3.5 Some eta invariants for lens spaces

3.5.1 The definitions

Consider the disk D^{4k} (and hence the sphere S^{4k-1}) endowed with the usual canonical Euclidean metrics and its associated Levi-Cività connection. Let C_p be the cyclic group of the p-th roots of unity, p odd prime. Consider a set $(q_1, \ldots, q_{2k})$ of integer numbers q_i, each coprime with p. We know that in this situation, one may define an action on $\mathbb{C}^{2k} \cong \mathbb{R}^{4k}$, for which both the disk D^{4k} and its boundary S^{4k-1} are invariant, by setting the generator $\zeta_p = e^{\frac{2\pi i}{p}}$ the primitive p root of unity to act on $\mathbb{C}^{2k}$ as

$$(z_1, \ldots, z_{2k}) \xmapsto{\zeta_p} \left(\zeta_p^{q_1} z_1, \ldots, \zeta_p^{q_{2k}} z_{2k}\right),$$

so that this generator rotates the jth copy of $\mathbb{R}^2$ through an angle $\theta_j = 2\pi q_j/p$, so that the action is by orientation-preserving isometries. Hence, one forms the lens space $L = L_p^{4k-1}(q_1, \ldots, q_{2k}) = S^{4k-1}/G$, which is a manifold which inherits metrics and connection from S^{4k-1} with the projection being a Riemannian submersion. Let $\eta_{B_{\xi|_L}}(s)$ denote the eta function for the signature operator $B_{\xi|_L}$ with respect to the Riemannian connection operator twisted with the bundle

$$\xi_{q|_L} = R_q(TL) = \otimes_{n=1}^{\infty} S_{q^n}(T_{\mathbf{C}}(L)) \otimes \otimes_{n=1}^{\infty} \Lambda_{q^n}(T_{\mathbf{C}}(L)).$$

Define the eta invariant of this operator by

$$\eta_{B_{\xi|_L}}(0) = \eta_{B_{\xi|_L}}(s) = \sum_{\lambda \neq 0,\ \lambda \in \mathrm{Spec}(D)} \frac{\mathrm{sign}(\lambda)}{|\lambda|^s}.$$

Denote, as in [HBJ92, pp. 26, 147],

$$\varphi(\tau, x) = (\wp(\tau, x) - e_1(\tau))^{\frac{1}{2}} = \frac{1}{2i} \frac{\theta'(\tau, 0)}{\theta_1(\tau, 0)} \frac{\theta_1\left(\tau, \frac{x}{2\pi i}\right)}{\theta\left(\tau, \frac{x}{2\pi i}\right)}.$$

3.5.2 The proposition

Proposition 3.32 *In the situation just described, one has*

$$\eta_{\alpha_s}\left(0, L_p^{4k-1}(q_1, \ldots, q_{2k}), \xi_q\right) = \frac{1}{p} \sum_{r=1}^{p-1} \left(\prod_{j=1}^{2k} \varphi\left(\tau, i\frac{2\pi q_j r}{p}\right)\right) \frac{1}{\varepsilon(\tau)^{\frac{k}{2}}} \chi_{\alpha_s}\left(e^{\frac{2\pi r}{p}}\right).$$

3.5.3 First part of the proof

The idea of the proof is just follow the procedure of [APS75II, Theorem 2.12]. Apply to our setting the Lefschetz–Atiyah–Bott formula for fixed points of the action of any nontrivial element in the group. We are then exactly in the situation described above, including the following remarks:

Remark 3.33 In the notation of [AB68], $G = C_p$ acts freely on $\widetilde{Y} = S^{4k-1}$; $Y = L_p^{4k-1}$ is a manifold; the projection is a Riemannian submersion; Y inherits a metric and a (Riemannian) connection from S^{4k-1}; $X = D^{4k}/C_p$; and, for all $g \in G - \{1\}$, $\widetilde{X} = D^{4k}$ has only a fixed point (the origin), $\widetilde{X}^g = \left(D^{4k}\right)^g = \{(0, 0, \ldots, 0)\}$.

Hence the invariant defined in [AS68III, p. 586ff], the "signature defect"

$$\sigma_g\left(\widetilde{Y}\right) = \sigma\left(g, \widetilde{Y}\right) = \mathbf{L}\left(g, \widetilde{X}\right) - \operatorname{sign}\left(g, \widetilde{X}\right)$$

(compare [Don78, p. 899]), where $\mathbf{L}\left(g, \widetilde{X}\right) = \int_{\widetilde{X}^g} \mathcal{L}\left(T\widetilde{X}, g\right)$, and in our case, we have for $g \neq 1, \operatorname{sign}\left(g, \widetilde{X}\right) = 0$, since it is the G-signature of the quadratic form induced by the cup product for $H^{2k}\left(D^{4k}\right) = 0$, and $\mathbf{L}\left(g, \widetilde{X}\right) = \int_{\widetilde{X}^g} \mathcal{L}\left(T\widetilde{X}, g\right) = L\left(T\widetilde{X}, g\right)(0)$.That was the classical case.

In the case of the disk, we will know that $\operatorname{sign}\left(g, \widetilde{X}, \widetilde{\xi}_q\right) = 0$ whenever $g \neq 1$, so one obtains

$$\eta_{\varepsilon,g}\left(0, \widetilde{Y}\right) = \sigma_{g,\xi_q}\left(\widetilde{Y}\right) = \int_{\widetilde{X}^g} \Phi_\varepsilon\left(T\widetilde{X}, g\right),$$

$$\begin{aligned}
\sigma_{g,\widetilde{\xi}_q}\left(S^{4k-1}\right) &= \sigma\left(g, S^{4k-1}, \widetilde{\xi}_q\right) = \Phi_\varepsilon\left(g, D^{4k}\right) - \operatorname{sign}\left(g, D^{4k}, \widetilde{\xi}_q\right) \\
&= \int_{\left(D^{4k}\right)^g} \Phi_\varepsilon\left(TD^{4k}, g\right) = \Phi_\varepsilon\left(TD^{4k}, g\right)(0) = \left(\prod_{j=1}^{2k} \Phi_{\varepsilon,\theta_j(g)}\left(TD^{4k}\right)\right)(0).
\end{aligned}$$

The classes above for our disk will give (cf. [AB68, LM89, HZ74])

$$\begin{aligned}
\Phi_{\varepsilon,g}\left(TD^{4k}\right) &= \left(\prod_{j=1}^{2k} \frac{1}{f_\varepsilon\left(\tau, i\theta_j\left(g\right)\right)}\right) \frac{1}{\varepsilon\left(\tau\right)^{\frac{n}{4}}} = \\
\frac{\prod_{j=1}^{2k}\left(\wp\left(\tau, i\frac{2\pi k_j}{p}\right) - e_1\left(\tau\right)\right)^{\frac{1}{2}}}{\varepsilon\left(\tau\right)^{\frac{n}{4}}} &= \left(\prod_{j=1}^{2k} \varphi\left(\tau, i\frac{2\pi q_j r}{p}\right)\right) \frac{1}{\varepsilon\left(\tau\right)^{\frac{k}{2}}}.
\end{aligned}$$

3.5.4 The twisted signature complex for D^{4k} and C_p actions

Lemma 3.34 $\operatorname{sign}\left(g, D^{4k}, \xi_q\right) \equiv 0$.

This lemma generalises the classical ones and holds because the flatness of the connection on TD^{4k} considered makes ξ_q a flat bundle in that case as well.

Proof of the lemma

Let C_p be the cyclic group of the p-th roots of unity, p an odd prime. Consider a set $S = (q_1, \dots, q_n)$ of integer numbers q_i, each coprime with p. We know that, in this situation, one may define an action on $\mathbb{C}^n \cong \mathbb{R}^n$, for which both D^{2n}, S^{2n-1} are invariant, by setting the generator $\zeta_p = e^{\frac{2\pi i}{p}}$ the primitive p root of unity to act on $\mathbb{C}^n$ as

$$(z_1, \dots, z_n) \overset{\zeta_p}{\longmapsto} \left(\zeta_p^{q_1} z_1, \dots, \zeta_p^{q_n} z_n\right).$$

So, each power of ζ_p is an isometry with respect to the canonical metrics on $\mathbb{C}^n$, so it makes sense to consider the subgroup inclusion $C_p \hookrightarrow \operatorname{Isom}\left(D^{4k}\right)$. One knows (see e.g. [LM89, p. 211]) that the operator $D^+ \colon \Gamma\left(C\ell^+\left(D^{4k}\right)\right) \to \Gamma\left(C\ell^-\left(D^{4k}\right)\right)$ is an $\operatorname{Isom}\left(D^{4k}\right)$-operator, i.e., for any $g \in \operatorname{Isom}\left(D^{4k}\right)$, it is possible to lift the action of $\operatorname{Isom}\left(D^{4k}\right)$ on D^{4k} to the tangent bundle as its differential dg and then extend it to the bundles associated to it, namely to $C\ell\left(D^{4k}\right)$ and it preserves both the even/odd and plus/minus splittings, and such action is compatible with the D^+ operator, i.e.,

$$g\left(D^+(\varphi)\right) = D^+(g(\varphi)), \quad \text{for all } \varphi \in \Gamma\left(C\ell\left(D^{4k}\right)\right), g \in \operatorname{Isom}\left(D^{4k}\right)$$

and in particular, with the considered inclusion, D^+ becomes a C_p-operator.

Now consider the twisting by the bundles ξ considered in the remark above. In any of the considered cases, we have a left action $g^* \colon \xi \to \xi$ of $g \in C_p$ so that g^* is a morphism of bundles which preserves both the metrics and the connection on ξ. How to define g^* is clear in our main target case, namely, $\xi = W \otimes V$, where W is a finite dimensional bundle associated to TD^{4k} and V is the representation space for some $\rho \colon C_p \to \mathrm{U}(r)$. In our situation, on indecomposable elements we have $D_\xi^+(g(\varphi \otimes \varepsilon)) = g\left(D_\xi^+(\varphi \otimes \varepsilon)\right)$, and in particular we may restrict it to, e.g., kernels of operators.

Since D_ξ^+ is a C_p-operator, for any $g \in C_p$ it makes sense to calculate

$$\operatorname{ind}_g\left(D_\xi^+\right) = \operatorname{tr}\left(g\,|_{\operatorname{Ker}\left(D_\xi^+\right)}\right) - \operatorname{tr}\left(g\,|_{\operatorname{Ker}\left(D_\xi^-\right)}\right),$$

where tr is the trace of the considered operator g restricted to $\operatorname{Ker}\left(D_\xi^\pm\right)$, which are now representation spaces for C_p themselves. This index can hence be considered as a difference of the characters for both representations evaluated on an element g. Recall now that we had seen that

$$\operatorname{Ker}(D_\xi) = \operatorname{Ker}(D) \otimes \xi, \qquad \operatorname{Ker}\left(D_\xi^\pm\right) = \operatorname{Ker}\left(D^\pm\right) \otimes \xi,$$

so

$$\operatorname{tr}\left(g\,|_{\operatorname{Ker}(D)\otimes\xi}\right) = \operatorname{tr}(\rho(g)) = (\operatorname{char}(\rho))(g)$$

for the considered representation

$$\rho \colon G \to \operatorname{Hom}_{\mathbb{C}}(\operatorname{Ker}(D) \otimes \xi, \operatorname{Ker}(D) \otimes \xi)$$

and since $\rho = \rho_1 \otimes \rho_2$, with $\rho_1 \colon G \to \mathrm{Hom}_{\mathbb{C}}(\mathrm{Ker}(D), \mathrm{Ker}(D))$, $\rho_2 \colon G \to \mathrm{Hom}_{\mathbb{C}}(\xi, \xi)$, one has the usual identities in the character ring, $\mathrm{char}(\rho) = \mathrm{char}(\rho_1 \otimes \rho_2) = \mathrm{char}(\rho_1) \cdot \mathrm{char}(\rho_2)$, and

$$\begin{aligned} \mathrm{tr}\left(g\mid_{\mathrm{Ker}(D^{\pm})\otimes\xi}\right) &= \left(\mathrm{char}\left(\rho^{\pm}\right)(g)\right) = \left(\mathrm{char}\left(\rho_1^{\pm}\right)(g)\right) \cdot \left(\mathrm{char}\left(\rho_2\right)(g)\right) \\ &= \mathrm{tr}\left(g\mid_{\mathrm{Ker}(D^{\pm})}\right) \mathrm{tr}\left(g\mid_{\xi}\right). \end{aligned}$$

According to [LM89, p. 265, Remark 14.6], we define

$$\mathrm{sign}\left(g, D^{4k}, \xi\right) = \mathrm{ind}_g\left(D_\xi^+\right)$$

and so

$$\begin{aligned} \mathrm{sign}\left(g, D^{4k}, \xi\right) &= \mathrm{ind}_g\left(D_\xi^+\right) = \mathrm{tr}\left(g\mid_{\mathrm{Ker}(D^+)\otimes\xi}\right) - \mathrm{tr}\left(g\mid_{\mathrm{Ker}(D^-)\otimes\xi}\right) \\ &= \mathrm{tr}\left(g\mid_{\mathrm{Ker}(D^+)}\right)\mathrm{tr}\left(g\mid_\xi\right) - \mathrm{tr}\left(g\mid_{\mathrm{Ker}(D^-)}\right)\mathrm{tr}\left(g\mid_\xi\right) \\ &= \mathrm{tr}\left(g\mid_{\mathrm{Ker}(D^+)}\right)\mathrm{tr}\left(g\mid_\xi\right) - \mathrm{tr}\left(g\mid_{\mathrm{Ker}(D^-)}\right)\mathrm{tr}\left(g\mid_\xi\right) \\ &= \mathrm{sign}\left(g, D^{4k}\right)\mathrm{tr}\left(g\mid_\xi\right). \end{aligned}$$

In particular, $\mathrm{sign}\left(g, D^{4k}, \xi\right)$ will vanish if so do $\mathrm{sign}\left(g, D^{4k}\right)$. And, as explained in [AS68III, p. 590], we are in a situation in which $\mathrm{sign}\left(g, D^{4k}\right)$ *is completely determined by the action of g on $\hat{H}^{2k}\left(D^{4k}; \mathbb{R}\right)$*, the image by the natural map $\varphi \colon H^{2k}\left(D^{4k}, S^{4k-1}; \mathbb{R}\right) \to H^{2k}\left(D^{4k}; \mathbb{R}\right)$. Hence, $\mathrm{sign}\left(g, D^{4k}\right) = 0$.

Lemma 3.35 *The antipodal isometry τ on S^{4k-1} and the operator A_{ξ_q} restricted to S^{4k-1} commute.*

Proof: This follows from the properties of the eta invariant described in [APS75II] (change of orientations implies change of sign of the invariant), so that all of $\eta_{\xi_q}\left(s, \widetilde{Y}\right) = \eta_{\varepsilon,1}\left(s, \widetilde{Y}\right)$ are identically zero. $\square$

3.5.5 Conclusion of the proof

In our case, $\eta_{\varepsilon,\alpha}(0, Y) = \dfrac{1}{|G|}\displaystyle\sum_{g\in G} \eta_{\varepsilon,g}\left(0, \widetilde{Y}\right)\chi_\alpha(g)$ translates into

Proposition 3.36

$$\begin{aligned} \eta_{\varepsilon,\alpha}(0, Y) &= \frac{1}{|G|}\sum_{g\neq 1} \sigma_{\varepsilon,g}\left(0, \widetilde{X}\right)\chi_\alpha(g), \quad \text{i.e.,} \\ \eta_{\varepsilon,\alpha}\left(0, L_p^{4k-1}\right) &= \frac{1}{|G|}\sum_{g\neq 1} \sigma_{\varepsilon,g}\left(0, D^{4k}\right)\chi_\alpha(g). \end{aligned}$$

Remark 3.37 Under the action of $g_r = \zeta_p^r \in G = C_p$, being

$$\left(TD^{4k}\right)_0 \cong \oplus_{j=1}^{2k} N\left(\frac{2\pi q_j}{p}\right),$$

we have

$$g_r \mid_{N\left(\frac{2\pi q_j}{p}\right)} = \zeta_p^r \mid_{N\left(\frac{2\pi q_j}{p}\right)} = \text{rotation by } \frac{2\pi q_j r}{p}$$

So, by [AS68III], we have

Proposition 3.38

$$\begin{aligned}\sigma_{\varepsilon,g_r}\left(0,D^{4k}\right) &= \Phi_{\varepsilon,g_r}\left(D^{4k}\right) = \left(\prod_{j=1}^{2k}\frac{1}{f_\varepsilon\left(\tau, i\frac{2\pi q_j r}{p}\right)}\right)\frac{1}{\varepsilon\left(\tau\right)^{\frac{k}{2}}},\\ \textit{where}\quad f_\varepsilon\left(\tau,x\right) &= \varphi_\varepsilon\left(\tau,x\right)^{-1} = \left(\wp\left(\tau,x\right)-e_1\left(\tau\right)\right)^{-\frac{1}{2}}.\end{aligned}$$

The lemmas above imply, by formal addition of the result of [Don78] in the twisted versions for a representation α of $\pi_1\left(L_p^{4k-1}\left(q_1,\ldots,q_{2k}\right)\right) = C_p$,

$$\eta_\alpha\left(0, L_p^{4k-1}\left(q_1,\ldots,q_{2k}\right),\xi_q\right) = \frac{1}{p}\sum_{r=1}^{p-1}\left(\Phi_{\varepsilon,\frac{2\pi r}{p}}\left(TD^{4k}\right)\chi_\alpha\left(e^{\frac{2\pi r}{p}}\right)\right),$$

$$\eta_\alpha\left(0, L_p^{4k-1}\left(q_1,\ldots,q_{2k}\right),\xi_q\right) = \frac{1}{p}\sum_{r=1}^{p-1}\left(\left(\prod_{j=1}^{2k}\frac{1}{f_\varepsilon\left(\tau, i\frac{2\pi q_j r}{p}\right)}\right)\frac{1}{\varepsilon\left(\tau\right)^{\frac{k}{2}}}\chi_\alpha\left(e^{\frac{2\pi r}{p}}\right)\right)$$

i.e.,

$$\eta_\alpha\left(0, L_p^{4k-1}\left(q_1,\ldots,q_{2k}\right),\xi_q\right) = \frac{1}{p}\sum_{r=1}^{p-1}\left(\left(\prod_{j=1}^{2k}\varphi\left(\tau, i\frac{2\pi q_j r}{p}\right)\right)\frac{1}{\varepsilon\left(\tau\right)^{\frac{k}{2}}}\chi_\alpha\left(e^{\frac{2\pi r}{p}}\right)\right),$$

and, being $\varphi\left(\tau,x\right)$ a Jacobi function in τ and $x/2\pi i$ of weight 1 and index 0 for $\Gamma_0\left(2\right)$, its evaluation at the rational point $x = 2\pi i q_j r/p$ will be modular of the same weight for $\Gamma = \Gamma_0\left(2\right)\cap\Gamma_1\left(p\right)$, compare e.g. [EZ85]. Since, as a modular form for $\Gamma_0\left(2\right)$, $\mathrm{wt}\left(\varepsilon\left(\tau\right)\right) = 4$, we see that $\Phi_{\varepsilon,g}\left(g_r, D^{4k}\right)$ is a modular function for Γ of weight 0. $\square$

3.6 Consequences

3.6.1 Eta invariants on lens spaces and number theory

In [HZ74] Hirzebruch and Zagier present a number of relations between index theory and number theory. In this section we will prove some generalisations of these results in the context of our *loop space* operators. We will outline some developments in this area, to be treated more extensively in [Gal01]. In summary we might state the following principle: all the relations in [HZ74] generalise to loop space operators. However as a detailed exposition of these generalisations would occupy as much space as the original book, we will present here only some motivating examples which serve to justify this principle.

The expression we have obtained for the invariants associated to the virtual loop space signature operator for our lens spaces is

$$\eta_{\alpha_s}\left(0, L_p^{4k-1}\left(q_1, \ldots, q_{2k}\right), \xi_q\right) = \frac{1}{p\varepsilon\left(\tau\right)^{\frac{k}{2}}} \sum_{r=1}^{p-1} \left(\prod_{j=1}^{2k} \varphi\left(\tau, i\frac{2\pi q_j r}{p}\right)\right) \chi_{\alpha_s}\left(e^{\frac{2\pi r}{p}}\right)$$

or

$$\eta_{\alpha_s}\left(0, L_p^{4k-1}\left(q_1, \ldots, q_{2k}\right), \xi_q\right) = \frac{1}{p} \sum_{r=1}^{p-1} \left(\prod_{j=1}^{2k} \chi_\varepsilon\left(\tau, i\frac{2\pi q_j r}{p}\right)\right) \chi_{\alpha_s}\left(e^{\frac{2\pi r}{p}}\right).$$

In [APS75II, Prop. 2.12] and [Don78, Prop. 4.1] it is shown that the operator corresponding to the constant term of our q-series has eta invariant given by

$$\eta_{\alpha_s}\left(0, L_p^{4k-1}\left(q_1, \ldots, q_{2k}\right), \xi_0\right) = \frac{(-1)^k}{p} \sum_{r=1}^{p-1} \left(\prod_{j=1}^{2k} \coth\left(\frac{i\pi q_j r}{p}\right)\right) \chi_{\alpha_s}\left(e^{\frac{2\pi r}{p}}\right). \tag{3.8}$$

It is well known that cotangent sums of this kind satisfy number-theoretic relations, discovered probably by Rademacher and Dedekind and related to index theory and generalised by Hirzebruch and Zagier, among others. We will briefly recall the classical situation for the simplest case, $k = 1$, that is, for lens spaces $L_p^3(q_1, q_2)$. The classical formula and our elliptic generalisation then have the following form:

$$\eta_{\alpha_s}\left(0, L_p^3\left(q_1, q_2\right), \xi_q\right) = \frac{1}{p} \sum_{r=1}^{p-1} \left(\chi_\varepsilon\left(\tau, i\frac{2\pi q_1 r}{p}\right) \chi_\varepsilon\left(\tau, i\frac{2\pi q_2 r}{p}\right)\right) \chi_{\alpha_s}\left(e^{\frac{2\pi r}{p}}\right)$$

with the corresponding eta invariant at the cusp $q = 0$ given by

$$\eta_{\alpha_s}\left(0, L_p^3\left(q_1, q_2\right), \xi_0\right) = \frac{(-1)^1}{p} \sum_{r=1}^{p-1} \left(\coth\left(\frac{i\pi q_1 r}{p}\right) \coth\left(\frac{i\pi q_2 r}{p}\right)\right) \chi_{\alpha_s}\left(e^{\frac{2\pi r}{p}}\right). \tag{3.9}$$

The sum

$$\sum \coth(i\pi q_1 r/p) \coth(i\pi q_2 r/p)) = -\sum \cot(\pi q_1 r/p) \cot(\pi q_2 r/p))$$

satisfies the *Rademacher formula* [HZ74]

$$-\sum_{r=1}^{|c|-1} \cot\left(\frac{\pi r}{c}\right) \cot\left(\frac{\pi a r}{c}\right) = 4\,|c|\,s(a, c)$$

where without loss of generality one takes $(q_1, q_2) = (1, a)$ and the *Dedekind sum* $s(p,q)$ for p, q coprime is defined by

$$s(p,q) = \sum_{k=1}^{p} \left(\left(\frac{k}{p}\right)\right) \left(\left(\frac{kq}{p}\right)\right)$$

in terms of the mod(1)-*first Bernoulli function*

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \in \mathbb{R} - \mathbb{Z} \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

also known as the Dedekind symbol or sawtooth function, with $[x]$ the greatest integer $\leq x$. Those sums also produce the Mordell numbers for counting rational polyhedra.

Thus we see that the Dedekind sum has an interpretation as a classical signature defect (3.9) corresponding to an isolated singularity, given by the origin of the disk when the quotiented by the action of a cyclic group.

The original interest of Dedekind functions $s(a,b)$ comes from the Dedekind–Riemann functional equation for the Dedekind η-function:

$$\begin{aligned} \eta\left(\frac{a\tau + b}{c\tau + d}\right) &= \epsilon(a,b,c,d) \left(\frac{c\tau + d}{i}\right)^{\frac{1}{2}} \eta(\tau) \\ \text{where } \epsilon(a,b,c,d) &= \exp\left(\pi i \left(\frac{a+d}{12c} - s(d,c)\right)\right). \end{aligned}$$

Dedekind sums have a *reciprocity property*

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right).$$

The Rademacher formula above is obtained by (finite) Fourier analysis of functions $x \mapsto \left(\left(\frac{x}{q}\right)\right)$ for fixed q.

Hirzebruch and Zagier generalised all the functions so far defined to higher dimensions and explained a good deal about their connections with topological and geometrical invariants in their book about the Atiyah–Singer theorem and elementary number theory, and obtained, at least in their number theoretical and combinatorial sense, a good deal of generalisations. Since they involve and imply reciprocity theorems, it will not be unusual to find many proofs for them, as well as many interpretations. Our own contribution to this fits with recent work by Fukuhara, Bayad [Bay01] and others and is a consequence of the eta invariant identies above and their relatives. Hence, our aim is to generalise the identities between Dedekind sums and the signature defect for lens spaces

$$s(a,c) = -\frac{2}{3}\operatorname{def}(a,c) = \eta_S\left(L_p^{4k-1}\right)$$

to a suitable arithmetic/geometric identity involving *elliptic* functions. The left hand side is the eta invariant for the operator considered; what lacks is a suitable arithmetic interpretation for the right hand side. As in the classical case, many equivalent definitions are available. We will state the most direct ones and indicate geometric descriptions of the spaces involved.

The most recent and thorough exposition in this area is [Bay01], where an elliptic analogue of the multiple Dedekind sums considered by Zagier [Zag88] is introduced; applying the same methods Bayad obtains quite similar results with the cotangent functions replaced by Jacobi forms. In particular, Bayad proves the reciprocity law for the new Dedekind sums and recovers the classical results as expected as the independent term in the q-series for the corresponding functions. Moreover, by a specialisation to the 2-division points, Bayad recovers as well Egami's results. The Jacobi forms $D_L(z,\varphi)$ considered by Bayad are exactly the building blocks of our invariants. We can use them to prove our generalisations of the results in [HZ74].

The Zagier generalisation of Dedekind sums is as follows: Given n even and $n+1$ pairwise coprime natural numbers $p, a_1, \ldots, a_n$, let

$$\delta(p; a_1, \ldots, a_n) = \frac{(-1)^n}{p} \sum_{j=1}^{p-1} \cot\left(\frac{\pi j a_1}{p}\right) \cdots \cot\left(\frac{\pi j a_n}{p}\right).$$

Zagier obtains a generalisation of the Dedekind reciprocity law,

$$\sum_{k=1}^{n} \delta(a_k; a_1, \ldots, \widehat{a}_k, \ldots, a_n) = 1 - \frac{\ell(a_0, \ldots, a_n)}{a_0 \cdots a_n}.$$

For $n = 2$ we then have $\delta(a; b, c) = -4s(b, c; a)$. From the definitions, it follows that

$$\begin{aligned}
\delta(a_k; a_1, \ldots, \widehat{a}_k, \ldots, a_n) &= \frac{(-1)^n}{p} \sum_{j=1}^{p-1} \cot\left(\frac{\pi j a_1}{p}\right) \cdots \cot\left(\frac{\pi j a_n}{p}\right) \\
&= \frac{(-1)^n}{p} \sum_{j=1}^{p-1} \eta_{g_j}\left(0, S^{4k-1}, \vec{a}\right)
\end{aligned}$$

with $\eta_{g_j}\left(0, S^{4k-1}, \vec{a}(k)\right)$ the classical invariant for the signature for $g_j = e^{\frac{2\pi i j}{p}}$ acting on S^{4k-1} via $\vec{a}(k)$. Since in general for a representation α we have

$$\eta_\alpha\left(0, L^{4k-1}_{\vec{a}(k)}\right) = \frac{(-1)^n}{p} \sum_{j=1}^{p-1} \eta_{g_j}\left(0, S^{4k-1}, \vec{a}\right) \widetilde{\chi}_\alpha(g_j)$$

we observe that if we pick the trivial representation $\alpha = \alpha_0$ we obtain

$$\eta_{\alpha_0}\left(0, L^{4k-1}_{\vec{a}(k)}\right) = \frac{(-1)^n}{p} \sum_{j=1}^{p-1} \eta_{g_j}\left(0, S^{4k-1}, \vec{a}\right)$$

so that

$$\begin{aligned}\delta\left(a_k;a_1,\ldots,\widehat{a}_k,\ldots,a_n\right) &= \frac{(-1)^n}{p}\sum_{j=1}^{p-1}\cot\left(\frac{\pi j a_1}{p}\right)\cdots\cot\left(\frac{\pi j a_n}{p}\right)\\ &= \frac{(-1)^n}{p}\sum_{j=1}^{p-1}\eta_{g_j}\left(0,S^{4k-1},\vec{a}\right)\\ &= \eta_{\alpha_0}\left(0,L^{4k-1}_{\vec{a}(k)}\right).\end{aligned}$$

The Zagier sums have the alternative expression

$$\delta\left(a;a_1,\ldots,a_n\right)=\sum_{k\geq 0}\sum_{\substack{0<b_1\cdots b_n<a\\ a\mid\sum_{i=1}^{n}a_ib_i}}\left(2\frac{b_1}{a_1}-1\right)\cdots\left(2\frac{b_n}{a_n}-1\right)$$

for n even.

Lemma 3.39 [HZ74, p. 216] *The Pontrjagin classes of the rational homology manifold* $\mathbb{C}\mathrm{P}^n/G$ *for* $G=\mathbb{Z}/\left(b_0\right)\times\cdots\times\mathbb{Z}/\left(b_n\right)$ *are given by*

$$\pi^*\left(p\left(\mathbb{C}\mathrm{P}^n/G\right)\right)=\prod_{k=0}^{n}\left(1+b_k^2x^2\right)$$

whenever the b_k *are pairwise coprime, for* n *odd. For* n *even, it only holds when the Diophantine equation*

$$b_0\cdots b_n=\ell_n\left(b_0,\ldots,b_n\right)$$

is satisfied, where

$$\ell\left(a_0,\ldots,a_n\right)=L_{\frac{n}{2}}\left(p_1\left(a_0,\ldots,a_n\right),\ldots,p_{\frac{n}{2}}\left(a_0,\ldots,a_n\right)\right)=\mathcal{L}\left(\xi\right)\in H^i\left(X\right),\quad \xi\downarrow X.$$

Perhaps the easiest elliptic example to look at is the 3-dimensional one provided by [Scz84] which replaces the cotangent sums by the Eisenstein series

$$E_k\left(x\right)=\sum_{\omega\in L}^{e}\left(\omega+x\right)^{-k}\left|\omega+x\right|^{-s}|_{s=0}\,,\quad k=0,1,\ldots.$$

Sczech defines the sums

$$D\left(a,c\right)=\frac{1}{c}\sum_{k\in L/cL}E_1\left(\frac{k}{c}\right)E_1\left(\frac{ak}{c}\right)$$

and shows that

$$\begin{aligned}D\left(a,c\right)+D\left(c,a\right) &= 2iE_2\left(0\right)\mathrm{Im}\left(\frac{a}{c}+\frac{1}{ac}+\frac{c}{a}\right),\ c\neq 0\\ \text{for all } a,c &\in \mathcal{O}_L=\left\{x\in\mathbb{C}\mid xL\subset L\right\}\end{aligned}$$

via the addition formula for the Weierstrass zeta function. From this reciprocity law he gives applications to the work by Harder on cohomology classes represented by Eisenstein series. In the Sczech series we have already the key feature of the elliptic Dedekind sums, namely that the summation is not over the group $\mathbb{Z}/(c)$ but over $\mathbb{Z}/(c) \times \mathbb{Z}/(c)$, a typical height two phenomenon.

3.6.2 Multiple elliptic Dedekind sums of level 2

Egami and Bayad [Bay01] define multiple Dedekind sums which are level 2 elliptic functions,

$$d_\tau(p; a_1, \dots, a_n) = \frac{(2\pi i)^n}{p} \sum_{\substack{0 \le m, n < p \\ (m,n) \ne (0,0)}} e^{\frac{2\pi i m}{p}} \prod_{k=1}^{r} \varphi\left(\tau, \frac{2\pi i a_k (m\tau + n)}{p}\right),$$

for which there is a corresponding reciprocity theorem

$$\begin{aligned} \sum_{k=0}^{n} d_\tau\left(a_k; a_1, \dots, \widehat{a}_k, \dots, a_n; \frac{\varphi}{a_k}\right) &= -\widetilde{M}_{n,\tau}\left(a_0; a_1, \dots, a_n; \frac{1}{2}\right) \\ &= (2\pi i)^{n+1} \operatorname{coef}\left(z^n, z^{n+1} \prod_{k=0}^{n} \varphi(a_k 2\pi i z)\right). \end{aligned}$$

Here $a_0, a_1, \dots, a_n$ are pairwise coprime natural numbers with $a_0 + \cdots + a_n$ even and φ the semiparameter for a 2-division point in $\mathbb{C}/L$.

We complete the picture by giving further elliptic eta invariants generalising the elliptic eta invariants for lens spaces we have been considering until now. The details on its geometrical realisation are to be explained in [Gal01]. We simply define the functions

$$\eta_{(m,n)}\left(0, S^{4k-1}(p; q_1, \dots, q_{2k}), \widetilde{\xi}_q\right) = \frac{\left(\prod_{j=1}^{2k} \varphi\left(\tau, i\frac{2\pi q_j (m\tau+n)}{p}\right)\right)}{\varepsilon(\tau)^{\frac{k}{2}}};$$

then the Bayad–Dedekind sums give

$$\frac{\delta_\tau\left(p; a_1, \dots, a_n; \frac{1}{2}\right)}{\varepsilon(\tau)^{\frac{k}{2}}} = \sum_{\substack{0 \le m, n < p \\ (m,n) \ne (0,0)}} \frac{e^{\frac{2\pi i m}{p}}}{p} \left(\eta_{(m,n)}\left(0, S^{4k-1}(p; q_1, \dots, q_{2k}), \widetilde{\xi}_q\right)\right).$$

In the case of the cyclic group $\mathbb{Z}/p$, the 2-characters are essentially the representations of $\mathbb{Z}/p \times \mathbb{Z}/p$, so that

$$\eta_\alpha\left(0, L^{4k-1}_{\vec{a}(k)}\right) = \frac{(-1)^n}{p} \sum_{j=1}^{p-1} \eta_{g_j}\left(0, S^{4k-1}, \vec{a}\right) \widetilde{\chi}_\alpha(g_j)$$

generalises to

$$\eta_{\alpha,\beta}\left(0, L^{4k-1}_{\vec{a}(k)}\right) = \frac{(-1)^n}{p} \sum_{\substack{0\le m,\, n<p \\ (m,n)\neq(0,0)}} \eta_{(m,n)}\left(0, S^{4k-1}, \vec{a}\right) \widetilde{\chi}_{\alpha,\beta}\left(e^{\frac{2\pi i(m\tau+n)}{p}}\right)$$

and the version of the inverse from the finite Fourier transform will be

$$\eta_{(m,n)}\left(0, S^{4k-1}, \vec{a}\right) = \frac{(-1)^n}{p} \sum_{(\alpha,\beta)\in \mathrm{Irr}(\mathbb{Z}/p\times\mathbb{Z}/p)} \eta_{\alpha,\beta}\left(0, L^{4k-1}_{\vec{a}(k)}\right) \chi_{\alpha,\beta}\left(e^{\frac{2\pi i(m\tau+n)}{p}}\right).$$

This may be interpreted as an identity between eta invariants for operators on $L^{4k-1}_{\vec{a}(k)} \times L^{4k-1}_{\vec{a}(k)}$ and $S^{4k-1}\times S^{4k-1}$ or heuristically in terms of loops spaces, as discussed later. Moreover, if one allows spaces with singularities as quotients, then the natural setting for those invariants is to consider them as corresponding to orbifolds of the form $S^{4k-1}/\mathbb{Z}_p$.

3.6.3 On topological proofs of reciprocity theorems for elliptic Dedekind sums

In [HZ74] the authors observe that their developments provide in fact a topological proof of the Zagier–Rademacher reciprocity by means of their result [10.3(4)] together with results of Bott on the Pontrjagin classes of the generalised rational homology manifold $\mathbb{C}\mathrm{P}^n/G$, sometimes called the symplectic action on the projective space. We let $\left(S^1\right)^{n+1}$ act on $\mathbb{C}\mathrm{P}^n$ via

$$\begin{array}{rcl}(\zeta_0,\ldots,\zeta_n)\cdot(z_0:\cdots:z_n) & = & (\zeta_0 z_0:\cdots:\zeta_n z_n) \\ (\zeta_0,\ldots,\zeta_n) & \in & \left(S^1\right)^{n+1},\quad (z_0:\cdots:z_n)\in\left(\mathbb{C}^{n+1}-\{0\}\right)/\mathbb{C} = \mathbb{C}\mathrm{P}^n.\end{array}$$

Consider in particular the action of the product of cyclic groups

$$\mu_{b_0}\times\cdots\times\mu_{b_n} \hookrightarrow S^1\times\cdots\times S^1, \qquad \mu_{b_k} = \text{group of } k\text{-th roots of unity}.$$

A consequence of the topological constructions for equivariant signatures on rational homology manifolds ensures that the coefficient of x^n in $\pi^*\left(\mathcal{L}\left(\mathbb{C}\mathrm{P}^n/G\right)\right)$ is $\deg(\pi)\times\operatorname{sign}\left(\mathbb{C}\mathrm{P}^n/G\right)$, n even. Those manifolds, although not having proper tangent bundles, were shown by Thom (ICM Mexico 1958) to have nevertheless Pontrjagin classes and hence Pontrjagin numbers as well as suitably valued genera provided those come given by multiplicative sequences based on the existing characteristic classes.

If we consider instead the generalised elliptic Dedekind sums

$$\delta_\tau\left(p; a_1,\ldots,a_n;\varphi\right) = \frac{1}{p}\sum_{\omega\in E_p} D_\tau\left(\frac{a_1\omega}{p};\varphi\right)\cdots D_\tau\left(\frac{a_n\omega}{p};\varphi\right)$$

the remarks above ensure that the Hizebruch–Zagier approach [HZ74, p.217] goes through, changing the Hirzebruch L-polynomials for those given by an oriented elliptic genus. If we prove that the coefficient of x^n in $\pi^*\left(\Phi_\varepsilon\left(\mathbb{C}\mathrm{P}^n/G\right)\right)$ is $\nu_{|G|} \times \operatorname{sign}\left(\mathcal{L}\mathbb{C}\mathrm{P}^n/G\right)$, n even, then the considerations in [HZ74] on the rational Pontrjagin classes of $\mathbb{C}\mathrm{P}^n/G$ apply. We then find that Bayad's elliptic version of this identity comes out from the substitution of the Hirzebruch L-polynomials with the appropriate elliptic genus, the signature of the space with that of its free smooth loop space, and the degree of the projection with corresponding multiplicative factor for the elliptic genus as described in [HBJ92].

Chapter 4

An algebraic Atiyah–Patodi–Singer construction in elliptic cohomology

Motivation for this chapter comes from the following construction performed in [APS75II], which gives rise to what those authors called Description II (the algebraic one), for some eta invariants. They establish a correspondence between the representation ring of the finite fundamental group G of an odd dimensional manifold Y bounding a manifold X and the equivalence classes of vector bundles on Y, taken modulo integers,

$$R\left(\pi_1 Y\right) \to K^{-1}\left(Y; \mathbb{Q}/\mathbb{Z}\right),$$

sending the class of a representation α to the class $[\alpha]$ of the the representation vector bundle V_α associated to α over Y, and then using the pushforward in K-theory to obtain an element in the coefficient group of K-theory modulo integers. This is done by means of the completion homomorphism, which relates K_G^* and $K^*\left(BG\right)$, and the pullback of the corresponding classifying map f from Y to BG.

To extend the construction in [APS75II], we first need the equivalent for $K^{-1}\left(\mathrm{pt}; \mathbb{Q}/\mathbb{Z}\right)$, but because of the particularities of the coefficients in elliptic cohomology, we are bound to consider $\mathcal{E}\ell\ell^*\left(\mathrm{pt}; \mathbb{Q}/\mathbb{Z}\right)$ as the coefficient group for our theory, being G in this case a finite group of odd order. We proceed to give the details in the first section of this chapter. In the second one, once produced the coefficients, we will proceed to reproduce the Atiyah–Patodi–Singer construction with the necessary adaptations to our case, obtaining in this way an element in $\mathcal{E}\ell\ell^*\left(\mathrm{pt}; \mathbb{Q}/\mathbb{Z}\right)$ associated to the chosen element $\mathcal{E}\ell\ell_G^*$ for the G-manifold Y classified by a map f. This invariant will be seen, in the third section, to be modular in the sense that the coefficients of the theory allow essentially the one of Katz's divided congruence rings [Kat75]. Next we will make clear how, in the case of cyclic groups of odd order and lens spaces, these invariants give the same as the ones obtained from operators in the previous chapter and they include all the generators for the equivariant cohomology ring $\mathcal{E}\ell\ell_G^*$ involved.

Finally, we will consider the problem of their meaning as geometrical and representation-theoretic invariants. In the case of elliptic cohomology, we still lack the sound correspondence between bundles and representations which makes the power of K-theory. However, the algebraic formal construction can be mimed as we will see, using the description in [Dev96b] for the completion relating $\mathcal{E}\ell\ell_G^*$ and $\mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^*(BG)$ in the cases considered. Then, the constructions of elliptic objects as in Baker–Thomas [BT96] allow to establish a correspondence between a given element in $\mathcal{E}\ell\ell_G^*$ and a Virasoro bundle over Y determined by it. This prepares the way for seeing them as given by representations of the Segal categories that we will introduce in the next chapter.

4.1 $\mathcal{E}\ell\ell^*$-theory with $\mathbb{Q}/\mathbb{Z}$ coefficients

In this section we will give some details on the construction of $\mathcal{E}\ell\ell^*\left[\frac{1}{|G|}\right]$-theory with $\mathbb{Q}/\mathbb{Z}$ coefficients, for G a finite group of odd order, checking that it is conveniently defined not only for finite CW-complexes, but as well for the space BG. This has to be checked, since the coefficient group is not finitely generated, and hence the usual short exact sequence only ensures the exactness of tensoring with the coefficients in the case of either the space is finite dimensional or the coefficient group is finitely generated. Hence, our aim in this section is to check that, for the groups considered,

$$\mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^*(BG;\mathbb{Q}/\mathbb{Z}) = \varprojlim_N \mathcal{E}\ell\ell^*\left[\frac{1}{|G|}\right](B_G^N;\ \mathbb{Q}/\mathbb{Z}),$$

where B_G^N is the N-th skeleton of the classifying space BG.

Now let Y be an odd-dimensional spin manifold with finite fundamental group $G = \pi_1(Y)$. Let BG be the classifying space of G, and let B_G^N be its N-skeleton. The following holds:

Proposition 4.1 $\quad \mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^*(BG;\mathbb{Q}/\mathbb{Z}) = \varprojlim_N \mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^*(B_G^N;\mathbb{Q}/\mathbb{Z}).$

Moreover, we have

Proposition 4.2 $\quad \mathcal{E}\ell\ell^{\,\mathrm{odd}}(B_G^{2n}) \otimes \mathbb{Q} = 0,\ \ \mathcal{E}\ell\ell^{\,\mathrm{even}}(B_G^{2n+1}) \otimes \mathbb{Q} = 0.$

From the equation, using the coefficient sequence for $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and letting $N \to \infty$, we deduce that

Proposition 4.3 $\quad \mathcal{E}\ell\ell^{\,\mathrm{odd}}\left[\frac{1}{|G|}\right](BG;\mathbb{Q}/\mathbb{Z}) \overset{\delta}{\cong} \widetilde{\mathcal{E}\ell\ell}^{\,\mathrm{even}}\left[\frac{1}{|G|}\right](BG).$

We will begin with the case of rational coefficients, $A = \mathbb{Q}$, and consider $\mathbb{Q}/\mathbb{Z}$ later. A lot of care is needed here, since for *infinite dimensional* CW-complexes there are many things which are false; e.g. one has $K^*(\mathbb{CP}^\infty;\mathbb{Q}) \cong \mathbb{Q}[[x]] \neq \mathbb{Z}[[x]] \otimes_{\mathbb{Z}} \mathbb{Q}$.

Recall that, given any tower of abelian groups of the form

$$A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow A_N \leftarrow A_{N+1} \leftarrow \cdots,$$

the *inverse limit* $\varprojlim_N A_N$ and the *derived limit* $\varprojlim_N^1 A_N$ are given by the following exact sequence,

$$0 \to \varprojlim_N A_N \to \prod_N A_N \xrightarrow{\Delta} \prod_N A_N \to \varprojlim_N^1 A_N \to 0.$$

Here

$$\Delta(a_0, a_1, \dots, a_N, a_{N+1}, \dots) = (a_0 - \overline{a_1}, a_1 - \overline{a_2}, \dots, a_N - \overline{a_{N+1}}, \dots),$$

where $\overline{a_{N+1}}$ is the image of a_{N+1} in A_N.

Lemma 4.4 *If the groups A_N are zero for infinitely many N, then it follows that both the limit and the derived limit of the system are zero.*

Proof: For $(a_0, a_1, a_2, \dots) \in \lim A_N = \ker \Delta$, we have $a_N = 0$ for infinitely many N, and $a_N = 0$ implies $a_{N-1} = \overline{a_N} = 0$. Therefore, $a_N = 0$ for all N. For the derived limit, we must show that Δ is surjective. For $(b_0, b_1, b_2, \dots) \in \prod A_N$, let $c_N = b_N + \sum_{m>N} \overline{b_m}$ for each N. Only finitely many of the images $\overline{b_m}$ are nonzero in A_N, so these summations make sense. Also, $\Delta(c_0, c_1, c_2, \dots) = (b_0, b_1, b_2, \dots)$, as required. $\square$

In particular, for the inverse system

$$\widetilde{\mathcal{E}\ell\ell}^q_{\mathbb{Q}} BG^0 \leftarrow \widetilde{\mathcal{E}\ell\ell}^q_{\mathbb{Q}} BG^1 \leftarrow \cdots \leftarrow \widetilde{\mathcal{E}\ell\ell}^q_{\mathbb{Q}} BG^N \leftarrow \widetilde{\mathcal{E}\ell\ell}^q_{\mathbb{Q}} BG^{N+1} \leftarrow \cdots$$

we have:

Lemma 4.5 *Because of the uniqueness of formal group laws over the rationals (at least for finite CW-complexes), the following holds:*

$$\widetilde{\mathcal{E}\ell\ell}^q_{\mathbb{Q}} \left(BG^N\right) \cong \begin{cases} 0, & \text{for } q \not\equiv 0 \bmod (4) \\ \widetilde{H}^N_{\mathbb{Q}} \left(BG^N\right) \otimes \mathcal{E}\ell\ell^{q-N}, & \text{for } q \equiv 0 \bmod (4). \end{cases}$$

We apply this result to obtain

Lemma 4.6 *There is a natural isomorphism*

$$\widetilde{\mathcal{E}\ell\ell}^* (BG; \mathbb{Q}) \cong \varprojlim_N \widetilde{\mathcal{E}\ell\ell}^* \left(BG^N; \mathbb{Q}\right).$$

Proof: Use the Milnor short exact sequence

$$0 \to \varprojlim_N^1 \widetilde{\mathcal{E}\ell\ell}^{q-1}_{\mathbb{Q}} \left(BG^N\right) \to \widetilde{\mathcal{E}\ell\ell}^q_{\mathbb{Q}} (BG) \to \varprojlim_N \widetilde{\mathcal{E}\ell\ell}^q_{\mathbb{Q}} \left(BG^N\right) \to 0$$

to prove the claim. $\square$

$\mathbb{Q}/\mathbb{Z}$ coefficients

We are really more interested in taking coefficients in

$$\mathbb{Q}/\mathbb{Z} \cong \lim_{\overrightarrow{m}} \mathbb{Z}/m\mathbb{Z},$$

where the colimit is taken over the partially ordered set of natural numbers, with "arrows" $m \leq m'$ if m divides m'.

We consider the short exact sequence of coefficient groups

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,$$

which induces for each N a long exact sequence for the cohomology of the N-skeleton BG^N:

$$\begin{array}{rcl} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^0 BG^N & \to & \widetilde{\mathcal{E}\ell\ell}^0_{\mathbb{Q}} BG^N \to \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^0_{\mathbb{Q}/\mathbb{Z}} BG^N \xrightarrow{\delta} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^1 BG^N \to \cdots \\ \cdots & \to & \widetilde{\mathcal{E}\ell\ell}^q_{\mathbb{Q}} BG^N \to \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^q_{\mathbb{Q}/\mathbb{Z}} BG^N \xrightarrow{\delta} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q+1} BG^N \to \widetilde{\mathcal{E}\ell\ell}^{q+1}_{\mathbb{Q}} BG^N \to \cdots \end{array}$$

Now Lemma 4.5 says that $\widetilde{\mathcal{E}\ell\ell}^q_{\mathbb{Q}} BG^N$ is zero unless $q-N$ is divisible by 4, and similarly $\widetilde{\mathcal{E}\ell\ell}^{q+1}_{\mathbb{Q}} BG^N$ is zero unless $q - N \equiv 3 \bmod (4)$. Therefore,

Lemma 4.7 *The connecting homomorphism*

$$\delta^q_N : \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^q_{\mathbb{Q}/\mathbb{Z}} BG^N \longrightarrow \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q+1} BG^N$$

is an isomorphism for $q - N \equiv 1, 2 \bmod (4)$. It is epi for $q - N \equiv 0 \bmod (4)$ and mono for $q - N \equiv 3 \bmod (4)$.

If we now fix q, we can think of $\{\delta^q_N\}$ as defining a map of inverse systems

$$\left(\begin{array}{ccccccccc} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^q_{\mathbb{Q}/\mathbb{Z}} BG^0 & \leftarrow & \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^q_{\mathbb{Q}/\mathbb{Z}} BG^1 & \leftarrow & \cdots & \leftarrow & \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^q_{\mathbb{Q}/\mathbb{Z}} BG^N & \leftarrow & \cdots \\ \delta^q_0 \downarrow & & \delta^q_1 \downarrow & & & & \delta^q_N \downarrow & & \\ \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q+1}_{\mathbb{Z}} BG^0 & \leftarrow & \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q+1}_{\mathbb{Z}} BG^1 & \leftarrow & \cdots & \leftarrow & \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q+1}_{\mathbb{Z}} BG^N & \leftarrow & \ldots \end{array} \right).$$

Hence, by Lemma 4.7, half of the vertical arrows in this diagram are isomorphisms. Taking inverse limits, we can show

Lemma 4.8 *For all q, the connecting maps δ^q_N induce natural isomorphisms*

$$\begin{array}{rcl} \lim\limits_{\overleftarrow{N}} \delta^q_N & : & \lim\limits_{\overleftarrow{N}} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^q (BG^N; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \lim\limits_{\overleftarrow{N}} \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{q+1} (BG^N) \\ \lim\limits_{\overleftarrow{N}}{}^1 \delta^q_N & : & \lim\limits_{\overleftarrow{N}}{}^1 \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^q (BG^N; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \lim\limits_{\overleftarrow{N}}{}^1 \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{q+1} (BG^N). \end{array}$$

Proof: The argument is: "Since δ_N^q is an isomorphism for infinitely many N, so are $\varprojlim_N \delta_N^q$ and $\varprojlim_N^1 \delta_N^q$". To be more precise, write

$$A_N = \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^q \left(BG^N; \mathbb{Q}/\mathbb{Z}\right), \qquad B_N = \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{q+1}(BG^N),$$

and consider the exact sequence of inverse systems

$$0 \to \{\ker \delta_N^q\} \to \{A_N\} \to \{B_N\} \to \{\mathrm{coker}\delta_N^q\} \to 0.$$

This can be broken into two short exact sequences in the usual way,

$$\begin{aligned} 0 &\to \{\ker \delta_N^q\} \to \{A_N\} \to \{\mathrm{Im}\,(\delta_N^q)\} \to 0 \\ 0 &\to \{\mathrm{Im}\ \delta_N^q\} \to \{B_N\} \to \{\mathrm{coker}\,(\delta_N^q)\} \to 0, \end{aligned}$$

and taking inverse limits gives exact sequences

$$\begin{aligned} 0 &\to \varprojlim_N \ker \delta_N^q \to \varprojlim_N A_N \to \varprojlim_N \mathrm{Im}\,(\delta_N^q) \to \varprojlim_N{}^1 \ker \delta_N^q \to \varprojlim_N{}^1 A_N \to \varprojlim_N{}^1 \mathrm{Im}\,(\delta_N^q) \to 0 \\ 0 &\to \varprojlim_N \mathrm{Im}\ \delta_N^q \to \varprojlim_N B_N \to \varprojlim_N \mathrm{coker}\,(\delta_N^q) \to \varprojlim_N{}^1 \mathrm{Im}\ \delta_N^q \to \varprojlim_N{}^1 B_N \to \varprojlim_N{}^1 \mathrm{coker}\,(\delta_N^q) \to 0. \end{aligned}$$

Now if δ_N^q is an isomorphism for infinitely many values of N, then $\ker \delta_N^q$ and $\mathrm{coker}\delta_N^q$ are zero for infinitely many values of N. Therefore, the corresponding lim and $\lim^1$ terms are zero, and we have

$$\begin{aligned} 0 &\to 0 \to \varprojlim_N A_N \to \varprojlim_N \mathrm{Im}\,(\delta_N^q) \to 0 \to \varprojlim_N{}^1 A_N \to \varprojlim_N{}^1 \mathrm{Im}\,(\delta_N^q) \to 0 \\ 0 &\to \varprojlim_N \mathrm{Im}\ \delta_N^q \to \varprojlim_N B_N \to 0 \to \varprojlim_N{}^1 \mathrm{Im}\ \delta_N^q \to \varprojlim_N{}^1 B_N \to 0 \to 0, \end{aligned}$$

so that

$$\varprojlim_N A_N \xrightarrow{\cong} \varprojlim_N \mathrm{Im}\,(\delta_N^q)$$

$$\varprojlim_N{}^1 A_N \xrightarrow{\cong} \varprojlim_N{}^1 \mathrm{Im}\,(\delta_N^q)$$

and

$$\varprojlim_N \mathrm{Im}\ \delta_N^q \xrightarrow{\cong} \varprojlim_N B_N$$

$$\varprojlim_N{}^1 \mathrm{Im}\ \delta_N^q \xrightarrow{\cong} \varprojlim_N{}^1 B_N.$$

In order to continue, we need an argument from Anderson's thesis:

Lemma 4.9 (Anderson) *A sufficient condition for the vanishing of* $\varprojlim_N^1 h^*(X^N)$ *is that in the spectral sequence with*

$$E_2^{p,q} = \widetilde{H}^p(X; h^q(\mathrm{pt}))$$

there exists, for each (p,q)*, some* r *such that* $E_r^{p,q} \cong E_\infty^{p,q}$.

Note that this spectral sequence converges to $\tilde{h}^{p+q}(X)$ if X is a finite CW-complex. It happens that Anderson's condition is fulfilled trivially for

$$h = \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^*, \qquad X = BG.$$

For all p, q,

$$E_2^{p,q} = \widetilde{H}^p(BG; \mathcal{E}\ell\ell^q\left(\mathrm{pt};\, \mathbb{Z}\left[\tfrac{1}{|G|}\right]\right)) = 0,$$

and therefore,

$$\varprojlim_N^1 \left(\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^* (BG^N)\right) = 0.$$

The proof would go then as follows. Let

$$E_{2,N}^{p,q} = \widetilde{H}^p\left(BG^N;\ \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^q(\mathrm{pt})\right).$$

Since the coefficients are torsion-free,

$$E_{2,N}^{p,q} \cong \widetilde{H}^p\left(BG^N\right) \otimes \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^q(\mathrm{pt}) = \widetilde{H}^p\left(BG^N\right) \otimes \mathbb{Z}\left[\tfrac{1}{|G|}\right] \otimes \mathcal{E}\ell\ell^q(\mathrm{pt}).$$

For $N > p$,

$$H^p\left(BG^N\right) \otimes \mathbb{Z}\left[\tfrac{1}{|G|}\right] \cong H^p(BG) \otimes \mathbb{Z}\left[\tfrac{1}{|G|}\right] = 0.$$

Now we can argue as before. Since $E_{2,N}^{p,q} = 0$ for infinitely many N, we have

$$\varprojlim_N E_{2,N}^{p,q} = 0 \text{ and } \varprojlim_N^1 E_{2,N}^{p,q} = 0,$$

and then the short exact sequence

$$0 \to \varprojlim_N^1 E_{2,N}^{p,q} \to E_2^{p,q} \to \varprojlim_N E_{2,N}^{p,q} \to 0$$

gives $E_2^{p,q} = 0$, which trivially satisfies Anderson's criterion. $\square$

Now we can prove Propositions 4.1 and 4.2.

Lemma 4.10 *The following hold*

$$\begin{aligned}
\widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{*}(BG;\mathbb{Q}/\mathbb{Z}) &\cong \lim_{\overleftarrow{N}} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{*}\left(BG^{N};\mathbb{Q}/\mathbb{Z}\right)\\
\widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{*}(BG) &\cong \lim_{\overleftarrow{N}} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{*}\left(BG^{N}\right)\\
\widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{*}(BG;\mathbb{Q}/\mathbb{Z}) &\cong \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{*+1}(BG)\,.
\end{aligned}$$

As in Lemma 4.6, we consider, for each q, the short exact sequences

$$0 \to \lim_{\overleftarrow{N}}{}^{1} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q} BG^{N} \to \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q+1} BG \to \lim_{\overleftarrow{N}} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q+1} BG^{N} \to 0,$$

$$0 \to \lim_{\overleftarrow{N}}{}^{1} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q-1}_{\mathbb{Q}/\mathbb{Z}} BG^{N} \to \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q}_{\mathbb{Q}/\mathbb{Z}} BG \to \lim_{\overleftarrow{N}} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q}_{\mathbb{Q}/\mathbb{Z}} BG^{N} \to 0.$$

Since $\lim_{\overleftarrow{N}}{}^{1} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q} BG^{N} = 0$ by Lemma 4.9 and therefore $\lim_{\overleftarrow{N}}{}^{1} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q-1}_{\mathbb{Q}/\mathbb{Z}} BG^{N} = 0$, by Lemma 4.8, we have the proof of Proposition 4.1 and we also proved that

$$\widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{*}(BG) \cong \lim_{\overleftarrow{N}} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{*}(BG^{N}).$$

By Lemma 4.8 again,

$$\lim_{\overleftarrow{N}} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q}(BG^{N};\mathbb{Q}/\mathbb{Z}) \cong \lim_{\overleftarrow{N}} \widetilde{\mathcal{E}\ell\ell}\left[\tfrac{1}{|G|}\right]^{q+1}(BG^{N};\mathbb{Q}/\mathbb{Z}).$$

Proposition 4.2 follows immediately.

4.2 The algebraic extension to elliptic cohomology

In this section we will perform the algebraic construction for our version of [APS75II] invariants adapted to elliptic cohomology. As in the classical Atiyah–Patodi–Singer construction, the essential tool will be a completion with respect to an augmentation ideal I_G relating $\mathcal{E}\ell\ell^{*}_{G}$ and $\mathcal{E}\ell\ell^{*}(BG)$. We will summarise before how this was done in the case of K-theory, since the logic is the same for us, despite of the expected technicalities.

For complex K-theory, we have the standard isomorphism $R(G) \to K^{0}(BG)$, for any finite group G. This isomorphism assigns to every representation α of G an associated bundle V_α over BG, determined by the representation module —in this case a complex vector space— associated to α for every point of BG. Atiyah's construction considers the augmentation ideal I_G, which is the kernel of the representation induced between representation rings by the inclusion in G of the trivial group. Every unitary representation of G in some $\mathrm{U}(k)$ is then associated to its rank k. One may then complete with respect to I_G by means of $c_{I_G}\colon I_G \to \widetilde{K}^{0}(BG)$, and then use the Bockstein homomorphism

$$\delta^{-1}\colon \widetilde{K}^{0}(BG) \to K^{-1}(BG;\mathbb{Q}/\mathbb{Z})$$

from the short exact sequence of coeficients $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, to get the composition

$$\delta^{-1} \circ c_{I_G} = \gamma\colon R(G)/I_G \to K^{-1}(BG; \mathbb{Q}/\mathbb{Z})$$

and then send the representation $\alpha\colon G \to \mathrm{U}(k)$ of rank k to the class $\gamma(\alpha - k)$. Then $\tilde{R}(G) = R(G)/I_G$ is called the reduced representation ring.

Now, if our G is the fundamental group of a smooth manifold Y, we can consider the pullback by $f\colon X \to BG$ of the classifying map for Y with respect to the $\pi_1(Y)$-action on Y. For Y odd dimensional, we get a map $f^*\colon K^{-1}(BG; \mathbb{Z}) \to K^{-1}(Y; \mathbb{Q}/\mathbb{Z})$ which, composed with γ, reads $f^* \circ \gamma\colon \tilde{R}(G) \to K^{-1}(Y; \mathbb{Q}/\mathbb{Z})$ and sends the reduced representation $\alpha - k$ to a class $[\alpha]$ in $K^{-1}(Y; \mathbb{Q}/\mathbb{Z})$. Now, Atiyah–Patodi–Singer use the direct image map known to be given when Y is spin^c by $K^{-1}(Y; \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$, by sending the class of $[\alpha]$ in $K^{-1}(Y; \mathbb{Q}/\mathbb{Z})$ to a number in $\mathbb{Q}/\mathbb{Z}$, which they prove to be the eta invariant of a corresponding Dirac operator twisted with the flat bundle V_α fournished by the representation.

The aim of this section is to develop the parallel construction for elliptic cohomology.

Consider Devoto's ring of coefficients $\mathcal{E}\ell\ell_G^{**}$ for equivariant elliptic cohomology, Inside $\mathcal{E}\ell\ell_G^{**}$ we have the augmentation ideal I_G. Recall that to consider the completion homomorphism which relates it to $\mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^*(BG)$, consider the kernel I_G of the restriction-induced map

$$\mathcal{E}\ell\ell_G^* \overset{\mathrm{res}_G^{\{e\}}}{\to} \mathcal{E}\ell\ell_{\{e\}}^*.$$

Then, by [BT96, Dev96b, HKR00], we get a *completion* homomorphism c

$$c\colon \mathcal{E}\ell\ell_G^* \to \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^*(BG).$$

Restricting c to I_G and composing with δ^{-1} (the coboundary map analysed in the former section) we get a map

$$\gamma\colon I_G \to \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{\mathrm{odd}}(BG;\ \mathbb{Q}/\mathbb{Z}).$$

If $\vartheta \in \mathcal{E}\ell\ell_G^{**}$, then we shall write $\widetilde{\vartheta}$ for the element

$$\widetilde{\vartheta}(g_1, g_2, \tau) = \vartheta(e, e, \tau) \quad \text{for } (g_1, g_2) \in TG.$$

The element $\vartheta - \widetilde{\vartheta}$ is in I_G for all $\vartheta \in \mathcal{E}\ell\ell_G^*$. (It admits a difference construction approach as for K-theory, but less intuitive from a geometric point of view, as we lack a good definition for relative elliptic objects.) Thus to any element $\vartheta \in \mathcal{E}\ell\ell_G^*$ we have associated an element

$$\gamma(\vartheta - \tilde{\vartheta}) \in \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{\mathrm{odd}}(BG;\ \mathbb{Q}/\mathbb{Z}).$$

We have restricted c to I_G and now composing with δ^{-1} we get a map

$$\gamma\colon I_G \to \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{\mathrm{odd}}(BG;\ \mathbb{Q}/\mathbb{Z}).$$

Let $f\colon Y \to BG$ be the canonical map classifying the fundamental group G-action on Y. Then, pulling back $\gamma(\vartheta - \widetilde{\vartheta})$ by this map, we get an element

$$f^*\left(\gamma(\vartheta - \widetilde{\vartheta})\right) = [\vartheta] \in \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{\mathrm{odd}}(Y;\mathbb{Q}/\mathbb{Z}).$$

As we have seen, our Y is orientable for elliptic cohomology and, for the coefficients considered, we know as well that we have defined a direct image map

$$\lambda_{\mathbb{Z}[1/n]}\colon \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{s}(Y;\mathbb{Z}) \to \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{s-2k-1}(\mathrm{pt};\ \mathbb{Z}),$$

where $\dim Y = 2k+1$. By the general construction recalled above, the map $\lambda_{\mathbb{Z}\left[\frac{1}{|G|}\right]}$ induces, for each $m \in \mathbb{Z}[n]$, a direct image map

$$\lambda[m]\colon \mathcal{E}\ell\ell^{s}(Y;\ \mathbb{Z}/m\mathbb{Z}) \to \mathcal{E}\ell\ell^{s-2k-1}(\mathrm{pt};\ \mathbb{Z}/m\mathbb{Z}).$$

These maps fit together to give a map

$$\lambda\colon \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{s}(Y;\ \mathbb{Q}/\mathbb{Z}) \to \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{s-2k-1}(\mathrm{pt};\ \mathbb{Q}/\mathbb{Z}).$$

Finally we get an element ($\mathrm{ind}_{\mathbb{F}}(\vartheta)$, in Devoto's suggested notation), that we will call a cohomological eta invariant,

$$\varkappa(\vartheta) = \lambda([\vartheta]) \in \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{\mathrm{even}}(\mathrm{pt};\ \mathbb{Q}/\mathbb{Z}).$$

4.3 Geometric interpretation. Modularity of cohomological eta invariants

In this section we will discuss the properties and interpretations of the constructed classes $\varkappa(\vartheta)$. We would give an insight of what is their geometric-topological meaning, in terms of Devoto's equivariant elliptic cohomology characteristic classes for some bundles. This requires to define the set of G-equivariant vector bundles over G considered itself as a G-space by conjugation. The definition of its elements $V = \{V_g\}$ and its properties includes a splitting of each of the components V_g with respect to (g,h) in TG. We will then use the approach in [FLM88] to introduce a multigrading induced by this splitting.

The Lie Algebra of $\left(\mathrm{Diff}^+(S^1_c)\right)_{\langle g\rangle}$ and of its (universal) central extensions —Virasoro extensions for $\mathrm{Vect}_{\mathbf{C}}(S^1_c)$, $\mathrm{Vect}_{\mathbf{C}}(S^1_c)_{\langle g\rangle}$— come into the picture now. We will concentrate in some of its elements in $\mathrm{Vect}_G(G)[[q^{\frac{1}{N}}]]$. Modularity will then come out as a consequence of the structure of the coefficient rings involved and the modularity of the complex orientation used.

This connects naturally the $\varkappa(\vartheta)$ invariants just defined which take elements from $\mathcal{E}\ell\ell_G^*$ to others in $\mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^*(X;\mathbb{Q}/\mathbb{Z})$ with the values of the elliptic genera associated to the theories considered and their corresponding Gysin maps and orientation class. Hence, the bundles appearing, being essentially the ones in Devoto's papers, when pulled back to the manifold whose fundamental group action we are analysing, give back the geometric interpretation. Their modularity comes out of the one of the coefficients of the elliptic cohomology theory involved in each case. This should be considered in the more general frame of the topological q-expansion principle as explained in [Lau99], but we will concentrate more heavily in some particular cases and analytical details, since they show up in our applications.

4.3.1 Virasoro algebras and finite groups

Let $\mathrm{Vect}_G(G)$ be the set of G-equivariant k-vector bundles over G (where k denotes the real field or the complex field depending on context). Here G is considered as a G-space by conjugation. An element $V \in \mathrm{Vect}_G(G)$ is a family V_g of vector spaces, for $g \in G$, and a family $x\colon V_g \to V_{gxg^{-1}}$ of linear maps compatible with the group product. In particular, each V_g affords a representation of $C_g(G)$. The set $\mathrm{Vect}_G(G)$ has two operations induced by $\oplus$ and $\otimes$. We shall say that an element $V \in \mathrm{Vect}_G(G)$ is *pseudo-irreducible* if each $g \in G$ acts on V_g by scalar multiplication.

Proposition 4.11 *Each element $V \in \mathrm{Vect}_G(G)$ has a decomposition into a sum of pseudo-irreducible elements.*

For each $g \in G$, the vector space V_g admits a decomposition $V_g = \bigoplus_{0\ \leq j\ <\ |g|} V_g[j]$, where g acts on $V_g[j]$ as $\exp\{2\pi i j/|g|\}$. Let $e = \{|g|,\ g \in G\}$. If $|G| \geq j > |g|$, we define $V_g[j] = 0$. Now, for each $0 \leq j \leq e$, we have a vector bundle $\pi\colon V_j \to G$ whose fibre over g is $V_g[j]$. Pick now any pair g, x of elements of G. If $v \in V_g[j] \subset V_g$ and we call $g\prime = hgh^{-1}$, then $g\prime(x(v)) = x(g(v)) = \exp\{2\pi i j/|g|\}$. This equation shows that the decomposition $V = \oplus V_j$ is really a decomposition of equivariant bundles which are pseudo-irreducible by definition.

Let $V = V_G(G)$ be an element in $\mathrm{Vect}_G(G)$. Take in it any element $g \in C_{\langle g_1, g_2\rangle}(G)$, where $(g_1, g_2) \in TG$, with $|g_1| = |g_2| = c$. Then $V\mid_{C_{\langle g_1,g_2\rangle}(G)}$ splits as

$$V\mid_{C_{\langle g_1,g_2\rangle}(G)} = \bigoplus_{k=0}^{c-1}\bigoplus_{j=0}^{c-1}\left(\ker\left(g_1 - e^{\frac{2\pi i\tau j}{c}}\mathrm{id}_{V_{g_1,g_2}}\right) \cap \ker\left(g_1 - e^{\frac{2\pi i k}{c}}\mathrm{id}_{V_{g_1,g_2}}\right)\right)$$

with $V\mid^{(j,k)}_{C_{\langle g_1,g_2\rangle}(G)} = \left(\ker\left(g_1 - e^{\frac{2\pi i\tau j}{c}}\mathrm{id}_{V_{g_1,g_2}}\right) \cap \ker\left(g_1 - e^{\frac{2\pi i k}{c}}\mathrm{id}_{V_{g_1,g_2}}\right)\right)$. Since V is a G-equivariant principal complex vector bundle over G as a G-set by conjugacy, it provides a representation for $C_g(G)$ for each $g \in G$, so it provides a representation for $C_g(G) \cap C_{\langle g_1,g_2\rangle}(G)$ by restriction. In particular, for $g = g_1$, $C_{\langle g_1,g_2\rangle}(G) \subseteq C_{g_1}(G)$.

Definition 4.12 *A* multi-grading m *on* $\mathbf{V} \in \mathrm{Vect}_G(G)$ *is a choice, for each $g \in G$, of a decomposition $V_g = \bigoplus_{j\in J(g)} V_g(j)$ where $V_g(j)$ is the part of V_g of degree j. The choice must* *be done in such a way that the sets $J(g)$ depend only on the conjugacy class of g and that the maps*

$$h\colon V_g \to V_{hgh^{-1}}$$

are graded maps of degree 0.

This makes that our variable in the $(V_g(j))_n$ have degree $\left(n+\frac{j}{|g|}\right)$. All right decompositions for h $\left(V_{[g,h]}(j,k)\right)_{n,m}$ there will have bigrading $\left(n+\frac{j}{|g|}\right), \left(m+\frac{k}{|g'|}\right)$.

Denote by $\mathrm{Vect}_{\mathbf{C}}\left(S_c^1\right)$ the complexified Lie algebra for $\mathrm{Diff}^+\left(S_c^1\right)$. We know that it will have as generators $\left\{d_{c,n}=-z_c^{n+1}\dfrac{d}{dz_c}, n\in Z\right\}$. Alternatively, if we set $z_c=e^{i\theta_c}$, then we may write $\left\{d_{c,n}=ie^{i\theta_c}\dfrac{d}{d\theta_c}, n\in\mathbb{Z}\right\}$, where $\dfrac{d}{d\theta_c}=iz_c\dfrac{d}{dz_c}$. We will consider then the Lie algebra of $\left(\mathrm{Diff}^+\left(S_c^1\right)\right)_{\langle g\rangle}$, which we will denote by $\left(\mathrm{Vect}_{\mathbf{C}}\left(S_c^1\right)\right)_{\langle g\rangle}$.

Definition 4.13 Let $\mathrm{V}\in \mathrm{Vect}_G(G)$ be any element and let $\mathrm{V}=\oplus_j V_j$ be its decomposition into pseudo-irreducible elements. We then consider the bundle of even positive Fourier coefficients $\mathcal{L}\mathrm{V}$. This bundle is an element of $\mathrm{Vect}_G(G)[[q^{\frac{1}{N}}]]$ for some N, which is an analogue of the space of positive Fourier coefficients, giving, for each $g\in G$,

$$(\mathcal{L}\mathrm{V})_g=\bigoplus_j\left(\bigoplus_{n\geq 0}\left(\ (V_g\left[j\right])_n\ \ q^{n+\frac{j}{|g|}}\right)\right).$$

Here, g acts on $V_g\left[j\right]$ as $\exp\left(2\pi ij/|g|\right)$.

Definition 4.14 In order to consider twisted fermions, we need to shift slightly the grading of the *Neveu–Schwartz bundle* $\mathcal{NS}\mathrm{V}\in K_G(G)[[q]]$, which is the bundle

$$\left(\mathcal{NS}\mathrm{V}\right)_g=\ \bigoplus_j\left(\bigoplus_{n>0}\left(\ V_g(j)\ \right)\left(-q^{n+\frac{j}{|g|}-\frac{1}{2}}\right)\right).$$

Definition 4.15 The *state space* $\mathcal{F}\in K_G(G)[[q^{\frac{1}{N}}]]$ of *twisted fermions* is the exterior algebra of the Neveu–Schwartz bundle,

$$\mathcal{F}_g=\Lambda_\bullet\left(\left(\mathcal{NS}\mathrm{V}\right)_g\right)=\bigotimes_{\substack{n\geq 0\\ j}}\ \wedge_{\left[-q^{n-\frac{1}{2}}\right]}V_g(j).$$

We are considering complex finite-dimensional $V_g\left[j\right]$. We give a basis $\left\{e^a_{g[j]}\right\}_{a=1}^{\dim_C(V_g[j])}$ which yields coordinates $\left\{z^a_{g[j]}\right\}_{a=1}^{\dim_C(V_g[j])}$. We have a splitting $V_g(j)\cong\oplus_{a=1}^{\dim_C(V_g[j])}V_g\left[j\right]^a$. Since g acts on $V_g\left[j\right]$ as $\exp\left(2\pi ij/|g|\right)$, this splitting is compatible with the g-action. Then,

$$(\mathcal{L}\mathrm{V})_g=\bigoplus_j\left(\bigoplus_{n\geq 0}\ \left(\oplus_{a=1}^{\dim_C(V_g[j])}V_g\left[j\right]^a_n\ \right)q^{n+\frac{j}{|g|}}\right),$$

which we can as well write as

$$(\mathcal{L}\mathrm{V})_g = \bigoplus_j q^{\frac{j}{|g|}} \left(\bigoplus_{n\geq 0} \left(\oplus_{a=1}^{\dim_C (V_g[j])} V_g\,[j]_n^a \right) q^n \right),$$

where $q^{\frac{j}{|g|}} V_g\,[j]^a$ is defined in [FFR91] or [Mil89].

Definition 4.16 The *Fock state space* $\mathcal{B} \in K_G(G)[[q^{\frac{1}{N}}]]$ of *twisted bosons* is the exterior algebra of the Ramond bundle,

$$\mathcal{B}_g = \mathrm{S}_\bullet \left((\mathcal{N}\mathcal{S}\mathrm{V})_g \right) = \bigotimes_{\substack{n\geq 0 \\ j}} \wedge_{[-q^{n-\frac{1}{2}}]} V_g(j),$$

where $\mathcal{B}_g\,[j] = \mathcal{B}\,(V_g\,[j]) = \mathrm{S}_\bullet \left(\bigoplus_j q^{\frac{j}{|g|}} \left(\bigoplus_{n\geq 0} \left(\oplus_{a=1}^{\dim_C (V_g[j])} V_g\,[j]_n^a \right) q^n \right) \right) =$

$\mathrm{S}_\bullet \left(\bigoplus_j \left(\bigoplus_{n\geq 0} \left(\oplus_{a=1}^{\dim_C (V_g[j])} V_g\,[j]_{n+\frac{j}{|g|}}^a \right) q^{n+\frac{j}{|g|}} \right) \right).$

4.3.2 Construction of some G-elliptic objects on odd-dimensional manifolds

We recall some well-known facts on the smooth loop space $\mathcal{L}BG$ of the classifying space of G a finite group; see [BT96].

Proposition 4.17

$$\mathcal{L}BG = \amalg_{[g]\in\mathcal{C}(G)}\, \mathcal{L}_{[g]}BG,$$

$$\mathcal{L}_{[g]}BG \simeq BC_g\,(G)\,.$$

Taking this into account, a general vector bundle $V \downarrow \mathcal{L}BG$ will be determined by its "components" $V_{[g]} \downarrow \mathcal{L}_{[g]}BG$. We will consider such vector bundles of the form $\oplus_{n\in Z} V_{[g]}^n \downarrow \mathcal{L}_{[g]}BG$, $\dim \left(V_{[g]}^n \right) < \infty$, Z a weighted copy of $\mathbb{Z}$, e.g. $\mathbb{Z} + \frac{j}{c}$, for some integer c. We will keep track of this weighting by using the by now just formal variable $q^{\frac{1}{c}}$. Suppose from now on that, if not otherwise stated, $c = |G|$. The grading applies as well to the bundle altogether $V = \oplus_{n\in Z} V^n \downarrow \mathcal{L}BG$.

Suppose we are given an element V^n in $\mathrm{Vect}_{G,\,\mathrm{fin}}\,(G)$. From classical K-theory we know that $\mathrm{Groth}(\mathrm{Vect}_{G,\,\mathrm{fin}}\,(G)) = K_G\,(G)$, so that

$$\mathrm{Groth}\,(\mathrm{Vect}_G\,(G)) = K_G\,(G) \left[\left[q^{\frac{1}{|G|}} \right] \right] \cong \oplus_{[g]\in\mathcal{C}(G)}\, R\,(C_g\,(G)) \left[\left[q^{\frac{1}{|C_g(G)|}} \right] \right].$$

The Baker–Thomas expressions belong to

$$\oplus_{[g]\in\mathcal{C}(G)}\, K\left(\mathcal{L}_{[g]}BG\right) \left[\left[q^{\frac{1}{|G|}} \right] \right] =$$

$$\oplus_{[g]\in\mathcal{C}(G)}\ K\left(BC_g\left(G\right)\right)\left[\left[q^{\frac{1}{|G|}}\right]\right]=$$

$$\oplus_{[g]\in\mathcal{C}(G)}\ R\left(\widehat{BC_g\left(G\right)}\right)\left[\left[q^{\frac{1}{|G|}}\right]\right],$$

by Atiyah completion.

Let G be a finite group of odd order. Let M be a compact connected spin manifold of odd dimension, and F a principal G-bundle over M. Such an F will be classified by a map $\ell_{[F]}\colon \pi_1(M)\to G$ defined up to homotopy of F, namely the *holonomy*.

Let $\mathcal{L}M$ be the free (smooth) loop space on M, and consider $\gamma\in\mathcal{L}M$. A loop in $\mathcal{L}M$ itself, $\sigma\in\mathcal{L}(\mathcal{L}M)$, will be given by a pair $\sigma=\left(\breve{\Sigma}\xrightarrow{\breve{f}}M\right)$, where $\breve{\Sigma}$ is a torus with a distinguished circle s in it, and $\breve{f}$ is a continuous map, in such a way that $\breve{f}\mid_s\equiv\gamma$, the base point of the loop $\breve{\sigma}$ in $\mathcal{L}M$. Since the manifold M is spin, we want to consider only $\breve{\sigma}$ compatible with the spin structure on M. Similarly, a path in $\mathcal{L}M$ itself, will be given by $\sigma=\left(\Sigma\xrightarrow{f}M\right)$, where Σ is an annulus with two parametrised connected components in its boundary, one incoming (S^1_0,s^1_0) and one outgoing (S^1_1,s^1_1), with a spin structure on it, and f is a continuous map preserving the spin structure, in such a way that $f\mid_{S^1_0}\equiv\gamma_0$ and $f\mid_{S^1_1}\equiv\gamma_1$ respectively, the beginning and the end point of the path σ in $\mathcal{L}M$. We want to consider both the loop and the path to have a complex structure on the interior of their domain, so a modulus, say τ, will exist for everyone of them. In the next chapter, we will call one such Σ one Segal annulus, and we will analyse how the complex structure on its interior determines a modulus $\tau\in\mathfrak{H}$. Collapsing the boundary of the Segal annulus by identifying both parametrisations gives a torus with the same modulus τ. Remark that for a given Σ, not always a convenient σ will exist; this is the way the geometry of M comes into our picture. The pullback bundle $f^*(F)$ of F over Σ is a principal G-bundle over Σ such that, restricted to its boundaries, covers their parametrisations, and the complex structure on the interior of Σ lifts to the interior of the total space, termed f^*F as well.

The G-action on $\pi\colon F\downarrow M$ induces a splitting in $\mathcal{L}M$ compatible with the one in the connected components induced by the $\pi_1(M)$-action on $\widetilde{M}$, the universal covering of M;

$$\mathcal{L}M=\amalg_{[\alpha]\in\mathcal{C}(\pi_1(M))}\ \mathcal{L}_{[\alpha]}M,\qquad g\in G,$$

with $[g]\in\mathcal{C}(G)$, $\mathcal{L}_{[g]}M=\amalg_{[\alpha]\in\mathcal{C}\left(\ell_{[F]}\right)^{-1}([g])}\mathcal{L}_{[\alpha]}M$, $\mathcal{L}F=\amalg_{g\in G}\mathcal{L}_gF$,

$$\mathcal{L}_gF=\left\{\hat{\gamma}:\hat{\gamma}\left(t+1\right)=g\cdot\hat{\gamma}\left(t\right),\forall t,\,\pi\circ\hat{\gamma}=\gamma\in\mathcal{L}M\right\}.$$

Since $\hat{\gamma}$ is a smooth map, $\hat{\gamma}(\mathbb{R})$ will be contained on a connected component of F, which we call $F_{\hat{\gamma}}$.

If $\widetilde{M}\downarrow M$ is the universal covering, then $\pi_1(M)\overset{\ell_{[G]}}{\twoheadrightarrow}G$, $G\curvearrowright F$, for every

$$\alpha\in\mathcal{L}_\alpha\widetilde{M}=\left\{\widetilde{\gamma}:\widetilde{\gamma}\left(t+1\right)=\alpha\cdot\widetilde{\gamma}\left(t\right),\forall t,\,\pi\circ\widetilde{\gamma}=\gamma\in\mathcal{L}M\right\}.$$

For every $\beta \in \pi_1(M)$ there is a map $\mathcal{L}_\alpha \widetilde{M} \overset{\mathcal{L}(\beta)}{\to} \mathcal{L}_{\beta\alpha\beta^{-1}} \widetilde{M}$, $p(t) = (\beta \bullet p)(t) = \beta(p(t))$. This induces for every pair g, h of elements in G a map $\mathcal{L}_g F \overset{\mathcal{L}(h)}{\to} \mathcal{L}_{hgh^{-1}} F$. If $h \in C_g(G)$, then we get a self map $\mathcal{L}_g F \overset{\mathcal{L}(h)}{\to} \mathcal{L}_g F$. So we get an action of $C_g(G)$ in $\mathcal{L}_g F$. Evaluation of loops in any given point, plus the fact that F is a G principal bundle, tells us that this action is principal. We have, at the homotopy level,

$$\mathcal{L}_g F / C_g(G) \simeq \mathcal{L}_{[g]} M,$$

i.e., $\mathcal{L}_g F \downarrow \mathcal{L}_{[g]} M$ is a principal $C_g(G)$-bundle. So, for every representation $\rho_{C_g(G)}$ of $C_g(G)$ on a (complex) vector space V_g, we may construct an associated bundle $\mathcal{L}_g F \times_{\rho_{C_g(G)}} V_g \downarrow \mathcal{L}_{[g]} M$ with fibre V_g and structure group $C_g(G)$. Thus, using $\rho_{C_g(G)}$ we construct an associated bundle

$$\xi_g = \mathcal{L}_g F \times_{\rho_{C_g(G)}} \mathcal{H}_g \downarrow \mathcal{L}_{[g]} M$$

with structure group $C_g(G)$ as well and fibre $\mathcal{H}_g$. Get $\xi = \amalg_{g \in \mathcal{C}(G)} \xi_g \downarrow \mathcal{L}M$ a global bundle over $\mathcal{L}M$. Then $\xi \mid_{\mathcal{L}_{[g]} M} = \xi_g$. In the next chapter, we will see that this construction turns out to give an admissible bundle ξ in the sense of Baker and Thomas and that, furthermore, it satisfies a modularity condition which makes it a sound *G-elliptic object* in the sense of Segal.

Let G be a finite group of odd order n. We have previously considered for such groups G-modular forms of level $(2, |G|)$, i.e., elements in $\mathcal{E}\ell\ell_G^*$,

$$\psi_{j,k}(\tau) \in \mathcal{E}\ell\ell^{-2}(\Gamma(g,h)).$$

It can be seen that under the maps Λ, Φ in [Dev98] we have a correspondence

$$\psi_{j,k}(\tau) \longleftrightarrow \psi_{jk} \in \mathcal{E}\ell\ell_G^{-2}$$

with the map $\psi_{jk}(\tau)\colon TG \times \mathfrak{H}_+ \to \mathbb{C}$ trivial for $(g_1, g_2) = (e, e)$. Hence, this element (and its inverse) will be termed as well

$$\psi_{jk}(\tau) \in \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{-2}(BG).$$

We have seen before, as a consequence of what happens for cyclic or bicyclic groups, that the element ψ_{jk} is the inverse of a primitive root of Igusa's polynomial $\Phi_n(X)$, say $z = 1/x$. Therefore, it can be seen as a modular form (see [Dev98]). On the other hand, we have seen that the Bockstein coboundary connecting map β is an isomorphism

$$\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}(BG; \mathbb{Q}/\mathbb{Z}) \cong \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*+1}(BG),$$

so we have $w = \beta^{-1}(z) \in \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{-3}(BG; \mathbb{Q}/\mathbb{Z})$. Using the same tools as in [Dev98], we can see naturally w as $\operatorname{ch}(\xi_g)$, for a bundle ξ_q over $\mathcal{L}BG$.

Consider now a spin manifold Y of dimension $4k-1$ which bounds a spin manifold X of dimension $4k$, with $\pi_1(Y)=G$. Then we have a classifying map

$$f_Y\colon Y \to BG$$

and pulling back ξ_q to Y we obtain a bundle

$$V_q = f_Y^*(\xi_q)$$

over Y which happens to be flat (recall it is q-ing from symmetric and exterior powers of the flat bundle V). The following assumption is modelled as usual on [APS75II, p. 415].

Assumption V_q can be extended to a bundle W_q over X, not necessarily flat. Choose on it any connection extending the flat one on V_q and choose any Riemannian metric on X. Suppose that we have a (not necessarily classifying) map $f_X\colon Y \to BG$ extending f_Y (locally constructible by transversality results).

Then for $t_{\mathcal{E}}^{\mathcal{K}}$ the Miller elliptic genus and ch the Chern character (both with suitable coefficients), applying the Riemann–Roch theorem for generalised complex orientable cohomology theories and their multiplicative transformations as in [Mil89] we obtain, by pushing forward,

$$\mathrm{ch}\left(t_{\mathcal{E}}^{\mathcal{K}}\left(p_!^{\mathcal{E}}\left(f_X^*(w)\right)\right)\right) =$$

$$\mathrm{ch}\left(p_!^{\mathcal{K}}\left(t_{\mathcal{E}}^{\mathcal{K}}\left(\left(f_X^*(w)\right)\right)\right)\cup R_q(TX)\right) =$$

$$p_!^{\mathcal{H}}\left(\mathrm{ch}\left(t_{\mathcal{E}}^{\mathcal{K}}\left(f_X^*(w)\right)\cup R_q(TX)\cup \hat{A}(TX)\right)\right),$$

where $R_q(TX)$ and $\hat{A}(TX)$ are as defined earlier. By evaluation and its properties (see e.g. [HBJ92]),

$$p_!^{\mathcal{H}}\left(\mathrm{ch}\left(t_{\mathcal{E}}^{\mathcal{K}}\left(f_X^*(y)\right)\right)\cup R_q(TX)\cup \hat{A}(TX)\right)([1]_{\mathcal{H}}) =$$

$$\left\{\mathrm{ch}\left(t_{\mathcal{E}}^{\mathcal{K}}\left(f_X^*(y)\right)\right)\cup R_q(TX)\cup \hat{A}(TX)\right\}([X]_{\mathcal{H}}),$$

and, since $R_q(TX)\cup \hat{A}(TX) = \Phi_{\mathcal{E}}(TX)$, we obtain

$$\left\{\mathrm{ch}\left(t_{\mathcal{E}}^{\mathcal{K}}\left(f_X^*(y)\right)\right)\cup \Phi_{\mathcal{E}}(X)\right\}([X]_{\mathcal{H}}),$$

being $\Phi_{\mathcal{E}}(TX)$ the cohomology class obtained by applying to the Pontrjagin class of X (i.e., of its tangent bundle) the multiplicative sequence associated to the considered elliptic genus. It is well known (e.g. [Zag88]) that it is determined by

$$Q(\tau,x) = \exp\left(\sum_{j=1}^{\infty}\frac{2\widetilde{G_{2j}(\tau)}}{(2j)!}x^{2j}\right),$$

so it is a series in the Pontrjagin classes whose coefficients are modular forms for the congruence subgroup $\Gamma_0(2)$. Since

$$\mathrm{ch}\left(t_{\mathcal{E}}^{\mathcal{K}}\left(f_X^*\left(y\right)\right)\right)$$

is a polynomial in the Chern classes of the pullback bundle W_q over X, if we see that the coefficients are modular forms for the right thing of the right weight, we are done. The important point is to see that "the pullback f^* preserves modularity"; the multiplicative property of ch and $t_{\mathcal{E}}^{\mathcal{K}}$ will do the rest,

$$\mathrm{ch}\left(t_{\mathcal{E}}^{\mathcal{K}}\left(f_X^*\left(y\right)\right)\right) = \mathrm{ch}\left(W_q\right),$$

with $W_q\mid_Y \sim V_q$, flat, so with trivial rational Chern classes,

$$f_X^* \colon \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{-2}(BG) \to \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{-2}(X).$$

4.4 Modularity of elements in elliptic cohomologies modulo localised integers

Our intention is to describe as much as possible $\mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^*(X;\mathbb{Q}/\mathbb{Z})$, for X a finite CW-complex (let us write $X \in \underline{\mathrm{CW}}^{<\infty}$). We know from [Ada74, p. 201] that we have the short exact sequence

$$0 \to \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^*(X)\otimes\mathbb{Q}/\mathbb{Z} \to \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^*(X;\mathbb{Q}/\mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}}\left(\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*+1}(X),\ \mathbb{Q}/\mathbb{Z}\right) \to 0$$

where the condition $X \in \underline{\mathrm{CW}}^{<\infty}$ is essential, since $\mathbb{Q}/\mathbb{Z}$ is not finitely generated. We can in any case consider this as the theory associated to the Moore spectrum $\mathbf{Ell}\left[\frac{1}{n}\right]\wedge\mathbb{Q}/\mathbb{Z}$ as described in [Ada74].

If we restrict to the case of the point, we have, since $\mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^* = \mathbb{Z}\left[\frac{1}{2}\right][\delta,\varepsilon]\left[\Delta^{-1}\right]$ is torsion-free and hence $\mathrm{Tor}_1^{\mathbb{Z}}\left(\mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^{*+1}(\mathrm{pt}),\ \mathbb{Q}/\mathbb{Z}\right) = 0$,

$$\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^*\otimes\mathbb{Q}/\mathbb{Z} \cong \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^*(\mathrm{pt};\mathbb{Q}/\mathbb{Z}).$$

We are interested in the modularity of this. We know from general algebra that since

$$\mathbb{Q}/\mathbb{Z} \cong \oplus_{p\in\mathfrak{P}}\,\mathbb{Z}/\left(p^\infty\right),$$

for $\mathfrak{P}$ the set of all primes, then

$$\begin{aligned}
\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^*\otimes\mathbb{Q}/\mathbb{Z} &\cong \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^*\otimes\left(\oplus_{\substack{p\in\mathfrak{P}\\ p\nmid n}}\mathbb{Z}/\left(p^\infty\right)\right)\\
&\cong \oplus_{p\in\mathfrak{P}}\left(\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^*\otimes\mathbb{Z}/\left(p^\infty\right)\right),
\end{aligned}$$

and hence we are interested in understanding each of the direct summands $\mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^{*}\otimes\mathbb{Z}/\left(p^{\infty}\right)$, with $\mathbb{Z}/\left(p^{\infty}\right)=\varinjlim_{i}\mathbb{Z}/\left(p^{i}\right)$.

We have that

$$\begin{aligned}\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}\otimes\mathbb{Z}/\left(p^{\infty}\right) &= \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}\otimes\varinjlim_{i}\left(\mathbb{Z}/\left(p^{i}\right)\right)\\ &\cong \varinjlim_{i}\left(\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}\otimes\mathbb{Z}/\left(p^{i}\right)\right).\end{aligned}$$

As explained in [Dev96b, p. 388], one has elliptic cohomology theories associated to mod $\left(p^{i}\right)$ elliptic genera such that their coefficients

$$\mathcal{E}\ell\ell^{*}\left(\mathrm{pt};\mathbb{Z}/\left(p^{i}\right)\right)\cong\mathcal{E}\ell\ell^{*}\otimes\mathbb{Z}/\left(p^{i}\right)$$

are the *modular forms mod* p^{i}, $\mathcal{M}_{-*}\left(\Gamma;p^{i}\right)$, as they appear described by Swinnerton-Dyer, Serre, and more schematically by Katz [Kat75]. We will turn on more detail on their description in the frame considered by [Dev96b], since it is relevant for our job in relation to the Atiyah–Patodi–Singer construction. So, we have got

$$\begin{aligned}\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}\otimes\mathbb{Q}/\mathbb{Z} &\cong \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}\otimes\left(\oplus_{p\in\mathfrak{P}}\mathbb{Z}/\left(p^{\infty}\right)\right)\\ &\cong \oplus_{p\in\mathfrak{P}}\left(\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}\otimes\mathbb{Z}/\left(p^{\infty}\right)\right)\\ &\cong \oplus_{p\in\mathfrak{P}}\left(\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}\otimes\varinjlim_{i}\left(\mathbb{Z}/\left(p^{i}\right)\right)\right)\\ &\cong \oplus_{p\in\mathfrak{P}}\left(\varinjlim_{i}\left(\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}\otimes\mathbb{Z}/\left(p^{i}\right)\right)\right)\\ &\cong \oplus_{p\in\mathfrak{P}}\left(\varinjlim_{i}\left(\mathcal{M}_{-*}^{\left[\frac{1}{|G|}\right]}\left(\Gamma_{0}\left(2,N\right);p^{i}\right)\right)\right),\end{aligned}$$

where $\mathcal{M}_{-*}^{\mathcal{E}}\left(\Gamma_{0}\left(2,N\right);p^{i}\right)$ are the Serre modular forms once localised at Δ.

Now we want to describe the summands $\mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^{*}\otimes\mathbb{Z}/\left(p^{\infty}\right)$ in terms of the divided congruences of [Kat75]. Since

$$\begin{aligned}\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}\otimes\mathbb{Z}/\left(p^{\infty}\right) &= \mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}\otimes\varinjlim_{i}\left(\mathbb{Z}/\left(p^{i}\right)\right)\\ &\cong \varinjlim_{i}\left(\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{*}\otimes\mathbb{Z}/\left(p^{i}\right)\right),\end{aligned}$$

it looks like we will have to be interested in considering congruences between modular forms mod p^{n} for all n altogether. This is the problem that Katz considered in [Kat75].

Chapter 5

Segal annuli and elliptic cohomology

We will describe an interpretation of the construction of the previous chapter in terms of conformal field theory.

The aim of this chapter is to obtain elements in the coefficient ring $\mathcal{E}\ell\ell_*^G$ for G-equivariant elliptic cohomology, as defined by Devoto [Dev96b, Dev98], from characters of certain representations of geometric objects proposed by Segal in his definition of conformal field theory.

We will not go deeply into the physical interpretation of these objects, although we will study carefully their representation theory once we have them defined. Nevertheless, we will give some motivation here. Recall that in theoretical physics one considers strings (one-dimensional compact smooth manifolds) propagating through a particular space-time. One requires this propagation to be conformal, which makes sense in both Minkowskian and Euclidean space-times. A description of the evolution of a string gives a conformal morphism between its incoming (initial) state and outgoing (final) state, and these morphisms must be composable. Suppose our string is closed and connected —a copy of the circle S^1. Then its evolution will be described by the image in our space-time of a complex manifold with boundary, in the form of an annulus or cylinder A. If we want our system to carry any information, we consider real or complex valued functions associated to the initial and final states. In fact one considers orientation-preserving diffeomorphisms $\phi\colon S^1 \to S^1$ defined on the boundary of the manifold A. The space A of all such evolutions, up to equivalence, will be

$$(0,1) \times \left(\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)\right) \;/\; S^1$$

where $r \in (0,1)$ represents the length of A, $\mathrm{Diff}^+(S^1)$ is the infinite dimensional Lie group of orientation-preserving diffeomorphisms of S^1, and the quotient by S^1 arises from the rotational symmetry of the situation.

Our "objects" A are in fact morphisms in Segal's category of manifolds. There is a composition operation defined by sewing the outgoing boundaries of one manifold to the incoming boundaries of another, according to their parametrisations ϕ. In our case, since we consider only manifolds which are topologically annuli, composition gives A the structure of a semigroup.

Segal also considers some extensions of A that we are interested in: A^{Spin}, the extension of A by spin structures; $A(G)$, its extension by the action of a finite group G; and finally $A^{\mathrm{Spin}}(G)$, encoding both simultaneously.

5.1 Segal categories of annuli

In the first part of this section we will define a (Lie) semigroup $\mathcal{A}$, whose elements are the *Segal annuli*. Intuitively, a Segal annulus may be thought of as an equivalence class of Riemann surfaces with boundary $S^1 \sqcup S^1$ and certain additional structure. Our exposition is based on Section 2 of an unpublished manuscript by Segal. We will then describe several extensions $\mathcal{A}(G)$, $\mathcal{A}^{\mathrm{Spin}}$, $\mathcal{A}^{\mathrm{Spin}}(G)$ of $\mathcal{A}$ by a finite group G and by possible spin structures.

5.1.1 Segal annuli

For $0 < r < 1$, consider the *standard annulus* A_r in $\mathbb{C}$ given by

$$A_r = \{z \in \mathbb{C};\ r \leq |z| \leq 1\}\,.$$

The boundary components of the standard annulus A_r are the standard circles $S^1, S^1_r \subset \mathbb{C}$ of radius 1 and r. We write $A_{>r}$ for the "half-open" annulus $A_r - S^1_r$.

Definition 5.1 A *surface with parametrised boundaries* is a compact smooth 2-dimensional manifold X with boundary ∂X, together with a diffeomorphism $s\colon S^1 \cong S$ for each connected component S of the boundary. We assume further that a there is a complex structure defined in the interior of X and each boundary circle has a neighbourhood diffeomorphic to some $A_{>r}$ by a map which is holomorphic in the interior.

Observe that the complex structure on the interior of X induces an orientation on each component of the boundary. A boundary circle S is usually called *outgoing* if the parametrisation s agrees with this orientation, or *incoming* otherwise.

Definition 5.2 A *Segal pre-annulus* is given by a surface with parametrised boundaries A which is diffeomorphic to some standard annulus and which has one incoming circle and one outgoing circle. A *Segal annulus* is then an isomorphism class of Segal pre-annuli, where two Segal pre-annuli are said to be *isomorphic* if they are diffeomorphic by a map which is holomorphic in the interior and which respects the boundary orientations and their parametrisations. We write $\mathcal{A}$ for the set of all Segal annuli.

A simple but important example is the Segal annulus A_q, for $q = re^{i\theta}$, $0 < r < 1$, represented by the surface A_r with its canonical complex structure and boundary parametrisations $S^1 \to S^1_r$, $S^1 \to S^1$ given by multiplication by q and the identity respectively.

To work with the object $\mathcal{A}$ it will be convenient to recall several alternative descriptions. Let $\mathrm{Diff}^+(S^1)$ be the set of orientation-preserving diffeomorphisms $s\colon S^1 \to S^1$, and let S^1 act on $\mathrm{Diff}^+(S^1)$ by pre-multiplication,

$$(s \cdot w)\colon \; z \;\mapsto\; s(wz), \quad \text{for } s \in \mathrm{Diff}^+(S^1),\; w, z \in S^1.$$

(With the operation induced by composition, $\mathrm{Diff}^+(S^1)$ is an infinite-dimensional simple Lie group. One of Segal's motivations for defining $\mathcal{A}$ is that it should play the rôle of the non-existent complexification of this Lie group. Note that $\pi_1\left(\mathrm{Diff}^+(S^1)\right) \cong \mathbb{Z}$.)

By a corollary of the Riemann Mapping Theorem, any Segal pre-annulus is isomorphic to some standard annulus A_r, $r \in (0,1)$, with boundary parametrisations which are given for some $s, t \in \mathrm{Diff}^+(S^1)$ by

$$\begin{array}{ccc} S^1 & \longrightarrow & S^1_r \\ e^{i\theta} & \mapsto & rs(e^{i\theta}), \end{array} \qquad\qquad \begin{array}{ccc} S^1 & \longrightarrow & S^1 \\ e^{i\theta} & \mapsto & t(e^{i\theta}) \end{array}$$

on the incoming (= inner) and outgoing (= outer) boundary circles respectively. Two such Segal pre-annuli (A_r, s, t), $(A_{r'}, s', t')$ may themselves be isomorphic, by a rigid rotation, if $r = r'$ and $s \cdot w = s'$, $t \cdot w = t'$ for some $w \in S^1$. We therefore have:

Proposition 5.3 *There is a bijection*

$$\mathcal{A} \;\cong\; (0,1) \;\times\; \left(\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)\right)/S^1$$

where S^1 acts diagonally on the product.

We can choose the rigid rotation such that $t(1) = 1$, and the parametrisations may be defined by

$$\begin{array}{ccc} S^1 & \longrightarrow & S^1_r \\ e^{i\theta} & \mapsto & q\, s(e^{i\theta}), \end{array} \qquad\qquad \begin{array}{ccc} S^1 & \longrightarrow & S^1 \\ e^{i\theta} & \mapsto & t(e^{i\theta}) \end{array}$$

with $s(1) = t(1) = 1$, $q = re^{i\theta}$. Thus any element of $\mathcal{A}$ may be expressed uniquely as (A_q, s, t) in this manner and we have a bijection

$$\mathcal{A} \;\cong\; \mathbb{C}^{\times}_{<1} \;\times\; \mathrm{Diff}^+_1(S^1) \times \mathrm{Diff}^+_1(S^1), \tag{5.1}$$

where $\mathrm{Diff}^+_1(S^1)$ is the subgroup of diffeomorphisms preserving the base point $1 \in S^1$.

To any Segal pre-annulus A we associate a closed surface $\widehat{A}$ by attaching disks

$$D = \{z \in \mathbb{C},\; |z| \le 1\} \qquad \text{and} \qquad D_\infty = \{z \in \mathbb{C},\; |z| \ge 1\} \cup \{\infty\}$$

to the incoming and outgoing boundary circles according to the their parametrisations. The resulting surface may be identified holomorphically with the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$,

$$\begin{tikzcd} S^1 \sqcup S^1 \arrow[r] \arrow[d] & D \sqcup D_\infty \arrow[d] & \\ A \arrow[r] & \widehat{A} \arrow[r, "\Theta", "\cong"'] & \mathbb{C} \cup \{\infty\}. \end{tikzcd}$$

(As Segal remarks, the existence of the complex structure on the sewn manifold "is a non-trivial theorem" which is "by no means obvious". We will not give the details here.)

Now the boundary parametrisations become maps $S^1 \to \mathbb{C} \cup \{\infty\}$ which extend to embeddings f_0, f_∞ of D, D_∞ in the Riemann sphere. The identification Θ may be chosen uniquely such that the images $f_0(0)$ and $f_\infty(\infty)$ are 0 and ∞ respectively and such that $f'_\infty(\infty) = 1$, for example, and so we have

Proposition 5.4 *Any element in $\mathcal{A}$ is uniquely representable as the annulus A in $\mathbb{C}$ bounded by curves $S^1 \to \mathbb{C}$*

$$\begin{array}{rcl} z \;\longmapsto\; f_0(z) & = & a_1\, z \;+\; a_2\, z^2 \;+\; a_3\, z^3 \;+\; \cdots \\ z \;\longmapsto\; f_\infty(z) & = & \left(z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots \right)^{-1} \end{array}$$

which extend respectively to holomorphic embeddings

$$\begin{array}{l} f_0 : D \hookrightarrow \mathbb{C} \subset S^2, \\ f_\infty : D_\infty \hookrightarrow \mathbb{C}^\times \cup \{\infty\} \subset S^2. \end{array}$$

A generalisation of this structure is given by Huang, who considers isomorphism classes of Riemann spheres with $n+1$ holomorphically embedded disks (one incoming and n outgoing) in order to give a geometric foundation for the theory of vertex operator algebras. However the theory we study here is slightly stricter since Huang does not assume that the embedded disks are disjoint.

Using the representation of Segal annuli embedded holomorphically in the complex plane, one can identify the tangent space of $\mathcal{A}$ similarly. A tangent vector at (A, f_0, f_∞) in $\mathcal{A}$ is determined by a complex tangent vector field X along $\partial A \subset A$, but represents the trivial vector if it can be extended to a holomorphic vector field on all of A. Using

$$(f_0^{-1}, f_\infty^{-1})\colon \partial A \cong S^1 \sqcup S^1,$$

the vector field X can be further identified with a pair of elements of $\mathrm{Vect}_{\mathbb{C}}(S^1)$. Thus we obtain

Proposition 5.5 *The tangent space to $\mathcal{A}$ is given, using the notation of Proposition 5.4, by*

$$T_{(A,f_0,f_\infty)}(\mathcal{A}) \;\cong\; \left(\mathrm{Vect}_{\mathbb{C}}(S^1) \oplus \mathrm{Vect}_{\mathbb{C}}(S^1)\right) \;/\; \mathrm{Vect}_{\mathbb{C}}(A)$$

where $\mathrm{Vect}_{\mathbb{C}}(A)$ *is considered as a subspace of* $\mathrm{Vect}_{\mathbb{C}}(S^1) \oplus \mathrm{Vect}_{\mathbb{C}}(S^1)$ *via* $Y \mapsto (f_0^{-1}, f_\infty^{-1}) Y|_{\partial A}$.

Denote by $\mathrm{Hol}(D)$ the set of holomorphic functions from the unit disk D to $\mathbb{C}$ with smooth boundary values. Any holomorphic map $f\colon D \to \mathbb{C}$ is determined by the restriction $f|_{S^1}\colon S^1 \to \mathbb{C}$, since its value at any interior point z_0 of D is given by the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0}\, dz.$$

In particular, $\mathrm{Hol}(D)$ may be regarded as a subspace of $C^\infty(S^1)$. If we consider the space

$$\mathrm{Hol}_1(D) = \{\, f \in \mathrm{Hol}(D);\ f(0) = 0,\ f'(0) = 1 \,\}$$

then with the notation of Proposition 5.4 we can identify Segal annuli with elements

$$(\, a_1,\, {a_1}^{-1} f_0,\, f_\infty^{\mathrm{inv}}) \ \in\ \mathbb{C} \times \mathrm{Hol}_1(D) \times \mathrm{Hol}_1(D)$$

where $f_\infty^{\mathrm{inv}} \colon D \to \mathbb{C}$ is given by $z \mapsto f_\infty(z^{-1})^{-1}$. In this way, $\mathcal{A}$ may be identified with a (bounded, open) domain in $\mathbb{C} \times \mathrm{Hol}_1(D) \times \mathrm{Hol}_1(D)$.

5.1.2 The torus, moduli, and composition

Choose $q = re^{i\theta}$ and consider the standard *torus with a preferred cycle*

$$\breve{A}_q = \mathbb{C}^\times \,/\, \{z \sim qz\}.$$

Considering a fundamental domain, $\breve{A}_q$ is obtained from the standard annulus A_r by identifying the boundary circle S^1 with the boundary circle S^1_r after a rotation by θ. For any $\tau \in \mathbb{C}$ with $q = e^{2\pi i \tau}$ we can identify the *standard torus* $T_\tau = \mathbb{C} \,/\, \mathbb{Z} + \tau\mathbb{Z}$ with $\breve{A}_q$ via the exponential map,

$$\begin{array}{ccc} T_\tau = \mathbb{C} \,/\, \mathbb{Z} + \tau\mathbb{Z} & \longrightarrow & \breve{A}_q = \mathbb{C}^\times \,/\, \{z \sim qz\} \\ w & \mapsto & e^{2\pi i w}. \end{array}$$

The canonical complex structure on T_τ gives the complex structure of $\breve{A}_q$.

Definition 5.6 The *modulus* of a Segal annulus A is the complex number q with $0 < |q| < 1$ such that the torus with preferred cycle $\breve{A}$ obtained by identifying the outgoing and incoming boundary circles according to their parametrisations is isomorphic to $\breve{A}_q$.

An alternative definition could be given using the notation of Proposition 5.4. A Segal annulus given by $A \subset \mathbb{C}$ bounded by the curves f_0 and f_∞ has modulus q if there exists an injective holomorphism $\Theta\colon A \to \mathbb{C}^\times$ such that $\Theta(f_0(z)) = q\,\Theta(f_\infty(z))$ for all $z \in S^1$. The Segal annulus A_q clearly has modulus q.

The modulus defines a holomorphism $\mathcal{A} \to \mathbb{C}^\times$, although we will not give a proof of this fact here.

Two Segal pre-annuli can be composed by identifying the outgoing boundary circle of one with the incoming boundary circle of the other, according to their parametrisations. This is termed the *sewing operation*. In the language of Proposition 5.4, the composite of two Segal annuli A and A', defined by curves f_0, f_∞ and f'_0, f'_∞, is the Segal annulus B defined by curves g_0, g_∞ if there exist injective holomorphisms $\Theta\colon A \to \mathbb{C}^\times$, $\Theta'\colon A' \to \mathbb{C}^\times$ such that $\Theta f_0 = g_0$, $\Theta' f'_\infty = g_\infty$ and $\Theta f_\infty = \Theta' f'_0$. Using Proposition 5.5, one can show

Theorem 5.7 *The sewing operation induces a well-defined associative multiplication on* $\mathcal{A}$ *such that the composition map*

$$\mu\colon \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

is a holomorphism.

Thus $\mathcal{A}$ may be termed a Lie semigroup. It is not a monoid since the obvious neutral element, S^1, is not a surface.

Remark 5.8 Segal also defines a category of surfaces $\mathcal{C}$. The objects C_n of $\mathcal{C}$ are given by disjoint unions of n copies of the circle S^1. A morphism $C_m \to C_n$ is given by an isomorphism class of Riemann surfaces X with boundary ∂X, together with an identification $C_n \sqcup C_m \cong \partial X$ which is orientation preserving on C_n and orientation reversing on C_m. Two surfaces are isomorphic if they are diffeomorphic by a map which respects the complex structures and the parametrisations. Composition of morphisms is defined by the sewing operation. (Segal's category also lacks identities, which one should formally add to the definition in order to use the word "category".) Of course $\mathcal{A}$ is just the subsemigroup of $\mathcal{C}$ consisting of the morphisms $C_1 \to C_1$ given by surfaces which are topologically annuli.

5.1.3 Principal G-bundles over Segal annuli

We now extend $\mathcal{A}$ by a group G by considering the isomorphism classes of regular G-coverings of the Segal annuli. For us, G will always be a finite discrete group. Recall [Bre72] that an action of G on a space X is *properly discontinuous* if each $x \in X$ has a neighbourhood U such that $Ug \cap U \neq \varnothing$ implies that g is the identity in G. In particular, a properly discontinuous action is free.

Let Y be a compact manifold with (possibly empty) boundary, with the trivial G-action.

Definition 5.9 A *principal G-bundle* over Y is a manifold P with a properly discontinuous right action of G and a smooth G-equivariant projection $\pi\colon P \to Y$ which induces a diffeomorphism $P/G \cong Y$. Two principal G-bundles $\pi\colon P \to Y$ and $\pi'\colon P' \to Y$ are *isomorphic* if there is a smooth G-equivariant map $H\colon P \to P'$ such that $\pi' H = \pi$.

If Y has a complex structure then we give P the canonical complex structure such that the projection is holomorphic. An isomorphism $H\colon P \to P'$ is then automatically an (injective) holomorphism.

We start by considering bundles over the boundary circles of an annulus. The fundamental example of a principal G-bundle over S^1 is the principal $\mathbb{Z}$-bundle $\mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$, where $\mathbb{Z}$ acts (on the right) on $\mathbb{R}$ by translation, $t \cdot n = t + n$. More generally, we recall:

Lemma 5.10 *The set* $\mathrm{Princ}_G(S^1)$ *of isomorphism classes of principal G-bundles over the circle is in bijection with the set $\mathcal{C}(G)$ of conjugacy classes in G.*

Proof: This is a special case of the classification of principal bundles:

$$\mathrm{Princ}_G(Y) \;\cong\; \mathrm{Hom}(\pi_1 Y, G) \;/\; \text{inner automorphisms of } G \tag{5.2}$$

with $Y = S^1$ and $\mathrm{Hom}(\pi_1 S^1, G) \cong \mathrm{Hom}(\mathbb{Z}, G) \cong G$. $\square$

From the universal example $\mathbb{R} \to S^1$ above there is an explicit definition for each conjugacy class (x) in G of a principal G-bundle $\pi\colon S^1_x \to S^1$ where

$$S^1_x \;=\; \mathbb{R} \times_{\mathbb{Z}} G \;=\; (\mathbb{R} \times G) \;/\; (t+1, g) \sim (t, xg)$$

with the canonical right G-action and the projection map defined by $\pi(t,g) = e^{2\pi i t}$. Given another representative $x' = y^{-1}xy$ of the conjugacy class (x) there is a G-bundle isomorphism $S^1_{x'} \to S^1_x$ induced by $(t,g) \mapsto (t, yg)$. Conversely we associate to any principal G-bundle $\pi\colon X \to S^1$ the conjugacy class (x) of G defined by $\tilde{\gamma}(1)\, x = \tilde{\gamma}(0)$, where $\tilde{\gamma}\colon [0,1] \to X$ is a lift of a generator $\gamma\colon [0,1] \to S^1$ of $\pi_1 S^1$. The element x is only defined up to conjugation in G since we have a choice of of lift $\tilde{\gamma}(0) \in X$ of the basepoint of S^1.

Example 5.11 Suppose G is the cyclic group $\mathbb{Z}_c$ of order c and $x = 1$ is the canonical generator. Then we can identify the bundle S^1_x with a copy of S^1 on which x acts by rotation by $2\pi/c$ and the projection is given by $\pi(z) = z^c$.

This example also generalises. Suppose G is a group of order n and x is an element of order c in G. Then the total space of the bundle S^1_x has n/c connected components each given by a copy of S^1 as before. The components are rotated and permuted by the G-action and may be identified with the cosets $G/\langle x \rangle$. Explicitly, the map $\mathbb{R} \times G \to S^1 \times G$ given by $(t, x^j g) \mapsto (e^{2\pi i (t+j)/c}, g)$ induces a bundle isomorphism

$$S^1_x \;\cong\; S^1 \times_{\mathbb{Z}_c} G \;=\; (S^1 \times G) \;/\; (e^{2\pi i/c} z, g) \sim (z, xg),$$

where we have identified $\mathbb{Z}_c$ with $\langle x \rangle \subseteq G$. The projection $\pi\colon S^1 \times_{\mathbb{Z}_c} G \to S^1$ is given by $\pi(z,g) = z^c$.

Lemma 5.12 *Let (x) be a conjugacy class of G represented by an element $x \in G$ of order c, and let $S^1_x \cong S^1 \times_{\mathbb{Z}_c} G$ be the principal G-bundle over S^1 described above. Then for each element y in the centraliser $C_G(x)$ of x there is a G-bundle automorphism $\sigma_y\colon S^1_x \to S^1_x$ over S^1 defined by*

$$\sigma_y : \; (z, g) \;\mapsto\; (z, yg).$$

The map from the centraliser of x to the group of G-bundle automorphisms of S^1_g over S^1,

$$\sigma : \; C_G(x) \;\to\; \mathrm{Aut}(S^1_g),$$

is an isomorphism.

Proof: The map $\sigma_y(z,g) = (z, yg)$ is clearly equivariant under the right G-action, respects the projection $\pi(z,g) = z^c$, and has inverse $\sigma_{y^{-1}}$. It is well defined on $S^1_x \cong S^1 \times_{\mathbb{Z}_c} G$,

$$\sigma_y(e^{2\pi i/c}z, g) = (e^{2\pi i/c}z, yg) = (z, xyg) = (z, yxg) = \sigma_y(z, xg),$$

if and only if x and y commute. Furthermore $\sigma_y\sigma_{y'}(z,g) = (z, yy'g) = \sigma_{yy'}(z,g)$, and so we have a homomorphism $\sigma\colon C_G(x) \to \mathrm{Aut}(S^1_g)$. This is clearly injective: $(z, yg) = (z, y'g)$ implies $y = y'$.

Conversely we can show explicitly that any G-bundle map $H\colon S^1_x \to S^1_x$ over S^1 arises as σ_y for some y. Since H is equivariant we have

$$H(z,g) = H(e^{-2\pi ij/c}z, 1) \cdot x^j g\ ,$$

where 1 is the identity element of G, and so H is determined by the values $H(z,1)$ for $z = e^{2\pi it/c}$, $t \in [0,1)$. Since H must induce the identity on S^1 under $\pi(z,g) = z^c$ we can write

$$H(e^{2\pi it/c}, 1) = (e^{2\pi it/c}, y(t)), \qquad t \in [0,1),$$

for some function $y\colon [0,1) \to G$ which by continuity must be constant, $y(t) = y$ for some $y \in G$. Now we have $H(z,1) = (z,y) = \sigma_y(z,1)$ for all $z = e^{2\pi it/c}$, $t \in [0,1)$, and so we conclude that $H = \sigma_y$. $\square$

We now turn to the definition of $\mathcal{A}(G)$, which is essentially a G-equivariant version of $\mathcal{A}$. The object $\mathcal{A}(G)$ will be a disjoint union of semigroups, indexed by the conjugacy classes (x) of G. Its elements are G-bundles over annuli, which are regarded as "structured endomorphisms" of the G-bundles over S^1 induced on the boundaries.

We give the following formal definition.

Definition 5.13 A *G-Segal pre-annulus* is a principal G-bundle P over a Segal pre-annulus A, together with a principal G-bundle X over S^1 and a bundle isomorphism $X \sqcup X \cong P|_{\partial A}$ over the boundary parametrisation $S^1 \sqcup S^1 \cong \partial A$,

$$\begin{tikzcd}
X \sqcup X \arrow[r, "\cong"] \arrow[d] & P|_{\partial A} \arrow[r] \arrow[d] & P \arrow[d] \\
S^1 \sqcup S^1 \arrow[r, "\cong"] & \partial A \arrow[r] & A.
\end{tikzcd}$$

A *G-Segal annulus* is then an isomorphism class of G-Segal pre-annuli, where two G-Segal pre-annuli (P, A, X) and (P', A', X') are said to be *isomorphic* if there is a smooth G-equivariant map $P \to P'$ which induces an isomorphism of Segal pre-annuli $A \cong A'$ and a bundle isomorphism $X \cong X'$. We write $\mathcal{A}(G)$ for the set of all G-Segal annuli.

Recall from (5.1) that a Segal annulus may be uniquely represented by (A_q, s, t) where $q \in \mathbb{C}^\times_{<1}$ and s, t are orientation-preserving pointed diffeomorphisms $S^1 \to S^1$.

Lemma 5.14 *Any G-Segal annulus may be represented by a Segal annulus $A = (A_q, s, t)$ together with bundles*

$$\begin{array}{rclcl} X & = & S^1 \times_{\mathbb{Z}_c} G & = & (S^1 \times G)\ /\ (e^{2\pi i/c}z, g) \sim (z, xg) \\ P & = & A_q \times_{\mathbb{Z}_c} G & = & (A_q \times G)\ /\ (e^{2\pi i/c}z, g) \sim (z, xg), \end{array}$$

where $x \in G$ is of order c, and an isomorphism $X \sqcup X \cong P|_{\partial A}$ given by

$$(z, g) \mapsto (qs(z), yg) \qquad\qquad (z, g) \mapsto (t(z), g)$$

on the incoming and outgoing bundles respectively, where $y \in C_G(x)$.

Proof: The boundary parametrisations $S^1 \hookrightarrow A$ of an annulus induce isomorphisms $\pi_1 S^1 \cong \pi_1 A$ of the corresponding fundamental groups, and the classification given in (5.2) tells us not only that the isomorphism class of a principal G-bundle P over A is determined by some conjugacy classes (x) in G, but also that the bundles X on the boundary are then determined by the same conjugacy class (x). We may therefore take X and P to be the G-bundles given. From Lemma 5.12 we know that G-bundle isomorphisms $S^1_x \to S^1_x$ over S^1 are determined by elements of the centraliser of x, and so the isomorphism $X \sqcup X \cong P|_{\partial A}$ over the given boundary parametrisations $(qs, t)\colon S^1 \sqcup S^1 \cong \partial A$ can always be expressed as $(z, g) \mapsto (qs(z), yg)$ and $(z, g) \mapsto (t(z), y'g)$ for some $y, y' \in C_G(x)$. But since we can still translate all of P by an isomorphism, determined by the element ${y'}^{-1}$ for example, we can assume that the element y' is in fact trivial. $\square$

Therefore we may consider a G-Segal annulus as a Segal annulus together with a conjugacy class (x) in G and an element y of the centraliser of x.

An alternative formulation and derivation of this result is given by the following.

Proposition 5.15 *Each G-Segal annulus may be uniquely represented by a Segal annulus A together with an isomorphism class of principal G-bundles over the associated torus $\breve{A}$. In particular there is a bijection*

$$\mathcal{A}(G) \ \cong\ \mathcal{A} \times TG/G,$$

where $TG = \{(x, y) \,:\, xy = yx\}$ is the set of pairs of commuting elements in G, and G acts diagonally on $TG \subseteq G \times G$ by conjugation, $(x, y)^g = (g^{-1}xg, g^{-1}yg)$.

Proof: Given a G-Segal annulus represented by principal G-bundles $X \downarrow S^1$ and $P \downarrow A$ and boundary parametrisations $X \rightrightarrows P$ over $S^1 \rightrightarrows A$, define $\breve{P}$ by identifying the two copies of X on the boundary in the same way that we defined $\breve{A}$ by "sewing" the two copies of S^1. The induced projection map $\breve{P} \to \breve{A}$ then gives a principal G-bundle $\breve{P} \downarrow \breve{A}$,

$$\begin{tikzcd} X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P \arrow[r] \arrow[d, "\pi"] & \breve{P} \arrow[d, "\breve{\pi}"] \\ S^1 \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r] & \breve{A}. \end{tikzcd}$$

Conversely, given A with boundary parametrisations $S^1 \rightrightarrows A$, any principal G-bundle $\breve{P} \downarrow \breve{A}$ over the associated torus may be pulled back along $A \to \breve{A}$ to a principal G-bundle $P \downarrow A$, and along $S^1 \to \breve{A}$ to a principal G-bundle $X \downarrow S^1$. The parametrisation $S^1 \sqcup S^1 \cong \partial A$ lifts to a bundle isomorphism $X \cup X \cong P|_{\partial A}$ as required:

$$\begin{tikzcd} X \sqcup X \arrow[r, "\cong"] \arrow[d] & P|_{\partial A} \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & \breve{P} \arrow[d] \\ S^1 \sqcup S^1 \arrow[r, "\cong"] & \partial A \arrow[r] & A \arrow[r] & \breve{A}. \end{tikzcd}$$

These processes respect isomorphic structures and are mutually inverse on isomorphism classes.

For the second part of the proposition, we recall that isomorphism classes of principal G-bundles over $\breve{A}$ are classified by conjugacy classes of homomorphisms $\pi_1 \breve{A} \to G$, and observe that

$$\mathrm{Hom}(\pi_1 \breve{A}, G) \;\cong\; \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}, G) \;\cong\; TG.$$

Therefore, $\mathrm{Princ}_G(\breve{A}) \cong TG/G$. $\square$

In the identification $\mathcal{A}(G) \cong \mathcal{A} \times TG/G$ of this proposition, an element (x, y) of TG/G corresponds to a conjugacy class (x) together with an element $y \in C_G(x)$, for some choice of representative x in the conjugacy class, which we may regard geometrically as given by a bundle S^1_x, as in Lemma 5.14, together with the parallel transport (or monodromy) along a path joining the base points of the boundary circles of the annulus.

The sewing operation on Segal annuli extends to define a partial composition on $\mathcal{A}(G)$, obtained by sewing the outgoing bundle $X \downarrow S^1$ of one G-Segal annulus to the incoming bundle of another. This will only make sense if these boundary bundles are isomorphic.

Definition 5.16 The composition of G-Segal annuli is given by

$$(A, (x, y)) \circ (A', (x', y')) \;=\begin{cases} (A \circ A', (x, yy'')) & \text{if } \exists\, g : (x', y')^g = (x, y'') \\ \text{undefined} & \text{if } (x) \neq (x'). \end{cases}$$

5.1.4 Spin structures

Once given spin structures on manifolds, there is a canonical way of getting one in connected sums or products of them. This gives a structure on the spin cobordism ring Ω_*^{Spin}.

Definition 5.17 A *spin structure on a standard* S^1 is a real line bundle L on S^1 together with an isomorphism $L \otimes L \cong TS^1$. Two choices are possible for L, periodic (or trivial), and antiperiodic (or Möbius), $S^1_P = (S^1, L_P)$, $S^1_A = (S^1, L_A)$.

We recall some special features of spin bundles over S^1 from [LM89, p. 90]. Remember that $P_{\mathrm{SO}}(S^1) \cong S^1$. Since $S^1 = \partial D$ and D admits a (unique) spin structure, S^1 as boundary of the disk gives a 2-fold connected covering of S^1 which corresponds to the trivial element in $\Omega_1^{\mathrm{Spin}} \cong \mathbb{Z}_2$. The 2-fold disconnected covering of S^1 (formed by two copies of S^1 with opposite orientations, as obtained when one applies the circle inversion) cannot be represented as the boundary of a spin manifold, and hence represents the generator σ of Ω_1^{Spin}. Obviously, 2σ is zero, since it bounds two cylinders. *Two copies of the trivial structure on the disk bound one cylinder.* One finds that $\sigma^2 \in \Omega_2^{\mathrm{Spin}}$ is nontrivial as well: it is the one corresponding to the square root of the tangent bundle of the complex manifold $\mathbb{C}\mathrm{P}^1 \cong S^2$ as a smooth manifold.

More formally: Since $S^1 \cong \mathrm{SO}(2)$, $S^1 \cong \mathrm{Spin}(2)$, the exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}\,(2) \to \mathrm{SO}\,(2) \to 0$$

takes $z \in \mathrm{Spin}\,(2)$ to z^2.

Recall from [LM89, p. 87] that for every $n \geq 1$, $N = \binom{n}{2}$, one has

$$P_{\mathrm{SO}(N)}\,(\mathrm{SO}\,(n)) = \mathrm{SO}\,(n) \times \mathrm{SO}\,(N)$$

and the two coverings are

$$\begin{aligned} P_{\mathrm{Spin}(N)}\,(\mathrm{SO}\,(n)\,, x) &= \mathrm{SO}\,(n) \times \mathrm{Spin}\,(N) \\ P_{\mathrm{Spin}(N)}\,(\mathrm{SO}\,(n)\,, y) &= (\mathrm{Spin}\,(n) \times \mathrm{Spin}\,(N))\,/\mathbb{Z}_2, \end{aligned}$$

where $\mathbb{Z}_2$ acts on $\mathrm{Spin}\,(n) \times \mathrm{Spin}\,(N)$ by the map $(\varepsilon_1, \varepsilon_2) \longmapsto (-\varepsilon_1, -\varepsilon_2)$. This gives, for $n = 2$, $N = \binom{2}{2} = 1$,

$$\begin{aligned} P_{\mathrm{Spin}(1)}\,(\mathrm{SO}\,(2)\,, x) &= \mathrm{SO}\,(2) \times \mathrm{Spin}\,(1) \\ P_{\mathrm{Spin}(1)}\,(\mathrm{SO}\,(2)\,, y) &= (\mathrm{Spin}\,(2) \times \mathrm{Spin}\,(1))\,/\mathbb{Z}_2 \end{aligned}$$

i.e.,

$$\begin{aligned} P_{\mathbb{Z}_2}\left(S^1, x\right) &= S^1 \times \mathbb{Z}_2 \\ P_{\mathbb{Z}_2}\left(S^1, y\right) &= \left(S^1 \times \mathbb{Z}_2\right)/\mathbb{Z}_2 \end{aligned}$$

where $\mathbb{Z}_2$ acts on $\mathrm{Spin}\,(2) \times \mathrm{Spin}\,(1)$ by the map $(w, \pm 1) \longmapsto (-w, \mp 1)$. The projections onto S^1 are given by

$$\begin{aligned} P_{\mathbb{Z}_2}\left(S^1, x\right) &\to S^1 \\ (z, \pm 1) &\longmapsto \pm z \end{aligned}$$

$$\begin{aligned} P_{\mathbb{Z}_2}\left(S^1, y\right) &= \left(S^1 \times \mathbb{Z}_2\right)/\mathbb{Z}_2 \\ (w, \pm 1) &\longmapsto w^2. \end{aligned}$$

One has

$$
\begin{array}{rcl}
S^1 \times \mathbb{Z}_2 & \to & S^1 \\
(z, \pm 1) & \longmapsto & \pm z \\
\left(S^1 \times \mathbb{Z}_2\right)/\mathbb{Z}_2 & = & \left(S^1 \times \mathbb{Z}_2\right)/\mathbb{Z}_2 \\
(w, \pm 1) & \longmapsto & w^2.
\end{array}
$$

The first total space of the bundle is not connected. The second one is connected, since $(S^1 \times \mathbb{Z}_2)/\mathbb{Z}_2 = \{[w, \varepsilon]\}$, with $[w, \varepsilon] = \{(w, \varepsilon), (-w, -\varepsilon)\}$.

The associated $\mathbb{R}$-bundles are obtained by $\varphi_{\alpha,\varepsilon}((z, \pm 1), v) = ((z, \pm 1) \cdot \varepsilon, \rho_\alpha(\varepsilon)(v))$. This gives for $\alpha = x$,

$$
\varphi_{x,1}((z, \pm 1), v) = ((z, \pm 1), v), \qquad \varphi_{x,-1}((z, \pm 1) \cdot -1, v) = ((z, \mp 1), -v),
$$

and for $\alpha = y$,

$$
\varphi_{y,1}((z, \pm 1), v) = ((z, \pm 1), v), \qquad \varphi_{y,-1}((z, \pm 1) \cdot -1, v) = ((-z, \mp 1), -v).
$$

Definition 5.18 A spin structure on a Riemann surface Σ is a complex line bundle S over Σ together with a smooth surjective fibre-preserving map $\mu \colon S \to K = T^{(*)}_{\text{hol}}(\Sigma)$, where $T^{(*)}_{\text{hol}}(\Sigma)$ is the holomorphic (co)tangent bundle of Σ, satisfying

$$
\mu(\lambda s) = \lambda^2 \mu(s)
$$

for any section s of S. Such sections s will be called *spinors*.

Now consider the infinitely long annulus $A_\infty = \mathbb{C}^\times$. This is a Riemann surface and its holomorphic (co)tangent bundle admits a global trivialisation given by

$$
a\, dz_p \longmapsto (p, a).
$$

Let $S_0 = S_1 = A \times \mathbb{C}$, and define maps

$$
\begin{array}{rcl}
\mu_0 & : & S_0 \to T_{\text{hol}}(A_\infty), \quad \mu_0(z, w) = \left(z, w^2\right) \\
\mu_1 & : & S_1 \to T_{\text{hol}}(A_\infty), \quad \mu_1(z, w) = \left(z, z w^2\right).
\end{array}
$$

Clearly S_0 and S_1 are isomorphic as complex bundles. The inexistence of a consistent square root of z in $\mathbb{C}^\times$ prevents them from being so as spin bundles. They restrict to inequivalent spin structures over any $A_q \hookrightarrow A_\infty$ and, since any spin structure on a manifold X induces one canonically on its boundary, one gets spin structures on S^1_r and S^1_1, the connected components of the boundary of A_q.

Since $[D^2, B\text{Spin}] = 0$, the disk admits a unique spin structure.

Definition 5.19 A *spin structure on a standard annulus* A_q is a holomorphic line bundle L with an isomorphism $L \otimes L \cong T_{\mathrm{hol}}\, A_q$, where T_{hol} denotes the complex tangent bundle. The spin structure L on A_q induces one on the boundary. Since $H^1(A_q; \mathbb{Z}_2) \cong \mathbb{Z}_2$, there are essentially two of them, $\{\varepsilon_0, \varepsilon_1\} = \{P, A\}$.

It is possible to represent them by the maps

$$\begin{aligned} P_{\mathrm{Spin}(2)}\,(A_q, \varepsilon_0) &\rightarrow P_{\mathrm{SO}(2)}\,(A_q) \\ (z, w) &\rightarrow \left(z, w^2\right), \end{aligned}$$

$$\begin{aligned} P_{\mathrm{Spin}(2)}\,(A_q, \varepsilon_1) &\rightarrow P_{\mathrm{SO}(2)}\,(A_q) \\ (z, w) &\rightarrow \left(z^2, w\right). \end{aligned}$$

Definition 5.20 Let ε be any of the two inequivalent spin structures on an annulus. Let A be a Segal annulus, with one spin structure ε. Then we will denote by (A, ε_i) the Segal annuli of its spin classes.

5.2 Representations of Segal annuli

5.2.1 The conformal group and the Virasoro algebra

Because of previous remarks, we know that in order study the representation theory of $\mathcal{A}$ and its extensions we must study the groups of orientation-preserving diffeomorphisms of the circle, $\mathrm{Diff}^+(S^1)$. In this section we discuss the relation of $\mathrm{Diff}^+(S^1)$ with loop groups, the Virasoro algebra, and the conformal group.

Let G be a Lie group and $\mathfrak{g}$ the Lie algebra of G. We consider

$$LG \;=\; \mathrm{Map}(S^1, G),$$

the infinite dimensional Lie group given by the set of all smooth maps from S^1 to G, with multiplication defined pointwise. An atlas for LG is given by $\{\mathcal{U} \cdot f\}$ for all elements f of LG, where $\mathcal{U} = LU = \mathrm{Map}(S^1, U)$, for U a neighbourhood of the identity in G, homeomorphic by the exponential map to an open set $\check{U}$ in $\mathfrak{g}$. The topology of LG is given by saying that the $\mathcal{U} \cdot f$ are open and homeomorphic to $\check{\mathcal{U}} = L\check{U} = \mathrm{Map}(S^1, \check{U})$ in the topological vector space $L\mathfrak{g} = \mathrm{Map}(S^1, \mathfrak{g})$. We remark that $L\mathfrak{g}$ is just the Lie algebra of LG; see [PS86] for more details. Furthermore, we may identify a smooth map $\gamma \colon S^1 \to \mathfrak{g}$ with its formal series expansion

$$\gamma(z) \;=\; \sum_{n=-\infty}^{\infty} \gamma_k\, z^n$$

for $\gamma_k \in \mathfrak{g}$, $z \in S^1$.

Consider the semi-Riemannian manifold $\mathbb{R}^{p,q} = \mathbb{R}^p \times \mathbb{R}^q$ with metric

$$\langle (x,y),(x'y')\rangle \;\; = \;\; x\cdot x' - y\cdot y'.$$

Then $\mathbb{R}^{3,1}$ is termed *Minkowski space* and $\mathbb{R}^{1,1}$ is the *Minkowski plane*. One also considers the compactification $S^{1,1}$ of the Minkowski plane, given by the submanifold $S^1\times S^1$ in $\mathbb{R}^{2,2}$.

Definition 5.21 The *conformal group* $\mathrm{Conf}(S^{1,1})$ is the group of all conformal diffeomorphisms of $S^{1,1}$. The conformal group $\mathrm{Conf}(\mathbb{R}^{1,1})$ is the connected component of the identity in $S^{1,1}$.

The condition $\phi^* g' = \Omega^2 g$ for a conformal transformation $\phi = (u,v)$ of the Minkowski plane is equivalent to

$$u_x^2 > v_x^2, \qquad u_x = \epsilon v_y \quad \text{and} \quad u_y = \epsilon v_x$$

for $\epsilon = \pm 1$. In the case $\epsilon = 1$, the map is termed *orientation preserving*. Any orientation-preserving conformal diffeomorphism of $\mathbb{R}^{1,1}$ is uniquely determined by a pair of diffeomorphisms of $\mathbb{R}$, given intuitively be rotating the plane by $\pi/4$. Explicitly, the function $\zeta\colon C^\infty(\mathbb{R})\times C^\infty(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$\zeta(f,g):\;(x+y,x-y) \;\;\mapsto\;\; (fx+gy, fx-gy)$$

restricts to an isomorphism of $\mathrm{Diff}^+(\mathbb{R})\times\mathrm{Diff}^+(\mathbb{R})$ to the connected component of the identity in the group of orientation-preserving conformal diffeomorphisms of $\mathbb{R}^{1,1}$.

5.2.2 The twisted group $\left(\mathrm{Diff}^+(S^1_c)\right)_{\langle g\rangle}$

Fix $\{g_1,\ldots,g_n\} = \mathcal{C}(G)$ throughout. Let $g\in G$ be an element of order g. Let S^1_c be the copy of the unit circle parametrised by $z_c = z^{\frac{1}{c}}$ (if the standard one is parametrised by z). Thus, it is (isomorphic to) a connected component for the standard bundle S^1_g before constructed, and is related to the standard one by the projection $z_c \mapsto z_c^c = z$. Term $\mathrm{Diff}^+(S^1_c)$ the group of oriented diffeomorphisms of S^1_c. We want to consider the subgroup in $\mathrm{Diff}^+(S^1_c)$ of the elements φ_c that are compatible with the bundle structure on S^1_c, where g acts as multiplication by $e^{\frac{2\pi i}{c}}$. That amounts to ask of φ_c such that $\varphi_c\left(z_c e^{\frac{2\pi i}{c}}\right) = \varphi_c(z_c)\, e^{\frac{2\pi i}{c}}$. So, define

$$\left(\mathrm{Diff}^+\left(S^1_c\right)\right)_{\langle g\rangle} = \left\{\varphi_c \in \mathrm{Diff}^+\left(S^1_c\right) \mid \varphi_c\left(z_c e^{\frac{2\pi i}{c}}\right) = \varphi_c(z_c)\, e^{\frac{2\pi i}{c}}\right\}$$

and refer to it as the set of g-diffeomorphisms of S^1_c.

Definition 5.22 The set of g-diffeomorphisms of S^1_c extended by the centraliser $C_g(G)$ of g in G is defined by

$$\left(\mathrm{Diff}^+\left(S^1_c\right)\right)_{\langle g\rangle} \times_{\mathbb{Z}_c} C_g(G) = \left(\left(\mathrm{Diff}^+\left(S^1_c\right)\right)_{\langle g\rangle} \times C_g(G)\right)/\sim_c$$

where we identify the copies of $\mathbb{Z}_c$ in $\left(\mathrm{Diff}^+(S^1_c)\right)_{\langle g\rangle}$, generated by $\varphi_{c,g}$, $\varphi_{c,g}(z_c) = z_c e^{\frac{2\pi i}{c}}$, and in $C_g(G)$ generated by g itself.

This same construction can be performed for any other representation of g in $\left(\mathrm{Diff}^+\left(S^1_c\right)\right)_{\langle g\rangle}$, generated by $\varphi_{c,g,j}$, where $\varphi_{c,g,j}\left(z_c\right) = z_c e^{\frac{2\pi i j}{c}}$, provided it generates a copy of $\mathbb{Z}_c$, i.e., if and only if $(j,c)=1$.

5.2.3 Representations of Segal annuli constructions

Definition 5.23 A *representation* ρ_G of $\mathcal{A}\left(G\right)$ will be a family of morphisms of semigroups $\rho_{g_i}\colon \mathcal{A}_{g_i}\left(G\right) \to \mathrm{End}\left(E_{G,g_i}\right)$, for some (pre, indefinite, Hilbert) topological vector spaces E_{G,g_i}, such that

$$\text{for all } \left[W_1\right],\left[W_2\right] \in \mathcal{A}_{g_i}(G), \quad \rho_{g_i}\left(\left[W_1\right]\cdot\left[W_2\right]\right) = \rho_{g_i}\left(\left[W_1\right]\right)\circ\rho_{g_i}\left(\left[W_2\right]\right).$$

We will then write $\rho_G \in \mathrm{Rep}\left(\mathcal{A}\left(G\right)\right)$.

Definition 5.24 A *projective representation* U_G of $\mathcal{A}\left(G\right)$ will be a family of maps of semigroups $U_{g_i}\colon \mathcal{A}_{g_i}\left(G\right) \to \mathrm{End}\left(E_{G,g_i}\right)$, for some (pre, indefinite, Hilbert) topological vector spaces E_{G,g_i}, such that

$$\text{for all } \left[W_1\right],\left[W_2\right] \in \mathcal{A}\left(G\right), \quad \omega\left(\left[W_1\right],\left[W_2\right]\right)\ U_{g_i}\left(\left[W_1\right]\cdot\left[W_2\right]\right) = U_{g_i}\left(\left[W_1\right]\right)\circ U_{g_i}\left(\left[W_2\right]\right),$$

where ω is a cocycle, i.e., the following identity holds for every triple $\left[W_1\right],\left[W_2\right],\left[W_3\right]$ of elements of $\mathcal{A}_{g_i}(G)$:

$$\omega\left(\left[W_1\right]\circ\left[W_2\right],\left[W_3\right]\right)\cdot\omega\left(\left[W_1\right]\circ\left[W_2\right]\right) = \omega\left(\left[W_1\right],\left[W_2\right]\circ\left[W_3\right]\right)\cdot\omega\left(\left[W_2\right],\left[W_3\right]\right).$$

We will then write $\rho_G \in \mathrm{Rep}\left(\mathcal{A}\left(G\right)\right)$.

Likewise, we denote by $\mathrm{HolPRep}\left(\mathcal{A}\left(G\right)\right)$ the set of holomorphic projective representations of $\mathcal{A}(G)$, and $\mathrm{PEPRep}\left(\mathrm{Diff}^+\left(S^1_c\right)\times_{\mathbb{Z}_c} C_g\left(G\right)\right)$ denotes the set of positive energy projective representations of $\mathrm{Diff}^+\left(S^1_c\right)\times_{\mathbb{Z}_c} C_g\left(G\right)$.

There is a bijection

$$\mathrm{HolPRep}\left(\mathcal{A}\left(G\right)\right) \longleftrightarrow \mathrm{PEPRep}\left(\mathrm{Diff}^+\left(S^1_c\right)\times_{\mathbb{Z}_c} C_g\left(G\right)\right),$$

which generalises the bijection

$$\mathrm{HolPRep}\left(\mathcal{A}\right) \longleftrightarrow \mathrm{PEPRep}\left(\mathrm{Diff}^+\left(S^1\right)\right).$$

5.2.4 Change-of-groups properties

In his definition of G-equivariant elliptic cohomology [Dev96b], Devoto gives explicit change-of-groups properties for the coefficient ring $\mathcal{E}\ell\ell^*_G$. For subgroups $K \leq H$ of G, there are notions of conjugation $\mathcal{E}\ell\ell^*_H \to \mathcal{E}\ell\ell^*_{H^g}$, restriction $\mathcal{E}\ell\ell^*_H \to \mathcal{E}\ell\ell^*_K$ and induction $\mathcal{E}\ell\ell^*_K \to \mathcal{E}\ell\ell^*_H$, satisfying some compatibility axioms, which give $G \mapsto \mathcal{E}\ell\ell^*_G$ the structure of a *Mackey functor*. In this section we will show that our construction $G \mapsto \mathcal{R}_G$ has these formal properties also, where $\mathcal{R}_G = \mathrm{HolPRep}\left(\mathcal{A}^{\mathrm{Spin}}(G)\right)$. More precisely, we must prove

Theorem 5.25 *Let G be a finite group. Then for all $K \leq H \leq G$ and $g \in G$ there are homomorphisms*

$$\begin{aligned} \mathrm{c}_g &: \mathcal{R}_H \longrightarrow \mathcal{R}_{H^g}, \\ \mathrm{rest}^H_K &: \mathcal{R}_H \longrightarrow \mathcal{R}_K, \\ \mathrm{ind}^H_K &: \mathcal{R}_K \longrightarrow \mathcal{R}_H, \end{aligned}$$

where $H^g = g^{-1}Hg$, satisfying the following conditions:

1. $c_{gh} = c_h c_g$ *if* $g, h \in G$, *and* $c_h = \mathrm{id}_{\mathcal{R}_H}$ *for* $h \in H$,
2. $\mathrm{res}^K_L \,\mathrm{res}^H_K = \mathrm{res}^H_L$ *for* $L \leq K \leq H$, *and* $\mathrm{res}^H_H = \mathrm{id}_{\mathcal{R}_H}$,
3. $c_g \,\mathrm{res}^H_K = \mathrm{res}^{H^g}_{K^g} c_g$ *and* $c_g \,\mathrm{ind}^H_K = \mathrm{ind}^{H^g}_{K^g} c_g$,
4. $\mathrm{ind}^H_K \,\mathrm{ind}^K_L = \mathrm{ind}^H_L$ *for* $L \leq K \leq H$, *and* $\mathrm{ind}^H_H = \mathrm{id}_{\mathcal{R}_H}$,
5. (Mackey's double coset formula) *if K and L are subgroups of H, then*
$$\mathrm{res}^H_L \,\mathrm{ind}^H_K = \sum_{K\backslash H/L} \mathrm{ind}^L_{K^h\cap L} c_h \,\mathrm{res}^K_{K\cap L^{h^{-1}}}.$$

Proof: We recall that

$$\mathcal{A}^{\mathrm{Spin}}(G) \cong TG/G \times \mathcal{A}^{\mathrm{Spin}},$$

where TG is the set of pairs of commuting elements $\{(x, y)\,;\; xy = yx\} \subseteq G \times G$, on which G acts diagonally by conjugation, $(x, y)^g = (g^{-1}xg, g^{-1}xg)$. We write $[x, y]_G$ or just $[x, y]$ for the equivalence class of (x, y) under the G-action. The composition on $\mathcal{A}^{\mathrm{Spin}}(G)$ is then given by

$$([x, y_1], A_1) \circ ([x, y_2], A_2) = ([x, y_1y_2], A_1 \circ A_2)\,.$$

If K is a subgroup of H and $g \in G$ then we consider the maps $\mathcal{A}^{\mathrm{Spin}}(K) \to \mathcal{A}^{\mathrm{Spin}}(H)$ and $\mathcal{A}^{\mathrm{Spin}}(H^g) \to \mathcal{A}^{\mathrm{Spin}}(H)$ given by

$$\begin{array}{ccccccc} TH^g/H^g & \xrightarrow{\cong} & TH/H & \qquad & TK/K & \longrightarrow & TH/H \\ [x^g, y^g]_{H^g} & \longmapsto & [x, y]_H & & [x, y]_K & \longmapsto & [x, y]_H \end{array} \tag{5.3}$$

and by the identity on $\mathcal{A}^{\mathrm{Spin}}$. These are well-defined and respect the composition, and so they induce the conjugation and restriction maps $c_g\colon \mathcal{R}_H \to \mathcal{R}_{H^g}$ and $\mathrm{res}^H_K\colon \mathcal{R}_H \to \mathcal{R}_K$ respectively. Conditions (1), (2) and the first part of (3) are clearly satisfied, since corresponding relations are satisfied by the maps in (5.3).

The induction (or *transfer*) maps $\mathrm{ind}^H_K\colon \mathcal{R}_K \to \mathcal{R}_H$ are less straightforward to define, since they are not induced by morphisms $\mathcal{A}^{\mathrm{Spin}}(H) \to \mathcal{A}^{\mathrm{Spin}}(K)$. Let

$$\mathcal{C}(K) = \{(k_1), \ldots, (k_m)\}$$

be the set of conjugacy classes of the group K, so that TK/K has elements $[k_i, y]$ for $1 \leq i \leq m$ and $y \in C_K(k_i)$, the centraliser of k_i in K. Then we can identify $\mathcal{A}^{\mathrm{Spin}}(K)$ with a disjoint union of semigroups

$$\begin{aligned} \mathcal{A}^{\mathrm{Spin}}(K) &\cong \coprod_{(k)\in\mathcal{C}(k)} \mathcal{A}^{\mathrm{Spin}}(K)_{(k)}, \\ \text{where} \quad \mathcal{A}^{\mathrm{Spin}}(K)_{(k)} &= \{([k,y],A)\,;\; y \in C_K(k),\; A \in \mathcal{A}^{\mathrm{Spin}}\}. \end{aligned}$$

An element $\rho \in \mathcal{R}_K$ is therefore given by a family of Hilbert spaces $E_{(k)}$ and morphisms $\rho_{(k)} \colon \mathcal{A}^{\mathrm{Spin}}(K)_{(k)} \to \mathrm{End}(E_{(k)})$, indexed by the conjugacy classes of the group K. Now given two subgroups $K \leq H$ of G we must consider how the conjugacy classes of K and H are related. For each conjugacy class (h) of H we define a Hilbert space $F_{(h)}$ from those $E_{(k)}$ for which $(k) \subseteq (h)$,

$$F_{(h)} = \bigoplus_{xK\in H/K;\, h^x\in K} E_{(h^x)}.$$

Then for $\rho \in \mathcal{R}_K$ given by representations $\rho_{(k)}$ on $E_{(k)}$ we define $\mathrm{ind}_K^H(\rho) \in \mathcal{R}_H$ by representations $(\mathrm{ind}_K^H\rho)_{(h)}$ on $F_{(h)}$, where

$$(\mathrm{ind}_K^H\rho)_{(h)}([h,y],A) = \sum_{xK\in H/K;\, (h^x,y^x)\in TK} \rho_{(h^x)}([h^x,y^x],A),$$

with the desired properties. $\square$

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