On low degree curves in $C^{(2)}$

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Al Yashin, que ha estat present tot el camí,
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CONTENTS

Introducció

En aquesta Tesi s'estudien corbes en el producte simètric d'una corba, $C^{(2)}$. Les caracteritzem, estudiem la seva immersió en $C^{(2)}$ i deduïm propietats de la corba C a partir de l'existència de corbes a $C^{(2)}$ d'un tipus concret. A més, donem una caracterització de $C^{(n)}$, per n general, a partir de l'existència de certes subvarietats amb unes propietats concretes.

Sigui C una corba llisa projectiva complexa de gènere g. Donat un enter $n \geq 1$, l'n-èssim producte simètric de C és el quocient del producte Cartesià per l'acció de l'n-èssim grup simètric, el qual actua en $C \times \cdots \times C$ permutant els factors, és a dir,

$$C^{(n)} = \frac{C \times \cdots \times C}{S_n}.$$

És ben sabut que $C^{(n)}$ és una varietat projectiva llisa de dimensió n que parametritza els divisors efectius de grau n en C, o el que és el mateix, les n-eples no ordenades de punts de C.

Els productes simètrics de corbes tenen un paper molt important tant pel que fa a la teoria de corbes algebraiques, com pel que fa a l'estudi de varietats algebraiques de dimensió superior. En el primer camp, els productes simètrics de corbes s'usen en la teoria de Brill-Noether per estudiar divisors especials en corbes. A més, l'i-èssim producte simètric determina C. Per $i = g - 1$ aquest fet és una conseqüència d'un famós teorema de Torelli, ja que $C^{(g-1)}$ determina la polarització de $J(C)$.

En el segon camp, els productes simètrics de corbes donen exemples particularment simples de varietats irregulars. A més, tenen dimensió d'Albanese màxima per $n \leq g$. De fet, $Alb(C^{(n)}) = J(C)$ i el morfisme d'Albanese envia birracionalment $C^{(n)}$ en l'n-èssim lloc de Brill-Noether $W_n(C)$ dins de $J(C)$ per $n \leq g$. Per $n = 1$ aquest morfisme no és més que la immersió estàndard de la corba en la seva Jacobiana. Per $n = g - 1$ la subvarietat $W_n(C)$ determina la polarització de $J(C)$ i per $n = g$ resulta que $C^{(g)}$ és birracional a $J(C)$. A més, per $n > 2g - 2$ la varietat $C^{(n)}$ és un fibrat projectiu sobre $J(C)$.

Els productes simètrics de corbes juguen un paper molt important en l'estudi i classificació de les varietats irregulars ja que són les varietats més simples amb dimensió d'Albanese màxima. Tenim, per exemple, els resultats a [CCM98] on es demostra que els productes simètrics de corbes de gènere 3 són justament les superfícies minimals birracionals a divisors theta de varietats abelianes principalment polaritzades de dimensió 3 no descomposables (es pot consultar [JLT13] per a resultats més generals).

A més, en l'estudi de superfícies algebraiques, es conjectura que els productes simètrics de corbes estan caracteritzats per la seva estructura de Hodge particularment simple.

Conjectura. *Les úniques superfícies minimals, S, amb irregularitat $q(S) > 2$ tals que $H^0(X, \Omega_X^2) \cong \bigwedge^2 H^0(X, \Omega_X^1)$, són els productes simètrics de corbes i les superfícies de Fano de rectes en un 3-fold cúbic llis.*

Aquesta conjectura va ser demostrada per $q = 3$ en [HP02] i de manera independent en [Pir02].

Els productes simètrics de corbes han estat bastament estudiats, tot i que encara hi ha moltes qüestions interesants obertes sobre la seva geometria. En aquesta tesi donem algunes eines que poden ser útils per progressar cap a la seva resposta. Per exemple, l'estructura del con nef, el qual està generat per les classes dels divisors nef, no és totalment coneguda. És ben sabut que per una corba C molt general aquest con està generat per la classe de les corbes coordenades i la classe de la diagonal. Un dels rajos que el delimiten està determinat per la classe dual a la diagonal via el producte d'intersecció, per tant, aquest costat és tancat. Per a valors de g concrets es tenen alguns resultats particulars, però per g general la seva descripció precisa és una pregunta oberta.

Destaquem dues caracteritzacions geomètriques de $C^{(2)}$ les quals juguen un paper important en el desenvolupament d'aquesta tesi. La primera apareix a [CCM98] on es demostra que $C^{(2)}$ és l'única superfície minimal amb irregularitat q que està recoberta per corbes de gènere q i auto-intersecció igual a 1.

La segona és el resultat principal a [MPP11b] on es troba una caracterització sorprenentment precisa de $C^{(2)}$ a partir de l'existència d'un únic divisor amb certes propietats. El teorema diu:

Teorema 1.1.12. *Sigui S una superfície llisa de tipus general amb irregularitat q que conté un divisor 1-connectat, D, tal que $p_a(D) = q$ i*

$D^2 > 0$. *Aleshores, el model minimal de S és o bé*

1. *el producte de dues corbes de gèneres g_1, $g_2 \geq 2$ ($g_1 + g_2 = q$) o bé*

2. *el producte simètric $C^{(2)}$, on C és una corba llisa de gènere q, i $D^2 = 1$.*

A més, si D és 2-connectat, només el segon cas és possible.

Una pregunta que sorgeix de manera natural d'aquest teorema és si una caracterització similar existeix per productes simètrics de corbes de d'ordre superior. Donarem una resposta afirmativa a aquesta pregunta en el Capítol 2.

Una corba $C \subset S$ amb $C^2 > 0$ satisfà $p_a(C) \geq q$, ja que si no, seria contreta pel morfisme d'Albanese i no podria tenir auto-intersecció positiva. Per un teorema de Xiao a [Xia87], una corba que es mou linealment satisfà $p_a(C) \geq 2q - 1$. Quan $p_a(C) = q$ tenim la superfície totalment determinada pel resultat anterior. Com que en un sentit ampli el fet que una corba tingui auto-intersecció positiva ens diu que la corba es mou, és natural preguntar-se si existeixen corbes tals que $q < p_a(C) < 2q - 1$ amb auto-intersecció positiva. A [MPP11a] els autors plantegen aquesta mateixa pregunta i anomenen a una corba en aquest rang de gènere, una corba de gènere baix. El propòsit principal d'aquesta tesi és estudiar aquesta qüestió quan la superfície S és el producte simètric d'una corba.

La superfície $C^{(2)}$ té molta geometria provinent de la corba, i per tant és un punt de partida natural on buscar un exemple tal. Al llarg de la tesi donarem una resposta parcial a la pregunta anterior.

En aquesta superfície tenim una família molt important de divisors efectius, les corbes coordenades, C_P, que són imatge de C pel morfisme que envia $Q \in C$ a $Q + P \in C^{(2)}$ amb P fixat. Aquesta família de corbes determina la superfície per la caracterització a [CCM98] ja mencionada. Definim el grau d'una corba a $C^{(2)}$ com el valor del producte d'intersecció amb una corba coordenada (el qual no depèn del punt P).

Els productes simètrics de corbes estan estretament relacionats amb varietats abelianes, com mostren alguns dels resultats ja mencionats. Recordem que el grau d'una corba C en una varietat polaritzada X (de dimensió n) es defineix com el producte d'intersecció, d, de la corba amb un divisor que representi la polarització. El possible gènere d'una corba que viu en una varietat abeliana i la seva relació amb els invariants de la corba han estat àmpliament estudiats. El

nostre treball segueix aquesta mateixa línia en el cas dels productes simètrics. En una varietat abeliana principalment polaritzada, quan el grau és n Ran va demostrar ([Ran80]) que la corba és llisa i que X és la seva Jacobiana, generalitzant un famós teorema de Matsusaka. A [Deb94] es demostra que $g(C) < \frac{(2d-1)^2}{2(n-1)}$, donant així una relació entre el grau i la geometria de la corba. Tot seguit mencionem alguns altres resultats interesants sobre corbes en varietats abelianes. A [Pir95] es demostra, per una varietat abeliana complexa A, genèrica de dimensió $n > 3$, que si existeix una aplicació no constant $f : C \to A$ aleshores $g(C) > \frac{n(n-1)}{2}$. A [NP94] els autors demostren que si P és una varietat de Prym genèrica de dimensió $n = 2$ ó $n \geq 4$, aleshores una corba D tal que existeix una aplicació no constant $f : D \to P$ satisfà $g(D) \geq 2n - 2$. A més, en una varietat de Prym genèrica de dimensió $n \geq 2$, per una corba D de gènere $g(D) \leq 2n - 2$, les úniques deformacions de f s'obtenen composant-la amb translacions, el que ells anomenen rigidesa de la corba. Finalment, conjecturen que en una Jacobiana genèrica $J(C)$ de dimensió $g \geq 4$, una corba a $J(C)$ és tal que o bé $g(D) \geq 2g - 2$ o $g(D) \leq g$. Recentment, a [Mar11] aquesta conjectura ha estat demostrada per Jacobianes molt genèriques, i l'autora conjectura que, de fet, no hi ha corbes de gènere $2g - 2$ en una Jacobiana molt genèrica.

Les corbes B a $C^{(2)}$ de grau 1 estan caracteritzades (a [ACGH85] i [Cha08]) per l'existència d'un morfisme de grau dos $f : C \to B$ de tal manera que la immersió de B en $C^{(2)}$ ve donada per les fibres d'aquest morfisme.

Per tal d'estudiar corbes de gènere baix a $C^{(2)}$, donem una caracterització de les corbes en aquesta superfície amb grau $d > 1$, a partir de l'existència d'un diagrama de corbes

$$
\begin{array}{ccc}
D & \xrightarrow{(2:1)} & B \\
{\scriptstyle (d:1)}\downarrow & & \\
C & &
\end{array}
$$

amb certes propietats.

Quan un està interessat en trobar corbes de grau 1 a $C^{(2)}$, l'únic que cal fer és trobar una involució en la corba C. Aleshores, el quocient de C per aquesta involució és una corba a $C^{(2)}$. Per tal de trobar corbes de grau superior, necessitem una tercera corba auxiliar per construir un diagrama. Amb aquesta idea, donarem un mètode per construir aquests diagrames utilitzant l'acció de grups no abelians en corbes.

Donats dos automorfismes d'una corba D que no commuten, prenent els quocients de D per aquests automorfismes obtenim un diagrama que defineix una corba a $C^{(2)}$.

Un cop tenim un diagrama tal, calcularem els invariants de les corbes a partir del nombre de punts fixos dels automorfismes involucrats, traduint així les nostres hipòtesis en les corbes, en hipòtesis sobre l'acció dels automorfismes. Pel Teorema d'Existència de Riemann serem capaços d'estudiar l'existència de diagrames satisfent el nostre conjunt de condicions.

RESUM EN CATALÀ

Presentem ara un breu repàs als resultats que recull aquesta tesi.

En el primer capítol introduïm alguns resultats bàsics sobre la geometria dels productes simètrics de corbes. Destaquem en aquest capítol el resultat (Teorema 1.1.12) de Mendes-Lopes, Pardini i Pirola que apareix a [MPP11b] on caracteritzen el segon producte simètric d'una corba a partir de l'existència d'un divisor efectiu amb certes propietats numèriques. Aquest resultat és la inspiració per al desenvolupament d'aquesta tesi.

En el segon capítol donem una caracterització de l'n-èssim producte simètric d'una corba a partir de l'existència d'una cadena de subvarietats amb certes propietats numèriques. El resultat és:

Teorema 2.0.14. *Sigui X una varietat projectiva llisa de dimensió n. Suposem que existeix una cadena d'inclusions*

$$X = V_n \supset V_{n-1} \supset \cdots \supset V_2 \supset V_1 = C$$

tals que

1. *V_i és una varietat llisa i irreductible amb $dim(V_i) = i$.*

2. *Per $i < n$, V_i és un divisor ample en V_{i+1}.*

3. *$V_i \cdot C = 1$ dins de V_{i+1}.*

4. *V_2 és una superfície de tipus general.*

5. *$q(X) = g(C)$.*

Aleshores, $X \cong C^{(n)}$. A més, $V_i \cong C^{(i)}$ i és un divisor coordenat dins de V_{i+1} per $i < n$.

En el tercer capítol introduïm alguns preliminars sobre corbes, principalment sobre morfismes finits. Una eina molt important en el nostre treball és l'estudi de l'acció d'un grup en una corba, i el morfisme quocient per l'acció d'aquest grup. En la segona part del capítol resumim alguns resultats ben coneguts sobre el grup d'automorfismes d'una corba de gènere almenys dos.

En el quart capítol presentem la teoria de grups Fuchsians i la seva relació amb les superfícies de Riemann, ja que el Teorema d'Existència de Riemann (Teorema 4.4.4) serà una eina crucial per al nostre treball. Considerem concretament l'existència de corbes d'un cert gènere amb l'acció d'un grup fixat, G, actuant d'una manera concreta. Expliquem amb detall que, per tal de tenir l'acció d'un grup en una corba de gènere g amb quocient $\mathbb{P}^1$, l'única cosa que necessitem és un conjunt d'elements del grup G, $\{\alpha_1, \ldots, \alpha_k\}$ tals que generin el grup, els seus ordres compleixin la fórmula de Riemann-Hurwitz per accions de grups i $\alpha_1 \cdots \alpha_k = 1$. Aquests elements α_i determinen el nombre de punts en el lloc discriminant, el nombre de punts fixats per l'acció de G i la monodromia de l'aplicació quocient.

Estem interessats no només en l'existència d'aquestes corbes, sinó també en el subespai que defineixen en l'espai de mòduli $\mathcal{M}_g$. Amb aquest propòsit definim l'espai de Hurwitz associat a una acció donada i repassem alguns resultats sobre la seva imatge en l'espai de mòduli de corbes de gènere g.

En el cinquè capítol demostrem una caracterització de les corbes a $C^{(2)}$ de grau donat a partir de l'existència de diagrames de corbes amb certes propietats. El resultat que resumeix aquesta caracterització és el següent

Teorema 5.1.5. *Sigui $\overline{B}$ una corba irreductible amb normalització B i tal que no existeix cap morfisme no trivial $B \to C$. Un morfisme de grau 1 de B en $C^{(2)}$ existeix, amb imatge $\tilde{B}$, i $\tilde{B} \cdot C_p = d$ si, i només si, existeix una corba llisa i irreductible D i un diagrama*

$$D \xrightarrow{(2:1)} B$$
$$\downarrow{\scriptstyle(d:1)}$$
$$C$$

que no redueix.

Diem que un diagrama de morfismes de corbes

$$
\begin{array}{ccc}
D & \xrightarrow{(e:1)} & B \\
{\scriptstyle (d:1)}\downarrow & & \\
C & &
\end{array}
$$

redueix si existeixen corbes F i H tals que existeix un diagrama

$$
\begin{array}{ccc}
D & \xrightarrow{(e:1)} & B \\
 & & \\
{\scriptstyle (d:1)}\Big(\begin{array}{cc} {\scriptstyle (k:1)} & {\scriptstyle (k:1)} \\ F \dashrightarrow H \\ {\scriptstyle (e:1)} \end{array} \\
 & & \\
C & &
\end{array}
$$

amb $k > 1$, de manera que el quadrat superior sigui un diagrama commutatiu i la fletxa vertical esquerra doni una factorització del morfisme de grau d original.

Quan $k = d$ direm que el diagrama completa, i tindrem un diagrama commutatiu

$$
\begin{array}{ccc}
D & \xrightarrow{(e:1)} & B \\
{\scriptstyle (d:1)}\downarrow & & \downarrow{\scriptstyle (d:1)} \\
C & \dashrightarrow & H. \\
 & {\scriptstyle (e:1)} &
\end{array}
$$

Per a d primer les dues definicions coincideixen. La qüestió de decidir si un diagrama de corbes completa no és nou en la literatura (veure per exemple [Acc06] i [BT09]). Nosaltres relacionem aquesta qüestió amb la geometria de productes simètrics $C^{(2)}$.

A més, com que estem interessats en l'auto-intersecció d'aquestes corbes, calculem l'auto-intersecció de $\tilde{B}$ dins de $C^{(2)}$ utilitzant únicament informació provinent del diagrama anterior. En l'última secció d'aquest capítol observem que per corbes de gènere baix tenim algunes cotes en el grau de les possibles corbes d'auto-intersecció positiva.

En el sisè capítol presentem un mètode per construir diagrames que no completen utilitzant l'acció d'un grup no abelià en una corba D. Tenint en compte els resultats previs, és també un mètode per trobar corbes a $C^{(2)}$. El resultat és

Proposició 6.0.3. *Sigui D una corba projectiva irreductible i llisa amb l'acció d'un grup finit G. Siguin $\alpha, \beta \in G$ amb $o(\alpha) = d \geq 2$ i $o(\beta) = e \geq 2$. Considerem el diagrama*

$$
\begin{array}{ccc}
D & \xrightarrow{\ (e:1)\ } & D/\langle \beta \rangle = B \\
\downarrow{\scriptstyle (d:1)} & & \\
D/\langle \alpha \rangle = C & &
\end{array}
\qquad .
$$

Aleshores,

1. *Si l'ordre de $\langle \alpha, \beta \rangle$ és igual a $e \cdot d$, el diagrama completa.*

2. *Si l'ordre de $\langle \alpha, \beta \rangle$ és estrictament més gran que $e \cdot d$, el diagrama no completa.*

Quan l'ordre de $\langle \alpha, \beta \rangle$ és estrictament més petit que $e \cdot d$, qualsevol cosa és possible.

Més endavant estudiarem corbes de grau dos. En aquest cas els dos morfismes que apareixen en el diagrama seran Galois, definits per involucions en D. Per tant, estem interessats en grups no abelians generats per dues involucions, és a dir, grups diedrals D_n d'ordre $2n \geq 6$.

Així doncs, dediquem en aquest capítol una secció a estudiar la inclusió de $\tilde{B}$ en $C^{(2)}$ usant la informació provinent del diagrama, en altres paraules, provinent de l'acció de D_n en la corba D. Calculem l'auto-intersecció de $\tilde{B}$ així com les seves singularitats. Trobem que la corba té $\frac{1}{4}\left(\nu((ij)^2) - \nu(ij)\right)$ singularitats nodals, on i i j són les involucions que generen D_n i $\nu(\cdot)$ denota el nombre de punts fixos de l'automorfisme. A més, l'auto-intersecció de la corba $\tilde{B} \subset C^{(2)}$ ve donada per la fórmula

$$
\tilde{B}^2 = g(D) - 1 - 2(2g(C) - 2) + \frac{1}{2}(\nu((ij)^2)).
$$

Estudiarem també alguns casos de corbes de grau 3 a $C^{(2)}$. Primer observem que per grau tres no tots els diagrames que poden aparèixer vindran de l'acció d'un grup en D, ja que no tots els morfismes de grau tres són Galois. Fins i tot en els casos Galois, com que hi ha una infinitat de grups d'ordre finit generats per una involució i un element d'ordre tres, no som capaços d'estudiar tots els casos. Considerarem els anomenats grups triangulars esfèrics S_4, A_4 i A_5 generats

per una involució (que denotem i) i un element d'ordre tres (que denotem α) que no commuten. Hem triat aquests grups perquè són els més simples amb aquesta propietat i són suficients per il·lustrar les nostres tècniques en aquesta situació. A més, els grups diedrals que apareixen de manera natural en el cas de grau dos són també grups triangulars esfèrics, i per tant, tot l'estudi s'engloba en una mateixa classe de grups. Com en el cas de grau dos, calculem les singularitats de $\tilde{B}$ i la seva auto-intersecció a partir dels punts fixos per l'acció del grup en la corba D. Demostrem que $\tilde{B}$ té $\frac{1}{2}\nu(i\alpha^2 i\alpha) + \frac{1}{2}(\nu((i\alpha)^2) - \nu(i\alpha))$ singularitats nodals i a més, que l'auto-intersecció de la corba $\tilde{B} \subset C^{(2)}$ ve donada per la fórmula

$$\tilde{B}^2 = g(D) - 1 - 3(2g(C) - 2) + \nu(i\alpha^2 i\alpha) + \nu((i\alpha)^2).$$

En el setè capítol estudiem corbes de grau baix i auto-intersecció positiva. Donem una resposta parcial a la pregunta sobre corbes de gènere baix i auto-intersecció positiva. Demostrem que

Teorema 7.1.3. *No hi ha corbes de grau dos $\tilde{B}$ a $C^{(2)}$ amb $\tilde{B}^2 > 0$ i $g(C) < p_a(\tilde{B}) < 2g(C) - 1$.*

I per a aquelles corbes construïdes per l'acció de S_4, A_4 i A_5 en una corba D demostrem que tenen també auto-intersecció no positiva. Un cop contestada aquesta pregunta, ens preguntem si es pot dir alguna cosa sobre corbes a $C^{(2)}$ amb auto-intersecció positiva i gènere més alt. En aquesta línia, hem trobat una classificació completa per a grau dos.

Teorema 7.1.6 (Classificació)**.** *Tots els parells de corbes llises (C, B) amb $B \xrightarrow{(1:1)} C^{(2)}$ i imatge $\tilde{B}$ tals que $p_a(\tilde{B}) \geq 2g(C) - 1$, $\tilde{B}^2 > 0$ i $\tilde{B} \cdot C_p = 2$ estan considerats en un dels casos següents:*

0. C és una corba de gènere 2 amb l'acció d'un automorfisme d'ordre 10, σ, tal que $\nu(\sigma) = 1$, $\nu(\sigma^2) = 3$, $\nu(\sigma^5) = 6$ i $\tilde{B}$ és la simetrització del graf de σ. Hi ha un nombre finit de classes d'isomorfia de corbes C amb un automorfisme tal.

1. Hi ha una corba D amb l'acció del grup diedral D_{10} tal que les corbes C i B són quocients de D per l'acció de certes involucions $i, j \in D_{10}$.

Hi ha tres famílies d'accions en D, donant lloc a tres famílies amb les següents propietats:

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	Dim. de D en mòduli	Dim. de C en mòduli	Altres propietats
5	2	3	1	1	1	D hiperel·líptica $\tilde{B}$ llisa
4	2	2	1	1	1	D hiperel·líptica $\tilde{B}$ té 1 node
6	2	3	1	2	2?	D biel·líptica $\tilde{B}$ llisa

A més, en les tres famílies la corba B és hiperel·líptica i tal que $p_a(\tilde{B}) = 2g(C) - 1$.

2. *Hi ha una corba D amb l'acció del grup diedral D_6 tal que les corbes C i B són quocients de D per l'acció de certes involucions $i, j \in D_6$. Hi ha deu famílies d'accions.*

Hi ha una família tal que

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	Dim. de D en mòduli	Dim. de C en mòduli	Altres propietats
5	2	3	2	2	2	$\tilde{B}$ té 1 node

En aquesta família les corbes D i B són hiperel·líptiques i tals que $p_a(\tilde{B}) = 2g(C)$.

A més, hi ha nou famílies amb les següents propietats

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	Dim. de D en mòduli	Dim. de C en mòduli	Altres propietats
5	2	3	2	1	1	$\tilde{B}$ llisa
7	3	4	1	2	2	$\tilde{B}$ té 1 node
6	3	3	1	2	2	$\tilde{B}$ té 2 nodes
4	2	2	2	1	1	$\tilde{B}$ té 1 node
6	2	3	2	2	2	$\tilde{B}$ llisa
9	3	5	1	3	3	$\tilde{B}$ llisa
8	3	4	1	3	3	$\tilde{B}$ té 1 node
5	2	2	2	2	2	$\tilde{B}$ té 1 node
7	2	3	2	3	2	$\tilde{B}$ llisa

En les nou famílies B i C són biel·líptiques i $p_a(\tilde{B}) = 2g(C) - 1$.

3. *Hi ha una corba D amb l'acció del grup diedral D_4 tal que les corbes C i B són quocients de D per l'acció de certes involucions*

$i, j \in D_4$. **Hi ha tres tipus de famílies amb les següents característiques:**

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	Dim. de D en mòduli	Altres propietats
$-1 + s + \frac{1}{2}k$	$\frac{s+k}{2}$	$\frac{2s+k}{4}$	4	$\frac{2s+k-4}{4}$	$\tilde{B}$ té $\frac{k}{4}$ nodes $s + k \geq 8$
$-2 + s + \frac{1}{2}k$	$\frac{s+k-2}{2}$	$\frac{2s+k-4}{4}$	4	$\frac{2s+k-4}{4}$	$\tilde{B}$ té $\frac{k}{4}$ nodes $s + k \geq 10$
$-3 + s + \frac{1}{2}k$	$\frac{s+k-4}{2}$	$\frac{2s+k-8}{4}$	4	$\frac{2s+k-4}{4}$	$\tilde{B}$ té $\frac{k}{4}$ nodes $s + k \geq 12$

A més, en les tres famílies C és hiperel·líptica, amb qualsevol gènere possible, i $p_a(\tilde{B}) = 2g(C)$.

El fet interessant és justament que del teorema podem deduir que l'existència d'una corba de grau dos a $C^{(2)}$ amb auto-intersecció positiva implica, per exemple, que C ha de ser biel·líptica o hiperel·líptica, i per tant, obtenim informació sobre la geometria de la corba C. A més, observem que gràcies al teorema obtenim que les possibles corbes de grau dos amb auto-intersecció positiva descriuen un conjunt molt restringit, ja que, per exemple, el seu gènere aritmètic és com a màxim $2g(C)$ i la seva auto-intersecció és igual a 1, 2 ó 4.

En la part final del capítol estudiem corbes de grau tres donades per l'acció d'un grup triangular esfèric en una corba D, donant una llista de les possibilitats on especifiquem el gènere de les corbes involucrades, les singularitats d'aquestes corbes i el valor de l'auto-intersecció de $\tilde{B}$ a $C^{(2)}$. Els resultats estan resumits en el Teorema 7.2.3 i les taules que s'hi mencionen:

Teorema 7.2.3. *Siguin (C, B) un parell de corbes llises i $B \xrightarrow{(1:1)} C^{(2)}$ amb imatge $\tilde{B}$ de grau tres, tals que $p_a(\tilde{B}) \geq 2g(C) - 1$ i $\tilde{B}^2 > 0$. Suposem que $\tilde{B}$ està definida per l'acció d'un grup esfèric triangular, $G = A_4, S_4, A_5$, en una corba D tal que B és el quocient de D per una involució i C és el quocient per un element d'ordre tres. Aleshores, el parell (C, B) està considerat en una de les taules que apareixen en la Secció 7.2.2.*

En particular, aquells donats per A_4 estan descrits a les Taules 7.1, 7.2, 7.3, 7.4, 7.5, 7.6, 7.7, 7.8, 7.9 i 7.10. Aquells donats per l'acció de S_4 estan descrits a les Taules 7.11 i 7.12. Finalment, aquells donats per A_5 estan descrits a la Taula 7.13.

En grau dos tots els diagrames estan descrits per l'acció d'un grup, però en grau tres això ja no es compleix. Hi ha morfismes de grau tres no Galois que poden formar part d'un diagrama que no completi. A més, fins i tot en els casos Galois, hi ha una infinitat de grups d'ordre finit generats per un element d'ordre tres i un d'ordre dos, per tant, un estudi complet no és possible amb les nostres tècniques, i en qualsevol cas, seria d'una gran complexitat.

Finalment, afegim un apèndix presentant una sèrie de conceptes i resultats de teoria de grups, a més de donar detalls sobre els grups que intervenen en el desenvolupament de la tesi. Les relacions concretes entre els elements en aquests grups són molt útils per al nostre estudi i els descrivim en l'apèndix per a possibles referències.

Els mètodes usats en aquesta tèsi avarquen diverses àrees incloses teoria de corbes, teoremes d'anul·lació genèrica i la topologia de recobriments ramificats. Utilitzem també propietats bàsiques d'espais de mòduli per extreure conclusions sobre les famílies de corbes amb accions tals i els seus quocients. A més, hem utilitzat tècniques de teoria de grups per estudiar amb detall les seves accions en corbes.

INTRODUCTION

> Algebraic geometry studies the delicate balance between
> the geometrically plausible and the algebraically possible.
>
> George R. Kempf.

This thesis studies curves in the symmetric square of a curve, $C^{(2)}$. We characterize them, study their immersion in $C^{(2)}$ and deduce properties of the curve C from the existence of curves in $C^{(2)}$ of specific type. In addition, we give a characterization of $C^{(n)}$, for n general, by the existence of certain subvarieties with particular properties.

Let C be a smooth complex projective curve of genus g. For an integer $n \geq 1$, the nth symmetric product of C is the quotient of the Cartesian product by the action of the nth symmetric group, which acts on $C \times \cdots \times C$ permuting the factors, that is

$$C^{(n)} = \frac{C \times \cdots \times C}{S_n}.$$

It is well known that $C^{(n)}$ is a smooth and projective variety of dimension n. It parametrizes the effective degree n divisors on C, or equivalently, the unordered n-tuples of points of C.

Symmetric products of curves play a very important role both in the theory of algebraic curves and in the theory of higher dimensional algebraic varieties. In the first area, they are exploited by Brill-Noether theory to study special divisors on curves. Moreover, the ith symmetric product determines C. For $i = g - 1$ this fact is a consequence of a famous theorem of Torelli, because $C^{(g-1)}$ determines the polarization in $J(C)$.

In the second area, they give particularly simple examples of irregular varieties. Moreover, they have maximal Albanese dimension for $n \leq g$. In fact, $Alb(C^{(n)}) = J(C)$ and the Albanese morphism maps $C^{(n)}$ birationally on the nth Brill-Noether locus $W_n(C)$ inside $J(C)$ for $n \leq g$. For $n = 1$ it is the standard embedding of a curve in its Jacobian. For $n = g-1$ the subvariety $W_n(C)$ gives the polarization of $J(C)$ and for $n = g$ we find that $C^{(g)}$ is birational to $J(C)$. Furthermore, for $n > 2g - 2$ the variety $C^{(n)}$ is a projective fiber bundle over $J(C)$.

Symmetric products play an important role in the study and classification of irregular varieties since they are the most simple maximal Albanese dimension varieties. We have, for instance, the results

in [CCM98] where it is proven that symmetric products of curves of genus 3 are precisely the minimal surfaces which are birational to theta divisors of indecomposable principally polarized abelian 3-folds (see also [JLT13] for more general results).

Furthermore, in the study of algebraic surfaces, it is a conjecture that symmetric products of curves are characterized by its particularly simple Hodge structure.

Conjecture. *The only minimal surfaces S of irregularity $q(S) > 2$ with $H^0(X, \Omega_X^2) \cong \bigwedge^2 H^0(X, \Omega_X^1)$, are the symmetric squares of curves and the Fano surfaces of lines in a smooth cubic 3-fold.*

It was proven for $q = 3$ in [HP02] and independently in [Pir02].

Symmetric products of curves have been widely studied, although there are still many very interesting open questions about their geometry. In this thesis we give some tools that might be useful to make progress towards their answering. For instance, the structure of the nef cone, which is spanned by the classes of nef divisors, is not completely known. It is well known that for a very general curve C this cone is spanned by the class of a coordinate curve and the class of the diagonal. A extremal ray of this cone is given by the dual class of the diagonal via the intersection pairing, thus this side is closed. For particular values of g there are some results on the other ray, but for general genus its concrete description is an open question.

We remark two geometric characterizations of $C^{(2)}$ that play an important role in the development of the present thesis. The first one appears in [CCM98] where it is proven that $C^{(2)}$ is the only minimal algebraic surface with irregularity q that is covered by curves of genus q and self-intersection equal to 1.

The second one is the main result in [MPP11b] where they prove a surprisingly precise characterization of $C^{(2)}$ by the existence of a single divisor with certain properties. The theorem reads:

Theorem 1.1.12. *Let S be a smooth surface of general type with irregularity q containing a 1-connected divisor D such that $p_a(D) = q$ and $D^2 > 0$. Then the minimal model of S is either*

1. *the product of two curves of genus g_1, $g_2 \geq 2$ ($g_1 + g_2 = q$) or*

2. *the symmetric product $C^{(2)}$, where C is a smooth curve of genus q, and $D^2 = 1$.*

Furthermore, if D is 2-connected, only the second case occurs.

A natural question that arises from this theorem is whether a similar characterization exists for higher symmetric products of curves. We will give a positive answer to this question in Chapter 2.

A curve $C \subset S$ with $C^2 > 0$ satisfies $p_a(C) \geq q$, since otherwise it would be contracted by the Albanese morphism, and could not have positive self-intersection. By a theorem of Xiao in [Xia87], a curve that moves linearly satisfies $p_a(C) \geq 2q - 1$. When $p_a(C) = q$ we have the surface completely determined by the previous result. Since in a naive sense the positive self-intersection of a curve in a surface tells us that this curve moves, it is natural to ask if there are curves with $q < p_a(C) < 2q - 1$ and positive self-intersection. In [MPP11a] the authors set out this question, and call a curve in this range of genus, a curve of low genus. The main purpose in this thesis is to study this question when the surface S is the symmetric square of a curve.

The surface $C^{(2)}$ has a lot of geometry coming from the curve, and therefore it is a natural starting point to search such an example. We are going to give a partial answer to the previous question.

In this surface we have a very important family of effective divisors, the coordinate curves, that are the image of C by the morphism that sends a point $Q \in C$ to $Q + P \in C^{(2)}$ with P fixed. This family of curves determines the surface by the characterization in [CCM98] already mentioned. We define the degree of a curve in $C^{(2)}$ as the intersection product with a coordinate curve (which does not depend on the point P).

Symmetric products of curves are closely related with abelian varieties by some of the results already mentioned. We remind the definition of the degree of a curve C in a polarized variety X (of dimension n) which is defined as the intersection product, d, of the curve with a divisor representing the polarization. The possible genus of a curve lying in an abelian variety and its relation with the invariants of the curve has been widely studied. Our work follows this same line in the case of symmetric products. In a principally polarized abelian variety, when the degree is n it was proven by Ran (see [Ran80]) that the curve is smooth and X is its Jacobian variety, generalizing a famous theorem of Matsusaka. In [Deb94] it is proven that $g(C) < \frac{(2d-1)^2}{2(n-1)}$, giving a relation between the degree and the geometry of the curve. Next, we mention some other interesting results on curves on abelian varieties. In [Pir95] it is proven that for a generic complex abelian variety A of dimension $n > 3$, if there exists a non-constant map $f : C \to A$ then $g(C) > \frac{n(n-1)}{2}$. In [NP94] the authors prove that if P is a generic Prym

variety of dimension $n = 2$ or $n \geq 4$, then a curve D such that there exists a non constant map $f : D \to P$ satisfies $g(D) \geq 2n-2$. Moreover, in a generic Prym variety of dimension $n \geq 2$, for a curve D of genus $g(D) \leq 2n - 2$, the only deformations of f are obtained by composing it with translations, which they call rigidity of the curve. Finally they conjecture that in a generic Jacobian variety $J(C)$ of dimension $g \geq 4$, a curve in $J(C)$ is such that either $g(D) \geq 2g - 2$ or $g(D) \leq g$. Recently, in [Mar11] this conjecture has been proven for a very generic Jacobian variety, and the author conjectures that, in fact, there are no curves of genus $2g - 2$ in a very generic Jacobian.

Curves B in $C^{(2)}$ with degree one are characterized (see [ACGH85] and [Cha08]) by the existence of a degree two morphism $f : C \to B$ in such a way that the immersion of B in $C^{(2)}$ is given by the fibers of this morphism.

In order to study low genus curves in $C^{(2)}$, we give a characterization of curves in the symmetric square with degree $d > 1$ by the existence of a diagram of curves

$$
\begin{array}{ccc}
D & \xrightarrow{(2:1)} & B \\
{\scriptstyle(d:1)}\big\downarrow & & \\
C & &
\end{array}
$$

with certain properties.

If one is interested in finding curves of degree one in $C^{(2)}$, the only thing to do is to find an involution in the curve C. Then, the quotient of C by this involution is a curve in $C^{(2)}$. In order to find higher degree curves, we need an auxiliary third curve to construct a diagram. With this idea, we will give a method to construct such diagrams using the action of non abelian groups on curves. Given two automorphisms of a curve D that do not commute, taking the quotients of D by these automorphisms we obtain a diagram which defines a curve in $C^{(2)}$.

Once we have such a diagram, we will compute the invariants of the curves from the number of fixed points of the automorphisms involved, hence translating our hypothesis on the curves on hypothesis on the action of the automorphisms. By the Riemann's Existence Theorem we will be able to study the existence of such diagrams satisfying our set of conditions.

SUMMARY

We present now a brief overview of the results proved in this Thesis.

In Chapter one we present some basic results on the geometry of symmetric products of curves. We emphasize in this chapter the result (Theorem 1.1.12) by Mendes-Lopes, Pardini and Pirola that appeared in [MPP11b] where they characterize the symmetric square of a curve by the existence of an effective divisor with certain numerical properties. This result is the inspiration for the development of this thesis.

In Chapter two we give a characterization of the nth symmetric product of a curve by the existence of a chain of subvarieties with certain numerical properties. The result is:

Theorem 2.0.14. *Let X be a smooth projective variety of dimension n. Assume there exists a chain of inclusions*

$$X = V_n \supset V_{n-1} \supset \cdots \supset V_2 \supset V_1 = C$$

such that

1. *V_i is a smooth irreducible variety with $dim(V_i) = i$.*

2. *For $i < n$, V_i is an ample divisor in V_{i+1}.*

3. *$V_i \cdot C = 1$ inside V_{i+1}.*

4. *V_2 is a surface of general type.*

5. *$q(X) = g(C)$.*

Then $X \cong C^{(n)}$. Moreover, $V_i \cong C^{(i)}$ and it is a coordinate divisor inside V_{i+1} for $i < n$.

In Chapter three we introduce some preliminars on curves, mainly on finite morphisms. A very important tool for our work is the study of the action of a group on a curve, and the quotient morphism by the action of this group. In the second part of this chapter we summarize some well known results on the automorphism group of a curve of genus at least two.

In Chapter four we introduce the theory of Fuchsian groups and its relation to Riemann surfaces, since the Riemann's Existence Theorem (Theorem 4.4.4) will be a crucial tool for our work. We consider specifically the existence of curves of a certain genus with the action of a

prescribed group G in a concrete way. We explain with detail that, in order to have the action of a group on a curve of genus g with quotient curve $\mathbb{P}^1$, the only thing we need is a set of elements in the group G, $\{\alpha_1, \ldots, \alpha_k\}$ such that they generate the group, their orders fulfill the Riemann-Hurwitz formula for group actions and $\alpha_1 \cdots \alpha_k = 1$. These elements α_i determine the number of branch points, the number of points fixed by the action of G and the monodromy of the quotient map.

We are interested not only in the existence of these curves, but also on the subspace they define in the moduli space $\mathcal{M}_g$. To this aim we define the Hurwitz space associated to a given action and give some results on its image in the moduli space of curves of genus g.

In Chapter five we prove a characterization of the curves in $C^{(2)}$ with given degree by the existence of a diagram of curves with certain properties. The result that summarizes this characterization is the following

Theorem 5.1.5. *Let $\overline{B}$ be an irreducible curve with normalization B and such that there are no non-trivial morphisms $B \to C$. A morphism of degree one from B to $C^{(2)}$ exists, with image $\tilde{B}$, and $\tilde{B} \cdot C_p = d$ if, and only if, there exists a smooth irreducible curve D and a diagram*

$$
\begin{array}{ccc}
D & \xrightarrow{(2:1)} & B \\
{\scriptstyle (d:1)}\downarrow & & \\
C & &
\end{array}
$$

which does not reduce.

We say that a diagram of morphisms of curves

$$
\begin{array}{ccc}
D & \xrightarrow{(e:1)} & B \\
{\scriptstyle (d:1)}\downarrow & & \\
C & &
\end{array}
$$

reduces if there exist curves F and H such that there exists a diagram

$$
\begin{array}{ccc}
D & \xrightarrow{(e:1)} & B \\
{\scriptstyle (d:1)}\Big| \;\; {\scriptstyle (k:1)}\downarrow & & \downarrow{\scriptstyle (k:1)} \\
F & \dashrightarrow{(e:1)} & H \\
& \searrow\downarrow & \\
& C &
\end{array}
$$

with $k > 1$, the upper square being a commutative diagram and the left vertical arrows giving a factorization of the original degree d morphism.

When $k = d$ we will say that the diagram completes, and we will have a commutative diagram

$$
\begin{array}{ccc}
D & \xrightarrow{(e:1)} & B \\
{\scriptstyle(d:1)}\downarrow & & \downarrow{\scriptstyle(d:1)} \\
C & \dashrightarrow{(e:1)} & H.
\end{array}
$$

For d a prime number both definitions coincide. The question of deciding whether a diagram of morphisms of curves completes is not new on the literature (see for instance [Acc06] and [BT09]). We relate this question with the geometry of symmetric products $C^{(2)}$.

Moreover, since we are interested in the self-intersection of these curves, we compute the self-intersection of $\tilde{B}$ inside $C^{(2)}$ using only information coming from the previous diagram. In the last section of this chapter we observe that for curves of low genus we have some bounds on the degree of possible curves with positive self-intersection.

In Chapter six we introduce a method to construct diagrams which do not complete using the action of a non abelian group on a curve D. Regarding the previous results, it is also a method to find curves on $C^{(2)}$. The result is

Proposition 6.0.3. *Let D be a projective smooth irreducible curve with the action of a finite group G. Let $\alpha, \beta \in G$ with $o(\alpha) = d \geq 2$ and $o(\beta) = e \geq 2$. Consider the diagram*

$$
\begin{array}{ccc}
D & \xrightarrow{(e:1)} & D/\langle\beta\rangle = B \\
{\scriptstyle(d:1)}\downarrow & & \\
D/\langle\alpha\rangle = C & &
\end{array}
\qquad .
$$

Then,

1. *If the order of $\langle\alpha, \beta\rangle$ equals $e \cdot d$ then the diagram completes.*

2. *If the order of $\langle\alpha, \beta\rangle$ is strictly greater than $e \cdot d$ then the diagram does not complete.*

When the order of $\langle \alpha, \beta \rangle$ is strictly less than $e \cdot d$ then anything could happen.

We will study curves of degree two, hence, both morphisms in the diagram above will be Galois, defined by involutions in D. Thus, we are interested in non abelian groups generated by two involutions, that is, dihedral groups D_n with order $2n \geq 6$.

We devote a section studying the inclusion of $\tilde{B}$ in $C^{(2)}$ using the information coming from the diagram, in other words, given by the action of D_n on the curve D. We compute the self-intersection of $\tilde{B}$ as well as its singularities. We find that the curve has $\frac{1}{4}\left(\nu((ij)^2) - \nu(ij)\right)$ nodal singularities, where i and j are the involutions generating D_n and $\nu(\cdot)$ denotes the number of fixed points of the automorphism. Moreover, the self-intersection of the curve $\tilde{B} \subset C^{(2)}$ is given by the formula

$$\tilde{B}^2 = g(D) - 1 - 2(2g(C) - 2) + \frac{1}{2}(\nu((ij)^2)).$$

We will study also some curves of degree three in $C^{(2)}$. First we observe that in this case not all diagrams that could appear would come from the action of a group on D, since not all degree three morphisms are Galois. Even in the Galois cases, since there is an infinity of groups of finite order generated by an involution and an order three element, we are not able to study all such cases. We take the so called spherical triangle groups S_4, A_4 and A_5 generated by an involution (that we denote i) and an order three element (that we denote α) that do not commute. We have chosen these three groups because they are the easiest in this context and are enough to illustrate our techniques in the degree three situation. Moreover, the dihedral groups that appear naturally in the degree two case are also spherical triangle groups, and hence, we complete in this way the study of a whole class of groups. As in the degree two case, we can compute the singularities of $\tilde{B}$ and its self-intersection from the fixed points of the action of the group on the curve D. We prove that $\tilde{B}$ has $\frac{1}{2}\nu(i\alpha^2 i\alpha) + \frac{1}{2}(\nu((i\alpha)^2) - \nu(i\alpha))$ nodal singularities and furthermore, that the self-intersection of the curve $\tilde{B} \subset C^{(2)}$ is given by the formula

$$\tilde{B}^2 = g(D) - 1 - 3(2g(C) - 2) + \nu(i\alpha^2 i\alpha) + \nu((i\alpha)^2).$$

In Chapter seven we study curves with low degree and positive self-intersection. We give a partial answer to the question concerning low genus curves with positive self-intersection. We prove that

Theorem 7.1.3. *There are no degree two curves $\tilde{B}$ lying in $C^{(2)}$ with $g(C) < p_a(\tilde{B}) < 2g(C) - 1$ and $\tilde{B}^2 > 0$.*

And for those curves constructed by the action of S_4, A_4 and A_5 on a curve D we prove that they also have non-positive self-intersection. Once answered this question, we wonder if something can be said about curves in $C^{(2)}$ with positive self-intersection and higher genus. In this sense, we have found a complete classification for degree two.

Theorem 7.1.6 (Classification). *All pairs of smooth curves (C, B) with $B \xrightarrow{(1:1)} C^{(2)}$ and image $\tilde{B}$ such that $p_a(\tilde{B}) \geq 2g(C) - 1$, $\tilde{B}^2 > 0$ and $\tilde{B} \cdot C_p = 2$ fall in one of the following cases:*

0. *C is a curve of genus 2 with an action of an automorphism of order 10, σ, such that $\nu(\sigma) = 1$, $\nu(\sigma^2) = 3$, $\nu(\sigma^5) = 6$ and $\tilde{B}$ is the symmetrization of the graph of σ. There is a finite number of isomorphism classes of curves C with such an automorphism.*

1. *There is a curve D with an action of the dihedral group D_{10} such that the curves C and B are the quotients of D by certain involutions $i, j \in D_{10}$.*

   There are three families of actions on D, giving three families with the following properties:

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	*Moduli dim. of D*	*Moduli dim. of C*	*Other properties*
5	2	3	1	1	1	*D hyperelliptic $\tilde{B}$ smooth*
4	2	2	1	1	1	*D hyperelliptic $\tilde{B}$ has 1 node*
6	2	3	1	2	2?	*D bielliptic $\tilde{B}$ smooth*

   Furthermore, in all three families the curve B is hyperelliptic and $p_a(\tilde{B}) = 2g(C) - 1$.

2. *There is a curve D with an action of the dihedral group D_6 such that the curves C and B are the quotients of D by certain involutions $i, j \in D_6$. There are ten families of actions.*

   There is one family such that

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	*Moduli dim. of D*	*Moduli dim. of C*	*Other properties*
5	2	3	2	2	2	*$\tilde{B}$ has 1 node*

Furthermore, in this family the curves D and B are hyperelliptic and $p_a(\tilde{B}) = 2g(C)$.

Moreover, there are nine families with the following properties

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	Moduli dim. of D	Moduli dim. of C	Other properties
5	2	3	2	1	1	$\tilde{B}$ smooth
7	3	4	1	2	2	$\tilde{B}$ has 1 node
6	3	3	1	2	2	$\tilde{B}$ has 2 nodes
4	2	2	2	1	1	$\tilde{B}$ has 1 node
6	2	3	2	2	2	$\tilde{B}$ smooth
9	3	5	1	3	3	$\tilde{B}$ smooth
8	3	4	1	3	3	$\tilde{B}$ has 1 node
5	2	2	2	2	2	$\tilde{B}$ has 1 node
7	2	3	2	3	2	$\tilde{B}$ smooth

Furthermore, in all nine families B and C are bielliptic and $p_a(\tilde{B}) = 2g(C) - 1$.

3. *There is a curve D with an action of the dihedral group D_4 such that the curves C and B are the quotients of D by certain involutions $i, j \in D_4$. There are three types of families with the following characteristics:*

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	Moduli dim. of D	Other properties
$-1+s+\frac{1}{2}k$	$\frac{s+k}{2}$	$\frac{2s+k}{4}$	4	$\frac{2s+k-4}{4}$	$\tilde{B}$ has $\frac{k}{4}$ nodes $s+k \geq 8$
$-2+s+\frac{1}{2}k$	$\frac{s+k-2}{2}$	$\frac{2s+k-4}{4}$	4	$\frac{2s+k-4}{4}$	$\tilde{B}$ has $\frac{k}{4}$ nodes $s+k \geq 10$
$-3+s+\frac{1}{2}k$	$\frac{s+k-4}{2}$	$\frac{2s+k-8}{4}$	4	$\frac{2s+k-4}{4}$	$\tilde{B}$ has $\frac{k}{4}$ nodes $s+k \geq 12$

Furthermore, in all three families C is hyperelliptic, with any possible genus, and $p_a(\tilde{B}) = 2g(C)$.

The interesting point is precisely that from the theorem we can deduce that the existence of a degree two curve in $C^{(2)}$ with positive self-intersection implies, for example, that C must be bielliptic or hyperelliptic, and therefore, we obtain information about the geometry

of the curve C. Moreover, we observe that by this theorem we obtain that the possible curves of degree two with positive self-intersection describe a very rigid set, since, for instance, its arithmetic genus is at most $2g(C)$ and its self-intersection is equal to 1, 2 or 4.

In the final part of the chapter we study curves of degree three given by the action of a spherical triangle group on a curve D, giving a list of possibilities where we specify the genus of all the curves involved, the singularities of these curves and the value of the self-intersection of $\tilde{B}$ in $C^{(2)}$. The results are summarized in Theorem 7.2.3 and the tables mentioned in it:

Theorem 7.2.3. *Let (C, B) be a pair of smooth curves with $B \xrightarrow{(1:1)} C^{(2)}$ and image $\tilde{B}$ of degree three such that $p_a(\tilde{B}) \geq 2g(C) - 1$ and $\tilde{B}^2 > 0$. Assume that it is defined by the action of a spherical triangle group, $G = A_4, S_4, A_5$, on a curve D such that B is the quotient of D by an involution and C is the quotient by an order three element. Then, the pair (C, B) is considered in one of the tables in Section 7.2.2.*

In particular, those given by A_4 are described in Tables 7.1, 7.2, 7.3, 7.4, 7.5, 7.6, 7.7, 7.8, 7.9 and 7.10. Those given by the action of S_4 are described in Tables 7.11 and 7.12. Finally, those given by A_5 are described in Table 7.13.

In the degree two case, all diagrams are described by a group action, but in the degree three case it is no longer true. There are non Galois morphisms of degree three which can be part of a diagram which does not complete. Moreover, even in the Galois cases, there is an infinity of groups with finite order generated by an element of order three and one of order two, hence a complete study is not possible with our techniques and in any case it would have a great complexity.

We provide finally an appendix presenting a series of concepts and results of group theory and giving details on the groups involved in the exposition of the thesis. The specific relations between the elements on these groups are very useful for our study, and we describe them in the appendix for possible reference.

The methods used in this thesis range over a variety of areas including theory of curves, generic vanishing theorems and the topology of ramified coverings. We also use basic properties of the moduli spaces to get conclusions on the families of curves with such actions and the family of their quotients. In addition, we have used techniques of group theory to study with detail their action on curves.

NOTATION

We work over the complex numbers. All varieties considered are projective and irreducible. We use the equivalence between compact Riemann surfaces and complex algebraic curves with no further mention.

Given a sheaf $\mathcal{F}$ in a variety X we put $h^i(X, \mathcal{F}) = dim H^i(X, \mathcal{F})$.

For a smooth variety X we denote by $q(X) = h^0(X, \Omega^1_X)$ its irregularity, its geometric genus by $p_g(X) = h^0(X, \omega_X)$ and its Euler Characteristic by $\chi(X) = \chi(\mathcal{O}_X) = \sum_i (-1)^i h^i(X, \mathcal{O}_X)$. Given a curve C, possibly singular, denote by $\hat{C}$ its normalization. Then, we denote by $g(C) = g(\hat{C}) = h^0(\hat{C}, \omega_{\hat{C}})$ the geometric genus (or topological genus) and by $p_a(C) = h^1(C, \mathcal{O}_C)$ the arithmetic genus. We will call node an ordinary singularity of order two, and cusp a non ordinary singularity of order two.

We recall that two Cartier divisors D_1, $D_2 \in Div(X)$ are said to be numerically equivalent if $D_1 \cdot C = D_2 \cdot C$ for every irreducible curve $C \subset X$. We denote the numerical equivalence relation by $\equiv_{num}$.

Let C be a smooth curve and $C^{(n)}$ its nth symmetric product. We denote by $\pi_C : C \times \overset{n}{\cdots} \times C \to C^{(n)}$ the natural map, and $C_P \subset C^{(n)}$ will denote the coordinate divisor parametrizing all degree n divisors in C through $P \in C$. We put Δ_C for the main diagonal in $C^{(n)}$. Moreover, $\Delta_{C \times C}$ denotes the diagonal of the Cartesian product $C \times C$.

Given a curve $B \subset C^{(2)}$ we say that B has degree d when $d = B \cdot C_P$.

The linear equivalence of two divisors in a variety X is denoted by $\equiv_{lin}$. The birational equivalence of two varieties is denoted by $\approx$.

We say that a curve is very general when it represents a very general point in the moduli space $\mathcal{M}_g$, that is, a point in the complement of a countable union of proper closed subvarieties. Furthermore, we say that a property is satisfied by a general curve when it is satisfied for all curves that correspond to the points belonging to a Zariski open non-empty subset of $\mathcal{M}_g$.

An irrational pencil on a surface S is a morphism $f : S \to B$ with connected fibers, where B is a smooth curve with genus > 0. The genus of the pencil is $g(B)$.

For $\sigma \in \mathrm{Aut}(C)$, we denote by $\nu(\sigma)$ the number of points fixed by σ. We put Γ_σ for the curve in $C \times C$ given by the graph of σ, that is, $\Gamma_\sigma = \{(x, \sigma(x)), \ x \in C\}$.

For G a finite group, we denote by $|G|$ the order of the group and by $o(g)$ the order of an element $g \in G$. In general, for a finite set X we denote by $|X|$ the cardinality of the set.

Chapter One

SYMMETRIC PRODUCTS OF CURVES

In this chapter we present several definitions and known results on symmetric products of curves. The geometry of symmetric products of curves is closely related to that of the curve itself and has been studied from different points of view, some more algebraic and some more topological.

1.1 ON THE GEOMETRY OF $C^{(n)}$

Let C be a smooth projective curve of genus g. For an integer $n \geq 1$, we consider the n-fold Cartesian product of C, the variety

$$C \times \cdots \times C.$$

Then, the n**th symmetric product** of C is the quotient of the Cartesian product by the action of the nth symmetric group that acts on $C \times \cdots \times C$ permuting the factors, that is

$$C^{(n)} = \frac{C \times \cdots \times C}{S_n}.$$

The morphism $\pi_C : C \times \cdots \times C \rightarrow C^{(n)}$ is the induced projection morphism. Hence, $C^{(n)}$ is the projective variety of dimension n parametrizing the effective degree n divisors on C, or equivalently, the unordered n-tuples of points of C.

The nth symmetric product of a curve is a smooth variety (see [ACGH85]).

Associated to a curve there is also its **Jacobian variety**, that is defined as

$$J(C) = \frac{H^0(C, K_C)^*}{H_1(C, \mathbb{Z})}.$$

It is a g-dimensional complex torus. Choosing a basis $\omega_1, \dots, \omega_g$ of $H^0(C, K_C)$ and fixing a point $P \in C$, we define the **Abel-Jacobi map**

$$
\begin{array}{ccc}
C^{(n)} & \xrightarrow{\;u\;} & J(C) \\[4pt]
P_1 + \cdots + P_n & \longrightarrow & \left(\sum_{i=1}^{n} \int_P^{P_i} \omega_1, \dots, \sum_{i=1}^{n} \int_P^{P_i} \omega_g \right).
\end{array}
$$

It is the Albanese morphism of $C^{(n)}$. It is well known that

Theorem 1.1.1 (Abel's Theorem). *Let $D, D' \in C^{(n)}$ be two effective divisors of degree n on a smooth curve C. Then, D is linearly equivalent to D' if and only if $u(D) = u(D')$.*

Let $Pic^n(C)$ denote the **Picard Variety** of C parametrizing the isomorphism classes of line bundles of degree n on C. Let $P \in C$, the **Picard map** is defined as

$$
\begin{array}{ccc}
C^{(n)} & \xrightarrow{\;v\;} & Pic^0(C) \\[4pt]
D & \longrightarrow & \mathcal{O}_C(D - nP).
\end{array}
$$

By the isomorphism $Pic^0(C) \cong J(C)$, for any fixed point $P \in C$, the following diagram commutes

$$
\begin{array}{ccc}
& C^{(n)} & \\
{\scriptstyle v}\swarrow & & \searrow{\scriptstyle u} \\
Pic^0(C) & \xrightarrow{\;\cong\;} & J(C).
\end{array}
$$

Furthermore,

Theorem 1.1.2 (Jacobi's inversion theorem). *([ACGH85]) The map $u : C^{(g)} \to J(C)$ is surjective.*

Given a point $P \in C$, we define the divisor C_P of $C^{(n)}$ as

$$C_P = \left\{ P + \mathcal{Q} \mid \mathcal{Q} \in C^{(n-1)} \right\}.$$

That is, C_P is the image of the inclusion map $i_P : C^{(n-1)} \to C^{(n)}$ with $i_P(\mathcal{Q}) = P + \mathcal{Q}$. The divisor C_P is ample in $C^{(n)}$ (see [Pol03]).

The numerical equivalence class of C_P is independent of P, and so, when talking about numerical classes, the subindex P will not be significant. We will call these divisors the **coordinate divisors**, and when $n = 2$, the **coordinate curves**. These coordinate divisors form a 1-dimensional family $\mathcal{C}$ of algebraically equivalent divisors in $C^{(n)}$ (not linearly equivalent), and its numerical class determines the family, that is:

Lemma 1.1.3. *([CS93]) If D is an effective divisor on $C^{(n)}$ which is numerically equivalent to C_P, then D belongs to the family $\mathcal{C}$.*

We consider the **main diagonal** divisor, Δ_C, in $C^{(n)}$ defined as

$$\Delta_C = \left\{ \mathcal{Q} + 2P \mid P \in C, \ \mathcal{Q} \in C^{(n-2)} \right\}.$$

This is precisely the branch divisor of π_C.

Furthermore, let Θ be the theta divisor on $J(C)$. We denote by θ the numerical equivalence class of $u^*(\Theta)$. Then,

Lemma 1.1.4. *In $C^{(n)}$ we have the following numerical equivalence*

$$\Delta_C \equiv_{num} 2((n + g - 1)C_P - \theta).$$

Since Δ_C is the branch divisor of π_C, it is numerically divisible by 2, and hence we can consider the numerical equivalence class of $\Delta_C/2$. Moreover, the numerical class of a canonical divisor in $C^{(n)}$ can be written as a linear combination of those of C_P and the diagonal as follows:

$$K_{C^{(n)}} \equiv_{num} (2g - 2)C_P - \frac{\Delta_C}{2}.$$

To compute intersection products in $C^{(n)}$ we have the following formula:

Lemma 1.1.5.

$$\theta^r C_P^{n-r} = \frac{g!}{(g - r)!}, \quad 0 \leq r \leq n \leq g.$$

We recall now some results on the cohomology of $C^{(n)}$:

Proposition 1.1.6. *([Mac62]) There exists the following relations between the holomorphic form on $C^{(n)}$ and the n-fold Cartesian product:*

$$\Omega^p_{C^{(n)}} = (\Omega^p_{C \times \cdots \times C})^{S_n}.$$

Where the superindex $(\cdot)^{S_n}$ states for the invariant part under the action of S_n.

Proposition 1.1.7. *([Mac62]) The Hodge numbers of $C^{(n)}$ are as follows:*

$$h^{p,q}(C^{(n)}) = \sum_{0 \le k \le p} \binom{g}{p-k} \binom{g}{q-k}, \quad 0 \le p \le q, \; p+q \le n.$$

Lemma 1.1.8. *([Mac62]) $H^*(C^{(n)}, \mathbb{Z})$ is torsion free.*

Proposition 1.1.9. *([Mac62])*

$$H^0(C^{(n)}, \omega_{C^{(n)}}) \cong \bigwedge^n H^0(C, \omega_C).$$

Now, we are going to introduce a subvariety of $Pic^d(C)$ induced by linear series in C closely related to symmetric products:

$$W_d^r(C) := \left\{ |D| \in Pic^d(C) \mid deg|D| = d, \; dim|D| \ge r \right\}$$

which parametrizes the complete linear systems on C of degree d and dimension at least r. The dimension of these varieties is bounded from below by the Brill-Noether number

$$\rho(g, r, d) = g - (r+1)(g - d + r).$$

The Brill-Noether number $\rho(g, r, d)$ is negative if and only if $W_d^r(C)$ is empty. When $\rho(g, r, d) \ge 0$ and C is a very general curve, we have that $dim W_d^r(C) = \rho(g, r, d)$.

Taking $r = 0$ and a fixed point $P \in C$ we can consider the easiest dth Brill-Noether locus inside $Pic^0(C)$ as

$$W_d = \{|D - dP| \in Pic^0(C) \mid deg|D| = d, \; dim|D| \ge 0\}.$$

For $n \le g$, the Picard map v gives a birational morphism with image precisely W_n. In fact, for $n = g-1$, the image of u gives the polarization on the Jacobian variety and moreover, for $n > 2g-2$, $C^{(n)}$ is a projective fiber bundle over $J(C)$ given by u.

1.1.1 Square symmetric products

In the main part of this thesis we will deal with $C^{(2)}$. In this section we recall some results on the surface $C^{(2)}$ and on curves contained in it, which will be our main object of study.

Proposition 1.1.10.

$$H^0(C^{(2)}, \Omega^1_{C^{(2)}}) \cong H^0(C, \omega_C) \quad \textit{and} \quad \bigwedge^2 H^0(C^{(2)}, \Omega^1_{C^{(2)}}) \cong H^0(C^{(2)}, \omega_{C^{(2)}}).$$

We remind the classical Castelnuovo-de Franchis Theorem:

Theorem 1.1.11 (Castelnuovo-de Franchis Theorem). *Let S be a surface with two linearly independent 1-forms ω_1, $\omega_2 \in H^0(S, \Omega^1_S)$ such that $\omega_1 \wedge \omega_2 = 0$. Then, there exists a smooth curve B with genus $g \geq 2$, an irrational pencil $p : S \to B$ and two 1-forms α_1, $\alpha_2 \in H^0(B, \Omega^1_B)$ such that $\omega_1 = p^*\alpha_1$ and $\omega_2 = p^*\alpha_2$.*

Thus, by Castelnuovo-de Franchis Theorem and Proposition 1.1.10 we deduce that $C^{(2)}$ has no irrational pencil of genus ≥ 2. Furthermore, we have that

$$p_g(C^{(2)}) = \frac{g(g-1)}{2}, \quad q(C^{(2)}) = g \quad \text{and} \quad \chi(C^{(2)}) = \frac{g(g-3)}{2} + 1.$$

For $g \geq 3$, since $p_g > 0(C^{(2)})$ and $K^2_{C^{(2)}} > 0$, we deduce that $C^{(2)}$ is a surface of general type. Instead, for $g = 2$, the surface $C^{(2)}$ is not of general type.

Next, we recall the intersection products of the main numerical classes in the square symmetric product of a curve C:

$$C_P^2 = 1, \quad \Delta_C^2 = 4 - 4g, \quad C_P \cdot \Delta = 2 \quad \text{and} \quad K^2_{C^{(2)}} = (g-1)(4g-9).$$

Consider now $a : C^{(2)} \to J(C)$, the Albanese map of $C^{(2)}$ with base point $P_0 + Q_0$, given by $a(P + Q) = P + Q - P_0 - Q_0$, that is, the Abel-Jacobi map v followed by the isomorphism $Pic^0(C) \cong J(C)$. By the Riemann-Roch Theorem, if C is not hyperelliptic then a is injective, while if C is hyperelliptic, a contracts to a point the rational curve given by the g^1_2, and is injective on its complement. In particular, for $g = 2$, $C^{(2)}$ is the blow-up of $J(C)$ in a point, and the exceptional divisor on $C^{(2)}$ is precisely $K_{C^{(2)}}$.

The square symmetric product of a curve can be described in a very precise geometric way, by the existence of a divisor with certain numerical properties.

Theorem 1.1.12. *([MPP11b]) Let S be a smooth surface of general type with irregularity q containing a 1-connected divisor D such that $p_a(D) = q$ and $D^2 > 0$. Then the minimal model of S is either*

 1. the product of two curves of genus g_1, $g_2 \geq 2$ ($g_1 + g_2 = q$) or

2. *the symmetric product $C^{(2)}$, where C is a smooth curve of genus q, and $C^2 = 1$.*

Furthermore, if D is 2-connected, only the second case occurs.

Recall that an effectice divisor D on a smooth surface S is m-connected if $D = D_1 + D_2$ with D_1, D_2 effective and nonzero, implies $D_1 \cdot D_2 \geq m$.

We remark that in the proof of this theorem it is used the characterization of $C^{(2)}$ given in [CCM98] where it is proven that $C^{(2)}$ is the only minimal algebraic surface with irregularity q that is covered by curves of genus q and self-intersection 1.

Finally, we observe that there is a natural system of local coordinates in $C \times C$ around a point (P, Q): let z_1, z_2 be local coordinates in C around P and Q respectively, then (z_1, z_2) are local coordinates around (P, Q) in $C \times C$. With this coordinates, π_C is expressed locally as follows:

$$
\begin{array}{ccc}
C \times C & \longrightarrow & C^{(2)} \\
(z_1, z_2) & \longrightarrow & (z_1 + z_2, z_1 \cdot z_2).
\end{array}
$$

That is, the coordinates in $C^{(2)}$ are the symmetric polynomials in the coordinates of $C \times C$. With this coordinates, the diagonal Δ_C is, locally, $(2z, z^2)$.

1.2   CURVES IN $C^{(n)}$

In this section we remind some known facts about curves in $C^{(n)}$, for $n \geq 2$. Given a curve $B \subset C^{(n)}$, we define the **degree** of B as the integer d if and only if $C_P \cdot B = d$.

The curves of degree one in $C^{(2)}$ are completely characterized by the following two results. One goal of this thesis is to give a characterization for higher degree curves.

Lemma 1.2.1. *([ACGH85]) Suppose $f : C \to B$ is a degree n morphism, with B a smooth irreducible curve. Then we have a curve*

$$
\Sigma := \{ f^{-1}(q) \mid q \in B \} \subset C^{(n)}
$$

such that

- *$C_P \cdot \Sigma = 1$ and*

- *$\Delta_C \cdot \Sigma = 2g(C) - 2 - n(2g(B) - 2)$, i.e. the degree of the ramification divisor of f.*

And conversely,

Lemma 1.2.2. *([Cha08]) Let $B \subset C^{(n)}$ be a curve such that*

i) $B \not\subset Supp(C_P) \; \forall P \in C$ *and*

ii) $B \cdot C_P = 1$.

Then, B is smooth and there exists a degree n map $f : C \to B$ such that $B = \{f^{-1}(Q) \mid Q \in B\} \subset C^{(n)}$.

In [Cil83] a similar result is proven. Assuming that B is not contained in the diagonal and that $B \cap C_P$ is a single point for all P, then there exists a morphism f as before.

Remark 1.2.3. *By the two previous lemmas, if $f : C \to B$ is a degree n morphism, then $\{f^{-1}(q) \mid q \in B\} \subset C^{(n)}$ is a smooth curve isomorphic to B, so we can consider B embedded in $C^{(n)}$. Moreover, by the proof of Lemma 1.2.2, the preimage of B by $\tau : C \times C^{(n-1)} \to C^{(n)}$ is isomorphic to C through the projection onto the first factor.*

Let $\tilde{B}$ be an irreducible curve in $C^{(n)}$ not contained in a coordinate divisor. Let B be its normalization and assume that C does not cover B with degree n. Then, since C_P is ample in $C^{(n)}$, by Lemma 1.2.2 we obtain that $\tilde{B} \cdot C_P \geq 2$.

We define a particular curve in $C^{(n)}$:

Definition 1.2.4. *The curve C lies in $C^{(n)}$ as a curve of the form $P + C$ with $P = P_1 + \cdots + P_{n-1} \in C^{(n-1)}$, that is, $C = C_{P_1} \cap \cdots \cap C_{P_{n-1}}$. We call this kind of curve a **coordinate curve**. Notice that for $n = 2$ this definition coincides with the usual one.*

We finish this section with a very interesting example that gives curves in $C^{(n)}$ of higher degree, and later we describe with detail this example for the surface $C^{(2)}$. The aim is to show a natural example of a curve in $C^{(2)}$ not considered in Lemmas 1.2.1 and 1.2.2 that motivates the study of curves with higher degree, and moreover, show that there exists an embedding of C in $C^{(2)}$ which image has non-positive self-intersection, and thus, it is not a coordinate curve.

Example 1.2.5. *If C has a degree $n + 1$ morphism on a curve B, then C lies in $C^{(n)}$ in a different way to that of the* coordinate curves:

Let $C \xrightarrow{f} B$ be such a map. If we consider the fiber over $f(P)$, $P \in C$, as a zero cycle, then the residue $f^{-1}(f(P)) \setminus \{P\}$ obtained subtracting

the point P is a divisor of degree n. We define a curve $\tilde{C}$ as the image of

$$
\begin{array}{ccc}
C & \longrightarrow & C^{(n)} \\
P & \longrightarrow & f^{-1}(f(P)) \setminus \{P\}.
\end{array}
$$

We notice that $\tilde{C} \subset C^{(n)}$ is not a coordinate curve, and furthermore, that $\tilde{C} \cdot C_P = n$, so it is not numerically equivalent to C_P^{n-1}.

In fact, as in Remark 1.2.3, we can consider $B \subset C^{(n+1)}$ as the set $\{f^{-1}(Q) \mid Q \in B\}$ and its preimage by $\tau : C \times C^{(n)} \to C^{(n+1)}$ is isomorphic to C through the projection onto the first factor. We observe that the image of $\tau^{-1}(B)$ in $C^{(n)}$ by the second projection is precisely $\tilde{C}$.

Example 1.2.6. *Now, we look again at the situation of Example 1.2.5 in the case of $C^{(2)}$.*

Assume that we have a morphism $f : C \overset{(3:1)}{\to} B$ with $g(C) = g$ and $g(B) = b$. Consider the morphism

$$
\begin{array}{ccc}
C & \overset{h}{\longrightarrow} & C^{(2)} \\
P & \longrightarrow & f^{-1}(f(P)) \setminus \{P\}.
\end{array}
$$

This morphism has degree one on its image. We denote by $\tilde{C}$ the image of h, the curve $\tilde{C} = \overline{\{x + y \mid f(x) = f(y)\}} \setminus \Delta$.

Regarding f, there are three kinds of points in C: those on which f does not ramify, those on which f is totally ramified and those on which it ramifies but not totally. We compute, locally, the image of each of them:

- *if $f^{-1}(f(P)) = x + y + P = f^{-1}(f(x)) = f^{-1}(f(y))$ with x, y, P pairwise different, then we get three points in $\tilde{C}$:* $\begin{cases} h(P) = x + y \\ h(x) = y + P \\ h(y) = x + P \end{cases}$.

  Locally, there are coordinates z_1, z_2, z_3 around x, y, P respectively, and f is $f(z_i) = z_i = t$ where t is a local coordinate around the point $f(P) \in B$.

  Hence, around P, for instance, h is

  $$
  h(z_1) = (2z_1, z_1^2) = (z_2 + z_3, z_2 \cdot z_3)
  $$

  so these are smooth points of $\tilde{C}$.

- *if $f^{-1}(f(P)) = f^{-1}(f(x)) = x + P + P$ then we get two different points in $\tilde{C}$:* $\begin{cases} h(P) = x + P \\ h(x) = P + P \end{cases}$ *. Locally, there are coordinates z_1, z_2*

around P, x respectively, and f is $f(z_1) = z_1^2 = t = z_2 = f(z_2)$, so $z_1 = \pm\sqrt{z_2}$.

Hence, around the point x, the morphism h is

$$h(z_2) = (\sqrt{z_2} + (-\sqrt{z_2}), \sqrt{z_2} \cdot (-\sqrt{z_2})) = (0, -z_2)$$

thus in this point $\tilde{C}$ is smooth and transversal to the diagonal.

And, around P, we see that

$$h(z_1) = (-z_1 + z_2, -z_1 \cdot z_2) = (z_1^2 - z_1, -z_1^3)$$

a smooth point.

- *if $f^{-1}(f(P)) = P + P + P$ then we get one point in $\tilde{C}$: $h(P) = P + P$.*

  Locally, there is a coordinate z around P, and f is $f(z) = z^3 = t$, so $z = \{\sqrt[3]{t}, \xi\sqrt[3]{t}, \xi^2\sqrt[3]{t}\}$ with ξ a primitive third root of unity.

  Hence, locally the image of P is $h(z) = ((\xi+\xi^2)z, \xi z \cdot \xi^2 z) = (-z, z^2)$, a smooth point.

  With these local coordinates, the diagonal is written as $(2z, z^2)$, i.e. the curve $y = \frac{x^2}{4}$. Therefore, the intersection of $\tilde{C}$ and Δ_C is, locally, $z^2 = \frac{z^2}{4}$ and thus, they are tangent with multiplicity 2.

In this way, for each point where f ramifies there is an intersection point of $\tilde{C}$ with the diagonal of multiplicity equal to the ramification index minus one. Hence,

$$\tilde{C} \cdot \Delta_C = \sum_{P \in C} (e_P - 1) = deg(\mathrm{Ram}(f)).$$

By the Riemann-Hurwitz formula we have that

$$2g - 2 = 3(2b - 2) + deg(\mathrm{Ram}(f)) \implies deg(\mathrm{Ram}(f)) = 2g - 6b + 4.$$

Thus, $\tilde{C}$ is smooth and isomorphic to C. In particular, $g(\tilde{C}) = g$. By the adjunction formula we have that

$$
\begin{aligned}
2g - 2 = \;& \tilde{C}^2 + \tilde{C} \cdot K_{C^{(2)}} = \tilde{C}^2 + \tilde{C} \cdot ((2g-2)C_P - \tfrac{\Delta_C}{2}) = \\
& \tilde{C}^2 + (2g-2)\underbrace{\tilde{C} \cdot C_P}_{2} - \tilde{C} \cdot \tfrac{\Delta_C}{2} \\
& \Rightarrow \tilde{C}^2 = 2 - 2g - \tfrac{1}{2}\tilde{C} \cdot \Delta_C.
\end{aligned}
$$

Therefore, we deduce that

$$\tilde{C}^2 = 2 - 2g - \frac{deg(\mathrm{Ram}(f))}{2} = 3(b - g).$$

And hence, we obtain that $\tilde{C}^2 \leq 0$ because $g \geq b$ by the Riemann-Hurwitz formula.

<div style="text-align: right">

2

Chapter Two

</div>

CHARACTERIZATION OF $C^{(n)}$

In this chapter, following the ideas introduced in the articles [CCM98] and [MPP11b] we prove a characterization of the nth symmetric product of a curve by the existence of a chain of subvarieties with certain properties. This generalizes the 2-dimensional case proved in the mentioned references. First, we remind some results that are useful for the proof of the main result in this chapter, Theorem 2.0.14.

Theorem 2.0.7 (Hodge Index Theorem). *([Har77]) Let H be an ample divisor in a surface S, and suppose that D is a divisor not numerically trivial with $D \cdot H = 0$. Then $D^2 < 0$.*

Theorem 2.0.8 (Lefschetz Theorem for Picard Groups). *([Laz04]) Let X be a smooth projective variety of dimension $n \geq 4$, and let $Y \subset X$ be a reduced effective ample divisor. Then, the map given by restriction $Pic(X) \to Pic(Y)$ is an isomorphism.*

For $n = 3$ the restriction map $Pic(X) \to Pic(Y)$ is injective.

Since $C^{(d-1)} \subset C^{(d)}$ is an ample divisor, this tells us that for $d > 3$ $s_P^{d\,*} : Pic(C^{(d)}) \to Pic(C^{(d-1)})$ is an isomorphism where the inclusion $s_P^d : C^{(d-1)} \to C^{(d)}$ is defined as $s_P^d(D) = P + D$. Moreover, we have an injective map $s_P^{3\,*} : Pic(C^{(3)}) \hookrightarrow Pic(C^{(2)})$ (in fact, in this situation it is an isomorphism, see [Pol03]).

Corollary 2.0.9. *Let X be an algebraic variety of dimension $n \geq 3$ and let D be a reduced effective ample divisor. Then, the restriction map $Pic^0(X) \to Pic^0(D)$ is injective.*

Proof. By the Lefschetz Theorem, we have that the restriction morphism $Pic^0(X) \to Pic^0(D)$ has trivial kernel. $\qquad\square$

We remind some results on generic vanishing theory. The main objects of interest are the cohomological support loci.

Definition 2.0.10. *Let X be an irregular (smooth) variety of dimension d. The cohomological support loci of $\mathcal{O}_X$ are the sets*

$$V^i(X) = V^i(X, \mathcal{O}_X) = \{\eta \in Pic^0(X) \mid h^i(X, \mathcal{O}_X \otimes \eta) \neq 0\},$$

where $i = 1, \ldots, d$.

The main result about the structure of these sets was proved by Green and Lazarsfeld, with an important addition due to Simpson (the fact that the translations are given by torsion elements).

Theorem 2.0.11. *([GL91], [Sim93]) Let X be an irregular variety of dimension d, and let W be an irreducible component of some $V^i(X)$. Then*

1. *there exist a subtorus $Z \subset Pic^0(X)$ and a torsion point of $Pic^0(X)$, β, such that $W = \beta + Z$, and*

2. *there exists a fibration $f : X \to Y$ onto a normal variety Y of dimension $dimY \leq d - i$, such that (any smooth model of) Y is of maximal Albanese dimension and $Z \subset f^* Pic^0(Y)$.*

As a corollary they obtained the following result, previously proved also by Green and Lazarsfeld in [GL87].

Theorem 2.0.12. *For any irregular variety X of dimension d,*

$$codim_{Pic^0(X)} V^i(X) \geq dim\, a(X) - i$$

where $a(X)$ is the image of X by its Albanese morphism.
   In particular, $h^i(L) = 0$ for general L and $i < dim\, a(X)$.

Before stating the theorem, we define the index of a family of divisors:

Definition 2.0.13. *Given an irreducible family $\mathcal{D} \subset B \times X$, with dimension 1 ($dimB = 1$), of effective divisors in a projective variety X, the **index** $i = i(\mathcal{D})$ of $\mathcal{D}$ is the degree of the projection, $p_2 : \mathcal{D} \to X$, or equivalently, the number of divisors of $\mathcal{D}$ passing through the general point of X.*

Notice that the family of coordinate divisors in $C^{(n)}$ has index n.

Now, we have all the necessary tools to state and prove the main theorem in this section.

Theorem 2.0.14. *Let X be a smooth projective variety of dimension n. Assume that there exists a chain of inclusions*

$$X = V_n \supset V_{n-1} \supset \cdots \supset V_2 \supset V_1 = C$$

such that

1. *V_i is a smooth irreducible variety with $dim(V_i) = i$.*

2. *For $i < n$, V_i is an ample divisor in V_{i+1}.*

3. *$V_i \cdot C = 1$ inside V_{i+1}.*

4. *V_2 is a surface of general type.*

5. *$q(X) = g(C)$.*

Then $X \cong C^{(n)}$. Moreover, $V_i \cong C^{(i)}$ and it is a coordinate divisor inside V_{i+1} for $i < n$.

Proof. We prove the theorem by induction. First, we observe that since C is an irreducible smooth curve, in particular, it is 2-connected, and hence by Theorem 1.1.12 we deduce that $S := V_2$ is birational to $C^{(2)}$. Hence, the case $n = 2$ is already known. By the proof of this theorem (see [MPP11b], Proposition 4.3) we have, moreover, that there exists a 1-dimensional family in $Pic^0(S)$

$$\mathcal{W} := \{\tilde\eta \in Pic^0(S) \mid h^0(S, \mathcal{O}_S(C) \otimes \tilde\eta) = 1\},$$

which is the image by the isomorphism $Pic^0(S) \cong Pic^0(C)$ given by the restriction map, of

$$W_1(C) = \{\eta \in Pic^0(C) \mid h^0(C, \mathcal{O}_C(C) \otimes \eta) = 1\}$$

which is the image of C by the natural map $C \to Pic^0(C)$ defined as $p \to \mathcal{O}_C(p - C|_C)$.

Furthermore, $\mathcal{C} = \{C_\eta, \ \eta \in \mathcal{W}\}$ is the family of coordinate curves in $C^{(2)}$, where C_η is the curve such that $\mathcal{O}_S(C_\eta) = \mathcal{O}_S(C) \otimes \eta$.

We observe, moreover, that by Lemma 1.1.3 the curve C is in fact a coordinate curve and thus $0 \in \mathcal{W}$. Since the divisor C is ample in S, in

fact $S \cong C^{(2)}$, because any exceptional divisor would have intersection product 0 with C.

We assume now that $n \geq 3$ and that the result is proven for all $dim(X) \leq n-1$. We are going to prove the theorem for $dim(X) = n$.

Since V_i is ample in V_{i+1}, by Corollary 2.0.9 and the hypothesis $q(X) = g(C)$, we obtain the following chain of isomorphisms given by the restriction maps:

$$Pic^0(X) \cong Pic^0(V_{n-1}) \cong \ldots \cong Pic^0(V_i) \cong \cdots \cong Pic^0(V_2) \cong Pic^0(C).$$

We assume, moreover, that the image of $\mathcal{W}$ in V_i by these isomorphisms parametrizes the family of coordinate divisors in $V_i \cong C^{(i)}$ for $i < n$, and we are going to prove that its image in X parametrizes the coordinate divisors in $C^{(n)}$, making this statement part of the inductive process.

We remind that by the induction hypothesis, V_{i-1} is a coordinate divisor in $V_i \cong C^{(i)}$ for all $i < n$.

Claim: There exists $\alpha \in Pic^0(X)$ such that

$$\alpha|_{V_{n-1}} = \mathcal{O}_{V_{n-1}}(V_{n-2} - V_{n-1}|_{V_{n-1}}).$$

Consider $\mathcal{O}_S(V_{n-2}|_S - (V_{n-1}|_{V_{n-1}})|_S)$. We observe that $V_{n-1} \cong C^{(n-1)}$ and $S \cong C^{(2)}$ with the inclusion $S \hookrightarrow V_{n-2}$ given by a point in $C^{(n-4)}$ (when $n = 3$, V_{n-2} is just C). Therefore, $V_{n-2}|_S$ is algebraically a coordinate curve C_Q in $S \cong C^{(2)}$.

Moreover, $V_{n-1}|_S \cdot C = 1$ and hence

$$(V_{n-2}|_S - (V_{n-1}|_{V_{n-1}})|_S) \cdot C = (C_Q - V_{n-1}|_S) \cdot C = 0$$

and

$$(V_{n-2}|_S - (V_{n-1}|_{V_{n-1}})|_S)^2 = (C_Q - V_{n-1}|_S)^2 =$$
$$C_Q^2 - 2 C_Q \cdot V_{n-1}|_S + (V_{n-1}|_S)^2 = -1 + (V_{n-1}|_S)^2 \geq 0$$

because V_{n-1} is ample in V_n.

Since C is ample in S, by the Hodge index Theorem, we deduce that $V_{n-2}|_S - (V_{n-1}|_{V_{n-1}})|_S$ is numerically trivial. As $S \cong C^{(2)}$, in fact it is algebraically trivial, because there is no torsion in $H^2(C^{(2)}, \mathbb{Z})$ (see Lemma 1.1.8).

By the Lefschetz Theorem for Picard Groups applied to the chain of V_i's we have that the restriction map gives an injective morphism $Pic(V_{n-1}) \hookrightarrow Pic(S)$, and from the isomorphism $Pic^0(V_{n-1}) \cong Pic^0(S)$, since

$$\mathcal{O}_S(V_{n-2}|_S - (V_{n-1}|_{V_{n-1}})|_S) \in Pic^0(S),$$

we deduce that

$$\mathcal{O}_{V_{n-1}}(V_{n-2} - V_{n-1}|_{V_{n-1}}) \in Pic^0(V_{n-1}).$$

Consequently, by the isomorphism between the Pic^0's, there exists an α as claimed. $\Diamond$

Furthermore, we obtain that V_{n-2} and $V_{n-1}|_{V_{n-1}}$ are numerically equivalent, so $1 = V_{n-2}^{n-1} = (V_{n-1}|_{V_{n-1}})^{n-1} = V_{n-1}^n$.

Now, consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(V_{n-1}) \to \mathcal{O}_{V_{n-1}}(V_{n-1}) \to 0.$$

Let $\mathcal{W}_n$ be the image of $\mathcal{W}$ by the isomorphism $Pic^0(V_{n-1}) \cong Pic^0(S)$ and $\eta \in \mathcal{W}_n$ a general element. We tensor the previous exact sequence with $\alpha \otimes \eta$ and get

$$0 \to \alpha \otimes \eta \to \alpha \otimes \eta \otimes \mathcal{O}_X(V_{n-1}) \to \mathcal{O}_{V_{n-1}}(V_{n-2}) \otimes \eta|_{V_{n-1}} \to 0.$$

We take cohomology and obtain

$$\begin{aligned} 0 \to \quad & H^0(X, \alpha \otimes \eta) \to H^0(X, \alpha \otimes \eta \otimes \mathcal{O}_X(V_{n-1})) \to \\ & H^0(V_{n-1}, \mathcal{O}_{V_{n-1}}(V_{n-2}) \otimes \eta|_{V_{n-1}}) \to H^1(X, \alpha \otimes \eta) \to \dots \end{aligned}$$

First of all, we observe that $H^0(X, \alpha \otimes \eta) = 0$ since $\alpha \otimes \eta \in Pic^0(X)$ is non trivial.

Second, we notice that the image of the Albanese morphism of X has dimension greater or equal than the minimum of $n-1$ and $g(C)$, which is in particular greater or equal than two. Indeed, we know that the image of the Albanese morphism of $C^{(n-1)}$ is of dimension $n-1$ if $n-1 \le g(C)$ and $g(C)$ otherwise, and by the identification of Pic^0's, this subvariety of $J(C)$ lives inside the image of the Albanese morphism of X, hence, it is of dimension at least $\min(n-1, g(C)) \ge 2$. Thus, applying generic vanishing results (Theorems 2.0.11 and 2.0.12) to $V^1(X) = \{\varsigma \in Pic^0(X) \mid h^1(X, \varsigma) > 0\}$ we deduce that it is the union of finitely many translates of proper abelian subvarieties.

Furthermore, we know that $W_1(C)$ generates $Pic^0(C)$, hence, its image by the identification $Pic^0(X) \cong Pic^0(C)$ generates $Pic^0(X)$, and when we translate it by a fixed element $\alpha \in Pic^0(X)$ it still generates. By the generic vanishing results, it cannot be contained in $V^1(X)$. So, for a general $\eta \in \mathcal{W}_n$ we obtain that $\alpha \otimes \eta \notin V^1(X)$ and thus $H^1(X, \alpha \otimes \eta) = 0$.

Therefore, for $\eta \in \mathcal{W}_n$ general we have that

$$h^0(X, \alpha \otimes \eta \otimes \mathcal{O}_X(V_{n-1})) = h^0(V_{n-1}, \mathcal{O}_{V_{n-1}}(V_{n-2}) \otimes \eta|_{V_{n-1}}) = 1 > 0.$$

And by semicontinuity, $h^0(X, \alpha \otimes \eta \otimes \mathcal{O}_X(V_{n-1})) > 0$ for all $\eta \in \mathcal{W}_n$.

In this way we have a 1-dimensional family $\mathcal{D}$ in X, of effective divisors algebraically equivalent to V_{n-1}. Let H_η denote the effective divisor in X such that $\mathcal{O}_X(H_\eta) = \mathcal{O}_X(V_{n-1}) \otimes \alpha \otimes \eta$. We observe that:

- $V_{n-1} \cdot H_\eta = (V_{n-2})_\eta$. Indeed,

$$\mathcal{O}_{V_{n-1}}(H_\eta) = \mathcal{O}_{V_{n-1}}(V_{n-1}|_{V_{n-1}}) \otimes \alpha|_{V_{n-1}} \otimes \eta = \mathcal{O}_{V_{n-1}}(V_{n-2}) \otimes \eta$$

  where we consider $\eta \in Pic^0(V_n)$ or $Pic^0(V_{n-1})$ indistinctively by the isomorphism given by the restriction map.

- Since H_η is algebraically equivalent to V_{n-2}, we have that H_η is ample and $H_\eta^n = 1$. Hence, $Pic^0(X) \cong Pic^0(H_\eta) \cong Pic^0(C)$. In particular, when H_η is smooth, $q(H_\eta) = g(C)$.

- If H_η is smooth, since $V_{n-1} \cdot H_\eta = (V_{n-2})_\eta \cong C^{(n-2)}$, we can apply the induction hypothesis to H_η and deduce that $H_\eta \cong C^{(n-1)}$. In addition we obtain that in $Pic^0(H_\eta)$ there is a 1-dimensional family $\{\varsigma \in Pic^0(H_\eta) \mid h^0(H_\eta, \mathcal{O}_{H_\eta}((V_{n-2})_\eta) \otimes \varsigma) > 0\}$ which is the image of $\mathcal{W}$ via the identification $Pic^0(H_\eta) \cong Pic^0(C)$.

Now, we study the possible singularities of the hypersurfaces H_η.

First, a divisor H_η does not contain a curve of singularities. Otherwise, since V_{n-1} is ample, this curve would cut V_{n-1} in a point, and then $(C^{(n-2)})_\eta$ would be singular, contradicting our hypothesis. Hence, each H_η has at most a finite number of singularities.

Second, the possible singularities do not move with the family. Otherwise, the singularities would move as the divisors giving curves $\{B_i\}$ such that the intersection point of B_i and H_η would be a singular point of H_η. Since V_{n-1} is ample, a curve B_i would intersect V_{n-1} in a point $P \in (C^{(n-2)})_\eta$ for a certain η, and then $(C^{(n-2)})_\eta$ would be singular, which is a contradiction.

Third, there is no base curve for the family $\mathcal{D}$. Otherwise, this curve would intersect V_{n-1} and then the family $\mathcal{D}_{n-2}$ of coordinate divisors in $C^{(n-1)}$ would have a base point which is not possible.

Finally, there is no singularity Q common to all $H_\eta \in \mathcal{D}$. Otherwise, the point Q would be a base point of the family, and with a local computation we deduce that then $V_{n-1}^n > 1$ contradicting $V_{n-1}^n = 1$.

Therefore, not all elements $H_\eta \in \mathcal{D}$ are singular, in fact, the general one is smooth, and those singular have at most isolated singularities.

Since the ith symmetric product of a curve deforms in an algebraic family only as the ith symmetric product of a curve, we deduce that

the general element in the family $\mathcal{D}$ is birational to $C^{(n-1)}$. Moreover, since $V_{n-1} \cdot H_\eta = (C^{n-2})_\eta$ we deduce that the image by the restriction map of $\mathcal{D}$ to a general divisor in the family gives the family of coordinate divisors in $H_\eta \approx C^{(n-1)}$.

Consequently, since the index of the family of coordinate divisors on $C^{(n-1)}$ is $n - 1$, we deduce that the index of $\mathcal{D}$ in X is n. Indeed, given a general point in H_η, we have $n-1$ other elements of $\mathcal{D}$ passing trough it, plus H_η, hence n elements of the family.

Finally, we see that indeed $X \cong C^{(n)}$.

Let $Q \in X$ be a general point and let $H_1, \ldots, H_n$ be the divisors in $\mathcal{D}$ passing through Q. Let $D_1 = V_{n-1} \cdot H_1$, then D_1 is a coordinate divisor in $V_{n-1} \cong C^{(n-1)}$, hence, it is of the form $C^{(n-2)} + P_1$, for certain $P_1 \in C$. In a similar way, $H_i \cdot V_{n-1} = C^{(n-2)} + P_i$, thus, we have a birational map

$$
\begin{array}{ccc}
X & \dashrightarrow & C^{(n)} \\
Q & \dashrightarrow & P_1 + \cdots + P_n.
\end{array}
$$

Since V_{n-1} is ample in X, we deduce that $X \cong C^{(n)}$ because any curve contracted by the birational map would have product 0 with V_{n-1}. Observe finally that if $V_{n-1} \cdot H_\eta = C^{(n-2)} + P$, then $H_\eta = C_P$, the coordinate divisor with base point P, and hence, $\mathcal{W}_n$ parametrizes the coordinate divisors in $C^{(n)}$. $\qquad\square$

Corollary 2.0.15. *Let X be a smooth projective variety of dimension n. Assume that there exists a divisor D isomorphic to $C^{(n-1)}$ such that, if C denotes a coordinate curve in $D \cong C^{(n-1)}$, then $D \cdot C = 1$ and, moreover, $q(X) = g(C)$. Then $X \cong C^{(n)}$.*

PRELIMINARS ON CURVES

In this chapter we remind some well known results about complex algebraic curves for sake of completeness and for this document to be self-contained. The main references for this chapter are [FK80] and [Mir95].

3.1 FINITE MORPHISMS OF CURVES

Let $f : C \to D$ be a finite morphism of curves. Let $P \in C$ be a point and $Q = f(P)$ be its image. It is well known that there are local coordinates around P and $f(P)$ such that the morphism f, locally, can be written as

$$z \to z^{e_P} = \xi.$$

Definition 3.1.1. *The **ramification index** of f at P is e_P. If $e_P > 1$ we say f is **ramified** at P and that Q is a **branch point** of f. If $e_P = 1$ we say f is **unramified** at P.*

*The **degree** of f is*

$$deg(f) = \sum_{f(P)=Q} e_P$$

for any $Q \in D$. This sum does not depend on Q.

For any point $Q \in D$ we put

$$f^*(Q) = \sum_{f(P)=Q} e_P \cdot P.$$

*The **ramification divisor** of f is*

$$\mathrm{Ram}(f) = \sum_{P \in C} (e_P - 1) \cdot P \in Div(C).$$

Notice that this sum is finite since only for a finite number of $P \in C$, $e_P \neq 1$.

*The **branch locus** of f is the image of $\mathrm{Ram}(f)$ by f without multiplicities, that is, $\mathrm{Branch}(f) = Q_1 + \cdots + Q_r$ with Q_i branch point of f. As a subset of D, it is finite.*

Proposition 3.1.2. *Let $f : C \to D$ be a finite morphism of curves. Then*

$$K_C \equiv_{lin} f^* K_D + \mathrm{Ram}(f).$$

Taking degrees we obtain that

Theorem 3.1.3 (Riemann-Hurwitz Formula)**.** *Let $f : C \to D$ be a finite morphism of curves. Denote $g = g(C)$, $\gamma = g(D)$ and $n = deg(f)$. Then,*

$$2g - 2 = n(2\gamma - 2) + \sum_{P \in C} (e_P - 1).$$

Given a finite morphism of curves $f : C \to D$ we will call **branching type of** f the set of numbers:

$$(\gamma; (e_{P_1^1}, \ldots, e_{P_1^{k_1}}), \ldots, (e_{P_r^1}, \ldots, e_{P_r^{k_r}}))$$

where $\mathrm{Branch}(f) = Q_1 + \cdots + Q_r$ and $f^*(Q_j) = \sum_{i=1}^{k_j} e_{P_j^i} \cdot P_j^i$. That is, the ramification indices of the points over the branch locus.

We consider now two morphisms of curves with a specially simple description.

Proposition 3.1.4. *([Har77]) A non constant morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d can be obtained as the composition*

$$\mathbb{P}^1 \hookrightarrow \mathbb{P}^d \xrightarrow{\pi_L} \mathbb{P}^1 \to \mathbb{P}^1$$

where the first arrow is a d-tuple embedding, the second is a linear projection with base a linear variety of dimension $d - 2$ (L), and the last one is an automorphism of $\mathbb{P}^1$.

Proposition 3.1.5. *A morphism $f : E \to \mathbb{P}^1$ with degree 3 from an elliptic curve E to $\mathbb{P}^1$ can be obtained as the composition of*

$$E \hookrightarrow \mathbb{P}^2 \xrightarrow{\pi_P} \mathbb{P}^1$$

where the first arrow is the embedding of E in $\mathbb{P}^2$, as a cubic plane curve, given by the linear series defined by the fibers of the morphism f. The second one, π_P, is the projection with base a point not in the image of E.

More in general, we have a topological description of morphisms of curves.

Let $f : C \to D$ be a degree d morphism of curves. We consider now $D^* = D \setminus \mathrm{Branch}(f)$, $C^* = C \setminus f^{-1}(\mathrm{Branch}(f))$ and the topological covering induced by f, $F : C^* \to D^*$. The monodromy of F gives a group homomorphism $\mu : \pi_1(D^*, x) \to S_d$. Take a branch point $Q \in D$ and a small open neighborhood W of Q in D. Let $P_1, \ldots, P_k$ be the preimages of Q ($k < d$). Then, the image by μ of a small loop around Q is a permutation with cycle structure $(e_{P_1}, \ldots, e_{P_k})$. That is, it is the product of disjoint cycles of length e_{P_i}.

The monodromy μ and the divisor $\mathrm{Branch}(f)$ determine the morphism due to the following result:

Proposition 3.1.6. *([Mir95]) Let D be a compact Riemann surface, let B be a finite subset of D, and let q be a base point of $D \setminus B$. Then, there is a $1-1$ correspondence*

$$
\left\{
\begin{array}{c}
\textit{isomorphism classes of} \\
\textit{maps } f : C \to D \\
\textit{of degree } d \textit{ whose} \\
\textit{branch points lie in } B
\end{array}
\right\}
\quad \leftrightarrow \quad
\left\{
\begin{array}{c}
\textit{group homomorphisms} \\
\mu : \pi_1(D \setminus B, x) \to S_d \\
\textit{with transitive image} \\
\textit{(up to conjugacy) in } S_d
\end{array}
\right\} .
$$

Moreover, at a point $Q \in B$, if γ is a small loop in $D \setminus B$ around Q based at x, and $\mu([\gamma])$ has cycle structure $(m_1, \ldots, m_k)$, then there are k preimages $P_1, \ldots, P_k$ of Q in the corresponding morphism f_μ with $mult_{P_i}(f_\mu) = m_i$ for each i.

For $D = \mathbb{P}^1$, since the fundamental group of $\mathbb{P}^1 \setminus B$ is the free group on $r = |B|$ elements, $\alpha_1, \ldots, \alpha_r$, subject to the relation $\alpha_1 \cdots \alpha_r = 1$, we have that

Proposition 3.1.7. *Fix a finite set $B = \{Q_1, \ldots, Q_r\} \subset \mathbb{P}^1$. Then there is a $1-1$ correspondence*

$$
\left\{
\begin{array}{c}
\textit{isomorphism classes of} \\
\textit{maps } f : C \to \mathbb{P}^1 \\
\textit{of degree } d \textit{ whose} \\
\textit{branch points lie in } B
\end{array}
\right\}
\quad \leftrightarrow \quad
\left\{
\begin{array}{c}
\textit{conjugacy classes of } r - \textit{tuples} \\
(\sigma_1, \ldots, \sigma_r) \textit{ of permutations in} \\
S_d \textit{ such that } \sigma_1 \cdots \sigma_r = 1 \\
\textit{and the subgroup generated} \\
\textit{by the } \sigma_i\textit{'s is transitive}
\end{array}
\right\} .
$$

Moreover, if σ_i has cycle structure $(m_1, \ldots, m_k)$ then there are k preimages $P_1, \ldots, P_k$ of Q_i in the corresponding cover $f : C \to \mathbb{P}^1$, with $mult_{P_j}(f) = m_j$ for each j.

We observe that the monodromy $\mu : \pi_1(D^*, q) \to S_d$ can be factored through a surjection onto a finite group G. When $D = C/G$ we say that f is a **Galois morphism** with group G. We are interested in the study of Galois morphisms.

Assume that we have a Galois morphism $f : C \to D$ with Galois group G. Then, the branch points of f have branch type $(m, \ldots, m)$ and we say that such a branch point has order m. In this case we say that f is **totally ramified**.

Taking a basis, we have a presentation

$$\pi_1(D \setminus B, x) = \langle \alpha_1, \beta_1, \ldots, \alpha_\gamma, \beta_\gamma, \delta_1, \ldots, \delta_r \mid \prod_{j=1}^{\gamma} [\alpha_j, \beta_j] \prod_{i=1}^{r} \delta_i = 1 \rangle$$

where $[\alpha_j, \beta_j] = \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1}$.

We define the orbifold fundamental group $\pi_1^{orb}(D \setminus B, q; m_1, \ldots m_r)$ as

$$\langle \alpha_1, \beta_1, \ldots, \alpha_\gamma, \beta_\gamma, \delta_1, \ldots, \delta_r \mid \prod_{j=1}^{\gamma} [\alpha_j, \beta_j] \prod_{i=1}^{r} \delta_i = 1, \ \delta_i^{m_i} = 1, \ 1 \leq i \leq r \rangle.$$

Then, we have an exact sequence

$$1 \to \pi_1(C, x_0) \to \pi_1^{orb}(D \setminus B, x; m_1, \ldots m_r) \to G \to 1$$

completely determined by the monodromy, that describes the topological action of G on C.

3.2   AUTOMORPHISM GROUP OF COMPACT RIEMANN SURFACES

Let C be a compact Riemann surface. Let $G \subset \operatorname{Aut}(C)$ be a finite subgroup. For $P \in C$, set

$$G_P = \{g \in G \mid g(P) = P\}$$

the **stabilizer** of P and

$$\mathcal{O}_G(P) = \{g(P), \ g \in G\}$$

the **orbit** of the point by the action of G.

Proposition 3.2.1. *Assume $g(C) \geq 2$. Then G_P is a cyclic subgroup of* $\operatorname{Aut}(C)$.

Given an automorphism σ of a curve C, we consider its graph inside $C \times C$, that is

$$\Gamma_\sigma = \{(x, \sigma(x))\} \subset C \times C.$$

We are interested in the intersection of a graph with the diagonal and also with other graphs. We have that

Proposition 3.2.2. *The diagonal in $C \times C$ cuts the graph of an automorphism transversally.*

For lack of a reference we include the proof of this result that probably is well known by the experts.

Proof. Let x be a point fixed by σ. Then, there exists a neighborhood U of x and a local coordinate z around x in such a way that σ is locally

$$
\begin{array}{ccc}
(U, x) & \xrightarrow{\sigma} & (U, x) \\
z & \longrightarrow & \xi z
\end{array}
$$

where, if $n = o(\sigma)$, ξ is a primitive nth root of unity.

Then, around (x, x) we have a local system of coordinates in $U \times U$ where the graph of σ is locally $\{(z, \xi z)\}$ and the diagonal is $\{(z, z)\}$. Therefore, taking derivatives, we see that since $\xi \neq 1$, because σ is not the identity, these two curves are transversal. $\qquad\square$

Corollary 3.2.3. *Let α and β be two automorphisms of a curve C. If $\alpha^{-1}\beta \neq 1$, then the graphs of α and β in $C \times C$ intersect transversally and moreover, $\Gamma_\alpha \cdot \Gamma_\beta$ equals the number of fixed points $\nu(\alpha^{-1}\beta)$ of the automorphism $\alpha^{-1}\beta$.*

Proof. We have in $C \times C$ the action of $1 \times \alpha^{-1}$. We transform the two considered graphs by this action:

$$
\begin{array}{ccc}
\Gamma_\alpha = \{(x, \alpha(x))\} & \xrightarrow{1 \times \alpha^{-1}} & \{(x, x)\} = \Delta_{C \times C} \\
\Gamma_\beta = \{(x, \beta(x))\} & \xrightarrow{1 \times \alpha^{-1}} & \{(x, \alpha^{-1}\beta(x))\} = \Gamma_{\alpha^{-1}\beta}.
\end{array}
$$

Since the diagonal intersects transversally the graph of any automorphism, we deduce that the two graphs intersect also transversally in $\nu(\alpha^{-1}\beta)$ points. $\qquad\square$

Now, we study the action of a group G on the curve C and the resulting orbit space.

Lemma 3.2.4. *Let G be a finite group of order n acting on a curve C. Given a point $P \in C$, let α be a generator of G_P. Then we have that*

$$n = |G_P| \cdot |\{\text{conjugates of } G_P\}| \cdot |\{\text{points fixed by } \alpha \text{ in } \mathcal{O}_G(P)\}|.$$

Proof. By Theorems A.0.5 and A.4.7 we obtain that

$$n = |G| = |G_P| \cdot [G : G_P] = |G_P| \cdot |\mathcal{O}_G(P)|.$$

Now, since the point P has stabilizer G_P, given a conjugate of G_P, we see that $\sigma G_P \sigma^{-1} = G_{\sigma(P)}$, that is, it is the stabilizer of $\sigma(P)$. Moreover, given any element $\beta \in G$, $\beta(P)$ has stabilizer G_P or one of its conjugates. Therefore, in the orbit of P there are the same number of points with stabilizer each conjugate of G_P and all conjugates of G_P are stabilizers of points in the orbit. Hence,

$$|\mathcal{O}_G(P)| = |\{\text{conjugates of } G_P\}| \cdot |\{\text{points fixed by } \alpha \text{ in } \mathcal{O}_G(P)\}|. \quad \square$$

Theorem 3.2.5. *Let C be a compact Riemann surface. Let $G \subseteq \mathrm{Aut}(C)$ be a subgroup. Then, the orbit space C/G is a compact Riemann surface and the natural projection $\pi : C \to C/G$ from a point onto the corresponding orbit is an open mapping. In fact, it is a degree $n = |G|$ morphism ramified on the points fixed by some element $\alpha \in G \setminus \{1\}$.*

Proposition 3.2.6. *Let C be a compact Riemann surface. Let G and G' be subgroups of $\mathrm{Aut}(C)$. If G and G' are conjugate subgroups inside $\mathrm{Aut}(C)$ then $C/G \cong C/G'$.*

We include the proof of the following two results because, during them, a precise study of the action of a group on a compact Riemann surface is made. We will use later some important pieces of these proofs to our purposes.

Theorem 3.2.7 (Hurwitz Theorem). *Let C be a compact Riemann surface of genus $g \geq 2$. Let N be the order of $\mathrm{Aut}(C)$, then*

$$N \leq 84(g - 1).$$

Proof. Let $G := \mathrm{Aut}(C)$. Consider the holomorphic projection morphism $\pi : C \to C/G$ and let $\gamma = g(C/G)$ be the genus of the quotient curve. We know that:

- π is of degree $N = |G|$.

- π is ramified only at the points fixed by elements of G and $\forall P \in C$

$$e_P = |G_P|.$$

Let $P_1, \ldots, P_r$ be a maximal set of inequivalent (that is, $P_j \neq h(P_k)$, for all $h \in G$, and all $j \neq k$) points fixed by elements of $G \setminus \{1\}$.

Let $t_j = |G_{P_j}|$. Then, there are N/t_j distinct points on C equivalent under G to P_j, each with a stabilizer subgroup of order t_j (if h takes P to Q then $G_Q = hG_Ph^{-1}$).

Thus, the degree of the ramification divisor of π is given by

$$deg(\mathrm{Ram}(\pi)) = \sum_{P \in R}(e_P - 1) = \sum_{j=1}^{r} \frac{N}{t_j}(t_j - 1) = N \sum_{j=1}^{r} \left(1 - \frac{1}{t_j}\right).$$

The Riemann-Hurwitz formula now reads

$$2g - 2 = N(2\gamma - 2) + N \sum_{j=1}^{r} \left(1 - \frac{1}{t_j}\right). \tag{3.1}$$

Notice that $t_j \geq 2$ and thus $\frac{1}{2} \leq 1 - \frac{1}{t_j} < 1$. The rest of the proof consists of an analysis of (3.1). It is clear (since we may assume that $N > 1$) that $g > \gamma$.

We consider the different possibilities:

- $\gamma \geq 2$

  In this case we obtain by (3.1) that

  $$N \leq g - 1.$$

- $\gamma = 1$

  In this case (3.1) becomes

  $$2g - 2 = N \sum_{j=1}^{r} \left(1 - \frac{1}{t_j}\right).$$

  If $r = 0$, then also $g = 1$ but we assumed that $g > 1$, so (3.1) implies that
  $$N \leq 4(g - 1).$$

- $\gamma = 0$

  We write (3.1) as

  $$2(g-1) = N\left(\sum_{j=1}^{r}\left(1 - \frac{1}{t_j}\right) - 2\right)$$

  and conclude that $r \geq 3$, since $2(g-1) > 0$, $N > 1$ and $1 - \frac{1}{t_j} < 1$ for each j.

  If $r \geq 5$ then (3.1) gives $N \leq 4(g-1)$.

  If $r = 4$, then it cannot be that all the t_j are equal to 2, thus at least one is ≥ 3 and (3.1) gives $N \leq 12(g-1)$.

  It remains to consider the case $r = 3$. Without lost of generality we assume that $2 \leq t_1 \leq t_2 \leq t_3$. Clearly $t_3 > 3$, otherwise the right hand side of (3.1) is negative. Furthermore, $t_2 \geq 3$.

  If $t_3 \geq 7$ then we obtain $N \leq 84(g-1)$. We conclude with the rest of cases:

  If $t_3 = 6$ and $t_1 = 2$, then $t_2 \geq 4$ and $N \leq 24(g-1)$.

  If $t_3 = 6$ and $t_1 \geq 3$, then $N \leq 12(g-1)$.

  If $t_3 = 5$ and $t_1 = 2$, then $t_2 \geq 4$ and $N \leq 40(g-1)$.

  If $t_3 = 5$ and $t_1 \geq 3$, then $N \leq 15(g-1)$.

  If $t_3 = 4$ then $t_1 \geq 3$ and $N \leq 24(g-1)$. $\square$

Regarding the number of fixed points of an automorphism of a compact Riemann surface we have the following bounds. We include the proofs because we find them illustrative.

Proposition 3.2.8. *For* $1 \neq \sigma \in \mathrm{Aut}(C)$,

$$\nu(\sigma) \leq 2 + \frac{2g}{o(\sigma) - 1}.$$

Proof. We apply the Riemann-Hurwitz formula to the natural projection $C \to C/\langle\sigma\rangle$. If $\gamma = g(C/\langle\sigma\rangle)$, then

$$2g - 2 = o(\sigma)(2\gamma - 2) + \sum_{j=1}^{o(\sigma)-1} \nu(\sigma^j).$$

We must explain the evaluation of the degree of the ramification divisor of the projection map appearing in the above formula. Clearly

$deg(\mathrm{Ram})$ is the weighted sum of the points fixed by $\langle\sigma\rangle$; each fixed point appearing one less time than the order of its stabilizer subgroup. This is exactly the contribution for $deg(\mathrm{Ram})$ in this sum. We use now the obvious inequality $\nu(\sigma) \leq \nu(\sigma^j)$, $j = 1, \ldots, o(\sigma) - 1$, and conclude that

$$2g - 2 \geq o(\sigma)(2\gamma - 2) + \nu(\sigma)(o(\sigma) - 1),$$

that is,

$$\nu(\sigma) \leq 2 + \frac{2g}{o(\sigma) - 1} - \frac{2\gamma o(\sigma)}{o(\sigma) - 1}$$

from which we deduce the proposition. $\qquad\square$

Corollary 3.2.9. *If $o(\sigma)$ is prime, then*

$$\nu(\sigma) = 2 + \frac{2g - 2\gamma o(\sigma)}{o(\sigma) - 1}$$

where γ is the genus of $C/\langle\sigma\rangle$.

Proof. Since $o(\sigma)$ is prime, $\nu(\sigma) = \nu(\sigma^j)$ for $j = 1, \ldots, o(\sigma) - 1$. $\qquad\square$

Theorem 3.2.10. *([FK80]) Let $\sigma \in \mathrm{Aut}(C)$ be of prime order n. If σ has a fixed point, it must have at least two.*

To finish this chapter we define a very important class of curves, characterized by the existence of an involution.

Definition 3.2.11. *A compact Riemann surface C will be called, for $\gamma \in \mathbb{Z}_{\geq 0}$, γ-**hyperelliptic** if there is a compact Riemann surface $\tilde{C}$ of genus γ and a holomorphic mapping of degree 2*

$$p : C \to \tilde{C}.$$

The change of sheet in the map p provides a γ-hyperelliptic involution $i_\gamma \in \mathrm{Aut}(C)$ with $o(i_\gamma) = 2$ and $\nu(i_\gamma) = 2g + 2 - 4\gamma$.

Theorem 3.2.12. *([FK80]) Let C be a γ-hyperelliptic compact Riemann surface of genus $g > 4\gamma + 1$. We have:*

- *If $1 \neq T \in \mathrm{Aut}(C)$ with $\nu(T) > 4(\gamma + 1)$, then $T = i_\gamma$.*

- *The involution i_γ is in the center of $\mathrm{Aut}(C)$.*

Corollary 3.2.13.   • *When $\gamma = 0$ we say that C is **hyperelliptic**, and we call $\iota = i_0$ the hyperelliptic involution.*

The hyperelliptic involution ι is unique for $g(C) > 1$.

- *When $\gamma = 1$ we say that C is **bielliptic**, and we call $\iota_1 = i_1$ the bielliptic involution.*

  The bielliptic involution ι_1 is unique for $g(C) > 5$.

Corollary 3.2.14. *Let C be a curve of genus $g > 3$. Then C cannot be both hyperelliptic and bielliptic.*

We remark that all curves of genus 2 are hyperelliptic, with the hyperelliptic map induced by the canonical linear series.

4

FUCHSIAN GROUPS

In this chapter we review the theory of Fuchsian groups and Riemann surfaces. Sections 1, 2 and 3 are introductory on the subject and are taken from [Kat92]. We refer the reader to this book for the proofs of the results and more detailed explanations. Section 4 derives from [FK80], [Mac73], [Mac74] and [BCGMG03]. The results in this section constitute a fundamental tool for the subsequent study of curves developed in Chapters 6 and 7. In Section 5 we introduce the Hurwitz space associated to a group action on curves of certain genus and give some results on its structure and geometry. This spaces will be useful for the study of families of curves that we make in Chapter 7.

4.1   THE GROUP $\mathrm{PSL}(2, \mathbb{R})$ ACTING ON $\mathbb{H}$

We consider the upper half-plane

$$\mathbb{H} = \{z = x + yi \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

equipped with the hyperbolic metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$.

Let us consider the unimodular group $\mathrm{SL}(2, \mathbb{R})$ of square matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(M) = ad - bc = 1$ and trace $\mathrm{tr}(M) = a + d$. We can consider it acting on $\mathbb{C}$ as $z \to \frac{az+b}{cz+d}$.

The set of fractional (or Möbius) transformations of $\mathbb{C}$ onto itself of the form

$$\left\{ z \xrightarrow{T} \frac{az + b}{cz + d} \mid a, b, c, d \in \mathbb{R}, \ ad - bc = 1 \right\}$$

forms a group such that the product of two transformations corresponds to the product of matrices and the inverse corresponds to the

inverse matrix. This group is called $\mathrm{PSL}(2,\mathbb{R})$. Each of these transformations T can be represented by a pair of matrices $\pm M \in \mathrm{SL}(2,\mathbb{R})$. Thus, the group $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to $\mathrm{SL}(2,\mathbb{R})/\pm Id$.

Moreover, $\mathrm{tr}(-M) = -\mathrm{tr}(M)$, so that $\mathrm{tr}^2(T) = \mathrm{tr}^2(M)$ and we define $\mathrm{Tr}(T) = |\mathrm{tr}(M)|$.

Theorem 4.1.1. *The group* $\mathrm{PSL}(2,\mathbb{R})$ *acts on* $\mathbb{H}$ *by isometries, that is, homeomorphisms that preserve the hyperbolic metric.*

The fixed points of a Möbius transformation are found solving the equation $z = \frac{az+b}{cz+d}$ that has discriminant $(a+d)^2 - 4$. We distinguish three types of elements in $\mathrm{PSL}(2,\mathbb{R})$ depending on the value of its trace ($\mathrm{Tr}(T) = |a+d|$) and hence on the number of solutions of the previous equation.

- **Elliptic elements**: $\mathrm{Tr}(T) < 2$. Then T is conjugate in $SL(2,\mathbb{R})$ to a unique matrix $\begin{pmatrix} cos\,\theta & sin\,\theta \\ -sin\,\theta & cos\,\theta \end{pmatrix}$. They are called elliptic because regarding the action on all $\mathbb{R}^2$ the invariant curves are ellipses. It has a pair of complex conjugate fixed points in $\mathbb{C}$, and therefore, one fixed point in $\mathbb{H}$.

- **Hyperbolic elements**: $\mathrm{Tr}(T) > 2$. Then T is conjugate in the group $SL(2,\mathbb{R})$ to a unique matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$, $\lambda \neq 1$. Regarding the action on all $\mathbb{R}^2$ the invariant curves are hyperbolas. It has two fixed points in $\mathbb{R} \cup \{\infty\}$, one repulsive and one attractive, and hence no fixed point in $\mathbb{H}$.

- **Parabolic elements**: $\mathrm{Tr}(T) = 2$. It has one fixed point in the projective line $\mathbb{R} \cup \{\infty\}$, and hence no fixed point in $\mathbb{H}$. They are called parabolic by analogy, as intermediate between hyperbolic and elliptic.

Besides being a group, $\mathrm{PSL}(2,\mathbb{R})$ is a topological space in which a transformation $z \to \frac{az+b}{cz+d}$ can be identified with a point $(a,b,c,d) \in \mathbb{R}^4$. More precisely, as a topological space, $SL(2,\mathbb{R})$ can be identified with the subset of $\mathbb{R}^4$

$$X = \left\{ (a,b,c,d) \in \mathbb{R}^4 \mid ad - bc = 1 \right\}.$$

If we define $\delta(a,b,c,d) = (-a,-b,-c,-d)$, then $\delta : X \to X$ is a homeomorphism and δ together with the identity forms a cyclic group

of order 2 acting on X. We topologize $\mathrm{PSL}(2, \mathbb{R})$ as the quotient space $X/\langle \delta \rangle$.

The group of all isometries of $\mathbb{H}$, $\mathrm{Isom}(\mathbb{H})$ is topologized similarly.

Definition 4.1.2. *For a subset $A \subset \mathbb{H}$ we define $\mu(A)$, the **hyperbolic area** of A, by*

$$\mu(A) = \int_A \frac{dxdy}{y^2}$$

if this integral exists.

Theorem 4.1.3. *The hyperbolic area is invariant under all transformations in $\mathrm{PSL}(2, \mathbb{R})$, that is, if $A \subset \mathbb{H}$ is such that $\mu(A)$ exists, and $T \in \mathrm{PSL}(2, \mathbb{R})$, then $\mu(T(A)) = \mu(A)$.*

We are going to center our attention in some special subgroups of $\mathrm{PSL}(2, \mathbb{R})$.

Definition 4.1.4. *A subgroup Γ of $\mathrm{Isom}(\mathbb{H})$ is called **discrete** if the induced topology on Γ is a discrete topology, i.e. if Γ is a discrete set in the topological space $\mathrm{Isom}(\mathbb{H})$.*

Definition 4.1.5. *A discrete subgroup of $\mathrm{Isom}(\mathbb{H})$ is called a **Fuchsian group** if it consists of orientation-preserving transformations, in other words, a Fuchsian group is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$.*

In a more general setting, if X denotes a metric space and G a group of homeomorphisms of X:

Definition 4.1.6. *We say that a group G **acts properly discontinuously** on X if the G-orbit of any point $x \in X$ is locally finite.*

It is clear by the definition that a group G acts properly discontinuously on X if and only if each orbit is discrete and the order of the stabilizer of each point is finite. An equivalent definition would be:

Theorem 4.1.7. *The group G acts properly discontinuously on X if each point $x \in X$ has a neighborhood V such that $T(V) \cap V \neq \emptyset$ for only finitely many $T \in G$.*

With this new concept we have the following characterization:

Theorem 4.1.8. *Let Γ be a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Then Γ is a Fuchsian group if and only if Γ acts properly discontinuously on $\mathbb{H}$.*

Corollary 4.1.9. *Let Γ be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Then Γ acts properly discontinuously on $\mathbb{H}$ if and only if for all $z \in \mathbb{H}$, $\mathcal{O}_\Gamma(z)$, the Γ-orbit of z, is a discrete subset of $\mathbb{H}$.*

We are also interested in the points fixed by the action of Γ on $\mathbb{H}$.

Lemma 4.1.10. *If $ST = TS$ then S maps the fixed-point set of T to itself.*

Theorem 4.1.11. *Let Γ be a Fuchsian group all of whose non-identity elements have the same fixed-point set. Then Γ is cyclic.*

Moreover, if Γ contains hyperbolic or parabolic elements it is infinite cyclic. If it contains elliptic elements it is finite.

Notice that in the previous theorem, if Γ is cyclic, then all its elements are of the same kind, since all of them are powers of one fixed element.

4.2   FUNDAMENTAL REGIONS

The space $\mathbb{H}$, with respect to the action of a Fuchsian group, can be divided in pieces as we explain during this section.

Definition 4.2.1. *Let X be a metric space, and G be a group of homeomorphisms acting properly discontinuously on X. A closed region $F \subset X$ is defined to be a **fundamental region** for G if*

$$\bigcup_{T \in G} T(F) = X$$

and

$$\mathring{F} \cap T(\mathring{F}) = \emptyset \text{ for all } T \in G \setminus \{Id\}.$$

*The family $\{T(F) \mid T \in G\}$ is called the **tessellation** of X.*

Theorem 4.2.2. *Let F_1 and F_2 be two fundamental regions for a Fuchsian group Γ, and $\mu(F_1) < \infty$. Suppose that the boundaries of F_1 and F_2 have zero hyperbolic area. Then $\mu(F_1) = \mu(F_2)$.*

Therefore, the area of a fundamental region (with nice boundary) is, if finite, a numerical invariant of the group Γ. Since the area on the quotient space $\mathbb{H}/\Gamma$ is induced by the hyperbolic area on $\mathbb{H}$, the hyperbolic area of $\mathbb{H}/\Gamma$, denoted by $\mu(\mathbb{H}/\Gamma)$, is well defined and equal to $\mu(F)$ for any fundamental region F.

Definition 4.2.3. *Let Γ be an arbitrary Fuchsian group and let $P \in \mathbb{H}$ be not fixed by any element of $\Gamma \setminus \{Id\}$. We define the **Dirichlet region** for Γ centered at P to be the set*

$$D_P(\Gamma) = \{z \in \mathbb{H} \mid \rho(z, P) \leq \rho(z, T(P)) \ \forall T \in \Gamma\}$$

where ρ denotes the hyperbolic metric.

By the invariance of the hyperbolic metric under the action of $\mathrm{PSL}(2, \mathbb{R})$, this region can also be define as

$$D_P(\Gamma) = \{z \in \mathbb{H} \mid \rho(z, P) \leq \rho(T(z), P) \ \forall T \in \Gamma\}.$$

Theorem 4.2.4. *If P is not fixed by any element of $\Gamma \setminus \{Id\}$, then $D_P(\Gamma)$ is a connected fundamental region for Γ.*

Dirichlet regions for Fuchsian groups can be quite complicated. They are bounded by geodesics in $\mathbb{H}$ and possibly by segments of the real axis. If two such geodesics intersect in $\mathbb{H}$, their point of intersection is called a vertex of the Dirichlet region. It can be shown that the vertices are isolated.

Definition 4.2.5. *A fundamental region, F, for a Fuchsian group Γ is called **locally finite** if the tessellation $\{T(F) \mid T \in \Gamma\}$ is locally finite.*

Theorem 4.2.6. *A Dirichlet region is locally finite.*

We call two points $u, v \in \mathbb{H}$ congruent if they belong to the same Γ-orbit. First, notice that two such points in a fundamental region F may be congruent only if they belong to the boundary of F.

Suppose now that F is a Dirichlet region for Γ, and let us consider congruent vertices of F. The congruence is an equivalence relation on the vertices of F and the equivalence classes are called **cycles**. If u is fixed by an elliptic element S, then $v = Tu$ is fixed by the elliptic element TST^{-1}. Thus, if one vertex of the cycle is fixed by an elliptic element, then all vertices of that cycle are fixed by conjugate elliptic elements.

Such a cycle is called an **elliptic cycle** and the vertices are called **elliptic vertices**. The number of elliptic cycles is equal to the number of non-congruent elliptic points in F.

Since the Dirichlet region F is a fundamental region, it is clear that every point $w \in \mathbb{H}$ fixed by an elliptic element S' of Γ lies on the boundary of $T(F)$ for some $T \in \Gamma$. Hence, $u = T^{-1}(w)$ lies on the

boundary of F and is fixed by the elliptic element $S = T^{-1}S'T$. If the order of S is at least 3, then u must be a vertex of F. But if S has order 2, its fixed point might lie in the interior of a side of F. We will include such elliptic fixed points as vertices of F.

If a point in $\mathbb{H}$ has a non-trivial stabilizer in Γ, this stabilizer is a finite cyclic subgroup of Γ (it is generated by an elliptic element), and moreover, by Lemma 4.1.10 it is a maximal finite cyclic subgroup of Γ. Conversely, every maximal finite cyclic subgroup of Γ is the stabilizer of a single point in $\mathbb{H}$. We can summarize the above as:

Theorem 4.2.7. *There is a one-to-one correspondence between the elliptic cycles of F and the conjugacy classes of non-trivial maximal finite cyclic subgroups of Γ.*

Definition 4.2.8. *The orders of non-conjugate maximal finite cyclic subgroups of Γ are called the **periods** of Γ.*

Each period is repeated as many times as there are conjugacy classes of maximal finite cyclic subgroups of that order.

A parabolic element can be considered as an elliptic element of infinite order; it has a unique fixed point in $\mathbb{R} \cup \infty$.

Definition 4.2.9. *A Fuchsian group Γ is called a **surface Fuchsian group** if it contains no non-trivial element of finite order.*

4.2.1 Connection with Riemann surfaces

Let Γ be a Fuchsian group acting on the upper half-plane $\mathbb{H}$ and let F be a fundamental region for this action, with $\mu(\mathbb{H}/\Gamma) < \infty$ (that is, $\mu(F) < \infty$). The group Γ induces a natural projection (continuous and open) $\pi : \mathbb{H} \to \mathbb{H}/\Gamma$, and the points of $\mathbb{H}/\Gamma$ are the Γ-orbits.

The restriction of π to F identifies the congruent points of F that necessarily belong to its boundary, and makes F/Γ an oriented Riemann surface with possibly some marked points (which correspond to the elliptic cycles of F) and cusps (which correspond to non-congruent vertices at infinity of F), also known as orbifold. Its topological type is determined by the number of cusps and by its genus.

If F is locally finite, the quotient space $\mathbb{H}/\Gamma$ is homeomorphic to F/Γ, hence, by choosing F to be a Dirichlet region we can find the topological type of $\mathbb{H}/\Gamma$.

If Γ has a compact Dirichlet region F, then F has finitely many sides, and the quotient space $\mathbb{H}/\Gamma$ is compact.

Definition 4.2.10. *A Fuchsian group is called **cocompact** if the quotient space $\mathbb{H}/\Gamma$ is compact.*

Therefore, if Γ is cocompact, the quotient is a compact Riemann surface. We have that

Theorem 4.2.11. *The quotient space of a Fuchsian group Γ, $\mathbb{H}/\Gamma$ is compact if and only if any Dirichlet region for Γ is compact.*

Theorem 4.2.12. *A Fuchsian group Γ is cocompact if and only if $\mu(\mathbb{H}/\Gamma) < \infty$ and Γ contains no parabolic elements.*

4.3 SIGNATURE OF A FUCHSIAN GROUP

Assume now that Γ has a compact fundamental region F. Then, F has finitely many sides, and hence finitely many vertices, finitely many elliptic cycles, and a finite number of periods, say $m_1, \ldots, m_r$. The quotient space $\mathbb{H}/\Gamma$ is a compact Riemann surface of genus g with exactly r marked points.

Definition 4.3.1. *In this case we say that the Fuchsian group Γ has **signature** $(g; m_1, \ldots, m_r)$. If $r = 0$, that is, there are no elliptic cycles, we will write $(g; -)$.*

Theorem 4.3.2. *Let Γ have signature $(g; m_1, \ldots, m_r)$. Then*

$$\mu(\mathbb{H}/\Gamma) = 2\pi \left((2g - 2) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right).$$

Theorem 4.3.3 (Poincaré's Theorem). *Let $g \geq 0$, $r \geq 0$ be integers, and for $r > 0$ and $1 \leq i \leq r$, let $m_i \geq 2$ be also integers. If*

$$2g - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) > 0$$

then there exists a Fuchsian group with signature $(g; m_1, \ldots, m_r)$.

From this theorem we can deduce the following result on uniformization of compact Riemann surfaces.

Corollary 4.3.4. *For any integer $g > 1$ there exists a Fuchsian group acting on $\mathbb{H}$ without fixed points such that $\mathbb{H}/\Gamma$ has genus g.*

Looking at the construction in the proof of Poincaré's Theorem, we can also extract information about the relations between the generators of the group Γ. Let x_k be an elliptic isometry fixing an elliptic vertex with order m_k, for $k = 1, \ldots r$, and let a_i, b_i be hyperbolic isometries mapping pairwise the (non-elliptic) sides of our fundamental region. Then we have that

$$x_1^{m_1} = \cdots = x_r^{m_r} = Id$$

and by the construction in Poincaré's Theorem

$$\prod_{j=1}^{g} [a_i, b_i] \prod_{i=1}^{r} x_i = Id$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

Thus, the presentation of a group Γ with signature $(g; m_1, \ldots, m_r)$ is

$$\langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_r \mid x_i^{m_i} = 1, \prod_{j=1}^{g} [a_j, b_j] \prod_{i=1}^{r} x_i = 1 \rangle. \qquad (4.1)$$

If $(2g-2) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) \leq 0$ there does not exist a Fuchsian group with signature $(g; m_1, \ldots, m_r)$.

Finally, we remind the general uniformization theorem for compact Riemann surfaces.

Theorem 4.3.5. *([FK80]) Every compact Riemann surface C of genus g is conformally equivalent to a quotient $\mathcal{U}/G$ with $\mathcal{U}$ simply connected and G a group of Möbius transformations acting discontinuously and fixed point free. In particular,*

1. *if $g = 0$ then $\mathcal{U} = \mathbb{P}^1$,*

2. *if $g = 1$ then $\mathcal{U} = \mathbb{C}$,*

3. *if $g > 1$ then $\mathcal{U} = \mathbb{H}$ and G is a Fuchsian group which contains only hyperbolic elements.*

Moreover, $G = \pi_1(C)$ and $\mathcal{U}$ is the universal covering space of C.

4.4 GROUP ACTIONS ON CURVES

In this section we use the language and results previously introduced to study the existence of compact Riemann surfaces with the action of a given group G with prescribed fixed points structure.

Let Γ be a Fuchsian group with periods $m_1, \ldots, m_r$ and with compact quotient space of genus g. If N is a torsion-free normal subgroup of finite index in Γ, then Γ/N acts as a group of automorphisms of the compact Riemann surface $\mathbb{H}/N$.

Theorem 4.4.1. *An epimorphism $\theta : \Gamma \to G$ has as kernel a surface group (see Definition 4.2.9) if and only if θ preserves the (finite) orders of the elements of Γ. Such a morphism is called a **smooth epimorphism**.*

Therefore, in the situation of the theorem, G acts as an automorphism group of the compact Riemann surface $\mathbb{H}/Ker(\theta)$.

Conversely, suppose that C is a compact Riemann surface and G a group of automorphisms of C. Then, there exists a Fuchsian group, Γ, and a torsion-free normal subgroup, N, of finite index in Γ such that there exists an isomorphism ϕ and a biholomorphism f compatible with the diagram

$$
\begin{array}{ccc}
\Gamma/N \times \mathbb{H}/N & \longrightarrow & \mathbb{H}/N \\
\downarrow{\scriptstyle(\phi,f)} & & \downarrow{\scriptstyle f} \\
G \times C & \longrightarrow & C
\end{array}
$$

where the horizontal arrows give the group action.

The group N is the group of covering transformations of the universal covering space $\mathbb{H}$ of C, and therefore, it is isomorphic to the fundamental group of C. The group Γ is obtained by lifting all the maps in G in all possible ways to obtain maps of the universal covering space into itself.

To summarize,

Theorem 4.4.2. *The group G is a group of automorphisms of the curve $C = \mathbb{H}/N$, with N a surface Fuchsian group, if and only if $G \cong \Gamma/N$ for some Fuchsian group Γ, with $C/G \cong \mathbb{H}/\Gamma$.*

There exists a holomorphic map π and a homomorphism $\rho = \pi^*$

compatible with the diagram

$$
\begin{array}{ccc}
\Gamma \times \mathbb{H} & \longrightarrow & \mathbb{H} \\
\downarrow {\scriptstyle (\rho,\pi)} & & \downarrow {\scriptstyle \pi} \\
G \times C & \longrightarrow & C.
\end{array}
$$

Proposition 4.4.3. *The homomorphism ρ maps the stabilizer Γ_z isomorphically onto $G_{\pi(z)}$.*

Let $x_i \in \Gamma$, for $1 \le i \le r$ be the elliptic generators of the Fuchsian group Γ in a given presentation (4.1). Each x_i fixes a single point, say z_i. The set of points of $\mathbb{H}$ with non-trivial stabilizer is the union of orbits $\mathcal{O}_\Gamma(z_1) \cup \cdots \cup \mathcal{O}_\Gamma(z_r)$ and the stabilizer of $\tilde{\gamma} z_i$ is $\tilde{\gamma} \langle x_i \rangle \tilde{\gamma}^{-1}$.

It follows from Proposition 4.4.3 that the points of C with non-trivial stabilizer by the action of G are those of the form $\gamma \pi(z_i)$ with $\gamma \in G$, and the stabilizer of such a point is $\gamma \langle \rho(x_i) \rangle \gamma^{-1}$, which is a cyclic group of order m_i.

Therefore, the signature of Γ gives us the branching type (see Section 3.1) of the morphism $C = \mathbb{H}/N \to C' = \mathbb{H}/\Gamma$. If $g = g(C)$ and $g' = g(C')$, then by the Riemann-Hurwitz formula we obtain that

$$
2g - 2 = |G| \left(2g' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right).
$$

And moreover, the epimorphism $\Gamma \to G$ gives us the structure of the monodromy. We remark that, in fact, we have an isomorphism between the Fuchsian group Γ and the orbifold fundamental group associated to the Galois morphism $C \to C'$ (see Section 3.1).

To summarize, we have that

Theorem 4.4.4 (Riemann's Existence Theorem). *The group G acts on a curve of genus g, with branching type $(g'; m_1, \ldots, m_r)$ if and only if the Riemann-Hurwitz formula is satisfied and G has a $(g'; m_1, \ldots, m_r)$ generating vector.*

Where a $(g'; m_1, \ldots, m_r)$ generating vector (or G-Hurwitz vector) is a $2g' + r$-tuple

$$
(a_1, b_1, \ldots, a_{g'}, b_{g'}; c_1, \ldots, c_r)
$$

of elements of G generating the group and such that $o(c_i) = m_i$ and $\prod_{j=1}^{g'} [a_i, b_i] \prod_{i=1}^{r} c_i = 1$. We call this last condition the **product one condition**.

We remark that Riemann's Existence theorem is not a constructive result. It states the existence of such a curve, nevertheless it gives no further information about it.

Now, we compute the fixed point number for any $\gamma \in G$ by considering each G-orbit separately. For brevity we write $\alpha_1, \ldots, \alpha_r$ instead of $\rho(x_1), \ldots, \rho(x_r)$ and $P_1, \ldots, P_r$ instead of $\pi(z_1), \ldots, \pi(z_r)$.

An element $\gamma \in G$ has a fixed point in the orbit $\mathcal{O}_G(P_i)$ if and only if $\langle \gamma \rangle$ is conjugate to a subgroup of $\langle \alpha_i \rangle$. Then, of course, the order d of γ must be a factor of m_i and $\langle \gamma \rangle$ will be conjugate to the unique subgroup of $\langle \alpha_i \rangle$ of order d. We define $\varepsilon_i(\gamma) = 1$ if $\langle \gamma \rangle$ is conjugate to this subgroup, $\varepsilon_i(\gamma) = 0$ otherwise.

Assume next that $\varepsilon_i(\gamma) = 1$. The number of fixed points of γ and $\beta\gamma\beta^{-1}$, $\beta \in G$, in the orbit $\mathcal{O}_G(P_i)$ are the same, so we may assume that $\langle \gamma \rangle$ is the unique cyclic subgroup $\langle \alpha_i^{m_i/d} \rangle$ of order d in $\langle \alpha_i \rangle$.

Then, for any $\beta \in G$, γ fixes βP_i if and only if $\beta^{-1}\gamma\beta$ fixes P_i, that is, $\beta^{-1}\langle \gamma \rangle \beta = \langle \gamma \rangle$. Equivalently β belongs to the normalizer (see Definition A.0.11) $N_G(\langle \gamma \rangle)$ of the cyclic subgroup $\langle \gamma \rangle$.

For each $P \in \mathcal{O}_G(P_i)$ there are precisely m_i elements $\beta \in G$ such that $P = \beta P$. Thus, when we enumerate the β in $N_G(\langle \gamma \rangle)$, we count each fixed point m_i times, and the number of γ-fixed points in the ith orbit is

$$|N_G(\langle \gamma \rangle)|/m_i.$$

Summing from $i = 0$ to r we obtain the following theorem:

Theorem 4.4.5. *([Mac73]) Let C be a compact Riemann surface and G a group of its automorphisms, $N = \pi_1(C)$, $\mathbb{H}$ the universal covering space of C ($\mathbb{H}/N = C$) and Γ a Fuchsian group such that $\Gamma/N = G$.*

Let $x_1, \ldots, x_r$ with orders $m_1, \ldots, m_r$ respectively, be generators of maximal finite cyclic subgroups of Γ, including exactly one for each conjugacy class.

Let ρ denote the natural homomorphism of Γ on G. For $1 \neq \gamma \in G$ let $\varepsilon_i(\gamma)$ be 1 or 0 according as γ is or is not conjugate to a power of $\rho(x_i)$.

Then the number $\nu(\gamma)$ of points of C fixed by γ is given by the formula

$$\nu(\gamma) = |N_G(\langle \gamma \rangle)| \sum_{i=1}^{r} \frac{\varepsilon_i(\gamma)}{m_i}.$$

Remark 4.4.6. *Therefore, ρ determines which conjugation classes in G correspond to monodromies in the morphism $C \to C' = C/G$. That*

is, $\rho(x_i)$ acts locally around its fixed points as multiplication by $e^{\frac{2\pi i}{m_i}}$, giving ramification points with $e_P = m_i$.

Hence, giving a generating vector we determine the action of G on the curve C. We remark that different generating vectors can specify the same action of G on C. The action of $Aut(G)$ on a generating vector changes the vector itself, whereas the action defined remains the same. Thus, when we say that ρ determines the topological action, we consider ρ as the action itself, and not the particular generating vector used to define it.

Definition 4.4.7. *Given a Galois morphism $f : C \to C/G = \mathbb{P}^1$, we call **branching data** of f the set of conjugacy classes of the non trivial stabilizers G_{P_i} for $P_i \in C$ with $f(P_i) \in \mathrm{Branch}(f)$ and $f(P_i) \neq f(P_j)$ for $i \neq j$.*

Given a generating vector, we can find the associated branching data taking the conjugacy classes on G of the generators. And given the branching data we find a generating vector taking one element for each conjugacy class in such a way that they generate and their ordered product is one.

4.5 HURWITZ SPACES

We are interested not only on the existence of the action on a curve, but also on the subspace of the moduli space of curves with such an action. With this goal we consider the Hurwitz space associated to an action.

Definition 4.5.1. *Given a curve C with $G \xhookrightarrow{i_j} Aut(C)$, $j = 1, 2$. We say that the actions of G on C given by i_1 and i_2 are **equivalent** if there exists an automorphism $h : C \to C$ that commutes with both actions of the group G.*

When a curve has two equivalent actions of G, the quotient curves by both actions are isomorphic.

For example, the actions on a curve C defined by conjugate subgroups of $Aut(C)$ are equivalent.

As we have seen in the previous section, given a curve C and a group $G \subset Aut(C)$, the action of G on C can be described by a smooth epimorphism $\rho : \Gamma \to G$ with Γ a Fuchsian group. The conjugacy

classes of the images of the elliptic generators of Γ, together with the genus, g, of the curve determine the topological type of the action.

We observe that the genus of the quotient curve C/G is determined by the Riemann-Hurwitz formula and the topological type of the action.

Definition 4.5.2. *The Hurwitz space $\mathcal{H}(g; G, \rho)$ associated to a triple $(g; G, \rho)$ is the set of equivalence classes of morphisms $C \to C/G$ determined by an action of type ρ.*

The Hurwitz space $\mathcal{H}(g; G, \rho)$ admits two natural maps

$$
\begin{array}{ccc}
\mathcal{H}(g; G, \rho) & \xrightarrow{\;\phi\;} & \mathcal{M}_g \\
\downarrow{\scriptstyle\psi} & & \\
M_{g',r} & &
\end{array}
$$

where the horizontal map sends a morphism $C \to C/G$ to the class of C, and the vertical one sends it to C/G with the r branch points marked.

Theorem 4.5.3. *([Vö05]) The Hurwitz space $\mathcal{H}(g; G, \rho)$ carries a structure of quasi-projective variety such that ϕ and ψ are morphisms.*

Assume now that $g \geq 2$ and that $\mathcal{H}(g; G, \rho)$ is non empty. By the previous section on covering space theory, we have that the morphism ψ is surjective and finite. Furthermore, also ϕ is finite since the automorphism group of a curve of genus ≥ 2 is of finite degree. We have the following result:

Theorem 4.5.4. *([CLP11]) The Hurwitz space $\mathcal{H}(g; G, \rho)$ is a connected complex manifold of dimension $3(g' - 1) + r$, where g' is the genus of $C' = C/G$, and r is the cardinality of the branch locus of $C \to C/G$.*
The image $\mathcal{M}_{g;G,\rho}$ inside the moduli space $\mathcal{M}_g$ is an irreducible closed subset of the same dimension $3(g' - 1) + r$.

Moreover, when the dimension of $\mathcal{H}(g; G, \rho)$ is at least four we have that

Theorem 4.5.5. *([Vö05]) If $3(g' - 1) + r \geq 4$ then $\mathrm{Aut}(C) = G$ for each curve C representing a general point of $\mathcal{M}_{g;G,\rho}$. Thus, the morphism $\mathcal{H}(g; G, \rho) \to \mathcal{M}_{g;G,\rho}$ is birational.*

Let $D_n = \langle i, j \rangle$ be the dihedral group of order $2n$ generated by the involutions i and j. An important tool in this thesis is the study of morphisms $C \to C/D_n = \mathbb{P}^1$. Given a D_n-Hurwitz vector v, in [CLP11] they associate a tuple of positive integers to this vector, depending on the parity of n as follows:

- If $n = 2l+1$ is odd then $\upsilon(\mathrm{v}) = (k, k_1, \ldots, k_{n'})$ where k (respectively k_p) is the number of elements in v in the conjugacy class of i (respectively $(ij)^p$).

- If $n = 2l$ is even then $\upsilon(\mathrm{v}) = (k_i, k_j, , k_1, \ldots, k_{n'})$ where k_i (respectively k_j, k_p) is the number of elements in v in the conjugacy class of i (respectively j, $(ij)^p$).

Since the group $\mathrm{Aut}(D_n) = \mathbb{Z}/n \rtimes (\mathbb{Z}/n)^*$ acts diagonally on the set of Hurwitz vectors, we have an action of $\mathrm{Aut}(D_n)$ on the set of tuples $\{\upsilon(\mathrm{v}) \mid \mathrm{v} \text{ Hurwitz vector}\}$. The automorphism may change the conjugacy class of c_i (the element in the vector), but it does not change the conjugacy class of $\langle c_i \rangle$, the subgroup generated by it, and hence, the action is the same.

Definition 4.5.6. *The equivalence class of $\upsilon(v)$ is denoted $[\upsilon(v)]$ and for $C/D_n = \mathbb{P}^1$, it is the **numerical type** of the vector.*

In this specific case we know more about the structure of the associated Hurwitz space.

Theorem 4.5.7. *([CLP11]) The dihedral Galois morphisms $C \to \mathbb{P}^1$ of a fixed numerical type form an irreducible subvariety of the moduli space.*

$$\begin{array}{ccc} & \scriptstyle(e:1) & \\ D & \longrightarrow & B \\ \scriptstyle(d:1)\Big\downarrow & & \\ C & & \end{array}$$

Chapter Five

CURVES IN $C^{(2)}$

In this chapter we present a characterization of curves $\tilde{B}$ in $C^{(2)}$ from the existence of a curve D in a diagram of morphisms of smooth curves with certain properties. After that, we compute the self-intersection of the curve $\tilde{B}$ using information coming from the diagram. Finally, we consider the case of low genus curves and give a bound on its degree.

5.1 CHARACTERIZATION

To begin with, we introduce two concepts on diagrams of curves in order to characterize curves lying in $C^{(2)}$.

Definition 5.1.1. *We say that a diagram of morphisms of curves*

$$\begin{array}{ccc} & \scriptstyle(e:1) & \\ D & \longrightarrow & B \\ \scriptstyle(d:1)\Big\downarrow & & \\ C & & \end{array}$$

reduces *if there exist curves F and H such that there exists a diagram*

$$\begin{array}{ccc} & \scriptstyle(e:1) & \\ D & \longrightarrow & B \\ & \scriptstyle(k:1) & \scriptstyle(k:1) \\ \scriptstyle(d:1) & F \xrightarrow{\scriptstyle(e:1)} H & \\ & C & \end{array}$$

with $k > 1$, the upper square being a commutative diagram and the left vertical arrows giving a factorization of the original degree d morphism.

*When $k = d$ we will say that the diagram **completes**, and we will obtain a commutative diagram*

$$
\begin{array}{ccc}
D & \xrightarrow{(e:1)} & B \\
{\scriptstyle (d:1)}\downarrow & & \downarrow{\scriptstyle (d:1)} \\
C & \dashrightarrow[(e:1)] & H.
\end{array}
$$

Notice that when d is a prime number both definitions are equivalent.

Now, we are going to relate these diagrams of morphisms with the existence of curves in the symmetric square $C^{(2)}$. We remind that by Remark 1.2.3 given a degree two morphism $f : D \to B$ there is a curve isomorphic to B in $D^{(2)}$, given by the pairs of points in D with the same image by f. Hence, we can consider B embedded in $D^{(2)}$.

Lemma 5.1.2. *We consider a diagram of morphisms of smooth irreducible curves*

$$
\begin{array}{ccc}
D & \xrightarrow[(2:1)]{f} & B \\
{\scriptstyle (d:1)}\downarrow{\scriptstyle g} & & \\
C & &
\end{array}
\qquad .
$$

The morphism $g^{(2)}$ restricted to $B \subset D^{(2)}$ (with the immersion given by the fibers of f) has image the diagonal $\Delta_C \subset C^{(2)}$ if and only if the morphism g factorizes through the curve B by f.

Proof. Let i be the involution in D that defines f, that is, the change of sheet. Since $B = \{x + y \mid f(x) = f(y)\} = \{x + i(x)\} \subset D^{(2)}$, then $Im(g^{(2)}|_B) = \{g(x) + g(i(x))\}$. It is contained in the diagonal Δ_C if and only if $g(x) = g(i(x))$ for all $x \in D$, that is, if and only if g factorizes through B by f. $\qquad\square$

In the following theorem we relate the existence of a diagram that does not reduce with the existence of a curve in $C^{(2)}$ defined, in a certain sense, by it.

Theorem 5.1.3. *Assume that there exists a diagram of morphisms of smooth irreducible curves*

$$
\begin{array}{ccc}
D & \xrightarrow[(2:1)]{f} & B \\
{\scriptstyle (d:1)}\downarrow{\scriptstyle g} & & \\
C & &
\end{array}
$$

which does not reduce and such that the morphism g does not factorize through B by f. Then, $g^{(2)}$ gives a degree one map $B \to C^{(2)}$ with reduced image a curve $\tilde{B}$ of degree precisely d.

Proof. Consider a diagram as above

$$
\begin{array}{ccc}
D & \xrightarrow[(2:1)]{f} & B \\
{\scriptstyle (d:1)}\Big\downarrow {\scriptstyle g} & & \\
C & &
\end{array}
$$

and look at the induced morphism $D^{(2)} \xrightarrow{g^{(2)}} C^{(2)}$. By Remark 1.2.3 we have an immersion $B \subset D^{(2)}$ as the set of pairs of points in D with the same image by f. Let ι be the involution in D that interchanges the two points in each fiber of f. Then, we consider D inside $D \times D$ as $\pi_D^{-1}(B) \cong D$, that is, ordered pairs of points with the same image by f.

Let $\tilde{B} = g^{(2)}(B)_{red}$, the reduced image curve in $C^{(2)}$, and consider the map $B \xrightarrow{(k:1)} \tilde{B}$ induced by $g^{(2)}$. We want to see that $k = 1$.

Notice that by Lemma 5.1.2 we can assume that $\tilde{B}$ is not Δ_C. We know that $B \cdot D_P = 1$, hence,

$$
1 = g^{(2)}{}_*(B \cdot D_P) = g^{(2)}{}_*(B) \cdot (\frac{1}{d}C_P) \Rightarrow g^{(2)}{}_*(B) \cdot C_P = d.
$$

In addition, since the map $B \xrightarrow{(k:1)} \tilde{B}$ is $g^{(2)}|_B$, we have that $d = (k\tilde{B}) \cdot C_p$, and thus $\tilde{B} \cdot C_p = \frac{d}{k}$, that is, k divides d.

Assume by contradiction that $k > 1$.

Let F be the preimage of $\tilde{B}$ by the morphism $\pi_C : C \times C \to C^{(2)}$. Then $F \to \tilde{B}$ has degree two and thus we obtain a diagram

$$(5.1)$$

Where the exterior arrows form a commutative diagram, and hence, also the interior arrows give a commutative diagram. Thus, the morphism $D \to F$ has degree k. Now, the restriction to D of $g \times g$ followed by the projection onto one factor of $C \times C$ is precisely $g : D \to C$ by construction. That is, we obtain the diagram

$$
\begin{array}{ccc}
D & \xrightarrow{(2:1)} & B \\
\downarrow{\scriptstyle(k:1)} & & \downarrow{\scriptstyle(k:1)} \\
F & \xrightarrow{(2:1)} & \tilde{B} \\
\downarrow & & \\
C & &
\end{array}
$$

Hence, the original diagram reduces, contradicting our hypothesis.

Consequently, $k = 1$ and thus we deduce that the curve $\tilde{B}$ has normalization B.

Moreover, looking at diagram (5.1) we deduce that $D \overset{(1:1)}{\to} F$, that is, the preimage of $\tilde{B}$ by π_C has normalization D, and we will denote it by $\tilde{D}$. So we have:

$$
\begin{array}{ccc}
D \times D & \longrightarrow & D^{(2)} \\
\end{array}
\tag{5.2}
$$

$$
\begin{array}{ccc}
D & \xrightarrow{f} & B \\
\downarrow{\scriptstyle\approx} & & \downarrow{\scriptstyle\approx} \\
\tilde{D} & \longrightarrow & \tilde{B} \\
& & \\
C \times C & \xrightarrow{\pi_C} & C^{(2)} \\
\downarrow{\scriptstyle pr} & & \\
C & &
\end{array}
$$

where the red arrows show the original diagram. $\qquad\square$

Conversely, we have also a theorem in the opposite direction, relating the existence of curves in $C^{(2)}$ with the existence of diagrams which do not reduce.

Theorem 5.1.4. *Given an irreducible curve $\tilde{B}$ lying in $C^{(2)}$ with degree d, let B be its normalization, and assume that there are no non trivial morphisms $B \to C$. Then, there exists a smooth irreducible curve D and a diagram*

$$
\begin{array}{ccc}
D & \xrightarrow{(2:1)} & B \\
{\scriptstyle(d:1)}\big\downarrow & & \\
C & &
\end{array}
$$

which does not reduce.

Proof. First of all, we observe that $\tilde{B}$ is not the diagonal in $C^{(2)}$ because we are assuming that there are no morphisms from B to C.

Let $\tilde{D} = \pi_C^*(\tilde{B}) \in Div(C \times C)$ and D its normalization. We notice that with our hypothesis $\tilde{D}$ is irreducible. Indeed, otherwise, one of its components would have as normalization the curve B, because we have a $(2 : 1)$ morphism from $\tilde{D}$ to B, and since $\tilde{D} \subset C \times C$ we would obtain a non trivial morphism from B to C contradicting our hypothesis.

Now, we are going to compute the degree of $\tilde{D} \to C$, given by the projection onto one factor:

$$
\tilde{D} \cdot (C \times P + P \times C) = \pi_{C*}(\pi_C^*(\tilde{B}) \cdot \pi_C^*(C_P)) = \\
\tilde{B} \cdot \pi_{C*}\pi_C^*(C_P) = 2\tilde{B} \cdot C_P = 2d.
$$

And therefore, since $\tilde{D}$ is symmetric with respect to the involution $(x, y) \to (y, x)$ by construction, $\tilde{D} \cdot (C \times P) = d$, and so, the degree of the morphism on C is precisely d. In this way, we have a diagram

$$
\begin{array}{ccc}
\tilde{D} & \xrightarrow{(2:1)} & \tilde{B} \\
{\scriptstyle(d:1)}\big\downarrow & & \\
C & &
\end{array}
\qquad
\begin{array}{c}
\text{and taking their normalizations} \\
\text{we obtain a diagram of} \\
\text{morphisms of smooth curves}
\end{array}
\qquad
\begin{array}{ccc}
D & \xrightarrow{(2:1)} & B \\
{\scriptstyle(d:1)}\big\downarrow & & \\
C & & .
\end{array}
$$

We call $f : D \to B$ the map coming from $\pi_C|_{\tilde{D}}$ and $g : D \to C$ the map coming from the projection onto one factor of $C \times C$.

Let α be the degree one morphism induced in B by the immersion of $\tilde{B}$ in $C^{(2)}$, that is, $\alpha : B \to \tilde{B} \subset C^{(2)}$. Since we have $D \xrightarrow{(2:1)} B$, by Remark 1.2.3 we obtain an immersion of B in $D^{(2)}$ as pairs of points with the same image by this morphism.

Since $D \xrightarrow{(1:1)} \tilde{D} \subset C \times C$ we can consider that a general point in D is a pair (x, y) with $x, y \in C$. Moreover, since $D \to B$ is induced by $\pi_C|_{\tilde{D}}$,

a general fiber of $D \to B$ will be two points (x,y) and (y,x). Hence, we can write a general point of $B \subset D^{(2)}$ as $(x,y) + (y,x)$.

Now, we consider the restriction to $B \subset D^{(2)}$ of $g^{(2)}$. We have by construction that this morphism is precisely α (we send each element of the pair to its first factor, which is the same as symmetrize in a general point, and therefore the image is the original $\tilde{B}$). In particular, $g^{(2)}|_B$ is generically of degree one.

We are going to see that the diagram does not reduce by contradiction. Assume that the diagram reduces, that is, that there exist curves F and H, and a diagram

$$
\begin{array}{ccc}
D & \overset{f}{\longrightarrow} & B \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle r} \\
F & \overset{s}{\longrightarrow} & H \\
\downarrow{\scriptstyle l} & & \\
C & &
\end{array}
$$

as in Definition 5.1.1. Then, by Remark 1.2.3 there is a curve isomorphic to H inside $F^{(2)}$ which points are the fibers of the morphism s. Hence, we have

$$
\begin{array}{ccccc}
 & & & & \tilde{B} \\
 & & \overset{\alpha}{\nearrow} & & \uparrow \\
B & \hookrightarrow & D^{(2)} & \overset{g^{(2)}}{\longrightarrow} & C^{(2)} \\
\downarrow & & \downarrow{\scriptstyle h^{(2)}} & \nearrow{\scriptstyle l^{(2)}} & \\
H & \hookrightarrow & F^{(2)} & &
\end{array}
$$

By definition, the image of $B \subset D^{(2)}$ by $h^{(2)}$ is $H \subset F^{(2)}$, that is, the embedding of H in $F^{(2)}$ given by s, and we know that $l \circ h = g$ so $l^{(2)} \circ h^{(2)} = g^{(2)}$, hence

$$
g^{(2)}|_B : \; B \overset{h^{(2)}=r}{\longrightarrow} H \overset{l^{(2)}}{\longrightarrow} \tilde{B} \qquad {\scriptstyle (1:1)}
$$

thus r, as well as h, have degree one. Consequently, our diagram does not reduce (see Definition 5.1.1). $\qquad\square$

Putting these two theorems together we find a characterization of curves in the symmetric square $C^{(2)}$.

Corollary 5.1.5. *Let $\overline{B}$ be an irreducible curve with normalization B and such that there are no non-trivial morphisms $B \to C$. A morphism of degree one from the curve B to the surface $C^{(2)}$ exists, with image $\tilde{B}$ of degree d if, and only if, there exists a smooth irreducible curve D and a diagram*

$$
\begin{array}{ccc}
D & \xrightarrow{(2:1)} & B \\
{\scriptstyle (d:1)}\downarrow & & \\
C & &
\end{array}
$$

which does not reduce.

Remark 5.1.6. *If we consider the case $d = 1$ we recover the result of Lemma 1.2.2.*

5.2 SELF-INTERSECTION OF $\tilde{B} \subset C^{(2)}$

Let $\tilde{B} \subset C^{(2)}$ be a curve of degree d which immersion is given by a diagram

$$
\begin{array}{ccc}
D & \xrightarrow[f]{(2:1)} & B \\
{\scriptstyle (d:1)}\downarrow {\scriptstyle g} & & \\
C & &
\end{array}
$$

as in Theorems 5.1.3 and 5.1.4. In this section we compute the self-intersection, $\tilde{B}^2$, using the information given by the diagram.

First, we remind that by Lemma 1.2.1 the ramification degree of $f : D \to B$ equals $B \cdot \Delta_D$, considering $B \subset D^{(2)}$ as the set of pairs of points with the same image by f. So, by the Riemann-Hurwitz formula we obtain that

$$
\begin{aligned}
2g(D) - 2 &= 2(2g(B) - 2) + deg(\mathrm{Ram}(f)) \Rightarrow \\
B \cdot \tfrac{\Delta_D}{2} &= g(D) - 2g(B) + 1.
\end{aligned} \tag{5.3}
$$

Second, from the adjunction formula for $B \subset D^{(2)}$, we deduce that $2g(B) - 2 = B^2 + B \cdot K_{D^{(2)}}$. Moreover, from $K_{D^{(2)}} \equiv_{num} (2g(D) - 2)D_P - \tfrac{\Delta_D}{2}$, we deduce that

$$
2g(B) - 2 = B^2 + 2g(D) - 2 - B \cdot \frac{\Delta_D}{2}. \tag{5.4}
$$

By the equalities (5.3) and (5.4) we obtain that the self-intersection of $\underline{B \text{ inside } D^{(2)}}$ is
$$B^2 = 1 - g(D).$$

Now, the adjunction formula reads $2p_a(\tilde{B}) - 2 = \tilde{B}^2 + \tilde{B} \cdot K_{C^{(2)}}$, and since $K_{C^{(2)}} \equiv_{num} (2g(C) - 2)C_P - \frac{\Delta_C}{2}$, we deduce that

$$\tilde{B}^2 = 2p_a(\tilde{B}) - 2 - d(2g(C) - 2) + \tilde{B} \cdot \frac{\Delta_C}{2}. \tag{5.5}$$

Furthermore, by the projection formula we obtain that

$$\tilde{B} \cdot \Delta_C = g_*^{(2)}(B) \cdot \Delta_C = g_*^{(2)}(B \cdot g^{(2)*}\Delta_C) = B \cdot g^{(2)*}\Delta_C. \tag{5.6}$$

Therefore, it remains to compute $g^{(2)*}\Delta_C \cdot B$.

We claim that
$$g^{(2)*}\Delta_C = \Delta_D + 2R$$

with

$$R = \overline{\{x + y \mid g(x) = g(y)\} \setminus \Delta_D}. \tag{5.7}$$

Proof of the claim. Consider the commutative diagram

$$
\begin{array}{ccc}
D \times D & \xrightarrow{\pi_D} & D^{(2)} \\
\downarrow{\scriptstyle g \times g} & & \downarrow{\scriptstyle g^{(2)}} \\
C \times C & \xrightarrow{\pi_C} & C^{(2)}.
\end{array}
$$

As a set, the preimage of Δ_C by $g^{(2)}$ is clearly formed by the divisors Δ_D and R, that is, $g^{(2)*}\Delta_C = n\Delta_D + mR$ with $m, n \in \mathbb{Z}_{>0}$. We want to determine m and n.

We know that $\pi_C^*(\Delta_C)$ in $C \times C$ is two times the diagonal $\Delta_{C \times C}$. The divisor $(g \times g)^*(\Delta_{C \times C})$ is clearly the diagonal in $D \times D$ plus a divisor $R_0 = \overline{\{(x,y) \mid g(x) = g(y)\} \setminus \Delta_{D \times D}}$.

Thus, the preimage of $g^{(2)*}(\Delta_C)$ in $D \times D$ is exactly $2(\Delta_{D \times D} + R_0)$. Since π_D ramifies with degree two only on the diagonal $\Delta_{D \times D}$, we deduce that $n = 1$ and $m = 2$ as claimed. $\diamondsuit$

Notice, moreover, that we have $R_0 \to \Delta_{C \times C}$ with degree $d^2 - d$. Since R_0 is not on the ramified locus of $g \times g$, there are exactly $d^2 - d$ points in a general fiber of this map and hence, we deduce that R_0 is a reduced divisor, and thus also R is a reduced divisor.

Then, by (5.3), (5.6) and the previous claim we obtain that

$$\tilde{B} \cdot \Delta_C = B \cdot \Delta_D + 2B \cdot R = 2g(D) - 4g(B) + 2 + \sigma \tag{5.8}$$

with

$$\sigma := 2B \cdot R = \pi_{D*}(\pi_D^* B) \cdot R = \pi_{D*}(D) \cdot R = $$
$$\pi_{D*}(D \cdot \pi_D^*(R)) = D \cdot \pi_D^*(R) \tag{5.9}$$

where the divisor $D \subset D \times D$ is the set of ordered pairs of points with the same image by the morphism f. Hence, in a naive sense, we can say that σ counts how the the fibers of f and g meet inside $D \times D$.

We notice that when we intersect $g^{(2)*}(\Delta_C)$ with the diagonal Δ_D we obtain that

$$\Delta_D \cdot g^{(2)*}(\Delta_C) = \Delta_D^2 + 2\Delta_D \cdot R = 4 - 4g(D) + 2\Delta_D \cdot R.$$

We push-forward this intersection product to $C^{(2)}$ by $g_*^{(2)}$, and from the projection formula we deduce that

$$g_*^{(2)}(\Delta_D \cdot g^{(2)*}(\Delta_C)) = g_*^{(2)}(\Delta_D) \cdot \Delta_C = d\Delta_C \cdot \Delta_C = d(4 - 4g(C)).$$

Moreover, we have that

$$4 - 4g(D) + 2\Delta_D \cdot R = d(4 - 4g(C)).$$

Thus, by the Riemann-Hurwitz formula for the morphism g we obtain the equality $\Delta_D \cdot R = deg(\mathrm{Ram}(g))$. We remind that we had already seen this for $d = 3$ in Example 1.2.6.

Finally, from formulas (5.5) and (5.8) we deduce that

$$\begin{aligned} \tilde{B}^2 &= 2p_a(\tilde{B}) - 2 - d(2g(C) - 2) + (g(D) - 2g(B) + 1 + \tfrac{\sigma}{2}) \\ &= g(D) - 1 - d(2g(C) - 2) + 2(p_a(\tilde{B}) - g(B)) + \tfrac{\sigma}{2}. \end{aligned} \tag{5.10}$$

Furthermore, by the adjunction formula for $\tilde{D} \subset C \times C$, we obtain that

$$\begin{aligned} p_a(\tilde{D}) &= 1 + \tfrac{1}{2}(\tilde{D}^2 + \tilde{D} \cdot K_{C \times C}) \\ &= 1 + \tfrac{1}{2}(2\tilde{B}^2 + \tilde{D} \cdot ((2g - 2)(C \times P + P \times C))) \\ &= 1 + \tfrac{1}{2}(2\tilde{B}^2 + (2g - 2)2d) = 1 + \tilde{B}^2 + d(2g - 2). \end{aligned}$$

Where the second equality is consequence of

$$\tilde{D}^2 = (\pi_C^* \tilde{B})^2 = \pi_C^*(\tilde{B}^2) = deg\pi_C \; \tilde{B}^2 = 2\tilde{B}^2.$$

Therefore, we have the following formula

$$\begin{aligned} \tilde{B}^2 &= p_a(\tilde{D}) - 1 - d(2g(C) - 2) \\ &= g(D) - 1 - d(2g(C) - 2) + (p_a(\tilde{D}) - g(D)). \end{aligned} \tag{5.11}$$

Moreover, from (5.10) and (5.11) we deduce that

$$p_a(\tilde{D}) - g(D) = 2(p_a(\tilde{B}) - g(B)) + \frac{\sigma}{2}. \tag{5.12}$$

Summarizing, we have seen that

Lemma 5.2.1. *Let $\tilde{B} \subset C^{(2)}$ be a curve given by a non completing diagram*

$$D \xrightarrow[f]{(2:1)} B$$
$$\Big\downarrow{}^{(d:1)}{}_{g}$$
$$C \qquad .$$

Then,

$$\begin{aligned}
\tilde{B}^2 &= g(D) - 1 - d(2g(C) - 2) + 2(p_a(\tilde{B}) - g(B)) + \tfrac{\sigma}{2} \\
&= g(D) - 1 - d(2g(C) - 2) + (p_a(\tilde{D}) - g(D)).
\end{aligned}$$

If $\tilde{D}$ is a smooth curve, the formula in Lemma 5.2.1 becomes much easier:

$$\tilde{B}^2 = g(D) - 1 - d(2g(C) - 2).$$

Notice that, if $\tilde{D}$ is smooth, then by (5.12) also $\tilde{B}$ will be smooth, but the convers is not true.

5.3 CURVES IN $C^{(2)}$ WITH LOW GENUS

In this section we consider curves $\tilde{B} \subset C^{(2)}$ such that

$$q(C^{(2)}) = g(C) = g < p_a(\tilde{B}) < 2g(C) - 1 \qquad (5.13)$$

with positive self-intersection.

We remind some known results about diagrams of curves that we will use to give a bound on the degree of possible such curves.

Theorem 5.3.1 (Castelnuovo-Severi)**.** *([Acc94]) Let B, C and D be curves with genera $g(B) = b$, $g(C) = g$ and $g(D) = h$ such that they lay in a diagram*

$$D \xrightarrow{(n_1:1)} B$$
$$\Big\downarrow{}^{(n_2:1)}$$
$$C$$

which is simple, that is, it does not exist a curve F such that the previous diagram factorizes via F in the following way:

$$D$$
$$\searrow$$
$$\qquad F \longrightarrow B$$
$$\qquad \downarrow$$
$$\qquad C \qquad .$$

Then,

$$h \leq n_1 b + n_2 g + (n_1 - 1)(n_2 - 1).$$

We are interested in the case $n_1 = 2$ and $n_2 = d$, hence we read from that theorem

$$g(D) \leq 2g(B) + dg(C) + d - 1.$$

Thus, we have a necessary condition for given curves C, B and D to give a diagram as in Theorem 5.1.3. Notice that since $n_1 = 2$, our hypothesis of the non existence of morphisms $B \to C$ implies that the diagram is simple.

Another result concerning diagrams of curves is the following theorem by Accola in [Acc06].

Theorem 5.3.2. *Assume that there exists a commutative diagram*

$$
\begin{array}{ccc}
D & \xrightarrow{(n_1:1)} & B \\
{\scriptstyle (n_2:1)} \downarrow & & \downarrow {\scriptstyle (n_2:1)} \\
C & \xrightarrow{(n_1:1)} & F
\end{array}
$$

with $g(B) = b$, $g(C) = g$, $g(D) = h$ and $g(F) = l$, then

$$h + n_1 n_2 l \leq n_1 b + n_2 g + (n_1 - 1)(n_2 - 1).$$

Therefore, given a diagram of curves we can use this result to determine if such a F can exist. In the case $n_1 = 2$ and n_2 a prime number, when the inequality is not satisfied, the diagram does not complete and hence by Theorem 5.1.3 B lays in $C^{(2)}$ (with possibly some singularities).

By Lemma 5.2.1 and Castelnuovo-Severi Theorem (Theorem 5.3.1) we get that

$$g(D) = \tilde{B}^2 + 1 + 2d(g(C) - 1) - (p_a(\tilde{D}) - g(D)) \leq 2g(B) + dg(C) + d - 1$$
$$\Leftrightarrow \tilde{B}^2 \leq 2g(B) + d(3 - g) - 2 + (p_a(\tilde{D}) - g(D)).$$

Since are assuming that the condition (5.13) is satisfied, necessarily $g(B) \leq p_a(\tilde{B}) \leq 2g - 2$, so we obtain from this inequality that $d(g - 3) \leq 4g - 6 - \tilde{B}^2 + (p_a(\tilde{D}) - g(D))$. Consequently, for $g \geq 4$ we have that

$$d \leq \frac{4g - 6 - \tilde{B}^2 + (p_a(\tilde{D}) - g(D))}{g - 3}.$$

Hence, for a fixed g, we have a relation between the self-intersection of $\tilde{B}$, its degree and the singularities of $\tilde{D}$.

If we assume $\tilde{D}$ smooth, that is, $p_a(\tilde{D}) - g(D) = 0$, since we are considering the case $\tilde{B}^2 \geq 1$, we deduce that $d \leq \frac{4g-7}{g-3}$. Thus, for $g \geq 9$ a curve of low genus and positive self-intersection should have degree at most 4.

This inequality motivates the study of curves in $C^{(2)}$ with low degree to look for such a curve.

Chapter Six

CONSTRUCTION OF CURVES

In this section we introduce a method to construct diagrams that do not complete (see Definition 5.1.1) using the action of a finite group on a curve (see appendix for generalities). Therefore, by Theorem 5.1.3, it is also a method to find curves of certain degrees in square symmetric products of curves. Next, we use this method to construct curves of degrees two and three. Furthermore, we study their singularities and give formulas to compute their self-intersection from the properties of the action on the curve D defining the diagram. We keep the notations introduced in the previous chapter.

Proposition 6.0.3. *Let D be a projective smooth irreducible curve with the action of a finite group G. Let $\alpha, \beta \in G$ with $o(\alpha) = d \geq 2$ and $o(\beta) = e \geq 2$. Consider the diagram*

$$
\begin{array}{ccc}
D & \xrightarrow{(e:1)} & D/\langle\beta\rangle = B \\
{\scriptstyle(d:1)}\Big\downarrow & & \\
D/\langle\alpha\rangle = C & &
\end{array}
\qquad .
$$

Then,

1. *If the order of $\langle\alpha,\beta\rangle$ equals $e \cdot d$ then the diagram completes.*

2. *If the order of $\langle\alpha,\beta\rangle$ is strictly greater than $e \cdot d$ then the diagram does not complete.*

Observation 6.0.4. *If the order of $\langle\alpha,\beta\rangle$ is strictly less than $e \cdot d$ then anything can happen. For instance, if $\beta = \alpha^k$, we can close the diagram*

as

$$D \xrightarrow{(e:1)} D/\langle \alpha^k \rangle = B$$
$$\downarrow^{(d:1)} \qquad \swarrow_{(d/e:1)}$$
$$D/\langle \alpha \rangle = C$$

.

In that case, the diagram completes if and only if C covers a curve H with degree e, because considering the composition of the projection of B onto C with the morphism to H we obtain the completed diagram:

$$D \xrightarrow{(e:1)} D/\langle \alpha^k \rangle = B$$
$$\downarrow^{(d:1)} \qquad \swarrow_{(d/e:1)} \qquad \downarrow$$
$$D/\langle \alpha \rangle = C \xrightarrow{(e:1)} H.$$

Proof of the Proposition.   1. Assume that $|\langle \alpha, \beta \rangle| = e \cdot d$. Let F be the quotient of D by the action of $\langle \alpha, \beta \rangle$. We get a diagram

$$D \xrightarrow{(e:1)} B$$
$$\downarrow^{(d:1)} \qquad \searrow^{(ed:1)}$$
$$C \qquad D/\langle \alpha, \beta \rangle = F.$$

Then, we can define morphisms from B and C to F in such a way that the diagram completes, because both B and C are quotients of D by subgroups of $\langle \alpha, \beta \rangle$.

2. Assume that $|\langle \alpha, \beta \rangle| > e \cdot d$. By contradiction we assume that the diagram completes. That is, there exists a curve H giving a commutative diagram

$$D \xrightarrow{(e:1)} B$$
$$\downarrow^{(d:1)} \quad \searrow^{(ed:1)} \quad \downarrow^{(d:1)}$$
$$C \xrightarrow{(e:1)} H.$$

Hence, the automorphisms α and β act on the fibers of $D \xrightarrow{(ed:1)} H$, and so the group $\langle \alpha, \beta \rangle$ acts on these fibers.

Therefore, the orbit of a general point of D by the action of $\langle \alpha, \beta \rangle$ must be contained in a fiber of $D \xrightarrow{(ed:1)} H$, but the cardinality of the first is strictly greater than the degree of the second, so this inclusion is not possible, and consequently such a curve H cannot exist. $\qquad \square$

6.1 DIHEDRAL CASE

In the next chapter we are going to study the self-intersection of curves in square symmetric products with degree 2. Regarding Theorem 5.1.3, the main tool will be to study diagrams which do not complete of the form

$$
\begin{array}{ccc}
D & \xrightarrow{(2:1)} & B \\
{\scriptstyle (2:1)}\big\downarrow & & \\
C & &
\end{array}
$$

Hence, in D there are two involutions (the changes of sheet) that by Proposition 6.0.3 generate a group of order at least five, which by Theorem A.2.3 will be a dihedral group. In this section we study the immersion of $\tilde{B}$ in $C^{(2)}$ given by this diagram, that is, how it intersects the diagonal, its number of singularities and which kind of singularities they are. Moreover, we study the singularities of $\tilde{D} = \pi_C^*(\tilde{B})$.

Let D be a curve with two involutions, i and j, that generate a dihedral group of order $2n > 4$. Using Proposition 6.0.3, we obtain a diagram that does not complete

$$
\begin{array}{ccc}
D & \xrightarrow{\quad f \quad} & B = D/\langle i \rangle \\
{\scriptstyle g}\big\downarrow & & \\
C = D/\langle j \rangle & &
\end{array}
$$

with $deg(g) = deg(f) = 2$.

Then, by Theorem 5.1.3, there is a curve $\tilde{B}$ in $C^{(2)}$ with normalization B and $\tilde{B} \cdot C_P = 2$. Its preimage by $\pi_C : C \times C \to C^{(2)}$ is $\tilde{D} := \pi_C^*(\tilde{B})$, which has normalization D. They lay in a diagram like (5.2). We make a detailed study of the action of D_n, and specially of the points fixed by certain automorphisms.

First of all, we compute the value of σ (see (5.7), (5.9), and Lemma 5.2.1), that is, the intersection of the graphs of f and g in $D \times D$. We remind that we consider $D \subset D \times D$ as the set of points $\{(x, f(x))\}$, that is, the graph of the morphism f.

Since g is the quotient by the action of the involution j, clearly

$$
\pi_D^*(R) = \overline{\{(x, y) \mid g(x) = g(y)\} \setminus \Delta_{D \times D}} = \{(x, j(x))\} = \Gamma_j
$$

hence, by Corollary 3.2.3 we obtain that

$$
\sigma = D \cdot \pi_D^*(R) = \Gamma_i \cdot \Gamma_j = \nu(ij). \tag{6.1}
$$

These are pairs of different points in $D \subset D \times D$ with the same image in $\tilde{D}$, so their images by $g \times g$ are singularities of $\tilde{D}$. Now, we are going to see that their images in $\tilde{B}$ are smooth points.

Lemma 6.1.1. *The image in $\tilde{B}$ by $\pi_C|_{\tilde{D}}$ of a point $(g \times g)(x, i(x))$ with $ij(x) = x$ is a smooth point where $\tilde{B}$ is tangent to the diagonal.*

Proof. To begin with, we are going to study the singularities on $\tilde{D}$ of the form $(g \times g)(x, i(x))$.

Consider the morphism $g \times g : D \times D \to C \times C$. It is Galois with group $\{1 \times 1,\ 1 \times j,\ j \times 1,\ j \times j\}$. The preimage of $\tilde{D}$ by $g \times g$ consists in four divisors:

$$
\begin{aligned}
D_0 &= \{(x, i(x))\} = (1 \times 1)(D), \\
D_1 &= \{(x, ji(x))\} = (1 \times j)(D), \\
D_2 &= \{(j(x), i(x))\} = (j \times 1)(D) \quad \text{and} \\
D_3 &= \{(j(x), ji(x))\} = (j \times j)(D).
\end{aligned}
\tag{6.2}
$$

The points $(x, i(x))$ and $(i(x), x)$ with $ij(x) = x$ are two intersections of the curves D_0 and D_3 with the same image by $g \times g$:

$$
\begin{array}{llll}
D_0 \ni & (x, i(x)) & = & (j(j(x)), j(i(j(x)))) \in D_3 \quad \text{and} \\
D_0 \ni & (i(x), x) = (j(x), i(j(x))) & = & (j(x), j(i(j(x)))) \in D_3.
\end{array}
$$

Since $g \times g$ is not ramified in these points, if we see that the divisors D_0 and D_3 are transversal, as the morphism is a local homeomorphism, $\tilde{D}$ will be transversal on the image, and therefore it will have a node.

We notice that the automorphism $1 \times i$ is acting on the product $D \times D$, and that D_0 and D_3 being transversal is equivalent to the transformed divisors $1 \times i(D_0)$ and $1 \times i(D_3)$ being transversal. So, since

$$
\begin{aligned}
(1, i)(D_0) &= \{(x, x) \mid x \in D\} = \Delta_{D \times D}, \\
(1, i)(D_3) &= \{(j(y), iji(y))\} = \{(z, (ij)^2(z))\} = \Gamma_{(ij)^2},
\end{aligned}
$$

and the diagonal intersects always transversally the graph of an automorphism (see Proposition 3.2.2), we deduce that D_0 and D_3 are transversal.

Now, we are interested in the image by π_C of one of the previous points $(g(x), g(i(x))) = (g(x), g(x))$. Since it is over the diagonal, which is the ramification divisor of π_C, there are no other points with the same image. Moreover, the points $(x, i(x))$ and $(i(x), x)$ have the same

image by π_D, and hence $g(x) + g(x)$ has only one preimage in B, which is the normalization of $\tilde{B}$, so $\tilde{B}$ has only one branch in $g(x) + g(x)$.

Consequently, we have a nodal singularity $(g(x), g(x))$ in $\tilde{D} \subset C \times C$ which image in $\tilde{B} \subset C^{(2)}$ has a single branch. We want to see that this branch is smooth.

Let (z_1, z_2) be a system of local coordinates in $C \times C$ with both z_i a local coordinate in C around $g(x)$. Using them, π_C is written locally as $(z_1, z_2) \to (z_1 + z_2, z_1 z_2) = (z, t)$ with (z, t) local coordinates in $C^{(2)}$ centered in $g(x) + g(x)$.

The curve $\tilde{B} = \pi_C(\tilde{D})$ around $g(x) + g(x)$ has a local expression $p(z, t) = 0$ such as

$$p(z, t) = az + bt + cz^2 + dzt + et^2 + \cdots$$

Then, its preimage by π_C, the curve $\tilde{D}$, around $(g(x), g(x))$ is

$$p(z_1 + z_2, z_1 z_2) = a(z_1 + z_2) + bz_1 z_2 + c(z_1 + z_2)^2 + d(z_1 + z_2)z_1 z_2 + e(z_1 z_2)^2 + \cdots$$

Since we know that in $(g(x), g(x))$ we have a node, we deduce that $a = 0$. In the same way, $b \neq 0$ because otherwise we would have $c(z_1 + z_2)^2$, that would be a cusp and not a node. Therefore, the linear term in $p(z, t)$ is non zero, and hence $g(x) + g(x)$ is a smooth point.

Since the intersection multiplicity in $(g(x), g(x))$ of $\tilde{D}$ and $\Delta_{C \times C}$ is two, also the intersection multiplicity in $g(x) + g(x)$ of $\tilde{B}$ and Δ_C is two, and therefore it is a tangent point of these two curves. $\qquad \square$

Now, we are going to study the rest of singularities of $\tilde{B}$ and $\tilde{D}$.

Proposition 6.1.2.

$$|\operatorname{Sing} \tilde{B}| = \frac{1}{4} \left(\nu((ij)^2) - \nu(ij) \right).$$

Proof. First, we want to know when two points of B ($x + y$ and $z + t$) have the same image in $\tilde{B}$ by $g^{(2)}$:

$$
\begin{array}{ll}
B & = \{x + y \in D^{(2)} \mid f(x) = f(y)\} \qquad \begin{array}{l} x + y \\ z + t \end{array} \\
\downarrow & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\tilde{B} & = \{g(x) + g(y) \in C^{(2)} \mid f(x) = f(y)\} \quad P + Q
\end{array}
$$

where $i(x) = y$, $i(z) = t$ and we assume that $j(x) = z$ and $j(y) = t$. Then, we obtain that

$$ijij(x) = iji(z) = ij(t) = i(y) = x.$$

And in a similar way,

$$ijij(y) = y,$$
$$ijij(z) = z \quad \text{and}$$
$$ijij(t) = t.$$

So x, y, z and t are fixed points of $(ij)^2$. Conversely, we are going to take one point fixed by $(ij)^2$ and see when it gives a singularity in $\tilde{B}$.

Let x be a point fixed by $(ij)^2$. Since there is no cyclic subgroup of D_n containing i and $(ij)^2$, we deduce that x is not fixed by i (see Proposition 3.2.1 and Corollary A.2.9). Let $y = i(x) \neq x$. We want to study the fiber of $g^{(2)}$. We begin considering the image of y by the automorphism j. We have three possibilities:

- If $j(y) = x$ then $ij(x) = i(y) = x$ and $ij(y) = i(x) = y$. So x, y are two points fixed by ij, that by Lemma 6.1.1 give a smooth point in $\tilde{B}$ and one singularity in $\tilde{D}$.

- If $j(y) = y$ then $x = jiji(x) = jij(y) = ji(y) = j(x)$. So x is fixed by j, and hence, by Proposition 3.2.1, j and $(ij)^2$ lay in a cyclic group, which contradicts Corollary A.2.9. This case does not occur.

- If $j(y) = z \notin \{x, y\}$ then $i(z) = t \notin \{x, y\}$ and we deduce that $j(t) = ji(z) = jij(y) = jiji(x) = x$. So, we have two different points in B with the same image in $\tilde{B}$, that is, a singularity in $\tilde{B}$. Notice that x, y, z, t are four points fixed by $(ij)^2$, and not by ij. They are giving one singularity in $\tilde{B}$ and two different singularities in $\tilde{D}$.

Therefore, $|\mathrm{Sing}\, \tilde{B}| \geq \frac{1}{4} \left(\nu((ij)^2) - \nu(ij) \right)$. We are going to see that there are no more.

For a point in B outside the ramification locus of $g^{(2)}$, its image will be a singularity only if there is another point with the same image, because $g^{(2)}$ is a local homeomorphism. We have already studied this case in the above discussion. Hence, it remains to consider the ramification points of $g^{(2)}$.

In B there are two types of points where $g^{(2)}$ ramifies: those in R (see (5.7)) and those in a coordinate curve D_Q with Q a ramification point of g. We have seen in Lemma 6.1.1 that the image of a point in $B \cap R$ is always smooth, so it remains only to study those in D_Q for $Q \in \mathrm{Ram}(g)$.

To do this, we study the intersection of $\tilde{B}$ with a coordinate curve C_P, $P \in \mathrm{Branch}(g)$ (the curve C_P is the image of D_Q for a certain point

$Q \in \mathrm{Ram}(g)$) and we will see that in all points of $\tilde{B} \cap C_P$ the curve $\tilde{B}$ is smooth.

Let $P \in C$ be a point, we know that $2 = \tilde{B} \cdot C_P$. If $\exists\, \eta_1, \eta_2 \in \tilde{B} \cap C_P$, $\eta_1 \neq \eta_2$ then

$$2 = \tilde{B} \cdot C_P \geq mult_{\eta_1} C_P \cdot mult_{\eta_1} \tilde{B} + mult_{\eta_2} C_P \cdot mult_{\eta_2} \tilde{B} =$$
$$mult_{\eta_1} \tilde{B} + mult_{\eta_2} \tilde{B} \Rightarrow mult_{\eta_i} \tilde{B} = 1 \Rightarrow \tilde{B} \text{ smooth at } \eta_i.$$

If $\eta \in \tilde{B} \cap C_P$, then $2 \geq mult_\eta \tilde{B}$, so the multiplicity of a singular point will be at most 2.

We consider the set of points

$$C_P \cap \tilde{B} = \{P + Q \mid \exists\, x, y \in D \text{ with } g(x) = P,\ g(y) = Q \text{ and } f(x) = f(y)\}.$$

If $P \in \mathrm{Branch}(g)$ then $\exists\, x \in D$ such that $j(x) = x$ and $g(x) = P$. Take the point $y = i(x) \neq x$ because i and j have no common fixed point (see Proposition 3.2.1 and Corollary A.2.9), thus, we deduce that $g(y) \neq P$.

Hence, we obtain that $\tilde{B} \cap C_P = P + g(y)$ with multiplicity 2. To prove that this is a smooth point of $\tilde{B}$ we consider the other coordinate curve passing through this point, that is, now we consider $\tilde{B} \cap C_{g(y)}$.

If $j(y) = y$, since $i(x) = y$, then $ijij(y) = iji(y) = ij(x) = i(x) = y$ so y would be fixed by $(ij)^2$, but j and $(ij)^2$ have no common fixed points (see Proposition 3.2.1 and Corollary A.2.9). Hence, $j(y) = z \neq y$ and in particular, $g(y) = g(z)$. Let $t = i(z) \neq x, y, z$. We obtain that $\tilde{B} \cap C_{g(y)} = \{g(y) + P, g(y) + g(t)\}$, two smooth different points of $\tilde{B}$.

Therefore,

$$|\mathrm{Sing}\,\tilde{B}| = \frac{1}{4}\left(\nu((ij)^2) - \nu(ij)\right). \qquad \square$$

Corollary 6.1.3.

$$|\mathrm{Sing}\,\tilde{D}| = \frac{1}{2}\nu((ij)^2).$$

Proof. By the arguments in the proof of Proposition 6.1.2 we have a relation between the singularities in $\tilde{B}$ and those in $\tilde{D}$ and we obtain the inequality

$$|\mathrm{Sing}\,\tilde{D}| \geq \frac{1}{2}\nu((ij)^2).$$

The curve $\tilde{D}$ could have other singularities if there were other tangencies between $\tilde{B}$ and Δ_C. We remind that $g^{(2)^*}(\Delta_C) = \Delta_D + 2R$. We have already seen that those intersections of $\tilde{B}$ and Δ_C which preimage in $D^{(2)}$ is a point in R correspond to singularities in $\tilde{D}$. It remains to consider those with preimage in Δ_D.

Such intersections are points $x + x$ with $i(x) = x$. Since i and j have no common fixed point (see Proposition 3.2.1 and Corollary A.2.9) we deduce that $g^{(2)}$ is not ramified at these points. Hence, if we see that B and Δ_D are transversal then we can conclude that $\tilde{B}$ and Δ_C are also transversal, and consequently, that there are no other singularities in the curve $\tilde{D}$.

The preimage of a point $x + x$ with $i(x) = x$ by π_D is an intersection between $D(= \Gamma_i)$ and $\Delta_{D \times D}$ that intersect transversally by Proposition 3.2.2. Hence, the intersection multiplicity is one, and in the image by π_D, the curves B and Δ_D intersect also with multiplicity one, and therefore, they are transversal. $\qquad\square$

Now, we study these singularities to determine their contribution to the arithmetic genus of $\tilde{D}$ and $\tilde{B}$.

Proposition 6.1.4. *All singularities in $\tilde{D}$ and $\tilde{B}$ are nodes.*

Proof. We will begin studying the singularities of $\tilde{D}$ and later their images in $\tilde{B}$.

Consider the morphism $g \times g : D \times D \to C \times C$. It is Galois with group $\langle 1 \times j, \, j \times 1 \rangle$. The preimage of $\tilde{D}$ by this morphism consists in the four divisors described in (6.2).

We have just seen in Corollary 6.1.3 that the singularities of $\tilde{D}$ come from points fixed by $(ij)^2$. Let x_0 be such that $(ij)^2(x_0) = x_0$ (i.e. $ij(x_0) = ji(x_0)$). In $D \times D$ it gives two points in $D_0 \cap D_3$:

$$
\begin{aligned}
D_0 \ni \ (x_0, i(x_0)) \quad &= \quad (j(j(x_0)), j(i(j(x_0)))) \ \in D_3 \ \text{ and} \\
D_0 \ni \ (j(x_0), i(j(x_0))) \quad &= \quad (j(x_0), j(i(j(x_0)))) \ \in D_3.
\end{aligned}
$$

These two points have the same image by $g \times g$, and are not ramified. Since D_0 and D_3 are transversal (as we have seen in the proof of Lemma 6.1.1) and the morphism $g \times g$ is a local homeomorphism around these points, $\tilde{D}$ is transversal on the image. Therefore, all singularities in $\tilde{D}$ are nodes.

Those singular points in $\tilde{D}$ that are image of a point $(x_0, i(x_0))$ with $ij(x_0) \neq x_0$, are not on the diagonal of $C \times C$ and hence π_C does not ramify on them, that is, it is a local homeomorphism, and therefore, their images on $C^{(2)}$ are also nodes.

We have already seen in Lemma 6.1.1 that those singular points in $\tilde{D}$ that are image of points of the form $(x_0, i(x_0))$ with $ij(x_0) = x_0$, lay on the diagonal of $C \times C$, that π_C ramifies on them and that their images are smooth points of $\tilde{B}$.

Consequently, all singularities in $\tilde{B}$ are nodes. $\qquad\square$

Therefore, by Propositions 6.1.2 and 6.1.4 we obtain that

Corollary 6.1.5.

$$p_a(\tilde{B}) - g(B) = \frac{1}{4}(\nu((ij)^2) - \nu(ij)).$$

Moreover, from (5.12) and (6.1) we deduce that

$$p_a(\tilde{D}) - g(D) = \frac{1}{2}\nu((ij)^2).$$

This is also a consequence of Corollary 6.1.3 and Proposition 6.1.4.

Remark 6.1.6. *By Lemma 5.2.1 we obtain that*

$$\tilde{B}^2 = g(D) - 1 - 2(2g - 2) + \frac{1}{2}(\nu((ij)^2)).$$

6.2   SPHERICAL TRIANGLE CASES

We are going to pay some attention also to curves of degree 3 in the square symmetric product. By Theorem 5.1.3, we will look for diagrams which do not complete with appropriate degrees. Our main tool to find such diagrams is Proposition 6.0.3, so we will consider only Galois morphisms. Let D be a curve with two automorphisms: i of order 2 and α of order 3 such that they do not commute, giving a diagram

$$
\begin{array}{ccc}
D & \xrightarrow{\;\;f\;\;} & B = D/\langle i \rangle \\
\Big\downarrow{g} & & \\
C = D/\langle \alpha \rangle & &
\end{array}
\quad .
$$

There are infinitely many groups of finite order generated by an involution and an element of order three. Here we are going to consider only the special cases listed in Lemma A.3.2, the so called spherical triangle groups, because of their simple structure. Their elements are well known and we have all the needed information about them. That is, we assume

$$\langle i, \, \alpha \rangle = A_4, S_4, A_5.$$

We do not consider the spherical triangle group S_3 because since it is such that $|S_3| = 3 \cdot 2 = o(\alpha) \cdot o(i)$, the diagram obtained would complete (see Proposition 6.0.3).

We will perform an analysis of the curve $\tilde{B} \subset C^{(2)}$ analogous to that of Section 6.1. We remind that we consider $D \subset D \times D$ as the graph of i.

To begin with, we are going to compute σ (see (5.7), (5.9), and Lemma 5.2.1), that is, the intersection of the graphs of g and f inside $D \times D$. Since g is the quotient by the action of α, we obtain that

$$\pi_D^* R = \overline{\{(x,y) \mid g(x) = g(y)\} \setminus \Delta_{D \times D}} =$$
$$\{(x, \alpha(x))\} + \{(x, \alpha^2(x))\} = \Gamma_\alpha + \Gamma_{\alpha^2}.$$

Therefore, by Corollary 3.2.3

$$\sigma = D \cdot \pi_D^* R = \Gamma_i \cdot (\Gamma_\alpha + \Gamma_{\alpha^2}) = \nu(i\alpha) + \nu(i\alpha^2).$$

Additionally, we notice that writing $\Gamma_{\alpha^2} = \{(z, \alpha^2(z))\}$ as $\{(\alpha(y), y)\}$ and applying $i \times 1$ we find the curve $\{(i\alpha(y), y)\}$. In the same way, if we apply this automorphism to $\Gamma_i = \{(z, i(z))\}$ we find the diagonal, and hence, $\nu(i\alpha) = \Gamma_i \cdot \Gamma_{\alpha^2} = \nu(i\alpha^2)$. We conclude that

$$\sigma = 2\nu(i\alpha) = 2\nu(i\alpha^2). \tag{6.3}$$

These are pairs of different points in $D \subset D \times D$ with the same image in $\tilde{D}$, so their images by $g \times g$ are singularities of $\tilde{D}$. Now, we are going to see that their images in $\tilde{B}$ are smooth points.

Lemma 6.2.1. *The image in $\tilde{B}$ by $\pi_C|_{\tilde{D}}$ of a point $(g \times g)(x, i(x))$ with $i\alpha(x) = x$ or $i\alpha^2(x) = x$ is a smooth point where $\tilde{B}$ is tangent to the diagonal.*

Proof. First, we are going to study the singular points of the form $(g \times g)(x, i(x))$ in $\tilde{D}$.

Consider the morphism $g \times g : D \times D \to C \times C$. It is Galois with group

$$\{1 \times 1, \alpha \times \alpha, \alpha \times \alpha^2, \alpha^2 \times \alpha, 1 \times \alpha, \alpha \times 1, \alpha^2 \times \alpha^2, 1 \times \alpha^2, \alpha^2 \times 1\}.$$

We consider $D_0 = \{(x, i(x))\} = D$ and all its images by the elements of

that group, that is, all preimages of $\tilde{D}$ by $g \times g$:

$$
\begin{aligned}
(1 \times 1)D &= & \{(x, i(x))\} & = D_0, \\
(\alpha \times \alpha)D &= & \{(\alpha(x), \alpha i(x))\} & = D_1, \\
(\alpha \times \alpha^2)D &= & \{(\alpha(x), \alpha^2 i(x))\} & = D_2, \\
(\alpha^2 \times \alpha)D &= & \{(\alpha^2(x), \alpha i(x))\} & = D_3, \\
(1 \times \alpha)D &= & \{(x, \alpha i(x))\} & = D_4, \\
(\alpha \times 1)D &= & \{(\alpha(x), i(x))\} & = D_5, \\
(\alpha^2 \times \alpha^2)D &= & \{(\alpha^2(x), \alpha^2 i(x))\} & = D_6, \\
(1 \times \alpha^2)D &= & \{(x, \alpha^2 i(x))\} & = D_7 \ \text{ and} \\
(\alpha^2 \times 1)D &= & \{(\alpha^2(x), i(x))\} & = D_8.
\end{aligned}
\tag{6.4}
$$

Each of the singular points in $\tilde{D}$ corresponding to $(x, i(x))$ with $i\alpha(x) = x$, has as preimages one point in $D_0 \cap D_2$ and one in $D_0 \cap D_3$, specifically

$$
\begin{aligned}
D_0 \ni \ (x, i(x)) &= \ (\alpha^2(i(x)), \alpha(i(i(x)))) &\in D_3 \ \text{ and} \\
D_0 \ni \ (i(x), x) &= \ (\alpha(x), \alpha^2 i(x)) &\in D_2.
\end{aligned}
$$

Since $g \times g$ is not ramified in these points, if we see that D_0 and D_3 are transversal, as the morphism is a local homeomorphism, $\tilde{D}$ will be transversal on the image, and therefore, the image will be a node in $\tilde{D}$.

We look at their transformations by the automorphism $1 \times i$ acting on the surface $D \times D$:

$$
\begin{aligned}
D_0 = \{(y, i(y))\} & \xrightarrow{1 \times i} \{(y, y)\} = \Delta_{D \times D}, \\
D_2 = \{(\alpha(x), \alpha^2 i(x))\} & \xrightarrow{1 \times i} \{(\alpha(x), i\alpha^2 i(x))\} = \{(z, i\alpha^2 i\alpha^2(z))\} = \Gamma_{(i\alpha^2)^2}, \\
D_3 = \{(\alpha^2(x), \alpha i(x))\} & \xrightarrow{1 \times i} \{(\alpha^2(x), i\alpha i(x))\} = \{(z, i\alpha i\alpha(z))\} = \Gamma_{(i\alpha)^2}.
\end{aligned}
$$

Consequently, since the diagonal meets transversally the graph of an automorphism (see Proposition 3.2.2), all intersections are transversal and these singularities are nodes on $\tilde{D}$.

Since the points $(x, i(x))$ and $(i(x), x) = (\alpha(x), i\alpha(x))$ have the same image by the morphism π_D, there is only one point for each of these singularities in B, the normalization of $\check{B}$. Then, with the same arguments of the proof of Lemma 6.1.1, we deduce that $\tilde{B}$ is smooth and tangent to Δ_C in each of these points.

Notice that since given a point x with $i\alpha(x) = x$ the images of $(x, i(x))$ and $(i(x), x)$ are equal, we have also proved the lemma for those $x \in D$ with $i\alpha^2(x) = x$, because in that case $i(x)$ is a point fixed by $i\alpha$. Indeed, $i\alpha^2(x) = x \ \Leftrightarrow x = \alpha i(x) \ \Leftrightarrow i(x) = i\alpha(i(x))$. $\qquad \square$

Now, we study the rest of singularities of $\tilde{D}$ and $\tilde{B}$.

Proposition 6.2.2.

$$|\operatorname{Sing} \tilde{B}| = \frac{1}{2}\nu(i\alpha^2 i\alpha) + \frac{1}{2}(\nu((i\alpha)^2) - \nu(i\alpha)).$$

Proof. First, we want to know when two different points in D have the same image in $\tilde{D}$. We remind that $D \to \tilde{D}$ is the normalization map.

Let (x, y) (with $i(x) = y$) and (z, t) (with $i(z) = t$) be two points with the same image by $g \times g$, that is, such that $\alpha^k(x) = z$ and $\alpha^r(y) = t$ for certain $k, r \in \{1, 2\}$. We assume that the two pairs are different, so we do not consider neither $k = 3$ nor $r = 3$.

Given such two pairs, we obtain that

$$
\begin{array}{rclclclcl}
x &=& i(y) &=& i\alpha^{3-r}(t) &=& i\alpha^{3-r}i(z) &=& i\alpha^{3-r}i\alpha^k(x),\\
y &=& i(x) &=& i\alpha^{3-k}(z) &=& i\alpha^{3-k}i(t) &=& i\alpha^{3-k}i\alpha^r(y),\\
z &=& i(t) &=& i\alpha^r(y) &=& i\alpha^r i(x) &=& i\alpha^r i\alpha^{3-k}(z) \ \ \textbf{and}\\
t &=& i(z) &=& i\alpha^k(x) &=& i\alpha^k i(y) &=& i\alpha^k i\alpha^{3-r}(t),
\end{array}
$$

i.e. they are points fixed by certain automorphisms. Notice that i and α have no common fixed point (see Proposition 3.2.1 and Section A.3).

We have four possibilities for k, r: $\begin{cases} k = r = 1 \\ k = r = 2 \\ k = 2 \ r = 1 \\ k = 1 \ r = 2 \end{cases}$, that can be gathered in two cases:

- **Case A**: if $k = r$, then the two points in each involution pair are fixed by the same automorphism, for instance, x and y are fixed by $i\alpha^2 i\alpha$ and the points z and t are fixed by $i\alpha i\alpha^2$.

- **Case B**: if $k \neq r$, then one of the two points in an involution pair is fixed by $i\alpha i\alpha$ and the other by $i\alpha^2 i\alpha^2$.

We study these two cases separately:

Case A: Assume that k = r.

Let $x \in D$ be such that $i\alpha^2 i\alpha(x) = x$, that is, $i\alpha(x) = \alpha i(x)$, and take $y := i(x)$. Since x is fixed by $i\alpha^2 i\alpha$, then it is not fixed by i (see Proposition 3.2.1 and Section A.3). We denote by

$$
\begin{array}{ll}
z_1 = \alpha(x) & t_1 = \alpha(y)\\
z_2 = \alpha^2(x) & t_2 = \alpha^2(y).
\end{array}
$$

If we consider $z_0 = x$ and $t_0 = y$ then the pairs (z_n, t_m) form a fiber of the morphism $g \times g$.

We notice that $i(z_1) = i\alpha(x) = \alpha i(x) = \alpha(y) = t_1$, so we obtain that $(z_1, t_1) \in D \subset D \times D$.

We claim that $i(z_2) \neq t_2$. Otherwise, $i(z_2) = t_2$, so $i\alpha^2(x) = \alpha^2 i(x)$ and hence $i\alpha i\alpha^2(x) = x$. This would imply that there exists a cyclic group containing both $i\alpha^2 i\alpha$ and $i\alpha i\alpha^2$ (see Proposition 3.2.1), which is not possible inside our groups $\langle i, \alpha \rangle = A_4, S_4, A_5$ (see Section A.3).

Therefore, we have two different pairs of points on D with image in $\tilde{D}$ pairwise equal, that is, two singularities with two branches. We notice that
$$\begin{aligned} i\alpha^2 i\alpha(y) &= y, \\ i\alpha i\alpha^2(z_1) &= z_1 \text{ and} \\ i\alpha i\alpha^2(t_1) &= t_1. \end{aligned}$$

Consequently, x and y are both fixed by $i\alpha^2 i\alpha$, and hence there are $\nu(i\alpha^2 i\alpha)$ singularities in $\tilde{D}$ coming from this kind of points. Notice that the image of $\{(x, y), (z_1, t_1)\}$ and that of $\{(y, x), (t_1, z_1)\}$ will be two different singularities in $\tilde{D}$ with the same image in $\tilde{B}$, so they give $\frac{1}{2}\nu(i\alpha^2 i\alpha)$ singularities in $\tilde{B}$ because they are not on the branch locus of π_C.

Case B: Assume that $\mathbf{k} \neq \mathbf{r}$.

Let $x \in D$ be a point such that $i\alpha i\alpha(x) = x$ i.e. $i\alpha(x) = \alpha^2 i(x)$ with $i\alpha(x) \neq x$. We notice that this is only possible when $\langle i, \alpha \rangle = S_4$ because in the other two cases the order of $i\alpha$ is prime, and hence the points fixed by it and its square are the same. Those points with $i\alpha(x) = x$ have been already considered in Lemma 6.2.1 where we have seen that their images in $\tilde{B}$ are smooth points.

Since x is fixed by $(i\alpha)^2$ it is not fixed by i because there is no cyclic group containing both in S_4 (see Proposition 3.2.1 and Section A.3). Let $y := i(x) \neq x$. Then, we denote by
$$\begin{aligned} z_1 &= \alpha(x) & t_1 &= \alpha(y) \\ z_2 &= \alpha^2(x) & t_2 &= \alpha^2(y). \end{aligned}$$

We notice that $i(z_1) = i\alpha(x) = \alpha^2 i(x) = \alpha^2(y) = t_2$, so we obtain that $(z_1, t_2) \in D \subset D \times D$.

We claim that $i(z_2) \neq t_1$. Otherwise, $i(z_2) = t_1$, so $i\alpha^2(x) = \alpha i(x)$ and hence $i\alpha^2 i\alpha^2(x) = x$. This would imply that there exists a cyclic group containing both $i\alpha i\alpha$ and $i\alpha^2 i\alpha^2$, which is not possible inside S_4 (see Proposition 3.2.1 and Section A.3).

Therefore, we have two different pairs of points in D with images in $\tilde{D}$ pairwise equal, that is, two singularities with two branches. We notice that

$$
\begin{aligned}
i\alpha^2 i\alpha^2(y) &= y, \\
i\alpha^2 i\alpha^2(z_1) &= z_1 \text{ and} \\
i\alpha i\alpha(t_2) &= t_2.
\end{aligned}
$$

Consequently, we have $\nu((i\alpha)^2) - \nu(i\alpha)$ singularities in $\tilde{D}$ coming from this kind of points (only when $\langle i, \alpha \rangle = S_4$). Notice again that since these singular points are not on the ramification divisor of π_C, they have images pairwise equal in $\tilde{B}$, so they give $\frac{1}{2}\left(\nu((i\alpha)^2) - \nu(i\alpha)\right)$ singularities in $\tilde{B}$.

Next, we will see that there are no other singularities in $\tilde{B}$. For a point in B outside the ramification locus of $g^{(2)}$, its image will be a singularity only if there is another point with the same image, because $g^{(2)}$ is a local homeomorphism around it. This is the case we have just studied. Then, it only remains to consider the ramification points of $g^{(2)}$.

In $B \subset D^{(2)}$ there are two types of points where $g^{(2)}$ ramifies: those in R (see (5.7)) and those in D_x with x a ramification point of g. We have seen in Lemma 6.2.1 that the image of a point in $B \cap R$ is always smooth, so it remains only to study those in D_x for $x \in \mathrm{Ram}(g)$.

To do this, we study the intersection of $\tilde{B}$ with a coordinate curve C_P, with $P \in \mathrm{Branch}(g)$ (the curve C_P is the image of D_x with the point $x \in \mathrm{Ram}(g)$). We remind that $C_P \cdot \tilde{B} = 3$.

Let $P \in \mathrm{Branch}(g)$ i.e. $\exists! \, x$ such that $g(x) = P$, that is, x is a point fixed by α. Let $y := i(x) \neq x$ because the automorphisms α and i have no common fixed point. Then, $g^{(2)}(x + y) = P + g(y) = P + Q$.

We know that C_P intersects $\tilde{B}$ in a single point $P + Q$ with multiplicity three. We want to know how C_Q intersects $\tilde{B}$ to prove that it is a smooth point. We distinguish two cases:

First, if $y \in \mathrm{Ram}(g)$, that is $\alpha(y) = y$ i.e. $\alpha i(x) = i(x)$ or equivalently, $i\alpha i(x) = x$, then, since we are assuming that $\alpha(x) = x$, this would imply that $\langle \alpha, i\alpha i \rangle$ is contained in a cyclic group, which is not possible (see Proposition 3.2.1 and Section A.3). So this case does not happen.

Second, if $y \notin \mathrm{Ram}(g)$ then there exist t and z such that $\alpha(y) = t$ and $\alpha^2(y) = z$.

If $\exists k$ such that $\alpha^k i(t) = i(z)$, then there would be a point in the intersection of C_Q and $\tilde{B}$ with multiplicity greater than 1, otherwise there would be two different points in this intersection: $g^{(2)}(t + i(t))$ and $g^{(2)}(z + i(z))$. In any case, these points do not belong to C_P, and

hence, in $P+Q$ the intersection multiplicity of C_Q and $\tilde{B}$ is one. Therefore, the curve $\tilde{B}$ is smooth at $P+Q$.

To sum up, the total amount of singularities is given by

$$|\mathrm{Sing}\,\tilde{B}| = \frac{1}{2}\nu(i\alpha^2 i\alpha) + \frac{1}{2}(\nu((i\alpha)^2) - \nu(i\alpha)).$$

Where the second addend is zero except for $\langle i, \alpha \rangle \cong S_4$. $\qquad\square$

Corollary 6.2.3.

$$|\mathrm{Sing}\,\tilde{D}| = \nu(i\alpha^2 i\alpha) + \nu((i\alpha)^2).$$

Proof. By the same arguments used in Corollary 6.1.3 (with α as j), the curve $\tilde{D}$ does not have more singularities than those considered during the proof of Proposition 6.2.2. $\qquad\square$

Now, we study which kind of singularities they are.

Proposition 6.2.4. *All singularities in $\tilde{D}$ and $\tilde{B}$ are nodes.*

Proof. We begin studying the singularities on $\tilde{D}$ and later their image by π_C.

We have by construction that $g \times g : D \times D \to C \times C$ is Galois with group $\langle 1 \times \alpha, \alpha \times 1 \rangle$. We consider $D_0 = D = \{(x, i(x))\} \subset D \times D$ and all its images by the elements of that group as described in (6.4). If we see that they intersect transversally, then we will be able to conclude that their image by $g \times g$, that is $\tilde{D}$, is also transversal because $g \times g$ does not ramify on the preimages of the singular points of $\tilde{D}$.

Taking the intersections of D_0 with its images by elements of the group $\langle 1 \times \alpha, \alpha \times 1 \rangle$, we recover the cases in the proof of Proposition 6.2.2 of possible singularities, that in this language are

$$
\begin{array}{llll}
i\alpha^2 i\alpha(x) = x & \Leftrightarrow i\alpha(x) = \alpha i(x) & \leftrightarrow D_0 \cap D_1 \\
i\alpha i\alpha^2(x) = x & \Leftrightarrow i\alpha^2(x) = \alpha^2 i(x) & \leftrightarrow D_0 \cap D_6
\end{array}
\left.\begin{array}{l} \\ \\ \end{array}\right\}
\begin{array}{c} \alpha^2 \times \alpha^2 \\ \text{permutes them.} \end{array}
$$

$$
\begin{array}{llll}
i\alpha^2 i\alpha^2(x) = x & \Leftrightarrow i\alpha^2(x) = \alpha i(x) & \leftrightarrow D_0 \cap D_3 \\
i\alpha i\alpha(x) = x & \Leftrightarrow i\alpha(x) = \alpha^2 i(x) & \leftrightarrow D_0 \cap D_2
\end{array}
\left.\begin{array}{l} \\ \\ \end{array}\right\}
\begin{array}{c} \alpha^2 \times \alpha \\ \text{permutes them.} \end{array}
$$

The first two correspond to Case A and the last two correspond to Case B. Now, we want to see that these intersections are transversal. We are going to look at their transformations by the automorphism

$1 \times i$ of $D \times D$.

$$D_0 = \{(y, i(y))\} \xrightarrow{1 \times i} \{(y, y)\} = \Delta_{D \times D},$$
$$D_1 = \{(\alpha(x), \alpha i(x))\} \xrightarrow{1 \times i} \{(\alpha(x), i\alpha i(x))\} = \{(z, i\alpha i\alpha^2(z))\} = \Gamma_{i\alpha i\alpha^2},$$
$$D_6 = \{(\alpha^2(x), \alpha^2 i(x))\} \xrightarrow{1 \times i} \{(\alpha^2(x), i\alpha^2 i(x))\} = \{(z, i\alpha^2 i\alpha(z))\} = \Gamma_{i\alpha^2 i\alpha},$$
$$D_2 = \{(\alpha(x), \alpha^2 i(x))\} \xrightarrow{1 \times i} \{(\alpha(x), i\alpha^2 i(x))\} = \{(z, i\alpha^2 i\alpha^2(z))\} = \Gamma_{i\alpha^2 i\alpha^2},$$
$$D_3 = \{(\alpha^2(x), \alpha i(x))\} \xrightarrow{1 \times i} \{(\alpha^2(x), i\alpha i(x))\} = \{(z, i\alpha i\alpha(z))\} = \Gamma_{i\alpha i\alpha}.$$

Consequently, since the diagonal meets transversally the graph of an automorphism (see Proposition 3.2.2), all intersections are transversal, and hence all singularities in $\tilde{D}$ are nodes.

Notice that, those singular points $(g(x_0), g(i(x_0)))$ with $i\alpha(x_0) \neq x_0$ are not on the diagonal of $C \times C$, and hence π_C does not ramify on them. Therefore, their images on $C^{(2)}$ are also nodes because π_C is a local homeomorphism around them.

Those singular points $(g(x_0), g(i(x_0)))$ with $i\alpha(x_0) = x_0$ are on the diagonal of $C \times C$, and hence π_C ramifies on them. We have already seen in Lemma 6.2.1 that their images in $\tilde{B}$ are smooth points.

Consequently, all singularities in $\tilde{B}$ are nodes. $\qquad \square$

Therefore, by Propositions 6.1.2 and 6.2.4 we obtain that

Corollary 6.2.5.

$$p_a(\tilde{B}) - g(B) = \frac{1}{2}\nu(i\alpha^2 i\alpha) + \frac{1}{2}(\nu((i\alpha)^2) - \nu(i\alpha)).$$

Moreover, from (5.12) and (6.3) we deduce that

$$p_a(\tilde{D}) - g(D) = \nu(i\alpha^2 i\alpha) + \nu((i\alpha)^2).$$

Remark 6.2.6. *By Lemma 5.2.1 we obtain that*

$$\tilde{B}^2 = g(D) - 1 - 3(2g - 2) + \nu(i\alpha^2 i\alpha) + \nu((i\alpha)^2).$$

Chapter Seven

CURVES WITH POSITIVE SELF-INTERSECTION IN $C^{(2)}$

In this chapter, using the previous results, we study curves $\tilde{B}$ in $C^{(2)}$ (with $g := g(C) \geq 2$) such that $\tilde{B}^2 > 0$ and $p_a(\tilde{B}) > g(C)$. We describe completely the degree two case ($\tilde{B} \cdot C_P = 2$) and some particular cases for curves of degree three ($\tilde{B} \cdot C_P = 3$).

We study the numerical conditions determined by our hypothesis, and next we define, when possible, a smooth epimorphism that gives a curve D with quotients C and B that lie on a diagram which does not complete.

7.1 DEGREE TWO

First of all we study those curves $\tilde{B} \subset C^{(2)}$ with $\tilde{B} \cdot C_P = 2$ such that $\pi_C^*(\tilde{B})$ is reducible. We begin with an example of a special case:

Example 7.1.1. *Let Γ be a Fuchsian group with signature $(0; 10, 5, 2)$. It exists because of Poincaré's Theorem. We consider the smooth epimorphism*

$$
\begin{aligned}
\Gamma &\rightarrow \mathbb{Z}/10 = \langle \sigma \rangle \\
x_1 &\rightarrow \sigma \\
x_2 &\rightarrow \sigma^4 \\
x_3 &\rightarrow \sigma^5
\end{aligned}
$$

which gives a morphism $C \rightarrow \mathbb{P}^1$, where C is a curve with an automorphism $\sigma \in Aut(C)$ of order 10 with $\nu(\sigma) = 1$, $\nu(\sigma^2) = 3$ and $\nu(\sigma^5) = 6$ (see Lemma 3.2.4). By the Riemann-Hurwitz formula we obtain that

$$2g(C) - 2 = 10(-2) + 1 \cdot 9 + 2 \cdot 4 + 5 \cdot 1 \Rightarrow g(C) = 2.$$

Consider now the graph of σ, Γ_σ, in $C \times C$. The image of Γ_σ by π_C is a curve $\tilde{B}$ of degree two in $C^{(2)}$ such that $\pi_C^*(\tilde{B}) = \Gamma_\sigma + \Gamma_{\sigma^{-1}}$.

In the following proposition we prove that it is the only curve of degree two in $C^{(2)}$ with reducible preimage by π_C and positive self-intersection.

Proposition 7.1.2. *The only curves $\tilde{B} \subset C^{(2)}$ of degree two, such that $\pi_C^*(\tilde{B})$ is reducible and $\tilde{B}^2 > 0$, are the symmetrization of the graph of σ on C, where C is a curve of genus 2 and σ is an automorphism of order 10 such that $\nu(\sigma) = 1$, $\nu(\sigma^2) = 3$ and $\nu(\sigma^5) = 6$.*

Proof. Let $\tilde{B}$ be such a curve. We are going to describe it in order to prove the proposition.

We observe that if $\pi_C^*(\tilde{B})$ reduces, then it consists of two copies of C, and the projections onto each factor are isomorphisms. This gives an automorphism of C, $C \xrightarrow{\sigma} C$. Notice that the order of σ must be at least 3 because otherwise, $\pi_C^*(\tilde{B})$ would have only one component.

Hence, we have in $C \times C \supset \pi_C^*(\tilde{B}) = C_1 + C_2$ with

$$C_1 = \{(x, \sigma(x))\} = \Gamma_\sigma \quad \text{and}$$
$$C_2 = \{(y, \sigma^{-1}(y))\} = \{(\sigma(z), z)\} = \Gamma_{\sigma^{-1}}$$

which by π_C go to $\tilde{B} = \{x + \sigma(x), \ x \in C\}$. The curve $\tilde{B}$ has normalization C and moreover,

- $\tilde{B} \cdot C_P = 2$.

- $\tilde{B} \cdot \Delta_C = 2\nu(\sigma)$. Indeed, consider

$$\tilde{B} \cap \Delta_C = \{x + \sigma(x) \mid \sigma(x) = x\}.$$

  The preimage of these points by π_C correspond to points were C_1 and C_2 meet (transversally by Corollary 3.2.3) over the diagonal. They intersect the diagonal transversally (by Proposition 3.2.2), and taking local coordinates, as in Lemma 6.1.1, we see that $\tilde{B}$ and Δ_C are tangent at $x + x$ for x a point fixed by σ and that they have no other intersection.

- $|\text{Sing } \tilde{B}| = \frac{1}{2}(\nu(\sigma^2) - \nu(\sigma))$. Indeed, a general curve C_P intersects $\tilde{B}$ in two different points $P + \sigma(P)$ and $P + \sigma^{-1}(P)$. Since $C_P \cdot \tilde{B} = 2$ when these two points are different they are smooth points on $\tilde{B}$. To determine the singularities of $\tilde{B}$ we need to study when these two points coincide. We have two possibilities:

Either $\sigma(P) = P$ and hence $\tilde{B}$ intersects the diagonal in a smooth (tangent) point as we have just seen.

Or $\sigma(P) = \sigma^{-1}(P) \neq P$, that is, P is fixed by σ^2 and not by σ. We observe that if P is fixed by σ^2, the point $\sigma(P)$ is also fixed by σ^2, and both give the same singularity $P + \sigma(P) = \sigma(P) + \sigma^2(P)$.

- All singularities of $\tilde{B}$ are nodes. Indeed, consider the normalization morphism

$$
\begin{array}{ccc}
C & \to & \tilde{B} \subset C^{(2)} \\
x & \to & x + \sigma(x).
\end{array}
$$

Let Σ be the set of points of C where the morphism is not $(1:1)$, that is, where two different points have the same image in $C^{(2)}$. In particular, the image of $P \in \Sigma$ is a singularity of $\tilde{B}$. That is, $\Sigma = \{x \in C \mid \exists\, y \in C,\ x \neq y \text{ with } y = \sigma(x) \text{ and } x = \sigma(y)\}$. Equivalently, x and y are fixed by σ^2 and not by σ.

Thus, those points $x + \sigma(x)$ with $\sigma^2(x) = x \neq \sigma(x)$ have two branches in the normalization of $\tilde{B}$, and since $C_P \cdot \tilde{B} = 2$, these are order two singularities. We conclude that these singularities are nodes.

Therefore, $p_a(\tilde{B}) = g(C) + \frac{1}{2}(\nu(\sigma^2) - \nu(\sigma))$. Since by hypothesis $p_a(\tilde{B}) > g(C)$ we obtain that $\tilde{B}$ is singular, so $o(\sigma)$ must be even (different from 2).

We call $g = g(C)$, $s = \nu(\sigma)$ and $r = \nu(\sigma^2) - \nu(\sigma)$. From the adjunction formula we deduce that

$$
\begin{aligned}
\tilde{B}^2 &= 2p_a(\tilde{B}) - 2 - 2(2g - 2) + \tilde{B} \cdot \tfrac{\Delta_C}{2} \\
&= 2(g + \tfrac{1}{2}r) - 2 - 2(2g - 2) + s = -2g + 2 + r + s > 0 \Leftrightarrow \quad (7.1) \\
2g - 2 &< r + s = \nu(\sigma^2).
\end{aligned}
$$

By hypothesis the inequality (7.1) is satisfied. By Proposition 3.2.8 we know that

$$
\nu(\sigma^2) \leq 2 + \frac{2g}{o(\sigma^2) - 1} \tag{7.2}
$$

and we remind that $o(\sigma) \geq 4$.

Next, we study separately the different possibilities for $o(\sigma^2)$.

Assume first that $o(\sigma^2) \geq 4$ (so $o(\sigma) \geq 8$), then, from (7.1) and (7.2) we deduce that

$$
2 + \frac{2g}{3} \geq 2 + \frac{2g}{o(\sigma^2) - 1} \geq r + s > 2g - 2. \tag{7.3}
$$

This implies that $3 > g$, so it remains to consider $g = 2$ with $r + s = 3$. From (7.3) we deduce that $5 \geq o(\sigma^2) \geq 4$. We have two cases:

1. First, assume that $o(\sigma^2) = 4$, then $o(\sigma) = 8$. We consider the morphism $C \xrightarrow{(8:1)} C/\langle\sigma\rangle$ and notice that the points fixed by σ^2 not fixed by σ come by pairs, and hence r is even. Therefore, we have that $r = 2$ and $s = 1$ (since we have seen that $r \neq 0$). There could be other ramification points coming from points fixed by σ^4 that would come by groups of four (see Lemma 3.2.4). Considering the Riemann-Hurwitz formula for this morphism with all our data we obtain that

$$2 \cdot 2 - 2 = 8(2\gamma - 2) + 1 \cdot 7 + 2 \cdot 3 + 4\lambda \cdot 1 \Leftrightarrow 5 = 16\gamma + 4\lambda,$$

which is not possible.

2. Second, assume that $o(\sigma^2) = 5$, then $o(\sigma) = 10$. We consider the morphism $C \xrightarrow{(10:1)} C/\langle\sigma\rangle$ and notice that the points fixed by σ^2 not fixed by σ come by pairs, and hence r is even. Therefore, we have that $r = 2$ and $s = 1$ (since we have seen that $r \neq 0$). There could be ramification points coming from points fixed by σ^5 that would come by groups of five (see Lemma 3.2.4). Considering the Riemann-Hurwitz formula for this morphism with all our data we obtain that

$$2 \cdot 2 - 2 = 10(2\gamma - 2) + 1 \cdot 9 + 2 \cdot 4 + 5\lambda \cdot 1 \Leftrightarrow$$
$$5 = 20\gamma + 5\lambda \Rightarrow \gamma = 0 \ \lambda = 1.$$

Therefore $\nu(\sigma^5) = 5 + 1 = 6$, the case described in Example 7.1.1. We notice that the Hurwitz space of such morphisms $C \to \mathbb{P}^1$ has dimension 0, and therefore, in moduli, we have a finite number of such curves C (see Section 4.5).

Now, it remains to take care of the cases $o(\sigma) = 6$ and $o(\sigma) = 4$.

- Assume that $o(\sigma) = 6$, then $o(\sigma^2) = 3$. From (7.1) and (7.2) we deduce that $g = 3$ and $r + s = 5$. As before, r must be even and non zero, so we have two options: either $r = 2$ and $s = 3$ or $r = 4$ and $s = 1$.

  We consider the morphism $C \xrightarrow{(6:1)} C/\langle\sigma\rangle$. Its ramification points will be points fixed by σ with ramification index 6, points fixed by σ^2 (not by σ), coming on pairs, with ramification index 3 and

points fixed by σ^3 (not by σ), coming in groups of three, with ramification index 2. With all these data we see that the Riemann-Hurwitz formula is not satisfied and therefore, such an action does not exist.

- Assume that $o(\sigma) = 4$, then $o(\sigma^2) = 2$. By (7.1) we have that

$$r + s \geq 2g - 1. \tag{7.4}$$

  If γ is the genus of the quotient of C by the action of σ, from the Riemann-Hurwitz formula, (7.4) and (7.2) we deduce that $\gamma = 0$ and $s = 3$ which is not compatible with r even. Hence, such an action does not exist.

And therefore, the only case for $\tilde{B}$ satisfying the hypothesis of the proposition is the one described in Example 7.1.1. $\qquad\square$

7.1.1 Curves with low genus

We are looking for a curve $\tilde{B}$ in $C^{(2)}$, for a certain C, with low genus and positive self-intersection. The goal of this section is to prove

Theorem 7.1.3. *There are no degree two curves $\tilde{B}$ lying in $C^{(2)}$ with $g(C) < p_a(\tilde{B}) < 2g(C) - 1$ and $\tilde{B}^2 > 0$.*

Proof. Let $\tilde{B}$ be a curve in the symmetric square of a curve C of genus $g = g(C)$ with $\tilde{B} \cdot C_P = 2$ such that $g < p_a(\tilde{B}) < 2g - 1$. Let B be the normalization of $\tilde{B}$. Notice that since $\Delta_C^2 < 0$ we can assume $\tilde{B} \neq \Delta_C$.

By Proposition 7.1.2, we know that if $\pi_C^*(\tilde{B})$ is reducible then the self-intersection of $\tilde{B}$ is not positive, except for the case described in Example 7.1.1 that does not satisfy the inequality for the genus. Hence, it remains to study the case $\tilde{D} = \pi_C^*(\tilde{B})$ irreducible.

By the proof of Theorem 5.1.4, we know that B lies in a diagram

$$\begin{array}{ccc} D & \xrightarrow{(2:1)} & B \\ {\scriptstyle(2:1)}\Big\downarrow & & \\ C & & \end{array} \tag{7.5}$$

that does not complete. Since the two morphisms have degree two, they are Galois, and thus, there exist two involutions i and j (the changes of sheet) such that $B = D/\langle i \rangle$ and $C = D/\langle j \rangle$.

If $i = j$ then $\tilde{B} = \Delta_C$, that has negative selfintersection, so $i \neq j$. Denoting $G = \langle i, j \rangle$ we obtain that $|G| > 4$ because the diagram does not complete (see Proposition 6.0.3). Therefore, $G = D_n$ for certain n (see Theorem A.2.3).

We use the following notation for the number of fixed points of the involved automorphisms of the curve D:

$$s = \nu(j), \quad t = \nu(i), \quad r = \nu(ij) \quad \text{and} \quad r + k = \nu((ij)^2).$$

Let $b = g(B)$, $g = g(C)$ and $h = g(D)$.

The strategy is to find the restrictions on the numbers s, t, r, k, b, g and h given by our hypothesis and see that there is no set of numbers compatible with all of them.

(a) By the Riemann-Hurwitz formula for the morphism $D \to C$ we obtain that $2h - 2 = 2(2g - 2) + s$, and so

$$g = \frac{2h + 2 - s}{4} \geq 3 \Leftrightarrow s \leq 2h - 10. \tag{7.6}$$

We impose $g \geq 3$ because otherwise there is no possible $p_a(\tilde{B})$ with $g < p_a(\tilde{B}) < 2g - 1$.

(b) By the Riemann-Hurwitz formula for the morphism $D \to B$ we obtain that $b = \frac{1}{4}(2h + 2 - t)$. So, from Corollary 6.1.5 we deduce that

$$p_a(\tilde{B}) = b + \frac{1}{4}(r + k - r) = \frac{1}{4}(2h + 2 - t + k). \tag{7.7}$$

(c) By (7.6) and (7.7) the condition $g < p_a(\tilde{B})$ translates into

$$\frac{2h + 2 - s}{4} < \frac{2h + 2 - t}{4} + \frac{1}{4}k \Leftrightarrow t < s + k. \tag{7.8}$$

(d) Again by (7.6) and (7.7), the inequality $p_a(\tilde{B}) \leq 2g - 2$ is equivalent to

$$\frac{2h + 2 - t}{4} + \frac{1}{4}k \leq 2\frac{2h + 2 - s}{4} - 2 \Leftrightarrow 6 + 2s + k \leq 2h + t. \tag{7.9}$$

(e) From Remark 6.1.6 and (7.6) we deduce that

$$\tilde{B}^2 = -h + 1 + s + \frac{1}{2}(r + k) > 0 \Leftrightarrow h \leq s + \frac{1}{2}(r + k). \tag{7.10}$$

(f) By (7.10) and (7.9) we obtain that

$$\frac{6 + 2s + k - t}{2} \le h \le s + \frac{1}{2}(r + k)$$

and hence

$$\frac{6 + 2s + k - t}{2} \le s + \frac{1}{2}(r + k) \Leftrightarrow t + r \ge 6. \qquad (7.11)$$

Finally, we want to see that either (7.10) or (7.11) is not possible.

Note: *We can assume that $\langle i, j \rangle = D_{2l}$. Indeed, if $\langle i, j \rangle = D_{2l+1}$* then the involutions i and j would be conjugate, and so $t = s$. Since 2 and $2l + 1$ are coprime, the automorphisms ij and $(ij)^2$ would have the same fixed points and thus $k = 0$, contradicting (7.8).

Let $\gamma = g(D/D_{2l})$. By the Riemann-Hurwitz formula for group quotients (see (3.1)) applied to $\tau : D \to D/D_{2l}$, we have that

$$h - 1 = 2l(2\gamma - 2) + 2l \sum_{P \in \mathrm{Br}} \left(1 - \frac{1}{m_P}\right) \qquad (7.12)$$

where $m_P = e_Q - 1$ with $\tau(Q) = P$ (we remind that since τ is Galois, it is totally ramified, and we call m_P the order of the branch point P).

We count the number of branch points of τ using Lemma 3.2.4 repetitively:

- Since i has t fixed points, order 2 and l conjugates (all generating subgroups conjugate to $\langle i \rangle$), in a fiber of τ there will be either $2 = \frac{4l}{2 \cdot l}$ fixed points of i or no fixed points at all. So we will have $\frac{t}{2}$ branch points of order 2 corresponding to the conjugacy class of the stabilizer $\langle i \rangle$.

- Since j has s fixed points and i and j are not conjugate, we will have $\frac{s}{2}$ branch points of order 2 corresponding to the conjugacy class of $\langle j \rangle$, which will be different from those in the previous point.

- Since ij has order $2l$ and r fixed points, we will have $\frac{r}{2}$ branch points of order $2l$.

- Since $(ij)^2$ has k fixed points not fixed by ij, we will have $\frac{k}{4}$ branch points of order l.

- We could have other branch points coming from powers of ij that do not generate the whole $\langle ij \rangle$.

All together, this gives that

$$2l \sum_{P \in Br} \left(1 - \frac{1}{m_P} \right) \geq 2l\left(\frac{s+t}{2} \cdot \frac{1}{2} + \frac{r}{2}(1 - \frac{1}{2l}) + \frac{k}{4}(1 - \frac{1}{l})\right)$$
$$= \frac{l}{2}(s+t) + r\frac{2l-1}{2} + k\frac{l-1}{2}.$$

Then, by (7.12) we have that

$$h - 1 \geq 2l(2\gamma - 2) + (s+t)\frac{l}{2} + r\frac{2l-1}{2} + k\frac{l-1}{2}. \qquad (7.13)$$

Claim: Necessarily $l = 2$, $\gamma = 0$ and $t + r = 6$.
Proof of the Claim: First, we prove that otherwise we would have

$$2l(2\gamma - 2) + (s+t)\frac{l}{2} + r\frac{2l-1}{2} + k\frac{l-1}{2} \geq s + \frac{1}{2}(r + k) \qquad (7.14)$$

which would contradict (7.10).
We observe that (7.14) is equivalent to

$$2l(2\gamma - 2) + t\frac{l}{2} + s(\frac{l}{2} - 1) + \frac{r}{2}(2l - 2) + \frac{k}{2}(l - 2) \geq 0. \qquad (7.15)$$

Since $l \geq 2$ it is always satisfied for $\gamma > 0$, and when $\gamma = 0$ the inequality (7.15) becomes

$$l(t + r - 8) + (l - 2)(s + r + k) \geq 0.$$

The second addend is always positive, so for $t + r \geq 8$ it is satisfied.
Since by (7.11) we have that $t + r \geq 6$ with t and r even, we deduce that to contradict (7.14) the only possibility is $t + r = 6$ and $\gamma = 0$.
Now, from (7.11) and (7.6) with this two new conditions, we deduce that $h = s + \frac{1}{2}(6 - t + k)$ and $t + 4 \leq s + k$.
Finally, we observe that the inequality (7.13) with these new restrictions becomes

$$s + \frac{1}{2}(6 - t + k) - 1 \geq -4l + \frac{l}{2}(t + s) + (6 - t)\frac{2l-1}{2} + k\frac{l-1}{2} \Leftrightarrow$$
$$(2 - l)(s - t + k) \geq -10 + 4l.$$

As we have just seen, we need $s + k - t \geq 4 > 0$, so the only possibility for this inequality to be satisfied is $l = 2$. $\Diamond$

Considering again (7.13) for $l = 2$, $\gamma = 0$, $h = s + \frac{1}{2}(6 - t + k)$ and $t + r = 6$ we observe that we do not have equality. Then, in order to

satisfy the Riemann-Hurwitz formula, there should be more branch points (n' of such) satisfying

$$1 = 4 \sum \left(1 - \frac{1}{m_i} \right) \geq 4n' \frac{1}{2}$$

which is not possible, and hence such a morphism $D \to D/D_{2l}$ does not exist.

Consequently, by all the previous calculations, the self-intersection of $\tilde{B}$ is not positive. This finishes the proof of Theorem 7.1.3. □

7.1.2 Curves with higher genus

Now, we are going to consider degree two curves $\tilde{B}$ in $C^{(2)}$ with positive self-intersection and $p_a(\tilde{B}) \geq 2g - 1$.

Let $\tilde{B}$ be a curve in the symmetric square of a curve C of genus g with $\tilde{B} \cdot C_P = 2$. Let B be the normalization of $\tilde{B}$. As in the proof of Theorem 7.1.3 we can assume $\tilde{B} \neq \Delta_C$.

By Proposition 7.1.2 we know exactly when $\pi_C^*(\tilde{B})$ is reducible and $\tilde{B}^2$ is positive, so it remains to study $\tilde{D} := \pi_C^*(\tilde{B})$ irreducible. As in the proof of Theorem 7.1.3, B lies in a diagram (7.5) that does not complete with $B = D/\langle i \rangle$ and $C = D/\langle j \rangle$, where i and j are involutions such that $\langle i, j \rangle = D_n$.

7.1.2.1 Numerical and geometrical conditions

We want to decide if there exists a curve D where D_n acts in such a way that the actions of i and j give a diagram

$$D \longrightarrow B = D/\langle i \rangle$$
$$\downarrow$$
$$C = D/\langle j \rangle$$

such that the image of B in $C^{(2)}$, $\tilde{B}$, has positive self-intersection.

Remark 7.1.4. *If $\langle i, j \rangle = D_{2l+1}$, then i and j are conjugated involutions and so $C \cong B$. Moreover, $\nu(ij) = \nu((ij)^2)$ and hence $g = g(B) = p_a(\tilde{B})$ by Corollary 6.1.5. Consequently, the condition $p_a(\tilde{B}) \geq 2g - 1$ is not satisfied in this case. Therefore, we assume that $\langle i, j \rangle = D_{2l}$.*

We introduce some notation and recall some equalities from Section 6.1:

$$g = g(C) \quad b = g(B) \quad h = g(D)$$
$$s = \nu(j) \quad t = \nu(i) \quad r = \nu(ij) \quad\quad r + k = \nu((ij)^2)$$
$$\tilde{B}^2 = -h + 1 + s + \tfrac{1}{2}(r + k) \hspace{3cm} (7.16)$$
$$g = \tfrac{2h+2-s}{4} \hspace{3cm} h = \tfrac{4g-2+s}{2}$$
$$b = \tfrac{2h+2-t}{4} = g + \tfrac{s-t}{4} \hspace{2cm} p_a(\tilde{B}) = b + \tfrac{1}{4}k = g + \tfrac{s-t+k}{4}$$

The strategy is to find the restrictions on the numbers s, t, r and k given by our hypothesis and then, using the techniques described in Section 4.4, construct an action of D_n (and hence a diagram) or prove that it does not exist.

Let $\gamma = g(D/D_{2l})$. By the Riemann-Hurwitz formula we have that

$$h - 1 = 2l(2\gamma - 2) + 2l \sum_{P \in \text{Br}} \left(1 - \frac{1}{m_P}\right).$$

We know that there are at least $\frac{t+s}{2}$ branch points of order 2, $\frac{r}{2}$ branch points of order $2l$ and $\frac{k}{4}$ of order l (See discussion after (7.12) for details). Hence, we deduce that

$$h - 1 \geq 2l(2\gamma - 2) + (s + t)\frac{l}{2} + r\frac{2l - 1}{2} + k\frac{l - 1}{2}.$$

Observe that if $\gamma > 0$ then $\tilde{B}^2 < 0$, so we obtain that $\gamma = 0$, that is, $D/D_{2l} = \mathbb{P}^1$.

As in the proof of Theorem 7.1.3, we must impose for $\gamma = 0$ that

$$(l - 2)(s + r + k) < l(8 - (t + r)), \hspace{2cm} (7.17)$$

which is possible only if $t + r < 8$. Since t and r are even, this gives us the condition $t + r \leq 6$. Moreover, we have that if $l \geq 4$, then

$$\frac{s + r + k}{8 - (t + r)} < \frac{l}{l - 2} \leq 2. \hspace{2cm} (7.18)$$

This inequality will help us to determine the possible values of our parameters. We are going to study separately the cases $l \geq 4$, $l = 3$ and $l = 2$.

We want to give an epimorphism from a Fuchsian group Γ to a dihedral group D_{2l} in such a way that, if N is the kernel of that epimorphism, the action of D_{2l} on $D = \mathbb{H}/N$ has exactly the number of

fixed points determined by (t, r, s, k) and possibly some other coming from other powers of the automorphism ij. Once we have such an action, we can construct a non completing diagram (7.5) satisfying all our conditions with $C = D/\langle j \rangle$ and $B = D/\langle i \rangle$. Since we have imposed that $\mathbb{H}/\Gamma = \mathbb{P}^1$, the signature of the Fuchsian group will be $(0; m_1, \ldots, m_n)$. That is, it will have only elliptic elements, that will give the fixed points in the action of D_{2l} on $\mathbb{H}/N$, or equivalently, the ramification points of $\mathbb{H}/N \to \mathbb{H}/\Gamma$. We refer to Section 4.4 for a more detailed explanation.

Since we define an epimorphism $\rho : \Gamma \to D_{2l}$ once we know the images of the $x_i \in \Gamma$, we take the Fuchsian group in such a way that ρ respects the orders of the generators and that the signature satisfies the conditions in Poincaré's Theorem to assert its existence and that $\mu(\mathbb{H}/\Gamma) < \infty$, so we will not check it again. For each ρ, the associated vector of elements of G will clearly generate the whole group. Nevertheless, we will check in each case that the product one condition is satisfied.

Assume first that $l \geq 4$. By the conditions $t + r \leq 6$ and (7.18) together with the information on the parity of s and k given by (7.16), the possibilities for t, r, s and k are enumerated in the following list:

$$
\begin{array}{llll}
t = 0 & r = 0 & s = 0 & k = 0, 4, 8, 12 \\
 & & s = 4 & k = 0, 4, 8 \\
 & & s = 8 & k = 0, 4 \\
 & & s = 12 & k = 0 \\
 & r = 2 & s = 0 & k = 0, 4, 8 \\
 & & s = 4 & k = 0, 4 \\
 & & s = 8 & k = 0 \\
 & r = 4 & s = 0 & k = 0 \\
t = 2 & r = 0 & s = 2 & k = 0, 4, 8 \\
 & & s = 6 & k = 0, 4 \\
 & & s = 10 & k = 0 \\
 & r = 2 & s = 2 & k = 0 \\
t = 4 & r = 0 & s = 0 & k = 0, 4 \\
 & & s = 4 & k = 0 \\
 & r = 2 & s = 0 & k = 0
\end{array}
$$

Therefore, we have 28 different possibilities for (t, r, s, k).

Once we have the possible values for the 4-tuples (t, r, s, k), we will decide if they are compatible with the possible action of a group D_{2l} on a curve D. When it is compatible, we will describe the smooth

epimorphism that gives us the curve D as a ramified covering of $\mathbb{P}^1$ as explained in Section 4.4. The curves C and B are the quotients of D by the involutions j and i respectively.

As in the proof of Theorem 7.1.3 our conditions translate, in particular, to

$$h \leq s + \frac{1}{2}(r + k). \tag{7.19}$$

Next, we list the different numerical possibilities we have obtained and either we give the corresponding smooth epimorphism or prove that there exists no such epimorphism.

t = s = 0 There is no epimorphism $\Gamma \to D_{2l}$ since the only ramification would come from powers of ij which does not generate the whole group. With this we discard 8 cases.

t = r = k = 0, s = 4 With this values (7.19) is not satisfied.

t = r = 0, s = 4, k = 4 By (7.19) the only possibility is $g = 2$ and $h = 5$.

By the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$ we obtain that
$$4 = -4l + 2l \left(2 \left(1 - \frac{1}{2} \right) + 1 - \frac{1}{l} + \sum \left(1 - \frac{1}{m_i} \right) \right)$$
$$\Leftrightarrow 6 = 2l \sum \left(1 - \frac{1}{m_i} \right).$$

Since $l \geq 4$ and $1 - \frac{1}{m_i} \geq \frac{1}{2}$ we have that

$$3 = l \sum \left(1 - \frac{1}{m_i} \right) \geq 4\frac{1}{2}n' = 2n'$$

where n' is the number of branch points that correspond to the points fixed by neither i, j, ij nor $(ij)^2$. Therefore, we see that $n' = 1$. With this, we are going to determine the possible values of l and $m := m_1$:

$$3 = l \left(1 - \frac{1}{m} \right) \Rightarrow m = \frac{l}{l-3} \Rightarrow (l-3)|l \Rightarrow$$
$$l - 3 \leq \frac{l}{2} \Rightarrow l \leq 6 \Rightarrow l = 4 \text{ or } 6. \tag{7.20}$$

⋄ If $l = 4$ then $m = 4$, and the corresponding ramification points are fixed by an element of order 4 in D_8, that is, the rotation $(ij)^2$. But we have already considered all points fixed by $(ij)^2$, consequently, this case is not possible.

$\diamond$ If $l = 6$, then $m = 2$, and the corresponding ramification points are fixed by an element of order 2 in D_{12}. The only element of order 2 which we have not considered yet is $(ij)^6$. Hence, we have points fixed by j (or one of its conjugates $(ij)^{2\alpha}j$), by $(ij)^2$ and by $(ij)^6$. These elements generate a proper subgroup of D_{12} and therefore there is no smooth epimorphism $\Gamma \to D_{12}$ giving the prescribed ramification.

t = r = 0, s = 4, k = 8 By (7.19) the only possibilities are $g = 2$ and $h = 5$ together with $g = 3$ and $h = 7$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we find that there are no other branch points in any of the two cases, and thus, no other conjugation classes on the monodromy. Therefore, there is no epimorphism $\Gamma \to D_{2l}$ giving the prescribed ramification, because any subgroup generated by $(ij)^2$ and a conjugate of j is proper.

t = r = 0, s = 8, k = 0 By (7.19) the only possibility is $g = 2$ and $h = 7$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we find that there is another branch point. With the same arguments that we used after (7.20) we deduce that this case do not define an epimorphism $\Gamma \to D_{2l}$ giving the prescribed ramification.

t = r = 0, s = 8, k = 4 By (7.19) the only possibilities are $g = 2$ and $h = 7$ together with $g = 3$ and $h = 9$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we find that there are no other branch points in any of the two cases, and thus, no other conjugation classes on the monodromy. Therefore, there is no epimorphism $\Gamma \to D_{2l}$ giving the prescribed ramification because the elements given do not generate the whole group.

t = r = 0, s = 12, k = 0 By (7.19) the only possibilities are $g = 2$ and $h = 9$ together with $g = 3$ and $h = 11$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we find that there are no other branch points in any of the two cases, and thus, no other conjugation classes on the monodromy. Therefore, there is no epimorphism $\Gamma \to D_{2l}$ giving the prescribed ramification because the elements given do not generate the whole group.

t = 0, r = 2, s = 4, k = 0 By (7.19) the only possibility is $g = 2$ together with $h = 5$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we find that there is another branch point, with $m = 2$ and that $l = 5$. Thus, the additional branch point needs to come from points fixed by $(ij)^5 \in D_{10}$.

Consequently we impose one branch point coming from ij (image of points fixed by it), 2 branch points coming from j (and its conjugates $(ij)^{2\alpha}j$) and one coming from $(ij)^5$. Hence, we take a Fuchsian group

$$\Gamma = \langle x_1, x_2, x_3, x_4 \mid x_1^{10} = x_2^2 = x_3^2 = x_4^2 = Id \rangle.$$

That is, a Fuchsian group with signature $(0; 10, 2, 2, 2)$, and we define the following epimorphism:

$$\begin{array}{rcl} \Gamma & \to & D_{10} \\ x_1 & \to & ij \\ x_2 & \to & (ij)^5 \\ x_3 & \to & (ij)^4 j \\ x_4 & \to & j. \end{array}$$

Notice that $ij(ij)^5(ij)^4jj = 1$.

t = 0, r = 2, s = k = 4 By (7.19) the only possibilities are $g = 2$ and $h = 5$ together with $g = 3$ and $h = 7$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we deduce that none of these cases is possible.

t = 0, r = 2, s = 8, k = 0 By (7.19) the only possibilities are $g = 2$ and $h = 7$ together with $g = 3$ and $h = 9$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we deduce that none of these cases is possible.

t = 2, r = 0, s = 2, k = 0 By (7.19) we find that these values do not satisfy our hypothesis.

t = 2, r = 0, s = 2, k = 4 By (7.19) the only possibility is $g = 2$ together with $h = 4$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we find that there is one more branch point

with $m = 2$ and that $l = 5$. Thus, the additional branch point needs to come from points fixed by $(ij)^5 \in D_{10}$.

Consequently, we have one branch point coming from j (or one of its conjugates $(ij)^{2\alpha}j$), one coming from i (or one of its conjugates $(ij)^{2\alpha}i$), one coming from $(ij)^2$ and one coming from $(ij)^5$. Hence, we take a Fuchsian group with signature $(0; 5, 2, 2, 2)$ and define the following epimorphism:

$$
\begin{array}{rcl}
\Gamma & \to & D_{10} \\
x_1 & \to & (ij)^2 \\
x_2 & \to & (ij)^5 \\
x_3 & \to & (ij)^2 i \\
x_4 & \to & j.
\end{array}
$$

Notice that $(ij)^2(ij)^5(ij)^2 ij = 1$.

t = 2, r = 0, s = 2, k = 8 By (7.19) the only possibilities are $g = 2$ and $h = 4$ together with $g = 3$ and $h = 6$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we deduce that none of these cases is possible.

t = 2, r = 0, s = 6, k = 0 By (7.19) the only possibility is $g = 2$ together with $h = 6$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we find that there is another branch point with $m = 2$ and that $l = 5$. Thus, the additional branch point needs to come from points fixed by $(ij)^5 \in D_{10}$.

Consequently, we have 3 branch points coming from j (or one of its conjugates $(ij)^{2\alpha}j$), one coming from i (or one of its conjugates $(ij)^{2\alpha}i$) and one coming from $(ij)^5$. Hence, we take a Fuchsian group with signature $(0; 2, 2, 2, 2, 2)$ and define the following epimorphism:

$$
\begin{array}{rcl}
\Gamma & \to & D_{10} \\
x_1 & \to & (ij)^5 \\
x_2 & \to & (ij)^4 i \\
x_3,\ x_4\ x_5 & \to & j.
\end{array}
$$

Notice that $(ij)^5(ij)^4 ijjj = 1$.

t = 2, r = 0, s = 6, k = 4 By (7.19) the only possibilities are $g = 2$ and $h = 6$ together with $g = 3$ and $h = 8$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we deduce that none of these cases is possible.

t = 2, r = 0, s = 10, k = 0 By (7.19) the only possibilities are $g = 2$ and $h = 8$ together with $g = 3$ and $h = 10$.

Putting all this information in the Riemann-Hurwitz formula for $D \to D/D_{2l} = \mathbb{P}^1$, we deduce that none of these cases is possible.

t = 2, r = 2, s = 2, k = 0 By (7.19) we find that these values do not satsfy our hypothesis.

t = 4, s = 0 or 4 In all these cases $b \leq g$ so our hypothesis are not satisfied.

From the previous 28 cases we have seen that only 3 of them are giving pairs (B, C) with $B \to C^{(2)}$ of degree one.

Assume now that l = 3.

We are going to consider now the case $\langle i, j \rangle = D_6$. For $l = 3$ the condition (7.17) becomes $s + r + k < 3(8 - (t + r))$, i.e. $24 > s + 4r + k + 3t$. Since s, r, k and t are even integers this is equivalent to

$$s + 4r + k + 3t \leq 22. \tag{7.21}$$

In D_6 there are six conjugacy classes:

$$\{Id\}, \ \{i, (ij)^2 i, (ij)^4 i\}, \ \{j, (ij)^2 j, (ij)^4 j\},$$
$$\{ij, (ij)^5\}, \ \{(ij)^2, (ij)^4\} \text{ and } \{(ij)^3\}.$$

Let p be the number of points fixed by $(ij)^3$ and not by ij. Since there are only six classes of conjugation, all branch points in the morphism $D \to D/D_6$ will be considered in either t, s, r, p or k, thus, the Riemann-Hurwitz formula reads:

$$\begin{aligned} 2h - 2 &= 12(-2) + 12\left(\frac{s+t}{2}\left(1 - \frac{1}{2}\right) + \frac{r}{2}\left(1 - \frac{1}{6}\right) + \frac{k}{4}\left(1 - \frac{1}{3}\right) + \frac{p}{6}\left(1 - \frac{1}{2}\right)\right) \\ &= -24 + 3t + 3s + 5r + 2k + p. \end{aligned}$$

Hence, $h = \frac{1}{2}(-22 + 3s + 3t + 5r + 2k + p)$. If we consider the second condition in (7.19) with these values, then we deduce that

$$\begin{aligned} \tfrac{1}{2}(-22 + 3s + 3t + 5r + 2k + p) &\leq s + \tfrac{1}{2}(r + k) \\ \Leftrightarrow s + 3t + 4r + k + p &\leq 22. \end{aligned} \tag{7.22}$$

Notice that if it is satisfied, then also (7.21) is satisfied.

Now, we observe that we can embed D_6 in S_6 in such a way that i is odd and j is even (thus ij is odd, $(ij)^2$ is even and $(ij)^3$ is odd). Since we will need the product one condition when constructing the morphism from a Fuchsian group, we need to impose $\frac{t}{2} + \frac{r}{2} + \frac{p}{6}$ to be even, or which is the same, $t + r + \frac{p}{3}$ multiple of four. By this and inequality (7.22) we can reduce the possible values of t, r and p to the following list:

$$
\begin{array}{lll}
t = 0 & r = 0 & p = 0, 12 \\
t = 0 & r = 2 & p = 6 \\
t = 0 & r = 4 & p = 0 \\
t = 2 & r = 0 & p = 6 \\
t = 2 & r = 2 & p = 0 \\
t = 4 & r = 0 & p = 0.
\end{array}
$$

Furthermore, we can also embed D_6 in S_6 in such a way that i is even and j is odd (thus ij is odd, $(ij)^2$ is even and $(ij)^3$ is odd). Since we will require the product one condition when constructing the morphism from a Fuchsian group, we need to impose $\frac{s}{2} + \frac{r}{2} + \frac{p}{6}$ to be even, or which is the same, $s + r + \frac{p}{3}$ multiple of four.

Next, we study each of these cases using both the numerical hypothesis and the conditions for the existence of a smooth epimorphism giving the prescribed ramification. We list the different cases and study them separately.

t = r = p = 0 We would have only points fixed by j or $(ij)^2$ which do not generate all D_6.

t = r = 0, p = 12 We need at least one branch point from $(ij)^2$ and one from j to generate, hence $k \geq 4$ and $s \geq 2$. Since we need $s + r + \frac{p}{3}$ to be multiple of four we deduce that s is multiple of four. By (7.22) we obtain that $s = 4$ and $k = 4$. In this case $h = 5$, $g = 2$ and $p_a(\tilde{B}) = 4 = 2g$. We define the smooth epimorphism

$$
\begin{array}{rcl}
\Gamma & \to & D_6 \\
x_1, \, x_2 & \to & (ij)^3 \\
x_3 & \to & (ij)^2 \\
x_4 & \to & (ij)^4 j \\
x_5 & \to & j.
\end{array}
$$

Notice that $(ij)^3(ij)^3(ij)^2(ij)^4jj = 1$.

t = 0, r = 2, p = 6 We need at least one branch point coming from j to generate, so $s \geq 2$. Since we need $s + r + \frac{p}{3}$ to be multiple of

four we obtain that s is multiple of four. From (7.22) we deduce that $s + k \leq 8$. Therefore, we have three possibilities: $s = 4$ with $k = 0$ or $k = 4$, and $s = 8$ with $k = 0$.

⋄ **s = 4 k = 0** In this case $h = 3$ and $g = 1$, hence this case is not considered in our discussion.

⋄ **s = 4 k = 4** In this case $h = 7$, $g = 3$ and $p_a(\tilde{B}) = 5 = 2g - 1$. We define the smooth epimorphism

$$
\begin{aligned}
\Gamma &\rightarrow D_6 \\
x_1 &\rightarrow ij \\
x_2 &\rightarrow (ij)^3 \\
x_3 &\rightarrow (ij)^2 \\
x_4,\, x_5 &\rightarrow j.
\end{aligned}
$$

Notice that $ij(ij)^3(ij)^2jj = 1$.

⋄ **s = 8 k = 0** In this case $h = 9$, $g = 3$ and $p_a(\tilde{B}) = 5 = 2g - 1$. We define the smooth epimorphism

$$
\begin{aligned}
\Gamma &\rightarrow D_6 \\
x_1 &\rightarrow ij \\
x_2 &\rightarrow (ij)^3 \\
x_3 &\rightarrow (ij)^2 j \\
x_4,\, x_5,\, x_6 &\rightarrow j.
\end{aligned}
$$

Notice that $ij(ij)^3(ij)^2jjjj = 1$.

t = 0, r = 4, p = 0 We need at least one branch point coming from j to generate. Therefore $s \geq 2$ and since we need $s + r + \frac{p}{3}$ to be multiple of four we obtain that s is multiple of four. From (7.22) we deduce that $s + k \leq 6$. Therefore, the only possible case is **s = 4 with k = 0**.

In this case $h = 5$, $g = 2$ and $p_a(\tilde{B}) = 3 = 2g - 1$. We define the smooth epimorphism

$$
\begin{aligned}
\Gamma &\rightarrow D_6 \\
x_1 &\rightarrow ij \\
x_2 &\rightarrow ji \\
x_3,\, x_4 &\rightarrow j.
\end{aligned}
$$

Notice that $ijjijj = 1$.

t = 2, r = 0, p = 6 Since we need $s + r + \frac{p}{3}$ to be multiple of four we obtain that $s = 2 + 4\alpha$ with $\alpha \in \mathbb{Z}_{\geq 0}$. From (7.22) we deduce that $s + k \leq 10$. We are going to study the different cases:

⬦ **s = 2 k = 0** Then $h = -2$, which is impossible.

⬦ **s = 2 k = 4** Then $h = 2$ and $g = 1$, hence this case is not considered in our discussion.

⬦ **s = 2 k = 8** Then $h = 6$, $g = 3$ and $p_a(\tilde{B}) = 5 = 2g - 1$. We define the smooth epimorphism

$$
\begin{aligned}
\Gamma &\rightarrow D_6 \\
x_1 &\rightarrow j \\
x_2 &\rightarrow i \\
x_3 &\rightarrow (ij)^3 \\
x_4,\ x_5 &\rightarrow (ij)^2.
\end{aligned}
$$

Notice that $ji(ij)^3(ij)^2(ij)^2 = 1$.

⬦ **s = 6 k = 0** Then $h = 4$ and $g = 1$, hence this case is not considered in our discussion.

⬦ **s = 6 k = 4** Then $h = 8$, $g = 3$ and $p_a(\tilde{B}) = 5 = 2g - 1$. We define the smooth epimorphism

$$
\begin{aligned}
\Gamma &\rightarrow D_6 \\
x_1 &\rightarrow (ij)^3 \\
x_2 &\rightarrow (ij)^2 \\
x_3 &\rightarrow i \\
x_4,\ x_5,\ x_6 &\rightarrow j.
\end{aligned}
$$

We notice that $(ij)^3(ij)^2ijjj = 1$.

t = 2, r = 2, p = 0 Since we need $s + r + \frac{p}{3}$ to be multiple of four we obtain that $s = 2 + 4\alpha$ with $\alpha \in \mathbb{Z}_{\geq 0}$. From (7.22) we deduce that $s + k \leq 8$. We are going to study the different cases:

⬦ **s = 2 k = 0** Then $h = 0$, hence this case is not considered in our discussion.

⬦ **s = 2 k = 4** Then $h = 4$, $g = 2$ and $p_a(\tilde{B}) = 3$. We define the smooth epimorphism

$$
\begin{aligned}
\Gamma &\rightarrow D_6 \\
x_1 &\rightarrow i \\
x_2 &\rightarrow j \\
x_3 &\rightarrow ij \\
x_4 &\rightarrow (ij)^4.
\end{aligned}
$$

Notice that $ijij(ij)^4 = 1$.

$\diamond$ **s = 6 k = 0** Then $h = 6$, $g = 2$ and $p_a(\tilde{B}) = 3$. We define the smooth epimorphism

$$
\begin{array}{rcl}
\Gamma & \to & D_6 \\
x_1,\ x_2,\ x_3 & \to & j \\
x_4 & \to & i \\
x_5 & \to & ij.
\end{array}
$$

Notice that $jjjiij = 1$.

t = 4, r = 0, p = 0 We need at least one branch point coming from j to generate. Since we need $s + r + \frac{p}{3}$ to be multiple of four we obtain that s is multiple of four. From (7.22) we deduce that $s + k \leq 10$. We are going to study the different cases:

$\diamond$ **s = 4 k = 0** Then $h = 1$ and $g = 0$, hence this case is not considered in our discussion.

$\diamond$ **s = 4 k = 4** Then $h = 5$, $g = 2$ and $p_a(\tilde{B}) = 3 = 2g - 1$. We define the smooth epimorphism

$$
\begin{array}{rcl}
\Gamma & \to & D_6 \\
x_1,\ x_2 & \to & i \\
x_3 & \to & (ij)^4 \\
x_4 & \to & (ij)^2 j \\
x_5 & \to & j.
\end{array}
$$

Notice that $ii(ij)^4(ij)^2 jj = 1$.

$\diamond$ **s = 8 k = 0** Then $h = 7$, $g = 2$ and $p_a(\tilde{B}) = 3 = 2g - 1$. We define the smooth epimorphism

$$
\begin{array}{rcl}
\Gamma & \to & D_6 \\
x_1,\ x_2 & \to & i \\
x_3,\ x_4,\ x_5,\ x_6 & \to & j.
\end{array}
$$

Notice that $iijjjj = 1$.

Therefore, we have found 10 more cases giving pairs (B, C) with $B \to C^{(2)}$ of degree one.

Assume finally that l = 2.

We consider now the case $\langle i, j \rangle = D_4$. For $l = 2$ the condition (7.17) becomes $0 < 2(8 - (t + r))$, i.e. $t + r < 8$ and since t and r are even integers, it is equivalent to $t + r \leq 6$.

In D_4 we have five conjugacy classes:

$$\{Id\},\ \{i,(ij)^2i\},\ \{j,(ij)^2j\},\ \{ij,(ij)^3\} \text{ and } \{(ij)^2\}.$$

First, we observe that we can embed D_4 in S_4 in such a way that i is odd and j is even (thus ij is odd and $(ij)^2$ is even). Since we will need the product one condition when constructing the morphism from a Fuchsian group, we need to impose $\frac{t}{2} + \frac{r}{2}$ to be even, or what is the same, $t+r$ to be multiple of four. Hence, we have only four possibilities for the integers t and r:

$$
\begin{aligned}
t &= 0 \quad r = 0,4 \\
t &= 2 \quad\ \ r = 2 \\
t &= 4 \quad\ \ r = 0.
\end{aligned}
$$

Moreover, we can also embed D_4 in S_4 in such a way that i is even and j is odd (thus ij is odd and $(ij)^2$ is even). Since we will need the product one condition when constructing the morphism from a Fuchsian group, we need to impose $\frac{s}{2} + \frac{r}{2}$ to be even, or what is the same, $s + r$ to be multiple of four.

Since there are only five conjugacy classes in D_4, all branch points in $D \to D/D_4$ will be considered in either t, s, r or k, thus the Riemann-Hurwitz formula reads:

$$
\begin{aligned}
2h - 2 &= 8(-2) + 8\left(\tfrac{s+t}{2}\left(1 - \tfrac{1}{2}\right) + \tfrac{r}{2}\left(1 - \tfrac{1}{4}\right) + \tfrac{k}{4}\left(1 - \tfrac{1}{2}\right)\right) \\
&= -16 + 2s + 2t + 3r + k.
\end{aligned}
$$

Hence, $h = -7 + s + t + \frac{3}{2}r + \frac{1}{2}k$. If we consider the second condition in (7.19) with these values, then we obtain that

$$h \le s + \frac{1}{2}(r+k) \Leftrightarrow -7 + s + t + \frac{3}{2}r + \frac{1}{2}k \le s + \frac{1}{2}(r+k) \Leftrightarrow t + r \le 7.$$

Thus, this condition will always be satisfied.

The condition $g \ge 2$ is equivalent to $s \le 2h - 6$, which with the expression of h becomes

$$s \le -14 + 2s + 2t + 3r + k - 6 \Leftrightarrow 20 \le s + 2t + 3r + k. \tag{7.23}$$

We study now each case using both the numerical hypothesis and the conditions for the existence of a smooth epimorphism giving the prescribed ramification. We list them and study them separately.

t = r = 0 There can only be points fixed by j and $(ij)^2$, that do not generate D_4.

t = 0, r = 4 We need that $s > 0$ to obtain a generating vector. Since $s + r$ should be multiple of four, we obtain that s is multiple of four. From (7.23) we deduce that $8 \leq s + k$. With this we have that

$$h = -1 + s + \frac{1}{2}k \quad \text{and} \quad g = \frac{s + k}{4}.$$

So, there are 2 branch points coming from ij, $\frac{s}{2}$ branch points coming from j (an even number) and $\frac{k}{4}$ branch points coming from $(ij)^2$. Depending on the parity of $\frac{k}{4}$ we define epimorphisms from a Fuchsian group:

if $\frac{k}{4}$ is even: if $\frac{k}{4}$ is odd:

$$
\begin{aligned}
\Gamma &\to D_4 \\
x_1 &\to ij \\
x_2 &\to ji \\
x_3, \ldots, x_{2+\frac{s}{2}} &\to j \\
x_{3+\frac{s}{2}}, \ldots, x_{2+\frac{s}{2}+\frac{k}{4}} &\to (ij)^2
\end{aligned}
$$

$$
\begin{aligned}
\Gamma &\to D_4 \\
x_1 &\to ij \\
x_2 &\to ji \\
x_3, \ldots, x_{1+\frac{s}{2}} &\to j \\
x_{2+\frac{s}{2}} &\to j(ij)^2 \\
x_{3+\frac{s}{2}}, \ldots, x_{3+\frac{s}{2}+\frac{k}{4}} &\to (ij)^2.
\end{aligned}
$$

Notice that in both cases the product one condition is satisfied.

t = 2, r = 2 We need that $s > 0$ to be able to have the product one condition. Since $s + r$ should be multiple of four, we obtain that $s = 4\alpha + 2$ with $\alpha \in \mathbb{Z}_{\geq 0}$. From (7.23) we deduce that $10 \leq s + k$. With this, we obtain that

$$h = -2 + s + \frac{1}{2}k \quad \text{and} \quad g = \frac{s + k - 2}{4}.$$

So, there are 1 branch point coming from i, 1 branch point coming from ij, $\frac{s}{2}$ branch points coming from j (an odd number) and other $\frac{k}{4}$ branch points coming from $(ij)^2$. Depending on the parity of $\frac{k}{4}$ we define epimorphisms from a Fuchsian group:

if $\frac{k}{4}$ is even: if $\frac{k}{4}$ is odd:

$$
\begin{aligned}
\Gamma &\to D_4 \\
x_1 &\to i \\
x_2 &\to ij \\
x_3 &\to j \\
x_4, \ldots, x_{3+\frac{s}{2}} &\to j \\
x_{4+\frac{s}{2}}, \ldots, x_{3+\frac{s}{2}+\frac{k}{4}} &\to (ij)^2
\end{aligned}
$$

$$
\begin{aligned}
\Gamma &\to D_4 \\
x_1 &\to i \\
x_2, \ldots, x_{1+\frac{s}{2}} &\to j \\
x_{2+\frac{s}{2}} &\to ij \\
x_{3+\frac{s}{2}}, \ldots, x_{2+\frac{s}{2}+\frac{k}{4}} &\to (ij)^2.
\end{aligned}
$$

t = 4, r = 0 We need that $s > 0$ to generate. Since $s + r$ should be multiple of four, we obtain that s is multiple of four. From (7.23) we deduce that $12 \leq s + k$. With this, we obtain that

$$h = -3 + s + \frac{1}{2}k \ \text{ and } \ g = \frac{s + k - 4}{4}.$$

So, there are 2 branch point coming from i, $\frac{s}{2}$ branch points coming from j (an even number) and other $\frac{k}{4}$ branch points coming from $(ij)^2$. Depending on the parity of $\frac{k}{4}$ we construct epimorphisms from a Fuchsian group:

if $\frac{k}{4}$ is even:

$$
\begin{aligned}
\Gamma &\to D_4 \\
x_1 &\to i \\
x_2 &\to i \\
x_3, \dots, x_{2+\frac{s}{2}} &\to j \\
x_{3+\frac{s}{2}}, \dots, x_{2+\frac{s}{2}+\frac{k}{4}} &\to (ij)^2
\end{aligned}
$$

if $\frac{k}{4}$ is odd:

$$
\begin{aligned}
\Gamma &\to D_4 \\
x_1 &\to i \\
x_2 &\to i \\
x_3, \dots, x_{1+\frac{s}{2}} &\to j \\
x_{2+\frac{s}{2}} &\to j(ij)^2 \\
x_{3+\frac{s}{2}}, \dots, x_{2+\frac{s}{2}+\frac{k}{4}} &\to (ij)^2.
\end{aligned}
$$

We have, therefore, 3 more families of morphisms giving pairs (B, C) with $B \to C^{(2)}$ of degree one.

7.1.2.2 Classification

Now that we have defined the curves D giving diagrams characterizing curves $\tilde{B} \subset C^{(2)}$ with our conditions, we are going to study the properties of the curves involved in each case. In particular, we compute the genus of D, C and B, the arithmetic genus of $\tilde{B}$ and its self-intersection. Moreover, we compute the dimension of each family, with special attention to the dimension of the image in the moduli space of curves of genus $g(C)$ for each family.

To begin with, we prove a lemma that we use repeatedly:

Lemma 7.1.5. *Let D be a curve with an action of D_{2l}. Let i and j be two involutions generating this dihedral group. Denote by $C = D/\langle j \rangle$. Since $(ij)^l$ is in the center of D_{2l}, its action descends to C. Then, if β is the induced automorphism in C, $\nu(\beta) = \frac{1}{2}(\nu(i) + \nu((ij)^l))$ if l is odd, and $\nu(\beta) = \frac{1}{2}(\nu(j) + \nu((ij)^l))$ if l is even.*

Proof. Consider C embedded in $D^{(2)}$ as $\{P + j(P), \ P \in D\}$. In this way, the action of β on C is just

$$\beta(P + j(P)) = (ij)^l(P) + (ij)^l(j(P)) = (ij)^l(P) + j((ij)^l(P)).$$

A point will be fixed by β when either $P = (ij)^l(P)$ or $P = (ij)^l j(P)$. From the former we obtain $\frac{1}{2}\nu((ij)^l)$ points of C fixed by β and from the later $\frac{1}{2}\nu((ij)^l j)$. Notice that when l is odd $(ij)^l j$ is conjugated to i in D_{2l}, so $\nu((ij)^l j) = \nu(i)$ and when l is even $(ij)^l j$ is conjugated to j in D_{2l} and hence $\nu((ij)^l j) = \nu(j)$. $\qquad\square$

Theorem 7.1.6 (Classification). *All pairs of smooth curves (C, B) with $B \xrightarrow{(1:1)} C^{(2)}$ and image $\tilde{B}$ such that $p_a(\tilde{B}) \geq 2g(C) - 1$, $\tilde{B}^2 > 0$ and $\tilde{B} \cdot C_P = 2$ fall in one of the following cases:*

0. *C is a curve of genus 2 with an action of an automorphism of order 10, σ, such that $\nu(\sigma) = 1$, $\nu(\sigma^2) = 3$, $\nu(\sigma^5) = 6$ and $\tilde{B}$ is the symmetrization of the graph of σ. There is a finite number of isomorphism classes of curves C with such an automorphism.*

1. *There is a curve D with an action of D_{10} such that C and B are the quotients of D by certain involutions $i, j \in D_{10}$.*

   There are three topological types of actions on D, giving three families with the following properties:

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	Moduli dim. of D	Moduli dim. of C	Other properties
5	2	3	1	1	1	*D hyperelliptic* *$\tilde{B}$ smooth*
4	2	2	1	1	1	*D hyperelliptic* *$\tilde{B}$ has 1 node*
6	2	3	1	2	2?	*D bielliptic* *$\tilde{B}$ smooth*

   Furthermore, in all three families the curve B is hyperelliptic and $p_a(\tilde{B}) = 2g(C) - 1$.

2. *There is a curve D with an action of D_6 such that C and B are the quotients of D by certain involutions $i, j \in D_6$. There are ten topological types of actions.*

   There is one topological type of action on D giving a family such that

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	Moduli dim. of D	Moduli dim. of C	Other properties
5	2	3	2	2	2	*$\tilde{B}$ has 1 node*

   Furthermore, in this family the curves D and B are hyperelliptic and $p_a(\tilde{B}) = 2g(C)$.

Moreover, there are nine topological types of actions on D, giving nine families with the following properties

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	Moduli dim. of D	Moduli dim. of C	Other properties
5	2	3	2	1	1	$\tilde{B}$ smooth
7	3	4	1	2	2	$\tilde{B}$ has 1 node
6	3	3	1	2	2	$\tilde{B}$ has 2 nodes
4	2	2	2	1	1	$\tilde{B}$ has 1 node
6	2	3	2	2	2	$\tilde{B}$ smooth
9	3	5	1	3	3	$\tilde{B}$ smooth
8	3	4	1	3	3	$\tilde{B}$ has 1 node
5	2	2	2	2	2	$\tilde{B}$ has 1 node
7	2	3	2	3	2	$\tilde{B}$ smooth

Furthermore, in all nine families B and C are bielliptic and $p_a(\tilde{B}) = 2g(C) - 1$.

3. *There is a curve D with an action of D_4 such that C and B are the quotients of D by certain involutions $i, j \in D_4$. There are three families of topological types of actions on D with the following characteristics:*

$g(D)$	$g(C)$	$g(B)$	$\tilde{B}^2$	Moduli dim. of D	Other properties
$-1 + s + \frac{1}{2}k$	$\frac{s+k}{2}$	$\frac{2s+k}{4}$	4	$\frac{2s+k-4}{4}$	$\tilde{B}$ has $\frac{k}{4}$ nodes $s + k \geq 8$
$-2 + s + \frac{1}{2}k$	$\frac{s+k-2}{2}$	$\frac{2s+k-4}{4}$	4	$\frac{2s+k-4}{4}$	$\tilde{B}$ has $\frac{k}{4}$ nodes $s + k \geq 10$
$-3 + s + \frac{1}{2}k$	$\frac{s+k-4}{2}$	$\frac{2s+k-8}{4}$	4	$\frac{2s+k-4}{4}$	$\tilde{B}$ has $\frac{k}{4}$ nodes $s + k \geq 12$

Furthermore, in all three families C is hyperelliptic, with any possible genus, and $p_a(\tilde{B}) = 2g(C)$.

Proof. We have already proven in Proposition 7.1.2 that the only pair (C, B) fulfilling all the conditions in the theorem and with $\pi_C^*(\tilde{B})$ reducible is the one described in point $0.$, and as we have seen, there is a finite number of curves C with such an automorphism.

From now on we assume that $\tilde{D} = \pi^*(\tilde{B})$ is irreducible. As we have explained in the beginning of the proof of Theorem 7.1.3, we need

to study the diagrams that we have found during the discussion in Section 7.1.2.1 to complete the proof of the theorem.

Given the smooth epimorphism from a Fuchsian group Γ, $\Gamma \to D_{2l}$ in each case, we find the curve D with the action of $D_{2l} = \langle i, j \rangle$ as described by the epimorphism. We consider $C = D/\langle j \rangle$ and $B = D/\langle i \rangle$. We will denote by $\beta_C \in \mathrm{Aut}(C)$ the action on C induced by $(ij)^l$ and by β_B the action induced on B.

By the discussion in Section 6.1, we are able to compute the genus of D, C and B, the arithmetic genus of $p_a(\tilde{B})$ and the self-intersection of $\tilde{B}$ once we know the number of fixed points corresponding to the different conjugacy classes in D_{2l}.

As we have seen in Section 4.5, the curves D with the prescribed action of D_n are parametrized by an algebraic variety. We call this variety $\mathcal{D}$. The image of $\mathcal{D}$ in the moduli space $\mathcal{M}_h$, given by forgetting the action, is an irreducible variety of the same dimension. We want to study the morphism η from $\mathcal{D}$ to $\mathcal{M}_g$ that sends (D, ρ) to $[C]$, and we wonder in which cases it has positive dimensional fibers. Since in each case the topological action ρ is fixed, we will omit it and denote a point $(D, \rho) \in \mathcal{D}$ simply as $[D]$.

We are going to study each case separately to finish the proof of our theorem. Before, we make some general remarks.
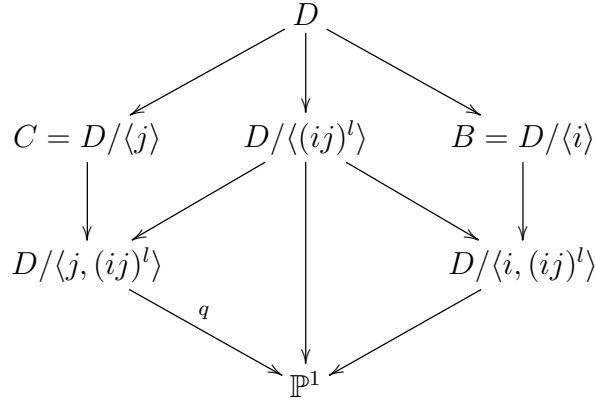
First, we consider the morphism $q : D/\langle j, (ij)^l \rangle \to D/D_{2l} \cong \mathbb{P}^1$. We observe that it is not Galois since $\langle j, (ij)^l \rangle$ is not normal in D_{2l} for $l \neq 2$, and it is Galois for $l = 2$.

Second, we note that to give a curve D with an action of D_{2l}, is equivalent to give $\mathbb{P}^1$ with a certain number, n, of marked points, and the branching data for the map. To avoid automorphisms, we can fix three of these points to be 0, 1 and ∞. As we change the rest of points, we change the pair (D, ρ) in the family $\mathcal{D}$, and therefore, $\mathcal{D}$ has dimension $n - 3$.

Third, we are going to see that in all cases the curve C is γ-hyperelliptic for $\gamma = 0, 1$ with $\beta_C = i_\gamma$, hence, to give the curve C is equivalent to give $\mathbb{P}^1$ or the curve E, with the branch points of p (the γ-hyperelliptic morphism) marked (m points).

We have the following diagram of curves for each described action

of $D_{2l} = \langle i, j \rangle$ on D:

$$
\begin{array}{ccc}
& D & \\
\swarrow & \downarrow & \searrow \\
C = D/\langle j \rangle \quad D/\langle (ij)^l \rangle \quad B = D/\langle i \rangle \\
\downarrow \qquad \swarrow \qquad \searrow \qquad \downarrow \\
D/\langle j, (ij)^l \rangle \qquad\qquad D/\langle i, (ij)^l \rangle \\
\searrow \quad\scriptstyle q \qquad \downarrow \qquad \swarrow \\
& \mathbb{P}^1 &
\end{array}
$$

We observe that for $l \neq 2$ the curves $D/\langle j, (ij)^l$ and $D/\langle i, (ij)^l \rangle$ are isomorphic because $\langle j, (ij)^l$ and $\langle i, (ij)^l \rangle$ are conjugate.

Hence, η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x_1, \ldots, x_{n-3}\}$ together with the monodromy description, to the curve determined by $\{F; x_1, \ldots, x_m\}$ were F is the genus γ curve given by the quotient of C by its γ-hyperelliptic involution. Therefore, study the fibers of η is equivalent to study the fibers of the morphism $\mathcal{M}_{0,n} \times \{\rho\} \to \mathcal{M}_{\gamma,m}$ defined by the previous correspondence.

Given a curve $[C]$ in the image of η, we consider the data determined by its γ-hyperelliptic involution, which is unique for $\gamma = 0$ and there is at most a finite number of possibilities for $\gamma = 1$. If we know the morphism q, we can recover the data which determines $[D]$ taking the images of the branch points of p together with the rest of branch points of q. Therefore, we can translate the question on the dimension of the fiber of η to a question on the number of possible morphisms q for a given curve $[C]$ in the image of η. We want to determine if given $\mathbb{P}^1$ (respectively E) with m marked points and some information about the branching type of q, then there are a finite number of possible q's, and hence a finite number of curves $[C]$.

We begin with the diagrams described by the action of D_{10}. We have seen that there are three types of topological actions given by D_{10} satisfying our hypothesis. We are going to study each of them separately.
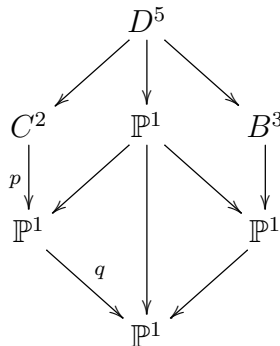
1. The first case is described by the smooth epimorphism of groups

$$
\begin{array}{rcl}
\rho: \ \Gamma & \to & D_{10} \\
x_1 & \to & ij \\
x_2 & \to & (ij)^5 \\
x_3 & \to & (ij)^4 j \\
x_4 & \to & j.
\end{array}
$$

The action of D_{10} on a curve of genus $g(D) = 5$ defined by ρ gives a curve of genus $g(B) = 3$ inside $C^{(2)}$ with $g(C) = 2$. Moreover, the curve $\tilde{B}$ is smooth and $\tilde{B}^2 = 1$.

Now, consider the action of the automorphism $(ij)^5$ on the curve D. Since $\nu((ij)^5) = 10 + 2 = 12$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle (ij)^5 \rangle) = \frac{1}{4}(2 \cdot 5 + 2 - 12) = 0$, so $D/\langle (ij)^5 \rangle = \mathbb{P}^1$ and therefore D is hyperelliptic.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 6$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle \beta_C \rangle) = \frac{1}{4}(4 + 2 - 6) = 0$ and therefore β_C is the hyperelliptic involution in C. In the same way, β_B is the hyperelliptic involution in B. We obtain the following diagram:

$$
\begin{array}{ccccc}
 & & D^5 & & \\
 & \swarrow & \downarrow & \searrow & \\
C^2 & & \mathbb{P}^1 & & B^3 \\
{\scriptstyle p}\downarrow & \swarrow & \downarrow & \searrow & \downarrow \\
\mathbb{P}^1 & & & & \mathbb{P}^1 \\
 & \searrow_{\ q} & \downarrow & \swarrow & \\
 & & \mathbb{P}^1 & &
\end{array}
$$

where the superindex denote the genus of the curves.

We observe that $\mathcal{D}$ is the 1-dimensional family of all curves of genus 5 with maximal dihedral symmetry (see [BCGMG03]).
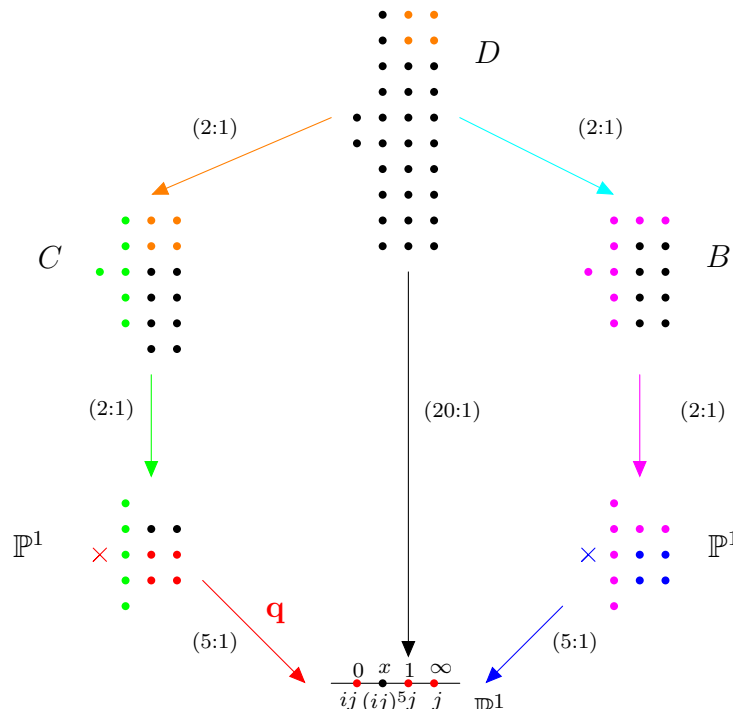
Claim: The map η is finite.

Proof of the claim: First, notice that the morphism $q : \mathbb{P}^1 \to \mathbb{P}^1$ has 3 branch points, and therefore, there are a finite number of such morphisms modulo automorphisms of $\mathbb{P}^1$.

Second, we observe that $D/D_{10} \cong \mathbb{P}^1$ has four marked points. To avoid automorphisms we fix x to be the branch point associated

to (image of) points fixed by $(ij)^5$, 0 to be associated to the points fixed by ij and $\{1, \infty\}$ to be associated to j.

Third, $C/\langle \beta_C \rangle \cong \mathbb{P}^1$ has six marked points, the points where the hyperelliptic morphism, p, is branched. We observe that since p is the projection given by the action of β_C, five of the branch points of p are a fiber of q, in particular, the images of points fixed by $(ij)^5$ in D, and the sixth has ramification index 5 in q, in particular, the image of the points fixed by ij.

In the following figure we show how the ramification of the morphism $D \to D/D_{10}$ is distributed along the factorizations by C and B respectively. The colored points denote ramification or branch points depending if they are before or after the corresponding colored arrow. The points marked with a cross are both branch and ramification points for the previous and next morphisms respectively.

Therefore, η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x\}$ to the one determined by $\{\mathbb{P}^1; q^{-1}(0), q^{-1}(x)\}$, where q is the morphism $D/\langle j, (ij)^5 \rangle \cong \mathbb{P}^1 \to D/D_{10} \cong \mathbb{P}^1$.

Finally, given a curve $[C]$ in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its hyperelliptic involution by a suitable q (they will be 0 and x) together with the other two branch points of q (they will be 1 and ∞). By a suitable q we mean that one of the branch points of p is a ramification point of q, and the other five are a fiber of q.

Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D. Moreover, there are only a finite number of possibilities for q, and hence, we can recover at most a finite number of $[D] \in \mathcal{D}$. $\Diamond$

2. The second case is described by

$$\begin{array}{rcl} \Gamma & \to & D_{10} \\ x_1 & \to & (ij)^2 \\ x_2 & \to & (ij)^5 \\ x_3 & \to & (ij)^2 i \\ x_4 & \to & j. \end{array}$$

We obtain the action of D_{10} on a curve of genus $g(D) = 4$ giving a curve of genus $g(B) = 2$ inside $C^{(2)}$ with $g(C) = 2$ and such that $\tilde{B}$ has one nodal singularity ($p_a(\tilde{B}) = 3$) and $\tilde{B}^2 = 1$.

Now, consider the action of the automorphism $(ij)^5$ on the curve D. Since $\nu((ij)^5) = 10$, by the Riemann-Hurwitz formula, we obtain that $g(D/\langle(ij)^5\rangle) = \frac{1}{4}(2 \cdot 4 + 2 - 10) = 0$ and therefore D is hyperelliptic.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 6$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle\beta_C\rangle) = \frac{1}{4}(2 \cdot 2 + 2 - 6) = 0$ and therefore β_C is the hyperelliptic involution. In the same way, β_B is the hyperelliptic involution in B. We obtain the following diagram:
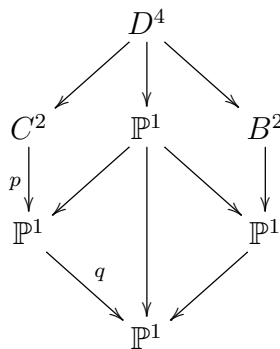
where the superindex denote the genus of the curves.

We observe that $\mathcal{D}$ is the 1-dimensional family of all curves of genus 4 with maximal dihedral symmetry (see [BCGMG03]).

Claim: The map η is finite.

Proof of the claim: In the following figure we show how the ramification of $D \to D/D_{10}$ is distributed along the factorization by C and B respectively. The colored points denote ramification or branch points depending if they are before or after the colored arrow.
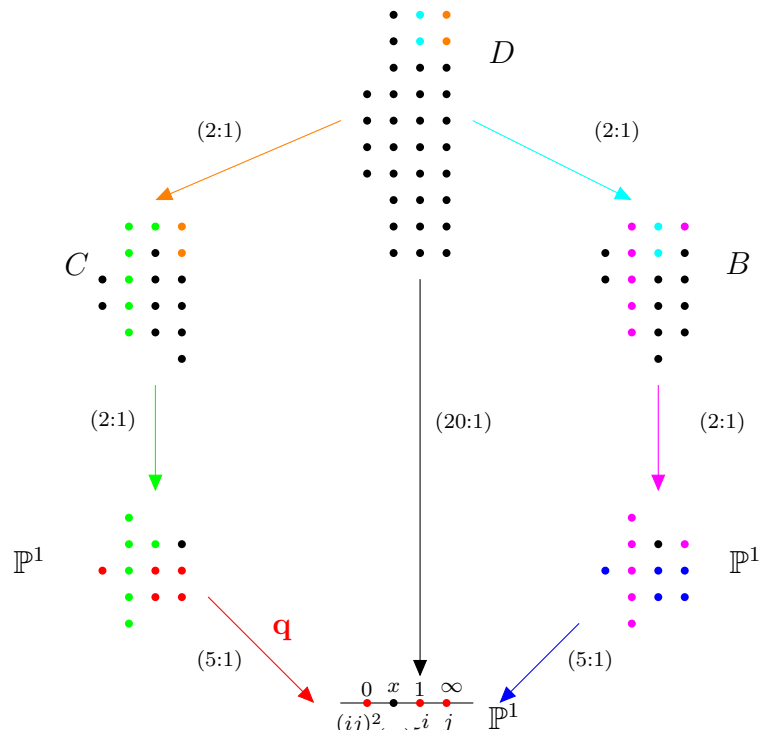
Since q has three branch points, there are a finite number of possible morphisms q, and therefore, following the proof of the previous claim we see that in this case η is also finite. $\diamondsuit$
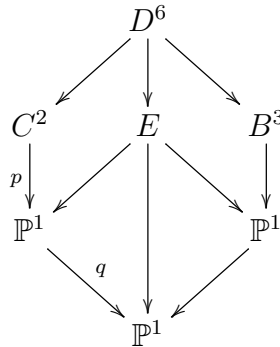
3. The third case is described by

$$
\begin{array}{rcl}
\Gamma & \to & D_{10} \\
x_1 & \to & (ij)^5 \\
x_2 & \to & (ij)^4 i \\
x_3,\ x_4,\ x_5 & \to & j.
\end{array}
$$

We obtain the action of D_{10} on a curve of genus $g(D) = 6$ giving a curve of genus $g(B) = 3$ inside $C^{(2)}$ with $g(C) = 2$ and such that $\tilde{B}$ is smooth and $\tilde{B}^2 = 1$.

Now, consider the action of the automorphism $(ij)^5$ on the curve D. Since $\nu((ij)^5) = 10$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle (ij)^5 \rangle) = \frac{1}{4}(2 \cdot 6 + 2 - 10) = 1$, that is, $D/\langle (ij)^5 \rangle$ is an elliptic curve, and therefore D is bielliptic.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 6$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle \beta_C \rangle) = \frac{1}{4}(2 \cdot 2 + 2 - 6) = 0$ and therefore β_C is the hyperelliptic involution. In the same way, β_B is the hyperelliptic involution of B. We obtain the following diagram:

$$
\begin{array}{ccccc}
 & & D^6 & & \\
 & \swarrow & \downarrow & \searrow & \\
C^2 & & E & & B^3 \\
{\scriptstyle p}\downarrow & \swarrow & & \searrow & \downarrow \\
\mathbb{P}^1 & & & & \mathbb{P}^1 \\
 & \searrow{\scriptstyle q} & \downarrow & \swarrow & \\
 & & \mathbb{P}^1 & &
\end{array}
$$

where the superindex denote the genus of the curves.

Notice that in this case $\mathcal{D}$ is 2-dimensional.

Claim: The map η is finite.

*Proof of the claim:* First, notice that $q : \mathbb{P}^1 \to \mathbb{P}^1$ has 4 branch points with branching type $(1, 2, 2)$, that is, a non ramified point on the fiber and two ramified points with ramification index 2.

Second, $D/D_{10} \cong \mathbb{P}^1$ has five marked points. We fix 0 to be the branch point associated to the points fixed by $(ij)^5$, ∞ to be the branch point associated to the points fixed by i and $\{1, x_1, x_2\}$ to be the branch points associated to the points fixed by j.

Third, $C/\beta_C \cong \mathbb{P}^1$ has six marked points. We observe that since p is the projection given by the action of β_C, they are non ramified by q. Five of them form a fiber, in particular, the images of points fixed by $(ij)^5$, and one of them, the image of a point fixed by a conjugate of i, lies over a branch point of q being non ramified.

In the following figure we show how the ramification of the morphism $D \to D/D_{10}$ is distributed along the factorizations by C

and B respectively. The colored points denote ramification or branch points depending if they are before or after the corresponding colored arrow.
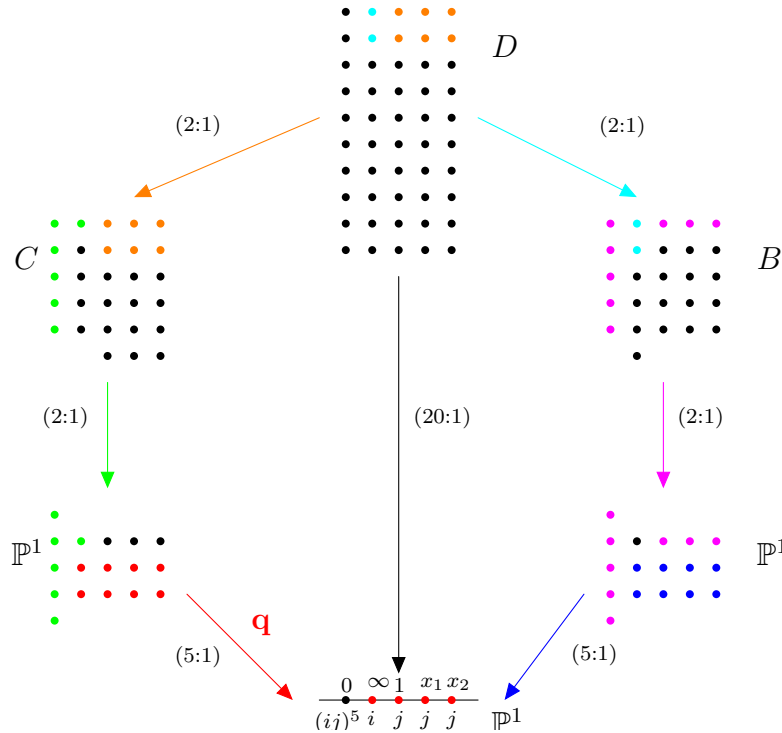
Therefore, the map η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x_1, x_2\}$ to the curve determined by the data $\{\mathbb{P}^1; q^{-1}(0), q^{-1}(\infty)_1\}$, where q is the morphism defined as $D/\langle j, (ij)^5\rangle \cong \mathbb{P}^1 \to D/D_{10} \cong \mathbb{P}^1$ and $q^{-1}(\infty)_1$ denotes the point on the fiber with $e_P = 1$.

Finally, given a curve $[C]$ in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its hyperelliptic involution by a suitable q (they will be 0 and ∞) together with the other three branch points of q (they will be 1, x_1 and x_2). By a suitable q we mean that five of the branch points of p are a fiber of q and the other lies over a branch point being non ramified.

Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D.

Moreover, we expect that there should be only a finite number of possibilities for q. Since q is a degree five morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$, it is given, in homogeneous coordinates by two degree five polynomials. Given five of the branch points of p, we assume that their image is $0 \in \mathbb{P}^1$ and we have one of the polynomials determined. Assuming that the sixth point has image $\infty \in \mathbb{P}^1$, we obtain one factor of the other polynomial. When we impose the branching type $(1, 2, 2)$ in the four branch points, we obtain a system of twelve equations with five unknowns.

The resolution of this system of equations has a very high computational cost[1] because of the high degree of the equations involved. We were not able to finish it. Probably a more refined algorithm would be needed. Nevertheless, the high number of equations compared to the number of unknowns takes us to conjecture that this system of equations has a finite number of solutions. If so, we could recover at most a finite number of $[D] \in \mathcal{D}$. ◇

Finally notice that in all three cases $p_a(\tilde{B}) = 2g(C) - 1$.

We have seen that we have ten types of topological actions given by D_6. We are going to study each of them separately.
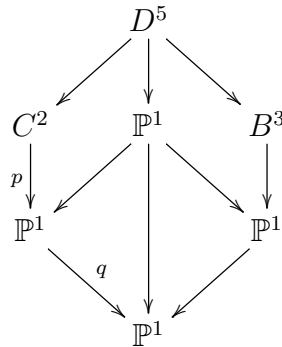
0. The first case is described by

$$
\begin{aligned}
\Gamma &\rightarrow D_6 \\
x_1,\, x_2 &\rightarrow (ij)^3 \\
x_3 &\rightarrow (ij)^2 \\
x_4 &\rightarrow (ij)^4 j \\
x_5 &\rightarrow j.
\end{aligned}
$$

We have the action of D_6 on a curve of genus $g(D) = 5$ giving a curve of genus $g(B) = 3$ inside $C^{(2)}$ with $g(C) = 2$ and such that $\tilde{B}$ has one node and $\tilde{B}^2 = 2$.

Now, consider the action of the automorphism $(ij)^3$ on the curve D. Since $\nu((ij)^3) = 12$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle(ij)^3\rangle) = \frac{1}{4}(2 \cdot 5 + 2 - 12) = 0$ and therefore D is hyperelliptic.

---

[1]System specifications: Processor: Intel Xeon W3520 @2.67GHz. 4 GB RAM. Using Windows 64 bits and Wolfram Mathematica 9

By Lemma 7.1.5 we know that $\nu(\beta_C) = 6$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle\beta_C\rangle) = \frac{1}{4}(4+2-6) = 0$ and therefore β_C is the hyperelliptic involution in C. In the same way, β_B is the hyperelliptic involution in B. We obtain the following diagram:

$$
\begin{array}{ccccc}
 & & D^5 & & \\
 & \swarrow & \downarrow & \searrow & \\
C^2 & & \mathbb{P}^1 & & B^3 \\
\downarrow{\scriptstyle p} & \swarrow & \downarrow & \searrow & \downarrow \\
\mathbb{P}^1 & & & & \mathbb{P}^1 \\
 & \searrow{\scriptstyle q} & \downarrow & \swarrow & \\
 & & \mathbb{P}^1 & &
\end{array}
$$

where the superindex denote the genus of the curves.

Notice that in this case $\mathcal{D}$ is 2-dimensional.

Claim: The map η is finite.

Proof of the claim: First, notice that the morphism $q : \mathbb{P}^1 \to \mathbb{P}^1$ has 3 branch points, and therefore, there are a finite number of such morphisms modulo automorphisms of $\mathbb{P}^1$.

Second, we observe that $D/D_6 \cong \mathbb{P}^1$ has 5 marked points. To avoid automorphisms we fix $\{0, 1\}$ to be the branch points associated to (the images of) the points fixed by $(ij)^3$, ∞ to be associated to the points fixed by $(ij)^2$ and $\{x_1, x_2\}$ to be associated to the points fixed by j.

Third, $C/\langle\beta_C\rangle \cong \mathbb{P}^1$ has six marked points, the points where the hyperelliptic morphism, p, is branched. We observe that since p is the projection given by the action of β_C, these points form two fibers of q, in particular, the images of the points fixed by the automorphism $(ij)^3$ on the curve D.

In the following figure we show how the ramification of the morphism $D \to D/D_6$ is distributed along the factorizations by C and B respectively. The colored points denote ramification or branch points depending if they are before or after the corresponding colored arrow.
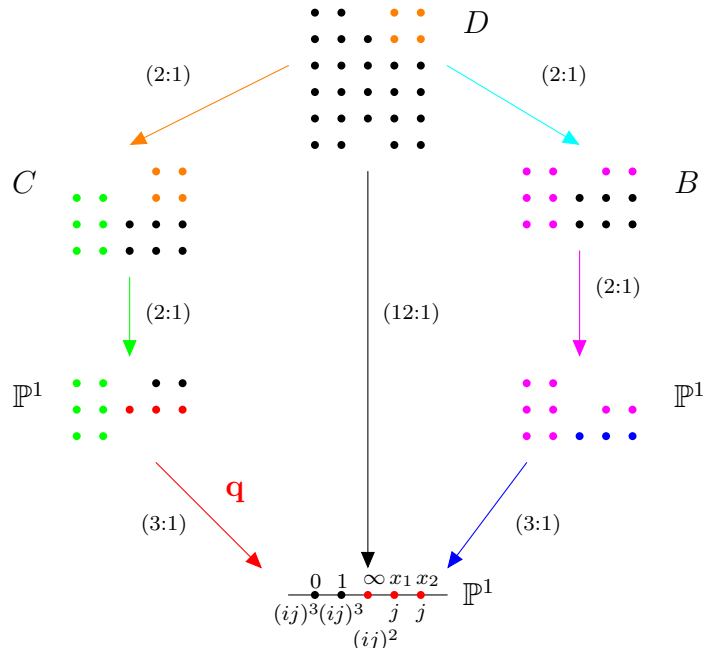
Therefore, the map η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x_1, x_2\}$ to the curve determined by the data $\{\mathbb{P}^1; q^{-1}(0), q^{-1}(1)\}$, where q denotes the morphism given by $D/\langle j, (ij)^3 \rangle \cong \mathbb{P}^1 \to D/D_6 \cong \mathbb{P}^1$.

Finally, given a curve $[C]$ in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its hyperelliptic involution by a suitable q (they will be 0 and 1) together with the other three branch points of q (they will be ∞, x_1 and x_2). By a suitable we q we mean that the branch points of p form to fibers of q, q has a point of ramification index three and two of index two.

Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D. Moreover, there are only a finite number of possibilities for q, and hence, we can recover at most a finite number of $[D] \in \mathcal{D}$. $\Diamond$

We observe that in this case $p_a(\tilde{B}) = 2g(C)$.

1. The second case is described by

$$
\begin{aligned}
\Gamma &\to D_6 \\
x_1 &\to ij \\
x_2 &\to ji \\
x_3, \ x_4 &\to j.
\end{aligned}
$$

We obtain the action of D_6 on a curve of genus $g(D) = 5$ giving a curve of genus $g(B) = 3$ inside $C^{(2)}$ with $g(C) = 2$ and such that $\tilde{B}$ is smooth and $\tilde{B}^2 = 2$.

Now, consider the action of the automorphism $(ij)^3$ on the curve D. Since $\nu((ij)^3) = 4$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle (ij)^3 \rangle) = \frac{1}{4}(2 \cdot 5 + 2 - 4) = 2$.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 2$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle \beta_C \rangle) = \frac{1}{4}(2 \cdot 2 + 2 - 2) = 1$ and therefore C is bielliptic, as well as B, with β_C and β_B the bielliptic involutions. We obtain the following diagram:
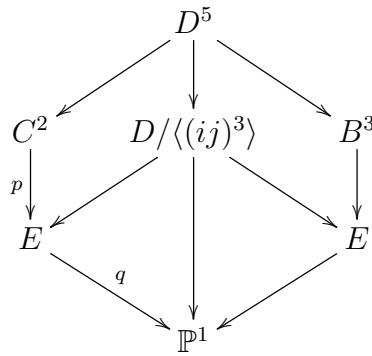
where the superindex denote the genus of the curves.

Notice that in this case $\mathcal{D}$ is 1-dimensional.

Claim: The map η is finite.

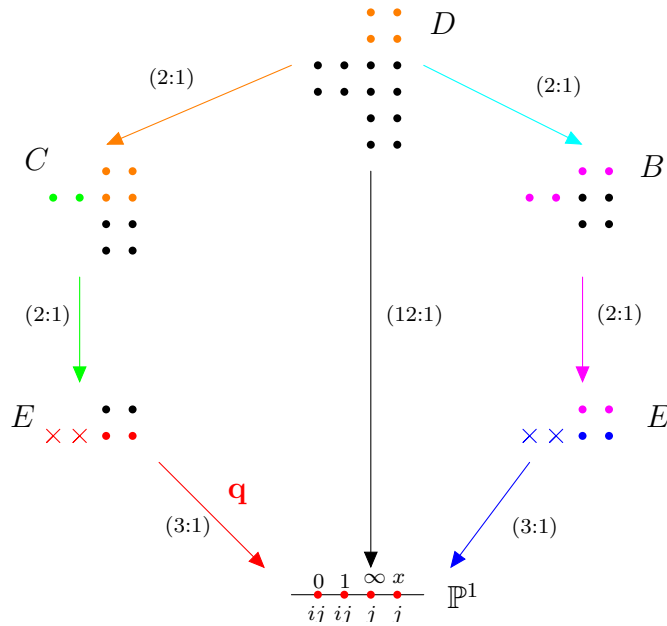*Proof of the claim:* First, notice that $q : E \to \mathbb{P}^1$ has 4 branch points and the branching type is:

- Two points totally ramified, that is, with branching type (3).
- Two points with branching type $(1, 2)$ that is, a non ramified point on the fiber and a ramified one with $e_P = 2$.

Second, we observe that $D/D_6 \cong \mathbb{P}^1$ has four marked points. We fix $\{0, 1\}$ to be the branch points associated to the points fixed by ij and $\{\infty, x\}$ to be the branch points associated to the points fixed by j.

Third, $C/\beta_C \cong E$ has two marked points. We observe that since p is the projection given by the action of β_C, the two branch points of p have ramification index 3 in q, in particular, they are image of the points fixed by ij.

In the following figure we show how the ramification of the morphism $D \to D/D_6$ is distributed along the factorizations by C and B respectively. The colored points denote ramification or branch points depending if they are before or after the colored arrow. The points marked with a cross are both branch and ramification points for the previous and next morphisms respectively.

Therefore, η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x\}$ to the one determined by $\{E; q^{-1}(0), q^{-1}(1)\}$, where $E := D/\langle j, (ij)^3 \rangle$ and q is the morphism $E \to D/D_6 \cong \mathbb{P}^1$.

Finally, given a curve $[C]$ in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic involution by a suitable q (they will be 0 and 1), together with the other two branch points of q (they will be ∞ and x). By a suitable q we mean that the two branch points of p have ramification index 3 in q.

Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D. Moreover, there are only a finite number of possibilities for q. Indeed, since we have the elliptic curve given as the quotient of C by its bielliptic involution, one of the branch points of p determine the immersion

of E in $\mathbb{P}^2$ in such a way that it is an inflexion point, and then necessarily the other point will be another inflexion. The projection point will be then the intersection of the respective tangent lines. Thus, we can recover at most a finite number of $[D] \in \mathcal{D}$. $\Diamond$

2. The third case is described by

$$\begin{aligned}
\Gamma &\to D_6 \\
x_1 &\to ij \\
x_2 &\to (ij)^3 \\
x_3 &\to (ij)^2 \\
x_4, x_5 &\to j.
\end{aligned}$$

We obtain the action of D_6 on a curve of genus $g(D) = 7$ giving a curve of genus $g(B) = 4$ inside $C^{(2)}$ with $g(C) = 3$ and such that $\tilde{B}$ has one nodal singularity ($p_a(\tilde{B}) = 5$) and $\tilde{B}^2 = 1$.

Now, consider the action of the automorphism $(ij)^3$ on the curve D. Since $\nu((ij)^3) = 6 + 2 = 8$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle(ij)^3\rangle) = \frac{1}{4}(2 \cdot 7 + 2 - 8) = 2$.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 4$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle\beta_C\rangle) = \frac{1}{4}(2 \cdot 3 + 2 - 4) = 1$ and therefore C is bielliptic as well as B with β_C and β_B the bielliptic involutions. We obtain the following diagram:
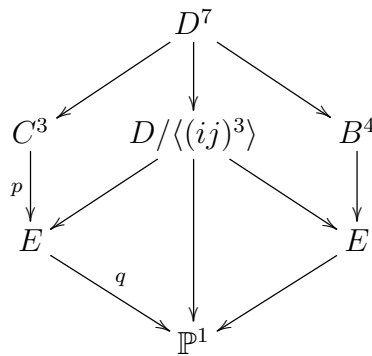
where the superindex denote the genus of the curves.

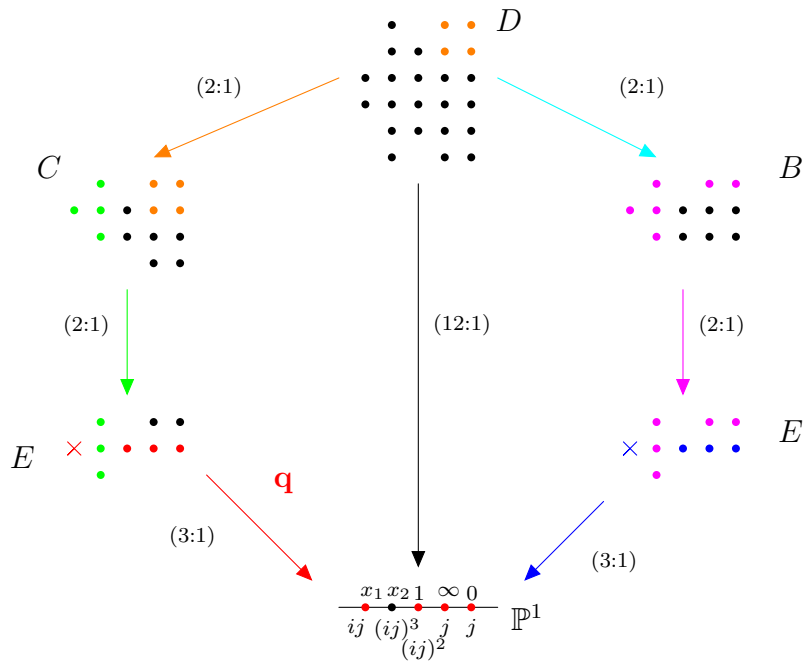Notice that in this case $\mathcal{D}$ is 2-dimensional.

Claim: The map η is finite.

Proof of the claim: First, notice that $q : E \to \mathbb{P}^1$ is branched over four points with the same branching type that in the previous point.

Second, $D/D_6 \cong \mathbb{P}^1$ has five marked points. We fix x_1 to be the branch point associated to the points fixed by ij, 1 to be associated to the points fixed by $(ij)^2$, $\{\infty, 0\}$ to be associated to j and x_2 to $(ij)^3$.

Third, $C/\beta_C \cong E$ has four marked points. We observe that since the morphism p is the projection given by the action of β_C, one of the branch points of p has ramification index 3 in q, in particular, the image of the points fixed by ij, and the other three form a fiber of q, in particular, the images of points fixed by $(ij)^3$.

In the following figure we show how the ramification of the morphism $D \to D/D_6$ is distributed along the factorizations by C and B respectively. The colored points denote ramification or branch points depending if they are before or after the corresponding colored arrow. The points marked with a cross are both branch and ramification points for the previous and next morphisms respectively.

Therefore, the map η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x_1, x_2\}$ to the curve determined by the data $\{E; q^{-1}(x_1), q^{-1}(x_2)\}$, where $E := D/\langle j, (ij)^3 \rangle$ and q is the morphism $E \to D/D_6 \cong \mathbb{P}^1$.

Finally, given a curve $[C]$ in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic involution by a suitable q (they will be x_1 and x_2, together with the other three branch points of q (they will be 1, 0 and ∞). By a suitable q we mean that one of the branch points of p has ramification index 3 in q, and the other 3 are a fiber of q.

Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D. Moreover, there are only a finite number of possibilities for q. Indeed, since the elliptic curve is given, one of the branch points determine the immersion of E in $\mathbb{P}^2$ in such a way that it is an inflexion point, and at least for one of the four options, the other three points will be over a line. Then, the projection point will be the intersection of the tangent to the inflexion and the line containing the other three. Thus, we can recover at most a finite number of $[D] \in \mathcal{D}$. $\Diamond$

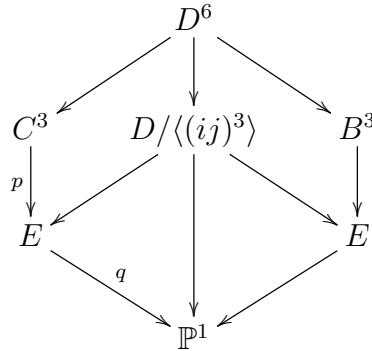3. The fourth case is described by

$$\begin{aligned}
\Gamma &\to D_6 \\
x_1 &\to j \\
x_2 &\to i \\
x_3 &\to (ij)^3 \\
x_4, x_5 &\to (ij)^2.
\end{aligned}$$

We obtain the action of D_6 on a curve of genus $g(D) = 6$ giving a curve of genus $g(B) = 3$ inside $C^{(2)}$ with $g(C) = 3$ and such that $\tilde{B}$ has two nodal singularities ($p_a(\tilde{B}) = 5$) and $\tilde{B}^2 = 1$.

Now, consider the action of the automorphism $(ij)^3$ on the curve D. Since $\nu((ij)^3) = 6$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle (ij)^3 \rangle) = \frac{1}{4}(2 \cdot 6 + 2 - 6) = 2$.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 4$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle \beta_C \rangle) = \frac{1}{4}(2 \cdot 3 + 2 - 4) = 1$ and therefore C is bielliptic as well as B with β_C and β_B the

bielliptic involutions. We obtain the following diagram:

$$
\begin{array}{ccccc}
& & D^6 & & \\
& \swarrow & \downarrow & \searrow & \\
C^3 & & D/\langle (ij)^3 \rangle & & B^3 \\
\downarrow p & \searchdown & \downarrow & \seardown & \downarrow \\
E & & & & E \\
& \searrow q & \downarrow & \swarrow & \\
& & \mathbb{P}^1 & &
\end{array}
$$

where the superindex denote the genus of the curves.

Notice that in this case $\mathcal{D}$ is 2-dimensional.

Claim: The map η is finite.

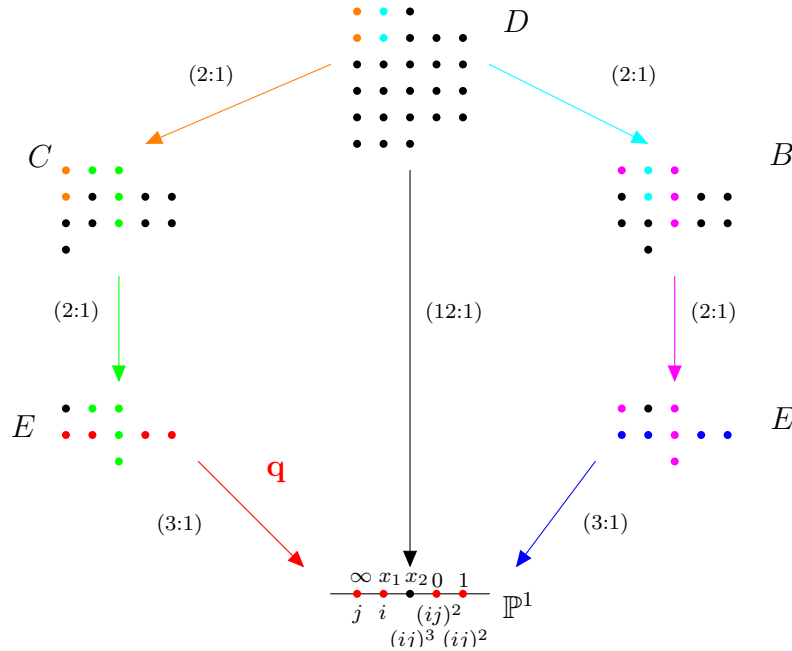*Proof of the claim:* First, notice that $q : E \to \mathbb{P}^1$ has 4 branch points and the branching type is:

- Two points totally ramified, that is, with branching type (3).

- Two points with branching type $(1, 2)$ that is, a non ramified point on the fiber and a ramified one with $e_P = 2$.

Second, $D/D_6 \cong \mathbb{P}^1$ has five marked points. We fix $\{0, 1\}$ to be the branch points associated to the points fixed by $(ij)^2$, ∞ to be associated to the points fixed by j, x_1 to the points fixed by i and x_2 to the points fixed by $(ij)^3$

Third, $C/\beta_C \cong E$ has four marked points. We observe that since the p is the projection given by the action of β_C, three of the branch of p form a fiber of q, in particular, the images of points fixed by $(ij)^3$, and the forth will be a non-ramified point for q lying over a branch point of q, in particular, the image of the points fixed by i.

In the following figure we show how the ramification of the morphism $D \to D/D_6$ is distributed along the factorization by C and B respectively. The colored points denote ramification or branch points depending if they are before or after the colored arrow.

Therefore, the map η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x_1, x_2\}$ to the curve determined by the data $\{E; q^{-1}(x_1), q^{-1}(x_2)\}$, where $E := D/\langle j, (ij)^3 \rangle$ and q is the morphism $E \to D/D_6 \cong \mathbb{P}^1$.

Finally, given a curve $[C]$ in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic involution by a suitable q (they will be x_1 and x_2, together with the other three branch points of q (they will be 0, 1 and ∞). By a suitable q we mean that one of the branch points of p is not ramified but lies over a branch point and the other 3 are a fiber of q.

Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D. Moreover, there are only a finite number of possibilities for q. Indeed, since the elliptic curve is given, taking three of the branch points we determine the immersion of E in $\mathbb{P}^2$, and taking a line passing through the fourth and tangent to E but not on this point (a finite number of such), we obtain a finite number of candidates for the projection point. Only those with two points with ramification index 3 are possible q's. Thus, we can recover a finite number of $[D] \in \mathcal{D}$. $\diamondsuit$

4. The fifth case is described by

$$
\begin{aligned}
\Gamma &\rightarrow D_6 \\
x_1 &\rightarrow i \\
x_2 &\rightarrow j \\
x_3 &\rightarrow ij \\
x_4 &\rightarrow (ij)^4.
\end{aligned}
$$

We obtain the action of D_6 on a curve of genus $g(D) = 4$ giving a curve of genus $g(B) = 2$ inside $C^{(2)}$ with $g(C) = 2$ and such that $\tilde{B}$ has one nodal singularity ($p_a(\tilde{B}) = 3$) and $\tilde{B}^2 = 2$.

Now, consider the action of the automorphism $(ij)^3$ on the curve D. Since $\nu((ij)^3) = 2$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle(ij)^3\rangle) = \frac{1}{4}(2 \cdot 4 + 2 - 2) = 2$.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 2$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle\beta_C\rangle) = \frac{1}{4}(2 \cdot 2 + 2 - 2) = 1$ and therefore C is bielliptic as well as B with β_C and β_B the bielliptic involutions. We obtain the following diagram:

$$
\begin{array}{ccccc}
 & & D^4 & & \\
 & \swarrow & \downarrow & \searrow & \\
C^2 & & D/\langle(ij)^3\rangle & & B^2 \\
\downarrow{\scriptstyle p} & \searrow & \downarrow & \swarrow & \downarrow \\
E & & & & E \\
 & \searrow{\scriptstyle q} & \downarrow & \swarrow & \\
 & & \mathbb{P}^1 & &
\end{array}
$$

where the superindex denote the genus of the curves.

Notice that in this case $\mathcal{D}$ is 2-dimensional.

Claim: The map η is finite.

Proof of the claim: First, notice that $q : E \rightarrow \mathbb{P}^1$ is branched over four points with the same branching type that in the previous point.

Second, $D/D_6 \cong \mathbb{P}^1$ has four marked points. We fix 0 to be the branch point associated to the points fixed by ij, 1 to be associated to the points fixed by i, ∞ to the points fixed by j and x to the points fixed by $(ij)^2$.

Third, $C/\beta_C \cong E$ has two marked points. We observe that since the p is the projection given by the action of β_C, one of the branch points of p has ramification index 3 in q, in particular, the image of the points fixed by ij, and the other is non ramified but lies over a branch point of q, the image of the points fixed by i.

In the following figure we show how the ramification of the morphism $D \to D/D_6$ is distributed along the factorizations by C and B respectively. The colored points denote ramification or branch points depending if they are before or after the colored arrow. The points marked with a cross are both branch and ramification points for the previous and next morphisms respectively.

Therefore, η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x\}$ to the one determined by $\{E; q^{-1}(0), q^{-1}(1)_1\}$, where $E := D/\langle j, (ij)^3 \rangle$ and q is the morphism $E \to D/D_6 \cong \mathbb{P}^1$ and $q^{-1}(1)_1$ denotes the point on the fiber with $e_P = 1$.

Finally, given a curve $[C]$ in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic involution by a suitable q (they will be 0 and 1), and the other two branch points of q (they will be ∞ and x). By a suitable q we mean that

one of the branch points of p has ramification index 3 in q and the other is non ramified with image a branch point of q.

Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D. Moreover, there are only a finite number of possibilities for q. Indeed, since the elliptic curve is given, one of the branch points determines the immersion on $\mathbb{P}^2$ in such a way that it is an inflexion, and taking a line passing through the other and tangent to E, but not on this point (a finite number of such), we obtain a finite number of candidates for the projection point. Only those with two points with ramification index 3 would be possible q's. Thus, we can recover at most a finite number of $[D] \in \mathcal{D}$. $\Diamond$
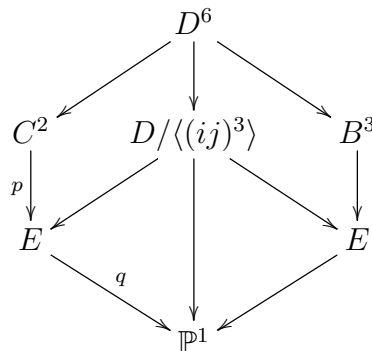
5. The sixth case is described by

$$\begin{aligned}
\Gamma &\to D_6 \\
x_1, x_2, x_3 &\to j \\
x_4 &\to i \\
x_5 &\to ij.
\end{aligned}$$

We obtain the action of D_6 on a curve of genus $g(D) = 6$ giving a curve of genus $g(B) = 3$ inside $C^{(2)}$ with $g(C) = 2$ and such that $\tilde{B}$ is smooth and $\tilde{B}^2 = 2$.

Now, consider the action of the automorphism $(ij)^3$ on the curve D. Since $\nu((ij)^3) = 2$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle (ij)^3 \rangle) = \frac{1}{4}(2 \cdot 6 + 2 - 2) = 3$.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 2$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle \beta_C \rangle) = \frac{1}{4}(2 \cdot 2 + 2 - 2) = 1$ and therefore C is bielliptic as well as B with β_C and β_B the bielliptic involutions. We obtain the following diagram:

where the superindex denote the genus of the curves.

Notice that in this case $\mathcal{D}$ is 2-dimensional.
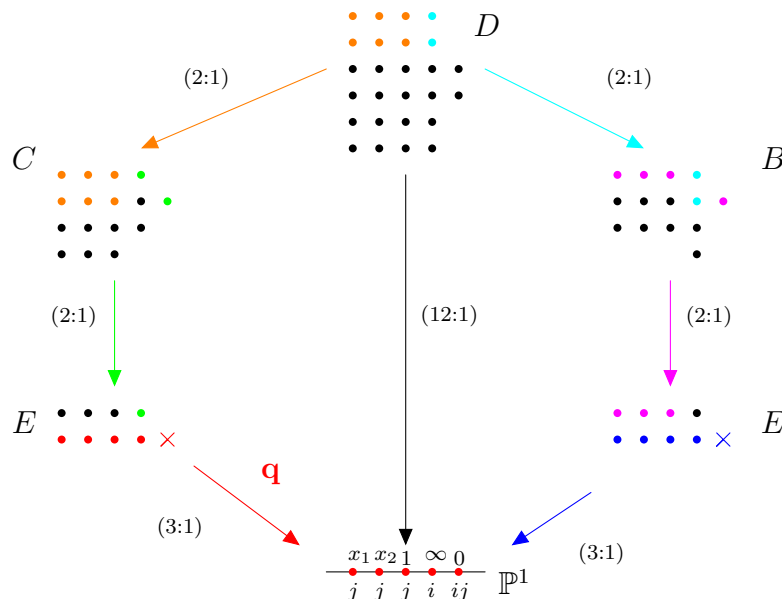
Claim: The map η is finite.

Proof of the claim: First, notice that $q : E \to \mathbb{P}^1$ has 5 branch points with branching type:

- One point totally ramified, that is, branching type (3).
- Four points with branching type $(1, 2)$ that is, a non ramified point on the fiber and a ramified one with $e_P = 2$.

Second, $D/D_6 \cong \mathbb{P}^1$ has five marked points. We fix 0 to be the branch point associated to the points fixed by ij, ∞ to be the branch point associated to the points fixed by i and the other three be associated to the points fixed by j.

Third, $C/\beta_C \cong E$ has two marked points. We observe that since the p is the projection given by the action of β_C, one of the branch points of p has ramification index 3 in q, in particular, the image of the points fixed by ij, and the other is non ramified but lies over a branch point of q, the image of the points fixed by i.

In the following figure we show how the ramification of the morphism $D \to D/D_6$ is distributed along the factorizations by C and B. The colored points denote ramification or branch points depending if they are before or after the corresponding colored arrow. The points marked with a cross are both branch and ramification points for the previous and next morphisms.

Therefore, the map η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x_1, x_2\}$ to the curve determined by the data $\{E; q^{-1}(0), q^{-1}(\infty)_1\}$, where $E := D/\langle j, (ij)^3 \rangle$ and q is the morphism $E \to D/D_6 \cong \mathbb{P}^1$ and $q^{-1}(\infty)_1$ denotes the point on the fiber with $e_P = 1$.

Now, given a curve $[C]$ in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic involution by a suitable q (they will be 0 and ∞), together with the other three branch points of q (they will be 1, x_1 and x_2). By a suitable q we mean that one of the branch points of p has ramification order 3, and the other is non ramified lying over a branch point of q.

Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D. Moreover, there are only a finite number of possibilities for q. Indeed, since the elliptic curve is given, one of the branch points determines the immersion on $\mathbb{P}^2$ in such a way that it is an inflexion, and taking a line passing through the other and tangent to E but not on this point (a finite number of such), we obtain a finite number of candidates for the projection point. Thus, we can recover at most a finite number of $[D] \in \mathcal{D}$. $\Diamond$

6. The seventh case is described by

$$
\begin{aligned}
\Gamma &\to D_6 \\
x_1 &\to ij \\
x_2 &\to (ij)^3 \\
x_3 &\to (ij)^2 j \\
x_4, x_5, x_6 &\to j.
\end{aligned}
$$

We obtain the action of D_6 on a curve of genus $g(D) = 9$ giving a curve of genus $g(B) = 5$ inside $C^{(2)}$ with $g(C) = 3$ and such that $\tilde{B}$ is smooth and $\tilde{B}^2 = 1$.

Now, consider the action of the automorphism $(ij)^3$ on the curve D. Since $\nu((ij)^3) = 6 + 2 = 8$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle (ij)^3 \rangle) = \frac{1}{4}(2 \cdot 9 + 2 - 8) = 3$.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 4$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle \beta_C \rangle) = \frac{1}{4}(2 \cdot 3 + 2 - 4) = 1$ and therefore C is bielliptic as well as B with β_C and β_B the

bielliptic involutions. We obtain the following diagram:

$$
\begin{array}{ccccc}
& & D^9 & & \\
& \swarrow & \downarrow & \searrow & \\
C^3 & & D/\langle (ij)^3 \rangle & & B^5 \\
\downarrow{\scriptstyle p} & \swarrow & \downarrow & \searrow & \downarrow \\
E & & & & E \\
& \searrow{\scriptstyle q} & \downarrow & \swarrow & \\
& & \mathbb{P}^1 & &
\end{array}
$$

where the superindex denote the genus of the curves.

Notice that in this case $\mathcal{D}$ is 3-dimensional.
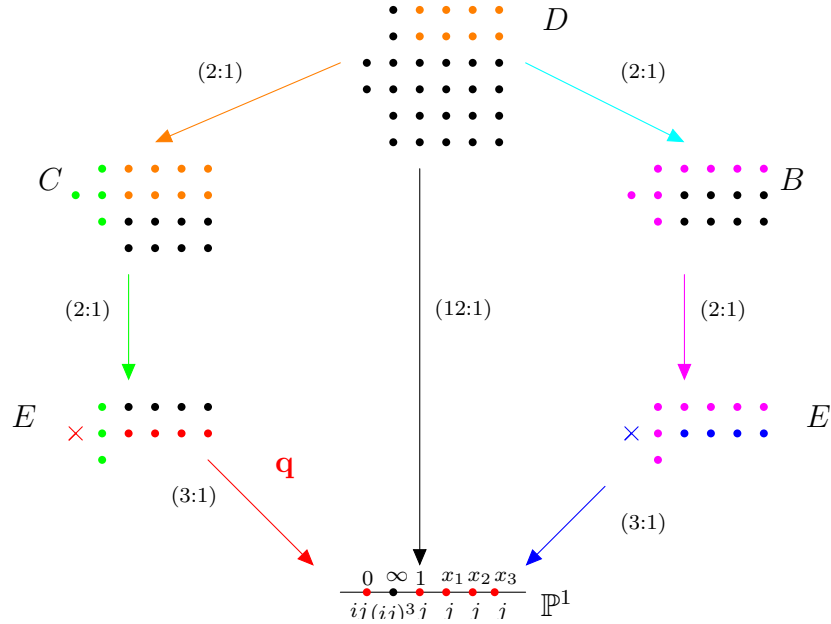
Claim: The map η is finite.

Proof of the claim: First, notice that $q : E \to \mathbb{P}^1$ has 5 branch points with branching type:

- One point totally ramified, that is, branching type (3).

- Four points with branching type $(1, 2)$ that is, a non ramified point on the fiber and a ramified one with $e_P = 2$.

Second, $D/D_6 \cong \mathbb{P}^1$ has six marked points. We fix 0 to be the branch point associated to the points fixed by ij, ∞ to be associated to the points fixed by $(ij)^3$ and the other four to the points fixed by j.

Third, $C/\beta_C \cong E$ has four marked points. We observe that since the p is the projection given by the action of β_C, one of the branch points of p has ramification index 3 in q, in particular, the image of the points fixed by ij, and the other three form a non ramified fiber, the images of the points fixed by $(ij)^3$.

In the following figure we show how the ramification of the morphism $D \to D/D_6$ is distributed along the factorization by C and B respectively. The colored points denote ramification or branch points depending if they are before or after the colored arrow. The points marked with a cross are both branch and ramification points for the previous and next morphisms respectively.

Therefore, η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x_1, x_2, x_3\}$ to the curve determined by the data $\{E; q^{-1}(0), q^{-1}(\infty)\}$, where $E := D/\langle j, (ij)^3 \rangle$ and q is the morphism $E \to D/D_6 \cong \mathbb{P}^1$.

Finally, given a curve C in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic involution by a suitable q (they will be 0 and ∞), together with the other four branch points of q (they will be 1, x_1, x_2 and x_3). By a suitable q we mean that one of the branch points of p has ramification order 3 in q, and the other three are a fiber of q.

Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D. Moreover ,there are only a finite number of possibilities for q. Indeed, since the elliptic curve is given, one of the branch points determines the immersion of E in $\mathbb{P}^2$ in such a way that it is an inflexion point, then, at least for one of the four options, the other three points will be over a line. The projection point will be the intersection of the tangent to the inflexion and the line containing the other three. Thus, we can recover at most a finite number of $[D] \in \mathcal{D}$. $\Diamond$
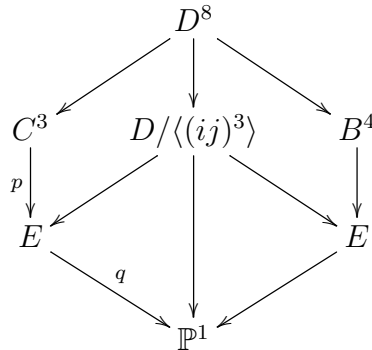
7. The eighth case is described by

$$\begin{aligned}
\Gamma &\to D_6 \\
x_1 &\to (ij)^3 \\
x_2 &\to (ij)^2 \\
x_3 &\to i \\
x_4, x_5, x_6 &\to j.
\end{aligned}$$

We obtain the action of D_6 on a curve of genus $g(D) = 8$ giving a curve of genus $g(B) = 4$ inside $C^{(2)}$ with $g(C) = 3$ and such that $\tilde{B}$ has one nodal singularity ($p_a(\tilde{B}) = 5$) and $\tilde{B}^2 = 1$.

Now, consider the action of the automorphism $(ij)^3$ on the curve D. Since $\nu((ij)^3) = 6$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle (ij)^3 \rangle) = \frac{1}{4}(2 \cdot 8 + 2 - 6) = 3$.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 4$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle \beta_C \rangle) = \frac{1}{4}(2 \cdot 3 + 2 - 4) = 1$ and therefore C is bielliptic as well as B with β_C and β_B the bielliptic involutions. We obtain the following diagram:

where the superindex denote the genus of the curves.

Notice that in this case $\mathcal{D}$ is 3-dimensional.
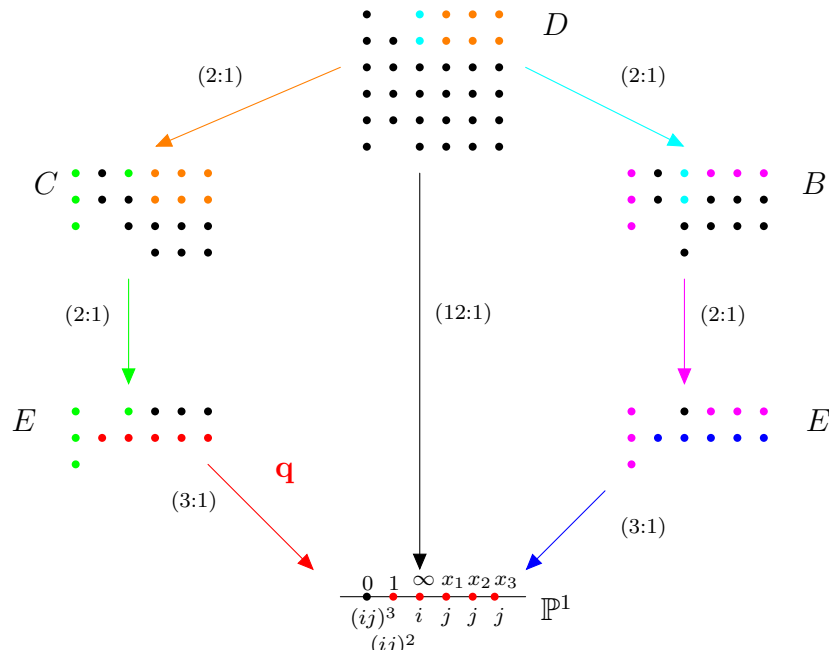
Claim: The map η is finite.

Proof of the claim: First, notice that $q : E \to \mathbb{P}^1$ has 5 branch points with branching type:

- One point totally ramified, that is, branching type (3).

- Four points with branching type $(1, 2)$ that is, a non ramified point on the fiber and a ramified one with $e_P = 2$.

Second, $D/D_6 \cong \mathbb{P}^1$ has six marked points. We fix 0 to be the branch point associated to the points fixed by $(ij)^3$, ∞ to be associated to the points fixed by i, 1 to the points fixed by $(ij)^2$ and the other three to the points fixed by j.

Third, $C/\beta_C \cong E$ has four marked points. We observe that since the p is the projection given by the action of β_C, one of the branch points of p is non ramified lying over a branch point of q, in particular, the image of the points fixed by a conjugate of i, and the other three form a non ramified fiber, the images of the points fixed by $(ij)^3$.

In the following figure we show how the ramification of the morphism $D \to D/D_6$ is distributed along the factorization by C and B respectively. The colored points denote ramification or branch points depending if they are before or after the colored arrow.

Therefore, η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x_1, x_2, x_3\}$ to the curve determined by the data $\{E; q^{-1}(0), q^{-1}(\infty)_1\}$, where $E := D/\langle j, (ij)^3 \rangle$ and q is the morphism $E \to D/D_6 \cong \mathbb{P}^1$ and $q^{-1}(\infty)_1$ denotes the point on the fiber with $e_P = 1$.

Now, given a curve $[C]$ in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic morphism by a suitable q (they will be 0 and ∞), together with the other four branch points of q (they will be 0, x_1, x_2 and x_3). By a suitable q we mean that one of the branch points of p is non ramified lying over a branch point and the other three are a fiber of q.

Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D. Moreover, there are only a finite number of possibilities for q. Indeed, since the elliptic curve is given, taking three of the four points we determine an immersion of E in $\mathbb{P}^2$, and taking a line passing through the other and tangent to E but not on this point (a finite number of such), we obtain a finite number of candidates for the projection point. Thus, we can recover at most a finite number of $[D] \in \mathcal{D}$. $\Diamond$

8. The ninth case is described by

$$
\begin{aligned}
\Gamma &\to D_6 \\
x_1, x_2 &\to i \\
x_3 &\to (ij)^4 \\
x_4 &\to (ij)^2 j \\
x_5 &\to j.
\end{aligned}
$$

We obtain the action of D_6 on a curve of genus $g(D) = 5$ giving a curve of genus $g(B) = 2$ inside $C^{(2)}$ with $g(C) = 2$ and such that $\tilde{B}$ has one nodal singularity ($p_a(\tilde{B}) = 3$) and $\tilde{B}^2 = 2$.

Now, consider the action of the automorphism $(ij)^3$ on the curve D. Since $\nu((ij)^3) = 0$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle(ij)^3\rangle) = \frac{1}{4}(2 \cdot 5 + 2) = 3$.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 2$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle\beta_C\rangle) = \frac{1}{4}(2 \cdot 2 + 2 - 2) = 1$ and therefore C is bielliptic as well as B with β_C and β_B the

bielliptic involutions. We obtain the following diagram:

$$
\begin{array}{ccccc}
& & D^5 & & \\
& \swarrow & \downarrow & \searrow & \\
C^2 & & D/\langle(ij)^3\rangle & & B^2 \\
{\scriptstyle p}\downarrow & \searrow & \downarrow & \swarrow & \downarrow \\
E & & & & E \\
& \searrow_{q} & \downarrow & \swarrow & \\
& & \mathbb{P}^1 & &
\end{array}
$$
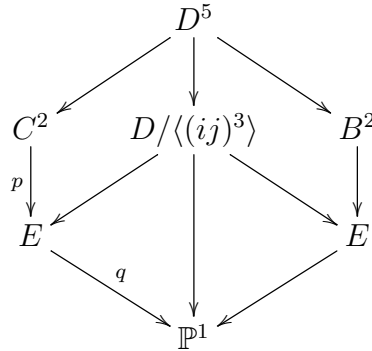
where the superindex denote the genus of the curves.

Notice that in this case $\mathcal{D}$ is 2-dimensional.

Claim: The map η is finite.

Proof of the claim: First, notice that $q : E \to \mathbb{P}^1$ has 5 branch points with branching type:

- Four points with branching type $(1, 2)$ that is, a non ramified point on the fiber and a ramified one with $e_P = 2$.

- One point with branch type (3), that is, a single point on the fiber with $e_P = 3$.

Second, $D/D_6 \cong \mathbb{P}^1$ has five marked points. We fix $\{0, 1\}$ to be the branch points associated to the points fixed by i, ∞ to be associated to the points fixed by $(ij)^2$ and $\{x_1, x_2\}$ to be associated to the points fixed by j.

Third, $C/\beta_C \cong E$ has two marked points. We observe that since the p is the projection given by the action of β_C, the two branch points of p are non ramified lying over a branch point of q, in particular, the images of the points fixed by a conjugate of i.

In the following figure we show how the ramification of the morphism $D \to D/D_6$ is distributed along the factorization by C and B respectively. The colored points denote ramification or branch points depending if they are before or after the corresponding colored arrow.
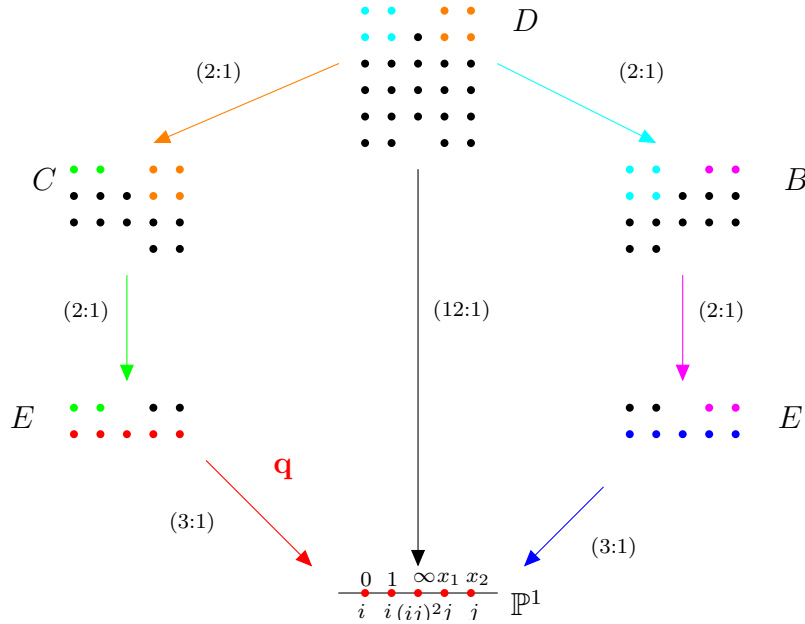
Therefore, the map η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x_1, x_2\}$ to the curve determined by the set of data $\{E; q^{-1}(0)_1, q^{-1}(1)_1\}$, where q is the morphism given by $D/\langle j, (ij)^3 \rangle \cong E \to D/D_6 \cong \mathbb{P}^1$.

Finally, given a curve $[C]$ in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic involution by a suitable q (they will be 0 and 1), together with the other three branch points of q (they will be ∞, x_1 and x_2). By a suitable q we mean that the branch points of p have images different branch points of q being non ramified. Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D.

Moreover, there are only a finite number of possibilities for q. Indeed, let x and y be the branch points of p. A suitable q can be described by the immersion of E in $\mathbb{P}^2$ given by the linear series of the fibers followed by the projection from a point not belonging to the image of E. Assume that we have such an immersion. The point with ramification index three is an inflection of the curve in $\mathbb{P}^2$ and the projection point lies over the tangent in this point.

Moreover, the lines linking x and y with the projection point are tangent to the curve in certain points x' and y' respectively.

Assume by contradiction that there is a positive dimensional family of pairs (x', y') as above. That is, assume that there is a positive dimensional family of suitable morphisms q. Given a particular immersion determined by one such q, we remind that the projection point is determined by the intersection of the tangent to an inflexion point and the lines through x and y tangent to the curve. If we move x and y by a point $z \in E \subset \mathbb{P}^2$, we change the immersion of the curve, but we keep the same planar equation. If there is a one dimensional family of suitable morphisms q, then the point where the new tangents through $x + z$ and $y + z$ intersect should be over the tangent to the inflexion point, giving another morphism q in the family. Doing the effective computations we find that for a general z it does not happen, and hence, there are only a finite number of suitable q's. Hence, we can recover at most a finite number of $[D] \in \mathcal{D}.\Diamond$

9. The tenth case is described by

$$\begin{aligned} \Gamma &\to D_6 \\ x_1,\, x_2 &\to i \\ x_3, \ldots, x_6 &\to j. \end{aligned}$$

We obtain the action of D_6 on a curve of genus $g(D) = 7$ giving a curve of genus $g(B) = 3$ inside $C^{(2)}$ with $g(C) = 2$ and such that $\tilde{B}$ is smooth and $\tilde{B}^2 = 2$.

Now, consider the action of the automorphism $(ij)^3$ on the curve D. Since $\nu((ij)^3) = 0$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle (ij)^3 \rangle) = \frac{1}{4}(2 \cdot 7 + 2) = 4$.

By Lemma 7.1.5 we know that $\nu(\beta_C) = 2$ and from the Riemann-Hurwitz formula we deduce that $g(C/\langle \beta_C \rangle) = \frac{1}{4}(2 \cdot 2 + 2 - 2) = 1$ and therefore C is bielliptic as well as B with β_C and β_B the bielliptic involutions. We obtain the following diagram:
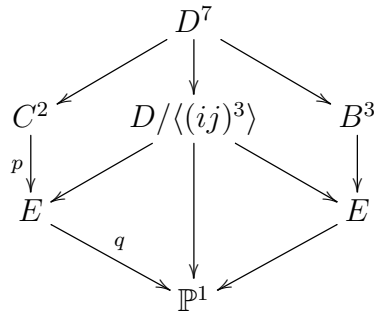
where the superindex denote the genus of the curves.

Notice that in this case $\mathcal{D}$ is 3-dimensional.

Claim: The map η has 1-dimensional fibers.

Proof of the claim: First, notice that $q : E \to \mathbb{P}^1$ has 6 branch points with branching type $(1, 2)$ that is, a non ramified point on the fiber and a ramified one with $e_P = 2$.

Second, $D/D_6 \cong \mathbb{P}^1$ has six marked points. We fix $\{0, 1\}$ to be the branch points associated to (the images of) the points fixed by i and the rest, $\{\infty, x_1, x_2, x_3\}$, to be associated to the points fixed by j.

Third, $C/\beta_C \cong E$ has two marked points. We observe that since the morphism p is the projection given by the action of β_C, the two branch points of p are non ramified lying over a branch point of q, in particular, the images of the points fixed by a conjugate of the automorphism i.

In the following figure we show how the ramification of the morphism $D \to D/D_6$ is distributed along the factorization by C and B respectively. The colored points denote ramification or branch points depending if they are before or after the corresponding colored arrow.
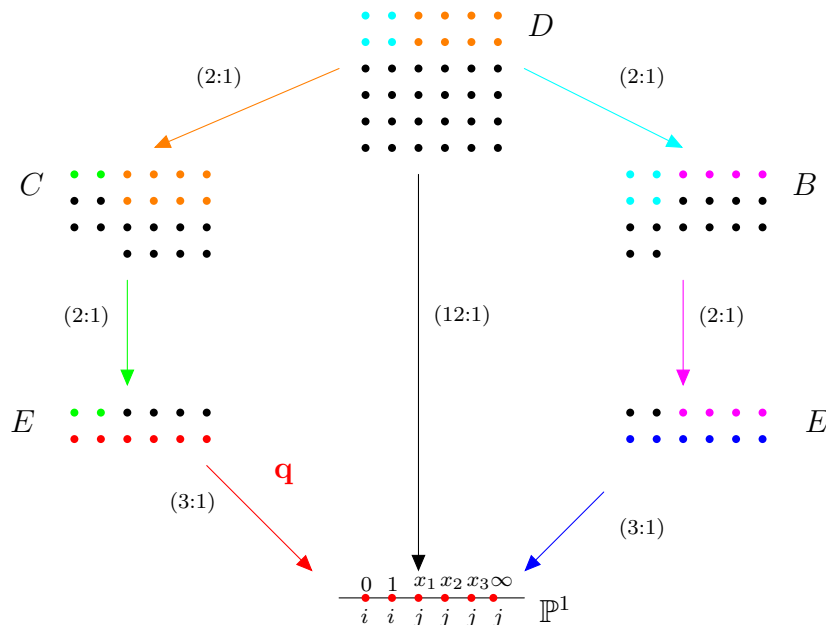
Therefore, η is equivalent to send the curve determined by the data $\{\mathbb{P}^1; 0, 1, \infty, x_1, x_2, x_3\}$ to the curve determined by the data $\{E; q^{-1}(0)_1, q^{-1}(1)_1\}$, where $E := D/\langle j, (ij)^3 \rangle$ and q is the morphism $E \to D/D_6 \cong \mathbb{P}^1$ and $q^{-1}(0)_1$ and $q^{-1}(0)_1$ denote the point on the fiber with $e_P = 1$.

Now, given a curve $[C]$ in the image of $\mathcal{D}$, we recover $\eta^{-1}([C])$ taking the image of the branch points of its bielliptic morphism by a suitable q (they will be x_1 and x_2), together with the other four branch points of q (they will be 0, 1, ∞ and x_3). By a suitable q we mean that the branch points of p are non ramified lying over a branch point, and q has generic ramification.

Such a morphism q exists because C is in the image of $\mathcal{D}$, and therefore it is the quotient of a D. Moreover, we claim that there is a one dimensional family of possibilities for q.

Indeed, the elliptic curve E is given, with two marked points x and y, and we are looking for a $q : E \to \mathbb{P}^1$ with generic ramification and the two marked points over a branch point but non-ramified.

Each immersion of E in $\mathbb{P}^2$ is given by a line bundle $a \in Pic^3(E)$. If we consider the projection $\pi : E^{(3)} \to Pic^3(E)$, the fibers of this morphism are $\mathbb{P}^2$'s given by the linear series. A morphism to $\mathbb{P}^1$ of order 3 can be seen a line in this $\mathbb{P}^2$ with no base point, that is, not contained in a divisor E_x.

Given two points $x, y \in E$, for each $\mathbb{P}^2 = \pi^{-1}(a)$ we have four points of type $x + 2x'$ and four of type $y + 2y'$, hence, there are 16 lines that contain one of each type. In this same fiber of π there are 9 points of type $3Q$. Therefore, for a general, at least one of the 16 lines through $x + 2x'$ and $y + 2y'$ will not contain a point of type $3Q$, and therefore, we deduce that given $x, y \in E$ there is a 1-dimensional family ($dim\, Pic(E) = 1$) of morphisms of degree 3 from E to $\mathbb{P}^1$ with generic branching type and x, y non ramified but with image a branch point.

Hence, we can recover a one dimensional family of $[D] \in \mathcal{D}$. And for each D we find a different B, therefore, we have a one dimensional family of curves $\tilde{B} \subset C^{(2)}$ for each $C \in \eta(\mathcal{D})$. $\Diamond$

Finally, we notice that in the last nine cases $p_a(\tilde{B}) = 2g(C) - 1$.

We have seen that we have three families of types of topological actions given by D_4. We are going to study each of them separately.

1. The first family, parametrized by s and k with $s \in 4\mathbb{Z}_{>0}$, $k \in 4\mathbb{Z}_{\geq 0}$ and $s + k \geq 8$, is described by

if $\frac{k}{4}$ is even: $\qquad\qquad$ if $\frac{k}{4}$ is odd:

$$
\begin{aligned}
\Gamma &\to D_4 \\
x_1 &\to ij \\
x_2 &\to ji \\
x_3, \ldots, x_{2+\frac{s}{2}} &\to j \\
x_{3+\frac{s}{2}}, \ldots, x_{2+\frac{s}{2}+\frac{k}{4}} &\to (ij)^2
\end{aligned}
$$

$$
\begin{aligned}
\Gamma &\to D_4 \\
x_1 &\to ij \\
x_2 &\to ji \\
x_3, \ldots, x_{1+\frac{s}{2}} &\to j \\
x_{2+\frac{s}{2}} &\to j(ij)^2 \\
x_{3+\frac{s}{2}}, \ldots, x_{3+\frac{s}{2}+\frac{k}{4}} &\to (ij)^2.
\end{aligned}
$$

For each such pair (s, k) we have the action of D_4 on a curve of genus

$$
g(D) = -1 + s + \frac{1}{2}k
$$

giving a curve of genus

$$
g(B) = \frac{2s + k}{4}
$$

inside $C^{(2)}$ with

$$
g(C) = \frac{s + k}{4}
$$

and such that $\tilde{B}$ has $\frac{k}{4}$ nodal singularities. Hence,

$$
p_a(\tilde{B}) = \frac{2s + k}{4} + \frac{k}{4} = \frac{s + k}{2} = 2g(C)
$$

and

$$
\tilde{B}^2 = 1 - s - \frac{1}{2}k + 1 + s + \frac{1}{2}(4 + k) = 4.
$$

Now, consider the action of the automorphism $(ij)^2$ on the curve D. Since $\nu((ij)^2) = k + 4$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle(ij)^2\rangle) = \frac{1}{4}(2(-1 + s + \frac{1}{2}k) + 2 - k - 4) = \frac{s-2}{2} \geq 1$.

By Lemma 7.1.5 we know that $\nu(\beta_C) = \frac{s+k+4}{2}$ and we deduce from the Riemann-Hurwitz formula that

$$
g(C/\langle\beta_C\rangle) = \frac{1}{4}\left(2\frac{s + k}{4} + 2 - \frac{s + k + 4}{2}\right) = 0
$$

and therefore C is hyperelliptic.

We also have that $\nu(\beta_B) = \frac{k+4}{2}$ and hence, we obtain that

$$g(B/\langle \beta_B \rangle) = \frac{1}{4}(2\frac{2s+k}{4} + 2 - \frac{k+4}{2}) = \frac{s}{4} \geq 1.$$

Notice that for each pair (s,k) $\mathcal{D}$ is $\frac{2s+k-4}{4}$-dimensional.

We remark that η is finite for each of these families. Indeed, the corresponding q for each pair (s,k) is a degree two morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$ which can be described as the immersion of $\mathbb{P}^1$ in $\mathbb{P}^2$ as a conic followed by the projection from a point not belonging to the conic. Since two of the branch points of the hyperelliptic morphism of C are ramification points of q, once we have an immersion of $\mathbb{P}^1$ as a conic, taking the intersection point of the tangent lines in two branch points of p we have determined a possible morphism q. Once we have a possible q, we need to check that $\frac{k}{2}$ of the branch points of p are $\frac{k}{4}$ of the fibers of q and that the rest are in separate fibers. Hence, given a $[C]$ in the image of $\mathcal{D}$ it has a finite number of possible preimages $[D]$.

2. The second family, parametrized by s and k with $s \in 2 + 4\mathbb{Z}_{\geq 0}$, $k \in 4\mathbb{Z}_{\geq 0}$ and $s + k \geq 10$, is described by

if $\frac{k}{4}$ is even: if $\frac{k}{4}$ is odd:

$$
\begin{aligned}
\Gamma &\rightarrow D_4 \\
x_1 &\rightarrow i \\
x_2 &\rightarrow ij \\
x_3 &\rightarrow j \\
x_4, \ldots, x_{3+\frac{s}{2}} &\rightarrow j \\
x_{4+\frac{s}{2}}, \ldots, x_{3+\frac{s}{2}+\frac{k}{4}} &\rightarrow (ij)^2
\end{aligned}
\qquad
\begin{aligned}
\Gamma &\rightarrow D_4 \\
x_1 &\rightarrow i \\
x_2, \ldots, x_{1+\frac{s}{2}} &\rightarrow j \\
x_{2+\frac{s}{2}} &\rightarrow ij \\
x_{3+\frac{s}{2}}, \ldots, x_{2+\frac{s}{2}+\frac{k}{4}} &\rightarrow (ij)^2.
\end{aligned}
$$

For each such a pair (s,k) we have the action of D_4 on a curve of genus

$$g(D) = -2 + s + \frac{1}{2}k$$

giving a curve of genus

$$g(B) = \frac{2s+k-4}{4}$$

inside $C^{(2)}$ with

$$g(C) = \frac{s+k-2}{4}$$

and such that $\tilde{B}$ has $\frac{k}{4}$ nodal singularities. Hence,

$$p_a(\tilde{B}) = \frac{2s + k - 4}{4} + \frac{k}{4} = \frac{s + k - 2}{2} = 2g(C)$$

and

$$\tilde{B}^2 = 2 - s - \frac{1}{2}k + 1 + s + \frac{1}{2}(2 + k) = 4.$$

Now, consider the action of the automorphism $(ij)^2$ on the curve D. Since $\nu((ij)^2) = k + 2$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle(ij)^2\rangle) = \frac{1}{4}(2(-2 + s + \frac{1}{2}k) + 2 - k - 2) = \frac{s-2}{2} \geq 0$. By Lemma 7.1.5 we know that $\nu(\beta_C) = \frac{s+k+2}{2}$ and we deduce from the Riemann-Hurwitz formula that

$$g(C/\langle\beta_C\rangle) = \frac{1}{4}\left(2\frac{s + k - 2}{4} + 2 - \frac{s + k + 2}{2}\right) = 0$$

and therefore C is hyperelliptic.

We also have that $\nu(\beta_B) = \frac{k+2+2}{2}$ and hence, we obtain that

$$g(B/\langle\beta_B\rangle) = \frac{1}{4}(2\frac{2s + k - 4}{4} + 2 - \frac{k + 4}{2}) = \frac{s - 2}{4} \geq 0.$$

Notice that for each pair (s, k) $\mathcal{D}$ is $\frac{2s+k-4}{4}$-dimensional.

We remark that η is finite for each of these families as soon as $k \geq 4$. Indeed, the corresponding q for each pair (s, k) is a degree two morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$ that can be described as the immersion of $\mathbb{P}^1$ in $\mathbb{P}^2$ as a conic followed by the projection from a point not belonging to the conic. Since one of the branch points of the hyperelliptic morphism of C is a ramification point of q, once we have an immersion of $\mathbb{P}^1$ as a conic, taking the intersection point of the tangent line in one of the branch points of p and the line linking other two points (for this we need $k \geq 4$) we have determined a possible morphism q. Once we have a possible q, we need to check that $\frac{k}{2}$ (in total) of the branch points of p are $\frac{k}{4}$ of the fibers of q and that the rest are in separate fibers. We observe that the other ramification point of q corresponds to the points fixed by i. Hence, given a $[C]$ in the image of $\mathcal{D}$ it has a finite number of possible preimages $[D]$.

3. The third family, parametrized by s and k with $s \in 4\mathbb{Z}_{>0}$, $k \in 4\mathbb{Z}_{\geq 0}$ and $s + k \geq 12$, is described by

if $\frac{k}{4}$ is even: if $\frac{k}{4}$ is odd:

$$
\begin{aligned}
\Gamma &\to D_4 \\
x_1 &\to i \\
x_2 &\to i \\
x_3, \ldots, x_{2+\frac{s}{2}} &\to j \\
x_{3+\frac{s}{2}}, \ldots, x_{2+\frac{s}{2}+\frac{k}{4}} &\to (ij)^2
\end{aligned}
$$

$$
\begin{aligned}
\Gamma &\to D_4 \\
x_1 &\to i \\
x_2 &\to i \\
x_3, \ldots, x_{1+\frac{s}{2}} &\to j \\
x_{2+\frac{s}{2}} &\to j(ij)^2 \\
x_{3+\frac{s}{2}}, \ldots, x_{2+\frac{s}{2}+\frac{k}{4}} &\to (ij)^2
\end{aligned}
$$

For each such a pair (s, k) we have the action of D_4 on a curve of genus

$$
g(D) = -3 + s + \frac{1}{2}k
$$

giving a curve of genus

$$
g(B) = \frac{2s + k - 8}{4}
$$

inside $C^{(2)}$ with

$$
g(C) = \frac{s + k - 4}{4}
$$

and such that $\tilde{B}$ has $\frac{k}{4}$ nodal singularities. Hence,

$$
p_a(\tilde{B}) = \frac{2s + k - 8}{4} + \frac{k}{4} = \frac{s + k - 4}{2} = 2g(C)
$$

and

$$
\tilde{B}^2 = 3 - s - \frac{1}{2}k + 1 + s + \frac{1}{2}(0 + k) = 4.
$$

Now, consider the action of the automorphism $(ij)^2$ on the curve D. Since $\nu((ij)^2) = k$, by the Riemann-Hurwitz formula we obtain that $g(D/\langle(ij)^2\rangle) = \frac{1}{4}(2(-3 + s + \frac{1}{2}k) + 2 - k) = \frac{s-2}{2} \geq 1$.

By Lemma 7.1.5 we know that $\nu(\beta_C) = \frac{s+k}{2}$ and we deduce from the Riemann-Hurwitz formula that

$$
g(C/\langle\beta_C\rangle) = \frac{1}{4}\left(2\frac{s + k - 4}{4} + 2 - \frac{s + k}{2}\right) = 0
$$

and therefore C is hyperelliptic.

We also have that $\nu(\beta_B) = \frac{k+4}{2}$ and hence, we obtain that

$$
g(B/\langle\beta_B\rangle) = \frac{1}{4}(2\frac{2s + k - 8}{4} + 2 - \frac{k + 4}{2}) = \frac{s - 4}{4} \geq 0.
$$

Notice that for each pair (s, k) the space $\mathcal{D}$ has dimension $\frac{2s+k-4}{4}$.

We remark that η is finite for each of these families as soon as $k \geq 8$. Indeed, the corresponding q for each pair (s, k) is a degree two morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$ that can be described as the immersion of $\mathbb{P}^1$ in $\mathbb{P}^2$ as a conic followed by the projection from a point not belonging to the conic. Once we have an immersion of $\mathbb{P}^1$ as a conic, taking the intersection point of two lines linking two of the branch points respectively (for this we need $k \geq 8$) we have determined a possible morphism q. Once we have a possible q, we need to check that $\frac{k}{2}$ (in total) of the branch points of p are $\frac{k}{4}$ of the fibers of q and that the rest are in separate fibers. We observe that the ramification points of q correspond to the points fixed by i. Hence, given a $[C]$ in the image of $\mathcal{D}$ it has a finite number of possible preimages $[D]$.

$\square$

Remark 7.1.7. *Notice that in all cases $\tilde{B}^2 \leq p_a(\tilde{B}) - g(C) + 2$, thus satisfying the inequality in Corollary 4.7 of [MPP11a] even if in the cases with $g(C) = 2$ the surface is not of general type, and therefore the hypothesis are not fulfilled.*

We can say something more about the curves D in relation to the curves C and B using the following lemma:

Lemma 7.1.8. *([KR89]) Given a curve C, let $G \subset Aut(C)$ be a finite group such that $G = H_1 \cup \cdots \cup H_t$ where the subgroups $H_i \subset G$ satisfy $H_i \cap H_j = 1_G$ if $i \neq j$. Then we have the following isogeny:*

$$J_C^{t-1} \times J_{C/G}^g \approx J_{C/H_1}^{h_1} \times \cdots \times J_{C/H_t}^{h_t}$$

where $g = |G|$ and $h_i = |H_i|$ and J^m means the product of J with itself m times.

Corollary 7.1.9. *We have the following isogeny for the Jacobian variety of one of the curves D that appear on the theorem:*

$$J_D \approx J_C \times J_B \times J_{D/\langle ij \rangle}.$$

Proof. We can decompose $G = D_n = \langle i, j \rangle$, with i and j two involutions, as

$$D_n = \langle ij \rangle \cup \langle i \rangle \cup \langle (ij)i \rangle \cup \langle (ij)^2 i \rangle \cup \cdots \cup \langle (ij)^{n-1} i \rangle.$$

Hence, following the notation of Lemma 7.1.8, we have that $t = n + 1$.

We remind that in our case $D/D_n \cong \mathbb{P}^1$, so by Lemma 7.1.8 we obtain that

$$J_D^n \approx J_{D/\langle ij \rangle}^n \times J_{D/\langle i \rangle}^2 \times J_{D/\langle (ij)i \rangle}^2 \times \cdots \times J_{D/\langle (ij)^{n-1}i \rangle}^2.$$

Since i is conjugated of all $(ij)^{2k}i$ we deduce that

$$J_{D/\langle i \rangle} \cong J_{D/\langle (ij)^{2k}i \rangle} \cong J_B$$

and since j is conjugated of all $(ij)^{2k+1}i$ we obtain that

$$J_{D/\langle j \rangle} \cong J_{D/\langle (ij)^{2k+1}i \rangle} \cong J_C.$$

Therefore,

$$J_D^n \approx J_{D/\langle ij \rangle}^n \times J_B^n \times J_C^n$$

and applying Poincaré duality we conclude that

$$J_D \approx J_{D/\langle ij \rangle} \times J_B \times J_C. \qquad \square$$

7.2   DEGREE THREE

Now, we are going to study some cases of curves of degree 3 and positive self-intersection. Specifically, those characterized by a non completing diagram coming from the action of a spherical triangle group ($G = A_4, S_4, A_5$) on a curve D, as explained in Section 6.2. We are going to consider once again separately those curves $\tilde{B} \subset C^{(2)}$ with low and high genus.

Before, we observe a general property of degree 3 curves in $C^{(2)}$.

Proposition 7.2.1. *Let $\tilde{B} \subset C^{(2)}$ be a degree 3 curve. Then $\pi_C^*(\tilde{B})$ is irreducible.*

Proof. Let B be the normalization of $\tilde{B}$. If $\pi_C^*(\tilde{B})$ were reducible, then $\pi_B^*(\tilde{B}) = B_1 + B_2$ with B_1 and B_2 two divisors with normalization B. Since we have a morphism from $\pi_C^*(\tilde{B})$ to C of degree 3, one, let us say B_1, would have a degree one morphism to C and B_2 would have a degree two morphism to C. But then, on the one hand, B and C are isomorphic and on the other hand there is a degree two morphism from B to C, a contradiction since we are assuming that $g(C) \geq 2$. $\square$

Therefore, all curves of degree 3 are characterized by a non completing diagram. Now, we are going to study the self-intersection of those coming from our specific kind of diagrams.

7.2.1   Curves with low genus

We consider curves of low genus, that is, $g(C) < p_a(\tilde{B}) < 2g(C) - 1$. The main result in this section is the following proposition.

Proposition 7.2.2. *Those curves $\tilde{B}$ in $C^{(2)}$ of degree 3 defined by the action of $G = A_4, S_4, A_5$ on a curve D with low genus have non positive self-intersection.*

Proof. We have by hypothesis

$$
\begin{array}{ccc}
D & \xrightarrow{\;(2:1)\;} & B = D/\langle i \rangle \\
\Big\downarrow {\scriptstyle (3:1)} & & \\
C = D/\langle \alpha \rangle & &
\end{array}
$$

with $\langle \alpha, i \rangle = A_4, S_4, A_5$. We denote by

$$b = g(B) \quad g = g(C) \quad h = g(D)$$
$$s = \nu(\alpha) \quad t = \nu(i) \quad r = \nu(i\alpha) \quad r + k = \nu((i\alpha)^2) \quad e = \nu(i\alpha^2 i\alpha).$$

With this notation, the equality in Remark 6.2.6 translates into

$$\tilde{B}^2 = h - 1 - 3(2g - 2) + e + k + r.$$

We proceed by contradiction. Assume that $\tilde{B}^2$ is positive.

First, we are going to use our hypothesis to give some restrictions for the possible values of b, g, h, s, t, r, k and e.

a) The Riemann-Hurwitz formula for $D \to C$ gives that:

$$2h - 2 = 3(2g - 2) + 2s \Rightarrow g = \frac{h + 2 - s}{3} \geq 3 \Leftrightarrow h - s \geq 7. \quad \textbf{(7.24)}$$

We impose $g \geq 3$ because otherwise there is no possible $p_a(\tilde{B})$ with $g < p_a(\tilde{B}) < 2g - 1$.

b) Since $\tilde{B}^2 = h + 5 - 6g + e + k + r > 0$, by (7.24) we obtain that

$$h + 5 - (2h + 4 - 2s) + e + k + r > 0 \Leftrightarrow h \leq 2s + e + k + r. \quad \textbf{(7.25)}$$

c) The Riemann-Hurwitz formula for $D \to B$ gives that

$$2h - 2 = 2(2b - 2) + t \Leftrightarrow b = \frac{2h + 2 - t}{4}.$$

Therefore, by Propositions 6.2.2 and 6.2.4 we obtain that

$$p_a(\tilde{B}) = \frac{2h + 2 - t}{4} + \frac{1}{2}(e + k). \quad \textbf{(7.26)}$$

d) By (7.24) and (7.26) the condition $g < p_a(\tilde{B})$ becomes

$$2 + 3t < 2h + 4s + 6e + 6k. \tag{7.27}$$

e) The condition $p_a(\tilde{B}) \leq 2g - 2$ becomes

$$8s + 14 + 6e + 6k \leq 2h + 3t. \tag{7.28}$$

f) From (7.25) and (7.27) we deduce that

$$\begin{aligned} 2 + 3t - 4s - 6e - 6k < 2h \leq & \quad 4s + 2e + 2k + 2r \Rightarrow \\ 2 + 3t < & \quad 8e + 8k + 2r + 8s. \end{aligned} \tag{7.29}$$

g) By (7.25) and (7.28) we obtain that

$$\begin{aligned} 14 + 8s + 6e + 6k - 3t \leq 2h \leq & \quad 4s + 2e + 2k + 2r \Rightarrow \\ 14 + 4s + 4e + 4k \leq & \quad 3t + 2r. \end{aligned} \tag{7.30}$$

h) And from the last two conditions we deduce that:

$$\begin{aligned} 14 + 4s + 4e + 4k - 2r \leq 3t < -2 + 8e + 8k + 2r + 8s \Rightarrow \\ 4 < s + e + k + r. \end{aligned} \tag{7.31}$$

Now, we are going to see that all the conditions cannot be satisfied for each of the considered groups $\langle i, \alpha \rangle = A_4, S_4, A_5$.

- $\langle i, \alpha \rangle \cong A_4$

  ◇ $o(i\alpha) = 3 \Rightarrow \nu(i\alpha) = \nu((i\alpha)^2) \Rightarrow k = 0$.
  ◇ $i\alpha$ is conjugated to α so $r = s$.
  ◇ $i\alpha^2 i\alpha$ is conjugated to i so $e = t$.

  Therefore, from (7.30) we deduce that $14 + 4s + 4t \leq 3t + 2s$, which implies $14 + 2s + t \leq 0$, which is impossible.

- $\langle i, \alpha \rangle \cong S_4$ (we remind that $|S_4| = 24$).

  We take $i = (1\ 2)$, $\alpha = (1\ 4\ 3)$, $i\alpha = (1\ 4\ 3\ 2)$, $(i\alpha)^2 = (1\ 3)(2\ 4)$ and $i\alpha^2 i\alpha = (1\ 2\ 3)$. Then, α and $i\alpha^2 i\alpha$ are conjugated, and we deduce that $s = e$.

  Let $\gamma = g(D/S_4)$ and consider the morphism $D \to D/S_4$. By Lemma 3.2.4 we obtain that

◇ The points fixed by i give $\frac{t}{2}$ branch points of order 2.

◇ The points fixed by α give $\frac{s}{2}$ branch points of order 3.

◇ The points fixed by $i\alpha$ give $\frac{r}{2}$ branch points of order 4.

◇ The points fixed by $(i\alpha)^2$ give $\frac{k}{4}$ branch points of order 2.

By the Riemann-Hurwitz formula for $D \to D/S_4$ we obtain that

$$h - 1 = 12(2\gamma - 2) + 12\left(\frac{t}{2}(\frac{1}{2}) + \frac{s}{2}(\frac{2}{3}) + \frac{r}{2}(\frac{3}{4}) + \frac{k}{4}(\frac{1}{2})\right)$$

$$\Leftrightarrow h = 24\gamma - 23 + 3t + 4s + \frac{9}{2}r + \frac{3}{2}k \qquad (7.32)$$

We claim that $24\gamma - 23 + 3t + 4s + \frac{9}{2}r + \frac{3}{2}k > 3s + k + r$ which contradicts $\tilde{B}^2 > 0$. That is, we claim that

$$24\gamma - 23 + 3t + s + \frac{7}{2}r + \frac{1}{2}k > 0. \qquad (7.33)$$

If $\gamma \geq 1$ our claim is satisfied, so it remains to study the case $\gamma = 0$. In this case, condition (7.33) becomes $6t + 2s + 7r + k > 46$.

We remind that in this case we have that $s = e$. By the condition (7.24) we know that $6t + 2s + 7r + k \geq 20 + 4t + 4r$ so, if $4t + 4r > 26$ our claim is satisfied. Assume, by contradiction that $4t + 4r \leq 26$. Since $t, r \in \mathbb{Z}$ it is equivalent to assume $t + r \leq 6$.

Then, from (7.30) together with the previous assumption we deduce that

$$14 + 8s + 4k \leq 3t + 2r \leq 18 - r \Rightarrow$$
$$r + 8s + 4k \leq 4 \Rightarrow s = 0 \text{ and } r + 4k \leq 4.$$

Then, by this condition and the inequality (7.31), we obtain that $4 < k + r \leq 4k + r \leq 4$, which is impossible. Hence, the claim is satisfied, which contradicts $\tilde{B}^2 > 0$.

• $\langle i, \alpha \rangle \cong A_5$ (we remind that $|A_5| = 60$).

We take A_5 embedded in S_5 as $i = (1\ 2)(4\ 5)$, $\alpha = (1\ 4\ 3)$ and $i\alpha = (1\ 5\ 4\ 3\ 2)$. Since $o(i\alpha) = 5$, a prime number, we deduce that $i\alpha$ and $(i\alpha)^2$ have the same fixed points and therefore $k = 0$. Moreover, $i\alpha^2 i\alpha = (1\ 4\ 5\ 2\ 3)$ that is conjugated of $i\alpha$ or $(i\alpha)^2$, so $e = r$.

Let $\gamma = g(D/A_5)$ and consider the morphism $D \to D/A_5$. By Lemma 3.2.4 we obtain that

$\diamond$ The points fixed by i give $\frac{t}{2}$ branch points of order 2.

$\diamond$ The points fixed by α give $\frac{s}{2}$ branch points of order 3.

$\diamond$ The points fixed by $i\alpha$ give $\frac{r}{2}$ branch points or order 5.

By the Riemann-Hurwitz formula for $D \to D/A_5$ we obtain that

$$2h - 2 = 60(2\gamma - 2) + 60 \left(\frac{t}{2}(1 - \frac{1}{2}) + \frac{s}{2}(1 - \frac{1}{3}) + \frac{r}{2}(1 - \frac{1}{5}) \right)$$

$$\Leftrightarrow h = 60\gamma - 59 + \frac{15}{2}t + 10s + 12r \qquad (7.34)$$

We claim that $60\gamma - 59 + \frac{15}{2}t + 10s + 12r > 2s + 2r$ which contradicts $\tilde{B}^2 > 0$. That is, we claim that $60\gamma - 59 + \frac{15}{2}t + 8s + 10r > 0$.

If $\gamma \geq 1$ our claim is satisfied, so it remains to study the case $\gamma = 0$. In this case our claim is equivalent to $15t + 16s + 20r > 118$.

We remind that $r = e$ and $k = 0$. By condition (7.27) we know that $15t + 16s + 20r > 7t + 80$ so, if $7t > 38$, our claim is satisfied. Assume, by contradiction that $t \leq \frac{38}{7}$. Since $t \in \mathbb{Z}$ it is equivalent to assume that $t \leq 5$.

Then, from (7.30) together with the previous assumption we deduce that

$$14 + 4s + 2r \leq 3t \leq 15 \Rightarrow s = r = 0.$$

Then, the condition (7.31) implies $4 < 0$, which is impossible. Therefore, the claim is satisfied, which contradicts $\tilde{B}^2 > 0$. $\square$

7.2.2   Curves with higher genus

Now, we are going to drop the condition of low genus and describe all smooth epimorphisms $\Gamma \to G$ with $G = A_4, \ S_4, \ A_5$ that give a non-completing diagram of morphisms of curves characterizing a curve $\tilde{B} \subset C^{(2)}$ with $\tilde{B}^2 > 0$ and $g(C) \geq 2$.

We are going to consider separately each group $G = A_4, S_4, A_5$. We keep the notation introduced in the proof of Proposition 7.2.2. We begin with a numerical analysis of our hypothesis and later, for those values compatible with the hypothesis, we give (or prove that it does not exist) a smooth epimorphism from a Fuchsian group defining a curve D with an action of G and the prescribed ramification.

7.2.2.1 **Alternate group of degree** 4

As we have seen in the proof of Proposition 7.2.2, in this case we have that $k = 0$, $r = s$ and $e = t$. Therefore, the conditions $g \geq 2$ and $\tilde{B}^2 > 0$ translate into

$$h - s \geq 4 \quad \text{and}$$
$$h \leq 3s + t.$$

Now, we consider the action of A_4 on a curve D. The group A_4 has three non identity conjugacy classes, those of i, α and α^2. Since α and α^2 have the same fixed points, the Riemann-Hurwitz formula for $D \to D/A_4$ reads:

$$2h - 2 = 12(2\gamma - 2) + 12(\frac{2}{3}s + \frac{1}{2}\frac{t}{2}) = 24\gamma - 24 + 8s + 3t.$$

Therefore, imposing $\tilde{B}^2 > 0$ we obtain that

$$2h = 24\gamma - 22 + 8s + 3t \leq 6s + 2t \Leftrightarrow$$
$$24\gamma + 2s + t \leq 22 \Rightarrow \gamma = 0 \qquad (7.35)$$
$$\Rightarrow h = 4s + \tfrac{3}{2}t - 11.$$

By (7.35), imposing $g \geq 2$ gives us the condition $4s + \frac{3}{2}t - 11 - s \geq 4$, that is,

$$3s + \frac{3}{2}t \geq 15. \qquad (7.36)$$

Moreover, by (7.35) the condition $\tilde{B}^2 > 0$ translates into

$$4s + \tfrac{3}{2}t - 11 \leq 3s + t \Leftrightarrow$$
$$s + \tfrac{1}{2}t \leq 11.$$

We are going to analyse all possible values of s and t such that $s + \frac{1}{2}t \leq 11$ satisfying the inequality (7.36). With this conditions we observe that we can discard the following cases:

s	t
2	0, 2, 4
3	0, 2
4	0

Given a pair (s, t) satisfying all the conditions, we find a curve D with the action of A_4 with the prescribed ramification, when possible, giving the smooth epimorphism $\Gamma \to A_4$. See Section 4.4 for a detailed explanation.

We remind that to give the smooth epimorphism is equivalent to give a set of elements in A_4 (a A_4-Hurwitz vector) which generate the whole group and such that their product in a certain order is the identity element of the group (product one condition). If one of the conditions is not satisfied, then there is no smooth epimorphism. These elements determine the branching data for the covering $D \to D/A_4$ in the following way: there is one branch point for each element, and the monodromy over this branch point is determined by the conjugacy class of the element.

According to this, if $s = 0$, then the only possible elements in the set of generators are i and its conjugates, that do not generate A_4.

Moreover, if $s = 1$, then in any possible set of elements of A_4 used to describe the action, there would be one element conjugated to α and the rest would be conjugated to i. We observe that all elements conjugated to i have zero or three copies of α on their expression, and all conjugates of α have one, two or four copies of α on their expression. Thus, we deduce that the product of all of them will have $3j \pm 1$ copies of α on the expression, and hence, the condition of product one is not possible to be satisfied.

For the rest of values satisfying $s + \frac{1}{2}t \leq 11$ we can find a curve with the described action of A_4. We study each value of s separately, giving the value of t and the invariants of the curves C and $\tilde{B} \subset C^{(2)}$. To simplify the notation, we describe the smooth epimorphism $\rho : \Gamma \to A_4$ giving a product one relation of elements of A_4. The image of each elliptic generator $x_i \in \Gamma$ is written in square brackets $[\cdot]$ and the exponent of the brackets denote the number of x_i with this image.

We list the possible values of s and give a table describing the different possible cases.

s = 2

t	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
6	$[i]^2[\alpha i\alpha^2][\alpha i\alpha][\alpha] = 1$	6	2	2	5	7
8	$[i]^3[\alpha i\alpha^2][\alpha i\alpha][\alpha i] = 1$	9	3	3	7	6
10	$[i]^4[\alpha i\alpha^2][\alpha i\alpha][\alpha] = 1$	12	4	4	9	5
12	$[i]^5[\alpha i\alpha^2][\alpha i\alpha][\alpha i] = 1$	15	5	5	11	4
14	$[i]^6[\alpha i\alpha^2][\alpha i\alpha][\alpha] = 1$	18	6	6	13	3
16	$[i]^7[\alpha i\alpha^2][\alpha i\alpha][\alpha i] = 1$	21	7	7	15	2
18	$[i]^8[\alpha i\alpha^2][\alpha i\alpha][\alpha] = 1$	24	8	8	17	1

Table 7.1: Action of A_4 with $s = 2$

s = 3

t	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
4	$[\alpha]^3[i]^2 = 1$	7	2	3	5	7
6	$[i\alpha][\alpha^2 i\alpha][\alpha]^2[i]^2 = 1$	10	3	4	7	6
8	$[\alpha]^3[i]^4 = 1$	13	4	5	9	5
10	$[i\alpha][\alpha^2 i\alpha][\alpha]^2[i]^4 = 1$	16	5	6	11	4
12	$[\alpha]^3[i]^6 = 1$	19	6	7	13	3
14	$[i\alpha][\alpha^2 i\alpha][\alpha]^2[i]^6 = 1$	22	7	8	15	2
16	$[\alpha]^3[i]^8 = 1$	25	8	9	17	1

Table 7.2: Action of A_4 with $s = 3$

s = 4

t	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
2	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha i][i] = 1$	8	2	4	5	7
4	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha][i]^2 = 1$	11	3	5	7	6
6	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha i][i]^3 = 1$	14	4	6	9	5
8	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha][i]^4 = 1$	17	5	7	11	4
10	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha i][i]^5 = 1$	20	6	8	13	3
12	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha][i]^6 = 1$	23	7	9	15	2
14	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha i][i]^7 = 1$	26	8	10	17	1

Table 7.3: Action of A_4 with $s = 4$

s = 5

t	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
0	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2]^2 = 1$	9	2	5	5	7
2	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2][\alpha^2 i][i] = 1$	12	3	6	7	6
4	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2]^2[i]^2 = 1$	15	4	7	9	5
6	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2][\alpha^2 i][i]^3 = 1$	18	5	8	11	4
8	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2]^2[i]^4 = 1$	21	6	9	13	3
10	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2][\alpha^2 i][i]^5 = 1$	24	7	10	15	2
12	$[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2]^2[i]^6 = 1$	27	8	11	17	1

Table 7.4: Action of A_4 with $s = 5$

s = 6

t	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
0	$[\alpha]^3[\alpha i]^3 = 1$	13	3	7	7	6
2	$[i\alpha][\alpha^2 i\alpha][\alpha]^5 = 1$	16	4	8	9	5
4	$[\alpha]^6[i]^2 = 1$	19	5	9	11	4
6	$[i\alpha][\alpha^2 i\alpha][\alpha]^5[i]^2 = 1$	22	6	10	13	3
8	$[\alpha]^6[i]^4 = 1$	25	7	11	15	2
10	$[i\alpha][\alpha^2 i\alpha][\alpha]^5[i]^4 = 1$	28	8	12	17	1

Table 7.5: Action of A_4 with $s = 6$

s = 7

t	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
0	$[\alpha]^3[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha] = 1$	13	3	7	7	6
2	$[\alpha]^3[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha i][i] = 1$	20	5	10	11	4
4	$[\alpha]^3[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha][i]^2 = 1$	23	6	11	13	3
6	$[\alpha]^3[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha i][i]^3 = 1$	26	7	12	15	2
8	$[\alpha]^3[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha][i]^4 = 1$	29	8	13	17	1

Table 7.6: Action of A_4 with $s = 7$

s = 8

t	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
0	$[\alpha]^3[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2]^2 = 1$	21	5	11	11	4
2	$[\alpha]^3[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2][\alpha^2 i][i] = 1$	24	6	12	13	3
4	$[\alpha]^3[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2]^2[i]^2 = 1$	27	7	13	15	2
6	$[\alpha]^3[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2][\alpha^2 i][i]^3 = 1$	30	8	14	17	1

Table 7.7: Action of A_4 with $s = 8$

s = 9

t	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
0	$[\alpha]^6[\alpha i]^3$	25	6	13	13	3
2	$[i\alpha][\alpha^2 i\alpha][\alpha]^8 = 1$	28	7	14	15	2
4	$[\alpha]^9[i]^2$	31	8	15	17	1

Table 7.8: Action of A_4 with $s = 9$

s = 10

t	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
0	$[\alpha]^6[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha] = 1$	29	7	15	15	2
2	$[\alpha]^6[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha i][i] = 1$	32	8	16	17	1

Table 7.9: Action of A_4 with $s = 10$

s = 11

t	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
0	$[\alpha]^6[\alpha i\alpha][\alpha][\alpha i\alpha][\alpha^2]^2 = 1$	33	8	17	17	1

Table 7.10: Action of A_4 with $s = 11$

7.2.2.2 Symmetric group of degree 4

As in the proof of Proposition 7.2.2, in this case we have that $e = s$, r, s and t are even and k is multiple of 4. Therefore, the conditions $g \geq 2$ and $\tilde{B}^2 > 0$ translate into

$$h - s \geq 4$$
$$h \leq 3s + r + k.$$

Now, we consider the action of S_4 on the curve D. The group S_4 has four non identity conjugacy classes, those of i (one transposition), α (cycles of order three), $\beta = i\alpha$ (cycles of order 4) and β^2 (double transpositions). Since β and β^2 have the same fixed points, the Riemann-Hurwitz formula for $D \to D/S_4$ reads:

$$2h - 2 = 24(2\gamma - 2) + 24\left(\frac{s}{2} \cdot \frac{2}{3} + \frac{t}{2} \cdot \frac{1}{2} + \frac{r}{2} \cdot \frac{3}{4} + \frac{k}{4} \cdot \frac{1}{2}\right)$$
$$\Rightarrow h = 24\gamma - 23 + 3t + 4s + \frac{9}{2}r + \frac{3}{2}k.$$

Therefore, imposing $\tilde{B}^2 > 0$ we obtain that

$$24\gamma - 23 + 3t + 4s + \frac{9}{2}r + \frac{3}{2}k \leq 3s + r + k \Leftrightarrow$$
$$24\gamma + 3t + s + \frac{7}{2}r + \frac{1}{2}k \leq 23 \Rightarrow \gamma = 0 \qquad (7.37)$$
$$\Rightarrow h = -23 + 3t + 4s + \frac{9}{2}r + \frac{3}{2}k.$$

By (7.37), the condition $\tilde{B}^2 > 0$ translates into

$$3t + s + \frac{7}{2}r + \frac{1}{2}k \leq 23 \qquad (7.38)$$

and $g \geq 2$ becomes

$$3t + 3s + \frac{9}{2}r + \frac{3}{2}k \geq 27. \qquad (7.39)$$

Now, we analyse all possible values for r, t, s and k satisfying the inequalities (7.38) and (7.39). With these conditions we can discard the following cases:

r	t	s	k
0	4	0	0, 4, 8
0	4	2	0, 4
0	4	4	0
2	2	0	0, 4
2	2	2	0

Given (r, t, s, k) satisfying all the conditions, we define a curve D with the action of S_4, with the prescribed ramification, giving the smooth epimorphism $\Gamma \rightarrow S_4$. See Section 4.4 for a detailed explanation. We remind that to give the smooth epimorphism is equivalent to give a set of elements in S_4 (S_4-generating vector) that generate the whole group and such that their product in a certain order is the identity element of the group (product one condition).

For $r = 0$ and $t = 0$ in any possible set of elements of S_4 used to describe the action there would be only elements conjugated to α and β^2 that have even index, and so they could not generate the whole S_4, where there are also odd index elements.

Since i and β have odd index, in order to have the product one condition we need $\frac{t}{2} + \frac{r}{2}$ to be even, or which is the same, $t + r$ to be multiple of four. With this condition we discard the cases

r	t
0	2 or 6
2	0 or 4
4	2
6	0

Since the product of two double transpositions is again a double transposition or 1, with a detailed study of the multiplication table of S_4 we deduce that we cannot have both the generation and the product one condition at the same time in the following cases.

r	t	s	k
0	4	0	12, 16, 20
2	2	0	8, 12, 16, 20
4	0	0	0, 4

For the rest of values satisfying $3t + s + \frac{7}{2}r + \frac{1}{2}k \leq 23$ we can find a curve with the action of S_4. We consider the two pairs of values (r, t) separately, and give a table with the values of s and k, the product one relation and the invariants of the curves C and $\tilde{B} \subset C^{(2)}$. To simplify the notation, we describe the smooth epimorphism $\rho : \Gamma \rightarrow A_5$ giving a product one relation of elements of A_5. The image of each $x_i \in \Gamma$ is written in square brackets $[\cdot]$ and the exponent of the brackets denote the number of x_i with this image. We prove in the notes after the tables that the elements taken generate the whole group when it is not absolutely clear.

For **r = 0** and **t = 4**

s	k	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
2	8	$[(12)][(23)][(132)][(13)(24)]^2 =$ $\quad$ (1) $[t_1][t_4][\alpha_1^2][\beta_1^2]^2 = 1$	9	3	4	9	6
2	12	$[(12)][(23)][(132)][(13)(24)]$ $[(12)(34)][(14)(23)] =$ $\quad$ (1) $[t_1][t_4][\alpha_1^2][\beta_1^2][\beta_3^2][\beta_2^2] = 1$	15	5	7	14	4
2	16	$[(12)][(23)][(132)][(13)(24)]^4 =$ $\quad$ (1) $[t_1][t_4][\alpha_1^2][\beta_1^2]^4 = 1$	21	7	10	19	2
4	4	$[i][\alpha][i][\alpha][(i\alpha)^2] = 1$	11	3	5	9	6
4	8	$[(23)][(34)][(234)]^2[(12)(34)]^2 =$ $\quad$ (2) $[t_4][t_6][\alpha_2]^2[\beta_3^2] = 1$	17	5	8	14	4
4	12	$[i][\alpha][i][\alpha][(i\alpha)^2]^3 = 1$	23	7	11	19	2
6	0	$[i]^2[\alpha]^3 = 1$	13	3	6	9	6
6	4	$[(12)][(34)][(12)(34)][(234)]^3 =$ $\quad$ (3) $[t_1][t_6][\beta_3^2][\alpha_2]^3 = 1$	19	5	9	14	4
6	8	$[i]^2[\alpha]^3[(i\alpha)^2]^2 = 1$	25	7	12	19	2
8	0	$[(234)]^3[(132)][(13)][(12)] =$ $[\alpha_2]^3[\alpha_1^2][t_2][t_1] = 1$	21	5	10	14	4
8	4	$[(243)]^4[(12)][(13)][(12)(34)] =$ $[\alpha_2^2]^4[t_1][t_2][\beta_3^2] = 1$	27	7	13	19	2
10	0	$[(234)]^3[(123)]^2[(13)][(12)] =$ $[\alpha_2]^3[\alpha_1]^2[t_2][t_1] = 1$	29	7	14	19	2

Table 7.11: Action of S_4 with $r = 0$, $t = 4$

Notes:

1. They generate since $\langle \alpha_1, \beta_1^2 \rangle \cong A_4$ and there is no single transposition in this subgroup, so $\langle t_1, \alpha_1, \beta_1^2 \rangle \cong S_4$.

2. They generate since $\langle (12)(34)(34), (234) \rangle \cong S_4$.

3. They generate since $\langle (12), (234) \rangle \cong S_4$.

For r = 2 and t = 2

s	k	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
2	4	$[i\alpha][i][\alpha][(i\alpha)^2] = 1$	6	2	3	6	7
2	8	$[i][i\alpha][\alpha^2][(i\alpha)^2]^2 = 1$	12	4	6	11	5
2	12	$[i\alpha][i][\alpha][(i\alpha)^2]^3 = 1$	18	6	9	16	3
2	16	$[i][i\alpha][\alpha^2][(i\alpha)^2]^4 = 1$	24	8	12	21	1
4	0	$[i][i\alpha][\alpha]^2 = 1$	8	2	4	6	7
4	4	$[(i\alpha)^2][i\alpha][i][\alpha^2]^2 = 1$	14	4	7	11	5
4	8	$[i][i\alpha][\alpha]^2[(i\alpha)^2]^2 = 1$	20	6	10	16	3
4	12	$[(i\alpha)^2][i\alpha][i][\alpha^2]^2[(i\alpha)^2]^2 = 1$	26	8	13	21	1
6	0	$[i][i\alpha][\alpha][\alpha^2]^2 = 1$	16	4	8	11	5
6	4	$[(i\alpha)^2][i\alpha][i][\alpha]^2[\alpha^2] = 1$	22	6	11	16	3
6	8	$[(i\alpha)^2]^2[i][i\alpha][\alpha][(\alpha)^2]^2 = 1$	28	8	14	21	1
8	0	$[i][i\alpha][\alpha]^3[\alpha^2] = 1$	24	6	12	16	3
8	4	$[(i\alpha)^2][i\alpha][i][\alpha]^4 = 1$	30	8	15	21	1
10	0	$[i][i\alpha][\alpha]^5 = 1$	32	8	16	21	1

Table 7.12: Action of S_4 with $r = 2$, $t = 2$

7.2.2.3   Alternate group of degree 5

As we have seen in the proof of Proposition 7.2.2, in this case we have that $k = 0$, $e = r$ and r, s and t even. Therefore, the conditions $g \geq 2$ and $\tilde{B}^2 > 0$ translate into

$$h - s \geq 4$$
$$h \leq 2s + 2r.$$

Now, we consider the action of A_5 on the curve D. The group A_5 has four non identity conjugacy classes, those of i (double transpositions),

α (cycles of order three), $\beta = i\alpha$ (cycles of order 5) and β^2, nevertheless, since β and β^2 have the same fixed points, the Riemann-Hurwitz formula for $D \to D/A_5$ reads:

$$2h - 2 = 60(2\gamma - 2) + 60\left(\frac{s}{2} \cdot \frac{2}{3} + \frac{t}{2} \cdot \frac{1}{2} + \frac{r}{2} \cdot \frac{4}{5}\right)$$
$$\Rightarrow h = 60\gamma - 59 + \frac{15}{2}t + 10s + 12r.$$

Therefore, imposing $\tilde{B}^2 > 0$ we obtain that

$$60\gamma - 59 + \frac{15}{2}t + 10s + 12r \leq 2s + 2r \Leftrightarrow$$
$$60\gamma + \frac{15}{2}t + 8s + 10r \leq 59 \Rightarrow \gamma = 0 \tag{7.40}$$
$$\Rightarrow h = -59 + \frac{15}{2}t + 10s + 12r.$$

By (7.40), the condition $\tilde{B}^2 > 0$ translates into

$$\frac{15}{2}t + 8s + 10r \leq 59 \tag{7.41}$$

and $g \geq 2$ becomes

$$\frac{15}{2}t + 9s + 12r \geq 63. \tag{7.42}$$

We are going to analyse all possible values for r, s and t such that $\frac{15}{2}t + 8s + 10r \leq 59$ satisfying the inequality (7.42). With this condition we observe that we can discard the following cases:

r	s	t
0	0, 2, 4, 6	any
2	0, 2, 4	any
4	0	0

It remains only the possibility $r = 4$. We describe in the following table the possible actions of A_5 on a curve D with the ramification determined by the values of r, s and t.

We give the value of s and t, the product one relation and the invariants of the curves C and $\tilde{B} \subset C^{(2)}$. To simplify the notation, we describe the smooth epimorphism $\rho : \Gamma \to A_5$ giving a product one relation of elements of A_5. The image of each $x_i \in \Gamma$ is written in square brackets $[\cdot]$ and the exponent of the brackets denote the number of x_i with this image. We prove in the notes after the tables that the elements taken generate the whole group when it is not absolutely clear.

r = 4

s	t	branching data	h	g	b	$p_a(\tilde{B})$	$\tilde{B}^2$
0	2	$[(12345)][(13)(24)][(15234)] = 1$ **(1)**	4	2	2	4	5
2	0	$[i\alpha][\alpha][\alpha i] = 1$	9	3	5	7	4

Table 7.13: Action of A_5 with $r = 4$

Note:

1. These three elements generate A_5 because in A_5 an element of order two and one of order five can only generate D_5 or A_5, but since $(15234) \neq (12345)^j$ it cannot be D_5 (since all elements of order five in D_5 are a cyclic group).

The discussion in this section is summarized by the following theorem.

Theorem 7.2.3. *Let (C, B) be a pair of smooth curves with $B \xrightarrow{(1:1)} C^{(2)}$ and image $\tilde{B}$ of degree three such that $p_a(\tilde{B}) \geq 2g(C) - 1$ and $\tilde{B}^2 > 0$. Assume that it is defined by the action of a spherical triangle group, $G = A_4, S_4, A_5$, on a curve D such that B is the quotient of D by an involution and C is the quotient by an order three element. Then, the pair (C, B) is considered in one of the tables in Section 7.2.2.*

In particular, those given by A_4 are described in Tables 7.1, 7.2, 7.3, 7.4, 7.5, 7.6, 7.7, 7.8, 7.9 and 7.10. Those given by the action of S_4 are described in Tables 7.11 and 7.12. Finally, those given by A_5 are described in Table 7.13.

A

Appendix One

GENERALITIES ON GROUPS

In this section we collect some basic results on finite groups. The main reference for this section is [Ros09]. We begin giving some definitions related with the structure of a group and some general results about them.

Definition A.0.4. *Let $H \subseteq G$ be a subgroup of the group G. The number (cardinality) of left cosets of H in G is called the **index** of H in G and it is denoted by $[G : H]$. It is also equal to the number of right cosets.*

The index of a subgroup and the order of the group are related by the following theorem.

Theorem A.0.5 (Lagrange's Theorem). *If $H \subseteq G$ is a subgroup of the group G, then $|G| = |H|[G : H]$.*

Next, we define important classes in a group.

Definition A.0.6. *1. Let $K \subseteq G$ be a subgroup of the group G. The subgroup K is called **normal** in G if, and only if,*

$$gK = Kg \ \forall g \in G.$$

It is denoted $K \lhd G$.

*2. If $g, h \in G$, the element $h^{-1}gh$ is called the **conjugate** of g by h in the group G.*

*3. For a fixed element $g \in G$, the set $Cl_G\{g\} = \{h^{-1}gh \mid h \in G\}$, the set of conjugates of g in G, is called the **conjugacy class** of g in G.*

Theorem A.0.7.   *1. If $K \subseteq G$ is a subgroup of the group G, then the following conditions are equivalent:*

1.a. $K \triangleleft G$;

1.b. for all $g \in G$, $g^{-1}Kg \subset K$;

1.c. for all $g \in G$ and all $k \in K$, $g^{-1}kg \in K$.

2. Suppose $K \triangleleft G$. If $k \in K$, then all conjugates of k in G belong to K, and K is the union of a collection of conjugacy classes of G.

Definition A.0.8. *In a group G, the set*

$$Z(G) = \{a \in G \mid ag = ga \; \forall g \in G\} = \{a \in G \mid g^{-1}ag = a \; \forall g \in G\}$$

*form a normal Abelian subgroup of G called the **center** of G.*

Definition A.0.9. *Given $g \in G$ we define the **centralizer** of g in G as the set $\{h \in G \mid h^{-1}gh = g\}$.*

With this, we have the following structure theorem:

Theorem A.0.10 (First Class Equation)**.** *Suppose that G is a finite group, and let $Cl_G\{g_1\}, \ldots, Cl_G\{g_k\}$ be a complete list of the conjugacy classes of G whose orders are larger than 1. Then,*

$$G = Z(G) \sqcup \bigsqcup_{i=1}^{k} Cl_G\{g_k\}.$$

Definition A.0.11. *For $H \subseteq G$ a subgroup of the group G, we define the **normalizer** of H in G as*

$$N_G(H) = \{g \in G \mid g^{-1}Hg = H\}.$$

Definition A.0.12. *A group G is called **simple** if it contains no proper non-neutral normal subgroup.*

Theorem A.0.13. *If $|G|$ is a prime number, then the group G is simple and cyclic.*

Let S_d denote the symmetric group of degree d, that is, the set of permutations of a set of d elements.

Theorem A.0.14 (Cayley's theorem)**.** *Every finite group is isomorphic to a subgroup of a symmetric group.*

This allows us to define:

Definition A.0.15. *The **degree** of the finite group G is the least integer d such that G can be embedded in S_d.*

In the following three sections we give properties of three classes of groups: cyclic groups, dihedral groups and triangle groups.

A.1 CYCLIC GROUPS

In this section we consider cyclic groups $G = \langle g \rangle$.

Theorem A.1.1.   *1. For each positive integer n there exists a cyclic group of order n.*

   2. All cyclic groups of order n are isomorphic.

   3. All infinite cyclic groups are isomorphic to the group $\mathbb{Z}$.

   4. Every homomorphic image of a cyclic group is cyclic.

Next, we describe the subgroups of $\langle g \rangle$.

Theorem A.1.2.   *1. The subgroups of $\mathbb{Z}$ are $\langle 0 \rangle$ and $n\mathbb{Z}$, one for each positive integer n.*

   2. All non-neutral subgroups of an infinite cyclic group are isomorphic to $\mathbb{Z}$.

Theorem A.1.3. *Suppose G is a cyclic group of order n. It contains a cyclic subgroup H of order m if and only if m divides n, and when this happens H is unique.*

Corollary A.1.4. *All subgroups of a cyclic group are cyclic.*

A.2 DIHEDRAL GROUPS

In this section we consider dihedral groups. For more details see [Cona] and [Conb].

Definition A.2.1. *For $n \geq 3$, the **dihedral group** D_n, is defined as the group of isometries that leave invariant a regular n-gon.*
   Equivalently,

$$D_n = \langle i, j \mid i^2 = j^2 = (ij)^n = 1 \rangle = \langle s, r \mid s^2 = r^n = 1, \ s^{-1}rs = r^{-1} \rangle.$$

Notice that with the first notation i and j are symmetries of the n-gon, and ij is a rotation, whereas with the second s is a symmetry (as well as rs), and r is a rotation. We will use both notations indistinctly.

Proposition A.2.2. *We list here some properties of dihedral groups.*

1. *D_n has degree n and order $2n$.*

2. *$sr^k = r^{-k}s$ (because $s^{-1}rs = r^{-1}$ is equivalent to $sr = r^{-1}s$).*

3. *$r^k s$ has order 2.*

4. *All elements with order greater than 2 are powers of r.*

5. *$(r^l s)r^k(r^l s)^{-1} = r^{-k}$ i.e. any rotation is conjugated to its inverse by any reflection.*

Dihedral groups play a very important role in this thesis because of the following result:

Theorem A.2.3. *Let G be a finite non-abelian group generated by two elements of order 2. Then G is isomorphic to a dihedral group.*

Now, we describe the conjugacy classes of D_n.

Theorem A.2.4. *The conjugacy classes in D_n are as follows, depending on the parity of n:*

1. *Odd n: $\{1\}, \{r^{\pm 1}\}, \{r^{\pm 2}\}, \dots, \{r^{\pm \frac{n-1}{2}}\}, \{r^k s \mid 0 \le k \le n-1\}$.*

2. *Even n: $\{1\}, \{r^{\pm 1}\}, \{r^{\pm 2}\}, \dots, \{r^{\frac{n}{2}}\}, \{r^{2k}s \mid 0 \le k \le \frac{n}{2} - 1\}$, $\{r^{2k+1}s \mid 0 \le k \le \frac{n}{2} - 1\}$.*

This says that a rotation is conjugate only to its inverse and the set of reflections falls either in one or two conjugacy classes depending on the parity of n. Geometrically, it makes sense because for odd n all reflections are symmetries with respect to a line linking a vertex with the midpoint of the opposite side, and for even n there are two types of reflections: those across a line through opposite vertices and those through the midpoints of opposite edges.

Next, we give some results on its structure.

Proposition A.2.5. *When $n \ge 3$ is odd, the center of D_n is trivial. When $n \ge 3$ is even, the center of D_n is $\{1, r^{\frac{n}{2}}\}$.*

Corollary A.2.6. *If $n \geq 6$ is twice an odd number then we can decompose D_n as the product $D_n \cong D_{n/2} \times \mathbb{Z}/2$.*

Finally, we describe the subgroups of D_n. Considering the special cases $D_1 = \langle r, s \rangle \cong \mathbb{Z}/2$ with $r = 1$ and s of order two, together with $D_2 = \langle r, s \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ with r and s of order two, different, and commuting, we have that

Theorem A.2.7. *Every subgroup of D_n is cyclic or dihedral. A complete list of subgroups is as follows:*

1. *Cyclic subgroups $\langle r^d \rangle$, where $d|n$, with index $2d$.*

2. *Dihedral subgroups $\langle r^d, r^k s \rangle$, where $d|n$ and $0 \leq k \leq d-1$, with index d.*

Every subgroup of D_n occurs exactly once in this listing.

Proposition A.2.8. *When n is odd, the proper normal subgroups of D_n are $\langle r^d \rangle$ for $d|n$; these are the subgroups with even index.*

When n is even, the proper normal subgroups of D_n are $\langle r^d \rangle$ with index d and $d|n$, together with $\langle r^2, s \rangle$ and $\langle r^2, rs \rangle$ with index 2.

Corollary A.2.9. *Let i and j be two involutions generating a dihedral group D_n, $n \geq 3$. Then, there is no cyclic subgroup containing $(ij)^2$ and one of the involutions i or j.*

Proof. With the previous notation, put $r = ij$ and s either i or j. Since neither i nor j are a power of ij, we are in the second case of Theorem A.2.7. Moreover, $(ij)^2 \neq 1$, so the subgroup is not cyclic. $\square$

A.3 TRIANGLE GROUPS

In this section we define and describe the so called triangle groups.

Definition A.3.1. *For natural numbers l, m and n, the **triangle (or von Dyck) group** $D(l, m, n)$ is defined by the following presentation:*

$$\langle a, b, c \mid a^l = b^m = c^n = abc = 1 \rangle.$$

There are three types:

- **Spherical triangle groups**

  The triple (l, m, n) in this case satisfies

  $$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1.$$

  The only possibilities for which are $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$ and $(2, 2, n)$.

  All of these groups turn out to be finite subgroups of $SO(3, \mathbb{R})$.

- **Euclidean triangle groups**

  The triple (l, m, n) in this case satisfies

  $$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1,$$

  for which the only solutions are $(4, 4, 2)$ and $(3, 3, 3)$. Both of these give wallpaper groups (plane crystallographic groups), that is, a discrete group of isometries of the Euclidean plane that contains two linearly independent translations. None of them is finite.

- **Hyperbolic triangle groups**

  The triple (l, m, n) in this case satisfies

  $$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1,$$

  for which there are infinitely many possibilities.

Some particular cases:

- $D(1, n, n) \cong \mathbb{Z}/n$ cyclic group of order n.

- $D(2, 2, n) \cong D_n$ dihedral group of degree n.

- $D(2, 3, 3) \cong A_4$.

- $D(2, 3, 4) \cong S_4$.

- $D(2, 3, 5) \cong A_5$.

Notice, moreover, that an element of order two and an element of order three can generate infinitely many different groups of finite oder ([Mil02]). Nevertheless, if the order of its product is 2, 3, 4 or 5 the group is uniquely determined.

Lemma A.3.2. *([Mil01]) Let σ_1, σ_2 be elements of orders 2 and 3 respectively, then*

- *if $o(\sigma_1\sigma_2) = 2$ then $\langle \sigma_1, \sigma_2 \rangle \cong S_3 \cong D_3$,*

- *if $o(\sigma_1\sigma_2) = 3$ then $\langle \sigma_1, \sigma_2 \rangle \cong A_4$,*

- *if $o(\sigma_1\sigma_2) = 4$ then $\langle \sigma_1, \sigma_2 \rangle \cong S_4$ and*

- *if $o(\sigma_1\sigma_2) = 5$ then $\langle \sigma_1, \sigma_2 \rangle \cong A_5$.*

We describe here the last three because we use their structure during the exposition of the rest of this thesis. We describe with detail their conjugacy classes and their subgroup structure.

A.3.1 Alternate group of degree 4

The group A_4 is the triangle group $D(2,3,3)$. It has order $|A_4| = 12$. There are 4 conjugacy classes:

1. Id.

2. $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, the set of double transpositions.

3. $\{(1\ 2\ 3), (4\ 3\ 2), (3\ 4\ 1), (2\ 1\ 4)\}$, formed by 4 order 3 cycles.

4. $\{(1\ 3\ 2), (4\ 2\ 3), (3\ 1\ 4), (2\ 4\ 1)\}$, formed by 4 order 3 cycles.

We denote by $\alpha = (1\ 2\ 3)$ and $i = (1\ 2)(3\ 4)$. With this, the subgroup structure of A_4 is described by:
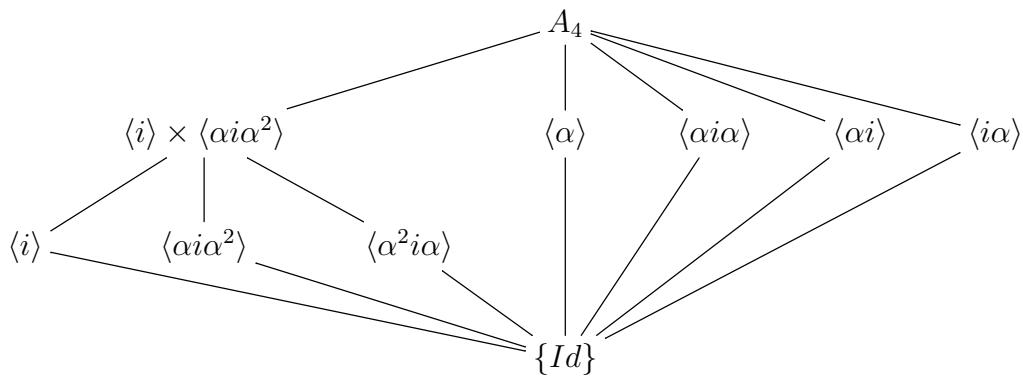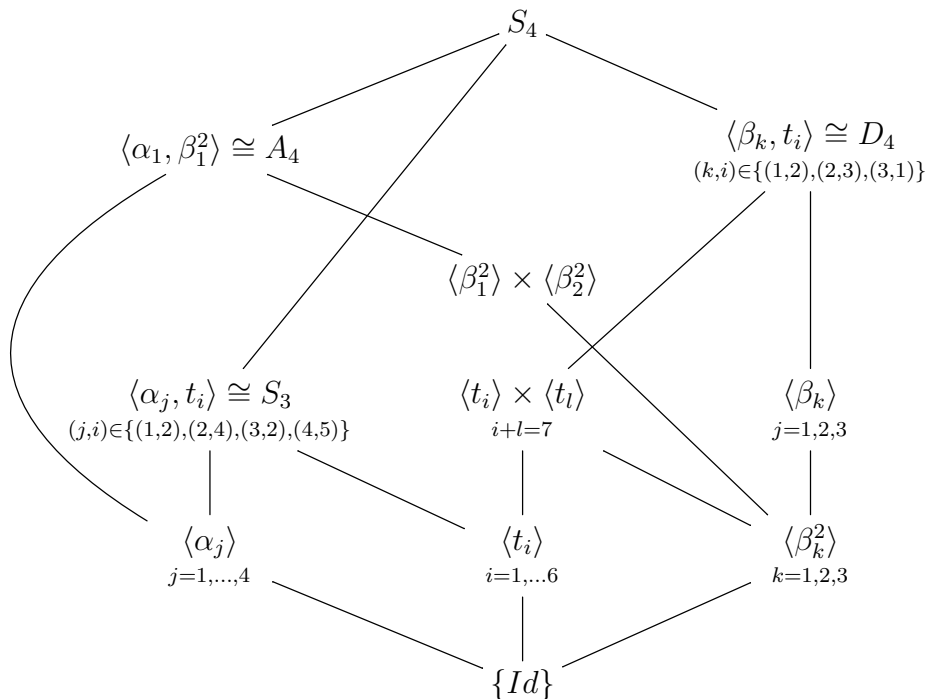

Those subgroups isomorphic are also conjugated with each other, and non conjugated with any other subgroup.

A.3.2 Symmetric group of degree 4

The group S_4 is the triangle group $D(2,3,4)$. It has order $|S_4| = 24$. There are 5 conjugacy classes:

1. Id.

2. $\{t_1 := (1\ 2), t_2 := (1\ 3), t_3 := (1\ 4), t_4 := (2\ 3), t_5 := (2\ 4), t_6 := (3\ 4)\}$, the set of single transpositions.

3. $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, the set of double transposition.

4. $\{\alpha_1 := (1\ 2\ 3), (1\ 3\ 2), \alpha_2 := (2\ 3\ 4), (2\ 4\ 3), \alpha_3 := (3\ 4\ 1), (3\ 1\ 4), \alpha_4 := (4\ 1\ 2), (4\ 2\ 1)\}$, the set of order 3 cycles.

5. $\{\beta_1 := (1\ 2\ 3\ 4), \beta_2 := (1\ 2\ 4\ 3), \beta_3 := (1\ 3\ 2\ 4), (1\ 3\ 4\ 2), (1\ 4\ 2\ 3), (1\ 4\ 3\ 2)\}$, the set of order 4 cycles.

Using the notation t_i, α_j, β_k, the subgroup structure of S_4 is described by the following diagram:


Each node of this graph is formed by subgroups of S_4 conjugated with each other, and non conjugated with any other subgroup.

A.3.3   Alternate group of degree 5

The group A_5 is the triangle group $D(2,3,5)$. It has order $|A_5| = 60$. There are 5 conjugacy classes:

1. *Id*.

2. $\{(1\ 2\ 3), (1\ 3\ 2), \dots\}$, formed by 20 order 3 cycles.

3. $\{(1\ 2)(3\ 4), \dots\}$, formed by 15 double transpositions.

4. $\{(1\ 2\ 3\ 4\ 5), \dots\}$, formed by 12 order 5 cycles.

5. $\{(1\ 3\ 5\ 2\ 4), \dots\}$, formed by 12 order 5 cycles.

In A_5 there are 6 subgroups isomorphic to $\mathbb{Z}/5$, generated by a cycle of order 5; 15 subgroups isomorphic to $\mathbb{Z}/2$, generated by a double transposition and 10 subgroups isomorphic to $\mathbb{Z}/3$, generated by a cycle of order 3. The subgroup structure of A_5 is:
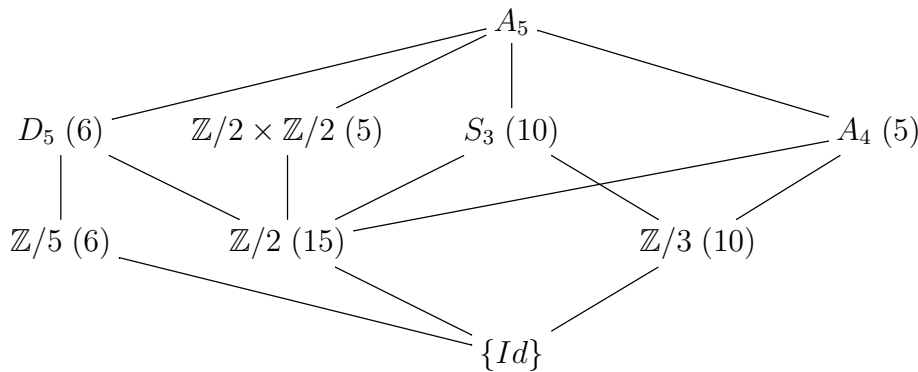
Those subgroups isomorphic are also conjugated with each other, and non conjugated with any other subgroup.

A.4   GROUP ACTION

Finally, we introduce some definitions and well known results on the action of a group G on a set X.

Definition A.4.1. *Given a non-empty set X and a group G, we say that G **acts** on X if, for each $g \in G$, there exists a map $g : X \to X$, and these maps satisfy:*

1. *$1(x) = x$ and*

   2. $gh(x) = g(h(x))$

for all $x \in X$ and $g, h \in G$.

Theorem A.4.2. *A map $g : X \to X$ as considered in the above definition is a permutation (bijection) of the set X.*

Definition A.4.3. *Let X be a set and G a group acting on X. We define an equivalence relation $\sim$ by: If $x, y \in X$ then*

$$x \sim y \Leftrightarrow g(x) = y$$

for some $g \in G$.

Definition A.4.4. *An equivalence class of the equivalence relation $\sim$ given above is called an **orbit** of the action of G on X. The orbit containing the element $x \in X$ is called the **orbit of** x, and it is denoted by $\mathcal{O}_G(x)$.*

Definition A.4.5. *Given a group G acting on a set X and $x \in X$, the subset of G defined as*

$$G_x := \{g \in G \mid g(x) = x\}$$

*is called the **stabilizer** of x in G.*

Lemma A.4.6. *Using the notation above, for $x \in X$, G_x is a subgroup of the group G.*

Theorem A.4.7 (Orbit-Stabilizer Theorem)**.** *Assume that G acts on a set X, $x \in X$ and $\mathcal{O}_G(x)$ is the orbit of x, then*

$$|\mathcal{O}_G(x)| = [G : G_x].$$

Lemma A.4.8.   *1. If $x, y \in X$, $g \in G$ and $y = g(x)$ then $G_y = g\, G_x\, g^{-1}$. In particular, $|G_y| = |G_x|$.*

   2. If $\theta : X \to X'$ is a G-isomorphism (θ is an isomorphism with $\theta(gx) = g\theta(x) \; \forall x \in X$) and $\theta(x) = y$ then $G_y = G_x$.

    We finish with a particular and very interesting example of the action of a group on a set.

Example A.4.9 (Conjugacy action)**.** *Let G be a group, and let X be the underlying set of G. We can define an action of G on X by conjugation as follows:*

$$G \times X \longrightarrow X$$
$$(g, x) \longrightarrow x^g := g^{-1}xg.$$

With this definition, the conjugacy class of an element x is the orbit of x by this action and its centralizer is the corresponding stabilizer.

We can also define an analogous action of G on its subgroups by $g^{-1}Hg$, in such a way that the normalizer of a subgroup is its stabilizer by this action.

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