



**Universitat
Autònoma
de Barcelona**

Geometric Differential Operators for Shape Modelling

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at Universitat Autònoma de Barcelona to fulfil
the degree of **Doctor en Matemàtiques**.

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Medical imaging feeds research in many computer vision and image processing fields: image filtering, segmentation, shape recovery, registration, retrieval and pattern matching. Because of their low contrast changes and large variety of artifacts and noise, medical imaging processing techniques relying on an analysis of the geometry of image level sets rather than on intensity values result in more robust treatment. From the starting point of treatment of intravascular images, this PhD thesis addresses the design of differential image operators based on geometric principles for a robust shape modelling and restoration. Among all fields applying shape recovery, we approach filtering and segmentation of image objects.

For a successful use in real images, the segmentation process should go through three stages: noise removing, shape modelling and shape recovery. This PhD addresses all three topics, but for the sake of algorithms as automated as possible, techniques for image processing will be designed to satisfy three main principles: a) convergence of the iterative schemes to non-trivial states avoiding image degeneration to a constant image and representing smooth models of the originals; b) smooth asymptotic behavior ensuring stabilization of the iterative process; c) fixed parameter values ensuring equal (domain free) performance of the algorithms whatever initial images/shapes. Our geometric approach to the generic equations that model the different processes approached enables defining techniques satisfying all the former requirements. First, we introduce a new curvature-based geometric flow for image filtering achieving a good compromise between noise removing and resemblance to original images. Second, we describe a new family of diffusion operators that restrict their scope to image level curves and serve to restore smooth closed models from unconnected sets of points. Finally, we design a regularization of snake (distance) maps that ensures its smooth convergence towards any closed shape. Experiments show that performance of the techniques proposed overpasses that of state-of-the-art algorithms.

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Introduction

The Goal of this Work

A main topic in image processing is the obtention of smooth accurate representations of image objects. There are two main steps related to the former image segmentation problem, extraction of points lying on the objects of interest and computation of a smooth model of such (possibly unconnected) set of points. This PhD focus on the following three points involved in a robust shape modelling:

1. **Image Smoothing.**

Image feature extraction algorithms [6], [28] mainly use the mechanisms of human perception that base on the detection of abrupt changes. Therefore some sort of image filtering is a compulsory step to remove noise and artifacts that hinder performance of the extractors in real images.

2. **Contour Closing.**

After image filtering the set of points conforming to the characteristics (edges, ridges) defining image objects are prone to be uncompleted. Line continuation is a possible approach [50], [55] in order to restore a closed model of the shape.

3. **Shape Recovery and Distance Maps.**

The final step in object segmentation is shape recovery, that is, the computation of a representation of the shape boundary as compact as possible. Because B-spline snakes are the ideal tools for such shape encoding, the definition of a map ensuring their convergence is crucial.

The Approach

The novelty of our work is that the operators performing any of the above tasks are designed to achieve an applicability to real automated procedures as efficient and reliable as possible. To such purpose, we consider that the algorithms should satisfy the following principles:

1. **Meaningful Convergence**

The above tasks imply the evolution of an initial function in time (either the image to be filtered or the snake modelling objects), which is computed using

an iterative scheme. The first requirement is that any operator converges to an image/shape yielding smooth accurate representations of the original images and objects.

2. Smooth Asymptotic Behavior

In real applications stabilization of the numeric scheme by means of the characteristics of the evolving function should be as important as its theoretic convergence. As if stopping of the iterative algorithm was dependent on the number of iterations, its applicability to real automated procedures would be very limited. A smooth asymptotic behavior ensures such process stabilization.

3. Robust Parameter Setting

The ideal automated-oriented algorithm should be parameter free. Because this is an utopian situation difficult to achieve, we will use the following relaxation. We require that the parameter values making the technique fulfill the first two points is invariant whatever the nature and degree of noise and the geometry of shapes.

The Tools

The mathematical tools to handle deformation/evolution processes lie in the framework of Partial Differential Equations (PDE's for short). Although any phenomena formulated in PDE terms is of analytic nature, the geometry of the underlying space where it takes place completely determines and characterizes it. In this work, we will see that an analysis of the PDE's related to image processing techniques from a geometric point of view, provides with the necessary tools to design operators satisfying the former requirements.

State-of-the-Art Overview

Image Filtering and Shape Restoration

Noise removal for an improving of image quality is an issue that has been addressed since the early years of image processing. The most common approach [1], [64] consists in filtering the initial image by means of a parabolic (time dependant) PDE. From this point of view, current image filtering operators can be split into two distinct families: the ones arising from heat diffusion processes and the ones based on geometric flows.

The first group filters an image by means of a PDE that describes the physical process of heat diffusion and admit a divergence-like formulation. Linear filtering by convolution with a Gaussian kernel is the classic low-pass image smoothing. Although, it enjoys from an efficient image regularization, it must, by its own design, tackle with blurring of the image descriptive features such as edges or ridges. This need of a selective image filtering has been frequently addressed in the last few years and leads to designing equations able to remove noise and preserve, as much as possible, image contrast changes.

A usual way of achieving this compromise in the filtering technique consists in including a term preventing diffusion across edges. Treating edges as insulators was first introduced in the Perona-Malik model [60] that adds an edge enhancing factor inside the image gradient divergence that describes the classic heat diffusion. In an ideal continuous setting, such factor would evolve images towards a piece-wise constant representation of regions of uniform gray-level separated by the original image edges. In practice, the scheme depends upon the choice of a threshold value determining what magnitude of the image gradient is to be considered an edge and constitutes a main problem when dealing with highly noisy images [53], [74], [81]. Since noise affects image contrast changes, it punches the theoretic closed edges, letting the heat out. This phenomena, known as the Pinhole Effect [53], blurs edge and yields constant final images. The other drawback comes from the backwards edge-enhancement diffusion of the model that might amplify noise and produces false step edges (the Stair-Casing Effects, [80], [81]). Anisotropic diffusion [72] provides with an efficient robust-to-noise model by using a regularized version of the gradient in the Perona-Malik equation, so that the heat diffusion is described by means of a diffusion tensor with the eigenvector of minimum eigenvalue oriented along the direction of the image features to be preserved. The well-posedness of the former formulation has been exploited hereinafter by other authors [8], [9].

Another approach to image feature preservation relies on adding a reaction term to the diffusion equation, the close-to-data constrain [62], forcing similarity to original images [43], [2], [7]. The trade-off between smoothing and similarity is controlled with an extra parameter that needs to be determined for every image as it strongly depends on the amount of noise.

Geometric flows base on shape/curve regularization ruled by deformations that depend exclusively on the geometry of the evolving curve. By their independence from the curve parameter, they admit an implicit formulation, which serves to apply them to image filtering through a level sets [57], [48] approach. As the latter smoothing bases on the image level curves geometry rather than on its intensity values, it does not produce any edge blurring [1], [64] as it alters edges shapes but not the contrast change defining them.

Current geometric flows derive from the mean curvature flow ([32], [33], [34] or [40]), which is considered the geometric equivalent to the heat equation [1] as in the curve arc length parameter the curvature corresponds to the curve second derivative. Curves evolving under their mean curvature reduce their number of oscillations, so that the flow enjoys from the same regularizing properties than heat diffusion. Unfortunately, as in the case of heat-based filtering, smoothing is indiscriminate and shapes converge to circles [32] [33] before collapsing to a point. Up to our knowledge there are two different solutions to prevent curve collapsing [49], [79].

The min-Max flow of [49] exploits that evolution under negative curvature stops as soon as the curve becomes convex. Assuming that the curve is embedded in the plane as a level set of a given image, the average gray level in a window centered at each point serves as a switch between positive and negative curvature. The target curve evolves until this average equals its own gray level, which it is satisfied for polygonal shapes of a side length depending on the window size. It follows that the stencil size used to compute averages must reach a compromise between resemblance to the original

curve and shape simplification (noise removing). The stochastic based geometric flow introduced in [79] suggests adding a stopping factor to the mean curvature equation. Such term penalizes given orientations of the curve unit tangent, so that the initial shape converges to a pre-given polygon.

A common drawback of most of current image filtering techniques is that their steady-state is trivial, that is a constant image for diffusion processes [72] and either a point or a straight line for curvature based ones [32], [33], [40]. This property, although desirable for a scale-space analysis of images/shapes [1], [41], is a main nuisance when using these techniques for segmenting purposes, as they require a stopping time to recover shapes of interest. The simplest way to avoid this feature-damaging effect is to rely on a given (image-dependant) number of iterations to stop the evolution ([60], [72], [32]). Another solution ([46], [9], [2], [43]) is to add a new term, the close-to-data constraint [62], that controls resemblance to the original data. A third option consists in including some previous knowledge about the geometry of the shape to be smoothed [79].

We conclude that current filtering techniques converge to trivial images (which, usually, achieve the minimum of a particular energy) unless we either admit an irregular image close to the original one ([2], [7], [43], [49]), rely on an image-dependant number of iterations ([60], [72], [32]), or prefer a (previously given) model of shapes [79]. In any case, some sort of user intervention is essential in order to stabilize the iterative process at images useful for a latter recognition or segmenting task.

Our Contribution

The first method we propose is a curvature based shape evolution with a factor preventing the flow from converging to a trivial curve. Selective smoothing by means of a stopping factor is not original, as other techniques such as the edge enhancement of [60] or the geometric flow of [79] also exploit this idea. The novelty of our work lies on the fact that the stopping factor relies on a roughness function, that measures the local irregularity of a curve in terms of the smoothness of its tangent direction along the curve, instead of basing on a threshold value [60] or a pre-given model of the shape [79]. Such roughness measure does not penalize the curve curvature magnitude but high variation and cancels on infinitely smooth curves. As a consequence, by adding it to the mean curvature flow, we obtain a regularized curvature flow (RCF) that converges to smooth curves preserving significant enough features as to identify the original noisy shape. The associated image filtering admits integration with a finite difference Euler scheme presenting a smooth enough asymptotic behavior as to stabilize by means of standard numeric criteria.

Another contribution is to handle the study of geometric flows by means of the curve unit tangent angle and curvature in the arc-length parameterization of the curve, which brings our work closer to [32], [33], [34] than to the viscosity solutions implicit parameterization approach of [15], [20], [57]. The new formulations developed turn into an ideal tool to detect the source of pathologies in curve evolution and distance maps.

Shape Restoration

Smooth line connection arises in so different image processing areas as segmentation, image in-painting, splines interpolation or object disocclusion. Most techniques [4], [12], [50] base on variational principles that lead to a diffusion-like equation that serves to smoothly evolve an image/curve until the completion is achieved.

Object disocclusion and image in-painting focus on filling in gaps in an image in such a way that level lines arriving at gaps boundaries are smoothly prolonged inside. Following the Gestalt principle of good continuation, recent techniques ([4], [13], [50], [55]) base on the way humans join unconnected curves to reconstruct the underlying shape. According to psychophysicist studies [37], the curves that reconstruct objects should minimize an energy functional involving length and curvature (Euler's elastica [54]). In the case of (the one dimensional) line continuation for object disocclusion ([55], [50]), curve gaps are interpolated (joint) by line segments minimizing the functional, for image in-painting ([4],[13]) one seeks for the function, which corresponds to the reconstructed image, that achieves a simultaneous minimum for all its level sets. The fact that the curvature term is of second order, introduces a non-linear fourth order term in the associated variational problem that troubles the numeric implementation. Although there are some numeric approaches to a weak formulation of the curvature term [14], the most common solution is either implementing a simplified version of the functional or introducing of an extra function to be minimized. Dropping the curvature term in the elastica functional ([50], [13]) in disocclusion algorithms results in a linear interpolation of curve gaps. This geodesic approach is easy to handle, but the reconstructed contours are prone to present corners at joint points, thus, violating the Gestalt principle. The image restoration of [4] bases on the computation of a vector field that smoothly extends the image unit gradient at gaps boundary to the whole gap to be filled-in. The restored image is the function of minimum variation which level sets are given by the vector extension. In analytic terms, the functional uses the angle of the vector to compute the curvature term in the Euler elastica, so that, a system of second order PDE's replaces the fourth order original equation.

A completely different approach to contour closing comes from mathematical morphology [1] [64], which basing on set operators achieves contour completion by a dilation at a suitable scale. Although the complexity of the computational algorithm is minimum compared to the iterative schemes of variational techniques, morphological closings might not achieve so high quality in restorations. First, as there is not any explicit control on curve smoothness, restorations do not need to satisfy the Gestalt principle. Second, the fact that the associated structural elements (typically, lineal or circular) are constant is a main drawback, since it may imply that the closed shapes differ significantly from the incomplete contours.

Our Contribution

Our second contribution consists in embedding contour closing into the framework of functional extension described by means of a diffusion tensor that prevents (degenerates) any diffusion in the normal space to the uncomplete contours. This natural requirement for operators designed to extend image level sets implies loosing the strong

ellipticity of the differential operator that ensures [22], [23] existence of a unique solution. We propose approaching diffusion processes from the point of Riemmanian geometry by means of the diffusion tensor that describes the mechanism. In this context, degeneracy implies restricting diffusion to the manifolds generated by the vector fields of positive eigenvalues, provided that they exist. A classic result (Frobenius theorem [67]) on differential geometry yields sufficient conditions on the vector fields for the unique existence of solutions to a restricted anisotropic extension/diffusion process.

The restricted diffusion applied to extending the characteristic function of the open curve results in our Anisotropic Contour Completion (ACC for short). The fact that a smooth vector field automatically satisfies the Frobenius conditions reduces contour completion to the definition of a vector field representing the tangent space of the contours to be closed. Instead of using a, computational expensive, iterative extension ([75], [4]) of vector fields we suggest using the Structure Tensor [36]. This matrix, commonly used for analyzing the direction of maximum contrast change [74] and texture orientation [5], [36], yields smooth extensions/regularizations of the image gradient direction. It follows that applied to images having the uncomplete contour as level set, the Structure Tensor produces smooth vector field representing a reliable model of the open curve. This strategy gives rise to Linear Vector Fields, if a mask of the open contour is used, and Distance Vector Fields in the case of using its distance map.

Shape Recovery

The preferred tool to detect and smoothly model image objects are active contours or snakes ([11], [12], [38], [47], [58]). Snakes are evolving curves that deform under the gradient descent of an energy that characterizes a smooth approach of the target contour as a minimal curve. Resemblance to the curve is guaranteed by means of an external potential, while internal constraints involving the norm of the snake first derivatives control its continuity and smoothness and determine the snake at parts where no information is available.

There are two different frameworks in current snakes formulations: the parametric [38] and the geometric [11], [12], depending on the way the energy is formulated. In classic snakes [38], the external energy splits from an internal constrains given in terms of minimum variation of the snake first and second derivatives. Minimization of the first derivative variation makes the snake linearly interpolate contours, meanwhile the second derivative term forces it to smoothly approach them. The former internal constrains are not intrinsic to the geometry of the curve and their values depend upon an explicit parameterization of the curve, which limits parametric snakes to segmentation of a single object. Geodesic snakes [11], [12], fuse both energies into a geometrically intrinsic potential that seeks for the curve of minimal length in a Riemmanian manifold with the external energy as metric. Since their formulation does not depend on the snake parameter, they admit a level set approach [48] able to detect more than one object. However as they only take into account the snake first derivatives (length) they are limited to contour linear interpolation.

In any case, poor convergence to concave shapes is a main limitation of snakes as a

standard segmentation and shape modelling technique. The external energy is crucial in order to ensure the snake convergence because it is the only force that pushes the snake into concave regions, as its internal energy increases when new inflexion points are created. The usual external potentials used in snake formulations are (Euclidean) distance and gradient maps for the parametric case and just gradient maps for the geodesic one. Curves of level zero correspond to the contour of interest and the snake moves in the direction opposite to the gradient of the distance/gradient map. Unfortunately, the geometry of the curve of level zero may produce maps with null gradients along some curves. Because the snake gets caught in these local minima and produces a wrong model of the shape, several ways of addressing this problem have been developed in the last years. The first naive solution is initializing the snake close to the final shape so that we make sure that it is far away from these local minima, which is certainly not a very elegant approach for automated procedures. Some authors [17], [68] suggest searching for the global minimum of the energy, but global minimums of real images are hard to find in an efficient way without manual intervention.

The most efficient solutions are given by the gradient vector flow technique [75], in the case of a parametric formulation, or [16] area minimizing balloon forces, in the case of a geodesic approach. Gradient vector flow techniques ([75], [77], [59]) use a vector value heat equation to compute a regularized version of the gradient of the external energy. By means of a reaction term weighted by the external potential gradient they yield a smooth approximation that only admits isolated zeros. The technique succeeds in producing smooth gradients in the whole image that guide the snake to the final contour for a large variety of geometries. However the vector field obtained with GVF may have saddle points which also trap the snake. Balloon-like forces [16] are added to the geodesic formulation in the form of a constant shrinking force that pushes the snake into concavities and helps the snake adapting to any geometry. However, in order to ensure that the scheme will stop at the boundary of interest, an equilibrium between the constant shrinking velocity and the static vector field given by the image object must be achieved. It follows that, should this condition be satisfied, incorporating the curvature term into the convergence scheme constitutes a significant drawback. For the balloon force must overpass the magnitude of the curvature to enter into concave regions but, at the same time, it must be kept under the minimum of magnitude of the external force gradient to guarantee non trivial steady states. This dichotomy, makes geodesic snake convergence to depend on parameters which values are case sensitive.

Our Contribution

Our last contribution concerns the development of a new external energy based on the geometry of the curve to be modelled that ensures a robust snake convergence whatever the concavity of the curve is. The first important novelty of our contribution is a geometric characterization of convex functionals in terms of smoothness of their level sets. Convexity of the external energy guarantees [23] the existence of a unique minimum that can be reached following the functional gradient descent. Unfortunately, the analytic formulation of convexity in terms of the functional second

derivatives is not robust to noise and irregularities and leads to ambiguities in real images. Our geometric criterion is a simple and robust way of detecting what makes current external energies fail to attract the snake. The usual external potentials used in parametric snake formulations may be regarded as a propagation of either the target curve (Euclidean) or a function defined on it (GVF). In any case their main source of error is due to shock formation during the evolution process that introduces singular level curves and inflexion points. Our second proposal is to use a smooth curve deformation to propagate the initial shape. By tracking back the deformation of a curve that evolves by minimum curvature flow, we construct a distance map that encapsulates the natural way of adapting to non convex shapes. The gradient of this map, which we call curvature vector flow (CVF), is a smooth external force that guides the snake towards any contour, whatever its geometry.

Chapter 1

Partial Differential Equations in Image Processing

All techniques discussed in this work handle evolution and transformation of images and curves. Because images and curves are represented by functions defined on a given domain, Ω of \mathbb{R} or \mathbb{R}^2 with values in either \mathbb{R}^2 (curves) or \mathbb{R} (images), in this first chapter we briefly introduce the general equation that describes evolution of a given initial function in time. We will discuss the different processes that such equation models as well as the special cases that arise in image processing techniques. We will give intuitive interpretations of each of the formulations, a suitable classification according to the issues we will address and examples of techniques fitting into the latter categories. An overview on general theory on PDE's may be found in [22], [23] or [71].

The general model for the evolution of an initial function $u_0(x)$, $x \in \mathbb{R}^n$, in time is formulated with a PDE of parabolic type:

$$u_t(x, t) = Lu \quad \text{with} \quad u(x, 0) = u_0(x) \quad (1.1)$$

The nature of the particular process that is modelled is determined by the values that the function u takes on the boundary, $\partial\Omega$, of the domain Ω holding the evolution process. The right-hand side of the equation, L , denotes a partial differential operator and it represents/describes the mechanisms that rule the transformation of the function. Let us classify the different phenomena governed by (1.1) and describe the mechanism itself.

Let u_{x_I} be the partial derivative with respect to the coordinates specified by the index set $I = (i_1, \dots, i_k)$ and define the order of the derivative to be the cardinal of the index set $|I| = k$. An operator of order N is given by:

$$Lu = \sum_{k=0}^{k=N} \sum_{|I|=k} a_I u_I$$

where the coefficients a_I are smooth functions depending on x and, eventually, on the function u and its derivatives. In this general setting, PDE's can be classified

according to either the nature of the differential operator or the domain of definition of the function:

OPERATOR TYPES

1. **Domain Types.** Depending on the dimensions of the spaces the function takes its values we can differentiate two cases:

- (a) **Image Evolutions.** This is the usual case where $u : \mathbb{R}^n \rightarrow \mathbb{R}$ represents mass density, heat, a chemical concentration or an image.
- (b) **Curve Deformations.** In the case that the function u is a curve parameterization $\gamma : [0, 1] \rightarrow \mathbb{R}^n$, equation (1.1) represents its deformation. Snakes and geometric flows are some of the examples found in image processing.

2. **Operator Types.** The differential operator, L , can be:

- (a) **Linear.** If for any two functions, u_1, u_2 , it satisfies additivity:

$$L(u_1 + u_2) = Lu_1 + Lu_2$$

This is a desirable property, both, in practical applications (signal processing) and from a theoretic (mathematical) point of view. On one hand, it implies that given the response for two different signals the response for their addition is the sum of responses. On the other one, the operator can be computed by means of convolution with a kernel (a filter in Fourier domain) [22] which is fast computational operation. Finally, from a mathematical point of view, linearity simplifies the study of the solution to (1.1) as there are some general rules to design operators yielding uniquely solvable problems [23].

It is not difficult to check that the condition is satisfied only if the coefficients in (1.2) depend exclusively on x . Furthermore, one can talk of **homogeneous** operators if they are constant and **inhomogeneous** otherwise. One way of understanding this last case is by thinking that the domain/space where the process takes place is not an Euclidean \mathbb{R}^n but a manifold/surface [21].

However, most of the techniques used in image processing depend upon operators whose nature is:

- (b) **Non-Linear.** In many cases the mechanisms of the deformation/evolution (1.1) depend on the values of the function u and its derivatives. We may interpret that the deformation process is not always taking place in the same manifold/space but on a space that also changes/transforms in time. Anisotropic diffusions and curvature flows fit into this category.

Any of the above operators describes 2 main phenomena depending on the boundary conditions used in equation (1.1):

PROCESSES MODELLED

1. **Extension Processes.** Dirichlet boundary conditions:

$$u(x, t)|_{\partial\Omega} = f(x)$$

mean that the solution $u(x, t)$ converges to a function (the equilibrium point of (1.1) given by $Lu \equiv 0$) that extends the boundary function f .

Periodic boundary conditions can also be included, as, in this case, the function u represents a surface/curve and equation (1.1) its deformation. In some sense, we may regard such deformation as a propagation of the original shape in the space \mathbb{R}^n .

2. **Diffusion Processes.** Neumann boundary conditions are given in terms of the scalar product of the function gradient, ∇u , and the unit normal, \vec{n} , to the boundary domain:

$$\langle \nabla u(x, t), \vec{n} \rangle|_{\partial\Omega} = 0$$

Since the above condition means that the flow of mass/heat across the boundary $\partial\Omega$ is zero, it implies that the values of the original function u_0 are re-distributed (diffused) in Ω .

We will give a more precise description of the former processes in the next Sections, which restrict to PDE's given by second order operators:

$$Lu = \underbrace{\sum a_{ij}u_{x_i x_j}}_{\text{Second Order}} + \underbrace{\sum b_i u_{x_i}}_{\text{First Order}} + \underbrace{cu}_{\text{Zero Order}} \quad (1.2)$$

A second order operator is called **elliptic** if for any vector $\xi = (\xi_1, \dots, \xi_n)$ the quadratic form given by $\sum a_{ij}\xi_i\xi_j$ is positive-definite. Under this condition it is guaranteed ([23]) that a unique solution to (1.1) exists. Besides, evolution equations governed by such operators enjoy from convenient properties for image processing purposes:

PROPERTIES OF ELLIPTIC OPERATORS

1. **Smoothness.** Solutions to (1.1) are infinitely smooth functions, so that the equation is a good noise removing tool. We remit the reader to Example 1.1.1 for an illustrative case where solutions fail to be smooth.
2. **Maximum Principle.** The extrema values are attained at time zero, which ensures simplification of shapes and images.

In the particular framework of Image Processing, we have split the general PDE (1.1) into heat-like equations and those ones that minimize an energy functional.

1.1 Heat Equations

In this context a heat equation will be any second order PDE (1.2) with its second order term given in divergence form:

$$u_t = \underbrace{\operatorname{div}(J\nabla u)}_{\text{Diffusion}} + \underbrace{F_1(u_x, u_y)}_{\text{Transport}} + \underbrace{F_0(u)}_{\text{Reaction}} \quad (1.3)$$

where J is a symmetric semi positive definite tensor (i.e. for each point, (x, y) , $J = J(x, y)$ is a 2×2 matrix), F_1, F_0 are the first and zero order terms and the divergence of a vector $j = (j_1, \dots, j_n)$ is given by $\text{div}(j) = \partial_{x_1}(j_1) + \dots + \partial_{x_n}(j_n)$.

In this case we can provide with a more accurate description of the mechanisms that govern the evolution of the initial function, as each of the summands in (1.3) has a specific role:

1. Diffusion Term

The second order term, $\text{div}(J\nabla u)$, is the main descriptor as it determines the way information/mass distributes (diffuses). For a better understanding, let us consider our function u to be a mass distribution and let us represent it as mass particles which density/concentration depends on the values of u , the higher they are the more mass density. Under this model, we have that the vector field $j = J\nabla u$ locally describes the direction towards the initial mass particles move, as the scheme drawn in fig.1.1(a) represents. The amount of mass that a given enclosed area, Ω , loses or gains only depends on how much mass escapes through its boundary. If j was tangent to the area boundary, $\partial\Omega$, the effect of the diffusion would be redistributing the mass particles along it (fig.1.1 (b)). This intuitive reasoning is mathematically formulated by means of the generalized version of integration by parts given by the divergence formula:

Divergence Formula

$$\int_{\Omega} v \text{div}(j) = - \int_{\Omega} \langle \nabla v, j \rangle + \int_{\partial\Omega} v \langle j, \vec{n} \rangle \quad (1.4)$$

for \vec{n} the outward normal vector to $\partial\Omega$. If we apply the latter equality to $v = 1$ we obtain that, in the absence of other terms, the evolution (1.3) fulfils:

$$\int_{\Omega} u_t = \int_{\Omega} \text{div}(j) = \int_{\partial\Omega} \langle j, \vec{n} \rangle$$

It follows that only the normal component of the flow affects the total amount of mass enclosed by the area Ω . The fact that tangent components do not change the shape of the boundary of the area enclosing mass will be especially relevant in the analysis of geometric flows.

Apart from the above physical interpretation, the diffusion term has a geometric meaning that will be useful at some parts of the foregoing discussion. The mass flow, \vec{j} , is univocally determined by the diffusion tensor, J . The scalar product that it defines corresponds to the metric of the (Riemmanian) surface where the diffusion process takes place. Geometrically, a metric is described by means of an ellipse with principal axes of lengths equal to the eigenvalues of J (λ_1, λ_2) oriented by its corresponding eigenvectors ξ and η . The ellipse associated to this metric locally describes the way mass distributes in the plane: an amount λ_1 of mass travels along ξ and an amount λ_2 , in the direction η . In the context of metrics, diffusion processes are classified into **isotropic**, when the eigenvalues of J are equal and **anisotropic**, in the case of different eigenvalues.

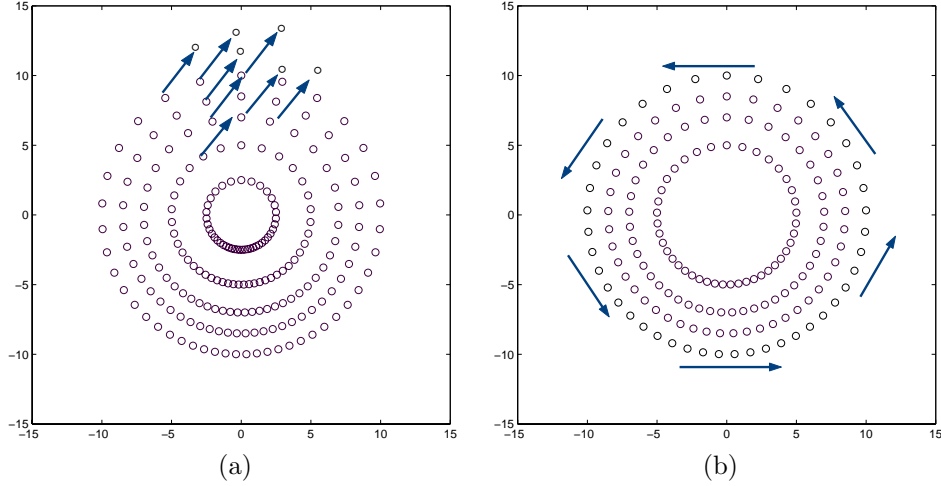


Figure 1.1: Mass diffusion across (a) and along (b) boundary domain

2. Transport Term

The transport first order term:

$$F_1(u_x, u_y) = A(x, y, u)u_x + B(x, y, u)u_y$$

dues its name to the way solutions to a pure first order PDE:

$$u_t = A(x, y, u)u_x + B(x, y, u)u_y$$

are computed. The classic way of solving the above equation is by finding the level curves (characteristic curves) of the solution, $u(t, x, y)$, in the space-time domain. Since the values of the initial function $u_0(x, y)$ travel along such curves, the whole process may be interpreted as moving/transporting the initial quantities. This way of propagating the initial function may lead to ambiguous (i.e. discontinuous solutions) situations each time two characteristic lines of different levels cross, since at such points the function takes more than one value. This pathology arises in the non-linear case:

Example 1.1.1 Example of a discontinuous solution

Consider the transport equation given by:

$$u_t = uu_x \quad \text{with} \quad u(x, 0) = x^2$$

The level curves, $\gamma(s) = (x(s), t(s))$, of the solution satisfy:

$$0 = \frac{d}{ds}(u(\gamma(s))) = u_x x_s + u_t t_s = u_x x_s + u t_s u_x = (x_s + t_s x_0^2)u_x$$

It follows that:

$$\left. \begin{aligned} x_s &= -x_0^2 \\ t_s &= 1 \end{aligned} \right\}$$

and that the solution is constant and equal to x_0^2 along the straight lines $x = -x_0^2 t + x_0$. Since non-parallel lines always cross, the potential solution, $u(x, t)$, is discontinuous for all times. This does not contradict the properties given at the beginning of the chapter as the equation lacks of the second order term. The effect of adding such a diffusion term is that the solution is not constant along characteristic lines, but follows a differential equation. In other words the level curves are not straight lines as diffusion bends them. The drawing in fig.1.2 (b) represents the jump discontinuity of the solution to the transport equation at a point where two characteristic curves (red lines in the x-t plane) meet. The regularizing effect of a diffusion term is illustrated in the surface mesh of fig.1.2 (a) representing the solution to $u_t = uu_x + u_{xx}$. Note that its level curves (lines on the surface and on the x-t plane) are not straight lines.

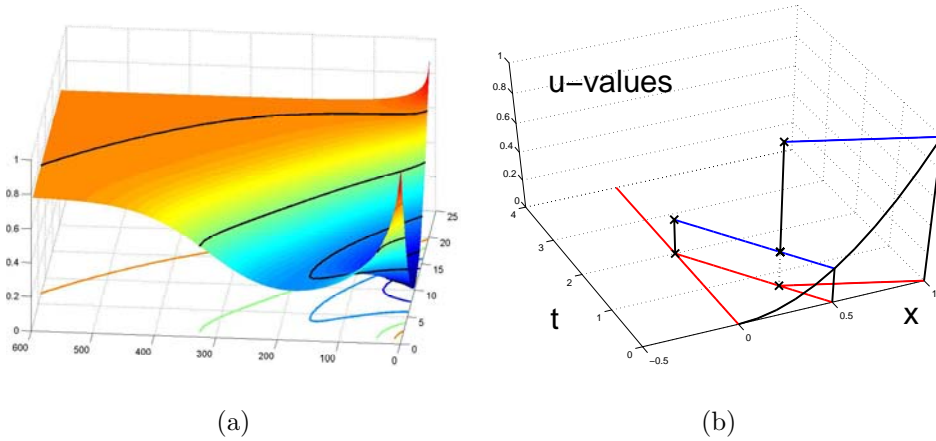


Figure 1.2: Regularizing effect of diffusion (a) in a transport equation (b)

3. Reaction Term

One of its usual roles is to include Dirichlet boundary conditions into the model, instead of imposing them at each step of the evolution. The name Reaction Term comes from the early days of diffusion theory, when as such processes were approach from a point of view of Newton mechanics, Dirichlet conditions were interpreted in terms of a reaction (response/force) of the boundary domain to diffusion.

1.1.1 Image Diffusions

Diffusion is the natural physical way of distributing information. The dynamic process of the evolution of an initial heat distribution, u_0 , in time is governed by a heat

equation in pure divergence form:

$$u_t(x, y, t) = \operatorname{div}(J\nabla u) \quad \text{with} \quad u(x, y, 0) = u_0(x, y) \quad (1.5)$$

with Neumann conditions in the boundary domain. The solution represents the heat or mass distribution in the plane at each time t . Concerning the final heat distribution, steady states of (1.5) can be described by means of their level sets. If we denote by Ω the region enclosed by a level curve γ then, the Divergence formula yields that final states must satisfy the following integral equation:

$$0 = \int_{\Omega} u_t = \int_{\Omega} \operatorname{div}(J\nabla u) = \int_{\gamma} \frac{\nabla u^t \cdot J\nabla u}{|\nabla u|} \quad (1.6)$$

Therefore the final heat distribution will be constant unless the metric given by J degenerates (i.e. cancels) on some closed curves. In this case, the final heat distribution will consist of closed regions of uniformly distributed heat separated by these curves. In thermodynamic terms we may think that these curves behave like insulators.

This property is commonly used in image processing. The original image is set to be the initial heat distribution and the metric is chosen in such a way that it degenerates on points that satisfy some conditions. In this manner the final state that we achieve is an image so that features of interest in the original image are easier to identify. Some filtering techniques conforming to equation (1.5) are the following:

Example 1.1.2 *Gaussian Filtering*

The heat equation in Euclidean space:

$$u_t = \operatorname{div}(\nabla u) = u_{xx} + u_{yy} = \Delta u$$

is the classic low-pass filter image smoothing. It is the only technique with an explicit solution given by convolution of the initial function with a Gaussian kernel G_t :

$$u(x, y, t) = G_t * u_0 = \frac{1}{2\pi t} \int e^{-(\tilde{x}^2 + \tilde{y}^2)/(2t)} u_0(\tilde{x} - x, \tilde{y} - y) d\tilde{x} d\tilde{y}$$

Its indiscriminate smoothing (blurring) of edges constitutes one of its main drawbacks.

Example 1.1.3 *Edge Enhancement Diffusion of Perona-Malik*

This method, introduced by Perona and Malik [60], was designed to avoid edge blurring and profit the smoothing effects of the Gaussian filtering at image areas where contrast changes are not significant enough. It is defined as:

$$u_t = \operatorname{div}(g(|\nabla u|)\nabla u) \quad (1.7)$$

with the edge enhancing function defined as $g(s) = \frac{\lambda^2}{\lambda^2 + s^2}$, for a constant λ .

In physical terms, one can interpret the function g as a stopping diffusion factor that prevents diffusion across edges, as by (1.6), steady states are characterized by:

$$\int_{\{u \geq \alpha\}} u_t = \int_{\{u \equiv \alpha\}} g(|\nabla u|)|\nabla u| ds = \int_{\{u \equiv \alpha\}} \frac{\lambda^2 |\nabla u|}{\lambda^2 + |\nabla u|^2} ds$$

It follows that there is not any diffusion across those curves where $|\nabla u| \rightarrow \infty$, that is, at ideal edges. For a different interpretation in image processing, let us develop the divergence term and write equation (1.7) in the form:

$$u_t = gu_{\xi\xi} + (g + g'|\nabla u|)u_{\eta\eta}$$

where $u_{\xi\xi}$ and $u_{\eta\eta}$ denote, respectively, the second derivatives in the tangent, ($\xi = \frac{\nabla u^\perp}{|\nabla u|}$), and normal, ($\eta = \frac{\nabla u}{|\nabla u|}$), directions. At those points, where $g + g'|\nabla u| < 0$ not only edges are preserved but even enhanced. This gives an interpretation to the parameter λ as a threshold value for $|\nabla u|$ since for $|\nabla u| < \lambda$, the edge is blurred and enhanced, otherwise.

However, the above properties constitute the main drawbacks of the technique. On one hand, as edges in real images are prone to be incomplete, some diffusion occurs at these places, producing the so known Pinhole Effect [53]. On the other one, backwards diffusion at significant edges makes the method potentially ill-posed [74],[81], [56] and prone to yield step images (Staircasing Effects, [80], [81]).

Example 1.1.4 *Anisotropic Diffusion of Weickert*

One solution to the problem consists in using a regularized version of the gradient in the formulation of the Perona-Malik method:

$$u_t = \operatorname{div}(J(St_\rho(\nabla u_\sigma))\nabla u) \quad (1.8)$$

where the diffusion tensor $J(St_\rho(\nabla u_\sigma))$ has the same eigenvectors as the Structure Tensor [36], St_ρ , with eigenvalues depending on the structures to be enhanced. In [72], [74] two different models are proposed to enhance edges and linear structures. Let λ_1 , λ_2 denote the biggest and the smallest eigenvalues of J_ρ and v_1 , v_2 the corresponding eigenvectors, then the formulations are:

1. Edge Enhancement

For edge enhancing the eigenvalues of the diffusion tensor are set to:

$$\mu_1 := g(\lambda_1), \quad \mu_2 := 1$$

with stopping diffusion function, g , defined as follows:

$$g(s) := \begin{cases} 1 & (s \leq 0) \\ 1 - \exp\left(\frac{-C}{(s/\lambda)^4}\right) & (s \geq 0) \end{cases}$$

where C and λ are constants. Like the case of the Perona Malik model, the magnitude of λ determines which edges are to be smoothed. Weickert determines [72] the optimal values for C and λ to be: $C = 3.31488$ and $\lambda = 3$. The Structure Tensor is over a regularized version of the gradient $\nabla u_\sigma := G_\sigma * \nabla u$, with $\sigma = 3$ and integration scale $\rho = 0$.

2. Line Enhancement

The variant for enhancement of linear structures profits that the eigenvector v_1 is oriented in the direction of maximum contrast change and that the difference

of the eigenvalues $(\lambda_1 - \lambda_2)^2$ (coherence) measures the reliability of the potential edge, being minimum in the isotropic case. Taking into account all these facts a diffusion tensor with eigenvalues:

$$\mu_1 := \alpha, \quad \mu_2 := \alpha + (1 - \alpha) * \exp \frac{-C}{(\lambda_1 - \lambda_2)^2};$$

will preferably diffuse along linear structures, so that their gray level uniformizes.

General arguments on non-linear parabolic PDE's [7], [72] yield that, as the diffusivity tensor is positive defined, the initial boundary problem has a unique smooth solution which continuously depends, in L^2 , on the initial data [74]. Besides, the existence of Lyapunov functionals provided a description of the asymptotic behavior as a constant function equal to the integral of the initial image. This is, indeed, the main drawback of the method, since it implies the need of a stopping time if one is to use it either to segment images or for shape recognition.

Still the generic good properties of the model (1.8) have inspired other methods [8], [9] aimed at enhancing other image structures:

Example 1.1.5 Adaptive Smoothing

In [9], the authors propose a variant formulation of divergence-like equations better suited for selective smoothing of image structures. If $\eta = \frac{\nabla u}{|\nabla u|}$ denotes the unit gradient and $\xi = \frac{\nabla u^\perp}{|\nabla u|}$ the tangent to the image level curves, then the general evolution equation:

$$u_t = c(|\nabla u|)(au_{\eta\eta} + bu_{\xi\xi}) \quad \text{with} \quad a, b \in [0, 1] \quad (1.9)$$

comprises the Perona-Malik and the anisotropic models and allows smoothing along other preferred directions different from edges. In particular, they suggest using directional derivatives in the principal directions of the function Hessian:

$$\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}$$

in order to enhance ridges and valleys.

1.1.2 Information Extension

Heat diffusion has another mathematical and physical use hardly exploited in image analysis. Heat diffusion (second order elliptic operators, in general) has the property of smoothly extending a function defined on a curve in the plane. If L denotes an elliptic operator, then the function that solves the PDE:

$$Lu = 0 \quad \text{with} \quad u|_\gamma = f \quad (1.10)$$

is the unique smooth extension [23] of the function f which was defined only on the curve γ . The equation is solved by seeking the steady states of the associated parabolic PDE:

$$\begin{aligned} u_t &= Lu \\ u|_\gamma &= f \end{aligned} \quad (1.11)$$

From the point of view of thermodynamics, we may think that the heat distribution on γ given by f never puts out. Boundary conditions can either be imposed at each iteration or added to the model as a reaction term by means of the characteristic function, χ_γ :

$$u_t = Lu + \chi_\gamma(u - f)$$

where:

$$\chi_{|\gamma} = \begin{cases} 1 & \text{if } x \in \gamma \\ 0 & \text{otherwise} \end{cases}$$

We will restrict to elliptic operators admitting a divergence form given in equation (1.5). The ellipse describing the metric given by J corresponds to the structural element of the associated dilation, so that isotropy leads to circular structural element and anisotropy to elliptic ones. In the case of classic mathematical morphology, the extension is based on the Laplacian operator and the function to be extended is the characteristic function of a set of points. The scale or radius of the dilation corresponds to time in equation (1.11). Some of the image processing fields and techniques handling functional extension are:

Example 1.1.6 *Image In-Painting and Contour Completion*

Image gap filling and curve completion can be viewed as particular cases of a functional extension process. In the case of multiple level sets completion (image gap filling shown in fig.1.3 (a),(c)), γ corresponds to the gap boundary and f to the image to be restored. For contour completion (fig.1.3(b)), γ is the unconnected curve to be closed and f its characteristic function. As example we have the image in-painting developed in [4] that uses high order extension to compute a vector field interpolating the image gradient at gap boundaries. In Chapter 4 we will also address contour closing and gap filling.

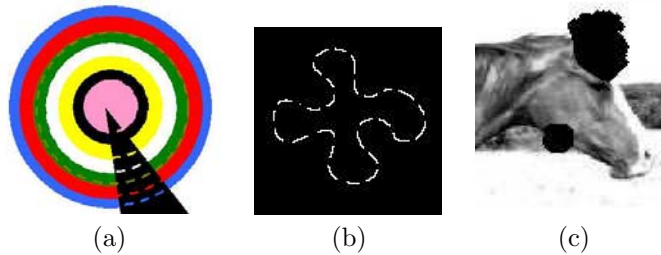


Figure 1.3: Examples of Extension Processes: level lines continuation or gap filling (a), single contour completion (b) and image in-painting (c)

Example 1.1.7 *Gradient Vector Flow*

These techniques [75], [77] serve to smoothly extend a vector field, (f_x, f_y) , defined on a curve of interest. The vector extensions, (u, v) are the steady-points of the

following diffusion-reaction equations:

$$\begin{aligned} u_t &= \Delta u - (f_x^2 + f_y^2)(u - f_x) \\ v_t &= \Delta v - (f_x^2 + f_y^2)(v - f_y) \end{aligned}$$

As GVF arises from energy minimization principles, we will describe it in detail in next Section.

1.1.3 Geometric Flows

Geometric flows encode shape evolutions, that is, those curve deformations that are independent of the parameterization. If $\gamma(v, 0) := \gamma_0(v) = (x(v), y(v))$ is the initial curve, then its evolution in time is given by:

$$\gamma_t(v, t) = \alpha(v, t) \vec{t}(v, t) + \beta(v, t) \vec{n}(v, t)$$

with \vec{t} , \vec{n} the curve tangent and normal vectors, respectively, and α , β smooth functions depending on the local geometry of the curve. This last requirement limits both functions to depend on the curve unit tangent, θ , and the curve curvature, κ .

Since a change of parameter in a curve does not alter its shape but only the speed we are travelling on the curve, we have infinitely many formulations representing the same curve evolution (although the function of the embedding, γ , may change). In fact, since tangent motion just redistributes (mass) points along the curve, we have that [29], up to a change of parameter, any geometric flow is given by:

$$\gamma_t(u, t) = \beta \vec{n}$$

Thanks to their independence upon the curve parameter, geometric flows can be applied to image filtering through an implicit level-set formulation [48], [65]:

$$u_t = |\nabla u| \beta(\theta, \kappa) = |\nabla u| \beta \left(\frac{\nabla u}{|\nabla u|}, \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) \right)$$

Although in their implicit formulation they are highly non-linear equations, geometric flows adjust to the heat-like model in a proper curve parameter. In [24] it is proved that, in the arc-length parameter, the angle of the curve unit tangent, θ , and the curve curvature, κ , follow PDE's:

$$\begin{aligned} \theta_t(s, t) &= \partial_s(\beta) + \left(\int_0^s \beta \theta_s ds \right) \theta_s \\ \kappa_t(s, t) &= \beta \kappa^2 + \beta_{ss} + \left(\int_0^s \beta \kappa ds \right) \kappa_s \end{aligned}$$

which are of heat diffusion type as $\beta = \beta(\theta, \kappa)$.

In order to guarantee that the original shape γ_0 will be smoothed, the total Gaussian curvature, $\bar{\kappa}(t) = \int_0^1 |\kappa| \sqrt{\dot{x}^2 + \dot{y}^2} du$, and the number of inflexion points must decrease in time. A main advantage of curvature based image filtering techniques over diffusion processes [48], [39] is that, as they are contrast preserving [1], they modify the shapes of edges without blurring. The most popular curvature based techniques are:

Example 1.1.8 *Mean Curvature Flow*

This flow, also known as Geometric Heat Equation, is one of the most widely studied geometric flows ([32], [33], [34] or [40]). Its formulation for curves is given by:

$$\gamma_t(u, t) = \kappa \vec{n} \quad (1.12)$$

The word Geometric Heat Equation applies because in arc length parameter, we have that the diffusion term of the PDE that the angle follows is the Laplacian:

$$\theta_t(s, t) = \partial_s(\theta_s) + \left(\int_0^s \theta_s^2 ds \right) \theta_s = \theta_{ss} + \left(\int_0^s \theta_s^2 ds \right) \theta_s$$

General arguments on non-linear PDE's [32] yield that, as far as the curvature (θ_s) remains bounded, a unique solution fulfilling the necessary conditions to smooth irregularities exists. Besides, we have a complete description of the asymptotic behavior of the mean curvature flow:

Proposition 1.1.1 [32] [33] *Let $\gamma(t, u)$ be the solution to the geometric heat equation:*

$$\gamma_t = \kappa \vec{n}$$

with initial condition a closed simple smooth curve. Then, $\gamma(t, u)$ becomes a circle and finally collapses to a point.

This Proposition supplies images and curves evolving under the curvature flow with a desirable property for a scale-space analysis [1], [41], which studies images/shapes at different level of detail by means of a progressive simplification of their shapes and features. However, as in the case of diffusion processes, this property represents a main drawback if one is to use such equation just to smooth irregularities, as it requires a stopping time in order to preserve enough features as to recognize the original shapes in the image. Up to this moment, techniques developed to cope with trivial final curves focus on converging to a particular geometric model of shapes:

Example 1.1.9 *Stochastic Geometric Flows*

Basing on the stochastic process that models evolution of a pixel, this geometric flow [79] adds a factor to the mean curvature flow that cancels on polygons of N-vertices:

$$\cos^2(N\theta)\kappa \vec{n} \quad \text{or} \quad \sin^2(N\theta)\kappa \vec{n}$$

The resulting image filtering drives all level sets to a polygon given a priori, which limits the technique applicability to images presenting different geometric designs.

Although, strictly speaking, it can not be considered a geometric flow (it depends on the embedding image), the next curvature based evolution achieves more accurate shape models:

Example 1.1.10 *Min/Max Flow*

In [49] Malladi and Sethian introduced the following curvature-based image evolution:

$$u_t = \begin{cases} \min(\kappa, 0) & \text{if } \Phi(x, y) < v \\ \max(\kappa, 0) & \text{if } \Phi(x, y) \geq v \end{cases}$$

where $\Phi(x, y)$ is the average of all pixel values in a window centered at (x, y) and v is the mean value in the window on the level curve to be smoothed. In the case of real images, the authors suggest using the image gradient to switch between evolution under positive and negative curvature:

$$u_t = \begin{cases} \min(\kappa, 0) & \text{if } \Phi(|\nabla u|) < v \\ \max(\kappa, 0) & \text{if } \Phi(|\nabla u|) \geq v \end{cases}$$

for $\Phi(|\nabla u|)$ an average of the norm $|\nabla u|$ and v a threshold value determining whether the curve is considered an edge or not.

In any case, the steady state of the equation is a polygonal curve that approaches the original shape.

Selective shape smoothing based on the curve regularity rather than on its geometry will be addressed in Chapter 2

1.2 Minimization Processes

Most of the former techniques minimize an energy functional and its formulation derives from the gradient descent of such energy. In fact, energy minimization flows are the main sources of PDE's that do not conform to the model of a heat-like equation.

Let us review the variational principles that rule functional minimization. For any function/curve, v , a functional, E , is defined as the integral:

$$E(v) = \int_0^1 L(v(x), \nabla u(x), x) dx = \int L(z, p, x) dx$$

where L is commonly known as *Lagrangian*.

As in the case of functions in \mathbb{R}^n , cancellation of E first derivative characterizes its minima/maxima. The functional first derivative or *first variation* is obtained as follows. For any function ν and any real number δ , we will call a perturbation of u any function of the form $\tilde{u} = u + \delta\nu$. It should be clear that the derivative with respect to δ evaluated at $\delta = 0$ yields the functional first variation at a point u provided that L is smooth in all its arguments (z, p, x) . Hence, if we denote:

$$E(\delta) = \int L(u + \delta\nu, \nabla u + \delta\nabla\nu, x)$$

Then the first variation of E at a point u is given by:

$$\begin{aligned} E'(0) &:= \partial_\delta \left(\int L(u(x) + \delta\nu(x), \nabla u(x) + \delta\nabla\nu(x), x) dx \right) = \\ &\int (\langle L_z(u, \nabla u, x), \nu \rangle + \langle L_p(u, \nabla u, x), \nabla\nu \rangle) dx \end{aligned} \quad (1.13)$$

The functional E has a critical point if the above integral cancels for any ν . Assuming the usual boundary conditions (Dirichlet, Neumann or periodic), integration by parts yields:

$$E'(0) = \int \langle L_z - (L_p)_x, \nu \rangle dx = \int \langle L_z - \operatorname{div}(L_p), \nu \rangle dx = 0 \quad (1.14)$$

which implies that extremal curves of E solve:

Euler-Lagrange equations

$$0 = L_z - (L_p)_x \quad (1.15)$$

or equivalently, that they correspond to the stationary points of the dynamical system, which is analogous to the gradient descent [30], given by:

$$u_t(x, t) = -L_z + (L_p)_x \quad (1.16)$$

Unfortunately, the converse does not always hold, that is, not all points cancelling (1.16) are extreme points of the functional. As in the case of functions in \mathbb{R}^n , an analysis of the functional second variation determines the false minima. This second derivative, $E''(0)$, is found by means of differentiating once again the expression (1.13) with respect to δ and evaluating at $\delta = 0$:

$$E''(0) := \int (\nu^t L_{zz}(u, \nabla u, x) \nu + (\nabla \nu)^t L_{pp}(u, \nabla u, x) \nabla \nu + 2\nu^t L_{zp}(u, \nabla u, x) \nabla \nu) du$$

where second derivatives should be understood in the context of symmetric 2-forms. In this framework we will use the notation $E''(\nu_1, \nu_2)$ to denote the former symmetric form applied to the functions ν_1 and ν_2 . If the second variation is (strictly) positive definite, the functional is called (strictly) convex. In this case, we can ensure [23] that any function cancelling (1.13) is a stable stationary point (if convexity is strict) and, further, a minima of E . In fact, convexity of the functional is equivalent to the ellipticity condition for second order operators:

Example 1.2.1 *Convexity and Ellipticity for Minimizing Diffusions*

Let $J(x)$ be an $n \times n$ symmetric matrix and, for each integrable smooth function $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ consider the energy functional given by:

$$E(u) := \int_{\Omega} |\nabla u|_J dx = \int_{\Omega} \nabla u^t J \nabla u dx = \int L(\nabla u, x) dx = \int L(p, x) dx$$

On one hand, the gradient descent of the former functional is:

$$u_t = \operatorname{div}(J \nabla u) = \operatorname{div}(L_p)$$

On the other hand, the ellipticity condition that guarantees uniqueness of a weak solution reads:

$$\int (\nabla v)^t J \nabla v \geq C \int |\nabla v|^2$$

which is exactly the condition that E is uniformly convex:

$$E''(v, v) = \int (\nabla v)^t L_{pp} \nabla v = \int (\nabla v)^t J \nabla v \geq C \int |\nabla v|^2$$

Apart from the Gaussian filtering, image in-painting and mean curvature flow described in the previous Section, there are some other techniques developed from variational principles:

1.2.1 Close-to-Data Constrains

This family of operators arise as a solution to avoid the convergence to trivial steady states that mares efficiency of diffusion processes. The idea is to constrain the solutions to a diffusion equation (1.5) to take equal values than the original function at the function descriptive curves (edges, ridges). Such (Dirichlet-like) restrictions easily are easily modelled in terms of energy minimization and result in adding a reaction term to the diffusion equation.

1. Half-Quadratic minimization

These approaches ([2], [43]) base on the minimization of the functional:

$$E(u) = \int \int \frac{\alpha}{2} (u - u_0)^2 + \Phi(|\nabla u|) dx dy$$

where u_0 is the image to be filtered. The corresponding Euler-Lagrange equations correspond to a gradient descent flow:

$$u_t = \operatorname{div}(g(|\nabla u|^2) \nabla u) + 2\alpha(u_0 - u) \quad \text{for} \quad g = \frac{\Phi'(|\nabla u|)}{|\nabla u|} \quad (1.17)$$

that is a diffusion equation with a reaction term weighted by α . Since this parameter weights the influence of the close-to-data constrain [62], it determines the similarity between the original noisy image, u_0 , and its filtered version. Although it prevents the flow from having a trivial (i.e. constant image) steady state, an inappropriate (high) value for α troubles the denoising process.

2. Gradient Vector Flow

As already mentioned in Section 1.11 these methods ([75], [76], [77]) regularize and extend vector fields. The smoothing diffusion term and the value constrain extension term arise from the minimization of the energy functional:

$$\begin{aligned} E(u(x, y), v(x, y)) = E(V) &= \int |V|^2 + \int |f|^2 |V - f|^2 = \\ &= \int (u_x^2 + u_y^2 + v_x^2 + v_y^2) + \int (f_x^2 + f_y^2) ((u - f_x)^2 + (v - f_y)^2) \end{aligned} \quad (1.18)$$

for $f = (f_x, f_y)$ the vector field to be extended and regularized. Using calculus of variations, it follows that GVF is the steady state of:

$$\begin{aligned} u_t &= \Delta u - (f_x^2 + f_y^2)(u - f_x) \\ v_t &= \Delta v - (f_x^2 + f_y^2)(v - f_y) \end{aligned}$$

The functional version (1.18) provides a simple interpretation of each of the terms. Because each diffusion Δu , Δv corresponds to minimizing the total variation, $\int(u_x^2 + u_y^2)$, $\int(v_x^2 + v_y^2)$ of the functions, it yields regularity to the solution. Meanwhile the reaction terms $(f_x^2 + f_y^2)(u - f_x)$, $(f_x^2 + f_y^2)(v - f_y)$ impose a restriction on GVF values, as at those points where the norm of the original vector field is large, GVF must minimize $\int(f_x^2 + f_y^2)((u - f_x)^2 + (v - f_y)^2)$, which is minimum when (u, v) coincides with the original vector field.

In most cases, the vector field to be extended/regularized is the gradient of an external potential that serves to guide a deformable curve to a given image curve:

1.2.2 Snakes

Snakes constitute another important family of techniques based on calculus of variations principles. Snakes are curves that serve to segment image objects whose boundaries are characterized by a lack of differentiability of the image (edges, ridges). A measure of such irregularity serves as an attractor external energy, E_{ext} , that it is minimum at the target object. The snake is, then, the curve that achieves a compromise between minimizing this energy and conforming to a given degree of regularity. Different ways of imposing the latter internal energy give rise to two main snake models:

1. Parametric Snakes

Parametric deformable models [38], [47] use Newton mechanics laws to define their internal constraints, which are given in terms of elasticity and stretching of the curve. In mathematical terms, the snake is the curve $\gamma(u) = (x(u), y(u))$ that minimizes the energy functional:

$$E(\gamma) = \int_{\gamma} (E_{int} + E_{ext}) du = \int_{\gamma} (\alpha \|\dot{\gamma}\|^2 + \beta \|\ddot{\gamma}\|^2 + E_{ext}) du ,$$

where $\alpha, \beta \in [0, 1]$ determine the trade-off between length and curvature minimization. The usual external potentials, E_{ext} , are either the gradient map, $g = \frac{1}{1+|\nabla u|^2}$, or the distance map to image edges. In the general case, the curves minimizing such potentials are unconnected, so that the snake uses its internal energy to obtain a close model. The α -term (elasticity) linearly interpolates object contours producing a piece wise linear model of the curve. A higher regularity is achieved thanks to the β -curvature minimizing term (rigidity) because it penalizes curves corners and undulations. This makes the snake smoothly approach the unconnected set of points and allows it swallowing isolated set of points (outliers) product of image noise.

A main inconvenience of physics-based snakes is that they handle an explicit curve parameterization (sampling), which difficulties modelling more than one object and requires dynamic updating during the snake evolution.

2. Geodesic Deformable Models

Geodesic snakes [11], [12] are an alternative to physics-based snakes based on the theory of curve evolution and level sets methods [57]. In particular, the solution to the problem is the curve (γ) of minimum length in a Riemannian surface with a metric (g) depending on the image contrast changes. It follows that if we note by u the image to be segmented, the geodesic functional is given by:

$$E_{geod} = \int_{\Gamma} g \, ds \quad \text{with} \quad g = \frac{1}{1 + |\nabla u|^2} \quad (1.19)$$

The normal component of the Euler-Lagrange formulation characterizes geodesic snakes as the steady curve of

$$\frac{\partial \gamma}{\partial t} = (g \cdot \kappa - \langle \nabla g, \vec{n} \rangle) \cdot \vec{n} \quad (1.20)$$

We can give the following interpretation to each of the terms involved in the above formula. The role of $\langle \nabla g, \vec{n} \rangle \cdot \vec{n}$ is pretty clear: it is a vector field pointing to the region of interest that attracts the snake to its boundary. Notice that in the standard formulation (1.19) the scope of this static term reduces to a narrow environ of the boundaries of interest. The curvature term, $g \cdot \kappa \vec{n}$, influences different aspects of the snake evolution. On one hand, it defines its motion when it is located far away from the object boundaries. On the other one, it serves as a curve regularizing term, ensuring continuity of the final segmenting snake in a similar fashion [78] the α -term of parametric snakes does. Finally, it gives the process a smooth behavior and ensures continuity during the deformation, in the sense that it prevents shock formation [23]. However, incorporating the curvature term into the convergence scheme difficulties the snake convergence to concave areas. In order to increase convergence to concavities and to speed up the evolution a constant velocity term in the direction of the normal component is usually added to the evolution equation giving rise to 'balloon'-like snakes [16]. The new term corresponds to area minimization and results in a constant dynamic speed in the gradient descent:

$$\frac{\partial \gamma}{\partial t} = (g \cdot \kappa + gV_0 - \langle \nabla g, \vec{n} \rangle) \cdot \vec{n} \quad (1.21)$$

The behavior of the above techniques will be discussed in deeper detail in Chapters 3 and 5.3 where we will describe a new external energy ensuring convergence of parametric snakes whatever the geometry of the set of points to be approached.

Chapter 2

Regularized Curvature Flow: The Geometric Perona-Malik Approach

A theoretical analysis of recent filtering techniques [24] points out that if an image smoothing operator is to be robust against strong noisy images, it should be independent of image intensity. Essential advantage in this context is represented by geometric flows [32], [49]. As image smoothing relies exclusively on the geometry of its level curves, geometric flows do not produce any edge blurring [1], [39], which makes these techniques more robust in very noisy images. However, the fact that they reduce high curvature values and the number of inflexion points whatever the regularity of the curve, limits their applicability to shape recovery. We argue that any filtering technique to be used within a shape recovery algorithm, should take into account differentiability of the curve. Smoothing should only be applied to those arc of the curve presenting a given degree of irregularity. This need of a selective shape filtering leads to the formulation of a function measuring lack of smoothness in a shape.

We propose a novel geometric flow that penalizes irregularity in the curvature rather than its magnitude. To this purpose we develop a simple criterion to measure the degree of local irregularity present in the curve, which is added as stopping factor in the mean curvature flow. The resulting evolution equation profits the smoothing effects of the mean curvature flow to smooth those curve arcs that have a significant irregular shape. We will call this geometric flow as Regularized Curvature Flow, denoted by RCF for short.

The first relevant property of RCF is that the evolution converges to a smooth curve that keeps high resemblance to the original noisy shape. This fact is one of RCF main advantages over other geometric flows yielding non-trivial curves. First because RCF final state is smoother than the polygonal shapes that [49], [79] produce. Second, because RCF formulation is simpler and faster than the 4th order [46] or level-sets decomposition techniques [11], [52]. Another significant improvement is that RCF parameters are independent of the geometry of the initial curve our method is applied to. In this fashion, the image operator, obtained through the level sets formulation [57], is suitable for shape recovery in non-user intervention applications. Furthermore,

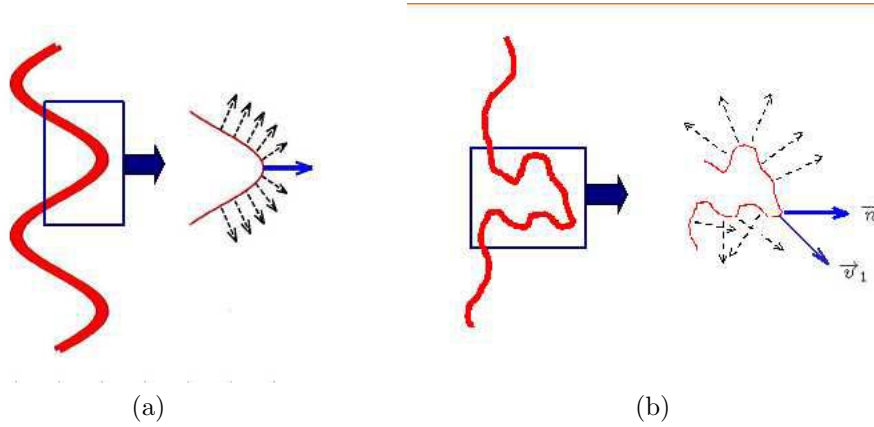


Figure 2.1: Measure of shape irregularity in a regular, (a), and irregular, (b), arc .

RCF smooth evolution makes the iterative scheme used in its implementation to stop by means of standard numeric stop criteria [18], [70], [30]. This also constitutes an advantage over most PDE based techniques that either present an irregular evolution speed [49], [79] or rapidly converge to constant images [60], [72], [32]. Consequently, in practice, they must rely on a given image-dependant number of iterations to stop the evolution, meanwhile RCF admits equal stop parameters for any image. We take special care in studying the error made in RCF numeric implementation and the way this error affects the values of the parameters that rule RCF stabilization. Indeed we propose a new way of setting parametric values in terms of the maximum accuracy in the implementation.

2.1 A Selective Curvature Flow

Let us begin with introducing a roughness function that measures the degree of noise of a plane curve.

2.1.1 Definition of a Local Measure of Shape Irregularity

A plane curve is determined (up to rotations and translations) by the angular orientation, θ , of its unit tangent. Therefore, a reasonable approach for a stopping motion term is to consider a measure of θ smoothness. It should be clear that lack of differentiability in the angle θ corresponds to the variability of the normal unit vector \vec{n} around each point. We argue that this rate of variability can be computed by means of the projection of \vec{n} onto a robust mean of \vec{n} in a neighborhood of each point. That is, it suffices to compare \vec{n} to a smooth approximation, \vec{v}_1 , of the vector. At regular/differential arcs, the former vectors should agree, meanwhile, in the presence of noise/irregularity, they will significantly differ (as the irregular shape of figure 2.1 illustrates).

We will make use of the structure tensor, J_ρ [36], computed over the unit normal in order to obtain the vector \vec{v}_1 . We recall the reader that, given a regularization scale ρ , the structure tensor is defined as the convolution of the projection matrices onto $\vec{n} = (-\sin(\theta), \cos(\theta))$ with a gaussian G_ρ , of variance ρ and zero mean:

$$\begin{aligned} J_\rho &= G_\rho * \left[\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} (-\sin(\theta), \cos(\theta)) \right] = \\ &= \begin{pmatrix} G_\rho * \cos^2(\theta) & G_\rho * \cos(\theta) \sin(\theta) \\ G_\rho * \cos(\theta) \sin(\theta) & G_\rho * \sin^2(\theta) \end{pmatrix} = \begin{pmatrix} a_{11}(u) & a_{12}(u) \\ a_{12}(u) & a_{22}(u) \end{pmatrix} \end{aligned}$$

We assert that the eigenvector of J_ρ of maximum eigenvalue suits our purposes. We define $v_1 = (\cos \psi, \sin \psi)$, in terms of the coefficients of J_ρ as:

$$A := \tan 2\psi = \frac{2a_{12}}{a_{11} - a_{22}} \quad (2.1)$$

The square norm of the vector product $\vec{v}_1 \times \vec{n}$ is the measure of curve irregularity we propose:

$$g(\theta) = \|\vec{v}_1 \times \vec{n}\|^2 = \sin^2(\psi - \theta) \quad (2.2)$$

We will refer to the function g as **Roughness Measure**.

Remark: First notice that the discrete numeric implementation of a convolution with a gaussian is approximated by means of a windowed (squared) kernel or mask. This implies that discrete gaussians are functions of compact support. Second, even in the continuous domain, gaussian kernels are approximations of the identity for the convolution product. That is they converge to the dirac delta as the variance tends to zero. For this (among others) reason they are commonly used in the area of Fourier analysis as a tool to study the "local" behavior of functions [?], [?]. Based on the above considerations, we will refer to the Roughness Measure as local measure of irregularity, in spite of being just an approximation to the local behavior of a function in the continuous domain.

Let us intuitively explain why the function g conforms to the idea of a curve irregularity measure. The statement follows easily if one realizes that the function A is, in fact, equal to:

$$A = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{2G_\rho * \cos(\theta) \sin(\theta)}{G_\rho * \cos^2(\theta) - G_\rho * \sin^2(\theta)} = \frac{G_\rho * \sin(2\theta)}{G_\rho * \cos(2\theta)}$$

That is, the factors of the quotient are the solutions to the heat equation at time $t = \rho$ with initial conditions $\sin(2\theta)$ and $\cos(2\theta)$. The function g compares, in a particular way, these quantities to the original functions $\sin(2\theta)$ and $\cos(2\theta)$. Therefore it is rather sensible to expect that those functions that cancel g will be smooth. For a rigorous mathematical justification we remit the reader to [24].

Properties of the Roughness Measure

These are the properties that make g suitable to our purposes:

1. Locally, it measures the degree of symmetry around each point, in particular it cancels on arcs of circles. This property makes our RCF stabilize finite unions of arcs of circles and straight lines.
2. Those curves that cancel g are \mathcal{C}^∞ curves, i.e. they are infinitely smooth. This is convenient for a latter image processing procedure, since it ensures that higher order operators applied to the image level curves will be accurate.

The above considerations turn our roughness measure into the perfect candidate for a stopping evolution term in a mean curvature flow motion.

2.1.2 Formulation of the Regularized Curvature Flow

The **Regularized Geometric Heat Equation** we suggest is the geometric flow defined as:

$$\gamma_t(u, t) = g(\theta)\kappa\vec{n} \quad \text{with } \gamma(u, 0) = \gamma_0(u) \quad (2.3)$$

where the function $g(\theta)$ is given by formula (2.2) and κ denotes the curvature. The numeric implementation through level sets formulation [65] is detailed in Section 2.2.1.

Properties of RCF

Let us enumerate those properties of RCF that ensure shape simplification and convergence to non trivial steady curves.

1. No new inflexion points (i.e. zeros of the curvature) are created.
2. The total Gaussian curvature, $\bar{\kappa}(t) = \int_0^1 |\kappa| \sqrt{\dot{x}^2 + \dot{y}^2} du$, decreases over the orbits of (2.3).

These two properties guarantee that the evolution under RCF will simplify shapes. Their geometric interpretation is that the curve oscillation, i.e. its total variation, reduces in time. This fact ensures that, given an initial curve γ_0 , its evolution will neither become more irregular (property 1), nor stretch (property 2): its only possible evolution is a progressive reduction of oscillations. This does not constitute a great novelty, since RCF shares this behavior with the other geometric flows [40], [32], [49], [79]. The differential feature that makes RCF more suitable for shape reconstruction purposes is its different asymptotic behavior, which is described by means of the following properties:

3. Whatever initial shape, its evolution under RCF converges, in time, to the steady states of (2.3) given by $g\kappa \equiv 0$.

First, notice that the above statement is not a triviality since the limit set of an orbit could, perfectly, be a periodic orbit. The existence of Lyapunov functionals for RCF (i.e. functions that decrease along the orbits) excludes this possibility. Second, observe that the former property guarantees that the numeric iterative implementation admits a stop criterion in terms of the magnitude of the speed $g\kappa \equiv 0$.

This final state will never be a single round point like in the evolution by mean curvature flow [32], [33]. This follows from the fact that the roughness measure cancels on circles, which prevents the evolution from collapsing to a point.

4. Steady states of (2.3) are simple closed curves.

Finally, we have that the amount of shape irregularity, according to our roughness measure, of the final state of (2.3) is smaller than the one of the initial curve. In fact, it can be shown (Section 2.3) that:

5. The roughness measure g tends to zero over the solutions to (2.3).

Therefore, by virtue of the second property of g , solutions to RCF tend asymptotically to a C^∞ curve, which is the fixed point of equation (2.3) given by $g \equiv 0$. Indeed, we have empirically checked that evolution under RCF converges to a smooth curve that conserves features significant enough to identify the original noisy shape. This already constitutes an advantage over other PDE-based techniques. Firstly, it ensures a higher order smoothness of final curves than the polygonal shapes yielded by [49], [79]. Secondly, the value of RCF unique parameter (ρ) is set according to the desired degree of differentiability. Differentiability is a property which is not linked to either resemblance to the original image/shape [46], [9], [62], geometry of its level sets [79] or level of detail [11], [52]. It follows that with a fixed set of parametric values RCF converges, by its own design, to smooth curves resembling the original ones. Finally, property 5 provides the technique with a natural stop criterion in terms of the magnitude of the roughness measure.

2.2 Numerical Issues

2.2.1 Level Sets Approximation

The level sets implicit formulation of RCF is given by:

$$u_t = g \left(\frac{\nabla u}{|\nabla u|} \right) |\nabla u| \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) = g \left(\frac{\nabla u}{|\nabla u|} \right) \frac{u_{xx}u_y^2 - 2u_{xy}u_xu_y + u_{yy}u_x^2}{|\nabla u|^2}$$

The exact implementation would imply tracking, for each image pixel, its level curve in order to perform the convolution with a one dimensional gaussian kernel along the level-line. Since this is computationally unfeasible we propose an approximate algorithm, which consists in computing the roughness measure using a gaussian in two variables, $G_\rho(x, y) = \frac{1}{2\pi\rho^2} e^{-(x^2/2\rho^2)-(y^2/2\rho^2)}$. That is, the structure tensor is computed by means of the formula:

$$\tilde{J}_\rho(x, y) = \int G_\rho(\tilde{x} - x, \tilde{y} - y) \cdot \left[\left(\frac{\nabla u}{|\nabla u|} \right) \otimes \left(\frac{\nabla u}{|\nabla u|} \right) \right] d\tilde{x}d\tilde{y}$$

Notice that in the discrete implementation the above integral is computed over a window centered at each image pixel of size $2\epsilon \times 2\epsilon$.

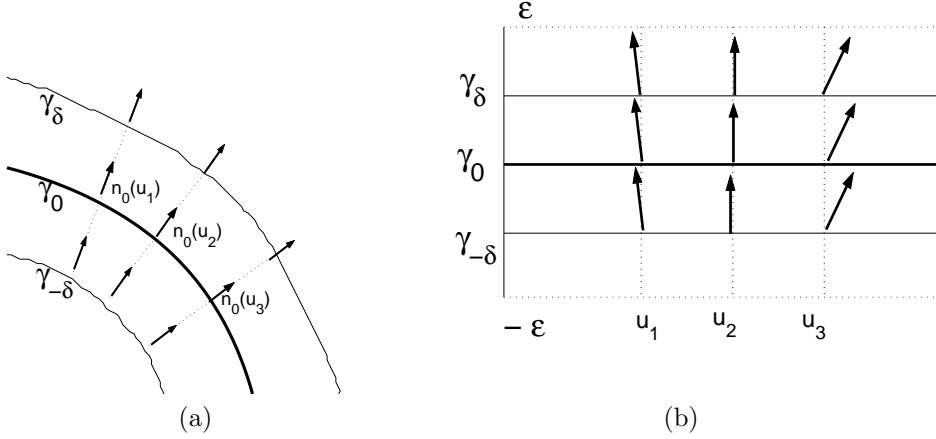


Figure 2.2: Normal vectors in a tubular neighborhood (a) and tube parameter domain (b)

Let γ_0 denote the target curve and δ the distance from an image pixel (x, y) to γ_0 . Then, the relation between $\tilde{J}(x, y)$ and the true structure tensor, $J(x)$, computed along γ_0 is given by:

$$\tilde{J} = J + \delta \int G(x) \cdot (\kappa_{Im} - \kappa_0 \tan(\theta))(t_0 \otimes n_0) dx + \mathcal{O}(\delta^2) \quad (2.4)$$

where κ_0 stands for the curvature of γ_0 and κ_{Im} for the flow lines curvature. It follows that the error made is bounded by:

$$\begin{aligned} Error = |J_\rho - \tilde{J}_\rho| &\leq \epsilon \left(C \int |\kappa_0| + \int |\kappa_{Im}| \right) \|\vec{t}_0 \otimes \vec{n}_0\| + \\ &+ \mathcal{O}(\epsilon^2) = E(\gamma_0) + E(Im) + \mathcal{O}(\epsilon^2) \end{aligned} \quad (2.5)$$

We remit the reader to the Appendix for the mathematical argumentation that leads to the above formula. Let us analyze the meaning of each of the terms involved in (2.5) and the way the error affects in the numeric implementation of RCF.

Two are the main sources of error in formula (2.5). The term $E(Im)$, proportional to the rate of variation of the unit normal to γ_0 along the flow lines, comes from the initial embedding function. First notice that, in the particular ideal case of the signed distance map, this term cancels. This follows because distance maps are, locally, the embedding of a tubular neighborhood of γ_0 . Hence, normal vectors to the level curves in a band around γ_0 are constant in the gradient direction as the drawing of fig.2.2(a) shows. Also notice that for images with uniform areas bounded by edges, $E(Im)$ is nearly negligible because border curves correspond to inflexion points in the image gradient direction and, hence, $\kappa_{Im} = 0$. Only images with regular level curves on a textured or noisy background may evolve in a way such that motion of level

curves differs from RCF due to an arbitrary huge $E(Im)$. In the first stages of the evolution, the roughness measure could cancel on the regular curve but not in the noisy neighborhood. However, as the image evolves, since background variation decreases, $E(Im)$ asymptotically decreases to a small positive value and the evolution tends to agree to that of the ideal signed distance map. We observe that some curvature based techniques successfully used in image processing (such as the image average based min/max flow [49]) present a similar pathology. Moreover our numerical experiments show that the dependency upon the initial embedding function does not significantly affect the final shapes achieved with RCF.

The first summand in (2.5), $E(\gamma_0)$, measures the error made in using the image cartesian coordinates, (x, y) , instead of the tubular coordinates given by the curve parameter, u , and the parameter, δ , of the normal line $n_0(u)$ (fig.2.2(b)). Because it corresponds to the term $-\kappa_0 \tan(\theta)(t_0 \otimes n_0)$ of formula (2.4) it vanishes when the curve is symmetric around the point. It follows that for finite unions of circles and straight lines the only source of error is the one introduced by the embedding function. For other curves, first notice that $E(\gamma_0)$ is bounded by the total gaussian curvature of the target curve γ_0 . Second that, in the case of a positive g , RCF level sets formulation agrees with the mean curvature flow. Curves evolving under the mean curvature flow [32], [33] converge to a circle of radius namely R , and then collapse to a point. Furthermore, their total gaussian curvature, before reaching the limit circle, is a decreasing function of time. The latter implies that $E(\gamma_0)$ only blows when a level curve collapses (which happens a finite number of times) and keeps bounded for the remains of the evolution:

$$2\pi R \leq E(\gamma_0) = \int |\kappa_0(u, t)| \leq \int |\kappa_0(u, 0)|$$

The former analysis of the level sets formulation of RCF points the following. Evolution of distance maps perfectly matches the curve formulation of RCF in the sense that all properties given in Section 2.1 are preserved and, hence, stabilization can be detected by means of the magnitude of g . In the general case, the roughness measure does not tend to zero but to the positive value that depends on the initial embedding image. The length of the time intervals where the former behavior holds hinges upon the level sets topology. This numerical error introduced in the computation of RCF difficulties using an evolution stop criterion in terms of the magnitude of g and motivates searching for an alternate.

2.2.2 RCF Best Parametric Values

Parameters involved in any numeric implementation can be split into the ones that are inherent to the method implemented and those concerning the numeric algorithm. In the case of RCF, these two sets reduce to:

- **RCF Parameters:** ρ

This parameter controls smoothness of the final curves and, by virtue of (2.5), it also influences in the error made in RCF implicit approximation. This fact limits, for the sake of error minimization, the magnitude of ρ to values less or equal to 1. Indeed we always use $\rho = 0.5$ in our experiments.

- **Numeric Parameters**

1. **Stop Parameters:** $\{\epsilon, T\}$

The value of ϵ determines the maximum amplitude of the roughness measure oscillations. Because g decreases smoothly on the orbits of RCF, ϵ can be arbitrarily small ($\epsilon = 10^{-3}, 10^{-4}$).

In (real) images presenting a rich level set topology, the length of the time interval T influences on the level of detail of the longest level curves (we remit the reader to Section 5.1.3 for examples on the impact of this parameter). Because these curves are usually descriptive of image features we recommend using $T < 50$.

Experiments in Sections 5.1.2 and 5.1.3 show that the set $\{\epsilon = 10^{-3}, T = 50\}$ guarantees noise removal and geometry preservation.

2. **Gray-Level of Target Curve:** α_0

Since geometric flows are designed for curves rather than for images, the proper way to apply them to image denoising is through a level set decomposition ([11], [52]). Because such decomposition adds an extra computational cost, we suggest filtering the image itself and computing any stop quantity over a curve (manually chosen) describing the image features. In this manner, stopping quantities are smoother in time (see Section 5.1.3) so that standard stabilization criteria work fine.

2.3 Mathematical Issues

2.3.1 Properties of the roughness measure

In this section, we will prove the local and global properties of the roughness function g (given in Section 2.1.1) which describe the curves that cancel this function. On one hand, we will show that, locally, our roughness measure g measures the degree of symmetry of the curve. And on the other, we will see that the function g really measures the amount of irregularity on the curve, as those curves that cancel g are C^∞ curves.

In order to analyze the local properties, we first notice that g cancels at those points such that $\psi - \theta = 0$. The latter quantity is nothing but the angle of v_1 with respect to the unit normal at each point. This fact motivates, for the sake of notational simplicity, using 'gauge' coordinates. That is, in order to determine what characterizes points $\gamma(u_0)$ such that $g(\theta(u_0), u_0) = 0$, we can take as coordinate system in \mathbb{R}^2 the one given by $\vec{t}(u_0), \vec{n}(u_0)$. Notice that taking angles with respect to the axis given by $\vec{n}(u_0)$, the roughness measure equals $\sin^2(\arctan(A))$. Thus if one is to prove that g cancels in symmetric situations it suffices to check that at a point $u = u_0$, such that $g = 0$, A computed with respect to $\vec{n}(u_0)$ cancels.

Proposition 2.3.1 *If the angle of the tangent does not turn more than $\pi/2$, the function g measures the degree of symmetry of the curve around each point.*

Proof. Using 'gauge' coordinates, it suffices to check that $A(u_0) = 0$. We assert that the latter equality is satisfied if and only if $a_{12}(u_0) = 0$. This follows because, by hypothesis, θ does not turn more than $\pi/2$ and we can assure that $a_{11} - a_{22} = \int_{-\infty}^{\infty} G_\rho(u) \cos(2\theta(u)) du \neq 0$. If we further take $u_0 = 0$, the coefficient a_{12} equals:

$$\begin{aligned} a_{12}(0) &= \frac{1}{2} \int_{-\infty}^{\infty} G_\rho(u) \sin(2\theta(u)) du = \frac{1}{2} \int_{-\infty}^0 G_\rho(u) \sin(2\theta(u)) du + \\ &+ \frac{1}{2} \int_0^{\infty} G_\rho(u) \sin(2\theta(u)) du \end{aligned}$$

By changing the integration variable of the second integral by $-u$, we obtain:

$$a_{12}(0) = \frac{1}{2} \int_{-\infty}^0 G_\rho(u) \sin(2\theta(u)) du + \frac{1}{2} \int_{-\infty}^0 G_\rho(-u) \sin(2\theta(-u)) du$$

Now, by assumption, the curve is symmetric around zero (i.e. $\theta(-u) = -\theta(u)$), so that by symmetry of the gaussian we obtain:

$$a_{12}(0) = \frac{1}{2} \int_{-\infty}^0 G_\rho(u) \sin(2\theta(u)) du - \frac{1}{2} \int_{-\infty}^0 G_\rho(u) \sin(2\theta(u)) du = 0$$

□

In the discrete version of RCF, since gaussian kernels are of compact support, the above result implies that g measures the local degree of symmetry in a curve. That is, for a given scale ρ , it cancels at those points, $u = u_0$ where the curve is symmetric in a neighborhood of length $2\rho + 1$ centered at u_0 . The former Proposition and symmetry of circles yields:

Corollary 2.3.1 *At whatever scale, ρ , the function (2.2) cancels on circles.*

Concerning those curves having null roughness measure, we have the following global result:

Theorem 2.3.1 *Let γ be a curve such that $g \equiv 0$. Then the angle of the tangent θ is a C^∞ function*

Proof. Notice that the roughness measure cancels if and only if:

$$0 \equiv \tan(\arctan(A) - 2\theta) = \frac{\tan(\arctan(A)) - \tan(2\theta)}{1 + \tan(\arctan(A)) \tan(2\theta)} = \frac{A - \tan(\theta)}{1 + A \tan(2\theta)}$$

Hence, it cancels on those functions such that:

$$\frac{G_\rho * \sin(2\theta)}{G_\rho * \cos(2\theta)} = \frac{\sin(2\theta)}{\cos(2\theta)}$$

Or equivalently:

$$\frac{G_\rho * \sin(2\theta)}{\sin(2\theta)} = \frac{G_\rho * \cos(2\theta)}{\cos(2\theta)} = f(\rho, s)$$

That means that we have the following two equalities:

$$G_\rho * \sin(2\theta) = f(\rho, s) \sin(2\theta) \quad \text{and} \quad G_\rho * \cos(2\theta) = f(\rho, s) \cos(2\theta) \quad (2.6)$$

Now, as noticed at the beginning of the section, the functions $\tilde{S} := G_\rho * \sin(2\theta)$ and $\tilde{C} := G_\rho * \cos(2\theta)$ are solutions to the heat equation at time $t = \rho$, so they are C^∞ . We will show that this fact makes the functions f and θ belong to the Sobolev spaces H^k for any k . General theory on Sobolev spaces [23] yields that $f, \theta \in C^\infty$. We will begin with proving that they belong to H^1 and, then, proceed by induction. In order to prove the assertion it suffices to check [23] that the L^2 -norm of the incremental quotients of f and θ is uniformly bounded. By definition, the incremental quotient of a function v is defined as:

$$D^h(v) := \frac{v(s) - v(s+h)}{h}$$

Taking difference quotients in equations (2.6), we obtain:

$$\left. \begin{aligned} D^h(\tilde{S}) &= D^h(f) \sin(\phi) + D^h(\phi) f \cos(\phi) \\ D^h(\tilde{C}) &= D^h(f) \cos(\phi) - D^h(\phi) f \sin(\phi) \end{aligned} \right\}$$

Therefore we have that:

$$\int D^h(f)^2 + D^h(\phi)^2 f^2 = \int D^h(\tilde{S})^2 + D^h(\tilde{C})^2 \leq \int \tilde{S}_s^2 + \tilde{C}_s^2 < M < \infty$$

If $f^2 > \epsilon > 0$, then the above inequality yields that $f, \phi \in H^1$. Taking difference quotients over first derivatives, we get:

$$\left. \begin{aligned} D^h(\tilde{S}_s) &= D^h(f_s) \sin(\phi) + 2\phi_s f_s \cos(\phi) + D^h(\phi_s) f \cos(\phi) - \phi_s^2 f \sin(\phi) \\ D^h(\tilde{C}_s) &= D^h(f_s) \cos(\phi) - 2\phi_s f_s \sin(\phi) - D^h(\phi_s) f \sin(\phi) - \phi_s^2 f \cos(\phi) \end{aligned} \right\}$$

Taking the sum of squares and integrating, we get that:

$$\begin{aligned} \infty > M > \int \tilde{S}_{ss}^2 + \tilde{C}_{ss}^2 &\geq \int D^h(f_s)^2 - 2\phi_s^2 f D^h(f_s) + 4\phi_s^2 f_s^2 + 4D^h(\phi_s) \phi_s f f_s + \\ &+ D^h(\phi_s)^2 f^2 + \phi_s^4 f^2 = \int D^h(f_s)^2 + \int D^h(\phi_s^2) \partial_s(f^2) + 2 \int \phi_s^2 f_s^2 + \\ &+ 4 \int \phi_s^2 f_s^2 + \int D^h(\phi_s^2) \partial_s(f^2) + \int D^h(\phi_s)^2 f^2 + \int \phi_s^4 f^2 > \int D^h(f_s)^2 + \\ &+ \int D^h(\phi_s)^2 f^2 + 2 \int D^h(\phi_s^2) \partial_s(f^2) = \int D^h(f_s)^2 + \int D^h(\phi_s)^2 f^2 - \\ &- 2 \int \phi_s^2 D^h(\partial_s(f^2)) \end{aligned} \quad (2.7)$$

Now, developing derivatives, the last integral equals:

$$\begin{aligned} -2 \int \phi_s^2 D^h(\partial_s(f^2)) &= -4 \int \phi_s^2 D^h(f f_s) = -4 \int \phi^2 f_s^2 - 4 \int f \phi_s^2 D^h(f_s) \geq \\ &\geq -4 \int \phi^2 f_s^2 - 4\epsilon \int D^h(f_s)^2 - \frac{4}{4\epsilon} \int f^2 \phi_s^4 \end{aligned}$$

Where we obtain the last inequality by applying Cauchy. Replacing this expression into (2.7) we get the following bound on the second incremental differences:

$$(1 - 4\epsilon) \int D^h(f_s)^2 + \int D^h(\phi_s)^2 f^2 < M + 4 \int \phi^2 f_s^2 + \frac{4}{\epsilon} \int f^2 \phi_s^4 < C < \infty$$

The last inequality follows from the bound on $\int f^2 \phi_s^4$ obtained from:

$$M > \int (\cos(\phi)\tilde{S} - \sin(\phi)\tilde{C})^4 = \int f^4 \phi_s^4 > \epsilon \int f^2 \phi_s^4$$

Hence, if $f^2 > 0$, we get that f and ϕ belong to H^2 . Taking higher order incremental differences, we have that:

$$\int D^h(f^n)^2 + \int f^2 D^h(\phi^n)^2 + \text{Previously bounded terms} \leq M < \infty$$

So both functions, f and ϕ belong to H^k for any k and, thus, they are smooth functions.

The last thing to prove is that $f^2 \neq 0$. But this follows from the fact that the curve γ does not have self-intersections. \square

2.3.2 Existence of Solutions to RCF

First notice that since a change of parameter in a curve does not alter its shape but only the speed we are travelling on the curve, we have infinitely many formulations representing the same curve evolution (although the function of the embedding, γ , may change). Secondly, since a curve in the plane is determined by its curvature, by means of the properties of the PDE associated to the curvature we can infer properties of the family of curves solving (2.3). We will derive equations for the curvature associated to the arc-length parameter, s , and the angle θ since they will be useful to prove existence of solutions and to describe their properties.

Equivalent formulations for RCF

Proposition 2.3.2 *The curvature of the solution to (2.3) parameterized by the angle θ satisfies:*

$$\kappa_T = g\kappa + \partial_{\theta\theta}(g\kappa) \quad \text{with} \quad \frac{dT}{d\tau} = \kappa^2$$

Proof. The change to the angular parameter induces a coordinate change in the domain of definition of the equation, $(u, t) \mapsto (\theta(u, t), \tau(t))$, given by the partial derivatives $\partial_u = \theta_u \partial_\theta$ and $\partial_t = \theta_t \partial_\theta + \tau_t \partial_\tau$. Now, following [32] and [33], we have that the angle, θ , and the curvature satisfy:

$$\theta_t(u, t) = \partial_s(g\kappa) \quad \text{and} \quad \kappa_t(u, t) = g\kappa^3 + \partial_{ss}(g\kappa)$$

Since, without loss of generality we can take $\tau_t = 1$, differentiating with respect to the new time variable, we have that the curvature fulfills:

$$\kappa_\tau = -\theta_t \kappa_\theta + \kappa_t = -\kappa_\theta \partial_s(g\kappa) + g\kappa^3 + \partial_{ss}(g\kappa)$$

Using the relation between partial derivatives $\partial_s = \kappa \partial_\theta$, we get that:

$$\kappa_\tau = g\kappa^3 + \kappa^2 \partial_{\theta\theta}(g\kappa)$$

Finally, consider a new time derivative, dT defined by $\frac{dT}{d\tau} = \kappa^2$ to obtain:

$$\kappa_T = \partial_T(\kappa(\theta, t(T))) = \kappa_t \frac{dt}{dT} = g\kappa + \partial_{\theta\theta}(g\kappa)$$

□

Proposition 2.3.3 *Let $\theta_0(s, t)$ be the angle between the tangent to the initial curve $\gamma_0(s, t)$ and a fixed axis. Then the problem given by (2.3) is equivalent to:*

$$\theta_\tau(s, \tau) = \partial_s(g\theta_s) + \left(\int_0^s g\theta_s^2 ds\right)\theta_s \tag{2.8}$$

with initial condition $\theta(s, 0) = \theta_0(s)$ and periodic boundary conditions $\theta(0, \tau) = \theta(L, \tau)$

Proof. In the case of the arc-length parameter and also taking $\tau_t = 1$, the coordinate change ([32], [33]) in the domain of definition of the solutions to (2.3) is given by $\partial_u = \|\gamma_u\| \partial_s$ and $\partial_t = -\left(\int_0^s g\theta_s^2 ds\right)\partial_s + \partial_\tau$. Therefore, we have that:

$$\theta_\tau(s, \tau) = \theta_t + \left(\int_0^s g\theta_s^2 ds\right)\theta_s = \partial_s(g\theta_s) + \left(\int_0^s g\theta_s^2 ds\right)\theta_s$$

where the last equality follows from the expression of θ_t ([32], [33]). This yields that angles of solutions to (2.3) satisfy (2.8). Conversely, for any solution to (2.8) the curves:

$$\gamma(s, t) = \left(\int_0^s \cos(\theta) d\tilde{s}, \int_0^s \sin(\theta) d\tilde{s}\right)$$

are closed curves that satisfy an equation that, up to a change of parameter, equals (2.3). □

We will prove existence of solutions to RCF by showing that the equivalent problem given in Proposition 2.3.3 has a weak solution provided that θ_0 is \mathcal{C}^2 . Under this assumption on the differentiability of the initial function we will construct a sequence of weak solutions to the associated linear problem:

$$\theta_\tau(s, \tau) = \partial_s(g(\psi)\theta_s) + \left(\int_0^s g(\psi)\psi_s^2 ds\right)\theta_s \tag{2.9}$$

uniformly bounded in H^1 . Then, as the space is reflexive, it exists a subsequence that weakly converges to a function $\theta \in H^1$, which is a weak solution to 2.8.

Notice that since the roughness measure can cancel at some points, the differential operator is not elliptic at every point. We recall that this condition guarantees existence of weak solutions to (2.9) in the space $\mathcal{C}([0, T], H^1)$ by means of Galerkin's method. The key point lies on having a uniform bound on the first derivative that can

be obtained if one applies the elliptic condition to the integral version of the equation. Hence the lack of ellipticity forces to consider the partial differential equation that the first derivative, $\theta_s = \kappa$, satisfies in order to obtain a uniform bound for its L^2 -norm. This forces the initial function, θ_0 to belong to H^1 and $\psi \in \mathcal{C}^1$.

We will begin with proving that the linear problem (2.9) has weak solutions. Next we will deduce that, in spite of the lack of ellipticity, these solutions are at least \mathcal{C}^2 almost everywhere, provided that the second derivative of the roughness measure, g_{ss} , stays bounded almost everywhere. Finally we will address existence of weak solutions to the non linear problem given in Proposition 2.3.3.

Existence of Solutions to the Linear Problem

We will show that the linear problem (2.9) has weak solutions by means of its approximations in certain finite dimensional spaces (Galerkin's method). Let $\{\omega^k(s)\}_k$ be a smooth periodic basis of H^1 , orthonormal in L^2 . We could take, for instance, the eigenfunctions of the Laplacian operator with a time dependent period, L , given by:

$$\frac{dL}{dt} = -g(\psi)\psi_s^2 L \Rightarrow L(t, s) = L_0(s)e^{-g(\psi)\psi_s^2 t}$$

which implies that the domain of integration (s, t) is not a rectangle, $[0, L_0] \times [0, T]$, but a trapezoid of basis L_0 at time 0 and $L(T)$ at time T .

The projection of equation (2.9) onto the finite dimensional spaces, Ω_m , spanned by $\{\omega^1, \dots, \omega^m\}$ is given by:

$$\int_0^L \theta_t^m \omega = - \int_0^L g(\psi)\theta_s^m \omega_s + \int_0^L \left(\int_0^s g(\psi)\psi_s^2 ds \right) \theta_s^m \omega \quad (2.10)$$

for $\omega \in \Omega_m$ and initial condition the projection of ψ onto Ω_m , $\psi_m = \sum (\int \psi \omega_k) \omega_k$. We assert that the former equation has a unique weak solution, θ^m , of the form:

$$\theta^m = \sum_k d_m^k(t) \omega^k(s)$$

This follows from the fact that the coefficients, d_m^k , satisfy the system of ordinary differential equations obtained by requiring that the functions θ^m satisfy (2.10) for each of the generators, ω^k , $k = 1, \dots, m$. We claim that θ^m are uniformly bounded in $L^2([0, T], H^1)$, provided that ψ is \mathcal{C}^1 . Because the operator is not elliptic, we need first a bound on θ_s^m :

Lemma 2.3.1 *Let $\kappa_0(s)$ be a periodic function on \mathbb{R} . For each $\psi \in \mathcal{C}^1(I, \mathbb{R})$, consider the initial boundary problem:*

$$\kappa_t(s, t) = g(\psi)\psi_s^2 \kappa + \partial_{ss}(g(\psi)\kappa) + \left(\int_0^s g(\psi)\psi_s^2 ds \right) \kappa_s \quad (2.11)$$

with $\kappa(s, 0) = \kappa_0$. If $\kappa \in H^1$ is a weak solution and g_{ss} is uniformly bounded by M , then the solution is unique and satisfies:

$$\max_{0 < t < T} \|\kappa\|_{L^2}^2 \leq e^{T(\|\psi_s^2\|_\infty + M)} \|\kappa_0\|_{L^2}^2$$

where $\|\psi_s\|_\infty = \max_s |\psi_s|^2$.

Proof. Let $\kappa \in H^1$ be a weak solution of the initial boundary problem (2.11), that is, a function satisfying the integral equation:

$$\int \kappa_t \omega = \int g(\psi) \psi_s^2 \kappa \omega + \int \left(\int_0^s g(\psi) \psi_s^2 ds \right) \kappa_s \omega - \int \partial_s (g(\psi) \kappa) \omega_s$$

for all $\omega \in H^1$. Denote by η the square of the L^2 -norm of κ . If $\kappa_s \neq 0$ almost every where, then η fulfills:

$$\begin{aligned} \eta_t &= \int \kappa_t \kappa = \int g(\psi) \psi_s^2 \kappa^2 + \int \left(\int_0^s g(\psi) \psi_s^2 ds \right) \kappa_s \kappa - \int \partial_s (g(\psi) \kappa) \kappa_s = \int g(\psi) \psi_s^2 \kappa^2 + \\ &+ \frac{1}{2} \int \left(\int_0^s g(\psi) \psi_s^2 ds \right) \partial_s (\kappa^2) - \int g(\psi) \kappa_s^2 - \int g_s(\psi) \partial_s (\kappa^2) \leq - \int g_s(\psi) \partial_s (\kappa^2) + \\ &= \frac{1}{2} \int \psi_s^2 \kappa^2 \end{aligned}$$

The last inequality obtained through integration by parts of $\int \left(\int_0^s g(\psi) \psi_s^2 ds \right) \partial_s (\kappa^2)$. Integrating once again by parts, we get that:

$$\eta_t \leq \int (g_{ss} + \psi_s^2) \kappa^2 \leq (M + \|\psi_s^2\|_\infty) \eta$$

For M a bound on g_{ss} . Finally Gronwall's Lemma yields that:

$$\|\kappa\|_{L^2}^2 \leq e^{t(M + \|\psi_s^2\|_\infty)} \|\kappa_0\|_{L^2}^2$$

On one hand, since the PDE is linear, this inequality implies uniqueness of the solutions. On the other hand, taking maximums over time we obtain the bound on $\|\kappa\|_{L^2}^2$. In the case $\kappa_s \equiv 0$, the equation for the L^2 -norm yields:

$$\eta_t = \int g(\psi) \psi_s^2 \kappa^2 \leq \|\psi_s^2\|_\infty \eta$$

Thus, in this case we also have:

$$\|\kappa\|_{L^2}^2 \leq e^{t\|\psi_s^2\|_\infty} \|\kappa_0\|_{L^2}^2$$

Splitting the domain of integration into the set $\{\kappa_s \equiv 0\}$ and its complementary, we get the result in the general case. \square

The above bound on the solutions to the equation associated to the first derivatives of a function solving (2.9) and standard arguments on linear PDE's gives existence to the linear problem:

Proposition 2.3.4 *Let $\psi \in C^1$ be a function such that g_{ss} is uniformly bounded almost everywhere. Then it exists a unique weak solution to:*

$$\theta_\tau(s, \tau) = \partial_s (g(\psi) \theta_s) + \left(\int_0^s g(\psi) \psi_s^2 ds \right) \theta_s$$

with initial condition $\theta(s, 0) = \theta_0$ a periodic function in the Sobolev space H^1 . Besides we have the following bound on the L^2 norm of the solutions:

$$\max_{0 \leq t < T} \|\theta\|_{L^2} \leq \|\theta_0\|_{L^2}$$

Proof. See [24]. □

Existence of Solutions to RCF

We recall that we will prove existence by means of a suitable sequence of weak solutions to the linear problem (2.9). Since the operator is not everywhere elliptic, we need solutions to (2.9) to be, at least C^2 , in order to explicitly derive a bound on $\theta_s = \kappa$.

Proposition 2.3.5 *Let ψ, θ_0 be C^2 functions. Then the weak solution to :*

$$\theta_t = \partial_s(g(\psi)\theta_s) + \left(\int_0^s g(\psi)\psi_s^2 ds\right)\theta_s$$

with initial condition $\theta(s, 0) = \theta_0$ is C^2 as well.

Proof.

If $g > \epsilon$ almost everywhere, we can assure that the solutions belong at least to H^4 , since the coefficients of (2.9) belong to C^2 , the initial function is C^2 and the second order operator is of elliptic type. Hence, on that set, by virtue of Morrey's inequality, we have that θ is C^3 . We may only have problems in a neighborhood of those points where g is continuous and cancels, as the equation resembles a transport one (i.e. a 1st order PDE):

$$\theta_t = g_s(\psi)\theta_s \tag{2.12}$$

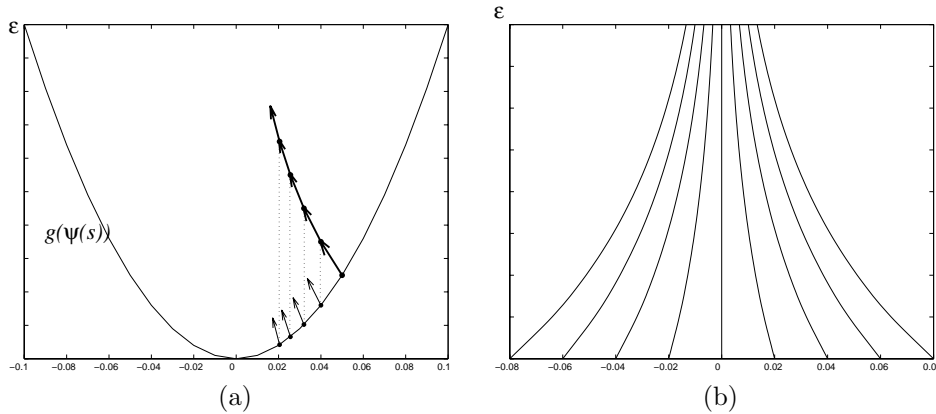


Figure 2.3: Behavior at a point of cancelling g , (a), and level lines of θ , (b)

By general theory of first order PDE's, we know that solutions, whatever the differentiability of the initial function, may develop singularities (shocks) either if

the equation is non linear or if its coefficients are not C^∞ . Our equation fits into the second group as we can only assure that g_s is continuous. We recall that shock formation is essentially due to crossing of two characteristics, that is, when two level curves of the function solving (2.12) cross. If we show characteristics do not cross for any positive time, then we can assure that the function will be as smooth as the initial function θ_0 . The key point lies on noticing that the function g is as smooth as ψ . Since $g = 0$ is a minimum and ψ is C^2 , we have that the graph of g , locally, is like the one shown in figure 2.3(a). Now, the characteristics of equation (2.12) are given by $\alpha(t) = (s(t), t)$ with $s_t = -g_s(\psi)$, that is, they correspond to the flux given by the normal vector to the graph $(s, g(\psi(s)))$, as it illustrates figure 2.3(a). Hence they correspond to the flux drawn in figure 2.3(b). The real equation can be viewed as a perturbation of (2.12), which not only does not alter the flux of figure 2.3(b) but also adds the regularizing effect of the diffusion $g\theta_{ss}$. This proves the statement. \square

Differentiability of solutions to (2.9), yields that their first derivative satisfies:

Proposition 2.3.6 *Let θ be a solution to the linear problem (2.9), with initial condition a C^2 periodic function. Then the L^2 norm of its first derivative κ is bounded by:*

$$\max_{0 \leq t < T} \|\kappa(t)\|_{L^2} \leq e^T \|\kappa(0)\|_{L^2}$$

where $\kappa_0 = \partial(\theta_0)/\partial_s$

Proof. We remit the reader to Proposition 2.3.8 for the idea of the proof and to [24] for the details. \square

Equipped with the above results, we are ready to prove the main result of this section:

Theorem 2.3.2 *For any C^2 periodic function $\theta_0(s)$, there exists a unique periodic weak solution to:*

$$\theta_\tau(s, \tau) = \partial_s(g\theta_s) + \left(\int_0^s g\theta_s^2 ds\right)\theta_s \quad \text{with} \quad \theta(s, 0) = \theta_0(s)$$

Proof. Let $X = \mathcal{C}([0, T], H^1(I, \mathbb{R}))$ be the space of continuous functions from $[0, T]$ to the Sobolev space of periodic real functions with first derivatives square integrable. Each function, $\psi \in X$, defines the linear parabolic PDE:

$$\theta_t(s, t) = \partial_s(a(s, t)\theta_s) + b(s, t)\theta_s \quad \text{with initial condition} \quad \theta(s, 0) = \theta_0(s)$$

with $b(s, t) := \left(\int_0^s g(\psi)\psi_s^2 ds\right)$ and $a(s, t) := g(\psi)$. By Proposition 2.3.4, we know that, at least, a C^2 solution satisfying the maximum principle exists. Furthermore, by virtue of the equation associated to its first derivative, $\theta_s = \kappa$, we have that

$$\max_{0 < t < T} \|\kappa\|_{L^2} \leq e^T \max \|\kappa_0\| \tag{2.13}$$

Consider the sequence in H^1 defined as:

$$\theta^1 := A(\theta_0) \quad ; \quad \theta^k := A(\theta^{k-1}) \quad k > 1$$

We claim that the sequences θ^k and θ_t^k are bounded in $L^2([0, T], H^1(I, \mathbb{R}))$ and $L^2([0, T], H(I, \mathbb{R})^{-1})$, respectively, where $H(I, \mathbb{R})^{-1}$ denotes the topological dual space of $H^1(I, \mathbb{R})$. Then, since the spaces are reflexive, there exists subsequences, θ^{k_m} and $\theta_t^{k_m}$ and a function $\theta \in L^2([0, T], H^1(I, \mathbb{R}))$, with $\theta_t \in L^2([0, T], H(I, \mathbb{R})^{-1})$ such that:

$$\theta^{k_m} \rightharpoonup \theta \text{ weakly } L^2([0, T], H^1(I, \mathbb{R})) \quad \text{and} \quad \theta_t^{k_m} \rightharpoonup \theta_t \text{ weakly } L^2([0, T], H^1(I, \mathbb{R})^*)$$

This function θ is a weak solution [23] (by virtue of Lipschitzity of g) to the initial boundary problem of Proposition 2.3.3 and produces a piece wise smooth curve that solves equation (2.3).

The maximum principle for solutions to (2.8) and the inequality (2.13), gives a bound for $\|\theta^k\|_{L^2([0, T], H^1(I, \mathbb{R}))}^2$. For a bound on θ_t^k in H^{-1} , consider, for each $\omega \in H^1$, the following:

$$\begin{aligned} \left| \int \theta_t^k \omega \right| &= \left| - \int a(s, t) \theta_s \omega_s + \int b(s, t) \theta_s \omega \right| \leq \|\theta_s\|_{L^2} \|\omega_s\|_{L^2} + C \|\theta_s\|_{L^2} \|\omega\|_{L^2} \leq \\ &\leq (e^T \|\kappa_0\|_{L^2} + C \|\kappa_0\|_{L^2}) \|\omega\|_{H^1} \end{aligned}$$

Where we have applied Holder to obtain the first inequality. It follows that $\|\theta_t^k\|_{H^{-1}} \leq e^T \|\kappa_0\|_{L^2} + C \|\kappa_0\|_{L^2}$. Weakly convergence for bounded sequences in reflexive spaces yields existence of weak solutions for finite time. \square

2.3.3 Differentiability and Uniqueness

Differentiability

Let us see that the weak solutions to (2.8) are as smooth as the initial function θ_0 . This fact will ensure uniqueness of solutions if θ_0 is \mathcal{C}^2 . Notice how lack of ellipticity forces a certain degree of differentiability in the initial condition in order to guarantee existence and uniqueness of weak solutions.

Proposition 2.3.7 *Consider the initial boundary problem:*

$$\theta_t = \partial_s(g(\theta)\theta_s) + \left(\int g(\theta)\theta_s^2 \right) \theta_s$$

with initial condition a periodic \mathcal{C}^k function θ_0 and periodic boundary conditions. Then any weak solution $\theta \in H^1$ is \mathcal{C}^k as well.

Proof. As in the case of differentiability of the solutions to the linear problem (Proposition 2.3.5), we must only bother in a neighborhood of those points that cancel g . In the open set where $g > \epsilon$, as the operator is elliptic, we have that the solutions are \mathcal{C}^n for $t > t_n$, thus improving the differentiability of the original function.

We will follow the arguments given in Proposition 2.3.5 and show that the characteristic lines of the non-linear transport equation:

$$\theta_t = g_s(\theta)\theta_s$$

are smooth and well defined (that is, curves of different levels do not cross). The level lines, $\alpha(t)$, are given by the normal vector to the graph of g : $(-g_s(\theta(s, t)), 1)$. The only difference with respect to the linear case is that the roughness measure depends on time. We will see that for small times, $t < \delta$, the flux is well defined. For larger times just apply the same argument to the problem with initial condition $\theta(s, t - \delta)$.

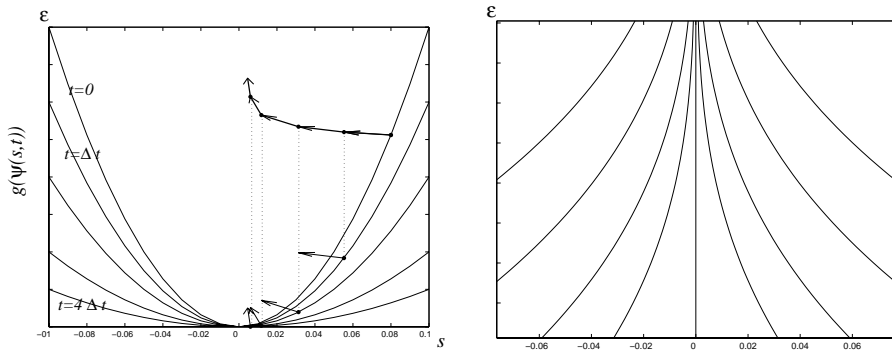


Figure 2.4: Flux given by the normal vector when g decreases and level lines when g

Since by assumption the initial function is C^2 , in a small enough neighborhood of a point cancelling the roughness measure, the graph of g at time zero is like a parabola. Now, by continuity of g with respect to time, for times $t < \delta$, g either increases or decreases uniformly in a neighborhood of the zero of $g(s, 0)$. If g increases uniformly, we are in the situation of figure 2.4 and the level curves (shown in figure ??) are as in Proposition 2.3.5. In the case of an increasing g , the second order term of the equation increases its weight with time. Since it is bounded by its value for time zero, $g(s_0, 0)$, we are diffusing the function along the curves given by $(-g_s(\theta(s, t)), 1)$. Thus we are adding some curvature to the level curves of the solutions producing a well defined flux as well. In fact, as $g \rightarrow 0$ over the solutions to RCF (Theorem 2.3.4 in Section 2.3.4), this last situation never occurs. \square

Uniqueness

In order to show uniqueness we will consider the curve in cartesian coordinates and require that points move to fix their x -coordinate, that is, locally, $\gamma = \gamma(x) = (x, y(x))$. Using this parameterization, the flow is given [32] by:

$$y_t = \frac{gy_{xx}}{1 + y_x^2}$$

where $g = g(\arctan(y_x))$. We will see that if the functions solving (2.8) are C^2 , then the solution to the above equation is unique. Since we can cover the curve with a finite number of cartesian maps, we obtain uniqueness for the RCF.

Theorem 2.3.3 *It exists a unique \mathcal{C}^2 solution to the PDE:*

$$y_t = \frac{gy_{xx}}{1 + y_x^2} = g\partial_x(\arctan(y_x)) \quad (2.14)$$

Proof. Gronwall's Lemma applied to the L^2 -norm of the difference of two solutions yields the result. We remit the reader to [24] for the computational details. \square

Since the initial function must be \mathcal{C}^2 in order to guarantee existence and uniqueness of solutions, solutions to (2.8) are at least \mathcal{C}^2 . This fact gives sense to the maximum principles and energy bounds for the curvature that we will derive in next section.

2.3.4 Properties of RCF

We devote this part to the analysis of the solutions to the RCF. Firstly, we will show that the RCF has the same shape simplification properties as the MCF. Secondly, we will prove the main advantage of RCF over MCF: convergence of the orbits to a non trivial \mathcal{C}^∞ curve that keeps significant inflexion points of the initial curve. This facts makes our flow suitable for shape recognition analysis and image filtering. We will begin with some bounds on the solutions to RCF and, then, analyze RCF asymptotic properties.

Energy Bounds

Proposition 2.3.8 *Let $\gamma(u, t)$ be a family of curves solving (2.3), for $(u, t) \in [0, 1] \times [0, T)$. Then the energy of the curvature $\|\kappa(t)\|_{L^2} := \int_0^1 \kappa^2 \sqrt{\dot{x}^2 + \dot{y}^2} du = \int_0^1 \kappa^2 \nu du$ is a decreasing function of time,*

$$\|\kappa(t)\|_{L^2} \leq \|\kappa(0)\|_{L^2}$$

Proof. We will see that the inequality is fulfilled between two consecutive inflexion points, θ_i, θ_{i+1} . Since κ belongs to the Sobolev space $H_0^1(\mathbb{R})$ of functions with compact supports with first derivatives in L^2 , it can be expressed as $\kappa = \sum d_n(t)\omega_n$, where the functions ω_n are the eigenvectors of the Laplacian operator $-\Delta$ and we may assume they are positive. We will prove that the coefficients $d_n(t) = \int_{\theta_i}^{\theta_{i+1}} \omega_n \kappa$ decrease in time. Let η denote d_n for an arbitrary n . If we differentiate with respect to time and integrate by parts, as $\kappa = 0$ at θ_i, θ_{i+1} , by Proposition 2.3.2, we obtain:

$$\eta_t = \int_{\theta_i}^{\theta_{i+1}} \omega_n \partial_{\theta\theta}(g\kappa) d\theta + \int_{\theta_i}^{\theta_{i+1}} \omega_n g \kappa d\theta$$

If $\partial_{\theta\theta}(g\kappa) \neq 0$, integrating again by parts the first integral we get

$$\eta_t = \int_{\theta_i}^{\theta_{i+1}} \partial_{\theta\theta}(\omega_n)(g\kappa) d\theta + \int_{\theta_i}^{\theta_{i+1}} \omega_n g \kappa d\theta = (1 - \lambda_n) \int_{\theta_i}^{\theta_{i+1}} \omega_n g \kappa d\theta$$

where $\lambda_n = (n\pi)^2$ are the eigenvalues of the Laplacian operator $-\Delta$. Since $(1 - \lambda_n) < 0$ and $\omega_n g \kappa$ is positive, if $\kappa > 0$, and negative otherwise, we have that $\eta_t < 0$ when

$\kappa > 0$ and $\eta_t > 0$ when $\kappa < 0$. This fact implies that $|\eta(t)|$ decreases in time, which proves the result if $\partial_{\theta\theta}(g\kappa) \neq 0$.

In the case that $\partial_{\theta\theta}(g\kappa) \equiv 0$ we are going to show that either the arc remains steady or we have a circle. In any case, since circles are steady curves we have that $\eta(t) \equiv \eta(0)$. The fact that $\partial_{\theta\theta}(g\kappa) \equiv 0$ implies that $\partial_\theta(g\kappa) = f(t)$. Now, since at the boundary points, we have that $g\kappa(t, \theta_i) = g\kappa(t, \theta_{i+1}) = 0$, there exists an interior extremum, that is the first derivative $\partial_\theta(g\kappa)$ cancels at some point. This implies that, indeed, $\partial_\theta(g\kappa) \equiv 0$ on the whole arc, hence we have that $g\kappa$ is constant and equal to zero, by virtue of the cancellation of κ at the boundary points. So let us assume that we have a convex curve and check that it necessarily must be a circle. Since in this case $\theta_i = \theta_{i+1}$ the first derivative $\partial_\theta(g\kappa)$ must cancel in the whole curve. Developing the derivative, we obtain that $\frac{-g\theta}{g} = \frac{\kappa\theta}{\kappa}$. Integrating this equation, we can write the curvature in the form $\kappa = F(t)e^{1/g}$. Substituting this expression in the PDE for the curvature, we get:

$$\kappa_t = F'e^{\frac{1}{g}} - F\frac{g_t}{g^2}e^{\frac{1}{g}} = gFe^{\frac{1}{g}}$$

Now, because $\theta_t = \partial_s(g\kappa) = \kappa\partial_\theta(g\kappa)$, we have that the derivative with respect to time of the factors of the function A given by (2.1) cancel:

$$\partial_t(a_{12}) = G_\rho * (2\theta_t \cos(2\theta)) \quad \text{and} \quad \partial_t(a_{11} - a_{22}) = G_\rho * (2\theta_t \sin(2\theta))$$

It follows that $g_t \sim \sin(0.5 * \arctan(A) - \theta)(A_t/(1 + A^2) - \theta_t) = 0$ and $F'e^{\frac{1}{g}} = gFe^{\frac{1}{g}}$. Hence F satisfies the differential equation $\frac{F'}{F} = g$. Finally, since g is independent of time, this last equality implies that, indeed, $g = C_1$ is constant. Thus $\kappa = F(t)e^{1/C_1}$ and the curve is a circle. \square

Remarks:

1. It can be proved that the length, $L(t)$, of a curve satisfying (2.3) decreases in time at the rate $L_t = - \int g\kappa^2 \nu du$. It follows that the total curvature also decreases in time:

$$\bar{\kappa}(t) = \int |\kappa| \sqrt{\dot{x}^2 + \dot{y}^2} du \leq L(t) \|\kappa(t)\|_{L^2} \leq L(0) \|\kappa(t)\|_{L^2}$$

which is one of the requirements for curve simplification.

2. The above Proposition ensures existence of weak solutions of the curvature equation for all time, hence we can assure that C^∞ piece-wise curves exist for all time.

Besides this bound on the L^2 -energy of κ , we have that κ grows at most exponentially.

Lemma 2.3.2 *Let $\kappa(\theta, t)$, $t \in [0, T)$, be the solution to (??) with initial condition $\kappa_0(\theta)$ and assume that $g_{\theta\theta}$ is bounded by a constant M . Then, the following inequality holds:*

$$\max_{0 < t < T} |\kappa(\theta, t)| \leq e^{T(M+1)} \max |\kappa_0|$$

Proof. The function $v = e^{-(M+1)t}\kappa$ fulfills the PDE:

$$v_t = -(M+1)v + e^{-(M+1)t}\kappa_t = -(M+1)v + gv + \partial_{\theta\theta}(gv)$$

If the function v had an interior extremum at the set where κ is positive, then:

$$0 = (g - (M+1) + g_{\theta\theta})v + g\partial_{\theta\theta}(v) \leq g\partial_{\theta\theta}(v)$$

Since $g \geq 0$, we get that the extremum is, indeed, a minimum. A similar argument at points with negative κ , yields that $|v(\theta, t)| \leq \max |v(\theta, 0)| = \max |\kappa_0|$. We conclude that $|\kappa| \leq e^{(M+1)t}|v|$ and, thus, $\max_{0 \leq t < T} |\kappa| \leq e^{(M+1)T} \max_{0 \leq t < T} |v| = e^{(M+1)T} \max |\kappa_0|$. \square

This maximum principle for the curvature yields that no new inflexion points are created.

Asymptotic Behavior

Proposition 2.3.9 *The number of inflexion points (i.e. zeros of the curvature) of a family of curves, $\gamma(u, t)$, satisfying (2.3) for $t \in [0, T]$ is a non-increasing function of time.*

Proof. We will prove that between two consecutive inflexion points the curvature does not change its sign. In an arc joining two consecutive inflexion points Lemma 2.3.2 holds and, so, we have that:

$$e^{T(M+1)} \min \kappa_0 \leq \kappa \leq e^{T(M+1)} \max \kappa_0$$

It follows that κ keeps the sign of κ_0 between consecutive inflexion points. \square

The former Propositions yield that, as all geometric flow, RCF ensures shape simplification. What turns RCF into a preferable tool for image filtering and shape recovery tasks is the fact that solutions tend asymptotically to a \mathcal{C}^∞ curve, which is the fixed point of equation (2.3) given by $g \equiv 0$. We can summarize this asymptotic behavior in the following results:

Proposition 2.3.10 *Steady states of (2.3) are simple closed curves.*

Proof. First notice that existence of a Lyapunov functional (the length functional, for instance) ensures that RCF has not periodic orbits. It follows that the limit of any orbit $\gamma(u, t)$ must be a steady point of equation (2.3). The inclusion principle [32] and stability of circles for RCF yield the statement as follows. Let C_1 be a circle containing γ_0 and C_2 a circle included in γ_0 . By the inclusion principle, we have that evolutions of these curves under RCF fulfill $C_2(u, t) \subset \gamma(u, t) \subset C_1(u, t)$. The fact that any circle is a steady point yields the result. \square

Before proceeding to the proof of the main property of RCF concerning convergence to zero of the roughness measure (Theorem 2.3.4), let us begin with some notations and considerations. The norm of the first derivative of a curve solving (2.3) will be noted by $\nu = \sqrt{\dot{x}^2 + \dot{y}^2}$. Following [32], [33] it is easy to check that ν solves:

$$\nu_t = -g\kappa^2\nu \quad \text{with initial condition} \quad \nu(u, 0) = \nu^0(u) \quad (2.15)$$

since ν is a decreasing function of time, it is bounded above by ν^0 . Furthermore, non triviality of steady states of curves following RCF (Proposition 2.3.10) yield that ν is also bounded from below by the minimum, $\nu_{min} > 0$, of the norms corresponding to the limit curve. Also by Proposition 2.3.10, we know that the orbits of (2.3) converge to a curve fulfilling $g\kappa \equiv 0$. Since the latter quantity is continuous (g is Lipschitz [24]) with respect the L^2 -norm, it follows that:

$$\int g\kappa^2 = \int g\kappa \cdot \kappa \leq \|g\kappa\|_{L^2}\|\kappa\|_{L^2} \leq \|g\kappa\|_{L^2}\|\kappa(s, 0)\|_{L^2} \xrightarrow{t \rightarrow \infty} 0$$

We will note t_0 a time such that $\int g\kappa^2 < \epsilon$. The lower bound ν_{min} and the former asymptotic behavior will be used in the proof of Theorem 2.3.4. Let us begin with a preliminary Lemma:

Lemma 2.3.3 *The norm, ν , satisfies:*

$$\max_{t \leq T} |\partial_u(\nu^2)| \leq TM\nu^2(u, 0) + \nu_u^2(u, 0)$$

Proof. Recall that solutions to (2.3.3) are \mathcal{C}^2 classic solutions that converge to a smooth function θ_{lim} by existence of Lyapunov functionals. It follows that $\partial_u(g\kappa^2) \rightarrow \partial_u(g_{lim}\kappa_{lim}^2)$ and, hence, $\partial_u(g\kappa^2)$ is uniformly bounded by a constant $M/2$. Integrating (2.15) and noting $I = \int g\kappa^2 dt$, we get that $\nu = e^{-I}\nu(u, 0)$, so that the first derivative:

$$|\partial_u(\nu^2)| = | -2\nu^2 \int \partial_u(g\kappa^2) dt | + |e^{-2I}\nu_u^2(u, 0)| \leq M\nu^2(u, 0) + \nu_u^2(u, 0)$$

□

Theorem 2.3.4 *The roughness measure g tends to zero over the solutions to RCF.*

Proof. We recall that one of the definitions of the roughness measure is the square norm of the vector product:

$$g(\theta) = \|\vec{v}_1 \times \vec{n}\|^2$$

where the vector $\vec{v}_1 = (\cos \psi, \sin \psi)$ is given by the quantity:

$$A := \tan 2\psi = \frac{2a_{12}}{a_{11} - a_{22}} = \frac{G_\rho * \sin(2\theta)}{G_\rho * \cos(2\theta)}$$

Therefore, by the above definition of A , g is close to zero in the measure that the vectors $\vec{v}_1 = \vec{T}_\rho = (G_\rho * \sin(2\theta), G_\rho * \cos(2\theta))$ and $T = (\sin(2\theta), \cos(2\theta))$ coincide. We will check that in the limit, when t goes to infinity, T converges to \vec{T}_ρ . To such purpose we will study the asymptotic behavior of the PDE's that the former vectors follow.

For a given parameter u for a curve following (2.3), we have that T solves the PDE:

$$T_t(u, t) = \partial_s(gT_s(u, t)) = \frac{1}{\nu} \partial_u(g \frac{T_u}{\nu}) \quad \text{with } T(s, 0) = T_0 \quad (2.16)$$

where $\nu = \sqrt{\dot{x}^2 + \dot{y}^2}$. On the other hand, \tilde{T}_ρ corresponds to the solution at time $t = \rho$ of:

$$\tilde{T}_t = \tilde{T}_{uu} \quad \text{with } \tilde{T}(u, 0) = T(u, t) \quad (2.17)$$

If we change to the arc length parameter of $\gamma(u, t)$, the above PDE's convert to:

$$T_t = \partial_s(gT_s) + \left(\int g\kappa^2 \right) T_s \quad \text{with } T(s, 0) = T_0 \quad (2.18)$$

$$\tilde{T}_t = \nu \partial_s(\nu \tilde{T}_s) + \left(\int g\kappa^2 \right) \tilde{T}_s \quad \text{with } \tilde{T}(s, 0) = T(s, t) \quad (2.19)$$

where $\nu = 1/u_s = \nu(s)$ denotes the derivative of the inverse change from parameter u to parameter s . First, notice that in (2.19) s does not need to correspond to the arc length of \tilde{T} . Second, the quantity $g\kappa^2$ is computed over the solution to RCF with the function g given by the parameter u . For a better handling of (2.19), we will develop the term of the second derivative and use the formulation:

$$\tilde{T}_t = \nu^2 \tilde{T}_{ss} + \frac{1}{2} \partial_s(\nu^2) \tilde{T}_s + \left(\int g\kappa^2 \right) \tilde{T}_s \quad (2.20)$$

Let t_0 be a time such that $\int g\kappa^2 < \epsilon$, and consider the solutions to (2.18), (2.20) with initial condition $T(s, t_0)$. Let $(T - \tilde{T})^2$ be the square of the norm of the difference vector and define $\eta(t)$ to be:

$$\eta(t) = \int (T - \tilde{T})^2 ds \quad \text{for times } t > t_0 \quad (2.21)$$

If we note $I_g = \int g\kappa^2$, differentiate (2.21) with respect to time and integrate by parts we get:

$$\begin{aligned} \eta_t &= \int (\partial_s(gT_s) - \nu^2 \partial_s(\tilde{T}_s))(T - \tilde{T}) + \int I_g(T_s - \tilde{T}_s)(T - \tilde{T}) - \frac{1}{2} \int \partial_s(\nu^2) \tilde{T}_s (T - \tilde{T}) = \\ &= \int \partial_s(gT_s - \tilde{T}_s)(T - \tilde{T}) + \int (1 - \nu^2) \tilde{T}_{ss} (T - \tilde{T}) + \frac{1}{2} \int I_g \partial_s((T - \tilde{T})^2) - \\ &\quad - \frac{1}{2} \int \partial_s(\nu^2) \tilde{T}_s (T - \tilde{T}) = - \int (gT_s - \tilde{T}_s)(T_s - \tilde{T}_s) - \int \partial_s(1 - \nu^2) \tilde{T}_s (T - \tilde{T}) - \\ &\quad - \int (1 - \nu^2) \tilde{T}_s (T_s - \tilde{T}_s) - \frac{1}{2} \int g\kappa^2 (T - \tilde{T})^2 - \frac{1}{2} \int \partial_s(\nu^2) \tilde{T}_s (T - \tilde{T}) \leq - \int g(T_s - \tilde{T}_s)^2 - \\ &\quad - \int (g - 1) \tilde{T}_s (T_s - \tilde{T}_s) - \int (1 - \nu^2) \tilde{T}_s (T_s - \tilde{T}_s) + \frac{1}{2} \int \partial_s(\nu^2) \tilde{T}_s (T - \tilde{T}) \leq \\ &\leq \int (\nu^2 - g) \tilde{T}_s (T_s - \tilde{T}_s) + \frac{1}{2} \int \partial_s(\nu^2) \tilde{T}_s (T - \tilde{T}) = - \int (\nu^2 - g) \tilde{T}_s^2 + \\ &\quad + \int (\nu^2 - g) \tilde{T}_s T_s + \frac{1}{2} \int \partial_s(\nu^2) \tilde{T}_s (T - \tilde{T}) \leq \int |\nu^2 - g| \tilde{T}_s^2 + \int (\nu^2 - g) \tilde{T}_s T_s + \\ &\quad + \frac{1}{2} \int \partial_s(\nu^2) \tilde{T}_s (T - \tilde{T}) \leq \int |\nu^2 - g| \tilde{T}_s^2 + \int (\nu^2 - g) \tilde{T}_s T_s + \frac{1}{4} P_2(t) \int \tilde{T}_s^2 + \frac{1}{4} \eta \end{aligned}$$

The last inequality follows from Cauchy's inequality combined with Lemma 2.3.3, provided that $P_2(t)$ denotes the polynomial of degree 2 given by the bound on $(\partial_s(\nu^2))^2 = (\partial_u(\nu^2)/\nu)^2$. Since s is the arc length, we have that $T_s = \kappa T^\perp$, where κ is the curvature of the underlying solution to RCF. Thus, by Proposition 2.3.8, its L^2 -norm is bounded by $\|T_s\|_{L^2}^2 = \|\kappa\|_{L^2}^2 \leq \|\kappa(s, 0)\|_{L^2}^2 = \|\kappa_0\|_{L^2}^2$. Applying Cauchy-Schwarz to $\int(\nu^2 - g)\tilde{T}_s T_s$, it follows that η_t fulfills:

$$\begin{aligned} \eta_t &\leq \int |\nu^2 - g| \tilde{T}_s^2 + \|(\nu^2 - g)\tilde{T}_s\|_{L^2} \|T_s\|_{L^2} \leq \\ &\leq K \|\tilde{T}_s\|_{L^2}^2 + K \|\kappa_0\|_{L^2} \|\tilde{T}_s\|_{L^2} + \frac{1}{4} P_2(t) \|\tilde{T}_s\|_{L^2}^2 + \frac{1}{4} \eta \end{aligned}$$

where K is a bound on $(g - \nu^2)^2$. Further, because equation (2.19) is of elliptic type with initial condition $T(s, t_0)$ it can be shown [23] that:

$$\nu_{min}^2 \max_t \|\tilde{T}_s\|_{L^2}^2 \leq \max_t \|g\kappa^2\|_{L^1}^2 \leq \epsilon^2$$

These considerations yield that $\eta_t \leq \epsilon(C_1 \|\kappa_0\|_{L^2}^2 + C_2 \epsilon + C_3 \epsilon P_2(t)) + C_4 \eta$ for suitable constants $C_1 \dots C_4$. Finally, applying Gronwall's Lemma we obtain that η is bounded by:

$$\begin{aligned} \eta(t) &\leq e^{C_4 t} \left(\eta(0) + \epsilon \int_0^t C_1 \|\kappa_0\|_{L^2}^2 + C_2 \epsilon + C_3 \epsilon P_2(t) dt \right) \leq \\ &\leq \epsilon T (C_1 \|\kappa_0\|_{L^2}^2 + C_2 \epsilon + C_3 \epsilon P_2(T)) e^{C_4 T} \end{aligned}$$

That is, it exists a time t_0 ensuring that solutions to (2.18) and (2.20) with initial condition $T(s, t_0)$ differ less than ϵ in L^2 , for finite times $0 \leq t \leq T$. Finally, Proposition 2.3.10 ensures that solutions to (2.16) and (2.17) computed over the parameter u also converge to each other in the latter sense since:

$$\eta(s, t) = \int (T(s, t) - \tilde{T}(s, t))^2 ds = \int (T(u, t) - \tilde{T}(u, t))^2 \nu(u, t) du \geq \nu_{min} \eta(u, t)$$

This asymptotic behavior, yields the following:

$$\|\tilde{T}_\rho - T\|_{L^2} = \|\tilde{T}(\rho) - T(t)\|_{L^2} \leq \|\tilde{T}(\rho) - T(\rho + t)\|_{L^2} + \|T(\rho + t) - T(t)\|_{L^2} \xrightarrow{t \rightarrow \infty} 0$$

The first summand converges to zero because $T(\rho + t)$ is the solution to (2.18) for initial condition $T_0 = T(s, t)$ at time $t = \rho$. Hence, $\|\tilde{T}(\rho) - T(\rho + t)\|_{L^2} = \eta(\rho) \leq \epsilon$, if t is large enough. The second, because solutions to equation (2.18) are Cauchy, as they converge to a curve in time. \square

2.3.5 Error in RCF Level Sets Formulation

Let us deduce the formula for the error in RCF level sets formulation given in Section 2.2.1. We recall the reader that the difference between the two dimensional structure tensor, \tilde{J} , and the exact value, J , computed along the level line was given in terms of

the curve curvature, κ_0 , and the curvature, κ_{Im} , of the image gradient integral curves by the formula:

$$|J_\rho - \tilde{J}_\rho| \leq \epsilon \left(C \int |\kappa_0| + \int |\kappa_{Im}| \right) |\vec{t}_0 \otimes \vec{n}_0| + \mathcal{O}(\epsilon^2) \quad (2.22)$$

where ϵ is the size of the window used to compute 2D-convolutions. For the sake of notational simplicity the scale ρ will be dropped, the projection matrix onto a vector v will be noted by P_v and convolutions will be evaluated at the origin. In a level sets formulation, the parameterization of a curve is the implicit one. Since this coordinate change is local, formula (2.22) only holds for bounded domains of integration. We recall that this is the case in the discrete numeric implementation, so that there is no loss of generality in assuming that the x -coordinate is the parameter and $\gamma_0(x) = (x, y_0(x))$.

For each image point (x, y) , denote by $(\bar{x}, y_0(\bar{x})) = (\bar{x}, \bar{y})$ the point achieving the distance, δ , from the image pixel to γ_0 . Notice that it coincides with the pixel projection onto γ_0 along the normal direction $\vec{n}_0(\bar{x})$. The Taylor development of P_n at (\bar{x}, \bar{y}) yields that:

$$\begin{aligned} P_n(x, y) &= P_n(\bar{x}, \bar{y}) + \langle \nabla P_n, (x - \bar{x}, y - \bar{y}) \rangle + \mathcal{O}(\delta^2) = P_n(\bar{x}, \bar{y}) + \delta \langle \nabla P_n, n_0(\bar{x}) \rangle + \\ &+ \mathcal{O}(\delta^2) = P_n(\bar{x}, \bar{y}) + \delta (\partial_{n_0} P_n)|_{x=\bar{x}} + \mathcal{O}(\delta^2) \end{aligned} \quad (2.23)$$

Trigonometric arguments (see fig.2.5 (b) for a graphical representation) yield that:

$$|\bar{x} - x| = \delta \sin(\theta(\bar{x})) \quad \Rightarrow \quad \bar{x} = x \pm \delta \sin(\theta(\bar{x}))$$

Using the Taylor development of $\sin(\theta(\bar{x}))$ and $\cos(\theta(\bar{x}))$ it follows that:

$$\bar{x} = x + \delta [\sin(\theta(x)) + \cos(\theta(x))\theta_x(x)(x - \bar{x}) + \mathcal{O}((x - \bar{x})^2)]$$

Further, since $|x - \bar{x}| = \delta \sin(\theta(\bar{x})) = \mathcal{O}(\delta)$, we conclude that the change of coordinates \bar{x} is given by:

$$\bar{x} = x + \delta \sin(\theta(x)) + \mathcal{O}(\delta^2) = x + \delta \sin(\theta) + \mathcal{O}(\delta^2)$$

This bounds the first summand of (2.23) as:

$$P_n(\bar{x}, \bar{y}) = P_{n_0}(\bar{x}) = P_{n_0}(x + \delta \sin(\theta) + \mathcal{O}(\delta^2)) = P_{n_0}(x) + \delta \sin(\theta) \partial_x(P_{n_0}) + \mathcal{O}(\delta^2) \quad (2.24)$$

Taking into account that in an implicit parameterization the norm of the curve first derivative equals $|\gamma'_0|^2 = 1 + (y'_0)^2 = 1/\cos(\theta)^2$, we have that the partial with respect to the arc length, s , is given by:

$$\partial_s = \cos(\theta) \partial_x$$

Then by the Frenet formulae, it follows that the first derivative in equation (2.24) equals:

$$\sin(\theta) \partial_x(P_{n_0}) = \tan(\theta) \partial_s(P_{n_0}) = \tan(\theta) \partial_s(\vec{n}_0 \otimes \vec{n}_0) = -\kappa_0 \tan(\theta) (\vec{t}_0 \otimes \vec{n}_0)$$

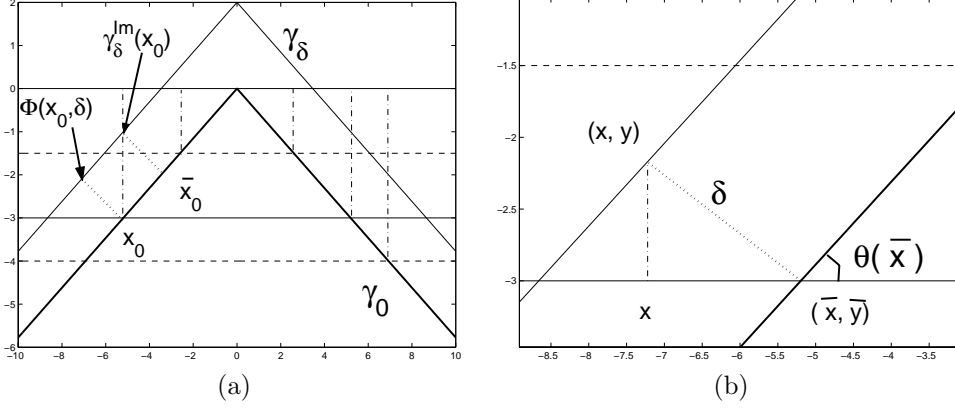


Figure 2.5: Tube parameterization versus implicit (a) and coordinate change (b).

For the second term of (2.23), because ∂_n corresponds to the unit tangent of the flow lines, again the Frenet yield:

$$(\partial_{n_0} P_n)|_{x=\bar{x}} = (\partial_n P_n)|_{x=\bar{x}} = \partial_n(n \otimes n)|_{x=\bar{x}} = \kappa_{Im}(\bar{x})(t_0 \otimes n_0) = \kappa_{Im}(x)(t_0 \otimes n_0) + \mathcal{O}(\delta) \quad (2.25)$$

where κ_{Im} stands for the flow lines curvature. Blending together (2.24) and (2.25), we conclude that the tensor \tilde{J} computed with a 2-D gaussian kernel G defined in a window of size ϵ equals:

$$\begin{aligned} \tilde{J} &= \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} G(x, y) P_n(x, y) = \int \int G(x, y) P_{n_0}(x) dx dy + \\ &+ \delta \int \int G(x, y) \kappa_{Im}(x) (t_0(x) \otimes n_0(x)) dx dy - \kappa_0(x) \tan(\theta(x)) (t_0(x) \otimes n_0(x)) dx dy + \\ &+ \mathcal{O}(\delta^2) = J + \delta \int_{-\epsilon}^{\epsilon} G(x) (\kappa_{Im} - \kappa_0 \tan(\theta)) (t_0 \otimes n_0) dx + \mathcal{O}(\epsilon^2) \end{aligned}$$

which proves the formula for the error because:

$$\begin{aligned} |\tilde{J} - J| &\leq \epsilon \int (|\kappa_{Im}| + |\kappa_0 \tan(\theta)|) \|t_0 \otimes n_0\| + \mathcal{O}(\epsilon^2) = \\ &= \epsilon \|t_0 \otimes n_0\| \left(\int |\kappa_{Im}| + C \int |\kappa_0| \right) + \mathcal{O}(\epsilon^2) \end{aligned}$$

provided that the angle of the unit tangent, θ does not turn around more than π in a neighborhood of each point in the target curve, so that we can ensure that $|\tan(\theta)|$ is bounded by a constant C .

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In this chapter, we have defined a novel geometric flow which is well suited for image filtering and shape recovery tasks. The latter are the usual applications of geometric flows. Still, because they model shape propagation/deformation, geometric flows and, in particular, curvature based ones can be used in image segmentation and shape modelling. In the next section we will use the mean curvature flow to design a new external energy for snakes that ensures their convergence, independently of the particular geometric features (concavity) of the contour to be approached.

Chapter 3

Curvature Vector Flow. Heading towards an efficient Shape Modeling

Shape modelling arises in many fields of computer vision and graphics [35], [31], [82], to mention just a few. The most efficient way of producing smooth models of shapes is by means of a snake [38], [11], [12]. Snakes are curves that minimize an energy functional by means of its gradient descent. The definition of the external energy characterizing the target object is crucial for a successful model of the shape. In spite of the improvement of the external energy given by the gradient vector flow technique [75] or [16] area minimizing balloon forces, highly non convex shapes can not be obtained, yet.

A study of distance maps from a geometric point of view clearly shows the limitations of the current external forces. Standard tools of calculus of variations link convergence to the minimum of the external potential to differentiability of its level sets. A back-tracking of the evolution of the contour of level zero by minimum curvature flow [49] is the natural geometric way of converging to non-convex regions. Since the PDE associated to this evolution is of elliptic type, we can assure that propagation of non convex regions will not develop shocks during the process. In this manner we build a distance map having the contour of level zero as unique local minimum. The gradient of the map, called Curvature Vector Flow, is capable of attracting any initial interior or exterior curve towards the contour of interest, independently of its geometric features.

3.1 Snake Convergence and Smoothness of the External Energy

A common feature to all existent snake techniques is that they deform until a minimum of an energy functional is achieved. Since cancellation of the first variation of this functional (Euler-Lagrange equation) is the criterion used to characterize local minima, snakes are ruled by calculus of variations principles. In this section we will use standard results of variational techniques to establish conditions on the external

energy that prevent snakes reaching concave regions. Such conditions are in terms of the convexity/ellipticity [23] of the functional second variation. Further, we will relate, in the case of parametric snakes, the former analytical requirements to the geometry of the external energy level sets. This will provide us with a criterion in terms of existence of singular points of the level curves easier to verify than the convexity condition.

3.1.1 Convexity and Smoothness of Level Sets

Before giving the main result of the section concerning convexity, let us establish what snake functionals ensure convexity exclusively by means of the external energy term. We must treat separately the case of parametric and geodesic snakes.

We recall the reader that the functional associated to a parametric snake [38], γ , splits energy into the following two terms:

$$\begin{aligned} E(\gamma, \dot{\gamma}, \ddot{\gamma}) &= E_{int} + E_{ext} = E_1 + E_2 + E_{ext} = \int_{\gamma} \alpha |\dot{\gamma}|^2 + \int_{\gamma} \beta |\ddot{\gamma}|^2 + E_{ext} = \\ &= \int_{\gamma} \alpha |\dot{\gamma}|^2 + \int_{\gamma} \beta |\ddot{\gamma}|^2 + \int_{\gamma} F(\gamma) du = \int_{\gamma} L^1(\dot{\gamma}) du + \int_{\gamma} L^2(\ddot{\gamma}) du + \int_{\gamma} F(\gamma) du = \\ &= \int L^1(p) du + \int L^2(P) du + \int F(z) du \end{aligned}$$

where the potential $F = F(x, y)$ is positive, depends on the object to be detected and, in the ideal case, should cancel on curves segmenting the target objects. Our first remark is that convexity of this functional relies exclusively on the external energy convexity. This fact follows from the convexity of the snake internal energy:

$$\begin{aligned} E_1''(q, q) &= \int q^t L_{pp}^1 q du = \int q^t \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} q du \\ E_2''(Q, Q) &= \int Q^t L_{PP}^2 Q du = \int Q^t \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} Q du \end{aligned}$$

Therefore convexity of E is ensured provided that for any curve ν the following inequality holds:

$$E_{ext}''(\nu, \nu) = \int \nu^t F_{zz} \nu du \geq 0$$

That is, the snake energy is convex if and only if the Hessian of the potential F is positive defined. This situation significantly differs from the convexity conditions of the functional guiding geodesic snakes. In this case, the functional to be minimized [11], [12] fuses the external potential F and the length minimizing term, L^1 :

$$E_{geod}(\gamma) = \int F(\gamma) \sqrt{\dot{x}^2 + \dot{y}^2} du = \int F(\gamma) |\dot{\gamma}| du = \int FL^1 du$$

This fact difficulties studying convexity of E_{geod} by means of the convexity of each of the factors, since its first variation is:

$$E_{geod}' = \int \langle F_z, \nu \rangle \cdot L^1 du + \int F \cdot \langle L_p^1, \dot{\nu} \rangle du$$

and, differentiating the above expression, its second equals:

$$E''(\nu, \nu) = \int (\dot{\nu}^t F_{pp} \dot{\nu}) L^1 du + \int (\nu^t L_{zz}^1 \nu) F du + 2 \int \langle F_z, \nu \rangle \langle L_p^1, \dot{\nu} \rangle du$$

Hence even in the case that both F and L_1 were convex, this will not suffice to guarantee convexity of E_{geod} as the second variation involves F and L^1 first derivatives. This constitutes a main inconvenience and motivates using the classic parametric snakes formulation and search for a computational robust way of detecting non-convex external potentials.

Convexity is a property not easy to detect analytically in a robust way as it requires precise high (second) order derivatives. In this Section we give a simple convexity criterion in terms of the geometry of the external energy level sets. Notice that because we are interested in avoiding false minima it suffices to guarantee that F is convex at all points in the image such that $\nabla F = 0$, that is, that there are not saddle points.

Theorem 3.1.1 *A function $F(x, y)$ has no saddle points such that $\nabla F = 0$ if and only if F level sets are smooth curves.*

Proof. Let us assume that there exists a point p such that $\nabla F(p) = 0$ and the Hessian:

$$HessF(p) = \begin{pmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{pmatrix}_p$$

is not positive defined. We will prove that this holds if and only if the level curve through p , namely γ , is singular at the point p . Via a translation we can take $p = (0, 0)$ and $F(0, 0) = 0$. Under the latter assumptions, we have that any function E can be written as:

$$\begin{aligned} E(x, y) &= \int_0^1 \frac{dE(tx, ty)}{dt} dt = x \int_0^1 E_x(tx, ty) dt + y \int_0^1 E_y(tx, ty) dt = \\ &= xf(x, y) + yg(x, y) \end{aligned}$$

Applying the above decomposition to the external energy F , we have that:

$$\begin{aligned} F(x, y) &= x^2 \int_0^1 F_{xx}(tx, ty) dt + 2xy \int_0^1 F_{xy}(tx, ty) dt + y^2 \int_0^1 F_{yy}(tx, ty) dt = \\ &= a(x, y)x^2 + 2b(x, y)xy + c(x, y)y^2 \end{aligned} \quad (3.1)$$

Furthermore, by differentiating the above equality with respect to (x, y) , we have that $a(0, 0) = E_{xx}(0, 0)$, $2b(0, 0) = E_{xy}(0, 0)$ and $c(0, 0) = E_{yy}(0, 0)$. The origin $p = (0, 0)$ is a saddle point, if and only if the determinant $E_{xx}(0, 0) * E_{yy}(0, 0) - E_{xy}^2(0, 0) = ac - 4b^2$ is negative, which implies that the curve given by:

$$0 = a(x, y)x^2 + 2b(x, y)xy + c(x, y)y^2 \quad (3.2)$$

is singular at the origin in the algebraic sense (i.e. there are more than one tangent lines). This follows because if $y = \lambda x$ is a line through the origin, then the polynomial in λ :

$$0 = a(x, y)x^2 + 2b(x, y)\lambda x + c(x, y)\lambda^2 x^2 = x^2(a(x, y) + 2b(x, y)\lambda + c(x, y)\lambda^2)$$

has real roots for $(x, y) = (0, 0)$. This values of λ correspond to the slopes of a line that cuts the curve given by (3.2) at the origin with contact order at least two. This proves the result since the curve of level zero of F is given by equation (3.2). \square

In figure 3.1 we have an illustrative example of the above theorem. The level set (fig.3.1 (a)) through the saddle point inside the red square of fig.3.1 (a) presents a singularity located at the latter, meanwhile the remaining level curves are smooth.

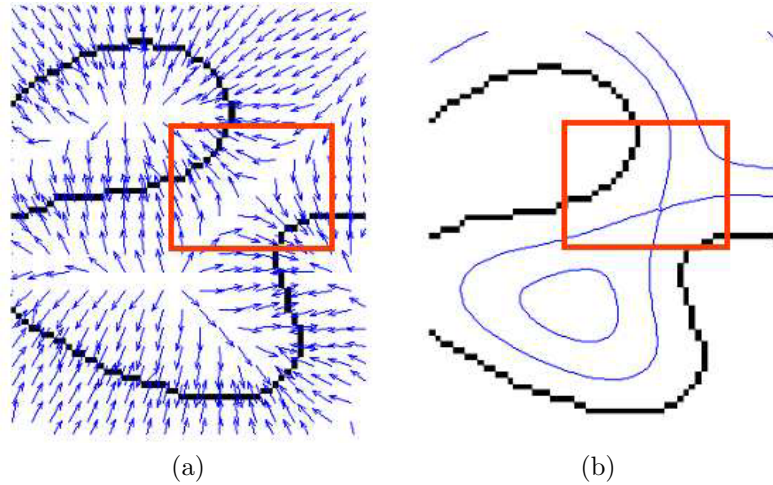


Figure 3.1: Inflexion point (a) and Singular Curve (b)

The above theorem is a valuable tool to analyze whether an external force guarantees snake convergence. We will use the above theorem to characterize, for each external energy, its particular pathology and define a new external potential ensuring convergence whatever the geometry of the target curve.

3.2 Shape Propagation

Distance maps encode the evolution of the curve of level zero, γ_0 , under a geometric flow defined, generically, by a parabolic PDE:

$$\gamma_t(u, t) = \beta(u, t) \vec{n}. \quad (3.3)$$

with initial condition $\gamma(u, 0) = \gamma_0(u)$ a closed curve and \vec{n} denoting the unit inward normal. Each level curve of a given distance, d , corresponds to the solution to (3.3) at time $t = d$. This point of view, reduces the study of distance maps to the analysis of the propagation of the zero level curve governed by means of a geometric flow. We will use the machinery developed in [24] in order to study the drawbacks of the Euclidean distance map and define a more natural way of propagating shapes that will produce distance maps capable of guiding a snake to any closed curve. Since a plane curve is defined, up to rigid transformations, by its unit tangent, a pleasant way of handling geometric flows is by means of the equation of the angle of the unit

tangent, θ , in the arc length, s , parameterization. The advantage of this formulation is that we reduce the study of the properties of (3.3) to the analysis of a single equation, so that standard results on PDE's can be applied. The parabolic PDE for θ when the curve solves (3.3) is given by:

$$\theta_t(s, t) = \partial_s(\beta) + \left(\int_0^s \beta \theta_s ds \right) \theta_s . \quad (3.4)$$

with initial condition the angle of the unit tangent, θ_0 , of the initial curve. An important remark is that the first order term arises due to the change of parameter, and, hence, it is present in any geometric flow.

3.2.1 Euclidean Distance Maps

In Euclidean distance maps, propagation of the initial curve, γ_0 , is equivalent to mathematical morphology with a circle of radius 1 as structural element. Erosion corresponds to the inward propagation and dilation to the outward one. Hence the geometric flow associated [1] [64] is given by:

$$\gamma_t = \pm \vec{n} .$$

The minus sign corresponds to the dilation and the plus to the erosion. Since, in this case, $\beta = \pm 1$ is constant, the corresponding equation (3.4) for θ is simply:

$$\theta_t(s, t) = \pm \left(\int_0^s \theta_s ds \right) \theta_s . \quad (3.5)$$

This equation is a first order non-linear PDE that is solved [23] by means of the computation of the characteristic curves, that is, those curves in the s - t plane, $\alpha(u)$, such that the function solving (3.5) keeps constant, that is $\theta(\alpha(u)) \equiv \text{const}$. Assuming this last condition for $\alpha(u) = (s(u), t(u))$, we obtain that:

$$0 = \frac{d}{du}(\theta(s(u), t(u))) = t_u \theta_t + s_u \theta_s = t_u \left(\pm \left(\int_0^s \theta_s ds \right) \theta_s \right) + s_u \theta_s = (\pm t_u \theta + s_u) \theta_s .$$

Since the equality holds for all points in the characteristic, we have that α solves the first order PDE:

$$\pm t_u \theta + s_u = 0$$

And, consequently, its tangent vector fulfils the following system of ODE's:

$$\left. \begin{array}{l} t_u = 1 \\ s_u = \pm \theta \end{array} \right\}$$

Therefore, the characteristics through a point $(s_0, 0)$ are parameterized as:

$$\alpha(u) = (\pm \theta_0 u + s_0, u) .$$

where θ_0 equals $\theta_0(s_0, 0)$ and is constant along the characteristics.

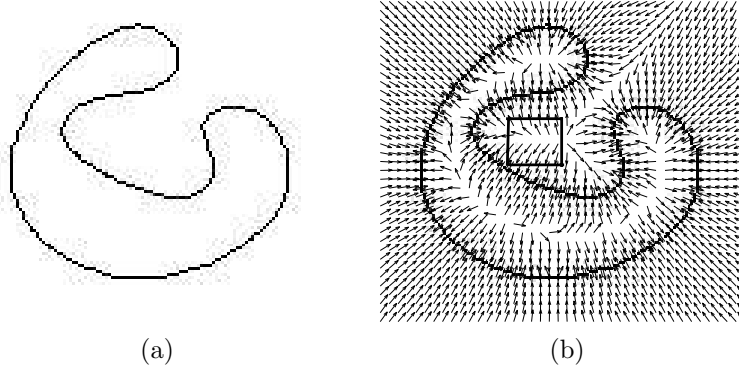


Figure 3.2: Highly non-convex curve (a) and gradient of Euclidean distance map (b).

This means that we have straight lines in a plane which slopes, $\theta_0(s_0)$, do not need to be a monotonous function of the curve parameter. Variation of the characteristics slope along the initial curve is given by the derivative of the angle θ'_0 . Since the initial curve is parameterized by its arc length, we have that the sign of the curvature of the initial curve determines whether the slopes increase or decrease. For convex curves, characteristics slope are either increasing (inward propagation) or decreasing (outward propagation) along the curve. Hence, two different characteristics never cross during the curve propagation and the distance map is a smooth map. However, for non-convex shapes, changes in the monotonicity of the slopes induced by the curve inflexion points make characteristics meet each other in finite positive time (squared region of fig. 3.2(b)). At this time, the evolution of the angle develops a discontinuity or shock and the corresponding curve is not smooth any more. Indeed shocks in the angle domain translate into points or, even, curves where the gradient of the distance map cancels, that is, they correspond to crests and valleys of the distance map. Although this property is used in computer vision for extraction of shape skeletons [66], [69], it constitutes a main hindrance for shape modelling with snakes. Highly non-convex shapes (see fig. 3.2(a)) with the angle turning around more than π between two consecutive inflexion points produce distance maps with crests of positive slope (fig. 3.4(a)). These crests and valleys induce local minima in the snake energy functional that our deformable model, which seeks for zeros of the energy gradient, will never cross.

The best approach up to our knowledge to overcome the null gradient problem along curves is by means of the use of a regularized gradient (that only cancels at isolated points) as external force. Such regularization is obtained by means of the GVF [75] or its generalized faster version GGVF [77].

3.2.2 Gradient Vector Flow and Saddle Points

The GVF/GGVF technique [75], [77] consists in substituting the gradient of the external energy, ∇E_e , by the vector field $v(x)$ that is the steady-state of the reaction-diffusion vector equation:

$$u_t = g(|\nabla E_e|)\Delta u - h(|\nabla E_e|)(u - \nabla E_e) \quad \text{with } u(x, 0) = \nabla E_e \quad (3.6)$$

The weighting functions, $g(\cdot)$ and $h(\cdot)$ are monotonically non-increasing and non-decreasing functions of the norm $|\nabla E_e|$, respectively. An important remark is that $h(0) = 0$. In this manner, the equilibrium vector field smoothly extends ∇E_e , thanks to the Laplacian, keeping close to the original gradient when the latter is significant enough. It can be used either to extend the edge map to the whole image or to regularize a gradient of a distance map. The main difference between GVF and GGVF is that the latter field keeps enough force as to drive the snake until the edge.

Notice that, in any of the two cases (GVF or GGVF), at parts of null gradient the equilibrium point is an harmonic function. Harmonic functions [61] do not admit accumulation of zeros and, thus, our vector field v will only have isolated points with $|v| = 0$. This important feature solves the problem of the distance map null gradient along curves (fig. 3.3 (b)). However, in both cases the geometry of the contours introduces saddle points in the vector field v , as the close up in fig. 3.3 (c) illustrates. These false minima of the snake energy trap, once again, the snake and prevent the deformable contour from entering into concave regions where the angle of the unit tangent, θ , turns around more than π .

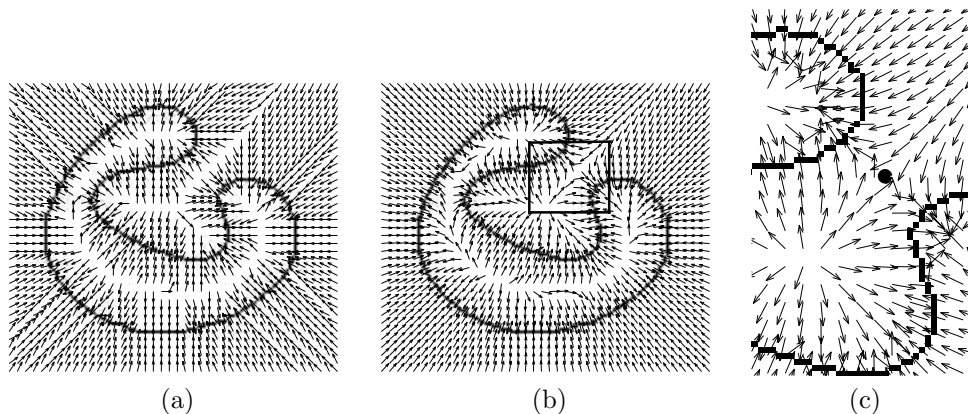


Figure 3.3: Gradient of Euclidean distance map to non-convex curve (a), GGVF (b) and saddle point of GGVF (c).

For regularization of gradients of Euclidean distance maps, saddle points appear because of shock formation during the propagation of the curve of level zero. In the case of extension of image gradients we find a similar problem. The Laplacian is an isotropic linear constant way of extending information. Therefore in every image region such that the contour/edge of interest is concave and the angle of the image

gradient (parallel to the unit normal of the curve) turns more than π , two opposite directions meet (fig. 3.3 (c)) and we have a saddle point.

In order to eliminate saddle points we need changing the propagation of shapes, so that the geometry of the contour to be extended is taken into account.

3.2.3 Curvature Vector Flow

Let us analyze the problem of the Euclidean distance map. From an analytic point of view, we see that equation (3.5) is a non-linear first order PDE, prone to develop shocks during the evolution. From a geometric one, it lies on the fact that we are propagating the curve at constant speed whatever its geometric features. In other words, the structural element used in the mathematical morphology is a circle of constant radius, which means that all points in the curve travel equal distance at the same time. We argue that the evolution should consider different metrics depending on the local geometry around each point so that structural circular elements of non constant radius are used. And what characterizes the local geometry of a curve better than curvature?. We propose a distance map based on the Mean Curvature Flow (MCF), that is, the evolution of the curve of level zero under the PDE given by:

$$\gamma_t = \kappa \vec{n} . \quad (3.7)$$

This equation makes points on the curve travel a distance that depends on the magnitude of the curvature, the higher its absolute value, the faster and further the point will move. From the mathematical morphology point of view we make the radius of the structural circle depend on the absolute value of the curvature. Now, can we assure, analytically speaking, that our evolution will stay smooth for all times?. On one hand we have that the equation (3.5) associated to θ in arc length parameter has turned into a PDE of parabolic elliptic type:

$$\theta_t = \theta_{ss} + \left(\int_0^s \theta_s^2 ds \right) \theta_s . \quad (3.8)$$

Hence by general theory on PDE's [23], we already know that our Curvature Distance Map (CDM) will be infinitely differentiable. Intuitively, the Laplacian that equation (3.8) contains introduces curvature into the characteristic lines, so that two characteristics do not intersect any more. Besides, the large amount of literature [32], [33], [34] on MCF, states that any initial curve evolves smoothly towards a convex shape, circular in the limit, before collapsing to a point. This is the key point to the definition of CDM.

Evolution by negative curvature

$$\gamma_t = \min((\kappa, 0)) \vec{n}$$

Evolution by positive curvature

$$\gamma_t = \max((\kappa, 0)) \vec{n}$$

(3.9)

We will define separately the outward and inward propagation in order to ensure maximal accuracy in the position of the snake. The analysis of Section 3.2.1 points out that convex shapes do not develop shocks during their propagation. An evolution of a non convex shape by negative curvature [49] stops as soon as the curve becomes

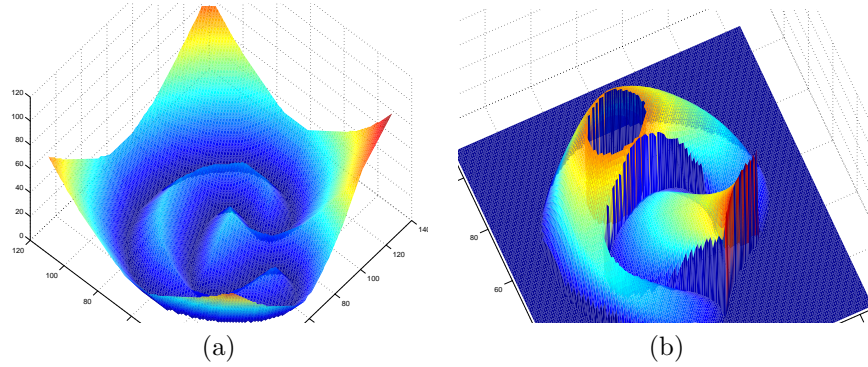


Figure 3.4: Detail of Euclidean distance map, showing a crest of positive slope (a) and detail of Elliptic Distance Map (b) .

convex. Therefore, for the outward propagation, we will evolve the initial shape under the flow given by (3.9) until it stabilizes. The tracking of the curve for each time produces the level sets of the outward CDM for the non convex regions. To complete the outward CDM, we use an outward Euclidean distance map to the stable state of the flow by negative curvature. For the inward propagation we use evolution under positive curvature (3.9) until the curve becomes circular and then we use the Euclidean distance map to this circle to complete the inward propagation.

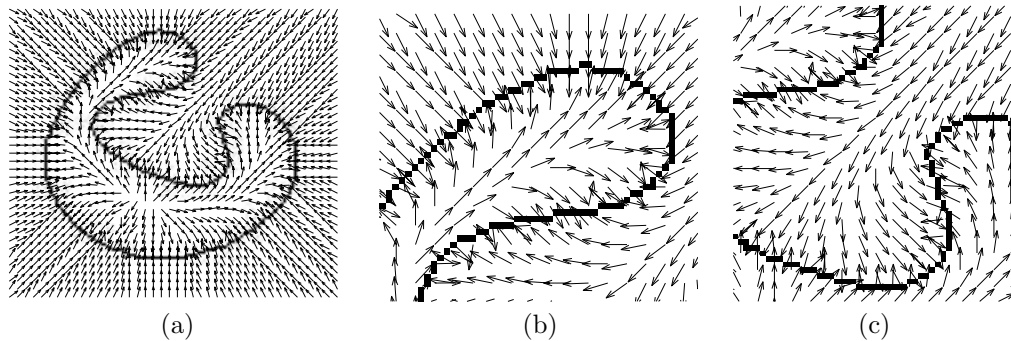


Figure 3.5: Gradient of the Elliptic Distance Map (a) and close-ups from the interior (b) and exterior (c).

Since there are not any self intersections between the level curves of CDM, by virtue of Theorem 3.1.1 we obtain maps (fig. 3.4 (b)) without curves of null gradient or saddle points (fig. 3.5 (a)). The gradient of this map, CVF, drives the snake to the zero level curve of CDM whatever its geometry. Details of CVF shown in figure 3.5 illustrate the absence of either saddle points or null gradients in both, the interior (fig. 3.5 (b)) and exterior (fig. 3.5 (c)) regions of a highly non-convex curve.

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The vector flow introduced in this chapter is an ideal tool for shape modelling, as far as shape contours are closed curves. Unfortunately, curves describing image main features, such as edges and ridges, are prone to be uncompleted, which difficulties applying CVF to object segmentation. It follows that in order to successfully use CVF in image segmentation, we need to be able to produce reliable closed models of unconnected sets of points. Once again, we will make use of the geometry of image level sets to define a new family of diffusion operators that uniformize the gray level of image characteristic curves, when applied to a gray level image, and complete unconnected curves if applied to their mask image.