1 - \eta)\big)$. Note that for $x \in B_{3/2}$ we have

$$(-\Delta)^s \left(w(1 - \eta)\right)(x) = c_{n,s} \int_{\mathbb{R}^n \setminus B_{3/2}} \frac{-\big(w(1 - \eta)\big)(y)}{|x - y|^{n+2s}} dy.$$

From this expression we obtain that

$$\|(-\Delta)^s \left(w(1 - \eta)\right)\|_{L^\infty(B_1)} \le C\|(1+|y|)^{-n-2s} w(y)\|_{L^1(\mathbb{R}^n)}$$

and for all $\gamma \in (0, \beta]$,

$$[(-\Delta)^s \left(w(1 - \eta)\right)]_{C^\gamma(\overline{B_1})} \le C\|(1+|y|)^{-n-2s-\gamma} w(y)\|_{L^1(\mathbb{R}^n)}$$
$$\le C\|(1+|y|)^{-n-2s} w(y)\|_{L^1(\mathbb{R}^n)}$$

for some constant C that depends only on n, s, β, and η. Therefore

$$\|\tilde{h}\|_{C^\beta(\overline{B_1})} \le C\big(\|h\|_{C^\beta(\overline{B_2})} + \|(1+|x|)^{-n-2s} w(x)\|_{L^1(\mathbb{R}^n)}\big),$$

while we also clearly have

$$\|\tilde{w}\|_{C^\beta(\overline{\mathbb{R}^n})} \le C\|w\|_{C^\beta(\overline{B_2})}.$$

The constants C depend only on n, s, β and η. Now, we finish the proof by applying Proposition 2.2 with w replaced by $\tilde{w}$. □

Finally, the fourth result is the analog of the second one, but instead of assuming $w \in L^\infty(\mathbb{R}^n)$, it only assumes $w \in L^\infty(B_2)$ and $(1 + |x|)^{-n-2s}w(x) \in L^1(\mathbb{R}^n)$.

Corollary 2.5. *Assume that $w \in C^\infty(\mathbb{R}^n)$ is a solution of $(-\Delta)^s w = h$ in B_2. Then, for every $\beta \in (0, 2s)$,*

$$\|w\|_{C^\beta(\overline{B_{1/2}})} \leq C\left(\|(1+|x|)^{-n-2s}w(x)\|_{L^1(\mathbb{R}^n)} + \|w\|_{L^\infty(B_2)} + \|h\|_{L^\infty(B_2)} \right)$$

where the constant C depends only on n, s, and β.

Proof. Analog to the proof of Corollary 2.4. $\qquad\square$

As a consequence of the previous results we next prove that every solution u of (1.1) is $C^s(\mathbb{R}^n)$. First let us find an explicit upper barrier for $|u|$ to prove that $|u| \leq C\delta^s$ in Ω. This is the first step to obtain the C^s regularity.

To construct this we will need the following result, which is proved in the Appendix.

Lemma 2.6 (Supersolution). *There exist $C_1 > 0$ and a radial continuous function $\varphi_1 \in H_{\mathrm{loc}}^s(\mathbb{R}^n)$ satisfying*

$$\begin{cases} (-\Delta)^s\varphi_1 \geq 1 & \text{in } B_4 \setminus B_1 \\ \varphi_1 \equiv 0 & \text{in } B_1 \\ 0 \leq \varphi_1 \leq C_1(|x| - 1)^s & \text{in } B_4 \setminus B_1 \\ 1 \leq \varphi_1 \leq C_1 & \text{in } \mathbb{R}^n \setminus B_4 . \end{cases} \tag{2.1}$$

The upper barrier for $|u|$ will be constructed by scaling and translating the supersolution from Lemma 2.6. The conclusion of this barrier argument is the following.

Lemma 2.7. *Let Ω be a bounded domain satisfying the exterior ball condition and let $g \in L^\infty(\Omega)$. Let u be the solution of (1.1). Then,*

$$|u(x)| \leq C\|g\|_{L^\infty(\Omega)}\delta^s(x) \quad \text{for all } x \in \Omega ,$$

where C is a constant depending only on Ω and s.

In the proof of Lemma 2.7 it will be useful the following

Claim 2.8. *Let Ω be a bounded domain and let $g \in L^\infty(\Omega)$. Let u be the solution of (1.1). Then,*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C(\operatorname{diam}\Omega)^{2s}\|g\|_{L^\infty(\Omega)}$$

where C is a constant depending only on n and s.

Proof. The domain Ω is contained in a large ball of radius $\operatorname{diam}\Omega$. Then, by scaling the explicit (super)solution for the ball given by (1.4) we obtain the desired bound. $\qquad\square$

We next give the

Proof of Lemma 2.7. Since Ω satisfies the exterior ball condition, there exists $\rho_0 > 0$ such that every point of $\partial\Omega$ can be touched from outside by a ball of radius ρ_0. Then, by scaling and translating the supersolution φ_1 from Lemma 2.6, for each of this exterior tangent balls B_{ρ_0} we find an upper barrier in $B_{2\rho_0} \setminus B_{\rho_0}$ vanishing in $\overline{B_{\rho_0}}$. This yields the bound $u \leq C\delta^s$ in a ρ_0-neighborhood of $\partial\Omega$. By using Claim 2.8 we have the same bound in all of $\overline{\Omega}$. Repeating the same argument with $-u$ we find $|u| \leq C\delta^s$, as wanted. $\qquad\square$

The following lemma gives interior estimates for u and yields, as a corollary, that every bounded weak solution u of (1.1) in a $C^{1,1}$ domain is $C^s(\mathbb{R}^n)$.

Lemma 2.9. *Let Ω be a bounded domain satisfying the exterior ball condition, $g \in L^\infty(\Omega)$, and u be the solution of (1.1). Then, $u \in C^\beta(\Omega)$ for all $\beta \in (0, 2s)$ and for all $x_0 \in \Omega$ we have the following seminorm estimate in $B_R(x_0) = B_{\delta(x_0)/2}(x_0)$:*

$$[u]_{C^\beta(\overline{B_R(x_0)})} \leq CR^{s-\beta}\|g\|_{L^\infty(\Omega)}, \tag{2.2}$$

where C is a constant depending only on Ω, s, and β.

Proof. Recall that if u solves (1.1) in the weak sense and η_ϵ is the standard mollifier then $(-\Delta)^s(u * \eta_\epsilon) = g * \eta_\epsilon$ in B_R for ϵ small enough. Hence, we can regularize u, obtain the estimates, and then pass to the limit. In this way we may assume that u is smooth.

Note that $B_R(x_0) \subset B_{2R}(x_0) \subset \Omega$. Let $\tilde{u}(y) = u(x_0 + Ry)$. We have that

$$(-\Delta)^s\tilde{u}(y) = R^{2s}g(x_0 + Ry) \quad \text{in } B_1. \tag{2.3}$$

Furthermore, using that $|u| \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)\delta^s$ in Ω —by Lemma 2.7— we obtain

$$\|\tilde{u}\|_{L^\infty(B_1)} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)R^s \tag{2.4}$$

and, observing that $|\tilde{u}(y)| \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)R^s(1 + |y|^s)$ in all of $\mathbb{R}^n$,

$$\|(1 + |y|)^{-n-2s}\tilde{u}(y)\|_{L^1(\mathbb{R}^n)} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)R^s, \tag{2.5}$$

with C depending only on Ω and s.

Next we use Corollary 2.5, which taking into account (2.3), (2.4), and (2.5), yields

$$\|\tilde{u}\|_{C^\beta(\overline{B_{1/4}})} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)R^s$$

for all $\beta \in (0, 2s)$, where $C = C(\Omega, s, \beta)$.

Finally, we observe that

$$[u]_{C^\beta(\overline{B_{R/4}(x_0)})} = R^{-\beta}[\tilde{u}]_{C^\beta(\overline{B_{1/4}})}.$$

Hence, by an standard covering argument, we find the estimate (2.2) for the C^β seminorm of u in $\overline{B_R(x_0)}$. $\qquad\square$

We now prove the C^s regularity of u.

Proof of Proposition 1.1. By Lemma 2.9, taking $\beta = s$ we obtain

$$\frac{|u(x) - u(y)|}{|x - y|^s} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big) \tag{2.6}$$

for all x, y such that $y \in B_R(x)$ with $R = \delta(x)/2$. We want to show that (2.6) holds, perhaps with a bigger constant $C = C(\Omega, s)$, for all $x, y \in \overline{\Omega}$, and hence for all $x, y \in \mathbb{R}^n$ (since $u \equiv 0$ outside Ω).

Indeed, observe that after a Lipschitz change of coordinates, the bound (2.6) remains the same except for the value of the constant C. Hence, we can flatten the boundary near $x_0 \in \partial\Omega$ to assume that $\Omega \cap B_{\rho_0}(x_0) = \{x_n > 0\} \cap B_1(0)$. Now, (2.6) holds for all x, y satisfying $|x - y| \leq \gamma x_n$ for some $\gamma = \gamma(\Omega) \in (0, 1)$ depending on the Lipschitz map.

Next, let $z = (z', z_n)$ and $w = (w', w_n)$ be two points in $\{x_n > 0\} \cap B_{1/4}(0)$, and $r = |z - w|$. Let us define $\bar{z} = (z', z_n + r)$, $\bar{z} = (z', z_n + r)$ and $z_k = (1 - \gamma^k)z + \gamma^k\bar{z}$ and $w_k = \gamma^k w + (1 - \gamma^k)\bar{w}$, $k \geq 0$. Then, using that bound (2.6) holds whenever $|x - y| \leq \gamma x_n$, we have

$$|u(z_{k+1}) - u(z_k)| \leq C|z_{k+1} - z_k|^s = C|\gamma^k(z - \bar{z})(\gamma - 1)|^s \leq C\gamma^k|z - \bar{z}|.$$

Moreover, since $x_n > r$ in all the segment joining $\bar{z}$ and $\bar{w}$, splitting this segment into a bounded number of segments of length less than γr, we obtain

$$|u(\bar{z}) - u(\bar{w})| \leq C|\bar{z} - \bar{w}|^s \leq Cr^s.$$

Therefore,

$$|u(z) - u(w)| \leq \sum_{k \geq 0} |u(z_{k+1}) - u(z_k)| + |u(\bar{z}) - u(\bar{w})| + \sum_{k \geq 0} |u(w_{k+1}) - u(w_k)|$$

$$\leq \left(C\sum_{k \geq 0} (\gamma^k r)^s + Cr^s \right) \big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)$$

$$\leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{L^\infty(\Omega)}\big)|z - w|^s,$$

as wanted. $\qquad\square$

The following lemma is similar to Proposition 2.2 but it involves the weighted norms introduced above. It will be used to prove Proposition 1.4 and Theorem 1.5.

Lemma 2.10. *Let s and α belong to $(0, 1)$, and $\beta > 0$. Let U be an open set with nonempty boundary. Assume that neither β nor $\beta + 2s$ is an integer, and $\alpha < 2s$. Then,*

$$\|w\|_{\beta+2s;U}^{(-\alpha)} \leq C\left(\|w\|_{C^\alpha(\mathbb{R}^n)} + \|(-\Delta)^s w\|_{\beta;U}^{(2s-\alpha)} \right) \tag{2.7}$$

for all w with finite right hand side. The constant C depends only on n, s, α, and β.

Proof. Step 1. We first control the $C^{\beta+2s}$ norm of w in balls $B_R(x_0)$ with $R = d_{x_0}/2$.

Let $x_0 \in U$ and $R = d_{x_0}/2$. Define $\tilde{w}(y) = w(x_0 + Ry) - w(x_0)$ and note that

$$\|\tilde{w}\|_{C^\alpha(B_1)} \le R^\alpha [w]_{C^\alpha(\mathbb{R}^n)}$$

and

$$\|(1 + |y|)^{-n-2s}\tilde{w}(y)\|_{L^1(\mathbb{R}^n)} \le C(n, s)R^\alpha [w]_{C^\alpha(\mathbb{R}^n)}.$$

This is because

$$|\tilde{w}(y)| = |w(x_0 + Ry) - w(x_0)| \le R^\alpha |y|^\alpha [w]_{C^\alpha(\mathbb{R}^n)}$$

and $\alpha < 2s$. Note also that

$$\|(-\Delta)^s \tilde{w}\|_{C^\beta(\overline{B_1})} = R^{2s+\beta}\|(-\Delta)^s w\|_{C^\beta(\overline{B_R(x_0)})} \le R^\alpha \|(-\Delta)^s w\|_{\beta;U}^{(2s-\alpha)}.$$

Therefore, using Corollary 2.4 we obtain that

$$\|\tilde{w}\|_{C^{\beta+2s}(\overline{B_{1/2}})} \le CR^\alpha \big([w]_{C^\alpha(\mathbb{R}^n)} + \|(-\Delta)^s w\|_{\beta;U}^{(2s-\alpha)}\big),$$

where the constant C depends only on n, s, α, and β. Scaling back we obtain

$$\sum_{l=1}^{k} R^{l-\alpha}\|D^l w\|_{L^\infty(B_{R/2}(x_0))} + R^{2s+\beta-\alpha}[w]_{C^{\beta+2s}(\overline{B_{R/2}(x_0)})} \le \tag{2.8}$$

$$\le C\big(\|w\|_{C^\alpha(\mathbb{R}^n)} + \|(-\Delta)^s w\|_{\alpha;U}^{(2s-\alpha)}\big),$$

where k denotes the greatest integer less that $\beta + 2s$ and $C = C(n, s)$. This bound holds, with the same constant C, for each ball $B_R(x_0)$, $x_0 \in U$, where $R = d_{x_0}/2$.

Step 2. Next we claim that if (2.8) holds for each ball $B_{d_x/2}(x)$, $x \in U$, then (2.7) holds. It is clear that this already yields

$$\sum_{l=1}^{k} d_x^{k-\alpha} \sup_{x\in U} |D^k u(x)| \le C\bigg(\|w\|_{C^\alpha(\mathbb{R}^n)} + \|(-\Delta)^s w\|_{\beta;U}^{(2s-\alpha)}\bigg) \tag{2.9}$$

where k is the greatest integer less than $\beta + 2s$.

To prove this claim we only have to control $[w]_{\beta+2s;U}^{(-\alpha)}$ —see Definition 1.3. Let $\gamma \in (0, 1)$ be such that $\beta + 2s = k + \gamma$. We next bound

$$\frac{|D^k w(x) - D^k w(y)|}{|x - y|^\gamma}$$

when $d_x \ge d_y$ and $|x - y| \ge d_x/2$. This will yield the bound for $[w]_{\beta+2s;U}^{(-\alpha)}$, because if $|x - y| < d_x/2$ then $y \in B_{d_x/2}(x)$, and that case is done in Step 1.

We proceed differently in the cases $k = 0$ and $k \ge 1$. If $k = 0$, then

$$d_x^{\beta+2s-\alpha}\frac{w(x) - w(y)}{|x - y|^{2s+\beta}} = \left(\frac{d_x}{|x - y|}\right)^{\beta+2s-\alpha}\frac{w(x) - w(y)}{|x - y|^\alpha} \le C\|w\|_{C^\alpha(\mathbb{R}^n)}.$$

If $k \ge 1$, then

$$d_x^{\beta+2s-\alpha}\frac{|D^k w(x) - D^k w(y)|}{|x - y|^\gamma} \le \left(\frac{d_x}{|x - y|}\right)^\gamma d_x^{\beta+2s-\alpha-\gamma}|D^k w(x) - D^k w(y)| \le C\|w\|_{k;U}^{(-\alpha)},$$

where we have used that $\beta + 2s - \alpha - \gamma = k - \alpha$.

Finally, noting that for $x \in B_R(x_0)$ we have $R \leq d_{x_0} \leq 3R$, (2.7) follows from (2.8), (2.9) and the definition of $\|w\|_{\alpha+2s;U}^{(-\alpha)}$ in (1.3).   □

Finally, to end this section, we prove Proposition 1.4.

Proof of Proposition 1.4. Set $\alpha = s$ in Lemma 2.10.   □

Remark 2.11. When g is continuous, the notions of bounded weak solution and viscosity solution of (1.1) —and hence of (1.6)— coincide.

Indeed, let $u \in H^s(\mathbb{R}^n)$ be a weak solution of (1.1). Then, from Proposition 1.1 it follows that u is continuous up to the boundary. Let u_ε and g_ε be the standard regularizations of u and g by convolution with a mollifier. It is immediate to verify that, for ε small enough, we have $(-\Delta)^s u_\varepsilon = g_\varepsilon$ in every subdomain $U \subset\subset \Omega$ in the classical sense. Then, noting that $u_\varepsilon \to u$ and $g_\varepsilon \to g$ locally uniformly in Ω, and applying the stability property for viscosity solutions [9, Lemma 4.5], we find that u is a viscosity solution of (1.1).

Conversely, every viscosity solution of (1.1) is a weak solution. This follows from three facts: the existence of weak solution, that this solution is a viscosity solution as shown before, and the uniqueness of viscosity solutions [9, Theorem 5.2].

As a consequence of this, if g is continuous, any viscosity solution of (1.1) belongs to $H^s(\mathbb{R}^n)$ —since it is a weak solution. This fact, which is not obvious, can also be proved without using the result on uniqueness of viscosity solutions. Indeed, it follows from Proposition 1.4 and Lemma 4.4, which yield a stronger fact: that $(-\Delta)^{s/2}u \in L^p(\mathbb{R}^n)$ for all $p < \infty$. Note that although we have proved Proposition 1.4 for weak solutions, its proof is also valid —with almost no changes— for viscosity solutions.

3. Boundary regularity

In this section we study the precise behavior near the boundary of the solution u to problem (1.1), where $g \in L^\infty(\Omega)$. More precisely, we prove that the function $u/\delta^s|_\Omega$ has a $C^\alpha(\overline{\Omega})$ extension. This is stated in Theorem 1.2.

This result will be a consequence of the interior regularity results of Section 2 and an oscillation lemma near the boundary, which can be seen as the nonlocal analog of Krylov's boundary Harnack principle; see Theorem 4.28 in [15].

The following proposition and lemma will be used to establish Theorem 1.2. They are proved in the Appendix.

Proposition 3.1 (1-D solution in half space, [7])**.** *The function φ_0, defined by*

$$\varphi_0(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^s & \text{if } x \geq 0, \end{cases} \tag{3.1}$$

satisfies $(-\Delta)^s \varphi_0 = 0$ in $\mathbb{R}_+$.

The lemma below gives a subsolution in $B_1 \setminus B_{1/4}$ whose support is $B_1 \subset \mathbb{R}^n$ and such that it is comparable to $(1 - |x|)^s$ in B_1.

Lemma 3.2 (Subsolution). *There exist $C_2 > 0$ and a radial function $\varphi_2 = \varphi_2(|x|)$ satisfying*

$$\begin{cases} (-\Delta)^s \varphi_2 \leq 0 & in \ B_1 \setminus B_{1/4} \\ \varphi_2 = 1 & in \ B_{1/4} \\ \varphi_2(x) \geq C_2(1 - |x|)^s & in \ B_1 \\ \varphi_2 = 0 & in \ \mathbb{R}^n \setminus B_1 \, . \end{cases} \tag{3.2}$$

To prove Hölder regularity of $u/\delta^s|_\Omega$ up to the boundary, we will control the oscillation of this function in sets near $\partial\Omega$ whose diameter goes to zero. To do it, we will set up an iterative argument as it is done for second order equations.

Let us define the sets in which we want to control the oscillation and also auxiliary sets that are involved in the iteration.

Definition 3.3. Let $\kappa > 0$ be a fixed small constant and let $\kappa' = 1/2 + 2\kappa$. We may take, for instance $\kappa = 1/16$, $\kappa' = 5/8$. Given a point x_0 in $\partial\Omega$ and $R > 0$ let us define

$$D_R = D_R(x_0) = B_R(x_0) \cap \Omega$$

and

$$D^+_{\kappa' R} = D^+_{\kappa' R}(x_0) = B_{\kappa' R}(x_0) \cap \{x \in \Omega : -x \cdot \nu(x_0) \geq 2\kappa R\} \, ,$$

where $\nu(x_0)$ is the unit outward normal at x_0; see Figure 3.1. By $C^{1,1}$ regularity of the domain, there exists $\rho_0 > 0$, depending on Ω, such that the following inclusions hold for each $x_0 \in \partial\Omega$ and $R \leq \rho_0$:

$$B_{\kappa R}(y) \subset D_R(x_0) \quad \text{for all } y \in D^+_{\kappa' R}(x_0) \, , \tag{3.3}$$

and

$$B_{4\kappa R}(y^* - 4\kappa R\nu(y^*)) \subset D_R(x_0) \quad \text{and} \quad B_{\kappa R}(y^* - 4\kappa R\nu(y^*)) \subset D^+_{\kappa' R}(x_0) \tag{3.4}$$

for all $y \in D_{R/2}$, where $y^* \in \partial\Omega$ is the unique boundary point satisfying $|y - y^*| = \text{dist}(y, \partial\Omega)$. Note that, since $R \leq \rho_0$, $y \in D_{R/2}$ is close enough to $\partial\Omega$ and hence the point $y^* - 4\kappa R\nu(y^*)$ lays on the line joining y and y^*; see Remark 3.4 below.

Remark 3.4. Throughout the paper, $\rho_0 > 0$ is a small constant depending only on Ω, which we assume to be a bounded $C^{1,1}$ domain. Namely, we assume that (3.3) and (3.4) hold whenever $R \leq \rho_0$, for each $x_0 \in \partial\Omega$, and also that every point on $\partial\Omega$ can be touched from both inside and outside Ω by balls of radius ρ_0. In other words, given $x_0 \in \partial\Omega$, there are balls of radius ρ_0, $B_{\rho_0}(x_1) \subset \Omega$ and $B_{\rho_0}(x_2) \subset \mathbb{R}^n \setminus \Omega$, such that $\overline{B_{\rho_0}(x_1)} \cap \overline{B_{\rho_0}(x_2)} = \{x_0\}$. A useful observation is that all points y in the segment that joins x_1 and x_2 —through x_0— satisfy $\delta(y) = |y - x_0|$. Recall that $\delta = \text{dist}(\cdot, \partial\Omega)$.
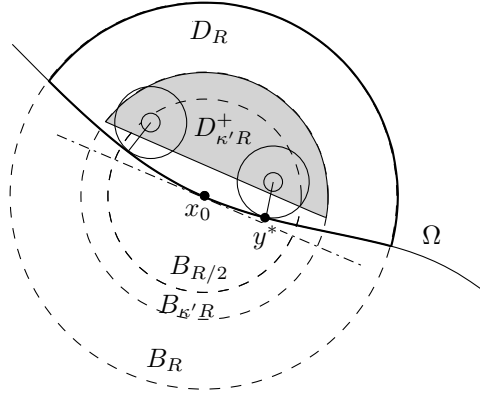
FIGURE 3.1.   The sets D_R and $D^+_{\kappa' R}$

In the rest of this section, by $|(-\Delta)^s u| \leq K$ we mean that either $(-\Delta)^s u = g$ in the weak sense for some $g \in L^\infty$ satisfying $\|g\|_{L^\infty} \leq K$ or that u satisfies $-K \leq (-\Delta)^s u \leq K$ in the viscosity sense.

The first (and main) step towards Theorem 1.2 is the following.

Proposition 3.5. *Let Ω be a bounded $C^{1,1}$ domain, and u be such that $|(-\Delta)^s u| \leq K$ in Ω and $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, for some constant K. Given any $x_0 \in \partial\Omega$, let D_R be as in Definition 3.3.*

Then, there exist $\alpha \in (0,1)$ and C depending only on Ω and s —but not on x_0— such that

$$\sup_{D_R} u/\delta^s - \inf_{D_R} u/\delta^s \leq CKR^\alpha \tag{3.5}$$

for all $R \leq \rho_0$, where $\rho_0 > 0$ is a constant depending only on Ω.

To prove Proposition 3.5 we need three preliminary lemmas. We start with the first one, which might be seen as the fractional version of Lemma 4.31 in [15]. Recall that $\kappa' \in (1/2, 1)$ is a fixed constant throughout the section. It may be useful to regard the following lemma as a bound by below for $\inf_{D_{R/2}} u/\delta^s$, rather than an upper bound for $\inf_{D^+_{\kappa' R}} u/\delta^s$.

Lemma 3.6. *Let Ω be a bounded $C^{1,1}$ domain, and u be such that $u \geq 0$ in all of $\mathbb{R}^n$ and $|(-\Delta)^s u| \leq K$ in D_R, for some constant K. Then, there exists a positive constant C, depending only on Ω and s, such that*

$$\inf_{D^+_{\kappa' R}} u/\delta^s \leq C\Big(\inf_{D_{R/2}} u/\delta^s + KR^s \Big) \tag{3.6}$$

for all $R \leq \rho_0$, where $\rho_0 > 0$ is a constant depending only on Ω.

Proof. Step 1. We do first the case $K = 0$. Let $R \leq \rho_0$, and let us call $m = \inf_{D^+_{\kappa' R}} u/\delta^s \geq 0$. We have $u \geq m\delta^s \geq m(\kappa R)^s$ on $D^+_{\kappa' R}$. The second inequality is a consequence of (3.3).

We scale the subsolution φ_2 in Lemma 3.2 as follows, to use it as lower barrier:

$$\psi_R(x) := (\kappa R)^s \varphi_2\left(\tfrac{x}{4\kappa R}\right).$$

By (3.2) we have

$$\begin{cases} (-\Delta)^s \psi_R \leq 0 & \text{in } B_{4\kappa R} \setminus B_{\kappa R} \\ \psi_R = (\kappa R)^s & \text{in } B_{\kappa R} \\ \psi_R \geq 4^{-s} C_2 (4\kappa R - |x|)^s & \text{in } B_{4\kappa R} \setminus B_{\kappa R} \\ \psi_R \equiv 0 & \text{in } \mathbb{R}^n \setminus B_{4\kappa R}. \end{cases}$$

Given $y \in D_{R/2}$, we have either $y \in D_{\kappa' R}^+$ or $\delta(y) < 4\kappa R$, by (3.4). If $y \in D_{\kappa' R}^+$ it follows from the definition of m that $m \leq u(y)/\delta(y)^s$. If $\delta(y) < 4\kappa R$, let y^* be the closest point to y on $\partial\Omega$ and $\tilde{y} = y^* + 4\kappa\nu(y^*)$. Again by (3.4), we have $B_{4\kappa R}(\tilde{y}) \subset D_R$ and $B_{\kappa R}(\tilde{y}) \subset D_{\kappa' R}^+$. But recall that $u \geq m(\kappa R)^s$ in $D_{\kappa' R}^+$, $(-\Delta)^s u = 0$ in Ω, and $u \geq 0$ in $\mathbb{R}^n$. Hence, $u(x) \geq m\psi_R(x - \tilde{y})$ in all $\mathbb{R}^n$ and in particular $u/\delta^s \geq 4^{-s} C_2 m$ on the segment joining y^* and $\tilde{y}$, that contains y. Therefore,

$$\inf_{D_{\kappa' R}^+} u/\delta^s \leq C \inf_{D_{R/2}} u/\delta^s. \tag{3.7}$$

Step 2. If $K > 0$ we consider $\tilde{u}$ to be the solution of

$$\begin{cases} (-\Delta)^s \tilde{u} = 0 & \text{in } D_R \\ \tilde{u} = u & \text{in } \mathbb{R}^n \setminus D_R. \end{cases}$$

By Step 1, (3.7) holds with u replaced by $\tilde{u}$.

On the other hand, $w = \tilde{u} - u$ satisfies $|(-\Delta)^s w| \leq K$ and $w \equiv 0$ outside D_R. Recall that points of $\partial\Omega$ can be touched by exterior balls of radius less than ρ_0. Hence, using the rescaled supersolution $KR^{2s}\varphi_1(x/R)$ from Lemma 2.6 as upper barrier and we readily prove, as in the proof of Lemma 2.7, that

$$|w| \leq C_1 K R^s \delta^s \quad \text{in } D_R.$$

Thus, (3.6) follows. $\qquad\square$

The second lemma towards Proposition 3.5, which might be seen as the fractional version of Lemma 4.35 in [15], is the following.

Lemma 3.7. *Let Ω be a bounded $C^{1,1}$ domain, and u be such that $u \geq 0$ in all of $\mathbb{R}^n$ and $|(-\Delta)^s u| \leq K$ in D_R, for some constant K. Then, there exists a positive constant C, depending on Ω and s, such that*

$$\sup_{D_{\kappa' R}^+} u/\delta^s \leq C\Big(\inf_{D_{\kappa' R}^+} u/\delta^s + KR^s \Big) \tag{3.8}$$

for all $R \leq \rho_0$, where $\rho_0 > 0$ is a constant depending only on Ω.

Proof. Step 1. Consider first the case $K = 0$. In this case (3.8) follows from the Harnack inequality for the fractional Laplacian [18] —note that we assume $u \geq 0$ in all $\mathbb{R}^n$. Indeed, by (3.3), for each $y \in D^+_{\kappa' R}$ we have $B_{\kappa R}(y) \subset D_R$ and hence $(-\Delta)^s u = 0$ in $B_{\kappa R}(y)$. Then we may cover $D^+_{\kappa' R}$ by a finite number of balls $B_{\kappa R/2}(y_i)$, using the same (scaled) covering for all $R \leq \rho_0$, to obtain

$$\sup_{B_{\kappa R/2}(y_i)} u \leq C \inf_{B_{\kappa R/2}(y_i)} u.$$

Then, (3.8) follows since $(\kappa R/2)^s \leq \delta^s \leq (3\kappa R/2)^s$ in $B_{\kappa R/2}(y_i)$ by (3.3).

Step 2. When $K > 0$, we prove (3.8) by using a similar argument as in Step 2 in the proof of Proposition 3.6. $\qquad \square$

Before proving Lemma 3.9 we give an extension lemma —see [11, Theorem 1, Section 3.1] where the case $\alpha = 1$ is proven in full detail.

Lemma 3.8. *Let $\alpha \in (0, 1]$ and $V \subset \mathbb{R}^n$ a bounded domain. There exists a (non-linear) map $E : C^{0,\alpha}(\overline{V}) \to C^{0,\alpha}(\mathbb{R}^n)$ satisfying*

$$E(w) \equiv w \quad in \ \overline{V}, \quad [E(w)]_{C^{0,\alpha}(\mathbb{R}^n)} \leq [w]_{C^{0,\alpha}(\overline{V})}, \quad and \quad \|E(w)\|_{L^\infty(\mathbb{R}^n)} \leq \|w\|_{L^\infty(V)}$$

for all $w \in C^{0,\alpha}(\overline{V})$.

Proof. It is immediate to check that

$$E(w)(x) = \min\left\{ \min_{z \in \overline{V}} \left\{ w(z) + [w]_{C^\alpha(\overline{V})} |z - x|^\alpha \right\}, \|w\|_{L^\infty(V)} \right\}$$

satisfies the conditions since, for all x, y, z in $\mathbb{R}^n$,

$$|z - x|^\alpha \leq |z - y|^\alpha + |y - x|^\alpha.$$

$\qquad \square$

We can now give the third lemma towards Proposition 3.5. This lemma, which is related to Proposition 3.1, is crucial. It states that $\delta^s|_\Omega$, extended by zero outside Ω, is an approximate solution in a neighborhood of $\partial\Omega$ inside Ω.

Lemma 3.9. *Let Ω be a bounded $C^{1,1}$ domain, and $\delta_0 = \delta\chi_\Omega$ be the distance function in Ω extended by zero outside Ω. Let $\alpha = \min\{s, 1 - s\}$, and ρ_0 be given by Remark 3.4. Then,*

$$(-\Delta)^s \delta_0^s \qquad belongs \ to \quad C^\alpha(\overline{\Omega_{\rho_0}}),$$

where $\Omega_{\rho_0} = \Omega \cap \{\delta < \rho_0\}$. In particular,

$$|(-\Delta)^s \delta_0^s| \leq C_\Omega \quad in \ \Omega_{\rho_0},$$

where C_Ω is a constant depending only on Ω and s.

Proof. Fix a point x_0 on $\partial\Omega$ and denote, for $\rho > 0$, $B_\rho = B_\rho(x_0)$. Instead of proving that

$$(-\Delta)^s \delta_0^s = c_{n,s} \mathrm{PV} \int_{\mathbb{R}^n} \frac{\delta_0(x)^s - \delta_0(y)^s}{|x - y|^{n+2s}} dy$$

is $C^\alpha(\overline{\Omega \cap B_{\rho_0}})$ —as a function of x—, we may equivalently prove that

$$\text{PV} \int_{B_{2\rho_0}} \frac{\delta_0(x)^s - \delta_0(y)^s}{|x - y|^{n+2s}} dy \qquad \text{belongs to} \qquad C^\alpha(\overline{\Omega \cap B_{\rho_0}}). \qquad (3.9)$$

This is because the difference

$$\frac{1}{c_{n,s}}(-\Delta)^s \delta_0^s - \text{PV} \int_{B_{2\rho_0}} \frac{\delta_0(x)^s - \delta_0(y)^s}{|x - y|^{n+2s}} dy = \int_{\mathbb{R}^n \setminus B_{2\rho_0}} \frac{\delta_0(x)^s - \delta_0(y)^s}{|x - y|^{n+2s}} dy$$

belongs to $C^s(\overline{B_{\rho_0}})$, since δ_0^s is $C^s(\mathbb{R}^n)$ and $|x|^{-n-2s}$ is integrable and smooth outside a neighborhood of 0.

To see (3.9), we flatten the boundary. Namely, consider a $C^{1,1}$ change of variables $X = \Psi(x)$, where $\Psi : B_{3\rho_0} \to V \subset \mathbb{R}^n$ is a $C^{1,1}$ diffeomorphism, satisfying that $\partial\Omega$ is mapped onto $\{X_n = 0\}$, $\Omega \cap B_{3\rho_0}$ is mapped into $\mathbb{R}^n_+$, and $\delta_0(x) = (X_n)_+$. Such diffeomorphism exists because we assume Ω to be $C^{1,1}$. Let us respectively call V_1 and V_2 the images of B_{ρ_0} and $B_{2\rho_0}$ under Ψ. Let us denote the points of $V \times V$ by (X, Y). We consider the functions x and y, defined in V, by $x = \Psi^{-1}(X)$ and $y = \Psi^{-1}(Y)$. With these notations, we have

$$x - y = -D\Psi^{-1}(X)(X - Y) + \mathcal{O}\left(|X - Y|^2\right),$$

and therefore

$$|x - y|^2 = (X - Y)^T A(X)(X - Y) + \mathcal{O}\left(|X - Y|^3\right), \qquad (3.10)$$

where

$$A(X) = \left(D\Psi^{-1}(X)\right)^T D\Psi^{-1}(X)$$

is a symmetric matrix, uniformly positive definite in $\overline{V_2}$. Hence,

$$\text{PV} \int_{B_{2\rho_0}} \frac{\delta_0(x)^s - \delta_0(y)^s}{|x - y|^{n+2s}} dy = \text{PV} \int_{V_2} \frac{(X_n)_+^s - (Y_n)_+^s}{|(X - Y)^T A(X)(X - Y)|^{\frac{n+2s}{2}}} g(X, Y) dY,$$

where we have denoted

$$g(X, Y) = \left(\frac{(X - Y)^T A(X)(X - Y)}{|x - y|^2}\right)^{\frac{n+2s}{2}} J(Y)$$

and $J = |\det D\Psi^{-1}|$. Note that we have $g \in C^{0,1}(\overline{V_2 \times V_2})$, since Ψ is $C^{1,1}$ and we have (3.10).

Now we are reduced to proving that

$$\psi_1(X) := \text{PV} \int_{V_2} \frac{(X_n)_+^s - (Y_n)_+^s}{|(X - Y)^T A(X)(X - Y)|^{\frac{n+2s}{2}}} g(X, Y) dY, \qquad (3.11)$$

belongs to $C^\alpha(\overline{V_1^+})$ (as a function of X), where $V_1^+ = V_1 \cap \{X_n > 0\}$.

To prove this, we extend the Lipschitz function $g \in C^{0,1}(\overline{V_2 \times V_2})$ to all $\mathbb{R}^n$. Namely, consider the function $g^* = E(g) \in C^{0,1}(\mathbb{R}^n \times \mathbb{R}^n)$ provided by Proposition 3.8, which satisfies

$$g^* \equiv g \text{ in } \overline{V_2 \times V_2} \quad \text{and} \quad \|g^*\|_{C^{0,1}(\mathbb{R}^n \times \mathbb{R}^n)} \leq \|g\|_{C^{0,1}(\overline{V_2 \times V_2})} \, .$$

By the same argument as above, using that $V_1 \subset\subset V_2$, we have that $\psi_1 \in C^\alpha(\overline{V_1^+})$ if and only if so is the function

$$\psi(X) = \text{PV} \int_{\mathbb{R}^n} \frac{(X_n)_+^s - (Y_n)_+^s}{|(X-Y)^T A(X)(X-Y)|^{\frac{n+2s}{2}}} g^*(X,Y) dY.$$

Furthermore, from g^* define $\tilde{g} \in C^{0,1}(\overline{V_2} \times \mathbb{R}^n)$ by $\tilde{g}(X,Z) = g^*(X, X+MZ) \det M$, where $M = M(X) = D\Psi(X)$. Then, using the change of variables $Y = X + MZ$ we deduce

$$\psi(X) = \text{PV} \int_{\mathbb{R}^n} \frac{(X_n)_+^s - \big(e_n \cdot (X+MZ)\big)_+^s}{|Z|^{n+2s}} \tilde{g}(X,Z) dZ.$$

Next, we prove that $\psi \in C^\alpha(\mathbb{R}^n)$, which concludes the proof. Indeed, taking into account that the function $(X_n)_+^s$ is s-harmonic in $\mathbb{R}_+^n$ —by Proposition 3.1— we obtain

$$\text{PV} \int_{\mathbb{R}^n} \frac{(e' \cdot X')_+^s - (e' \cdot (X'+Z))_+^s}{|Z|^{n+2s}} dZ = 0$$

for every $e' \in \mathbb{R}^n$ and for every X' such that $e' \cdot X' > 0$. Thus, letting $e' = e_n^T M$ and $X' = M^{-1}X$ we deduce

$$\text{PV} \int_{\mathbb{R}^n} \frac{(X_n)_+^s - \big(e_n \cdot (X+MZ)\big)_+^s}{|Z|^{n+2s}} dZ = 0$$

for every X such that $(e_n^T M) \cdot (M^{-1}X) > 0$, that is, for every $X \in \mathbb{R}_+^n$.

Therefore, it holds

$$\psi(X) = \int_{\mathbb{R}^n} \frac{\phi(X,0) - \phi(X,Z)}{|Z|^{n+2s}} \big(\tilde{g}(X,Z) - \tilde{g}(X,0)\big) dZ,$$

where

$$\phi(X,Z) = (e_n \cdot (X+MZ))_+^s$$

satisfies $[\phi]_{C^s(\overline{V_2} \times \mathbb{R}^n)} \leq C$, and $\|\tilde{g}\|_{C^{0,1}(\overline{V_2} \times \mathbb{R}^n)} \leq C$.

Let us finally prove that ψ belongs to $C^\alpha(\overline{V_1^+})$. To do it, let X and $\bar{X}$ be in $\overline{V_1^+}$. Then, we have

$$\psi(X) - \psi(\bar{X}) = \int_{\mathbb{R}^n} \frac{\Theta(X, \bar{X}, Z)}{|Z|^{n+2s}} dZ,$$

where

$$\begin{aligned}
\Theta(X,\bar{X},Z) &= \big(\phi(X,0) - \phi(X,Z)\big)\big(\tilde{g}(X,Z) - \tilde{g}(X,0)\big) \\
&\quad - \big(\phi(\bar{X},0) - \phi(\bar{X},Z)\big)\big(\tilde{g}(\bar{X},Z) - \tilde{g}(\bar{X},0)\big) \\
&= \big(\phi(X,0) - \phi(X,Z) - \phi(\bar{X},0) + \phi(\bar{X},Z)\big)\big(\tilde{g}(X,Z) - \tilde{g}(X,0)\big) \\
&\quad - \big(\phi(\bar{X},0) - \phi(\bar{X},Z)\big)\big(\tilde{g}(X,Z) - \tilde{g}(X,0) - \tilde{g}(\bar{X},Z) + \tilde{g}(\bar{X},0)\big).
\end{aligned} \tag{3.12}$$

Now, on the one hand, it holds

$$|\Theta(X,\bar{X},Z)| \le C|Z|^{1+s}, \tag{3.13}$$

since $[\phi]_{C^s(\overline{V_2}\times\mathbb{R}^n)} \le C$ and $\|\tilde{g}\|_{C^{0,1}(\overline{V_2}\times\mathbb{R}^n)} \le C$.

On the other hand, it also holds

$$|\Theta(X,\bar{X},Z)| \le C|X-\bar{X}|^s \min\{|Z|,|Z|^s\}. \tag{3.14}$$

Indeed, we only need to observe that

$$\begin{aligned}
\big|\tilde{g}(X,Z) - \tilde{g}(X,0) - \tilde{g}(\bar{X},Z) + \tilde{g}(\bar{X},0)\big| &\le C\min\big\{\min\{|Z|,1\},|X-\bar{X}|\big\} \\
&\le C\min\{|Z|^{1-s},1\}|X-\bar{X}|^s.
\end{aligned}$$

Thus, letting $r = |X-\bar{X}|$ and using (3.13) and (3.14), we obtain

$$\begin{aligned}
|\psi(X) - \psi(\bar{X})| &\le \int_{\mathbb{R}^n} \frac{|\Theta(X,\bar{X},Z)|}{|Z|^{n+2s}}dZ \\
&\le \int_{B_r} \frac{C|Z|^{1+s}}{|Z|^{n+2s}}dZ + \int_{\mathbb{R}^n\setminus B_r} \frac{Cr^s\min\{|Z|,|Z|^s\}}{|Z|^{n+2s}}dZ \\
&\le Cr^{1-s} + C\max\{r^{1-s},r^s\},
\end{aligned}$$

as desired. $\qquad\square$

Next we prove Proposition 3.5.

Proof of Proposition 3.5. By considering u/K instead of u we may assume that $K = 1$, that is, that $|(-\Delta)^s u| \le 1$ in Ω. Then, by Claim 2.8 we have $\|u\|_{L^\infty(\mathbb{R}^n)} \le C$ for some constant C depending only on Ω and s.

Let $\rho_0 > 0$ be given by Remark 3.4. Fix $x_0 \in \partial\Omega$. We will prove that there exist constants $C_0 > 0$, $\rho_1 \in (0,\rho_0)$, and $\alpha \in (0,1)$, depending only on Ω and s, and monotone sequences (m_k) and (M_k) such that, for all $k \ge 0$,

$$M_k - m_k = 4^{-\alpha k}, \quad -1 \le m_k \le m_{k+1} < M_{k+1} \le M_k \le 1, \tag{3.15}$$

and

$$m_k \le C_0^{-1}u/\delta^s \le M_k \quad \text{in } D_{R_k} = D_{R_k}(x_0), \quad \text{where } R_k = \rho_1 4^{-k}. \tag{3.16}$$

Note that (3.16) is equivalent to the following inequality in B_{R_k} instead of D_{R_k} — recall that $D_{R_k} = B_{R_k} \cap \Omega$.

$$m_k\delta_0^s \le C_0^{-1}u \le M_k\delta_0^s \quad \text{in } B_{R_k} = B_{R_k}(x_0), \quad \text{where } R_k = \rho_1 4^{-k}. \tag{3.17}$$

If there exist such sequences, then (3.5) holds for all $R \leq \rho_1$ with $C = 4^\alpha C_0/\rho_1^\alpha$. Then, by increasing the constant C if necessary, (3.5) holds also for every $R \leq \rho_0$.

Next we construct $\{M_k\}$ and $\{m_k\}$ by induction.

By Lemma 2.7, we find that there exist m_0 and M_0 such that (3.15) and (3.16) hold for $k = 0$ provided we pick C_0 large enough depending on Ω and s.

Assume that we have sequences up to m_k and M_k. We want to prove that there exist m_{k+1} and M_{k+1} which fulfill the requirements. Let

$$u_k = C_0^{-1}u - m_k\delta_0^s. \tag{3.18}$$

We will consider the positive part u_k^+ of u_k in order to have a nonnegative function in all of $\mathbb{R}^n$ to which we can apply Lemmas 3.6 and 3.7. Let $u_k = u_k^+ - u_k^-$. Observe that, by induction hypothesis,

$$u_k^+ = u_k \quad \text{and} \quad u_k^- = 0 \quad \text{in } B_{R_k}. \tag{3.19}$$

Moreover, $C_0^{-1}u \geq m_j\delta_0^s$ in B_{R_j} for each $j \leq k$. Therefore, by (3.18) we have

$$u_k \geq (m_j - m_k)\delta_0^s \geq (m_j - M_j + M_k - m_k)\delta_0^s \geq (-4^{-\alpha j} + 4^{-\alpha k})\delta_0^s \quad \text{in } B_{R_j}.$$

But clearly $0 \leq \delta_0^s \leq R_j^s = \rho_1^s 4^{-js}$ in B_{R_j}, and therefore using $R_j = \rho_1 4^{-j}$

$$u_k \geq -\rho_1^{-\alpha}R_j^s(R_j^\alpha - R_k^\alpha) \quad \text{in } B_{R_j} \text{ for each } j \leq k.$$

Thus, since for every $x \in B_{R_0} \setminus B_{R_k}$ there is $j < k$ such that

$$|x - x_0| < R_j = \rho_1 4^{-j} \leq 4|x - x_0|,$$

we find

$$u_k(x) \geq -\rho_1^{-\alpha}R_k^{\alpha+s}\left|\frac{4(x-x_0)}{R_k}\right|^s\left(\left|\frac{4(x-x_0)}{R_k}\right|^\alpha - 1\right) \quad \text{outside } B_{R_k}. \tag{3.20}$$

By (3.20) and (3.19), at $x \in B_{R_k/2}(x_0)$ we have

$$0 \leq -(-\Delta)^s u_k^-(x) = c_{n,s}\int_{x+y\notin B_{R_k}} \frac{u_k^-(x+y)}{|y|^{n+2s}}\,dy$$

$$\leq c_{n,s}\,\rho_1^{-\alpha}\int_{|y|\geq R_k/2} R_k^{\alpha+s}\left|\frac{8y}{R_k}\right|^s\left(\left|\frac{8y}{R_k}\right|^\alpha - 1\right)|y|^{-n-2s}\,dy$$

$$= C\rho_1^{-\alpha}R_k^{\alpha-s}\int_{|z|\geq 1/2} \frac{|8z|^s(|8z|^\alpha - 1)}{|z|^{n+2s}}\,dz$$

$$\leq \varepsilon_0\rho_1^{-\alpha}R_k^{\alpha-s},$$

where $\varepsilon_0 = \varepsilon_0(\alpha) \downarrow 0$ as $\alpha \downarrow 0$ since $|8z|^\alpha \to 1$.

Therefore, writing $u_k^+ = C_0^{-1}u - m_k\delta_0^s + u_k^-$ and using Lemma 3.9, we have

$$|(-\Delta)^s u_k^+| \leq C_0^{-1}|(-\Delta)^s u| + m_k|(-\Delta)^s\delta_0^s| + |(-\Delta)^s(u_k^-)|$$

$$\leq (C_0^{-1} + C_\Omega) + \varepsilon_0\rho_1^{-\alpha}R_k^{\alpha-s}$$

$$\leq \left(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\right)R_k^{\alpha-s} \quad \text{in } D_{R_k/2}.$$

In the last inequality we have just used $R_k \leq \rho_1$ and $\alpha \leq s$.

Now we can apply Lemmas 3.6 and 3.7 with u in its statements replaced by u_k^+, recalling that

$$u_k^+ = u_k = C_0^{-1}u - m_k\delta^s \quad \text{in } D_{R_k}$$

to obtain

$$\sup_{D_{\kappa'R_k/2}^+} (C_0^{-1}u/\delta^s - m_k) \leq C\left(\inf_{D_{\kappa'R_k/2}^+}(C_0^{-1}u/\delta^s - m_k) + \left(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\right)R_k^\alpha\right)$$

$$\leq C\left(\inf_{D_{R_k/4}}(C_0^{-1}u/\delta^s - m_k) + \left(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\right)R_k^\alpha\right) \quad (3.21)$$

Next we can repeat all the argument "upside down", that is, with the functions $u^k = M_k\delta^s - u$ instead of u_k. In this way we obtain, instead of (3.21), the following:

$$\sup_{D_{\kappa'R_k/2}^+} (M_k - C_0^{-1}u/\delta^s) \leq C\left(\inf_{D_{R_k/4}}(M^k - C_0^{-1}u/\delta^s) + \left(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\right)R_k^\alpha\right). \quad (3.22)$$

Adding (3.21) and (3.22) we obtain

$$M_k - m_k \leq C\left(\inf_{D_{R_k/4}}(C_0^{-1}u/\delta^s - m_k) + \inf_{D_{R_k/4}}(M_k - C_0^{-1}u/\delta^s) + \left(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\right)R_k^\alpha\right)$$

$$= C\left(\inf_{D_{R_{k+1}}}C_0^{-1}u/\delta^s - \sup_{D_{R_{k+1}}}C_0^{-1}u/\delta^s + M_k - m_k + \left(C_1\rho_1^{s-\alpha} + \varepsilon_0\rho_1^{-\alpha}\right)R_k^\alpha\right),$$

$$(3.23)$$

and thus, using that $M_k - m_k = 4^{-\alpha k}$ and $R_k = \rho_1 4^{-k}$,

$$\sup_{D_{R_{k+1}}}C_0^{-1}u/\delta^s - \inf_{D_{R_{k+1}}}C_0^{-1}u/\delta^s \leq \left(\tfrac{C-1}{C} + C_1\rho_1^s + \varepsilon_0\right)4^{-\alpha k}.$$

Now we choose α and ρ_1 small enough so that

$$\frac{C-1}{C} + C_1\rho_1^s + \varepsilon_0(\alpha) \leq 4^{-\alpha}.$$

This is possible since $\varepsilon_0(\alpha) \downarrow 0$ as $\alpha \downarrow 0$ and the constants C and C_1 do not depend on α nor ρ_1 —they depend only on Ω and s. Then, we find

$$\sup_{D_{R_{k+1}}}C_0^{-1}u/\delta^s - \inf_{D_{R_{k+1}}}C_0^{-1}u/\delta^s \leq 4^{-\alpha(k+1)},$$

and thus we are able to choose m_{k+1} and M_{k+1} satisfying (3.15) and (3.16). $\qquad\square$

Finally, we give the:

Proof of Theorem 1.2. Define $v = u/\delta^s|_\Omega$ and $K = \|g\|_{L^\infty(\Omega)}$. As in the proof of Proposition 3.5, by considering u/K instead of u we may assume that $|(-\Delta)^s u| \leq 1$ in Ω and that $\|u\|_{L^\infty(\Omega)} \leq C$ for some constant C depending only on Ω and s.

First we claim that there exist constants C, $M > 0$, $\widetilde{\alpha} \in (0, 1)$ and $\beta \in (0, 1)$, depending only on Ω and s, such that

(i) $\|v\|_{L^\infty(\Omega)} \leq C$.

(ii) For all $x \in \Omega$, it holds the seminorm bound

$$[v]_{C^\beta(\overline{B_{R/2}(x)})} \leq C\left(1 + R^{-M}\right),$$

where $R = \text{dist}(x, \mathbb{R}^n \setminus \Omega)$.

(iii) For each $x_0 \in \partial\Omega$ and for all $\rho > 0$ it holds

$$\sup_{B_\rho(x_0)\cap\Omega} v - \inf_{B_\rho(x_0)\cap\Omega} v \leq C\rho^{\widetilde{\alpha}}.$$

Indeed, it follows from Lemma 2.7 that $\|v\|_{L^\infty(\Omega)} \leq C$ for some C depending only on Ω and s. Hence, (i) is satisfied.

Moreover, if $\beta \in (0, 2s)$, it follows from Lemma 2.9 that for every $x \in \Omega$,

$$[u]_{C^\beta(B_{R/2}(x))} \leq CR^{-\beta}, \qquad \beta \in (0, 2s),$$

where $R = \delta(x)$. But since Ω is $C^{1,1}$, then provided $\delta(x) < \rho_0$ we will have

$$\|\delta^{-s}\|_{L^\infty(B_{R/2}(x))} \leq CR^{-s} \quad \text{and} \quad [\delta^{-s}]_{C^{0,1}(B_{R/2}(x))} \leq CR^{-s-1}$$

and hence, by interpolation,

$$[\delta^{-s}]_{C^\beta(B_{R/2}(x))} \leq CR^{-s-\beta}$$

for each $\beta \in (0, 1)$. Thus, since $v = u\delta^{-s}$, we find

$$[v]_{C^\beta(B_{R/2}(x))} \leq C\left(1 + R^{-s-\beta}\right)$$

for all $x \in \Omega$ and $\beta < \min\{1, 2s\}$. Therefore hypothesis (ii) is satisfied. The constants C depend only on Ω and s.

In addition, using Proposition 3.5 and that $\|v\|_{L^\infty(\Omega)} \leq C$, we deduce that hypothesis (iii) is satisfied.

Now, we claim that (i)-(ii)-(iii) lead to

$$[v]_{C^\alpha(\overline{\Omega})} \leq C,$$

for some $\alpha \in (0, 1)$ depending only on Ω and s.

Indeed, let $x, y \in \Omega$, $R = \text{dist}(x, \mathbb{R}^n \setminus \Omega) \geq \text{dist}(y, \mathbb{R}^n \setminus \Omega)$, and $r = |x - y|$. Let us see that $|v(x) - v(y)| \leq Cr^\alpha$ for some $\alpha > 0$.

If $r \geq 1$ then it follows from (i). Assume $r < 1$, and let $p \geq 1$ to be chosen later. Then, we have the following dichotomy:

Case 1. Assume $r \geq R^p/2$. Let $x_0, y_0 \in \partial\Omega$ be such that $|x - x_0| = \text{dist}(x, \mathbb{R}^n \setminus \Omega)$ and $|y - y_0| = \text{dist}(y, \mathbb{R}^n \setminus \Omega)$. Then, using (iii) and the definition of R we deduce

$$|v(x) - v(y)| \leq |v(x) - v(x_0)| + |v(x_0) - v(y_0)| + |v(y_0) - v(y)| \leq CR^{\widetilde{\alpha}} \leq Cr^{\widetilde{\alpha}/p}.$$

Case 2. Assume $r \leq R^p/2$. Hence, since $p \geq 1$, we have $y \in B_{R/2}(x)$. Then, using (ii) we obtain

$$|v(x) - v(y)| \leq C(1 + R^{-M})r^\beta \leq C\left(1 + r^{-M/p}\right)r^\beta \leq Cr^{\beta - M/p}.$$

To finish the proof we only need to choose $p > M/\beta$ and take $\alpha = \min\{\widetilde{\alpha}/p, \beta - M/p\}$. $\qquad\square$

4. Interior estimates for u/δ^s

The main goal of this section is to prove the C^γ bounds in Ω for the function u/δ^s in Theorem 1.5.

To prove this result we find an equation for the function $v = u/\delta^s|_\Omega$, that is derived below. This equation is nonlocal, and thus, we need to give values to v in $\mathbb{R}^n \setminus \Omega$, although we want an equation only in Ω. It might seem natural to consider u/δ^s, which vanishes outside Ω since $u \equiv 0$ there, as an extension of $u/\delta^s|_\Omega$. However, such extension is discontinuous through $\partial\Omega$, and it would lead to some difficulties.

Instead, we consider a $C^\alpha(\mathbb{R}^n)$ extension of the function $u/\delta^s|_\Omega$, which is $C^\alpha(\overline{\Omega})$ by Theorem 1.2. Namely, throughout this section, let v be the $C^\alpha(\mathbb{R}^n)$ extension of $u/\delta^s|_\Omega$ given by Lemma 3.8.

Let $\delta_0 = \delta\chi_\Omega$, and note that $u = v\delta_0^s$ in $\mathbb{R}^n$. Then, using (1.1) we have

$$g(x) = (-\Delta)^s(v\delta_0^s) = v(-\Delta)^s\delta_0^s + \delta_0^s(-\Delta)^s v - I_s(v, \delta_0^s)$$

in $\Omega_{\rho_0} = \{x \in \Omega : \delta(x) < \rho_0\}$, where

$$I_s(w_1, w_2)(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{\big(w_1(x) - w_1(y)\big)\big(w_2(x) - w_2(y)\big)}{|x - y|^{n+2s}} \, dy \qquad (4.1)$$

and ρ_0 is a small constant depending on the domain; see Remark 3.4. Here, we have used that $(-\Delta)^s(w_1 w_2) = w_1(-\Delta)^s w_2 + w_2(-\Delta)^s w_1 - I_s(w_1, w_2)$, which follows easily from (1.2). This equation is satisfied pointwise in Ω_{ρ_0}, since g is C^α in Ω. We have to consider Ω_{ρ_0} instead of Ω because the distance function is $C^{1,1}$ there and thus we can compute $(-\Delta)^s\delta_0^s$. In all Ω the distance function δ is only Lipschitz and hence $(-\Delta)^s\delta_0^s$ is singular for $s \geq \frac{1}{2}$.

Thus, the following is the equation for v:

$$(-\Delta)^s v = \frac{1}{\delta_0^s}\left(g(x) - v(-\Delta)^s\delta_0^s + I_s(v, \delta_0^s)\right) \quad \text{in } \Omega_{\rho_0}. \qquad (4.2)$$

From this equation we will obtain the interior estimates for v. More precisely, we will obtain a priori bounds for the interior Hölder norms of v, treating $\delta_0^{-s}I_s(v, \delta_0^s)$ as a lower order term. For this, we consider the weighted Hölder norms given by Definition 1.3.

Recall that, in all the paper, we denote C^β the space $C^{k,\beta'}$, where $\beta = k + \beta'$ with k integer and $\beta' \in (0, 1]$.

In Theorem 1.2 we have proved that $u/\delta^s|_\Omega$ is $C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$, with an estimate. From this C^α estimate and from the equation for v (4.2), we will find next the estimate for $\|u/\delta^s\|_{\gamma;\Omega}^{(-\alpha)}$ stated in Theorem 1.5.

The proof of this result relies on some preliminary results below.

Next lemma is used to control the lower order term $\delta_0^{-s}I_s(v, \delta_0^s)$ in the equation (4.2) for v.

Lemma 4.1. *Let Ω be a bounded $C^{1,1}$ domain, and $U \subset \Omega_{\rho_0}$ be an open set. Let s and α belong to $(0,1)$ and satisfy $\alpha + s \leq 1$ and $\alpha < s$. Then,*

$$\|I_s(w, \delta_0^s)\|_{\alpha;U}^{(s-\alpha)} \leq C\left([w]_{C^\alpha(\mathbb{R}^n)} + [w]_{\alpha+s;U}^{(-\alpha)}\right), \tag{4.3}$$

for all w with finite right hand side. The constant C depends only on Ω, s, and α.

To prove Lemma 4.1 we need the next

Lemma 4.2. *Let $U \subset \mathbb{R}^n$ be a bounded open set. Let $\alpha_1, \alpha_2, \in (0,1)$ and $\beta \in (0,1]$ satisfy $\alpha_i < \beta$ for $i = 1,2$, $\alpha_1 + \alpha_2 < 2s$, and $s < \beta < 2s$. Assume that $w_1, w_2 \in C^\beta(U)$. Then,*

$$\|I_s(w_1, w_2)\|_{2\beta-2s;U}^{(2s-\alpha_1-\alpha_2)} \leq C\left([w_1]_{C^{\alpha_1}(\mathbb{R}^n)} + [w_1]_{\beta;U}^{(-\alpha_1)}\right)\left([w_2]_{C^{\alpha_2}(\mathbb{R}^n)} + [w_2]_{\beta;U}^{(-\alpha_2)}\right), \tag{4.4}$$

for all functions w_1, w_2 with finite right hand side. The constant C depends only on α_1, α_2, n, β, and s.

Proof. Let $x_0 \in U$ and $R = d_{x_0}/2$, and denote $B_\rho = B_\rho(x_0)$. Let

$$K = \left([w_1]_{C^{\alpha_1}(\mathbb{R}^n)} + [w_1]_{\beta;U}^{(-\alpha_1)}\right)\left([w_2]_{C^{\alpha_2}(\mathbb{R}^n)} + [w_2]_{\beta;U}^{(-\alpha_2)}\right).$$

First we bound $|I_s(w_1, w_2)(x_0)|$.

$$\begin{aligned}
|I_s(w_1, w_2)(x_0)| &\leq C\int_{\mathbb{R}^n} \frac{|w_1(x_0) - w_1(y)||w_2(x_0) - w_2(y)|}{|x_0 - y|^{n+2s}}\, dy \\
&\leq C\int_{B_R(0)} \frac{R^{\alpha_1+\alpha_2-2\beta}[w_1]_{\beta;U}^{(-\alpha_1)}[w_2]_{\beta;U}^{(-\alpha_2)}|z|^{2\beta}}{|z|^{n+2s}}\, dz + \\
&\qquad + C\int_{\mathbb{R}^n \setminus B_R(0)} \frac{[w_1]_{C^{\alpha_1}(\mathbb{R}^n)}[w_2]_{C^{\alpha_2}(\mathbb{R}^n)}|z|^{\alpha_1+\alpha_2}}{|z|^{n+2s}}\, dz \\
&\leq CR^{\alpha_1+\alpha_2-2s}K.
\end{aligned}$$

Let $x_1, x_2 \in B_{R/2}(x_0) \subset B_{2R}(x_0)$. Next, we bound $|I_s(w_1, w_2)(x_1) - I_s(w_1, w_2)(x_2)|$. Let η be a smooth cutoff function such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ outside $B_{3/2}(0)$. Define

$$\eta^R(x) = \eta\left(\frac{x - x_0}{R}\right) \quad \text{and} \quad \bar{w}_i = (w_i - w_i(x_0))\eta^R, \quad i = 1, 2.$$

Note that we have

$$\|\bar{w}_i\|_{L^\infty(\mathbb{R}^n)} = \|\bar{w}_i\|_{L^\infty(B_{3R/2})} \leq \left(\frac{3R}{2}\right)^{\alpha_i}[w_i]_{C^{\alpha_i}(\mathbb{R}^n)}$$

and

$$[\bar{w}_i]_{C^\beta(\mathbb{R}^n)} \le C\left([w_i]_{C^\beta(\overline{B_{3R/2}})}\|\eta\|_{L^\infty(B_{3R/2})} + \|w_i - w_i(0)\|_{L^\infty(B_{3R/2})}[w_i]_{C^\beta(\overline{B_{3R/2}})}\right)$$

$$\le CR^{\alpha_i-\beta}\left([w_i]_{C^{\alpha_i}(\mathbb{R}^n)} + [w_i]_{\beta;U}^{(-\alpha_i)}\right).$$

Let

$$\varphi_i = w_i - w_i(x_0) - \bar{w}_i$$

and observe that φ_i vanishes in B_R. Hence, $\varphi_i(x_1) = \varphi_i(x_2) = 0$, $i = 1, 2$. Next, let us write

$$I_s(w_1, w_2)(x_1) - I_s(w_1, w_2)(x_2) = c_{n,s}\left(J_{11} + J_{12} + J_{21} + J_{22}\right),$$

where

$$J_{11} = \int_{\mathbb{R}^n} \frac{\left(\bar{w}_1(x_1) - \bar{w}_1(y)\right)\left(\bar{w}_2(x_1) - \bar{w}_2(y)\right)}{|x_1 - y|^{n+2s}}\, dy$$

$$- \int_{\mathbb{R}^n} \frac{\left(\bar{w}_1(x_2) - \bar{w}_1(y)\right)\left(\bar{w}_2(x_2) - \bar{w}_2(y)\right)}{|x_2 - y|^{n+2s}}\, dy,$$

$$J_{12} = \int_{\mathbb{R}^n \setminus B_R} \frac{-\left(\bar{w}_1(x_1) - \bar{w}_1(y)\right)\varphi_2(y)}{|x_1 - y|^{n+2s}} + \frac{\left(\bar{w}_1(x_2) - \bar{w}_1(y)\right)\varphi_2(y)}{|x_2 - y|^{n+2s}}\, dy,$$

$$J_{21} = \int_{\mathbb{R}^n \setminus B_R} \frac{-\left(\bar{w}_2(x_1) - \bar{w}_2(y)\right)\varphi_1(y)}{|x_1 - y|^{n+2s}} + \frac{\left(\bar{w}_2(x_2) - \bar{w}_2(y)\right)\varphi_1(y)}{|x_2 - y|^{n+2s}}\, dy,$$

and

$$J_{22} = \int_{\mathbb{R}^n \setminus B_R} \frac{\varphi_1(y)\varphi_2(y)}{|x_1 - y|^{n+2s}} - \frac{\varphi_1(y)\varphi_2(y)}{|x_2 - y|^{n+2s}}\, dy.$$

We now bound separately each of these terms.

Bound of J_{11}. We write $J_{11} = J_{11}^1 + J_{11}^2$ where

$$J_{11}^1 = \int_{\mathbb{R}^n} \frac{\left(\bar{w}_1(x_1) - \bar{w}_1(x_1 + z) - \bar{w}_1(x_2) + \bar{w}_1(x_2 + z)\right)\left(\bar{w}_2(x_1) - \bar{w}_2(x_1 + z)\right)}{|z|^{n+2s}}\, dz,$$

$$J_{11}^2 = \int_{\mathbb{R}^n} \frac{\left(\bar{w}_1(x_2) - \bar{w}_1(x_2 + z)\right)\left(\bar{w}_2(x_1) - \bar{w}_2(x_1 + z) - \bar{w}_2(x_2) + \bar{w}_2(x_2 + z)\right)}{|z|^{n+2s}}\, dz.$$

To bound $|J_{11}^1|$ we proceed as follows

$$|J_{11}^1| \le \int_{B_r(0)} \frac{R^{\alpha_1-\beta}[w_1]_{\beta;U}^{(-\alpha_1)}|z|^\beta R^{\alpha_2-\beta}[w_2]_{\beta;U}^{(-\alpha_2)}|z|^\beta}{|z|^{n+2s}}\, dz +$$

$$+ \int_{\mathbb{R}^n \setminus B_r(0)} \frac{R^{\alpha_1-\beta}[w_1]_{\beta;U}^{(-\alpha_1)}r^\beta R^{\alpha_2-\beta}[w_2]_{\beta;U}^{(-\alpha_2)}|z|^\beta}{|z|^{n+2s}}\, dz$$

$$\le CR^{\alpha_1+\alpha_2-2\beta}r^{2\beta-2s}K.$$

Similarly, $|J_{11}^2| \le CR^{\alpha_1+\alpha_2-2\beta}r^{2\beta-2s}K$.

Bound of J_{12} and J_{21}. We write $J_{12} = J_{12}^1 + J_{12}^2$ where

$$J_{12}^1 = \int_{\mathbb{R}^n \setminus B_R} -\varphi_2(y) \frac{\bar{w}_1(x_1) - \bar{w}_1(x_2)}{|x_1 - y|^{n+2s}} \, dy$$

and

$$J_{12}^2 = \int_{\mathbb{R}^n \setminus B_R} -\varphi_2(y) \big(\bar{w}_1(x_2) - \bar{w}_1(y)\big) \left\{ \frac{1}{|x_1 - y|^{n+2s}} - \frac{1}{|x_2 - y|^{n+2s}} \right\} dy \,.$$

To bound $|J_{12}^1|$ we recall that $\varphi_2(x_1) = 0$ and proceed as follows

$$|J_{12}^1| \leq C \int_{\mathbb{R}^n \setminus B_R} |x_1 - y|^{\alpha_2} [\varphi_2]_{C^{0,\alpha_2}(\mathbb{R}^n)} \frac{R^{\alpha_1 - \beta} [w_1]_{\beta;U}^{(-\alpha_1)} r^{\beta}}{|x_1 - y|^{n+2s}} \, dy$$

$$\leq C R^{\alpha_1 + \alpha_2 - \beta - 2s} r^{\beta} K \leq C R^{\alpha_1 + \alpha_2 - 2\beta} r^{2\beta - 2s} K.$$

We have used that $[\varphi_2]_{C^{\alpha_2}(\mathbb{R}^n)} = [w - \bar{w}]_{C^{\alpha_2}(\mathbb{R}^n)} \leq 2[w]_{C^{\alpha_2}(\mathbb{R}^n)}$, $r \leq R$, and $\beta < 2s$.

To bound $|J_{12}^2|$, let $\Phi(z) = |z|^{-n-2s}$. Note that, for each $\gamma \in (0,1]$, we have

$$|\Phi(z_1 - z) - \Phi(z_2 - z)| \leq C|z_1 - z_2|^{\gamma} |z|^{-n-2s-\gamma} \tag{4.5}$$

for all z_1, z_2 in $B_{R/2}(0)$ and $z \in \mathbb{R}^n \setminus B_R(0)$. Then, using that $\varphi_2(x_2) = 0$,

$$|J_{12}^2| \leq C \int_{\mathbb{R}^n \setminus B_R} |x_2 - y|^{\alpha_1 + \alpha_2} [\varphi_2]_{C^{\alpha_2}(\mathbb{R}^n)} [\varphi_2]_{C^{\alpha_2}(\mathbb{R}^n)} \frac{|x_1 - x_2|^{2\beta - 2s}}{|x_2 - y|^{n+2\beta}} \, dy$$

$$\leq C R^{\alpha_1 + \alpha_2 - 2\beta} r^{2\beta - 2s} K \,.$$

This proves that $|J_{12}| \leq C R^{\alpha_1 + \alpha_2 - 2\beta} r^{2\beta - 2s} K$. Changing the roles of α_1 and α_2 we obtain the same bound for $|J_{21}|$.

Bound of J_{22}. Using again $\varphi_i(x_i) = 0$, $i = 1, 2$, we write

$$J_{22} = \int_{\mathbb{R}^n \setminus B_R} \big(\varphi_1(x_1) - \varphi_1(y)\big) \big(\varphi_2(x_1) - \varphi_2(y)\big) \left(\frac{1}{|x_1 - y|^{n+2s}} - \frac{1}{|x_2 - y|^{n+2s}} \right) dy \,.$$

Hence, using again (4.5),

$$|J_{22}| \leq C \int_{\mathbb{R}^n \setminus B_R} |x_1 - y|^{\alpha_1 + \alpha_2} [\varphi_2]_{C^{0,\alpha_2}(\mathbb{R}^n)} [\varphi_2]_{C^{0,\alpha_2}(\mathbb{R}^n)} \frac{|x_1 - x_2|^{2\beta - 2s}}{|x_1 - y|^{n+2\beta}} \, dy$$

$$\leq C R^{\alpha_1 + \alpha_2 - 2\beta} r^{2\beta - 2s} K \,.$$

Summarizing, we have proven that for all x_0 such that $d_x = 2R$ and for all $x_1, x_2 \in B_{R/2}(x_0)$ it holds

$$|I_s(\delta_0^s, w)(x_0)| \leq C R^{\alpha_1 - \alpha_2 - 2s} K$$

and

$$\frac{|I_s(\delta_0^s, w)(x_1) - I_s(\delta_0^s, w)(x_2)|}{|x_1 - x_2|^{2\beta - 2s}} \leq C R^{\alpha_1 + \alpha_2 - 2\beta} \big([w]_{\alpha+s;U}^{(-\alpha)} + [w]_{C^{\alpha}(\mathbb{R}^n)}\big) \,.$$

This yields (4.4), as shown in Step 2 in the proof of Lemma 2.10. $\qquad\square$

Next we prove Lemma 4.1.

Proof of Lemma 4.1. The distance function δ_0 is $C^{1,1}$ in $\overline{\Omega_{\rho_0}}$ and since $U \subset \Omega_{\rho_0}$ we have $d_x \leq \delta_0(x)$ for all $x \in U$. Hence, it follows that

$$[\delta_0^s]_{C^s(\mathbb{R}^n)} + [\delta_0^s]_{\beta;U}^{(-s)} \leq C(\Omega, \beta)$$

for all $\beta \in [s, 2]$.

Then, applying Lemma 4.2 with $w_1 = w$, $w_2 = \delta_0^s$, $\alpha_1 = \alpha$, $\alpha_2 = s$, and $\beta = s + \alpha$, we obtain

$$\|I_s(w, \delta_0^s)\|_{2\alpha;U}^{(s-\alpha)} \leq C\left([w]_{C^\alpha(\mathbb{R}^n)} + [w]_{\alpha+s;U}^{(-\alpha)}\right),$$

and hence (4.3) follows. $\qquad\square$

Using Lemma 4.1 we can now prove Theorem 1.5 and Corollary 1.6.

Proof of Theorem 1.5. Let $U \subset\subset \Omega_{\rho_0}$. We prove first that there exist $\alpha \in (0, 1)$ and C, depending only on s and Ω —and not on U—, such that

$$\|u/\delta^s\|_{\alpha+2s;U}^{(-\alpha)} \leq C\left(\|g\|_{L^\infty(\Omega)} + \|g\|_{\alpha;\Omega}^{(s-\alpha)}\right).$$

Then, letting $U \uparrow \Omega_{\rho_0}$ we will find that this estimate holds in Ω_{ρ_0} with the same constant.

To prove this, note that by Theorem 1.2 we have

$$\|u/\delta^s\|_{C^\alpha(\overline{\Omega})} \leq C(s, \Omega)\|g\|_{L^\infty(\Omega)}.$$

Recall that v denotes the $C^\alpha(\mathbb{R}^n)$ extension of $u/\delta^s|_\Omega$ given by Lemma 3.8, which satisfies $\|v\|_{C^\alpha(\mathbb{R}^n)} = \|u/\delta^s\|_{C^\alpha(\overline{\Omega})}$. Since $u \in C^{\alpha+2s}(\Omega)$ and $\delta \in C^{1,1}(\Omega_{\rho_0})$, it is clear that $\|v\|_{\alpha+2s;U}^{(-\alpha)} < \infty$ —it is here where we use that we are in a subdomain U and not in Ω_{ρ_0}. Next we obtain an a priori bound for this seminorm in U. To do it, we use the equation (4.2) for v:

$$(-\Delta)^s v = \frac{1}{\delta^s}\left(g(x) - v(-\Delta)^s\delta_0^s + I(\delta_0^s, v)\right) \quad \text{in } \Omega_{\rho_0} = \{x \in \Omega : \delta(x) < \rho_0\}.$$

Now we will se that this equation and Lemma 2.10 lead to an a priori bound for $\|v\|_{\alpha+2s;U}^{(-\alpha)}$. To apply Lemma 2.10, we need to bound $\|(-\Delta)^s v\|_{\alpha;U}^{(2s-\alpha)}$. Let us examine the three terms on the right hand side of the equation.

First term. Using that

$$d_x = \text{dist}(x, \partial U) < \text{dist}(x, \partial\Omega) = \delta(x)$$

for all $x \in U$ we obtain that, for all $\alpha \leq s$,

$$\|\delta^{-s}g\|_{\alpha;U}^{(2s-\alpha)} \leq C(s, \Omega)\|g\|_{\alpha;\Omega}^{(s-\alpha)}.$$

Second term. We know from Lemma 3.9 that, for $\alpha \leq \min\{s, 1-s\}$,

$$\|(-\Delta)^s\delta_0^s\|_{C^\alpha(\overline{\Omega_{\rho_0}})} \leq C(s, \Omega).$$

Hence,

$$\|\delta^{-s}v(-\Delta)^s\delta_0^s\|_{\alpha;U}^{(2s-\alpha)} \leq \mathrm{diam}(\Omega)^s\|\delta^{-s}v(-\Delta)^s\delta_0^s\|_{\alpha;U}^{(s-\alpha)} \leq C\big(s,\Omega\big)\|v\|_{C^\alpha(\mathbb{R}^n)}$$
$$\leq C\big(s,\Omega\big)\|g\|_{L^\infty(\Omega)}\,.$$

Third term. From Lemma 4.1 we know that

$$\|I(v,\delta_0^s)\|_{\alpha;U}^{(s-\alpha)} \leq C(n,s,\alpha)\bigg(\|v\|_{C^\alpha(\mathbb{R}^n)} + [v]_{\alpha+s;U}^{(-\alpha)}\bigg),$$

and hence

$$\|\delta^{-s}I(v,\delta_0^s)\|_{\alpha;U}^{(2s-\alpha)} \leq C(n,s,\Omega,\alpha)\bigg(\|v\|_{C^\alpha(\mathbb{R}^n)} + [v]_{\alpha+s;U}^{(-\alpha)}\bigg)$$
$$\leq C(n,s,\Omega,\alpha,\varepsilon_0)\|v\|_{C^\alpha(\mathbb{R}^n)} + \varepsilon_0\|v\|_{\alpha+2s;U}^{(-\alpha)}$$

for each $\epsilon_0 > 0$. The last inequality is by standard interpolation.

Now, using Lemma 2.10 we deduce

$$\|v\|_{\alpha+2s;U}^{(-\alpha)} \leq C\left(\|v\|_{C^\alpha(\mathbb{R}^n)} + \|(-\Delta)^s v\|_{\alpha;U}^{(2s-\alpha)}\right)$$
$$\leq C\left(\|v\|_{C^\alpha(\mathbb{R}^n)} + \|\delta^{-s}g\|_{\alpha;U}^{(2s-\alpha)} + \|\delta^{-s}v(-\Delta)^s\delta_0^s\|_{\alpha;U}^{(2s-\alpha)} + \|I(v,\delta_0^s)\|_{\alpha;U}^{(s-\alpha)}\right)$$
$$\leq C(s,\Omega,\alpha,\varepsilon_0)\left(\|g\|_{L^\infty(\Omega)} + \|g\|_{\alpha;\Omega}^{(s-\alpha)}\right) + C\varepsilon_0\|v\|_{\alpha+2s;U}^{(-\alpha)},$$

and choosing ε_0 small enough we obtain

$$\|v\|_{\alpha+2s;U}^{(-\alpha)} \leq C\left(\|g\|_{L^\infty(\Omega)} + \|g\|_{\alpha;\Omega}^{(s-\alpha)}\right).$$

Furthermore, letting $U \uparrow \Omega_{\rho_0}$ we obtain that the same estimate holds with U replaced by Ω_{ρ_0}.

Finally, in $\Omega \setminus \Omega_{\rho_0}$ we have that u is $C^{\alpha+2s}$ and δ^s is uniformly positive and $C^{0,1}$. Thus, we have $u/\delta^s \in C^\gamma(\Omega \setminus \Omega_{\rho_0})$, where $\gamma = \min\{1,\alpha+2s\}$, and the theorem follows. $\qquad\square$

Next we give the

Proof of Corollary 1.6. (a) It follows from Proposition 1.1 that $u \in C^s(\mathbb{R}^n)$. The interior estimate follow by applying repeatedly Proposition 1.4.

(b) It follows from Theorem 1.2 that $u/\delta^s|_\Omega \in C^\alpha(\overline{\Omega})$. The interior estimate follows from Theorem 1.5. $\qquad\square$

The following two lemmas are closely related to Lemma 4.2 and are needed in [20] and in Remark 2.11 of this paper.

Lemma 4.3. *Let U be an open domain and α and β be such that $\alpha \leq s < \beta$ and $\beta - s$ is not an integer. Let k be an integer such that $\beta = k + \beta'$ with $\beta' \in (0,1]$. Then,*

$$[(-\Delta)^{s/2}w]_{\beta-s;U}^{(s-\alpha)} \leq C\big(\|w\|_{C^\alpha(\mathbb{R}^n)} + \|w\|_{\beta;U}^{(-\alpha)}\big), \qquad (4.6)$$

for all w with finite right hand side. The constant C depends only on n, s, α, and β.

Proof. Let $x_0 \in U$ and $R = d_{x_0}/2$, and denote $B_\rho = B_\rho(x_0)$. Let η be a smooth cutoff function such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ outside $B_{3/2}(0)$. Define

$$\eta^R(x) = \eta\left(\frac{x - x_0}{R}\right) \quad \text{and} \quad \bar{w} = (w - w(x_0))\eta^R.$$

Note that we have

$$\|\bar{w}\|_{L^\infty(\mathbb{R}^n)} = \|\bar{w}\|_{L^\infty(\overline{B_{3R/2}})} \leq \left(\frac{3R}{2}\right)^\alpha [w]_{C^\alpha(\mathbb{R}^n)}.$$

In addition, for each $1 \leq l \leq k$

$$\|D^l \bar{w}\|_{L^\infty(\mathbb{R}^n)} \leq C \sum_{m=0}^{l} \|D^m(w - w(x_0))D^{l-m}\eta^R\|_{L^\infty(\overline{B_{3R/2}})}$$

$$\leq C R^{-l+\alpha}\left([w]_{C^\alpha(\mathbb{R}^n)} + \sum_{m=1}^{l}[w]_{m,U}^{(-\alpha)}\right).$$

Hence, by interpolation, for each $0 \leq l \leq k - 1$

$$\|D^l \bar{w}\|_{C^{l+\beta'}(\mathbb{R}^n)} \leq C R^{-l-\beta'+\alpha}\left([w]_{C^\alpha(\mathbb{R}^n)} + \sum_{m=1}^{l}[w]_{m,U}^{(-\alpha)}\right),$$

and therefore

$$[D^k \bar{w}]_{C^{\beta'}(\mathbb{R}^n)} \leq C R^{-\beta+\alpha}\|w\|_{\beta;U}^{(-\alpha)}. \tag{4.7}$$

Let $\varphi = w - w(x_0) - \bar{w}$ and observe that φ vanishes in B_R and, hence, $\varphi(x_1) = \varphi(x_2) = 0$.

Next we proceed differently if $\beta' > s$ or if $\beta' < s$. This is because $C^{\beta-s}$ equals either $C^{k,\beta'-s}$ or $C^{k-1,1+\beta'-s}$.

Case 1. Assume $\beta' > s$. Let $x_1, x_2 \in B_{R/2}(x_0) \subset B_{2R}(x_0)$. We want to bound $|D^k(-\Delta)^{s/2}w(x_1) - D^k(-\Delta)^{s/2}w(x_2)|$, where D^k denotes any k-th derivative with respect to a fixed multiindex. We have

$$(-\Delta)^{s/2}w = (-\Delta)^{s/2}\bar{w} + (-\Delta)^{s/2}\varphi \quad \text{in } B_{R/2}.$$

Then,

$$D^k(-\Delta)^{s/2}w(x_1) - D^k(-\Delta)^{s/2}w(x_2) = c_{n,\frac{s}{2}}(J_1 + J_2),$$

where

$$J_1 = \int_{\mathbb{R}^n}\left\{\frac{D^k\bar{w}(x_1) - D^k\bar{w}(y)}{|x_1 - y|^{n+s}} - \frac{D^k\bar{w}(x_2) - D^k\bar{w}(y)}{|x_2 - y|^{n+s}}\right\}dy$$

and

$$J_2 = D^k\int_{\mathbb{R}^n \setminus B_R}\frac{-\varphi(y)}{|x_1 - y|^{n+s}}\,dy - D^k\int_{\mathbb{R}^n \setminus B_R}\frac{-\varphi(y)}{|x_2 - y|^{n+s}}\,dy.$$

To bound $|J_1|$ we proceed as follows. Let $r = |x_1 - x_2|$. Then, using (4.7),

$$|J_1| = \left| \int_{\mathbb{R}^n} \frac{D^k \bar{w}(x_1) - D^k \bar{w}(x_1 + z) - D^k \bar{w}(x_2) + D^k \bar{w}(x_2 + z)}{|z|^{n+s}} \, dz \right|$$

$$\leq \int_{B_r} \frac{R^{\alpha-\beta} \|w\|_{\beta;U}^{(-\alpha)} |z|^{\beta'}}{|z|^{n+s}} \, dz + \int_{\mathbb{R}^n \setminus B_r} \frac{R^{\alpha-\beta} \|w\|_{\beta;U}^{(-\alpha)} r^{\beta'}}{|z|^{n+s}} \, dz$$

$$\leq C R^{\alpha-\beta} r^{\beta'-s} \|w\|_{\beta;U}^{(-\alpha)}.$$

Let us bound now $|J_2|$. Writing $\Phi(z) = |z|^{-n-s}$ and using that $\varphi(x_0) = 0$,

$$|J_2| = \left| \int_{\mathbb{R}^n \setminus B_R} \varphi(y) \left(D^k \Phi(x_1 - y) - D^k \Phi(x_2 - y) \right) dy \right|$$

$$\leq C \int_{\mathbb{R}^n \setminus B_R} |x_0 - y|^\alpha [w]_{C^\alpha(\mathbb{R}^n)} \frac{|x_1 - x_2|^{\beta'-s}}{|x_0 - y|^{n+\beta}} \, dy$$

$$\leq C R^{\alpha-\beta} r^{\beta'-s} [w]_{C^\alpha(\mathbb{R}^n)},$$

where we have used that

$$|D^k \Phi(z_1 - z) - D^k \Phi(z_2 - z)| \leq C |z_1 - z_2|^{\beta'-s} |z|^{-n-\beta}$$

for all z_1, z_2 in $B_{R/2}(0)$ and $z \in \mathbb{R}^n \setminus B_R$.

Hence, we have proved that

$$[(-\Delta)^{s/2} w]_{C^{\beta-s}(\overline{B_R(x_0)})} \leq C R^{\alpha-\beta} \|w\|_{\beta;U}^{(-\alpha)}.$$

Case 2. Assume $\beta' < s$. Let $x_1, x_2 \in B_{R/2}(x_0) \subset B_{2R}(x_0)$. We want to bound $|D^{k-1}(-\Delta)^{s/2} w(x_1) - D^{k-1}(-\Delta)^{s/2} w(x_2)|$. We proceed as above but we now use

$$|D^{k-1} \bar{w}(x_1) - D^{k-1} \bar{w}(x_1 + y) - D^{k-1} \bar{w}(x_2) + D^{k-1} \bar{w}(x_2 + y)| \leq$$

$$\leq \left| D^k \bar{w}(x_1) - D^k \bar{w}(x_2) \right| |y| + |y|^{1+\beta'} \|\bar{w}\|_{C^\beta(\mathbb{R}^n)}$$

$$\leq \left(|x_1 - x_2|^{\beta'} |y| + |y|^{1+\beta'} \right) R^{\alpha-\beta} \|w\|_{\beta;U}^{(-\alpha)}$$

in B_r, and

$$|D^{k-1} \bar{w}(x_1) - D^{k-1} \bar{w}(x_1 + y) - D^{k-1} \bar{w}(x_2) + D^{k-1} \bar{w}(x_2 + y)| \leq$$

$$\leq \left| D^k \bar{w}(x_1) - D^k \bar{w}(x_1 + y) \right| |x_1 - x_2| + |x_1 - x_2|^{1+\beta'} \|\bar{w}\|_{C^\beta(\mathbb{R}^n)}$$

$$\leq \left(|y|^{\beta'} |x_1 - x_2| + |x_1 - x_2|^{1+\beta'} \right) R^{\alpha-\beta} \|w\|_{\beta;U}^{(-\alpha)}$$

in $\mathbb{R}^n \setminus B_r$. Then, as in Case 1 we obtain $[(-\Delta)^{s/2} w]_{C^{\beta-s}(\overline{B_R(x_0)})} \leq C R^{\alpha-\beta} \|w\|_{\beta;U}^{(-\alpha)}$.

This yields (4.6), as in Step 2 of Lemma 2.10. $\qquad \square$

Next lemma is a variation of the previous one and gives a pointwise bound for $(-\Delta)^{s/2} w$. It is used in Remark 2.11.

Lemma 4.4. *Let $U \subset \mathbb{R}^n$ be an open set, and let $\beta > s$. Then, for all $x \in U$*

$$|(-\Delta)^{s/2} w(x)| \leq C(\|w\|_{C^s(\mathbb{R}^n)} + \|w\|_{\beta;U}^{(-s)}) \Big(1 + |\log \operatorname{dist}(x, \partial U)| \Big),$$

whenever w has finite right hand side. The constant C depends only on n, s, and β.

Proof. We may assume $\beta < 1$. Let $x_0 \in U$ and $R = d_{x_0}/2$, and define $\bar{w}$ and φ as in the proof of the previous lemma. Then,

$$(-\Delta)^{s/2} w(x_0) = (-\Delta)^{s/2} \bar{w}(x_0) + (-\Delta)^{s/2} \varphi(x_0) = c_{n,\frac{s}{2}} (J_1 + J_2),$$

where

$$J_1 = \int_{\mathbb{R}^n} \frac{\bar{w}(x_0) - \bar{w}(x_0 + z)}{|z|^{n+s}} \, dz \quad \text{and} \quad J_2 = \int_{\mathbb{R}^n \setminus B_R} \frac{-\varphi(x_0 + z)}{|z|^{n+s}} \, dz.$$

With similar arguments as in the previous proof we readily obtain $|J_1| \leq C(1 + |\log R|)\|w\|_{\beta;U}^{(-s)}$ and $|J_2| \leq C(1 + |\log R|)\|w\|_{C^s(\mathbb{R}^n)}$. $\qquad \square$

Appendix A. Basic tools and barriers

In this appendix we prove Proposition 3.1 and Lemmas 3.2 and 2.6. Proposition 3.1 is well-known (see [7]), but for the sake of completeness we sketch here a proof that uses the Caffarelli-Silvestre extension problem [8].

Proof of Proposition 3.1. Let (x, y) and (r, θ) be Cartesian and polar coordinates of the plane. The coordinate $\theta \in (-\pi, \pi)$ is taken so that $\{\theta = 0\}$ on $\{y = 0, \ x > 0\}$. Use that the function $r^s \cos(\theta/2)^{2s}$ is a solution in the half-plane $\{y > 0\}$ to the extension problem [8],

$$\operatorname{div}(y^{1-2s} \nabla u) = 0 \quad \text{in } \{y > 0\},$$

and that its trace on $y = 0$ is φ_0. $\qquad \square$

The fractional Kelvin transform has been studied thoroughly in [5].

Proposition A.1 (Fractional Kelvin transform). *Let u be a smooth bounded function in $\mathbb{R}^n \setminus \{0\}$. Let $x \mapsto x^* = x/|x|^2$ be the inversion with respect to the unit sphere. Define $u^*(x) = |x|^{2s-n} u(x^*)$. Then,*

$$(-\Delta)^s u^*(x) = |x|^{-2s-n} (-\Delta)^s u(x^*), \tag{A.1}$$

for all $x \neq 0$.

Proof. Let $x_0 \in \mathbb{R}^n \setminus \{0\}$. By subtracting a constant to u^* and using $(-\Delta)^s |x|^{2s-n} = 0$ for $x \neq 0$, we may assume $u^*(x_0) = u(x_0^*) = 0$. Recall that

$$|x - y| = \frac{|x^* - y^*|}{|x^*||y^*|}.$$

Thus, using the change of variables $z = y^* = y/|y|^2$,

$$
\begin{aligned}
(-\Delta)^s u^*(x_0) &= c_{n,s}\, \mathrm{PV} \int_{\mathbb{R}^n} \frac{-u^*(y)}{|x_0 - y|^{n+2s}}\, dy \\
&= c_{n,s}\, \mathrm{PV} \int_{\mathbb{R}^n} \frac{-|y|^{2s-n} u(y^*)}{|x_0^* - y^*|^{n+2s}} |x_0^*|^{n+2s} |y^*|^{n+2s}\, dy \\
&= c_{n,s} |x_0|^{-n-2s}\, \mathrm{PV} \int_{\mathbb{R}^n} \frac{-|z|^{n-2s} u(z)}{|x_0^* - z|^{n+2s}} |z|^{n+2s}\, |z|^{-2n} dz \\
&= c_{n,s} |x_0|^{-n-2s}\, \mathrm{PV} \int_{\mathbb{R}^n} \frac{-u(z)}{|x_0^* - z|^{n+2s}} dz \\
&= |x_0|^{-n-2s} (-\Delta)^s u(x_0^*)\,.
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, using Proposition A.1 we prove Lemma 2.6.

Proof of Lemma 2.6. Let us denote by ψ (instead of u) the explicit solution (1.4) to problem (1.3) in B_1, which satisfies

$$
\begin{cases}
(-\Delta)^s \psi = 1 & \text{in } B_1 \\
\psi \equiv 0 & \text{in } \mathbb{R}^n \setminus B_1 \\
0 < \psi < C(1 - |x|)^s & \text{in } B_1\,.
\end{cases}
\tag{A.2}
$$

From ψ, the supersolution φ_1 in the exterior of the ball is readily built using the fractional Kelvin transform. Indeed, let ξ be a radial smooth function satisfying $\xi \equiv 1$ in $\mathbb{R}^n \setminus B_5$ and $\xi \equiv 0$ in B_4, and define φ_1 by

$$
\varphi_1(x) = C|x|^{2s-n} \psi(1 - |x|^{-1}) + \xi(x)\,.
\tag{A.3}
$$

Observe that $(-\Delta)^s \xi \geq -C_2$ in B_4, for some $C_2 > 0$. Hence, if we take $C \geq 4^{2s+n}(1 + C_2)$, using (A.1), we have

$$
(-\Delta)^s \varphi_1(x) \geq C|x|^{-2s-n} + (-\Delta)^s \xi(x) \geq 1 \quad \text{in } B_4\,.
$$

Now it is immediate to verify that φ_1 satisfies (2.1) for some $c_1 > 0$.

To see that $\varphi_1 \in H_{\mathrm{loc}}^s(\mathbb{R}^n)$ we observe that from (A.3) it follows

$$
|\nabla \varphi_1(x)| \leq C(|x| - 1)^{s-1} \quad \text{in } \mathbb{R}^n \setminus B_1
$$

and hence, using Lemma 4.4, we have $(-\Delta)^{s/2} \varphi_1 \in L_{\mathrm{loc}}^p(\mathbb{R}^n)$ for all $p < \infty$. $\qquad\square$

Next we prove Lemma 3.2.

Proof of Lemma 3.2. We define

$$
\psi_1(x) = (1 - |x|^2)^s \chi_{B_1}(x)\,.
$$

Since (1.4) is the solution of problem (1.3), we have $(-\Delta)^s \psi_1$ is bounded in B_1. Hence, for $C > 0$ large enough the function $\psi = \psi_1 + C\chi_{\overline{B_{1/4}}}$ satisfies $(-\Delta)^s \psi \leq 0$ in $B_1 \setminus \overline{B_{1/4}}$ and it can be used as a viscosity subsolution. Note that ψ is upper

semicontinuous, as required to viscosity subsolutions, and it satisfies pointwise (if C is large enough)

$$\begin{cases} \psi \equiv 0 & \text{in } \mathbb{R}^n \setminus B_1 \\ (-\Delta)^s \psi \leq 0 & \text{in } B_1 \setminus \overline{B_{1/4}} \\ \psi = 1 & \text{in } \overline{B_{1/4}} \\ \psi(x) \geq c(1 - |x|)^s & \text{in } B_1. \end{cases}$$

If we want a subsolution which is continuous and $H^s(\mathbb{R}^n)$ we may construct it as follows. We consider the viscosity solution (which is also a weak solution by Remark 2.11) of

$$\begin{cases} (-\Delta)^s \varphi_2 = 0 & \text{in } B_1 \setminus B_{1/4} \\ \varphi_2 \equiv 0 & \text{in } \mathbb{R}^n \setminus B_1 \\ \varphi_2 = 1 & \text{in } \overline{B_{1/4}}. \end{cases}$$

Using ψ as a lower barrier, it is now easy to prove that φ_2 satisfies (3.2) for some constant $c_2 > 0$. $\qquad \square$

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Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain
   E-mail address: xavier.ros.oton@upc.edu

Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain
   E-mail address: joaquim.serra@upc.edu

C. Nonexistence results for nonlocal equations with critical and supercritical nonlinearities

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Collaboration with X. Ros-Oton

NONEXISTENCE RESULTS FOR NONLOCAL EQUATIONS WITH CRITICAL AND SUPERCRITICAL NONLINEARITIES

XAVIER ROS-OTON AND JOAQUIM SERRA

ABSTRACT. We prove nonexistence of nontrivial bounded solutions to some nonlinear problems involving nonlocal operators of the form

$$Lu(x) = \sum a_{ij}\partial_{ij}u + \mathrm{PV}\int_{\mathbb{R}^n}(u(x) - u(x+y))K(y)dy.$$

These operators are infinitesimal generators of symmetric Lévy processes. Our results apply to even kernels K satisfying that $K(y)|y|^{n+\sigma}$ is nondecreasing along rays from the origin, for some $\sigma \in (0,2)$ in case $a_{ij} \equiv 0$ and for $\sigma = 2$ in case that (a_{ij}) is a positive definite symmetric matrix.

Our nonexistence results concern Dirichlet problems for L in star-shaped domains with critical and supercritical nonlinearities (where the criticality condition is in relation to n and σ).

We also establish nonexistence of bounded solutions to semilinear equations involving other nonlocal operators such as the higher order fractional Laplacian $(-\Delta)^s$ (here $s > 1$) or the fractional p-Laplacian. All these nonexistence results follow from a general variational inequality in the spirit of a classical identity by Pucci and Serrin.

1. INTRODUCTION AND RESULTS

The aim of this paper is to prove nonexistence results for the following type of nonlinear problems

$$\begin{cases} Lu &= f(x,u) &\text{in } \Omega \\ u &= 0 &\text{in } \mathbb{R}^n\backslash\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, f is a critical or supercritical nonlinearity (as defined later), and L is an integro-differential elliptic operator. Our main results concern operators of the form

$$Lu(x) = \mathrm{PV}\int_{\mathbb{R}^n}\big(u(x) - u(x+y)\big)K(y)dy \tag{1.2}$$

and

$$Lu(x) = \sum_{i,j} a_{ij}\partial_{ij}u + \mathrm{PV}\int_{\mathbb{R}^n}\big(u(x) - u(x+y)\big)K(y)dy, \tag{1.3}$$

where (a_{ij}) is a positive definite matrix (independent of $x \in \Omega$) and K is a nonnegative kernel satisfying

$$K(y) = K(-y) \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{|y|^2}{1 + |y|^2} K(y) dy < \infty. \tag{1.4}$$

These operators are infinitesimal generators of symmetric Lévy processes.

We will state two different nonexistence results, one corresponding to (1.2) and the other to (1.3).

On the one hand, we consider operators (1.2) that may not have a definite order but only satisfy, for some $\sigma \in (0, 2)$,

$$K(y)|y|^{n+\sigma} \quad \text{is nondecreasing along rays from the origin.} \tag{1.5}$$

Heuristically, (1.5) means that even if the order is not defined, σ acts as an upper bound for the order of the operator —see Section 2 for some examples. For these operators we prove, under some additional technical assumptions on the kernel, nonexistence of nontrivial bounded solutions to (1.1) in star-shaped domains for supercritical nonlinearities. When $f(x, u) = |u|^{q-1}u$, the critical power for this class of operators is $q = \frac{n+\sigma}{n-\sigma}$.

On the other hand, we establish the analogous result for second order integro-differential elliptic operators (1.3) with kernels K satisfying (1.5) with $\sigma = 2$. In this case, the critical power is $q = \frac{n+2}{n-2}$.

Moreover, we can use the same ideas to prove an abstract variational inequality that applies to more general problems. For instance, we can obtain nonexistence results for semilinear equations involving the higher order fractional Laplacian $(-\Delta)^s$ (i.e., with $s > 1$) or the fractional p-Laplacian.

When L is the Laplacian $-\Delta$, the nonexistence of nontrivial solutions to (1.1) for critical and supercritical nonlinearities in star-shaped domains follows from the celebrated Pohozaev identity [12]. For positive solutions, this result can also be proved with the moving spheres method [20, 14]. For more general elliptic operators (such as the p-Laplacian, the bilaplacian Δ^2, or k-hessian operators), the nonexistence of regular solutions usually follows from Pohozaev-type or Pucci-Serrin identities [13].

When L is the fractional Laplacian $(-\Delta)^s$ with $s \in (0, 1)$, which corresponds to $K(y) = c_{n,s}|y|^{-n-2s}$ in (1.2), this nonexistence result for problem (1.1) was first obtained by Fall-Weth for positive solutions [8] (by using the moving spheres method). In $C^{1,1}$ domains, the nonexistence of nontrivial solutions (not necessarily positive) can be deduced from the Pohozaev identity for the fractional Laplacian, recently established by the authors in [17, 15].

Both the local operator $-\Delta$ and the nonlocal operator $(-\Delta)^s$ satisfy a property of invariance under scaling. More precisely, denoting $w_\lambda(x) = w(\lambda x)$, these operators satisfy $Lw_\lambda(x) = \lambda^\sigma Lw(\lambda x)$, with $\sigma = 2$ in case $L = -\Delta$ and $\sigma = 2s$ in case $L = (-\Delta)^s$. These scaling exponents are strongly related to the critical powers $q = \frac{n+2}{n-2}$ and $q = \frac{n+2s}{n-2s}$ obtained for power nonlinearities $f(x, u) = |u|^{q-1}u$ in (1.1).

Here, we prove a nonexistence result for problem (1.1) with operators L that may not satisfy a scale invariance condition but satisfy (1.5) instead. Our arguments are in the same philosophy as Pucci-Serrin [13], where they proved a general variational identity that applies to many second order problems. Here, we prove a variational inequality that applies to the previous integro-differential problems.

Before stating our results recall that, given $\sigma > 0$ and $\Omega \subset \mathbb{R}^n$, the nonlinearity $f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})$ is said to be *supercritical* if

$$\frac{n-\sigma}{2} t\, f(x,t) > nF(x,t) + x \cdot F_x(x,t) \quad \text{for all} \ \ x \in \Omega \ \text{ and } \ t \neq 0, \tag{1.6}$$

where $F(x,t) = \int_0^t f(x,\tau)d\tau$. When $f(x,u) = |u|^{q-1}u$, this corresponds to $q > \frac{n+\sigma}{n-\sigma}$.

As explained later on in this Introduction, by bounded solution of (1.1) we mean a critical point $u \in L^\infty(\Omega)$ of the associated energy functional.

Our first nonexistence result reads as follows. Note that it applies not only to positive solutions but also to changing-sign ones.

In the first two parts of the theorem, we assume the solution u to be $W^{1,r}$ for some $r > 1$. This is a natural assumption that is satisfied when L is a pure fractional Laplacian and also for those operators L with kernels K satisfying an additional assumption on its "order", as stated in part (c).

Theorem 1.1. *Let K be a nonnegative kernel satisfying* (1.4), (1.5) *for some* $\sigma \in (0,2)$, *and*

$$K \text{ is } C^1(\mathbb{R}^n \setminus \{0\}) \text{ and } |\nabla K(y)| \leq C \frac{K(y)}{|y|} \quad \text{for all } y \neq 0 \tag{1.7}$$

for some constant C. Let L be given by (1.2). *Let $\Omega \subset \mathbb{R}^n$ be any bounded star-shaped domain, and $f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})$ be a supercritical nonlinearity, i.e., satisfying* (1.6). *Let u be any bounded solution of* (1.1). *The following statements hold:*

(a) *If $u \in W^{1,r}(\Omega)$ for some $r > 1$, then $u \equiv 0$.*

(b) *Assume that $K(y)|y|^{n+\sigma}$ is not constant along some ray from the origin, and that the nonstrict inequality*

$$\frac{n-\sigma}{2} t\, f(x,t) \geq nF(x,t) + x \cdot F_x(x,t) \quad \text{for all} \ \ x \in \Omega \ \ \text{and} \ \ t \in \mathbb{R} \tag{1.8}$$

holds instead of (1.6). *If $u \in W^{1,r}(\Omega)$ for some $r > 1$, then $u \equiv 0$.*

(c) *Assume that in addition Ω is convex, that the kernel K satisfies*

$$K(y)|y|^{n+\epsilon} \quad \text{is nonincreasing along rays from the origin} \tag{1.9}$$

for some $\epsilon \in (0, \sigma)$, and that

$$\max_{\partial B_r} K(y) \leq C \min_{\partial B_r} K(y) \quad \text{for all } r \in (0,1) \tag{1.10}$$

for some constant C. Then, $u \in W^{1,r}(\Omega)$ for some $r > 1$, and therefore statements (a) and (b) hold without the assumption $u \in W^{1,r}(\Omega)$.

Note that in part (c) we have the additional assumption that the domain Ω is convex. This is used to prove the $W^{1,r}$ regularity of bounded solutions to (1.1) (and it is not needed for example when the operator is the fractional Laplacian, see Remark 6.7). Note also that condition (1.5) means in some sense that L has order at most σ, while (1.9) means that L is at least of order ϵ for some small $\epsilon > 0$.

Some examples to which our result applies are sums of fractional Laplacians of different orders, anisotropic operators (i.e., with nonradial kernels), and also operators whose kernels have a singularity different of a power at the origin. More examples are given in Section 2.

Note that for $f(x,u) = |u|^{q-1}u$, part (a) gives nonexistence for supercritical powers $q > \frac{n+\sigma}{n-\sigma}$, while part (b) establishes nonexistence also for the critical power $q = \frac{n+\sigma}{n-\sigma}$. The nonexistence of nontrivial solutions for the critical power in case that $K(y)|y|^{n+\sigma}$ is constant along all rays from the origin remains an open problem. Even for the fractional Laplacian $(-\Delta)^s$, this has been only established for positive solutions, and it is not known for changing-sign solutions.

The existence of nontrivial solutions in (1.1) for subcritical nonlinearities was obtained by Servadei and Valdinoci [18] by using the mountain pass theorem. Their result applies to nonlocal operators of the form (1.2) with symmetric kernels K satisfying $K(y) \geq \lambda|y|^{-n-\sigma}$.

As stated in Theorem 1.1, the additional hypotheses of part (c) lead to the $W^{1,r}(\Omega)$ regularity of bounded solutions for some $r > 1$. This is a consequence of the following proposition.

Proposition 1.2. *Let $\Omega \subset \mathbb{R}^n$ be any bounded and convex domain. Let L be an operator satisfying the hypotheses of Theorem 1.1 (c), i.e., satisfying (1.2), (1.4), (1.5), (1.7), (1.9), and (1.10). Let $f \in C^{0,1}_{loc}(\overline{\Omega} \times \mathbb{R})$, and let u be any bounded solution of (1.1). Then,*

$$\|u\|_{C^{\epsilon/2}(\mathbb{R}^n)} \leq C \qquad and \qquad |\nabla u(x)| \leq C\delta(x)^{\frac{\epsilon}{2}-1} \quad in \ \ \Omega, \qquad (1.11)$$

where $\delta(x) = dist(x, \partial\Omega)$ and C is a constant that depends only on Ω, ϵ, σ, f, and $\|u\|_{L^\infty(\Omega)}$.

Note that (1.11) and the fact that Ω is convex imply $u \in W^{1,r}(\Omega)$ for all $1 < r < \frac{1}{1-\epsilon/2}$. In (1.11) the exponents $\epsilon/2$ are optimal, as seen when $L = (-\Delta)^{\epsilon/2}$ (see [16]).

Our second nonexistence result, stated next, deals with operators of the form (1.3). Here, the additional assumptions on Ω and K leading to the $W^{1,r}$ regularity of solutions are not needed thanks to the presence of the second order constant coefficients regularizing term.

Theorem 1.3. *Let L be an operator of the form (1.3), where (a_{ij}) is a positive definite symmetric matrix and K is a nonnegative kernel satisfying (1.4). Assume in addition that (1.7) holds, and that*

$$K(y)|y|^{n+2} \ is \ nondecreasing \ along \ rays \ from \ the \ origin. \qquad (1.12)$$

Let $\Omega \subset \mathbb{R}^n$ be any bounded star-shaped domain, $f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})$, and u be any bounded solution of (1.1). If (1.8) holds with $\sigma = 2$, then $u \equiv 0$.

Note that for $f(x, u) = |u|^{q-1}u$ we obtain nonexistence for critical and supercritical powers $q \geq \frac{n+2}{n-2}$.

The proofs of Theorems 1.1 and 1.3 follow some ideas introduced in our proof of the Pohozaev identity for the fractional Laplacian [17]. The key ingredient in all these proofs is the scaling properties both of the bilinear form associated to L and of the potential energy associated to f. These two terms appear in the variational formulation of (1.1), as explained next.

Recall that solutions to problem (1.1), with L given by (1.2) or (1.3), are critical points of the functional

$$\mathcal{E}(u) = \frac{1}{2}(u, u) - \int_\Omega F(x, u) \tag{1.13}$$

among all functions u satisfying $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Here, $F(x, u) = \int_0^u f(x, t)dt$, and $(\cdot, \cdot)$ is the bilinear form associated to L. More precisely, in case that L is given by (1.2), we have

$$(u, v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big(u(x) - u(x + y)\big)\big(v(x) - v(x + y)\big)K(y)dx\, dy, \tag{1.14}$$

while in case that L is given by (1.3), we have

$$(u, v) = \int_\Omega A(\nabla u, \nabla v)dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big(u(x) - u(x + y)\big)\big(v(x) - v(x + y)\big)K(y)dx\, dy, \tag{1.15}$$

where $A(p, q) = p^T A q$ and $A = (a_{ij})$ is the matrix in (1.3).

Both Theorems 1.1 and 1.3 are particular cases of the more general result that we state next. This result establishes nonexistence of bounded solutions $u \in W^{1,r}(\Omega)$, $r > 1$, to problems of the form (1.1) with variational operators L satisfying a scaling inequality.

Proposition 1.4. *Let E be a Banach space contained in $L^1_{\text{loc}}(\mathbb{R}^n)$, and $\|\cdot\|$ be a seminorm in E. Assume that for some $\alpha > 0$ the seminorm $\|\cdot\|$ satisfies*

$$w_\lambda \in E \quad and \quad \|w_\lambda\| \leq \lambda^{-\alpha}\|w\| \ \ for\ every\ w \in E\ and\ \lambda > 1, \tag{1.16}$$

where $w_\lambda(x) = w(\lambda x)$.

Let $\Omega \subset \mathbb{R}^n$ be any bounded star-shaped domain with respect to the origin, $p > 1$, and $f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})$. Consider the energy functional

$$\mathcal{E}(u) = \frac{1}{p}\|u\|^p - \int_\Omega F(x, u), \tag{1.17}$$

where $F(x, u) = \int_0^u f(x, t)dt$, and let u be a critical point of $\mathcal{E}$ among all functions $u \in E$ satisfying $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$.

Assume that f is supercritical, in the sense that

$$\alpha t\, f(x,t) > n F(x,t) + x \cdot F_x(x,t) \quad \text{for all } x \in \Omega \text{ and } t \neq 0. \qquad (1.18)$$

If $u \in L^\infty(\Omega) \cap W^{1,r}(\Omega)$ for some $r > 1$, then $u \equiv 0$.

Some examples to which this result applies are second order variational operators such as the Laplacian or the p-Laplacian, the nonlocal operators in Theorems 1.1 or 1.3, or the higher order fractional Laplacian $(-\Delta)^s$ (here $s > 1$). See Section 2 for more examples.

Remark 1.5. Proposition 1.4 establishes nonexistence of nontrivial bounded solutions belonging to $W^{1,r}(\Omega)$, $r > 1$. In general, removing the $W^{1,r}$ assumption may be done in two different situations:

First, it may happen that the space $E_\Omega = \{u \in E : u \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$ is embedded in $W^{1,r}(\Omega)$, $r > 1$. This happens for instance when considering the natural functional spaces associated to the Laplacian, the p-Laplacian with $p > 1$, the higher order fractional Laplacian $(-\Delta)^s$ (with $s \geq 1$), and of the nonlocal operators considered in Theorem 1.3.

Second, even if the space E_Ω is not embedded in $W^{1,r}$, it is often the case that by some regularity estimates one can prove that critical points of (1.17) belong to $W^{1,r}$, $r > 1$. This occurs when the operator if the fractional Laplacian, and also in Theorem 1.1 (c), thanks to Proposition 1.2.

As said before, for local operators of order 2, the nonexistence of regular solutions usually follows from Pohozaev-type or Pucci-Serrin identities [13]. Our proofs are in the spirit of these identities. However, for nonlocal operators this type of identity is only known for the fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$ [17], and requires a precise knowledge of the boundary behavior of solutions to (1.1) [16] (that are not available for most L). To overcome this, instead of proving an identity we prove an inequality which is sufficient to prove nonexistence. This approach allows us to require much less regularity on the solution u and, thus, to include a wide class of operators in our results.

The paper is organized as follows. In Section 2 we give a list of examples of operators to which our results apply. In Section 3 we present the main ideas appearing in the proofs of our results. In Section 4 we prove Proposition 1.4. In Section 5 we prove Theorems 1.1 and 1.3. Finally, in Section 6 we prove Proposition 1.2.

2. EXAMPLES

In this Section we give a list of examples to which our results apply.

(i) First, note that if $K_1, ..., K_m$ are kernels satisfying the hypotheses of Theorem 1.1, and $a_1, ..., a_m$ are nonnegative numbers, then $K = a_1 K_1 + \cdots + a_m K_m$ also satisfies the hypotheses. In particular, our nonexistence result applies to operators of the form

$$L = a_1(-\Delta)^{\alpha_1} + \cdots + a_m(-\Delta)^{\alpha_m},$$

with $a_i \geq 0$ and $\alpha_i \in (0,1)$. The critical exponent is $q = \frac{n+2\max\alpha_i}{n-2\max\alpha_i}$.

(ii) Theorem 1.1 may be applied to anisotropic operators L of the form (1.2) with nonradial kernels such as

$$K(y) = H(y)^{-n-\sigma},$$

where H is any homogeneous function of degree 1 whose restriction to S^{n-1} is positive and C^1. These operators are infinitesimal generators of σ-stable symmetric Lévy processes. The critical exponent is $q = \frac{n+\sigma}{n-\sigma}$.

(iii) Theorem 1.1 applies also to operators with kernels that do not have a power-like singularity at the origin. For example, the one given by the kernel

$$K(y) = \frac{c}{|y|^{n+\sigma}\log\left(2 + \frac{1}{|y|}\right)}, \qquad \sigma \in (0,2),$$

whose singularity at $y = 0$ is comparable to $|y|^{-n-\sigma}\big|\log|y|\big|^{-1}$. In this example we also have that the critical exponent is $q = \frac{n+\sigma}{n-\sigma}$.

Other examples of operators that may not have a definite order are given by infinite sums of fractional Laplacians, such as $L = \sum_{k\geq1}\frac{1}{k^2}(-\Delta)^{s-\frac{1}{k}}$.

(iv) Theorem 1.3 applies to operators such as $L = -\Delta + (-\Delta)^s$, with $s \in (0,1)$, and also anisotropic operators whose nonlocal part is given by nonradial kernels, as in example (ii). For all these operators, the critical power is $q = \frac{n+2}{n-2}$.

(v) One may take in (1.17) the $W^{s,p}(\mathbb{R}^n)$ seminorm

$$\|u\|^p = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy.$$

This leads to nonexistence results for the s-fractional p-Laplacian operator, considered for example in [4, 9]. The critical power for this operator is $q = \frac{n+ps}{n-ps}$.

(vi) Our results can also be used to obtain a generalization of Theorem 8 in [13], where Pucci and Serrin proved nonexistence results for the bilaplacian Δ^2 and the polylaplacian $(-\Delta)^K$, with K positive integer. More precisely, Proposition 1.4 can be applied to the $H^s(\mathbb{R}^n)$ seminorm to obtain nonexistence of bounded solutions u to (1.1) with $L = (-\Delta)^s$, $s > 1$. Note that the hypotheses $u \in W^{1,r}(\Omega)$ is always satisfied, since the fractional Sobolev embeddings yield that any function $u \in H^s(\mathbb{R}^n)$ that vanishes outside Ω belongs to $W^{1,r}(\Omega)$ for $r = 2$ (see Remark 1.5).

As an example, when $n > 2s$ and $f(u) = \lambda u + |u|^{q-1}u$, one obtains nonexistence of bounded solutions for $\lambda < 0$ and $q \geq \frac{n+2s}{n-2s}$ and also for $\lambda \leq 0$ and $q > \frac{n+2s}{n-2s}$, as in [13].

(vii) Proposition 1.4 can be applied to the usual $W^{1,p}(\Omega)$ norm to obtain nonexistence of bounded weak solutions to (1.1) with $L = -\Delta_p$, the p-Laplacian.

These nonexistence results were obtained by Otani in [11] via a Pohozaev-type inequality.

More generally, we may consider nonlinear anisotropic operators that come from setting

$$\|u\|^p = \int_\Omega H(\nabla u)^p |x|^\gamma dx$$

in (1.17), where H is any norm in $\mathbb{R}^n$. In this case, the critical power is $q = \frac{n+\gamma+p}{n+\gamma-p}$. For $\gamma = 0$, some problems involving this class of operators were studied in [2, 10, 6]. For $\gamma \neq 0$, nonexistence results for these type of problems were studied in [1].

(viii) From Proposition 1.4 one may obtain also nonexistence results for k-Hessian operators $S_k(D^2u)$ with $2k < n$. Recall that $S_k(D^2u)$ are defined in terms of the elementary symmetric polynomials acting on the eigenvalues of D^2u, and that these are variational operators. In the two extreme cases $k = 1$ and $k = n$, we have $S_1(D^2u) = \Delta u$ and $S_n(D^2u) = \det D^2u$.

Tso studied this problem in [21], and obtained nonexistence of solutions $u \in C^4(\Omega) \cap C^1(\overline{\Omega})$ in smooth star-shaped domains via a Pohozaev identity. Our results give only nonexistence for supercritical powers $q > \frac{(n+2)k}{n-2k}$, and not for the critical one. As a counterpart, we only need to assume the solution u to be $L^\infty(\Omega) \cap W^{1,r}(\Omega)$.

3. Sketch of the proof

In this section we sketch the proof of the nonexistence of critical points to functionals of the form

$$\mathcal{E}(u) = \frac{1}{2}(u, u) + \int_\Omega F(u), \tag{3.1}$$

where $(\cdot, \cdot)$ is a bilinear form satisfying, for some $\alpha > 0$,

$$u_\lambda \in E \quad \text{and} \quad \|u_\lambda\| := (u_\lambda, u_\lambda)^{1/2} \leq \lambda^{-\alpha}(u, u)^{1/2} \quad \text{for all} \quad \lambda \geq 1, \tag{3.2}$$

where $u_\lambda(x) = u(\lambda x)$. Of course, this is a particular case of Proposition 1.4, in which $p = 2$, E is a Hilbert space, and f does not depend on x. Note that in this case condition (1.16) reads as (3.2). In case of Theorems 1.1 and 1.3, the bilinear form is given by (1.14) and (1.15), respectively.

The proof goes as follows. Since u is a critical point of (3.1), then we have that

$$(u, \varphi) = \int_\Omega f(u)\varphi \, dx \quad \text{for all } \varphi \in E \text{ satisfying } \varphi \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega.$$

Next we use $\varphi = u_\lambda$, with $\lambda > 1$, as a test function. Note that, by (3.2), we have $u_\lambda \in E$, and since Ω is star-shaped, then $u_\lambda \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Hence u_λ is indeed an admissible test function. We obtain

$$(u, u_\lambda) = \int_\Omega f(u)u_\lambda \, dx \quad \text{for all} \quad \lambda \geq 1. \tag{3.3}$$

Now, we differentiate with respect to λ in both sides of (3.3). On the one hand, since $u \in L^\infty(\Omega) \cap W^{1,r}(\Omega)$, one can show —see Lemma 4.2— that

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_\Omega f(u)u_\lambda \, dx = \int_\Omega (x \cdot \nabla u)f(u) \, dx = \int_\Omega x \cdot \nabla F(u)dx = -n \int_\Omega F(u)dx.$$

On the other hand,

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} (u, u_\lambda) = \frac{d}{d\lambda}\bigg|_{\lambda=1^+} \{\lambda^{-\alpha}I_\lambda\} = -\alpha(u, u) + \frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda,$$

where

$$I_\lambda = \lambda^\alpha(u, u_\lambda). \tag{3.4}$$

We now claim that

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda \leq 0. \tag{3.5}$$

Indeed, using (3.2) and the Cauchy-Schwarz inequality, we deduce

$$I_\lambda \leq \lambda^\alpha \|u\|\|u_\lambda\| \leq \|u\|^2 = I_1,$$

and thus (3.5) follows. Therefore, we find

$$-n \int_\Omega F(u)dx = -\alpha (u, u) + \frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda \leq -\alpha (u, u),$$

and since $(u, u) = \int_\Omega uf(u)dx$,

$$\int_\Omega uf(u)dx \leq \frac{n}{\alpha} \int_\Omega F(u)dx.$$

From this, the nonexistence of nontrivial solutions for supercritical nonlinearities follows immediately.

In case of Theorem 1.1 (b) and Theorem 1.3, with a little more effort we will be able to prove that (3.5) holds with strict inequality, and this will yield the nonexistence result for critical nonlinearities.

When the previous bilinear form is invariant under scaling, in the sense that (3.2) holds with an equality instead of an inequality, then one has $I_\lambda = (u_{\sqrt{\lambda}}, u_{1/\sqrt{\lambda}})$. In the case $L = (-\Delta)^s$, it is proven in [17] that

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda = \Gamma(1 + s) \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu)dS,$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. This gives the boundary term in the Pohozaev identity for the fractional Laplacian.

Remark 3.1. This method can also be used to prove nonexistence results in star-shaped domains with respect to infinity or in the whole space $\Omega = \mathbb{R}^n$. However, one need to assume some decay on u and its gradient ∇u, which seems a quite restrictive hypothesis. More precisely, when $f(u) = |u|^{q-1}u$ and the operator is the fractional Laplacian $(-\Delta)^s$, this proof yields nonexistence of bounded solutions

(decaying at infinity) for subcritical nonlinearities $q < \frac{n+2s}{n-2s}$ in star-shaped domains with respect to infinity, and for noncritical nonlinearities $q \neq \frac{n+2s}{n-2s}$ in the whole $\mathbb{R}^n$. The classification of entire solutions in $\mathbb{R}^n$ for the critical power $q = \frac{n+2s}{n-2s}$ was obtained in [5].

4. Proof of Proposition 1.4

In this section we prove Proposition 1.4. For it, we will need the following lemma, which can be viewed as a Hölder-type inequality in normed spaces. For example, for $\|u\| = \left(\int_{\mathbb{R}^n} |u|^p\right)^{1/p}$, we recover the usual Hölder inequality (assuming that the Minkowski inequality holds).

Lemma 4.1. *Let E be a normed space, and $\|\cdot\|$ a seminorm in E. Let $p > 1$, and define $\Phi = \frac{1}{p}\|\cdot\|^p$. Assume that Φ is Gateaux differentiable at $u \in E$, and let $D\Phi(u)$ be the Gateaux differential of Φ at u. Then, for all v in E,*

$$\langle D\Phi(u), v\rangle \leq p\,\Phi(u)^{1/p'}\,\Phi(v)^{1/p},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, equality holds whenever $v = u$.

Proof. Since $\Phi^{1/p}$ is a seminorm, then by the triangle inequality we find that

$$\Phi(u + \varepsilon v) \leq \left\{\Phi(u)^{1/p} + \varepsilon\,\Phi(v)^{1/p}\right\}^p$$

for all u and v in E and for all $\varepsilon \in \mathbb{R}$. Hence, since these two quantities coincide for $\varepsilon = 0$, we deduce

$$\langle D\Phi(u), v\rangle = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Phi(u+\varepsilon v) \leq \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left\{\Phi(u)^{1/p} + \varepsilon\,\Phi(v)^{1/p}\right\}^p = p\,\Phi(u)^{1/p'}\Phi(v)^{1/p},$$

and the lemma follows. $\square$

Before giving the proof of Proposition 1.4, we also need the following lemma.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain, and let $u \in W^{1,r}(\Omega)$, $r > 1$. Then,*

$$\frac{u_\lambda - u}{\lambda - 1} \rightharpoonup x \cdot \nabla u \quad \text{weakly in } L^1(\Omega),$$

where $u_\lambda(x) = u(\lambda x)$.

Proof. Similarly to [7, Theorem 5.8.3], it can be proved that

$$\int_\Omega \left|\frac{u_\lambda - u}{\lambda - 1}\right|^r dx \leq C\int_\Omega |\nabla u|^r dx.$$

Thus, since $1 < r \leq \infty$, then $L^r \cong (L^{r'})'$ and hence there exists a sequence $\lambda_k \to 1$, and a function $v \in L^r(\Omega)$, such that

$$\frac{u_{\lambda_k} - u}{\lambda_k - 1} \rightharpoonup v \quad \text{weakly in } L^r(\Omega).$$

On the other hand note that, for each $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega u\,(x \cdot \nabla\phi)\,dx = -\int_\Omega (x \cdot \nabla u)\,\phi\,dx - n\int_\Omega u\phi\,dx.$$

Moreover, it is immediate to see that, for λ sufficiently close to 1,

$$\int_\Omega u\,\frac{\phi_\lambda - \phi}{\lambda - 1}\,dx = -\lambda^{-n-1}\int_\Omega \frac{u_{1/\lambda} - u}{1/\lambda - 1}\,\phi\,dx + \frac{\lambda^{-n} - 1}{\lambda - 1}\int_\Omega u\phi\,dx.$$

Therefore,

$$\begin{aligned}
\int_\Omega u\,(x \cdot \nabla\phi)\,dx &= \lim_{k\to\infty}\int_\Omega u\,\frac{\phi_{1/\lambda_k} - \phi}{1/\lambda_k - 1}\,dx \\
&= \lim_{k\to\infty} -\int_\Omega \frac{u_{\lambda_k} - u}{\lambda_k - 1}\,\phi\,dx - n\int_\Omega u\phi\,dx \\
&= -\int_\Omega v\phi\,dx - n\int_\Omega u\phi\,dx.
\end{aligned}$$

Thus, it follows that $v = x \cdot \nabla u$.

Now, note that this argument yields also that for each sequence $\mu_k \to 1$ there exists a subsequence $\lambda_k \to 1$ such that

$$\frac{u_{\lambda_k} - u}{\lambda_k - 1} \rightharpoonup x \cdot \nabla u \quad \text{weakly in } L^r(\Omega).$$

Since this can be done for any sequence μ_k, then this implies that

$$\frac{u_\lambda - u}{\lambda - 1} \rightharpoonup x \cdot \nabla u \quad \text{weakly in } L^r(\Omega).$$

Finally, since $L^r(\Omega) \subset L^1(\Omega)$, the lemma follows. $\qquad\square$

We can now give the:

Proof of Proposition 1.4. Define $\Phi = \frac{1}{p}\|\cdot\|^p$. Since u is a critical point of (1.17), then

$$\langle D\Phi(u), \varphi\rangle = \int_\Omega f(x, u)\varphi\,dx \tag{4.1}$$

for all $\varphi \in E$ satisfying $\varphi \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Since Ω is star-shaped, we may choose $\varphi = u_\lambda$, with $\lambda \geq 1$, as a test function in (4.1). We find

$$\langle D\Phi(u), u_\lambda\rangle = \int_\Omega f(x, u)u_\lambda dx \quad \text{for all } \lambda \geq 1. \tag{4.2}$$

We compute now the derivative with respect to λ at $\lambda = 1^+$ in both sides of (4.2). On the one hand, using Lemma 4.2 we find that

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_\Omega u_\lambda f(x,u)dx = \int_\Omega (x \cdot \nabla u)f(x,u)\,dx$$

$$= \int_\Omega \big\{x \cdot \nabla\big(F(x,u)\big) - x \cdot F_x(x,u)\big\}dx \qquad (4.3)$$

$$= -\int_\Omega \big\{nF(x,u) + x \cdot F_x(x,u)\big\}dx.$$

Note that here we have used also that $F(x,u) \in W^{1,1}(\Omega)$, which follows from $u \in L^\infty(\Omega)$, $(x \cdot \nabla u)f(x,u) \in L^r(\Omega)$, and $x \cdot F_x(x,u) \in L^\infty$.

On the other hand, let

$$I_\lambda = \lambda^\alpha \langle D\Phi(u), u_\lambda \rangle. \qquad (4.4)$$

Then,

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \langle D\Phi(u), u_\lambda \rangle = -\alpha \langle D\Phi(u), u \rangle + \frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda$$

$$= -\alpha \int_\Omega uf(x,u)dx + \frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda, \qquad (4.5)$$

where we have used that $\langle D\Phi(u), u \rangle = \int_\Omega uf(x,u)dx$, which follows from (4.2).

Now, using Lemma 4.1 and the scaling condition (1.16), we find

$$I_\lambda = \lambda^\alpha \langle D\Phi(u), u_\lambda \rangle \le p\,\lambda^\alpha \Phi(u)^{1/p'}\Phi(u_\lambda)^{1/p} = \lambda^\alpha \|u\|^{p/p'}\|u_\lambda\|$$

$$\le \|u\|^{p/p'+1} = \|u\|^p = p\,\Phi(u) = \langle D\Phi(u), u \rangle = I_1,$$

where $1/p + 1/p' = 1$. Therefore,

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda \le 0.$$

Thus, it follows from (4.2), (4.3), and (4.5) that

$$-\int_\Omega \big\{nF(x,u) + x \cdot F_x(x,u)\big\}dx \le -\alpha \int_\Omega uf(x,u)dx,$$

which contradicts (1.18) unless $u \equiv 0$. $\qquad\qquad \square$

5. Proof of Theorems 1.1 and 1.3

This section is devoted to give the

Proof of Theorem 1.1. Recall that u is a weak solution of (1.1) if and only if

$$(u, \varphi) = \int_\Omega f(x,u)\varphi\,dx \qquad (5.1)$$

for all φ satisfying $(\varphi, \varphi) < \infty$ and $\varphi \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, where $(\cdot, \cdot)$ is given by (1.14). Note that (1.16) is equivalent to (1.5). Thus, part (a) follows from Proposition 1.4, where $\alpha = \frac{n-\sigma}{2}$.

Moreover, it follows from the proof of Proposition 1.4 that

$$-\int_\Omega \{nF(x,u) + x \cdot F_x(x,u)\} dx = \frac{\sigma - n}{2} \int_\Omega uf(x,u)dx + \frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda, \qquad (5.2)$$

where

$$I_\lambda = \lambda^{\frac{n-\sigma}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \big(u(x) - u(x+y)\big)\big(u_\lambda(x) - u_\lambda(x+y)\big)K(y)dx\,dy.$$

Thus, to prove part (b), it suffices to show that

$$\frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda < 0. \qquad (5.3)$$

Following the proof of Proposition 1.4, by the Cauchy-Schwarz inequality we find

$$I_\lambda \leq \lambda^{\frac{n-\sigma}{2}} \|u\| \, \|u_\lambda\|$$

$$= \sqrt{I_1}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(u(x)-u(x+z)\big)^2\lambda^{-n-\sigma}K(z/\lambda)dx\,dz\right)^{1/2}$$

$$= \frac{I_1}{2} + \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(u(x)-u(x+z)\big)^2\lambda^{-n-\sigma}K(z/\lambda)dx\,dz$$

$$\leq I_1.$$

Denote now $K(y) = g(y)/|y|^{n+\sigma}$. Then,

$$I_1 - I_\lambda \geq \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(u(x)-u(x+y)\big)^2\left\{K(y) - \lambda^{-n-\sigma}K(y/\lambda)\right\}dx\,dy$$

$$= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\big(u(x)-u(x+y)\big)^2}{|y|^{n+\sigma}}\{g(y) - g(y/\lambda)\}dx\,dy,$$

and therefore, by the Fatou lemma

$$-\frac{d}{d\lambda}\bigg|_{\lambda=1^+} I_\lambda \geq \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\big(u(x)-u(x+y)\big)^2}{|y|^{n+\sigma}}\,y \cdot \nabla g(y)dx\,dy.$$

Now, recall that $g \in C^1(\mathbb{R}^n \setminus \{0\})$ is nondecreasing along all rays from the origin and nonconstant along some of them. Then, we have that $y \cdot \nabla g(y) \geq 0$ for all y, with strict inequality in a small ball B. This yields that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\big(u(x)-u(x+y)\big)^2}{|y|^{n+\sigma}}\,y \cdot \nabla g(y)dx\,dy > 0$$

unless $u \equiv 0$. Indeed, if $u(x) - u(x+y) = 0$ for all $x \in \mathbb{R}^n$ and $y \in B$ then u is constant in a neighborhood of x, and thus u is constant in all of $\mathbb{R}^n$.

Therefore, using (5.2) we find that if u is a nontrivial bounded solution then

$$\frac{n-\sigma}{2}\int_\Omega uf(x,u)dx < \int_\Omega \big\{nF(x,u) + x\cdot F_x(x,u)\big\}dx,$$

which is a contradiction with (1.8).

Finally, part (c) follows from (a), (b), and Proposition 1.2. $\qquad\square$

To end this section, we give the

Proof of Theorem 1.3. As explained in the Introduction, weak solutions to problem (1.1) with L given by (1.3) are critical points to (1.17) with $p = 2$ and with

$$\|u\|^2 = \int_\Omega A(\nabla u, \nabla u)dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(u(x) - u(x+y)\big)^2 K(y)dxdy,$$

where $A(p, q) = p^T A q$ and $A = (a_{ij})$ is the matrix in (1.3). It is immediate to see that this norm satisfies (1.16) with $\alpha = \frac{n-2}{2}$ whenever (1.12) holds. Moreover, since A is positive definite by assumption, then $\|u\|_{W^{1,2}(\Omega)} \leq c\|u\|^2$, and hence $u \in W^{1,r}(\Omega)$ with $r = 2$.

Then, it follows from the proof of Proposition 1.4 that

$$\frac{n-2}{2}\int_\Omega uf(x,u)dx = \int_\Omega \{nF(x,u) + x\cdot F_x(x,u)\}\,dx + \left.\frac{d}{d\lambda}\right|_{\lambda=1^+} I_\lambda,$$

where

$$\begin{aligned}I_\lambda = \lambda^{\frac{n-2}{2}}&\int_\Omega A(\nabla u, \nabla u_\lambda)dx + \\ + \lambda^{\frac{n-2}{2}}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\big(u(x) - u(x+y)\big)\big(u_\lambda(x) - u_\lambda(x+y)\big)K(y)dxdy.\end{aligned}\qquad(5.4)$$

Now, as in the proof of Theorem 1.1, we find

$$I_1 - I_\lambda \geq \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\big(u(x) - u(x+y)\big)^2}{|y|^{n+2}}\big\{g(y) - g(y/\lambda)\big\}dy,$$

where $g(y) = K(y)|y|^{n+2}$. Thus, differentiating with respect to λ, we find that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=1^+} I_\lambda \geq \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\big(u(x) - u(x+y)\big)^2}{|y|^{n+2}}y\cdot\nabla g(y)dy.$$

Moreover, since $\int_{\mathbb{R}^n}\frac{|y|^2}{1+|y|^2}K(y)dy < \infty$ and g is radially nondecreasing, then it follows that $\lim_{t\to 0}g(t\tau) = 0$ for almost all $\tau \in S^{n-1}$. Thus, if K is not identically zero then $y\cdot\nabla g(y)$ is positive in a small ball B, and hence

$$\left.\frac{d}{d\lambda}\right|_{\lambda=1^+} I_\lambda > 0$$

unless $u \equiv 0$, which yields the desired result. $\qquad\square$

6. Proof of Proposition 1.2

In this section we prove Proposition 1.2. To prove it, we follow the arguments used in [16], where we studied the regularity up to the boundary for the Dirichlet problem for the fractional Laplacian. The main ingredients in the proof of this result are the interior estimates of Silvestre [19] and the supersolution given by the next lemma.

Lemma 6.1. *Let L be an operator of the form* (1.2), *with K symmetric, positive, and satisfying* (1.9). *Let $\psi(x) = (x_n)_+^{\epsilon/2}$. Then,*

$$L\psi \geq 0 \quad in \quad \mathbb{R}^n_+,$$

where $\mathbb{R}^n_+ = \{x_n > 0\}$.

Proof. Assume first $n = 1$. Let $x \in \mathbb{R}_+$. Since K is symmetric, we have

$$L\psi(x) = \frac{1}{2} \int_{-\infty}^{+\infty} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big) K(y) dy.$$

Then, it is immediate to see that there exists $\rho > 0$ such that

$$2\psi(x) - \psi(x+y) - \psi(x-y) > 0 \quad \text{for } |y| < \rho$$

and

$$2\psi(x) - \psi(x+y) - \psi(x-y) < 0 \quad \text{for } |y| > \rho.$$

Thus, using that $K(y)|y|^{1+\epsilon}$ is nonincreasing in $(0, +\infty)$, and that $(-\Delta)^{\epsilon/2}\psi = 0$ in $\mathbb{R}_+$, we find

$$
\begin{aligned}
L\psi(x) &= \frac{1}{2} \int_{|y|<\rho} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big) K(y) dy \\
&\quad + \frac{1}{2} \int_{|y|>\rho} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big) K(y) dy \\
&\geq \frac{1}{2} \int_{|y|<\rho} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big) \frac{K(\rho)|\rho|^{1+\epsilon}}{|y|^{1+\epsilon}} dy \\
&\quad + \frac{1}{2} \int_{|y|>\rho} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big) \frac{K(\rho)|\rho|^{1+\epsilon}}{|y|^{1+\epsilon}} dy \\
&= K(\rho)|\rho|^{1+\epsilon} \frac{1}{2} \int_{-\infty}^{+\infty} \frac{2\psi(x) - \psi(x+y) - \psi(x-y)}{|y|^{1+\epsilon}} dy \\
&= K(\rho)|\rho|^{1+\epsilon} (-\Delta)^{\epsilon/2}\psi(x) = 0.
\end{aligned}
$$

Thus, the lemma is proved for $n = 1$.

Assume now $n > 1$, and let $x \in \mathbb{R}^n_+$. Then,

$$
\begin{aligned}
L\psi(x) &= \frac{1}{2} \int_{\mathbb{R}^n} \big(2\psi(x) - \psi(x+y) - \psi(x-y)\big) K(y) dy \\
&= \frac{1}{4} \int_{S^{n-1}} \left(\int_{-\infty}^{+\infty} \big(\psi(x) - \psi(x+t\tau) - \psi(x-t\tau)\big) t^{n-1} K(t\tau) dt \right) d\tau.
\end{aligned}
\tag{6.1}
$$

Now, for each $\tau \in S^{n-1}$, the kernel $K_1(t) := t^{n-1}K(t\tau)$ satisfies $K_1(t)t^{1+\epsilon}$ is nonincreasing in $(0, +\infty)$, and in addition

$$
\psi(x + \tau t) = (x_n + \tau_n t)_+^{\epsilon/2} = \tau_n^{\epsilon/2}(x_n/\tau_n + t)_+^{\epsilon/2}.
$$

Thus, by using the result in dimension $n = 1$, we find

$$
\int_{-\infty}^{+\infty} \big(\psi(x) - \psi(x+t\tau) - \psi(x-t\tau)\big) t^{n-1} K(t\tau) dt \geq 0.
\tag{6.2}
$$

Therefore, we deduce from (6.1) and (6.2) that $L\psi(x) \geq 0$ for all $x \in \mathbb{R}^n_+$, and the lemma is proved. $\qquad\square$

The following result is the analog of Lemma 2.7 in [16].

Lemma 6.2. *Under the hypotheses of Proposition 1.2, it holds*

$$
|u(x)| \leq C\delta(x)^{\epsilon/2} \quad \text{for all } x \in \Omega,
$$

where C is a constant depending only on Ω, ϵ, and $\|u\|_{L^\infty(\Omega)}$.

Proof. By Lemma 6.1, we have that $\psi(x) = (x_n)_+^{\epsilon/2}$ satisfies $L\psi \geq 0$ in $\mathbb{R}^n_+$. Thus, we can truncate this $1D$ supersolution in order to obtain a strict supersolution ϕ satisfying $\phi \equiv \psi$ in $\{x_n < 1\}$, $\phi \equiv 1$ in $\{x_n > 1\}$, and $L\phi \geq c_0$ in $\{0 < x_n < 1\}$.

We can now use $C\phi$ as a supersolution at each point of the boundary $\partial\Omega$ to deduce $|u| \leq C\delta^{\epsilon/2}$ in Ω; see Lemma 2.7 in [16] for more details. $\qquad\square$

We next prove the following result, which is the analog of Proposition 2.3 in [16].

Proposition 6.3. *Under the hypotheses of Proposition 1.2, assume that $w \in L^\infty(\mathbb{R}^n)$ solves $Lw = g$ in B_1, with $g \in L^\infty$. Then, there exists $\alpha > 0$ such that*

$$
\|w\|_{C^\alpha(B_{1/2})} \leq C \left(\|g\|_{L^\infty(B_1)} + \|w\|_{L^\infty(\mathbb{R}^n)} \right),
\tag{6.3}
$$

where C depends only on n, ϵ, σ, and the constant in (1.10).

Proof. With slight modifications, the results in [19] yield the desired result.

Indeed, given $\delta > 0$ conditions (1.5), (1.9), and (1.10) yield

$$
\kappa Lb(x) + 2 \int_{\mathbb{R}^n \setminus B_{1/4}} \big(|8y|^\eta - 1\big) K(y) dy < \frac{1}{2} \inf_{A \subset B_2, \, |A| > \delta} \int_A K(y) dy
\tag{6.4}
$$

for some κ and η depending only on n, ϵ, σ, and the constant in (1.10). Moreover, since our hypotheses are invariant under scaling, then (6.4) holds at every scale. Note that (6.4) is exactly hypothesis (2.1) in [19].

Then, as mentioned by Silvestre in [19, Remark 4.3], Lemma 4.1 in [19] holds also with (4.1) therein replaced by $Lw \leq \nu_0$ in B_1, with ν_0 depending on κ. Therefore, the Hölder regularity of w with the desired estimate (6.3) follows from [19, Theorem 5.1].

Note that it is important to have σ strictly less than 2, since otherwise condition (6.4) does not hold. □

The following is the analog of Proposition 2.2 in [16].

Proposition 6.4. *Under the same hypotheses of Proposition 1.2, assume that $w \in C^\beta(\mathbb{R}^n)$ solves $Lw = g$ in B_1, with $g \in C^\beta$, $\beta \in (0,1)$. Then, there exists $\alpha > 0$ such that*

$$\|w\|_{C^{\beta+\alpha}(B_{1/2})} \leq C \left(\|g\|_{C^\beta(B_1)} + \|w\|_{C^\beta(\mathbb{R}^n)} \right) \quad if \quad \beta + \alpha < 1,$$

$$\|w\|_{C^{0,1}(B_{1/2})} \leq C \left(\|g\|_{C^\beta(B_1)} + \|w\|_{C^\beta(\mathbb{R}^n)} \right) \quad if \quad \beta + \alpha > 1,$$

where C and α depend only on n, ϵ, σ, and the constants in (1.10) and (1.7).

Proof. It follows from the previous Proposition applied to the incremental quotients $(w(x+h) - w(x))/|h|^\beta$ and from Lemma 5.6 in [3]. □

As a consequence of the last two propositions, we find the following corollaries. The first one is the analog of Corollary 2.5 in [16].

Corollary 6.5. *Under the same hypotheses of Proposition 1.2, assume that $w \in L^\infty(\mathbb{R}^n)$ solves $Lw = g$ in B_1, with $g \in L^\infty$. Then, there exists $\alpha > 0$ such that*

$$\|w\|_{C^\alpha(B_{1/2})} \leq C \left(\|g\|_{L^\infty(B_1)} + \|w\|_{L^\infty(B_2)} + \|(1+|y|)^{-n-\epsilon} w(y)\|_{L^1(\mathbb{R}^n)} \right),$$

where C depends only on n, ϵ, σ, and the constants in (1.7) and (1.10).

Proof. Using (1.7), the proof is exactly the same as the one in [16, Corollary 2.5]. □

The second one is the analog of Corollary 2.4 in [16].

Corollary 6.6. *Under the same hypotheses of Proposition 1.2, assume that $w \in C^\beta(\mathbb{R}^n)$ solves $Lw = g$ in B_1, with $g \in C^\beta$, $\beta \in (0,1)$. Then, there exists $\alpha > 0$ such that*

$$\|w\|_{C^{\beta+\alpha}(B_{1/2})} \leq C \left(\|g\|_{C^\beta(B_1)} + \|w\|_{C^\beta(\overline{B_2})} + \|(1+|y|)^{-n-\epsilon} w(y)\|_{L^1(\mathbb{R}^n)} \right)$$

if $\beta + \alpha < 1$, while

$$\|w\|_{C^{0,1}(B_{1/2})} \leq C \left(\|g\|_{C^\beta(B_1)} + \|w\|_{C^\beta(\overline{B_2})} + \|(1+|y|)^{-n-\epsilon} w(y)\|_{L^1(\mathbb{R}^n)} \right)$$

if $\beta + \alpha > 1$. The constant C depends only on n, ϵ, σ and the constants in (1.7) and (1.10).

Proof. Using (1.7), the proof is the same as the one in [16, Corollary 2.4]. □

We can finally give the

Proof of Proposition 1.2. Let now $x \in \Omega$, and $2R = \text{dist}(x, \partial\Omega)$. Then, one may rescale problem (1.1)-(1.2) in $B_R = B_R(x)$, to find that $w(y) := u(x + Ry)$ satisfies $\|w\|_{L^\infty(B_2)} \leq CR^{\epsilon/2}$, $|w(y)| \leq CR^{\epsilon/2}(1 + |y|^{\epsilon/2})$ in $\mathbb{R}^n$, and $\|L_R w\|_{L^\infty(B_1)} \leq CR^\epsilon$, where

$$L_R w(y) = \int_{\mathbb{R}^n} \big(w(y) - w(y + z)\big) K_R(y) dy$$

and $K_R(y) = K(Ry)R^{n+\epsilon}$.

Moreover, it is immediate to check that (1.7) yields

$$|\nabla K_R(y)| \leq C \frac{K_R(y)}{|y|},$$

with the same constant C for each $R \in (0, 1)$. The other hypotheses of Proposition (1.2) are clearly satisfied by the kernels K_R for each $R \in (0, 1)$.

Hence, one may apply Corollaries 6.5 and 6.6 (repeatedly) to obtain

$$|\nabla w(0)| \leq CR^{\epsilon/2}.$$

From this, we deduce that $|\nabla u(x)| \leq CR^{\frac{\epsilon}{2} - 1}$, and since this can be done for any $x \in \Omega$, we find

$$|\nabla u(x)| \leq C\delta(x)^{\frac{\epsilon}{2} - 1} \quad \text{in } \Omega,$$

as desired. The $C^{\epsilon/2}(\mathbb{R}^n)$ regularity of u follows immediately from this gradient bound. $\qquad\square$

Remark 6.7. The convexity of the domain has been only used in the construction of the supersolution. To establish Proposition 1.2 in general $C^{1,1}$ domains, one only needs to construct a supersolution which is not $1D$ but it is radially symmetric and with support in $\mathbb{R}^n \setminus B_1$, as in [16, Lemma 2.6], where it is done for the fractional Laplacian.

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Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain
E-mail address: xavier.ros.oton@upc.edu

Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain
E-mail address: joaquim.serra@upc.edu

D. Regularity for fully nonlinear nonlocal parabolic equation with rough kernels

REGULARITY FOR FULLY NONLINEAR NONLOCAL PARABOLIC EQUATIONS WITH ROUGH KERNELS

JOAQUIM SERRA

ABSTRACT. We prove space and time regularity for solutions of fully nonlinear parabolic integro-differential equations with rough kernels. We consider parabolic equations $u_t = \mathrm{I}u$, where I is translation invariant and elliptic with respect to the class $\mathcal{L}_0(\sigma)$ of Caffarelli and Silvestre, $\sigma \in (0,2)$ being the order of I. We prove that if u is a viscosity solution in $B_1 \times (-1,0]$ which is merely bounded in $\mathbb{R}^n \times (-1,0]$, then u is C^β in space and $C^{\beta/\sigma}$ in time in $\overline{B_{1/2}} \times [-1/2,0]$, for all $\beta < \min\{\sigma, 1+\alpha\}$, where $\alpha > 0$. Our proof combines a Liouville type theorem —relaying on the nonlocal parabolic C^α estimate of Chang and Dávila— and a blow up and compactness argument.

1. INTRODUCTION

In [2], Caffarelli and Silvestre introduced the ellipticity class $\mathcal{L}_0 = \mathcal{L}_0(\sigma)$, with order $\sigma \in (0,2)$. The class $\mathcal{L}_0$ contains all linear operators L of the form

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) K(y)\, dy,$$

where the kernels $K(y)$ satisfy the ellipticity bounds

$$0 < \lambda \frac{2-\sigma}{|y|^{n+\sigma}} \leq K(y) \leq \Lambda \frac{2-\sigma}{|y|^{n+\sigma}}.$$

This includes kernels that may be very oscillating and irregular. That is why the words *rough kernels* are sometimes used to refer to $\mathcal{L}_0$. The extremal operators M_σ^+ and M_σ^- for $\mathcal{L}_0$ are

$$M_\sigma^+ u(x) = \sup_{L \in \mathcal{L}_0} Lu(x) \quad \text{and} \quad M_\sigma^- u(x) = \inf_{L \in \mathcal{L}_0} Lu(x).$$

If $u \in L^\infty(\mathbb{R}^n)$ satisfies the two viscosity inequalities $M_\sigma^+ u \geq 0$ and $M_\sigma^- u \leq 0$ in B_1, then u belongs to $C^\alpha(\overline{B_{1/2}})$. More precisely, one has the estimate

$$\|u\|_{C^\alpha(B_{1/2})} \leq C \|u\|_{L^\infty(\mathbb{R}^n)}. \tag{1.1}$$

This estimate, with constants that remain bounded as the $\sigma \nearrow 2$, is one of the main results in [2].

For second order equations ($\sigma = 2$) the analogous of (1.1) is the classical estimate of Krylov and Safonov, and differs from (1.1) only from the fact that it has $\|u\|_{L^\infty(B_1)}$ instead of $\|u\|_{L^\infty(\mathbb{R}^n)}$ on the right hand side. This apparently harmless difference comes from the fact that elliptic equations of order $\sigma < 2$ are nonlocal. By analogy

with second order equations, from (1.1) one expects to obtain $C^{1,\alpha}$ interior regularity of solutions to translation invariant elliptic equations $\mathrm{I}u = 0$ in B_1. When $\sigma = 2$, this is done by applying iteratively the estimate (1.1) to incremental quotients of u, improving at each step by α the Hölder exponent in a smaller ball (see [1]). However, in the case $\sigma < 2$ the same iteration does not work since, right after the first step, the L^∞ norm of the incremental quotient of u is only bounded in $B_{1/2}$, and not in the whole $\mathbb{R}^n$.

The previous difficulty is very related to the fact that the operator will "see" possible distant high frequency oscillations in the exterior Dirichlet datum. In [2], this issue is bypassed by restricting the ellipticity class, i.e., introducing a new class $\mathcal{L}_1 \subset \mathcal{L}_0$ of operators with C^1 kernels (away from the origin). The additional regularity of the kernels has the effect of averaging distant high frequency oscillations, balancing out its influence. This is done with an integration by parts argument. Hence, the $C^{1+\alpha}$ estimates in [2] are "only" proved for elliptic equations with respect to $\mathcal{L}_1$ (instead of $\mathcal{L}_0$).

Very recently, Kriventsov [7] succeeded in proving the same $C^{1+\alpha}$ estimates for elliptic equations of order $\sigma > 1$ with rough kernels, that is, for $\mathcal{L}_0$. The proof in [7] is quite involved and combines fine new estimates with a compactness argument. In [7] the same methods are used to obtain other interesting applications, including nearly sharp Schauder type estimates for linear, non translation invariant, nonlocal elliptic equations.

Here, we extend the main result in [7] in two ways, providing in addition a new proof of it. First, we pass from elliptic to parabolic equations. Second, we allow also $\sigma \leq 1$, proving in this case $C^{\sigma-\epsilon}$ regularity in space and $C^{1-\epsilon}$ in time (for all $\epsilon > 0$) for solutions to nonlocal translation invariant parabolic equations with rough kernels. Our proof follows a new method, different from that in [7]. As explained later in this introduction, our strategy is to prove first a Liouville type theorem for global solutions, and to deduce later the interior estimates from this Liouville theorem, using a blow up and compactness argument. That a regularity estimate and a Liouville theorem are in some way equivalent is an old principle in PDEs, but here it turns out to be very useful to bypass the difficulty iterating the "nonlocal" estimate (1.1).

Therefore, a main interest of this paper lies precisely on the method that we introduce here. It is very flexible and can be useful in different contexts with nonlocal equations. For instance, the method can be used to study equations which are nonlocal also in time, and also to analyze boundary regularity for nonlocal equations (see Remark 1.1).

To have a local $C^{1+\alpha}$ estimate for solutions that are merely bounded in $\mathbb{R}^n$, it is necessary that the order of the equation be greater than one. Indeed, for nonlocal equations of order σ with rough kernels there is no hope to prove a local Hölder estimate of order greater than σ for solutions that are merely bounded in $\mathbb{R}^n$. The reason being that influence of the distant oscillations is too strong. Counterexamples

can be easily constructed even for linear equations. That is why the condition $\sigma > 1$ is necessary for the $C^{1,\alpha}$ estimates of Kriventsov [7]. Also, this is why we prove C^β estimates in space only for $\beta < \sigma$.

As explained above, the difficulty of nonlocal equations with rough kernels, with respect to local ones, is that the estimate (1.1) is not immediately useful to prove higher order Hölder regularity for solutions of $Iu = 0$ in B_1. Recall that the classical iteration fails because, after the first step, the L^∞ norm of the incremental quotient of order α is only controlled in $B_{1/2}$, and not in the whole $\mathbb{R}^n$. The idea in our approach is that the iteration does work if one considers a solution in the whole space. If we have a global solution u, then we can apply (1.1) at every scale and deduce that u is C^α in all space. Then, we consider the incremental quotients of order α of u, which we control in the whole $\mathbb{R}^n$, and we prove that u is $C^{2\alpha}$. And so on. When this is done with estimates, taking into account the growth at infinity of the function u and the scaling of the estimates, we obtains a Liouville theorem. Using it, we deduce the higher order interior regularity of u directly, using a blow up argument and compactness argument. In order to have compactness of sequences of viscosity solutions we only need the C^α estimate (1.1).

For local translation invariant elliptic equations like $F(D^2u) = 0$ in B_1 it would be a unnecessary complication to first prove the Liouville theorem and then obtain the interior estimate by the blow up and compactness argument in this paper. Indeed, as said above, the iteration already works in the bounded domain B_1. Nevertheless, it is worth noting that equations of the type $F(D^2u, Du, x) = 0$, with continuous dependence on x, become $\tilde{F}(D^2u) = 0$ after blow up at some point. By this reason, one can see that the second order Liouville theorem and the blow up method provide a $C^{1,\alpha}$ bound for solutions to $F(D^2u, Du, x) = 0$ in B_1. However, this approach gives nothing new with respect to classical perturbative methods (as in [1]).

For nonlocal equations, we could also have considered non translation invariant equations —with continuous dependence on x—, and having also lower order terms. This is because in our argument we blow up the equation. Translation and scale invariances are only needed in the limit equation (after blow up), to which we apply the Liouville theorem. And, in a typical situation, when one blows up a non translation invariant equations with lower order terms one gets a translation invariant equation with no lower order terms. Hence, in the appropriate setting, we could certainly extend our results to these equations. In this paper, however, we do not include this since we are not interested in pushing the method to its limits, but rather in giving a clear example of its use.

In the following remark we give two examples of different contexts in which the method of this paper is useful.

Remark 1.1. *Nonlocal dependence also on time.* Let us point out that it is not essential to our argument that that the equation is local in time. Hence, the same ideas could be useful when considering nonlinear parabolic-like equations which

have a nonlocal dependence on the past time. For instance, it could be useful when studying the nonlinear versions of the generalized master equations [4].

Boundary regularity. A boundary version of the method in the present paper turns out to be a powerful tool in the study of the boundary regularity for fully nonlinear integro-differential elliptic equations; this is done in the work of Ros-Oton and the author [8]. In this case, the Liouville theorem to be used is for solutions in a half space $\{x_n > 0\}$, which clearly corresponds to the blow up of a smooth domain at a given boundary point. Interestingly, the possible solutions in this Liouville theorem are not planes, but instead they are of the type $c(x_n)_+^s$, for some constant c. Once one has this "boundary" Liouville theorem —its proof is more involved than that of the "interior" one in this paper—, then the blow up and compactness argument in this paper can be adapted to obtain fine boundary regularity results.

2. MAIN RESULT

The basic parabolic C^α estimate on which all our argument relies has been obtained by Chang and Dávila [5] —this is the parabolic version of (1.1) and we state it below.

In order that the statements of the results naturally include their classical second order versions, it is convenient to define the ellipticity class $\mathcal{L}_0(2)$, as the set of second order linear operators

$$Lu(x) = a_{ij}\partial_{ij}u(x)$$

with (a_{ij}) satisfying

$$0 < c_n\lambda\mathrm{Id} \le (a_{ij}) \le c_n\Lambda\mathrm{Id}, .$$

The constant c_n is a appropriately chosen so that the operators in $\mathcal{L}_0(\sigma)$ converge to operators in $\mathcal{L}_0(2)$ (when applied to bounded smooth functions).

Throughout the paper, ω_{σ_0} denotes the weight

$$\omega_{\sigma_0}(x) = \frac{2 - \sigma_0}{1 + |x|^{n+\sigma_0}}.$$

Theorem 2.1 (Regularity in space from [5, Theorem 5.1]). *Let $\sigma_0 \in (0, 2]$ and $\sigma \in [\sigma_0, 2]$. Let $u \in C\big(\overline{B_1} \times [-1, 0]\big)$ with $\sup_{t \in [-1,0]} \int_{\mathbb{R}^n} u(x, t)\omega_{\sigma_0}(x)\, dx < \infty$ satisfy the following two inequalities in the viscosity sense*

$$u_t - M_\sigma^+ u \le C_0 \quad and \quad u_t - M_\sigma^- u \ge -C_0 \quad in\ B_1 \times (-1, 0].$$

Then, for some $\alpha \in (0, 1)$ and $C > 0$, depending only on σ_0, ellipticity constants, and dimension, we have

$$\sup_{t \in [-1/2, 0]}\big[u(\,\cdot\,, t)\big]_{C^\alpha(B_{1/2})} \le C\left(\|u\|_{L^\infty(B_1 \times (-1, 0])} + \sup_{t \in [-1,0]}\|u(\,\cdot\,, t)\|_{L^1(\mathbb{R}^n, \omega_{\sigma_0})} + C_0\right).$$

Theorem 5.1 of [5] contains also a $C^{\alpha/\sigma}$ estimate in time, for some $\alpha > 0$. However, for our argument we only need the estimate in space from [5], which is the one stated above.

Before stating our main result let us briefly recall some definitions (translation invariant elliptic operator, viscosity solution, etc.), which are by now standard in the context of integro-differential equations. They can be found in detail in [2, 5].

As in [2], an operator I is said to be elliptic with respect to $\mathcal{L}_0(\sigma)$, $\sigma \in [\sigma_0, 2]$, if

$$M_\sigma^-(u-v)(x) \le \mathrm{I}u(x) - \mathrm{I}v(x) \le M_\sigma^+(u-v)(x),$$

for all elliptic test functions u, v at x, which are C^2 functions in a neighborhood of x and having finite integral against the weight ω_{σ_0}. Recall that I is defined as a "black box" acting on test functions, with the only assumption that if u is a test function at x, then $\mathrm{I}u$ is continuous near x. The operator we have in mind is

$$\mathrm{I}u(x) = \inf_\alpha \sup_\beta \big(L_{\alpha\beta}u + c_{\alpha,\beta}\big)$$

where $L_{\alpha\beta} \in \mathcal{L}_0(\sigma)$ and $\inf_\alpha \sup_\beta c_{\alpha,\beta} = 0$.

That I is translation invariant clearly means

$$\mathrm{I}\big(u(x_0 + \cdot)\big)(x) = (\mathrm{I}u)(x_0 + x).$$

The definition we use of viscosity solutions (and inequalities) for parabolic equations is the one in [5]. Namely, let f and u such be continuous functions in a parabolic domain. Assume that $\int_{\mathbb{R}^n} u(x,t)\omega_{\sigma_0}(x)\,dx$ is locally bounded for all t in the domain. Then, we say that u is a viscosity solution of

$$u_t - \mathrm{I}u = f$$

if whenever a test function $v(x,t)$ touches by above (below) u at (x_0, t_0) we have $\big(v_{t^-} - \mathrm{I}v\big)(x_0, t_0) \le f(x_0, t_0)$ ($\ge$). For parabolic equations v touching u by above at (x_0, t_0) means $v(x,t) \ge u(x,t)$ for all $x \in \mathbb{R}^n$ and for all $t \le t_0$. As in [5], test functions v are quadratic functions in some small cylinder and outside they have the same type of growth as the solutions u. That is,

$$v(x,t) = a_{ij}x_i x_j + b_i x_i + ct + d \quad \text{in the cylinder } \overline{B_\epsilon(x_0)} \times [t_0 - \epsilon, t_0]$$

for some $\epsilon > 0$ and

$$\|v(\,\cdot\,,t)\|_{L^1(\mathbb{R}^n, \omega_{\sigma_0})} = \int_{\mathbb{R}^n} |v(x,t)|\omega_{\sigma_0}(x)\,dx$$

if locally bounded for t in the domain of the equation. As explained in [5], in order to have left continuity in time of $(\partial_t - \mathrm{I})v(x,t)$, one additionally requires test functions to satisfy $\lim_{t \nearrow t_0} \|v(\,\cdot\,,t) - v(\,\cdot\,,t_0)\|_{L^1(\mathbb{R}^n, \omega_{\sigma_0})} = 0$ for all t_0 in the domain.

Our main result is the following.

Theorem 2.2. *Let $\sigma_0 \in (0,2)$ and $\sigma \in [\sigma_0, 2]$. Let $u \in L^\infty\big(\mathbb{R}^n \times (-1,0)\big)$ be a viscosity solution of $u_t - \mathrm{I}u = f$ in $B_1 \times (-1,0]$, where I is a translation invariant*

elliptic operator with respect to the class $\mathcal{L}_0(\sigma)$ with $\mathrm{I}0 = 0$. *Let* $\alpha = \alpha(\sigma_0)$ *be given by Theorem 2.1.*

Then, for all $\epsilon > 0$, *letting*

$$\beta = \min\{\sigma, 1 + \alpha\} - \epsilon,$$

$u(\,\cdot\,, t)$ *belongs to* $C^\beta\left(\overline{B_{1/2}}\right)$ *for all* $t \in [-1/2, 0]$, *and* $u(x, \,\cdot\,)$ *belongs to* $C^{\beta/\sigma}\left([-1/2, 0]\right)$ *for all* $x \in B_{1/2}$. *Moreover, the following estimate holds*

$$\sup_{t \in [-1/2, 0]} \left\|u(\,\cdot\,, t)\right\|_{C^\beta(B_{1/2})} + \sup_{x \in B_{1/2}} \left\|u(x, \,\cdot\,)\right\|_{C^{\beta/\sigma}([-1/2, 0])} \leq CC_0,$$

where

$$C_0 = \left\|u\right\|_{L^\infty(\mathbb{R}^n \times (-1, 0))} + \left\|f\right\|_{L^\infty(B_1 \times (-1, 0))}$$

and C is a constant depending only on σ_0, ϵ, ellipticity constants, and dimension.

3. Liouville type theorem

As said in the introduction, Theorem 2.2 will follow from a Liouville type theorem, which we state below, and a blow up and compactness augment.

In all the paper, given $\sigma \in (0, 2]$ and $R > 0$, Q_R^σ denotes the parabolic cylinder

$$Q_R^\sigma := \left\{(x, t) \,:\, |x| \leq R \text{ and } -R^\sigma < t < 0\right\}. \tag{3.1}$$

For $z \in \mathbb{R}^n \times (\infty, 0]$, the cylinder $z + Q_R^\sigma$ is denoted as $Q_R^\sigma(z)$.

Theorem 3.1. *Let $\sigma_0 \in (0, 2)$, $\sigma \in [\sigma_0, 2]$, and $\alpha = \alpha(\sigma_0)$ be given by Theorem 2.1. Let $0 < \beta < \min\{\sigma_0, 1 + \alpha\}$. Let I be a translation invariant operator, elliptic with respect to $\mathcal{L}_0(\sigma)$, with $\mathrm{I}0 = 0$. Assume that u in $C\left(\mathbb{R}^n \times (-\infty, 0]\right)$ satisfies the growth control*

$$\|u\|_{L^\infty(Q_R^\sigma)} \leq CR^\beta \quad \text{for all } R \geq 1 \tag{3.2}$$

and that it is a viscosity solution of

$$u_t = \mathrm{I}u \quad \text{in all of } \mathbb{R}^n \times (-\infty, 0].$$

Then, if $\beta < 1$, u is constant. And if $\beta \geq 1$, $u(x, t) = a \cdot x + b$ is an affine function of the x variables only.

Proof. For all $\rho \geq 1$ consider $v_\rho(x, t) = \rho^{-\beta} u(\rho x, \rho^\sigma t)$. Note that the growth control on u is transferred to v_ρ. Indeed,

$$\|v_\rho\|_{L^\infty(Q_R^\sigma)} = \rho^{-\beta}\|u\|_{L^\infty(Q_{\rho R}^\sigma)} \leq C\rho^{-\beta}(\rho R)^\beta = CR^\beta \quad \text{for all } R \geq 1$$

Hence, since $\beta < \sigma_0$,

$$\sup_{t \in [-1, 0]} \|v_\rho(\,\cdot\,, t)\|_{L^1(\mathbb{R}^n, \omega_{\sigma_0})} \leq C(n, \sigma, \beta).$$

Moreover, since u is satisfies $u_t \leq M_\sigma^+ u$ and $u_t \geq M_\sigma^- u$ in $\mathbb{R}^n \times (-\infty, 0]$, also v_ρ satisfies the same inequalities.

By applying Theorem 2.1 to the function v_ρ we obtain

$$\sup_{t\in[-1/2,0]} \big[v_\rho(\,\cdot\,,t)\big]_{C^\alpha(B_{1/2})} \leq C.$$

Scaling back to u the previous estimate (setting $\rho = 2^{1/\sigma}R$) we obtain

$$\sup_{t\in(-R^\sigma,0]} \big[u(\,\cdot\,,t)\big]_{C^\alpha(B_R)} \leq CR^{\beta-\alpha}.$$

In this way for all $h \in \mathbb{R}^n$ we have an improved growth control for the incremental quotient

$$v_h^{(\alpha)}(x,t) = \frac{u(x+h,t) - u(x,t)}{|h|^\alpha}.$$

Namely,

$$\|v_h^{(\alpha)}\|_{L^\infty(Q_R^\sigma)} \leq CR^{\beta-\alpha} \quad \text{for all } R \geq 1.$$

Now, $v_h^{(\alpha)}$ satisfies again $\big(v_h^{(\alpha)}\big)_t \leq M_\sigma^+ v_h^{(\alpha)}$ and $\big(v_h^{(\alpha)}\big)_t \geq M_\sigma^- v_h^{(\alpha)}$. Hence, we may repeat the previous scaling augment and use Theorem 2.1 to obtain

$$\sup_{t\in[-R^\sigma,0]} \big[v_h^{(\alpha)}(\,\cdot\,,t)\big]_{C^\alpha(B_R)} \leq CR^{\beta-2\alpha}.$$

And this provides an improved growth control for

$$v_h^{(2\alpha)}(x,t) = \frac{u(x+h,t) - u(x,t)}{|h|^{2\alpha}},$$

that is,

$$\|v_h^{(2\alpha)}\|_{L^\infty(Q_R^\sigma)} \leq R^{\beta-2\alpha} \quad \text{for all } R \geq 1.$$

It is clear that we may keep iterating in this way, improving the growth control by α at each step.

After a bounded number of N of iterations we will have $(N+1)\alpha > 1$ and we will obtain

$$\sup_{t\in[-R^\sigma,0]} \big[v_h^{(N\alpha)}(\,\cdot\,,t)\big]_{C^\alpha(B_R)} \leq CR^{\beta-N\alpha},$$

which implies

$$\|v_h^{(1)}\|_{L^\infty(Q_R^\sigma)} \leq R^{\beta-1} \quad \text{for all } R \geq 1. \tag{3.3}$$

As usual with fully nonlinear equations we may do a last iteration to obtain a $C^{1,\alpha}$ estimate by using that $v_h^{(1)}$ satisfies the two viscosity inequalities. Thus, using one more time Theorem 2.1 at every scale and (3.3) we obtain

$$\left\| \frac{D_x u(x+h,t) - D_x u(x,t)}{|h|^\alpha} \right\|_{L^\infty(Q_R)} \leq R^{\beta-1-\alpha} \quad \text{for all } R \geq 1.$$

Above D_x denotes any derivative with respect to some of the space variables.

Therefore, since by assumption $\beta < 1 + \alpha$, sending $R \nearrow \infty$ we obtain

$$D_x u(x+h,t) = D_x u(x,t) \quad \text{for all } h \in \mathbb{R}^n.$$

Hence, $D_x u$ depends on the variable t only. But since $D_x u$ satisfies

$$(D_x u)_t \leq M_\sigma^+(D_x u) = 0 \quad \text{and} \quad (D_x u)_t \geq M_\sigma^-(D_x u) = 0$$

then it is $(D_x u)_t = 0$ in all of $\mathbb{R}^n \times (-\infty, 0]$. Therefore,

$$u(x, t) = a \cdot x + \psi(t).$$

Finally, since $u_t = \mathrm{I}u = 0$ we have $\psi(t) = b$ for some constant $b \in \mathbb{R}$. Moreover, in the case $\beta < 1$ the growth control yields $a = 0$. $\qquad\square$

4. Preliminary lemmas and proof of Theorem 2.2

As said above the proof of Theorem 2.2 is by compactness. The following result is a consequence of the theory in [3] and provides compactness under weak convergence of sequences of elliptic operators which are elliptic with respect to some $\mathcal{L}_0(\sigma)$, with σ varying in the interval $[\sigma_0, 2]$. We use the definition from [3] of weak convergence of operators. Namely, a sequence of translation invariant operators I_m is said to converge weakly to I if for all $\epsilon > 0$ and test function v, which is a quadratic polynomial in B_ϵ and belongs to $L^1(\mathbb{R}^n, \omega_{\sigma_0})$, we have

$$\mathrm{I}_m v(x) \to \mathrm{I}v(x) \quad \text{uniformly in } \overline{B_{\epsilon/2}}.$$

Lemma 4.1. *Let $\sigma_0 \in (0, 2)$, $\sigma_m \in [\sigma_0, 2]$, and I_m such that*

- *I_m is translation invariant and elliptic with respect to $\mathcal{L}_0(\sigma_m)$.*
- *$\mathrm{I}_m 0 = 0$.*

Then, a subsequence of $\sigma_m \to \sigma \in [\sigma_0, 2]$ and a subsequence of I_m converges weakly to some translation invariant operator I elliptic with respect to $\mathcal{L}_0(\sigma)$.

Proof. We may assume by taking a subsequence that $\sigma_m \to \sigma \in [\sigma_0, 2]$. Consider the class $\mathcal{L} = \bigcup_{\sigma \in [\sigma_0, 2]} \mathcal{L}_0(\sigma)$. This class satisfies Assuptions 23 and 24 of [3]. Also, each I_m is elliptic with respect to $\mathcal{L}$. Hence using Theorem 42 in [3] there is a subsequence of I_m converging weakly (with respect to the weight ω_{σ_0}) to a translation invariant operator I, also elliptic with respect to $\mathcal{L}$. To see that I is in fact elliptic with respect to $\mathcal{L}_0(\sigma) \subset \mathcal{L}$ we just observe that for test functions u and v that are quadratic polynomials in a neighborhood of x and that belong to $L^1(\mathbb{R}^n, \omega_{\sigma_0})$, the inequalities

$$M_{\sigma_m}^- v(x) \leq \mathrm{I}_m(u + v)(x) - \mathrm{I}_m u(x) \leq M_{\sigma_m}^+ v(x)$$

pass to the limit to obtain

$$M_\sigma^- v(x) \leq \mathrm{I}(u + v)(x) - \mathrm{I}u(x) \leq M_\sigma^+ v(x).$$

$\qquad\square$

The following result from [6] is a parabolic version of Lemma 5 in [3]. It is the basic stability result which is needed in compactness arguments.

Lemma 4.2 (Reduced version of [6, Theorem 5.3]). *Let $R > 0$, $T > 0$, and I_m be a sequence of translation invariant elliptic operators. Let $u_m \in C\left(\overline{B_R} \times [-T, 0]\right)$ be viscosity solutions of*

$$\partial_t u_m - I_m u_m = f_m \quad in \ B_R \times (-T, 0].$$

Assume that

$$I_m \to I \quad weakly \ with \ respect \ to \ \omega_{\sigma_0},$$

$$u_m(x, t) \to u(x, t) \quad uniformly \ in \ \overline{B_R} \times [-T, 0],$$

$$\sup_{t \in [-T, 0]} \int_{\mathbb{R}^n} |u_m - u|(x, t) \omega_{\sigma_0}(x) \, dx \longrightarrow 0,$$

and

$$f_m \to 0 \quad uniformly \ in \ \overline{B_R} \times [-T, 0].$$

Then, u is a viscosity solution of $\partial_t u = Iu$ in $B_R \times (-T, 0]$.

The following useful lemma states that if in a sequence of nested sets a function u is close enough to its "least squares" fitting plane, then u is C^β with $\beta \in (1, 2)$.

Lemma 4.3. *Let $\sigma \in (1, 2]$, $\beta \in (1, \sigma)$, and let u be a continuous function belonging to $L^\infty(Q_\infty)$, where $Q_\infty = \mathbb{R}^n \times (-\infty, 0]$. For $z = (z', z_{n+1}) \in \mathbb{R}^n \times (-\infty, 0]$ and $r > 0$, define the constant in t affine function*

$$\ell_{r,z}(x, t) := a^* \cdot (x - z') + b^*, \tag{4.1}$$

where

$$a_i^* = a_i^*(r, z) = \frac{\int_{Q_r^\sigma(z)} u(x, t)(x_i - z_i) \, dx \, dt}{\int_{Q_r^\sigma(z)} (x_i - z_i)^2 \, dx \, dt}, \quad 1 \le i \le n, \tag{4.2}$$

and

$$b^* = b^*(r, z) = \fint_{Q_r^\sigma(z)} u(x, t) \, dx \, dt, \tag{4.3}$$

where $Q_\sigma^r(z)$ was defined in (3.1) Equivalently,

$$(a^*, b^*) = \arg\min \int_{Q_r^\sigma(z)} \left(u(x, t) - a \cdot (x - z') + b\right)^2 dx \, dt.$$

If for some constant C_0 we have

$$\sup_{r > 0} \sup_{z \in Q_{1/2}} r^{-\beta} \left\| u - \ell_{r,z} \right\|_{L^\infty(Q_r^\sigma(z))} \le C_0, \tag{4.4}$$

where $Q_{1/2} = B_{1/2} \times (-1/2, 0]$, then

$$\sup_{t \in [-1/2, 0]} \left\| u(\cdot, t) \right\|_{C^\beta(B_{1/2})} + \sup_{x \in B_{1/2}} \left\| u(x, \cdot) \right\|_{C^{\beta/\sigma}([-1/2, 0])} \le C\left(\|u\|_{L^\infty(Q_\infty)} + C_0\right), \tag{4.5}$$

where C depends only on β.

Proof. We may assume $\|u\|_{L^\infty(Q_\infty)} = 1$. Recall the definition of Q_r^σ in (3.1). Note that (4.4) implies that for all $z \in \overline{Q_{1/2}}$, $r > 0$, and $\bar{z} \in Q_r^\sigma(z)$ we have

$$\left|\ell_{2r,z}(\bar{z}) - \ell_{r,z}(\bar{z})\right| \le \left|u(\bar{z}) - \ell_{2r,z}(\bar{z})\right| + \left|u(\bar{z}) - \ell_{r,z}(\bar{z})\right|$$
$$\le CC_0 r^\beta.$$

But this happening for every $\bar{z} \in Q_r^\sigma(z)$ means

$$\left|a^*(2r, z) - a^*(r, z)\right| \le CC_0 r^{\beta-1}$$

and

$$\left|b^*(2r, z) - b^*(r, z)\right| \le CC_0 r^\beta.$$

In addition since $\|u\|_{L^\infty(Q_\infty)} = 1$ we clearly have that

$$|a^*(1, z)| \le C \quad \text{and} \quad |b^*(r, z)| \le 1 \quad \text{for all } r > 0. \tag{4.6}$$

Since $\beta > 1$ this implies the existence of the limits

$$a(z) := \lim_{r\searrow 0} a^*(r, z) \quad \text{and} \quad b(z) := \lim_{r\searrow 0} b^*(r, z).$$

Moreover,

$$\left|a(z) - a^*(r, z)\right| \le \sum_{m=0}^\infty \left|a^*(2^{-m}r, z) - a^*(2^{-m-1}r, z)\right|$$
$$\le \sum_{m=0}^\infty CC_0 2^{-(\beta-1)m} r^{\beta-1} \le C(\beta) C_0 r^{\beta-1}.$$

And similarly

$$\left|b(z) - b^*(r, z)\right| \le C(\beta) C_0 r^\beta.$$

Using (4.6) we obtain

$$|a(z)| \le C(\beta)(C_0 + 1) \quad \text{and} \quad |b(z)| \le 1. \tag{4.7}$$

We have thus proven that for all $z \in \overline{Q_{1/2}}$ there are $a(z) \in \mathbb{R}^n$ and $b(z) \in \mathbb{R}$ satisfying the bounds (4.7) such that for all $r > 0$

$$\left\|u - a(z) \cdot x - b(z)\right\|_{L^\infty(Q_r^\sigma(z))} \le C(\beta) C_0 r^\beta$$

This implies that u is differentiable in the x directions, that $a(z) = D_x u(z)$ and $b(z) = u(z)$, and that (4.5) holds. $\qquad\square$

The following standard lemma will be used to show that rescaled functions in the blow up argument also satisfy elliptic equations with the same ellipticity constants.

Lemma 4.4. *Let $\sigma > 1$ and* I *be a translation invariant operator with respect to* $\mathcal{L}_0(\sigma)$ *with* I0 = 0. *Given $x_0 \in \mathbb{R}^n$, $r > 0$, $c > 0$, and $\ell(x) = a \cdot x + b$, define* $\tilde{\text{I}}$ *by*

$$\tilde{\text{I}}\left(\frac{w(x_0 + r\,\cdot) - \ell(x_0 + r\,\cdot)}{c}\right) = \frac{r^\sigma}{c}(\text{I}w)(x_0 + r\,\cdot).$$

Then $\tilde{I}$ is also translation invariant and elliptic with respect to $\mathcal{L}_0(\sigma)$ (with the same ellipticity constants) with $\tilde{I}0 = 0$.

Proof. We have

$$\tilde{I}u(x) = \frac{r^\sigma}{c}I\left(cu\left(\frac{\cdot - x_0}{r}\right) + \ell(\cdot)\right)(x_0 + rx)$$

$$= \frac{r^\sigma}{c}I\left(cu\left(\frac{\cdot - x_0}{r}\right)\right)(x_0 + rx),$$

where we have used $M_\sigma^+ \ell = M_\sigma^- \ell = 0$.

We clearly see from the second expression that $\tilde{I}$ is translation invariant.

Also,

$$\tilde{I}0 = \frac{r^\sigma}{c}I0 = 0.$$

Moreover,

$$\{\tilde{I}u - \tilde{I}v\}(x) = \frac{r^\sigma}{c}\left\{I\left(cu\left(\frac{\cdot - x_0}{r}\right) + \ell\right) - I\left(cv\left(\frac{\cdot - x_0}{r}\right) + \ell\right)\right\}(x_0 + rx)$$

$$\leq \frac{r^\sigma}{c}M_\sigma^+\left(cu\left(\frac{\cdot - x_0}{r}\right) - cv\left(\frac{\cdot - x_0}{r}\right)\right)(x_0 + rx)$$

$$= M_\sigma^+(u - v)(x),$$

since I is elliptic with respect to $\mathcal{L}_0(\sigma)$ and M_σ^+ is translation invariant, positively homogeneous of degree one, and scale invariant of order σ. Similarly,

$$M_\sigma^-(u - v)(x) \leq \{\tilde{I}u - \tilde{I}v\}(x).$$

$\square$

The following proposition immediately implies Theorem 2.2. However the statement of the proposition is better suited for a proof by contradiction.

Proposition 4.5. *Let $\sigma_0 \in (0, 2)$ and $\sigma \in [\sigma_0, 2]$. Let $u \in L^\infty\left(\mathbb{R}^n \times (-1, 0)\right)$ be a viscosity solution of $u_t - Iu = f$ in $B_1 \times (-1, 0]$, where I is a translation invariant elliptic operator with respect to the class $\mathcal{L}_0(\sigma)$. Let $\alpha = \alpha(\sigma_0)$ be given by Theorem 2.1.*

Then, for all $\beta < \min\{\sigma_0, 1 + \alpha\}$, $u(\cdot, t)$ belongs to $C^\beta\left(\overline{B_{1/2}}\right)$ for all $t \in [-1/2, 0]$, and $u(x, \cdot)$ belongs to $C^{\beta/\sigma}([-1/2, 0])$ for all $x \in B_{1/2}$. Moreover, the following estimate holds

$$\sup_{t \in [-1/2, 0]}\|u(\cdot, t)\|_{C^\beta(B_{1/2})} + \sup_{x \in B_{1/2}}\|u(x, \cdot)\|_{C^{\beta/\sigma'}([-1/2, 0])} \leq CC_0$$

where

$$C_0 = \|u\|_{L^\infty(\mathbb{R}^n \times (-1, 0))} + \|f\|_{L^\infty(B_1 \times (-1, 0))}$$

and C depends only on σ_0, β, ellipticity constants, and dimension.

Proof. For $r \in (0, +\infty]$, we denote

$$Q_r = B_r \times (-r, 0].$$

Note that we may consider u to be defined in the whole Q_∞ and not only in $\mathbb{R}^n \times (-1, 0]$ by extending u by zero. This is only by notational convenience and there is no difference in doing it since the equation is local in time and its domain will still be Q_1.

The proof is by contradiction. Suppose that the statement is false, i.e., there are sequences of functions u_k, f_k, operators I_k, and orders $\sigma_k \in [\sigma_0, 2]$ such that

- $\partial_t u_k - I_k u_k = f_k$ in $B_1 \times (-1, 0]$
- I_k is translation invariant and elliptic with respect to $\mathcal{L}_0(\sigma_k)$
- $\|u_k\|_{L^\infty(Q_\infty)} + \|f_k\|_{L^\infty(Q_1)} = 1$ (by scaling to make $C_0 = 1$)

but

$$\sup_{t \in [-1/2, 0]} \left\| u_k(\cdot, t) \right\|_{C^\beta(B_{1/2})} + \sup_{x \in B_{1/2}} \left\| u_k(x, \cdot) \right\|_{C^{\beta/\sigma_k}([-1/2, 0])} \nearrow \infty. \qquad (4.8)$$

We split the proof in two cases: $\sigma_0 \leq 1$ and $\sigma_0 > 1$.

Case $\sigma_0 \leq 1$. Since in this case we have $\beta < 1$, (4.8) is equivalent to

$$\sup_k \sup_{r > 0} \sup_{z \in Q_{1/2}} r^{-\beta} \left\| u_k - u_k(z) \right\|_{L^\infty(Q_r^{\sigma_k}(z))} = \infty. \qquad (4.9)$$

Define the quantity

$$\Theta(r) := \sup_k \sup_{r' \geq r} \sup_{z \in Q_{1/2}} (r')^{-\beta} \left\| u_k - u_k(z) \right\|_{L^\infty\left(Q_{r'}^{\sigma_k}(z)\right)},$$

which is monotone in r. Note that we have $\Theta(r) < \infty$ for $r > 0$ and $\Theta(r) \nearrow \infty$ as $r \searrow 0$. Clearly, there are sequences $r_m \searrow 0$, and k_m, and $z_m \in \overline{Q_{1/2}}$ for which

$$(r_m)^{-\beta} \left\| u_{k_m} - u_{k_m}(z_m) \right\|_{L^\infty(Q_{r_m}^{\sigma_{k_m}}(z_m))} \geq \Theta(r_m)/2. \qquad (4.10)$$

In this situation, let us denote $z_m = (x_m, t_m)$, $\sigma_m = \sigma_{k_m}$, and consider the blow up sequence

$$v_m(x, t) = \frac{u_{k_m}(x_m + r_m x, t_m + (r_m)^{\sigma_m} t) - u_{k_m}(z_m)}{(r_m)^\beta \Theta(r_m)}.$$

Note that we will have, for all $m \geq 1$,

$$v_m(0) = 0 \quad \text{and} \quad \|v_m\|_{L^\infty(Q_1)} \geq 1/2. \qquad (4.11)$$

The last inequality is a consequence of (4.10)

For all $R \geq 1$, v_m satisfies the growth control

$$
\begin{aligned}
\|v_m\|_{L^\infty(Q_R^{\sigma_m})} &= \frac{1}{(r_m)^\beta \Theta(r_m)} \big\|u_{k_m} - u_{k_m}(z_m)\big\|_{L^\infty\left(Q_{r_m R}^{\sigma_m}(z_m)\right)} \\
&= \frac{R^\beta}{\Theta(r_m)(r_m R)^\beta} \big\|u_{k_m} - u_{k_m}(z_m)\big\|_{L^\infty\left(Q_{r_m R}^{\sigma_m}(z_m)\right)} \\
&\leq \frac{R^\beta \Theta(r_m R)}{\Theta(r_m)} \\
&\leq R^\beta,
\end{aligned}
\tag{4.12}
$$

where we have used the definition of $\Theta(r)$ and its montonicity.

For all fixed $\rho \leq (1 - 2^{-\sigma_0})/r_m \nearrow \infty$, then v_m solves

$$
\big(\partial_t v_m - \tilde{I}_m v_m\big)(x,t) = \frac{(r_m)^{\sigma_m}}{(r_m)^\beta \Theta(r_m)} f(x_m + r \cdot, t_m + r^{\sigma_m} t) \quad \text{in } B_\rho \times (-\rho^{\sigma_0}, 0], \tag{4.13}
$$

where $\tilde{I}_m$ is the operator I_{k_m} appropriately rescaled. More precisely, given an elliptic test function $w : \mathbb{R}^n \to \mathbb{R}$ it is

$$
\tilde{I}_m \left(\frac{w(x_m + r \cdot) - u_{k_m}(z_m)}{(r_m)^\beta \Theta(r_m)} \right) = \frac{(r_m)^{\sigma_m}}{(r_m)^\beta \Theta(r_m)} \big(I_{k_m} w\big)(\cdot).
$$

By the proof of Lemma 4.4, $\tilde{I}_m$ is elliptic with respect to $\mathcal{L}_0(\sigma_m)$ with the same ellipticity constants.

Note that since $\beta < \sigma_0 \leq \sigma_m$ and $\Theta(r_m) \nearrow \infty$, the right hand sides of (4.13) converge uniformly to 0, and in particular they are uniformly bounded. Then, using the C^α estimate in Theorem 2.1 (rescaled) in every cylinder $B_\rho \times (-\rho^{\sigma_0}, 0]$, $\rho > 1$, we obtain a subsequence v_m converging locally uniformly in all of $\mathbb{R}^n \times (-\infty, 0]$ to some function v. Note that, although these C^α estimates for v_m clearly depend on ρ, the important fact is that they are independent of m. This is enough to obtain local uniform convergence by the Arzelà-Ascoli Theorem and the typical diagonal argument. Moreover, since all the v_m's satisfy the growth control

$$
\|v_m\|_{L^\infty(Q_R^{\sigma_0})} \leq \|v_m\|_{L^\infty(Q_R^{\sigma_m})} \leq R^\beta
$$

and $\beta < \sigma_0$, by dominated convergence we obtain that

$$
\sup_{t \in [-\rho^{\sigma_0}, 0]} \int \big|v_m - v\big|(x,t) \omega_{\sigma_0}(x) \, dx \to 0.
$$

In addition, by Lemma 4.1 there is a subsequence of $\tilde{I}_m$ which converges weakly to some operator $\tilde{I}$, translation invariant and elliptic with respect to $\mathcal{L}_0(\sigma)$ for some $\sigma \in [\sigma_0, 2]$ in every ball B_R. Hence, it follows from Lemma 4.2 that v satisfies

$$
v_t - \tilde{I}v = 0 \quad \text{in all of } \mathbb{R}^n \times (-\infty, 0].
$$

On the other hand, by local uniform convergence, passing to the limit the growth controls (4.18) for each v_m we obtain that $\|v\|_{L^\infty(Q_R^\sigma)} \leq R^\beta$. Hence, by Theorem

3.1, v must be constant. But passing (4.11) to the limit we obtain $v(0) = 0$ and $\|v\|_{L^\infty(Q_1)} \geq 1/2$ and hence v is not constant; a contradiction.

Case $\sigma_0 > 1$. In this case it is enough to consider $1 < \beta < \min\{\sigma_0, 1 + \alpha\}$. By Lemma 4.3, (4.8) implies

$$\sup_k \sup_{r>0} \sup_{z \in Q_{1/2}} r^{-\beta} \|u_k - \ell_{k,r,z}\|_{L^\infty(Q_r^{\sigma_k}(z))} = \infty, \tag{4.14}$$

where, as in Lemma 4.3, $\ell_{k,r,z}$ is the affine function of the variables x only which best fits u_k in $Q_r^{\sigma_k}(z)$ by least squares. Namely,

$$\ell_{k,r,z}(x) = a^*(k,r,z) \cdot (x - z') + b^*(k,r,z)$$

for

$$\big(a^*(k,r,z), b^*(k,r,z)\big) = \arg\min_{(a,b) \in \mathbb{R}^n \times \mathbb{R}} \int_{Q_r^{\sigma_k}(z)} \big(u_k(x,t) - a \cdot (x - z') + b\big)^2 \, dx \, dt,$$

where z' denotes the first n components of z.

Now we define the quantity

$$\Theta(r) := \sup_k \sup_{r' \geq r} \sup_{z \in Q_{1/2}} (r')^{-\beta} \|u_k - \ell_{k,r',z}\|_{L^\infty\big(Q_{r'}^{\sigma_k}(z)\big)},$$

which is monotone in r. Notice that we have $\Theta(r) < \infty$ for $r > 0$ and $\Theta(r) \nearrow \infty$ as $r \searrow 0$. Again, there are sequences $r_m \searrow 0$, and k_m, and $z_m \in \overline{Q_{1/2}}$ for which one (or more) of the following three possibilities happen

$$(r_m)^{-\beta} \|u_{k_m} - \ell_{k_m,r_m,z_m}\|_{L^\infty(Q_{r_m}^{\sigma_{k_m}}((z_m))} \geq \Theta(r_m)/2 \tag{4.15}$$

We then denote $z_m = (x_m, t_m)$, $\sigma_m = \sigma_{k_m}$, $\ell_m = \ell_{k_m,r_m,z_m}$, and consider the blow up sequence

$$v_m(x,t) = \left(\frac{u_{k_m} - \ell_m}{(r_m)^\beta \Theta(r_m)}\right)(x_m + r_m x, t_m + (r_m)^{\sigma_m} t).$$

Note that we will have, for all $m \geq 1$,

$$\int_{Q_1} v_m \, dx \, dt = 0, \quad \int_{Q_1} v_m x_i \, dx \, dt = 0, \ 1 \leq i \leq n, \tag{4.16}$$

which are the optimality conditions of least squares.

Translating (4.15) to v_m we obtain that

$$\|v_m\|_{L^\infty(Q_1)} \geq 1/2 \tag{4.17}$$

Next we prove the growth control

$$\|v_m\|_{L^\infty(Q_R^{\sigma_m})} \leq CR^\beta, \quad \text{for all } R \geq 1.$$

Indeed, for all k, $z \in Q_{1/2}$ and $r' \geq r$ we have, by definition of $\Theta(r)$, By definition of Θ, for all $z \in \overline{Q_{1/2}}$, $r > 0$, and $\bar{z} \in Q_{r'}^{\sigma_k}(z)$ we have

$$\left| \ell_{k,2r',z}(\bar{z}) - \ell_{k,r',z}(\bar{z}) \right| \leq \left| u(\bar{z}) - \ell_{k,2r',z}(\bar{z}) \right| + \left| u(\bar{z}) - \ell_{r,z}(\bar{z}) \right|$$
$$\leq C\Theta(r)(r')^{\beta}.$$

This happening for all $\bar{z} \in Q_{r'}^{\sigma_k}(z)$ implies

$$\frac{r'\left| a^*(k,2r',z) - a^*(k,r',z) \right|}{(r')^{\beta}\Theta(r)} \leq C \quad \text{and} \quad \frac{\left| b^*(k,2r',z) - b^*(k,r',z) \right|}{(r')^{\beta}\Theta(r)} \leq C.$$

And thus, setting $R = 2^N$, where $N \geq 1$ is an integer, we have

$$\frac{r\left| a^*(k,Rr,z) - a^*(k,r,z) \right|}{r^{\beta}\Theta(r)} \leq C \sum_{j=0}^{N-1} 2^{j(\beta-1)} \frac{2^j r\left| a^*(k,2^{j+1}r,z) - a^*(k,2^j r,z) \right|}{(2^j r)^{\beta}\Theta(r)}$$
$$\leq C2^{(\beta-1)N} = CR^{\beta-1}.$$

Similarly,

$$\frac{\left| b^*(k,Rr,z) - b^*(k,r,z) \right|}{r^{\beta}\Theta(r)} \leq CR^{\beta}.$$

Therefore, for all $R \geq 1$

$$\|v_m\|_{L^\infty(Q_R^{\sigma_m})} = \frac{1}{(r_m)^{\beta}\Theta(r_m)} \left\| u_{k_m} - \ell_{k_m,r_m,z_m} \right\|_{L^\infty\left(Q_{Rr_m}^{\sigma_m}(z_m)\right)}$$
$$\leq \frac{1}{(r_m)^{\beta}\Theta(r_m)} \left\| u_{k_m} - \ell_{k_m,Rr_m,z_m} \right\|_{L^\infty\left(Q_{Rr_m}^{\sigma_m}(z_m)\right)} +$$
$$+ \frac{1}{(r_m)^{\beta}\Theta(r_m)} \left\| \ell_{k_m,Rr_m,z_m} - \ell_{k_m,r_m,z_m} \right\|_{L^\infty\left(Q_{Rr_m}^{\sigma_m}(z_m)\right)} \quad (4.18)$$
$$\leq \frac{R^{\beta}\Theta(Rr_m)}{\Theta(r_m)} + CR^{\beta}$$
$$\leq CR^{\beta},$$

Next, for all fixed $\rho \leq (1 - 2^{-\sigma_0})/r_m \nearrow \infty$, v_m solves

$$\left(\partial_t v_m - \tilde{\mathrm{I}}_m v_m\right)(x,t) = \frac{(r_m)^{\sigma_m}}{(r_m)^{\beta}\Theta(r_m)} f(x_m + r\cdot, t_m + (r_m)^{\sigma_m}t) \quad \text{in } B_\rho \times (-\rho^{\sigma_0}, 0],$$
$$(4.19)$$

where $\tilde{\mathrm{I}}_m$ is defined by

$$\tilde{\mathrm{I}}_m\left(\frac{w(x_m + r_m\cdot) - \ell_m(x_m + r_m\cdot)}{(r_m)^{\beta}\Theta(r_m)}\right) = \frac{(r_m)^{\sigma_m}}{(r_m)^{\beta}\Theta(r_m)}\left(\mathrm{I}_{k_m}w\right)(\cdot).$$

By Lemma 4.4 $\tilde{\mathrm{I}}_m$ is elliptic with respect to $\mathcal{L}_0(\sigma_m)$,.

As a consequence, repeating the reasoning in the first part of the proof, a subsequence of v_m converges locally uniformly in $\mathbb{R}^n \times (-\infty, 0]$ to a function v which satisfies $v_t = \tilde{\mathrm{I}}v$ for some $\tilde{\mathrm{I}}$ in translation invariant and elliptic with respect to $\mathcal{L}_0(\sigma)$

with $\sigma \in [\sigma_0, 2]$. Hence, v satisfies the limit growth control of the v_m's, and thus by Theorem 3.1, $v = a \cdot x + b$. But passing (4.16) and (4.17) to the limit we reach a contradiction. $\qquad\square$

We finally give the

Proof of Theorem 2.2. Let $\delta = \epsilon/4$ and divide $[\sigma_0, 2]$ into $N = \lceil (2 - \sigma_0)/\delta \rceil$ intervals $[\sigma_j, \sigma_{j+1}]$, $j = 0, 1, 2, \ldots, N$, where $\sigma_N = 2$ and $0 \le \sigma_{j+1} - \sigma_j \le \delta$. For each of the intervals $[\sigma_j, \sigma_{j+1}]$ we use Proposition 4.5, with σ_0 replaced by σ_j. We obtain that the estimate of the Proposition holds for $\beta = \min\{\sigma_j, 1 + \alpha\} - \delta$ with a constant C_j that depends only δ, σ_j, ellipticity constants, and dimension. In particular, given $\sigma \in [\sigma_0, 2]$ the estimate of the Theorem holds for all $\beta \le \min\{\sigma, 1 + \alpha \ - 2\delta\}$ with constant $C = \max C_j$. $\qquad\square$

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Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain

E. Boundary regularity for fully nonlinear integro-differential equations

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BOUNDARY REGULARITY FOR FULLY NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

XAVIER ROS-OTON AND JOAQUIM SERRA

ABSTRACT. We study fine boundary regularity properties of solutions to fully nonlinear elliptic integro-differential equations of order $2s$, with $s \in (0, 1)$.

We consider the class of nonlocal operators $\mathcal{L}_* \subset \mathcal{L}_0$, which consists of all the infinitesimal generators of stable Lévy processes belonging to the class $\mathcal{L}_0$ of Caffarelli-Silvestre. For fully nonlinear operators I elliptic with respect to $\mathcal{L}_*$, we prove that solutions to $\mathrm{I}u = f$ in Ω, $u = 0$ in $\mathbb{R}^n \setminus \Omega$, satisfy $u/d^s \in C^{s-\epsilon}(\overline{\Omega})$ for all $\epsilon > 0$, where d is the distance to $\partial\Omega$ and $f \in L^\infty$.

We expect the Hölder exponent $s - \epsilon$ to be optimal (or almost optimal) for general right hand sides $f \in L^\infty$. Moreover, we also expect the class $\mathcal{L}_*$ to be the largest scale invariant subclass of $\mathcal{L}_0$ for which this result is true. In this direction, we show that the class $\mathcal{L}_0$ is too large for all solutions to behave like d^s.

The constants in all the estimates in this paper remain bounded as the order of the equation approaches 2.

CONTENTS

173

1. Introduction and results

This paper is concerned with boundary regularity for fully nonlinear elliptic integro-differential equations.

Since the foundational paper of Caffarelli and Silvestre [14], ellipticity for a nonlinear integro-differential operator is defined relatively to a given set $\mathcal{L}$ of linear translation invariant elliptic operators. This set $\mathcal{L}$ is called the ellipticity class.

The reference ellipticity class from [14] is the class $\mathcal{L}_0 = \mathcal{L}_0(s)$, containing all operators L of the form

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) K(y)\, dy \tag{1.1}$$

with even kernels $K(y)$ bounded between two positive multiples of $(1-s)|y|^{-n-2s}$, which is the kernel of the fractional Laplacian $(-\Delta)^s$.

In the three papers [14, 15, 16], Caffarelli and Silvestre studied the interior regularity for solutions u to

$$\begin{cases} Iu &= f \quad \text{in } \Omega \\ u &= g \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.2}$$

being I a translation invariant fully nonlinear integro-differential operator of order $2s$ (see the definition later on in this Introduction). They proved existence of viscosity solutions, established $C^{1+\alpha}$ interior regularity of solutions [14], $C^{2s+\alpha}$ regularity in case of convex equations [16], and developed a perturbative theory for non translation invariant equations [15]. Thus, the interior regularity for these equations is well understood.

However, very little is known about the boundary regularity for fully nonlinear problems of fractional order.

When I is the fractional Laplacian $-(-\Delta)^s$, the boundary regularity of solutions u to (1.2) is now well understood. The first result in this direction was obtained by Bogdan, who established the boundary Harnack principle for s-harmonic functions [5] —i.e., for solutions to $(-\Delta)^s u = 0$. More recently, we proved in [45] that if $f \in L^\infty$, $g \equiv 0$, and Ω is $C^{1,1}$ then $u \in C^s(\mathbb{R}^n)$ and $u/d^s \in C^\alpha(\overline{\Omega})$ for some small $\alpha > 0$, where d is the distance to the boundary $\partial\Omega$. Moreover, the limit of $u(x)/d^s(x)$ as $x \to \partial\Omega$ is typically nonzero (in fact it is positive if $f < 0$), and thus the C^s regularity of u is optimal. After this, Grubb [23] showed that when $f \in C^\beta$ with $\beta > 0$ (resp. $f \in L^\infty$), $g \equiv 0$, and Ω is smooth, then $u/d^s \in C^{\beta+s-\epsilon}(\overline{\Omega})$ (resp. $u/d^s \in C^{s-\epsilon}(\overline{\Omega})$) for all $\epsilon > 0$. In particular, $f \in C^\infty$ leads to $u/d^s \in C^\infty(\overline{\Omega})$. Thus, the correct notion of boundary regularity for equations of order $2s$ is the Hölder regularity of the quotient u/d^s.

The results of Grubb [23] apply not only to the fractional Laplacian, but to all linear pseudo-differential operators of order $2s$ satisfying the so called μ-transmission property. As explained later on in this Introduction, these results apply in particular to linear equations with operators of the form (1.3)-(1.4) whenever $a \in C^\infty(S^{n-1})$.

Here, we obtain boundary regularity for fully nonlinear integro-differential problems of the form (1.2) which are elliptic with respect to the class $\mathcal{L}_* \subset \mathcal{L}_0$ defined as follows. $\mathcal{L}_*$ consists of all linear operators of the form

$$Lu(x) = (1-s) \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) \frac{a(y/|y|)}{|y|^{n+2s}} \, dy, \qquad (1.3)$$

with

$$a \in L^\infty(S^{n-1}) \quad \text{satisfying} \quad a(\theta) = a(-\theta) \quad \text{and} \quad \lambda \leq a \leq \Lambda, \qquad (1.4)$$

where $0 < \lambda \leq \Lambda$ are called ellipticity constants. The class $\mathcal{L}_*$ consists of all infinitesimal generators of *stable* Lévy processes belonging to $\mathcal{L}_0$. Our main result establishes that when $f \in L^\infty$, $g \equiv 0$, and Ω is $C^{1,1}$, viscosity solutions u satisfy

$$u/d^s \in C^{s-\epsilon}(\overline{\Omega}) \text{ for all } \epsilon > 0. \qquad (1.5)$$

We also obtain boundary regularity for problem (1.2) with exterior data $g \in C^2$, and also for non translation invariant operators $\mathfrak{I}(u, x)$.

We believe the Hölder exponent $s - \epsilon$ in (1.5) to be optimal (or almost optimal) for merely bounded right hand sides f. Moreover, we expect the class $\mathcal{L}_*$ to be the largest scale invariant subclass of $\mathcal{L}_0$ for which this result is true.

For general elliptic equations with respect to $\mathcal{L}_0$, no fine boundary regularity results like (1.5) hold. In fact, the class $\mathcal{L}_0$ is too large for all solutions to be comparable to d^s near the boundary. Indeed, we show in Section 2 that there are powers $0 < \beta_1 < s < \beta_2$ for which the functions $(x_n)_+^{\beta_1}$ and $(x_n)_+^{\beta_2}$ satisfy

$$M_{\mathcal{L}_0}^+ (x_n)_+^{\beta_1} = 0 \quad \text{and} \quad M_{\mathcal{L}_0}^- (x_n)_+^{\beta_2} = 0 \quad \text{in } \{x_n > 0\},$$

where $M_{\mathcal{L}_0}^+$ and $M_{\mathcal{L}_0}^-$ are the extremal operators for the class $\mathcal{L}_0$; see their definition in Section 2. Hence, since $(-\Delta)^s(x_n)_+^s = 0$ in $\{x_n > 0\}$, we have at least three functions which solve fully nonlinear elliptic equations with respect to $\mathcal{L}_0$ but which are not even comparable near the boundary $\{x_n = 0\}$. As we show in Section 2, the same happens for the subclasses $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{L}_0$, which have more regular kernels and were considered in [14, 15, 16].

1.1. The class $\mathcal{L}_*$.

The class $\mathcal{L}_*$ consists of all infinitesimal generators of stable Lévy processes belonging to $\mathcal{L}_0$. This type of Lévy processes are well studied in probability, as explained next. In that context, the function $a \in L^\infty(S^{n-1})$ is called the spectral measure.

Stable processes are by several reasons a natural extension of Gaussian processes. For instance, the Generalized Central Limit Theorem states that the distribution of a sum of independent identically distributed random variables with heavy tails converges to a stable distribution; see [47], [33], or [3] for a precise statement of this result. Thus, stable processes are often used to model sums of many random independent perturbations with heavy-tailed distributions —i.e.,

when large outcomes are not unlikely. In particular, they arise frequently in financial mathematics, internet traffic statistics, or signal processing; see for instance [42, 34, 35, 37, 38, 39, 1, 29, 41, 26] and the books [36, 47].

Linear equations $Lu = f$ with L in the class $\mathcal{L}_*$ have been already studied, specially by Sztonyk and Bogdan; see for instance [55, 6, 43, 7, 8, 56]. Although there were some results on the boundedness of u/d^s, the Hölder regularity for the quotient u/d^s was not known. When the spectral measure a in (1.3)-(1.4) belongs to $C^\infty(S^{n-1})$, the regularity of u/d^s follows from the recent results of Grubb [23].

Notice that all second order linear uniformly elliptic operators are recovered as limits of operators in $\mathcal{L}_* = \mathcal{L}_*(s)$ as $s \to 1$. In particular, all second order fully nonlinear equations $F(D^2u, x) = f(x)$ are recovered as limits of the fully nonlinear integro-differential equations that we consider. Furthermore, when $s < 1$ the class of translation invariant linear operators $\mathcal{L}_*(s)$ is much richer than the one of second order uniformly elliptic operators. Indeed, while any operator in the latter class is determined by a positive definite $n \times n$ matrix, a function $a : S^{n-1} \to \mathbb{R}^+$ is needed to determine an operator in $\mathcal{L}_*(s)$.

A key feature of the class $\mathcal{L}_*$ for boundary regularity issues is that

$$L(x_n)_+^s = 0 \quad \text{in } \{x_n > 0\} \quad \text{for all } L \in \mathcal{L}_*.$$

This is essential first to construct barriers which are comparable to d^s, and later to prove finer boundary regularity.

1.2. **Equations with "bounded measurable coefficients".** The first result of in this paper, and on which all the other results rely, is Proposition 1.1 below.

Here, and throughout the article, we use the definition of viscosity solutions and inequalities of [14]. Moreover, for $r > 0$ we denote

$$B_r^+ = B_r \cap \{x_n > 0\} \quad \text{and} \quad B_r^- = B_r \cap \{x_n < 0\},$$

and the constants λ and Λ in (1.4) are called ellipticity constants.

The extremal operators associated to the class $\mathcal{L}_*$ are denoted by $M_{\mathcal{L}_*}^+$ and $M_{\mathcal{L}_*}^-$,

$$M_{\mathcal{L}_*}^+ u = \sup_{L \in \mathcal{L}_*} Lu \quad \text{and} \quad M_{\mathcal{L}_*}^- u = \inf_{L \in \mathcal{L}_*} Lu.$$

Note that, since $\mathcal{L}_* \subset \mathcal{L}_0$, then $M_{\mathcal{L}_0}^- \leq M_{\mathcal{L}_*}^- \leq M_{\mathcal{L}_*}^+ \leq M_{\mathcal{L}_0}^+$.

Proposition 1.1. *Let $s_0 \in (0, 1)$ and $s \in [s_0, 1)$. Assume that $u \in C(B_1) \cap L^\infty(\mathbb{R}^n)$ is a viscosity solution of*

$$\begin{cases} M_{\mathcal{L}_*}^+ u \geq -C_0 & in \ B_1^+ \\ M_{\mathcal{L}_*}^- u \leq C_0 & in \ B_1^+ \\ u = 0 & in \ B_1^-, \end{cases} \tag{1.6}$$

for some nonnegative constant C_0. Then, u/x_n^s is $C^\alpha(\overline{B_{1/2}^+})$ for some $\alpha > 0$, with the estimate

$$\|u/x_n^s\|_{C^\alpha(B_{1/2}^+)} \leq C \left(C_0 + \|u\|_{L^\infty(\mathbb{R}^n)} \right). \tag{1.7}$$

The constants α and C depend only on n, s_0, and the ellipticity constants.

It is important to remark that the constants in our estimate remain bounded as $s \to 1$. This means that from Proposition 1.1 we can recover the classical boundary Harnack inequality of Krylov [31].

The estimate of Proposition 1.1 is only a first step towards our results. It is obtained via a nonlocal version of the method of Krylov [31] for second order equations with bounded measurable coefficients; see also Section 9.2 in [10]. This method has been adapted to nonlocal equations by the authors in [45], where we proved estimate (1.7) for the fractional Laplacian $(-\Delta)^s$ in $C^{1,1}$ domains.

As explained before, our main result is the $C^{s-\epsilon}$ regularity of u/d^s in $C^{1,1}$ domains for solutions u to fully nonlinear integro-differential equations (see the next subsection). Thus, for solutions to the nonlinear equations we push the small Hölder exponent $\alpha > 0$ in (1.7) up to the exponent $s - \epsilon$ in (1.5). To achieve this, new ideas are needed, and the procedure that we develop differs substantially from that in second order equations. We use a new compactness method and the "boundary" Liouville-type Theorem 1.5, stated later on in the Introduction. This Liouville theorem relies on Proposition 1.1.

1.3. **Main result.** Before stating our main result, let us recall the definition and motivations of fully nonlinear integro-differential operators.

As defined in [14], a fully nonlinear operator I is said to be elliptic with respect to a subclass $\mathcal{L} \subseteq \mathcal{L}_0$ when

$$M_{\mathcal{L}}^-(u-v)(x) \leq \mathrm{I}u(x) - \mathrm{I}v(x) \leq M_{\mathcal{L}}^+(u-v)(x)$$

for all test functions u, v which are C^2 in a neighborhood of x and having finite integral against $\omega_s(x) = (1-s)(1+|x|^{-n-2s})$. Moreover, if

$$\mathrm{I}\left(u(x_0 + \cdot)\right)(x) = (\mathrm{I}u)(x_0 + x),$$

then we say that I is translation invariant.

Fully nonlinear elliptic integro-differential equations naturally arise in stochastic control and games. In typical examples, a single player or two players control some parameters (e.g. the volatilities of the assets in a portfolio) affecting the joint distribution of the random increments of n variables $X(t) \in \mathbb{R}^n$. The game ends when $X(t)$ exits for the first time a certain domain Ω (as when having automated orders to sell assets when their prices cross certain limits).

The *value* or *expected payoff* of these games $u(x)$ depends on the starting point $X(0) = x$ (initial prices of all assets in the portfolio). A remarkable fact is that value $u(x)$ solves an equation of the type $\mathrm{I}u = 0$, where

$$\mathrm{I}u(x) = \sup_{\alpha}\left(L_\alpha u + c_\alpha\right) \quad \text{or} \quad \mathrm{I}u(x) = \inf_{\beta}\sup_{\alpha}\left(L_{\alpha\beta} u + c_{\alpha\beta}\right). \tag{1.8}$$

The first equation, known as the Bellman equation, arises in control problems (a single player), while the second one, known as the Isaacs equations, arises in zero-sum games (two players). The linear operators L_α and $L_{\alpha\beta}$ are infinitesimal generators

of Lévy processes, standing for all the possible choices of the distribution of time increments of $X(t)$. The constants c_α and $c_{\alpha\beta}$ are costs associated to the choice of the operators L_α and $L_{\alpha\beta}$. More involved equations with zeroth order terms and right hand sides have also meanings in this context as interest rates or running costs. See [11, 52, 40, 20, 14], and references therein for more information on these equations.

When all L_α and $L_{\alpha\beta}$ belong to $\mathcal{L}_*$, then (1.8) are fully nonlinear translation invariant operators elliptic with respect to $\mathcal{L}_*$, as defined above.

A fractional Monge-Ampère operator has been recently introduced by Caffarelli-Charro [12]. It is a fully nonlinear integro-differential operator which, by the main result in [12], is elliptic with respect to $\mathcal{L}_*$ whenever the right hand side is uniformly positive.

The interior regularity for fully nonlinear integro-differential elliptic equations was mainly established by Caffarelli and Silvestre in the well-known paper [14]. More precisely, for some small $\alpha > 0$, they obtain $C^{1+\alpha}$ interior regularity for fully nonlinear elliptic equations with respect to the class $\mathcal{L}_1$ made of kernels in $\mathcal{L}_0$ which are C^1 away from the origin. For $s > \frac{1}{2}$, the same result in the class $\mathcal{L}_0$ has been recently proved by Kriventsov [30]. These estimates are uniform as the order of the equations approaches two, so they can be viewed as a natural extension of the interior regularity for fully nonlinear equations of second order. There were previous interior estimates by Bass and Levin [4] and by Silvestre [49] which are not uniform as the order of the equation approaches 2. An interesting aspect of [49] is that its proof is short and uses only elementary analysis tools, taking advantage of the nonlocal character of the equations. This is why same ideas have been used in other different contexts [18, 51].

For convex equations elliptic with respect to $\mathcal{L}_2$ (i.e., with kernels in $\mathcal{L}_0$ which are C^2 away from the origin), Caffarelli and Silvestre obtained $C^{2s+\alpha}$ interior regularity [16]. This is the nonlocal extension of the Evans-Krylov theorem. Other important references concerning interior regularity for nonlocal equations in nondivergence form are [44, 27, 19, 2, 25].

To give local boundary regularity results for $C^{1,1}$ domains it is useful the following:

Definition 1.2. We say that Γ is $C^{1,1}$ surface with radius $\rho_0 > 0$ splitting B_1 into Ω^+ and Ω^- if the following happens.

- The two disjoint domains Ω^+ and Ω^- partition B_1, i.e., $\overline{B_1} = \overline{\Omega^+} \cup \overline{\Omega^-}$.
- The boundary $\Gamma := \partial\Omega^+ \setminus \partial B_1 = \partial\Omega^- \setminus \partial B_1$ is $C^{1,1}$ surface with $0 \in \Gamma$.
- All points on $\Gamma \cap \overline{B_{3/4}}$ can be touched by two balls of radii ρ_0, one contained in Ω^+ and the other contained in Ω^-.

Our main result reads as follows.

Theorem 1.3. *Let Γ be a $C^{1,1}$ surface with radius ρ_0 splitting B_1 into Ω^+ and Ω^-; see Definition 1.2. Let $d(x) = \mathrm{dist}\,(x, \Gamma)$.*

Let $s_0 \in (0, 1)$ and $s \in [s_0, 1)$. Assume that I is a fully nonlinear and translation invariant operator, elliptic with respect to $\mathcal{L}_*(s)$, with $I0 = 0$. Let $f \in C(\overline{\Omega^+})$, and $u \in L^\infty(\mathbb{R}^n) \cap C(\overline{\Omega^+})$ be a viscosity solution of

$$\begin{cases} Iu = f & \text{in } \Omega^+ \\ u = 0 & \text{in } \Omega^-. \end{cases}$$

Then, u/d^s belongs to $C^{s-\epsilon}(\overline{\Omega^+ \cap B_{1/2}})$ for all $\epsilon > 0$ with the estimate

$$\left\| u/d^s \right\|_{C^{s-\epsilon}(\Omega^+ \cap B_{1/2})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega^+)} \big),$$

where the constant C depends only on ρ_0, s_0, ϵ, ellipticity constants, and dimension.

Remark 1.4. As in the case of the fractional Laplacian, under the hypotheses of Theorem 1.3 we have that $u \in C^s(\overline{\Omega^+ \cap B_{1/2}})$, with the estimate $\|u\|_{C^s(\Omega^+ \cap B_{1/2})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega^+)} \big)$. Indeed, one only needs to combine the interior estimates in [14, 30, 48] (stated in Theorem 2.6) with the supersolution in Lemma 3.3, exactly as we did in [45, Proposition 1.1] for $(-\Delta)^s$.

It is important to notice that our result is not only an a priori estimate for classical solutions but also applies to viscosity solutions. For local equations of second order $F(D^2u, Du, x) = f(x)$, the boundary regularity for viscosity solutions to fully nonlinear equations has been recently obtained by Silvestre-Sirakov [53]. The methods that we introduce here to prove Theorem 1.3 can be used also to give a new proof of the results for such second order fully nonlinear equations; see Section 8 for more details.

Besides its own interest, the boundary regularity of solutions to integro-differential equations plays an important role in different contexts. For example, it is needed in overdetermined problems arising in shape optimization [21, 22] and also in Pohozaev-type or integration by parts identities [46]. Moreover, boundary regularity issues appear naturally in free boundary problems [13, 50].

Theorem 1.3 is, to our knowledge, the first boundary regularity result for fully nonlinear integro-differential equations. It was only known that solutions u to these equations are C^α up to the boundary for some small $\alpha > 0$ (a result for u but not for the quotient u/d^s). For solutions u to elliptic equations with respect to $\mathcal{L}_*$, our result gives a quite accurate description of the boundary behavior. Namely, u/d^s is $C^{s-\epsilon}$ for all $\epsilon > 0$, where d is the distance to the boundary.

This result is close to being optimal, at least when the right-hand sides f are just bounded. Indeed, let us compare it with the best known boundary regularity results for the fractional Laplacian $(-\Delta)^s$, due to Gerd Grubb [23]. These results use powerful machinery from Hörmander's theory. One of the main results in [23] applies to solutions u of the linear problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } U \\ u = 0 & \text{in } \mathbb{R}^n \setminus U \end{cases} \tag{1.9}$$

in a C^∞ domain U. It states that if f is C^β for some $\beta \in (0, +\infty]$ (resp. $f \in L^\infty$), then u/d^s is also $C^{\beta+s-\epsilon}$ (resp. $C^{s-\epsilon}$) up to the boundary for all $\epsilon > 0$. These estimates in Hölder spaces are actually particular cases of sharp estimates in Hörmander's μ-spaces. These remarkable results are a major improvement of the previously available results by the authors [45]. The results in [23] apply to all pseudo-differential operators satisfying the so called μ-transmission property. In case of linear operators of the form (1.3)-(1.4), the μ-transmission property is satisfied when $a \in C^\infty(S^{n-1})$; see also [24]. Of course, these techniques are only available for linear operators, while our results are for fully nonlinear equations. We find thus interesting to have reached, when f is just L^∞, the same boundary regularity as in [23].

In a future work we plan to use the methods of the present paper to obtain higher order Hölder regularity of u/d^s for solutions to linear equations with $f \in C^k$ in C^{k+2} domains.

1.4. A Liouville theorem and other ingredients of the proof.
Theorem 1.3 follows by combining an estimate *on* the boundary, (1.10) below, with the known interior regularity estimates in [14, 30]. The estimate on the boundary reads as follows. If u satisfies the hypotheses of Theorem 1.3, then for all $z \in \Gamma \cap \overline{B_{1/2}}$ there exists $Q(z) \in \mathbb{R}$ for which

$$\left| u(x) - Q(z)\big((x-z) \cdot \nu(z)\big)_+^s \right| \leq C|x-z|^{2s-\epsilon} \quad \text{for all } x \in B_1. \tag{1.10}$$

Here, $\nu(z)$ is the unit normal vector to Γ at z pointing towards Ω^+.

Our proof of (1.10) differs substantially from boundary regularity methods in second order equations. A main reason for this is not only the nonlocal character of the estimates, but also that tangential and normal derivatives of the solution behave differently on the boundary; recall that the solution is C^s but cannot be Lipschitz up to the boundary.

The estimate on the boundary (1.10) relies heavily on two ingredients, as explained next.

The first ingredient is the following Liouville-type theorem for solutions in a half space.

Theorem 1.5. *Let $u \in C(\mathbb{R}^n)$ be a viscosity solution of*

$$\begin{cases} \mathrm{I}u = 0 & in \ \{x_n > 0\} \\ u = 0 & in \ \{x_n < 0\}, \end{cases}$$

where I is a fully nonlinear and translation invariant operator, elliptic with respect to $\mathcal{L}_*$ and with $\mathrm{I}0 = 0$. Assume that for some positive $\beta < 2s$, u satisfies the growth control at infinity

$$\|u\|_{L^\infty(B_R)} \leq CR^\beta \quad for \ all \ R \geq 1. \tag{1.11}$$

Then,

$$u(x) = K(x_n)_+^s$$

for some constant $K \in \mathbb{R}$.

To prove Theorem 1.5, we apply Proposition 1.1 to incremental quotients of u in the first $(n-1)$-variables. After this, rescaling the obtained estimates and using (1.11), we find that such incremental quotients are zero, and thus that u is a 1D solution. Then, we use that for 1D functions all operators $L \in \mathcal{L}_*$ coincide up to a multiplicative constant with the fractional Laplacian $(-\Delta)^s$; see Lemma 2.1. Therefore, we only need to prove a Liouville theorem for solutions to $(-\Delta)^s w = 0$ in $\mathbb{R}_+$, $w = 0$ in $\mathbb{R}_-$ satisfying a growth control at infinity, which is done in Lemma 5.2.

The second ingredient towards (1.10) is the following compactness argument. With u as in Theorem 1.3, we suppose by contradiction that (1.10) does not hold, and we blow up the fully nonlinear equation at a boundary point (after subtracting appropriate terms to the solution). We then show that the blow up sequence converges to an entire solution in $\{x \cdot \nu > 0\}$ for some unit vector ν. For this, we need to develop a boundary version of a method introduced by the second author in [48]. The method was conceived there to prove interior regularity for integro-differential equations with rough kernels. Finally, the contradiction is reached by applying the Liouville-type theorem stated above to the entire solution in $\{x \cdot \nu > 0\}$.

These are the main ideas used to prove (1.10). A byproduct of this blow-up method is that the same proof yields results for non translation invariant equations; see Theorem 1.6 below.

Finally, Theorem 1.3 follows by combining (1.10) with the interior regularity estimates in [14, 30].

1.5. **Non translation invariant equations.** An interesting feature of the blow up and compactness argument used in this paper is that it allows to deal also with equations depending continuously on the x variable. For example, consider

$$\mathfrak{I}(u, x) = f(x) \quad \text{in } \Omega^+,$$

where $\mathfrak{I}$ is an operator of the form

$$\mathfrak{I}(u, x) = \inf_{\beta} \sup_{\alpha} \left(\int_{\mathbb{R}^n} \{ u(x+y) + u(x-y) - 2u(x) \} K_{\alpha\beta}(x, y) \, dy + c_{\alpha\beta}(x) \right).$$
$$(1.12)$$

The kernels $K_{\alpha\beta}$ are of the form

$$K_{\alpha\beta}(x, y) = (1-s) \frac{a_{\alpha\beta}(x, y/|y|)}{|y|^{n+2s}}, \tag{1.13}$$

and satisfy, for all α and β,

$$0 < \frac{\lambda}{|y|^{n+2s}} \leq K_{\alpha\beta}(x, y) \leq \frac{\Lambda}{|y|^{n+2s}} \quad \text{for all } x \in \Omega^+ \text{ and } y \in \mathbb{R}^n, \tag{1.14}$$

$$\inf_{\beta} \sup_{\alpha} c_{\alpha\beta}(x) = 0 \quad \text{for all } x \in \Omega^+, \qquad \|c_{\alpha\beta}\|_{L^\infty} \leq \Lambda \tag{1.15}$$

and

$$\left| a_{\alpha\beta}(x_1, \theta) - a_{\alpha\beta}(x_2, \theta) \right| \leq \mu\big(|x_1 - x_2| \big) \tag{1.16}$$

for all $x_1, x_2 \in \Omega^+$ and $\theta \in S^{n-1}$, where μ is some modulus of continuity.

As proved in [15], the operator $\mathfrak{I}$ defined above satisfies the ellipticity condition

$$M_{\mathcal{L}_*}^-(u-v)(x) \leq \mathfrak{I}(u,x) - \mathfrak{I}(v,x) \leq M_{\mathcal{L}_*}^+(u-v)(x).$$

The assumption (1.15) guarantees that $\mathfrak{I}(0,x) = 0$.

The following is our result for non translation invariant equations. In this result, we also consider a nonzero Dirichlet condition $g(x)$.

Theorem 1.6. *Let Γ be a $C^{1,1}$ hypersurface with radius $\rho_0 > 0$ splitting B_1 into Ω^+ and Ω^-; see Definition 1.2.*

Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. Assume that $\mathfrak{I}$ is an operator of the form (1.12)-(1.16). Let $f \in C(\overline{\Omega^+})$, $g \in C^2(B_1)$, and $u \in L^\infty(\mathbb{R}^n) \cap C(\overline{\Omega^+})$ be a viscosity solution of

$$\begin{cases} \mathfrak{I}(u,x) = f(x) & in \ \Omega^+ \\ \qquad u = g(x) & in \ \Omega^-. \end{cases}$$

Then, given $\epsilon > 0$, for all $z \in \Gamma \cap \overline{B_{1/2}}$ there exists $Q(z) \in \mathbb{R}$ with $|Q(z)| \leq CC_0$ for which

$$\left| u(x) - g(x) - Q(z)\big((x-z) \cdot \nu(z)\big)_+^s \right| \leq CC_0|x-z|^{2s-\epsilon} \quad for \ all \ x \in B_1,$$

where

$$C_0 = \|f\|_{L^\infty(\Omega^+)} + \|g\|_{C^2(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}$$

and $\nu(z)$ is the unit normal vector to Γ at z pointing towards Ω^+. The constant C depends only on n, ρ_0, s_0, ϵ, μ, and ellipticity constants.

In case $g \equiv 0$, the proof of Theorem 1.6 is almost the same as that of Theorem 1.3. On the other hand, the full Theorem 1.6 follows from the case $g \equiv 0$ by applying it to the function $\tilde{u} = u - g$.

In Theorem 1.6, the C^2 norm of g may be replaced by the $C^{2s+\epsilon}$ norm for any $\epsilon > 0$. This easily follows from the proof of the result.

Remark 1.7. When the kernels $K_{\alpha\beta}$ belong to $\mathcal{L}_1$, interior regularity estimates for the operators $\mathfrak{I}$ are proved in [15]. For operators $\mathfrak{I}$ elliptic with respect to $\mathcal{L}_0$, these interior estimates can be proved by using the methods of the second author [48]. Once proved these interior estimates, it follows from Theorem 1.6 that $(u-g)/d^s \in C^{s-\epsilon}(\overline{\Omega^+ \cap B_{1/2}})$, as in Theorem 1.3.

The paper is organized as follows. In Section 2 we give some important results on $\mathcal{L}_*$ and $\mathcal{L}_0$. In Section 3 we construct some sub and supersolutions that will be used later. In Section 4 we prove Proposition 1.1. In Section 5 we show Theorem 1.5. Then, in Section 6 we prove our main result, Theorem 1.3. Finally, in Section 7 we prove results for non-translation-invariant equations.

2. Properties of $\mathcal{L}_*$ and $\mathcal{L}_0$

This section has two main purposes: to show that the class $\mathcal{L}_* \subset \mathcal{L}_0$ is the appropriate one to obtain fine boundary regularity results, and to give some important results on $\mathcal{L}_*$ and $\mathcal{L}_0$.

2.1. The class $\mathcal{L}_*$.

For $s \in (0,1)$, we define the ellipticity class $\mathcal{L}_* = \mathcal{L}_*(s)$ as the set of all linear operators L of the form (1.3)-(1.4).

Throughout the paper, the extremal operators (as defined in [14]) for the class $\mathcal{L}_*$ are denoted by M^+ and M^-, that is,

$$M^+ u(x) = M^+_{\mathcal{L}_*} u(x) = \sup_{L \in \mathcal{L}_*} Lu(x) \quad \text{and} \quad M^- u(x) = M^-_{\mathcal{L}_*} u(x) = \inf_{L \in \mathcal{L}_*} Lu(x).$$
$$(2.1)$$

The following useful formula writes an operator $L \in \mathcal{L}_*$ as a weighted integral of one dimensional fractional Laplacians in all directions.

$$\begin{aligned}
Lu &= (1-s) \int_{S^{n-1}} d\theta \, \frac{1}{2} \int_{-\infty}^{\infty} dr \left(\frac{u(x+r\theta) + u(x-r\theta)}{2} - u(x) \right) \frac{a(\theta)}{|r|^{n+2s}} r^{n-1} \\
&= -\frac{1-s}{2c_{1,s}} \int_{S^{n-1}} d\theta \, a(\theta) \, (-\partial_{\theta\theta})^s u(x),
\end{aligned}$$
$$(2.2)$$

where

$$-(-\partial_{\theta\theta})^s u(x) = c_{1,s} \int_{-\infty}^{\infty} \left(\frac{u(x+\theta r) + u(x-\theta r)}{2} - u(x) \right) \frac{dr}{|r|^{1+2s}}$$

is the one-dimensional fractional Laplacian in the direction θ, whose Fourier symbol is $-|\theta \cdot \xi|^{2s}$.

The following is an immediate consequence of the formula (2.2).

Lemma 2.1. *Let u be a function depending only on variable x_n, i.e. $u(x) = w(x_n)$, where $w : \mathbb{R} \to \mathbb{R}$. Then,*

$$Lu(x) = -\frac{1-s}{2c_{1,s}} \left(\int_{S^{n-1}} |\theta_n|^{2s} a(\theta) \, d\theta \right) (-\Delta)^s_{\mathbb{R}} w(x_n),$$

where $(-\Delta)^s_{\mathbb{R}}$ denotes the fractional Laplacian in dimension one.

Proof. Using (2.2) we find

$$\begin{aligned}
Lu(x) &= \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} -(-\Delta)^s_{\mathbb{R}} \big(w(x_n + \theta_n \cdot) \big) a(\theta) \, d\theta \\
&= \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} -|\theta_n|^{2s} (-\Delta)^s_{\mathbb{R}} \big(w(x_n + \cdot) \big) a(\theta) \, d\theta,
\end{aligned}$$

as wanted. $\qquad\qquad\square$

Another consequence of (2.2) is that M^+ and M^- admit the following "closed formulae":

$$M^+u(x) = \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} \left\{ \Lambda\big(-(-\partial_{\theta\theta})^s w(x)\big)^+ - \lambda\big(-(-\partial_{\theta\theta})^s w(x)\big)^- \right\} d\theta$$

and

$$M^-u(x) = \frac{1-s}{2c_{1,s}} \int_{S^{n-1}} \left\{ \lambda\big(-(-\partial_{\theta\theta})^s w(x)\big)^+ - \Lambda\big(-(-\partial_{\theta\theta})^s w(x)\big)^- \right\} d\theta.$$

In all the paper, given $\nu \in S^{n-1}$ and $\beta \in (0, 2s)$ we denote by $\varphi^\beta : \mathbb{R} \to \mathbb{R}$ and $\varphi_\nu^\beta : \mathbb{R}^n \to \mathbb{R}$ the functions

$$\varphi^\beta(x) := (x_+)^\beta \qquad \text{and} \qquad \varphi_\nu^\beta(x) := (x \cdot \nu)_+^\beta. \tag{2.3}$$

A very important property of $\mathcal{L}_*$ is the following.

Lemma 2.2. *For any unit vector $\nu \in S^{n-1}$, the function φ_ν^s satisfies $M^+\varphi_\nu^s = M^-\varphi_\nu^s = 0$ in $\{x \cdot \nu > 0\}$ and $\varphi_\nu^s = 0$ in $\{x \cdot \nu < 0\}$.*

Proof. We use Lemma 2.1 and the well-known fact that the function $\varphi^s(x) = (x_+)^s$ satisfies $(-\Delta)_{\mathbb{R}}^s \varphi^s = 0$ in $\{x > 0\}$; see for instance [45, Proposition 3.1]. $\square$

Next we give a useful property of M^+ and M^-.

Lemma 2.3. *Let $\beta \in (0, 2s)$, and let M^+ and M^- be defined by (2.1). For any unit vector $\nu \in S^{n-1}$, the function φ_ν^β satisfies $M^+\varphi_\nu^\beta(x) = \bar{c}(s,\beta)(x \cdot \nu)^{\beta-2s}$ and $M^-\varphi_\nu^\beta(x) = \underline{c}(s,\beta)(x \cdot \nu)^{\beta-2s}$ in $\{x \cdot \nu > 0\}$, and $\varphi_\nu^\beta = 0$ in $\{x \cdot \nu < 0\}$. Here, $\bar{c}$ and $\underline{c}$ are constants depending only on s, β, n, and ellipticity constants.*

Moreover, $\bar{c}$ and $\underline{c}$ satisfy $\bar{c} \geq \underline{c}$, and they are continuous as functions of the variables (s,β) in $\{0 < s \leq 1,\ 0 < \beta < 2s\}$. In addition, we have

$$\bar{c}(s,\beta) > \underline{c}(s,\beta) > 0 \quad \text{for all } \beta \in (s, 2s). \tag{2.4}$$

and

$$\lim_{\beta \nearrow 2s} \underline{c}(s,\beta) = \begin{cases} +\infty & \text{for all } s \in (0,1) \\ C > 0 & \text{for } s = 1. \end{cases} \tag{2.5}$$

Proof. Given $L \in \mathcal{L}_*$, by Lemma 2.1 we have

$$L\varphi_\nu^\beta(x) = -\frac{1-s}{2c_{1,s}} \left(\int_{S^{n-1}} |\theta_n|^{2s} a(\theta)\, d\theta \right) (-\Delta)_{\mathbb{R}}^s \varphi^\beta(x \cdot \nu).$$

Hence, using the scaling properties of the fractional Laplacian and of the function φ^β we obtain that, for $x \cdot \nu > 0$,

$$M^+\varphi_\nu^\beta(x) = C\,(x \cdot \nu)^{\beta-2s} \max\left\{ -\Lambda(-\Delta)_{\mathbb{R}}^s \varphi^\beta(1), -\lambda(-\Delta)_{\mathbb{R}}^s \varphi^\beta(1) \right\}$$

and

$$M^-\varphi_\nu^\beta(x) = C\,(x \cdot \nu)^{\beta-2s} \min\left\{ -\Lambda(-\Delta)_{\mathbb{R}}^s \varphi^\beta(1), -\lambda(-\Delta)_{\mathbb{R}}^s \varphi^\beta(1) \right\},$$

where $C = (1-s)/(2c_{1,s}) > 0$.

Therefore, to prove that the two functions $\bar{c}$ and $\underline{c}$ are continuous in the variables (s, β) in $\{0 < s \leq 1, 0 < \beta < 2s\}$, and that (2.4)-(2.5) holds, it is enough to prove the same for

$$(s, \beta) \longmapsto -(-\Delta)^s_{\mathbb{R}} \varphi^\beta(1).$$

We first prove continuity in β. If β and β' belong to $(0, 2s)$, then as $\beta' \to \beta$, we have $\varphi^{\beta'} \to \varphi^\beta$ in $C^2([1/2, 3/2])$ and

$$\int_{\mathbb{R}} |\varphi^{\beta'} - \varphi^\beta|(x)\,(1 + |x|)^{-1-2s}\,dx \to 0.$$

As a consequence, $(-\Delta)^s_{\mathbb{R}} \varphi^{\beta'}(1) \to (-\Delta)^s_{\mathbb{R}} \varphi^\beta(1)$. It is easy to see that if s and s' belong to $(0, 1]$, and $\beta < 2s$, then $(-\Delta)^{s'}_{\mathbb{R}} \varphi^\beta(1) \to (-\Delta)^s_{\mathbb{R}} \varphi^\beta(1)$ as $s' \to s$.

Moreover, note that whenever $\beta > s$, the function φ^β is touched by below by the function $\varphi^s - C$ at some point $x_0 > 0$ for some constant $C > 0$. Hence, we have $(-\Delta)^s_{\mathbb{R}} \varphi^\beta(x_0) > (-\Delta)^s_{\mathbb{R}} \varphi^s(x_0) = 0$. This yields (2.4).

Finally, (2.5) follows from an easy computation using the definition of $(-\Delta)^s_{\mathbb{R}}$, and thus the proof is finished. $\qquad\square$

2.2. The class $\mathcal{L}_0$.

As defined in [14], for $s \in (0, 1)$ the ellipticity class $\mathcal{L}_0 = \mathcal{L}_0(s)$ consists of all operators L of the form

$$Lu(x) = (1 - s) \int_{\mathbb{R}^n} \left(\frac{u(x + y) + u(x - y)}{2} - u(x) \right) \frac{b(y)}{|y|^{n+2s}}\,dy.$$

where

$$b \in L^\infty(\mathbb{R}^n) \quad \text{satisfies} \quad \lambda \leq b \leq \Lambda.$$

It is clear that

$$\mathcal{L}_* \subsetneq \mathcal{L}_0.$$

The extremal operators for the class $\mathcal{L}_0$ are denoted here by $M^+_{\mathcal{L}_0}$ and $M^-_{\mathcal{L}_0}$. Since $\mathcal{L}_* \subset \mathcal{L}_0$, we have

$$M^-_{\mathcal{L}_0} \leq M^- \leq M^+ \leq M^+_{\mathcal{L}_0}.$$

Hence, all elliptic equations with respect to $\mathcal{L}_*$ are elliptic with respect to $\mathcal{L}_0$ and all the definitions and results in [14] apply to the elliptic equations considered in this paper.

As in [14, 15] we consider the weighted L^1 spaces $L^1(\mathbb{R}^n, \omega_s)$, where

$$\omega_s(x) = (1 - s)(1 + |x|)^{-n-2s}. \tag{2.6}$$

The utility of this weighted space is that, if $L \in \mathcal{L}_0(s)$, then $Lu(x)$ can be evaluated classically and is continuous in $B_{\epsilon/2}$ provided $u \in C^2(B_\epsilon) \cap L^1(\mathbb{R}^n, \omega_s)$. One can then consider viscosity solutions to elliptic equations with respect to $\mathcal{L}_0(s)$ which are not bounded but belong to $L^1(\mathbb{R}^n, \omega_s)$. The weighted norm appears in stability results; see [15].

As said in the Introduction, the definitions we follow of viscosity solutions and viscosity inequalities are the ones in [14].

Next we state the interior Harnack inequality and the C^α estimate from [14].

Theorem 2.4 ([14])**.** *Let* $s_0 \in (0,1)$ *and* $s \in [s_0, 1]$*. Let* $u \geq 0$ *in* $\mathbb{R}^n$ *satisfy in the viscosity sense* $M_{\mathcal{L}_0}^- u \leq C_0$ *and* $M_{\mathcal{L}_0}^+ u \geq -C_0$ *in* B_R*. Then,*

$$u(x) \leq C\big(u(0) + C_0 R^{2s}\big) \quad \text{for every } x \in B_{R/2},$$

for some constant C *depending only on* n*,* s_0*, and ellipticity constants.*

Theorem 2.5 ([14])**.** *Let* $s_0 \in (0,1)$ *and* $s \in [s_0, 1]$*. Let* $u \in C(\overline{B_1}) \cap L^1(\mathbb{R}^n, \omega_s)$ *satisfy in the viscosity sense* $M_{\mathcal{L}_0}^- u \leq C_0$ *and* $M_{\mathcal{L}_0}^+ u \geq -C_0$ *in* B_1*. Then,* $u \in C^\alpha\big(\overline{B_{1/2}}\big)$ *with the estimate*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\big(C_0 + \|u\|_{L^\infty(B_1)} + \|u\|_{L^1(\mathbb{R}^n, \omega_s)}\big),$$

where α *and* C *depend only on* n*,* s*, and ellipticity constants.*

The following result is a consequence of the results in [30] in the case $s \in (1/2, 1)$. In the case $s \leq 1/2$ it follows as a particular case of the results for parabolic equations in [48].

Theorem 2.6 ([30], [48])**.** *Let* $s_0 \in (0,1)$ *and* $s \in [s_0, 1]$*. Let* $f \in C(\overline{B_1})$ *and* $u \in C(\overline{B_1}) \cap L^\infty(\mathbb{R}^n)$ *be a viscosity solution of* $\mathrm{I}u = f(x)$ *in* B_1*, where* I *is translation invariant and elliptic with respect to* $\mathcal{L}_0(s)$*, with* $\mathrm{I}0 = 0$*. Then,* $u \in C^s\big(\overline{B_{1/2}}\big)$ *with the estimate*

$$\|u\|_{C^s(B_{1/2})} \leq C\big(\|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}\big),$$

where C *depends only on* n*,* s_0*, and ellipticity constants.*

In fact, [30, 48] establish not only a C^s estimate, but also a C^β one, for all $\beta < \min\{2s, 1 + \alpha\}$. However, in this paper we only need the C^s estimate.

2.3. **No fine boundary regularity for** $\mathcal{L}_0$**.** The aim of this subsection is to show that the class $\mathcal{L}_0$ is too large for all solutions to behave comparably near the boundary. Moreover, we give necessary conditions on a subclass $\mathcal{L} \subset \mathcal{L}_0$ to have comparability of all solutions near the boundary. These necessary conditions lead us to the class $\mathcal{L}_*$.

In the next result we show that, for any scale invariant class $\mathcal{L} \subseteq \mathcal{L}_0$ that contains the fractional Laplacian $(-\Delta)^s$, and any unit vector ν, there exist powers $0 \leq \beta_1 \leq s \leq \beta_2$ such that $M_{\mathcal{L}}^+ \varphi_\nu^{\beta_1} = 0$ and $M_{\mathcal{L}}^- \varphi_\nu^{\beta_2} = 0$ in $\{x \cdot \nu > 0\}$. Before stating this result, we give the following

Definition 2.7. We say that a class of operators $\mathcal{L}$ is *scale invariant* of order $2s$ if for each operator L in $\mathcal{L}$, and for all $R > 0$, the rescaled operator L_R, defined by

$$(L_R u)(R\,\cdot\,) = R^{-2s} L\big(u(R\,\cdot\,)\big),$$

also belongs to $\mathcal{L}$.

The proposition reads as follows.

Proposition 2.8. *Assume that* $\mathcal{L} \subset \mathcal{L}_0(s)$ *is scale invariant of order* $2s$*. Then,*

(a) *For every $\nu \in S^{n-1}$ and $\beta \in (0, 2s)$ the function φ_ν^β defined in (2.3) satisfies*

$$\begin{aligned}
M_{\mathcal{L}}^+ \varphi_\nu^\beta(x) &= \overline{C}(\beta, \nu)(x \cdot \nu)^{\beta - 2s} \quad in \quad \{x \cdot \nu > 0\}, \\
M_{\mathcal{L}}^- \varphi_\nu^\beta(x) &= \underline{C}(\beta, \nu)(x \cdot \nu)^{\beta - 2s} \quad in \quad \{x \cdot \nu > 0\}.
\end{aligned} \tag{2.7}$$

Here, $\overline{C}$ and $\underline{C}$ are constants depending only on s, β, ν, n, and ellipticity constants.

(b) *The functions $\overline{C}$ and $\underline{C}$ are continuous in β and, for each unit vector ν, there are $\beta_1 \leq \beta_2$ in $(0, 2s)$ such that*

$$\overline{C}(\beta_1, \nu) = 0 \quad and \quad \underline{C}(\beta_2, \nu) = 0. \tag{2.8}$$

Moreover, for all $\beta \in (0, 2s)$,

$$\overline{C}(\beta, \nu) - \overline{C}(\beta_1, \nu) \text{ has the same sign as } \beta - \beta_1 \tag{2.9}$$

and

$$\underline{C}(\beta, \nu) - \underline{C}(\beta_2, \nu) \text{ has the same sign as } \beta - \beta_2. \tag{2.10}$$

(c) *If in addition the fractional Laplacian $-(-\Delta)^s$ belongs to $\mathcal{L}$, then we have $\beta_1 \leq s \leq \beta_2$.*

Proof. The scale invariance of $\mathcal{L}$ is equivalent to a scaling property of the extremal operators $M_{\mathcal{L}}^+$ and $M_{\mathcal{L}}^-$. Namely, for all $R > 0$, we have

$$M_{\mathcal{L}}^\pm\big(u(R\cdot)\big) = R^{2s}(M_{\mathcal{L}}^\pm u)(R\cdot).$$

(a) By this scaling property it is immediate to prove that given $\beta \in (0, 2s)$ and $\nu \in S^{n-1}$, the function φ_ν^β satisfies (2.7), where

$$\overline{C}(\beta, \nu) := M_{\mathcal{L}}^+ \varphi_\nu^\beta(\nu) \quad and \quad \underline{C}(\beta, \nu) := M_{\mathcal{L}}^- \varphi_\nu^\beta(\nu).$$

Of course, $\overline{C}$ and $\underline{C}$ depend also on s and the ellipticity constants, but these are fixed constants in this proof.

(b) Note that, as $\beta' \to \beta \in [0, 2s)$, we have $\varphi_\nu^{\beta'} \to \varphi_\nu^\beta$ in $C^2(\overline{B_{1/2}(\nu)})$ and in $L^1(\mathbb{R}^n, \omega_s)$. As a consequence, $\underline{C}$ and $\overline{C}$ are continuous in β in the interval $[0, 2s)$. Since $\varphi_\nu^\beta \to \chi_{\{x \cdot \nu > 0\}}$ as $\beta \to 0$, we have that

$$\underline{C}(\nu, 0) \leq \overline{C}(\nu, 0) < 0.$$

On the other hand, it is easy to see that

$$M_{\mathcal{L}_0}^- \varphi_\nu^\beta(\nu) \longrightarrow +\infty \quad as \ \beta \nearrow 2s.$$

Hence, using that $M_{\mathcal{L}_0}^- \leq M_{\mathcal{L}}^-$, we obtain

$$0 < \underline{C}(\nu, \beta) \leq \overline{C}(\nu, \beta) \quad for \ \beta \text{ close to } 2s.$$

Therefore, by continuity, there are β_1 and β_2 in $(0, 2s)$ such that

$$\overline{C}(\beta_1, \nu) = 0 \quad and \quad \underline{C}(\beta_2, \nu) = 0.$$

To prove (2.9), we observe that if $\beta > \beta_1$ the function φ_ν^β is be touched by below by $\varphi_\nu^{\beta_1} - C$ at some $x_0 \in \{x \cdot \nu > 0\}$ for some $C > 0$. It follows that

$$M_{\mathcal{L}}^+ \varphi_\nu^\beta(x_0) - M_{\mathcal{L}}^+ \varphi_\nu^{\beta_1}(x_0) \geq M_{\mathcal{L}_0}^-\big(\varphi_\nu^\beta - \varphi_\nu^{\beta_1}\big)(x_0) > 0.$$

Since the sign of $M_{\mathcal{L}}^+ \varphi_\nu^\beta$ is constant in $\{x \cdot \nu > 0\}$ it follows that $\overline{C}(\nu, \beta) > 0$ when $\beta > \beta_1$. Similarly one proves that $\overline{C}(\nu, \beta) < 0$ when $\beta < \beta_1$, and hence (2.10).

(c) It is an immediate consequence of the results in parts (a) and (b) and the fact that $-(-\Delta)^s \varphi_\nu^s = 0$ in $\{x \cdot \nu > 0\}$. $\qquad\square$

Clearly, to hope for some good description of the boundary behavior of solutions to all elliptic equations with respect to a scale invariant class $\mathcal{L}$, it must be $\beta_1 = \beta_2$ for every direction ν. Typical classes $\mathcal{L}$ contain the fractional Laplacian $-(-\Delta)^s$. Thus, for them, we must have $\beta_1 = \beta_2 = s$ for all $\nu \in S^{n-1}$. If this happens, then

$$L\varphi_\nu^s = 0 \quad \text{in } \{x \cdot \nu > 0\} \quad \text{for all } L \in \mathcal{L}, \text{ and for all } \nu \in S^{n-1}, \tag{2.11}$$

since $M_{\mathcal{L}}^- \leq L \leq M_{\mathcal{L}}^+$ for all $L \in \mathcal{L}$.

As a consequence, we find the following.

Corollary 2.9. *Let β_1, β_2 be given by (2.8) in Proposition 2.8. Then, for the classes $\mathcal{L}_0$, $\mathcal{L}_1$, and $\mathcal{L}_2$ we have $\beta_1 < s < \beta_2$.*

Proof. Let us show that for $\mathcal{L} = \mathcal{L}_0$ the condition (2.11) is not satisfied. Indeed, we may easily cook up $L \in \mathcal{L}_0$ so that $L\varphi_{e_n}^s(x', 1) \neq 0$ for $x' \in \mathbb{R}^{n-1}$. Namely, if we take

$$b(y) = \Big(\lambda + (\Lambda - \lambda)\chi_{B_{1/2}}(y)\Big),$$

then at points $x = (x', 1)$ we have

$$0 > L\varphi_{e_n}^s(x) = (1 - s) \int_{\mathbb{R}^n} \left(\frac{u(x + y) + u(x - y)}{2} - u(x)\right) \frac{b(y)}{|y|^{n+2s}} \, dy,$$

since $\varphi_{e_n}^s$ is concave in $B_{1/2}(x', 1)$ and $(-\Delta)^s \varphi_{e_n}^s = 0$ in $\{x_n > 0\}$.

By taking an smoothed version of $b(y)$, we obtain that both $\mathcal{L}_1$ and $\mathcal{L}_2$ fail to satisfy (2.11). $\qquad\square$

By the results in Subsection 2.1, we have that the class $\mathcal{L}_*$ satisfies the necessary condition (2.11). Although we do not have a rigorous mathematical proof, we believe that $\mathcal{L}_*$ is actually the largest scale invariant subclass of $\mathcal{L}_0$ satisfying (2.11).

3. Barriers

In this section we construct supersolutions and subsolutions that are needed in our analysis. From now on, all the results are for the class $\mathcal{L}_*$ (and not for $\mathcal{L}_0$).

First we give two preliminary lemmas.

Lemma 3.1. *Let $s_0 \in (0, 1)$ and $s \in [s_0, 1)$. Let*

$$\varphi^{(1)}(x) = \big(\mathrm{dist}(x, B_1)\big)^s \quad and \quad \varphi^{(2)}(x) = \big(\mathrm{dist}(x, \mathbb{R}^n \setminus B_1)\big)^s.$$

Then,

$$0 \le M^- \varphi^{(1)}(x) \le M^+ \varphi^{(1)}(x) \le C \left\{ 1 + (1-s) \left| \log(|x|-1) \right| \right\} \quad \text{in } B_2 \setminus B_1. \quad (3.1)$$

and

$$0 \ge M^+ \varphi^{(2)}(x) \ge M^- \varphi^{(2)}(x) \ge -C \left\{ 1 + (1-s) \left| \log(1-|x|) \right| \right\} \text{ in } B_1 \setminus B_{1/2}. \quad (3.2)$$

The constant C depends only on s_0, n, and ellipticity constants.

Note that the above bounds are much better than $\left| |x| - 1 \right|^{-s}$, which would be the expected bound given by homogeneity. This is since $\varphi^{(1)}$ and $\varphi^{(2)}$ are in some sense close to the 1D solution $(x_+)^s$.

Proof of Lemma 3.1. Let $L \in \mathcal{L}_*$. For points $x \in \mathbb{R}^n$ we use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. To prove (3.1) let us estimate $L\varphi^{(1)}(x_\rho)$ where $x_\rho = (0, 1+\rho)$ for $\rho \in (0,1)$ and for a generic $L \in \mathcal{L}_*$. To do it, we subtract the function $\psi(x) = (x_n - 1)_+^s$, which satisfies $L\psi(x_\rho) = 0$. Note that

$$\left(\varphi^{(1)} - \psi \right)(x_\rho) = 0 \quad \text{for all } \rho > 0$$

and that, for $|y| < 1$,

$$\left| \text{dist} \left(x_\rho + y, B_1 \right) - (1 + \rho + y_n)_+ \right| \le C |y'|^2.$$

This is because the level sets of the two previous functions are tangent on $\{y' = 0\}$.

Thus,

$$0 \le \left(\varphi_1^{(1)} - \psi \right)(x_\rho + y) \le \begin{cases} C\rho^{s-1}|y'|^2 & \text{for } y = (y', y_n) \in B_{\rho/2} \\ C|y'|^{2s} & \text{for } y = (y', y_n) \in B_1 \setminus B_{\rho/2} \\ C|y|^s & \text{for } y \in \mathbb{R}^n \setminus B_1. \end{cases}$$

The bound in $B_{\rho/2}$ follows from the inequality $a^s - b^s \le (a-b)b^{s-1}$ for $a > b > 0$.

Therefore, we have

$$0 \le L\varphi^{(1)}(x_\rho) = L\left(\varphi^{(1)} - \psi \right)(x_\rho)$$

$$= (1-s) \int \frac{\left(\varphi_1^{(1)} - \psi \right)(x_\rho + y) + \left(\varphi_1^{(1)} - \psi \right)(x_\rho - y)}{2} \frac{a(y/|y|)}{|y|^{n+2s}} \, dy$$

$$\le C(1-s)\Lambda \left(\int_{B_{\rho/2}} \frac{\rho^{s-1}|y'|^2 dy}{|y|^{n+2s}} + \int_{B_1 \setminus B_{\rho/2}} \frac{|y'|^{2s} dy}{|y|^{n+2s}} + \int_{\mathbb{R}^n \setminus B_1} \frac{|y|^s dy}{|y|^{n+2s}} \right)$$

$$\le C\left(1 + (1-s)|\log \rho| \right).$$

This establishes (3.1). The proof of (3.2) is similar. $\qquad\square$

In the next result, instead, the bounds are those given by the homogeneity. In addition, the constant in the bounds has the right sign to construct (together with the previous lemma) appropriate barriers.

Lemma 3.2. *Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. Let*

$$\varphi^{(3)}(x) = \big(\text{dist}(x, B_1)\big)^{3s/2} \quad \text{and} \quad \varphi^{(4)}(x) = \big(\text{dist}(x, \mathbb{R}^n \setminus B_1)\big)^{3s/2}.$$

Then,

$$M^- \varphi^{(3)}(x) \geq c(|x| - 1)^{-s/2} \quad \text{for all } x \in B_2 \setminus B_1. \tag{3.3}$$

and

$$M^- \varphi^{(4)}(x) \geq c(1 - |x|)^{-s/2} - C \quad \text{for all } x \in B_1 \setminus B_{1/2}. \tag{3.4}$$

The constants $c > 0$ and C depend only on n, s_0, and ellipticity constants.

Proof. Let $L \in \mathcal{L}_*$. For points $x \in \mathbb{R}^n$ we use the notation $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. To prove (3.4) let us estimate $L\varphi^{(4)}(x_\rho)$ where $x_\rho = (0, 1 + \rho)$ for $\rho \in (0, 1)$ and for a generic $L \in \mathcal{L}_*$. To do it we subtract the function $\psi(x) = (1 - x_n)_+^{3s/2}$, which by Lemma 2.3 satisfies $L\psi(x_\rho) = c\rho^{-s/2}$ for some $c > 0$. We note that

$$\big(\varphi^{(4)} - \psi\big)(x_\rho) = 0$$

and, similarly as in the proof of Lemma 3.1,

$$0 \geq \big(\varphi^{(4)} - \psi\big)(x_\rho + y) \geq \begin{cases} -C\rho^{3s/2-1}|y'|^2 & \text{for } y = (y', y_n) \in B_{\rho/2} \\ -C|y'|^{3s} & \text{for } y = (y', y_n) \in B_1 \setminus B_{\rho/2} \\ -C|y|^{3s/2} & \text{for } y \in \mathbb{R}^n \setminus B_1. \end{cases}$$

Hence,

$$L\varphi^{(4)}(x_\rho) - c\rho^{-s/2} = L\big(\varphi^{(4)} - \psi\big)(x_\rho)$$

$$\geq -C(1-s)\Lambda \left(\int_{B_{\rho/2}} \frac{\rho^{3s/2-1}|y'|^2 dy}{|y|^{n+2s}} + \int_{B_1 \setminus B_{\rho/2}} \frac{|y'|^{3s} dy}{|y|^{n+2s}} + \int_{\mathbb{R}^n \setminus B_1} \frac{|y|^{s/2} dy}{|y|^{n+2s}} \right)$$

$$\geq -C.$$

This establishes (3.4). To prove (3.3), we now define $\psi(x) = (x_n - 1)_+^{3s/2}$, and we use Lemma 2.3 and the fact that $\varphi^{(3)} - \psi$ is nonnegative in all of $\mathbb{R}^n$ and vanishes on the positive x_n axis. $\qquad \square$

We can now construct the sub and supersolutions that will be used in the next section.

Lemma 3.3. *Let $s_0 \in (0, 1)$ and $s \in [s_0, 1)$. There are positive constants ϵ and C, and a radial, bounded, continuous function φ_1 which is $C^{1,1}$ in $B_{1+\epsilon} \setminus \overline{B_1}$ and satisfies*

$$\begin{cases} M^+ \varphi_1(x) \leq -1 & \text{in } B_{1+\epsilon} \setminus \overline{B_1} \\ \varphi_1(x) = 0 & \text{in } B_1 \\ \varphi_1(x) \leq C\big(|x| - 1\big)^s & \text{in } \mathbb{R}^n \setminus B_1 \\ \varphi_1(x) \geq 1 & \text{in } \mathbb{R}^n \setminus B_{1+\epsilon} \end{cases}$$

The constants ϵ, c and C depend only on n, s_0, and ellipticity constants.

Proof. Let

$$\psi = \begin{cases} 2\varphi^{(1)} - \varphi^{(3)} & \text{in } B_2 \\ 1 & \text{in } \mathbb{R}^n \setminus B_2. \end{cases}$$

By Lemmas 3.1 and 3.2, for $|x| > 1$ it is

$$M^+\psi \leq C\left\{1 + (1-s)\big|\log(|x|-1)\big|\right\} - c(|x|-1)^{-s/2} + C.$$

Hence, we may take $\epsilon > 0$ small enough so that $M^+\psi \leq -1$ in $B_{1+\epsilon} \setminus \overline{B_1}$. We then set $\varphi_1 = C\psi$ with $C \geq 1$ large enough so that $\varphi_1 \geq 1$ outside $B_{1+\epsilon}$. $\qquad\square$

Lemma 3.4. *Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. There is $c > 0$, and a radial, bounded, continuous function φ_2 that satisfies*

$$\begin{cases} M^-\varphi_2(x) \geq c & \text{in } B_1 \setminus B_{1/2} \\ \varphi_2(x) = 0 & \text{in } \mathbb{R}^n \setminus B_1 \\ \varphi_2(x) \geq c\big(1 - |x|\big)^s & \text{in } B_1 \\ \varphi_2(x) \leq 1 & \text{in } \overline{B_{1/2}}. \end{cases}$$

The constants ϵ, c and C depend only on n, s_0, and ellipticity constants.

Proof. We first construct a subsolution ψ in the annulus $B_1 \setminus \overline{B_{1-\epsilon}}$, for some small $\epsilon > 0$. Then, using it, we will construct the desired subsolution in $B_1 \setminus B_{1/2}$. Let

$$\psi = \varphi^{(2)} + \varphi^{(4)}.$$

By Lemmas 3.1 and 3.2, for $1/2 < |x| < 1$ it is

$$M^-\psi \geq -C\left\{1 + (1-s)\big|\log(1-|x|)\big|\right\} + c(1-|x|)^{-s/2} - C.$$

Hence, we can take $\epsilon > 0$ small enough so that $M^-\psi \geq 1$ in $B_1 \setminus \overline{B_{1-\epsilon}}$.

Let us now construct a subsolution in $B_1 \setminus \overline{B_{1/2}}$ from ψ, which is a subsolution only in $B_1 \setminus \overline{B_{1-\epsilon}}$. We consider

$$\Psi(x) = \max_{0 \leq k \leq N} C^k \psi(2^{k/N}x),$$

where N is a large integer and $C > 1$. Notice that, for C large enough, the set $\{x \in B_1 \; : \; \Psi(x) = \psi(x)\}$ is an annulus contained in $B_1 \setminus \overline{B_{1-\epsilon}}$.

Consider, for $k \geq 0$,

$$A_k = \left\{x \in B_1 \; : \; \Psi(x) = C^k\psi(2^{k/N}x)\right\}.$$

Since $A_0 \subset B_1 \setminus \overline{B_{1-\epsilon}}$, then Ψ satisfies $M^-\Psi \geq 1$ in A_0.

Observe that $A_k = 2^{-k/N}A_0$, since $C^{-1}\Psi(2^{1/n}x) = \Psi(x)$ in the annulus $\{1/2 < |x| < 2^{-1/n}\}$. Hence, for $x \in A_k$ we have $2^{k/N}x \in A_0 \subset B_1 \setminus \overline{B_{1-\epsilon}}$ and

$$M^-\Psi(x) > M^-\big(C^k\psi(2^{k/N}\cdot)\big)(x) = C^k 2^{2sk/N} M^-\psi(2^{k/N}x) > 1.$$

We then set $\varphi_2 = c\Psi$ with $c > 0$ small enough so that $\varphi_2(x) \leq 1$ in $\overline{B_{1/2}}$. $\qquad\square$

Remark 3.5. Notice that the subsolution φ_2 constructed above is $C^{1,1}$ by below in $B_1 \setminus \overline{B}_{1/2}$, in the sense that it can be touched by below by paraboloids. This is important when considering non translation invariant equations for which a comparison principle for viscosity solutions is not available.

4. KRYLOV'S METHOD

The goal of this section is to prove Proposition 1.1. Its proof combines the interior Hölder regularity results of Caffarelli and Silvestre [14] and the next key Lemma.

Lemma 4.1. *Let* $s_0 \in (0,1)$, $s \in [s_0, 1)$, *and* $u \in C\left(\overline{B_1^+}\right)$ *be a viscosity solution of* (1.6). *Then, there exist* $\alpha \in (0,1)$ *and* C *depending only on* n, s_0, *and ellipticity constants, such that*

$$\sup_{B_r^+} u/x_n^s - \inf_{B_r^+} u/x_n^s \leq Cr^\alpha \left(C_0 + \|u\|_{L^\infty(\mathbb{R}^n)} \right) \tag{4.1}$$

for all $r \leq 3/4$.

To prove Lemma 4.1 we need two preliminary lemmas.

We start with the first, which is a nonlocal version of Lemma 4.31 in [28]. Throughout this section we denote

$$D_r^* := B_{9r/10} \cap \{x_n > 1/10\}.$$

Lemma 4.2. *Let* $s_0 \in (0,1)$ *and* $s \in [s_0, 1)$. *Assume that* u *satisfies* $u \geq 0$ *in all of* $\mathbb{R}^n$ *and*

$$M^- u \leq C_0 \quad \text{in } B_r^+,$$

for some $C_0 > 0$. *Then,*

$$\inf_{D_r^*} u/x_n^s \leq C \left(\inf_{B_{r/2}^+} u/x_n^s + C_0 r^s \right) \tag{4.2}$$

for all $r \leq 1$, *where* C *is a constant depending only on* s_0, *ellipticity constants, and dimension.*

Proof. Step 1. Assume $C_0 = 0$. Let us call

$$m = \inf_{D_r^*} u/x_n^s \geq 0.$$

We have

$$u \geq m x_n^s \geq m(r/10)^s \quad \text{in } D_r^*. \tag{4.3}$$

Let us scale and translate the subsolution φ_2 in Lemma 3.4 as follows to use it as lower barrier:

$$\psi_r(x) := (r/10)^s \, \varphi_2\left(\tfrac{10(x-x_0)}{2r}\right). \tag{4.4}$$

We then have, for some $c > 0$,

$$
\begin{cases}
M^-\psi_r \geq 0 & \text{in } B_{2r/10}(x_0) \setminus B_{r/10}(x_0) \\
\psi_r = 0 & \text{in } \mathbb{R}^n \setminus B_{2r/10}(x_0) \\
\psi_r \geq c\big(\tfrac{2r}{10} - |x|\big)^s & \text{in } B_{2/10}(x_0) \\
\psi_r \leq (r/10)^s & \text{in } B_{r/10}(x_0).
\end{cases}
$$

It is immediate to verify that $B_{r/2}^+$ is covered by balls of radius $2r/10$ such that the concentric ball of radius $r/10$ is contained in D_r^*, that is,

$$
B_{r/2}^+ \subset \bigcup \left\{ B_{2r/10}(x_0) \; : \; B_{r/10}(x_0) \subset D_r^* \right\}.
$$

Now, if we choose some ball $B_{r/10}(x_0) \subset D_r^*$ and define ψ_r by (4.4), then by (4.3) we have $u \geq m\psi_r$ in $B_{r/10}(x_0)$. On the other hand $u \geq m\psi_r$ outside $B_{2r/10}(x_0)$, since ψ_r vanishes there and $u \geq 0$ in all of $\mathbb{R}^n$ by assumption. Finally, $M^+\psi_r \leq 0$, and since $C_0 = 0$, $M^- u \geq 0$ in the annulus $B_{2r/10}(x_0) \setminus B_{r/10}(x_0)$.

Therefore, it follows from the comparison principle that $u \geq m\psi_r$ in $B_{2r/10}(x_0)$. Since these balls of radius $2r/10$ cover $B_{r/2}^+$ and $\psi_r \geq c\big(\tfrac{2r}{10} - |x|\big)^s$ in $B_{2/10}(x_0)$, we obtain

$$
u \geq cmx_n^s \qquad \text{in } B_{r/2}^+,
$$

which yields (4.2).

Step 2. If $C_0 > 0$ we argue as follows. First, let

$$
\phi(x) = \min\big\{ 1, \; 2(x_n)_+^s - (x_n)_+^{3s/2} \big\}.
$$

By Lemma 2.3, we have that $M^+\phi \leq -c$ in $\{0 < x_n < \epsilon\}$ for some $\epsilon > 0$ and some $c > 0$. By scaling ϕ and reducing c, we may assume $\epsilon = 1$.

We then consider

$$
\tilde{u}(x) = u(x) + \frac{C_0}{c} r^{2s} \phi(x/r).
$$

The function $\tilde{u}$ satisfies in $\{0 < x_n < r\}$

$$
M^-\tilde{u} - M^- u \leq M^+\left(\frac{C_0}{c} r^{2s} \phi(x/r) \right) \leq -C_0
$$

and hence

$$
M^-\tilde{u} \leq 0.
$$

Using that $u(x) \leq \tilde{u}(x) \leq u(x) + CC_0 r^s (x_n)_+^s$ and applying Step 1 to $\tilde{u}$, we obtain (4.2). $\qquad\square$

The second lemma towards Proposition 4.1 is a nonlocal version of Lemma 4.35 in [28]. It is an immediate consequence of the Harnack inequality of Caffarelli and Silvestre [14].

Lemma 4.3. *Let $s_0 \in (0,1)$, $s \in [s_0, 1)$, $r \leq 1$, and u satisfy $u \geq 0$ in all of $\mathbb{R}^n$ and*

$$M^+ u \geq -C_0 \quad and \quad M^- u \leq C_0 \quad in \ B_r^+ .$$

Then,

$$\sup_{D_r^*} u/x_n^s \leq C \left(\inf_{D_r^*} u/x_n^s + C_0 r^s \right),$$

for some constant C depending only on n, s_0, and ellipticity constants.

Proof. The lemma is a consequence of Theorem 2.4. Indeed, covering the set D_r^* with balls contained in B_r^+ and with radii comparable to r —using the same (scaled) covering for all r—, Theorem 2.4 yields

$$\sup_{D_r^*} u \leq C \left(\inf_{D_r^*} u + C_0 r^{2s} \right).$$

Then, the lemma follows by noting that x_n^s is comparable to r^s in D_r^*.   □

Next we prove Lemma 4.1.

Proof of Lemma 4.1. First, dividing u by a constant, we may assume that $C_0 + \|u\|_{L^\infty(\mathbb{R}^n)} \leq 1$.

We will prove that there exist constants $C_1 > 0$ and $\alpha \in (0, s)$, depending only on n, s_0, and ellipticity constants, and monotone sequences $(m_k)_{k \geq 1}$ and $(\overline{m}_k)_{k \geq 1}$ satisfying the following. For all $k \geq 1$,

$$\overline{m}_k - m_k = 4^{-\alpha k}, \quad -1 \leq m_k \leq m_{k+1} < \overline{m}_{k+1} \leq \overline{m}_k \leq 1, \tag{4.5}$$

and

$$m_k \leq C_1^{-1} u/x_n^s \leq \overline{m}_k \quad in \ B_{r_k}^+, \quad where \ r_k = 4^{-k} . \tag{4.6}$$

Note that since $u = 0$ in B_1^- then we have that (4.6) is equivalent to the following inequality in B_{r_k} instead of $B_{r_k}^+$

$$m_k (x_n)_+^s \leq C_1^{-1} u \leq \overline{m}_k (x_n)_+^s \quad in \ B_{r_k}, \quad where \ r_k = 4^{-k} . \tag{4.7}$$

Clearly, if such sequences exist, then (4.1) holds for all $r \leq 1/4$ with $C = 4^\alpha C_1$. Moreover, for $1/4 < r \leq 3/4$ the result follows from (4.8) below. Hence, we only need to construct $\{m_k\}$ and $\{\overline{m}_k\}$.

Next we construct these sequences by induction.

Using the supersolution φ_1 in Lemma 3.3 we find that

$$-\frac{C_1}{2} (x_n)_+^s \leq u \leq \frac{C_1}{2} (x_n)_+^s \quad in \ B_{3/4}^+ \tag{4.8}$$

whenever C_1 is large enough. Thus, we may take $m_1 = -1/2$ and $\overline{m}_1 = 1/2$.

Assume now that we have sequences up to m_k and $\overline{m}_k$. We want to prove that there exist m_{k+1} and $\overline{m}_{k+1}$ which fulfill the requirements. Let

$$u_k = C_1^{-1} u - m_k (x_n)_+^s .$$

We will consider the positive part u_k^+ of u_k in order to have a nonnegative function in all of $\mathbb{R}^n$ to which we can apply Lemmas 4.2 and 4.3. Let $u_k = u_k^+ - u_k^-$. Observe that, by induction hypothesis,

$$u_k^+ = u_k \quad \text{and} \quad u_k^- = 0 \quad \text{in } B_{r_k}.$$

Moreover, $C_1^{-1}u \geq m_j(x_n)_+^s$ in B_{r_j} for each $j \leq k$. Therefore, we have

$$u_k \geq (m_j - m_k)(x_n)_+^s \geq (m_j - \overline{m}_j + \overline{m}_k - m_k)(x_n)_+^s = (-4^{-\alpha j} + 4^{-\alpha k})(x_n)_+^s \quad \text{in } B_{r_j}.$$

But clearly $0 \leq (x_n)_+^s \leq r_j^s$ in B_{r_j}, and therefore using $r_j = 4^{-j}$

$$u_k \geq -r_j^s(r_j^\alpha - r_k^\alpha) \quad \text{in } B_{r_j} \text{ for each } j \leq k.$$

Thus, since for every $x \in B_1 \setminus B_{r_k}$ there is $j < k$ such that

$$|x| < r_j = 4^{-j} \leq 4|x|,$$

we find

$$u_k(x) \geq -r_k^{\alpha+s}\left|\frac{4x}{r_k}\right|^s\left(\left|\frac{4x}{r_k}\right|^\alpha - 1\right) \quad \text{outside } B_{r_k}. \tag{4.9}$$

Now let $L \in \mathcal{L}_*$. Using (4.9) and that $u_k^- \equiv 0$ in B_{r_k}, then for all $x \in B_{r_k/2}$ we have

$$0 \leq Lu_k^-(x) = (1-s)\int_{x+y\notin B_{r_k}} u_k^-(x+y)\frac{a(y/|y|)}{|y|^{n+2s}}\,dy$$

$$\leq (1-s)\int_{|y|\geq r_k/2} r_k^{\alpha+s}\left|\frac{8y}{r_k}\right|^s\left(\left|\frac{8y}{r_k}\right|^\alpha - 1\right)\frac{\Lambda}{|y|^{n+2s}}\,dy$$

$$= (1-s)\Lambda r_k^{\alpha-s}\int_{|z|\geq 1/2}\frac{|8z|^s(|8z|^\alpha - 1)}{|z|^{n+2s}}\,dz$$

$$\leq \varepsilon_0 r_k^{\alpha-s},$$

where $\varepsilon_0 = \varepsilon_0(\alpha) \downarrow 0$ as $\alpha \downarrow 0$ since $|8z|^\alpha \to 1$. Since this can be done for all $L \in \mathcal{L}_*$, u_k^- vanishes in B_{r_k} and satisfies pointwise

$$0 \leq M^- u_k^- \leq M^+ u_m^- \leq \varepsilon_0 r_k^{\alpha-s} \quad \text{in } B_{r_k/2}^+.$$

Therefore, recalling that

$$u_k^+ = C_1^{-1}u - m_k(x_n)_+^s + u_k^-,$$

and using that $M^+(x_n)_+^s = M^-(x_n)_+^s = 0$ in $\{x_n > 0\}$, we obtain

$$M^- u_k^+ \leq C_1^{-1}M^- u + M^+(u_k^-)$$

$$\leq C_1^{-1} + \varepsilon_0 r_k^{\alpha-s} \quad \text{in } B_{r_k/2}^+.$$

Also clearly

$$M^+ u_k^+ \geq M^+ u_k \geq -C_1^{-1} \quad \text{in } B_{r_k/2}^+.$$

Now we can apply Lemmas 4.2 and 4.3 with u in its statements replaced by u_k^+. Recalling that

$$u_k^+ = u_k = C_1^{-1}u - m_k x_n^s \quad \text{in } B_{r_k}^+,$$

we obtain

$$\sup_{D_{r_k/2}^*} (C_1^{-1}u/x_n^s - m_k) \le C\left(\inf_{D_{r_k/2}^*} (C_1^{-1}u/x_n^s - m_k) + C_1^{-1}r_k^s + \varepsilon_0 r_k^\alpha\right)$$

$$\le C\left(\inf_{B_{r_k/4}^+} (C_1^{-1}u/x_n^s - m_k) + C_1^{-1}r_k^s + \varepsilon_0 r_k^\alpha\right). \tag{4.10}$$

On the other hand, we can repeat the same reasoning "upside down", that is, considering the functions $\overline{u}_k = \overline{m}_k(x_n)_+^s - u$ instead of u_k. In this way we obtain, instead of (4.10), the following

$$\sup_{D_{r_k/2}^*} (\overline{m}_k - C_1^{-1}u/x_n^s) \le C\left(\inf_{B_{r_k/4}^+} (\overline{m}^k - C_1^{-1}u/x_n^s) + C_1^{-1}r_k^s + \varepsilon_0 r_k^\alpha\right). \tag{4.11}$$

Adding (4.10) and (4.11) we obtain

$$\overline{m}_k - m_k \le C\left(\inf_{B_{r_k/4}^+} (C_1^{-1}u/x_n^s - m_k) + \inf_{B_{r_k/4}^+} (\overline{m}_k - C_1^{-1}u/x_n^s) + C_1^{-1}r_k^s + \varepsilon_0 r_k^\alpha\right)$$

$$= C\left(\inf_{B_{r_{k+1}}^+} C_1^{-1}u/x_n^s - \sup_{B_{r_{k+1}}^+} C_1^{-1}u/x_n^s + \overline{m}_k - m_k + C_1^{-1}r_k^s + \varepsilon_0 r_k^\alpha\right).$$

Thus, using that $\overline{m}_k - m_k = 4^{-\alpha k}$, $\alpha < s$, and $r_k = 4^{-k} \le 1$, we obtain

$$\sup_{B_{r_{k+1}}^+} C_1^{-1}u/x_n^s - \inf_{B_{r_{k+1}}^+} C_1^{-1}u/x_n^s \le \left(\tfrac{C-1}{C} + C_1^{-1} + \varepsilon_0\right)4^{-\alpha k}.$$

Now we choose α small and C_1 large enough so that

$$\frac{C-1}{C} + C_1^{-1} + \varepsilon_0(\alpha) \le 4^{-\alpha}.$$

This is possible since $\varepsilon_0(\alpha) \downarrow 0$ as $\alpha \downarrow 0$ and the constant C depends only on n, s_0, and ellipticity constants. Then, we find

$$\sup_{B_{r_{k+1}}^+} C_1^{-1}u/x_n^s - \inf_{B_{r_{k+1}}^+} C_1^{-1}u/x_n^s \le 4^{-\alpha(k+1)},$$

and thus we are able to choose m_{k+1} and $\overline{m}_{k+1}$ satisfying (4.5) and (4.6).   □

To end this section, we give the

Proof of Proposition 1.1. Let $x \in B_{1/2}^+$ and let x_0 be its nearest point on $\{x_n = 0\}$. Let

$$d = \text{dist}\,(x, x_0) = x_n = \text{dist}\,(x, B_1^-).$$

By Theorem 2.5 (rescaled), we have

$$\|u\|_{C^\alpha\left(B_{d/2}(x)\right)} \leq Cd^{-\alpha}\left(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\right).$$

Hence, since $\|(x_n)^{-s}\|_{C^\alpha\left(B_{d/2}(x)\right)} \leq Cd^{-s}$, then for $r \leq d/2$

$$\mathrm{osc}_{B_r(x)}u/x_n^s \leq Cr^\alpha d^{-s-\alpha}\left(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\right). \tag{4.12}$$

On the other hand, by Lemma 4.1, for all $r \geq d/2$ we have

$$\mathrm{osc}_{B_r(x)\cap B_{3/4}^+}u/x_n^s \leq Cr^\alpha\left(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\right). \tag{4.13}$$

In both previous estimates $\alpha \in (0,1)$ depends only on n, s_0, and ellipticity constants. Let us call

$$M = \left(\|u\|_{L^\infty(\mathbb{R}^n)} + C_0\right).$$

Then, given $\theta > 1$ we have the following alternatives

(i) If $r \leq d^\theta/2$ then, by (4.12),

$$\mathrm{osc}_{B_r(x)}u/x_n^s \leq Cr^\alpha d^{-s-\alpha}M \leq Cr^{\alpha-(s+\alpha)/\theta}M.$$

(ii) If $d^\theta/2 < r \leq d/2$ then, by (4.13),

$$\mathrm{osc}_{B_r(x)}u/x_n^s \leq \mathrm{osc}_{B_{d/2}(x)}u/x_n^s \leq Cd^\alpha M \leq Cr^{\alpha/\theta}M.$$

(iii) If $d/2 < r$, then by (4.13)

$$\mathrm{osc}_{B_r(x)\cap B_{3/4}^+}u/x_n^s \leq Cr^\alpha M.$$

Choosing $\theta > \frac{s+\alpha}{\alpha}$ (so that the exponent in (i) is positive), we obtain

$$\mathrm{osc}_{B_r(x)\cap B_{3/4}^+}u/x_n^s \leq Cr^{\alpha'}M \quad \text{whenever} \quad x \in B_{1/2}^+ \quad \text{and} \quad r > 0, \tag{4.14}$$

for some $\alpha' \in (0, \alpha)$. This means that $\|u/x_n^s\|_{C^{\alpha'}(B_{1/2}^+)} \leq CM$, as desired. $\square$

5. Liouville type theorems

The goal of this section is to prove Theorem 1.5.

First, as a consequence of Proposition 1.1 we obtain the following Liouville-type result involving here the extremal operators (in contrast with Theorem 1.5). Note also that the growth condition CR^β in this lemma holds for $\beta < s + \alpha$ (with α small), whereas we have $\beta < 2s$ in the Liouville Theorem 1.5.

Proposition 5.1. *Let $s_0 \in (0,1)$ and $s \in [s_0, 1)$. Let $\alpha > 0$ be the exponent given by Proposition 1.1. Assume that $u \in C(\mathbb{R}^n)$ is a viscosity solution of*

$$M^+u \geq 0 \quad and \quad M^-u \leq 0 \quad in \quad \{x_n > 0\},$$
$$u = 0 \quad in \quad \{x_n < 0\}.$$

Assume that, for some positive $\beta < s + \alpha$, u satisfies the growth control at infinity

$$\|u\|_{L^\infty(B_R)} \leq CR^\beta \quad for\ all\ R \geq 1. \tag{5.1}$$

Then,
$$u(x) = K(x_n)_+^s$$
for some constant $K \in \mathbb{R}$.

Proof. Given $\rho \geq 1$, let $v_\rho(x) = \rho^{-\beta} u(\rho x)$. Note that for all $\rho \geq 1$ the function v_ρ satisfies the same growth control (5.1) as u. Indeed,
$$\|v_\rho\|_{L^\infty(B_R)} = \rho^{-\beta}\|u\|_{L^\infty(B_{\rho R})} \leq \rho^{-\beta} C(\rho R)^\beta = CR^\beta.$$
In particular $\|v_\rho\|_{L^\infty(B_1)} \leq C$ and $\|v_\rho\|_{L^1(\mathbb{R}^n, \omega_s)} \leq C$, with C independent of ρ. Hence, the function $\tilde{v}_\rho = v_\rho \chi_{B_1}$ satisfies $M^+\tilde{v}_\rho \geq -C$ and $M^-\tilde{v}_\rho \leq C$ in $B_{1/2} \cap \{x_n > 0\}$, and $\tilde{v}_\rho = 0$ in $\{x_n < 0\}$. Also, $\|\tilde{v}_\rho\|_{L^\infty(B_{1/2})} \leq C$. Therefore, by Proposition 1.1 we obtain that
$$\left\| v_\rho/x_n^s \right\|_{C^\alpha(B_{1/4}^+)} = \left\| \tilde{v}_\rho/x_n^s \right\|_{C^\alpha(B_{1/4}^+)} \leq C.$$
Scaling this estimate back to u we obtain
$$\left[u/x_n^s \right]_{C^\alpha(B_{\rho/4}^+)} = \rho^{-\alpha} \left[u(\rho x)/(\rho x_n)^s \right]_{C^\alpha(B_{1/4}^+)} = \rho^{\beta-s-\alpha} \left[v_\rho/(x_n)^s \right]_{C^\alpha(B_{1/4}^+)} \leq C\rho^{\beta-s-\alpha}.$$
Using that $\beta < s + \alpha$ and letting $\rho \to \infty$ we obtain
$$\left[u/x_n^s \right]_{C^\alpha(\mathbb{R}^n \cap \{x_n > 0\})} = 0,$$
which means $u = K(x_n)_+^s$. $\square$

The previous proposition will be applied to tangential derivatives of a solution to $Iu = 0$ as in the situation of Theorem 1.5. It gives that u is in fact a function of x_n alone. To proceed, we need the following crucial lemma It is a Liouville-type result for the fractional Laplacian in dimension 1, and classifies all functions which are s-harmonic in $\mathbb{R}_+$, vanish in $\mathbb{R}_-$, and grow at infinity less than $|x|^\beta$ for some $\beta < 2s$.

Lemma 5.2. *Let u satisfy $(-\Delta)^s u = 0$ in $\mathbb{R}_+$ and $u = 0$ in $\mathbb{R}_-$. Assume that, for some $\beta \in (0, 2s)$, u satisfies the growth control $\|u\|_{L^\infty(0,R)} \leq CR^\beta$ for all $R \geq 1$. Then $u(x) = K(x_+)^s$.*

To establish the lemma, we will need the following result. It classifies all homogeneous solutions (with no growth condition) that vanish in a half line of the extension problem of Caffarelli and Silvestre [17] in dimension $1 + 1$.

Lemma 5.3. *Let $s \in (0, 1)$. Let (x, y) denote a point in $\mathbb{R}^2$, and $r > 0$, $\theta \in (-\pi, \pi)$ be polar coordinates defined by the relations $x = r\cos\theta$, $y = r\sin\theta$. Assume that $\nu > -s$, and $q_\nu = r^{s+\nu}\Theta_\nu(\theta)$ is even with respect y (or equivalently with respect to θ) and solves*
$$\begin{cases} \text{div}\,(|y|^{1-2s}\nabla q_\nu) = 0 & \text{in } \{y \neq 0\} \\ \lim_{y\to 0} |y|^{1-2s}\partial_y q_\nu = 0 & \text{on } \{y = 0\} \cap \{x > 0\} \\ q_\nu = 0 & \text{on } \{y = 0\} \cap \{x < 0\}. \end{cases} \tag{5.2}$$
Then,

(a) ν *belongs to* $\mathbb{N} \cup \{0\}$ *and*

$$\Theta_\nu(\theta) = K \, |\sin \theta|^s \, P_\nu^s(\cos \theta),$$

   where P_ν^μ *is the associated Legendre function of first kind. Equivalently,*

$$\Theta_\nu(\theta) = C \left|\cos\left(\frac{\theta}{2}\right)\right|^{2s} {}_2F_1\left(-\nu, \nu+1; 1-s; \frac{1-\cos\theta}{2}\right),$$

   where ${}_2F_1$ *is the hypergeometric function.*
(b) *The functions* $\{\Theta_\nu\}_{\nu \in \mathbb{N} \cup \{0\}}$ *are a complete orthogonal system in the subspace of even functions of the weighted space* $L^2\big((-\pi, \pi), |\sin\theta|^{1-2s}\big)$.

Proof. We differ the proof to the Appendix.   □

We can now give the

Proof of Lemma 5.2. Let

$$P_s(x,y) = \frac{p_{1,s}}{y} \frac{1}{\left(1 + (x/y)^2\right)^{\frac{1+2s}{2}}}$$

be the Poisson kernel for the extension problem of Caffarelli and Silvestre; see [17, 9].
   Given the growth control $u(x) \le C|x|^\beta$ at infinity and $\beta < 2s$, the convolution

$$v(\,\cdot\,, y) = u * P_s(\,\cdot\,, y)$$

is well defined and is a solution of the extension problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } \{y > 0\} \\ v(x,0) = u(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Since $(-\Delta)^s u = 0$ in $\{x > 0\}$ and $u = 0$ in $\{x < 0\}$, the function v satisfies

$$\lim_{y \searrow 0} y^{1-2s}\partial_y v(x,y) = 0 \quad \text{for } x > 0 \quad \text{and} \quad v(x,0) = 0 \quad \text{for } x < 0.$$

Hence, v solves (5.2).
   Let Θ_ν, $\nu \in \mathbb{N} \cup \{0\}$, be as in Lemma 5.3. Recall that $r^{s+\nu}\Theta_\nu(\theta)$ also solve (5.2). By standard separation of variables, in every ball $B_R^+(0)$ of $\mathbb{R}^2$ the function v can be written as a series

$$v(x,y) = v(r\cos\theta, r\sin\theta) = \sum_{\nu=0}^{\infty} a_\nu r^{s+\nu}\Theta_\nu(\theta). \tag{5.3}$$

To obtain this expansion we use that, by Lemma 5.3 (b), the functions $\{\Theta_\nu\}_{\nu \in \mathbb{N} \cup \{0\}}$ are a complete orthogonal system in the subspace of even functions in the weighted space $L^2\big((-\pi, \pi), |\sin\theta|^{1-2s}\big)$, and hence are complete in $L^2\big((0, \pi), (\sin\theta)^{1-2s}\big)$.
   Moreover, by uniqueness, the coefficients a_ν are independent of R and hence the series (5.3) provides a representation formula for $v(x,y)$ in the whole $\{y > 0\}$.

Now, we claim that the growth control $\|u\|_{L^\infty(-R,R)} \leq CR^\beta$ with $\beta \in (0, 2s)$ is transferred to v (perhaps with a bigger constant C), that is,

$$\|v\|_{L^\infty(B_R^+)} \leq CR^\beta.$$

To see this, consider the rescaled function $u_R(x) = R^{-\beta}u(Rx)$, which satisfy the same growth control of u. Then,

$$v_R = R^{-\beta}v(R\,\cdot\,) = u_R * P_s.$$

Since the growth control for u_R is independent of R we find a bound for $\|v_R\|_{L^\infty(B_1^+)}$ that is independent of R, and this means that v is controlled by CR^β in B_R^+, as claimed.

Next, since we may assume that $\int_0^\pi |\Theta_\nu(\theta)|^2|\sin\theta|^a\, d\theta = 1$ for all $\nu \geq 0$, Parseval's identity yields

$$\int_{\partial^+ B_R} |v(x,y)|^2 y^a\, d\sigma = \sum_{\nu=0}^\infty |a_\nu|^2 R^{2s+2\nu+1+a},$$

where $\partial^+ B_R = \partial B_R \cap \{y > 0\}$. But by the growth control, we have

$$\int_{\partial^+ B_R} |v(x,y)|^2 y^a\, d\sigma \leq CR^{2\beta}\int_{\partial^+ B_R} y^a\, d\sigma = CR^{2\beta+1+a}.$$

Finally, since $2\beta < 4s < 2s + 2$, this implies $a_\nu = 0$ for all $\nu \geq 1$, and hence $u(x) = K(x_+)^s$, as desired. $\qquad\square$

The following basic Hölder estimate up to the boundary follows from [15, Section 3]. It is also a consequence of Lemma 6.4, which we prove in Section 6.

Lemma 5.4 ([15])**.** *Let $s_0 \in (0,1)$ and $s \in [s_0, 1]$. Let u be a solution of $M^+u \geq 0$ and $M^-u \leq 0$ in B_1^+, $u = 0$ in B_1^- and assume that $u \in L^1(\mathbb{R}^n, \omega_s)$. Then, for some $\alpha > 0$ it is $u \in C^\alpha\left(\overline{B}_{1/2}\right)$ and*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C\big(\|u\|_{L^\infty(B_1)} + \|u\|_{L^1(\mathbb{R}^n, \omega_s)}\big).$$

The constants α and C depend only on n, s_0, and ellipticity constants.

To end this section, we finally prove Theorem 1.5.

Proof of Theorem 1.5. Note that, since $\beta < 2s$, the growth control (1.11) yields $u \in L^1(\mathbb{R}^n, \omega_s)$.

Given $\rho \geq 1$, let $v_\rho = \rho^{-\beta}u(\rho\,\cdot\,)$. As in the proof of Proposition 5.1, v_ρ satisfies the same growth control as u, namely, $\|v_\rho\|_{L^\infty(B_R)} \leq CR^\beta$. Hence,

$$\|v_\rho\|_{L^\infty(B_1)} \leq C \quad \text{and} \quad \|v_\rho\|_{L^1(\mathbb{R}^n, \omega_s)} \leq C.$$

Moreover, since u satisfies $Iu = 0$ in $\{x_n > 0\}$ and $I0 = 0$ we have that $M^+u \geq 0$ and $M^-u \leq 0$ in $\{x_n > 0\}$. This implies that $M^+v_\rho \geq 0$ and $M^-v_\rho \leq 0$ in B_1^+. Then it follows from Lemma 5.4 that

$$\|v_\rho\|_{C^\alpha(B_{1/2})} \leq C.$$

Scaling the previous estimate back to u and setting $\rho = R$, we obtain

$$[u]_{C^\alpha(B_R)} \leq CR^{\beta-\alpha}.$$

Next, given $\tau \in S^{n-1}$ with $\tau_n = 0$ and given $h > 0$, we consider the "tangential" incremental quotients $v^{(1)}(x) = \frac{u(x+h\tau)-u(x)}{h^\alpha}$. We have shown that

$$\|v^{(1)}\|_{L^\infty(B_R)} \leq CR^{\beta-\alpha}.$$

Moreover, since I is translation invariant, $v^{(1)}$ satisfies $M^+v^{(1)} \geq 0$ and $M^-v^{(1)} \leq 0$ in $\{x_n > 0\}$. Hence, we can apply again the previous scaling argument to $v^{(1)}$ and obtain

$$[v^{(1)}]_{C^\alpha(B_R)} \leq CR^{\beta-2\alpha} \quad \text{for all } R \geq 1.$$

Thus, we have a new growth control for $v^{(2)}(x) = \frac{u(x+h\tau)-u(x)}{h^{2\alpha}}$. We can keep iterating in this way until we obtain (after a finite number N of iterations)

$$\left\| \frac{u(x+h\tau)-u(x)}{h} \right\|_{L^\infty(B_R)} \leq CR^{\beta-1}. \tag{5.4}$$

Now, $v^{(N)} = \frac{u(x+h\tau)-u(x)}{h}$ satisfies $M^+v^{(N)} \geq 0$, $M^-v^{(N)} \leq 0$ in $\{x_n > 0\}$ and $v^{(N)} = 0$ in $\{x_n < 0\}$. Moreover, $v^{(N)}$ satisfies the growth control (5.4) with exponent $\beta - 1 < 2s - 1 < s$. Hence, using Proposition 5.1 we conclude that $v^{(N)} \equiv 0$. Therefore, $u(x+h\tau) = u(x)$ for all $h > 0$ and for all unit vector τ with $\tau_n = 0$. This means that u depends only on the variable x_n. That is, $u(x) = w(x_n)$ for some function $w : \mathbb{R} \longrightarrow \mathbb{R}$.

Now, if $\tilde{u}$ is a test function of the form $\tilde{u}(x) = \tilde{w}(x_n)$, Lemma 2.1 yields

$$\begin{aligned}
M^+\tilde{u}(x) &= \sup_{L\in\mathcal{L}_*} L\tilde{u} \\
&= \sup_{\lambda\leq a\leq\Lambda} \frac{1-s}{2c_{1,s}} \left(\int_{S^{n-1}} |\theta_n|^{2s} a(\theta)\, d\theta \right) (-\Delta)^s_\mathbb{R} \tilde{w}(x_n) \tag{5.5} \\
&= C\left\{ \Lambda\big(-(-\Delta)^s_\mathbb{R}\tilde{w}(x_n)\big)^+ - \lambda\big(-(-\Delta)^s_\mathbb{R}\tilde{w}(x_n)\big)^- \right\}.
\end{aligned}$$

Similarly,

$$M^-\tilde{u}(x) = C\left\{ \lambda\big(-(-\Delta)^s\tilde{w}(x_n)\big)^+ - \Lambda\big(-(-\Delta)^s\tilde{w}(x_n)\big)^- \right\}. \tag{5.6}$$

Finally, recall that u solves $Iu = 0$ in $\mathbb{R}^n_+$, and $I0 = 0$. In particular we have $M^+u \geq 0$ and $M^-u \leq 0$ in $\mathbb{R}^n_+$ in the viscosity sense. Note that, since $u(x) = w(x_n)$, then we may test the viscosity inequalities using only test functions of the type $\tilde{u}(x) = \tilde{w}(x_n)$. Hence, using (5.5) and (5.6) we deduce that w is a viscosity solution of $(-\Delta)^s w = 0$ in $\mathbb{R}_+$ and $w = 0$ in $\mathbb{R}_-$. Clearly, w satisfies the growth control $\|w\|_{L^\infty(0,R)} \leq CR^\beta$. Therefore we deduce, using Lemma 5.2, that $u(x) = w(x_n) = K(x_n^+)^s$. $\qquad\square$

6. Regularity by compactness

In this section we prove the main result of the paper: the boundary regularity in $C^{1,1}$ domains for fully nonlinear elliptic equations with respect to the class $\mathcal{L}_*$, given by Theorem 1.3.

As explained in the Introduction, the following result is the main ingredient in the proof of Theorem 1.3.

Proposition 6.1. *Let $s_0 \in (0,1)$, $\delta \in (0, s_0/4)$, $\rho_0 > 0$, and $\beta = 2s_0 - \delta$ be given constants.*

Let Γ be a $C^{1,1}$ hypersurface with radius ρ_0 splitting B_1 into Ω^+ and Ω^-; see Definition 1.2.

Let $s \in [s_0, \max\{1, s_0 + \delta\}]$ and $f \in C(\overline{\Omega^+})$. Assume that $u \in C(\overline{B_1}) \cap L^\infty(\mathbb{R}^n)$ is a solution of $\mathrm{I}u = f$ in Ω^+ and $u = 0$ in Ω^-, where I is a fully nonlinear translation invariant operator elliptic with respect to $\mathcal{L}_*(s)$.

Then, for all $z \in \Gamma \cap \overline{B_{1/2}}$ there is a constant $Q(z)$ with $|Q(z)| \le CC_0$ for which

$$\left| u(x) - Q(z)\big((x-z)\cdot\nu(z)\big)_+^s \right| \le CC_0 |x-z|^\beta \quad \text{for all } x \in B_1,$$

where $\nu(z)$ is the unit normal vector to Γ at z pointing towards Ω^+ and

$$C_0 = \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega^+)}.$$

The constant C depends only on n, ρ_0, s_0, δ, and ellipticity constants.

The proof of Proposition 6.1 is by contradiction, using a blow up and compactness argument. In order to fix ideas, we prove first the following reduced version of the statement.

> Let $u \in C(\overline{B_1}) \cap L^\infty(\mathbb{R}^n)$ be a viscosity solution of $\mathrm{I}u = 0$ in B_1^+ and $u = 0$ in B_1^-. Then, given $\beta \in (s, 2s)$, there are $Q \in \mathbb{R}$ and $C > 0$ such that
>
> $$\left| u(x) - Q(x_n)_+^s \right| \le C|x|^\beta \quad \text{for all } x \in B_1. \tag{6.1}$$
>
> The constant C is independent of x, but it could depend on everything else, also on u.

We next prove (6.1) by contradiction. If (6.1) were false then it would be (by the contraposition of Lemma 6.2 below)

$$\sup_{r>0} r^{-\beta} \left\| u - Q_*(r)(x_n)_+^s \right\|_{L^\infty(B_r)} = +\infty,$$

where

$$Q_*(r) := \arg\min_{Q \in \mathbb{R}} \int_{B_r} \big(u(x) - Q(x_n)_+^s \big)^2 \, dx = \frac{\int_{B_r} u(x)\,(x_n)_+^s \, dx}{\int_{B_r} (x_n)_+^{2s} \, dx}. \tag{6.2}$$

Then, a useful trick is to define the monotone in r quantity

$$\theta(r) = \sup_{r'>r}(r')^{-\beta} \max\left\{ \left\| u - Q_*(r')(x_n)_+^s \right\|_{L^\infty(B_{r'})}, \; (r')^s\big|Q_*(2r') - Q_*(r')\big| \right\},$$

which satisfies $\theta(r) \nearrow \infty$ as $r \searrow 0$. Then, there is a sequence $r_m \searrow 0$ such that

$$(r_m)^{-\beta} \max \left\{ \left\| u - Q_*(r_m)(x_n)_+^s \right\|_{L^\infty(B_{r_m})}, (r_m)^s \left| Q_*(2r_m) - Q_*(r_m) \right| \right\} \geq \frac{\theta(r_m)}{2}.$$

$$(6.3)$$

We then consider the blow up sequence

$$v_m(x) = \frac{u(r_m x) - (r_m)^s Q_*(r_m)(x_n)_+^s}{(r_m)^\beta \theta(r_m)}.$$

Note that (6.3) is equivalent to

$$\max \left\{ \|v_m\|_{L^\infty(B_1)}, \left| \frac{\int_{B_2} v_m(x)(x_n)_+^s \, dx}{\int_{B_2} (x_n)_+^{2s} \, dx} - \frac{\int_{B_1} v_m(x)(x_n)_+^s \, dx}{\int_{B_1} (x_n)_+^{2s} \, dx} \right| \right\} \geq 1/2.$$

$$(6.4)$$

Also, by definition of $Q_*(r_m)$, we have

$$\int_{B_1} v_m(x)(x_n)_+^s \, dx = 0,$$

$$(6.5)$$

which is the optimality condition of "least squares".

In addition, by definition of θ, we have

$$\frac{(r')^{s-\beta} |Q_*(2r') - Q_*(r')|}{\theta(r)} \leq 1 \quad \text{for all } r' \geq r.$$

Thus, for $R = 2^N$ we have

$$\frac{r^{s-\beta} |Q_*(rR) - Q_*(r)|}{\theta(r)} \leq \sum_{j=0}^{N-1} 2^{j(\beta-s)} \frac{(2^j r)^{s-\beta} |Q_*(2^{j+1}r) - Q_*(2^j r)|}{\theta(r)}$$

$$\leq \sum_{j=0}^{N-1} 2^{j(\beta-s)} \leq C 2^{N(\beta-s)} = C R^{\beta-s}.$$

Moreover, v_m satisfy the growth control

$$\|v_m\|_{L^\infty(B_R)} = \frac{1}{\theta(r_m)(r_m)^\beta} \left\| u - Q_*(r_m)(x_n)_+^s \right\|_{L^\infty(B_{r_m R})}$$

$$\leq \frac{R^\beta}{\theta(r_m)(r_m R)^\beta} \left\| u - Q_*(r_m R)(x_n)_+^s \right\|_{L^\infty(B_{r_m R})} +$$

$$+ \frac{1}{\theta(r_m)(r_m)^\beta} |Q_*(r_m R) - Q_*(r_m)| (r_m R)^s$$

$$(6.6)$$

$$\leq \frac{R^\beta \theta(r_m R)}{\theta(r_m)} + C R^\beta$$

$$\leq C R^\beta,$$

for all $R \geq 1$, where we have used the definition θ and its monotonicity.

In addition, since $M^+(x_n)^s_+ = M^-(x_n)^s_+ = 0$ in $\{x_n > 0\}$, and $Iu = 0$ in B_1^+, we obtain that

$$\tilde{I}_m v_m = 0 \qquad \text{in } B_{1/r_m}^+,$$

for some $\tilde{I}_m$ translation invariant and elliptic with respect to $\mathcal{L}_*$. It follows, using the basic C^α estimate up to the boundary of Lemma 5.4 that (up to taking a subsequence)

$$v_m \longrightarrow v \quad \text{locally uniformly in } \mathbb{R}^n.$$

Moreover, since all the v_m's satisfy the growth control (6.19), and $\beta < 2s$, by the dominated convergence theorem we obtain that

$$\int_{\mathbb{R}^n} \big|v_m - v\big|(x)\,\omega_s(x)\,dx \to 0.$$

Also, by Theorem 42 in [15] a subsequence of $\tilde{I}_m$ converges weakly to some translation invariant operator $\tilde{I}$ elliptic with respect to $\mathcal{L}_*$. Hence, the stability result in [15] yields

$$\tilde{I}v = 0 \quad \text{in } \{x_n > 0\} \quad \text{and} \quad v = 0 \quad \text{in } \{x_n < 0\}.$$

Furthermore, passing to the limit the growth control (6.19) we obtain $\|v\|_{L^\infty(B_R)} \leq R^\beta$ for all $R \geq 1$. Thus, the Liouville type Theorem 1.5 implies

$$v(x) = K(x_n)^s_+.$$

Passing (6.5) to the limit (using uniform convergence) we find

$$\int_{B_1} v(x)(x_n)^s_+\,dx = 0.$$

But passing (6.4) to the limit, we obtain a contradiction. $\qquad\square$

To prove Proposition 6.1 we will need a more involved version of this argument, but the main idea is essentially contained in the previous reduced version. Before proving Proposition 6.1, let us give some preliminary results.

The following lemma is for general continuous functions u, not necessarily solutions to some equation.

Lemma 6.2. *Let $\beta > s$ and $\nu \in S^{n-1}$ be some unit vector. Let $u \in C(B_1)$ and define*

$$\phi_r(x) := Q_*(r)\,(x \cdot \nu)^s_+, \tag{6.7}$$

where

$$Q_*(r) := \arg\min_{Q \in \mathbb{R}} \int_{B_r} \big(u(x) - Q(x \cdot \nu)^s_+\big)^2\,dx = \frac{\int_{B_r} u(x)\,(x \cdot \nu)^s_+\,dx}{\int_{B_r} (x \cdot \nu)^{2s}_+\,dx}.$$

Assume that for all $r \in (0,1)$ we have

$$\big\|u - \phi_r\big\|_{L^\infty(B_r)} \leq C_0 r^\beta. \tag{6.8}$$

Then, there is $Q \in \mathbb{R}$ satisfying $|Q| \le C\big(C_0 + \|u\|_{L^\infty(B_1)}\big)$ such that

$$\big\|u - Q(x \cdot \nu)_+^s\big\|_{L^\infty(B_r)} \le CC_0 r^\beta$$

for some constant C depending only on β and s.

Proof. We may assume $\|u\|_{L^\infty(B_1)} = 1$. By (6.8), for all $x' \in B_r$ we have

$$\big|\phi_{2r}(x') - \phi_r(x')\big| \le \big|u(x') - \phi_{2r}(x')\big| + \big|u(x') - \phi_r(x')\big| \le CC_0 r^\beta.$$

But this happening for every $x' \in B_r$ yields, recalling (6.7),

$$\big|Q_*(2r) - Q_*(r)\big| \le CC_0 r^{\beta-s}.$$

In addition, since $\|u\|_{L^\infty(B_1)} = 1$, we clearly have that

$$|Q_*(1)| \le C. \tag{6.9}$$

Since $\beta > s$, this implies the existence of the limit

$$Q := \lim_{r \searrow 0} Q_*(r).$$

Moreover, using again $\beta - s > 0$,

$$\big|Q - Q_*(r)\big| \le \sum_{m=0}^\infty \big|Q_*(2^{-m}r) - Q_*(2^{-m-1}r)\big| \le \sum_{m=0}^\infty CC_0 2^{-m(\beta-s)} r^{\beta-s} \le CC_0 r^{\beta-s}.$$

In particular, using (6.9) we obtain

$$|Q| \le C(C_0 + 1). \tag{6.10}$$

We have thus proven that for all $r \in (0, 1)$

$$\begin{aligned}
\big\|u - Q(x \cdot \nu)_+^s\big\|_{L^\infty(B_r)} &\le \|u - Q_*(r)(x \cdot \nu)_+^s\|_{L^\infty(B_r)} + \\
&\quad + \|Q_*(r)(x \cdot \nu)_+^s - Q(x \cdot \nu)_+^s\|_{L^\infty(B_r)} \\
&\le C_0 r^\beta + |Q_*(r) - Q| r^s \le C(C_0 + 1) r^\beta.
\end{aligned}$$

$\square$

The following lemma will be used in the proof of Theorem 1.3 to obtain compactness for sequences of elliptic operators of variable order. Its proof is almost the same as the proof of Lemma 3.1 of [48].

Lemma 6.3. *Let $s_0 \in (0, 1)$, $s_m \in [s_0, 1]$, and I_m such that*

- *I_m is a fully nonlinear translation invariant operator elliptic with respect to $\mathcal{L}_*(s_m)$.*
- *$\mathrm{I}_m 0 = 0$.*

Then, a subsequence of $s_m \to s \in [s_0, 1]$ and a subsequence of I_m converges weakly (with the weight ω_{s_0}) to some fully nonlinear translation invariant operator I elliptic with respect to $\mathcal{L}_*(s)$.

Proof. We may assume by taking a subsequence that $s_m \to s \in [s_0, 1]$. Consider the class $\mathcal{L} = \bigcup_{s \in [s_0,1]} \mathcal{L}_*(s)$. This class satisfies Assuptions 23 and 24 of [15]. Also, each I_m is elliptic with respect to $\mathcal{L}$. Hence using Theorem 42 in [15] there is a subsequence of I_m converging weakly (with the weight ω_{s_0}) to a translation invariant operator I, also elliptic with respect to $\mathcal{L}$. Let us see next that I is in fact elliptic with respect to $\mathcal{L}_*(s) \subset \mathcal{L}$. Indeed, for test functions u and v that are quadratic polynomials in a neighborhood of x and that belong to $L^1(\mathbb{R}^n, \omega_{s_0})$, the inequalities

$$M_{s_m}^- v(x) \leq I_m(u + v)(x) - I_m u(x) \leq M_{s_m}^+ v(x)$$

pass to the limit to obtain

$$M_s^- v(x) \leq I(u + v)(x) - Iu(x) \leq M_s^+ v(x).$$

$\square$

The following lemma will be used to obtain a C^γ estimate up to the boundary for solutions to fully nonlinear integro-differential equations. This estimate will be useful in the proof of Proposition 6.1. It is essentially a consequence of the proof of Theorem 3.3 in [15]. Note that, in contrast with Proposition 6.1, in this lemma the assumption of regularity of the domain is only "from the exterior". Namely, we only assume that the exterior ball condition is satisfied.

Lemma 6.4. *Assume that B_1 is divided into two disjoint subdomains Ω_1 and Ω_2 such that $\overline{B_1} = \overline{\Omega_1} \cup \overline{\Omega_2}$. Assume that $\Gamma := \partial\Omega_1 \setminus \partial B_1 = \partial\Omega_2 \setminus \partial B_1$ is a $C^{0,1}$ surface and that $0 \in \Gamma$. Moreover assume that, for some $\rho_0 > 0$, all the points on $\Gamma \cap \overline{B_{3/4}}$ can be touched by a ball of radius $\rho_0 \in (0, 1/4)$ contained in Ω_2.*

Let $s_0 \in (0, 1)$ and $s \in [s_0, 1]$. Let $\alpha \in (0, 1)$, $g \in C^\alpha(\overline{\Omega_2})$, and $u \in C(\overline{B_1}) \cap L^1(\mathbb{R}^n, \omega_s)$ satisfy in the viscosity sense

$$M^+ u \geq -C_0 \quad \text{and} \quad M^- u \leq C_0 \quad \text{in } \Omega_1, \quad u = g \quad \text{in } \Omega_2.$$

Then, there is $\gamma \in (0, \alpha)$ such that $u \in C^\gamma(\overline{B_{1/2}})$ with the estimate

$$\|u\|_{C^\gamma(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|g\|_{C^\alpha(\Omega_2)} + \|u\|_{L^1(\mathbb{R}^n, \omega_s)} + C_0).$$

The constants C and γ depend only on n, s_0, α, ρ_0, and ellipticity constants.

Proof. Let $\tilde{u} = u\chi_{B_1}$. Then $\tilde{u}$ satisfies $M^+ \tilde{u} \geq -C_0'$ and $M^- \tilde{u} \leq C_0'$ in $\Omega_1 \cap B_{3/4}$ and $\tilde{u} = g$ in Ω_2, where $C_0' \leq C(C_0 + \|u\|_{L^1(\mathbb{R}^n, \omega_s)})$. Here, the constant C depends only on n, s_0, and ellipticity constants.

The proof consists of two steps.

First step. We next prove that there are $\delta > 0$ and C such that for all $z \in \Gamma \cap \overline{B_{1/2}}$ it is

$$\|\tilde{u} - g(z)\|_{L^\infty(B_r(z))} \leq Cr^\delta \quad \text{for all } r \in (0, 1), \tag{6.11}$$

where δ and C depend only on n, s_0, C_0', $\|u\|_{L^\infty(B_1)}$, $\|g\|_{C^\alpha(\Omega_2)}$, and ellipticity constants.

Let $z \in \Gamma \cap \overline{B_{1/2}}$. By assumption, for all $R \in (0, \rho_0)$ there $y_R \in \Omega_2$ such that a ball $B_R(y_R) \subset \Omega_2$ touches Γ at z, i.e., $|z - y_R| = R$.

Let φ_1 and $\epsilon > 0$ be the supersolution and the constant in Lemma 3.3. Take

$$\psi(x) = g(y_R) + \|g\|_{C^\alpha(\Omega_2)}\big((1+\epsilon)R\big)^\alpha + \big(C_0' + \|u\|_{L^\infty(B_1)}\big)\varphi_1\left(\frac{x - y_R}{R}\right).$$

Note that ψ is above $\tilde{u}$ in $\Omega_2 \cap B_{(1+\epsilon)R}$. On the other hand, from the properties of φ_1, it is $M^+\psi \leq -\big(C_0' + \|u\|_{L^\infty(B_1)}\big)R^{-2s} \leq -C_0'$ in the annulus $B_{(1+\epsilon)R}(y_R) \setminus B_R(y_R)$, while $\psi \geq \|u\|_{L^\infty(B_1)} \geq \tilde{u}$ outside $B_{(1+\epsilon)R}(y_R)$. It follows that $\tilde{u} \leq \psi$ and thus we have

$$\tilde{u}(x) - g(z) \leq C\big(R^\alpha + (r/R)^s\big) \quad \text{for all } x \in B_r(z) \quad \text{and for all } r \in (0, \epsilon R) \text{ and } R \in (0, \rho_0).$$

Here, C denotes a constant depending only on n, s_0, C_0', $\|u\|_{L^\infty(B_1)}$, $\|g\|_{C^\alpha(\Omega_2)}$, and ellipticity constants. Taking $R = r^{1/2}$ and repeating the argument up-side down we obtain

$$|\tilde{u}(x) - g(z)| \leq C\big(r^{\alpha/2} + r^{s/2}\big) \leq Cr^\delta \quad \text{for all } x \in B_r(z) \text{ and } r \in (0, \epsilon^{1/2})$$

for $\delta = \frac{1}{2}\min\{\alpha, s_0\}$. Taking a larger constant C, (6.11) follows.

    Second step. We now show that (6.11) and the interior estimates in Theorem 2.5 imply $\|u\|_{C^\gamma(B_{1/2})} \leq C$, where C depends only on the same quantities as above.

    Indeed, given $x_0 \in \Omega_1 \cap B_{1/2}$, let $z \in \Gamma$ and $r > 0$ be such that

$$d = \text{dist}\,(x_0, \Gamma) = \text{dist}\,(x_0, z).$$

Let us consider

$$v(x) = \tilde{u}\left(x_0 + \frac{d}{2}x\right) - g(z).$$

We clearly have

$$\|v\|_{L^\infty(B_1)} \leq C \quad \text{and} \quad \|v\|_{L^1(\mathbb{R}^n, \omega_s)} \leq C.$$

On the other hand, v satisfies

$$M^+v(x) = (d/2)^{2s}M^+\tilde{u}(x_0 + rx) \leq C_0' \quad \text{in } B_1$$

and

$$M^-v(x) = (d/2)^{2s}M^-\tilde{u}(x_0 + rx) \geq -C_0' \quad \text{in } B_1.$$

Therefore, Theorem 2.5 yields

$$\|v\|_{C^\alpha(B_{1/2})} \leq C$$

or equivalently

$$[u]_{C^\alpha(B_{d/4}(x_0))} \leq Cd^{-\alpha}. \tag{6.12}$$

    Combining (6.11) and (6.12), using a similar argument as in the proof of Proposition 1.1, we obtain

$$\|u\|_{C^\gamma(\Omega_1 \cap B_{1/2})} \leq C,$$

as desired. $\square$

    We can now give the

Proof of Proposition 6.1. The proof is by contradiction. Assume that there are sequences Γ_k, Ω_k^+, Ω_k^-, s_k, f_k, u_k, and I_k that satisfy the assumptions of the proposition. That is, for all $k \geq 1$:

- Γ_k is a $C^{1,1}$ hyper surface with radius ρ_0 splitting B_1 into Ω_k^+ and Ω_k^-.
- $s_k \in [s_0, \max\{1, s_0 + \delta\}]$.
- I_k is translation invariant and elliptic with respect to $\mathcal{L}_*(s_k)$.
- $\|u_k\|_{L^\infty(\mathbb{R}^n)} + \|f_k\|_{L^\infty(\Omega_k^+)} = 1$ (by scaling we may assume $C_0 = 1$).
- u_k is a solution of $I_k u_k = f_k$ in Ω_k^+ and $u_k = 0$ in Ω_k^-.

Suppose for a contradiction that the conclusion of the proposition does not hold. That is, for all $C > 0$, there are k and $z \in \Gamma_k \cap \overline{B_{1/2}}$ for which no constant $Q \in \mathbb{R}$ satisfies

$$\left| u_k(x) - Q\big((x - z) \cdot \nu_k(z)\big)_+^{s_k} \right| \leq C|x - z|^\beta \quad \text{for all } x \in B_1. \tag{6.13}$$

Above, $\nu_k(z)$ denotes the unit normal vector to Γ_k at z, pointing towards Ω_k^+.

In particular, noting that $s_k \in [s_0, s_0 + \delta]$ and $\beta \geq s_0 + 2\delta$ by assumption, and using Lemma 6.2, we obtain

$$\sup_k \sup_{z \in \Gamma_k \cap B_{1/2}} \sup_{r > 0} r^{-\beta} \|u_k - \phi_{k,z,r}\|_{L^\infty(B_r(z))} = \infty, \tag{6.14}$$

where

$$\phi_{k,z,r}(x) = Q_{k,z}(r)\big((x - z) \cdot \nu_k(z)\big)_+^{s_k} \tag{6.15}$$

and

$$Q_{k,z}(r) := \arg\min_{Q \in \mathbb{R}} \int_{B_r(z)} \left| u_k(x) - Q\big((x - z) \cdot \nu_k(z)\big)_+^{s_k} \right|^2 dx$$

$$= \frac{\int_{B_r(z)} u_k(x)\big((x - z) \cdot \nu_k(z)\big)_+^{s_k} dx}{\int_{B_r(z)} \big((x - z) \cdot \nu_k(z)\big)_+^{2s_k} dx}.$$

Next define the monotone in r quantity

$$\theta(r) := \sup_k \sup_{z \in \Gamma_k \cap B_{1/2}} \sup_{r' > r} (r')^{-\beta} \max\Bigg\{ \|u_k - \phi_{k,z,r'}\|_{L^\infty(B_{r'}(x_0))},$$

$$(r')^s |Q_{k,z}(2r') - Q_{k,z}(r')| \Bigg\}.$$

We have $\theta(r) < \infty$ for $r > 0$ and $\theta(r) \nearrow \infty$ as $r \searrow 0$. Clearly, there are sequences $r_m \searrow 0$, k_m, and $z_m \to z \in \overline{B_{1/2}}$, for which

$$(r_m)^{-\beta} \max\Bigg\{ \|u_{k_m} - \phi_{k_m,z_m,r_m}\|_{L^\infty(B_{r_m}(x_m))},$$

$$(r_m)^s |Q_{k_m,z_m}(2r_m) - Q_{k_m,z_m}(r_m)| \Bigg\} \geq \theta(r_m)/2. \tag{6.16}$$

From now on in this proof we denote $\phi_m = \phi_{k_m,z_m,r_m}$, $\nu_m = \nu_{k_m}(z_m)$, and $s_m = s_{k_m}$.

In this situation we consider

$$v_m(x) = \frac{u_{k_m}(z_m + r_m x) - \phi_m(z_m + r_m x)}{(r_m)^\beta \theta(r_m)}.$$

Note that, for all $m \geq 1$,

$$\int_{B_1} v_m(x) (x \cdot \nu_m)_+^{s_m} \, dx = 0. \tag{6.17}$$

This is the optimality condition for least squares.

Note also that (6.16) is equivalent to

$$\max \left\{ \|v_m\|_{L^\infty(B_1)}, \left| \frac{\int_{B_2} v_m(x) (x \cdot \nu_m)_+^{s_m} \, dx}{\int_{B_2} (x \cdot \nu_m)_+^{2s_m} \, dx} - \frac{\int_{B_1} v_m(x) (x \cdot \nu_m)_+^{s_m} \, dx}{\int_{B_1} (x \cdot \nu_m)_+^{2s_m} \, dx} \right| \right\} \geq 1/2, \tag{6.18}$$

which holds for all $m \geq 1$.

In addition, by definition of θ, for all k and z we have

$$\frac{(r')^{s-\beta} |Q_{k,z}(2r') - Q_{k,z}(r')|}{\theta(r)} \leq 1 \quad \text{for all } r' \geq r > 0.$$

Thus, for $R = 2^N$ we have

$$\frac{r^{s_k-\beta} |Q_{k,z}(rR) - Q_{k,z}(r)|}{\theta(r)} \leq \sum_{j=0}^{N-1} 2^{j(\beta - s_k)} \frac{(2^j r)^{s_k - \beta} |Q_{k,z}(2^{j+1}r) - Q_{k,z}(2^j r)|}{\theta(r)}$$

$$\leq \sum_{j=0}^{N-1} 2^{j(\beta - s_k)} \leq C 2^{N(\beta - s_k)} = C R^{\beta - s_k},$$

where we have used $\beta - s_k \geq \delta$.

Moreover, we have

$$\|v_m\|_{L^\infty(B_R)} = \frac{1}{\theta(r_m)(r_m)^\beta} \left\| u_{k_m} - Q_{k_m, z_m}(r_m) \big((x - z_m) \cdot \nu_m\big)_+^{s_m} \right\|_{L^\infty(B_{r_m R})}$$

$$\leq \frac{R^\beta}{\theta(r_m)(r_m R)^\beta} \left\| u_{k_m} - Q_{k_m, z_m}(r_m R) \big((x - z_m) \cdot \nu_m\big)_+^{s_m} \right\|_{L^\infty(B_{r_m R})} +$$

$$+ \frac{1}{\theta(r_m)(r_m)^\beta} |Q_{k_m, z_m}(r_m R) - Q_{k_m, z_m}(r_m)| \, (r_m R)^{s_m}$$

$$\leq \frac{R^\beta \theta(r_m R)}{\theta(r_m)} + C R^\beta,$$

and hence v_m satisfy the growth control

$$\|v_m\|_{L^\infty(B_R)} \leq C R^\beta \quad \text{for all } R \geq 1. \tag{6.19}$$

We have used the definition $\theta(r)$ and its monotonicity.

Now, without loss of generality (taking a subsequence), we assume that

$$\nu_m \longrightarrow \nu \in S^{n-1}.$$

Then, the rest of the proof consists mainly in showing the following Claim.

Claim. *A subsequence of v_m converges locally uniformly in $\mathbb{R}^n$ to some function v which satisfies $\tilde{I}v = 0$ in $\{x \cdot \nu > 0\}$ and $v = 0$ in $\{x \cdot \nu < 0\}$, for some $\tilde{I}$ translation invariant and elliptic with respect to $\mathcal{L}_*$.*

Once we know this, a contradiction is immediately reached using the Liouville type Theorem 1.5, as seen at the end of the proof.

To prove the Claim, given $R \geq 1$ and m such that $r_m R < 1/2$ define

$$\Omega^+_{R,m} = \left\{x \in B_R \ : \ (z_m + r_m x) \in \Omega^+_{k_m} \quad \text{and} \quad x \cdot \nu_m(z_m) > 0\right\}.$$

Notice that for all R and k, the origin 0 belongs to the boundary of $\Omega^+_{R,m}$.

We will use that v_m satisfies an elliptic equation in $\Omega^+_{R,m}$. Namely,

$$\tilde{I}_m v_m(x) = \frac{(r_m)^{2s_m}}{(r_m)^\beta \theta(r_m)} f_{k_m}(z_m + r_m x) \quad \text{in } \Omega^+_{R,m}. \tag{6.20}$$

where $\tilde{I}_m$ is defined by

$$\tilde{I}_m \left(\frac{w(z_m + r \cdot) - \phi_m(z_m + r \cdot)}{(r_m)^\beta \theta(r_m)}\right)(x) = \frac{(r_m)^{2s_m}}{(r_m)^\beta \theta(r_m)} \left(I_{k_m} w\right)(z_m + rx),$$

for all test function w. Equivalently, for all test function v,

$$\tilde{I}_m v(x) : \stackrel{(*)}{=} \frac{(r_m)^{2s_m}}{(r_m)^\beta \theta(r_m)} I_{k_m} \left((r_m)^\beta \theta(r_m) \ v\left(\frac{\cdot - z_m}{r}\right) + \phi_m(\cdot)\right)(z_m + r_m x)$$

$$\stackrel{(**)}{=} \frac{(r_m)^{2s_m}}{(r_m)^\beta \theta(r_m)} I_{k_m} \left((r_m)^\beta \theta(r_m) \ v\left(\frac{\cdot - z_m}{r_m}\right)\right)(z_m + r_m x),$$

the last identity being valid only in $\{x \cdot \nu_m > 0\}$ since $M^+ \phi_m = M^- \phi_m = 0$ in $\{(x - z) \cdot \nu_m > 0\}$.

Note that the right hand side of (6.20) converges uniformly to 0 as $r_m \searrow 0$, since $\beta = 2s_0 - \delta < 2s_m$ and $\theta(r_m) \nearrow \infty$.

Using that I_{k_m} is translation invariant and elliptic with respect to $\mathcal{L}_*(s_m)$ and that $I_{k_m} 0 = 0$ we readily show that $\tilde{I}_m$ is also elliptic with respect to $\mathcal{L}_*(s_m)$ (i.e., with the same ellipticity constants Λ and λ, which are always fixed). Also, since the domains $\Omega^+_{R,m}$ are always contained in $\{(x - z_m) \cdot \nu_m > 0\}$ we may define $\tilde{I}_m$ by $(**)$, and hence it is a translation invariant operator.

In order to prove the convergence of a subsequence of v_m we first obtain, for every fixed $R \geq 1$, a uniform in m bound for $\|v_m\|_{C^\delta(B_R)}$, for some small $\delta > 0$. Then the local uniform convergence of a subsequence of v_m follows from the Arzelà-Ascoli theorem. Let us fix $R \geq 1$ and consider that m is always large enough so that $r_m R < 1/4$.

Let Σ^-_m be the half space which is "tangent" to $\Omega^-_{k_m}$ at z_m, namely,

$$\Sigma^-_m := \left\{(x - z_m) \cdot \nu(z_m) < 0\right\}.$$

The first step is showing that, for all m and for all $r < 1/4$,

$$\left\| u_{k_m} - \phi_m \right\|_{L^\infty \left(B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-) \right)} \leq C r^{2s_m} \leq C r^{2s_0} \tag{6.21}$$

for some constant C depending only on s_0, ρ_0, ellipticity constants, and dimension.

Indeed, we may rescale and slide the supersolution φ_1 from Lemma 3.3 and use the fact that all points of $\Gamma_{k_m} \cap B_{3/4}$ can be touched by balls of radius ρ_0 contained in $\Omega_{k_m}^-$. We obtain that

$$|u_{k_m}| \leq C \big(\text{dist} \, (x, \Omega_{k_m}^-) \big)^{s_m},$$

with C depending only on n, s_0, ρ_0, and ellipticity constants. On the other hand, by definition of ϕ_m we have

$$|\phi_m| \leq C \big(\text{dist} \, (x, \Sigma_m^-) \big)^{s_m}.$$

But by assumption, points on $\Gamma_k \cap B_{3/4}$ can be also touched by balls of radius ρ_0 from the $\Omega_{k_m}^+$ side, and hence we have a quadratic control (depending only on ρ_0) on on how Γ_{k_m} separates from the hyperplane $\partial \Sigma_m^-$. As a consequence, in $B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-)$ we have

$$C \big(\text{dist} \, (x, \Omega_{k_m}^-) \big)^{s_m} \leq C r^{2s_m} \quad \text{and} \quad C \big(\text{dist} \, (x, \Sigma_m^-) \big)^{s_m} \leq C r^{2s_m}.$$

Hence, (6.21) holds.

We use now Lemma 6.4 to obtain that, for some small $\gamma \in (0, s_0)$,

$$\| u_{k_m} \|_{C^\gamma(B_{1/8}(z_m))} \leq C \quad \text{for all } m.$$

On the other hand, clearly

$$\| \phi_m \|_{C^\gamma(B_{1/8}(z_m))} \leq C \quad \text{for all } m.$$

Hence,

$$\left\| u_{k_m} - \phi_m \right\|_{C^\gamma \left(B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-) \right)} \leq C. \tag{6.22}$$

Next, interpolating (6.21) and (6.22) we obtain, for some positive $\delta < \gamma$ small enough (depending on γ, s_0, and δ),

$$\left\| u_{k_m} - \phi_m \right\|_{C^\delta \left(B_r(z_m) \cap (\Omega_{k_m}^- \cup \Sigma_m^-) \right)} \leq C r^{2s_0 - \delta} = C r^\beta. \tag{6.23}$$

Therefore, scaling (6.23) we find that

$$\left\| v_m \right\|_{C^\delta \left(B_R \setminus \Omega_{R,m}^+ \right)} \leq C \quad \text{for all } m \text{ with } r_m R < 1/4. \tag{6.24}$$

Next we observe that the boundary points on $\partial \Omega_{R,m}^+ \cap B_{3R/4}$ can be touched by balls of radius $(\rho_0/r_m) \geq \rho_0$ contained in $B_R \setminus \Omega_{R,m}^+$. We then apply Lemma 6.4 (rescaled) to v_m. Indeed, we have that v_m solves (6.20) and satisfies (6.24). Thus, we obtain, for some $\delta' \in (0, \delta)$,

$$\left\| v_m \right\|_{C^{\delta'}(B_{R/2})} \leq C(R), \quad \text{for all } m \text{ with } r_m R < 1/4, \tag{6.25}$$

where we write $C(R)$ to emphasize the dependence on R of the constant, which also depends on s_0, ρ_0, ellipticity constants, and dimension, but not on m.

As said above, the Arzelà-Ascoli theorem and the previous uniform (in m) $C^{\delta'}$ estimate (6.25) yield the local uniform convergence in $\mathbb{R}^n$ of a subsequence of v_m to some function v.

Next, since all the v_m's satisfy the growth control (6.19), and $2s_0 > \beta$, by the dominated convergence theorem we have $v_m \longrightarrow v$ in $L^1(\mathbb{R}^n, \omega_{s_0})$.

In addition, by Lemma 6.3 there is a subsequence of s_m converging to some $s \in [s_0, \min\{1, s_0 + \delta\}]$ and a subsequence of $\tilde{I}_m$ which converges weakly to some translation invariant operator $\tilde{I}$, which is elliptic with respect to $\mathcal{L}_*(s)$. Hence, it follows from the stability result in [15, Lemma 5] that $\tilde{I}v = 0$ in all of $\mathbb{R}^n$. Thus, the Claim is proved.

Finally, passing to the limit the growth control (6.19) on v_m we find $\|v\|_{L^\infty(B_R)} \leq R^\beta$ for all $R \geq 1$. Hence, by Theorem 1.5, it must be

$$v(x) = K\big(x \cdot \nu(z)\big)_+^s.$$

Passing (6.17) to the limit, we find

$$\int_{B_1} v(x)\big(x \cdot \nu(z)\big)_+^s \, dx = 0.$$

But passing (6.18) to the limit, we reach the contradiction. $\qquad\square$

Before giving the proof of Theorem 1.3, we prove the following.

Lemma 6.5. *Let Γ be a $C^{1,1}$ surface of radius $\rho_0 > 0$ splitting B_1 into Ω^+ and Ω^-; see Definition 1.2. Let $d(x) = \mathrm{dist}\,(x, \Omega^-)$. Let $x_0 \in B_{1/2}$ and $z \in \Gamma$ be such that*

$$\mathrm{dist}\,(x_0, \Gamma) = \mathrm{dist}\,(x_0, z) =: 2r.$$

Then,

$$\left\| \big((x - z) \cdot \nu(z)\big)_+^s - d^s(x) \right\|_{L^\infty(B_r(x_0))} \leq Cr^{2s}, \tag{6.26}$$

$$\left[d^s - \big((x - z) \cdot \nu(z)\big)_+^s \right]_{C^{s-\epsilon}(B_r(x_0))} \leq Cr^s, \tag{6.27}$$

and

$$\left[d^{-s} \right]_{C^{s-\epsilon}(B_r(x_0))} \leq Cr^{-2s+\epsilon}. \tag{6.28}$$

The constant C depends only on ρ_0.

Proof. Let us denote

$$\bar{d}(x) = \big((x - z) \cdot \nu(z)\big)_+.$$

First, since Γ is $C^{1,1}$ with curvature radius bounded below by ρ_0, we have that $|\bar{d} - d| \leq Cr^2$ in $B_r(x_0)$, and thus (6.26) follows.

To prove (6.27) we use on the one hand that

$$\left\| \nabla d - \nabla \bar{d} \right\|_{L^\infty(B_r(x_0))} \leq Cr, \tag{6.29}$$

which also follows from the fact that Γ is $C^{1,1}$. On the other hand, using the inequality $|a^{s-1} - b^{s-1}| \leq |a-b| \max\{a^{s-2}, b^{s-2}\}$ for $a, b > 0$, we find

$$\left\| d^{s-1} - \bar{d}^{s-1} \right\|_{L^\infty(B_r(x_0))} \leq Cr^2 \max \left\{ \left\| d^{s-2} \right\|_{L^\infty(B_r(x_0))}, \left\| \bar{d}^{s-2} \right\|_{L^\infty(B_r(x_0))} \right\} \leq Cr^s. \tag{6.30}$$

Thus, using (6.29) and (6.30), we deduce

$$\left[d^s - \bar{d}^s \right]_{C^{0,1}(B_r(x_0))} = \left\| d^{s-1}\nabla d - \bar{d}^{s-1}\nabla\bar{d} \right\|_{L^\infty(B_r(x_0))} \leq Cr^s.$$

Therefore, (6.27) follows.

Finally, interpolating the inequalities

$$\left[d^{-s} \right]_{C^{0,1}(B_r(x_0))} = \| d^{-s-1}\nabla d \|_{L^\infty(B_r(x_0))} \leq Cr^{-s-1} \quad \text{and} \quad \| d^{-s} \|_{L^\infty(B_r(x_0))} \leq Cr^{-s},$$

(6.28) follows. $\qquad \square$

We can finally give the

Proof of Theorem 1.3. As usual, we may assume that

$$\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\Omega^+)} \leq 1.$$

First, note that by Proposition 6.1 we have that, for all $z \in \Gamma \cap \overline{B_{1/2}}$, there is $Q = Q(z)$ such that

$$|Q(z)| \leq C \quad \text{and} \quad \left\| u - Q\left((x-z)\cdot\nu(z) \right)_+^s \right\|_{L^\infty(B_R(z))} \leq CR^{2s-\epsilon} \tag{6.31}$$

for all $R > 0$, where C depends only on n, s_0, ρ_0, ϵ, and ellipticity constants.

Indeed, let $\delta = \min\{\epsilon/2, s_0/4\}$ and take a partition $s_0 < s_1 < \cdots < s_N = 1$ of $[s_0, 1]$ satisfying $|s_{j+1} - s_j| \leq \delta$. Then, using Proposition 6.1 with s_0 replaced by s_j, (6.31) holds for all $s \in [s_j, s_{j+1}]$ with a constant C_j depending only on n, s_j, ρ_0, and ellipticity constants. Taking $C = \max_j C_j$, (6.31) holds for all $s \in [s_0, 1]$.

Now, to prove the $C^{s-\epsilon}$ estimate up to the boundary for u/d^s we must combine the C^s interior estimate for u in Theorem 2.6 with (6.31). To do it, we will use a similar argument for "glueing estimates" as in the proof of Proposition 1.1. However, here we need to be more precise in the argument because we want to obtain the best possible Hölder exponent.

Let x_0 be a point in $\Omega^+ \cap B_{1/4}$, and let $z \in \Gamma$ be such that

$$2r := \text{dist}(x_0, \Gamma) = \text{dist}(x_0, z) < \rho_0.$$

Note that $B_r(x_0) \subset B_{2r}(x_0) \subset \Omega^+$ and that $z \in \Gamma \cap B_{1/2}$ (since $0 \in \Gamma$).

We claim now that there is $Q = Q(x_0)$ such that $|Q(x_0)| \leq C$,

$$\|u - Qd^s\|_{L^\infty(B_r(x_0))} \leq Cr^{2s-\epsilon}, \tag{6.32}$$

and

$$\left[u - Qd^s \right]_{C^{s-\epsilon}(B_r(x_0))} \leq Cr^s, \tag{6.33}$$

where the constant C depends only on n, s_0, ϵ, ρ_0, and ellipticity constants.

Indeed, (6.32) follows immediately combining (6.31) and (6.26).

To prove (6.33), let

$$v_r(x) = r^{-s}u(z + rx) - Q\left(x \cdot \nu(z)\right)_+^s.$$

Then, (6.31) implies

$$\|v_r\|_{L^\infty(B_4)} \le Cr^{s-\epsilon}$$

and

$$\|v_r\|_{L^1(\mathbb{R}^n, \omega_s)} \le Cr^{s-\epsilon}.$$

Moreover, v_r solves the equation

$$\tilde{I}v_r = r^s f(z + rx) \quad \text{in } B_2(\tilde{x}_0),$$

where $\tilde{x}_0 = (x_0 - z)/r$ satisfies $|\tilde{x}_0 - z| = 2$ and $\tilde{I}$ is translation invariant and elliptic with respect to $\mathcal{L}_*$. Hence, using the interior estimate in Theorem 2.6 we obtain $[v_r]_{C^{s-\epsilon}(B_1(\tilde{x}_0))} \le Cr^{s-\epsilon}$. This yields that

$$r^{s-\epsilon}\left[u - Q\left((x-z)\cdot\nu(z)\right)_+^s\right]_{C^{s-\epsilon}(B_r(x_0))} = r^s[v]_{C^{s-\epsilon}(B_1(\tilde{x}_0))} \le Cr^s r^{s-\epsilon}.$$

Therefore, using (6.27), (6.33) follows.

Let us finally show that (6.32)-(6.33) yield the desired result. Indeed, note that, for all x_1 and x_2 in $B_r(x_0)$,

$$\frac{u}{d^s}(x_1) - \frac{u}{d^s}(x_2) = \frac{\left(u - Qd^s\right)(x_1) - \left(u - Qd^s\right)(x_2)}{d^s(x_1)} + \left(u - Qd^s\right)(x_2)\left(d^{-s}(x_1) - d^{-s}(x_2)\right).$$

By (6.33), and using that d is comparable to r in $B_r(x_0)$, we have

$$\frac{\left|\left(u - Qd^s\right)(x_1) - \left(u - Qd^s\right)(x_2)\right|}{d^s(x_1)} \le C|x_1 - x_2|^{s-\epsilon}.$$

Also, by (6.32) and (6.28),

$$\left|u - Qd^s\right|(x_2)\left|d^{-s}(x_1) - d^{-s}(x_2)\right| \le C|x_1 - x_2|^{s-\epsilon}.$$

Therefore,

$$[u/d^s]_{C^{s-\epsilon}(B_r(x_0))} \le C.$$

From this, we obtain the desired estimate for $\|u/d^s\|_{C^{s-\epsilon}(\Omega^+ \cap B_{1/2})}$ by summing a geometric series, as in the proof of Proposition 1.1 in [45]. $\qquad\square$

7. Non translation invariant versions of the results

Proposition 7.1. *Let $s_0 \in (0,1)$, $\delta \in (0, s_0/4)$, $\rho_0 > 0$, and $\beta = 2s_0 - \delta$ be given constants.*

Let Γ be a $C^{1,1}$ hypersurface with radius $\rho_0 > 0$ splitting B_1 into Ω^+ and Ω^-; see Definition 1.2.

Let $s \in [s_0, \max\{1, s_0 + \delta\}]$, and $f \in C(\overline{\Omega^+})$. Assume that $u \in C(\overline{B_1}) \cap L^\infty(\mathbb{R}^n)$ is a viscosity solution of $\mathfrak{I}(u, x) = f(x)$ in Ω^+ and $u = 0$ in Ω^-, where $\mathfrak{I}$ is an operator of the form (1.12)-(1.16).

Then, for all $z \in \Gamma \cap \overline{B_{1/2}}$ there exists $Q(z) \in \mathbb{R}$ with $|Q(z)| \leq C$ for which

$$\left| u(x) - Q(z)\big((x - z) \cdot \nu(z)\big)_+^s \right| \leq C|x - z|^\beta \quad \text{for all } x \in B_1,$$

where $\nu(z)$ is the unit normal vector to Γ at x pointing towards Ω^+. The constant C depends only on n, ρ_0, s_0, δ, $\|u\|_{L^\infty(\mathbb{R}^n)}$, $\|f\|_{L^\infty(\Omega^+)}$, the modulus of continuity μ, and ellipticity constants.

Proof. It is a variation of the Proof of Proposition 6.1. Hence, it is again by contradiction. Assume that there are sequences Γ_k, Ω_k^+, Ω_k^-, s_k, $\mathfrak{I}_k$, f_k, and u_k that satisfy the assumptions of the proposition. That is, for all $k \geq 1$:

- Γ_k is a $C^{1,1}$ hyper surface with radius ρ_0 splitting B_1 into Ω_k^+ and Ω_k^-.
- $s_k \in [s_0, \max\{1, s_0 + \delta\}]$.
- $\mathfrak{I}_k$ is elliptic with respect to $\mathcal{L}_*(s_k)$ and satisfies (1.12)-(1.16) (with $\mathfrak{I}$ and s replaced by $\mathfrak{I}_k$ and s_k, respectively).
- $\|u_k\|_{L^\infty(\mathbb{R}^n)} + \|f_k\|_{L^\infty(\Omega_k^+)} = 1$.
- u_k is a solution of $\mathfrak{I}_k(u_k, x) = f_k(x)$ in Ω_k^+ and $u_k = 0$ in Ω_k^-.

But suppose that the conclusion of the proposition does not hold. That is, for all $C > 0$, there are k and $z \in \Gamma_k \cap \overline{B_{1/2}}$ for which no constant $Q \in \mathbb{R}$ satisfies

$$\left| u_k(x) - Q\big((x - z) \cdot \nu_k(z)\big)_+^{s_k} \right| \leq C|x - z|^\beta \quad \text{for all } x \in B_1. \tag{7.1}$$

Above, $\nu_k(z)$ denotes the unit normal vector to Γ_k at z, pointing towards Ω_k^+.

As in the proof of Proposition 6.1, using Lemma 6.2, we have that

$$\sup_k \sup_{z \in \Gamma_k \cap B_{1/2}} \sup_{r > 0} r^{-\beta} \|u_k - \phi_{k,z,r}\|_{L^\infty(B_r(z))} = \infty. \tag{7.2}$$

where $\phi_{k,z,r}$ is given by (6.15).

We next define $\theta(r)$ and the sequences $r_m \searrow 0$, k_m, ϕ_m, ν_m, and $z_m \to z \in \overline{B_{1/2}}$ as in the proof of Proposition 6.1.

Again, we also define

$$v_m(x) = \frac{u_{k_m}(z_m + r_m x) - \phi_m(z_m + r_m x)}{(r_m)^\beta \theta(r_m)},$$

which satisfies (6.17), (6.18), and the growth control (6.19).

Note that, up to a subsequence, we may assume that $\nu_m \to \nu \in S^{n-1}$.

The rest of the proof consists in showing

Claim. *A subsequence of v_m converges locally uniformly in $\mathbb{R}^n$ to some function v which satisfies $\tilde{\mathfrak{I}}v = 0$ in $\{x \cdot \nu > 0\}$ and $v = 0$ in $\{x \cdot \nu < 0\}$, for some $\tilde{\mathfrak{I}}$ translation invariant and elliptic with respect to $\mathcal{L}_*$.*

Once we know this, a contradiction is immediately reached using the Liouville type Theorem 1.5, as seen at the end of the proof.

To prove the Claim, given $R \geq 1$ and m such that $r_m R < 1/2$ define

$$\Omega_{R,m}^+ = \left\{ x \in B_R \ : \ (z_m + r_m x) \in \Omega_{k_m}^+ \quad \text{and} \quad x \cdot \nu_m(z_m) > 0 \right\}.$$

Notice that for all R and k, the origin 0 belongs to the boundary of $\Omega_{R,m}^+$.

We will use that v_m satisfies an elliptic equation in $\Omega_{R,m}^+$. Namely,

$$\tilde{\mathfrak{I}}_m(v_m, x) = \frac{(r_m)^{2s_{k_m}}}{(r_m)^\beta \theta(r_m)} f(z_m + r_m x) \quad \text{in } \Omega_{R,m}^+. \tag{7.3}$$

where $\tilde{\mathfrak{I}}_m$ is defined by

$$\tilde{\mathfrak{I}}_m \left(\frac{w(z_m + r \cdot) - \phi_m(z_m + r \cdot)}{(r_m)^\beta \theta(r_m)} , \ x \right) = \frac{(r_m)^{2s_{k_m}}}{(r_m)^\beta \theta(r_m)} \mathfrak{I}_{k_m}(w , \ z_m + rx),$$

for all test function w. Equivalently, for all test function v,

$$\tilde{\mathfrak{I}}_m(v, x) \overset{(*)}{=} \frac{(r_m)^{2s_{k_m}}}{(r_m)^\beta \theta(r_m)} \mathfrak{I}_{k_m} \left((r_m)^\beta \theta(r_m) v \left(\frac{\cdot - z_m}{r_m} \right) + \phi_m(\cdot) , \ z_m + r_m x \right)$$

$$\overset{(**)}{=} \frac{(r_m)^{2s_{k_m}}}{(r_m)^\beta \theta(r_m)} \mathfrak{I}_{k_m} \left((r_m)^\beta \theta(r_m) v \left(\frac{\cdot - z_m}{r_m} \right) \right) (z_m + r_m x)$$

$$\overset{(***)}{=} \inf_\beta \sup_\alpha \left(\int_{\mathbb{R}^n} \left\{ v(x+y) + v(x-y) - 2v(x) \right\} K_{\alpha\beta}^{(m)}(z_m + r_m x, y) \, dy + \right.$$

$$\left. + \frac{(r_m)^{2s_{k_m}} c_{\alpha\beta}^{(m)}(z_m + rx)}{(r_m)^\beta \theta(r_m)} \right).$$

The last two identities hold only in $\{ x \cdot \nu_m > 0 \}$ since $M^+ \phi_m = M^- \phi_m = 0$ in $\{ (x - z) \cdot \nu_m > 0 \}$.

Note that the right hand side of (7.3) converges uniformly to 0 as $r_m \searrow 0$ since $\beta = 2s_0 - \delta < 2s_{k_m}$ and $\theta(r_m) \nearrow \infty$.

Using that $\mathfrak{I}_{k_m}$ is elliptic with respect to $\mathcal{L}_*(s_{k_m})$ and that $\mathfrak{I}_{k_m}(0, x) = 0$, we readily show that $\tilde{\mathfrak{I}}_m$ is also elliptic with respect to $\mathcal{L}_*(s_{k_m})$.

Note that, since $\mathfrak{I}_m$ is elliptic with respect to $\mathcal{L}_*(s_{k_m})$, and $\|f_{k_m}\|_{L^\infty} \leq 1$, then

$$M_{s_{k_m}}^+ u_{k_m} \geq -1 \quad \text{and} \quad M_{s_{k_m}}^- u_{k_m} \leq 1 \quad \text{in } \Omega^+,$$

and the same inequalities hold for v_m. Hence, by the same argument as in the proof of Proposition 6.1, we find that

$$\|v_m\|_{C^{\delta'}(B_{R/2})} \leq C(R), \quad \text{for all } m \text{ with } r_m R < 1/4,$$

where $C(R)$ depends only on R, n, s_0, ρ_0, and ellipticity constants, but not on m.

Then, the Arzelà-Ascoli theorem yields the local uniform convergence in $\mathbb{R}^n$ of a subsequence of v_m to some function v. Thus, the Claim is proved.

Next, since all the v_m's satisfy the growth control (6.19), and $2s_0 > \beta$, by the dominated convergence theorem $v_m \to v$ in $L^1(\mathbb{R}^n, \omega_{s_0})$.

Let now $\tilde{I}_m$ be the sequence of translation invariant operators defined by

$$\tilde{I}_m w = \inf_\beta \sup_\alpha \left(\int_{\mathbb{R}^n} \{w(x+y) + w(x-y) - 2w(x)\} K_{\alpha\beta}^{(m)}(z_m, y)\, dy \right).$$

Note that, for all test functions w,

$$\tilde{\mathfrak{I}}_m(w,x) - \tilde{I}_m(w) \longrightarrow 0 \quad \text{uniformly in compact sets of } \{(x-z) \cdot \nu > 0\}. \quad (7.4)$$

Indeed, by (1.16),

$$\left| K_{\alpha\beta}^{(m)}(z_m + r_m x, y) - K_{\alpha\beta}^{(m)}(z_m, y) \right| \leq \left(1 - s_{k_m}\right) \frac{\mu(Cr_m)}{|y|^{n+2s_{k_m}}} \longrightarrow 0$$

and

$$\left| \frac{(r_m)^{2s_{k_m}} c_{\alpha\beta}^{(m)}(z_m + rx)}{(r_m)^\beta \theta(r_m)} \right| \leq \Lambda(r_m)^{2s_{k_m} - \beta} \longrightarrow 0,$$

where μ is the modulus of continuity of the kernels $K_{\alpha\beta}(x,y)$ with respect to x.

On the other hand, by Lemma 6.3 there is a subsequence of s_{k_m} converging to some $s \in [s_0, \min\{1, 2s_0 - \delta\}]$ and a subsequence of $\tilde{I}_m$ which converges weakly to some translation invariant operator $\tilde{I}$, which is elliptic with respect to $\mathcal{L}_*(s)$. Hence, by (7.4), it follows that $\tilde{\mathfrak{I}}_m \to \tilde{I}$ weakly in compact subsets of $\{x \cdot \nu > 0\}$. Therefore, using the stability result in [15, Lemma 5], $\tilde{I}v = 0$ in $\{x \cdot \nu > 0\}$.

Finally, passing to the limit the growth control (6.19) on v_m, we find $\|v\|_{L^\infty(B_R)} \leq CR^\beta$ for all $R \geq 1$. Hence, by Theorem 1.5, it must be

$$v(x) = K\left(x \cdot \nu(z)\right)_+^s.$$

But passing (6.17) and (6.18) to the limit we find a contradiction. $\qquad\square$

We next prove Theorem 1.6.

Proof of Theorem 1.6. In case that $g \equiv 0$, the result follows from Proposition 7.1 by using the same argument as is the proof of Theorem 1.3 (partition of $[s_0, 1]$ into intervals of length smaller than $\epsilon/2$).

When g is not zero, we consider $\bar{u} = u - g\chi_{B_1}$. Then $\bar{u}$ satisfies $\bar{u} \equiv 0$ in Ω^- and

$$\bar{\mathfrak{I}}(\bar{u}, x) = \bar{f}(x) \quad \text{in } \Omega^+ \cap B_{3/4},$$

where

$$\bar{\mathfrak{I}}(w, x) = \mathfrak{I}(w + g\chi_{B_1}, x) - \mathfrak{I}(g\chi_{B_1}, x)$$

and

$$\bar{f}(x) = \mathfrak{I}(g\chi_{B_1}, x) + f(x).$$

Then, applying the result for $g \equiv 0$ to the function $\tilde{u}$, the theorem follows. $\qquad\square$

8. Final comments and remarks

Here we would like to make a few remarks and talk about some open problems and future research directions.

Higher regularity of u/d^s. In the proof of the Liouville-type Theorem 1.5, one starts with a solution satisfying $|u(x)| \leq C(1 + |x|^\beta)$. Then, one proves that the tangential derivatives satisfy $|\partial_\tau u(x)| \leq C(1 + |x|^{\beta-1})$. Hence, if $\beta - 1 < s$, Proposition 5.1 implies that $\partial_\tau u \equiv 0$, and thus u is 1D.

The fact that we only use $\beta < 1 + s$ seems to indicate that the quotient u/d^s could belong to $C^{1-\epsilon}$, and not only to $C^{s-\epsilon}$. However, for functions with growth at infinity $2s \leq \beta < 1 + s$, the integro-differential operators cannot be evaluated.

In fact, only having $\beta - 1 < s + \alpha$ would suffice to obtain $\partial_\tau u = c(x_n)_+^s$, and this seems enough to classify solutions in the half space. However, as before, such approach would require to give sense to the equation for functions that grow "too much" at infinity.

Therefore, the following question remains open. Is it possible to prove that u/d^s belongs to $C^{1+\alpha}$ when considering more regular kernels and right hand sides?

More general linear equations. In a future work we are planning to obtain $C^{s-\epsilon}$ regularity up to the boundary of u/d^s for linear equations involving general operators L of the form (1.3), where a is any measure (not supported in an hyperplane) which does not necessarily satisfy (1.4). We will also obtain higher order regularity of u/d^s for linear equations when $a \in C^k(S^{n-1})$, $f \in C^k(\overline{\Omega})$, and Ω is C^{k+2}.

Equations with lower order terms. We could have included lower order terms in the equations. Indeed, the compactness methods in Section 6 involve a blow up procedure. We have seen in Section 7 that non translation invariant equations with continuous dependence on x become translation invariant after blow up, and hence our methods still apply. Similarly, we could have considered equations with certain lower order terms, which disappear after blow up.

Second order fully nonlinear equations. As said in the introduction, with the methods developed in this paper one can prove the $C^{1,\alpha}$ and $C^{2,\alpha}$ boundary estimates for fully nonlinear equations $F(D^2u, Du, x) = f(x)$.

Obstacle and free boundary problems. The regularity theory for the obstacle problem (or other free boundary problems) is related to the boundary regularity of solutions to fully nonlinear elliptic equations. In this paper we have shown that $\mathcal{L}_*$ is the appropriate class to obtain fine regularity properties up to the boundary. We therefore wonder if one could obtain regularity results for free boundary problems involving operators in $\mathcal{L}_*$ similar to those for the fractional Laplacian [50].

9. Appendix

In this appendix we give the

Proof of Lemma 5.3. Let us show first the statement (a). Denote

$$a = 1 - 2s.$$

We first note that the Caffarelli-Silvestre extension equation $\Delta u + \frac{a}{y}\partial_y u = 0$ is written in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$, $\theta \in (0, \pi)$ as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{a}{r\sin\theta}\left(\sin\theta\, u_r + \cos\theta\, \frac{u_\theta}{r}\right) = 0.$$

Note the homogeneity of the equation in the variable r. If we seek for (bounded at 0) solutions of the form $u = r^{s+\nu}\Theta_\nu(\theta)$, then it must be $\nu > -s$ and

$$\Theta_\nu'' + a\cotg\theta\, \Theta_\nu' + (s+\nu)(s+\nu+a)\Theta_\nu = 0.$$

If we want u to satisfy the boundary conditions

$$u(x,0) = 0 \quad \text{for } x < 0 \quad \text{and} \quad |y|^a\partial_y u(x,y) \to 0 \quad \text{as } y \to 0,$$

then Θ_ν must satisfy

$$\begin{cases} \Theta_\nu(\theta) = \Theta_\nu(0) + o\big((\sin\theta)^{2s}\big) \to 0 & \text{as } \theta \searrow 0 \\ \Theta_\nu(\pi) = 0. \end{cases} \tag{9.1}$$

We have used that, for $x > 0$

$$\lim_{y\searrow 0} y^a\partial_y u(x,y) = 0 \quad \Rightarrow \quad u(x,y) = u(x,0) + o(y^{2s}),$$

since $a = 1 - 2s$.

To solve this ODE, consider

$$\Theta_\nu(\theta) = (\sin\theta)^s h(\cos\theta).$$

After some computations and the change of variable $z = \cos\theta$ one obtains the following ODE for $h(z)$:

$$(1 - z^2)h''(z) - 2zh'(z) + \left(\nu + \nu^2 - \frac{s^2}{1 - z^2}\right)h(z) = 0.$$

This is the so called "associated Legendre differential equation". All solutions to this second order ODE solutions are given by

$$h(z) = C_1 P_\nu^s(z) + C_2 Q_\nu^s(z),$$

where P_ν^s and Q_ν^s are the "associated Legendre functions" of first and second kind, respectively.

Translating (9.1) to the function h, using that $\sin\theta \sim (1 - \cos\theta)^{1/2}$ as $\theta \searrow 0$ and $\sin\theta \sim (1 + \cos\theta)^{1/2}$ as $\theta \nearrow \pi$, we obtain

$$\begin{cases} (1-z)^{s/2}h(z) = c + o\big((1-z)^s\big) & \text{as } z \nearrow 1 \\ \lim_{z\searrow -1}(1+z)^{s/2}h(z) = 0. \end{cases} \tag{9.2}$$

Let us prove that P_ν^s fulfill all these requirements only for $\nu = 0, 1, 2, 3, \ldots$, while Q_ν^s have to be discarded. To have a good description of the singularities of $P_\nu^s(z)$ at $z = \pm 1$ we use its expression as an hypergeometric function

$$P_\nu^s(z) = \frac{1}{\Gamma(1-s)} \frac{(1+z)^{s/2}}{(1-z)^{s/2}} \,_2F_1\left(-\nu, \nu+1; 1-s; \frac{1-z}{2}\right).$$

Using this and the definition of $_2F_1$ as a power series we obtain

$$P_\nu^s(z) = \frac{1}{\Gamma(1-s)} \frac{2^{s/2}}{(1-z)^{s/2}} \left\{1 - \frac{\nu(\nu+1)}{1-s} \frac{1-z}{2} + o\big((1-z)^2\big)\right\} \quad \text{as } z \nearrow 1.$$

Hence, $(1-z)^{s/2} P_\nu^s(z) = c + O(1-z) = c + o\big((1-z)^s\big)$ as desired.

For the analysis as $z \searrow -1$ we need to use Euler's transformation

$$_2F_1(a, b; c; x) = (1-x)^{c-b-a} \,_2F_1(c-a, c-b; c; x),$$

obtaining

$$P_\nu^s(z) = \frac{1}{\Gamma(1-s)} \frac{(1+z)^{s/2}}{2^{s/2}} \left(\frac{1+z}{2}\right)^{-s} \left\{\,_2F_1(1-s-\nu, -s-\nu; 1-s; 1) + o(1)\right\}$$

as $z \searrow -1$. It follows that the zero boundary condition is satisfied if and only if

$$_2F_1(1-s-\nu, -s-\nu; 1-s; 1) = \frac{\Gamma(1-s)\Gamma(s)}{\Gamma(-\nu)\Gamma(1+\nu)} = 0.$$

This implies $\nu = 0, 1, 2, 3, \ldots$, so that $\Gamma(-\nu) = \infty$.

With a similar analysis one easily finds that the functions $Q_\nu^s(x)$ do not satisfy (9.2) for any $\nu \geq -s$.

The statement (b) of the Lemma could be proved for example by using singular Sturm-Liouville theory after observing that the ODE

$$\Theta_\nu'' + a \cotg \theta \, \Theta_\nu' - \lambda \Theta_\nu = 0$$

can be written as

$$\big(|\sin\theta|^a \, \Theta_\nu'\big)' = \lambda |\sin\theta|^a \Theta_\nu.$$

However, it is not necessary to do it because we have already computed the eigenfunctions to this ODE, and they are given by

$$\Theta_k(\theta) = (\sin\theta)^s P_k^s(\cos\theta),$$

where P_ν^s are the associated Legendre functions of first kind. The functions $\{P_k^s(x)\}_{k\geq 0}$ have been well studied, and they are known to be a complete orthogonal system in $L^2\big((0,1), dx\big)$; see [32, 57]. Therefore, it immediately follows (after a change of variables) that $\{\Theta_k(\theta)\}_{k\geq 0}$ are a complete orthogonal system in $L^2\big((0,\pi), (\sin\theta)^a d\theta\big)$. Thus, the Lemma is proved. □

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Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain
   E-mail address: xavier.ros.oton@upc.edu

Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain
   E-mail address: joaquim.serra@upc.edu

F. The extremal solution for the fractional Laplacian

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Collaboration with X. Ros-Oton

THE EXTREMAL SOLUTION FOR THE FRACTIONAL LAPLACIAN

XAVIER ROS-OTON AND JOAQUIM SERRA

ABSTRACT. We study the extremal solution for the problem $(-\Delta)^s u = \lambda f(u)$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, where $\lambda > 0$ is a parameter and $s \in (0,1)$. We extend some well known results for the extremal solution when the operator is the Laplacian to this nonlocal case. For general convex nonlinearities we prove that the extremal solution is bounded in dimensions $n < 4s$. We also show that, for exponential and power-like nonlinearities, the extremal solution is bounded whenever $n < 10s$. In the limit $s \uparrow 1$, $n < 10$ is optimal. In addition, we show that the extremal solution is $H^s(\mathbb{R}^n)$ in any dimension whenever the domain is convex.

To obtain some of these results we need L^q estimates for solutions to the linear Dirichlet problem for the fractional Laplacian with L^p data. We prove optimal L^q and C^β estimates, depending on the value of p. These estimates follow from classical embedding results for the Riesz potential in $\mathbb{R}^n$.

Finally, to prove the H^s regularity of the extremal solution we need an L^∞ estimate near the boundary of convex domains, which we obtain via the moving planes method. For it, we use a maximum principle in small domains for integro-differential operators with decreasing kernels.

CONTENTS

1. Introduction and results

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $s \in (0,1)$, and consider the problem

$$\begin{cases} (-\Delta)^s u = \lambda f(u) & \text{in } \Omega \\ \qquad\quad u = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where λ is a positive parameter and $f : [0, \infty) \longrightarrow \mathbb{R}$ satisfies

$$f \text{ is } C^1 \text{ and nondecreasing, } f(0) > 0, \text{ and } \lim_{t \to +\infty} \frac{f(t)}{t} = +\infty. \tag{1.2}$$

Here, $(-\Delta)^s$ is the fractional Laplacian, defined for $s \in (0,1)$ by

$$(-\Delta)^s u(x) = c_{n,s} \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \tag{1.3}$$

where $c_{n,s}$ is a constant.

It is well known —see [4] or the excellent monograph [16] and references therein— that in the classical case $s = 1$ there exists a finite extremal parameter λ^* such that if $0 < \lambda < \lambda^*$ then problem (1.1) admits a minimal classical solution u_λ, while for $\lambda > \lambda^*$ it has no solution, even in the weak sense. Moreover, the family of functions $\{u_\lambda : 0 < \lambda < \lambda^*\}$ is increasing in λ, and its pointwise limit $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a weak solution of problem (1.1) with $\lambda = \lambda^*$. It is called the extremal solution of (1.1).

When $f(u) = e^u$, we have that $u^* \in L^\infty(\Omega)$ if $n \le 9$ [12], while $u^*(x) = \log \frac{1}{|x|^2}$ if $n \ge 10$ and $\Omega = B_1$ [23]. An analogous result holds for other nonlinearities such as powers $f(u) = (1 + u)^p$ and also for functions f satisfying a limit condition at infinity; see [30]. In the nineties H. Brezis and J.L. Vázquez [4] raised the question of determining the regularity of u^*, depending on the dimension n, for general nonlinearities f satisfying (1.2). The first result in this direction was proved by G. Nedev [26], who obtained that the extremal solution is bounded in dimensions $n \le 3$ whenever f is convex. Some years later, X. Cabré and A. Capella [7] studied the radial case. They showed that when $\Omega = B_1$ the extremal solution is bounded for all nonlinearities f whenever $n \le 9$. For general nonlinearities, the best known result at the moment is due to X. Cabré [6], and states that in dimensions $n \le 4$ then the extremal solution is bounded for any convex domain Ω. Recently, S. Villegas [36] have proved, using the results in [6], the boundedness of the extremal solution in dimension $n = 4$ for all domains, not necessarily convex. The problem is still open in dimensions $5 \le n \le 9$.

The aim of this paper is to study the extremal solution for the fractional Laplacian, that is, to study problem (1.1) for $s \in (0,1)$.

The closest result to ours was obtained by Capella-Dávila-Dupaigne-Sire [10]. They studied the extremal solution in $\Omega = B_1$ for the spectral fractional Laplacian A^s. The operator A^s, defined via the Dirichlet eigenvalues of the Laplacian in Ω, is related to (but different from) the fractional Laplacian (1.3). We will state their result later on in this introduction.

Let us start defining weak solutions to problem (1.1).

Definition 1.1. We say that $u \in L^1(\Omega)$ is a *weak solution* of (1.1) if

$$f(u)\delta^s \in L^1(\Omega), \tag{1.4}$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$, and

$$\int_\Omega u(-\Delta)^s \zeta dx = \int_\Omega \lambda f(u)\zeta dx \tag{1.5}$$

for all ζ such that ζ and $(-\Delta)^s\zeta$ are bounded in Ω and $\zeta \equiv 0$ on $\partial\Omega$.

Any bounded weak solution is a *classical solution*, in the sense that it is regular in the interior of Ω, continuous up to the boundary, and (1.1) holds pointwise; see Remark 2.1.

Note that for $s = 1$ the above notion of weak solution is exactly the one used in [5, 4].

In the classical case (that is, when $s = 1$), the analysis of singular extremal solutions involves an intermediate class of solutions, those belonging to $H^1(\Omega)$; see [4, 25]. These solutions are called [4] energy solutions. As proved by Nedev [27], when the domain Ω is convex the extremal solution belongs to $H^1(\Omega)$, and hence it is an energy solution; see [8] for the statement and proofs of the results in [27].

Similarly, here we say that a weak solution u is an *energy solution* of (1.1) when $u \in H^s(\mathbb{R}^n)$. This is equivalent to saying that u is a critical point of the energy functional

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_{\mathring{H}^s}^2 - \int_\Omega \lambda F(u)dx, \qquad F' = f, \tag{1.6}$$

where

$$\|u\|_{\mathring{H}^s}^2 = \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}u\right|^2 dx = \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}dxdy = (u, u)_{\mathring{H}^s} \tag{1.7}$$

and

$$(u, v)_{\mathring{H}^s} = \int_{\mathbb{R}^n}(-\Delta)^{s/2}u(-\Delta)^{s/2}v \, dx = \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)}{|x-y|^{n+2s}}dxdy. \tag{1.8}$$

Our first result, stated next, concerns the existence of a minimal branch of solutions, $\{u_\lambda, \ 0 < \lambda < \lambda^*\}$, with the same properties as in the case $s = 1$. These solutions are proved to be positive, bounded, increasing in λ, and semistable. Recall that a weak solution u of (1.1) is said to be *semistable* if

$$\int_\Omega \lambda f'(u)\eta^2 dx \leq \|\eta\|_{\mathring{H}^s}^2 \tag{1.9}$$

for all $\eta \in H^s(\mathbb{R}^n)$ with $\eta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. When u is an energy solution this is equivalent to saying that the second variation of energy $\mathcal{E}$ at u is nonnegative.

Proposition 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $s \in (0, 1)$, and f be a function satisfying (1.2). Then, there exists a parameter $\lambda^* \in (0, \infty)$ such that:*

(i) *If $0 < \lambda < \lambda^*$, problem (1.1) admits a minimal classical solution u_λ.*

(ii) *The family of functions $\{u_\lambda : 0 < \lambda < \lambda^*\}$ is increasing in λ, and its pointwise limit $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a weak solution of (1.1) with $\lambda = \lambda^*$.*

(iii) *For $\lambda > \lambda^*$, problem (1.1) admits no classical solution.*

(iv) *These solutions u_λ, as well as u^*, are semistable.*

The weak solution u^* is called the extremal solution of problem (1.1).

As explained above, the main question about the extremal solution u^* is to decide whether it is bounded or not. Once the extremal solution is bounded then it is a classical solution, in the sense that it satisfies equation (1.1) pointwise. For example, if $f \in C^\infty$ then u^* bounded yields $u^* \in C^\infty(\Omega) \cap C^s(\overline{\Omega})$.

Our main result, stated next, concerns the regularity of the extremal solution for problem (1.1). To our knowledge this is the first result concerning extremal solutions for (1.1). In particular, the following are new results even for the unit ball $\Omega = B_1$ and for the exponential nonlinearity $f(u) = e^u$.

Theorem 1.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $s \in (0,1)$, f be a function satisfying (1.2), and u^* be the extremal solution of (1.1).*

(i) *Assume that f is convex. Then, u^* is bounded whenever $n < 4s$.*

(ii) *Assume that f is C^2 and that the following limit exists:*

$$\tau := \lim_{t \to +\infty} \frac{f(t) f''(t)}{f'(t)^2}. \tag{1.10}$$

Then, u^* is bounded whenever $n < 10s$.

(iii) *Assume that Ω is convex. Then, u^* belongs to $H^s(\mathbb{R}^n)$ for all $n \geq 1$ and all $s \in (0,1)$.*

Note that the exponential and power nonlinearities e^u and $(1+u)^p$, with $p > 1$, satisfy the hypothesis in part (ii) whenever $n < 10s$. In the limit $s \uparrow 1$, $n < 10$ is optimal, since the extremal solution may be singular for $s = 1$ and $n = 10$ (as explained before in this introduction).

Note that the results in parts (i) and (ii) of Theorem 1.3 do not provide any estimate when s is small (more precisely, when $s \leq 1/4$ and $s \leq 1/10$, respectively). The boundedness of the extremal solution for small s seems to require different methods from the ones that we present here. Our computations in Section 3 suggest that the extremal solution for the fractional Laplacian should be bounded in dimensions $n \leq 7$ for all $s \in (0,1)$, at least for the exponential nonlinearity $f(u) = e^u$. As commented above, Capella-Dávila-Dupaigne-Sire [10] studied the extremal solution for the *spectral* fractional Laplacian A^s in $\Omega = B_1$. They obtained an L^∞ bound for the extremal solution in a ball in dimensions $n < 2\left(2 + s + \sqrt{2s+2}\right)$, and hence they proved the boundedness of the extremal solution in dimensions $n \leq 6$ for all $s \in (0,1)$.

To prove part (i) of Theorem 1.3 we borrow the ideas of [26], where Nedev proved the boundedness of the extremal solution for $s = 1$ and $n \leq 3$. To prove part (ii)

we follow the approach of M. Sanchón in [30]. When we try to repeat the same arguments for the fractional Laplacian, we find that some identities that in the case $s = 1$ come from local integration by parts are no longer available for $s < 1$. Instead, we succeed to replace them by appropriate inequalities. These inequalities are sharp as $s \uparrow 1$, but not for small s. Finally, part (iii) is proved by an argument of Nedev [27], which for $s < 1$ requires the Pohozaev identity for the fractional Laplacian, recently established by the authors in [29]. This argument requires also some boundary estimates, which we prove using the moving planes method; see Proposition 1.8 at the end of this introduction.

An important tool in the proofs of the results of Nedev [26] and Sanchón [30] is the classical L^p to $W^{2,p}$ estimate for the Laplace equation. Namely, if u is the solution of $-\Delta u = g$ in Ω, $u = 0$ in $\partial\Omega$, with $g \in L^p(\Omega)$, $1 < p < \infty$, then

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|g\|_{L^p(\Omega)}.$$

This estimate and the Sobolev embeddings lead to $L^q(\Omega)$ or $C^\alpha(\overline{\Omega})$ estimates for the solution u, depending on whether $1 < p < \frac{n}{2}$ or $p > \frac{n}{2}$, respectively.

Here, to prove Theorem 1.3 we need similar estimates but for the fractional Laplacian, in the sense that from $(-\Delta)^s u \in L^p(\Omega)$ we want to deduce $u \in L^q(\Omega)$ or $u \in C^\alpha(\overline{\Omega})$. However, L^p to $W^{2s,p}$ estimates for the fractional Laplace equation, in which $-\Delta$ is replaced by the fractional Laplacian $(-\Delta)^s$, are not available for all p, even when $\Omega = \mathbb{R}^n$; see Remarks 7.1 and 7.2.

Although the L^p to $W^{2s,p}$ estimate does not hold for all p in this fractional framework, what will be indeed true is the following result. This is a crucial ingredient in the proof of Theorem 1.3.

Proposition 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain, $s \in (0,1)$, $n > 2s$, $g \in C(\overline{\Omega})$, and u be the solution of*

$$\begin{cases} (-\Delta)^s u &= g \quad in \ \Omega \\ u &= 0 \quad in \ \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.11}$$

(i) *For each $1 \leq r < \frac{n}{n-2s}$ there exists a constant C, depending only on n, s, r, and $|\Omega|$, such that*

$$\|u\|_{L^r(\Omega)} \leq C\|g\|_{L^1(\Omega)}, \quad r < \frac{n}{n-2s}.$$

(ii) *Let $1 < p < \frac{n}{2s}$. Then there exists a constant C, depending only on n, s, and p, such that*

$$\|u\|_{L^q(\Omega)} \leq C\|g\|_{L^p(\Omega)}, \quad where \quad q = \frac{np}{n - 2ps}.$$

(iii) *Let $\frac{n}{2s} < p < \infty$. Then, there exists a constant C, depending only on n, s, p, and Ω, such that*

$$\|u\|_{C^\beta(\mathbb{R}^n)} \leq C\|g\|_{L^p(\Omega)}, \quad where \quad \beta = \min\left\{s, 2s - \frac{n}{p}\right\}.$$

We will use parts (i), (ii), and (iii) of Proposition 1.4 in the proof of Theorem 1.3. However, we will only use part (iii) to obtain an L^∞ estimate for u, we will not need the C^β bound. Still, for completeness we prove the C^β estimate, with the optimal exponent β (depending on p).

Remark 1.5. Proposition 1.4 does not provide any estimate for $n \leq 2s$. Since $s \in (0,1)$, then $n \leq 2s$ yields $n = 1$ and $s \geq 1/2$. In this case, any bounded domain is of the form $\Omega = (a, b)$, and the Green function $G(x, y)$ for problem (1.14) is explicit; see [2]. Then, by using this expression it is not difficult to show that $G(\cdot, y)$ is $L^\infty(\Omega)$ in case $s > 1/2$ and $L^p(\Omega)$ for all $p < \infty$ in case $s = 1/2$. Hence, in case $n < 2s$ it follows that $\|u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^1(\Omega)}$, while in case $n = 2s$ it follows that $\|u\|_{L^q(\Omega)} \leq C\|g\|_{L^1(\Omega)}$ for all $q < \infty$ and $\|u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^p(\Omega)}$ for $p > 1$.

Proposition 1.4 follows from Theorem 1.6 and Proposition 1.7 below. The first one contains some classical results concerning embeddings for the Riesz potential, and reads as follows.

Theorem 1.6 (see [34]). *Let $s \in (0, 1)$, $n > 2s$, and g and u be such that*

$$u = (-\Delta)^{-s}g \quad in \ \mathbb{R}^n, \tag{1.12}$$

in the sense that u is the Riesz potential of order $2s$ of g. Assume that u and g belong to $L^p(\mathbb{R}^n)$, with $1 \leq p < \infty$.

(i) *If $p = 1$, then there exists a constant C, depending only on n and s, such that*

$$\|u\|_{L^q_{\text{weak}}(\mathbb{R}^n)} \leq C\|g\|_{L^1(\mathbb{R}^n)}, \quad where \quad q = \frac{n}{n - 2s}.$$

(ii) *If $1 < p < \frac{n}{2s}$, then there exists a constant C, depending only on n, s, and p, such that*

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}, \quad where \quad q = \frac{np}{n - 2ps}.$$

(iii) *If $\frac{n}{2s} < p < \infty$, then there exists a constant C, depending only on n, s, and p, such that*

$$[u]_{C^\alpha(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}, \quad where \quad \alpha = 2s - \frac{n}{p},$$

where $[\,\cdot\,]_{C^\alpha(\mathbb{R}^n)}$ denotes the C^α seminorm.

Parts (i) and (ii) of Theorem 1.6 are proved in the book of Stein [34, Chapter V]. Part (iii) is also a classical result, but it seems to be more difficult to find an exact reference for it. Although it is not explicitly stated in [34], it follows for example from the inclusions

$$I_{2s}(L^p) = I_{2s-n/p}(I_{n/p}(L^p)) \subset I_{2s-n/p}(\text{BMO}) \subset C^{2s-\frac{n}{p}},$$

which are commented in [34, p.164]. In the more general framework of spaces with non-doubling n-dimensional measures, a short proof of this result can also be found in [19].

Having Theorem 1.6 available, to prove Proposition 1.4 we will argue as follows. Assume $1 < p < \frac{n}{2s}$ and consider the solution v of the problem

$$(-\Delta)^s v = |g| \ \text{ in } \mathbb{R}^n,$$

where g is extended by zero outside Ω. On the one hand, the maximum principle yields $-v \leq u \leq v$ in $\mathbb{R}^n$, and by Theorem 1.6 we have that $v \in L^q(\mathbb{R}^n)$. From this, parts (i) and (ii) of the proposition follow. On the other hand, if $p > \frac{n}{2s}$ we write $u = \tilde{v} + w$, where $\tilde{v}$ solves $(-\Delta)^s \tilde{v} = g$ in $\mathbb{R}^n$ and w is the solution of

$$\begin{cases} (-\Delta)^s w & = & 0 & \text{in } \Omega \\ w & = & \tilde{v} & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases}$$

As before, by Theorem 1.6 we will have that $\tilde{v} \in C^\alpha(\mathbb{R}^n)$, where $\alpha = 2s - \frac{n}{p}$. Then, the C^β regularity of u will follow from the following new result.

Proposition 1.7. *Let Ω be a bounded $C^{1,1}$ domain, $s \in (0,1)$, $h \in C^\alpha(\mathbb{R}^n \setminus \Omega)$ for some $\alpha > 0$, and u be the solution of*

$$\begin{cases} (-\Delta)^s u & = & 0 & \text{in } \Omega \\ u & = & h & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.13}$$

Then, $u \in C^\beta(\mathbb{R}^n)$, with $\beta = \min\{s, \alpha\}$, and

$$\|u\|_{C^\beta(\mathbb{R}^n)} \leq C\|h\|_{C^\alpha(\mathbb{R}^n \backslash \Omega)},$$

where C is a constant depending only on Ω, α, and s.

To prove Proposition 1.7 we use similar ideas as in [28]. Namely, since u is harmonic then it is smooth inside Ω. Hence, we only have to prove C^β estimates near the boundary. To do it, we use an appropriate barrier to show that

$$|u(x) - u(x_0)| \leq C\|h\|_{C^\alpha}\delta(x)^\beta \quad \text{in } \Omega,$$

where x_0 is the nearest point to x on $\partial\Omega$, $\delta(x) = \text{dist}(x, \partial\Omega)$, and $\beta = \min\{s, \alpha\}$. Combining this with the interior estimates, we obtain C^β estimates up to the boundary of Ω.

Finally, as explained before, to show that when the domain is convex the extremal solution belongs to the energy class $H^s(\mathbb{R}^n)$ —which is part (iii) of Theorem 1.3— we need the following boundary estimates.

Proposition 1.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, $s \in (0,1)$, f be a locally Lipschitz function, and u be a bounded positive solution of*

$$\begin{cases} (-\Delta)^s u & = & f(u) & \text{in } \Omega \\ u & = & 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.14}$$

Then, there exists constants $\delta > 0$ and C, depending only on Ω, such that

$$\|u\|_{L^\infty(\Omega_\delta)} \leq C\|u\|_{L^1(\Omega)},$$

where $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$.

This estimate follows, as in the classical result of de Figueiredo-Lions-Nussbaum [14], from the moving planes method. There are different versions of the moving planes method for the fractional Laplacian (using the Caffarelli-Silvestre extension, the Riesz potential, the Hopf lemma, etc.). A particularly clean version uses the maximum principle in small domains for the fractional Laplacian, recently proved by Jarohs and Weth in [22]. Here, we follow their approach and we show that this maximum principle holds also for integro-differential operators with decreasing kernels.

The paper is organized as follows. In Section 2 we prove Proposition 1.2. In Section 3 we study the regularity of the extremal solution in the case $f(u) = e^u$. In Section 4 we prove Theorem 1.3 (i)-(ii). In Section 5 we show the maximum principle in small domains and use the moving planes method to establish Proposition 1.8. In Section 6 we prove Theorem 1.3 (iii). Finally, in Section 7 we prove Proposition 1.4.

2. Existence of the extremal solution

In this section we prove Proposition 1.2. For it, we follow the argument from Proposition 5.1 in [7]; see also [16].

Proof of Proposition 1.2. Step 1. We first prove that there is no weak solution for large λ.

Let $\lambda_1 > 0$ be the first eigenvalue of $(-\Delta)^s$ in Ω and $\varphi_1 > 0$ the corresponding eigenfunction, that is,

$$\begin{cases} (-\Delta)^s \varphi_1 & = & \lambda_1 \varphi_1 & \text{in } \Omega \\ \varphi_1 & > & 0 & \text{in } \Omega \\ \varphi_1 & = & 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The existence, simplicity, and boundedness of the first eigenfunction is proved in [31, Proposition 5] and [32, Proposition 4]. Assume that u is a weak solution of (1.1). Then, using φ_1 as a test function for problem (1.1) (see Definition 1.1), we obtain

$$\int_\Omega \lambda_1 u \, \varphi_1 dx = \int_\Omega u(-\Delta)^s \varphi_1 dx = \int_\Omega \lambda f(u) \varphi_1 dx. \tag{2.1}$$

But since f is superlinear at infinity and positive in $[0, \infty)$, it follows that $\lambda f(u) > \lambda_1 u$ if λ is large enough, a contradiction with (2.1).

Step 2. Next we prove the existence of a classical solution to (1.1) for small λ. Since $f(0) > 0$, $\underline{u} \equiv 0$ is a strict subsolution of (1.1) for every $\lambda > 0$. The solution $\overline{u}$ of

$$\begin{cases} (-\Delta)^s \overline{u} & = & 1 & \text{in } \Omega \\ \overline{u} & = & 0 & \text{on } \mathbb{R}^n \setminus \Omega \end{cases} \tag{2.2}$$

is a bounded supersolution of (1.1) for small λ, more precisely whenever $\lambda f(\max \overline{u}) < 1$. For such values of λ, a classical solution u_λ is obtained by monotone iteration starting from zero; see for example [16].

Step 3. We next prove that there exists a finite parameter λ^* such that for $\lambda < \lambda^*$ there is a classical solution while for $\lambda > \lambda^*$ there does not exist classical solution.

Define λ^* as the supremum of all $\lambda > 0$ for which (1.1) admits a classical solution. By Steps 1 and 2, it follows that $0 < \lambda^* < \infty$. Now, for each $\lambda < \lambda^*$ there exists $\mu \in (\lambda, \lambda^*)$ such that (1.1) admits a classical solution u_μ. Since $f > 0$, u_μ is a bounded supersolution of (1.1), and hence the monotone iteration procedure shows that (1.1) admits a classical solution u_λ with $u_\lambda \leq u_\mu$. Note that the iteration procedure, and hence the solution that it produces, are independent of the supersolution u_μ. In addition, by the same reason u_λ is smaller than any bounded supersolution of (1.1). It follows that u_λ is minimal (i.e., the smallest solution) and that $u_\lambda < u_\mu$.

Step 4. We show now that these minimal solutions u_λ, $0 < \lambda < \lambda^*$, are semistable.

Note that the energy functional (1.6) for problem (1.1) in the set $\{u \in H^s(\mathbb{R}^n) : u \equiv 0$ in $\mathbb{R}^n \setminus \Omega, \ 0 \leq u \leq u_\lambda\}$ admits an absolute minimizer $u_{\min}$. Then, using that u_λ is the minimal solution and that f is positive and increasing, it is not difficult to see that $u_{\min}$ must coincide with u_λ. Considering the second variation of energy (with respect to nonpositive perturbations) we see that $u_{\min}$ is a semistable solution of (1.1). But since $u_{\min}$ agrees with u_λ, then u_λ is semistable. Thus u_λ is semistable.

Step 5. We now prove that the pointwise limit $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a weak solution of (1.1) for $\lambda = \lambda^*$ and that this solution u^* is semistable.

As above, let $\lambda_1 > 0$ the first eigenvalue of $(-\Delta)^s$, and $\varphi_1 > 0$ be the corresponding eigenfunction. Since f is superlinear at infinity, there exists a constant $C > 0$ such that

$$\frac{2\lambda_1}{\lambda^*} t \leq f(t) + C \quad \text{for all} \quad t \geq 0. \tag{2.3}$$

Using φ_1 as a test function in (1.5) for u_λ, we find

$$\int_\Omega \lambda f(u_\lambda)\varphi_1 dx = \int_\Omega \lambda_1 u_\lambda \varphi_1 dx \leq \frac{\lambda^*}{2} \int_\Omega \left(f(u_\lambda) + C\right)\varphi_1 dx.$$

In the last inequality we have used (2.3). Taking $\lambda \geq \frac{3}{4}\lambda^*$, we see that $f(u_\lambda)\varphi_1$ is uniformly bounded in $L^1(\Omega)$. In addition, it follows from the results in [28] that

$$c_1 \delta^s \leq \varphi_1 \leq C_2 \delta^s \quad \text{in } \Omega$$

for some positive constants c_1 and C_2, where $\delta(x) = \text{dist}(x, \partial\Omega)$. Hence, we have that

$$\lambda \int_\Omega f(u_\lambda)\delta^s dx \leq C$$

for some constant C that does not depend on λ. Use now $\bar{u}$, the solution of (2.2), as a test function. We obtain that

$$\int_\Omega u_\lambda dx = \lambda \int_\Omega f(u_\lambda)\bar{u} dx \leq C_3 \lambda \int_\Omega f(u_\lambda)\delta^s dx \leq C$$

for some constant C depending only on f and Ω. Here we have used that $\bar{u} \leq C_3 \delta^s$ in Ω for some constant $C_3 > 0$, which also follows from [28].

Thus, both sequences, u_λ and $\lambda f(u_\lambda)\delta^s$ are increasing in λ and uniformly bounded in $L^1(\Omega)$ for $\lambda < \lambda^*$. By monotone convergence, we conclude that $u^* \in L^1(\Omega)$ is a weak solution of (1.1) for $\lambda = \lambda^*$.

Finally, for $\lambda < \lambda^*$ we have $\int_\Omega \lambda f'(u_\lambda)|\eta|^2 dx \le \|\eta\|_{\mathring{H}^s}^2$, where $\|\eta\|_{\mathring{H}^s}^2$ is defined by (1.7), for all $\eta \in H^s(\mathbb{R}^n)$ with $\eta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Since $f' \ge 0$, Fatou's lemma leads to

$$\int_\Omega \lambda^* f'(u^*)|\eta|^2 dx \le \|\eta\|_{\mathring{H}^s}^2,$$

and hence u^* is semistable. $\qquad\square$

Remark 2.1. As said in the introduction, the study of extremal solutions involves three classes of solutions: classical, energy, and weak solutions; see Definition 1.1. It follows from their definitions that any classical solution is an energy solution, and that any energy solution is a weak solution.

Moreover, any weak solution u which is bounded is a classical solution. This can be seen as follows. First, by considering $u * \eta^\epsilon$ and $f(u) * \eta^\epsilon$, where η^ϵ is a standard mollifier, it is not difficult to see that u is regular in the interior of Ω. Moreover, by scaling, we find that $|(-\Delta)^{s/2}u| \le C\delta^{-s}$, where $\delta(x) = \text{dist}(x, \partial\Omega)$. Then, if $\zeta \in C_c^\infty(\Omega)$, we can integrate by parts in (1.5) to obtain

$$(u, \zeta)_{\mathring{H}^s} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\big(u(x) - u(y)\big)\big(\zeta(x) - \zeta(y)\big)}{|x - y|^{n+2s}} dx\, dy = \int_\Omega \lambda f(u)\zeta dx \qquad (2.4)$$

for all $\zeta \in C_c^\infty(\Omega)$. Hence, since $f(u) \in L^\infty$, by density (2.4) holds for all $\zeta \in H^s(\mathbb{R}^n)$ such that $\zeta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, and therefore u is an energy solution. Finally, bounded energy solutions are classical solutions; see Remark 2.11 in [28] and [33].

3. An example case: the exponential nonlinearity

In this section we study the regularity of the extremal solution for the nonlinearity $f(u) = e^u$. Although the results of this section follow from Theorem 1.3 (ii), we exhibit this case separately because the proofs are much simpler. Furthermore, this exponential case has the advantage that we have an explicit unbounded solution to the equation in the whole $\mathbb{R}^n$, and we can compute the values of n and s for which this singular solution is semistable.

The main result of this section is the following.

Proposition 3.1. *Let Ω be a smooth and bounded domain in $\mathbb{R}^n$, and let u^* the extremal solution of (1.1). Assume that $f(u) = e^u$ and $n < 10s$. Then, u^* is bounded.*

Proof. Let α be a positive number to be chosen later. Setting $\eta = e^{\alpha u_\lambda} - 1$ in the stability condition (1.9) (note that $\eta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$), we obtain that

$$\int_\Omega \lambda e^{u_\lambda}(e^{\alpha u_\lambda} - 1)^2 dx \le \|e^{\alpha u_\lambda} - 1\|_{\mathring{H}^s}^2. \qquad (3.1)$$

Next we use that

$$\left(e^b - e^a\right)^2 \leq \frac{1}{2}\left(e^{2b} - e^{2a}\right)(b - a) \tag{3.2}$$

for all real numbers a and b. This inequality can be deduced easily from the Cauchy-Schwarz inequality, as follows

$$\left(e^b - e^a\right)^2 = \left(\int_a^b e^t dt\right)^2 \leq (b - a)\int_a^b e^{2t} dt = \frac{1}{2}\left(e^{2b} - e^{2a}\right)(b - a).$$

Using (3.2), (1.8), and integrating by parts, we deduce

$$
\begin{aligned}
\|e^{\alpha u_\lambda} - 1\|_{\mathring{H}^s}^2 &= \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left(e^{\alpha u_\lambda(x)} - e^{\alpha u_\lambda(y)}\right)^2}{|x - y|^{n+2s}}dxdy \\
&\leq \frac{c_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\frac{1}{2}\left(e^{2\alpha u_\lambda(x)} - e^{2\alpha u_\lambda(y)}\right)\left(\alpha u_\lambda(x) - \alpha u_\lambda(y)\right)}{|x - y|^{n+2s}}dxdy \\
&= \frac{\alpha}{2}\int_\Omega e^{2\alpha u_\lambda}(-\Delta)^s u_\lambda dx.
\end{aligned}
$$

Thus, using that $(-\Delta)^s u_\lambda = \lambda e^{u_\lambda}$, we find

$$\|e^{\alpha u_\lambda} - 1\|_{\mathring{H}^s}^2 \leq \frac{\alpha}{2}\int_\Omega e^{2\alpha u_\lambda}(-\Delta)^s u_\lambda dx = \frac{\alpha}{2}\int_\Omega \lambda e^{(2\alpha+1)u_\lambda}dx. \tag{3.3}$$

Therefore, combining (3.1) and (3.3), and rearranging terms, we get

$$\left(1 - \frac{\alpha}{2}\right)\int_\Omega e^{(2\alpha+1)u_\lambda} - 2\int_\Omega e^{(\alpha+1)u_\lambda} + \int_\Omega e^{\alpha u_\lambda} \leq 0.$$

From this, it follows from Hölder's inequality that for each $\alpha < 2$

$$\|e^{u_\lambda}\|_{L^{2\alpha+1}} \leq C \tag{3.4}$$

for some constant C which depends only on α and $|\Omega|$.

Finally, given $n < 10s$ we can choose $\alpha < 2$ such that $\frac{n}{2s} < 2\alpha + 1 < 5$. Then, taking $p = 2\alpha + 1$ in Proposition 1.4 (iii) (see also Remark 1.5) and using (3.4) we obtain

$$\|u_\lambda\|_{L^\infty(\Omega)} \leq C_1\|(-\Delta)^s u_\lambda\|_{L^p(\Omega)} = C_1\lambda\|e^{u_\lambda}\|_{L^p(\Omega)} \leq C$$

for some constant C that depends only on n, s, and Ω. Letting $\lambda \uparrow \lambda^*$ we find that the extremal solution u^* is bounded, as desired. $\qquad\square$

The following result concerns the stability of the explicit singular solution $\log\frac{1}{|x|^{2s}}$ to equation $(-\Delta)^s u = \lambda e^u$ in the whole $\mathbb{R}^n$.

Proposition 3.2. *Let $s \in (0, 1)$, and let*

$$u_0(x) = \log\frac{1}{|x|^{2s}}.$$

Then, u_0 is a solution of $(-\Delta)^s u = \lambda_0 e^u$ in all of $\mathbb{R}^n$ for some $\lambda_0 > 0$. Moreover, u_0 is semistable if and only if

$$\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1+s)}{\Gamma\left(\frac{n-2s}{2}\right)} \leq \frac{\Gamma^2\left(\frac{n+2s}{4}\right)}{\Gamma^2\left(\frac{n-2s}{4}\right)}. \tag{3.5}$$

As a consequence:

- *If $n \leq 7$, then u is unstable for all $s \in (0,1)$.*
- *If $n = 8$, then u is semistable if and only if $s \lesssim 0'28206....$*
- *If $n = 9$, then u is semistable if and only if $s \lesssim 0'63237....$*
- *If $n \geq 10$, then u is semistable for all $s \in (0,1)$.*

Proposition 3.2 suggests that the extremal solution for the fractional Laplacian should be bounded whenever

$$\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1+s)}{\Gamma\left(\frac{n-2s}{2}\right)} > \frac{\Gamma^2\left(\frac{n+2s}{4}\right)}{\Gamma^2\left(\frac{n-2s}{4}\right)}, \tag{3.6}$$

at least for the exponential nonlinearity $f(u) = e^u$. In particular, u^* should be bounded for all $s \in (0,1)$ whenever $n \leq 7$. This is an open problem.

Remark 3.3. When $s = 1$ and when $s = 2$, inequality (3.6) coincides with the expected optimal dimensions for which the extremal solution is bounded for the Laplacian Δ and for the bilaplacian Δ^2, respectively. In the unit ball $\Omega = B_1$, it is well known that the extremal solution for $s = 1$ is bounded whenever $n \leq 9$ and may be singular if $n \geq 10$ [7], while the extremal solution for $s = 2$ is bounded whenever $n \leq 12$ and may be singular if $n \geq 13$ [13]. Taking $s = 1$ and $s = 2$ in (3.6), one can see that the inequality is equivalent to $n < 10$ and $n \lesssim 12.5653...$, respectively.

We next give the

Proof of Proposition 3.2. First, using the Fourier transform, it is not difficult to compute

$$(-\Delta)^s u_0 = (-\Delta)^s \log\frac{1}{|x|^{2s}} = \frac{\lambda_0}{|x|^{2s}},$$

where

$$\lambda_0 = 2^{2s}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1+s)}{\Gamma\left(\frac{n-2s}{2}\right)}.$$

Thus, u_0 is a solution of $(-\Delta)^s u_0 = \lambda_0 e^{u_0}$.

Now, since $f(u) = e^u$, by (1.9) we have that u_0 is semistable in $\Omega = \mathbb{R}^n$ if and only if

$$\lambda_0 \int_{\mathbb{R}^n} \frac{\eta^2}{|x|^{2s}}dx \leq \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}\eta\right|^2 dx$$

for all $\eta \in H^s(\mathbb{R}^n)$.

The inequality

$$\int_\Omega \frac{\eta^2}{|x|^{2s}}dx \leq H_{n,s}^{-1} \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}\eta\right|^2 dx$$

is known as the fractional Hardy inequality, and the best constant

$$H_{n,s} = 2^{2s} \frac{\Gamma^2\left(\frac{n+2s}{4}\right)}{\Gamma^2\left(\frac{n-2s}{4}\right)}$$

was obtained by Herbst [24] in 1977; see also [18]. Therefore, it follows that u_0 is semistable if and only if

$$\lambda_0 \leq H_{n,s},$$

which is the same as (3.5). $\qquad\square$

4. BOUNDEDNESS OF THE EXTREMAL SOLUTION IN LOW DIMENSIONS

In this section we prove Theorem 1.3 (i)-(ii).

We start with a lemma, which is the generalization of inequality (3.2). It will be used in the proof of both parts (i) and (ii) of Theorem 1.3.

Lemma 4.1. *Let f be a $C^1([0,\infty))$ function, $\widetilde{f}(t) = f(t) - f(0)$, $\gamma > 0$, and*

$$g(t) = \int_0^t \widetilde{f}(s)^{2\gamma-2} f'(s)^2 ds. \tag{4.1}$$

Then,

$$\left(\widetilde{f}(a)^\gamma - \widetilde{f}(b)^\gamma\right)^2 \leq \gamma^2 \big(g(a) - g(b)\big)(a-b)$$

for all nonnegative numbers a and b.

Proof. We can assume $a \leq b$. Then, since $\frac{d}{dt}\left\{\widetilde{f}(t)^\gamma\right\} = \gamma \widetilde{f}(t)^{\gamma-1} f'(t)$, the inequality can be written as

$$\left(\int_a^b \gamma \widetilde{f}(t)^{\gamma-1} f'(t) dt\right)^2 \leq \gamma^2 (b-a) \int_a^b \widetilde{f}(t)^{2\gamma-2} f'(t)^2 dt,$$

which follows from the Cauchy-Schwarz inequality. $\qquad\square$

The proof of part (ii) of Theorem 1.3 will be split in two cases. Namely, $\tau \geq 1$ and $\tau < 1$, where τ is given by (1.10). For the case $\tau \geq 1$, Lemma 4.2 below will be an important tool. Instead, for the case $\tau < 1$ we will use Lemma 4.3. Both lemmas are proved by Sanchón in [30], where the extremal solution for the p-Laplacian operator is studied.

Lemma 4.2 ([30]). *Let f be a function satisfying (1.2), and assume that the limit in (1.10) exists. Assume in addition that*

$$\tau = \lim_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2} \geq 1.$$

Then, any $\gamma \in (1, 1+\sqrt{\tau})$ satisfies

$$\limsup_{t\to+\infty} \frac{\gamma^2 g(t)}{f(t)^{2\gamma-1} f'(t)} < 1, \tag{4.2}$$

where g is given by (4.1).

Lemma 4.3 ([30])**.** *Let f be a function satisfying* (1.2)*, and assume that the limit in* (1.10) *exists. Assume in addition that*

$$\tau = \lim_{t \to \infty} \frac{f(t)f''(t)}{f'(t)^2} < 1.$$

Then, for every $\epsilon \in (0, 1 - \tau)$ there exists a positive constant C such that

$$f(t) \le C(1 + t)^{\frac{1}{1-(\tau+\epsilon)}}, \qquad \text{for all } t > 0.$$

The constant C depends only on τ and ϵ.

The first step in the proof of Theorem 1.3 (ii) in case $\tau \ge 1$ is the following result.

Lemma 4.4. *Let f be a function satisfying* (1.2)*. Assume that $\gamma \ge 1$ satisfies* (4.2)*, where g is given by* (4.1)*. Let u_λ be the solution of* (1.1) *given by Proposition 1.2 (i), where $\lambda < \lambda^*$. Then,*

$$\|f(u_\lambda)^{2\gamma} f'(u_\lambda)\|_{L^1(\Omega)} \le C$$

for some constant C which does not depend on λ.

Proof. Recall that the seminorm $\| \cdot \|_{\mathring{H}^s}$ is defined by (1.7). Using Lemma 4.1, (1.8), and integrating by parts,

$$
\begin{aligned}
\left\| \widetilde{f}(u_\lambda)^\gamma \right\|_{\mathring{H}^s}^2 &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left(\widetilde{f}(u_\lambda(x))^\gamma - \widetilde{f}(u_\lambda(y))^\gamma \right)^2}{|x - y|^{n+2s}} dx dy \\
&\le \gamma^2 \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\left(g(u_\lambda(x)) - g(u_\lambda(y)) \right) \left(u_\lambda(x) - u_\lambda(y) \right)}{|x - y|^{n+2s}} dx dy \\
&= \gamma^2 \int_{\mathbb{R}^n} (-\Delta)^{s/2} g(u_\lambda)(-\Delta)^{s/2} u_\lambda \, dx \\
&= \gamma^2 \int_\Omega g(u_\lambda)(-\Delta)^s u_\lambda \, dx \\
&= \gamma^2 \int_\Omega f(u_\lambda) g(u_\lambda) dx.
\end{aligned}
\tag{4.3}
$$

Moreover, the stability condition (1.9) applied with $\eta = \widetilde{f}(u_\lambda)^\gamma$ yields

$$\int_\Omega f'(u_\lambda) \widetilde{f}(u_\lambda)^{2\gamma} \le \left\| \widetilde{f}(u_\lambda)^\gamma \right\|_{\mathring{H}^s}^2.$$

This, combined with (4.3), gives

$$\int_\Omega f'(u_\lambda) \widetilde{f}(u_\lambda)^{2\gamma} \le \gamma^2 \int_\Omega f(u_\lambda) g(u_\lambda). \tag{4.4}$$

Finally, by (4.2) and since $\widetilde{f}(t)/f(t) \to 1$ as $t \to +\infty$, it follows from (4.4) that

$$\int_\Omega f(u_\lambda)^{2\gamma} f'(u_\lambda) \leq C \tag{4.5}$$

for some constant C that does not depend on λ, and thus the proposition is proved. □

We next give the proof of Theorem 1.3 (ii).

Proof of Theorem 1.3 (ii). Assume first that $\tau \geq 1$, where

$$\tau = \lim_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2}.$$

By Lemma 4.4 and Lemma 4.2, we have that

$$\int_\Omega f(u_\lambda)^{2\gamma} f'(u_\lambda)dx \leq C \tag{4.6}$$

for each $\gamma \in (1, 1 + \sqrt{\tau})$.

Now, for any such γ, we have that $\widetilde{f}^{2\gamma}$ is increasing and convex (since $2\gamma \geq 1$), and thus

$$\widetilde{f}(a)^{2\gamma} - \widetilde{f}(b)^{2\gamma} \leq 2\gamma f'(a)\widetilde{f}(a)^{2\gamma-1}(a-b).$$

Therefore, we have that

$$
\begin{aligned}
(-\Delta)^s \widetilde{f}(u_\lambda)^{2\gamma}(x) &= c_{n,s} \int_{\mathbb{R}^n} \frac{\widetilde{f}(u_\lambda(x))^{2\gamma} - \widetilde{f}(u_\lambda(y))^{2\gamma}}{|x-y|^{n+2s}}dy \\
&\leq 2\gamma f'(u_\lambda(x))\widetilde{f}(u_\lambda(x))^{2\gamma-1}c_{n,s}\int_{\mathbb{R}^n}\frac{u_\lambda(x)-u_\lambda(y)}{|x-y|^{n+2s}}dy \\
&= 2\gamma f'(u_\lambda(x))\widetilde{f}(u_\lambda(x))^{2\gamma-1}(-\Delta)^s u_\lambda(x) \\
&\leq 2\gamma\lambda f'(u_\lambda(x))f(u_\lambda(x))^{2\gamma},
\end{aligned}
$$

and thus,

$$(-\Delta)^s \widetilde{f}(u_\lambda)^{2\gamma} \leq 2\gamma\lambda f'(u_\lambda)f(u_\lambda)^{2\gamma} := v(x). \tag{4.7}$$

Let now w be the solution of the problem

$$\begin{cases} (-\Delta)^s w = v & \text{in } \Omega \\ w = 0 & \text{in } \mathbb{R}^n\backslash\Omega, \end{cases} \tag{4.8}$$

where v is given by (4.7). Then, by (4.6) and Proposition 1.4 (i) (see also Remark 1.5),

$$\|w\|_{L^p(\Omega)} \leq \|v\|_{L^1(\Omega)} \leq C \quad \text{for each } p < \frac{n}{n-2s}.$$

Since $\widetilde{f}(u_\lambda)^{2\gamma}$ is a subsolution of (4.8) —by (4.7)—, it follows that

$$0 \leq \widetilde{f}(u_\lambda)^{2\gamma} \leq w.$$

Therefore, $\|f(u_\lambda)\|_{L^p} \leq C$ for all $p < 2\gamma \frac{n}{n-2s}$, where C is a constant that does not depend on λ. This can be done for any $\gamma \in (1, 1 + \sqrt{\tau})$, and thus we find

$$\|f(u_\lambda)\|_{L^p} \leq C \quad \text{for each} \quad p < \frac{2n(1+\sqrt{\tau})}{n-2s}. \tag{4.9}$$

Hence, using Proposition 1.4 (iii) and letting $\lambda \uparrow \lambda^*$ it follows that

$$u^* \in L^\infty(\Omega) \quad \text{whenever} \quad n < 6s + 4s\sqrt{\tau}.$$

Hence, the extremal solution is bounded whenever $n < 10s$.

Assume now $\tau < 1$. In this case, Lemma 4.3 ensures that for each $\epsilon \in (0, 1 - \tau)$ there exist a constant C such that

$$f(t) \leq C(1+t)^m, \qquad m = \frac{1}{1 - (\tau + \epsilon)}. \tag{4.10}$$

Then, by (4.9) we have that $\|f(u_\lambda)\|_{L^p} \leq C$ for each $p < p_0 := \frac{2n(1+\sqrt{\tau})}{n-2s}$.

Next we show that if $n < 10s$ by a bootstrap argument we obtain $u^* \in L^\infty(\Omega)$. Indeed, by Proposition 1.4 (ii) and (4.10) we have

$$f(u^*) \in L^p \quad \Longleftrightarrow \quad (-\Delta)^s u^* \in L^p \quad \Longrightarrow \quad u^* \in L^q \quad \Longrightarrow \quad f(u^*) \in L^{q/m},$$

where $q = \frac{np}{n-2sp}$. Now, we define recursively

$$p_{k+1} := \frac{np_k}{m(n - 2sp_k)}, \qquad p_0 = \frac{2n(1 + \sqrt{\tau})}{n - 2s}.$$

Now, since

$$p_{k+1} - p_k = \frac{p_k}{n - 2sp_k}\left(2sp_k - \frac{m-1}{m}n\right),$$

then the bootstrap argument yields $u^* \in L^\infty(\Omega)$ in a finite number of steps provided that $(m-1)n/m < 2sp_0$. This condition is equivalent to $n < 2s + 4s\frac{1+\sqrt{\tau}}{\tau+\epsilon}$, which is satisfied for ϵ small enough whenever $n \leq 10s$, since $\frac{1+\sqrt{\tau}}{\tau} > 2$ for $\tau < 1$. Thus, the result is proved. □

Before proving Theorem 1.3 (i), we need the following lemma, proved by Nedev in [26].

Lemma 4.5 ([26]). *Let f be a convex function satisfying (1.2), and let*

$$g(t) = \int_0^t f'(\tau)^2 d\tau. \tag{4.11}$$

Then,

$$\lim_{t\to+\infty} \frac{f'(t)\widetilde{f}(t)^2 - \widetilde{f}(t)g(t)}{f(t)f'(t)} = +\infty,$$

where $\widetilde{f}(t) = f(t) - f(0)$.

As said above, this lemma is proved in [26]. More precisely, see equation (6) in the proof of Theorem 1 in [26] and recall that $\widetilde{f}/f \to 1$ at infinity.

We can now give the

Proof of Theorem 1.3 (i). Let g be given by (4.11). Using Lemma 4.1 with $\gamma = 1$ and integrating by parts, we find

$$
\begin{aligned}
\|f(u_\lambda)\|_{\overset{\circ}{H}^s}^2 &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(u_\lambda(x)) - f(u_\lambda(y)))^2}{|x-y|^{n+2s}} dx dy \\
&\leq \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(g(u_\lambda(x)) - g(u_\lambda(y)))(u_\lambda(x) - u_\lambda(y))}{|x-y|^{n+2s}} dx dy \\
&= \int_{\mathbb{R}^n} (-\Delta)^{s/2} g(u_\lambda)(-\Delta)^{s/2} u_\lambda dx \\
&= \int_{\mathbb{R}^n} g(u_\lambda)(-\Delta)^s u_\lambda dx \\
&= \int_\Omega f(u_\lambda) g(u_\lambda).
\end{aligned}
\tag{4.12}
$$

The stability condition (1.9) applied with $\eta = \widetilde{f}(u_\lambda)$ yields

$$
\int_\Omega f'(u_\lambda)\widetilde{f}(u_\lambda)^2 \leq \|\widetilde{f}(u_\lambda)\|_{\overset{\circ}{H}^s}^2,
$$

which combined with (4.12) gives

$$
\int_\Omega f'(u_\lambda)\widetilde{f}(u_\lambda)^2 \leq \int_\Omega f(u_\lambda) g(u_\lambda).
\tag{4.13}
$$

This inequality can be written as

$$
\int_\Omega \left\{ f'(u_\lambda)\widetilde{f}(u_\lambda)^2 - \widetilde{f}(u_\lambda) g(u_\lambda) \right\} \leq f(0) \int_\Omega g(u_\lambda).
$$

In addition, since f is convex we have

$$
g(t) = \int_0^t f'(s)^2 ds \leq f'(t) \int_0^t f'(s) ds \leq f'(t) f(t),
$$

and thus,

$$
\int_\Omega \left\{ f'(u_\lambda)\widetilde{f}(u_\lambda)^2 - \widetilde{f}(u_\lambda) g(u_\lambda) \right\} \leq f(0) \int_\Omega f'(u_\lambda) f(u_\lambda).
$$

Hence, by Lemma 4.5 we obtain

$$
\int_\Omega f(u_\lambda) f'(u_\lambda) \leq C.
\tag{4.14}
$$

Now, on the one hand we have that

$$
f(a) - f(b) \leq f'(a)(a-b),
$$

since f is increasing and convex. This yields, as in (4.7),

$$(-\Delta)^s \widetilde{f}(u_\lambda) \leq f'(u_\lambda)(-\Delta)^s u_\lambda = f'(u_\lambda)f(u_\lambda) := v(x).$$

On the other hand, let w the solution of the problem

$$\begin{cases} (-\Delta)^s w & = & v & \text{in } \Omega \\ w & = & 0 & \text{on } \partial\Omega. \end{cases} \qquad (4.15)$$

By (4.14) and Proposition 1.4 (i) (see also Remark 1.5),

$$\|w\|_{L^p(\Omega)} \leq \|v\|_{L^1(\Omega)} \leq C \text{ for each } p < \frac{n}{n-2s}.$$

Since $\widetilde{f}(u_\lambda)$ is a subsolution of (4.15), then $0 \leq \widetilde{f}(u_\lambda) \leq w$. Therefore,

$$\|f(u^*)\|_{L^p(\Omega)} \leq C \quad \text{for each} \quad p < \frac{n}{n-2s},$$

and using Proposition 1.4 (iii), we find

$$u^* \in L^\infty(\Omega) \quad \text{whenever} \quad n < 4s,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

5. Boundary estimates: the moving planes method

In this section we prove Proposition 1.8. This will be done with the celebrated moving planes method [21], as in the classical boundary estimates for the Laplacian of de Figueiredo-Lions-Nussbaum [14].

The moving planes method has been applied to problems involving the fractional Laplacian by different authors; see for example [11, 1, 17]. However, some of these results use the specific properties of the fractional Laplacian —such as the extension problem of Caffarelli-Silvestre [9], or the Riesz potential expression for $(-\Delta)^{-s}$—, and it is not clear how to apply the method to more general integro-differential operators. Here, we follow a different approach that allows more general nonlocal operators.

The main tool in the proof is the following maximum principle in small domains.

Recently, Jarohs and Weth [22] obtained a parabolic version of the maximum principle in small domains for the fractional Laplacian; see Proposition 2.4 in [22]. The proof of their result is essentially the same that we present in this section. Still, we think that it may be of interest to write here the proof for integro-differential operators with decreasing kernels.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying $\Omega \subset \mathbb{R}^n_+ = \{x_1 > 0\}$. Let K be a nonnegative function in $\mathbb{R}^n$, radially symmetric and decreasing, and satisfying*

$$K(z) \geq c|z|^{-n-\nu} \quad \text{for all} \quad z \in B_1$$

for some positive constants c and ν, and let

$$L_K u(x) = \int_{\mathbb{R}^n} \big(u(y) - u(x)\big)K(x-y)dy.$$

Let $V \in L^\infty(\Omega)$ be any bounded function, and $w \in H^s(\mathbb{R}^n)$ be a bounded function satisfying

$$\begin{cases} L_K w = V(x)w & in\ \Omega \\ w \geq 0 & in\ \mathbb{R}^n_+ \setminus \Omega \\ w(x) \geq -w(x^*) & in\ \mathbb{R}^n_+, \end{cases} \tag{5.1}$$

where x^* is the symmetric to x with respect to the hyperplane $\{x_1 = 0\}$. Then, there exists a positive constant C_0 such that if

$$\left(1 + \|V^-\|_{L^\infty(\Omega)}\right) |\Omega|^{\frac{\nu}{n}} \leq C_0, \tag{5.2}$$

then $w \geq 0$ in Ω.

Remark 5.2. When L_K is the fractional Laplacian $(-\Delta)^s$, then the condition (5.2) can be replaced by $\|V^-\|_{L^\infty}|\Omega|^{\frac{2s}{n}} \leq C_0$.

Proof of Lemma 5.1. The identity $L_K w = V(x)w$ in Ω written in weak form is

$$(\varphi, w)_K := \int\int_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\setminus\Omega)^2} (\varphi(x) - \varphi(y))(w(x) - w(y))K(x-y)dx\,dy = \int_\Omega Vw\varphi \tag{5.3}$$

for all φ such that $\varphi \equiv 0$ in $\mathbb{R}^n \setminus \Omega$ and $\int_{\mathbb{R}^n} (\varphi(x) - \varphi(y))^2 K(x-y)dx\,dy < \infty$. Note that the left hand side of (5.3) can be written as

$$\begin{aligned} (\varphi, w)_K = &\int_\Omega \int_\Omega (\varphi(x) - \varphi(y))(w(x) - w(y))K(x-y)dx\,dy \\ &+ 2\int_\Omega \int_{\mathbb{R}^n_+\setminus\Omega} \varphi(x)(w(x) - w(y))K(x-y)dx\,dy \\ &+ 2\int_\Omega \int_{\mathbb{R}^n_+} \varphi(x)(w(x) - w(y^*))K(x - y^*)dx\,dy, \end{aligned}$$

where y^* denotes the symmetric of y with respect to the hyperplane $\{x_1 = 0\}$.

Choose $\varphi = -w^- \chi_\Omega$, where w^- is the negative part of w, i.e., $w = w^+ - w^-$. Then, we claim that

$$\int\int_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\setminus\Omega)^2} (w^-(x)\chi_\Omega(x) - w^-(y)\chi_\Omega(y))^2 K(x-y)dx\,dy \leq (-w^-\chi_\Omega, w)_K. \tag{5.4}$$

Indeed, first, we have

$$(-w^-\chi_\Omega, w)_K = \int_\Omega \int_\Omega \{(w^-(x)-w^-(y))^2 + w^-(x)w^+(y) + w^+(x)w^-(y)\}K(x-y)dxdy +$$

$$+ 2\int_\Omega \int_{\mathbb{R}^n_+\setminus\Omega} \{w^-(x)(w^-(x) - w^-(y)) + w^-(x)w^+(y)\}K(x-y)dx\,dy$$

$$+ 2\int_\Omega \int_{\mathbb{R}^n_+} \{w^-(x)(w^-(x) - w^-(y^*)) + w^-(x)w^+(y^*)\}K(x - y^*)dx\,dy,$$

where we have used that $w^+(x)w^-(x) = 0$ for all $x \in \mathbb{R}^n$.

Thus, rearranging terms and using that $w^- \equiv 0$ in $\mathbb{R}^n_+ \setminus \Omega$,

$$(-w^- \chi_\Omega, w)_K = \int \int_{\mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2} (w^-(x)\chi_\Omega(x) - w^-(y)\chi_\Omega(y))^2 K(x-y) dx\, dy$$

$$+ \int_\Omega \int_\Omega 2w^-(x)w^+(y) K(x-y) dx\, dy +$$

$$+ 2 \int_\Omega \int_{\mathbb{R}^n_+ \setminus \Omega} \{w^-(x)w^+(y) - w^-(x)w^-(y)\} K(x-y) dx\, dy$$

$$+ 2 \int_\Omega \int_{\mathbb{R}^n_+} \{w^-(x)w^+(y^*) - w^-(x)w^-(y^*)\} K(x-y^*) dx\, dy$$

$$\geq \int \int_{\mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2} (w^-(x)\chi_\Omega(x) - w^-(y)\chi_\Omega(y))^2 K(x-y) dx\, dy +$$

$$+ 2 \int_\Omega \int_{\mathbb{R}^n_+} w^-(x)w^+(y) K(x-y) dx\, dy +$$

$$+ 2 \int_\Omega \int_{\mathbb{R}^n_+} -w^-(x)w^-(y^*) K(x-y^*) dx\, dy.$$

We next use that, since K is radially symmetric and decreasing, $K(x-y^*) \leq K(x-y)$ for all x and y in $\mathbb{R}^n_+$. We deduce

$$(-w^- \chi_\Omega, w)_K \geq \int \int_{\mathbb{R}^{2n} \setminus (\mathbb{R}^n \setminus \Omega)^2} (w^-(x)\chi_\Omega(x) - w^-(y)\chi_\Omega(y))^2 K(x-y) dx\, dy +$$

$$+ 2 \int_\Omega \int_{\mathbb{R}^n_+} w^-(x)w^+(y) - w^-(x)w^-(y^*) K(x-y) dx\, dy,$$

and since $w^-(y^*) \leq w^+(y)$ for all y in $\mathbb{R}^n_+$ by assumption, we obtain (5.4).

Now, on the one hand note that from (5.4) we find

$$\int_\Omega \int_\Omega (w^-(x) - w^-(y))^2 K(x-y) dx\, dy \leq (-w^- \chi_\Omega, w)_K.$$

Moreover, since $K(z) \geq c|z|^{-n-\nu} \chi_{B_1}(z)$, then

$$\|w^-\|^2_{\overset{\circ}{H}^{\nu/2}(\Omega)} := \frac{c_{n,s}}{2} \int_\Omega \int_\Omega \frac{(w^-(x) - w^-(y))^2}{|x-y|^{-n-\nu}} dx\, dy$$

$$\leq C\|w^-\|_{L^2(\Omega)} + C \int_\Omega \int_\Omega (w^-(x) - w^-(y))^2 K(x-y) dx\, dy,$$

and therefore

$$\|w^-\|^2_{\overset{\circ}{H}^{\nu/2}(\Omega)} \leq C_1 \|w^-\|_{L^2(\Omega)} + C_1(-w^- \chi_\Omega, w)_K. \qquad (5.5)$$

On the other hand, it is clear that

$$\int_\Omega V w w^- = \int_\Omega V(w^-)^2 \leq \|V^-\|_{L^\infty(\Omega)} \|w^-\|_{L^2(\Omega)}. \qquad (5.6)$$

Thus, it follows from (5.3), (5.5), and (5.6) that

$$\|w^-\|^2_{\mathring{H}^{\nu/2}(\Omega)} \leq C_1 \left(1 + \|V^-\|_{L^\infty}\right) \|w^-\|_{L^2(\Omega)}.$$

Finally, by the Hölder and the fractional Sobolev inequalities, we have

$$\|w^-\|^2_{L^2(\Omega)} \leq |\Omega|^{\frac{\nu}{n}} \|w^-\|^2_{L^q(\Omega)} \leq C_2 |\Omega|^{\frac{\nu}{n}} \|w^-\|^2_{\mathring{H}^{\nu/2}(\Omega)},$$

where $q = \frac{2n}{n-\nu}$. Thus, taking C_0 such that $C_0 < (C_1 C_2)^{-1}$ the lemma follows. □

Now, once we have the nonlocal version of the maximum principle in small domains, the moving planes method can be applied exactly as in the classical case.

Proof of Proposition 1.8. Replacing the classical maximum principle in small domains by Lemma 5.1, we can apply the moving planes method to deduce $\|u\|_{L^\infty(\Omega_\delta)} \leq C\|u\|_{L^1(\Omega)}$ for some constants C and $\delta > 0$ that depend only on Ω, as in de Figueiredo-Lions-Nussbaum [14]; see also [3].

Let us recall this argument. Assume first that all curvatures of $\partial\Omega$ are positive. Let $\nu(y)$ be the unit outward normal to Ω at y. Then, there exist positive constants s_0 and α depending only on the convex domain Ω such that, for every $y \in \partial\Omega$ and every $e \in \mathbb{R}^n$ with $|e| = 1$ and $e \cdot \nu(y) \geq \alpha$, $u(y - se)$ is nondecreasing in $s \in [0, s_0]$. This fact follows from the moving planes method applied to planes close to those tangent to Ω at $\partial\Omega$. By the convexity of Ω, the reflected caps will be contained in Ω. The previous monotonicity fact leads to the existence of a set I_x, for each $x \in \Omega_\delta$, and a constant $\gamma > 0$ that depend only on Ω, such that

$$|I_x| \geq \gamma, \qquad u(x) \leq u(y) \quad \text{for all} \quad y \in I_x.$$

The set I_x is a truncated open cone with vertex at x.

As mentioned in page 45 of de Figuereido-Lions-Nussbaum [14], the same can also be proved for general convex domains with a little more of care. □

Remark 5.3. When $\Omega = B_1$, Proposition 1.8 follows from the results in [1], where Birkner, López-Mimbela, and Wakolbinger used the moving planes method to show that any nonnegative bounded solution of

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } B_1 \\ u = 0 & \text{in } \mathbb{R}^n \setminus B_1 \end{cases} \tag{5.7}$$

is radially symmetric and decreasing.

When u is a bounded semistable solution of (5.7), there is an alternative way to show that u is radially symmetric. This alternative proof applies to all solutions (not necessarily positive), but does not give monotonicity. Indeed, one can easily show that, for any $i \neq j$, the function $w = x_i u_{x_j} - x_j u_{x_i}$ is a solution of the linearized problem

$$\begin{cases} (-\Delta)^s w = f'(u)w & \text{in } B_1 \\ w = 0 & \text{in } \mathbb{R}^n \setminus B_1. \end{cases} \tag{5.8}$$

Then, since $\lambda_1 \left((-\Delta)^s - f'(u); B_1\right) \geq 0$ by assumption, it follows that either $w \equiv 0$ or $\lambda_1 = 0$ and w is a multiple of the first eigenfunction, which is positive —see the

proof of Proposition 9 in [31, Appendix A]. But since w is a tangential derivative then it can not have constant sign along a circumference $\{|x| = r\}$, $r \in (0, 1)$, and thus it has to be $w \equiv 0$. Therefore, all the tangential derivatives $\partial_t u = x_i u_{x_j} - x_j u_{x_i}$ equal zero, and thus u is radially symmetric.

6. H^s REGULARITY OF THE EXTREMAL SOLUTION IN CONVEX DOMAINS

In this section we prove Theorem 1.3 (iii). A key tool in this proof is the Pohozaev identity for the fractional Laplacian, recently obtained by the authors in [29]. This identity allows us to compare the interior H^s norm of the extremal solution u^* with a boundary term involving u^*/δ^s, where δ is the distance to $\partial\Omega$. Then, this boundary term can be bounded by using the results of the previous section by the L^1 norm of u^*, which is finite.

We first prove the boundedness of u^*/δ^s near the boundary.

Lemma 6.1. *Let Ω be a convex domain, u be a bounded solution of* (1.14), *and $\delta(x) = dist(x, \partial\Omega)$. Assume that*

$$\|u\|_{L^1(\Omega)} \leq c_1$$

for some $c_1 > 0$. Then, there exists constants $\delta > 0$, c_2, and C such that

$$\|u/\delta^s\|_{L^\infty(\Omega_\delta)} \leq C \left(c_2 + \|f\|_{L^\infty([0,c_2])} \right),$$

where $\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) < \delta\}$. Moreover, the constants δ, c_2, and C depend only on Ω and c_1.

Proof. The result can be deduced from the boundary regularity results in [28] and Proposition 1.8, as follows.

Let $\delta > 0$ be given by Proposition 1.8, and let η be a smooth cutoff function satisfying $\eta \equiv 0$ in $\Omega \setminus \Omega_{2\delta/3}$ and $\eta \equiv 1$ in $\Omega_{\delta/3}$. Then, $u\eta \in L^\infty(\Omega)$ and $u\eta \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Moreover, we claim that

$$(-\Delta)^s(u\eta) = f(u)\chi_{\Omega_{\delta/4}} + g \qquad \text{in } \Omega \tag{6.1}$$

for some function $g \in L^\infty(\Omega)$, with the estimate

$$\|g\|_{L^\infty(\Omega)} \leq C \left(\|u\|_{C^{1+s}(\Omega_{4\delta/5} \setminus \Omega_{\delta/5})} + \|u\|_{L^1(\Omega)} \right). \tag{6.2}$$

To prove that (6.1) holds pointwise we argue separately in $\Omega_{\delta/4}$, in $\Omega_{3\delta/4} \setminus \Omega_{\delta/4}$, and in $\Omega \setminus \Omega_{3\delta/4}$, as follows:

- In $\Omega_{\delta/4}$, $g = (-\Delta)^s(u\eta) - (-\Delta)^s u$. Since $u\eta - u$ vanishes in $\Omega_{\delta/3}$ and also outside Ω, g is bounded and satisfies (6.2).
- In $\Omega_{3\delta/4} \setminus \Omega_{\delta/4}$, $g = (-\Delta)^s(u\eta)$. Then, using

$$\|(-\Delta)^s(u\eta)\|_{L^\infty(\Omega_{3\delta/4} \setminus \Omega_{\delta/4})} \leq C \left(\|u\eta\|_{C^{1+s}(\Omega_{4\delta/5} \setminus \Omega_{\delta/5})} + \|u\eta\|_{L^1(\mathbb{R}^n)} \right)$$

  and that η is smooth, we find that g is bounded and satisfies (6.2).
- In $\Omega \setminus \Omega_{3\delta/4}$, $g = (-\Delta)^s(u\eta)$. Since $u\eta$ vanishes in $\Omega \setminus \Omega_{2\delta/3}$, g is bounded and satisfies (6.2).

Now, since u is a solution of (1.14), by classical interior estimates we have

$$\|u\|_{C^{1+s}(\Omega_{4\delta/5}\setminus\Omega_{\delta/5})} \leq C\left(\|u\|_{L^\infty(\Omega_\delta)} + \|u\|_{L^1(\Omega)}\right); \tag{6.3}$$

see for instance [28]. Hence, by (6.1) and Theorem 1.2 in [28], $u\eta/\delta^s \in C^\alpha(\overline\Omega)$ for some $\alpha > 0$ and

$$\|u\eta/\delta^s\|_{C^\alpha(\overline\Omega)} \leq C\|f(u)\chi_{\Omega_{\delta/4}} + g\|_{L^\infty(\Omega)}.$$

Thus,

$$\|u/\delta^s\|_{L^\infty(\Omega_{\delta/3})} \leq \|u\eta/\delta^s\|_{C^\alpha(\overline\Omega)} \leq C\left(\|g\|_{L^\infty(\Omega)} + \|f(u)\|_{L^\infty(\Omega_{\delta/4})}\right)$$
$$\leq C\left(\|u\|_{L^1(\Omega)} + \|u\|_{L^\infty(\Omega_\delta)} + \|f(u)\|_{L^\infty(\Omega_{\delta/4})}\right).$$

In the last inequality we have used (6.2) and (6.3). Then, the result follows from Proposition 1.8. $\qquad\square$

We can now give the

Proof of Theorem 1.3 (iii). Recall that u_λ minimizes the energy $\mathcal{E}$ in the set $\{u \in H^s(\mathbb{R}^n) : 0 \leq u \leq u_\lambda\}$ (see Step 4 in the proof of Proposition 1.2 in Section 2). Hence,

$$\|u_\lambda\|_{\mathring{H}^s}^2 - \int_\Omega \lambda F(u_\lambda) = \mathcal{E}(u_\lambda) \leq \mathcal{E}(0) = 0. \tag{6.4}$$

Now, the Pohozaev identity for the fractional Laplacian can be written as

$$s\|u_\lambda\|_{\mathring{H}^s}^2 - n\mathcal{E}(u_\lambda) = \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u_\lambda}{\delta^s}\right)^2(x\cdot\nu)d\sigma, \tag{6.5}$$

see [29, page 2]. Therefore, it follows from (6.4) and (6.5) that

$$\|u_\lambda\|_{\mathring{H}^s}^2 \leq \frac{\Gamma(1+s)^2}{2s}\int_{\partial\Omega}\left(\frac{u_\lambda}{\delta^s}\right)^2(x\cdot\nu)d\sigma.$$

Now, by Proposition 6.1, we have that

$$\int_{\partial\Omega}\left(\frac{u_\lambda}{\delta^s}\right)^2(x\cdot\nu)d\sigma \leq C$$

for some constant C that depends only on Ω and $\|u_\lambda\|_{L^1(\Omega)}$. Thus, $\|u_\lambda\|_{\mathring{H}^s} \leq C$, and since $u^* \in L^1(\Omega)$, letting $\lambda \uparrow \lambda^*$ we find

$$\|u^*\|_{\mathring{H}^s} < \infty,$$

as desired. $\qquad\square$

7. L^p AND C^β ESTIMATES FOR THE LINEAR DIRICHLET PROBLEM

The aim of this section is to prove Propositions 1.4 and 1.7. We prove first Proposition 1.4.

Proof of Proposition 1.4. (i) It is clear that we can assume $\|g\|_{L^1(\Omega)} = 1$.
Consider the solution v of

$$(-\Delta)^s v = |g| \ \text{ in } \mathbb{R}^n$$

given by the Riesz potential $v = (-\Delta)^{-s}|g|$. Here, g is extended by 0 outside Ω.

Since $v \geq 0$ in $\mathbb{R}^n \setminus \Omega$, by the maximum principle we have that $|u| \leq v$ in Ω. Then, it follows from Theorem 1.6 that

$$\|u\|_{L^q_{\text{weak}}(\Omega)} \leq C, \quad \text{where} \quad q = \frac{n}{n - 2s},$$

and hence we find that

$$\|u\|_{L^r(\Omega)} \leq C \quad \text{for all} \quad r < \frac{n}{n - 2s}$$

for some constant that depends only on n, s, and $|\Omega|$.

(ii) The proof is analogous to the one of part (i). In this case, the constant does not depend on the domain Ω.

(iii) As before, we assume $\|g\|_{L^p(\Omega)} = 1$. Write $u = \tilde{v} + w$, where $\tilde{v}$ and w are given by

$$\tilde{v} = (-\Delta)^{-s}g \ \text{ in } \mathbb{R}^n, \tag{7.1}$$

and

$$\begin{cases} (-\Delta)^s w &= 0 \ \text{ in } \Omega \\ w &= \tilde{v} \ \text{ in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{7.2}$$

Then, from (7.1) and Theorem 1.6 we deduce that

$$[\tilde{v}]_{C^\alpha(\mathbb{R}^n)} \leq C, \quad \text{where} \quad \alpha = 2s - \frac{n}{p}. \tag{7.3}$$

Moreover, since the domain Ω is bounded, then g has compact support and hence $\tilde{v}$ decays at infinity. Thus, we find

$$\|\tilde{v}\|_{C^\alpha(\mathbb{R}^n)} \leq C \tag{7.4}$$

for some constant C that depends only on n, s, p, and Ω.

Now, we apply Proposition 1.7 to equation (7.2). We find

$$\|w\|_{C^\beta(\mathbb{R}^n)} \leq C\|\tilde{v}\|_{C^\alpha(\mathbb{R}^n)}, \tag{7.5}$$

where $\beta = \min\{\alpha, s\}$. Thus, combining (7.4), and (7.5) the result follows. □

Note that we have only used Proposition 1.7 to obtain the C^β estimate in part (iii). If one only needs an L^∞ estimate instead of the C^β one, Proposition 1.7 is not needed, since the L^∞ bound follows from the maximum principle.

As said in the introduction, the L^p to $W^{2s,p}$ estimates for the fractional Laplace equation, in which $-\Delta$ is replaced by the fractional Laplacian $(-\Delta)^s$, are not true for all p, even when $\Omega = \mathbb{R}^n$. This is illustrated in the following two remarks.

Recall the definition of the fractional Sobolev space $W^{\sigma,p}(\Omega)$ which, for $\sigma \in (0,1)$, consists of all functions $u \in L^p(\Omega)$ such that

$$\|u\|_{W^{\sigma,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+p\sigma}} \, dx \, dy \right)^{\frac{1}{p}}$$

is finite; see for example [15] for more information on these spaces.

Remark 7.1. Let $s \in (0,1)$. Assume that u and g belong to $L^p(\mathbb{R}^n)$, with $1 < p < \infty$, and that

$$(-\Delta)^s u = g \quad \text{in } \mathbb{R}^n.$$

(i) If $p \geq 2$, then $u \in W^{2s,p}(\mathbb{R}^n)$.

(ii) If $p < 2$ and $2s \neq 1$ then u may not belong to $W^{2s,p}(\mathbb{R}^n)$. Instead, $u \in B^{2s}_{p,2}(\mathbb{R}^n)$, where $B^\sigma_{p,q}$ is the Besov space of order σ and parameters p and q.

For more details see the books of Stein [34] and Triebel [35].

By the preceding remark we see that the L^p to $W^{2s,p}$ estimate does not hold in $\mathbb{R}^n$ whenever $p < 2$ and $s \neq \frac{1}{2}$. The following remark shows that in bounded domains Ω this estimate do not hold even for $p \geq 2$.

Remark 7.2. Let us consider the solution of $(-\Delta)^s u = g$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. When $\Omega = B_1$ and $g \equiv 1$, the solution to this problem is

$$u_0(x) = \left(1 - |x|^2\right)^s \chi_{B_1}(x);$$

see [20]. For p large enough one can see that u_0 does not belong to $W^{2s,p}(B_1)$, while $g \equiv 1$ belongs to $L^p(B_1)$ for all p. For example, when $s = \frac{1}{2}$ by computing $|\nabla u_0|$ we see that u_0 does not belong to $W^{1,p}(B_1)$ for $p \geq 2$.

We next prove Proposition 1.7. For it, we will proceed similarly to the C^s estimates obtained in [28, Section 2] for the Dirichlet problem for the fractional Laplacian with L^∞ data.

The first step is the following:

Lemma 7.3. *Let Ω be a bounded domain satisfying the exterior ball condition, $s \in (0,1)$, h be a $C^\alpha(\mathbb{R}^n \setminus \Omega)$ function for some $\alpha > 0$, and u be the solution of (1.13). Then*

$$|u(x) - u(x_0)| \leq C\|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} \delta(x)^\beta \quad \text{in } \Omega,$$

where x_0 is the nearest point to x on $\partial\Omega$, $\beta = \min\{s, \alpha\}$, and $\delta(x) = \text{dist}(x, \partial\Omega)$. The constant C depends only on n, s, and α.

Lemma 7.3 will be proved using the following supersolution. Next lemma (and its proof) is very similar to Lemma 2.6 in [28].

Lemma 7.4. *Let* $s \in (0,1)$. *Then, there exist constants* ϵ, c_1, *and* C_2, *and a continuous radial function* φ *satisfying*

$$\begin{cases} (-\Delta)^s\varphi \geq 0 & in \ B_2 \setminus B_1 \\ \varphi \equiv 0 & in \ B_1 \\ c_1(|x|-1)^s \leq \varphi \leq C_2(|x|-1)^s & in \ \mathbb{R}^n \setminus B_1. \end{cases} \tag{7.6}$$

The constants c_1 *and* C_2 *depend only on* n, s, *and* β.

Proof. We follow the proof of Lemma 2.6 in [28]. Consider the function

$$u_0(x) = (1 - |x|^2)_+^s.$$

It is a classical result (see [20]) that this function satisfies

$$(-\Delta)^s u_0 = \kappa_{n,s} \ \text{ in } B_1$$

for some positive constant $\kappa_{n,s}$.

Thus, the fractional Kelvin transform of u_0, that we denote by u_0^*, satisfies

$$(-\Delta)^s u_0^*(x) = |x|^{-2s-n}(-\Delta)^s u_0 \left(\frac{x}{|x|^2}\right) \geq c_0 \ \text{ in } B_2 \setminus B_1.$$

Recall that the Kelvin transform u_0^* of u_0 is defined by

$$u_0^*(x) = |x|^{2s-n} u_0 \left(\frac{x}{|x|^2}\right).$$

Then, it is clear that

$$a_1(|x|-1)^s \leq u_0^*(x) \leq A_2(|x|-1)^s \ \text{ in } B_2 \setminus B_1,$$

while u_0^* is bounded at infinity.

Let us consider now a smooth function η satisfying $\eta \equiv 0$ in B_3 and

$$A_1(|x|-1)^s \leq \eta \leq A_2(|x|-1)^s \ \text{ in } \mathbb{R}^n \setminus B_4.$$

Observe that $(-\Delta)^s\eta$ is bounded in B_2, since $\eta(x)(1+|x|)^{-n-2s} \in L^1$. Then, the function

$$\varphi = Cu_0^* + \eta,$$

for some big constant $C > 0$, satisfies

$$\begin{cases} (-\Delta)^s\varphi \geq 1 & in \ B_2 \setminus B_1 \\ \varphi \equiv 0 & in \ B_1 \\ c_1(|x|-1)^s \leq \varphi \leq C_2(|x|-1)^s & in \ \mathbb{R}^n \setminus B_1. \end{cases}$$

Indeed, it is clear that $\varphi \equiv 0$ in B_1. Moreover, taking C big enough it is clear that we have that $(-\Delta)^s\varphi \geq 1$. In addition, the condition $c_1(|x|-1)^s \leq \varphi \leq C_2(|x|-1)^s$ is satisfied by construction. Thus, φ satisfies (7.7), and the proof is finished. $\square$

Once we have constructed the supersolution, we can give the

Proof of Lemma 7.3. First, we can assume that $\|h\|_{C^\alpha(\mathbb{R}^n\setminus\Omega)} = 1$. Then, by the maximum principle we have that $\|u\|_{L^\infty(\mathbb{R}^n)} = \|h\|_{L^\infty(\mathbb{R}^n)} \leq 1$. We can also assume that $\alpha \leq s$, since

$$\|h\|_{C^s(\mathbb{R}^n)} \leq C\|h\|_{C^\alpha(\mathbb{R}^n\setminus\Omega)} \quad \text{whenever } s < \alpha.$$

Let $x_0 \in \partial\Omega$ and $R > 0$ be small enough. Let B_R be a ball of radius R, exterior to Ω, and touching $\partial\Omega$ at x_0. Let us see that $|u(x) - u(x_0)|$ is bounded by CR^β in $\Omega \cap B_{2R}$.

By Lemma 7.4, we find that there exist constants c_1 and C_2, and a radial continuous function φ satisfying

$$\begin{cases} (-\Delta)^s\varphi \geq 0 & \text{in } B_2 \setminus B_1 \\ \varphi \equiv 0 & \text{in } B_1 \\ c_1(|x| - 1)^s \leq \varphi \leq C_2(|x| - 1)^s & \text{in } \mathbb{R}^n \setminus B_1. \end{cases} \tag{7.7}$$
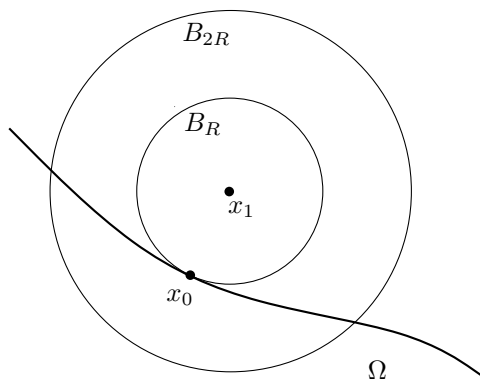

FIGURE 1.

Let x_1 be the center of the ball B_R. Since $\|h\|_{C^\alpha(\mathbb{R}^n\setminus\Omega)} = 1$, it is clear that the function

$$\varphi_R(x) = h(x_0) + 3R^\alpha + C_3 R^s \varphi\left(\frac{x - x_1}{R}\right),$$

with C_3 big enough, satisfies

$$\begin{cases} (-\Delta)^s\varphi_R \geq 0 & \text{in } B_{2R} \setminus B_R \\ \varphi_R \equiv h(x_0) + 3R^\alpha & \text{in } B_R \\ h(x_0) + |x - x_0|^\alpha \leq \varphi_R & \text{in } \mathbb{R}^n \setminus B_{2R} \\ \varphi_R \leq h(x_0) + C_0 R^\alpha & \text{in } B_{2R} \setminus B_R. \end{cases} \tag{7.8}$$

Here we have used that $\alpha \leq s$.

Then, since

$$\begin{aligned} (-\Delta)^s u &\equiv 0 \leq (-\Delta)^s\varphi_R & \text{in } \Omega \cap B_{2R}, \\ h &\leq h(x_0) + 3R^\alpha \equiv \varphi_R & \text{in } B_{2R} \setminus \Omega, \end{aligned}$$

and
$$h(x) \le h(x_0) + |x - x_0|^\alpha \le \varphi_R \quad \text{in } \mathbb{R}^n \setminus B_{2R},$$
it follows from the comparison principle that
$$u \le \varphi_R \quad \text{in } \Omega \cap B_{2R}.$$
Therefore, since $\varphi_R \le h(x_0) + C_0 R^\alpha$ in $B_{2R} \setminus B_R$,
$$u(x) - h(x_0) \le C_0 R^\alpha \quad \text{in } \Omega \cap B_{2R}. \tag{7.9}$$

Moreover, since this can be done for each x_0 on $\partial\Omega$, $h(x_0) = u(x_0)$, and we have $\|u\|_{L^\infty(\Omega)} \le 1$, we find that
$$u(x) - u(x_0) \le C\delta^\beta \quad \text{in } \Omega, \tag{7.10}$$
where x_0 is the projection on $\partial\Omega$ of x.

Repeating the same argument with u and h replaced by $-u$ and $-h$, we obtain the same bound for $h(x_0) - u(x)$, and thus the lemma follows. $\qquad\square$

The following result will be used to obtain C^β estimates for u inside Ω. For a proof of this lemma see for example Corollary 2.4 in [28].

Lemma 7.5 ([28]). *Let $s \in (0,1)$, and let w be a solution of $(-\Delta)^s w = 0$ in B_2. Then, for every $\gamma \in (0, 2s)$*
$$\|w\|_{C^\gamma(\overline{B_{1/2}})} \le C\left(\|(1 + |x|)^{-n-2s} w(x)\|_{L^1(\mathbb{R}^n)} + \|w\|_{L^\infty(B_2)} \right),$$
where the constant C depends only on n, s, and γ.

Now, we use Lemmas 7.3 and 7.5 to obtain interior C^β estimates for the solution of (1.13).

Lemma 7.6. *Let Ω be a bounded domain satisfying the exterior ball condition, $h \in C^\alpha(\mathbb{R}^n \setminus \Omega)$ for some $\alpha > 0$, and u be the solution of (1.13). Then, for all $x \in \Omega$ we have the following estimate in $B_R(x) = B_{\delta(x)/2}(x)$*
$$\|u\|_{C^\beta(\overline{B_R(x)})} \le C\|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)}, \tag{7.11}$$
where $\beta = \min\{\alpha, s\}$ and C is a constant depending only on Ω, s, and α.

Proof. Note that $B_R(x) \subset B_{2R}(x) \subset \Omega$. Let $\tilde{u}(y) = u(x + Ry) - u(x)$. We have that
$$(-\Delta)^s \tilde{u}(y) = 0 \quad \text{in } B_1. \tag{7.12}$$
Moreover, using Lemma 7.3 we obtain
$$\|\tilde{u}\|_{L^\infty(B_1)} \le C\|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} R^\beta. \tag{7.13}$$
Furthermore, observing that $|\tilde{u}(y)| \le C\|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} R^\beta(1 + |y|^\beta)$ in all of $\mathbb{R}^n$, we find
$$\|(1 + |y|)^{-n-2s} \tilde{u}(y)\|_{L^1(\mathbb{R}^n)} \le C\|h\|_{C^\alpha(\mathbb{R}^n \setminus \Omega)} R^\beta, \tag{7.14}$$
with C depending only on Ω, s, and α.

Now, using Lemma 7.5 with $\gamma = \beta$, and taking into account (7.12), (7.13), and (7.14), we deduce

$$\|\tilde{u}\|_{C^\beta\left(\overline{B_{1/4}}\right)} \leq C\|h\|_{C^\alpha(\mathbb{R}^n\setminus\Omega)}R^\beta,$$

where $C = C(\Omega, s, \beta)$.

Finally, we observe that

$$[u]_{C^\beta\left(\overline{B_{R/4}(x)}\right)} = R^{-\beta}[\tilde{u}]_{C^\beta\left(\overline{B_{1/4}}\right)}.$$

Hence, by an standard covering argument, we find the estimate (7.11) for the C^β norm of u in $\overline{B_R(x)}$. □

Now, Proposition 1.7 follows immediately from Lemma 7.6, as in Proposition 1.1 in [28].

Proof of Proposition 1.7. This proof is completely analogous to the proof of Proposition 1.1 in [28]. One only have to replace the s in that proof by β, and use the estimate from the present Lemma 7.6 instead of the one from [28, Lemma 2.9].   □

Acknowledgements

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Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain
  E-mail address: xavier.ros.oton@upc.edu

Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain
  E-mail address: joaquim.serra@upc.edu

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AN EXTENSION PROBLEM FOR SUMS OF FRACTIONAL LAPLACIANS AND 1-D SYMMETRY OF PHASE TRANSITIONS

XAVIER CABRÉ AND JOAQUIM SERRA

ABSTRACT. We prove, in low dimensions, 1-D symmetry of layer solutions to Allen-Cahn type equations involving a sum of fractional Laplacians. These operators are not scale invariant and hence they are infinitesimal generators of Lévy processes which are not stable. Still, we can set up a useful extension problem for these operators consisting in a system of PDEs coupled by a Dirichlet "common trace" condition and one Neumann type boundary condition.

1. INTRODUCTION

In this paper we study *layer solutions* of phase transition problems with a nonlocal diffusion. The main novelty is that the diffusion operator that we consider does not have self-similarity properties. For instance, we consider the nonlocal Allen-Cahn type equation

$$\sum_{i=1}^{K} \mu_i (-\Delta)^{s_i} u + W'(u) = 0 \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $\mu_i > 0$, $\sum \mu_i = 1$, $0 < s_1 < \cdots < s_K < 1$, and W is a double-well potential with wells of the same height located at ± 1. By definition, a *layer solution* is a solution which is monotone in the x_n direction with limits ± 1 as $x_n \to \pm\infty$. That is,

$$u_{x_n} \geq 0 \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1}. \tag{1.2}$$

Having always (1.1) in mind, we actually consider the more general equation

$$Lu + W'(u) = 0 \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

where, for some $s_* \in (0, 1)$, we have

$$Lu = \int_{[s_*, 1)} (-\Delta)^s u \, d\mu(s), \quad \mu \geq 0, \quad \mu\big([s_*, 1)\big) = 1. \tag{1.4}$$

We assume that μ is a probability measure supported in $[s_*, 1)$, i.e.,

$$\mu \geq 0 \quad \text{and} \quad \mu\big([s_*, 1)\big) = \mu(\mathbb{R}) = 1.$$

The operator L is the infinitesimal generator of a Lévy process $Y(t)$ which is isotropic but not stable. It has different behaviors at large and small time scales. Heuristically,

for a very small time step h, the probability that the increment $Y(t + h) - Y(t)$ coincides with that of a $2s$-stable Lévy process is given by $\mu(ds)$.

Recall that the fractional Laplacian is defined by

$$(-\Delta)^s u(x) = c_n(s) \, \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy \,, \tag{1.5}$$

where

$$c_n(s) = \pi^{-\frac{n}{2}} 2^{2s} \frac{\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma(2 - s)} s(1 - s) \,. \tag{1.6}$$

Equivalently, $(-\Delta)^s$ is the operator whose Fourier symbol is $|\xi|^{2s}$.

We may assume that

$$s_* = \max\{s \, : \, \mathrm{support} \, \mu \subset [s, 1)\} \,. \tag{1.7}$$

In the case of problem (1.1), we have $s_* = s_1$, which is the relevant exponent in a blow-down of the equation.

The double-well potential W is assumed to satisfy

$$W \in C^3(\mathbb{R}) \,, \quad W(\pm 1) = 0 \quad \text{and} \quad W(t) > 0 \text{ for } t \neq \pm 1 \,. \tag{1.8}$$

Similarly as for scale invariant diffusions in [19, 20], the appropriate energy functional for our problem is

$$\mathcal{E}(u, \Omega) = \mathcal{K}(u, \Omega) + \int_\Omega W(u) \, dx, \quad \text{with } \mathcal{K}(u, \Omega) = \int \mathcal{K}^s(u, \Omega) \, d\mu(s) \,, \tag{1.9}$$

where, for $0 < s < 1$,

$$\mathcal{K}^s(u, \Omega) = \frac{c_n(s)}{2} \iint \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \chi_\Omega(x) \left(\tfrac{1}{2}\chi_\Omega(y) + \chi_{\mathcal{C}\Omega}(y)\right) dx dy \,. \tag{1.10}$$

In this paper we establish an extension problem for the operator L. As a main application we obtain the following 1-D symmetry result for layer solutions to (1.3).

Theorem 1.1. *Assume that $u \in L^\infty(\mathbb{R}^n)$ is a layer solution of (1.3), that is, satisfying (1.2). Assume that either $n = 2$ and $s_* > 0$, or that $n = 3$ and $s_* \geq 1/2$, where s_* is given by (1.7).*

Then, u has 1-D symmetry. That is, $u(x) = u_0(a \cdot x)$ where $u_0 : \mathbb{R} \to \mathbb{R}$ is a layer solution in dimension one of $Lu_0 + W'(u_0) = 0$ in $\mathbb{R}$ and $a \in \mathbb{R}^n$ is some unit vector.

The existence of a 1-D solution relies on interior estimates for the operator L in bounded domains. These estimates are not very simple by the following two reasons. First, since the support of μ may arrive all the way to $s = 1$ we can not take advantage of the operator begin nonlocal to show Hölder continuity of solutions in a bounded domain. Second, since the measure μ can be continuous (not discrete) then the operator L may not have a well definite leading order. Therefore, the Hölder regularity in bounded domains for L requires some analysis based on the smoothness and growth of the Fourier multiplier. It will be established in a future work. Here, we use a factorization trick to deduce estimates in the whole $\mathbb{R}^n$; see Section 2.

When L is of the form (1.1) the interior estimates in a bounded domain are very elementary and the existence of a 1-D solution follows from a similar argument as in Palatuci, Savin, and Valdinoci [17].

Theorem 1.1 is clearly inspired in a conjecture of De Giorgi [11] for the Allen-Cahn equation: $-\Delta u = u - u^3$ in all $\mathbb{R}^n$. This conjecture states that, if $n \leq 8$, then solutions u which are monotone in one variable have 1-D symmetry. This has been proved in dimensions $n = 2$ by Ghoussoub and Gui [14], $n = 3$ by Ambrosio and Cabré [2], and for $4 \leq n \leq 8$, when one assumes in addition that u is a layer solution, by Savin [18].

For the related nonlocal equation, $(-\Delta)^s u + W'(u) = 0$ in all $\mathbb{R}^n$, analog results have been found for $n = 2$ and $s = 1/2$ by Cabré and Solà-Morales [7], for $n = 2$ and $s \in (0,1)$ by Cabré and Sire [5], and for $n = 3$ and $s \in [1/2, 1)$ by Cabré and Cinti [3, 4].

In this paper, we show how several arguments in [14, 2, 7, 3, 4, 5] can be adapted to equation (1.3) to obtain 1-D symmetry results. In these papers, symmetry is deduced from a Liouville type theorem. Provided that u satisfies certain energy estimates, this Liouville type theorem implies that any two directional derivatives of u coincide up to a multiplicative constant. This is equivalent to the 1-D symmetry. All the known symmetry results for fractional equations [3, 4, 5, 7] were proven using the extension problem of Caffarelli and Silvestre [10], which seems necessary to prove and even to state the Liouville theorem. The main novelty of the present paper is that we have an non scale invariant operator and the existence of an extension problem is a priori unclear. Here, we show what is the natural extension problem, and how one can prove the symmetry result using it. This new extension problem, discussed in Section 4, consists of a "system" of (possibly infinitely many) singular elliptic PDEs which are coupled by a single Neumann type boundary condition and a common trace constrain.

The ideas of this paper could be useful in other contexts where an extension operator is known for a family of operators and one needs to consider also sums (or integrals) of these operators.

A crucial step towards the 1-D symmetry is the obtention of a sharp estimate for the energy of minimizers and of monotone solutions in a ball of radius $R \geq 1$. Let us define

$$\Phi_{n,s}(R) = \begin{cases} R^{n-1}(R^{1-2s} - 1)(1 - 2s)^{-1} & \text{if } s \neq 1/2, \\ R^{n-1} \log R & \text{if } s = 1/2. \end{cases} \tag{1.11}$$

The function $\Phi_{n,s}(R)$ will be useful since it is continuous and decreasing in s of $R > 1$.

The following result is proven in Section 3. Throughout the paper we use the notation $B_R = \{x \in \mathbb{R}^n, \ |x| < R\}$.

Proposition 1.2. *Let u be a layer solution of* (1.3), *i.e., a solution satisfying* (1.2). *Then, for all $R \geq 1$,*

$$\mathcal{E}(u, B_R) \leq C\Phi_{n,s_*}(R),$$

where $\Phi_{n,s}$ is given by (1.11) and C depends only on n, s_* and W.

Related energy estimates have been obtained in [2, 4, 5, 20].

An strategy to prove Proposition 1.2 could be to show first that layer solutions are minimizers of the energy $\mathcal{E}$ in every ball and to compare the energy of u with some explicit competitor. This was done by Savin and Valdinoci in [20] for $L = (-\Delta)^s$ and their proof (with minor modifications) would give also the correct energy estimate for minimizers of our energy $\mathcal{E}$. However, in order to prove that layer solutions are minimizers via the standard foliation argument we need regularity estimates for solutions to (1.3) in a *bounded domain.* These estimates, in bounded domains and for general L of the form (1.4), turn out to be more intricate than the estimate in the whole space, given by Proposition 1.3 below. By this reason we follow a different approach à la Ambrosio-Cabré [2], which allows to obtain Proposition 1.2 in a more straight-forward way.

Let us now quickly link the energy functional $\mathcal{E}$ with problem (1.3) and make precise our notion of solution to (1.3). The quadratic form $\mathcal{K}(\cdot, \Omega)$ comes from a scalar product, which we denote by $\langle \cdot, \cdot \rangle_\Omega$. Namely,

$$\mathcal{K}(u, \Omega) = \frac{1}{2}\langle u, u \rangle_\Omega. \tag{1.12}$$

This scalar product is defined by

$$\langle u, v \rangle_\Omega = \int \langle u, v \rangle_{\Omega,s} \, d\mu(s) \tag{1.13}$$

where

$$\langle u, v \rangle_{\Omega,s} = c_n(s) \iint \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \chi_\Omega(x) \left(\tfrac{1}{2}\chi_\Omega(y) + \chi_{\mathcal{C}\Omega}(y)\right) dx dy. \tag{1.14}$$

Minimizers of $\mathcal{E}$ (with respect to compact perturbations) are functions in $u : \mathbb{R}^n \to \mathbb{R}$ that satisfy, for every $\xi \in C_c^\infty(\Omega)$,

$$\mathcal{E}(u, \Omega) \leq \mathcal{E}(u + \varepsilon\xi, \Omega)$$

$$= \mathcal{K}(u, \Omega) + \varepsilon^2 \mathcal{K}(\xi, \Omega) + \varepsilon\langle u, \xi \rangle_\Omega + \int_\Omega W(u + \varepsilon\xi) \, dx.$$

Equivalently,

$$0 \leq \varepsilon\mathcal{K}(\xi, \Omega) + \langle u, \xi \rangle_\Omega + \int_\Omega \frac{1}{\varepsilon}(W(u + \varepsilon\xi) - W(u)) \, dx$$

for every bounded domain $\Omega \subset \mathbb{R}^n$ and $\xi \in C_c^\infty(\Omega)$. Letting $\varepsilon \searrow 0$, we obtain

$$\langle u, \xi \rangle_\Omega + \int_\Omega W'(u)\xi \, dx = 0 \quad \text{for every } \Omega \subset\subset \mathbb{R}^n \text{ and } \xi \in C_c^\infty(\Omega). \tag{1.15}$$

Equation (1.15) is the weak version of equation (1.3). We will say that a function $u \in L^\infty(\mathbb{R}^n)$ is a *weak solution* of (1.3) if $\mathcal{E}(u, \Omega) < \infty$ and (1.15) is satisfied for all $\Omega \subset\subset \mathbb{R}^n$ and $\xi \in C_c^\infty(\Omega)$.

The relation between the weak and the strong formulations of the problem is given by the integration by parts type formula

$$\langle u, v \rangle_\Omega = \int_\Omega Lu(x)v(x)\,dx + \int c_n(s)\,d\mu(s) \int_{C\Omega} dx \int_\Omega dy \frac{u(x) - u(y)}{|x - y|^{n+2s}} v(x)\,, \quad (1.16)$$

that holds for $u, v \in C^2(\mathbb{R}^n)$ bounded. This formula is found integrating with respect to $d\mu(s)$ the well-known identities

$$\langle u, v \rangle_{\Omega,s} = \int_\Omega (-\Delta)^s u(x)v(x)\,dx + c_n(s) \int_{C\Omega} dx \int_\Omega dy \frac{u(x) - u(y)}{|x - y|^{n+2s}} v(x)\,. \quad (1.17)$$

These identities are very elementary but useful, for instance in the proof of an energy estimate for monotone solutions. Note the last term on the right side can be interpreted as a nonlocal flux. The identity (1.17) is easily proven by writing $(-\Delta)^s u$ as a singular integral and rearranging some terms. One needs only to observe that

$$\mathrm{PV} \int_\Omega dx \int_\Omega dy \frac{(u(x) - u(y))v(x)}{|x - y|^{n+2s}} = -\mathrm{PV} \int_\Omega dx \int_\Omega dy \frac{(u(x) - u(y))v(y)}{|x - y|^{n+2s}}.$$

On the one hand, using the integration by parts formula in (1.15) we find that, when u is smooth enough, we have

$$\int_\Omega Lu\,\xi\,dx + \int_\Omega W'(u)\xi\,dx = 0 \quad \text{for every } \Omega \subset\subset \mathbb{R}^n \text{ and } \xi \in C_c^\infty(\Omega),$$

and hence u is a solution of (1.3).

On the other hand, if u is merely a measurable function $u : \mathbb{R} \to [-1, 1]$ we can also give a notion of solution to (1.3), now integrating by parts in the opposite direction. Since $\xi \in C_c^\infty(\Omega)$ in (1.15), we find that $\langle u, v \rangle_\Omega = \int_{\mathbb{R}^n} uL\xi\,dx$ and thus

$$\int_{\mathbb{R}^n} uL\xi\,dx + \int_{\mathbb{R}^n} W'(u)\xi\,dx = 0 \quad \text{for every } \xi \in C_c^\infty(\Omega)\,. \quad (1.18)$$

This is the notion of solution to (1.3) *in the sense of distributions.*

Next proposition concerns $C^{2,\gamma}$ regularity of weak solutions to (1.3). It is proved in Section 2. In the following proposition we can prove regularity not only for weak solutions but also for solutions of the equation in the whole $\mathbb{R}^n$ in the sense of distributions.

Proposition 1.3. *Let $u \in L^\infty(\mathbb{R}^n)$ with $|u| \leq 1$ in all $\mathbb{R}^n$. Assume that u satisfies (1.18). Then, $u \in C^{2,\gamma}(\mathbb{R}^n)$ and*

$$\|u\|_{C^{2,\gamma}(\mathbb{R}^n)} \leq C$$

for some $\gamma > 0$ and C depending only on n, s_* and W.

According to Proposition 1.3, layer solutions always satisfy equation (1.3) in the classical sense. Indeed, the proofs of Theorems 2.5, 2.6 and 2.7 in [21] yield the estimate

$$\|(-\Delta)^s u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{C^{2,\gamma}(\mathbb{R}^n)}\,,$$

for every $u \in C^{2,\gamma}(\mathbb{R}^n)$, with C uniform for $s \in [s_*, 1)$ (depending only on n and s_*). Thus, since μ is a probability measure, $Lu = \int (-\Delta)^s u \, d\mu(s)$ is still in $C^{0,\gamma}(\mathbb{R}^n)$ and thus the equation is satisfied classically.

The paper is organized as follows: In Section 2 we prove the regularity Proposition 1.3. In Section 3 we prove the energy estimate of Proposition 1.2. In Section 4 we introduce the extension problem for the operator L that allows us to reformulate problem (1.3) as a system of PDEs. In Section 5, the last one, we obtain a Liouville type theorem within the framework of the extension problem and we us it to prove the 1-D symmetry result, Theorem 1.1.

2. Regularity

In this section we prove Proposition 1.3. It will be obtained by iterating the following

Lemma 2.1. *Let $u \in L^\infty(\mathbb{R}^n)$ satisfy $Lu = w$ in all of $\mathbb{R}^n$ in the sense of distributions. Assume that $w \in C^\beta(\mathbb{R}^n)$, $\beta \geq 0$. Then, there exist $\alpha > 0$ and C depending only on n and s_* such that $u \in C^{\beta+\alpha}(\mathbb{R}^n)$ and*

$$\|u\|_{C^{\beta+\alpha}(\mathbb{R}^n)} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{C^\beta(\mathbb{R}^n)}\big)$$

Proof. Since L is linear and translation invariant, it commutes with convolution. Thus, by considering convolutions of u and w with a smooth approximation of the identity, we may assume that u and w are smooth and that the equation holds in strong sense.

Let us consider first the case $\beta = 0$. Let $\epsilon = s_*/2$ and $v = (-\Delta)^\epsilon u$. Then v satisfies

$$\tilde{L}v = w \quad \text{in } \mathbb{R}^n,$$

where $\tilde{L} = \int_{[\epsilon, 1-\epsilon]} d\mu(\epsilon + t)(-\Delta)^t$. By the results of Silvestre in [22] —which apply to $\tilde{L}$ but not to L— we have

$$\|v\|_{C^{\bar{\alpha}}(\mathbb{R}^n)} \leq C\big(\|v\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{L^\infty(\mathbb{R}^n)}\big), \tag{2.1}$$

where $\bar{\alpha}$ and C depend only on n and s_* (we are using that μ is a probability measure).

But by classical Riesz potential estimates [15], since $(-\Delta)^\epsilon u = v$, we have

$$\|u\|_{C^{\bar{\alpha}+2\epsilon}} \leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|v\|_{C^{\bar{\alpha}}(\mathbb{R}^n)}\big) \tag{2.2}$$

and, since $\bar{\alpha}/2 + 2\epsilon > 2\epsilon$,

$$\|v\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{C^{\bar{\alpha}/2+2\epsilon}}. \tag{2.3}$$

Therefore it follows from (2.1), (2.2), and (2.3) that

$$\begin{aligned}
\|u\|_{C^{\bar{\alpha}+2\epsilon}(\mathbb{R}^n)} &\leq C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|v\|_{L^\infty(\mathbb{R}^n)} + \|w\|_{L^\infty(\mathbb{R}^n)}\big) \\
&\leq C\big(\|u\|_{C^{\bar{\alpha}/2+2\epsilon}(\mathbb{R}^n)} + \|w\|_{L^\infty(\mathbb{R}^n)}\big).
\end{aligned}$$

Thus, the estimate of the lemma with $\alpha = \bar{\alpha} + 2\epsilon$ and $\beta = 0$ follows using the typical interpolation inequality trick.

The cases $\beta > 0$ follow applying the previous case to incremental quotients (of derivatives) of u and w. $\qquad\square$

Finally, we prove Proposition 1.3.

Proof of Proposition 1.3. Since u and $W'(u)$ belong to $L^\infty(\mathbb{R}^n)$, Lemma 2.1 applied with $\beta = 0$ yields the bound $\|u\|_{C^\alpha(\mathbb{R}^n)} \leq C$ for some α depending only on L and some C depending on L, W and $\|u\|_{L^\infty}$. But f is a C^2 function and hence we find also a bound for $\|f(u)\|_{C^\alpha(\mathbb{R}^n)}$. This starts an standard bootstrap argument that leads, after using Lemma 2.1 $\lceil 2/\alpha \rceil$ times, to the estimate $\|u\|_{C^{2,\gamma}(\mathbb{R}^n)} \leq C$, where $\gamma = \lceil 2/\alpha \rceil \alpha - 2$. $\qquad\square$

3. ENERGY ESTIMATES

In this section we obtain an energy estimate for layer solutions of (1.3).

Next Claim will be used to prove the energy estimates. Recall the definition of $\Phi_{n,s}(R)$ from (1.11). Its proof is a simple calculation and we differ it to the Appendix.

Claim 3.1. *For every $R \geq 1$, we have*

$$c_n(s) \int_{B_R} \int_{\mathcal{C}B_R} \frac{\min\{1, |x - y|\}}{|x - y|^{n+2s}} \, dx \, dy \leq C\Phi_{n,s}(R)$$

where C depends only on n (but not on s).

The following proposition establishes the energy estimate for layer solutions in every dimension. Since there is no extra effort in doing it, we prove a slightly more general statement that can be used to show energy estimates for monotone solutions (without limits) in dimension three.

Proposition 3.2. *Let u be a solution of (1.1) which is monotone in the x_n direction. Define $\bar{u} : \mathbb{R}^n \to \mathbb{R}$ by $\bar{u}(x', x_n) = \bar{u}(x') = \lim_{x_n \to +\infty} u(x', x_n)$. Then, there exists a constant C depending only on n, s_*, and W, such that*

$$\mathcal{E}(u, B_R) - \mathcal{E}(\bar{u}, B_R) \leq C\Phi_{n,s_*}(R) \tag{3.1}$$

for every $R \geq 1$.

Proof. Consider, as in [2], the slided function u^t, $t \geq 0$, defined by $u^t(x', x_n) = u^t(x', x_n + t)$.

Using the integration by parts formula (1.16) and the equation satisfied by u^t we find

$$\frac{d}{dt} \mathcal{E}(u^t, B_R) = \int d\mu(s) c_n(s) \int_{\mathcal{C}B_R} dx \int_{B_R} dy \frac{u^t(x) - u^t(y)}{|x - y|^{n+2s}} \partial_t u^t(x) \,. \tag{3.2}$$

Indeed, we have

$$\frac{d}{dt}\mathcal{E}(u^t, B_R) = \langle u^t, \partial_t u^t \rangle_{B_R} + \int_\Omega W'(u)\partial_t u^t \, dx$$

$$= \int_{B_R} Lu^t \partial_t u^t + \int d\mu(s)c_n(s) \int_{\mathcal{C}B_R} dx \int_{B_R} dy \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\partial_t u^t(x) +$$

$$+ \int_{B_R} W'(u^t)\partial_t u^t(u) \, dx \,,$$

and note that $Lu^t + W'(u^t) \equiv 0$.

Using the bound $\|u^t\|_{C^{2,\gamma}(\mathbb{R}^n)} \leq C$ in Proposition 1.3 with C depending only on n, s_*, and W —thus, C independent of t— we find, by monotone convergence, that $u^t \to \bar{u}$ in $C^{2,\gamma}_{\text{loc}}(\mathbb{R}^n)$. We also find that $|u^t(x) - u^t(y)| \leq C\min\{1, |x-y|\}$.

Therefore, we have

$$\mathcal{E}(u, B_R) - \mathcal{E}(\bar{u}, B_R) = \mathcal{E}(u^t, B_R)\big|^0_{+\infty} = -\int_0^{+\infty} \frac{d}{dt}\mathcal{E}(u^t, B_R) \, dt \,.$$

Integrating (3.2) we obtain

$$\mathcal{E}(u, B_R) - \mathcal{E}(\bar{u}, B_R) = -\int_0^{+\infty} \frac{d}{dt}\mathcal{E}(u^t, B_R) \, dt$$

$$= -\int_0^{+\infty} dt \int d\mu(s)c_n(s) \int_{\mathcal{C}B_R} dx \int_{B_R} dy \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\partial_t u^t(x)$$

$$\leq \int_0^{+\infty} dt \int d\mu(s)c_n(s) \int_{\mathcal{C}B_R} dx \int_{B_R} dy \frac{C\min\{1, |x-y|\}}{|x-y|^{n+2s}}\partial_t u^t(x)$$

$$= \int d\mu(s)c_n(s) \int_{\mathcal{C}B_R} dx \int_{B_R} dy \frac{C\min\{1, |x-y|\}}{|x-y|^{n+2s}} \int_0^\infty \partial_t u^t(x) \, dt$$

$$\leq C\|u\|_{L^\infty(\mathbb{R}^n)} \int d\mu(s)c_n(s) \int_{\mathcal{C}B_R} dx \int_{B_R} dy \frac{\min\{1, |x-y|\}}{|x-y|^{n+2s}}$$

$$\leq C \int d\mu(s)\Phi_{n,s}(R)$$

$$\leq C\Phi_{n,s_*}(R) \,,$$

for some C depending only on n, s_*, and W. We have used Claim 3.1 and the fact that $\Phi_{n,s}$ is decreasing in s. $\qquad\square$

We finally give the

Proof of Proposition 1.2. It is an immediate consequence of Proposition 3.2, observing that, for layer solutions, we have $\bar{u} \equiv 1$ and clearly $\mathcal{E}(1, B_R) = 0$ for all $R > 0$. $\qquad\square$

4. Extension problem

In this section we give a local formulation of problem (1.3):

$$\int (-\Delta)^s u \, d\mu(s) = f(u) \, .$$

This can be done by working, at the same time, with several (or possibly infinitely many) extension problems of type Caffarelli-Silvestre [10].

Given $s \in (0,1)$ and $u \in L^\infty(\mathbb{R}^n)$, the *s-extension of u to $\mathbb{R}^{n+1}_+$* is defined by

$$\tilde{u}_s(\cdot, \lambda) = P_s(\cdot, \lambda) * u \quad \text{in } \mathbb{R}^n$$

for all $\lambda > 0$, where P_s is

$$P_s(x, \lambda) = p_{n,s} \frac{\lambda^{2s}}{\left(|x|^2 + \lambda^2\right)^{\frac{n+2s}{2}}} \, .$$

The function $\tilde{u}_s$ solves the extension problem of Caffarelli and Silvestre [10]:

$$\begin{cases} \nabla \cdot (\lambda^{1-2s} \nabla \tilde{u}_s) = 0 & \text{in } \mathbb{R}^{n+1}_+ = \{(x, \lambda), \ x \in \mathbb{R}^n, \ \lambda > 0\}, \\ \tilde{u}_s(x, 0) = u(x) & \text{on } \{\lambda = 0\} \, . \end{cases} \quad (4.1)$$

Moreover, from results in [10] we have that, for u regular enough,

$$(-\Delta)^s u(x) = -d(s) \lim_{\lambda \to 0^+} \lambda^{1-2s} \partial_\lambda \tilde{u}_s(x, \lambda) \, .$$

From the considerations above, to every solution of problem (1.3), it corresponds a solution of the following system of PDEs:

$$\begin{cases} \nabla \cdot (\lambda^{1-2s} \nabla \tilde{u}_s) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \tilde{u}_s(x, 0) = u(x) & \text{on } \{\lambda = 0\}, \\ -\int d(s) \lim_{\lambda \to 0^+} \lambda^{1-2s} \partial_\lambda \tilde{u}_s(x, \lambda) \, d\mu(s) = f(u) & \text{on } \{\lambda = 0\}. \end{cases} \quad (4.2)$$

This system possibly involves infinitely many unknowns, the functions $\{\tilde{u}_s\}_{s \in \text{supp}\,\mu}$. Note that if we consider the operator $L = \sum_{i=1}^K \mu_i (-\Delta)^s$ the number of unknowns appearing in the system is K (plus the common boundary value u).

This leads us to consider the energy functional

$$\tilde{\mathcal{E}}(\boldsymbol{w}, \Omega) = \tilde{\mathcal{K}}(\boldsymbol{w}, \Omega) + \int_{\underline{\Omega}} W(\underline{\boldsymbol{w}}) \, dx \, ,$$

with

$$\tilde{\mathcal{K}}(\boldsymbol{w}, \Omega) = \frac{1}{2} \int d\mu(s) \int_{\Omega^+} d(s) \lambda^{1-2s} |\nabla w_s|^2 \, dx \, d\lambda \, ,$$

where $\Omega \subset \mathbb{R}^{n+1}$ is any Lipschitz domain and Ω^+ and $\underline{\Omega}$ are, respectively, $\Omega \cap \{\lambda > 0\}$ and $\Omega \cap \{\lambda = 0\}$. Here, $\boldsymbol{w} = \{w_s\}_{s \in \text{supp}\,\mu}$ denotes a family of bounded functions in $C(\overline{\mathbb{R}^{n+1}_+})$ with the property that the traces in $\mathbb{R}^n$ of all the $w_s \in \boldsymbol{w}$ coincide. We will say that such a family $\boldsymbol{w}$ has *common trace* in $\mathbb{R}^n$ and will denote by $\underline{\boldsymbol{w}}$ the function $w_s|_{\{\lambda=0\}}$ (which is the same for all s).

Formally, the Euler-Lagrange equations for minimizers of $\tilde{\mathcal{E}}$ are (4.2). The following claim relates the kinetic parts of the energies $\mathcal{E}$ in $\mathbb{R}^n$ and $\tilde{\mathcal{E}}$ in $\mathbb{R}^{n+1}_+$. We outline its proof even if we will not use the claim in the rest of the paper.

Claim 4.1. *Let φ be such that $\mathcal{K}(\varphi, \mathbb{R}^n) < \infty$, and for each $s \in \operatorname{supp} \mu$, let $\tilde{\varphi}_s$ be the s-extension of φ to $\mathbb{R}^{n+1}_+$. Then, the family of s-extensions $\tilde{\boldsymbol{\varphi}} = \{\tilde{\varphi}_s\}_{s \in \operatorname{supp} \mu}$ satisfies*

$$\tilde{\mathcal{K}}(\tilde{\boldsymbol{\varphi}}, \mathbb{R}^{n+1}_+) = \mathcal{K}(\varphi, \mathbb{R}^n) < \infty .$$

Moreover, for every pair of functions φ, ψ such that $\mathcal{K}(\varphi, \mathbb{R}^n) < \infty$, $\mathcal{K}(\psi, \mathbb{R}^n) < \infty$, and $u \equiv v$ outside B_R, we have

$$\tilde{\mathcal{K}}(\tilde{\boldsymbol{\varphi}}, \mathbb{R}^{n+1}_+) - \tilde{\mathcal{K}}(\tilde{\boldsymbol{\psi}}, \mathbb{R}^{n+1}_+) = \mathcal{K}(\varphi, B_R) - \mathcal{K}(\psi, B_R) ,$$

where $\tilde{\boldsymbol{\varphi}} = \{\varphi_s\}$ and $\tilde{\boldsymbol{\psi}} = \{\psi_s\}$ are the families of s-extensions.

Proof. We may assume that $\varphi \in C^\infty_c(\mathbb{R}^n)$ by an approximation argument.
Integrating by parts we obtain

$$\int_{\mathbb{R}^{n+1}_+} d(s) \lambda^{1-2s} |\nabla \varphi_s|^2 \, dx \, d\lambda = - \int_{\mathbb{R}^{n+1}_+} d(s) \operatorname{div}\left(\lambda^{1-2s} \nabla \varphi_s\right) \varphi_s \, dx \, d\lambda$$

$$- \lim_{\lambda \searrow 0} \int_{\mathbb{R}^n} d(s) \lambda^{1-2s} (\partial_\lambda \varphi_s) \varphi_s \, dx \quad (4.3)$$

$$= 0 + \int_{\mathbb{R}^n} \varphi (-\Delta)^s \varphi \, dx = 2 \mathcal{K}^s(\varphi, \mathbb{R}^n).$$

The first part of the Claim follows integrating (4.3) with respect to $d\mu(s)$.
The second part of the Claim is proven similarly. $\qquad\square$

From here, by reproducing almost exactly the arguments in [9], next proposition is proven. It extends Lemma 7.2 for nonlocal minimal surfaces in [9] to our situation. Let us point out that the next proposition is not used in the sequel, but we state it since it gives an important structural property of our extension property. As a consequence of it, we could obtain the close relation between minimizers of $\mathcal{E}$ in $\mathbb{R}^n$ and of $\tilde{\mathcal{E}}$ in $\mathbb{R}^{n+1}_+$.

Proposition 4.2. *Assume that u be such that $\mathcal{K}(u, B_1) < \infty$ and let $\tilde{\boldsymbol{u}} = \{\tilde{u}_s\}_{s \in \operatorname{supp} \mu}$ be the family of s-extensions. Let $\varphi \in C^\infty_c(B_1)$. Then,*

$$\inf_{\Omega, \boldsymbol{w}} \int d(s) d\mu \int_{\Omega^+} \lambda^{1-2s} (|\nabla w_s|^2 - |\nabla \tilde{u}_s|^2) \, dx \, d\lambda = \mathcal{K}(u + \varphi, B_1) - \mathcal{K}(u, B_1) ,$$

where the infimum is taken among all bounded Lipschitz domains $\Omega \subset \mathbb{R}^{n+1}$ with $\underline{\Omega} \subset B_1$, and all families $\boldsymbol{w} = \{w_s\}$ having common trace $\underline{\boldsymbol{w}} = u + \varphi$ in $\mathbb{R}^n$ and such that $w_s - \tilde{u}_s$ are compactly supported in $\Omega^+ \cup \underline{\Omega}$.

Let us define the notion of minimizers of $\mathcal{E}$ and $\tilde{\mathcal{E}}$.

Definition 4.3. We say that $u \in L^\infty(\mathbb{R}^n)$ is a minimizer of $\mathcal{E}$ —given by (1.9), (1.10)— if $|u| \leq 1$ in all of $\mathbb{R}^n$ and for every $\Omega \subset\subset \mathbb{R}^n$ we have $\mathcal{E}(u, \Omega) < \infty$ and

$$\mathcal{E}(u, \Omega) \leq \mathcal{E}(u + \xi, \Omega) \quad \text{for every } \xi \in C_c^\infty(\Omega) \,.$$

Definition 4.4. We say that a family $\boldsymbol{v} = \{v_s\}_{s \in \text{supp}\,\mu} \subset C(\overline{\mathbb{R}_+^{n+1}})$ having common trace $\underline{\boldsymbol{v}}$ in $\mathbb{R}^n$ is a minimizer of $\tilde{\mathcal{E}}$ if $|\boldsymbol{v}| \leq 1$ on all of $\mathbb{R}^n$ and for every $\Omega \subset\subset \mathbb{R}^{n+1}$ we have $\mathcal{E}(\boldsymbol{v}, \Omega) < \infty$ and

$$\tilde{\mathcal{E}}(\boldsymbol{v}, \Omega) \leq \tilde{\mathcal{E}}(\boldsymbol{v} + \boldsymbol{\xi}, \Omega) \quad \text{for every } \boldsymbol{\xi} = \{\xi_s\} \subset C_c^\infty(\Omega) \text{ with common trace } \underline{\boldsymbol{\xi}} \text{ in } \mathbb{R}^n \,.$$

As a consequence of Proposition 4.2, we have the following link between minimizers of $\mathcal{E}$ and of $\tilde{\mathcal{E}}$. This is related to Proposition 7.3 in [9].

Proposition 4.5. *A function u is a minimizer of $\mathcal{E}$ if, and only if, the family of s-extensions $\tilde{\boldsymbol{u}} = \{\tilde{u}_s\}_{s \in \text{supp}\,\mu}$ is a minimizer of $\tilde{\mathcal{E}}$.*

The following further relation between $\mathcal{E}$ and $\tilde{\mathcal{E}}$ is the only one that we will use in the rest of the paper. It applies to functions possibly having infinite energy in all of $\mathbb{R}^n$. It states that an estimate on the energy $\mathcal{E}$ in balls for a function $u : \mathbb{R}^n \to [-1, 1]$ (satisfying regularity estimates) is immediately translated into an estimate of the energy $\tilde{\mathcal{E}}$ in cylinders for the family of s-extensions $\{u_s\}$. In the remaining part of the paper we denote by C_R the open cylinder in $\mathbb{R}_+^{n+1}$ having as bottom $B_R \subset \mathbb{R}^n$ and height R in the λ direction:

$$C_R = \{(x, \lambda), \ |x| < R, \ 0 < \lambda < R\} \,. \tag{4.4}$$

Lemma 4.6. *Let $u \in C^{2,\gamma}(\mathbb{R}^n)$ with $\|u\|_{C^{2,\gamma}(\mathbb{R}^n)} \leq C_0$. Let $\tilde{\boldsymbol{u}} = \{\tilde{u}_s\}$ be the family of s-extensions —note that $\underline{\tilde{\boldsymbol{u}}} = u$. Then, for $R \geq 1$ we have*

$$|\tilde{\mathcal{K}}(\tilde{\boldsymbol{u}}, C_R) - \mathcal{K}(u, B_R)| \leq C C_0^2 \Phi_{n,s_*}(R) \,,$$

where C depends only on n and s_*.

In the proof of Lemma 4.6 we will need the following elementary bounds for the extension problems.

Lemma 4.7. *Assume that $|u| \leq C_1$ and $|\nabla u| \leq C_2$ in $\mathbb{R}^n$. Then, for $s \in (0, 1)$, the extension of u, $\tilde{u}_s$, satisfies*

$$|\tilde{u}_s| \leq C_1 \quad and \quad |\nabla_x \tilde{u}_s| \leq C_2 \,, \tag{4.5}$$

in all $\mathbb{R}_+^{n+1}$. Moreover,

$$|\nabla_x \tilde{u}_s| + |\partial_\lambda \tilde{u}_s| \leq \frac{C C_1}{\lambda} \quad for \ \lambda > 0 \,, \tag{4.6}$$

where C depends only on n (but not on s).

Proof. These bounds are established in [5, Proposition 4.6.]. The bounds (4.5) follow from the maximum principle. The bound (4.6) follows by interior elliptic estimates, using a scaling argument and observing that for $s \in (0, 1)$ the weight λ^{1-2s} is uniformly bounded between universal constants in the domain $\{1 \leq \lambda \leq 2\}$. $\square$

We next give the

Proof of Lemma 4.6. By definition

$$2\tilde{\mathcal{K}}(\tilde{\boldsymbol{u}}, C_R) = \int d\mu(s) \int_{B_R} \int_0^R d(s)\lambda^{1-2s}|\nabla \tilde{u}_s|^2 \, dx \, d\lambda \,.$$

Integrating by parts,

$$\int_{B_R} \int_0^R d(s)\lambda^{1-2s}|\nabla \tilde{u}_s|^2 \, dx \, d\lambda = \int_{\partial B_R} \int_0^R d(s)\lambda^{1-2s} \tilde{u}_s \frac{\partial \tilde{u}_s}{\partial \nu} \, dS \, d\lambda +$$

$$+ \int_{B_R} d(s)R^{1-2s}(\tilde{u}_s \partial_\lambda \tilde{u}_s)|_{\lambda=R} \, dx + \int_{B_R} \left(\lim_{\lambda \to 0} d(s)\lambda^{1-2s} \partial_\lambda \tilde{u}_s \right) \tilde{u}_s \, dx \,. \quad (4.7)$$

Using the bounds (4.5) and (4.6) —note that the constants C_1, C_2 and C_3 appearing in these bounds are controled by C_0— we obtain

$$\left| \int_{\partial B_R} \int_0^R d(s)\lambda^{1-2s} \tilde{u}_s \frac{\partial \tilde{u}_s}{\partial \nu} \, dS \, d\lambda \right| \le d(s)C_0^2 R^{n-1} \int_0^R \min\{\lambda^{1-2s}, \lambda^{-2s}\} \, d\lambda$$

$$\le CC_0^2 \Phi_{n,s_*}(R)$$

for every $s \ge s^*$. Here we have used that $d(s)/(1-s) \le C$ as $s \nearrow 1$.

Similarly, still using (4.5) and (4.6),

$$\left| \int_{B_R} d(s)R^{1-2s}(\tilde{u}_s \partial_\lambda \tilde{u}_s)(x, R) \, dx \right| \le R^n d(s)R^{1-2s}C_0^2 R^{-1} \le CC_0^2 \, \Phi_{n,s_*}(R) \,.$$

On the other hand, recall that

$$\lim_{\lambda \to 0} d(s)\lambda^{1-2s} \tilde{u}_s(x, \lambda)\partial_\lambda \tilde{u}_s(x, \lambda) = u(x)(-\Delta)^s u(x) \,.$$

Therefore, integrating (4.7) with respect to $d\mu(s)$ and using the previous bounds, we have proven

$$\left| 2\tilde{\mathcal{K}}(\tilde{\boldsymbol{u}}, C_R) - \int_{B_R} uLu \, dx \right| \le CC_0^2 \Phi_{n,s_*}(R) \,.$$

Finally, by the formula of integration by parts (1.16) we have

$$\langle u, u \rangle_\Omega - \int_\Omega Lu(x)u(x) \, dx = \int c_n(s) \, d\mu(s) \int_{C\Omega} dx \int_\Omega dy \frac{u(x) - u(y)}{|x - y|^{n+2s}} u(x) \,.$$

Thus, using Claim 3.1 we obtain

$$\left| \int_{B_R} uLu \, dx - \langle u, u \rangle_{B_R} \right| \le \left| \int c_n(s) \, d\mu(s) \int_{CB_R} dx \int_{B_R} dy \frac{u(x) - u(y)}{|x - y|^{n+2s}} u(x) \right|$$

$$\le CC_0^2 \left| \int c_n(s) \, d\mu(s) \int_{CB_R} dx \int_{B_R} dy \frac{\min\{1, |x - y|\}}{|x - y|^{n+2s}} \right|$$

$$\le CC_0^2 \Phi_{n,s_*}(R) \,.$$

Since by definition $\langle u, u \rangle_\Omega = 2\mathcal{K}(u, \Omega)$, the lemma is proved .   □

Next we obtain $\tilde{\mathcal{E}}$-energy estimates for the family $\tilde{\boldsymbol{u}}$ of s-extensions of a layer solution to (1.3).

Lemma 4.8. *Assume that μ is compactly supported in $[s_*, 1)$. Let u be a layer solution in $\mathbb{R}^n$ of (1.3). Let $\tilde{\boldsymbol{u}} = \{\tilde{u}_s\}_{s\in\operatorname{supp}\mu}$ be the family of s-extensions of u to $\mathbb{R}^{n+1}_+$. Then,*

$$\tilde{\mathcal{E}}(\tilde{\boldsymbol{u}}, C_R) \leq C\Phi_{n,s_*}(R),$$

where $\Phi_{n,s}(R)$ is given by (1.11) and C depends only on n, s_*, and W.

Proof. It is a consequence of Proposition 1.2, combined with Proposition 1.3 and Lemma 4.6. $\qquad\square$

5. Liouville-type theorem and 1-D symmetry

In this section we obtain a Liouville theorem within the frame of the extension system (4.2).

Theorem 5.1. *Let $\boldsymbol{\sigma} = \{\sigma_s\}_{s\in\operatorname{supp}\mu}$ satisfy*

$$\begin{cases} -\sigma_s\nabla\cdot(\lambda^{1-2s}\varphi_s^2\nabla\sigma_s) \leq 0 & \text{in } \mathbb{R}^{n+1}_+, \quad \text{for each } s, \\ \sigma_s(x,0) = \underline{\boldsymbol{\sigma}}(x) & \text{on } \mathbb{R}^n, \quad \text{for each } s, \\ -\int d(s)\,\underline{\boldsymbol{\sigma}\boldsymbol{\varphi}}^2 \lim_{\lambda\to 0} \lambda^{1-2s}\partial_\lambda\sigma_s\,d\mu(s) \leq 0 & \text{on } \mathbb{R}^n, \end{cases} \qquad (5.1)$$

where $\boldsymbol{\varphi} = \{\varphi_s\}_{s\in\operatorname{supp}\mu}$ is a family of positive continuous functions having common trace on $\mathbb{R}^n$. Assume that $\lambda^{1-2s}\varphi_s^2|\nabla\sigma_s|^2 \in L^1_{loc}(\mathbb{R}^{n+1}_+)$, for every $s \in \operatorname{supp}\mu$.

Suppose, in addition, that for $R > 1$,

$$\int d(s)\,d\mu(s)\int_{C_R} \lambda^{1-2s}(\varphi_s\sigma_s)^2\,dxdy \leq CR^2F(R), \qquad (5.2)$$

for some constant C independent of R, and some nondecreasing function $F: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{j=1}^{\infty} \frac{1}{F(2^{j+1})} = +\infty.$$

Then, σ is constant.

Proof. We adapt the proof of Moschini [16, Theorem 5.1]. Since σ satisfies (5.1), we have

$$\nabla\cdot(\sigma_s\lambda^{1-2s}\varphi_s^2\nabla\sigma_s) \geq \lambda^{1-2s}\varphi_s^2|\nabla\sigma_s|^2, \qquad (5.3)$$

for each s. On the other hand,

$$\int_{\partial^+C_R} \sigma_s\lambda^{1-2s}\varphi_s^2\frac{\partial\sigma_s}{\partial\nu}\,dS \leq \left(\int_{\partial^+C_R} \lambda^{1-2s}\varphi_s^2|\nabla\sigma_s|^2\,dS\right)^{\frac{1}{2}}\left(\int_{\partial^+C_R} \lambda^{1-2s}(\varphi_s\sigma_s)^2\,dS\right)^{\frac{1}{2}},$$
$$(5.4)$$

where $\partial^+C_R = \partial C_R \setminus \{\lambda = 0\}$, and ν is the unit outer normal to ∂^+C_R. Now, set

$$D(R) = \int d(s)\,d\mu(s)\int_{C_R} \lambda^{1-2s}\varphi_s^2|\nabla\sigma_s|^2\,dx\,dy.$$

Let us write $d\tilde{\mu}(s) = d(s)d\mu(s)$. Using (5.3), the boundary condition in (5.1), (5.4), and Schwartz inequality we obtain

$$D(R) \leq \int d\tilde{\mu}(s) \int_{C_R} \nabla \cdot (\sigma_s \lambda^{1-2s} \varphi_s^2 \nabla \sigma_s)\, dx\, dy$$

$$\leq \int d\tilde{\mu}(s) \int_{\partial^+ C_R} \sigma_s \lambda^{1-2s} \varphi_s^2 \frac{\partial \sigma_s}{\partial \nu}\, dS$$

$$\leq \int d\tilde{\mu}(s) \left(\int_{\partial^+ C_R} \lambda^{1-2s} \varphi_s^2 |\nabla \sigma_s|^2\, dS \right)^{\frac{1}{2}} \left(\int_{\partial^+ C_R} \lambda^{1-2s} (\varphi_s \sigma_s)^2\, dS \right)^{\frac{1}{2}}$$

$$\leq \left(\int d\tilde{\mu}(s) \int_{\partial^+ C_R} \lambda^{1-2s} \varphi_s^2 |\nabla \sigma_s|^2\, dS \right)^{\frac{1}{2}} \left(\int d\tilde{\mu}(s) \int_{\partial^+ C_R} \lambda^{1-2s} (\varphi_s \sigma_s)^2\, dS \right)^{\frac{1}{2}}$$

$$= D'(R)^{\frac{1}{2}} \left(\int d\tilde{\mu}(s) \int_{\partial^+ C_R} \lambda^{1-2s} (\varphi_s \sigma_s)^2\, dS \right)^{\frac{1}{2}}.$$

Therefore, if $D(R) > 0$,

$$\left(\int d\tilde{\mu}(s) \int_{\partial^+ C_R} \lambda^{1-2s} (\varphi_s \sigma_s)^2\, dS \right)^{-1} \leq \frac{D'(R)}{D(R)^2} \tag{5.5}$$

Suppose by contradiction that $\boldsymbol{\sigma}$ were not constant. Then, for some $R_0 > 0$, $D(R) > 0$ for every $R > R_0$. Integrating (5.5) and using Schwartz inequality, we get that, for every $r_2 > r_1 > R_0$,

$$\frac{1}{D(r_1)} - \frac{1}{D(r_2)} \geq \int_{r_1}^{r_2} dR \left(\int d\tilde{\mu}(s) \int_{\partial^+ C_R} \lambda^{1-2s} (\varphi_s \sigma_s)^2\, dS \right)^{-1}$$

$$\geq (r_2 - r_1)^2 \left(\int d\tilde{\mu}(s) \int_{r_1}^{r_2} dR \int_{\partial^+ C_R} \lambda^{1-2s} (\varphi_s \sigma_s)^2\, dS \right)^{-1} \tag{5.6}$$

$$\geq (r_2 - r_1)^2 \left(\int d\tilde{\mu}(s) \int_{C_{r_2} \backslash C_{r_1}} \lambda^{1-2s} (\varphi_s \sigma_s)^2\, dx\, dy \right)^{-1}$$

Next, choose $r_2 = 2^{j+1}$ and $r_1 = 2^j$ with $j \geq N_0$ such that $2^{N_0} > R_0$. Using (5.2), (5.6) and suming over j, $N_0 \leq j \leq N$, we find

$$\frac{1}{D(2^{N_0})} \geq \frac{1}{4C} \sum_{j=N_0}^{N} \frac{1}{F(2^{j+1})}.$$

But, by the hypothesis on F, the sum

$$\sum_{j=N_0}^{\infty} \frac{1}{F(2^{j+1})} = +\infty,$$

which is a contradiction. $\square$

We finally prove 1-D symmetry of layer solutions to (1.3) in dimension two and, with the additional hypothesis $s_* \geq 1/2$, in dimension three.

Proof of Theorem 1.1. From u we construct the family of s-extensions $\{\tilde{u}_s\}$. Given $i < n$, we consider the families $\boldsymbol{\sigma^i} = \{\partial_i \tilde{u}_s / \partial_n \tilde{u}_s\}$ and $\boldsymbol{\varphi} = \{\partial_n \tilde{u}_s\}$. Observe that both families have common trace, namely, $\underline{\sigma}^i = \partial_i u / \partial_n u$ and $\underline{\varphi} = \partial_n u$ on $\mathbb{R}^n$. Let us show that these families satisfy the assumptions of Theorem 5.1.

Indeed, we have

$$
\begin{aligned}
\nabla \cdot (\lambda^{1-2s} \varphi_s^2 \nabla \sigma_s^i) &= \nabla \cdot \left(\lambda^{1-2s} (\partial_n \tilde{u}_s \partial_i \nabla \tilde{u}_s - \partial_i \tilde{u}_s \partial_n \nabla \tilde{u}_s) \right) \\
&= \lambda^{1-2s} \left(\partial_n \nabla \tilde{u}_s \cdot \partial_i \nabla \tilde{u}_s - \partial_i \nabla \tilde{u}_s \cdot \partial_n \nabla \tilde{u}_s \right) \\
&\qquad + \partial_n \tilde{u}_s \partial_i (\nabla \cdot (\lambda^{1-2s} \nabla \tilde{u}_s)) - \partial_i \tilde{u}_s \partial_n (\nabla \cdot (\lambda^{1-2s} \nabla \tilde{u}_s)) \\
&= 0, \quad \text{in } \mathbb{R}_+^{n+1},
\end{aligned}
$$

for each $s \in \operatorname{supp} \mu$. We now compute the flux on $\mathbb{R}^n = \{\lambda = 0\}$. Here we also use the notation $\underline{\lambda^{1-2s} \partial_\lambda v_s}$ for its limit as $\lambda \searrow 0$ (even in cases in which these limits are not common for all s). Denoting $d\tilde{\mu}(s) = d(s)\, d\mu(s)$ we have

$$
\begin{aligned}
\int \underline{\sigma_i}\, \underline{\varphi}^2\, \underline{\lambda^{1-2s} \partial_\lambda \sigma_s^i}\, d\tilde{\mu}(s) &= \int \underline{\sigma}^i (\partial_n u)^2 \left(\frac{\partial_n \underline{\lambda^{1-2s} \partial_\lambda \tilde{u}_s} \partial_i u - \partial_i \underline{\lambda^{1-2s} \partial_\lambda \tilde{u}_s} \partial_n u}{(\partial_n u)^2} \right) d\tilde{\mu}(s) \\
&= \underline{\sigma}^i (\partial_i u \partial_n - \partial_n u \partial_i) \int \underline{\lambda^{1-2s} \partial_\lambda \tilde{u}_s}\, d\tilde{\mu}(s) \\
&= \underline{\sigma}^i (\partial_i u \partial_n - \partial_n u \partial_i) f(u) \\
&= \underline{\sigma}^i (\partial_i u \partial_n u - \partial_n u \partial_i u) f'(u) \\
&\equiv 0, \quad \text{on } \mathbb{R}^n.
\end{aligned}
$$

Moreover, by Lemma 4.8 and by the assumptions of the theorem, we have that

$$
\begin{aligned}
\int d\tilde{\mu}(s) \int_{C_R} \lambda^{1-2s} (\varphi_s \sigma_s^i)^2\, dxdy &= \int d\tilde{\mu}(s) \int_{C_R} \lambda^{1-2s} (\partial_i u)^2\, dxdy \\
&\leq \tilde{\mathcal{E}}(u, C_R) \\
&\leq \Phi_{n,s_*}(R).
\end{aligned}
$$

Next, either if $n = 2$ and $s_* \in (0,1)$, of if $n = 3$ and $s_* \geq 1/2$, we have

$$
\Phi_{n,s_*}(R) \leq C R^2 \log(R).
$$

Finally the function $F(R) = R^2 \log(R)$ satisfies the assumption of Theorem 5.1, and we have seen that $\boldsymbol{\sigma^i} = \{\partial_i \tilde{u}_s / \partial_n \tilde{u}_s\}$ and $\boldsymbol{\varphi} = \{\partial_n \tilde{u}_s\}$ also satisfy the assumptions of Theorem 5.1. It follows that $\boldsymbol{\sigma^i}$ is equal to a constant a^i, for $i < n$. That is, $\nabla u = (a^1, 1)\partial_2 u$, if $n = 2$, or $\nabla u = (a^1, a^2, 1)\partial_3 u$, if $n = 3$. Equivalently, u has 1-D symmetry. $\qquad \square$

Appendix

We give the

Proof of Claim 3.1. Observe that

$$\int_{B_R}\int_{\mathcal{C}B_R}\frac{\min\{1,|x-y|\}}{|x-y|^{n+2s}}\,dx\,dy \le \int_{B_{R-1}}\int_{\mathcal{C}B_R}\frac{dx\,dy}{|x-y|^{n+2s}}+$$

$$+\int_{B_R}\int_{\mathcal{C}B_{R+1}}\frac{dx\,dy}{|x-y|^{n+2s}}+\int_{B_R\setminus B_{R-1}}\int_{B_{R+1}\setminus B_R}\frac{dx\,dy}{|x-y|^{n+2s-1}}\,.$$

The first term is bounded as follows: for $x\in B_{R-1}$ we have

$$\phi(x):=\int_{\mathcal{C}B_R}\frac{dy}{|x-y|^{n+2s}} \le \int_{R-|x|}^{\infty}\frac{r^{n-1}dr}{r^{n+2s}} = \frac{1}{2s}(R-|x|)^{-2s}\,.$$

Therefore,

$$\int_{B_{R-1}}\int_{\mathcal{C}B_R}\frac{dx\,dy}{|x-y|^{n+2s}} = \int_{B_{R-1}}\phi(x)dx \le \frac{C}{s}\int_0^{R-1}\frac{r^{n-1}dr}{(R-r)^{2s}}$$

$$\le \frac{CR^{n-1}}{s}\int_0^{R-1}\frac{dr}{(R-r)^{2s}} = \frac{C}{s}\Phi_{n,s}(R)\,.$$

The second term is identical with $R+1$ instead of R. Thus, it is also bounded by $\frac{C}{s}\Phi_{n,s}(R)$.

The third term is easily bounded in dimension $n=1$. Indeed, if $s\ne 1/2$,

$$\int_{R-1}^{R}\int_{R}^{R+1}\frac{dx\,dy}{|x-y|^{2s}} = \int_{-1}^{0}\int_0^1\frac{dx\,dy}{(x-y)^{2s}}$$

$$= \frac{1}{1-2s}\int_{-1}^{0}\left((1-y)^{1-2s}-(-y)^{1-2s}\right)dy$$

$$= \frac{1}{2(1-s)}\frac{2^{2-2s}-2}{1-2s}$$

$$\le \frac{C}{1-s}$$

with C independent of s. For $s=1/2$ we have

$$\int_{R-1}^{R}\int_{R}^{R+1}\frac{dx\,dy}{|x-y|} = \int_{-1}^{0}\int_0^1\frac{dx\,dy}{(x-y)} = \int_{-1}^{0}\log\left(\tfrac{1-y}{-y}\right)dy \le C.$$

It remains to bound the third term for $n>1$. We proceed as follows:

$$\int_{B_R\setminus B_{R-1}}\int_{B_{R+1}\setminus B_R}\frac{dx\,dy}{|x-y|^{n+2s-1}} = \int_{R-1}^{R}\int_{R}^{R+1}\int_0^{\pi}\frac{Cr_1^{n-1}r_2^{n-1}(\sin\theta)^{n-2}\,d\theta\,dr_1\,dr_2}{(r_1^2+r_2^2-2r_1r_2\cos\theta)^{\frac{n+2s-1}{2}}}$$

$$\le \int_{R-1}^{R}\int_{R}^{R+1}\int_0^{\pi}\frac{CR^{n-2}r_1r_2\,d\theta\,dr_1\,dr_2}{(r_1^2+r_2^2-2r_1r_2\cos\theta)^{\frac{2s+1}{2}}}\,,$$

where we have used that, for all r_1, r_2 and $\theta \in (0, \pi)$ in the domain of integration, we have

$$\frac{r_1 r_2 \sin \theta}{(r_1^2 + r_2^2 - 2r_1 r_2 \cos \theta)^{\frac{1}{2}}} \leq CR.$$

Next, we bound

$$\int_0^\pi \frac{CR^{n-2} r_1 r_2 \, d\theta}{(r_1^2 + r_2^2 - 2r_1 r_2 \cos \theta)^{\frac{2s+1}{2}}} = \int_0^\pi \frac{CR^{n-2} r_1 r_2 \, d\theta}{((r_1 - r_2)^2 + 2r_1 r_2 (1 - \cos \theta))^{\frac{2s+1}{2}}}$$

$$\leq \int_0^\infty \frac{CR^n \, dt}{((r_1 - r_2)^2 + R^2 t^2)^{\frac{2s+1}{2}}}$$

$$\leq \frac{CR^n}{(r_1 - r_2)^{2s+1}} \int_0^\infty \frac{dt}{\left(1 + \left(\frac{Rt}{(r_1 - r_2)}\right)^2\right)^{\frac{2s+1}{2}}}$$

$$= \frac{CR^{n-1}}{(r_1 - r_2)^{2s}} \int_0^\infty \frac{d\xi}{(1 + \xi^2)^{\frac{2s+1}{2}}}$$

$$\leq \frac{1}{s} \frac{CR^{n-1}}{(r_1 - r_2)^{2s}}.$$

We have thus come back to the situation of dimension $n = 1$. Indeed, from the previous inequalities

$$\int_{B_R \setminus B_{R-1}} \int_{B_{R+1} \setminus B_R} \frac{dx \, dy}{|x - y|^{n+2s-1}} \leq \frac{CR^{n-1}}{s} \int_{R-1}^R \int_R^{R+1} \frac{dr_1 \, dr_2}{(r_1 - r_2)^{2s}}$$

$$\leq \frac{CR^{n-1}}{s(1-s)},$$

where C depends only on n.

Putting together the bounds for the three terms, we have proved that

$$\int_{B_R} \int_{\mathcal{C} B_R} \frac{\min\{1, |x - y|\}}{|x - y|^{n+2s}} \, dx \, dy \leq \frac{C}{s(1-s)} \Phi_{n,s}(R), \tag{5.7}$$

with C independent of s.

Multiplying (5.7) by $c_n(s)$ —as in the statement of the claim— and using that $\frac{c_n(s)}{s(1-s)}$ is uniformly bounded for $s \in [0, 1)$ —as it is immediate to check in (1.5)— we conclude the proof. $\qquad \square$

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ICREA and Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain
  E-mail address: xavier.cabre@upc.edu

Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain
  E-mail address: joaquim.serra@upc.edu

H. Sharp isoperimetric inequalities via the ABP method

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SHARP ISOPERIMETRIC INEQUALITIES VIA THE ABP METHOD

XAVIER CABRÉ, XAVIER ROS-OTON, AND JOAQUIM SERRA

ABSTRACT. We prove some old and new isoperimetric inequalities with the best constant using the ABP method applied to an appropriate linear Neumann problem. More precisely, we obtain a new family of sharp isoperimetric inequalities with weights (also called densities) in open convex cones of $\mathbb{R}^n$. Our result applies to all nonnegative homogeneous weights satisfying a concavity condition in the cone. Remarkably, Euclidean balls centered at the origin (intersected with the cone) minimize the weighted isoperimetric quotient, even if all our weights are nonradial —except for the constant ones.

We also study the anisotropic isoperimetric problem in convex cones for the same class of weights. We prove that the Wulff shape (intersected with the cone) minimizes the anisotropic weighted perimeter under the weighted volume constraint.

As a particular case of our results, we give new proofs of two classical results: the Wulff inequality and the isoperimetric inequality in convex cones of Lions and Pacella.

1. INTRODUCTION AND RESULTS

In this paper we study isoperimetric problems with weights —also called densities. Given a weight w (that is, a positive function w), one wants to characterize minimizers of the weighted perimeter $\int_{\partial E} w$ among those sets E having weighted volume $\int_E w$ equal to a given constant. A set solving the problem, if it exists, is called an isoperimetric set or simply a minimizer. This question, and the associated isoperimetric inequalities with weights, have attracted much attention recently; see for example [46], [40], [19], [25], and [44].

The solution to the isoperimetric problem in $\mathbb{R}^n$ with a weight w is known only for very few weights, even in the case $n = 2$. For example, in $\mathbb{R}^n$ with the Gaussian weight $w(x) = e^{-|x|^2}$ all the minimizers are half-spaces [6, 18], and with $w(x) = e^{|x|^2}$ all the minimizers are balls centered at the origin [50]. Instead, mixed Euclidean-Gaussian densities lead to minimizers that have a more intricate structure of revolution [28]. The radial homogeneous weight $|x|^\alpha$ has been considered very recently.

In the plane ($n = 2$), minimizers for this homogeneous weight depend on the values of α. On the one hand, Carroll-Jacob-Quinn-Walters [16] showed that when $\alpha < -2$ all minimizers are $\mathbb{R}^2 \setminus B_r(0)$, $r > 0$, and that when $-2 \leq \alpha < 0$ minimizers do not exist. On the other hand, when $\alpha > 0$ Dahlberg-Dubbs-Newkirk-Tran [21] proved that all minimizers are circles passing through the origin (in particular, not centered at the origin). Note that this result shows that even radial homogeneous weights may lead to nonradial minimizers.

Weighted isoperimetric inequalities in cones have also been considered. In these results, the perimeter of E is taken relative to the cone, that is, not counting the part of ∂E that lies on the boundary of the cone. In [22] Díaz-Harman-Howe-Thompson consider again the radial homogeneous weight $w(x) = |x|^\alpha$, with $\alpha > 0$, but now in an open convex cone Σ of angle β in the plane $\mathbb{R}^2$. Among other things, they prove that there exists $\beta_0 \in (0, \pi)$ such that for $\beta < \beta_0$ all minimizers are $B_r(0) \cap \Sigma$, $r > 0$, while these circular sets about the origin are not minimizers for $\beta > \beta_0$.

Also related to the weighted isoperimetric problem in cones, the following is a recent result by Brock-Chiacchio-Mercaldo [7]. Assume that Σ is any cone in $\mathbb{R}^n$ with vertex at the origin, and consider the isoperimetric problem in Σ with any weight w. Then, for $B_R(0) \cap \Sigma$ to be an isoperimetric set for every $R > 0$ a necessary condition is that w admits the factorization

$$w(x) = A(r)B(\Theta), \tag{1.1}$$

where $r = |x|$ and $\Theta = x/r$. Our main result —Theorem 1.3 below— gives a sufficient condition on $B(\Theta)$ whenever Σ is convex and $A(r) = r^\alpha$, $\alpha \geq 0$, to guarantee that $B_R(0) \cap \Sigma$ are isoperimetric sets.

Our result states that Euclidean balls centered at the origin solve the isoperimetric problem in any open convex cone Σ of $\mathbb{R}^n$ (with vertex at the origin) for a certain class of nonradial weights. More precisely, our result applies to all nonnegative continuous weights w which are positively homogeneous of degree $\alpha \geq 0$ and such that $w^{1/\alpha}$ is concave in the cone Σ in case $\alpha > 0$. That is, using the previous notation, $w = r^\alpha B(\Theta)$ must be continuous in $\overline{\Sigma}$ and $rB^{1/\alpha}(\Theta)$ must be concave in Σ. We also solve weighted *anisotropic* isoperimetric problems in cones for the same class of weights. In these weighted anisotropic problems, the perimeter of a domain Ω is given by

$$\int_{\partial\Omega \cap \Sigma} H(\nu(x))w(x)dS,$$

where $\nu(x)$ is the unit outward normal to $\partial\Omega$ at x, and H is a positive, positively homogeneous of degree one, and convex function. Our results were announced in the recent note [13].

In the isotropic case, making the first variation of weighted perimeter (see [50]), one sees that the (generalized) mean curvature of $\partial\Omega$ with the density w is

$$H_w = H_{\text{eucl}} + \frac{1}{n}\frac{\partial_\nu w}{w}, \tag{1.2}$$

where ν is is the unit outward normal to $\partial\Omega$ and H_{eucl} is the Euclidean mean curvature of $\partial\Omega$. It follows that balls centered at the origin intersected with the cone have constant mean curvature whenever the weight is of the form (1.1). However, as we have seen in several examples presented above, it is far from being true that the solution of the isoperimetric problem for all the weights satisfying (1.1) are balls centered at the origin intersected with the cone. Our result provides a large class of nonradial weights for which, remarkably, Euclidean balls centered at the origin (intersected with the cone) solve the isoperimetric problem.

In Section 2 we give a list of weights w for which our result applies. Some concrete examples are the following:

$$\mathrm{dist}(x, \partial\Sigma)^\alpha \quad \text{in } \Sigma \subset \mathbb{R}^n,$$

where Σ is any open convex cone and $\alpha \geq 0$ (see example (ii) in Section 2);

$$x^a y^b z^c, \ (ax^r + by^r + cz^r)^{\alpha/r}, \ \text{ or } \ \frac{xyz}{xy + yz + zx} \quad \text{in } \Sigma = (0, \infty)^3,$$

where a, b, c are nonnegative numbers, $r \in (0, 1]$ or $r < 0$, and $\alpha > 0$ (see examples (iv), (v), and (vii), respectively);

$$\frac{x - y}{\log x - \log y}, \ \frac{x^{a+1} y^{b+1}}{(x^p + y^p)^{1/p}}, \ \text{ or } \ x \log\left(\frac{y}{x}\right) \quad \text{in } \Sigma = (0, \infty)^2,$$

where $a \geq 0$, $b \geq 0$, and $p > -1$ (see examples (viii) and (ix));

$$\left(\frac{\sigma_l}{\sigma_k}\right)^{\frac{\alpha}{l-k}}, \quad 1 \leq k < l < n, \quad \text{in } \Sigma = \{\sigma_1 > 0, \ ..., \sigma_l > 0\},$$

where σ_k is the elementary symmetric function of order k, defined by $\sigma_k(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$, and $\alpha > 0$ (see example (vii)).

Our isoperimetric inequality with an homogeneous weight w of degree α in a convex cone $\Sigma \subset \mathbb{R}^n$ yields as a consequence the following Sobolev inequality with best constant. If $D = n + \alpha$, $1 \leq p < D$, and $p_* = \frac{pD}{D-p}$, then

$$\left(\int_\Sigma |u|^{p_*} w(x)dx\right)^{1/p_*} \leq C_{w,p,n} \left(\int_\Sigma |\nabla u|^p w(x)dx\right)^{1/p} \tag{1.3}$$

for all smooth functions u with compact support in $\mathbb{R}^n$ —in particular, not necessarily vanishing on $\partial\Sigma$. This is a consequence of our isoperimetric inequality, Theorem 1.3, and a weighted radial rearrangement of Talenti [53], since these two results yield the radial symmetry of minimizers.

The proof of our main result follows the ideas introduced by the first author [9, 10] in a new proof of the classical isoperimetric inequality (the classical isoperimetric inequality corresponds to the weight $w \equiv 1$ and the cone $\Sigma = \mathbb{R}^n$). Our proof consists of applying the ABP method to an appropriate linear Neumann problem

involving the operator

$$w^{-1}\mathrm{div}(w\nabla u) = \Delta u + \frac{\nabla w}{w} \cdot \nabla u,$$

where w is the weight.

1.1. THE SETTING.

The classical isoperimetric problem in convex cones was solved by P.-L. Lions and F. Pacella [37] in 1990. Their result states that among all sets E with fixed volume inside an open convex cone Σ, the balls centered at the vertex of the cone minimize the perimeter relative to the cone (recall that the part of the boundary of E that lies on the boundary of the cone is not counted).

Throughout the paper Σ is an open convex cone in $\mathbb{R}^n$. Recall that given a measurable set $E \subset \mathbb{R}^n$ the relative perimeter of E in Σ is defined by

$$P(E;\Sigma) := \sup\left\{ \int_E \mathrm{div}\,\sigma\,dx \,:\, \sigma \in C_c^1(\Sigma, \mathbb{R}^n),\ |\sigma| \leq 1 \right\}.$$

When E is smooth enough,

$$P(E;\Sigma) = \int_{\partial E \cap \Sigma} dS.$$

The isoperimetric inequality in cones of Lions and Pacella reads as follows.

Theorem 1.1 ([37]). *Let Σ be an open convex cone in $\mathbb{R}^n$ with vertex at 0, and $B_1 := B_1(0)$. Then,*

$$\frac{P(E;\Sigma)}{|E \cap \Sigma|^{\frac{n-1}{n}}} \geq \frac{P(B_1;\Sigma)}{|B_1 \cap \Sigma|^{\frac{n-1}{n}}} \tag{1.4}$$

for every measurable set $E \subset \mathbb{R}^n$ with $|E \cap \Sigma| < \infty$.

The assumption of convexity of the cone can not be removed. In the same paper [37] the authors give simple examples of nonconvex cones for which inequality (1.4) does not hold. The idea is that for cones having two disconnected components, (1.4) is false since it pays less perimeter to concentrate all the volume in one of the two subcones. A connected (but nonconvex) counterexample is then obtained by joining the two components by a conic open thin set.

The proof of Theorem 1.1 given in [37] is based on the Brunn-Minkowski inequality

$$|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}},$$

valid for all nonempty measurable sets A and B of $\mathbb{R}^n$ for which $A + B$ is also measurable; see [31] for more information on this inequality. As a particular case of our main result, in this paper we provide a totally different proof of Theorem 1.1. This new proof is based on the ABP method.

Theorem 1.1 can be deduced from a degenerate case of the classical Wulff inequality stated in Theorem 1.2 below. This is because the convex set $B_1 \cap \Sigma$ is the Wulff

shape (1.6) associated to some appropriate anisotropic perimeter. As explained below in Section 3, this idea is crucial in the proof of our main result. This fact has also been used recently by Figalli and Indrei [24] to prove a quantitative isoperimetric inequality in convex cones. From it, one deduces that balls centered at the origin are the unique minimizers in (1.4) up to translations that leave invariant the cone (if they exist). This had been established in [37] in the particular case when $\partial\Sigma \setminus \{0\}$ is smooth (and later in [49], which also classified stable hypersurfaces in smooth cones).

The following is the notion of anisotropic perimeter. We say that a function H defined in $\mathbb{R}^n$ is a *gauge* when

$$H \text{ is nonnegative, positively homogeneous of degree one, and convex.} \qquad (1.5)$$

Somewhere in the paper we may require a function to be homogeneous; by this we always mean positively homogeneous.

Any norm is a gauge, but a gauge may vanish on some unit vectors. We need to allow this case since it will occur in our new proof of Theorem 1.1 —which builds from the cone Σ a gauge that is not a norm.

The anisotropic perimeter associated to the gauge H is given by

$$P_H(E) := \sup\left\{ \int_E \operatorname{div} \sigma \, dx \, : \, \sigma \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), \ \sup_{H(y) \leq 1} (\sigma(x) \cdot y) \leq 1 \text{ for } x \in \mathbb{R}^n \right\},$$

where $E \subset \mathbb{R}^n$ is any measurable set. When E is smooth enough one has

$$P_H(E) = \int_{\partial E} H\big(\nu(x)\big) dS,$$

where $\nu(x)$ is the unit outward normal at $x \in \partial E$.

The Wulff shape associated to H is defined as

$$W = \{x \in \mathbb{R}^n \, : \, x \cdot \nu < H(\nu) \ \text{ for all } \nu \in S^{n-1}\}. \qquad (1.6)$$

We will always assume that $W \neq \varnothing$. Note that W is an open set with $0 \in \overline{W}$. To visualize W, it is useful to note that it is the intersection of the half-spaces $\{x \cdot \nu < H(\nu)\}$ among all $\nu \in S^{n-1}$. In particular, W is a convex set.

From the definition (1.6) of the Wulff shape it follows that, given an open convex set $W \subset \mathbb{R}^n$ with $0 \in \overline{W}$, there is a unique gauge H such that W is the Wulff shape associated to H. Indeed, it is uniquely defined by

$$H(x) = \inf\big\{t \in \mathbb{R} : \ W \subset \{z \in \mathbb{R}^n \, : \, z \cdot x < t\}\big\}. \qquad (1.7)$$

Note that, for each direction $\nu \in S^{n-1}$, $\{x \cdot \nu = H(\nu)\}$ is a supporting hyperplane of W. Thus, for almost every point x on ∂W —those for which the outer normal $\nu(x)$ exists— it holds

$$x \cdot \nu(x) = H(\nu(x)) \quad \text{a.e. on } \partial W. \qquad (1.8)$$

Note also that, since W is convex, it is a Lipschitz domain. Hence, we can use the divergence theorem to find the formula

$$P_H(W) = \int_{\partial W} H(\nu(x))dS = \int_{\partial W} x \cdot \nu(x)dS = \int_W \operatorname{div}(x)dx = n|W|, \qquad (1.9)$$

relating the volume and the anisotropic perimeter of W.

When H is positive on S^{n-1} then it is natural to introduce its dual gauge H°, as in [1]. It is defined by

$$H^\circ(z) = \sup_{H(y) \leq 1} z \cdot y.$$

Then, the last condition on σ in the definition of $P_H(\cdot)$ is equivalent to $H^\circ(\sigma) \leq 1$ in $\mathbb{R}^n$, and the Wulff shape can be written as $W = \{H^\circ < 1\}$.

Some typical examples of gauges are

$$H(x) = \|x\|_p = \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p}, \qquad 1 \leq p \leq \infty.$$

Then, we have that $W = \{x \in \mathbb{R}^n : \|x\|_{p'} < 1\}$, where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

The following is the celebrated Wulff inequality.

Theorem 1.2 ([59, 54, 55]). *Let H be a gauge in $\mathbb{R}^n$ which is positive on S^{n-1}, and let W be its associated Wulff shape. Then, for every measurable set $E \subset \mathbb{R}^n$ with $|E| < \infty$, we have*

$$\frac{P_H(E)}{|E|^{\frac{n-1}{n}}} \geq \frac{P_H(W)}{|W|^{\frac{n-1}{n}}}. \qquad (1.10)$$

Moreover, equality holds if and only if $E = aW + b$ for some $a > 0$ and $b \in \mathbb{R}^n$ except for a set of measure zero.

This result was first stated without proof by Wulff [59] in 1901. His work was followed by Dinghas [23], who studied the problem within the class of convex polyhedra. He used the Brunn-Minkowski inequality. Some years later, Taylor [54, 55] finally proved Theorem 1.2 among sets of finite perimeter; see [56, 27, 42] for more information on this topic. As a particular case of our technique, in this paper we provide a new proof of Theorem 1.2. It is based on the ABP method applied to a linear Neumann problem. It was Robert McCann (in a personal communication around 2000) who pointed out that the first author's proof of the classical isoperimetric inequality also worked in the anisotropic case.

1.2. RESULTS.

The main result of the present paper, Theorem 1.3 below, is a weighted isoperimetric inequality which extends the two previous classical inequalities (Theorems 1.1 and 1.2). In particular, in Section 4 we will give a new proof of the classical Wulff theorem (for smooth domains) using the ABP method.

Before stating our main result, let us define the weighted anisotropic perimeter relative to an open cone Σ. The weights w that we consider will always be continuous functions in $\overline{\Sigma}$, positive and locally Lipschitz in Σ, and homogeneous of degree $\alpha \geq 0$.

Given a gauge H in $\mathbb{R}^n$ and a weight w, we define (following [3]) the weighted anisotropic perimeter relative to the cone Σ by

$$P_{w,H}(E;\Sigma) := \sup\left\{ \int_{E\cap\Sigma} \text{div}(\sigma w)dx \,:\, \sigma \in X_{w,\Sigma}, \, \sup_{H(y)\leq 1}(\sigma(x)\cdot y) \leq 1 \text{ for } x \in \Sigma \right\},$$

where $E \subset \mathbb{R}^n$ is any measurable set with finite Lebesgue measure and

$$X_{w,\Sigma} := \left\{ \sigma \in \left(L^\infty(\Sigma)\right)^n \,:\, \text{div}(\sigma w) \in L^\infty(\Sigma) \text{ and } \sigma w = 0 \text{ on } \partial\Sigma \right\}.$$

It is not difficult to see that

$$P_{w,H}(E;\Sigma) = \int_{\partial E\cap\Sigma} H\big(\nu(x)\big)w(x)dS \tag{1.11}$$

whenever E is smooth enough.

The definition of $P_{w,H}$ is the same as the one given in [3]. In their notation, we are taking $d\mu = w\chi_\Sigma \, dx$ and $X_{w,\Sigma} = X_\mu$.

Moreover, when H is the Euclidean norm we denote

$$P_w(E;\Sigma) := P_{w,\|\cdot\|_2}(E;\Sigma).$$

When $w \equiv 1$ in Σ and H is the Euclidean norm we recover the definition of $P(E;\Sigma)$; see [3].

Given a measurable set $F \subset \Sigma$, we denote by $w(F)$ the weighted volume of F

$$w(F) := \int_F w \, dx.$$

We also denote

$$D = n + \alpha.$$

Note that the Wulff shape W is independent of the weight w. Next we use that if ν is the unit outward normal to $W \cap \Sigma$, then $x \cdot \nu(x) = H(\nu(x))$ a.e. on $\partial W \cap \Sigma$, $x \cdot \nu(x) = 0$ a.e. on $\overline{W} \cap \partial\Sigma$, and $x \cdot \nabla w(x) = \alpha w(x)$ in Σ. This last equality follows from the homogeneity of degree α of w. Then, with a similar argument as in (1.9), we find

$$\begin{aligned}
P_{w,H}(W;\Sigma) &= \int_{\partial W\cap\Sigma} H(\nu(x))w(x)dS = \int_{\partial W\cap\Sigma} x \cdot \nu(x)\, w(x)dS \\
&= \int_{\partial(W\cap\Sigma)} x \cdot \nu(x)w(x)dS = \int_{W\cap\Sigma} \text{div}(xw(x))dx \\
&= \int_{W\cap\Sigma} \{nw(x) + x \cdot \nabla w(x)\}\, dx = D\, w(W \cap \Sigma).
\end{aligned} \tag{1.12}$$

Here —and in our main result that follows— for all quantities to make sense we need to assume that $W \cap \Sigma \neq \varnothing$. Recall that both W and Σ are open convex sets but that $W \cap \Sigma = \varnothing$ could happen. This occurs for instance if $H|_{S^{n-1}\cap\Sigma} \equiv 0$. On the other hand, if $H > 0$ on all S^{n-1} then $W \cap \Sigma \neq \varnothing$.

The following is our main result.

Theorem 1.3. *Let H be a gauge in $\mathbb{R}^n$, i.e., a function satisfying (1.5), and W its associated Wulff shape defined by (1.6). Let Σ be an open convex cone in $\mathbb{R}^n$ with vertex at the origin, and such that $W \cap \Sigma \neq \varnothing$. Let w be a continuous function in $\overline{\Sigma}$, positive in Σ, and positively homogeneous of degree $\alpha \geq 0$. Assume in addition that $w^{1/\alpha}$ is concave in Σ in case $\alpha > 0$.*

Then, for each measurable set $E \subset \mathbb{R}^n$ with $w(E \cap \Sigma) < \infty$,

$$\frac{P_{w,H}(E;\Sigma)}{w(E \cap \Sigma)^{\frac{D-1}{D}}} \geq \frac{P_{w,H}(W;\Sigma)}{w(W \cap \Sigma)^{\frac{D-1}{D}}}, \tag{1.13}$$

where $D = n + \alpha$.

Remark 1.4. Our key hypothesis that $w^{1/\alpha}$ is a concave function is equivalent to a natural curvature-dimension bound (in fact, to the nonnegativeness of the Bakry-Émery Ricci tensor in dimension $D = n + \alpha$). This was suggested to us by Cédric Villani, and has also been noticed by Cañete and Rosales (see Lemma 3.9 in [15]). More precisely, we see the cone $\Sigma \subset \mathbb{R}^n$ as a Riemannian manifold of dimension n equipped with a reference measure $w(x)dx$. We are also given a "dimension" $D = n + \alpha$. Consider the Bakry-Émery Ricci tensor, defined by

$$\mathrm{Ric}_{D,w} = \mathrm{Ric} - \nabla^2 \log w - \frac{1}{D-n}\nabla \log w \otimes \nabla \log w.$$

Now, our assumption $w^{1/\alpha}$ being concave is equivalent to

$$\mathrm{Ric}_{D,w} \geq 0. \tag{1.14}$$

Indeed, since $\mathrm{Ric} \equiv 0$ and $D - n = \alpha$, (1.14) reads as

$$-\nabla^2 \log w^{1/\alpha} - \nabla \log w^{1/\alpha} \otimes \nabla \log w^{1/\alpha} \geq 0,$$

which is the same condition as $w^{1/\alpha}$ being concave. Condition (1.14) is called a curvature-dimension bound; in the terminology of [58] we say that $\mathrm{CD}(0,D)$ is satisfied by $\Sigma \subset \mathbb{R}^n$ with the reference measure $w(x)dx$.

In addition, C. Villani pointed out that optimal transport techniques could also lead to weighted isoperimetric inequalities in convex cones; see Section 1.3.

Note that the shape of the minimizer is $W \cap \Sigma$, and that W depends only on H and not on the weight w neither on the cone Σ. In particular, in the isotropic case $H = \| \cdot \|_2$ we find the following surprising fact. Even that the weights that we consider are not radial (unless $w \equiv 1$), still Euclidean balls centered at the origin (intersected with the cone) minimize this isoperimetric quotient. The only explanation that one has a priori for this fact is that Euclidean balls centered at 0 have constant generalized mean curvature when the weight is homogeneous, as pointed out in (1.2). Thus, they are candidates to minimize perimeter for a given volume.

Note also that we allow w to vanish somewhere (or everywhere) on $\partial\Sigma$.

Equality in (1.13) holds whenever $E \cap \Sigma = rW \cap \Sigma$, where r is any positive number. However, in this paper we do not prove that $W \cap \Sigma$ is the unique minimizer of (1.13).

The reason is that our proof involves the solution of an elliptic equation and, due to an important issue on its regularity, we need to approximate the given set E by smooth sets. In a future work with E. Cinti and A. Pratelli we will refine the analysis in the proof of the present article and obtain a quantitative version of our isoperimetric inequality in cones. In particular, we will deduce uniqueness of minimizers (up to sets of measure zero). The quantitative version will be proved using the techniques of the present paper (the ABP method applied to a linear Neumann problem) together with the ideas of Figalli-Maggi-Pratelli [26].

In the isotropic case, a very recent result of Cañete and Rosales [15] deals with the same class of weights as ours. They allow not only positive homogeneities $\alpha > 0$, but also negative ones $\alpha \leq -(n-1)$. They prove that if a smooth, compact, and orientable hypersurfaces in a smooth convex cone is stable for weighted perimeter (under variations preserving weighted volume), then it must be a sphere centered at the vertex of the cone. In [15] the stability of such spheres is proved for $\alpha \leq -(n-1)$, but not for $\alpha > 0$. However, as pointed out in [15], when $\alpha > 0$ their result used together with ours give that spheres centered at the vertex are the unique minimizers among smooth hypersurfaces.

Theorem 1.3 contains the classical isoperimetric inequality, its version for convex cones, and the classical Wulff inequality. Indeed, taking $w \equiv 1$, $\Sigma = \mathbb{R}^n$, and $H = \| \cdot \|_2$ we recover the classical isoperimetric inequality with optimal constant. Still taking $w \equiv 1$ and $H = \| \cdot \|_2$ but now letting Σ be any open convex cone of $\mathbb{R}^n$ we have the isoperimetric inequality in convex cones of Lions and Pacella (Theorem 1.1). Moreover, if we take $w \equiv 1$ and $\Sigma = \mathbb{R}^n$ but we let H be some other gauge we obtain the Wulff inequality (Theorem 1.2).

A criterion of concavity for homogeneous functions of degree 1 can be found for example in [43, Proposition 10.3], and reads as follows. A nonnegative, C^2, and homogeneous of degree 1 function Φ on $\mathbb{R}^n$ is concave if and only if the restrictions $\Phi(\theta)$ of Φ to one-dimensional circles about the origin satisfy

$$\Phi''(\theta) + \Phi(\theta) \leq 0.$$

Therefore, it follows that a nonnegative, C^2, and homogeneous weight of degree $\alpha > 0$ in the plane $\mathbb{R}^2$, $w(x) = r^\alpha B(\theta)$, satisfies that $w^{1/\alpha}$ is concave in Σ if and only if

$$(B^{1/\alpha})'' + B^{1/\alpha} \leq 0.$$

Remark 1.5. Let w be an homogeneous weight of degree α, and consider the isotropic isoperimetric problem in a cone $\Sigma \subset \mathbb{R}^n$. Then, by the proofs of Proposition 3.6 and Lemma 3.8 in [50] the set $B_1(0) \cap \Sigma$ is stable if and only if

$$\int_{S^{n-1} \cap \Sigma} |\nabla_{S^{n-1}} u|^2 w \, dS \geq (n - 1 + \alpha) \int_{S^{n-1} \cap \Sigma} |u|^2 w \, dS \qquad (1.15)$$

for all functions $u \in C_c^\infty(S^{n-1} \cap \Sigma)$ satisfying

$$\int_{S^{n-1} \cap \Sigma} uw \, dS = 0. \tag{1.16}$$

Stability being a necessary condition for minimality, from Theorem (1.3) we deduce the following. If $\alpha > 0$, Σ is convex, and $w^{1/\alpha}$ is concave in Σ, then (1.15) holds.

For instance, in dimension $n = 2$, inequality (1.15) reads as

$$\int_0^\beta (u')^2 w \, d\theta \geq (1 + \alpha) \int_0^\beta u^2 w \, d\theta \qquad \text{whenever} \qquad \int_0^\beta uw \, d\theta = 0, \tag{1.17}$$

where $0 < \beta \leq \pi$ is the angle of the convex cone $\Sigma \subset \mathbb{R}^2$. This is ensured by our concavity condition on the weight w,

$$\left(w^{1/\alpha}\right)'' + w^{1/\alpha} \leq 0 \qquad \text{in } (0, \beta). \tag{1.18}$$

Note that, even in this two-dimensional case, it is not obvious that this condition on w yields (1.15)-(1.16). The statement (1.17) is an extension of Wirtinger's inequality (which corresponds to the case $w \equiv 1$, $\alpha = 0$, $\beta = 2\pi$). It holds, for example, with $w = \sin^\alpha \theta$ on S^1 —since (1.18) is satisfied by this weight. Another extension of Wirtinger's inequality (coming from the density $w = r^\alpha$) is given in [21].

In Theorem 1.3 we assume that w is homogeneous of degree α. In our proof, this assumption is essential in order that the paraboloid in (3.4) solves the PDE in (3.2), as explained in Section 3. Due to the homogeneity of w, the exponent $D = n + \alpha$ can be found just by a scaling argument in our inequality (1.13). Note that this exponent D has a dimension flavor if one compares (1.13) with (1.4) or with (1.10). Also, it is the exponent for the volume growth, in the sense that $w(B_r(0) \cap \Sigma) = Cr^D$ for all $r > 0$. The interpretation of D as a dimension is more clear in the following example that motivated our work.

Remark 1.6. The monomial weights

$$w(x) = x_1^{A_1} \cdots x_n^{A_n} \qquad \text{in} \quad \Sigma = \{x \in \mathbb{R}^n : x_i > 0 \text{ whenever } A_i > 0\}, \tag{1.19}$$

where $A_i \geq 0$, $\alpha = A_1 + \cdots + A_n$, and $D = n + A_1 + \cdots + A_n$, are important examples for which (1.13) holds. The isoperimetric inequality —and the corresponding Sobolev inequality (1.3)— with the above monomial weights were studied by the first two authors in [11, 12]. These inequalities arose in [11] while studying reaction-diffusion problems with symmetry of double revolution. A function u has symmetry of double revolution when $u(x, y) = u(|x|, |y|)$, with $(x, y) \in \mathbb{R}^D = \mathbb{R}^{A_1+1} \times \mathbb{R}^{A_2+1}$ (here we assume A_i to be positive integers). In this way, $u = u(x_1, x_2) = u(|x|, |y|)$ can be seen as a function in $\mathbb{R}^2 = \mathbb{R}^n$, and it is here where the Jacobian for the Lebesgue measure in $\mathbb{R}^D = \mathbb{R}^{A_1+1} \times \mathbb{R}^{A_2+1}$, $x_1^{A_1} x_2^{A_2} = |x|^{A_1} |y|^{A_2}$, appears. A similar argument under multiple axial symmetries shows that, when w and Σ are given by (1.19) and all A_i are nonnegative integers, and H is the Euclidean norm, Theorem 1.3 follows from the classical isoperimetric inequality in $\mathbb{R}^D$; see [12] for more details.

In [11] we needed to show a Sobolev inequality of the type (1.3) in $\mathbb{R}^2$ with the weight and the cone given by (1.19). As explained above, when A_i are all nonnegative integers this Sobolev inequality follows from the classical one in dimension D. However, in our application the exponents A_i were not integers —see [11]—, and thus the Sobolev inequality was not known. We showed a nonoptimal version (without the best constant) of that Sobolev inequality in dimension $n = 2$ in [11], and later we proved in [12] the optimal one in all dimensions n, obtaining the best constant and extremal functions for the inequality. In both cases, the main tool to prove these Sobolev inequalities was an isoperimetric inequality with the same weight.

An immediate consequence of Theorem 1.3 is the following weighted isoperimetric inequality in $\mathbb{R}^n$ for symmetric sets and even weights. It follows from our main result taking $\Sigma = (0, +\infty)^n$.

Corollary 1.7. *Let w be a nonnegative continuous function in $\mathbb{R}^n$, even with respect to each variable, homogeneous of degree $\alpha > 0$, and such that $w^{1/\alpha}$ is concave in $(0, \infty)^n$. Let $E \subset \mathbb{R}^n$ be any measurable set, symmetric with respect to each coordinate hyperplane $\{x_i = 0\}$, and with $|E| < \infty$. Then,*

$$\frac{P_w(E; \mathbb{R}^n)}{|E|^{\frac{D-1}{D}}} \geq \frac{P_w(B_1; \mathbb{R}^n)}{|B_1|^{\frac{D-1}{D}}}, \tag{1.20}$$

where $D = n + \alpha$ and B_1 is the unit ball in $\mathbb{R}^n$.

The symmetry assumption on the sets that we consider in Corollary 1.7 is satisfied in some applications arising in nonlinear problems, such as the one in [11] explained in Remark 1.6. Without this symmetry assumption, isoperimetric sets in (1.20) may not be the balls. For example, for the monomial weight $w(x) = |x_1|^{A_1} \cdots |x_n|^{A_n}$ in $\mathbb{R}^n$, with all A_i positive, $B_1 \cap (0, \infty)^n$ is an isoperimetric set, while the whole ball B_r having the same weighted volume as $B_1 \cap (0, \infty)^n$ is not an isoperimetric set (since it has longer perimeter).

We know only of few results where nonradial weights lead to radial minimizers. The first one is the isoperimetric inequality by Maderna-Salsa [38] in the upper half plane $\mathbb{R}_+^2$ with the weight x_2^α, $\alpha > 0$. To establish their isoperimetric inequality, they first proved the existence of a minimizer for the perimeter functional under constraint of fixed area, then computed the first variation of this functional, and finally solved the obtained ODE to find all minimizers. The second result is due to Brock-Chiacchio-Mercaldo [7] and extends the one in [38] by including the weights $x_n^\alpha \exp(c|x|^2)$ in $\mathbb{R}_+^n$, with $\alpha \geq 0$ and $c \geq 0$. In both papers it is proved that half balls centered at the origin are the minimizers of the isoperimetric quotient with these weights. Another one, of course, is our isoperimetric inequality with monomial weights [12] explained above (see Remark 1.6). At the same time as us, and using totally different methods, Brock, Chiacchio, and Mercaldo [8] have proved an isoperimetric inequality in $\Sigma = \{x_1 > 0, ..., x_n > 0\}$ with the weight $x_1^{A_1} \cdots x_n^{A_n} \exp(c|x|^2)$, with $A_i \geq 0$ and $c \geq 0$.

In all these results, although the weight $x_1^{A_1} \cdots x_n^{A_n}$ is not radial, it has a very special structure. Indeed, when all $A_1, ..., A_n$ are nonnegative *integers* the isoperimetric problem with the weight $x_1^{A_1} \cdots x_n^{A_n}$ is equivalent to the isoperimetric problem in $\mathbb{R}^{n+A_1+\cdots+A_n}$ for sets that have symmetry of revolution with respect to the first A_1+1 variables, the next $A_2 + 1$ variables, ..., and so on until the last $A_n + 1$ variables; see Remark 1.6. By this observation, the fact that half balls centered at the origin are the minimizers in $\mathbb{R}_+^n$ with the weight $x_1^{A_1} \cdots x_n^{A_n}$ or $x_1^{A_1} \cdots x_n^{A_n} \exp(c|x|^2)$, for $c \geq 0$ and A_i nonnegative integers, follows from the isoperimetric inequality in $\mathbb{R}^{n+A_1+\cdots+A_n}$ with the weight $\exp(c|x|^2)$, $c \geq 0$ (which is a radial weight). Thus, it was reasonable to expect that the same result for noninteger exponents $A_1, ..., A_n$ would also hold —as it does.

After announcing our result and proof in [13], Emanuel Milman showed us a nice geometric construction that yields the particular case when α is a nonnegative *integer* in our weighted inequality of Theorem 1.3. Using this construction, the weighted inequality in a convex cone is obtained as a limit case of the unweighted Lions-Pacella inequality in a narrow cone of $\mathbb{R}^{n+\alpha}$. We reproduce it in Remark 6.1 —see also the blog of Frank Morgan [45].

1.3. The proof. Related works.

The proof of Theorem 1.3 consists of applying the ABP method to a linear Neumann problem involving the operator $w^{-1}\mathrm{div}(w\nabla u)$, where w is the weight. When $w \equiv 1$, the idea goes back to 2000 in the works [9, 10] of the first author, where the classical isoperimetric inequality in all of $\mathbb{R}^n$ (here $w \equiv 1$) was proved with a new method. It consisted of solving the problem

$$\begin{cases} \Delta u = b_\Omega & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega \end{cases}$$

for a certain constant b_Ω, to produce a bijective map with the gradient of u, $\nabla u : \Gamma_{u,1} \longrightarrow B_1$, which leads to the isoperimetric inequality. Here $\Gamma_{u,1} \subset \Gamma_u \subset \Omega$ and $\Gamma_{u,1}$ is a certain subset of the lower contact set Γ_u of u (see Section 3 for details). The use of the ABP method is crucial in the proof.

Previously, Trudinger [57] had given a proof of the classical isoperimetric inequality in 1994 using the theory of Monge-Ampère equations and the ABP estimate. His proof consists of applying the ABP estimate to the Monge-Ampère problem

$$\begin{cases} \det D^2 u = \chi_\Omega & \text{in } B_R \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

where χ_Ω is the characteristic function of Ω and $B_R = B_R(0)$, and then letting $R \to \infty$.

Before these two works ([57] and [9]), there was already a proof of the isoperimetric inequality using a certain map (or coupling). This is Gromov's proof, which used the Knothe map; see [58].

After these three proofs, in 2004 Cordero-Erausquin, Nazaret, and Villani [20] used the Brenier map from optimal transportation to give a beautiful proof of the anisotropic isoperimetric inequality; see also [58]. More recently, Figalli-Maggi-Pratelli [26] established a sharp quantitative version of the anisotropic isoperimetric inequality, using also the Brenier map. In the case of the Lions-Pacella isoperimetric inequality, this has been done by Figalli-Indrei [24] very recently. As mentioned before, the proof in the present article is also suited for a quantitative version, as we will show in a future work with Cinti and Pratelli.

After announcing our result and proof in [13], we have been told that optimal transportation techniques à la [20] could also be used to prove weighted isoperimetric inequalities in certain cones. C. Villani pointed out that this is mentioned in the Bibliographical Notes to Chapter 21 of his book [58]. A. Figalli showed it to us with a computation when the cone is a halfspace $\{x_n > 0\}$ equipped with the weight x_n^α.

1.4. Applications.

Now we turn to some applications of Theorems 1.3 and Corollary 1.7.

First, our result leads to weighted Sobolev inequalities with best constant in convex cones of $\mathbb{R}^n$. Indeed, given any smooth function u with compact support in $\mathbb{R}^n$ (we do not assume u to vanish on $\partial\Sigma$), one uses the coarea formula and Theorem 1.3 applied to each of the level sets of u. This establishes the Sobolev inequality (1.3) for $p = 1$. The constant $C_{w,1,n}$ obtained in this way is optimal, and coincides with the best constant in our isoperimetric inequality (1.20).

When $1 < p < D$, Theorem 1.3 also leads to the Sobolev inequality (1.3) with best constant. This is a consequence of our isoperimetric inequality and a weighted radial rearrangement of Talenti [53], since these two results yield the radial symmetry of minimizers. See [12] for details in the case of monomial weights $w(x) = |x_1|^{A_1} \cdots |x_n|^{A_n}$.

If we use Corollary 1.7 instead of Theorem 1.3, with the same argument one finds the Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{p_*} w(x)dx \right)^{1/p_*} \leq C_{w,p,n} \left(\int_{\mathbb{R}^n} |\nabla u|^p w(x)dx \right)^{1/p}, \qquad (1.21)$$

where $p_* = \frac{pD}{D-p}$, $D = n + \alpha$, and $1 \leq p < D$. Here, w is any weight satisfying the hypotheses of Corollary 1.7, and u is any smooth function with compact support in $\mathbb{R}^n$ which is symmetric with respect to each variable x_i, $i = 1, ..., n$.

We now turn to applications to the symmetry of solutions to nonlinear PDEs. It is well known that the classical isoperimetric inequality yields some radial symmetry results for semilinear or quasilinear elliptic equations. Indeed, using the Schwartz rearrangement that preserves $\int F(u)$ and decreases $\int \Phi(|\nabla u|)$, it is immediate to show

that minimizers of some energy functionals (or quotients) involving these quantities are radially symmetric; see [48, 53]. Moreover, P.-L. Lions [36] showed that in dimension $n = 2$ the isoperimetric inequality yields also the radial symmetry of all positive solutions to the semilinear problem $-\Delta u = f(u)$ in B_1, $u = 0$ on ∂B_1, with $f \geq 0$ and f possibly discontinuous. This argument has been extended in three directions: for the p-Laplace operator, for cones of $\mathbb{R}^n$, and for Wulff shapes, as explained next.

On the one hand, the analogue of Lions radial symmetry result but in dimension $n \geq 3$ for the p-Laplace operator was proved with $p = n$ by Kesavan and Pacella in [35], and with $p \geq n$ by the third author in [52]. Moreover, in [35] it is also proved that positive solutions to the following semilinear equation with mixed boundary conditions

$$\begin{cases} -\Delta_p u = f(u) & \text{in } B_1 \cap \Sigma \\ \quad u = 0 & \text{on } \partial B_1 \cap \Sigma \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } B_1 \cap \partial \Sigma \end{cases} \tag{1.22}$$

have radial symmetry whenever $p = n$. Here, B_1 is the unit ball and Σ any open convex cone. This was proved by using Theorem 1.1 and the argument of P.-L. Lions mentioned above.

On the other hand, Theorem 1.2 is used to construct a Wulff shaped rearrangement in [1]. This yields that minimizers to certain nonlinear variational equations that come from anisotropic gradient norms have Wulff shaped level sets. Moreover, the radial symmetry argument in [36] was extended to this anisotropic case in [4], yielding the same kind of result for positive solutions of nonlinear equations involving the operator $Lu = \text{div}\,(H(\nabla u)^{p-1}\nabla H(\nabla u))$ with $p = n$. In the same direction, in a future paper [51] we will use Theorem 1.3 to obtain Wulff shaped symmetry of critical functions of weighted anisotropic functionals such as

$$\int \big\{ H^p(\nabla u) - F(u) \big\} w(x)\, dx.$$

Here, w is an homogeneous weight satisfying the hypotheses of Theorem 1.3 and H is any norm in $\mathbb{R}^n$. As in [52], we will allow $p \neq n$ but with some conditions on F in case $p < n$.

Related to these results, when f is Lipschitz, Berestycki and Pacella [5] proved that any positive solution to problem (1.22) with $p = 2$ in a convex spherical sector Σ of $\mathbb{R}^n$ is radially symmetric. They used the moving planes method.

1.5. Plan of the paper.

The rest of the article is organized as follows. In Section 2 we give examples of weights for which our result applies. In Section 3 we introduce the elements appearing in the proof of Theorem 1.3. To illustrate these ideas, in Section 4 we give the proof of the classical Wulff theorem via the ABP method. In Section 5 we

prove Theorem 1.3 in the simpler case $w \equiv 0$ on $\partial\Sigma$ and $H = \|\cdot\|_2$. Finally, in Section 6 we present the whole proof of Theorem 1.3.

2. Examples of weights

When $w \equiv 1$ our main result yields the classical isoperimetric inequality, its version for convex cones, and also the Wulff theorem. On the other hand, given an open convex cone $\Sigma \subset \mathbb{R}^n$ (different than the whole space and a half-space) there is a large family of functions that are homogeneous of degree one and concave in Σ. Any positive power of one of these functions is an admissible weight for Theorem 1.3. Next we give some concrete examples of weights w for which our result applies. The key point is to check that the homogeneous function of degree one $w^{1/\alpha}$ is concave.

(i) Assume that w_1 and w_2 are concave homogeneous functions of degree one in an open convex cone Σ. Then, $w_1^a w_2^b$ with $a \geq 0$ and $b \geq 0$, $(w_1^r + w_2^r)^{\alpha/r}$ with $r \in (0,1]$ or $r < 0$, and $\min\{w_1, w_2\}^\alpha$, satisfy the hypotheses of Theorem 1.3 (with $\alpha = a + b$ in the first case). More generally, if $F : [0,\infty)^2 \to \mathbb{R}_+$ is positive, concave, homogeneous of degree 1, and nondecreasing in each variable, then one can take $w = F(w_1, w_2)^\alpha$, with $\alpha > 0$.

(ii) The distance function to the boundary of any convex set is concave when defined in the convex set. On the other hand, the distance function to the boundary of any cone is homogeneous of degree 1. Thus, for any open convex cone Σ and any $\alpha \geq 0$,

$$w(x) = \operatorname{dist}(x, \partial\Sigma)^\alpha$$

is an admissible weight. When the cone is $\Sigma = \{x_i > 0, \ i = 1, ..., n\}$, this weight is exactly $\min\{x_1, ..., x_n\}^\alpha$.

(iii) If the concavity condition is satisfied by a weight w in a convex cone Σ' then it is also satisfied in any convex subcone $\Sigma \subset \Sigma'$. Note that this gives examples of weights w and cones Σ in which w is positive on $\partial\Sigma \setminus \{0\}$.

(iv) Let $\Sigma_1, ..., \Sigma_k$ be convex cones and $\Sigma = \Sigma_1 \cap \cdots \cap \Sigma_k$. Let

$$\delta_i(x) = \operatorname{dist}(x, \partial\Sigma_i).$$

Then, the weight

$$w(x) = \delta_1^{A_1} \cdots \delta_k^{A_k}, \quad x \in \Sigma,$$

with $A_1 \geq 0, ..., A_k \geq 0$, satisfies the hypotheses of Theorem 1.3. This follows from (i), (ii), and (iii). Note that when $k = n$ and $\Sigma_i = \{x_i > 0\}$, $i = 1, ..., n$, then $\Sigma = \{x_1 > 0, ..., x_n > 0\}$ and we obtain the monomial weight

$$w(x) = x_1^{A_1} \cdots x_n^{A_n}.$$

(v) In the cone $\Sigma = (0, \infty)^n$, the weights

$$w(x) = \left(A_1 x_1^{1/p} + \cdots + A_n x_n^{1/p} \right)^{\alpha p},$$

for $p \geq 1$, $A_i \geq 0$, and $\alpha > 0$, satisfy the hypotheses of Theorem 1.3. Similarly, one may take the weights

$$w(x) = \left(\frac{A_1}{x_1^r} + \cdots + \frac{A_n}{x_n^r} \right)^{-\alpha/r},$$

with $r > 0$, or the limit case

$$w(x) = \min\{A_1 x_1, \cdots, A_n x_n\}^\alpha.$$

This can be showed using the Minkowski inequality. More precisely, the first one can be showed using the classical Minkowski inequality with exponent $p \geq 1$, while the second one using a reversed Minkowski inequality that holds for exponents $p = -r < 0$.

In these examples $\Sigma = (0, \infty)^n$ and therefore by Corollary 1.7 we find that among all sets $E \subset \mathbb{R}^n$ which are symmetric with respect to each coordinate hyperplane, Euclidean balls centered at the origin minimize the isoperimetric quotient with these weights.

(vi) Powers of *hyperbolic polynomials* also provide examples of weights. An homogeneous polynomial $P(x)$ of degree k defined in $\mathbb{R}^n$ is called hyperbolic with respect to $a \in \mathbb{R}^n$ provided $P(a) > 0$ and for every $\lambda \in \mathbb{R}^n$ the polynomial in t, $P(ta + \lambda)$, has exactly k real roots. Let Σ be the component in $\mathbb{R}^n$, containing a, of the set $\{P > 0\}$. Then, Σ is a convex cone and $P(x)^{1/k}$ is a concave function in Σ; see for example [30] or [14, Section 1]. Thus, for any hyperbolic polynomial P, the weight

$$w(x) = P(x)^{\alpha/k}$$

satisfies the hypotheses of Theorem 1.3. Typical examples of hyperbolic polynomials are

$$P(x) = x_1^2 - \lambda_2 x_2^2 - \cdots - \lambda_n x_n^2 \ \text{ in } \Sigma = \left\{ x_1 > \sqrt{\lambda_2 x_2^2 + \cdots + \lambda_n x_n^2} \right\},$$

with $\lambda_2 > 0,...,\lambda_n > 0$, or the elementary symmetric functions

$$\sigma_k(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k} \ \text{ in } \Sigma = \{\sigma_1 > 0, ..., \sigma_k > 0\}$$

(recall that Σ is defined above as a component of $\{P > 0\}$). Other examples are

$$P(x) = \prod_{1 \leq i_1 < \cdots < i_r \leq n} \sum_{j=1}^r x_{i_j} \ \text{ in } \Sigma = \{x_i > 0, \ i = 1, ..., n\},$$

which have degree $k = \binom{n}{r}$ (this follows by induction from the first statement in example (i); see also [2]), or the polynomial $\det(X)$ in the convex cone of symmetric positive definite matrices —which we consider in the space $\mathbb{R}^{n(n+1)/2}$.

The interest in hyperbolic polynomials was originally motivated by an important paper of Garding on linear hyperbolic PDEs [29], and it is known that they form a rich class; see for example [30], where the same author showed various ways of constructing new hyperbolic polynomials from old ones.

(vii) If σ_k and σ_l are the elementary symmetric functions of degree k and l, with $1 \le k < l \le n$, then $(\sigma_l/\sigma_k)^{\frac{1}{l-k}}$ is concave in the cone $\Sigma = \{\sigma_1 > 0, ..., \sigma_k > 0\}$; see [39]. Thus,

$$w(x) = \left(\frac{\sigma_l}{\sigma_k} \right)^{\frac{\alpha}{l-k}}$$

is an admissible weight. For example, setting $k = n$ and $l = 1$ we find that we can take

$$w(x) = \left(\frac{x_1 \cdots x_n}{x_1 + \cdots + x_n} \right)^{\frac{\alpha}{n-1}}$$

in Theorem 1.3 or in Corollary 1.7.

(viii) If $f : \mathbb{R} \to \mathbb{R}_+$ is any continuous function which is concave in (a, b), then

$$w(x) = x_1 f\left(\frac{x_2}{x_1} \right)$$

is an admissible weight in $\Sigma = \{x = (r, \theta) : \arctan a < \theta < \arctan b\}$.

(ix) In the cone $\Sigma = (0, \infty)^2 \subset \mathbb{R}^2$ one may take

$$w(x) = \left(\frac{x_1 - x_2}{\log x_1 - \log x_2} \right)^{\alpha}$$

for $\alpha > 0$. In addition, in the same cone one may also take

$$w(x) = \frac{1}{e} \left(x_1^{x_1} x_2^{-x_2} \right)^{\frac{\alpha}{x_1 - x_2}}.$$

This can be seen by using (viii) and computing f in each of the two cases. When $\alpha = 1$, these functions are called the *logarithmic mean* and the *identric mean* of the numbers x_1 and x_2, respectively.

Using also (viii) one can check that, in the cone $\Sigma = (0, \infty)^2$, the weight $w(x) = xy(x^p + y^p)^{-1/p}$ is admissible whenever $p > -1$. Then, using (i) it follows that

$$w(x) = \frac{x^{a+1} y^{b+1}}{(x^p + y^p)^{1/p}}$$

is an admissible weight whenever $a \ge 0$, $b \ge 0$, and $p > -1$.

3. Description of the proof

The proof of Theorem 1.3 follows the ideas introduced by the first author in a new proof of the classical isoperimetric inequality; see [9, 10] or the last edition of Chavel's book [17]. This proof uses the ABP method, as explained next.

The Alexandroff-Bakelman-Pucci (or ABP) estimate is an L^∞ bound for solutions of the Dirichlet problem associated to second order uniformly elliptic operators written in nondivergence form,

$$Lu = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u,$$

with bounded measurable coefficients in a domain Ω of $\mathbb{R}^n$. It asserts that if Ω is bounded and $c \leq 0$ in Ω then, for every function $u \in C^2(\Omega) \cap C(\overline{\Omega})$,

$$\sup_\Omega u \leq \sup_{\partial\Omega} u + C \operatorname{diam}(\Omega) \|Lu\|_{L^n(\Omega)},$$

where C is a constant depending only on the ellipticity constants of L and on the L^n-norm of the coefficients b_i. The estimate was proven by the previous authors in the sixties using a technique that is nowadays called "the ABP method". See [10] and references therein for more information on this estimate.

The proof of the classical isoperimetric inequality in [9, 10] consists of applying the ABP method to an appropriate Neumann problem for the Laplacian —instead of applying it to a Dirichlet problem as customary. Namely, to estimate from below $|\partial\Omega|/|\Omega|^{\frac{n-1}{n}}$ for a smooth domain Ω, one considers the problem

$$\begin{cases} \Delta u = b_\Omega & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

The constant $b_\Omega = |\partial\Omega|/|\Omega|$ is chosen so that the problem has a solution. Next, one proves that $B_1 \subset \nabla u(\Gamma_u)$ with a contact argument (for a certain "contact" set $\Gamma_u \subset \Omega$), and then one estimates the measure of $\nabla u(\Gamma_u)$ by using the area formula and the inequality between the geometric and arithmetic means. Note that the solution of (3.1) is

$$u(x) = |x|^2/2 \qquad \text{when } \Omega = B_1,$$

and in this case one verifies that all the inequalities appearing in this ABP argument are equalities. After having proved the isoperimetric inequality for smooth domains, an standard approximation argument extends it to all sets of finite perimeter.

As pointed out by R. McCann, the same method also yields the Wulff theorem. For this, one replaces the Neumann data in (3.1) by $\partial u/\partial \nu = H(\nu)$ and uses the same argument explained above. This proof of the Wulff theorem is given in Section 4.

We now sketch the proof of Theorem 1.3 in the isotropic case, that is, when $H = \|\cdot\|_2$. In this case, optimizers are Euclidean balls centered at the origin intersected with the cone. First, we assume that $E = \Omega$ is a bounded smooth domain. The key idea is to consider a similar problem to (3.1) but where the

Laplacian is replaced by the operator

$$w^{-1}\operatorname{div}(w\nabla u) = \Delta u + \frac{\nabla w}{w}\cdot\nabla u.$$

Essentially (but, as we will see, this is not exactly as we proceed —because of a regularity issue), we solve the following Neumann problem in $\Omega \subset \Sigma$:

$$\begin{cases} w^{-1}\operatorname{div}(w\nabla u) = b_{\Omega} & \text{in } \Omega \\ \quad \dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega \cap \Sigma \\ \quad \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \cap \partial\Sigma, \end{cases} \tag{3.2}$$

where the constant b_{Ω} is again chosen depending on weighted perimeter and volume so that the problem admits a solution. Whenever u belongs to $C^1(\overline{\Omega})$ —which is not always the case, as discussed below in this section—, by touching the graph of u by below with planes (as in the proof of the classical isoperimetric inequality explained above) we find that

$$B_1 \cap \Sigma \subset \nabla u(\Omega). \tag{3.3}$$

Then, using the area formula, an appropriate weighted geometric-arithmetic means inequality, and the concavity condition on the weight w, we obtain our weighted isoperimetric inequality. Note that the solution of (3.2) is

$$u(x) = |x|^2/2 \qquad \text{when } \Omega = B_1 \cap \Sigma. \tag{3.4}$$

In this case, all the chain of inequalities in our proof become equalities, and this yields the sharpness of the result.

In the previous argument there is an important technical difficulty that comes from the possible lack of regularity up to the boundary of the solution to the weighted Neumann problem (3.2). For instance, if $\Omega \cap \Sigma$ is a smooth domain that has some part of its boundary lying on $\partial\Sigma$ —and hence $\partial\Omega$ meets tangentially $\partial\Sigma$—, then u can not be C^1 up to the boundary. This is because the Neumann condition itself is not continuous and hence $\partial_\nu u$ would jump from 1 to one 0 where $\partial\Omega$ meets $\partial\Sigma$.

The fact that u could not be C^1 up to the boundary prevents us from using our contact argument to prove (3.3). Nevertheless, the argument sketched above does work for smooth domains Ω well contained in Σ, that is, satisfying $\overline{\Omega} \subset \Sigma$. If, in addition, $w \equiv 0$ on $\partial\Sigma$ we can deduce the inequality for all measurable sets E by an approximation argument. Indeed, if $w \in C(\overline{\Omega})$ and $w \equiv 0$ on $\partial\Sigma$ then for any domain U with piecewise Lipschitz boundary one has

$$P_w(U;\Sigma) = \int_{\partial U \cap \Sigma} w\,dS = \int_{\partial U} w\,dS.$$

This fact allows us to approximate any set with finite measure $E \subset \Sigma$ by bounded smooth domains Ω_k satisfying $\overline{\Omega_k} \subset \Sigma$. Thus, the proof of Theorem 1.3 for weights w vanishing on $\partial\Sigma$ is simpler, and this is why we present it first, in Section 5.

Instead, if $w > 0$ at some part of (or everywhere on) $\partial\Sigma$ it is not always possible to find sequences of smooth sets with closure contained in the open cone and approximating in relative perimeter a given measurable set $E \subset \Sigma$. This is because the relative perimeter does not count the part of the boundary of E which lies on $\partial\Sigma$. To get around this difficulty (recall that we are describing the proof in the isotropic case, $H \equiv 1$) we need to consider an anisotropic problem in $\mathbb{R}^n$ for which approximation is possible. Namely, we choose a gauge H_0 defined as the gauge associated to the convex set $B_1 \cap \Sigma$; see (1.7). Then we prove that $P_{w,H_0}(\,\cdot\,;\Sigma)$ is a calibration of the functional $P_w(\,\cdot\,;\Sigma)$, in the following sense. For all $E \subset \Sigma$ we will have

$$P_{w,H_0}(E;\Sigma) \leq P_w(E;\Sigma),$$

while for $E = B_1 \cap \Sigma$,

$$P_{w,H_0}(B_1;\Sigma) = P_w(B_1 \cap \Sigma;\Sigma).$$

As a consequence, the isoperimetric inequality with perimeter $P_{w,H_0}(\cdot;\Sigma)$ implies the one with the perimeter $P_w(\cdot;\Sigma)$. For $P_{w,H_0}(\cdot;\Sigma)$ approximation results are available and, as in the case of $w \equiv 0$ on $\partial\Sigma$, it is enough to consider smooth sets satisfying $\overline{\Omega} \subset \Sigma$ —for which there are no regularity problems with the solution of the elliptic problem.

To prove Theorem 1.3 for a general anisotropic perimeter $P_{w,H}(\cdot;\Sigma)$ we also consider a "calibrated" perimeter $P_{w,H_0}(\cdot;\Sigma)$, where H_0 is now the gauge associated to the convex set $W \cap \Sigma$. Note that, as explained above, even for the isotropic case $H = \|\cdot\|_2$ we have to consider an anisotropic perimeter (associated to $B_1 \cap \Sigma$) in order to prove Theorem 1.3.

4. Proof of the classical Wulff inequality

In this section we prove the classical Wulff theorem for smooth domains by using the ideas introduced by the first author in [9, 10]. When H is smooth on S^{n-1}, we show also that the Wulff shapes are the only smooth sets for which equality is attained.

Proof of Theorem 1.2. We prove the Wulff inequality only for smooth domains, that we denote by Ω instead of E. By approximation, if (1.10) holds for all smooth domains then it holds for all sets of finite perimeter.

By regularizing H on S^{n-1} and then extending it homogeneously, we can assume that H is smooth in $\mathbb{R}^n \setminus \{0\}$. For non-smooth H this approximation argument will yield inequality (1.10), but not the equality cases.

Let u be a solution of the Neumann problem

$$\begin{cases} \Delta u = \dfrac{P_H(\Omega)}{|\Omega|} & \text{in } \Omega \\[2mm] \dfrac{\partial u}{\partial \nu} = H(\nu) & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Δ denotes the Laplace operator and $\partial u/\partial \nu$ the exterior normal derivative of u on $\partial\Omega$. Recall that $P_H(\Omega) = \int_{\partial\Omega} H(\nu(x))\, dS$. The constant $P_H(\Omega)/|\Omega|$ has been chosen so that the problem has a unique solution up to an additive constant. Since $H|_{S^{n-1}}$ and Ω are smooth, we have that u is smooth in $\overline{\Omega}$. See [47] for a recent exposition of these classical facts and for a new Schauder estimate for (4.1).

Consider the lower contact set of u, defined by

$$\Gamma_u = \left\{ x \in \Omega \ : \ u(y) \geq u(x) + \nabla u(x) \cdot (y - x) \ \text{ for all } y \in \overline{\Omega} \right\}. \qquad (4.2)$$

It is the set of points where the tangent hyperplane to the graph of u lies below u in all $\overline{\Omega}$. We claim that

$$W \subset \nabla u(\Gamma_u), \qquad (4.3)$$

where W denotes the Wulff shape associated to H, given by (1.6).

To show (4.3), take any $p \in W$, i.e., any $p \in \mathbb{R}^n$ satisfying

$$p \cdot \nu < H(\nu) \ \text{ for all } \nu \in S^{n-1}.$$

Let $x \in \overline{\Omega}$ be a point such that

$$\min_{y \in \overline{\Omega}} \left\{ u(y) - p \cdot y \right\} = u(x) - p \cdot x$$

(this is, up to a sign, the Legendre transform of u). If $x \in \partial\Omega$ then the exterior normal derivative of $u(y) - p \cdot y$ at x would be nonpositive and hence $(\partial u/\partial\nu)(x) \leq p \cdot \nu < H(\nu)$, a contradiction with the boundary condition of (4.1). It follows that $x \in \Omega$ and, therefore, that x is an interior minimum of the function $u(y) - p \cdot y$. In particular, $p = \nabla u(x)$ and $x \in \Gamma_u$. Claim (4.3) is now proved. It is interesting to visualize geometrically the proof of the claim, by considering the graphs of the functions $p \cdot y + c$ for $c \in \mathbb{R}$. These are parallel hyperplanes which lie, for c close to $-\infty$, below the graph of u. We let c increase and consider the first c for which there is contact or "touching" at a point x. It is clear geometrically that $x \notin \partial\Omega$, since $p \cdot \nu < H(\nu)$ for all $\nu \in S^{n-1}$ and $\partial u/\partial\nu = H(\nu)$ on $\partial\Omega$.

Now, from (4.3) we deduce

$$|W| \leq |\nabla u(\Gamma_u)| = \int_{\nabla u(\Gamma_u)} dp \leq \int_{\Gamma_u} \det D^2 u(x)\, dx. \qquad (4.4)$$

We have applied the area formula to the smooth map $\nabla u : \Gamma_u \to \mathbb{R}^n$, and we have used that its Jacobian, $\det D^2 u$, is nonnegative in Γ_u by definition of this set.

Next, we use the classical inequality between the geometric and the arithmetic means applied to the eigenvalues of $D^2 u(x)$ (which are nonnegative numbers for $x \in \Gamma_u$). We obtain

$$\det D^2 u \leq \left(\frac{\Delta u}{n} \right)^n \quad \text{in } \Gamma_u. \qquad (4.5)$$

This, combined with (4.4) and $\Delta u \equiv P_H(\Omega)/|\Omega|$, gives

$$|W| \leq \left(\frac{P_H(\Omega)}{n|\Omega|} \right)^n |\Gamma_u| \leq \left(\frac{P_H(\Omega)}{n|\Omega|} \right)^n |\Omega|. \qquad (4.6)$$

Finally, using that $P_H(W) = n|W|$ —see (1.9)—, we conclude that

$$\frac{P_H(W)}{|W|^{\frac{n-1}{n}}} = n|W|^{\frac{1}{n}} \le \frac{P_H(\Omega)}{|\Omega|^{\frac{n-1}{n}}}. \tag{4.7}$$

Note that when $\Omega = W$ then the solution of (4.1) is $u(x) = |x|^2/2$ since $\Delta u = n$ and $u_\nu(x) = x \cdot \nu(x) = H(\nu(x))$ a.e. on ∂W —recall (1.8). In particular, $\nabla u = \text{Id}$ and all the eigenvalues of $D^2 u(x)$ are equal. Therefore, it is clear that all inequalities (and inclusions) in (4.3)-(4.7) are equalities when $\Omega = W$. This explains why the proof gives the best constant in the inequality.

Let us see next that, when $H|_{S^{n-1}}$ is smooth, the Wulff shaped domains $\Omega = aW + b$ are the only smooth domains for which equality occurs in (1.10). Indeed, if (4.7) is an equality then all the inequalities in (4.4), (4.5), and (4.6) are also equalities. In particular, we have $|\Gamma_u| = |\Omega|$. Since $\Gamma_u \subset \Omega$, Ω is an open set, and Γ_u is closed relatively to Ω, we deduce that $\Gamma_u = \Omega$.

Recall that the geometric and arithmetic means of n nonnegative numbers are equal if and only if these n numbers are all equal. Hence, the equality in (4.5) and the fact that Δu is constant in Ω give that $D^2 u = a\text{Id}$ in all $\Gamma_u = \Omega$, where Id is the identity matrix and $a = P_H(\partial\Omega)/(n|\Omega|)$ is a positive constant. Let $x_0 \in \Omega$ be any given point. Integrating $D^2 u = a\text{Id}$ on segments from x_0, we deduce that

$$u(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) + \frac{a}{2}|x - x_0|^2$$

for x in a neighborhood of x_0. In particular, $\nabla u(x) = \nabla u(x_0) + a(x - x_0)$ in such a neighborhood, and hence the map $\nabla u - a\text{I}$ is locally constant. Since Ω is connected, we deduce that the map $\nabla u - a\text{I}$ is indeed a constant, say $\nabla u - a\text{I} \equiv y_0$.

It follows that $\nabla u(\Gamma_u) = \nabla u(\Omega) = y_0 + a\Omega$. By (4.3) we know that $W \subset \nabla u(\Gamma_u) = y_0 + a\Omega$. In addition, these two sets have the same measure since equalities occur in (4.4). Thus, $y_0 + a\Omega$ is equal to W up to a set of measure zero. In fact, in the present situation, since W is convex and $y_0 + a\Omega$ is open, one easily proves that $W = y_0 + a\Omega$. Hence, Ω is of the form $\tilde{a}W + \tilde{b}$ for some $\tilde{a} > 0$ and $\tilde{b} \in \mathbb{R}^n$. $\qquad\square$

5. Proof of Theorem 1.3: the case $w \equiv 0$ on $\partial\Sigma$ and $H = \|\cdot\|_2$

For the sake of clarity, we present in this section the proof of Theorem 1.3 under the assumptions $w \equiv 0$ on $\partial\Sigma$ and $H = \|\cdot\|_2$. The proof is simpler in this case. Within the proof we will use the following lemma.

Lemma 5.1. *Let w be a positive homogeneous function of degree $\alpha > 0$ in an open cone $\Sigma \subset \mathbb{R}^n$. Then, the following conditions are equivalent:*

- *For each $x, z \in \Sigma$, it holds the following inequality:*

$$\alpha \left(\frac{w(z)}{w(x)}\right)^{1/\alpha} \le \frac{\nabla w(x) \cdot z}{w(x)}.$$

- *The function $w^{1/\alpha}$ is concave in Σ.*

Proof. Assume first $\alpha = 1$. A function w is concave in Σ if and only if for each $x, z \in \Sigma$ it holds

$$w(x) + \nabla w(x) \cdot (z - x) \geq w(z). \qquad (5.1)$$

Now, since w is homogeneous of degree 1, we have

$$\nabla w(x) \cdot x = w(x). \qquad (5.2)$$

This can be seen by differentiating the equality $w(tx) = tw(x)$ and evaluating at $t = 1$. Hence, from (5.1) and (5.2) we deduce that an homogeneous function w of degree 1 is concave if and only if

$$w(z) \leq \nabla w(x) \cdot z.$$

This proves the lemma for $\alpha = 1$.

Assume now $\alpha \neq 1$. Define $v = w^{1/\alpha}$, and apply the result proved above to the function v, which is homogeneous of degree 1. We obtain that v is concave if and only if

$$v(z) \leq \nabla v(x) \cdot z.$$

Therefore, since $\nabla v(x) = \alpha^{-1} w(x)^{\frac{1}{\alpha}-1} \nabla w(x)$, we deduce that $w^{1/\alpha}$ is concave if and only if

$$w(z)^{1/\alpha} \leq \frac{\nabla w(x) \cdot z}{\alpha w(x)^{1-\frac{1}{\alpha}}},$$

and the lemma follows. □

We give now the

Proof of Theorem 1.3 in the case $w \equiv 0$ on $\partial\Sigma$ and $H = \|\cdot\|_2$. For the sake of simplicity we assume here that $E = U \cap \Sigma$, where U is some bounded smooth domain in $\mathbb{R}^n$. The case of general sets will be treated in Section 6 when we prove Theorem 1.3 in its full generality.

Observe that since $E = U \cap \Sigma$ is piecewise Lipschitz, and $w \equiv 0$ on $\partial\Sigma$, it holds

$$P_w(E; \Sigma) = \int_{\partial U \cap \Sigma} w(x) dx = \int_{\partial E} w(x) dx. \qquad (5.3)$$

Hence, using that $w \in C(\overline{\Sigma})$ and (5.3), it is immediate to prove that for any $y \in \Sigma$ we have

$$\lim_{\delta \downarrow 0} P_w(E + \delta y; \Sigma) = P_w(E; \Sigma) \quad \text{and} \quad \lim_{\delta \downarrow 0} w(E + \delta y) = w(E).$$

We have denoted $E + \delta y = \{x + \delta y, \ x \in E\}$. Note that $P_w(E + \delta y; \Sigma)$ could not converge to $P_w(E; \Sigma)$ as $\delta \downarrow 0$ if w did not vanish on the boundary of the cone Σ.

By this approximation property and a subsequent regularization of $E + \delta y$ (a detailed argument can be found in the proof of Theorem 1.3 in next section), we see that it suffices to prove (1.13) for smooth domains whose closure is contained in Σ.

Thus, from now on in the proof we denote by Ω, instead of E, any smooth domain satisfying $\overline{\Omega} \subset \Sigma$. We next show (1.13) with E replaced by Ω.

At this stage, it is clear that by approximating $w|_{\overline{\Omega}}$ we can assume $w \in C^\infty(\overline{\Omega})$.

Let u be a solution of the linear Neumann problem

$$
\begin{cases}
w^{-1}\mathrm{div}(w\nabla u) = b_\Omega & \text{in } \Omega \quad (\text{with } \overline{\Omega} \subset \Sigma) \\
\quad\dfrac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega.
\end{cases}
\tag{5.4}
$$

The Fredholm alternative ensures that there exists a solution of (5.4) (which is unique up to an additive constant) if and only if the constant b_Ω is given by

$$
b_\Omega = \frac{P_w(\Omega; \Sigma)}{w(\Omega)}.
\tag{5.5}
$$

Note also that since w is positive and smooth in $\overline{\Omega}$, (5.4) is a uniformly elliptic problem with smooth coefficients. Thus, $u \in C^\infty(\overline{\Omega})$. For these classical facts, see Example 2 in Section 10.5 of [34], or the end of Section 6.7 of [32].

Consider now the lower contact set of u, Γ_u, defined by (4.2) as the set of points in Ω at which the tangent hyperplane to the graph of u lies below u in all $\overline{\Omega}$. Then, as in the proof of the Wulff theorem in Section 4, we touch by below the graph of u with hyperplanes of fixed slope $p \in B_1$, and using the boundary condition in (5.4) we deduce that $B_1 \subset \nabla u(\Gamma_u)$. From this, we obtain

$$
B_1 \cap \Sigma \subset \nabla u(\Gamma_u) \cap \Sigma
\tag{5.6}
$$

and thus

$$
\begin{aligned}
w(B_1 \cap \Sigma) &\leq \int_{\nabla u(\Gamma_u) \cap \Sigma} w(p)\,dp \\
&\leq \int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} w(\nabla u(x)) \det D^2 u(x)\,dx \\
&\leq \int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} w(\nabla u) \left(\frac{\Delta u}{n}\right)^n dx.
\end{aligned}
\tag{5.7}
$$

We have applied the area formula to the smooth map $\nabla u : \Gamma_u \to \mathbb{R}^n$ and also the classical arithmetic-geometric means inequality —all eigenvalues of $D^2 u$ are nonnegative in Γ_u by definition of this set.

Next we use that, when $\alpha > 0$,

$$
s^\alpha t^n \leq \left(\frac{\alpha s + nt}{\alpha + n}\right)^{\alpha+n} \quad \text{for all } \ s > 0 \text{ and } t > 0,
$$

which follows from the concavity of the logarithm function. Using also Lemma 5.1, we find

$$\frac{w(\nabla u)}{w(x)}\left(\frac{\Delta u}{n}\right)^n \leq \left(\frac{\alpha\left(\frac{w(\nabla u)}{w(x)}\right)^{1/\alpha}+\Delta u}{\alpha+n}\right)^{\alpha+n} \leq \left(\frac{\frac{\nabla w(x)\cdot\nabla u}{w(x)}+\Delta u}{D}\right)^D.$$

Recall that $D = n + \alpha$. Thus, using the equation in (5.4), we obtain

$$\frac{w(\nabla u)}{w(x)}\left(\frac{\Delta u}{n}\right)^n \leq \left(\frac{b_\Omega}{D}\right)^D \quad \text{in } \Gamma_u \cap (\nabla u)^{-1}(\Sigma). \tag{5.8}$$

If $\alpha = 0$ then $w \equiv 1$, and (5.8) is trivial.

Therefore, since $\Gamma_u \subset \Omega$, combining (5.7) and (5.8) we obtain

$$w(B_1 \cap \Sigma) \leq \int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} \left(\frac{b_\Omega}{D}\right)^D w(x)dx = \left(\frac{b_\Omega}{D}\right)^D w(\Gamma_u \cap (\nabla u)^{-1}(\Sigma))$$
$$\leq \left(\frac{b_\Omega}{D}\right)^D w(\Omega) = D^{-D}\frac{P_w(\Omega;\Sigma)^D}{w(\Omega)^{D-1}}. \tag{5.9}$$

In the last equality we have used the value of the constant b_Ω, given by (5.5).

Finally, using that, by (1.12), we have $P_w(B_1;\Sigma) = D\,w(B_1 \cap \Sigma)$, we obtain the desired inequality (1.13).

An alternative way to see that (5.9) is equivalent to (1.13) is to analyze the previous argument when $\Omega = B_1 \cap \Sigma$. In this case $\overline{\Omega} \not\subset \Sigma$ and therefore, as explained in Section 3, we must solve problem (3.2) instead of problem (5.4). When $\Omega = B_1 \cap \Sigma$ the solution to problem (3.2) is $u(x) = |x|^2/2$. For this function u we have $\Gamma_u = B_1 \cap \Sigma$ and $b_{B_1 \cap \Sigma} = P_w(B_1;\Sigma)/w(B_1 \cap \Sigma)$ —as in (5.5). Hence, for these concrete Ω and u one verifies that all inclusions and inequalities in (5.6), (5.7), (5.8), (5.9) are equalities, and thus (1.13) follows. $\square$

6. Proof of Theorem 1.3: the general case

In this section we prove Theorem 1.3 in its full generality. At the end of the section, we include the geometric argument of E. Milman that provides an alternative proof of Theorem 1.3 in the case that the exponent α is an integer.

Proof of Theorem 1.3. Let

$$W_0 := W \cap \Sigma,$$

an open convex set, and nonempty by assumption. Since $\lambda W_0 \subset W_0$ for all $\lambda \in (0,1)$, we deduce that $0 \in \overline{W}_0$. Therefore, as commented in subsection 1.1, there is a unique gauge H_0 such that its Wulff shape is W_0. In fact, H_0 is defined by expression (1.7) (with W and H replaced by W_0 and H_0).

Since $H_0 \leq H$ we have

$$P_{w,H_0}(E; \Sigma) \leq P_{w,H}(E; \Sigma) \quad \text{for each measurable set } E,$$

while, using (1.11),

$$P_{w,H_0}(W_0; \Sigma) = P_{w,H}(W; \Sigma) \quad \text{and} \quad w(W_0) = w(W \cap \Sigma).$$

Thus, it suffices to prove that

$$\frac{P_{w,H_0}(E; \Sigma)}{w(E)^{\frac{D-1}{D}}} \geq \frac{P_{w,H_0}(W_0; \Sigma)}{w(W_0)^{\frac{D-1}{D}}} \tag{6.1}$$

for all measurable sets $E \subset \Sigma$ with $w(E) < \infty$.

The definition of H_0 is motivated by the following reason. Note that H_0 vanishes on the directions normal to the cone Σ. Thus, by considering H_0 instead of H, we will be able (by an approximation argument) to assume that E is a smooth domain whose closure is contained in Σ. This approximation cannot be done when H does not vanish on the directions normal to the cone —since the relative perimeter does not count the part of the boundary lying on $\partial \Sigma$, while when $\overline{E} \subset \Sigma$ the whole perimeter is counted.

We split the proof of (6.1) in three cases.

Case 1. Assume that $E = \Omega$, where Ω is a smooth domain satisfying $\overline{\Omega} \subset \Sigma$.

At this stage, it is clear that by regularizing $w|_{\overline{\Omega}}$ and $H_0|_{S^{n-1}}$ we can assume $w \in C^\infty(\overline{\Omega})$ and $H_0 \in C^\infty(S^{n-1})$.

Let u be a solution to the Neumann problem

$$\begin{cases} w^{-1}\text{div}(w\nabla u) = b_\Omega & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = H_0(\nu) & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

where $b_\Omega \in \mathbb{R}$ is chosen so that the problem has a unique solution up to an additive constant, that is,

$$b_\Omega = \frac{P_{w,H_0}(\Omega; \Sigma)}{w(\Omega)}. \tag{6.3}$$

Since w is positive and smooth in $\overline{\Omega}$, and H_0, ν, and Ω are smooth, we have that $u \in C^\infty(\overline{\Omega})$. See our comments following (5.4)-(5.5) for references of these classical facts.

Consider the lower contact set of u, defined by

$$\Gamma_u = \{x \in \Omega \ : \ u(y) \geq u(x) + \nabla u(x) \cdot (y - x) \ \text{ for all } y \in \overline{\Omega}\}.$$

We claim that

$$W_0 \subset \nabla u(\Gamma_u) \cap \Sigma. \tag{6.4}$$

To prove (6.4), we proceed as in the proof of Theorem 1.2 in Section 4. Take $p \in W_0$, that is, $p \in \mathbb{R}^n$ satisfying $p \cdot \nu < H_0(\nu)$ for each $\nu \in S^{n-1}$. Let $x \in \overline{\Omega}$ be a point such that

$$\min_{y \in \overline{\Omega}} \{u(y) - p \cdot y\} = u(x) - p \cdot x.$$

If $x \in \partial\Omega$ then the exterior normal derivative of $u(y) - p \cdot y$ at x would be nonpositive and, hence, $(\partial u/\partial\nu)(x) \le p \cdot \nu < H_0(p)$, a contradiction with (6.2). Thus, $x \in \Omega$, $p = \nabla u(x)$, and $x \in \Gamma_u$ —see Section 4 for more details. Hence, $W_0 \subset \nabla u(\Gamma_u)$, and since $W_0 \subset \Sigma$, claim (6.4) follows.

Therefore,

$$w(W_0) \le \int_{\nabla u(\Gamma_u) \cap \Sigma} w(p) dp \le \int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} w(\nabla u) \det D^2 u \; dx. \qquad (6.5)$$

We have applied the area formula to the smooth map $\nabla u : \Gamma_u \to \mathbb{R}^n$, and we have used that its Jacobian, $\det D^2 u$, is nonnegative in Γ_u by definition of this set.

We proceed now as in Section 5. Namely, we first use the following weighted version of the inequality between the arithmetic and the geometric means,

$$a_0^\alpha a_1 \cdots a_n \le \left(\frac{\alpha a_0 + a_1 + \cdots + a_n}{\alpha + n} \right)^{\alpha+n},$$

applied to the numbers $a_0 = \left(\frac{w(\nabla u)}{w(x)} \right)^{1/\alpha}$ and $a_i = \lambda_i(x)$ for $i = 1, ..., n$, where $\lambda_1, ..., \lambda_n$ are the eigenvalues of $D^2 u$. We obtain

$$\frac{w(\nabla u)}{w(x)} \det D^2 u \le \left(\frac{\alpha \left(\frac{w(\nabla u)}{w(x)} \right)^{1/\alpha} + \Delta u}{\alpha + n} \right)^{\alpha+n} \le \left(\frac{\frac{\nabla w(x) \cdot \nabla u}{w(x)} + \Delta u}{\alpha + n} \right)^{\alpha+n}. \qquad (6.6)$$

In the last inequality we have used Lemma 5.1. Now, the equation in (6.2) gives

$$\frac{\nabla w(x) \cdot \nabla u}{w(x)} + \Delta u = \frac{\operatorname{div}(w(x)\nabla u)}{w(x)} \equiv b_\Omega,$$

and thus using (6.3) we find

$$\int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} w(\nabla u) \det D^2 u \; dx \le \int_{\Gamma_u \cap (\nabla u)^{-1}(\Sigma)} w(x) \left(\frac{b_\Omega}{D} \right)^D dx$$

$$\le \int_{\Gamma_u} w(x) \left(\frac{b_\Omega}{D} \right)^D dx = \left(\frac{P_{w,H_0}(\Omega; \Sigma)}{D \, w(\Omega)} \right)^D w(\Gamma_u). \qquad (6.7)$$

Therefore, from (6.5) and (6.7) we deduce

$$w(W_0) \le \left(\frac{P_{w,H_0}(\Omega; \Sigma)}{D \, w(\Omega)} \right)^D w(\Gamma_u) \le \left(\frac{P_{w,H_0}(\Omega; \Sigma)}{D \, w(\Omega)} \right)^D w(\Omega). \qquad (6.8)$$

Finally, using that, by (1.12), we have $P_{w,H_0}(W; \Sigma) = D \, w(W_0)$, we deduce (6.1).

An alternative way to see that (6.8) is equivalent to (6.1) is to analyze the previous argument when $\Omega = W_0 = W \cap \Sigma$. In this case $\overline{\Omega} \not\subseteq \Sigma$ and therefore, as explained

in Section 3, we must solve problem

$$
\begin{cases}
w^{-1}\mathrm{div}\,(w\nabla u) = b_\Omega & \text{in } \Omega \\[2mm]
\dfrac{\partial u}{\partial \nu} = H_0(\nu) & \text{on } \partial\Omega \cap \Sigma \\[2mm]
\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \cap \partial\Sigma
\end{cases}
\tag{6.9}
$$

instead of problem (6.2). When $\Omega = W_0$, the solution to problem (6.9) is

$$
u(x) = |x|^2/2.
$$

For this function u we have $\Gamma_u = W_0$ and $b_{W_0} = P_{w,H_0}(W_0; \Sigma)/w(W_0)$ —as in (6.3). Hence, for these concrete Ω and u one verifies that all inclusions and inequalities in (6.4), (6.5), (6.6), (6.7), and (6.8) are equalities, and thus (6.1) follows.

Case 2. Assume now that $E = U \cap \Sigma$, where U is a bounded smooth open set in $\mathbb{R}^n$. Even that both U and Σ are Lipschitz sets, their intersection might not be Lipschitz (for instance if ∂U and $\partial\Sigma$ meet tangentially at a point). As a consequence, approximating $U \cap \Sigma$ by smooth sets converging in perimeter is a more subtle issue. However, we claim that there exists a sequence $\{\Omega_k\}_{k\geq 1}$ of smooth bounded domains satisfying

$$
\overline{\Omega_k} \subset \Sigma \qquad \text{and} \qquad \lim_{k\to\infty} \frac{P_{w,H_0}(\Omega_k; \Sigma)}{w(\Omega_k)^{\frac{D-1}{D}}} \leq \frac{P_{w,H_0}(E; \Sigma)}{w(E)^{\frac{D-1}{D}}}.
\tag{6.10}
$$

Case 2 follows immediately using this claim and what we have proved in Case 1. We now proceed to prove the claim.

It is no restriction to assume that e_n, the n-th vector of the standard basis, belongs to the cone Σ. Then, $\partial\Sigma$ is a convex graph (and therefore, Lipschitz in every compact set) over the variables $x_1, \ldots, x_{n-1}$. That is,

$$
\Sigma = \{x_n > g(x_1, \ldots, x_{n-1})\}
\tag{6.11}
$$

for some convex function $g : \mathbb{R}^{n-1} \to \mathbb{R}$.

First we construct a sequence

$$
F_k = \{x_n > g_k(x_1, \ldots, x_{n-1})\}, \quad k \geq 1
\tag{6.12}
$$

of convex smooth sets whose boundary is a graph $g_k : \mathbb{R}^{n-1} \to \mathbb{R}$ over the first $n-1$ variables and satisfying:

(i) $g_1 > g_2 > g_3 > \ldots$ in $\overline{B}$, where B is a large ball $B \subset \mathbb{R}^{n-1}$ containing the projection of $\overline{U}$.

(ii) $g_k \to g$ uniformly in $\overline{B}$.

(iii) $\nabla g_k \to \nabla g$ almost everywhere in $\overline{B}$ and $|\nabla g_k|$ is bounded independently of k.

(iv) The smooth manifolds $\partial F_k = \{x_n = g_k(x_1, \ldots, x_{n-1})\}$ and ∂U intersect transversally.

To construct the sequence g_k, we consider the convolution of g with a standard mollifier

$$\tilde{g}_k = g * k^{n-1} \eta(kx) + \frac{C}{k}$$

with C is a large constant (depending on $\|\nabla g\|_{L^\infty(\mathbb{R}^{n-1})}$) to guarantee $\tilde{g}_k > g$ in $\overline{B}$. It follows that a subsequence of $\tilde{g}_k$ will satisfy (i)-(iii). Next, by a version of Sard's Theorem [33, Section 2.3] almost every small translation of the smooth manifold $\{x_n = \tilde{g}_k(x_1, \ldots, x_{n-1})\}$ will intersect ∂U transversally. Thus, the sequence

$$g_k(x_1, \ldots, x_{n-1}) = \tilde{g}_k(x_1 - y_1^k, \ldots, x_{n-1} - y_{n-1}^k) + y_n^k$$

will satisfy (i)-(iv) if $y^k \in \mathbb{R}^n$ are chosen with $|y^k|$ sufficiently small depending on k —in particular to preserve (i).

Let us show now that $P_{w,H_0}(U \cap F_k; \Sigma)$ converges to $P_{w,H_0}(E; \Sigma)$ as $k \uparrow \infty$. Note that (i) yields $F_k \subset F_{k+1}$ for all $k \geq 1$. This monotonicity will be useful to prove the convergence of perimeters, that we do next.

Indeed, since we considered the gauge H_0 instead of H, we have the following property

$$P_{w,H_0}(E; \Sigma) = \int_{\partial U \cap \Sigma} H_0(\nu(x)) w(x) dx = \int_{\partial E} H_0(\nu(x)) w(x) dx. \qquad (6.13)$$

This is because $\partial E = \partial(U \cap E) \subset (\partial U \cap \Sigma) \cup (\overline{U} \cap \partial \Sigma)$ and

$$H_0(\nu(x)) = 0 \quad \text{for almost all } x \in \partial \Sigma. \qquad (6.14)$$

Now, since $\partial(U \cap F_k) \subset (\partial U \cap F_k) \cup (\overline{U} \cap \partial F_k)$ we have

$$0 \leq P_{w,H_0}(U \cap F_k; \Sigma) - \int_{\partial U \cap F_k} H_0(\nu(x)) w(x) dx \leq \int_{\overline{U} \cap \partial F_k} H_0(\nu_{F_k}(x)) w(x) dx.$$

On one hand, using dominated convergence, (6.11), (6.12), (ii)-(iii), and (6.14), we readily prove that

$$\int_{\overline{U} \cap \partial F_k} H_0(\nu_{F_k}(x)) w(x) dx \to 0.$$

On the other hand, by (i) and (ii), $F_k \cap (B \times \mathbb{R})$ is an increasing sequence exhausting $\Sigma \cap (B \times \mathbb{R})$. Hence, by monotone convergence

$$\int_{\partial U \cap F_k} H_0(\nu(x)) w(x) dx \to \int_{\partial U \cap \Sigma} H_0(\nu(x)) w(x) dx = P_{w,H_0}(E; \Sigma).$$

Therefore, the sets $U \cap F_k$ approximate $U \cap \Sigma$ in L^1 and in the (w,H_0)-perimeter. Moreover, by (iv), $U \cap F_k$ are Lipschitz open sets.

Finally, to obtain the sequence of smooth domains Ω_k in (6.10), we use a partition of unity and local regularization of the Lipschitz sets $U \cap F_k$ to guarantee the convergence of the (w,H_0)-perimeters. In case that the regularized sets had more than one connected component, we may always choose the one having better isoperimetric quotient.

Case 3. Assume that E is any measurable set with $w(E) < \infty$ and $P_{w,H_0}(E;\Sigma) \leq P_{w,H}(E;\Sigma) < \infty$. As a consequence of Theorem 5.1 in [3], $C_c^\infty(\mathbb{R}^n)$ is dense in the space BV_{μ,H_0} of functions of bounded variation with respect to the measure $\mu = w\chi_\Sigma$ and the gauge H_0. Note that our definition of perimeter $P_{w,H_0}(E;\Sigma)$ coincides with the (μ, H_0)-total variation of the characteristic function χ_E, that is, $|D_\mu\chi_E|_{H_0}$ in notation of [3]. Hence, by the coarea formula in Theorem 4.1 in [3] and the argument in Section 6.1.3 in [41], we find that for each measurable set $E \subset \Sigma$ with finite measure there exists a sequence of bounded smooth sets $\{U_k\}$ satisfying

$$\lim_{k\to\infty} w(U_k \cap \Sigma) = w(E) \quad \text{and} \quad \lim_{k\to\infty} P_{w,H_0}(U_k;\Sigma) = P_{w,H_0}(E;\Sigma).$$

Then we are back to Case 2 above, and hence the proof is finished.   □

After the announcement of our result and proof in [13], Emanuel Milman showed us a nice geometric construction that yields the weighted inequality in Theorem 1.3 in the case that α is a nonnegative *integer*. We next sketch this construction.

Remark 6.1 (Emanuel Milman's construction). When α is a nonnegative integer the weighted isoperimetric inequality of Theorem 1.3 (when $H = \|\cdot\|_2$) can be proved as a limit case of the Lions-Pacella inequality in convex cones of $\mathbb{R}^{n+\alpha}$. Indeed, let $w^{1/\alpha} > 0$ be a concave function, homogeneous of degree 1, in an open convex cone $\Sigma \subset \mathbb{R}^n$. For each $\varepsilon > 0$, consider the cone

$$\mathcal{C}_\varepsilon = \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^\alpha : x \in \Sigma, \ |y| < \varepsilon w(x)^{1/\alpha}\right\}.$$

From the convexity of Σ and the concavity of $w^{1/\alpha}$ we have that $\mathcal{C}_\varepsilon$ is a convex cone. Hence, by Theorem 1.1 we have

$$\frac{P(\tilde{E};\mathcal{C}_\varepsilon)}{|\tilde{E} \cap \mathcal{C}_\varepsilon|^{\frac{n+\alpha-1}{n+\alpha}}} \geq \frac{P(\mathcal{B}_1;\mathcal{C}_\varepsilon)}{|\mathcal{B}_1 \cap \mathcal{C}_\varepsilon|^{\frac{n+\alpha-1}{n+\alpha}}} \quad \text{for all } \tilde{E} \text{ with } |\tilde{E} \cap \mathcal{C}_\varepsilon| < \infty, \qquad (6.15)$$

where $\mathcal{B}_1$ is the unit ball of $\mathbb{R}^{n+\alpha}$. Now, given a Lipschitz set $E \subset \mathbb{R}^n$, consider the cylinder $\tilde{E} = E \times \mathbb{R}^\alpha$ one finds

$$|\tilde{E} \cap \mathcal{C}_\varepsilon| = \int_{E\cap\Sigma} dx \int_{\{|y|<\varepsilon w(x)^{1/\alpha}\}} dy = \omega_\alpha \varepsilon^\alpha \int_{E\cap\Sigma} w(x)dx = \omega_\alpha \varepsilon^\alpha w(E \cap \Sigma)$$

and

$$P(\tilde{E};\mathcal{C}_\varepsilon) = \int_{\partial E\cap\Sigma} dS(x) \int_{\{|y|<\varepsilon w(x)^{1/\alpha}\}} dy = \omega_\alpha \varepsilon^\alpha \int_{\partial E\cap\Sigma} w(x)dS = \omega_\alpha \varepsilon^\alpha P_w(E;\Sigma).$$

On the other hand, one easily sees that, as $\varepsilon \downarrow 0$,

$$\frac{P(\mathcal{B}_1;\mathcal{C}_\varepsilon)}{|\mathcal{B}_1 \cap \mathcal{C}_\varepsilon|^{\frac{n+\alpha-1}{n+\alpha}}} = (\omega_\alpha\varepsilon^\alpha)^{\frac{1}{n+\alpha}} \left(\frac{P_w(B_1;\Sigma)}{w(B_1 \cap \Sigma)^{\frac{n+\alpha-1}{n+\alpha}}} + o(1)\right),$$

where B_1 is the unit ball of $\mathbb{R}^n$. Hence, letting $\varepsilon \downarrow 0$ in (6.15) one obtains

$$\frac{P_w(E;\Sigma)}{w(E \cap \Sigma)^{\frac{n+\alpha-1}{n+\alpha}}} \geq \frac{P_w(B_1;\Sigma)}{w(B_1 \cap \Sigma)^{\frac{n+\alpha-1}{n+\alpha}}},$$

which is the inequality of Theorem 1.3 in the case that $H = \|\cdot\|_2$ and α is an integer.

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ICREA and Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain

E-mail address: xavier.cabre@upc.edu

Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain

E-mail address: xavier.ros.oton@upc.edu

Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain

E-mail address: joaquim.serra@upc.edu

I. Radial symmetry for diffusion equations with discontinuous nonlinearities

RADIAL SYMMETRY OF SOLUTIONS TO DIFFUSION EQUATIONS WITH DISCONTINUOUS NONLINEARITIES

JOAQUIM SERRA

ABSTRACT. We prove a radial symmetry result for bounded nonnegative solutions to the p-Laplacian semilinear equation $-\Delta_p u = f(u)$ posed in a ball of $\mathbb{R}^n$ and involving discontinuous nonlinearities f. When $p = 2$ we obtain a new result which holds in every dimension n for certain positive discontinuous f. When $p \geq n$ we prove radial symmetry for every locally bounded nonnegative f. Our approach is an extension of a method of P. L. Lions for the case $p = n = 2$. It leads to radial symmetry combining the isoperimetric inequality and the Pohozaev identity.

1. INTRODUCTION

We consider positive solutions of

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \subset \mathbb{R}^n\,, \\ u > 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where Ω is a ball. A classical theorem of Gidas-Ni-Nirenberg [10] states that if $f = f_1 + f_2$ with f_1 Lipschitz and f_2 nondecreasing, then a solution $u \in C^2(\overline{\Omega})$ to (1.1) has radial symmetry. Since f_2 might be any nondecreasing function, this result allows f to be discontinuous, but only with increasing jumps. Besides this, the only other general result for f discontinuous is, to our knowledge, the one of P. L. Lions [13], that states radial symmetry of solutions for every locally bounded $f \geq 0$ in dimension $n = 2$.

In this paper we establish radial symmetry of solutions to (1.1) in every dimension $n \geq 3$ under the assumption

$$\phi \leq f \leq \frac{2n}{n-2}\,\phi$$

for some nonincreasing function $\phi \geq 0$. In addition, we also obtain results for the p-Laplacian equation

$$\begin{cases} -\Delta_p u := -\nabla \cdot (|\nabla u|^{p-2}\nabla u) = f(u) & \text{in } \Omega\,, \\ u \geq 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a ball. For instance, under the assumption $p \geq n$, we establish radial symmetry of bounded solutions to (1.2) for every $f \geq 0$ locally bounded but possibly discontinuous.

The result to be proved in this paper is the following:

Theorem 1. *Let Ω be a ball in $\mathbb{R}^n$, $n \geq 2$, and let $1 < p < \infty$. Assume that $f \in L^\infty_{loc}([0, +\infty))$ is nonnegative. Let $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ be a solution of (1.2) in the weak sense. Assume that either*

    (a) *$p \geq n$,*

or

    (b) *$p < n$ and, for some nonincreasing function $\phi \geq 0$, we have $\phi \leq f \leq \frac{np}{n-p}\phi$.*

    Then, u is a radially symmetric and nonincreasing function. Moreover, $\frac{\partial u}{\partial r} < 0$ in $\{0 < u < \max_{\overline{\Omega}} u\}$, that will be an annulus or a punctured ball.

This result follows the approach introduced in 1981 by P. L. Lions within the paper [13], where the case $p = n = 2$ of Theorem 1 is proved (also with the hypothesis $f \geq 0$). In the same direction, Kesavan and Pacella [11] established the cases $p = n \geq 2$ of Theorem 1. In Lions' method, the isoperimetric inequality and the Pohozaev identity are combined to conclude the symmetry of u.

For some nonlinearities f which change sign, there exist positive solutions of (1.2) in a ball which are not radially symmetric, even with $p = 2$ and f Hölder continuous (see [3] for an example).

For $1 < p < \infty$, assuming that f is locally Lipschitz and positive, and that $u \in C^1(\overline{\Omega})$ is a positive solution of (1.2) in a ball, Damascelli and Pacella [5] ($1 < p < 2$) and Damascelli and Sciunzi [6] ($p > 2$) succeeded in applying the moving planes method to prove the radial symmetry of u.

Another symmetry result for (1.1) with possibly non-Lipschitz f is due to Dolbeault and Felmer [7]. They assume that f is continuous and that, in a neighborhood of each point of its domain, f is either decreasing, or is the sum of a Lipschitz and a nondecreasing functions. If, in addition, $f \geq 0$, solutions $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ to (1.1) in a ball are radially symmetric. A similar result for the p-Laplacian equation (1.2) is found in [8]. These results use a local version of the moving planes technique.

Under the weaker assumption that $f \geq 0$ is only continuous, for $1 < p < \infty$, Brock [2] proved that $C^1(\overline{\Omega})$ positive solutions of (1.2) are radially symmetric using the so called "continuous Steiner symmetrization".

The radial symmetry results in [2] (via continuous symmetrization) and in [7, 8] (via local moving planes) follow from more general local symmetry results [3, 7, 8] which do not require $f \geq 0$. These describe the only way in which radial symmetry may be broken through the formation of "plateaus" and radially symmetric cores placed arbitrarily on the top of them. The notion of local symmetry, introduced by Brock in [3], is very related to rearrangements. Nevertheless, in [7, 8], local symmetry results are proved using a local version of the moving planes method.

Our technique leads to symmetry only when Ω is a ball. Instead, the technique used in [2], as well as the moving planes method used in [10, 6, 7, 8] are still applicable when the domain is not a ball, but is symmetric about some hyperplane and convex in the normal direction to this hyperplane. See [1] for an improved version of the moving planes method that allowed to treat domains with corners.

A feature of the original moving planes method in [10] and [6] is that, in addition to the radial symmetry, leads to $\frac{\partial u}{\partial r} < 0$, for $r = |x| \in (0, R)$, R being the radius of the ball Ω. However, with discontinuous f we cannot expect so much, even with $p = 2$. A simple counterexample is constructed as follows: let v be the solution of

$$\begin{cases} -\Delta_p v = 1 & \text{in } A = \{1/2 < r < 1\}, \\ v = 0 & \text{on } \partial A. \end{cases}$$

Then, v is radial and positive, and thus it attains its maximum on a sphere $\{r = \rho_0\}$, for some $\rho_0 \in (1/2, 1)$. We readily check that $u = v\chi_{\{r>\rho_0\}} + (\max_{\overline{A}} v)\chi_{\{r\leq\rho_0\}}$ is a solution of (1.2) for $\Omega = \{r < 1\}$ and $f = \chi_{[0,\max_{\overline{A}} v)} \geq 0$, and u is constant on the ball $\{r \leq \rho_0\}$.

Related to this, Theorem 1 states that u is radial with $\frac{\partial u}{\partial r} < 0$ in the annulus or punctured ball $\{0 < u < \max_{\overline{\Omega}} u\}$ (see Lemma 6). Nevertheless, u might attain its maximum in a concentric ball of positive radius $\{u = \max_{\overline{\Omega}} u\}$, as occurs in the preceding example.

The following three distribution-type functions will play a central role in our proof:

$$I(t) = \int_{\{u>t\}} f(u) \, d\mathcal{H}^n, \quad J(t) = \mathcal{H}^n(\{u > t\}), \quad K = I^\alpha J^\beta. \tag{1.3}$$

These functions are defined for $t \in (-\infty, M)$, where $M = \max_\Omega u$. The parameters α, β in (1.3), that are appropriately chosen depending on p and n, are given by

$$\alpha = p' = \frac{p}{p-1}, \quad \beta = \frac{p-n}{n(p-1)}. \tag{1.4}$$

Lions [13] in the case $p = n = 2$ and Kesavan-Pacella [11] in the cases $p = n \geq 2$ used the distribution type function $K = I^\alpha$ (note that our $\beta = 0$ in these cases). By considering the function $K = I^\alpha J^\beta$ we are able to treat the cases $p \neq n$.

Observe that for any $t < 0$ the value of $K(t)$ is equal to the constant

$$K(0^-) = \lim_{t \to 0^-} K(t) = \left(\int_\Omega f(u) \, d\mathcal{H}^n \right)^\alpha (\mathcal{H}^n(\Omega))^\beta. \tag{1.5}$$

Remark 2. As we shall see, it is essential for our argument to work that the function K in (1.3) be nonincreasing. This is trivially the case when α, β given by (1.4) are nonnegative, and thus this occurs when $p \geq n$.

However, it may happen that, even with β being negative, K could be nonincreasing. This situation occurs under assumption (b) of Theorem 1, i.e., $1 < p < n$ and $\phi \leq f \leq \frac{pn}{n-p}\phi$ for some nonincreasing function $\phi \geq 0$. Indeed, in Lemma 4 (iii) we will prove that, in this case, K is absolutely continuous. Thus, to verify that K is nonincreasing we need to prove that $-K' \geq 0$ a.e.

Now, using statement (2.1) of the lemma, we obtain

$$-K' = \left\{\alpha I^{\alpha-1}J^\beta f + \beta I^\alpha J^{\beta-1}\right\}(-J') = \left\{\alpha f + \beta I/J\right\}I^{\alpha-1}J^\beta(-J') \quad \text{a.e.}$$

From this we see that $-K'(t)$ has the same sign as $\left\{\alpha f(t) + \beta I(t)/J(t)\right\}$, since $I, J, -J'$ are nonnegative by definition. Thus, since $\beta < 0$, we need $I(t)/J(t) + (\alpha/\beta)f(t) \leq 0$ a.e. Observing that $I(t)/J(t)$ is the mean of $f(u)$ over the superlevel set $\{u > t\}$, we easily conclude that a sufficient condition for $I/J + (\alpha/\beta)f \leq 0$ is that $f(s) \leq -(\alpha/\beta)f(t)$, whenever $s > t$. And this is satisfied if $\phi \leq f \leq -(\alpha/\beta)\phi$ for some nonincreasing $\phi \geq 0$. Replacing α, β by their values in (1.4) we obtain the condition (b) in Theorem 1 since $-\alpha/\beta = pn/(n-p)$.

Remark 3. Although the statement of Theorem 1 concerns solutions of (1.2) that are $C^1(\Omega)$, the arguments we shall use in its proof are often performed, in a standard way, with functions that are only of bounded variation. Nevertheless, from regularity results for degenerate elliptic equations of the type (1.2), we have that every bounded solution to (1.2) is $C^{1,\alpha}(\Omega)$ for some $\alpha > 0$. See, for instance, Lieberman [12]. Thus, there is no loss of generality in assuming, in Theorem 1, that $u \in C^1(\Omega)$ and this will turn some parts of its proof less technical.

2. Preliminaries and proof of Theorem 1

All the technical details that will be needed in the proof of Theorem 1 are contained in the following three lemmas. The two first of them would be immediate if we assumed that u and its level sets were regular enough. The third one leads to the radial symmetry of u and the property $\frac{\partial u}{\partial r} < 0$ in the annulus $\{0 < u < \max_\Omega u\}$. The arguments used in their proofs are rather standard: for example, a finer version of inequality (2.2) can be found in [4]. Nevertheless, we include them here to give a more self-contained treatment.

Lemma 4. *Let α, β be arbitrary real numbers and $\Omega \subset \mathbb{R}^n$ a bounded smooth domain. Assume that $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ is nonnegative and $u|_{\partial\Omega} \equiv 0$. Let $f \in L^\infty_{loc}([0,+\infty))$ and let I, J, K are defined by (1.3). Let $M = \max_{\overline{\Omega}} u$. Then:*
(i) The functions I, J and K are a.e. differentiable and

$$- K'(t) = \left\{\alpha I(t)^{\alpha-1}J(t)^\beta f(t) + \beta I(t)^\alpha J(t)^{\beta-1}\right\}(-J'(t)) \quad \text{for a.e. } t. \qquad (2.1)$$

(ii) For a.e. $t \in (0, M)$, we have $\mathcal{H}^{n-1}\big(u^{-1}(t) \cap \{|\nabla u| = 0\}\big) = 0$ and

$$- J'(t) \geq \int_{u^{-1}(t)} \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1}. \qquad (2.2)$$

(iii) Assume furthermore that hypothesis (b) in Theorem 1 holds and that u is a weak solution of (1.2). Then, I, J and K are absolutely continuous functions for $t < M$.

Proof. (i) The functions I and J are nonincreasing by definition and hence differentiable almost everywhere. Furthermore, they define nonpositive Lebesgue-Stieltjes

measures dI and dJ on $(0, M)$. By definition of Lebesgue integral, using approximation by step functions, we find that

$$I(t) = \int_{\{u \geq t\}} f(u) \, d\mathcal{H}^n = - \int_t^{M^+} f(t) dJ(t)$$

and hence $dI = f dJ$. From this, it follows that $I'(t) = f(t) J'(t)$ for dJ-a.e. t in $(0, M)$. But since $|\nabla u|$ is bounded in $\{u \geq t\}$, J is strictly decreasing. This leads to $\mathcal{L} \ll dJ$, where $\mathcal{L}$ is the Lebesgue measure in $(0, M)$. Therefore we have $I'(t) = f(t) J'(t)$ for a.e. t (in all this paper, unless otherwise indicated, a.e. is with respect to the Lebesgue measure). As a consequence, (2.1) holds.

(ii) Start defining

$$J_0(t) = \mathcal{H}^n \big(\{u > t, |\nabla u| > 0\} \big) .$$

Let $\epsilon > 0$ and $T \in (0, M)$. Let $u_T = \max(u, T)$. We extend u_T outside Ω by the constant T, to obtain a Lipschitz function defined in all $\mathbb{R}^n$. Applying to u_T the coarea formula for Lipschitz functions (see, for example, Theorem 2 in sec. 3.4.3 of [9]), we can compute

$$\mathcal{H}^n(\{u > T, |\nabla u| > \epsilon\}) = \int_{\mathbb{R}^n} |\nabla u_T| \frac{\chi_{\{u > T, |\nabla u| > \epsilon\}}}{|\nabla u|} \, d\mathcal{H}^n$$

$$= \int_T^M \int_{u^{-1}(t)} \frac{\chi_{\{u > T, |\nabla u| > \epsilon\}}}{|\nabla u|} \, d\mathcal{H}^{n-1} \, dt \qquad (2.3)$$

$$= \int_T^M \int_{u^{-1}(t) \cap \{|\nabla u| > \epsilon\}} \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1} \, dt .$$

For any given $\epsilon > 0$, $|\nabla u|^{-1} \chi_{\{u > T, |\nabla u| > \epsilon\}}$ is $\mathcal{H}^n$-summable. Now, by monotone convergence, letting $\epsilon \to 0$ in (2.3) we find that (2.3) also holds for $\epsilon = 0$ (and arbitrary T). We deduce that $J_0(t)$ is an absolutely continuous function and that

$$- J_0'(t) = \int_{u^{-1}(t) \cap \{|\nabla u| > 0\}} \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1}, \quad \text{for a.e. } t \in (0, M) . \qquad (2.4)$$

Applying one more time the coarea formula to u_T we obtain

$$0 = \int_{\mathbb{R}^n} |\nabla u_T| \chi_{\{u > T, |\nabla u| = 0\}} = \int_T^M \mathcal{H}^{n-1} \big(u^{-1}(t) \cap \{|\nabla u| = 0\} \big) \, dt .$$

We conclude that, for a.e. t, the set $u^{-1}(t) \cap \{|\nabla u| = 0\}$ has zero $\mathcal{H}^{n-1}$-measure. Having this into account we may change (2.4) for the apparently finer

$$- J_0'(t) = \int_{u^{-1}(t)} \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1}, \quad \text{for a.e. } t \in (0, M) . \qquad (2.5)$$

Next, observe that for a.e. $t \in (0, M)$ (where both $J'(t)$ and $J_0'(t)$ exist) we have the inequality

$$-J'(t) = \lim_{s \to t^+} \frac{\mathcal{H}^n \big(\{s \geq u > t\} \big)}{s - t} \geq \lim_{s \to t^+} \frac{\mathcal{H}^n \big(\{s \geq u > t, |\nabla u| > 0\} \big)}{s - t} = -J_0'(t) .$$

Combining this with (2.5) we get finally (2.2). It is easily verified that equality holds in (2.2) for a.e. t if the set $\{|\nabla u| = 0\}$ has zero $\mathcal{H}^n$-measure.

(iii) If $p < n$ and $\phi \leq f \leq \frac{np}{n-p}\phi$ for some nonincreasing $\phi \geq 0$, then a solution of $-\Delta_p u = f(u)$ will be p-harmonic in $\{u \geq t_0\}$, where $t_0 \in [0, +\infty]$ satisfies that $\phi(t) > 0$ for $t < t_0$ and $\phi(t) \equiv 0$ for $t > t_0$. Hence, if $t_0 < +\infty$, we will have that $u \equiv t_0 = M = \max_{\overline{\Omega}} u$ in $\{u \geq t_0\}$. Therefore, for every $t < M = \max_{\overline{\Omega}} u$ we have that $-\Delta_p u = f(u) \geq \phi(u) \geq \phi(t) > 0$ in $\{0 \leq u < t\}$. But since $f(u) \in L^\infty(\Omega)$, we can apply a result of H. Lou, Theorem 1.1 in [Lo] and find that $f(u)$ vanishes a.e. in the set $\{|\nabla u| = 0\} \cap \{0 \leq u < t\}$. Since $f(u) \geq \phi(t) > 0$ in $\{0 \leq u < t\}$, this is only possible if the singular set $\{|\nabla u| = 0, u < t\}$ has zero measure. Therefore, we have $J(t) = J_0(t) + \mathcal{H}^n(\{u = M\})$ for every $t < M$ and thus J is an absolutely continuous function (since we have shown that J_0 is absolutely continuous), at least for $t < M$. From this, it is immediate to see that also I and K are absolutely continuous for $t < M$. □

Lemma 5. *Under the assumptions of Theorem 1, let $M = \max_{\overline{\Omega}} u$. We have the following:*

(i) It holds the Gauss-Green type identity

$$I(t) = \int_{u^{-1}(t)} |\nabla u|^{p-1} d\mathcal{H}^{n-1}, \quad \text{for a.e. } t \in (0, M). \tag{2.6}$$

(ii) It holds the isoperimetric inequality

$$\mathcal{H}^{n-1}(u^{-1}(t)) \geq c_n \big(\mathcal{H}^n(\{u > t\})\big)^{\frac{n-1}{n}} = c_n J(t)^{\frac{n-1}{n}}, \quad \text{for a.e. } t \in (0, M), \tag{2.7}$$

where c_n is the optimal isoperimetric constant in $\mathbb{R}^n$, $c_n = \mathcal{H}^{n-1}(\partial B)\big(\mathcal{H}^n(B)\big)^{\frac{1-n}{n}}$ with B being a ball in $\mathbb{R}^n$.

Proof. (i) Since the function u is of bounded variation locally in Ω, we know from the coarea theorem for BV functions (see Theorem 1, sec. 5.5 of [9]) that the sets $\{u > t\}$ have finite perimeter for a.e. t. For the measure theoretic boundary $\partial_*\{u > t\}$ (see section 5.8 of [9]), we readily check that $\{u = t, |\nabla u| > 0\} \subset \partial_*\{u > t\} \subset u^{-1}(t)$. But recall from Lemma 4 (ii) that $\mathcal{H}^{n-1}\big(u^{-1}(t) \cap \{|\nabla u| = 0\}\big) = 0$ for a.e. t. We conclude that $\big(\int_{u^{-1}(t)} |\nabla u|^{p-1} d\mathcal{H}^{n-1}\big)$ and $\big(\int_{\partial_*\{u>t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1}\big)$ are equal for a.e. t.

On the other hand, the vector field $-\nabla u$ is perpendicular to the regular surface $u^{-1}(t) \cap \{|\nabla u| > 0\}$. We have just seen that this regular surface fills almost all $\partial_*\{u > t\}$, in the sense of $\mathcal{H}^{n-1}$-measure. As a conclusion, if ν is the measure theoretical normal vector for $\{u > t\}$ then $-\nabla u \cdot \nu = |\nabla u|$ $\mathcal{H}^{n-1}$-a.e. on $\partial_*\{u > t\}$.

Since u solves (1.2), by the generalized Gauss-Green theorem (Theorem 1, sec. 5.8. of [9]), we have

$$I(t) = \int_{\{u>t\}} f(u) d\mathcal{H}^n = \int_{\partial_*\{u>t\}} |\nabla u|^{p-1} d\mathcal{H}^{n-1} = \int_{u^{-1}(t)} |\nabla u|^{p-1} d\mathcal{H}^{n-1}, \tag{2.8}$$

for a.e. $t \in (0, M)$. Although that the precise version of Gauss-Green theorem we cite applies to a $C_c^1(\Omega)$ vector field, and we only have $|\nabla u|^{p-2} \nabla u \in C^0(\Omega)$, this can easily be handled as follows. Given t, we approximate uniformly in $\{u \geq t\}$ the continuous vector field $|\nabla u|^{p-2} \nabla u$ by a sequence of $C_c^1(\Omega)$ vector fields (ϕ_n) to which we can apply the theorem. Doing so $\nabla \cdot \phi_n$ converges weakly to $f(u)$, and this is enough for our purposes. Indeed, for each ϕ_n we have

$$\int_{\{u>t\}} \nabla \cdot \phi_n = \int_{\partial_*\{u>t\}} \phi_n \cdot \nu \, d\mathcal{H}^{n-1} \,, \quad \text{for a.e. } t \in (0, M) \,.$$

Now, letting $n \to \infty$ we obtain (2.8), and hence (2.6).

(ii) We have seen that $\{u > t\}$ is a bounded set of finite perimeter for a.e. $t \in (0, M)$. Thus the isoperimetric inequality (2.7) with the best constant follows immediately from Theorem 2 and the Remark that follows it in Section 5.6.2 of [9]. $\qquad \square$

Radial symmetry will follow from next lemma after having proved that hypothesis (1) and (2) on it hold. For a detailed discussion on a very similar question see the article of Brothers and Ziemer [4]. Here, we present an ad hoc argument inspired by this article.

Lemma 6. *Assume that $f \in L_{loc}^\infty([0, +\infty))$ is nonnegative. Let Ω be a ball in $\mathbb{R}^n$ and $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$ be a solution of (1.2) in the weak sense. Let $M = \max_{\overline{\Omega}} u$. Suppose that for a.e. $t \in (0, M)$*

(1) $\{u > t\}$ is a ball (centered at some point in $\mathbb{R}^n$, possibly depending on t)

and

(2) $|\nabla u|$ is constant on $\partial\{u > t\}$.

Then, u is radially symmetric. In addition, $\frac{\partial u}{\partial r} < 0$ in the annulus or punctured ball $\{0 < u < \max_{\overline{\Omega}} u\}$, but u could achieve its maximum in a ball of positive radius.

Proof. Denote by Θ the set of $t \in (0, M)$ for which $\{u > t\}$ is a ball. As Θ is a dense subset (its complementary has zero measure), for any $t \in (0, M)$ we have $\{u > t\} = \bigcup_{s>t,\, s\in\Theta}\{u > s\}$. Thus, every superlevel set $\{u > t\}$ is a increasing union of balls with bounded diameter, hence it is also a ball. Therefore $\Theta = (0, M)$ and we have

$$\{u > t\} = B(x(t); \rho(t))$$

for some $x(t)$, $\rho(t)$ defined for every $t \in (0, M)$.

From the continuity of ∇u and hypothesis (2) we deduce that $|\nabla u|$ is constant on $\partial B(x(t); \rho(t))$ for every $t \in (0, M)$. Besides, as u is a solution of (1.2), the Gauss-Green theorem leads to

$$\mathcal{H}^{n-1}\big(\partial B(x(t); \rho(t))\big)|\nabla u|^{p-1}\big(\partial B(x(t); \rho(t))\big) = \int_{B(x(t);\rho(t))} f(u) \, d\mathcal{H}^n \,.$$

But $f(u) \geq 0$ and, by the maximum principle, it is impossible that $f(u) \equiv 0$ on some $\{u > t\}$. We conclude that ∇u does not vanish in the open set $\{0 < u < M\}$.

Having now that u is a C^1 function whose gradient never vanishes in the open set $\{u < M\}$, it is easily shown that $J(t) = \mathcal{H}^n(\{u > t\})$ is locally Lipschitz in $(0, M)$. Therefore, also $\rho(t) = \big(J(t)/\omega_n\big)^{1/n}$ is locally Lipschitz ($\omega_n = \mathcal{H}^n(B_1)$ is the volume of a unit ball in $\mathbb{R}^n$). Moreover, since $B(x(t); \rho(t)) = \{u > t\} \supset \{u > s\} = B(x(s); \rho(s))$ for $t < s$, we deduce that

$$|x(t) - x(s)| \leq \rho(t) - \rho(s) \quad \text{for } t < s. \tag{2.9}$$

Thus, $x = x(t)$ is also locally Lipschitz.

Now suppose that u were not radially symmetric. Then x would not be identically constant in $(0, M)$ and hence we could find some $t_0 \in (0, M)$ such that the velocity vector $y = \frac{d}{dt}x(t_0)$ would exist and be nonzero. But in such case, setting $z = y/|y|$, $P(t) = x(t) + \rho(t)z$ and $Q(t) = x(t) - \rho(t)z$, by hypothesis, we would have

$$u(P(t)) \equiv u(Q(t)) \equiv t \quad \text{for all } t,$$

and $\nabla u(P(t_0)) \cdot z = -|\nabla u(P(t_0))|$ while $\nabla u(Q(t_0)) \cdot z = |\nabla u(Q(t_0))|$. This would lead to

$$1 = \frac{d}{dt}\Big|_{t_0} u(P(t)) = \nabla u(P(t_0)) \cdot (|y| + \rho'(t_0))z = -|\nabla u(P(t_0))|(|y| + \rho'(t_0))$$

and

$$1 = \frac{d}{dt}\Big|_{t_0} u(Q(t)) = \nabla u(Q(t_0)) \cdot (|y| - \rho'(t_0))z = |\nabla u(Q(t_0))|(|y| - \rho'(t_0)).$$

But we must have $|\nabla u(P(t_0))| = |\nabla u(Q(t_0))|$ since both $P(t_0)$ and $Q(t_0)$ belong to $\partial B(x(t_0); \rho(t_0)) = \partial\{u > t_0\}$. Then, it would follow that $|y| = 0$, which is a contradiction.

As a consequence, u is to be is radially symmetric. We already justified that $|\nabla u|$ does not vanish in $\{0 < u < M\}$, hence $\frac{\partial u}{\partial r} < 0$ in this open ring. However we may not discard the possibility of u being constant on a closed non-degenerate ball $\{u = M\}$, as happens in the example given in Section 1. $\qquad\square$

Finally we present the proof of the result in this paper.

Proof of Theorem 1. We first note that, under the assumptions of the theorem, $K(t)$ is nonincreasing for $t \in (0, M)$, where $M = \max_{\overline{\Omega}} u$, K is given by (1.3) and α, β are given by (1.4). Indeed, under hypothesis (a) of the theorem it is obvious because $\alpha, \beta \geq 0$. On the other hand, under hypothesis (b) Lemma 4 (iii) applies and hence K is absolutely continuous. But, as shown in Remark 2, $-K' \geq 0$ a.e. in this case. Therefore, K is nonincreasing again.

Since $K(t)$ is nonnegative and nonincreasing, we have (even if K could have jumps)

$$K(0^-) \geq K(0^+) - K(M^-) \geq \int_0^M -K'(t)\,dt. \tag{2.10}$$

Combining (2.10) and (2.1) in Lemma 4 (i), we are lead to

$$K(0^-) \geq \int_0^M \left\{ \alpha I(t)^{\alpha-1} J(t)^\beta f(t) + \beta I(t)^\alpha J(t)^{\beta-1} \right\} (-J'(t)) \, dt \, .$$

The integrand on the right equals $-K'(t)$ and hence is nonnegative for a.e. t. Also $-J'(t)$ is nonnegative. Therefore so is the factor in brackets and we can use inequality (2.2) to obtain a further estimate:

$$K(0^-) \geq \int_0^M \left\{ \alpha I(t)^{\alpha-1} J(t)^\beta f(t) + \beta I(t)^\alpha J(t)^{\beta-1} \right\} \left(\int_{u^{-1}(t)} \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1} \right) dt \, . \quad (2.11)$$

Equalities are obtained when K is absolutely continuous.

Next we derive the following isoperimetric-Hïżżlder type inequality:

$$I(t)^{\frac{1}{p-1}} \left(\int_{u^{-1}(t)} |\nabla u|^{-1} \, d\mathcal{H}^{n-1} \right) \geq c_n^{\frac{p}{p-1}} J(t)^{\frac{p(n-1)}{(p-1)n}} \, , \quad \text{for a.e. } t \in (0, M) \quad (2.12)$$

with c_n as in (2.7). To prove (2.12), we use (2.6) in Lemma 5 to conclude that, for a.e. t,

$$I(t)^{\frac{1}{p}} \left(\int_{u^{-1}(t)} |\nabla u|^{-1} \, d\mathcal{H}^{n-1} \right)^{\frac{p-1}{p}} =$$

$$= \left(\int_{u^{-1}(t)} |\nabla u|^{p-1} \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \left(\int_{u^{-1}(t)} |\nabla u|^{-1} \, d\mathcal{H}^{n-1} \right)^{\frac{p-1}{p}}$$

$$\geq \mathcal{H}^{n-1}\left(u^{-1}(t)\right) \geq c_n \mathcal{H}^n\left(\{u > t\}\right)^{\frac{n-1}{n}} = c_n J(t)^{\frac{n-1}{n}} \, ,$$

where the first inequality is a consequence of Hölder's inequality, and the second one of the isoperimetric inequality (2.7). We emphasize that both equalities hold simultaneously if and only if $\{u > t\}$ is a ball and $|\nabla u|$ is constant on $u^{-1}(t)$.

Returning to (2.11), we deduce from (2.12)

$$K(0^-) \geq \int_0^M \left\{ \alpha I^{\alpha-1} J^\beta f + \beta I^\alpha J^{\beta-1} \right\} \left(\int_{u^{-1}(t)} \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1} \right) dt$$

$$= \int_0^M \left\{ \alpha I^{\alpha-1-\frac{1}{p-1}} J^\beta f + \beta I^{\alpha-\frac{1}{p-1}} J^{\beta-1} \right\} I^{\frac{1}{p-1}} \left(\int_{u^{-1}(t)} \frac{1}{|\nabla u|} \, d\mathcal{H}^{n-1} \right) dt \quad (2.13)$$

$$\geq \int_0^M c_n^{\frac{p}{p-1}} \left\{ \alpha I^{\alpha-1-\frac{1}{p-1}} J^\beta f + \beta I^{\alpha-\frac{1}{p-1}} J^{\beta-1} \right\} J(t)^{\frac{p(n-1)}{(p-1)n}} \, dt \, .$$

For the last inequality we are using (for second time) that the factor in brackets in (2.11) is nonnegative, since $-K' \geq 0$.

Note that in order to obtain (2.13) we are integrating the isoperimetric-Hölder inequality (2.12) over almost all the levels. Accordingly,

Remark 7. A necessary condition for having equalities in (2.13) is that, for a.e. $t \in (0, M)$, $\{u > t\}$ is a ball and $|\nabla u|$ is constant on $u^{-1}(t)$.

Next, the values of α and β in (1.4) are set to satisfy that $\alpha - 1 - \frac{1}{p-1} = 0$ and $\beta - 1 + \frac{p(n-1)}{(p-1)n} = 0$. Then (2.13) becomes

$$
\begin{aligned}
K(0^-) &\geq \int_0^M c_n^{p'}\left(p' f(t)\,\mathcal{H}^n(\{u > t\}) + \frac{p-n}{n(p-1)}\int_{\{u>t\}} f(u)\,d\mathcal{H}^n \right) dt \\
&= c_n^{p'}\int_0^M \int_\Omega \chi_{\{u>t\}}\left(p' f(t) + \frac{p-n}{n(p-1)}f(u) \right) d\mathcal{H}^n\,dt \\
&= c_n^{p'}\left(p'\int_\Omega F(u)\,d\mathcal{H}^n + \frac{p-n}{n(p-1)}\int_\Omega u f(u)\,d\mathcal{H}^n \right) \\
&= c_n^{p'}\frac{p'}{n}\left(n\int_\Omega F(u)\,d\mathcal{H}^n + \frac{p-n}{p}\int_\Omega u f(u)\,d\mathcal{H}^n \right)
\end{aligned}
$$

for $F(s) = \int_0^s f(s')\,ds'$. Recalling (1.5) we obtain finally the inequality

$$
\frac{n}{p' c_n^{p'}}\mathcal{H}^n(\Omega)^{\frac{p-n}{n(p-1)}}\left(\int_\Omega f(u) \right)^{p'} \geq n\int_\Omega F(u) + \frac{p-n}{p}\int_\Omega u f(u). \qquad (2.14)
$$

Now we use for first time that Ω is a ball. As in [11], a combination of Pohozaev's identity

$$
n\int_\Omega F(u) + \frac{p-n}{p}\int_\Omega u f(u) = \frac{1}{p'}\int_{\partial\Omega}(x \cdot \nu)|\nabla u|^p \qquad (2.15)
$$

and Hölder inequality gives, when Ω is a ball of radius R, the inequality

$$
n\int_\Omega F(u) + \frac{p-n}{p}\int_\Omega u f(u) \geq \frac{1}{(n\omega_n R^{n-1})^{p'/p}}\frac{R}{p'}\left(\int_\Omega f(u) \right)^{p'}, \qquad (2.16)
$$

where $\omega_n = \mathcal{H}^n(B_1)$ is the volume of the unit ball in $\mathbb{R}^n$. Indeed, we only need to use (2.15), that for $\Omega = B_R$ we have $(x \cdot \nu) = R$ on $\partial\Omega$ and $\mathcal{H}^{n-1}(\partial\Omega) = n\omega_n R^{n-1}$, and the inequality

$$
\left(\int_{\partial\Omega}|\nabla u|^p \right)^{\frac{p-1}{p}}\left(\int_{\partial\Omega} 1^p \right)^{\frac{1}{p}} \geq \int_{\partial\Omega}|\nabla u|^{p-1} = \int_{\partial\Omega} -\nu \cdot \nabla u |\nabla u|^{p-1} = \int_\Omega -\Delta_p u = \int_\Omega f(u).
$$

To conclude, a straightforward computation (there is no magic behind this: note that all the inequalities obtained throughout this proof are equalities when u is radial) and recalling the value of c_n given in Lemma 5 (ii), we check that

$$
\begin{aligned}
\frac{n}{p' c_n^{p'}}\mathcal{H}^n(\Omega)^{\frac{p-n}{n(p-1)}} &= \frac{n\mathcal{H}^n(B_R)^{\frac{p'(n-1)}{n}}}{p'\mathcal{H}^{n-1}(\partial B_R)^{p'}}\mathcal{H}^n(B_R)^{\frac{p-n}{n(p-1)}} = \frac{n(\omega_n R^n)^{\frac{p'(n-1)}{n}+\frac{p-n}{n(p-1)}}}{p'(n\omega_n R^{n-1})^{p'}} \\
&= \frac{1}{(n\omega_n R^{n-1})^{p'/p}}\frac{R}{p'}.
\end{aligned}
$$

This enlightens that (2.14) and (2.16) are opposite inequalities. Therefore they must be, in fact, equalities.

It follows, recalling Remark 7 within this proof, that for a.e. $t \in (0, M)$,

(1) the level $\{u > t\}$ is a ball

and

(2) $|\nabla u|$ is constant on $\partial\{u > t\}$.

But then from Lemma 6 we conclude that u is a nonincreasing function of the radius and with $\frac{\partial u}{\partial r} < 0$ in $\{0 < u < \max_{\overline{\Omega}} u\}$. $\qquad\square$

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Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain

E-mail address: joaquim.serra@upc.edu