Resum: Capacitat analítica i nuclis de Riesz.

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En aquesta tesi estudiem diverses qüestions relatives a γ_α, que és la capacitat natural associada als nuclis vectorials de Riesz $x/|x|^{1+\alpha}$ a $\mathbb{R}^n$, amb $0 < \alpha < n$. La definició la donem a continuació. Sigui $E \subset \mathbb{R}^n$ un conjunt compacte i $0 < \alpha < n$, llavors

$$\gamma_\alpha(E) = \sup |< T, 1 >| \,,$$

on el suprem es pren sobre les distribucions reals T suportades a E i tals que per $1 \leq i \leq n$, $\|T * \frac{x_i}{|x|^{1+\alpha}}\|_\infty \leq 1$.

Aquestes capacitats es poden entendre com versions reals de la capacitat analítica. La noció de capacitat analítica va ser introduïda per Lars Ahlfors al 1947 per estudiar singularitats evitables de funcions analíques i acotades.

Si volem estudiar com la capacitat γ_α actua sobre conjunts, hem de tenir en compte la mida del conjunt. Més exactament, en termes de la dimensió de Hausdorff tenim:

1. Si $\dim(E) > \alpha$ llavors $\gamma_\alpha(E) > 0$.
2. Si $\dim(E) < \alpha$ llavors $\gamma_\alpha(E) = 0$.

això ens diu que la situació crítica passa quan la dimensió és α.

Un fet interessant, és que aquestes capacitats γ_α, es comporten diferent quan l'índex α és enter o no. Si α és enter, existeixen conjunts amb mesura de Hausdorff α finita i capacitat γ_α positiva. En canvi, aquest no és el cas quan $0 < \alpha < 1$.

Teorema A. *Sigui $0 < \alpha < 1$ i $E \subset \mathbb{R}^n$ un compacte amb $\mathcal{H}^\alpha(E) < \infty$. Llavors $\gamma_\alpha(E) = 0$.*

Les tècniques que usem no extenen per $\alpha > 1$, però en aquest cas tenim el següent resultat:

Teorema B. *Sigui $E \subset \mathbb{R}^n$ un compacte Ahlfors-David regular i de dimensió α, $\alpha \notin \mathbb{Z}$ i $0 < \alpha < n$. Llavors $\gamma_\alpha(E) = 0$.*

Usant un resultat d'H. Pajot sobre recobriments per conjunts Ahlfors-David regular, podem extendre el Teorema B a conjunts més generals:

Teorema C. *Sigui* $0 < \alpha < n$, $\alpha \notin \mathbb{Z}$ *i* $E \subset \mathbb{R}^n$ *un compacte amb* $\mathcal{H}^\alpha(E) < \infty$, *tal que per tot* $x \in E$,

$$0 < \theta_*^\alpha(x, E) \leq \theta^{*\alpha}(x, E) < \infty.$$

Llavors $\gamma_\alpha(E) = 0$.

Aquí $\theta_*^\alpha(x, E)$ i $\theta^{*\alpha}(x, E)$ denoten les $\alpha-$densitats inferior i superior de la mesura $\mathcal{H}^\alpha_{|E}$ en x, respectivament.

Per $0 < \alpha < 1$, també donem una caracterització de la capacitat γ_α en termes de la ja ben coneguda capacitat de Riesz $C_{s,p}$ de teoria del potencial no lineal.

Teorema D. *Sigui* $K \subset \mathbb{R}^n$ *un compacte i* $0 < \alpha < 1$, *llavors*

$$C^{-1} C_{\frac{2}{3}(n-\alpha), \frac{3}{2}}(K) \leq \gamma_\alpha(K) \leq C C_{\frac{2}{3}(n-\alpha), \frac{3}{2}}(K).$$

També deduïm que la capacitat γ_α és numerablement semiaditiva i és un invariant bilipschitz, quan $0 < \alpha < 1$.

Summary: Analytic Capacity and Riesz kernels.

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In this dissertation we study several questions concerning the natural capacity γ_α related to the signed vector valued Riesz kernels $x/|x|^{1+\alpha}$ in $\mathbb{R}^n$, where $0 < \alpha < n$. It is defined as follows. For a compact set $E \subset \mathbb{R}^n$ and $0 < \alpha < n$, set

$$\gamma_\alpha(E) = \sup |< T, 1 >| , \tag{1}$$

where the supremum is taken over all real distributions T supported on E such that for $1 \leq i \leq n$, the i-th Riesz potential of T, $T * \frac{x_i}{|x|^{1+\alpha}}$ is a function in $L^\infty(\mathbb{R}^n)$ and $\sup\limits_{1\leq i\leq n} \left\| T * \frac{x_i}{|x|^{1+\alpha}} \right\|_\infty \leq 1$.

These capacities can be understood as being certain real variable versions of analytic capacity. The notion of analytic capacity was introduced in 1947 by L. Ahlfors to study removable singularities of bounded analytic functions.

According to the classical case, if we want to study how the set function γ_α behaves, we first have to take into account the role played by the "size" of the set. More precisely in terms of Hausdorff dimension (denoted by dim) we have:

1. If $\dim(E) > \alpha$ then $\gamma_\alpha(E) > 0$.
2. If $\dim(E) < \alpha$ then $\gamma_\alpha(E) = 0$.

This says that the critical situation occurs in dimension α, in accordance with the classical case.

An interesting fact of the capacities γ_α, $0 < \alpha < n$, is that they behave differently when dealing with integer or non-integer indexes α. If α is an integer, there exist sets with finite α-dimensional Hausdorff measure and positive γ_α.

In constrast to this, in the first part of this dissertation we show that for $0 < \alpha < 1$, the capacity γ_α vanishes on compact sets $E \subset \mathbb{R}^n$ with finite α-Hausdorff measure, namely we show that

Theorem A. *Let $0 < \alpha < 1$ and let $E \subset \mathbb{R}^n$ be a compact set with $\mathcal{H}^\alpha(E) < \infty$. Then $\gamma_\alpha(E) = 0$.*

The techniques that we use do not extend to indexes $\alpha > 1$. However, we can extend the previous result to indexes $1 < \alpha < n$, $\alpha \notin \mathbb{Z}$, assuming Ahlfors-David regularity of the sets we are dealing with. The statement of the precise result is

Theorem B. *Let $E \subset \mathbb{R}^n$ be a compact Ahlfors-David regular set of non-integer dimension α, $0 < \alpha < n$. Then $\gamma_\alpha(E) = 0$.*

By using an adaptation of a covering argument by Ahlfors-David regular sets of H. Pajot, we can extend Theorem B as follows,

Theorem C. *Let* $0 < \alpha < n$, $\alpha \notin \mathbb{Z}$ *and let* $E \subset \mathbb{R}^n$ *be a compact set with* $\mathcal{H}^\alpha(E) < \infty$, *such that for all* $x \in E$,

$$0 < \theta^\alpha_*(x, E) \leq \theta^{*\alpha}(x, E) < \infty.$$

Then $\gamma_\alpha(E) = 0$.

Here $\theta^\alpha_*(x, E)$ and $\theta^{*\alpha}(x, E)$ denote the lower and upper α-densities of the measure $\mathcal{H}^\alpha_{|E}$ at x, respectively.

For $0 < \alpha < 1$, we give a characterization of the capacity γ_α in terms of the well known Riesz capacity $C_{s,p}$ of non-linear potential theory.

Theorem D. *Given a compact set* $K \subset \mathbb{R}^n$ *and* $0 < \alpha < 1$,

$$C^{-1} C_{\frac{2}{3}(n-\alpha), \frac{3}{2}}(K) \leq \gamma_\alpha(K) \leq C C_{\frac{2}{3}(n-\alpha), \frac{3}{2}}(K).$$

From Theorem D one can deduce the semiadditivity of the capacity γ_α and also the bilipschitz invariance of this quantity, for $0 < \alpha < 1$.