Contribution to the qualitative study of planar differential systems.

Maria Teresa Grau Montaña

Memoria presentada para aspirar
al grado de Doctor en Matemáticas.

Departament de Matemàtiques
Universitat Autònoma de Barcelona

Directores: Dr. Javier CHAVARRIGA
Dr. Héctor GIACOMINI

Bellaterra, Septiembre de 2004.

Contribution to the qualitative study of planar differential systems.

Maria Teresa Grau Montaña

Departament de Matemàtica
Universitat de Lleida
E-mail: mtgrau@matematica.udl.es

Los Drs. Javier Chavarriga y Héctor Giacomini CERTIFICAN que la presente memoria ha sido realizada por Maria Teresa Grau Montaña bajo su supervisión y constituye su Tesis para aspirar al grado de Doctor en Matemáticas por la Universitat Autònoma de Barcelona.

Dr. Javier Chavarriga Soriano
Departament de Matemàtica

Universitat de Lleida

Dr. Héctor Giacomini
Laboratoire de Mathématiques
et Physique Théorique
Université de Tours (Francia)

Contents

Agraïments

Tal com dicta el seu nom, aquesta memòria pretén recordar tots aquests anys i ser reflex de molts dies d'estudi i d'esforç per tal d'anar descobrint part de l'entramat de propietats que descriuen els sistemes dinàmics. Tantmateix, aquest temps no ha estat tan sols això. Cercant en la memòria trobo moltes altres descobertes, que resten llindant amb les matemàtiques. Com ja he sentit dir a diferents personatges, *la ciència no es fa en solitud*, es fa ciència discutint i aprenent dels teus "coł.legues" que són els millors professors.

Per aquest motiu, vull començar aquesta memòria agraint als meus companys de doctorat la seva amistat. Allò que he après en la seva companyia té a veure tant amb la ciència com amb la humanitat. També voldria agrair als meus companys de la Universitat de Lleida, en Jaume Giné, l'Isaac García i en Jordi Sorolla, pel temps que m'han regalat, bé sigui amb les nostres riques discussions, bé sigui amb el seu suport en els moments més baixos. Aquest treball l'he dut a terme bàsicament en dues localitzacions: Lleida i Tours. És per aquest motiu que voldria mencionar als meus companys en aquestes dues ciutats, per la seva cordialitat, simpatia i amistat.

No podria continuar sense destacar el suport dels meus directors de tesi, sense els quals és evident que aquest treball no hauria vist la llum. Vull també fer menció dels professors de tots els llocs que he visitat, agraint-los el temps i els coneixements que m'han ofert, començant per l'Armengol Gasull i en Jaume Llibre de la UAB, en Felipe Cano de la UVA, la Dana Schlomiuk de la U. Montreal (Canadà), l'Ivan Szántó de la UTFSM (Valparaíso, Xile), en Luis P. Lara de la UNR (Argentina), en Lubomir Gavrilov de la U. Paul Sabatier (Toulouse, França) i en Jean Pierre Françoise de la UPMC (França).

Finalment, voldria donar també les gràcies a tots els meus familiars i amics que m'han recolzat durant tot aquest temps. Dedico aquest treball als meus pares, que sempre he sentit tant a prop. Moltes gràcies.

Obro els ulls i veig més arbres,
les mans em toquen escorça
feta amb escarpi i molt temps.
Mes arrels solquen profundes
buscant la pluja caiguda.
Tinc arrels plenes de fruit
pro ma llavor encara és presa.

Escolta, calma't, respira
i intueix la teva essència.
Pi entre pins, aire de tots
més arrels entrellaçades.
Un dia el bri veu la llum,
llavors la branca s'il.lumina.
Oh, bosc insondable i perfecte!

Preface

The main purpose of this work has been to learn the most important tools, facts and problems concerning systems of differential equations in the real plane and to give some original results and some applications of them in order to solve some of these problems. We first give the most basic definitions and results of general type in the introduction, including the main topics discussed in this thesis. We motivate each problem and we describe the methods used to tackle it in each chapter.

Since we apply many results due to famous mathematicians, we would like to make a small chronological review of the biography of these scientists in order to historically situate the subject of this thesis and to render a tribute to them. This small review does not intend to be exhaustive and we only want to mention the most important mathematicians who contributed to the development of the qualitative theory of differential equations and chosen because they are cited in the text of this thesis.

Joseph LIOUVILLE. (1809 Saint-Omer, France – 1882 Paris, France.)

Joseph Liouville entered the École Polytechnique in 1825 and attended courses from Ampère and Arago. After graduating in 1827 he entered the École des Ponts et Chaussées, although due to some health problems, he was obliged to resign. He was appointed to many different posts during all his life. However, other mathematicians won the post in many occasions. This fact contributed to make worse his bad character. He had a heavy schedule of classes (40 hours/week) and some courses would not go particularly well and it appears that he lectured at too high a level. In 1836 Liouville founded the *Journal de Mathématiques Pures et Appliquées*. This journal, sometimes known as Journal de Liouville, did much for the quality of mathematics in France throughout the 19^{th} century.

Another aspect of Liouville's life was his involvement in politics. Encouraged by Arago, Liouville was elected to the Constituting Assembly in 1848 among the moderate republican majority. However he was not elected in 1849 due to the social events. This political defeat made Liouville's personality even bitter, also towards his old friends. Although Liouville's mathematical output had been greatly reduced while he was involved with politics, it picked up again in the 1850s.

Liouville's mathematical work is extremely wide ranging, from mathematical physics to astronomy to pure mathematics. Liouville investigated criteria for integrals of algebraic functions to be algebraic and he went on to investigate the general problem of integration of algebraic functions in finite terms. In particular, in this thesis we use some results on the so-called Liouvillian functions. He is also remembered for proving the existence of transcendental numbers. His work on boundary value problems on differential equations is remembered because of what is called today Sturm-Liouville theory. He contributed to differential geometry studying conformal transformations. He proved a major theorem concerning the measure preserving property of Hamiltonian dynamics. He wrote over 400 papers in total during his life.

Gaston DARBOUX. (1842 Nîmes, France – 1917 Paris, France.)

Gaston Darboux entered the École Polytechnique and then the École Normale Supérieure in 1861. He awarded his doctorate in 1866 with his doctoral thesis *Sur les surfaces orthogonales*. Darboux was appointed to the Collège de France for the academic year 1866 − 1867, then he taught at the Lycée Louis le Grand (where Galois was educated) between 1867 and 1872. In 1872 he was appointed to the École Normale Supérieure where he taught until 1881. From 1873 to 1878 he was suppléant to Liouville in the chair of rational mechanics at the Sorbonne. Then, in 1878 he became suppléant to Chasles in the chair of higher geometry, also at the Sorbonne. Two years later Chasles died and Darboux succeeded him to the chair of higher geometry, holding this chair until his death.

Darboux made important contributions to differential geometry and analysis. He might be best known for the Darboux integral which was introduced in a paper on differential equations of the second order written in 1870. In 1875 he gave his way of looking at the Riemann integral, defining upper and lower sums and defining a function to be integrable if the difference between the upper and lower sums tends to zero as the mesh size gets smaller.

Moreover, Darboux was also renowned as an exceptional teacher, writer and administrator. In 1870 he founded the *Bulletin des Sciences Mathématiques*, a journal that publishes original articles covering all branches of pure mathematics.

In relation with the topics included in this thesis, he studied the integrability problem and he gave a method to give an explicit expression of a first integral for a planar polynomial differential system in case that sufficient invariant algebraic curves are known, as it is explained in Chapter 1 and cited in [38].

Henri POINCARÉ (1854 Nancy, France – 1912 Paris, France.)

The importance on Poincaré in the development of the qualitative theory of dynamical systems and in mathematics in general is so huge that we must include all of a thesis of his life and research to make these notes coherent. However, since this is not our aim, we will only remark some of his main features.

Poincaré entered the École Polytechnique in 1873, graduating in 1875. He continued his studies at the École des Mines and after that, he was a mining engineer at Vesoul while completing his doctoral work. As a student of Charles Hermite, Poincaré received his doctorate in mathematics from the University of Paris in 1879. His thesis was on differential equations and the examiners were somewhat critical of the work. Immediately after receiving his doctorate, Poincaré was appointed to teach mathematical analysis at the University of Caen. His teaching is referred by his sometimes disorganized lecturing style. He won a chair in the Faculty of Science in Paris in 1881. In 1886 Poincaré was nominated for the chair of mathematical physics and probability at the Sorbonne and he also was appointed to a chair at the École Polytechnique, due to the intervention of Hermite. Changing his lectures every year, he would review optics, electricity, the equilibrium of fluid masses, the mathematics of electricity, astronomy, thermodynamics, light, and probability. Poincaré held these chairs in Paris until his death.

He is considered as one of the great geniuses of all time, surely because of his way of thinking and working. He was also concerned with the explanation of thought processes, in particular his own ones, which led to his major mathematical discoveries. An interesting aspect of Poincaré's work is that he tended to carefully develop his results from basic first principles and then proceed by sudden blows. Poincaré was a scientist concerned with many aspects of mathematics, physics and philosophy, and he is often described as the last universalist in mathematics.

His contributions in pure mathematics fall into such different areas such as topology, homotopy theory, algebra, geometry, analysis, number theory, and differential equations. In applied mathematics he studied optics, electricity, telegraphy, capillarity, elasticity, thermodynamics, potential theory, quantum theory, theory of relativity and cosmology. He is acknowledged as a co-discoverer, with Albert Einstein and Hendrik Lorentz, of the special theory of relativity.

His important work on the 3-body problem was consolidated when he was awarded the prize founded by Oscar II, King of Sweden and Norway, to celebrate his sixtieth birthday in 1889. In this memoir Poincaré gave the first description of homoclinic points, gave the first mathematical description of chaotic motion, and was the first to make major use of the idea of invariant integrals. However, when the memoir was about to be published in Acta Mathematica, edited by Mittag-Leffler, Poincaré found an error. They both discussed the problem concerning the error and it is interesting that this error is now regarded as marking the birth of chaos theory. A revised version of Poincaré's memoir appeared in 1890.

After Poincaré achieved prominence as a mathematician, he wrote several books describing for the general public the meaning and importance of science and mathematics. We should note that, despite his great influence on the mathematics of his time, he never founded his own school since he did not have any students. Although his contemporaries used his results they seldom used his techniques. In any case, Poincaré achieved the highest honors for his contributions of true genius. He was elected to the Académie des Sciences in 1887 and in 1906 was elected President of the Academy. He was honored by a large number of learned societies around the world and he won numerous prizes, medals and awards.

We cite Poincaré in this thesis mainly for his contribution to the *algebraic integrability* problem, which consists in finding necessary and sufficient conditions for the existence of a rational first integral of a planar polynomial differential system. Poincaré studied the memory authored by G. Darboux [38] and published in 1878, and he gave two main works in this subject [76, 77]. After these works, Poincaré observed how this problem was abandoned until 1890, when the Académie des Sciences de Paris offered the Grand Prix des Sciences Mathématiques on this subject. The contemporary mathematician to Poincaré, Paul Painlevé [71] awarded this price and Léon Autonne [9] awarded an Honor Mention. In [71], Painlevé gives a formula relating the gender of the algebraic invariant curves of a planar polynomial system with a rational first integral with degree of the system. The work of Autonne [9] is quite difficult to understand due to the author's personal notation. Another mathematician, also contemporary to Poincaré, who contributed to the algebraic integrability problem is M. N. Lagutinskii. This mathematician wrote his results in Russian and this is the main reason why he is almost unknown by the mathematical community. Some historical research on him is done in the work [39], where the authors found up Lagutinskii's result on the characterization of systems with a rational first integral by means of the so-called extactic curves. This result has been recently rediscovered by J. V. Pereira in [73].

Until the last quarter of the XXth century no other main contribution to the algebraic theory of integrability is known. This problem was retaken by J. P. Jouanolou with his work published in 1979, see [61]. Many other mathematicians, which we do not cite due to their number, have given their contribution to the problem since then.

Abraham SEIDENBERG (1916, Washington, USA – 1988, Milan, Italy)

He received his Ph.D. at John Hopkins University in 1943 and he became a professor in the Department of Mathematics at Berkeley in 1958.

His publications in pure mathematics include some influential work in commutative algebra and algebraic geometry. His papers on differential algebra contain some remarkable results on the Picard-Vessiot theory of homogeneous linear differential equations. We use one of his results when he studies formal differential equations, cf. Chapter 3 and citation [85].

Seidenberg was the author of two textbooks one in projective geometry, and the other on algebraic curves. He was also the editor of a collection *Studies in Algebraic Geometry* published under the auspices of the Mathematical Association of America. Many of his publications are related with the history of mathematics, in particular on ancient mathematics.

Michael F. SINGER (1950, New York, USA)

Michael F. Singer awarded his Ph. D. at the University of California, Berkeley in 1974 with the thesis *Functions satisfying elementary relations.* He is a current professor at the Department of Mathematics in North Carolina State University.

His works are mainly devoted to differential and difference algebra, symbolic computation and polynomial vector fields. In particular, we mention his results on elementary first integrals [79] coauthored with M.J. Prelle and on Liouvillian first integrals [86] for polynomial vector fields.

These biographies have been extracted from the following web pages.

http://www.biografiasyvidas.com/

http://www.britannica.com/

http://www-gap.dcs.st-and.ac.uk/~history/

http://dynaweb.oac.cdlib.org:8088/dynaweb/uchist/public/inmemoriam/inmemoriam1989/@Generic_BookTextView/2838

http://www4.ncsu.edu/~singer/

Chapter 1

Introduction

This thesis deals with planar polynomial differential systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

where $P(x, y), Q(x, y) \in \mathbb{R}[x, y]$ are coprime polynomials, that is, there is no non-constant polynomial which divides both P and Q. The dot denotes derivation with respect to the independent variable t usually called *time*, that is $\dot{} = \frac{d}{dt}$.

We call d the maximum degree of P and Q and we say that system (1.1) is of degree d. When d = 2, we say that (1.1) is a *quadratic system*. If p is a point such that $P(p) = Q(p) = 0$, then we say that p is a *singular point* of system (1.1).

Our interest in each chapter is different. Chapters 2 and 3 are more related with the integrability problem and we explain in the following section of this introduction all the definitions, preliminary results and classical Darboux theory related to this problem. Chapter 4 is related to the study of stability of a periodic orbit. In Chapter 5, we consider all the quadratic systems with an algebraic limit cycle known by the literature and, in order to exemplify the results obtained in previous chapters, we show two main features of this family of systems. Firstly, we show that these systems are not Liouvillian integrable by using the results of Chapter 3. Secondly, we prove that the algebraic limit cycles of these systems are hyperbolic, using the results of Chapter 4. Finally, Chapter 6 is about the isochronicity of a singular point. In Section 1.2 we describe the definitions and known results when studying stability of either a singular point, to motivate Chapter 6, or a periodic orbit, to introduce Chapter 4.

A very important definition through almost all the chapters is the notion of invariant curve. An *invariant curve* is a curve given by $f(x, y) = 0$, where $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^1$ function in the open set $\mathcal{U}$, non locally constant and such that there exists a $\mathcal{C}^1$ function in $\mathcal{U}$, denoted by $k(x, y)$ and called *cofactor*, which satisfies:

$$P(x,y)\,\frac{\partial f}{\partial x}(x,y) \,+\, Q(x,y)\,\frac{\partial f}{\partial y}(x,y) \;=\; k(x,y)\,f(x,y), \qquad (1.2)$$

for all $(x, y) \in \mathcal{U}$. The notion of invariant curve was first introduced in [51].

In order to simplify notation, we may always represent system (1.1) by the associated vector field at each point $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$. The identity (1.2) can be rewritten by $\nabla f \cdot \mathbf{F} = kf$, where, as usual, ∇f denotes the gradient vector related to $f(x, y)$, that is, $\nabla f(x, y) = (\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y))$ and $\cdot$ denotes the scalar product. We will denote by $\frac{df}{dt}$ or by $\dot{f}$ the function $\nabla f \cdot \mathbf{F}$ once evaluated on a solution of system (1.1). In case $f(x, y) = 0$ defines a curve in the real plane, this definition implies that the function $\nabla f \cdot \mathbf{F}$ is equal to zero on the points such that $f(x, y) = 0$. In the article [51] an invariant curve is defined as a $\mathcal{C}^1$ function $f(x, y)$ defined in the open set $\mathcal{U} \subseteq \mathbb{R}^2$, such that, the function $\nabla f \cdot \mathbf{F}$ is zero in all the points $\{(x, y) \in \mathcal{U} \,|\, f(x, y) = 0\}$. We notice that our definition of invariant curve is a particular case of the previous one but, for the sake of our results, the cofactor is very important and that's why we always assume its existence.

When the cofactor $k(x, y)$ is a polynomial, we say that $f(x, y) = 0$ is an invariant curve with polynomial cofactor. We only admit invariant curves with polynomial cofactor of degree lower or equal than $\mathrm{d} - 1$, that is $\deg k(x, y) \leq \mathrm{d} - 1$, where d is the degree of system (1.1).

The notion of invariant curve appears in almost all the chapters and it is a generalization of the notion of invariant algebraic curve. An *invariant algebraic curve* is an algebraic curve $f(x, y) = 0$, where $f(x, y) \in \mathbb{C}[x, y]$, which is invariant by the flow of system (1.1). This condition equals to $\nabla f \cdot \mathbf{F} = kf$, where the cofactor of an invariant algebraic curve is always a polynomial of degree $\deg k(x, y) \leq \mathrm{d} - 1$.

We cite [64, 83, 84] as compendiums of the results on invariant algebraic curves. For instance, in [64], it is shown that if $f(x, y) = 0$ and $g(x, y) = 0$

are two invariant algebraic curves of system (1.1) with cofactors $k_f(x,y)$ and $k_g(x,y)$, respectively, then the product of the two polynomials gives rise to the curve $(fg)(x,y)=0$ which is also an invariant algebraic curve of system (1.1) and whose cofactor is $k_f(x,y)+k_g(x,y)$.

In order to state the known results of integrability using invariant algebraic curves, we need to consider complex algebraic curves $f(x,y)=0$, where $f(x,y)\in\mathbb{C}[x,y]$. Since system (1.1) is defined by real polynomials, if $f(x,y)=0$ is an invariant algebraic curve with cofactor $k(x,y)$, then its conjugate $\bar{f}(x,y)=0$ is also an invariant algebraic curve with cofactor $\bar{k}(x,y)$. Hence, its product $f(x,y)\bar{f}(x,y)\in\mathbb{R}[x,y]$ gives rise to a real invariant algebraic curve with a real cofactor $k(x,y)+\bar{k}(x,y)$. For a sake of simplicity, we consider invariant algebraic curves defined by polynomials in $\mathbb{C}[x,y]$, although we always keep in mind the previous observation. In $\mathbb{R}^2$, the curve given by $f(x,y)=0$, although $f(x,y)$ is a real function, may only contain a finite number of isolated singular points or be the null set.

An algebraic curve $f(x,y)=0$ is called *irreducible* when $f(x,y)$ is an irreducible polynomial in the ring $\mathbb{C}[x,y]$. We can assume, without loss of generality, that $f(x,y)$ is an irreducible polynomial in $\mathbb{C}[x,y]$, because if $f(x,y)$ is reducible, then all its proper factors give rise to invariant algebraic curves. Given an algebraic curve $f(x,y)=0$, we can always assume that the polynomial $f(x,y)$ has no multiple factors, that is, its decomposition in the ring $\mathbb{C}[x,y]$ is of the form $f(x,y)=f_1(x,y)f_2(x,y)\ldots f_\ell(x,y)$, where $f_i(x,y)$ are irreducible polynomials and $f_i(x,y)\neq cf_j(x,y)$ if $i\neq j$ and for any $c\in\mathbb{C}$. The assumption that given an algebraic curve $f(x,y)=0$, the polynomial $f(x,y)$ has no multiple factors is mainly used to ensure that we do not consider "false" singular points. If p is a point such that $f(p)=0$ and $\nabla f(p)=0$, and $f(x,y)$ has no multiple factors, then p is a singular point of the curve $f(x,y)=0$. But, if $f(x,y)$ has multiple factors, for instance, $f(x,y)=f_1(x,y)^2$ where $f_1(x,y)$ is an irreducible polynomial in $\mathbb{C}[x,y]$, then all the points of the curve $\{p \mid f_1(p)=0\}$ satisfy the property that $f(p)=0$ and $\nabla f(p)=0$ although they are not all singular points.

We recall that if p is a singular point of an invariant algebraic curve $f(x,y)=0$ of a system (1.1), then p is a singular point of the system. Given an algebraic curve $f(x,y)=0$, we will always assume that the decomposition of $f(x,y)$ in the ring $\mathbb{C}[x,y]$ has no multiple factors. We want to generalize this property to invariant curves, that's why we will always assume that, given an invariant curve $f(x,y)=0$, if $p\in\mathcal{U}$ is such that $f(p)=0$ and $\nabla f(p)=0$, then p is a singular point of system (1.1). This technical hypothesis generalizes the notion of not having multiple factors for algebraic curves.

1.1 The integrability problem

A $\mathcal{C}^j$ function $H : \mathcal{U} \to \mathbb{R}$ such that it is constant on each trajectory of (1.1) and it is not locally constant is called a *first integral* of system (1.1) of class j defined on $\mathcal{U} \subseteq \mathbb{R}^2$. The equation $H(x, y) = c$ for a fixed $c \in \mathbb{R}$ gives a set of trajectories of the system, but in an implicit way. When $j \geq 1$, these conditions are equivalent to $P(x, y)\frac{\partial H}{\partial x} + Q(x, y)\frac{\partial H}{\partial y} = 0$ and H not locally constant. The problem of finding such a first integral and the functional class it must belong to is what we call the *integrability problem*.

To find an integrating factor or an inverse integrating factor for system (1.1) is closely related to finding a first integral for it. When considering the integrability problem we are also addressed to study whether an (inverse) integrating factor belongs to a certain given class of functions.

Given $\mathcal{W}$ an open set of $\mathbb{R}^2$, the function $\mu : \mathcal{W} \to \mathbb{R}$ of class $\mathcal{C}^j(\mathcal{W})$, $j > 1$, that satisfies the linear partial differential equation

$$P(x, y)\frac{\partial \mu}{\partial x} + Q(x, y)\frac{\partial \mu}{\partial y} = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\mu(x, y) \tag{1.3}$$

is called an *integrating factor* of system (1.1) defined on $\mathcal{W}$. The expression $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is called the *divergence* of system (1.1) and we denote it by $\mathrm{div}(x, y)$.

An easier function to find which also gives additional properties for a differential system (1.1) is the inverse of an integrating factor, that is, $V = 1/\mu$, which is called *inverse integrating factor*.

We note that $\{V = 0\}$ is formed by orbits of system (1.1). If V is defined on $\mathcal{W}$, then the function $\mu = 1/V$ defines on $\mathcal{W} \setminus \{V = 0\}$ an integrating factor of system (1.1), which allows the computation of a first integral of the system on $\mathcal{W} \setminus \{V = 0\}$. The *first integral H associated to the inverse integrating factor V* can be computed through the integral $H(x, y) = \int (Q(x, y)dx - P(x, y)dy)/V(x, y)$, and the condition (1.3) for $\mu = 1/V$ ensures that this line integral is well defined.

The inverse integrating factors play an important role in two of the most difficult open problems of qualitative theory of planar polynomial vector fields, which are the center problem and the 16^{th} Hilbert problem. The center problem is explained in Section 1.2.1 and the statement and history of

16^{th} Hilbert problem can be found in [59, 89]. In [19], it has been noticed that for many polynomial differential systems with a center at the origin there is always an inverse integrating factor V globally defined in all $\mathbb{R}^2$, which is usually a polynomial. However, the first integral for a polynomial differential system with a center at the origin can be very complicated. The relation between the inverse integrating factor and the 16^{th} Hilbert problem is mainly stated and proved in [54], where it is shown that if there exists an inverse integrating factor defined in a neighborhood of a limit cycle, then the limit cycle must be contained in the set $\{(x,y)\,|\,V(x,y)=0\}$. A shorter proof of this result can be found in [65], where it is also shown that any configuration of limit cycles is realizable in the set of polynomial vector fields.

Invariant algebraic curves are the main objects used in the Darboux theory of integrability. In [38], G. Darboux gives a method for finding an explicit first integral for a system (1.1) in case that d(d+1)/2+1 different irreducible invariant algebraic curves are known, where d is the degree of the system. In this case, a first integral of the form $H = f_1^{\lambda_1} f_2^{\lambda_2} \dots f_s^{\lambda_s}$, where each $f_i(x,y)=0$ is an invariant algebraic curve for system (1.1) and $\lambda_i \in \mathbb{C}$ not all of them null, for $i=1,2,\dots,s$, $s \in \mathbb{N}$ can be constructed. The functions of this type are called *Darboux* functions.

As we have already stated, given an invariant algebraic curve $f(x,y)=0$ whose imaginary part is not null, then its conjugate is also an invariant algebraic curve. Moreover, as system (1.1) is real, if $f(x,y)$ appears in the expression of a first integral of the form given by Darboux with exponent λ, then $\bar{f}(x,y)$ appears in the same expression with exponent $\bar{\lambda}$. We call $\mathrm{Re}f$ the real part of the polynomial f and by $\mathrm{Im}f$ its imaginary part. Analogously, let us call $\mathrm{Re}\lambda$ the real part of the complex number λ and by $\mathrm{Im}\lambda$ its imaginary part. We call $\mathbf{i}=\sqrt{-1}$ and we use the following formula for complex numbers:

$$\arctan(z) = \log\left[\left(\frac{1-\mathbf{i}z}{1+\mathbf{i}z}\right)^{\mathbf{i}/2}\right], \quad z \in \mathbb{C},$$

to show that

$$\begin{aligned} f^{\lambda}\bar{f}^{\bar{\lambda}} &= (\mathrm{Re}f + \mathrm{Im}f\,\mathbf{i})^{\mathrm{Re}\lambda + \mathrm{Im}\lambda\,\mathbf{i}}\ (\mathrm{Re}f - \mathrm{Im}f\,\mathbf{i})^{\mathrm{Re}\lambda - \mathrm{Im}\lambda\,\mathbf{i}} \\ &= \left((\mathrm{Re}f)^2 + (\mathrm{Im}f)^2\right)^{\mathrm{Re}\lambda}\ \exp\left\{-2\mathrm{Im}\lambda \arctan\left(\frac{\mathrm{Im}f}{\mathrm{Re}f}\right)\right\}. \end{aligned}$$

We deduce that the product $f(x,y)^{\lambda}\bar{f}(x,y)^{\bar{\lambda}}$ is a real function and so it is any Darboux function $H = f_1^{\lambda_1} f_2^{\lambda_2} \ldots f_s^{\lambda_s}$.

We have that the Darboux function H can be defined in the open set $\mathbb{R}^2 \setminus \Sigma$, where $\Sigma = \{(x,y) \in \mathbb{R}^2 \mid (f_1 \cdot f_2 \cdot \ldots \cdot f_r)(x,y) = 0\}$. We remark that, particularly, if $\lambda_i \in \mathbb{Z}$, $\forall i = 1,2,\ldots,r$, H is a *rational first integral* for system (1.1). In this sense J. P. Jouanoulou [61], showed that if at least $\mathrm{d}(\mathrm{d}+1)+2$ different irreducible invariant algebraic curves are known, then there exists a rational first integral.

The main fact used to prove Darboux's theorem (and Jouanoulou's improvement) is that the cofactor corresponding to each invariant algebraic curve is a polynomial of degree $\leq \mathrm{d}-1$. Invariant curves with polynomial cofactor can also be used in order to find a first integral for the system. This observation permits a generalization of the Darboux's theory which is given in [50], where, for instance, non-algebraic invariant curves with an algebraic cofactor for a polynomial system of degree 4 are presented. In Chapter 2 we give other examples of such invariant curves with polynomial cofactor for some families of systems and the way they are used to construct explicit first integrals and inverse integrating factors for the corresponding systems.

Some generalizations of the classical Darboux theory of integrability may be found in the literature. For instance, independent singular points can be taken into account to reduce the number of invariant algebraic curves necessary to ensure the Darbouxian integrability of the system, see [28]. A good summary of many of these generalizations can be found in [72] and a survey on the integrability of two-dimensional systems can be found in [19]. One of the most important definitions in this sense is the notion of exponential factor which is given by C. Christopher in [32], when he studies the multiplicity of an invariant algebraic curve. The notion of exponential factor is a particular case of invariant curve for system (1.1). Given two coprime polynomials $h, g \in \mathbb{R}[x,y]$, the function $e^{h/g}$ is called an *exponential factor* for system (1.1) if for some polynomial k of degree at most $\mathrm{d}-1$, where d is the degree of the system, the following relation is fulfilled:

$$P\left(\frac{\partial\, e^{h/g}}{\partial x}\right) + Q\left(\frac{\partial\, e^{h/g}}{\partial y}\right) = k(x,y)\; e^{h/g}.$$

As before, we say that $k(x,y)$ is the *cofactor* of the exponential factor $e^{h/g}$.

The next proposition, proved in [32], gives the relationship between the notion of invariant algebraic curve and exponential factor.

Proposition 1.1 [32] *If $F = e^{h/g}$ is an exponential factor and g is not a constant, then $g=0$ is an invariant algebraic curve, and h satisfies the equation $P\frac{\partial h}{\partial x} + Q\frac{\partial h}{\partial y} = h\, k_g + g\, k_F$ where k_g and k_F are the cofactors of g and F, respectively.*

The notion of exponential factor is very important in the Darboux theory of integrability since it not only allows the construction of first integrals following the same method described by Darboux, but it also explains the meaning of the multiplicity of an invariant algebraic curve in relation with the differential system (1.1). A recent and complete work on this subject can be found in [35].

In the same way as with invariant algebraic curves, given an exponential factor $F = \exp\{h/g\}$, since system (1.1) is a real system, there is no lack of generality in considering that $h(x,y), g(x,y) \in \mathbb{R}[x,y]$. If $F = \exp\{h/g\}$ is an exponential factor with non-null imaginary part, then its complex conjugate, $\bar{F} = \exp\{\bar{h}/\bar{g}\}$ is also an exponential factor, as it can be easily checked by its defining equation. Moreover, the product $F\,\bar{F} = \exp\{h/g + \bar{h}/\bar{g}\}$ is a real exponential factor with a real cofactor.

Since the notion of exponential factor is the most current generalization in the Darboux theory of integrability, any function of the form:

$$f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_r^{\lambda_r} \left(\exp\left(\frac{h_1}{g_1^{n_1}}\right)\right)^{\mu_1} \left(\exp\left(\frac{h_2}{g_2^{n_2}}\right)\right)^{\mu_2} \cdots \left(\exp\left(\frac{h_\ell}{g_\ell^{n_\ell}}\right)\right)^{\mu_\ell}, \quad (1.4)$$

where $r, \ell \in \mathbb{N}$, $f_i(x,y) = 0$ $(1 \leq i \leq r)$ and $g_j(x,y) = 0$ $(1 \leq j \leq \ell)$ are invariant algebraic curves of system (1.1), $h_j(x,y)$ $(1 \leq j \leq \ell)$ are polynomials in $\mathbb{C}[x,y]$, λ_i $(1 \leq i \leq r)$ and $\mu_j = 0$ $(1 \leq j \leq \ell)$ are complex numbers and n_j $(1 \leq j \leq \ell)$ are non-negative integers, is called a *(generalized) Darbouxian function* .

We recall that the integrability problem consists in finding the class of functions a first integral of a given system (1.1) must belong to. We have the system (1.1) defined in a certain class of functions, in this case, the polynomials with real coefficients $\mathbb{R}[x,y]$ and we consider the problem whether there is a first integral in another, possibly larger, class. For instance in [77], H. Poincaré stated the problem of determining when a system (1.1) has a rational first integral. The works of M.J. Prelle and M.F. Singer [79] and M.F. Singer [86] go on this direction since they give a characterization of when a polynomial system (1.1) has an elementary or a Liouvillian first integral.

An important fact of their results is that invariant algebraic curves play a distinguished role in this characterization. Moreover, this characterization is expressed in terms of the inverse integrating factor.

Roughly speaking, an *elementary function* is a function constructed from rational functions by using algebraic operations, composition and exponentials, applied a finite number of times, and a *Liouvillian function* is a function constructed from rational functions by using algebraic operations, composition, exponentials and integration, applied a finite number of times. A precise definition of these classes of functions is given in [79, 86]. We are mainly concerned with Liouvillian functions but we will state some results related to integration of a system (1.1) by means of elementary functions.

Theorem 1.2 [79] *If the system* (1.1) *has an elementary first integral, then there exist* $w_0, w_1, \ldots, w_n$ *algebraic over the field* $\mathbb{C}(x, y)$ *and* $c_1, c_2, \ldots, c_n$ *in* $\mathbb{C}$ *such that the elementary function*

$$\tilde{H} = w_0 + \sum_{i=1}^{n} c_i \ln(w_i) \tag{1.5}$$

is a first integral of system (1.1)*.*

The existence of an elementary first integral is intimately related to the existence of an algebraic inverse integrating factor, as the following result shows.

Theorem 1.3 [79] *If the system* (1.1) *has an elementary first integral, then there is an inverse integrating factor of the form*

$$V = \left(\frac{A(x, y)}{B(x, y)} \right)^{1/N},$$

where $A, B \in \mathbb{C}[x, y]$ *and* N *is an integer number.*

In the work [20], the systems (1.1) with a (generalized) Darboux first integral of the form (1.4) are studied and the following result is accomplished.

Theorem 1.4 [20] *If the system* (1.1) *has a (generalized) Darboux first integral of the form* (1.4)*, then there is a rational inverse integrating factor, that is, an inverse integrating factor of the form:*

$$V = \frac{A(x, y)}{B(x, y)},$$

where $A, B \in \mathbb{C}[x, y]$*.*

Unfortunately, not all the elementary functions of the form (1.5) are of (generalized) Darboux type. That's why, we can find systems with an elementary first integral and without a rational inverse integrating factor. The following example is of this type. The system appears in the works of Jean Moulin-Ollagnier [67, 68], although he does not give an explicit expression for the first integral. This example corresponds to $\ell = 1$ in the family of systems (2.10) studied in Section 2.3.2 in Chapter 2, where we give an explicit expression of the first integral for any value of $\ell \in \mathbb{R} - \{\frac{1}{2}(1-2r) \mid r \in \mathbb{N}\}$. The Lotka-Volterra system:

$$\dot{x} = x\left(1 - \frac{x}{2} + y\right), \quad \dot{y} = y\left(-3 + \frac{x}{2} - y\right), \tag{1.6}$$

has the irreducible invariant algebraic curves $x = 0$, $y = 0$ and $f(x,y) = 0$, where $f(x,y) := (x-2)^2 - 2xy$. Applying, for instance, the results described in Chapter 3, it can be shown that this system has no other irreducible invariant algebraic curve. The function $V(x,y) = x^{-1/2}y^{1/2}f(x,y)$ is the only algebraic inverse integrating factor of system (1.6) (modulus multiplication by non null constants). Since there is no rational inverse integrating factor, we deduce, by Theorem 1.4, that there is no (generalized) Darboux first integral. An elementary first for this system, which is of the form (1.5), is given by:

$$H(x,y) := \sqrt{2}\sqrt{x}\sqrt{y} + \ln(x - 2 + \sqrt{2}\sqrt{x}\sqrt{y}) - \ln(x - 2 - \sqrt{2}\sqrt{x}\sqrt{y}).$$

We remark that both Theorems 1.3 and 1.4 give a necessary condition to have an elementary or (generalized) Darbouxian, respectively, first integral. The reciprocals to the statements of Theorems 1.3 and 1.4 are not true. A result to clarify the easiest functional class of the first integral once we know the inverse integrating factor appears in [45], where the following theorem is stated and proved:

Theorem 1.5 [45] *If the system* (1.1) *has a polynomial inverse integrating factor, then the system has a (generalized) Darbouxian first integral.*

In any case, the following Theorem 1.6 ensures that given an algebraic inverse integrating factor, there is a Liouvillian first integral. The Liouvillian class of functions contains the rational, algebraic, Darbouxian and elementary classes of functions.

M.F. Singer shows in [86] the characterization of the existence of a Liouvillian first integral for a system (1.1) by means of its invariant algebraic curves.

Theorem 1.6 [86] *System* (1.1) *has a Liouvillian first integral if, and only if, there is an inverse integrating factor of the form* $V = \exp\left\{\int_{(x_0,y_0)}^{(x,y)} \eta\right\}$, *where* η *is a rational* 1*–form such that* $d\eta \equiv 0$*.*

Taking into account Theorem 1.6, C. Christopher in [33] gives the following result, which makes precise the form of the inverse integrating factor.

Theorem 1.7 [33] *If the system* (1.1) *has an inverse integrating factor of the form* $\exp\left\{\int_{(x_0,y_0)}^{(x,y)} \eta\right\}$, *where* η *is a rational* 1*–form such that* $d\eta \equiv 0$*, then there exists an inverse integrating factor of system* (1.1) *of the form*

$$V = \exp\{D/E\} \prod C_i^{l_i},$$

where D*,* E *and the* C_i *are polynomials in* x *and* y *and* $l_i \in \mathbb{C}$*.*

We notice that $C_i = 0$ are invariant algebraic curves and $\exp\{D/E\}$ is an exponential factor for system (1.1). In fact, since system (1.1) is a real system, we can assume, without loss of generality, that V is a real function.

Theorem 1.7 states that the search for Liouvillian first integrals can be reduced to the search of invariant algebraic curves and exponential factors. Therefore, if we characterize the possible cofactors, we have the invariant algebraic curves of a system and, hence, its Liouvillian or non Liouvillian integrability. In Chapter 3, we give a set of necessary conditions for a system (1.1) to have an invariant algebraic curve.

1.2 On the stability of singular points and periodic orbits

The stability of a singular point or a periodic orbit is a local characteristic, that is, we only need to study the behavior of the solutions of a differential system in a neighborhood of the singular point or the periodic orbit. This is the reason why we can state the results in a wider domain than planar polynomial differential systems. We can consider systems defined by functions of class $\mathcal{C}^1$ in a neighborhood $\mathcal{U} \subseteq \mathbb{R}^2$ of the singular point or the periodic orbit. We will always assume that the singular points of the considered system are isolated. We state and prove a sharp result for singular points, Theorem 6.8, which is related to the period function. Therefore, when describing the stability of singular points we need to consider analytic systems, a stronger hypothesis than to be a $\mathcal{C}^1$ system. In any case, polynomial differential systems as (1.1) are a particular and interesting enough case to be considered.

We take a planar differential system:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.7}$$

where P and Q are $\mathcal{C}^1$ functions in some open set $\mathcal{U} \subseteq \mathbb{R}^2$ and we consider the *flow* of system (1.7), which we denote by $\Phi_t(p)$.

The flow $\Phi_t(p)$ represents the unique solution of system (1.7) passing through the point $p \in \mathcal{U} \subseteq \mathbb{R}^2$. We notice that for each $p \in \mathcal{U}$ there exists an $\epsilon_p > 0$ (which may be $\epsilon_p = +\infty$) such that $t \in (-\epsilon_p, \epsilon_p)$ is the maximal symmetric interval of existence of the solution of (1.7) passing through p. We have that $\frac{d\Phi_t}{dt}(p) = (P(\Phi_t(p)), Q(\Phi_t(p)))$, for all $p \in \mathcal{U}$ and $t \in (-\epsilon_p, \epsilon_p)$, and $\Phi_0(p) = p$. Given $p \in \mathcal{U}$, the function $\Phi(\cdot, p) : (-\epsilon_p, \epsilon_p) \to \mathbb{R}^2$, where $\Phi(t, p) := \Phi_t(p)$, defines a *solution curve* or *orbit* of (1.7) with initial condition the point p.

We cite [60, 74, 88] for the definitions, summary of known results and further reading on the stability of planar differential systems as (1.7).

1.2.1 Singular points

Given a real number $\varepsilon > 0$ and a point $p \in \mathbb{R}^2$, we define $\mathcal{B}_\varepsilon(p)$ as the open ball of center p and radius ε, i.e.,

$$\mathcal{B}_\varepsilon(p) := \left\{ q \in \mathbb{R}^2 \;\mid\; \|q - p\| < \varepsilon \right\},$$

where $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^2$. We will always assume that ε is small enough such that $\mathcal{B}_\varepsilon(p) \subseteq \mathcal{U}$.

A singular point p of system (1.7) is said to be *stable* if for all $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for all $q \in \mathcal{B}_{\delta(\varepsilon)}(p)$ and $t \geq 0$ we have $\Phi_t(q) \in \mathcal{B}_\varepsilon(p)$.

A singular point p of system (1.7) is said to be *unstable* if it is not stable. The singular point p is *asymptotically stable* if it is stable and if there exists a $\delta > 0$ such that for all $q \in \mathcal{B}_\delta(p)$ we have $\lim_{t \to +\infty} \Phi_t(q) = p$.

The stability of a singular point p is determined by the behavior of the solutions of system (1.7) in a neighborhood of p. This behavior is, in general, determined by the linear approximation of the system at the point. Let $A(p)$ be the following matrix:

$$A(p) = \begin{pmatrix} \dfrac{\partial P}{\partial x}(p) & \dfrac{\partial P}{\partial y}(p) \\ \dfrac{\partial Q}{\partial x}(p) & \dfrac{\partial Q}{\partial y}(p) \end{pmatrix}, \tag{1.8}$$

which gives the linear approximation of the system (1.7) at p.

A singular point p is called *degenerate* if the matrix $A(p)$ is degenerate, that is, $\det A(p) = 0$. Otherwise, we will say that the singular point p is *non–degenerate*. The eigenvalues of the matrix $A(p)$, which we denote by λ and μ, determine, in general, the stability of the singular point.

We say that the singular point p is *hyperbolic* if none of the real parts of its corresponding eigenvalues is null, that is $(\text{Re}\lambda)(\text{Re}\mu) \neq 0$. As is it proved in, for instance, [74, 88], a hyperbolic singular point is either asymptotically stable if $\text{Re}\lambda < 0$ and $\text{Re}\mu < 0$ or unstable, otherwise. Hence, the stability of a hyperbolic singular point is completely characterized.

We will be concerned with non-degenerate real singular points which are not hyperbolic, that is, singular points whose eigenvalues are $\lambda = b\,\mathbf{i}$ and $\mu = -b\,\mathbf{i}$, with $b \neq 0$ and $\mathbf{i} = \sqrt{-1}$. Moreover, since we need some additional properties for the sake of our results, we will assume that the system is analytic in a neighborhood of the singular point. If system (1.7) has one of these singular points, by an affine change of variables and a change of the scale of time, the system may be brought to the form:

$$\dot{x} = -y + P_2(x, y), \quad \dot{y} = x + Q_2(x, y), \tag{1.9}$$

where $P_2(x, y)$ and $Q_2(x, y)$ are analytic functions in a neighborhood $\mathcal{U}$ of the origin of order equal or higher than 2. The origin of system (1.9) may be a center (stable) or a focus (unstable or asymptotically stable). We recall that an isolated singular point of (1.7), or the origin of (1.9) which is a particular case, is said to be a *focus* if it has a punctured neighborhood where all the orbits spiral in forward or backward time. An isolated singular point of (1.7) is said to be a *center* if it has a punctured neighborhood filled of periodic orbits. The *center problem* consists in determining if the origin of a system (1.9) is a focus or if it is a center.

In the local study of systems of the form (1.9) we find three problems closely related to one another: the determination of the origin's stability, the existence and the number of local limit cycles around the origin and the determination of first integrals when they exist.

H. Poincaré [75] developed an important technique for the general solution of these problems: it consists in finding a formal power series of the form

$$H(x,y) = \sum_{n=2}^{\infty} H_n(x,y), \tag{1.10}$$

where $H_2(x,y) = (x^2+y^2)/2$ and $H_n(x,y)$ are homogeneous polynomials of degree n, so that

$$\dot{H} = \frac{\partial H}{\partial x}(-y + P_2(x,y)) + \frac{\partial H}{\partial y}(x + Q_2(x,y)) = \sum_{k=1}^{\infty} V_{2k+1}(x^2+y^2)^k,$$

where V_{2k+1}, $k \geq 1$, are real numbers called *Liapunov constants.* The determination of these constants allows the solution of the three mentioned problems. These constants are determined in a recursive way explained, for instance, in [57, 87].

The vanishing of all Liapunov constants is a necessary and sufficient condition to have a center at the origin for system (1.9). If for some k we have $V_3 = V_5 = \ldots = V_{2k-1} = 0$ and $V_{2k+1} \neq 0$ then the origin of system (1.9) is a focus (stable if $V_{2k+1} < 0$ and unstable if $V_{2k+1} > 0$). We say that it is a focus of order k. In case all Liapunov constants are zero, the series $H(x,y)$ would be a first integral of the system if it was convergent. Poincaré proved that, if all Liapunov constants vanish, then it is always possible to find a power series of the form (1.10) convergent in some neighborhood of the origin. Then, this power series is an analytical first integral defined in some neighborhood of the origin. However, it is not always possible to express this first integral (convergent in some neighborhood of the origin) by means of elementary functions.

Theorem 1.8 [75] *System* (1.9) *has a center at the origin if, and only if, there exists a local analytical first integral of the form* $H(x,y) = x^2 + y^2 +$

$F(x, y)$ defined in a neighborhood of the origin, where $F(x, y)$ is an analytic function of order greater than 2.

When the Liapunov constants are computed from a family of systems, they are polynomials on the coefficients of the family. Hilbert's Nullstellensatz ensures that there always exists a finite number of independent polynomials which generates the whole ideal made up with all these Liapunov-Poincaré polynomials. The zero-set of these independent polynomials gives place to the center subfamilies. The reader is referred to [83, 84] for a survey on this subject.

Besides the stability of the origin of system (1.9), which is solved by Theorem 1.8, a main problem is that of studying the existence and properties of periodic solutions in a neighborhood of the origin O of (1.9). In this field, different methods have been used to study isolated periodic solutions, i.e. limit cycles, or non-isolated ones, i.e. period annulus. The stability of the singular point O does not imply the stability of the cycles close to the singular point. In fact, a non-isolated cycle is stable if and only if every neighboring cycle has the same period. This fact motivates the definition of isochronicity. We give a precise definition of isochronicity in the forthcoming paragraph. Isochronicity has been widely studied not only for its physical meaning and for its role in stability theory, but also for its relationship with bifurcation problems and to boundary value problems.

An essential tool to study the stability of the origin of system (1.9) is the *Poincaré map*, see [74, 78]. Let us consider a neighborhood $\mathcal{U}$ of the origin and let Σ be a section of system (1.9) through the origin, that is, a transversal curve through the origin for the flow of system (1.9). More precisely, we define a *section through the origin* as a simple arc without contact with the origin O as an endpoint. See the book of Andronov *et al.* [4], page 55, for a precise definition of simple arc without contact. We also need some assumptions on its regularity for technical reasons. Given a section through the origin $\Sigma \subset \mathbb{R}^2$, we consider a parameterization $c : \mathbb{R} \to \mathbb{R}^2$ such that $\Sigma = \{c(\sigma) \,|\, \sigma \in \mathbb{R}\}$ and $\lim_{\sigma \to -\infty} c(\sigma) = O$. We assume that $c(\sigma)$ is analytic for all $\sigma \in \mathbb{R}$.

For each point $p \in \Sigma$, the flow of system (1.9) through p will cross Σ again at a point $\mathcal{P}(p) \in \Sigma$ near p. The map $p \mapsto \mathcal{P}(p)$ is called the Poincaré map. As before, we denote by $\Phi_t(p)$ the flow of system (1.9) with the initial condition $\Phi_0(p) = p$ and we can define the *Poincaré map* in the following way. Given $p \in \Sigma$, there is a unique analytic function $\tau(p)$ such that $\Phi_{\tau(p)}(p) \in \Sigma$ and $\Phi_t(p) \notin \Sigma$ for any $0 < t < \tau(p)$, see [74]. In these terms, we have $\mathcal{P}(p) = \Phi_{\tau(p)}(p)$. We remark that both functions $\mathcal{P}$ and τ depend on the chosen section Σ. The function $\tau : \Sigma \to \mathbb{R}^+$ is called the *period function*. As usual $\mathbb{R}^+$ denotes the set of positive real numbers. In Chapter 6 we study the existence of a section Σ such that $\tau : \Sigma \to \mathbb{R}^+$ is constant. When such a Σ exists, we say that the origin O of (1.9) is *isochronous* and that Σ is an *isochronous section*.

1.2.2 Periodic orbits

Analogously to the definition for singular points, a periodic orbit γ of a system (1.7) is called *stable* if for each $\varepsilon > 0$ there is a neighborhood $\mathcal{U}_\varepsilon$ of γ such that for all $q \in \mathcal{U}_\varepsilon$ and $t \geq 0$, $d(\Phi_t(q), \gamma) < \varepsilon$, where d is the distance between Hausdorff sets.

A periodic orbit γ is said to be *unstable* if it is not stable. And γ is said to be *asymptotically stable* if it is stable and if there exists a neighborhood $\mathcal{U}_\gamma$ of γ such that for all $q \in \mathcal{U}_\gamma$ we have $\lim_{t\to+\infty} d(\Phi_t(q), \gamma) = 0$.

We distinguish the periodic orbits of a system (1.7) depending on the behavior of the solutions in a neighborhood of it. A *limit cycle* of system (4.1) is an isolated periodic orbit. Let γ be a limit cycle for system (4.1). We say that γ is *stable* if there exists a neighborhood $\mathcal{U}_\gamma \subseteq \mathcal{U}$ of γ such that for all $p \in \mathcal{U}_\gamma$, we have $\lim_{t\to+\infty} d(\Phi_t(p), \gamma) = 0$. Analogously, we say that γ is *unstable* if there exists a neighborhood $\mathcal{U}_\gamma \subseteq \mathcal{U}$ of γ such that for all $p \in \mathcal{U}_\gamma$, we have $\lim_{t\to-\infty} d(\Phi_t(p), \gamma) = 0$.

There might be limit cycles which are neither stable nor unstable. Using the Jordan curve theorem, which states that any simple closed curve as the limit cycle γ separates any neighborhood $\mathcal{U}_\gamma$ of γ into two disjoint sets having γ as a boundary, we can consider $\mathcal{U}_\gamma$ as the disjoint union of $\mathcal{U}_i \cup \gamma \cup \mathcal{U}_e$, where $\mathcal{U}_i$ and $\mathcal{U}_e$ are open sets situated, respectively, in the interior and exterior of γ. When for any $p \in \mathcal{U}_i$ we have $\lim_{t\to+\infty} d(\Phi_t(p), \gamma) = 0$ whereas for any $q \in \mathcal{U}_e$ we have $\lim_{t\to-\infty} d(\Phi_t(q), \gamma) = 0$ (or, the other way round, for any $p \in \mathcal{U}_i$ we have $\lim_{t\to-\infty} d(\Phi_t(p), \gamma) = 0$ whereas for any $q \in \mathcal{U}_e$ we have $\lim_{t\to+\infty} d(\Phi_t(q), \gamma) = 0$), we say that γ is *semi-stable*.

Any limit cycle γ of a system (1.7) is either stable, unstable or semi-stable as it is stated in [74]. For a detailed description of the classical known results on limit cycles see also [74].

The following result, which is stated as a corollary in page 214 of [74], gives a formula to distinguish the stability of a limit cycle.

Theorem 1.9 *Let $\gamma(t)$ be a periodic orbit of system* (1.7) *of period T. Then, γ is a stable limit cycle if*

$$\int_0^T \operatorname{div}(\gamma(t))\, dt < 0,$$

and it is an unstable limit cycle if

$$\int_0^T \operatorname{div}(\gamma(t))\, dt > 0.$$

It may be stable, unstable or semi-stable limit cycle or it may belong to a continuous band of cycles if this quantity is zero.

We recall that div is the *divergence* of system (1.7), that is, $\operatorname{div}(x, y) := \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$. When the quantity $\int_0^T \operatorname{div}(\gamma(t))\, dt$ is different from zero, we say that the limit cycle γ is *hyperbolic*.

Since we are considering differential systems (1.7) in the class of functions $\mathcal{C}^1$, we may have a limit cycle γ belonging to a sequence of periodic orbits $\{\gamma_n\, ,\, n \in \mathbb{N}\}$ with γ_{n+1} in the interior of γ_n, such that the sequence accumulates to a singular point, a periodic orbit or a graphic and such that every trajectory between γ_n and γ_{n+1} spirals towards γ_n or γ_{n+1} as $t \to \pm\infty$. This kind of phenomena does not exist for analytic systems.

In Chapter 4, we give another quantity which equals to $\int_0^T \operatorname{div}(\gamma(t))\, dt$ for a periodic orbit γ. So, we give an alternative method to study its stability.

Chapter 2

Integrability via second order linear homogeneous ode's

2.1 Motivation of the method

In this chapter, we consider a system (1.1) but in the form of a rational ordinary differential equations such as

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}\,. \tag{2.1}$$

Our aim is to show a relationship between solutions of a class of systems (1.1), or equivalently equations (2.1), and linear homogeneous ordinary differential equations of order 2 of the form:

$$A_2(x)\,w''(x) + A_1(x)\,w'(x) + A_0(x)\,w(x) = 0, \tag{2.2}$$

where $x \in \mathbb{R}$, $w'(x) = dw(x)/dx$ and $w''(x) = dw'(x)/dx$. We only consider ordinary differential equations (2.2) where $A_0(x)$, $A_1(x)$ and $A_2(x)$ are polynomials with real coefficients and $A_2(x) \not\equiv 0$.

By means of a change of variable we rewrite an equation (2.2) as a polynomial differential system such that it has an invariant curve related to $w(x)$. In case $w(x)$ is a polynomial we get an invariant algebraic curve.

Moreover, we give an explicit first integral for all the systems built up by this method by means of two independent solutions of equation (2.2).

We give analogous results for a linear homogeneous ordinary differential equation of order 1 such as

$$w'(x) + A(x)w(x) = 0, \tag{2.3}$$

where $x \in \mathbb{R}$, $w'(x) = dw(x)/dx$ and $A(x)$ is a rational function.

We give the explicit expression for the first integral of a certain system (1.1) by means of invariant curves for it, and applying the Generalized Darboux's Theory as explained in [50] where a new kind of first integrals, not only the Liouvillian ones as in classical theories, is described. We exemplify this result with the families of systems depending on parameters described in Section 2.3. We remark that the first integrals that we give in Section 2.2 are not, in general, of Liouvillian type. However, these first integrals are Liouvillian at the values of parameters which correspond to the systems with algebraic solutions.

All these results are stated in Section 2.2. In order to exemplify them, we consider the counterexamples to a conjecture stated by A. Lins Neto concerning the integrability problem and the degree of an irreducible invariant algebraic curve. In order to motivate this conjecture, we must cite the article [77] in which H. Poincaré stated the following problem concerning the integration of an equation (2.1): *Give conditions on the polynomials P and Q to recognize when there exists a rational first integral.* As the same H. Poincaré noticed, a sufficient condition to solve this problem consists in finding an upper bound for the degree of the invariant algebraic curves for a given system (1.1). From Darboux and Jouanolou's results, cf. [38, 61], it is known that for every polynomial vector field, there exists an upper bound for the possible degrees of irreducible invariant algebraic curves. However, it is a hard problem to explicitly determine such an upper bound. Some bounds have been given under certain conditions on the invariant curves, see D. Cerveau and A. Lins Neto's work [15], or on the local behavior of critical points, see M. Carnicer's work [13] and S. Walcher's work [91].

In this sense, A. Lins Neto conjectured in [62] that a polynomial system (1.1) of degree d with an invariant algebraic curve of degree high enough (where this bound only depends on d) would have a rational first integral. This conjecture has been shown to be false by several counterexamples. In [67], J. Moulin-Ollagnier gives a family of quadratic Lotka-Volterra systems, each with an invariant algebraic curve of degree 2ℓ, where ℓ is the parameter of the family, without rational first integral. A simpler example is given in [34] by C. Christopher and J. Llibre. In [24] a family of quadratic systems with an invariant algebraic curve of arbitrarily high degree without a Darboux first integral nor a Darboux inverse integrating factor is given. However, the family given in [24] has a (generalized) Darboux inverse integrating factor.

All these counterexamples exhibit a Liouvillian first integral.

The natural conjecture at this step, also given by A. Lins Neto, see [63], after the counterexample of J. Moulin-Ollagnier appeared, is that a polynomial system (1.1) of degree d with an invariant algebraic curve of degree high enough (where this bound only depends on d) has a Liouvillian first integral.

In Section 2.3 we consider all the families of quadratic systems with an algebraic curve of arbitrarily high degree known by these counterexamples to A. Lins Neto conjecture and we show that they all belong to the construction explained in Section 2.2. The families of quadratic systems with an algebraic curve of arbitrarily high degree studied in this chapter are the ones appearing in [23, 24, 34, 68] and one example more appearing in Section 2.3. This new example consists in a biparametrical family of quadratic systems, which we give an explicit expression of a first integral for, such that when one of the parameters is a natural number, say n, the system exhibits an irreducible invariant algebraic curve of degree n.

In Subsection 2.3.4, we give an example of a 3-parameter family of quadratic systems with a center at the origin which can be constructed by the method appearing in Section 2.2 from an equation (2.3).

A question suggested by these examples is whether there are polynomial systems which are not reversible nor Liouvillian integrable which have a center and can be integrated by means of Theorem 2.1, see Section 2.2. The work [11] is related to this question as it gives an example of an analytic system, not polynomial, with a center which is not reversible nor Liouvillian integrable. All the known families of polynomial vector fields with a center at the origin are either Liouvillian integrable or reversible, see [97, 98] for the definition of reversibility. In [97, 98], H. Żołądek classifies all the reversible cubic systems with a center.

Many reversible systems may have a first integral not given by Liouvillian functions or no explicit form of a first integral may be known. For instance the reversible system $\dot{x} = -y + x^4$, $\dot{y} = x$ has a first integral composed by Airy functions, see [50], and no Liouvillian first integral exists. The system $\dot{x} = -y^3 + x^2y^2/2$, $\dot{y} = x^3$ is an example given by Moussu, see [69], which has a center at the origin since it is a monodromic and reversible singular point and no explicit first integral is known for this system.

Several authors are working on the problem of finding a polynomial system not reversible and without a Liouvillian first integral. In a private communication with J. Llibre and H. Żołądek, they showed us a polynomial system with an integrable saddle whose first integral is not of Liouvillian type and this system may be not reversible. This example is still a work in progress and we describe it in Subsection 2.3.5 because its first integral can be constructed by means of the method described in Theorem 2.1.

Since some examples of polynomial systems, which can be integrated by the method described in Section 2.2, appear after a birrational transformation, another suggested open question is to determine whether a given polynomial system is birrationally equivalent to one derived from Theorem 2.1 or from Theorem 2.6. We recall that a *birrational transformation* is a rational change of variables whose inverse is also rational. This kind of transformations brings polynomial systems such (1.1) to polynomial systems and do not change the Liouvillian or non Liouvillian character of the first integral.

2.2 Homogeneous linear differential equations of order ≤ 2 and planar polynomial systems

We consider a homogeneous linear differential equation of order 2:

$$A_2(x)\, w''(x) + A_1(x)\, w'(x) + A_0(x)\, w(x) = 0, \tag{2.4}$$

where $w'(x) = dw(x)/dx$, $w''(x) = dw'(x)/dx$, $A_i(x) \in \mathbb{R}[x]$, $i = 0, 1, 2$, and $A_2(x) \not\equiv 0$.

Theorem 2.1 *Given $g(x, y) = g_0(x, y)/g_1(x, y)$ with $g_i(x, y) \in \mathbb{R}[x, y]$, satisfying $g_1(x, y) \not\equiv 0$ and $\partial g/\partial y \not\equiv 0$, each nonzero solution $w(x)$ of equation* (2.4) *is related to a finite number of solutions $y = y(x)$ of the rational equation*

$$\frac{dy}{dx} = \frac{A_0(x)\, g_1^2 + A_1(x)\, g_1\, g_0 + A_2(x)\, g_0^2 + A_2(x)\left(g_1 \frac{\partial g_0}{\partial x} - g_0 \frac{\partial g_1}{\partial x}\right)}{A_2(x)\left(g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y}\right)}, \tag{2.5}$$

by the functional change $dw/dx = g(x, y)\, w(x)$, which implicitly defines y as a function of x.

Proof. Let us consider equation (2.4) and the functional change $dw/dx = g(x, y)w(x)$ where $y = y(x)$, that is, y is implicitly defined as a function of x. This change may also be written as $w(x) = \exp(\int_{x_0}^{x} g(s, y(s))ds)$, where x_0 is any constant, and it is injective. We see that it is not necessarily bijective unless the maximum degree of $g_1(x, y)$ and $g_0(x, y)$ in the variable y equals to 1. In any case, it defines a finite number of functions $y(x)$.

By this functional change, equation (2.4) becomes

$$w(x)\left(A_0(x) + g\,A_1(x) + g^2\,A_2(x) + A_2(x)\frac{dy}{dx}\frac{\partial g}{\partial y} + A_2(x)\frac{\partial g}{\partial x}\right) = 0.$$

We have that $w(x)$ is a nonzero solution of (2.4) so this equation is equivalent to the ordinary differential equation of first order (2.5). Therefore, each non-zero solution $w(x)$ of equation (2.4) corresponds to a finite number of solutions $y = y(x)$ of the planar polynomial system (2.5). ∎

Theorem 2.2 *We consider the vector field* $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ *related to equation* (2.5)

$$\begin{aligned} P(x,y) &= A_2(x)\left(g_0\frac{\partial g_1}{\partial y} - g_1\frac{\partial g_0}{\partial y}\right),\\ Q(x,y) &= \left(A_0(x)\,g_1^2 + A_1(x)\,g_1\,g_0 + A_2(x)\,g_0^2 + A_2(x)\left(g_1\frac{\partial g_0}{\partial x} - g_0\frac{\partial g_1}{\partial x}\right)\right). \end{aligned}$$

Let $w(x)$ *be any nonzero solution of equation* (2.4)*. Then the curve defined by* $f(x, y) = 0$*, with* $f(x, y) := g_1(x, y)w'(x) - g_0(x, y)w(x)$ *is an invariant curve for system* (2.5) *and has the polynomial cofactor*

$$\begin{aligned} k(x,y) = &\left(A_0(x)\,\frac{\partial g_1}{\partial y} + A_1(x)\,\frac{\partial g_0}{\partial y}\right)g_1 + A_2(x)\,g_0\frac{\partial g_0}{\partial y}\\ &+A_2(x)\left(\frac{\partial g_1}{\partial y}\frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y}\frac{\partial g_1}{\partial x}\right). \end{aligned}$$

Proof. Let us consider $f(x,y)$ as defined above and let us compute $\nabla f \cdot \mathbf{F}$:

$$\begin{aligned}\nabla f \cdot \mathbf{F} &= \left(\frac{\partial g_1}{\partial x} w'(x) + g_1 w''(x) - \frac{\partial g_0}{\partial x} w(x) - g_0 w'(x)\right) A_2(x) \left(g_0 \frac{\partial g_1}{\partial y}\right.\\ &\left. - g_1 \frac{\partial g_0}{\partial y}\right) + \left(\frac{\partial g_1}{\partial y} w'(x) - \frac{\partial g_0}{\partial y} w(x)\right) \left[A_0(x)\, g_1^2 + A_1(x)\, g_1\, g_0\right.\\ &\left. + A_2(x)\, g_0^2 + A_2(x) \left(g_1 \frac{\partial g_0}{\partial x} - g_0 \frac{\partial g_1}{\partial x}\right)\right]\\ &= g_1 \left(g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y}\right) A_2(x)\, w''(x) +\\ & g_1 \left(A_1(x)\, g_0 + A_0(x)\, g_1\right) \left(\frac{\partial g_1}{\partial y} w'(x) - \frac{\partial g_0}{\partial y} w(x)\right) +\\ & \left(A_2(x) \left(\frac{\partial g_0}{\partial x}\frac{\partial g_1}{\partial y} - \frac{\partial g_1}{\partial x}\frac{\partial g_0}{\partial y}\right) + A_2(x)\, g_0 \frac{\partial g_0}{\partial y}\right) (g_1 w'(x) - g_0 w(x))\end{aligned}$$

Since $w(x)$ is a solution of (2.4), we can substitute $A_2(x) w''(x)$ by $-A_1(x) w'(x) - A_0(x) w(x)$. Therefore,

$$\begin{aligned}\nabla f \cdot \mathbf{F} &= \left[\left(A_0(x) \frac{\partial g_1}{\partial y} + A_1(x) \frac{\partial g_0}{\partial y}\right) g_1 + \right.\\ & \left. A_2(x)\, g_0 \frac{\partial g_0}{\partial y} + A_2(x) \left(\frac{\partial g_1}{\partial y}\frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y}\frac{\partial g_1}{\partial x}\right)\right] f(x,y).\end{aligned}$$

Then, we have that $f(x,y) = 0$ is an invariant curve for system (2.5) and has the written polynomial cofactor. ∎

Theorem 2.3 *Let* $\{w_1(x), w_2(x)\}$ *be a set of fundamental solutions of equation* (2.4). *We define* $f_i(x,y) := g_1(x,y) w_i'(x) - g_0(x,y) w_i(x)$, $i = 1,2$. *Then, system* (2.5) *has a first integral* $H(x,y)$ *defined by*

$$H(x,y) := \frac{f_1(x,y)}{f_2(x,y)} = \frac{g_1(x,y)\, w_1'(x) \;-\; g_0(x,y)\, w_1(x)}{g_1(x,y)\, w_2'(x) \;-\; g_0(x,y)\, w_2(x)}.$$

Proof. By Theorem 2.2, we have that $f_i(x,y) := g_1(x,y) w_i'(x) - g_0(x,y) w_i(x)$, $i = 1,2$, define invariant curves for system (2.5), both with the same polynomial cofactor $k(x,y)$ which is equal to:

$$\left(A_0(x) \frac{\partial g_1}{\partial y} + A_1(x) \frac{\partial g_0}{\partial y}\right) g_1 + A_2(x)\, g_0 \frac{\partial g_0}{\partial y} + A_2(x) \left(\frac{\partial g_1}{\partial y}\frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y}\frac{\partial g_1}{\partial x}\right).$$

We remark that f_1/f_2 cannot be constant since the two solutions $w_i(x)$, $i = 1, 2$, are independent. Therefore,

$$\nabla H \cdot \mathbf{F} = \frac{f_2(\nabla f_1 \cdot \mathbf{F}) - f_1(\nabla f_2 \cdot \mathbf{F})}{f_2^2} = \frac{f_2 k f_1 - f_1 k f_2}{f_2^2} \equiv 0.$$

So, $H(x, y)$ is a first integral of system (2.5). ■

Lemma 2.4 *The function defined by*

$$q(x) := A_2(x) \exp\left(\int_{x_0}^{x} \frac{A_1(s)}{A_2(s)} ds \right)$$

gives rise to an invariant curve for system (2.5)*, with cofactor* $(A_1(x) + A_2'(x)) \left(g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y} \right)$.

We notice that $q(x)$ is a product of invariant algebraic curves and exponential factors for system (2.4), with complex exponents.
Proof. We compute $\nabla q \cdot \mathbf{F}$ and we have

$$\nabla q \cdot \mathbf{F} = P(x, y) \frac{A_1(x) + A_2'(x)}{A_2(x)} q = (A_1(x) + A_2'(x)) \left(g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y} \right) q.$$

We notice that this algebraic cofactor has degree ≤ d − 1 provided that system (2.5) has degree d. ■

Proposition 2.5 *We use the same notation as in Theorem* 2.1*. Let* $w(x)$ *be a nonzero solution of* (2.4) *and we define* $f(x, y) := w'(x) - g(x, y) w(x)$ *and* $q(x)$ *as in Lemma* 2.4*. The function* $V(x, y) = q(x) f(x, y)^2$ *is an inverse integrating factor of system* (2.5)*.*

Proof. We only need to verify that $\nabla V \cdot \mathbf{F} = \mathrm{div}(x, y)\, V$, where $\mathrm{div}(x, y)$ is the divergence of system (2.5). We have that

$$\begin{aligned} \mathrm{div}(x, y) = \;& 2\, A_0(x)\, g_1 \frac{\partial g_1}{\partial y} + A_1(x) \left(g_0 \frac{\partial g_1}{\partial y} + g_1 \frac{\partial g_0}{\partial y} \right) + 2\, A_2(x)\, g_0 \frac{\partial g_0}{\partial y} \\ & + 2\, A_2(x) \left(\frac{\partial g_1}{\partial y} \frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y} \frac{\partial g_1}{\partial x} \right) + A_2'(x) \left(g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y} \right). \end{aligned}$$

Then,

$$
\begin{aligned}
\nabla V \cdot \mathbf{F} = &\ (\nabla q(x)\cdot \mathbf{F})\, f(x,y)^2 + 2 f(x,y) q(x)\, (\nabla f(x,y)\cdot \mathbf{F}) \\
= &\ \Bigg[2\,A_0(x)\,g_1\,\frac{\partial g_1}{\partial y} + 2\,A_1(x)\,g_1\,\frac{\partial g_0}{\partial y} + 2\,A_2(x)\,g_0\,\frac{\partial g_0}{\partial y} \\
&+2\,A_2(x)\left(\frac{\partial g_1}{\partial y}\frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y}\frac{\partial g_1}{\partial x}\right) + A_1(x)\,g_0\,\frac{\partial g_1}{\partial y} - A_1(x)\,g_1\,\frac{\partial g_0}{\partial y} \\
&+A_2'(x)\left(g_0\frac{\partial g_1}{\partial y} - g_1\frac{\partial g_0}{\partial y}\right)\Bigg]\; V \\
= &\ \Bigg[2\,A_0(x)\,g_1\,\frac{\partial g_1}{\partial y} + A_1(x)\,\left(g_0\frac{\partial g_1}{\partial y} + g_1\frac{\partial g_0}{\partial y}\right) + 2\,A_2(x)\,g_0\,\frac{\partial g_0}{\partial y} \\
&+2\,A_2(x)\left(\frac{\partial g_1}{\partial y}\frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y}\frac{\partial g_1}{\partial x}\right) + A_2'(x)\left(g_0\frac{\partial g_1}{\partial y} - g_1\frac{\partial g_0}{\partial y}\right)\Bigg]\; V \\
= &\ \mathrm{div}(x,y)\,V.
\end{aligned}
$$

■

We remark that Theorem 2.3 gives, in general, a non Liouvillian first integral for the planar polynomial systems (2.5). In Section 2.3 we analyze some polynomial systems constructed from Theorem 2.3 that have no Liouvillian first integral.

We consider now a linear homogeneous ordinary differential equation of order 1 such as

$$w'(x) + A(x)\,w(x) = 0, \tag{2.6}$$

where $x \in \mathbb{R}$, $w'(x) = dw(x)/dx$ and $A(x) = A_0(x)/A_1(x)$ with $A_i(x) \in \mathbb{R}[x]$ and $A_1(x) \not\equiv 0$. We give analogous results for this case whose proofs are not given to avoid non useful repetitions.

Theorem 2.6 *Given $g(x,y) = g_0(x,y)/g_1(x,y)$ with $g_i(x,y) \in \mathbb{R}[x,y]$, satisfying $g_1(x,y) \not\equiv 0$ and $\partial g/\partial y \not\equiv 0$ and $h(x) = h_0(x)/h_1(x)$ with $h_i(x) \in \mathbb{R}[x]$ and $h_1(x) \not\equiv 0$, each nonzero solution $w(x)$ of equation* (2.6) *is related to a finite number of solutions $y = y(x)$ of the rational equation*

$$\frac{dy}{dx} = \frac{A_1(x)\,h_0(x)\,g_1^2 - A_0(x)\,h_1(x)\,g_0\,g_1 - A_1(x)\,h_1(x)\left(g_1\frac{\partial g_0}{\partial x} - g_0\frac{\partial g_1}{\partial x}\right)}{A_1(x)\,h_1(x)\left(g_1\frac{\partial g_0}{\partial y} - g_0\frac{\partial g_1}{\partial y}\right)}, \tag{2.7}$$

by the functional change

$$w(x) = g(x,y) - \exp\left(-\int_0^x A(s)ds\right)\left[\int_0^x \exp\left(\int_0^s A(r)dr\right) h(s)ds\right].$$

Theorem 2.7 *We consider the vector field* $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ *related to equation* (2.7):

$$\begin{aligned} P(x,y) &= A_1(x)\,h_1(x)\left(g_1\frac{\partial g_0}{\partial y} - g_0\frac{\partial g_1}{\partial y}\right), \\ Q(x,y) &= A_1(x)\,h_0(x)\,g_1^2 - A_0(x)\,h_1(x)\,g_0\,g_1 - A_1(x)\,h_1(x)\left(g_1\frac{\partial g_0}{\partial x} - g_0\frac{\partial g_1}{\partial x}\right). \end{aligned}$$

Let $w(x)$ *be any nonzero solution of equation* (2.6)*, that is, for* $C \in \mathbb{R} - \{0\}$ *we have* $w(x) = C\exp\left(-\int_0^x A(s)ds\right)$*. Then, the function*

$$f(x,y) := g_1\,w(x) - g_0 + g_1\,\exp\left(-\int_0^x A(s)ds\right)\left[\int_0^x \exp\left(\int_0^s A(r)dr\right) h(s)ds\right]$$

gives an invariant curve for the polynomial system (2.7)*, with the polynomial cofactor*

$$\begin{aligned} k(x,y) = \; & -A_0(x)\,h_1(x)\,g_1\,\frac{\partial g_0}{\partial y} + A_1(x)\,h_0(x)\,g_1\,\frac{\partial g_1}{\partial y} \\ & + A_1(x)\,h_1(x)\left(\frac{\partial g_0}{\partial y}\frac{\partial g_1}{\partial x} - \frac{\partial g_1}{\partial y}\frac{\partial g_0}{\partial x}\right). \end{aligned}$$

Lemma 2.8 *The function* $q(x,y) = g_1(x,y)\exp\left(\int_0^x -A(s)ds\right)$ *gives rise to an invariant curve for system* (2.7) *with the same polynomial cofactor as* $f(x,y)$*.*

Theorem 2.9 *We use the same notation as in Theorem* 2.7 *and Lemma* 2.8*. The function* $H(x,y)$ *defined by* $H(x,y) := f(x,y)/q(x,y)$ *is a first integral for system* (2.7) *and the function* $V(x,y) := A_1(x)\,h_1(x)\,g_1(x,y)\,q(x,y)$ *is an inverse integrating factor.*

We remark that $H(x,y)$ is a Liouvillian function and, therefore, a system (2.7) has always a Liouvillian first integral.

In Section 2.3 we give an example of a 3-parameter family of quadratic systems with a center at the origin which can be constructed following Theorem 2.6.

2.3 Examples of families of quadratic systems

2.3.1 Quadratic systems with invariant algebraic curves of arbitrarily high degree linear in one variable

We first consider the examples of families of quadratic systems with algebraic solutions of arbitrarily high degree appearing in [23]. In that work all the invariant algebraic curves linear in the variable y, that is, defined by $f(x, y) = p_1(x)y + p_2(x)$, where p_1 and p_2 are polynomials, for quadratic systems, are determined.

The example appearing in [24] is a further study of an example appearing in [23] and the example given in [34] is also described in [23]. We show that all these quadratic systems, with an invariant algebraic curve of arbitrary degree can be constructed by the method explained in the previous section. Moreover, we give the explicit expression of a first integral for any value of the parameter n, even in the case when n is not a natural number. If n is a natural number, we obtain the invariant algebraic curves of arbitrary degree and a Liouvillian first integral. However, when $n \notin \mathbb{N}$ we obtain polynomial systems with a non Liouvillian first integral.

As it is shown in [23], all these families of systems can be written, after an affine change of variables if necessary, in the form:

$$\begin{aligned} \dot{x} &= \Omega_1(x), \\ \dot{y} &= (2n+1)\, L'(x)\, \Omega_1(x) - \frac{n(n+1)}{2}\, \Omega_1(x)\, \Omega_1''(x) \; - L(x)^2 + y^2, \end{aligned} \tag{2.8}$$

where $\Omega_1(x)$ is any quadratic polynomial, $L(x)$ is any linear polynomial and $' = d/dx$. We have that system (2.8) has an invariant curve $f(x, y) = 0$, where $f(x, y) := p_1(x)y + \Omega_1(x)p_1'(x) - L(x)p_1(x)$, with a cofactor $y + L(x)$, where $p_1(x)$ is a solution of the second order linear differential equation

$$\Omega_1(x)w''(x)+(\Omega_1'(x)-2L(x))w'(x)+\frac{n}{2}(4L'(x)-(n+1)\Omega_1''(x))w(x) = 0. \tag{2.9}$$

In [23] it is shown that, in case $n \in \mathbb{N}$, an irreducible polynomial of degree n belonging to a family of orthogonal polynomials is a solution of equation (2.9). For instance, when $\Omega_1(x) = 1$, we get the Hermite polynomials, when $\Omega_1(x) = x$, we get the Generalized Laguerre polynomials and when $\Omega_1(x) = 1 - x^2$, we get the Jacobi polynomials.

We consider again the general case in which $n \in \mathbb{R}$ and we define $A_2(x) := \Omega_1(x)$, $A_1(x) := \Omega_1'(x) - 2L(x)$ and $A_0(x) := \frac{n}{2}(4L'(x) - (n+1)\Omega_1''(x))$. We have the linear differential equation (2.9) in the same notation as in Theorem 2.1 and we consider $g(x,y) := \dfrac{L(x) - y}{A_2(x)}$.

The system obtained by the method explained in Section 2.2 exactly coincides with system (2.8). We consider a set of fundamental solutions of equation (2.9) $\{w_1(x), w_2(x)\}$ and applying Theorem 2.3, we have a first integral for system (2.8) for any value of the parameter $n \in \mathbb{R}$.

In case $n \in \mathbb{N}$ we have that $w_1(x)$ degenerates to a polynomial and $w_1'(x) - g(x,y)w_1(x) = 0$ coincides with the algebraic curve given in the work [23].

We explicitly give the first integral for each of the families described in [23] and for $n \in \mathbb{R}$. We have that $A_2(x)$ is a non-null quadratic polynomial in this case, and depending on its number of roots, we can transform it by a real affine change of variable to one of the following forms: $A_2(x) = 1, x, x^2, 1 - x^2, 1 + x^2$.

If $A_2(x) = 1$, we can choose $L(x) = x$ by an affine change of coordinates. We denote by $\Gamma(x)$ the Euler's–Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ and by ${}_1F_1(a;b;x)$ the confluent hypergeometric function defined by the series

$$ {}_1F_1(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^k}{k!}, $$

with $(a)_k = a(a+1)(a+2)\dots(a+k-1)$, the Pochhammer symbol. See [1] for further information about these functions.

A set of fundamental solutions $\{w_1(x), w_2(x)\}$ for (2.9) with $n \in \mathbb{R}$ is

$$ \begin{aligned} w_1(x) = &\ 2^n\sqrt{\pi}\left(\frac{1}{\Gamma\left(\frac{1-n}{2}\right)}\ {}_1F_1\left(-\frac{n}{2};\frac{1}{2};x^2\right) - \frac{2x}{\Gamma\left(-\frac{n}{2}\right)}\ {}_1F_1\left(\frac{1-n}{2};\frac{3}{2};x^2\right)\right), \\ w_2(x) = &\ 2^n\sqrt{\pi}\left(\frac{1}{\Gamma\left(\frac{1-n}{2}\right)}\ {}_1F_1\left(-\frac{n}{2};\frac{1}{2};x^2\right) + \frac{2x}{\Gamma\left(-\frac{n}{2}\right)}\ {}_1F_1\left(\frac{1-n}{2};\frac{3}{2};x^2\right)\right). \end{aligned} $$

So, a first integral for this system is the expression given in Theorem 2.3:

$H(x,y) := f_1(x,y)/f_2(x,y)$, where

$$\begin{aligned} f_{1,2}(x,y) = \; & \pm\Gamma\left(\frac{1-n}{2}\right)\left[6\,(xy-x^2+1)\;{}_1F_1\left(\frac{1-n}{2};\frac{3}{2};x^2\right)\right.\\ & \left. -\,4\,(n-1)\,x^2\;{}_1F_1\left(\frac{3-n}{2};\frac{5}{2};x^2\right)\right]+\\ & 3\,\Gamma\left(-\frac{n}{2}\right)\left[2nx\;{}_1F_1\left(1-\frac{n}{2};\frac{3}{2};x^2\right)+(x-y)\;{}_1F_1\left(-\frac{n}{2};\frac{1}{2};x^2\right)\right]. \end{aligned}$$

When $n \in \mathbb{N}$, we have that (2.9) corresponds to the equation for Hermite polynomials and $w_1(x)$ coincides with the Hermite polynomial of degree n. The invariant algebraic curve given in [23] corresponds to $f_1(x,y) = 0$.

If $A_2(x) = x$, we choose $L(x) = \frac{1}{2}(x-\alpha)$, where α is an arbitrary real constant, and a set of fundamental solutions for (2.9) is:

$$w_1(x) = \frac{(\alpha+1)_n}{\Gamma(n+1)}\;{}_1F_1\left(-n;\alpha+1;x\right), \quad w_2(x) = x^{-\alpha}\;{}_1F_1\left(-\alpha-n;1-\alpha;x\right).$$

The first integral for this system is $H(x,y) = x^{\alpha}h_1(x,y)/h_2(x,y)$ with:

$$\begin{aligned} h_1(x,y) = \; & (2y-x+\alpha)\,(\alpha+1)\;{}_1F_1\left(-n;\alpha+1;x\right)\\ & -2\,n\,x\;{}_1F_1\left(1-n;\alpha+2;x\right),\\ h_2(x,y) = \; & (2y-x+\alpha)\,(\alpha-1)\;{}_1F_1\left(-\alpha-n;1-\alpha;x\right)\\ & -2\,(\alpha+n)\,x\;{}_1F_1\left(1-\alpha-n;2-\alpha;x\right). \end{aligned}$$

The first integral as given in Theorem 2.3 is $f_1(x,y)/f_2(x,y)$ and we notice that $H(x,y) = cf_1(x,y)/f_2(x,y)$ where $c \in \mathbb{R}-\{0\}$. We do not write c in terms of the parameters of the system to simplify notation.

When $n \in \mathbb{N}$, we have that (2.9) in this case corresponds to the equation of Generalized Laguerre polynomials and $w_1(x)$ coincides with the Generalized Laguerre polynomial $L_n^{(\alpha)}(x)$. The invariant algebraic curve given in [23] corresponds to $f_1(x,y) = 0$, where $f_1(x,y) := w_1'(x) - g(x,y)w_1(x)$.

If $A_2(x) = x^2$, the birrational transformation already given in [23], $x = 1/X$ and $y = (1/X)(1/2-Y)$, makes this case equivalent to the previous one.

If $A_2(x) = 1 - x^2$, we choose $L(x) = \frac{1}{2}((\alpha+\beta)x + (\alpha-\beta))$, where α, β are two arbitrary real constants, and a set of fundamental solutions for (2.9) is:

$$\begin{aligned} w_1(x) &= \frac{(\alpha+1)_n}{\Gamma(n+1)}\, {}_2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-x}{2}\right), \\ w_2(x) &= (1-x)^{-\alpha}\, {}_2F_1\left(-\alpha-n, 1+\beta+n; 1-\alpha; \frac{1-x}{2}\right), \end{aligned}$$

where ${}_2F_1\left(a_1, a_2; b; x\right)$ is the hypergeometric function defined by

$${}_2F_1\left(a_1, a_2; b; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b)_k} \frac{x^k}{k!}.$$

The first integral given in Theorem 2.3 is $H(x,y) = (1-x)^{\alpha} h_1(x,y)/h_2(x,y)$, where:

$$\begin{aligned} h_1 = \; & n\,(1+\alpha+\beta+n)\,(x^2-1)\; {}_2F_1\left(1-n, 2+\alpha+\beta+n; 2+\alpha; u\right) + \\ & (\alpha+1)\,((\alpha+\beta)x + (\alpha-\beta) - 2y)\; {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; u\right), \\ h_2 = \; & (\alpha-1)\,((\alpha-\beta)x + (\alpha+b) + 2y)\; {}_2F_1\left(-\alpha-n, 1+\beta+n; 1-\alpha; u\right) + \\ & (\alpha+n)\,(1+\beta+n)\,(x^2-1)\; {}_2F_1\left(1-\alpha-n, 2+\beta+n; 2-\alpha; u\right), \end{aligned}$$

writing $u = (1-x)/2$. The first integral is $f_1(x,y)/f_2(x,y)$, as given in Theorem 2.3, and we notice that $H(x,y) = cf_1(x,y)/f_2(x,y)$ where $c \in \mathbb{R} - \{0\}$. As before, we do not write c in terms of the parameters of the system to simplify notation.

When $n \in \mathbb{N}$, we have that (2.9) corresponds to the equation of Jacobi polynomials and $w_1(x)$ coincides with the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ and the invariant algebraic curve given in [23] corresponds to $f_1(x,y) = 0$, where $f_1(x,y) := w_1'(x) - g(x,y)w_1(x)$.

If $A_2(x) = 1 + x^2$ the complex affine change of variable $x = \mathbf{i}X$, $\mathbf{i} = \sqrt{-1}$, makes this case equivalent to the previous one, as it is shown in [23].

We have re-encountered by this method all the examples appearing in [23] from a unified point of view. In addition, in this section we have given an explicit expression of a first integral for each case and for any value of the parameter $n \in \mathbb{R}$. To this end, we have found invariants for the system and we have applied the generalization of Darboux's method as explained in [50] to be able to construct a first integral which is, in general, of non Liouvillian type.

2.3.2 A Lotka-Volterra system

As it has been explained in Section 2.1, the first counterexample to Lins Neto conjecture was given by J. Moulin-Ollagnier in [67]. His example is a quadratic system with two invariant straight lines and an irreducible invariant algebraic curve $f(x,y)=0$ of degree 2ℓ when $\ell \in \mathbb{N}$. This gives a family of systems depending on the parameter ℓ which have a Darboux inverse integrating factor when $\ell \in \mathbb{N}$ but no rational first integral. The method used in [67] only shows the existence of such invariant algebraic curve but no closed formula to compute it is given. We give an explicit expression for an invariant by means of Bessel functions for any value of $\ell \in \mathbb{R} - \{\frac{1}{2}\}$ which, in the particular case $\ell \in \mathbb{N}$ degenerates to the algebraic curve encountered in [67].

We show that after a birrational transformation, this example coincides with a system constructed with the method explained in Section 2.2.

Let us consider the system appearing in [67] but assuming that $\ell \in \mathbb{R} - \{\frac{1}{2}\}$

$$\dot{x} = x\left(1 - \frac{x}{2} + y\right), \quad \dot{y} = y\left(-\frac{2\ell+1}{2\ell-1} + \frac{x}{2} - y\right). \tag{2.10}$$

We make the birrational transformation

$$x = \frac{4uv}{1-2\ell}, \quad y = \frac{1-2\ell}{4v},$$

whose inverse is

$$u = xy, \quad v = \frac{1-2\ell}{4y}.$$

By this transformation, system (2.10) becomes

$$\dot{u} = \frac{2u}{1-2\ell}, \quad \dot{v} = \frac{1-2\ell}{4} + \frac{2\ell+1}{2\ell-1}\,v + \frac{2u}{2\ell-1}\,v^2. \tag{2.11}$$

We notice that the equation for the orbits satisfied by the variable v as a function of u is a Ricatti equation.

Let us consider $g(u,v) := v$ and the linear differential equation of order 2 given by

$$u\,w''(u) + \frac{1}{2}(1+2\ell)\,w'(u) - \frac{1}{8}(1-2\ell)^2\,w(u) = 0. \tag{2.12}$$

Applying the method given in the previous section, this linear differential equation gives system (2.11) modulus a change of time.

A set of two fundamental solutions for equation (2.12) is given by

$$\begin{aligned} w_1(u) &= u^{(1-2\ell)/4} I_{\frac{1}{2}-\ell}\left((1-2\ell)\sqrt{\frac{u}{2}}\right), \\ w_2(u) &= u^{(1-2\ell)/4} I_{\ell-\frac{1}{2}}\left((1-2\ell)\sqrt{\frac{u}{2}}\right), \end{aligned} \tag{2.13}$$

provided that ℓ is not of the form $\frac{1}{2}(1-2r)$, with r an integer number, because in this case w_1 and w_2 are linearly dependent. The function $I_\nu(u)$ is the Modified Bessel function defined by the solution of the second order differential equation

$$u^2\,w''(u) + u\,w'(u) - (u^2+\nu^2)\,w(u) = 0, \tag{2.14}$$

and being bounded when $u \to 0$ in any bounded range of $\arg(u)$ with $\mathbb{R}e(u) \geq 0$. See [1] for further information about this function.

Hence, by Theorem 2.3 we have that $H(u,v) = f_1(u,v)/f_2(u,v)$, where $f_i(u,v) := w_i'(u) - v w_i(u)$ for $i = 1,2$, is a first integral for system (2.11). For a sake of simplicity we consider the following renaming of the independent variable $u = 2z^2/(1-2\ell)^2$. This is not a birrational transformation and that's why we only use it to simplify notation. The function H writes as:

$$H = \frac{(1-2\ell)^2 I_{\left(\frac{1+2\ell}{2}\right)}(z) - 4vzI_{\left(\frac{2\ell-1}{2}\right)}(z)}{(1-2\ell)^2 I_{-\left(\frac{1+2\ell}{2}\right)}(z) - 4vzI_{-\left(\frac{2\ell-1}{2}\right)}(z)}. \tag{2.15}$$

By Theorem 2.2 we have that $f_i(u,v) = 0$, $i = 1,2$ are invariant curves with the same polynomial cofactor k for system (2.11), therefore the curve $f(u,v) = 0$ given by $f(u,v) = \pi z^{2\ell+1}(f_1^2(u,v) - f_2^2(u,v))$ is also an invariant curve. We multiply by π only for esthetic reasons.

Now we assume that $\ell \in \mathbb{N}$ and we show that $f = 0$ defines an invariant *algebraic* curve. To this end we use the following formulas appearing in

[1, 93]. When $\nu-\frac{1}{2}=n\in\mathbb{Z}$, we define $c(n)=-n\pi\sqrt{-1}/2$ and the following relation is satisfied:

$$
\begin{aligned}
I_\nu(z) = & -\frac{1}{\sqrt{z}}\,\mathrm{e}^{c(n)}\sqrt{\frac{2}{\pi}}\left\{\sinh\left(c(n)-z\right)\sum_{k=0}^{\lfloor\frac{2|\nu|-1}{4}\rfloor}\frac{(|\nu|+2k-\frac{1}{2})!}{(2k)!(|\nu|-2k-\frac{1}{2})!(2z)^{2k}}\right.\\
& \left.+\ \cosh\left(c(n)-z\right)\sum_{k=0}^{\lfloor\frac{2|\nu|-3}{4}\rfloor}\frac{(|\nu|+2k+\frac{1}{2})!}{(2k+1)!(|\nu|-2k-\frac{3}{2})!(2z)^{2k+1}}\right\},
\end{aligned}
\tag{2.16}
$$

where $\lfloor x\rfloor$ stands for the greatest integer k such that $k\leq x$ and $|\nu|$ stands for the absolute value.

From the former equation we obtain the following two equalities, with $\nu-\frac{1}{2}=n\in\mathbb{Z}$ and $\ell\in\mathbb{N}$,

$$
I_\nu^2(z)-I_{-\nu}^2(z) = \frac{2}{\pi z}\sum_{k=0}^{n}(-1)^{k+1}\frac{(2n-k)!(2n-2k)!}{k!((n-k)!)^2}\left(\frac{1}{2z}\right)^{2(n-k)}, \tag{2.17}
$$

$$
\begin{aligned}
& I_{\ell+\frac{1}{2}}(z)I_{\ell-\frac{1}{2}}(z)\ -\ I_{-(\ell+\frac{1}{2})}(z)\ I_{-(\ell-\frac{1}{2})}(z) = (-1)^\ell\frac{2}{\pi z}\\
& \left[\left(\sum_{i=0}^{\lfloor\frac{\ell}{2}\rfloor}\frac{(\ell+2i)!}{(2i)!(\ell-2i)!}\left(\frac{1}{2z}\right)^{2i}\right)\left(\sum_{j=0}^{\lfloor\frac{\ell-2}{2}\rfloor}\frac{(\ell+2j)!}{(2j+1)!(\ell-2j-2)!}\left(\frac{1}{2z}\right)^{2j-1}\right)-\right.\\
& \left.\left(\sum_{i=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{(\ell+2i+1)!}{(2i+1)!(\ell-2i-1)!}\left(\frac{1}{2z}\right)^{2i-1}\right)\left(\sum_{j=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{(\ell+2j-1)!}{(2j)!(\ell-2j-1)!}\left(\frac{1}{2z}\right)^{2j}\right)\right].
\end{aligned}
\tag{2.18}
$$

Then, we have that $f_1(z,v)=(1-2\ell)^2\,I_{\ell+\frac{1}{2}}(z)-4vz\,I_{\ell-\frac{1}{2}}(z)$ and $f_2(z,v)=(1-2\ell)^2\,I_{-(\ell+\frac{1}{2})}(z)-4vz\,I_{-(\ell-\frac{1}{2})}(z)$, and we write f arranged in powers of v:

$$
\begin{aligned}
f(z,v) = & \ \pi z^{2\ell+1}\Big((1-2\ell)^4(I_{\ell+\frac{1}{2}}^2(z)-I_{-(\ell+\frac{1}{2})}^2(z))\\
& -\ 8vz(1-2\ell)^2(I_{\ell+\frac{1}{2}}(z)I_{\ell-\frac{1}{2}}(z)-I_{-(\ell+\frac{1}{2})}(z)I_{-(\ell-\frac{1}{2})}(z))\\
& +16v^2z^2(I_{\ell-\frac{1}{2}}^2(z)-I_{-(\ell-\frac{1}{2})}^2(z))\Big).
\end{aligned}
$$

Let us consider each coefficient of v in $f(z,v)$ separately and we will show that it is an even polynomial in the variable z. The coefficient in $f(z,v)$ of

v^0 is:

$$\pi z^{2\ell+1}(1-2\ell)^4(I^2_{\ell+\frac{1}{2}}(z) - I^2_{-(\ell+\frac{1}{2})}(z)),$$

which by equation (2.17) is an even polynomial in the variable z of degree 2ℓ. The coefficient in $f(z,v)$ of v^2 is:

$$16\pi z^{2\ell+3}(I^2_{\ell-\frac{1}{2}}(z) - I^2_{-(\ell-\frac{1}{2})}(z)),$$

which also by equation (2.17) is an even polynomial in the variable z of degree $2\ell+2$. Finally, the coefficient in $f(z,v)$ of v^1 is:

$$8\pi(1-2\ell)^2 z^{2\ell+2}(I_{\ell+\frac{1}{2}}(z)I_{\ell-\frac{1}{2}}(z) - I_{-(\ell+\frac{1}{2})}(z)I_{-(\ell-\frac{1}{2})}(z)), \qquad (2.19)$$

which by equation (2.18) is an even polynomial in the variable z of degree 2ℓ.

Hence, we have that $f(z,v)$ is an even polynomial in the variable z of total degree $2\ell+4$. When rewriting $z=(1-2\ell)\sqrt{u}/\sqrt{2}$ we have that $f(u,v)$ is a polynomial of total degree $\ell+2$ which is irreducible. The fact of being irreducible is easily seen because it is a polynomial of degree two in v and it cannot be decomposed in linear factors (the discriminant is not a polynomial raised to an even power) and the coefficients of v^0 and v^2 do not have any root in common.

Undoing the birrational change of variables we get that $f(x,y)$ is an irreducible polynomial of degree 2ℓ given by:

$$\begin{aligned} f(x,y) &= x^{\ell+\frac{1}{2}}y^{\ell-\frac{1}{2}}\Big[2y\left(I^2_{\ell+\frac{1}{2}}(z) - I^2_{-(\ell+\frac{1}{2})}(z)\right) - \\ &\quad 2\sqrt{2}\sqrt{xy}\left(I_{\ell+\frac{1}{2}}(z)I_{\ell-\frac{1}{2}}(z) - I_{-(\ell+\frac{1}{2})}(z)I_{-(\ell-\frac{1}{2})}(z)\right) + \\ &\quad x\left(I^2_{\ell-\frac{1}{2}}(z) - I^2_{-(\ell-\frac{1}{2})}(z)\right)\Big], \end{aligned}$$

where z is the same variable as before, that is, $z=\frac{(1-2\ell)\sqrt{xy}}{\sqrt{2}}$.

By equation (2.15) we can write the first integral for system (2.10) for any value of $\ell \in \mathbb{R} - \{\frac{1}{2}(1-2r) \mid r \in \mathbb{N}\}$:

$$H(x,y) = \frac{\sqrt{2y}\, I_{\left(\frac{1+2\ell}{2}\right)}(z) - \sqrt{x}\, I_{\left(\frac{2\ell-1}{2}\right)}(z)}{\sqrt{2y}\, I_{-\left(\frac{1+2\ell}{2}\right)}(z) - \sqrt{x}\, I_{-\left(\frac{2\ell-1}{2}\right)}(z)}.$$

We have studied system (2.10) for any value of the parameter $\ell \in \mathbb{R} - \{\frac{1}{2}(1-2r) \mid r \in \mathbb{N}\}$ giving an explicit expression for a first integral using

Theorem 2.1 and the Generalized Darboux's theory as explained in [50]. This first integral is not of Liouvillian type. Moreover, we give one of its invariants with a polynomial cofactor. In the particular case $\ell \in \mathbb{N}$, this invariant is the invariant algebraic curve whose existence was proved in [67].

2.3.3 A new example of a family of quadratic systems with an invariant algebraic curve of arbitrarily high degree

We give another example of a family of quadratic systems with an irreducible invariant algebraic curve of degree 2ℓ when $\ell \in \mathbb{N}$, where ℓ is a parameter of the family. This family also depends on the parameter $a \in \mathbb{R}$.

Let us consider the quadratic system

$$\begin{aligned} \dot{x} &= (2a-1)\ell x - a(2\ell-1)y + 2a(a-\ell)(2\ell-1)x^2 - 2a^2(2\ell-1)^2xy, \\ \dot{y} &= y(2(2a-1)\ell + 2a(2a-2\ell-1)(2\ell-1)x - 4a^2(2\ell-1)^2y), \end{aligned} \tag{2.20}$$

where $a, \ell \in \mathbb{R}$ which satisfy $a \neq 0$, $\ell \neq \frac{1}{2}$ and $(2\ell-1)a^2 - 2\ell \neq 0$. An straightforward computation shows that the system (2.20) has $y = 0$ and $y - x^2 = 0$ as invariant algebraic curves.

Let us consider the following birrational transformation $x = Y$, $y = XY^2$ whose inverse is $X = y/x^2$ and $Y = x$. In these new variables system (2.20) becomes

$$\begin{aligned} \dot{X} &= 2a(2\ell-1)(X-1)XY, \\ \dot{Y} &= ((2a-1)\ell + a(2\ell-1)(2a-2\ell-X)Y - 2a^2(2\ell-1)^2XY^2)\,Y. \end{aligned} \tag{2.21}$$

By a change of the time variable we can divide this system by Y and the resulting system coincides with the one described in Theorem 2.1 taking $A_2(X) := 2X(X-1)^2$, $A_1(X) := (2\ell - 2a + 3X)(X-1)$, $A_0(X) := \ell(1-2a)$ and $g(X,Y) := a(2\ell-1)Y/(X-1)$. The equation $A_2(X)w''(X) + A_1(X)w'(X) + A_0(X)w(X) = 0$ has the following set of fundamental solutions in this case:

$$\begin{aligned} w_1(X) &= (X-1)^{-\ell}\, {}_2F_1\left(\frac{1}{2}-\ell, -\ell; a-\ell; X\right), \\ w_2(X) &= (X-1)^{-\ell} X^{1-a+\ell}\, {}_2F_1\left(1-a, \frac{3}{2}-\ell; 2-a+\ell; X\right). \end{aligned}$$

By Theorem 2.2, $f_i(X,Y) = w_i'(X) - g(X,Y)w_i(X)$, $i = 1,2$, define invariants with a polynomial cofactor for system (2.21). Moreover, by Theorem 2.3 we have a non Liouvillian first integral given by $H(X,Y) = f_1(X,Y)/f_2(X,Y)$.

In the particular case $\ell \in \mathbb{N}$, we notice that $f_1(X,Y) = 0$ is a rational function. It is an easy computation to show that this rational function is a polynomial when rewritten in coordinates x and y. This polynomial gives place to an invariant algebraic curve of degree 2ℓ for system (2.20). That is, by undoing the birrational transformation, we deduce that $f_1(x,y) = 0$ is an irreducible invariant algebraic curve for system (2.20), given by:

$$\begin{aligned} f_1(x,y) &= 2(a-\ell)(\ell+(2\ell-1)ax)x^{2\ell-1}\ {}_2F_1\left(\frac{1}{2}-\ell, -\ell; a-\ell; \frac{y}{x^2}\right) \\ &\quad +\ell(2\ell-1)x^{2\ell-3}(x^2-y)\ {}_2F_1\left(\frac{3}{2}-\ell, 1-\ell; 1+a-\ell; \frac{y}{x^2}\right). \end{aligned}$$

It is easy to see that the polynomial $f_1(x,y)$ has degree 2ℓ and the cofactor associated to the invariant algebraic curve $f_1(x,y) = 0$ is $\ell(2\ell-1)((2a-1) + 4a(a-\ell)x - 4(2\ell-1)a^2y)$.

The first integral for (2.20) is given by $H(x,y) = y^{a-\ell} f_1(x,y)/h(x,y)$, where

$$\begin{aligned} h(x,y) = {} & 2\,(a-\ell-2)\ \big[(a-\ell-1)\,x^2 + \\ & (1-a-a(2\ell-1)x)\,y\big]\ x^{7-2a}\ {}_2F_1\left(1-a, \frac{3}{2}-a; 2-a-\ell; \frac{y}{x^2}\right) + \\ & (a-1)\,(2a-3)\,x^{5-2a}\,(x^2-y)\,y\ {}_2F_1\left(2-a, \frac{5}{2}-a; 3-a+\ell; \frac{y}{x^2}\right). \end{aligned}$$

We notice that when both a and ℓ belong to the set of natural numbers, we have that $h(x,y) = 0$ is an invariant algebraic curve different from the curve $f_1(x,y) = 0$. Then we have a quadratic system with a rational first integral $H(x,y)$ with arbitrary degree.

2.3.4 A complete family of quadratic systems with a center at the origin

In this subsection we give an example of a 3-parameter family of quadratic systems with a center at the origin which can be constructed using Theorem

2.6. The family encountered corresponds to the reversible case, see [83]. We refer to Subsection 1.2.1 for the definition of center and the computation of the center cases and to [97] for the definition of reversibility.

The family of quadratic systems depends on twelve parameters, but up to affine transformations and positive time rescaling, we get a family of five essential parameters. We have taken a system (2.7) and we have chosen $g(x,y) := y^2$, $h(x) := 2x(d\,x-1)/(1+ax)$ and $A(x) := 2b/(1+ax)$, where a, b, d are real parameters. Using Theorem 2.6, we have encountered the 3-parameter family of quadratic systems next described. We remark that in spite of the simplicity of the chosen polynomials $g(x,y)$, $A(x)$ and $h(x)$, we amazingly obtain the complete family of quadratic systems with a reversible center at the origin. We notice that other choices of the functions $g(x,y)$, $A(x)$ and $h(x)$ would give place to other families of polynomial systems.

The computation of the center cases for the family of quadratic systems was done by Dulac [40] for the case of complex systems and a proof for real systems is given in [66]. We also refer Bautin [10] who showed the existence of only three independent constants. The computation of the zero set of these three independent values gives place to four complete families of quadratic systems with a center at the origin which are described in [84]. These four complete families are the Hamiltonian case, the reversible case, the Lotka-Volterra case and the reversible case.

Let us now consider an equation (2.7) such as $(1+ax)\,w'(x)+2b\,w(x)=0$ and $g(x,y)$ and $h(x)$ as formerly defined. The rational equation as constructed in Theorem 2.6 is:

$$\frac{dy}{dx} = \frac{-x+dx^2-by^2}{y+axy},$$

which gives the corresponding quadratic planar system

$$\dot{x} = y + axy, \quad \dot{y} = -x + dx^2 - by^2. \tag{2.22}$$

We suppose that $ab(a+b)(a+2b)(a+b+d) \neq 0$. In case this value is zero, the origin of system (2.22) is still a center but with a Darboux integrating factor instead of a Darboux first integral. This particular case can also be studied by our method, but we do not write it to avoid giving examples without essential differences.

By Theorem 2.7 we have that $f(x,y)=0$ is an invariant curve of system (2.22) with cofactor $-2by$, where $f(x,y)$ is given by

$$\begin{aligned} f(x,y) &= b\,(a+b)\,(a+2b)\,w(x) - (a+b+d)\,(1+ax)^{-\frac{2b}{a}} \\ &\quad -b\,(a+b)\,(a+2b)\,y^2 \;+\; b\,(a+2b)\,d\,x^2 \\ &\quad -2\,b\,(a+b+d)\,x \;+\; a \;+\; b \;+\; d\,, \end{aligned} \tag{2.23}$$

with $w(x)$ any non-zero solution of $(1+ax)w'(x)+2bw(x)=0$, that is, $w(x)=C(1+ax)^{-\frac{2b}{a}}$.

Choosing $C=(a+b+d)\,(b(a+b)(a+2b))^{-1}$ we get an invariant conic. System (2.22) has two invariant algebraic curves, the former conic with cofactor $-2by$ and an invariant straight line given by $1+ax=0$ with cofactor y. The Darboux first integral

$$H(x,y)=(1+ax)^{\frac{2b}{a}}f(x,y)$$

coincides with the first integral described in Theorem 2.9.

The origin of this system is a center since it is a monodromic singular point with a continuous first integral defined in a neighborhood of it. This example addresses to the thought that other families of polynomial systems of higher degree with a center at the origin can be easily obtained by this method, avoiding the cumbersome computation of Poincaré-Liapunov constants.

2.3.5 An integrable saddle

As described in the first section of this chapter, an open question related to the integrability of planar polynomial differential systems is whether there exists a system like (1.1) with a center, not reversible and without a Liouvillian first integral. An example of an analytic system (not polynomial) with a center, which is not reversible and without a Liouvillian first integral is given in [11].

There is an analogy between saddle points which are integrable (as centers) and saddle points which are not (as foci), so an analogous open problem may be referred to an integrable saddle point. Moreover, center or focus points and saddle points are not distinguished in the domain of two–dimensional complex differential systems. J. Llibre and H. Żołądek are working in this problem and they have found the following system:

$$\dot{z}=-z(1-z^2),\quad \dot{y}=y+y^2+Lyz^2+kz^4, \tag{2.24}$$

where $k, L \in \mathbb{R}$. By means of the rational change of variables $x = z^2$, this quartic system is transformed into the following quadratic system:

$$\dot{x} = -2x(1-x), \quad \dot{y} = y + kx^2 + Lxy + y^2. \tag{2.25}$$

If we make the following change of variables to system (2.25):

$$x = \frac{1-u}{2}, \quad y = v + \frac{L}{4}u - \frac{L+2}{4},$$

we get the already described system (2.8), with variables u and v, with $A_2(u) = 1 - u^2$, and with parameters given by $a = 1/2$,

$$L = -(2b+1) - 4n - \frac{8n}{3}(b+n), \; k = \frac{4}{9}n(b+n)(3+2n)(3+2b+2n).$$

The integrability of this system is described in page 41. We can, therefore, construct a first integral for system (2.24).

We have an explicit expression for a first integral for system (2.24), which is, in general, not a Liouvillian function. It is easy to see that this first integral is continuous in the saddle point $(0,0)$. Hence, the origin of system (2.24) is an integrable saddle point. To solve the stated open problem, it must be shown that system (2.24) has no Liouvillian first integral and that it is not reversible. As we have already stated, this is a work in process of J. Llibre and H. Żołądek, who have kindly shown us this example.

The contents of this chapter, with the exception of Subsection 2.3.5, belong to the paper entitled *Integrability of planar polynomial differential systems through linear differential equations*, authored by H. Giacomini, J. Giné and M. Grau and accepted for publication in Rocky Mountain J. Math.

Abstract. In this work, we consider rational ordinary differential equations $dy/dx = Q(x,y)/P(x,y)$, with $Q(x,y)$ and $P(x,y)$ coprime polynomials with real coefficients. We give a method to construct equations of this type for which a first integral can be expressed from two independent solutions of a second–order homogeneous linear differential equation. This first integral is, in general, given by a non Liouvillian function.

We show that all the known families of quadratic systems with an irreducible invariant algebraic curve of arbitrarily high degree and without a rational first integral can be constructed by using this method. We also present a new example of this kind of families.

We give an analogous method for constructing rational equations but by means of a linear differential equation of first order.

Chapter 3

Necessary conditions for the existence of invariant algebraic curves

3.1 Preliminaries on invariant algebraic curves

We consider a planar polynomial differential system (1.1) and in this chapter we are mainly concerned with its invariant algebraic curves. We recall that we define an *invariant algebraic curve* as an algebraic curve $f(x,y) = 0$, where $f(x,y) \in \mathbb{R}[x,y]$, such that:

$$P(x,y)\,\frac{\partial f}{\partial x}(x,y) \,+\, Q(x,y)\,\frac{\partial f}{\partial y}(x,y) \,=\, k(x,y)\,f(x,y)\,, \tag{3.1}$$

where $k(x,y)$ is a polynomial called the *cofactor* of $f(x,y)$. It is easy to see that if P and Q are polynomials of degree at most d, then the cofactor is of degree at most d $-$ 1.

In Chapter 1, we have defined an invariant algebraic curve as a polynomial with complex coefficients but we have noticed that, since the polynomial system (1.1) is real, then we can assume that any invariant algebraic curve is real. In this chapter we will only consider real invariant algebraic curves $f(x,y) = 0$, where $f(x,y)$ is an irreducible polynomial in $\mathbb{R}[x,y]$, unless otherwise stated.

The importance of invariant algebraic curves to understand the dynamics of a system (1.1) has been remarked by several authors. We refer to [83, 84], and the references therein, for an exhaustive survey on this topic as well as being two of the initial works on this subject. We also consider exponential factors due to their relation with the multiplicity of invariant algebraic curves, see [35], and their role in the Liouvillian integrability, as stated in Theorems 1.6 and 1.7.

The main goal of this chapter consists in giving a set of necessary conditions for polynomials of degree lower or equal than $\mathrm{d}-1$ to be the cofactor of an invariant algebraic curve or an exponential factor for a system (1.1). These conditions consist in the value of the cofactor at a non-degenerate or an elementary degenerate singular point whose ratio of eigenvalues does not equal 1. These results are given in Section 3.2.

A generic system (1.1) of degree d has $\mathrm{d}^2+\mathrm{d}+1$ non-degenerate and different singular points (finite and infinite) whose ratio of eigenvalues is not a rational number and which are not all contained in an algebraic curve of degree $\leq \mathrm{d}-1$. The value of the cofactor at one of these points brings forth a linear equation with $\mathrm{d}(\mathrm{d}+1)/2+1$ unknowns, which correspond to the coefficients of the cofactor and the degree of the curve. Therefore, in general, we have much more independent linear equations than unknowns. Hence, this set of necessary conditions is wide enough to, in general, completely characterize the cofactors and the degree related to invariant algebraic curves of a given system.

The characterization of the invariant algebraic curves of a system (1.1) gives, in most cases, its dynamics since these curves are usually made of graphics. Moreover, the knowledge of invariant algebraic curves is directly connected with the existence of a Liouvillian first integral as stated in Theorems 1.6 and 1.7. Hence, we apply the aforementioned result in relation to the integrability problem.

In Section 3.3, we use these results on Liouvillian integrability of a polynomial system (1.1) to prove the non-existence of a Liouvillian first integral for a family of quadratic systems. In order to show the power of the method, we study a family of quadratic Lotka-Volterra systems and we show that no Liouvillian first integral can exist for this family.

In Chapter 5, we apply the results given in Section 3.2 to some families of quadratic systems with an algebraic limit cycle of degrees 2 and 4. We recall that a *limit cycle* for system (1.1) is an isolated periodic orbit and a limit cycle is said to be *algebraic* when it is contained in one of the ovals of an invariant algebraic curve. We show that the first integral of these systems cannot belong to the functional class of Liouvillian functions. To this end we will first prove that there are no other invariant algebraic curves for each system but the one which defines the algebraic limit cycle. As a consequence, and applying Theorem 1.7, we will conclude with the desired result.

3.2 Necessary conditions for a cofactor

We notice that the existence of an invariant algebraic curve depends on the existence of its corresponding cofactor. Moreover, while the degree of an invariant algebraic curve is not uniformly bounded for all the systems of degree d, the degree of its cofactor must be lower or equal than $\mathrm{d}-1$.

The main result of this chapter is a set of necessary conditions for a polynomial $k(x,y) \in \mathbb{R}[x,y]$, of degree lower or equal than $\mathrm{d}-1$, to be the cofactor of an invariant algebraic curve or an exponential factor for a planar polynomial system (1.1) of degree d.

These conditions on the cofactor correspond to its possible values at the critical points of the considered system. The hypothesis of coprimality between P and Q implies that all the singular points of a system (1.1) are isolated, that is, there always exists a neighborhood for each singular point so that no other singular point belongs to it.

When a singular point (x_0, y_0) is such that $(x_0, y_0) \in \mathbb{R}^2$, we say that it is a *real* singular point. We notice that since both $P(x,y)$ and $Q(x,y)$ have real coefficients, if (x_0, y_0) is a singular point with non-null imaginary part, then its conjugate $(\bar{x}_0, \bar{y}_0)$ is also a singular point. A singular point (x_0, y_0) with non-null imaginary part will be called a *complex* singular point, to be distinguished from a real one.

Real singular points for a system (1.1) are classified in terms of the behavior in $\mathbb{R}^2$ of the solutions of the system in their neighborhood, see for instance [74]. As it has been described in Subsection 1.2.1, the behavior of the solutions in a neighborhood of the singular point p is given by the matrix $A(p)$ defined in (1.8). Assume that p is a non–degenerate singular point. Let $\lambda, \mu \in \mathbb{C}$ be the eigenvalues of $A(p)$. We have that $\lambda\,\mu \neq 0$.

If $\lambda, \mu \in \mathbb{R}$ and $\lambda/\mu < 0$, the point p is called a *saddle*. If $\lambda, \mu \in \mathbb{R}$ and $\lambda/\mu > 0$, the point p is called a *node*.

As we are considering a real system and a real singular point, the characteristic polynomial of the matrix $A(p)$ is quadratic and with real coefficients. If one of the eigenvalues is a complex number with non-null imaginary part, the other eigenvalue is its conjugate. If $\lambda = a + b\mathbf{i}$ and $\mu = a - b\mathbf{i}$, where $\mathbf{i} = \sqrt{-1}$ and $b \neq 0$, then if $a \neq 0$ the point is called a *strong focus* and if $a = 0$ then it can be a *center* or a *weak focus*.

For a degenerate singular point the local study of the solutions in its neighborhood can be made by using the blow-up technique. If $A(p)$ has only one eigenvalue equal to zero then p is an *elementary degenerate* singular point and its local behavior is studied in [4]. If zero is a double eigenvalue of $A(p)$ but $A(p)$ is not identically zero then the degenerate singular point p is called *nilpotent* and a good characterization of its local behavior is given in [3]. Finally, for degenerate singular points p with $A(p)$ identically zero, see [41] for a detailed description of the blow-up technique, which can be rather complicated.

Complex singular points are also classified in degenerate and non–degenerate in the same manner as real singular points. The local behavior of the solutions in a neighborhood of a complex singular point has not the same sense than in the real case. However, the eigenvalues of the matrix $A(p)$ defined in (1.8) will be of great importance for our results, both for real and for complex singular points.

We take advantage of the location of each singular point of system (1.1) and the local behavior of the solutions in its neighborhood. In order to enlarge the set of conditions on the cofactor $k(x, y)$ at each singular point, we also consider infinite singular points. That is, we extend an equation $Q(x, y)\,dx - P(x, y)\,dy = 0$, to the complex projective plane $\mathbb{CP}(2)$, and we consider all the singular points of the extended equation in $\mathbb{CP}(2)$. We recall that considering the equation $\omega = 0$ defined by the 1-form $\omega := Q(x, y)\,dx - P(x, y)\,dy$, is equivalent to consider the differential system (1.1).

The next subsection, numbered 3.2.1, consists in a brief summary of the process of extending the equation in the affine plane to the projective plane, as well as its most interesting features like invariant algebraic curves and exponential factors.

In Subsection 3.2.2 a result due to A. Seidenberg [85], which describes the local behavior of the analytic solutions in a neighborhood of a singular point, is given. By using this result, the main contribution of this chapter is given in Subsection 3.2.3, as well as its proof.

3.2.1 Critical points at infinity

We first define a polynomial differential equation in $\mathbb{CP}(2)$ and the notion of invariant algebraic curve and critical point for the equation. We describe the planar polynomial systems obtained when taking local coordinates and we show the coherence between immersing an equation in the affine plane to the projective plane and the other way round, that is, submersing the differential equation of the projective plane in the affine plane by means of taking local coordinates. We only give an introductory summary of all the facts related to differential equations in $\mathbb{CP}(2)$.

Critical points at infinity may also be studied by submerging our equation in the Poincaré's sphere $\mathbb{S}^2$, see [58]. Both ways to study critical points at infinity are equivalent.

We recall that $\mathbb{CP}(2) = \{\mathbb{C}^3 - \{(0,0,0)\}\}/\sim$ with the equivalence relation $[X,Y,Z] \sim [X',Y',Z']$ if, and only if, there exists $\nu \in \mathbb{C} - \{0\}$ such that $[X',Y',Z'] = \nu[X,Y,Z]$.

We consider $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, three homogeneous polynomials of degree $\mathrm{d}+1$ in the variables (X,Y,Z) and the 1-form:

$$\Omega := \mathcal{P}\, dX + \mathcal{Q}\, dY + \mathcal{R}\, dZ.$$

We always assume that $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ are coprime polynomials, that is, that there is no non–constant polynomial which divides $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$.

We say that the 1-form Ω is *projective* if

$$X\mathcal{P} + Y\mathcal{Q} + Z\mathcal{R} \equiv 0.$$

Proposition 3.1 *The 1-form $\Omega = \mathcal{P}dX + \mathcal{Q}dY + \mathcal{R}dZ$ is projective if, and only if, there exist polynomials L, M, N of degree d such that Ω reads for $\Omega = (MZ - NY)\, dX + (NX - LZ)\, dY + (LY - MX)\, dZ$.*

Equally, the 1-form $\Omega = \mathcal{P}dX + \mathcal{Q}dY + \mathcal{R}dZ$ is projective if, and only if, there exist polynomials L, M, N of degree d such that:

$$\Omega = \det \begin{bmatrix} L & M & N \\ X & Y & Z \\ dX & dY & dZ \end{bmatrix}. \tag{3.2}$$

This proposition is proved in [25]. We notice that the polynomials L, M and N are not uniquely determined by $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. These polynomials can be replaced by $L' = L + \Delta X$, $M' = M + \Delta Y$ and $N' = N + \Delta Z$, where Δ is any homogeneous polynomial with variables X, Y, Z and degree $\mathrm{d} - 1$.

The projective 1-form Ω defines a *differential equation in* $\mathbb{CP}(2)$ given by $\Omega = 0$, which may be written, by Proposition 3.1,

$$(MZ - NY)\, dX + (NX - LZ)\, dY + (LY - MX)\, dZ = 0. \tag{3.3}$$

In this context, an *invariant algebraic curve* for equation (3.3) is an algebraic curve $F(X, Y, Z) = 0$, where F is a homogeneous polynomial, such that:

$$L\, \frac{\partial F}{\partial X} + M\, \frac{\partial F}{\partial Y} + N\, \frac{\partial F}{\partial Z} = K\, F\,,$$

for a certain homogeneous polynomial $K(X, Y, Z)$ of degree $\mathrm{d} - 1$, called the cofactor.

We notice that the cofactor is not uniquely determined since it depends on L, M, N. By changing (L, M, N) to $(L + \Delta X, M + \Delta Y, N + \Delta Z)$ with Δ a homogeneous polynomial of degree $\mathrm{d} - 1$, which define the same differential equation (3.3), an easy application of Euler's Theorem on homogeneous functions shows that K changes to $K + n\Delta$ where n is the degree of F. Let us prove this statement. We apply Euler's theorem on homogeneous functions to the polynomial $F(X, Y, Z)$, i.e.,

$$X\, \frac{\partial F}{\partial X} \;+\; Y\, \frac{\partial F}{\partial Y} \;+\; Z\, \frac{\partial F}{\partial Z} \;=\; nF\,.$$

So, let $K(X, Y, Z)$ be its cofactor for L, M and N, i.e.,

$$L\, \frac{\partial F}{\partial X} \;+\; M\, \frac{\partial F}{\partial Y} \;+\; N\, \frac{\partial F}{\partial Z} \;=\; K\, F\,,$$

and we change L, M and N by $L + \Delta X$, $M + \Delta Y$ and $N + \Delta Z$, where Δ is a homogeneous polynomial of degree $\mathrm{d} - 1$. Hence,

$$\begin{array}{l} (L + \Delta X)\, \dfrac{\partial F}{\partial X} \;+\; (M + \Delta Y)\, \dfrac{\partial F}{\partial Y} \;+\; (N + \Delta Z)\, \dfrac{\partial F}{\partial Z} \;= \\ \left(L\, \dfrac{\partial F}{\partial X} \;+\; M\, \dfrac{\partial F}{\partial Y} \;+\; N\, \dfrac{\partial F}{\partial Z} \right) + \Delta \left(X\, \dfrac{\partial F}{\partial X} \;+\; Y\, \dfrac{\partial F}{\partial Y} \;+\; Z\, \dfrac{\partial F}{\partial Z} \right) \;= \\ (K + n\Delta)\, F. \end{array}$$

A point $[X_0, Y_0, Z_0]$ belonging to $\mathbb{CP}(2)$ is a *singular point* of the projective equation $\mathcal{P}\, dX + \mathcal{Q}\, dY + \mathcal{R}\, dZ = 0$ if $\mathcal{P}(X_0, Y_0, Z_0) = \mathcal{Q}(X_0, Y_0, Z_0) = \mathcal{R}(X_0, Y_0, Z_0) = 0$.

An equation (3.3) is a planar polynomial system (1.1) when taking local coordinates in a chart. Let us consider a point $p := [X_0, Y_0, Z_0] \in \mathbb{CP}(2)$ and, without loss of generality, we assume that $Z_0 \neq 0$. We define the local coordinates in p by $x = X/Z$ and $y = Y/Z$. So, in local coordinates, we have $p = (x_0, y_0)$ with $x_0 = X_0/Z_0$ and $y_0 = Y_0/Z_0$. We consider an equation (3.3) and we define $P(x, y) := L(x, y, 1) - xN(x, y, 1)$ and $Q(x, y) := M(x, y, 1) - yN(x, y, 1)$. We notice that replacing (L, M, N) by $(L + \Delta X, M + \Delta Y, N + \Delta Z)$, where Δ is any homogeneous polynomial of degree $\mathrm{d} - 1$, gives the same polynomials $P(x, y)$ and $Q(x, y)$. We say that the equation $Q(x, y)\, dx - P(x, y)\, dy = 0$ is the differential equation (3.3) at the local chart at p. An invariant algebraic curve $F(X, Y, Z) = 0$ for equation (3.3) becomes $f(x, y) = 0$ with $f(x, y) := F(x, y, 1)$. It is easy to show that if $F(X, Y, Z) = 0$ is an invariant algebraic curve for (3.3) with cofactor $K(X, Y, Z)$, then $f(x, y) = 0$ is an invariant algebraic curve for $Q(x, y)dx - P(x, y)dy = 0$ with cofactor $k(x, y) = K(x, y, 1) - nN(x, y, 1)$. Moreover, if p is a singular point for (3.3), then (x_0, y_0) is a singular point for $Q(x, y)\, dx - P(x, y)\, dy = 0$.

We describe now the process of extending an equation in the plane given by $Q(x, y)\, dx - P(x, y)\, dy = 0$, to the projective space $\mathbb{CP}(2)$. We consider the change to projective coordinates $x = X/Z$ and $y = Y/Z$, from which $dx = (ZdX - XdZ)/Z^2$ and $dy = (ZdY - YdZ)/Z^2$ are deduced. The coordinates (x, y) are usually called finite coordinates and the set of points $[X, Y, Z] \in \mathbb{CP}(2)$ with $Z = 0$ is called the line at infinity.

Writing $P(x, y) = P_0 + P_1(x, y) + P_2(x, y) + \ldots + P_{\mathrm{d}}(x, y)$ where $P_i(x, y)$ is a homogeneous polynomial of degree i, and expressing (x, y) in terms of (X, Y, Z) we have:

$$\begin{aligned} P\left(\frac{X}{Z}, \frac{Y}{Z}\right) &= P_0 + \frac{1}{Z} P_1(X, Y) + \frac{1}{Z^2} P_2(X, Y) + \ldots + \frac{1}{Z^{\mathrm{d}}} P_{\mathrm{d}}(X, Y) \\ &= \frac{1}{Z^{\mathrm{d}}} (Z^{\mathrm{d}} P_0 + Z^{\mathrm{d}-1} P_1(X, Y) + \ldots + P_{\mathrm{d}}(X, Y)). \end{aligned}$$

We define $L(X, Y, Z) = Z^{\mathrm{d}} P_0 + Z^{\mathrm{d}-1} P_1(X, Y) + \ldots + P_{\mathrm{d}}(X, Y)$, which is a homogeneous polynomial of degree d. Analogously, we define the homogeneous polynomial of degree d, $M(X,Y,Z)$, from $Q(x,y)$, such that $M(X,Y,Z) = Z^{\mathrm{d}} Q(X/Z, Y/Z)$. Substituting in $Q(x,y)dx - P(x,y)dy = 0$ we have $L(YdZ - ZdY) + M(ZdX - XdZ) = 0$, which is an equation (3.3) with $N \equiv 0$.

An invariant algebraic curve $f(x,y) = 0$, with cofactor $k(x,y)$, for an equation $Q(x,y)\,dx - P(x,y)\,dy = 0$ defines the invariant algebraic curve $F(X,Y,Z) = 0$ in $\mathbb{CP}(2)$ with $F(X,Y,Z) = Z^n f(X/Z, Y/Z)$, where n is the degree of f. We have that the associated cofactor of $F(X,Y,Z) = 0$ is $K(X,Y,Z) = Z^{\mathrm{d}-1} k(X/Z, Y/Z)$.

3.2.2 Local behavior of solutions

In this subsection we give a brief summary of definitions and results concerning formal differential equations and their solutions. These results concern the local behavior of solutions in a neighborhood of a critical point of a system (1.1). Formal differential equations were studied by Seidenberg in [85]. We explicitly state only the necessary result for our aims. In [91], S. Walcher states the same result included in his Theorem 2.3. We refer the reader to [85, 91] for a further description.

Let $\mathbb{C}[[x,y]]$ be the ring of formal power series in two variables with coefficients in $\mathbb{C}$, that is,

$$\mathbb{C}[[x,y]] = \left\{ \varphi(x,y) = \sum_{i,j \geq 0} \varphi_{ij} x^i y^j \;\mid\; \varphi_{ij} \in \mathbb{C} \right\},$$

with the usual operations of addition and multiplication. This ring is factorial.

We are also interested in the ring $\mathbb{C}\{x,y\}$ of convergent power series, that is the subring of $\mathbb{C}[[x,y]]$ of elements φ with a positive radius of convergence. When $\varphi \in \mathbb{C}\{x,y\}$, we say that it is *analytic*.

We describe some properties of the elements of $\mathbb{C}[[x,y]]$. The *order* of $\varphi(x,y)$ is $\min_{i,j\geq 0}\{i+j \mid \varphi_{ij} \neq 0\}$. The set of units for this ring corresponds to all the $\varphi(x,y) \in \mathbb{C}[[x,y]]$ of order 0, that is, such that $\varphi_{00} \neq 0$. A unit element of this ring will be denoted by $\upsilon(x,y)$.

Two formal series $\varphi(x,y), \psi(x,y) \in \mathbb{C}[[x,y]]$ are said to be equal, $\varphi = \psi$, when $\varphi_{ij} = \psi_{ij}$ for all $i,j \geq 0$. A formal power series $\varphi(x,y)$ is said to be *constant* if $\varphi_{ij} = 0$ for all $i,j \geq 1$. We denote the formal power series whose coefficients are all null by $\mathbf{0}$.

We define the partial derivatives of $\varphi(x, y)$ with respect to x and y in the following way

$$\frac{\partial \varphi}{\partial x} := \sum_{i,j\geq 0} i\varphi_{ij} x^{i-1} y^j \, , \quad \frac{\partial \varphi}{\partial y} := \sum_{i,j\geq 0} j\varphi_{ij} x^{i} y^{j-1} \, .$$

Let $\varphi(x, y) \in \mathbb{C}[[x, y]] - \{\mathbf{0}\}$ be an irreducible non-unit element. An *analytic branch centered at* $(0, 0)$ is the equivalence class of φ under the equivalence relation $\varphi \sim \psi$ if $\varphi = \upsilon\psi$. Since $\varphi(x, y)$ is non-unit, we have that its order is greater or equal than 1.

A branch $\varphi(x, y)$ is said to be *linear* if its order equals 1 and *non-linear* if its order is strictly greater than 1.

Let $\varphi(x, y) \in \mathbb{C}[[x, y]] - \{\mathbf{0}\}$ a non-unit element of order s. The homogeneous polynomial of degree s given by $\varphi_s := \sum_{i=0}^{s} \varphi_{i,s-i} x^i y^{s-i}$ is called the *tangents* of φ at the origin. This homogeneous polynomial φ_s factorizes in $\mathbb{C}[x, y]$ in exactly s linear factors: $\varphi_s = \ell_1 \, \ell_2 \, \ldots \, \ell_s$. If all these linear factors are x (resp. y), we say that φ has vertical (resp. horizontal) tangent.

Consider the formal differential equation $Q(x, y)\, dx - P(x, y)\, dy = 0$, where $Q(x, y), P(x, y) \in \mathbb{C}[[x, y]] - \{\mathbf{0}\}$ are non-unit elements. Equally, this formal differential equation can be given by

$$\dot{x} \, = \, P(x, y), \quad \dot{y} \, = \, Q(x, y).$$

By a *solution* of this formal differential equation, we mean an analytic branch $\varphi(x, y)$ centered at the origin such that there exists $\kappa(x, y) \in \mathbb{C}[[x, y]]$ satisfying

$$P \frac{\partial \varphi}{\partial x} + Q \frac{\partial \varphi}{\partial y} \, = \, \kappa \, \varphi \, .$$

Theorem 3.2 [85, 91] *Consider the formal differential system*

$$\dot{x} = \lambda x + X_2(x, y), \quad \dot{y} = \mu y + Y_2(x, y), \tag{3.4}$$

where $X_2, Y_2 \in \mathbb{C}[[x, y]]$ are of order greater or equal than two and with $\lambda \neq 0$.

(a) If $\mu \neq 0$ and λ, μ are rationally independent or $\lambda/\mu < 0$, or $\mu = 0$, then there are exactly two solutions, a linear branch with horizontal tangent and a linear branch with vertical tangent.

(b) If $\mu \neq 0$ and λ, μ are rationally dependent and $\lambda/\mu > 0$, then

- *(i) if $\lambda/\mu = 1$, then for any direction there is a linear solution,*
- *(ii) if $\lambda/\mu > 1$ and $\lambda/\mu \in \mathbb{N}$, then there is a unique linear solution with horizontal tangent and either there are no solutions with vertical tangent or there are an infinite number of solutions with vertical tangent, all of them linear,*
- *(iii) if $\lambda/\mu > 1$ and $\lambda/\mu \notin \mathbb{N}$, we have that $\lambda/\mu = p/q$ with $p, q \in \mathbb{N}$ and $1 < q < p$. Then there is a unique linear solution with horizontal tangent and there is one solution with vertical tangent, which is linear. There are an infinite number of solutions with vertical tangent, all of them are non-linear and their tangents are given by $\varphi_q = x^q$.*

The expressions X_2 and Y_2 are formal complex series and λ and μ are complex numbers which can be real as a particular case.

We remark that except in the case $\lambda/\mu = 1$, the solutions of system (3.4) can only have horizontal or vertical tangent.

In [85], the proof of this result is given by means of blowing-up the origin of system (3.4). In [91] system (3.4) is supposed to be analytic in a neighborhood of the origin and the normal form technique is used to prove Theorem 3.2.

By using the normal form theory, Walcher in [91] also proves the following result, which lets us distinguish between a unique solution and an infinite number of solutions in case $\lambda/\mu > 1$ and $\lambda/\mu \in \mathbb{N}$. We do not intend to give a survey on normal form theory and we only state the following proposition for the sake of completeness. We refer the reader to [6] for an exhaustive description of this classical theory due to Poincaré and the proof of the following proposition.

Proposition 3.3 [6] *Let us consider a system* (3.4). *Let us assume that $\lambda/\mu = m$ with $m \in \mathbb{N}$ and $m > 1$. Then, there is a formal change of variables which transforms the system to*

$$\dot{x} = \lambda x + c y^m, \quad \dot{y} = \mu y, \tag{3.5}$$

with $c \in \mathbb{C}$.

Moreover, if system (3.4) *is analytic in a neighborhood of the origin, then the change of variables is analytic in a neighborhood of the origin.*

When $c = 0$ in (3.5), we say that system (3.4) is *linearizable*. If $c \neq 0$, we say that it is *non linearizable*.

The distinction between linearizable and non linearizable systems allows the distinction between an infinite number of solutions or a unique solution, as stated and proved in [91].

Proposition 3.4 *Consider a system* (3.4)*, analytic in a neighborhood of the origin, where* $\lambda/\mu = m$ *with* $m \in \mathbb{N}$ *and* $m > 1$*.*

- *If the system is linearizable, then there is a unique solution with horizontal tangent and an infinite number of solutions with vertical tangent, all of them linear.*
- *If the system is non linearizable, there is exactly one solution which is linear and with horizontal tangent.*

Let (x_0, y_0) be a non-degenerate or elementary degenerate singular point for system (1.1). Let λ and μ be the eigenvalues of the matrix $A(x_0, y_0)$. Throughout this chapter we will always assume that $\lambda \neq 0$ and $\lambda \neq \mu$. We notice that by an affine change of coordinates we can always write system (1.1) in the form (3.4) with this critical point at the origin. Moreover, the eigenvectors $\mathbf{v}_\lambda$ and $\mathbf{v}_\mu$ of the matrix $A(x_0, y_0)$ are related to the horizontal and vertical directions for system (3.4). Therefore, Theorem 3.2 and Proposition 3.4 apply for the critical point (x_0, y_0) of system (1.1).

3.2.3 Values of the cofactor at singular points

The main result in this section is split up in Theorems 3.8, 3.9 and 3.10, which are consequences of Theorem 3.2 and Proposition 3.4. These theorems give necessary conditions for a polynomial of degree lower or equal than $\mathrm{d}-1$ to be a cofactor of an invariant algebraic curve.

Given $p_0 := (x_0, y_0)$ a singular point for equation (1.1) and $f(x, y) = 0$ an invariant algebraic curve, only two possibilities hold: either $f(p_0) \neq 0$ or $f(p_0) = 0$. The next lemma deals with the first possibility and Theorems 3.8, 3.9 and 3.10 with the second one.

Lemma 3.5 *Let us consider a system* (1.1)*,* (x_0, y_0) *one of its critical points and* $f(x, y) = 0$ *an invariant algebraic curve with cofactor* $k(x, y)$*.*
If $f(x_0, y_0) \neq 0$*, then* $k(x_0, y_0) = 0$*.*

Proof. Assume that $f(x,y)=0$ is an invariant algebraic curve for system (1.1) with cofactor $k(x,y)$. Let (x_0,y_0) be a singular point of system (1.1). Since the left hand side of the equality

$$P(x,y)\,\frac{\partial f}{\partial x}(x,y)+Q(x,y)\,\frac{\partial f}{\partial y}(x,y)\;=\;k(x,y)\,f(x,y)$$

is zero at (x_0,y_0) and $f(x_0,y_0)\neq 0$, we deduce that $k(x_0,y_0)=0$. ∎

We remark that since we can assume that $f(x,y)$ and its cofactor $k(x,y)$ belong to $\mathbb{R}[x,y]$, then if (x_0,y_0) is a complex singular point such that $f(x_0,y_0)\neq 0$, Lemma 3.5 implies that $k(x_0,y_0)=0$ and $k(\bar{x}_0,\bar{y}_0)=0$.

Let $f(x,y)=0$ be an algebraic curve and (x_0,y_0) a point such that $f(x_0,y_0)=0$. We may expand $f(x,y)$ in powers of $(x-x_0)$ and $(y-y_0)$: $f(x,y)=f_s(x,y)+f_{s+1}(x,y)+\ldots+f_n(x,y)$, where $f_j(x,y)$ are homogeneous polynomials of degree j in powers of $(x-x_0)$ and $(y-y_0)$. Let s be the lowest degree in this expansion with $f_s(x,y)\not\equiv 0$. Since $f(x_0,y_0)=0$, we have $s\geq 1$. As $f_s(x,y)$ is a homogeneous polynomial of degree s in powers of $(x-x_0)$ and $(y-y_0)$ it factorizes in s linear homogeneous polynomials, that is, $f_s(x,y)=\ell_1\ell_2\ldots\ell_s$ with $\ell_i=a_i(x-x_0)+b_i(y-y_0)$, $a_i,b_i\in\mathbb{C}$, $i=1,2,\ldots,s$. We say that $f_s(x,y)=0$ is the equation of the *tangents* of the curve $f(x,y)=0$ in (x_0,y_0).

Given a polynomial $f(x,y)$ we denote by $\nabla f(x,y)$ the gradient vector at the point (x,y), that is $\nabla f(x,y):=(\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial y}(x,y))$. As usual, if $\mathbf{u},\mathbf{v}\in\mathbb{C}^2$ we denote by $\mathbf{u}\cdot\mathbf{v}$ its Euclidean scalar product.

The following three theorems are a consequence of Theorem 3.2 and deal with a non-degenerate or an elementary degenerate critical point (x_0,y_0), whose ratio of eigenvalues does not equal 1. We provide the tangents of an invariant algebraic curve $f(x,y)=0$ irreducible in $\mathbb{C}[x,y]$ such that $f(x_0,y_0)=0$. Once these tangents are described we deduce the value of the cofactor $k(x,y)$ at (x_0,y_0). We also describe the possible existence of another invariant algebraic curve $\tilde{f}(x,y)=0$ irreducible in $\mathbb{C}[x,y]$, with cofactor $\tilde{k}(x,y)$ such that $f(x,y)$ and $\tilde{f}(x,y)$ are relatively coprime and $\tilde{f}(x_0,y_0)=0$.

As before, we denote by (x_0,y_0) a singular point of system (1.1) which is supposed to be non-degenerate or elementary degenerate. Let A be the matrix of the linear approximation of system (1.1) at (x_0,y_0). Since we fix the point (x_0,y_0) we do not explicit the dependence on it. We denote by λ

and μ the eigenvalues of the linear approximation at this point and by $\mathbf{v}_\lambda$ and $\mathbf{v}_\mu$ two corresponding eigenvectors. We assume that $\lambda \neq \mu$ and $\lambda \neq 0$.

We denote by ℓ_λ any non-null homogeneous polynomial of degree 1 such that $\nabla \ell_\lambda \cdot \mathbf{v}_\lambda = 0$. An analogous definition stands for ℓ_μ. We use this notation throughout the chapter.

Let us consider an invariant algebraic curve $f(x,y) = 0$ irreducible in $\mathbb{C}[x,y]$ such that $f(x_0,y_0) = 0$. Let us move the point (x_0,y_0) to the origin $(0,0)$. The polynomial $f(x,y)$ is also an element of the ring $\mathbb{C}\{x,y\}$ and, consequently, of $\mathbb{C}[[x,y]]$. Since, $f(0,0) = 0$, it is not a unit element. In this ring $f(x,y)$ can be a reducible element. The following lemma describes the decomposition of $f(x,y)$ in the ring $\mathbb{C}[[x,y]]$ which coincides with its decomposition in $\mathbb{C}\{x,y\}$. The proof of the following lemma makes use of Newton-Puiseux algorithm which is described, for instance, in Chapter 1 of [14], see Corollary 1.5.5 (page 25) and Theorem 1.8.3 (page 32).

Lemma 3.6 *Let $f(x,y) \in \mathbb{C}[[x,y]]$ of positive order. Then,*

- *there are m irreducible elements $\varphi_1, \varphi_2, \ldots, \varphi_m \in \mathbb{C}[[x,y]]$, $m \geq 0$, such that f decomposes in the form*

$$f = \upsilon x^r \varphi_1 \ldots \varphi_m, \tag{3.6}$$

  where $r \in \mathbb{N}$, $r \geq 0$, and υ a unit element in $\mathbb{C}[[x,y]]$.

- *The elements φ_i of (3.6) can be taken in $\mathbb{C}[[x]][y]$, that is, they are polynomials in y.*

- *Such a decomposition is uniquely determined, up to order, by f.*

- *If $f \in \mathbb{C}\{x,y\}$ then the elements $\upsilon, \varphi_1, \varphi_2, \ldots, \varphi_m$ of (3.6) belong to $\mathbb{C}\{x,y\}$. In fact, $\varphi_i \in \mathbb{C}\{x\}[y]$, $i = 1, 2, \ldots, m$.*

Since we consider algebraic curves given by $f(x,y) = 0$ with f as an irreducible element of $\mathbb{C}[x,y]$, which is a subring of $\mathbb{C}\{x,y\}$, we can strengthen the thesis of the previous lemma.

Lemma 3.7 *Let $f(x,y)$ be an irreducible non constant polynomial in $\mathbb{C}[x,y]$ such that $f(0,0) = 0$. Then, the decomposition given in (3.6) is square free.*

Proof. Taking into account Lemma 3.6, we only need to prove that there is no repeated element φ_i neither $r > 1$ in the decomposition (3.6).

Assume that either there is a repeated element φ_i or x. Then, this element divides both f and $\frac{\partial f}{\partial y}$ in $\mathbb{C}\{x, y\}$. Therefore, f and $\frac{\partial f}{\partial y}$ in $\mathbb{C}\{x, y\}$ intersect in an infinite number of points inside the disk of convergence of this repeated element.

However, by Bézout's theorem, if f and $\frac{\partial f}{\partial y}$ have an infinite number of intersection points, there is a polynomial both dividing f and $\frac{\partial f}{\partial y}$. Since f is an irreducible polynomial, this divisor must coincide with f. So f divides $\frac{\partial f}{\partial y}$, that is, there exists $g \in \mathbb{C}[x, y]$ such that $\frac{\partial f}{\partial y} = gf$. Hence, the degree of $\frac{\partial f}{\partial y}$ equals the sum of degrees of f and g. But this is not possible because if f has degree n, $n \geq 1$, then $\frac{\partial f}{\partial y}$ has degree at most $n - 1$. ■

We do not give an explicit statement of Bézout's theorem since it is a classical and well known result. See, for instance, [48, 92] for a rigorous statement.

Let us consider $f(x, y) = 0$ an invariant algebraic curve of system (1.1) with $f(x, y) \in \mathbb{C}[x, y]$ as an irreducible polynomial. Let (x_0, y_0) be a non degenerate or elementary degenerate critical point of system (1.1) with eigenvalues λ and μ such that $\lambda \neq \mu$ and with $f(x_0, y_0) = 0$. Without loss of generality we can move (x_0, y_0) to the origin. An easy reasoning, based on the fact that $\mathbb{C}[[x, y]]$ is a factorial ring, shows that each of the irreducible elements appearing in the decomposition of $f(x, y)$ written in (3.6) is a solution of system (1.1). Therefore, we notice that $f(x, y) = 0$ defines a finite number of branches in (x_0, y_0) corresponding to its irreducible non-unit factors in $\mathbb{C}[[x, y]]$. The tangents of these branches are given by $f_s(x, y) = 0$ as defined above. The following theorem describes these tangents and the value of the cofactor at the singular point.

Theorem 3.8 *With the described notation, we have* $f_s(x, y) = (\ell_\lambda)^r \, (\ell_\mu)^{s-r}$ *with* $r, s \in \mathbb{N}$ *and* $r \leq s$*. Moreover,* $k(x_0, y_0) = r\mu + (s - r)\lambda$*.*

Proof. The curve $f(x, y) = 0$ defines a finite set of branches at (x_0, y_0) and the tangents $f_s(x, y) = 0$ at this point correspond to the tangents of each branch. Since only ℓ_λ and ℓ_μ are allowed tangents by Seidenberg's result, the first thesis of the theorem is easily concluded.

By a move of the point (x_0, y_0) to the origin and a linear change of variables, we write system (1.1) in the form (3.4). We have that the hypothesis of Theorem 3.2 are satisfied. Since we have moved (x_0, y_0) to the origin, now $k(x_0, y_0)$ is $k(0,0)$.

We expand the equation of the invariant algebraic curve

$$P(x,y)\,\frac{\partial f}{\partial x}(x,y) \,+\, Q(x,y)\,\frac{\partial f}{\partial y}(x,y) \,=\, k(x,y)\,f(x,y)$$

in powers of x and y and equating the non-null terms of lowest degree, which corresponds to degree s, we have that $\nabla f_s \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = k(0,0)\,f_s$. Since $f_s(x,y) = (\ell_\lambda)^r\,(\ell_\mu)^{s-r}$, and dividing both members of the equation by $(\ell_\lambda)^{r-1}\,(\ell_\mu)^{s-r-1}$ we get

$$r\,\ell_\mu \left[\nabla \ell_\lambda \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right] + (s-r)\,\ell_\lambda \left[\nabla \ell_\mu \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right] = k(0,0)\,\ell_\lambda\,\ell_\mu. \quad (3.7)$$

Let us first assume that $r \neq 0$ and $s \neq r$, then from equation (3.7) we deduce that there exists $a \in \mathbb{C}$ such that $\nabla \ell_\lambda \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = a\,\ell_\lambda$. Using the identity $\ell_\lambda = \nabla \ell_\lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ and equating the coefficients of x and y we get $\nabla \ell_\lambda \cdot A = a\,\nabla \ell_\lambda$. We multiply both terms of this equality by $\mathbf{v}_\mu$ noticing that since $\nabla \ell_\lambda \cdot \mathbf{v}_\lambda = 0$ then $\nabla \ell_\lambda \cdot \mathbf{v}_\mu \neq 0$. Hence, $\nabla \ell_\lambda \cdot A \cdot \mathbf{v}_\mu = a\,\nabla \ell_\lambda \cdot \mathbf{v}_\mu$ and this gives $\mu\,\nabla \ell_\lambda \cdot \mathbf{v}_\mu = a\,\nabla \ell_\lambda \cdot \mathbf{v}_\mu$ from which we obtain $a = \mu$. Analogous reasonings show that $\nabla \ell_\mu \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \lambda\,\ell_\mu$. Substituting in equation (3.7) and dividing by $\ell_\lambda\,\ell_\mu$ we have that $k(x_0, y_0) = r\mu + (s-r)\lambda$.

If $r = 0$ then $s - r = s \geq 1$. From equation (3.7) we get that there exists $b \in \mathbb{C}$ such that $\nabla \ell_\mu \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = b\,\ell_\mu$ and $k(x_0, y_0) = (s-r)\,b$. As before, it is easy to show that $b = \lambda$.

If $s = r$ then $r \geq 1$ and from equation (3.7) we have again $\nabla \ell_\lambda \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = a\,\ell_\lambda$, with $a \in \mathbb{C}$ and $k(x_0, y_0) = ra$. The equality $a = \mu$ is achieved as before.

■

The following theorem precises more accurately the form of the equation of the tangents $f_s(x,y) = 0$.

Theorem 3.9 *Let p_0 be a singular point for system* (1.1) *with associated eigenvalues λ and μ, with $\lambda \neq 0$ and let $f(x,y) = 0$ be an invariant algebraic* *curve irreducible in* $\mathbb{C}[x,y]$ *such that* $f(p_0)=0$. *We assume that* $f_s(x,y)=0$ *is the equation of the tangents of* $f=0$ *at* p_0.

- *If either* $\mu \neq 0$ *and* λ *and* μ *are rationally independent or* $\lambda/\mu < 0$, *or* $\mu = 0$, *then*
  - *either* $s=2$ *and* $f_2 = \ell_\lambda \, \ell_\mu$,
  - *or* $s=1$ *and* $f_1 = \ell_\lambda$,
  - *or* $s=1$ *and* $f_1 = \ell_\mu$.
- *If* $\mu \neq 0$ *and* λ *and* μ *are rationally dependent and* $\lambda/\mu > 0$, *we assume that* $\lambda/\mu > 1$.
  - *If* $\lambda/\mu = m$ *with* $m \in \mathbb{N}$, $m > 1$, *and system* (1.1) *is linearizable in* (x_0,y_0), *then there exists* $r \in \mathbb{N}$, $r \geq 0$ *and* $\epsilon \in \{0,1\}$ *such that* $s = r+\epsilon$ *and* $f_s = (\ell_\lambda)^\epsilon \, (\ell_\mu)^r$.
  - *If* $\lambda/\mu = m$ *with* $m \in \mathbb{N}$, $m > 1$, *and system* (1.1) *is non linearizable in* (x_0,y_0), *then* $s=1$ *and* $f_1 = \ell_\lambda$.
  - *If* $\lambda/\mu = p/q$ *with* $p,q \in \mathbb{N}$ *and* $1 < q < p$, *then*
    * *either* $s=1$ *and* $f_1 = \ell_\lambda$,
    * *or there exists* $r \in \mathbb{N}$, $r \geq 0$, *and* $\epsilon \in \{0,1\}$ *such that* $s = rq+\epsilon$ *and* $f_s = (\ell_\lambda)^\epsilon \, (\ell_\mu)^{rq}$,
    * *or there exists* $r \in \mathbb{N}$, $r \geq 0$, *and* $\epsilon \in \{0,1\}$ *such that* $s = rq+\epsilon+1$ *and* $f_s = (\ell_\lambda)^\epsilon \, (\ell_\mu)^{rq+1}$.

In the work [68], J. Moulin-Ollagnier also gives a set of necessary conditions for a system (1.1) to have an invariant algebraic curve. His set of necessary conditions correspond to the value of the cofactor at singular points with eigenvalues λ, μ such that $\lambda\,\mu \neq 0$ and $\lambda/\mu < 0$ or λ/μ rationally independent. His conditions coincide with the ones we give for these values of the eigenvalues. However, his proof uses other techniques, such as Levelt's method.

Proof. The fact that a unique linear branch is defined in a given direction depending on the value of the eigenvalues λ and μ plays a fundamental role in this proof. To simplify notation, we consider the point (x_0,y_0) moved to the origin $(0,0)$. Let us consider the polynomial $f(x,y)$ and factorize it in linear branches belonging to the ring of formal power series $\mathbb{C}[[x,y]]$.

First we assume that either $\mu \neq 0$ and λ and μ are rationally independent or $\lambda/\mu < 0$, or $\mu = 0$, then $f(x,y)$ can factorize at most in two linear branches because only two linear branches are defined in a neighborhood of (x_0, y_0). That is, $f(x,y)$ can be $f(x,y) = \varphi_\lambda \varphi_\mu \upsilon$ or $f(x,y) = \varphi_\lambda \upsilon$ or $f(x,y) = \varphi_\mu \upsilon$ with $\varphi_\lambda, \varphi_\mu, \upsilon \in \mathbb{C}[[x,y]]$, υ is a unit element, $\varphi_\lambda = \ell_\lambda + h.o.t.$ and $\varphi_\mu = \ell_\mu + h.o.t.$ where $h.o.t.$ denotes terms of order greater or equal than two. These factorizations give the form of f_s.

Now, we assume that $\mu \neq 0$ and λ and μ are rationally dependent and $\lambda/\mu > 1$. In this case non-linear branches can also appear in the factorization of $f(x,y)$ in $\mathbb{C}[[x,y]]$. However, all the irreducible non-linear branches that may appear are of the form $(\ell_\mu)^q + h.o.t.$, where here $h.o.t.$ means terms of order greater or equal than $q+1$. Let s be the number of irreducible non-unit linear branches which appear in the factorization of $f(x,y)$ plus the number of non-linear branches each one multiplied by its order q at the origin. We have that $s \in \mathbb{N}$ and $s \geq 1$. Since only one linear branch is defined with tangent ℓ_λ by Theorem 3.2, we have that either $f(x,y) = \varphi_\lambda \varphi_{\mu_1} \ldots \varphi_{\mu_r} \upsilon$, or $f(x,y) = \varphi_\lambda \varphi_\mu \varphi_{\mu_1} \ldots \varphi_{\mu_r} \upsilon$, or $f(x,y) = \varphi_\mu \varphi_{\mu_1} \ldots \varphi_{\mu_r} \upsilon$ or $f(x,y) = \varphi_{\mu_1} \ldots \varphi_{\mu_r} \upsilon$, with φ_λ a linear branch with tangent ℓ_λ, φ_μ a linear branch with tangent ℓ_μ, φ_{μ_i} a non linear branch with tangent $(\ell_\mu)^q$ and υ a unit element. These factorizations give the described form of f_s. ∎

In the following theorem we assume the existence of an invariant algebraic curve $f(x,y) = 0$ irreducible in $\mathbb{C}[x,y]$ and such that $f(x_0, y_0) = 0$. We study the possible existence of another invariant algebraic curve $\tilde{f}(x,y)$ irreducible in $\mathbb{C}[x,y]$, such that f and $\tilde{f}$ are relatively coprime and $\tilde{f}(x_0, y_0) = 0$. We assume that this curve $\tilde{f}(x,y) = 0$ exists and we describe the form of its tangents at (x_0, y_0), that is $\tilde{f}_{\tilde{s}}(x,y) = 0$, depending on the tangents of $f(x,y) = 0$ at this point.

Theorem 3.10 *Let p_0 be a singular point for equation* (1.1) *with associated eigenvalues λ and μ, with $\lambda \neq 0$ and $\lambda \neq \mu$. We assume that $f = 0$ and $\tilde{f} = 0$ are two coprime invariant algebraic curves irreducibles in $\mathbb{C}[x,y]$ for system* (1.1) *such that $f(p_0) = \tilde{f}(p_0) = 0$. Let $f_s = 0$ and $\tilde{f}_{\tilde{s}} = 0$ be the equations of the tangents of these curves at p_0.*

- *If either $\mu \neq 0$ and λ and μ are rationally independent or $\lambda/\mu < 0$, or $\mu = 0$, then*
  - *if $s = 2$, then no such $\tilde{f}(x,y) = 0$ can exist.*
  - *If $s = 1$ and $f_1 = \ell_\lambda$, then $\tilde{s} = 1$ and $\tilde{f}_1 = \ell_\mu$.*

  - *If* $s = 1$ *and* $f_1 = \ell_\mu$, *then* $\tilde{s} = 1$ *and* $\tilde{f}_1 = \ell_\lambda$.

- *If* $\mu \neq 0$ *and* λ *and* μ *are rationally dependent and* $\lambda/\mu > 0$, *we assume that* $\lambda/\mu > 1$. *Then there exists* $\tilde{s} \in \mathbb{N}$, $\tilde{s} \geq 1$, *such that*

  - *If* $\lambda/\mu = m$ *with* $m \in \mathbb{N}$, $m > 1$, *and system* (1.1) *is linearizable in* (x_0, y_0), *then*
    * *if* $f_s = \ell_\lambda \, (\ell_\mu)^{s-1}$, *then* $\tilde{f}_{\tilde{s}} = (\ell_\mu)^{\tilde{s}}$.
    * *If* $f_s = (\ell_\mu)^s$, *then* $\tilde{f}_{\tilde{s}} = (\ell_\lambda)^\epsilon \, (\ell_\mu)^{\tilde{s}-\epsilon}$ *with* $\epsilon \in \{0, 1\}$.
  - *If* $\lambda/\mu = m$ *with* $m \in \mathbb{N}$, $m > 1$, *and system* (1.1) *is non linearizable in* (x_0, y_0), *then no such* $\tilde{f}$ *can exist.*
  - *If* $\lambda/\mu = p/q$ *with* $p, q \in \mathbb{N}$, $1 < q < p$, *then*
    * *if* $f_s = \ell_\lambda \, (\ell_\mu)^{s-1}$, *then there exist* $\tilde{r} \in \mathbb{N}$ *and* $\epsilon \in \{0, 1\}$ *such that* $\tilde{s} = rq + \epsilon$ *and* $\tilde{f}_{\tilde{s}} = (\ell_\mu)^{\tilde{s}}$.
    * *If* $f_s = (\ell_\mu)^s$, *then there exist* $\tilde{r} \in \mathbb{N}$ *and* $\epsilon, \tilde{\epsilon} \in \{0, 1\}$ *such that* $\tilde{s} = rq + \epsilon + \tilde{\epsilon}$ *and* $\tilde{f}_{\tilde{s}} = (\ell_\lambda)^{\tilde{\epsilon}} \, (\ell_\mu)^{\tilde{s}-\tilde{\epsilon}}$.

Proof. The proof is a straightforward consequence of Theorem 3.2. For instance, if λ and μ are rationally independent or $\lambda/\mu < 0$ and $s = 2$, no other $\tilde{f}(x, y)$ can exist as long as it owns the described features since it would define another linear branch in (x_0, y_0). The other cases use exactly the same reasoning with the addition of the result given in Theorem 3.9. ∎

In case of considering an invariant algebraic curve $f(x, y) = 0$ irreducible in $\mathbb{R}[x, y]$ and (x_0, y_0) a real singular point with eigenvalues $a \pm bi$ with $b \neq 0$, we can reduce the number of possibilities.

Lemma 3.11 *Let* $f(x, y) = 0$ *be an invariant algebraic curve irreducible in* $\mathbb{R}[x, y]$ *and* (x_0, y_0) *a real singular point with eigenvalues* $a \pm b\mathbf{i}$ *with* $b \neq 0$, *then* $s = 2$, $f_2 = \ell_\lambda \, \ell_\mu$, $k(x_0, y_0) = \lambda + \mu$ *and no other invariant algebraic curve* $\tilde{f}(x, y) = 0$ *irreducible in* $\mathbb{R}[x, y]$ *with* $\tilde{f}(x_0, y_0) = 0$ *can exist.*

Proof. Let us consider the singular point (x_0, y_0) such that the matrix $A(x_0, y_0)$, with $A(x, y)$ previously defined, has eigenvalues with non–null real and imaginary part: $\lambda = a + b\mathbf{i}$, $\mu = a - b\mathbf{i}$ with $a, b \in \mathbb{R}$ and $b \neq 0$. Let us consider a real, irreducible in $\mathbb{R}[x, y]$, invariant algebraic curve $f(x, y) = 0$ with cofactor $k(x, y)$ and such that $f(x_0, y_0) = 0$. If this polynomial was irreducible in the ring $\mathbb{C}[[x, y]]$ by Theorems 3.8 and 3.9 its tangents in (x_0, y_0)

would be $a_0(x-x_0)+b_0(y-y_0)$ with $(-b_0,a_0)$ one of the eigenvectors of $A(x_0,y_0)$. The eigenvectors of $A(x_0,y_0)$ have non–null imaginary part because $A(x_0,y_0)$ is a matrix with real coefficients and eigenvalues with non–null imaginary part. Hence, the coefficients of $a_0(x-x_0)+b_0(y-y_0)$ have non–null imaginary part, in contradiction to the fact that $f(x,y)\in\mathbb{R}[x,y]$. Therefore, $f(x,y)$ is a reducible element of $\mathbb{C}[[x,y]]$ and equals the product of two complex conjugate branches. Its tangents in (x_0,y_0) equal the product $(a_0(x-x_0)+b_0(y-y_0))(\bar{a}_0(x-x_0)+\bar{b}_0(y-y_0))$. By the application of Theorem 3.8, we have that the value of the cofactor in (x_0,y_0) equals the sum of the two eigenvalues, that is $k(x_0,y_0)=\lambda+\mu=2a$. Theorem 3.10 shows that no other invariant algebraic curve can pass through (x_0,y_0). ■

Lemma 3.5 and Theorems 3.8, 3.9 and 3.10 give the possible values of the cofactor $k(x,y)$ of an invariant algebraic curve at a non-degenerate or elementary degenerate singular point (x_0,y_0) whose ratio of eigenvalues does not equal one. Given an equation (1.1) of degree d, we can extend this equation to $\mathbb{CP}(2)$. If $p_0 := [X_0,Y_0,Z_0]$ is a singular point of the extended equation, we can take local coordinates at this point and the hypothesis of Lemma 3.5 and Theorems 3.8, 3.9 and 3.10 are satisfied. We obtain, in this way, a condition on the value at p_0 of a polynomial $k(x,y)$ of degree $\leq \mathrm{d}-1$ to be a cofactor of an invariant algebraic curve. For an infinite point, the coefficients of the cofactor also depend on the degree n of the curve. So, we also give conditions on the degree of the algebraic curve. Therefore, the union of all these conditions for each non-degenerate or elementary degenerate singular point, finite or infinite, gives a set of necessary conditions on $k(x,y)$ to be a possible cofactor, and on the degree of the curve.

The following two lemmas also give conditions on a polynomial of degree lower or equal than $\mathrm{d}-1$ to be a cofactor, but associated with an exponential factor instead of an invariant algebraic curve.

Lemma 3.12 *Let* $g=\exp\{h/f\}$ *be an exponential factor for system* (1.1) *with cofactor* $k_g(x,y)$ *and let* (x_0,y_0) *be a critical point such that* $f(x_0,y_0)\neq 0$*, then* $k_g(x_0,y_0)=0$.

Proof. The left hand side of the defining equation of the exponential factor

$$P(x,y)\,\frac{\partial g}{\partial x}(x,y)\;+\;Q(x,y)\,\frac{\partial g}{\partial y}(x,y)\;=\;k_g(x,y)\,g(x,y)$$

equals zero at (x_0, y_0) and since $f(x_0, y_0) \neq 0$, we have that $g(x,y)$ is a non-null well defined function in a neighborhood of this point. Hence, we deduce that $k_g(x_0, y_0) = 0$. ■

The following lemma is a generalization of Lemma 3.12 and gives the form of an exponential factor for equation (1.1) in any chart of the extended differential equation $\Omega = 0$, where $\Omega = L(YdZ - ZdY) + M(ZdX - XdZ)$ as formerly defined.

Lemma 3.13 *Let $g(x,y) = \exp\{\phi(x,y)\}$ be an exponential factor of an equation $\omega = 0$, where $\omega = Q(x,y)\,dx - P(x,y)\,dy$ is a 1–form of degree* d*, where $\phi(x,y)$ is either a polynomial or a rational function. Let*

$$k_g(x,y) := P(x,y)\,\frac{\partial \phi}{\partial x}(x,y) \;+\; Q(x,y)\,\frac{\partial \phi}{\partial y}(x,y)$$

be the cofactor of this exponential factor g and we define

$$K_G(X,Y,Z) := Z^{\mathrm{d}-1}\, k_g\left(\frac{X}{Z}, \frac{Y}{Z}\right), \quad \Phi(X,Y,Z) := \phi\left(\frac{X}{Z}, \frac{Y}{Z}\right).$$

Let $[X_0, Y_0, Z_0] \in \mathbb{CP}(2)$ be a critical point of the 1-form Ω such that $\Phi(X_0, Y_0, Z_0)$ is well-defined (it is not a point vanishing its denominator) then $K_G(X_0, Y_0, Z_0) = 0$.

Proof. We define $G(X,Y,Z) := \exp\{\Phi(X,Y,Z)\}$. We take local coordinates at the point $[X_0, Y_0, Z_0]$. At least one of X_0, Y_0 and Z_0 is not null. We first assume that $Z_0 \neq 0$ and we show how this result coincides with the one given in Lemma 3.12. We assume, for instance, that $Y_0 \neq 0$ (if it is $X_0 \neq 0$ analogous reasonings work).

If $Z_0 \neq 0$, we define $x_0 = X_0/Z_0$ and $y_0 = Y_0/Z_0$. We have that (x_0, y_0) is a critical point for the 1-form ω and $g(x,y) = G(x,y,1)$ is an exponential factor with cofactor $k_g(x,y) = K_G(x,y,1)$. If $\Phi(X_0,Y_0,Z_0)$ is well-defined then $\phi(x,y) = \Phi(x,y,1)$ is well defined in (x_0,y_0) and the same proof of Lemma 3.12 shows that $k_g(x_0,y_0) = 0$ and then, $K_G(X_0,Y_0,Z_0) = 0$.

If $Y_0 \neq 0$, we define $u_0 = X_0/Y_0$, $v_0 = Z_0/Y_0$, $u = X/Y$ and $v = Z/Y$. We consider $\tilde{P}(u,v) = L(u,1,v) - uM(u,1,v)$, $\tilde{Q}(u,v) = -vM(u,1,v)$ and $\tilde{Q}(u,v)\,du - \tilde{P}(u,v)\,dv = 0$ is equation $\Omega = 0$ in this local chart. The point (u_0, v_0) is a critical point for this equation. We define $\tilde{\phi}(u,v) := \Phi(u,1,v)$ and $\tilde{g}(u,v) := G(u,1,v)$, which is an exponential factor of system $\dot{u} = \tilde{P}(u,v)$,

$\dot{v} = \tilde{Q}(u,v)$, with cofactor $\tilde{k}_g(u,v) := K_G(u,1,v)$. Let us prove this statement. We have that:

$$L(X,Y,Z)\,\frac{\partial \Phi}{\partial X}(X,Y,Z) + M(X,Y,Z)\,\frac{\partial \Phi}{\partial Y}(X,Y,Z) = K_G(X,Y,Z),$$

for the definition of Φ and k_g. Since $\Phi(X,Y,Z)$ is a homogeneous function of degree 0, then

$$\begin{aligned}\left(L(X,Y,Z) - \frac{X}{Y}M(X,Y,Z)\right)\,\frac{\partial \Phi}{\partial X}(X,Y,Z)\qquad\qquad&\\ -\frac{Z}{Y}M(X,Y,Z)\,\frac{\partial \Phi}{\partial Z}(X,Y,Z) \;&=\; K_G(X,Y,Z).\end{aligned}$$

Taking local coordinates $u = X/Y$ and $v = Z/Y$ we deduce that

$$\tilde{P}(u,v)\,\frac{\partial \tilde{\phi}}{\partial u}(u,v) + \tilde{Q}(u,v)\,\frac{\partial \tilde{\phi}}{\partial v}(u,v) = \tilde{k}_g(u,v),$$

which implies that $\tilde{g}(u,v)$ is an exponential factor for $\tilde{Q}(u,v)\,du - \tilde{P}(u,v)\,dv = 0$ with cofactor $\tilde{k}_g(u,v)$. The same reasoning given in Lemma 3.12 shows that $\tilde{k}_g(u_0,v_0) = 0$ and then, $K_G(X_0,Y_0,Z_0) = 0$. ∎

3.3 An application of the result: a Lotka-Volterra system

Let us consider the following Lotka-Volterra system

$$\dot{x} \;=\; x(ax + by + 1), \;\; \dot{y} \;=\; y(x+y), \tag{3.8}$$

with $0 < a < 1$ and $b > 1$. This family of quadratic systems is shown to have no Liouvillian first integral in [12]. However, once more, we prove this fact in this section in order to show the power of Theorems 3.8, 3.9 and 3.10. Indeed, in [12] the integrability of the system (3.8) for all $a, b \in \mathbb{R}$ is studied. We focus on parameters satisfying $0 < a < 1$ and $b > 1$ because in this parameter region the nature of the six singular points of the system does not change and we can directly apply our results without considering repetitive cases. Our aim is to show how easy computations show that no other invariant algebraic curve different from $x = 0$ and $y = 0$ can exist for system (3.8) with $0 < a < 1$ and $b > 1$.

Moreover, we have chosen this example to show that our method and Levelt's method do not coincide. All the quadratic Lotka-Volterra systems are studied by J. Moulin Ollagnier in [68], where the Levelt's method is described. In this article, J. Moulin Ollagnier studies the integrability of all the quadratic Lotka-Volterra systems using the Levelt's method and he encounters that system (3.8) with $0 < a < 1$ and $b > 1$ cannot be studied with this technique.

Theorem 3.14 *System* (3.8) *with* $0 < a < 1$ *and* $b > 1$ *has only two invariant algebraic curves which correspond to the invariant straight lines* $x = 0$ *and* $y = 0$.

Proof. We denote by $k_x(x, y) := ax + by + 1$ the cofactor of the invariant straight line $x = 0$ and by $k_y(x, y) := x + y$ the cofactor of $y = 0$. Let us consider the critical points of system (3.8) and its eigenvalues. The following table also contains for a critical point p_i, $i = 1, 2, \ldots, 6$, whether the invariant straight lines of the system pass through it or not. If an invariant straight line passes through a critical point, the value of the associated cofactor at this point is also given. For the infinite singular points we have taken local coordinates (u_1, v_1) with $u_1 = X/Y$ and $v_1 = Z/Y$ for p_4 and p_5. We denote by $k_{u_1}(u_1, v_1) := (a-1)u_1 + v_1 + b - 1$ and by $k_{v_1}(u_1, v_1) := -1 - u_1$ the cofactors of the invariant straight lines $u_1 = 0$ and $v_1 = 0$, respectively. For p_6 we take local coordinates (u_2, v_2) with $u_2 = Y/X$ and $v_2 = Z/X$ and we denote by $k_{u_2}(u_2, v_2) := (1-b)u_2 - v_2 + 1 - a$ and by $k_{v_2}(u_2, v_2) := -bu_2 - v_2 - a$ the cofactors of the invariant straight lines $u_2 = 0$ and $v_2 = 0$, respectively.

- $p_1 = [0, 0, 1]$ has eigenvalues $\lambda_1 = 1$ and $\mu_1 = 0$. Both x and y are null at this point and $k_x(p_1) = \lambda_1$, $k_y(p_1) = \mu_1$.

- $p_2 = [-\frac{1}{a}, 0, 1]$ has eigenvalues $\lambda_2 = -\frac{1}{a}$ and $\mu_2 = -1$. The polynomial x does not vanish at this point but y does and $k_y(p_2) = \lambda_2$.

- $p_3 = [1, -1, b-a]$ has eigenvalues $\lambda_3 = (1 - a + \sqrt{(1+a)^2 - 4b})/(2(a-b))$ and $\mu_3 = (1 - a - \sqrt{(1+a)^2 - 4b})/(2(a-b))$. No x nor y is null at this point.

- $p_4 = [0, 1, 0]$ has eigenvalues $\lambda_4 = -1$ and $\mu_4 = b - 1$. Both u_1 and v_1 are null at this point, $k_{u_1}(p_4) = \mu_4$ and $k_{v_1}(p_4) = \lambda_4$.

- $p_5 = [1-b, a-1, 0]$ has eigenvalues $\lambda_5 = (b-a)/(a-1)$ and $\mu_5 = 1-b$. Here, u_1 is not null at this point but v_1 is, and $k_{v_1}(p_5) = \lambda_5$.

- $p_6 = [1, 0, 0]$ has eigenvalues $\lambda_6 = 1 - a$ and $\mu_6 = -a$. Both u_2 and v_2 are null at this point, $k_{u_2}(p_6) = \lambda_6$ and $k_{v_2}(p_6) = \mu_6$.

We notice that these six singular points are always different in the range of the parameters considered.

Assume that $f(x, y) = 0$ is an irreducible invariant algebraic curve of degree n for system (3.8), different from $x = 0$ and $y = 0$ and with cofactor $k(x, y) := k_{00} + k_{10}x + k_{01}y$. In local coordinates (u_1, v_1) this cofactor may be written $k(u_1, v_1) = (k_{10} - n)u_1 + k_{00}v_1 + k_{01} - n$ and in local coordinates (u_2, v_2) this cofactor becomes $k(u_2, v_2) = (k_{01} - bn)u_2 + (k_{00} - n)v_2 + k_{10} - an$.

Now we apply Theorems 3.8, 3.9 and 3.10. It is clear that the curve $f = 0$ must satisfy that $f(p_1) \neq 0$, $f(p_4) \neq 0$ and $f(p_6) \neq 0$. Then $k(p_1) = k(p_4) = k(p_6) = 0$. The only polynomial of degree 1 which satisfies these three conditions is $k(x, y) := n(ax + y)$.

At the focus point p_3 we may have $f(p_3) \neq 0$ or $f(p_3) = 0$. If $f(p_3) \neq 0$, then $k(p_3) = 0$ by Lemma 3.5. If $f(p_3) = 0$ then $k(p_3) = \mu_3 + \lambda_3 = \frac{1-a}{a-b}$, by Lemma 3.11. This last dichotomy can be codified by $k(p_3) = \epsilon_3 \frac{1-a}{a-b}$ with $\epsilon_3 \in \{0, 1\}$. This last linear equation gives $n = \epsilon_3$, from which we deduce that if such $f(x, y) = 0$ exists, then it is an invariant straight line ($n = 1$). Easy calculations show that there is no invariant straight line with cofactor $ax + y$. We conclude that no invariant algebraic curve $f(x, y) = 0$ different from $x = 0$ and $y = 0$ can exist. ■

At the light of this proof, we give a shorter one using Bézout's Theorem and our Theorem 3.10, which can be applied even for a wider range of the parameters a and b. We only assume that $b > 1$. The invariant straight line $x = 0$ contains the singular points p_1, whose eigenvalues are 1 and 0, and p_4, whose eigenvalues are -1 and $b - 1$, and no other singular points. The invariant straight line $y = 0$ also contains the point p_1 an no other irreducible invariant algebraic curve (different from $x = 0$ and $y = 0$) can contain this point by virtue of Theorem 3.10. Analogously, the invariant straight line of infinity $z = 0$ also contains p_4 and if $b > 1$ no other irreducible invariant algebraic curve (different from $x = 0$ and $z = 0$) can contain this point by Theorem 3.10. If there exists an irreducible invariant algebraic curve $f(x, y) = 0$ of degree n and different from $x = 0$ and $y = 0$, it must cut the straight line $x = 0$ in n points of $\mathbb{CP}(2)$ by Bézout's Theorem. These n points are singular points of system (3.8). Hence, this curve $f(x, y) = 0$ cannot exist. Therefore, an alternative proof to Theorem 3.14 is completed. We are grateful to professor C. Christopher who pointed out this proof.

The following theorem is related to the Liouvillian integrability of system (3.8).

Theorem 3.15 *There is no Liouvillian first integral for system* (3.8) *with* $0 < a < 1$ *and* $b > 1$.

Proof. Assume that there is a Liouvillian first integral. By applying Theorem 1.7 and Theorem 3.14, we conclude that there exists an inverse integrating factor of the form

$$V(x,y) := \exp\left\{ \frac{h(x,y)}{x^{n_1} y^{n_2}} \right\} x^{c_1} y^{c_2},$$

where $h(x,y) \in \mathbb{R}[x,y]$, $n_1, n_2 \in \mathbb{N}$ and $c_1, c_2 \in \mathbb{R}$. We notice that, eventually, h, n_1, n_2, c_1 or c_2 can be null.

Since p_3 is a focus point for system (3.8) with $0 < a < 1$ and $b > 1$, and neither x nor y is null at this point, we have that $V(p_3)$ is a non-null well-defined real number and, by continuity, there is a neighborhood of p_3 in which V has no zeros. So, the first integral computed from V is continuous in a neighborhood of this focus point p_3, which gives a contradiction. Hence, no such first integral can exist. ∎

The contents of this chapter (and Section 5.1 of Chapter 5) belong to the paper entitled *Necessary conditions for the existence of invariant algebraic curves for planar polynomial systems*, authored by J. Chavarriga, H. Giacomini and M. Grau and accepted for publication in Bull. Sci. Math.

Abstract. This work deals with planar polynomial differential systems $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ and its invariant algebraic curves. We give a set of necessary conditions for a system to have an invariant algebraic curve. These conditions correspond to the value of the cofactor at the singular points of the system when considered in a compact space. In general, we completely characterize the invariant algebraic curves for a given system. We apply these results to determine the non Liouvillian integrability of several families of quadratic systems.

Chapter 4

On the stability of periodic orbits

As in the previous chapters, we consider planar polynomial differential systems, but in this chapter we can state the results in a wider domain, the planar polynomial differentials systems defined by $\mathcal{C}^1$ functions in some open set $\mathcal{U} \subseteq \mathbb{R}^2$. Obviously, planar polynomial differential systems as (1.1) are a particular case defined over all the plane $\mathbb{R}^2$.

In this chapter we consider a planar differential system:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{4.1}$$

where P and Q are $\mathcal{C}^1$ functions in some open set $\mathcal{U} \subseteq \mathbb{R}^2$. We assume that all the singular points of (4.1) are isolated. As for polynomial systems, given a system (4.1), we can always consider its vector field representation $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$.

In this chapter, we are concerned with the stability problem for periodic orbits, which we have described in Subsection 1.2.2, and we give an alternative formula to the one given in Theorem 1.9 to study the stability of a periodic orbit γ defined in an implicit way, as explained below. This is the main result of this chapter and it is stated in Theorem 4.1 in the following section. We can, therefore, distinguish the hyperbolicity of a limit cycle using two different quantities.

We have already defined the notion of invariant curve for a planar polynomial differential system and we can generalize this definition for a $\mathcal{C}^1(\mathcal{U})$

system like (4.1) as a curve $f(x, y) = 0$ satisfying the same hypothesis and following the identity (1.2).

Our main result in this chapter, namely Theorem 4.1, can only be applied when the periodic orbit γ is given in an implicit way, that is, when there exists an invariant curve $f(x, y) = 0$ such that $\gamma \subseteq \{(x, y) \mid f(x, y) = 0\}$.

In order to clarify the type of systems and limit cycles where we can apply our result, let us consider the following $\mathcal{C}^1$ system defined in all $\mathbb{R}^2$:

$$\begin{aligned} \dot{x} &= (x + y)\cos(x) - y(x^2 + xy + 2y^2), \\ \dot{y} &= (y - x)(\cos(x) - y^2) + \frac{x^2 + y^2}{2}\sin(x), \end{aligned} \tag{4.2}$$

which has $y^2 - \cos(x) = 0$ as invariant curve. We define $f(x, y) := y^2 - \cos(x)$ and we have that $f \in \mathcal{C}^1(\mathbb{R}^2)$ and that $\nabla f(x, y) = (\sin(x), 2y)$. Therefore, there is no $p \in \mathbb{R}^2$ such that both $f(p) = 0$ and $\nabla f(p) = 0$. Moreover, $f(x, y) = 0$ satisfies equation (1.2) with $k(x, y) = 2y(x - y) - (x + y)\sin(x)$. The divergence of this system is $\mathrm{div}(x, y) = -4y^2 + 2\cos(x) - x\sin(x)$ and $V(x, y) = (x^2 + y^2)f(x, y)$ is an inverse integrating factor. We denote by γ_n, $n \in \mathbb{Z}$, the oval of $f(x, y) = 0$ belonging to the strip $-\pi/2 + 2\pi n \leq x \leq \pi/2 + 2\pi n$. The oval γ_0 is a hyperbolic stable limit cycle for system (4.2), which can be shown just applying Theorem 1.9. We have, after some easy computations, that

$$\oint_{\gamma_n} \mathrm{div} \;=\; -4\arctan\left(\frac{x}{\sqrt{\cos(x)}}\right)\Bigg|_{x=-\pi/2+2\pi n}^{x=\pi/2+2\pi n}$$

which is zero when $n \neq 0$ and it is -4π for γ_0. Each one of the other ovals of $f(x, y) = 0$, γ_n with $n \neq 0$, belongs to the period annulus of a center as it can be shown from the fact that the function $H(x, y) = f(x, y)(x^2 + y^2)\exp\{2\arctan(y/x)\}$ is a first integral for system (4.2). Our result can be applied for any of the periodic orbits γ_n of this example.

When considering a polynomial system, as far as we know, only algebraic limit cycles are known in this implicit way.

We apply the result described in Theorem 4.1 for some distinguished systems, in order to show that all the known algebraic limit cycles of a quadratic system are hyperbolic. We give the proof the hyperbolicity of these limit cycles in Section 5.2 of Chapter 5, where we also show another feature of these systems.

4.1 Statement and proof of Theorem 4.1

Theorem 4.1 *We consider a system* (4.1) *and* $\gamma(t)$ *a periodic orbit of period* $T > 0$. *We assume that* $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ *is an invariant curve with* $\gamma \subseteq \{(x,y) \,|\, f(x,y) = 0\}$ *and that* $k(x,y)$ *is the* $\mathcal{C}^1$ *function given in* (1.2), *that is, the cofactor of the invariant curve* $f(x,y) = 0$. *We assume that if* $p \in \mathcal{U}$ *is such that* $f(p) = 0$ *and* $\nabla f(p) = 0$, *then* p *is a singular point of system* (4.1). *Then,*

$$\int_0^T k(\gamma(t))\, dt \;=\; \int_0^T \operatorname{div}(\gamma(t))\, dt. \tag{4.3}$$

In order to prove Theorem 4.1, we need to recall the definition and some properties of the Poincaré map, defined for the periodic orbit γ. Let us consider γ a periodic orbit with minimal period $T > 0$ for system (4.1) and $p_0 \in \gamma$. Let $\mathcal{U}_\gamma \subseteq \mathcal{U}$ be a neighborhood of γ not containing any singular point and $\Sigma = \{q \in \mathcal{U}_\gamma \mid (q - p_0) \cdot \mathbf{F}(p_0) = 0\}$, where $\cdot$ denotes the scalar product between the vectors $q - p_0$ and $\mathbf{F}(p_0)$. We have that Σ is a section of the vector field $\mathbf{F}$ in a neighborhood of γ.

As stated and proved in pages 210 and 211 in [74], we have that there exists a $\delta > 0$ and a unique function $\tau : \Sigma \to \mathbb{R}$, which is defined continuously and differentiable for any $q \in \Sigma \cap \mathcal{B}_\delta(p_0)$ such that $\tau(p_0) = T$ and $\Phi_{\tau(q)}(q) \in \Sigma$. As before, $\mathcal{B}_\delta(p_0)$ is the ball of center p_0 and radius δ. Then, for any $q \in \Sigma \cap \mathcal{B}_\delta(p_0)$, the function $\mathcal{P}(q) = \Phi_{\tau(q)}(q)$ is called the *Poincaré map* for γ at p_0. It is clear that fixed points of the Poincaré map, $\mathcal{P}(q) = q$, give rise to periodic orbits for system (4.1). Moreover, it can be shown that $\mathcal{P} : \Sigma \to \Sigma$ is a $\mathcal{C}^1$ diffeomorphism.

We notice that the derivative of $\mathcal{P}$ at p_0 can be represented by a 2×2 matrix, which we denote by $D\mathcal{P}(p_0)$. The following theorem, stated and proved in [5] page 118, is very useful to establish the stability of γ.

Theorem 4.2 *Let* $\mathbf{v}$ *be a non-null vector normal to* $\mathbf{F}(p_0)$. *Then,*

$$\mathbf{v} \cdot D\mathcal{P}(p_0) = \exp\left(\int_0^T \operatorname{div}(\gamma(t))\, dt \right) \mathbf{v}. \tag{4.4}$$

In order to show that the stability of γ is determined by the value of $\mathbf{v} \cdot D\mathcal{P}(p_0)$, as stated in Theorem 1.9, we consider the *displacement function* and we follow the reasoning of page 213 in [74]. For any $q \in \Sigma \cap \mathcal{B}_\delta(p_0)$, we have that $q = p_0 + s\mathbf{v}$, with $s \in (-\delta/|\mathbf{v}|, \delta/|\mathbf{v}|)$. Since $\mathcal{P}(q) \in \Sigma$, we have that given $s \in (-\delta/|\mathbf{v}|, \delta/|\mathbf{v}|)$, there exists a $\sigma(s) \in \mathbb{R}$ such that $\mathcal{P}(p_0 + s\mathbf{v}) = p_0 + \sigma(s)\mathbf{v}$. So, we have defined a $\mathcal{C}^1$ function $\sigma : (-\delta/|\mathbf{v}|, \delta/|\mathbf{v}|) \to \mathbb{R}$ and the displacement function is given by $\mathrm{d} : (-\delta/|\mathbf{v}|, \delta/|\mathbf{v}|) \to \mathbb{R}$ with $\mathrm{d}(s) = \sigma(s) - s$. It is clear that $\mathrm{d}(0) = 0$, $\mathrm{d}'(s) = \sigma'(s) - 1$ and $\mathbf{v} \cdot D\mathcal{P}(p_0 + s\mathbf{v}) = \sigma'(s)\mathbf{v}$. Since $\mathrm{d}(s)$ is $\mathcal{C}^1$, we have that the sign of $\mathrm{d}'(s)$ coincides with the sign of $\mathrm{d}'(0)$ for $|s|$ sufficiently small as long as $\mathrm{d}'(0) \neq 0$. By mean value theorem, we have that given $|s|$ sufficiently small there exists a $\xi \in (0, s)$ such that $\mathrm{d}(s) = \mathrm{d}'(\xi)s$. Therefore, if $\mathrm{d}'(0) > 0$, we have that $\mathrm{d}(s) > 0$ for $s > 0$ and $\mathrm{d}(s) < 0$ for $s < 0$, which implies that the periodic orbit γ is an unstable limit cycle. Similar reasonings show that if $\sigma'(0) > 1$ then γ is an unstable limit cycle and if $\sigma'(0) < 1$ then γ is a stable limit cycle. Theorem 1.9 clearly follows from Theorem 4.2 and the fact that $\sigma'(0)\mathbf{v} = \mathbf{v} \cdot D\mathcal{P}(p_0)$.

Lemma 4.3 *We consider a system* (4.1)*, its associated vector field representation* $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ *and* $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ *a non-null* $\mathcal{C}^1(\mathcal{U})$*-function. There exists a* $\mathcal{C}^1$ *function* $k(x, y)$ *such that* $\nabla f(q) \cdot \mathbf{F}(q) = k(q) f(q)$ *for any* $q \in \mathcal{U}$ *if, and only if, for any* $q \in \mathcal{U}$ *and any* $t \in \mathbb{R}$ *such that* $\Phi_t(q) \in \mathcal{U}$*, the following identity is satisfied:*

$$f(\Phi_t(q)) = f(q) \ \exp\left(\int_0^t k(\Phi_s(q))\, ds \right). \tag{4.5}$$

Proof. Assume that $\nabla f(q) \cdot \mathbf{F}(q) = k(q) f(q)$ for any $q \in \mathcal{U}$. We fix a point $q \in \mathcal{U}$ and we define $\varphi(t) = f(\Phi_t(q))$ for any $t \in \mathbb{R}$ such that $\Phi_t(q) \in \mathcal{U}$. We have that t belongs to an open interval $(-\epsilon_q, \epsilon_q)$ with $\epsilon_q > 0$ (and it may be that $\epsilon_q = +\infty$). We have, using some of the properties of the flow and the fact $\dot{f}(\Phi_t(q)) = k(\Phi_t(q))\, f(\Phi_t(q))$, that:

$$\dot{\varphi}(t) = \nabla f(\Phi_t(q)) \cdot \frac{d\Phi_t}{dt}(q) = \nabla f(\Phi_t(q)) \cdot \mathbf{F}(\Phi_t(q)) = \dot{f}(\Phi_t(q)) = k(\Phi_t(q))\, \varphi(t).$$

We have that $\frac{d\varphi}{dt}(t) = k(\Phi_t(q))\ \varphi(t)$ and $\varphi(0) = f(q)$. Solving this linear equation in the function $\varphi(t)$ we get $\varphi(t) = f(q) \exp\left(\int_0^t k(\Phi_s(q))\, ds \right)$. As we can consider the same reasoning for any $q \in \mathcal{U}$, we obtain identity (4.5). The reciprocal is proved by the same reasoning. ∎

Lemma 4.4 *We consider a system* (4.1) *and* $\gamma(t)$ *a periodic orbit of period* $T > 0$. *We assume that* $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ *is an invariant curve with* $\gamma \subseteq \{(x, y) \,|\, f(x, y) = 0\}$ *and that* $k(x, y)$ *is the* $\mathcal{C}^1$ *function given in* (1.2), *that is, the cofactor of the invariant curve* $f(x, y) = 0$. *We take any* p_0 *in* γ. *Then,*

$$\nabla f(p_0) \cdot D\mathcal{P}(p_0) = \exp\left(\int_0^T k(\gamma(t))\, dt\right) \nabla f(p_0). \tag{4.6}$$

Proof. We consider the Poincaré map defined in an interval of the straight line Σ containing p_0, $\mathcal{P}(q) = \Phi_{\tau(q)}(q)$. Since $f(x, y) = 0$ is an invariant curve defined in $\mathcal{U} \subseteq \mathbb{R}^2$, it is clear that for any $q \in \mathcal{U}$ and any $t \in \mathbb{R}$ such that $\Phi_t(q) \in \mathcal{U}$, identity (4.5) is satisfied as proved in Lemma 4.3. Hence,

$$f(\mathcal{P}(q)) = f(q) \exp\left(\int_0^{\tau(q)} k(\Phi_s(q))\, ds\right),$$

and differentiating this identity with respect to q we get

$$\nabla f(\mathcal{P}(q)) \cdot D\mathcal{P}(q) = \exp\left(\int_0^{\tau(q)} k(\Phi_s(q))\, ds\right) \nabla f(q) \, + f(q)$$

$$\exp\left(\int_0^{\tau(q)} k(\Phi_s(q)) ds\right)\left[\int_0^{\tau(q)} (\nabla k)\,(\Phi_s(q)) \cdot D\Phi_s(q)\, ds + k(\mathcal{P}(q)) \nabla \tau(q)\right],$$

where $D\mathcal{P}(q)$ and $D\Phi_s(q)$ stand for the Jacobian matrix with respect to q of the functions $\mathcal{P}$ and Φ_s, respectively, in the point q.

We evaluate the previous identity in $q = p_0$, taking into account that $f(p_0) = 0$ and $\tau(p_0) = T$, and we get identity (4.6). ∎

Proof of Theorem 4.1. The vector $\nabla f(p_0)$ is a non-null vector that is normal to the vector $\mathbf{F}(p_0)$ since $f(x, y) = 0$ is an invariant curve that contains γ, and $p_0 \in \gamma$. The fact of $\nabla f(p_0)$ to be a non-null vector is ensured by the assumption that if $p \in \mathcal{U}$ is such that $f(p) = 0$ and $\nabla f(p) = 0$, then p is a singular point of system (4.1). Since p_0 belongs to a periodic orbit, it cannot be a singular point.

Therefore, the vector $\mathbf{v}$ in the identity (4.4) of Theorem 4.2 can be replaced by $\nabla f(p_0)$. Using the identity (4.6) of Lemma 4.4, we deduce that

$$\exp\left(\int_0^T \operatorname{div}(\gamma(t))\, dt\right) = \exp\left(\int_0^T k(\gamma(t))\, dt\right),$$

from which (4.3) follows. ∎

In this chapter we prove that if γ is a periodic orbit of system (4.1) with period $T > 0$ and such that $\gamma \subseteq \{(x,y) \,|\, f(x,y) = 0\}$ where $f(x,y) = 0$ is an invariant curve of the system, then $\int_0^T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)(\gamma(t))\, dt = \int_0^T k(\gamma(t))\, dt$. We always assume that if $p \in \mathcal{U}$ is such that $f(p) = 0$ and $\nabla f(p) = 0$, then p is a singular point.

We deduce that if γ is hyperbolic, then $\int_0^T k(\gamma(t))\, dt$ is different from zero. In a recent work of C. Christopher, J. Llibre and J.V. Pereira [35], this integral also appears. In that work, the authors study the meaning of the multiplicity for invariant algebraic curves in planar polynomial differential systems. In their Definition 3.2 they say that the invariant algebraic curve $f(x,y) = 0$ of a planar polynomial system has *integrable multiplicity* m with respect to the system if m is the largest integer for which the following is true: there are $m-1$ exponential factors $\exp\{g_j/f^j\}$, $j = 1, \ldots, m-1$, with $\deg g_j \leq j \deg f$ such that each g_i is not a multiple of f.

In their Proposition 6.7, they consider a planar polynomial differential system with a regular invariant algebraic curve $f(x,y) = 0$ with cofactor $k_f(x,y)$ and they show that given any path γ on $f = 0$, if $\int_\gamma k_f\, dt = 0$, then the multiplicity of the curve $f(x,y) = 0$ is at least two.

Therefore, using our result, in case that we have an algebraic limit cycle γ given by a regular invariant algebraic curve $f(x,y) = 0$, we deduce that if γ is not hyperbolic, then the system has an exponential factor of the form $F = \exp\{h/f\}$.

The major part of the contents of this chapter (and Section 5.2 of Chapter 5) belong to the paper entitled *On the stability of limit cycles for planar differential systems*, authored by H. Giacomini and M. Grau and which is a preprint, 2004.

Abstract. We consider a planar differential system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, where P and Q are $\mathcal{C}^1$ functions in some open set $\mathcal{U} \subseteq \mathbb{R}^2$, and $\dot{} = \frac{d}{dt}$. Let γ be a periodic orbit of the system in $\mathcal{U}$. Let $f(x,y) : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that

$$P(x,y)\,\frac{\partial f}{\partial x}(x,y) \,+\, Q(x,y)\,\frac{\partial f}{\partial y}(x,y) \,=\, k(x,y)\,f(x,y),$$

where $k(x,y)$ is a $\mathcal{C}^1$ function in $\mathcal{U}$ and $\gamma \subseteq \{(x,y) \,|\, f(x,y) = 0\}$. We assume that if $p \in \mathcal{U}$ is such that $f(p) = 0$ and $\nabla f(p) = 0$, then p is a singular point.
We prove that $\int_0^T \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) (\gamma(t))\, dt = \int_0^T k(\gamma(t))\, dt$, where $T > 0$ is the period of γ. As an application, we take profit from this equality to show the hyperbolicity of the known algebraic limit cycles of quadratic systems.

Chapter 5

Two features of the known quadratic systems with an algebraic limit cycle

Our purpose in this chapter is to apply the results given in Chapters 3 and 4 for some distinguished systems, in this case the quadratic systems with an algebraic limit cycle. By the way, we show two features of these systems which are interesting by themselves. We will show that none of these quadratic systems has a Liouvillian first integral and that all these limit cycles are hyperbolic.

We consider all the families of quadratic systems with an algebraic limit cycle defined by polynomials of degrees 2 and 4. It is shown in [42, 43, 44] that there are no algebraic limit cycles of degree 3 for a quadratic system. For a shorter proof of this fact, see [17, 26]. All the families we consider are given in [27]. In [36], two examples of quadratic systems with an algebraic limit cycle of degree 5 and 6 are described. These examples are birrationally equivalent to one of the examples given in [27] and, since birrational transformations do not change the Liouvillian integrability of a system or the hyperbolic character of a limit cycle, we are also considering these examples in our results.

We first describe all the families of quadratic systems with algebraic limit cycles known by the literature.

The following result is due to Ch'in Yuan-shün [30] and characterizes the algebraic limit cycles of degree 2 for a quadratic system.

Theorem 5.1 [30] *If a quadratic system has an algebraic limit cycle of degree 2, then after an affine change of variables, the limit cycle becomes the circle*

$$\Gamma := x^2 + y^2 - 1 = 0. \tag{5.1}$$

Moreover, Γ is the unique limit cycle of the quadratic system which can be written in the form

$$\begin{array}{rcl} \dot{x} & = & -y\,(ax + by + c) - (x^2 + y^2 - 1), \\ \dot{y} & = & x\,(ax + by + c), \end{array} \tag{5.2}$$

with $a \neq 0$, $c^2 + 4(b+1) > 0$ and $c^2 > a^2 + b^2$.

In [42, 43, 44], Evdokimenko proves that there are no quadratic systems having limit cycles of degree 3. An easier proof of this fact can be found in the works [17, 26].

The study of algebraic limit cycles of degree 4 was initiated by A.I. Yablonskii, who found the first family, see [94] and followed by V.F. Filipstov, see [46], who found another family affine-independent of the previous one. A third family was found by J. Chavarriga [16] and in the work by J. Chavarriga, J. Llibre and J. Sorolla [27] a fourth family is found and it is proved that any quadratic system with an algebraic limit cycle of degree 4 is affine-equivalent to one of the four encountered families. The fact that the algebraic limit cycle for these four families of quadratic systems does not coexist with any other limit cycle is proved by J. Chavarriga, H. Giacomini and J. Llibre in [22].

We summarize these four families of algebraic limit cycles for quadratic systems in the following result.

Theorem 5.2 [27] *After an affine change of variables the only quadratic systems having an algebraic limit cycle of degree 4 are*

(a) *Yablonskii's system*

$$\begin{array}{rcl} \dot{x} & = & -4abcx - (a + b)y + 3(a + b)cx^2 + 4xy, \\ \dot{y} & = & (a + b)abx - 4abcy + (4abc^2 - \frac{3}{2}(a + b)^2 + 4ab)x^2 \\ & & +8(a + b)cxy + 8y^2, \end{array} \tag{5.3}$$

with $abc \neq 0$, $a \neq b$, $ab > 0$ and $4c^2(a-b)^2 + (3a-b)(a-3b) < 0$. This system has the invariant algebraic curve

$$(y + c\,x^2)^2 + x^2\,(x - a)(x - b) = 0, \tag{5.4}$$

whose oval is a limit cycle for system (5.3).

(b) *Filipstov's system*

$$\begin{aligned} \dot{x} &= 6\,(1+a)\,x + 2\,y - 6\,(2+a)\,x^2 + 12\,x\,y, \\ \dot{y} &= 15\,(1+a)\,y + 3\,a\,(1+a)\,x^2 - 2\,(9+5\,a)\,x\,y + 16\,y^2, \end{aligned} \tag{5.5}$$

with $0 < a < \frac{3}{13}$*. This system has the invariant algebraic curve*

$$3(1+a)(a\,x^2+y)^2 + 2\,y^2(2\,y - 3(1+a)x) = 0, \tag{5.6}$$

whose oval is a limit cycle for system (5.5)*.*

(c) *Chavarriga's system*

$$\begin{aligned} \dot{x} &= 5\,x + 6\,x^2 + 4(1+a)\,x\,y + a\,y^2, \\ \dot{y} &= x + 2\,y + 4\,x\,y + (2+3\,a)\,y^2, \end{aligned} \tag{5.7}$$

with $\frac{-71+17\sqrt{17}}{32} < a < 0$ *has the invariant algebraic curve*

$$x^2 + x^3 + x^2\,y + 2\,a\,x\,y^2 + 2\,a\,x\,y^3 + a^2\,y^4 = 0, \tag{5.8}$$

whose oval is a limit cycle for system (5.7)*.*

(d) *Chavarriga, Llibre and Sorolla's system*

$$\begin{aligned} \dot{x} &= 2\,(1 + 2\,x - 2\,a\,x^2 + 6\,x\,y), \\ \dot{y} &= 8 - 3\,a - 14\,a\,x - 2\,a\,x\,y - 8\,y^2, \end{aligned} \tag{5.9}$$

with $0 < a < \frac{1}{4}$ *has the invariant algebraic curve*

$$\frac{1}{4} + x - x^2 + a\,x^3 + x\,y + x^2\,y^2 = 0, \tag{5.10}$$

whose oval is a limit cycle for system (5.9)*.*

Furthermore, in [27] it is proved that the curve (5.4) has genus 0 and the curves (5.6), (5.8) and (5.10) have genus 1. We recall that the *genus* $\mathcal{G}$ of an algebraic curve is defined by $\mathcal{G} = \frac{1}{2}(n-1)(n-2) - (\delta + \kappa)$ where n is the degree of the irreducible polynomial which defines the curve, δ is the number of *nodes* (ordinary double points) and κ is the number of *cusps* (double points such that the tangent vector reverses sign as the curve is transversed). See [48, 92] for further information on planar algebraic curves.

In a work due to C. Christopher, J. Llibre and G. Świrszcz [36] two families of quadratic systems with an algebraic limit cycle of degrees five and six, respectively, are given. These two families are constructed by means of a birrational transformation of system (5.9). As defined in the introductory Chapter 1, a *birrational transformation* is a rational change of variables such that its inverse is also rational. Moreover, they prove that there is also a birrational transformation which converts Yablonskii's system (5.3) into the system with a limit cycle of degree 2, (5.2). At the time of the composition of this thesis, no other algebraic limit cycles for quadratic systems are known.

5.1 The first integral is not Liouvillian

In order to prove that none of the aforementioned quadratic systems with an algebraic limit cycle has a Liouvillian first integral, we only need to study the integrability of systems (5.2), (5.5), (5.7) and (5.9). All the other known cases are birrationally equivalent to one of these ones and if one of them has a Liouvillian first integral, then the birrational transformation of this first integral is a Liouvillian first integral for the transformed system. Therefore, since we will show that the systems (5.2), (5.5), (5.7) and (5.9) do not have a Liouvillian first integral, we conclude that none of the quadratic systems with an algebraic limit cycle, known until the moment of composition of this thesis, has a Liouvillian first integral.

We first state and proof the following theorem, which is a consequence of the results stated in Chapter 3.

Theorem 5.3 *Each one of the systems* (5.2), (5.3), (5.5), (5.7) *and* (5.9) *has only one invariant algebraic curve, when the limit cycle exists.*

Theorem 5.3 is proved using analogous reasonings for each system (5.2), (5.5), (5.7) and (5.9). Since system (5.3) is birrationally equivalent to system (5.2), then we do not need to study it.

The computation of the coordinates and nature of each singular point and the study of all the possibilities due to Theorems 3.8, 3.9 and 3.10 is easy but long. We only explicit these computations for systems (5.2) and (5.5), which exhaustively show all the encountered tricks.

Proof [for system (5.5)]. We first list all the singular points (finite and infinite) and the type they belong to, depending on their ratio of eigenvalues.

The singular point p_i has associated eigenvalues λ_i and μ_i, $i = 1, 2, \ldots, 7$. We always assume that the parameter of the system belongs to the interval in which the limit cycle exists, that is, $0 < a < \frac{3}{13}$. We also point out that whether $f_0(p_i) = 0$ or not, where $f_0 = 0$ is the algebraic curve given in (5.6). In case $f_0(p_i) = 0$ we give the value of $k_0(p_i)$, where $k_0(x, y)$ is the cofactor of $f_0(x, y) = 0$. For the singular points at infinity ($Z_0 = 0$), we notice that all of them have the coordinate $Y_0 \neq 0$, so local coordinates (u, v), where $u = X/Y$ and $v = Z/Y$, are taken at these points. The system in coordinates (u, v) is:

$$\begin{aligned} \dot{u} &= -4\,u + 2\,v + 2\,(2a+3)\,u^2 - 9(1+a)\,uv - 3\,a\,(1+a)u^3, \\ \dot{v} &= v(-16 + 2\,(9+5a)\,u - 15\,(1+a)\,v - 3\,a\,(1+a)\,u^2). \end{aligned} \tag{5.11}$$

We denote by $k_v(u, v) = -16 + 2\,(9+5a)\,u - 15\,(1+a)\,v - 3\,a\,(1+a)\,u^2$ the cofactor of the invariant straight line $v = 0$. The curve $f_0 = 0$ in these coordinates is given by $f_0(u, v) = 3\,a^2\,(1+a)\,u^4 + 6\,a\,(1+a)\,u^2 v + 3\,(1+a)\,v^2 - 6\,(1+a)\,uv + 4\,v$ and its cofactor is $k_0(u, v) = -2(8 - 4\,(3+2a)\,u + 15\,(1+a)\,v + 6\,a\,(1+a)\,u^2)$.

We list all the singular points of equation (5.5):

- $p_1 = [0, 0, 1]$ is a node point with $\lambda_1 = 6(1+a)$, $\mu_1 = 15(1+a)$, $f_0(p_1) = 0$ and $k_0(p_1) = 2\mu_1$.
- $p_2 = [20(1+a), 15(1+a)^2, 8(1-a)]$ is a focus point with $f_0(p_2) \neq 0$.
- $p_3 = [2(a - w_1), -18 - 33a - 16a^2 - (3+2a)w_1, 24(a+1)]$ is a node point with $f_0(p_3) = 0$ and $k_0(p_3) = \lambda_3$.
- $p_4 = [2(a + w_1), -18 - 33a - 16a^2 + (3+2a)w_1, 24(a+1)]$ is a saddle point with $f_0(p_4) = 0$ and $k_0(p_4) = \lambda_4$.
- $p_5 = [0, 1, 0]$ is a node point with $f_0(p_5) = 0$ and $k_0(p_5) = k_v(p_5) = -16$.
- $p_6 = [3 + 2a + w_2, 3a(a+1), 0]$ is a saddle point with $f_0(p_6) \neq 0$ and $k_v(p_6) = \lambda_6$.
- $p_7 = [3 + 2a - w_2, 3a(a+1), 0]$ is a saddle point with $f_0(p_7) \neq 0$ and $k_v(p_7) = \lambda_7$.

We have used the notation $w_1 = \sqrt{36 + 72a + 37a^2}$ and $w_2 = \sqrt{9 - 8a^2}$.

Assume that there is another invariant algebraic curve $f(x, y) = 0$ with $f(x, y) \in \mathbb{R}[x, y]$ which we suppose to be irreducible in $\mathbb{R}[x, y]$ and the polynomials $f_0(x, y)$ and $f(x, y)$ relatively coprime. We denote by $k(x, y) :=$

$k_{00}+k_{10}x+k_{01}y$ its cofactor. Let n be the degree of the polynomial $f(x,y)$, then its corresponding cofactor for the system with coordinates (u,v) is $k(u,v)=k_{01}-16n+(k_{10}+2(9+5a)n)u+(k_{00}-15(1+a)n)v-3a(1+a)nu^2$.

Let us consider the singular points for which the ratio of the eigenvalues is either rationally independent or negative, that is, p_2, p_4, p_6 and p_7. Let us consider one of these points, say p_i with $i=2,4,6,7$. The invariant algebraic curve $f=0$ either satisfies $f(p_i)\neq 0$ or $f(p_i)=0$. If $f(p_i)\neq 0$ then $k(p_i)=0$ by Lemma 3.5. On the other hand, if $f(p_2)=0$ then $k(p_2)=\lambda_2+\mu_2$ by Lemma 3.11. If $f(p_4)=0$, then $k(p_4)=\mu_4$ because we have that $f_0(p_4)=0$ and $k_0(p_4)=\lambda_4$, and we apply Theorems 3.8, 3.9 and 3.10. Analogously, if $f(p_6)=0$ then $k(p_6)=\mu_6$ because $k_v(p_6)=\lambda_6$, and if $f(p_7)=0$ then $k(p_7)=\mu_7$ because $k_v(p_7)=\lambda_7$.

We codify these conditions by the following equations $k(p_2)=\epsilon_2(\lambda_2+\mu_2)$, $k(p_4)=\epsilon_4\mu_4$, $k(p_6)=\epsilon_6\mu_6$ and $k(p_7)=\epsilon_7\mu_7$, where $\epsilon_i\in\{0,1\}$, $i=2,4,6,7$. These equations give a total of sixteen cases to study.

We solve these four linear equations for k_{00}, k_{10}, k_{01} and n and we get:

$$\begin{aligned} n \;=\; & \frac{1}{6(a^2-1)(9+11a)}\Big[(-18+48a+118a^2+52a^3+(26a^2+20a \\ & -6)\,w_1)\,\epsilon_2+(72+72a-70a^2-74a^3+(24+2a-26a^2)\,w_1)\,\epsilon_4 \\ & +(-27a-51a^2-22a^3+(18+49a+33a^2)\,w_2-(9+11a)\,w_1 \\ & +(9+11a)\,w_1\,w_2)\,\epsilon_6+(-27a-51a^2-22a^3 \\ & -(18+49a+33a^2)\,w_2-(9+11a)\,w_1-(9+11a)\,w_1\,w_2)\,\epsilon_7\Big]\,. \end{aligned}$$

For each one of the cases $\epsilon_i\in\{0,1\}$, $i=2,4,6,7$, we have that n is an algebraic function of the parameter a. This algebraic function can be studied numerically without loss of precision, due to its simplicity. This study for each value of ϵ_i gives that there is no natural number in the range of the function $n(a)$ when $0<a<3/13$, (except in two cases which will be carefully remarked) and, hence, no invariant algebraic curve $f(x,y)=0$, different from $f_0(x,y)=0$, can exist. However, we give a rigorous algebraic proof of this fact, which can be obtained from the characteristics of the function $n(a)$.

We differentiate the function $n(a)$ with respect to a and we get a rational function of the form

$$\frac{\partial n}{\partial a}=\frac{\alpha_0(a)+\alpha_1(a)w_1+\alpha_2(a)w_2+\alpha_3(a)w_1w_2}{(a^2-1)^2(9+11a)^2(8a^2-9)(37a^2+72a+36)},$$

where $\alpha_i(a)$ are polynomials in a. We notice that the denominator of this expression is strictly negative for $a\in(0,3/13)$. Let us consider the lowest degree polynomial in a which has the numerator as a factor. We denote it by $N(a)$. We encounter that $N(a)$ is a polynomial of degree 16 in a. We compute it for any of the possible values $\epsilon_i \in \{0,1\}$, $i = 2,4,6,7$, and we prove that $N(a)$ has a strictly defined sign except when $\epsilon_i = 1$ for all i or when $\epsilon_i = 0$ for all i.

In the other fourteen cases, $N(a)$ is a polynomial with definite sign in the interval $a \in (0, 3/13)$, which can be easily seen by Sturm's algorithm. For instance, when $\{\epsilon_2, \epsilon_4, \epsilon_6, \epsilon_7\} = \{1,1,1,0\}$, we get

$$\begin{aligned} N(a) &= (a-1)^4\,(a+1)^2\,(59049 + 389286\,a + 1048059\,a^2 \\ &\quad + 1479654\,a^3 + 1158795\,a^4 + 478608\,a^5 + 81649\,a^6), \end{aligned}$$

which is obviously strictly positive when $a \in (0, 3/13)$.

Therefore, the function n is a strictly increasing or decreasing function of a. We compute its value in $a = 0$ and $a = 3/13$ for the fourteen cases and we deduce that there is no natural number in the range of this function except for two cases which correspond to $\{\epsilon_2, \epsilon_4, \epsilon_6, \epsilon_7\} = \{0,1,0,1\}$ and $\{\epsilon_2, \epsilon_4, \epsilon_6, \epsilon_7\} = \{0,0,0,1\}$. Let us particularly study these cases.

In the case $\{\epsilon_2, \epsilon_4, \epsilon_6, \epsilon_7\} = \{0,1,0,1\}$, the function $n(a)$ can be seen to be strictly increasing in the interval $a \in (0, 3/13)$ by the described method. We have $n(0) = 1$ and $n(3/13) = -73 - 13\sqrt{41} + 7\sqrt{161} + 3\sqrt{6601} \simeq 2.754985$. So, there is a value $a^* \in (0, 3/13)$ such that $n(a^*) = 2$. We notice that we have computed the cofactor for this case. Straightforward computations show that system (5.5) has no invariant conic. These easy computations correspond to a linear system of equations on the coefficients of a conic.

In the case $\{\epsilon_2, \epsilon_4, \epsilon_6, \epsilon_7\} = \{0,0,0,1\}$, the function $n(a)$ can be seen to be strictly increasing in the interval $a \in (0, 3/13)$, $n(0) = 5$ and $n(3/13) = (9+13\sqrt{41}+7\sqrt{161}+3\sqrt{6601})/64 \simeq 6.6375$. So, there is a value $a_1^* \in (0, 3/13)$ such that $n(a_1^*) = 6$. Let us consider the singular point p_3 which is a node point with $\lambda_3 < \mu_3 < 0$. We deduce the equation $k(p_3) = \epsilon_3 \mu_3 + (s_3 - \epsilon_3)\lambda_3$, where $\epsilon_3 \in \{0,1\}$ and s_3 is an integer number with $s_3 \geq \epsilon_3$. We notice that we have the cofactor $k(x,y)$ and, once evaluated in p_3, we have a condition on ϵ_3, s_3 and a. When $\epsilon_3 = 0$, we can compute $s_3(a)$ from this equation and we have that for $a \in (0, 3/13)$ there is no integer number in the range on this function. When $\epsilon_3 = 1$, we compute $s_3(a)$ from this equation and we have that there is an $a_2^* \in (0, 3/13)$ for which $s(a_2^*) = 3$. An easy computation using resultants shows that $a_1^* \neq a_2^*$.

We consider the case when $\epsilon_i = 1$ for $i = 2,4,6,7$, then $N(a) \equiv 0$ and $n \equiv -1$. So, no invariant algebraic curve $f(x,y) = 0$ can exist in this case.

In case $\epsilon_i = 0$ for $i = 2, 4, 6, 7$, we have that $N(a) \equiv 0$ and $n \equiv 0$. So, no invariant algebraic curve $f(x, y) = 0$ can exist in this case either.

The non-existence of the invariant algebraic curve $f(x, y) = 0$ has been shown in the sixteen possible cases given by $\epsilon_i \in \{0, 1\}$, $i = 2, 4, 6, 7$. ■

Proof [for system (5.2)]. We consider system (5.2) with the invariant algebraic curve $f_0(x, y) = 0$ defined by $f_0(x, y) := x^2 + y^2 - 1$ and with cofactor $k_0(x, y) = -2x$. System (5.2) depends on three parameters a, b and c which satisfy $a \neq 0$, $c^2 + 4(b + 1) > 0$ and $c^2 > a^2 + b^2$ for the existence of the limit cycle. In order to show that there is no other invariant algebraic curve, different from $f_0(x, y) = 0$, we study two cases in which the nature of the singular points changes, that is, $b < -1$ and $b \geq -1$. We do not need to study all the singular points since only some of them are used. As before, the singular points at the line of infinity satisfy that $Y_0 \neq 0$ so we take local coordinates $u = X/Y$ and $v = Z/Y$ at these points. System (5.2) with local coordinates (u, v) is

$$\begin{array}{rcl} \dot{u} & = & -(1+b) - au - cv - (1+b)u^2 + v^2 - au^3 - cu^2v, \\ \dot{v} & = & -uv(b + au + cv). \end{array} \tag{5.12}$$

The cofactor of the invariant straight line $v = 0$ is $k_v(u, v) = -u(b+au+cv)$.

Assume $b < -1$ and consider the following three singular points, which correspond to the three singular points at the line of infinity.

- $p_5 = [-(1 + b), a, 0]$, with eigenvalues $\lambda_5 = -(1 + b)/a$ and $\mu_5 = -(a^2 + (1+b)^2)/a$, so it is a saddle point. At this point $f_0(p_5) \neq 0$ and $k_v(p_5) = \lambda_5$.

- $p_6 = [\mathbf{i}, 1, 0]$ with eigenvalues $\lambda_6 = 2(a-(b+1)\mathbf{i})$ and $\mu_6 = a-b\mathbf{i}$, so the ratio λ_6/μ_6 is not a real number. At this point $f_0(p_6) = 0$, $k_0(p_6) = \lambda_6$ and $k_v(p_6) = \mu_6$.

- $p_7 = [-\mathbf{i}, 1, 0]$ with eigenvalues $\lambda_7 = 2(a + (b + 1)\mathbf{i})$ and $\mu_7 = a + b\mathbf{i}$, so the ratio λ_7/μ_7 is not a real number. At this point $f_0(p_7) = 0$, $k_0(p_7) = \lambda_7$ and $k_v(p_7) = \mu_7$.

Let $f(x, y) = 0$ be an invariant algebraic curve of degree n of system (5.2) with cofactor $k(x, y) := k_{00} + k_{10}x + k_{01}y$ such that the polynomials $f(x, y)$ and $f_0(x, y)$ are relatively coprime, with $f(x, y) \in \mathbb{R}[x, y]$ and irreducible in $\mathbb{R}[x, y]$. The cofactor of this irreducible curve in local coordinates (u, v)

is $k(u,v) = k_{01} + (k_{10} - bn)u + k_{00}v - anu^2 - cnuv$. By Theorem 3.10 we have that the only possibility is $f(p_6) \neq 0$ and $f(p_7) \neq 0$. Therefore, by Lemma 3.5 we have that $k(p_6) = k(p_7) = 0$. In addition, the intersection of the curve $f = 0$ and the invariant straight line $v = 0$ must be one of the singular points, and the only possibility is that $f(p_5) = 0$. Since p_5 is a saddle point and $k_v(p_5) = \lambda_5$, by Theorems 3.8, 3.9 and 3.10, we get the equation $k(p_5) = \mu_5$. The combination of these three equations gives $n = 1$ and straightforward computations show that there is no invariant straight line for this system.

Assume that $b \geq -1$ and consider the same three singular points at infinity. But, p_5 is a node point in this case. Let us now consider the following two finite singular points.

- $p_3 = [-ac + b\sqrt{a^2+b^2-c^2}, -bc - a\sqrt{a^2+b^2-c^2}, a^2+b^2]$ which is a complex point with eigenvalues $\lambda_3 = 2(ac - b\sqrt{a^2+b^2-c^2})/(a^2+b^2)$ and $\mu_3 = \sqrt{a^2+b^2-c^2}$, whose ratio is never a real number. At this point $f_0(p_3) = 0$ and $k_0(p_3) = \lambda_3$.

- $p_4 = [-ac - b\sqrt{a^2+b^2-c^2}, -bc + a\sqrt{a^2+b^2-c^2}, a^2+b^2]$ which is a complex point with eigenvalues $\lambda_4 = 2(ac + b\sqrt{a^2+b^2-c^2})/(a^2+b^2)$ and $\mu_4 = -\sqrt{a^2+b^2-c^2}$, whose ratio is never a real number. At this point $f_0(p_4) = 0$ and $k_0(p_4) = \lambda_4$.

Assume that there is another invariant algebraic curve $f(x,y) = 0$ with cofactor $k(x,y)$ as in the previous case. As before, we have that $k(p_6) = k(p_7) = 0$. Since the polynomial $f(x,y)$ is assumed to be real, the behavior of the curve $f(x,y) = 0$ must coincide at the points p_3 and p_4. So, either $f(p_3) \neq 0$ and $f(p_4) \neq 0$ and then $k(p_3) = k(p_4) = 0$ or $f(p_3) = f(p_4) = 0$ and then $k(p_3) = \mu_3$ and $k(p_4) = \mu_4$. This condition can be codified by $k(p_3) = \epsilon_3\mu_3$ and $k(p_4) = \epsilon_3\mu_4$, with $\epsilon_3 = \{0,1\}$.

If $\epsilon_3 = 0$, the resolution of these four linear equations on the coefficients of $k(x,y)$ and on n give that $n = 0$, so no other invariant algebraic curve exists in this case.

If $\epsilon_3 = 1$, the resolution of these four linear equations give $n = 1$ and easy computations show that there is no invariant straight lines for system (5.2).

■

Theorem 5.3 is crucial in order to prove the following result. In view of Theorem 1.7 the information relating a polynomial system with its invariant algebraic curves and exponential factors gives the Liouvillian or non Liouvillian integrability of the system.

Theorem 5.4 *None of the systems* (5.2)*,* (5.3)*,* (5.5)*,* (5.7) *and* (5.9) *has a Liouvillian first integral.*

Proof. Theorem 5.3 states that each of these systems only has one invariant algebraic curve, namely $f_0(x,y)=0$. Assume that there is a Liouvillian first integral, then, by Theorem 1.7, we have that it has an inverse integrating factor of the form

$$V(x,y)=\exp\left\{\frac{h(x,y)}{f_0(x,y)^n}\right\} f_0(x,y)^c,$$

where $h(x,y)\in\mathbb{R}[x,y]$, $n\in\mathbb{N}$ and $c\in\mathbb{R}$. We eventually may have that $h(x,y)$ is constant and/or $n=0$ and/or $c=0$. The form of this inverse integrating factor is given by Theorem 1.7 and the fact that the only invariant algebraic curve of the system is $f_0(x,y)=0$. All these systems have a strong focus point in the region bounded by the limit cycle, namely p. For any of them it is easy to see that $f_0(p)\neq 0$, for the values of the parameters in which the limit cycle exists. Then, $V(p)\neq 0$ and, hence, the first integral constructed by this inverse integrating factor is continuous in a neighborhood of this focus point p. But, a first integral cannot be continuous in a neighborhood of a focus point without being constant on all the neighborhood. We have, then, a contradiction and we deduce that no such Liouvillian first integral can exist. ∎

5.2 Hyperbolicity of the limit cycles

The fact of the limit cycle of degree 2 being hyperbolic is stated in [95] (see pages 256–258) following the proof of [30]. As a consequence, and taking into account the forthcoming Lemma 5.6, one of the limit cycles of degree 4 (the one due to Yablonskii) is also hyperbolic, because this limit cycle of degree 4 is birrationally equivalent to the one of degree 2, as it is shown in [36]. Our contribution is the proof of the hyperbolicity of the other known limit cycles of quadratic systems.

The following lemmas show that the birrational transformations do not change the hyperbolicity character of a limit cycle. In the same way as in

Chapter 4, we state the results for planar differential systems defined in the class of $\mathcal{C}^1(\mathcal{U})$ functions, where $\mathcal{U} \subseteq \mathbb{R}^2$ is some open set, like system (4.1). Nevertheless, we will use them only for polynomial systems like (1.1), which are a particular case.

Lemma 5.5 *We consider a differential system* (4.1) *and a change of variables $x = F(u, v)$ and $y = G(u, v)$, where F, G are $\mathcal{C}^2$ functions in $\mathcal{U}$. We denote by $\dot{u} = R(u, v)$, $\dot{v} = S(u, v)$ the transformed differential system. Let*

$$J(u,v) := \frac{\partial F}{\partial u}(u,v)\,\frac{\partial G}{\partial v}(u,v)\;-\;\frac{\partial F}{\partial v}(u,v)\,\frac{\partial G}{\partial u}(u,v),$$

be the jacobian of the transformation. Then,

$$\begin{aligned}\frac{\partial P}{\partial x}(F(u,v),G(u,v))\;+\;\frac{\partial Q}{\partial y}(F(u,v),G(u,v))\;=\;\frac{\partial R}{\partial u}(u,v)\;+\;\frac{\partial S}{\partial v}(u,v)+\\ +\frac{1}{J(u,v)}\left(\frac{\partial J}{\partial u}(u,v)\,R(u,v)\;+\;\frac{\partial J}{\partial v}(u,v)\,S(u,v)\right).\end{aligned} \tag{5.13}$$

Lemma 5.5 is a computational result whose proof is clear after some easy manipulations. We use it to prove the following result which states that the value of the integral of the divergence on the limit cycle does not change under transformations of dependent variables.

Lemma 5.6 *We consider a differential system* (4.1) *with a periodic orbit γ of period $T > 0$ and a change of variables $x = F(u, v)$ and $y = G(u, v)$ which is well-defined in a neighborhood of γ. We denote by $\dot{u} = R(u, v)$, $\dot{v} = S(u, v)$ the transformed differential system and by ϑ the corresponding periodic orbit. Then,*

$$\int_0^T \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)(\gamma(t))\,dt = \int_0^T \left(\frac{\partial R}{\partial u}+\frac{\partial S}{\partial v}\right)(\vartheta(t))\,dt.$$

Proof. Using the same notation as in Lemma 5.5, we have that the integral $\int_0^T \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)(\gamma(t))\,dt$ becomes, under the transformation of dependent variables $x = F(u, v)$ and $y = G(u, v)$,

$$\int_0^T \left(\frac{\partial P}{\partial x}(F(u,v),G(u,v))+\frac{\partial Q}{\partial y}(F(u,v),G(u,v))\right)(\vartheta(t))\,dt$$

which, by Lemma 5.5, equals to:

$$\int_0^T \left(\frac{\partial R}{\partial u}(u,v) + \frac{\partial S}{\partial v}(u,v) \right) (\vartheta(t))\, dt + \\ + \int_0^T \frac{1}{J(u,v)} \left(\frac{\partial J}{\partial u}(u,v) R(u,v) + \frac{\partial J}{\partial v}(u,v) S(u,v) \right) (\vartheta(t))\, dt.$$

We notice that the integrand of the second integral in the former expression can be rewritten as $d(J(u,v))/J(u,v)$ and, since the change of variables is well defined in a neighborhood of γ, we have that this expression is a well defined, exact 1-form which is integrated over the closed curve ϑ, so $\oint_\vartheta d(J(u,v))/J(u,v) = 0$. ■

Therefore, in order to prove that all these families of limit cycles are hyperbolic, we only need to study the stability of the limit cycles of systems (5.5), (5.7) and (5.9). The hyperbolicity of the two limit cycles described in [36] is shown by the fact that they are birrationally equivalent to (5.9).

Theorem 5.7 *Each one of the limit cycles of systems* (5.5), (5.7) *and* (5.9) *is hyperbolic.*

Proof. In order to prove the hyperbolicity of the limit cycles of systems (5.5), (5.7) and (5.9) we use the same process for all of them. These systems depend on a parameter a which belong to a certain open interval when the limit cycle γ exists. Hence, the integral of the divergence of the system on the limit cycle is a function of this parameter a, which we denote by $\mathcal{D}(a)$. We denote by T the period of the limit cycle. Moreover, using Theorem 4.1, we have that:

$$\mathcal{D}(a) = \int_0^T \mathrm{div}(\gamma(t))\, dt + w \left(\int_0^T \mathrm{div}(\gamma(t))\, dt - \int_0^T k(\gamma(t))\, dt \right),$$

where k is the cofactor of the invariant algebraic curve containing the limit cycle and w is any real number.

We show that the function $\mathcal{D}(a)$ has no zero when a belongs to the interval of existence of limit cycle by choosing an adequate $w \in \mathbb{R}$ and parameterizing the limit cycle γ. We notice that other parameterizations may make the computations easier or more difficult, but we are not concerned with this fact. The way of choosing the adequate value of w is purely heuristic, although we expect that this choice is related with some geometric property.

Hyperbolicity of the limit cycle given by the algebraic curve (5.10) *for system* (5.9).
The stability of the limit cycle γ, as stated in Theorem 1.9, is given by the following function of the parameter a of the system, $\mathcal{D}(a) := \int_0^T \text{div}(\gamma(t))\, dt$, where $\text{div}(x, y) = 2(2 - 5ax - 2y)$ is the divergence of system (5.9) and $T > 0$ the period of the limit cycle. By Theorem 4.1, we have that

$$\int_0^T \text{div}(\gamma(t))\, dt = \int_0^T k(\gamma(t))\, dt,$$

where $k(x, y) = 4(2 - 3ax + 2y)$ is the cofactor of the invariant algebraic curve (5.10). So, given any real number w, we have that:

$$\begin{aligned} \mathcal{D}(a) &= \int_0^T \text{div}(\gamma(t))\, dt + w \int_0^T (\text{div} - k)(\gamma(t))\, dt \\ &= \int_0^T ((1 + w)\text{div} - wk)\, (\gamma(t))\, dt. \end{aligned}$$

We consider the following parameterization of the oval of the algebraic curve (5.10):

$$x(\tau) = \tau, \quad y_{\pm}(\tau) = \frac{-1 \pm 2\sqrt{(-a)\tau(\tau - \tau_1)(\tau - \tau_2)}}{2\tau}, \tag{5.14}$$

where $\tau_1 = \frac{1-\sqrt{1-4a}}{2a}$, $\tau_2 = \frac{1+\sqrt{1-4a}}{2a}$ and the parameter $\tau \in (\tau_1, \tau_2)$. The positive sign $y_+(\tau)$ gives a half of the oval and the negative sign $y_-(\tau)$ the other half. One of the endpoints of both parameterizations is $(x_1, y_1) = (\frac{1-\sqrt{1-4a}}{2a}, -\frac{1+\sqrt{1-4a}}{4})$ and the other endpoint is $(x_2, y_2) = (\frac{1+\sqrt{1-4a}}{2a}, \frac{-1+\sqrt{1-4a}}{4})$. We have that the vector field in (x_1, y_1) is $(0, 6\sqrt{1 - 4a})$ and in (x_2, y_2) is $(0, -6\sqrt{1 - 4a})$, so the flow on the limit cycle is clockwise. The line $2ax = 1$ cuts the limit cycle in two points with ordinates $\pm\sqrt{\frac{1-4a}{2}} - a$, which are given respectively by $y_{\pm}(1/2a)$. We have the following relation between the differentials: $d\tau = P(x(\tau), y_{\pm}(\tau))\, dt$ where $P(x, y) = 2(1 + 2x - 2ax^2 + 6xy)$. Then,

$$\mathcal{D}(a) = \int_0^T ((1 + w)\, \text{div} - wk)\, (\gamma(t))\, dt$$

$$\begin{aligned} &= \int_{\tau_1}^{\tau_2}\left(\frac{((1+w)\,\mathrm{div}-wk)}{P}\right)(\tau,y_+(\tau))\,d\tau + \\ &\qquad + \int_{\tau_2}^{\tau_1}\left(\frac{((1+w)\,\mathrm{div}-wk)}{P}\right)(\tau,y_-(\tau))\,d\tau \\ &= \int_{\tau_1}^{\tau_2}\left[\left(\frac{((1+w)\,\mathrm{div}-wk)}{P}\right)(\tau,y_+(\tau)) \right. \\ &\qquad \left. - \left(\frac{((1+w)\,\mathrm{div}-wk)}{P}\right)(\tau,y_-(\tau))\right]\,d\tau. \end{aligned}$$

For $w=-3$ and substituting by the parameterization, we get,

$$\mathcal{D}(a) = 8\int_{\tau_1}^{\tau_2} \frac{\sqrt{a\tau(\tau-\tau_1)(\tau_2-\tau)}}{\tau(1+8\tau+a\tau^2)}\,d\tau.$$

Since $\tau_1>0$ and $\tau_2>\tau_1$ for any $a\in(0,1/4)$ and the integrand $\frac{\sqrt{a\tau(\tau-\tau_1)(\tau_2-\tau)}}{\tau(1+8\tau+a\tau^2)}$ is also strictly positive and well defined for any $\tau\in(\tau_1,\tau_2)$ and $a\in(0,1/4)$, we have that $\mathcal{D}(a)>0$ for all $a\in(0,1/4)$, which implies that the limit cycle in system (5.9) is hyperbolic (and unstable).

Hyperbolicity of the limit cycle given by the algebraic curve (5.8) *for system* (5.7).
As before, in order to determine the stability of the limit cycle γ, we consider the function $\mathcal{D}(a) := \int_0^T \mathrm{div}(\gamma(t))\,dt$, where $\mathrm{div}(x,y) = 7+16x+2(4+5a)y$ is the divergence of system (5.7). Using Theorem 4.1, we get that

$$\mathcal{D}(a) = \int_0^T ((1+w)\mathrm{div}-wk)\,(\gamma(t))\,dt$$

for any $w\in\mathbb{R}$ and where $k(x,y)=2(5+9x+(5+6a)y)$ is the cofactor of the invariant curve (5.8) for system (5.7). We define $g(a,\tau) := (1-\tau)^2 + 4a\tau^2(\tau+1)$ and the following parameterization for the oval of (5.8):

$$x_\pm(\tau) = \frac{1-\tau+2a\tau^2 \pm \sqrt{g(a,\tau)}}{2a\tau^3(2-a\tau)},\ \ y_\pm(\tau) = \frac{1-\tau+2a\tau^2 \pm \sqrt{g(a,\tau)}}{2a\tau^2(a\tau-2)},$$

where $\tau\in(\tau_1,\tau_2)$ with $\tau_1<\tau_2$ the two biggest roots of the polynomial $g(a,\tau)$ in τ. We notice that $g(a,0)=1>0$, $g(a,1)=8a<0$, $g(a,(3+\sqrt{17})/2) = 4(29+7\sqrt{17})(a-\frac{-71+17\sqrt{17}}{32})>0$ and the coefficient of degree 3 in τ for g is $4a < 0$, when a belongs to the interval $(\frac{-71+17\sqrt{17}}{32}, 0)$ of existence of limit cycle. Hence, the polynomial $g(a, \tau)$ has three distinct real roots $0 < \tau_0 < 1 < \tau_1 < (3 + \sqrt{17})/2 < \tau_2$ in τ for any $a \in (\frac{-71+17\sqrt{17}}{32}, 0)$.

In Figure 5.1, we show the values of a and τ of existence of limit cycle, this is, the pointed region R. The curve $g(a, \tau) = 0$ gives the value of both roots τ_1 and τ_2.

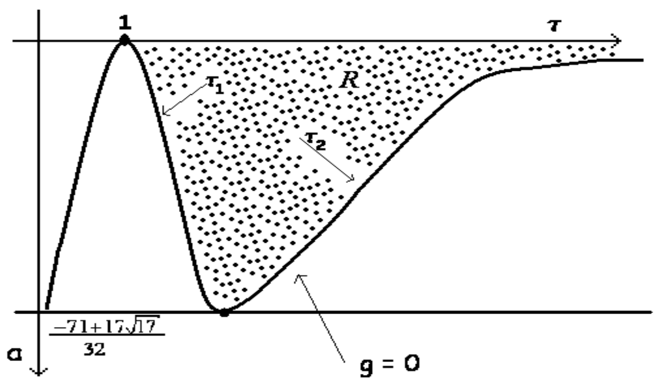

Figure 5.1: Roots τ_1 and τ_2 for $a \in (\frac{-71+17\sqrt{17}}{32}, 0)$.

When $a = \alpha(\tau)$ with $\alpha(\tau) = -\frac{(1-\tau)^2}{4\tau^2(1+\tau)}$, we have that $g(\alpha(\tau), \tau) \equiv 0$. That is, the function $\alpha(\tau)$ is defined implicitly by $g(\alpha(\tau), \tau) \equiv 0$. This function is the inverse of $\tau_1(a)$ and $\tau_2(a)$ for $a \in (\frac{-71+17\sqrt{17}}{32}, 0)$.

For this value of $a = \alpha(\tau)$, we have $x_\pm(\tau) = \frac{4(1+\tau)}{(\tau-1)(1+3\tau)}$, which is strictly decreasing in τ for $\tau > 1$ and $y_\pm(\tau) = \frac{-4\tau(1+\tau)}{(\tau-1)(1+3\tau)}$, which is strictly increasing in τ for $\tau > 1$, so, in the endpoints we have that $x_\pm(\tau_1) > x_\pm(\tau_2)$ and $y_\pm(\tau_1) < y_\pm(\tau_2)$. Moreover, $x_+(\tau) - x_-(\tau) = \frac{\sqrt{g}}{a\tau^3(2-a\tau)} < 0$, $y_+(\tau) - y_-(\tau) = \frac{\sqrt{g}}{a\tau^2(a\tau-2)} > 0$ for any $\tau \in (\tau_1, \tau_2)$. It is easy to see that the flow in a neighborhood of the focus point surrounded by the limit cycle is clockwise. We deduce the sense of the flow and the structure of the parameterization given in Figure 5.2.

In order to compute $\mathcal{D}(a)$, we have the following relation between the differentials $P(x_\pm(\tau), y_\pm(\tau))\, dt = x'_\pm(\tau)\, d\tau$, where $x'_\pm(\tau)$ denotes the derivative

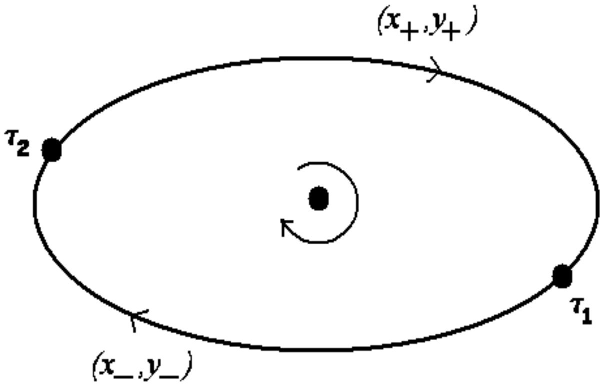

Figure 5.2: Flow and parameterization on the oval.

of the function $x_\pm(\tau)$ with respect to τ. Hence,

$$\begin{aligned}\mathcal{D}(a) = &\int_{\tau_1}^{\tau_2} \frac{(1+w)\,\mathrm{div} - wk}{P}(x_-(\tau), y_-(\tau))\; x_-'(\tau)\; d\tau \\ &+ \int_{\tau_2}^{\tau_1} \frac{(1+w)\,\mathrm{div} - wk}{P}(x_+(\tau), y_+(\tau))\; x_+'(\tau)\; d\tau \\ = &\int_{\tau_1}^{\tau_2} \left[\frac{2(w-8) - 2(13 - 5a + (1+a)w)\tau + (10 + 23a - 3aw)\tau^2}{\tau(1+\tau)(a\tau - 2)\sqrt{g}} \right. \\ &\left. - \frac{a(3w-7)\tau^3}{\tau(1+\tau)(a\tau-2)\sqrt{g}} \right] \; d\tau\,. \end{aligned} \tag{5.15}$$

This integrand is well-defined for any $\tau \in (\tau_1, \tau_2)$. It is not possible to find a value of w to make the integrand of definite sign in the interval (τ_1, τ_2).

As stated in [22], the limit cycle of system (5.7) is born in a Hopf bifurcation when $a = (-71 + 17\sqrt{17})/32$ and when a increases the algebraic limit cycle grows in size ending asymptotically at the curve $x^2(1 + x + y)$. By the form of (5.15), we have that $\mathcal{D}(a)$ is an analytic function in $a \in$

$[(-71+17\sqrt{17})/32, 0)$. When $a = \frac{-71+17\sqrt{17}}{32}$, the two biggest roots of $g(a,\tau) = 0$ are $\tau_1 = \tau_2 = (3+\sqrt{17})/2$ (the smallest root is $(7+\sqrt{17})/16$). The integrand in (5.15) for $\tau = (3+\sqrt{17})/2$ and $a = \frac{-71+17\sqrt{17}}{32}$ equals to zero, so $\mathcal{D}\left(\frac{-71+17\sqrt{17}}{32}\right) = 0$. We are going to prove that $\mathcal{D}(a)$ is a strictly decreasing function of a for $a \in (\frac{-71+17\sqrt{17}}{32}, 0)$, from which we deduce that $\mathcal{D}(a) < 0$ for the values of the parameter considered. Hence, the limit cycle is hyperbolic (and stable).

We only need to show that $\mathcal{D}'(a) < 0$ for $a \in (\frac{-71+17\sqrt{17}}{32}, 0)$ to be done. We compute the derivative of $\mathcal{D}(a)$ with respect to the parameter a as stated in [7, 8]. In particular we use formula (4) appearing in page 285 of Chapter 10 in [8]. We may consider the integral which gives $\mathcal{D}(a)$ as the integral of the rational function $\delta(a,\tau,v)$ on the curve $v^2 - g(a,\tau) = 0$, where

$$\begin{aligned}\delta(a,\tau,v) \;&=\; \frac{2(w-8) - 2(13-5a+(1+a)w)\tau + (10+23a-3aw)\tau^2}{2\tau(1+\tau)(a\tau-2)v} \\ &\quad - \frac{a(3w-7)\tau^3}{2\tau(1+\tau)(a\tau-2)v}.\end{aligned}$$

We have that:

$$\mathcal{D}'(a) = \oint_\gamma \left(\frac{\partial \delta}{\partial a}(a,\tau,v) + \frac{\partial \delta}{\partial v}(a,\tau,v)\, \frac{\partial v}{\partial a} \right) d\tau, \tag{5.16}$$

where this integral is done over the oval γ of the curve $v^2 - g(a,\tau) = 0$. The formula (4) in page 285 of [8] states that given a 1–form as $\delta(a,\tau,v)\,d\tau$ and an oval γ of a curve $f(a,\tau,v) = 0$, where a is a parameter, then

$$\frac{\partial}{\partial a} \oint_\gamma \delta(a,\tau,v)\,d\tau = \oint_\gamma \left(\frac{\partial \delta}{\partial a}(a,\tau,v)\,d\tau - \frac{\partial f}{\partial a}(a,\tau,v)\, \frac{d_{\tau,v}(\delta(a,\tau,v)\,d\tau)}{d_{\tau,v} f(a,\tau,v)} \right),$$

where $d_{\tau,v}$ means the differential with respect to τ and v and $\frac{d_{\tau,v}(\delta(a,\tau,v)\,d\tau)}{d_{\tau,v} f(a,\tau,v)}$ is the Gelfand-Leray form of $d_{\tau,v}(\delta(a,\tau,v)\,d\tau)$ with respect to $d_{\tau,v} f(a,\tau,v)$. We remark that the Gelfand-Leray form is not uniquely determined. The quotient $\frac{d_{\tau,v}(\delta(a,\tau,v)\,d\tau)}{d_{\tau,v} f(a,\tau,v)}$ reads for the class of 1-forms ϖ that satisfy $d_{\tau,v} f(a,\tau,v) \wedge \varpi = d_{\tau,v}(\delta(a,\tau,v)\,d\tau)$. Any representant ϖ of the class gives rise to the same value of the integral.

Since $\dfrac{\partial f}{\partial a} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial a} \equiv 0,$

$$d_{\tau,v}(\delta(a,\tau,v)\,d\tau) = \frac{\partial \delta}{\partial v}(a,\tau,v)\,dv \wedge d\tau, \quad d_{\tau,v} f = \frac{\partial f}{\partial \tau}\,d\tau + \frac{\partial f}{\partial v}\,dv$$

and

$$\left(\frac{\partial f}{\partial \tau}\,d\tau + \frac{\partial f}{\partial v}\,dv\right) \wedge \left(\frac{\partial \delta}{\partial v}\,\frac{\partial v}{\partial a}\,d\tau\right) = \frac{\partial \delta}{\partial v}\,\frac{\partial v}{\partial a}\,\frac{\partial f}{\partial v}\,dv \wedge d\tau = -\frac{\partial f}{\partial a}\frac{\partial \delta}{\partial v}\,dv \wedge d\tau,$$

we deduce that:

$$-\frac{\partial f}{\partial a}(a,\tau,v)\,\frac{d_{\tau,v}(\delta(a,\tau,v)\,d\tau)}{d_{\tau,v} f(a,\tau,v)} = \frac{\partial \delta}{\partial v}(a,\tau,v)\,\frac{\partial v}{\partial a}\,d\tau,$$

and we can write expression (5.16).

From the equation of the curve $v^2 - g(a,\tau) = 0$, we deduce $2v\frac{\partial v}{\partial a} = \frac{\partial g(a,\tau)}{\partial a}$, so $\frac{\partial v}{\partial a} = 2\tau^2(1+\tau)/v$. Hence,

$$\frac{\partial \delta}{\partial a}(a,\tau,v) + \frac{\partial \delta}{\partial v}(a,\tau,v)\,\frac{\partial v}{\partial a} =$$
$$= \frac{1}{(\tau+1)(a\tau-2)^2}\left[\frac{1}{v}\,(1+3\tau)(w-2+w\tau-4\tau) + \frac{q(\tau)}{v^3}(1+\tau)(a\tau-2)\tau\right],$$

where

$$\begin{aligned} q(\tau) \; &:= \; 2(w-8) + 2(13-5a+(a+1)w)\tau + \\ &\quad +(3aw-23a-10)\tau^2 + a(3w-7)\tau^3. \end{aligned}$$

We are going to show that, taking $w = -9-2\sqrt{17}$, the integrand of (5.16) is strictly negative for all $(\tau, a) \in R$, where R is the open region where the limit cycle exists, see Figure 5.1. Since the function $(1+\tau)(a\tau-2)^2 g^{3/2}$ is strictly positive in the region R, we are going to show that the product of the integrand by this function, denoted by $\beta(a,\tau)$, is strictly negative in the region R.

In the upper border of the region R, that is $a = 0$ and $\tau \geq 1$, we have that:

$$\beta(0,\tau) = -[11 + 92\tau + 42\tau^2 - 36\tau^3 + 19\tau^4 + 2\sqrt{17}(\tau^2-1)(3\tau^2-6\tau-1)],$$

whose real roots are all negative. We deduce that it is strictly negative for $\tau \geq 1$.

In the lower border of the region R, that is $g(a,\tau) = 0$ and $\tau \geq 1$, we have that

$$\beta(\alpha(\tau),\tau) = -\frac{(1+3\tau)^3}{8\tau(1+\tau)}\,(\tau^2-3\tau-2)\,[(\sqrt{17}-1)\tau-(\sqrt{17}+7)].$$

This expression has a definite negative sign in the region R since its zeroes are $-1/3$, $(3-\sqrt{17})/2 < 0$ and the double zero $(3+\sqrt{17})/2$. This double zero corresponds to the minimum point in the border of R.

If we show that $\beta(a,\tau)$ has no extremal values in the open region R, we have that it is strictly negative in this domain. We are going to compute $\frac{\partial\beta}{\partial a}$ and ensure that the curve $\frac{\partial\beta}{\partial a} = 0$ does not contain any point in R. We define the curve $a = \mu(\tau)$ by the zero set of

$$\begin{aligned}\frac{\partial\beta}{\partial a} &= -2\tau^2(1+\tau)\left[2\tau\left(14+25\tau+17\tau^2+\sqrt{17}(2+3\tau+3\tau^2)\right)a\right.\\ &\quad \left.-23+38\tau+49\tau^2+2\sqrt{17}(-1+6\tau+3\tau^2)\right].\end{aligned}$$

Here, $a = \mu(\tau)$ is a continuous strictly increasing function of τ for $\tau \geq 1$, has a horizontal asymptote to 0 when $\tau \to +\infty$ and $\mu(1) = -(4+\sqrt{17})/(7+\sqrt{17}) < 0$, hence it always takes negative values. It is easy to show that the greatest root of the equation $\mu(\tau) = \alpha(\tau)$ is $(\sqrt{17}-1)/8 < 1$. Hence, $\frac{\partial\beta}{\partial a} = 0$ does not contain any point in R and there is no extremal point of $\beta(a,\tau)$ in R. We deduce that $\beta(a,\tau) < 0$ in R and hence $D'(a) < 0$ for the values of a considered.

Hyperbolicity of the limit cycle given by the algebraic curve (5.6) *for system* (5.5).
In order to simplify our computations, we consider the following birrational change of the parameter, $a = 3c/(4+5c)$, with inverse $c = 4a/(3-5a)$ and we have that $c \in (0,1/2)$ is the interval which gives the limit cycle. The divergence is $\mathrm{div}(x,y) = 4[21(1+2c)-3(14+23c)x+11(4+5c)y]/(4+5c)$ and the cofactor related to the invariant algebraic curve (5.6) for system (5.5) is $k(x,y) = 24[5(1+2c)-(8+13c)x+2(4+5c)y]/(4+5c)$.

We define $g(c,\tau) := -\tau(c-\tau+c\tau^2) = c\tau(\tau-\tau_1)(\tau_2-\tau)$ where $\tau_1 = \frac{1-\sqrt{1-4c^2}}{2c}$ and $\tau_2 = \frac{1+\sqrt{1-4c^2}}{2c}$ and we consider the following parameterization of the oval of (5.6):

$$
\begin{aligned}
x_\pm(\tau) &= -\frac{2(1+2c)}{c^2(1+\tau)^3}\left(c-2\tau-2c\tau+c\tau^2 \pm 2\sqrt{(1+2c)g(c,\tau)}\right),\\
y_\pm(\tau) &= \frac{-12(1+2c)^2}{c^2(4+5c)(1+\tau)^4}\left(c-2\tau-2c\tau+c\tau^2 \pm 2\sqrt{(1+2c)g(c,\tau)}\right),
\end{aligned}
\tag{5.17}
$$

where $\tau \in (\tau_1, \tau_2)$. It is clear that $0 < \tau_1 < 1 < \tau_2$ for $c \in (0, 1/2)$. We have that $g(c,\tau)$ is strictly positive for all $\tau \in (\tau_1, \tau_2)$. In the two endpoints of the parameterization:

$$
\begin{aligned}
p_1 = \quad (x_\pm(\tau_1), y_\pm(\tau_1)) &= \left(\frac{1+2c+\sqrt{1-4c^2}}{c}, \frac{6(1+2c)(1+\sqrt{1-4c^2})}{c(4+5c)}\right),\\
p_2 = \quad (x_\pm(\tau_2), y_\pm(\tau_2)) &= \left(\frac{1+2c-\sqrt{1-4c^2}}{c}, \frac{6(1+2c)(1-\sqrt{1-4c^2})}{c(4+5c)}\right),
\end{aligned}
$$

an easy computation shows that $P(p_1) > 0$ and $P(p_2) < 0$, where $P(x, y)$ is the polynomial defining $\dot{x}$ in system (5.5). Moreover $x_+(\tau) - x_-(\tau) < 0$ and $y_+(\tau) - y_-(\tau) < 0$ for $\tau \in (\tau_1, \tau_2)$. Hence, the flow and the parameterizations are given as in Figure 5.3.

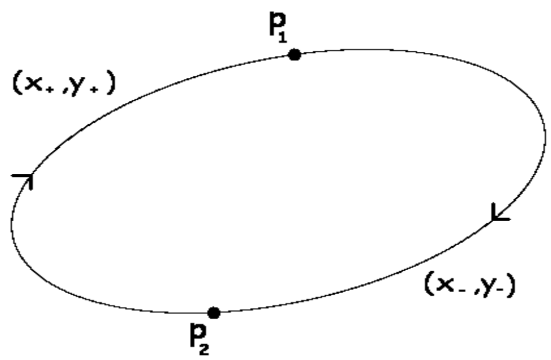

Figure 5.3: Flow and parameterization on the oval.

In the same way as in the previous cases, we consider

$$\begin{aligned}\mathcal{D}(c) = & \int_{\tau_1}^{\tau_2} \frac{(1+w)\,\mathrm{div} - wk}{P}(x_-(\tau), y_-(\tau))\; x_-'(\tau)\; d\tau \\ & + \int_{\tau_2}^{\tau_1} \frac{(1+w)\,\mathrm{div} - wk}{P}(x_+(\tau), y_+(\tau))\; x_+'(\tau)\; d\tau,\end{aligned} \tag{5.18}$$

where $x_\pm'(\tau)$ is the derivative of $x_\pm(\tau)$ with respect to τ. We choose $w = 37$ and we have:

$$\mathcal{D}(c) = 8\sqrt{1+2c}\int_{\tau_1}^{\tau_2} \frac{4+18c-(16+7c)\tau+7(2c-1)\tau^2+13c\tau^3}{(1+\tau)(4+8c+8\tau+17c\tau+c\tau^2)\sqrt{g(c,\tau)}}\, d\tau. \tag{5.19}$$

The roots in τ of $(1+\tau)(4+8c+8\tau+17c\tau+c\tau^2)$ are always negative for $c \in (0, 1/2)$, so, this product is always positive in the region of definition of the limit cycle. As stated in [22], the limit cycle of system (5.5) is born in a Hopf bifurcation when $c = 1/2$ (that is $a = 3/13$ and $\tau_1 = \tau_2 = 1$) and when c decreases the algebraic limit cycle grows in size ending asymptotically at the curve $y^2(3-6x+4y)$ when $c = 0$. By the form of (5.19), we have that $\mathcal{D}(c)$ is an analytic function in $c \in (0, 1/2]$. Moreover, when $c = 1/2$, the integrand in (5.19) is zero for $\tau = 1$, so we deduce $\mathcal{D}(1/2) = 0$.

We are going to compute the derivative of $\mathcal{D}(c)$ with respect to c. We consider the integral which gives $\mathcal{D}(c)$ as the integral of the rational function $\delta(c,\tau,v)$ on the curve $v^2 - g(c,\tau) = 0$, where

$$\delta(c,\tau,v) = 4\,\sqrt{1+2c}\,\frac{4+18c-(16+7c)\tau+7(2c-1)\tau^2+13c\tau^3}{(1+\tau)(4+8c+8\tau+17c\tau+c\tau^2)v}.$$

We define $q_0(c,\tau) := 4+18c-(16+7c)\tau+7(2c-1)\tau^2+13c\tau^3$. Hence,

$$\mathcal{D}'(c) = \oint_\gamma \left(\frac{\partial\delta}{\partial c}(c,\tau,v) + \frac{\partial\delta}{\partial v}(c,\tau,v)\,\frac{\partial v}{\partial c}\right)\, d\tau, \tag{5.20}$$

where this integral is done over the oval γ of the curve $v^2 - g = 0$. From the equation of the curve $v^2 - g = 0$, we deduce $\frac{\partial v}{\partial c} = -\tau(1+\tau^2)/(2v)$. Hence,

$$\frac{\partial\delta}{\partial c}(c,\tau,v) + \frac{\partial\delta}{\partial v}(c,\tau,v)\,\frac{\partial v}{\partial c} = \frac{2\,\sqrt{1+2c}\,\tau\,(1+\tau^2)q_0(c,\tau)}{(1+\tau)\,(4+8c+8\tau+17c\tau+c\tau^2)\,v^3} +$$

$$+\frac{4}{\sqrt{1+2c}\,(1+\tau)\,(4+8c+8\tau+17c\tau+c\tau^2)^2\,v}\,[\,8(1+2c)(7+9c)+$$
$$+2(72+204c+125c^2)\tau+(168+324c+11c^2)\tau^2+$$
$$+(243+627c+335c^2)\tau^3+(111+319c+235c^2)\tau^4+13c^2\tau^5]\,.$$

We are going to show that the integrand of (5.20) is strictly positive for all (c,τ) belonging to the region where the limit cycle exists, that is $R=\{(c,\tau)\in\mathbb{R}^2\,|\,\tau_1<\tau<\tau_2,\ 0<c<1/2\}$.

It is easy to see that $R=\{(c,\tau)\in\mathbb{R}^2\,|\,0<c<\tau/(1+\tau^2),\ \tau>0\}$. Since the function $\sqrt{1+2c}\,(4+8c+8\tau+17c\tau+c\tau^2)^2 g(c,\tau)^{3/2}/(4\tau)$ is strictly positive for all $(c,\tau)\in R$, we only need to show that the product of the integrand of (5.20) and the former function is positive. We denote by $\beta(c,\tau)$ this product. In the lower border of R, that is $c=0$ and $\tau\geq 0$, we have that:

$$\beta(0,\tau)=2(8+32\tau+42\tau^2+82\tau^3+83\tau^4),$$

which is strictly positive for all $\tau\geq 0$. In the upper border of R, that is $c=\tau/(1+\tau^2)$ and $\tau\geq 0$ we have:

$$\beta\left(\frac{\tau}{1+\tau^2},\tau\right) = \frac{2(1-\tau)^2(1+\tau)^3(2+3\tau)^3}{(1+\tau^2)^2},$$

which is strictly positive for all $\tau\geq 0$. Since $\beta(c,\tau)$ is a $\mathcal{C}^1$ function in R with its border, we only need to see that there are no extreme points in the region R and we will have that it is strictly positive in R.

The expression of the derivative of $\beta(c,\tau)$ with respect to c is:

$$\begin{aligned}\frac{\partial\beta}{\partial c} &= 24+48\tau-180\tau^2-37\tau^3+194\tau^4-125\tau^5\\ &\quad -2c\left(16+150\tau+651\tau^2+642\tau^3+182\tau^4+196\tau^5-13\tau^6\right).\end{aligned}$$

Let us call $\zeta(\tau)$ the function such that $(\partial\beta/\partial c)(\zeta(\tau),\tau)=0$. It is easy to see that $\zeta(\tau)$ is a continuous function for all $\tau\geq 0$, except for one point ν where there is a vertical asymptote. We have that $16<\nu<17$, $\zeta(\tau)$ is strictly decreasing for all $\tau\geq 0$, and it has a horizontal asymptote to 0 when τ tends to $+\infty$. An easy analysis of the signs of a rational function shows that the curves $\zeta(\tau)$ only intersect the border of the region R in two positive points

$0 < \nu_1 < \nu_2 < 1$, one intersection occurs with the curve $\tau/(1+\tau^2)$ and the other with $c = 0$. Moreover, $\beta\left(\zeta(\tau), \tau\right)$ is given by the following fraction:

$$\frac{(1+\tau)^3\,(2+3\tau)^2\,(400+1600\tau+4476\tau^2+6040\tau^3+4757\tau^4+777\tau^5)}{4\,(16+150\tau+651\tau^2+642\tau^3+182\tau^4+196\tau^5-13\tau^6)},$$

which is strictly positive for all $\tau \in [0, \nu]$. Hence, we deduce that if $\beta(c, \tau)$ has an extreme point inside the region R, then this point would belong to the curve $(\zeta(c), c)$ intersecting R in the region given by $\tau \in (\nu_1, \nu_2)$, but in this part of the curve the function $\beta(c, \tau)$ is strictly positive. In short, $\beta(c, \tau)$ is strictly positive in all the region R of existence of the limit cycle.

We deduce that the integrand of (5.20) is strictly positive in all R, from all these reasonings, and, therefore, $\mathcal{D}'(c) > 0$ for all $c \in (0, 1/2)$. Since $\mathcal{D}(c)$ is a strictly increasing continuous function of c for $c \in (0, 1/2)$ and $\mathcal{D}(1/2) = 0$, we have that $\mathcal{D}(c) < 0$ for all $c \in (0, 1/2)$. Hence, the limit cycle of system (5.5) is hyperbolic (and stable). ∎

Relations among elliptic functions

The aim of this subsection is to present the astonishing relations among elliptic functions which we obtained by using the identity given by Theorem 4.1 for systems (5.5) and (5.9).

Before the presented proof of Theorem 5.7, we got its proof for systems (5.5) and (5.9) by computing the corresponding integrals which give place to elliptic integrals. The identity given in Theorem 4.1 was used to encounter a Fuchs equation for the function $\mathcal{D}(a)$. After some thorough analysis of this Fuchs equation, we deduce the non-vanishing of the function $\mathcal{D}(a)$ for any value of the parameter in which the limit cycle exists. We are not going to give this proof, but we think that the relations among elliptic integrals obtained by the former reasoning are interesting by themselves. Hence, we give the identities obtained which, as far as we know, do not appear in any book of tables of integrals and relations between classical functions. On the other hand, we also give the obtention of the Fuchs equation for the function $\mathcal{D}(a)$ in the case of system (5.9).

Identities among elliptic integrals

The functions involved in this subsubsection are the complete elliptic integrals of first, second and third kinds, denoted by $\mathrm{K}(\omega)$, $\mathrm{E}(\omega)$ and $\Pi(\kappa,\omega)$, respectively. We recall the definition of these functions:

$$\begin{aligned}
\mathrm{K}(\omega) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\omega\sin^2(\theta)}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\omega t^2)}},\\
\mathrm{E}(\omega) &= \int_0^{\pi/2} \sqrt{1-\omega\sin^2(\theta)}\, d\theta = \int_0^1 \frac{\sqrt{1-\omega t^2}}{\sqrt{(1-t^2)}}\, dt,\\
\Pi(\kappa,\omega) &= \int_0^{\pi/2} \frac{d\theta}{(1-\kappa\sin^2(\theta))\sqrt{1-\omega\sin^2(\theta)}}\\
&= \int_0^1 \frac{dt}{(1-\kappa t^2)\sqrt{(1-t^2)(1-\omega t^2)}},
\end{aligned}$$

and their derivatives:

$$\begin{aligned}
\mathrm{K}'(\omega) &= \frac{1}{2(1-\omega)\omega}\,\mathrm{E}(\omega) - \frac{1}{2\omega}\,\mathrm{K}(\omega),\\
\mathrm{E}'(\omega) &= \frac{1}{2\omega}\,\mathrm{E}(\omega) - \frac{1}{2\omega}\mathrm{K}(\omega),\\
\frac{\partial\Pi(\kappa,\omega)}{\partial\kappa} &= \frac{1}{2\kappa(\kappa-1)}\,\mathrm{K}(\omega) + \frac{1}{2(\kappa-1)(\omega-\kappa)}\,\mathrm{E}(\omega)\\
&\quad + \frac{\kappa^2-\omega}{2\kappa(\kappa-1)(\omega-\kappa)}\,\Pi(\kappa,\omega),\\
\frac{\partial\Pi(\kappa,\omega)}{\partial\omega} &= \frac{1}{2(\kappa-\omega)(\omega-1)}\,\mathrm{E}(\omega) + \frac{1}{2(\kappa-\omega)}\,\Pi(\kappa,\omega).
\end{aligned}$$

For further information on elliptic integrals, see, for instance, [1, 93].

By explicit computation of the integrals for the system (5.9) using the parameterization given in (5.14), we get that the identity (4.3) stated in Theorem 4.1 reads for:

$$-9\,\mathrm{K}(\omega_0) + c_+\,\Pi(\omega_+,\omega_0) + c_-\,\Pi(\omega_-,\omega_0) \equiv 0, \tag{5.21}$$

which is valid for $a \in (0, 1/4)$, where

$$\omega_0 = \frac{2\sqrt{1-4a}}{1+\sqrt{1-4a}},\ \omega_\pm = \frac{2\sqrt{1-4a}}{9+\sqrt{1-4a}\pm 2\sqrt{16-a}},$$

$$c_\pm = \frac{9-\sqrt{1-4a}}{2} \pm \sqrt{16-a}.$$

The derivative of the expression in (5.21) with respect to a gives place to the same identity (5.21). In fact, when computing the derivative with respect to a of the expression given in (5.21), using the described formulas of derivation for these elliptic integrals, we get $-1/(1-4a+\sqrt{1-4a})$ times the same expression (5.21). This simple factor is different from zero when $a \in (0,1/4)$.

In the same way, the explicit computation of the integrals involved in the identity (4.3) stated in Theorem 4.1 for the system (5.5), via using the parameterization given in (5.17), gives:

$$5\,\mathrm{K}(\varsigma_0) \;+\; C_+\,\Pi(\varsigma_+,\varsigma_0) \;+\; C_-\,\Pi(\varsigma_-,\varsigma_0) \equiv 0, \tag{5.22}$$

which is valid for $c \in (0,1/2)$, where

$$\varsigma_0 = \frac{2\sqrt{1-4c^2}}{1+\sqrt{1-4c^2}}, \quad \varsigma_\pm = \frac{2\sqrt{1-4c^2}}{9+17c+\sqrt{1-4c^2}\pm\sqrt{64(1+2c)^2+c^2}},$$

$$C_\pm = \frac{-2(24+47c\pm 3\sqrt{64(1+2c)^2+c^2}}{9+17c+\sqrt{1-4c^2}\pm\sqrt{64(1+2c)^2+c^2}}.$$

The derivative of the expression (5.22) with respect to c gives place to the same identity (5.22).

We have not been able to give an analogous identity related to system (5.7) due to the fact that the corresponding integrals require much more computations to be identified with the elliptic integrals.

Fuchs equation for $\mathcal{D}(a)$ in system (5.9)

In this part of the subsection we develop the way we obtained a Fuchs equation for the function $\mathcal{D}(a)$ in system (5.9), via using the relation (5.21). We think that the fact of obtaining a Fuchs equation satisfied by this function is interesting to further understand the stability of algebraic limit cycles for polynomial systems. We obtained a similar Fuchs equation for system (5.5), but we do not state it because the equation itself does not give any further information about the properties of system (5.5) and the way it was obtained is completely analogous to the way equation (5.23) for system (5.9) is obtained.

Let us consider system (5.9) and we parameterize the oval which contains the limit cycle by (5.14). Taking the notation described in the previous subsection:

$$\omega_0 = \frac{2\sqrt{1-4a}}{1+\sqrt{1-4a}},\ \omega_\pm = \frac{2\sqrt{1-4a}}{9+\sqrt{1-4a}\pm 2\sqrt{16-a}},$$

$$c_\pm = \frac{9-\sqrt{1-4a}}{2} \pm \sqrt{16-a},$$

and

$$\mu = \sqrt{1+\sqrt{1-4a}}, \quad b_\pm = 2(4\pm\sqrt{16-a})\, c_\pm,$$

we explicitly compute the value of $\mathcal{D}(a)$:

$$\begin{aligned}\mathcal{D}(a) &= \int_0^T \operatorname{div}(\gamma(t))\, dt \\ &= \frac{\sqrt{2}}{\mu\sqrt{16-a}} \left[-34\sqrt{16-a}\,\mathrm{K}(\omega_0) + b_+\,\Pi(\omega_+,\omega_0) - b_-\,\Pi(\omega_-,\omega_0)\right].\end{aligned}$$

We compute the successive derivatives of $\mathcal{D}(a)$:

$$\begin{aligned}\mathcal{D}'(a) &= \frac{-4\sqrt{2}}{\mu\,(16-a)^{3/2}}\Bigg[\sqrt{16-a}\,\mathrm{K}(\omega_0) + \frac{2\mu^2\sqrt{16-a}}{a}\,\mathrm{E}(\omega_0) + \\ &\quad - c_+\,\Pi(\omega_+,\omega_0) + c_-\,\Pi(\omega_-,\omega_0)\Bigg],\end{aligned}$$

$$\begin{aligned}\mathcal{D}''(a) &= \frac{6\sqrt{2}}{\mu\,(16-a)^{5/2}}\Bigg[\frac{(10a^2+33a-64)\sqrt{16-a}}{3a(1-4a)}\,\mathrm{K}(\omega_0) + \\ &\quad \frac{(73a^2-420a+128)\mu^2\sqrt{16-a}}{6a^2(1-4a)}\,\mathrm{E}(\omega_0) + \\ &\quad c_+\,\Pi(\omega_+,\omega_0) - c_-\,\Pi(\omega_-,\omega_0)\Bigg],\end{aligned}$$

$$\begin{aligned}\mathcal{D}'''(a) &= \frac{-\sqrt{2}}{\mu(16-a)^{7/2}}\Bigg[\frac{q_1(a)\sqrt{16-a}}{a^2(1-4a)^2}\,\mathrm{K}(\omega_0) + \frac{q_2(a)\mu^2\sqrt{16-a}}{2a^3(1-4a)^2}\,\mathrm{E}(\omega_0) \\ &\quad - 15c_+\,\Pi(\omega_+,\omega_0) + 15c_-\Pi(\omega_-,\omega_0)\Bigg],\end{aligned}$$

where $q_1(a) := 180a^4+1347a^3-9685a^2+25664a-4096$ and $q_2(a) := 1812a^4-20259a^3+102164a^2-60544a+8192$. By elimination of independent functions and using the identity (5.21) we obtain the following third order homogeneous differential equation of Fuchs type for $\mathcal{D}(a)$:

$$8(a-16)a(4a-1)(17a+8)\,\mathcal{D}'''(a) + 4(612a^3-4119a^2-2600a +512)\,\mathcal{D}''(a) + 6(a-2)(289a+528)\,\mathcal{D}'(a) + 3(17a+64)\,\mathcal{D}(a) = 0. \tag{5.23}$$

An easy computation shows that $\mathcal{D}(1/4) = 0$, $\mathcal{D}'(1/4) = -8\sqrt{2}\pi/9$ and $\mathcal{D}''(1/4) = 98\sqrt{2}\pi/27$. Hence, equation (5.23) univocally determines the function $\mathcal{D}(a)$ defined in $a \in (0, 1/4]$. A thorough analysis of the properties of $\mathcal{D}(a)$ gives that $\mathcal{D}(a) > 0$ for $a \in (0, 1/4)$.

We remark that using identity (5.21) we get a Fuchs equation of order 3 for $\mathcal{D}(a)$. If we did not have this relation, we would get an equation of order 4, which would make the analysis of properties much more difficult. We notice that this Fuchs equation is an interesting alternative method to prove the hyperbolicity of the limit cycle in system (5.9). This kind of equation may exist for all algebraic limit cycle of a planar polynomial system and may let distinguish its hyperbolic character.

Section 5.1 belongs to the paper entitled *Necessary conditions for the existence of invariant algebraic curves for planar polynomial systems*, authored by J. Chavarriga, H. Giacomini and M. Grau and accepted for publication in Bull. Sci. Math.

Section 5.2 belongs to the paper entitled *On the stability of limit cycles for planar differential systems*, authored by H. Giacomini and M. Grau and which is a preprint, 2004.

Chapter 6

Isochronous points

In this chapter we are interested in the study of a characteristic of the period function around a singular point of focus–center type. Since this characteristic only depends on the behavior of the solutions in a neighborhood of this singular point, we state the results in a wider domain than planar polynomial differential systems. We consider planar differential systems defined by analytic functions in some neighborhood $\mathcal{U} \subseteq \mathbb{R}^2$ of the singular point. Without loss of generality, we always assume that the singular point is the origin of coordinates $O := (0, 0)$ and the autonomous differential system:

$$\dot{x} = \lambda x - y + P_2(x, y), \quad \dot{y} = x + \lambda y + Q_2(x, y), \tag{6.1}$$

where $P_2(x, y)$ and $Q_2(x, y)$ are analytic functions in a neighborhood $\mathcal{U}$ of the origin O and of order greater or equal than two. We assume that O is an isolated singular point of (6.1). We denote by $\mathcal{X}$ the equivalent vector field:

$$\mathcal{X} \; = \; (\lambda x - y + P_2(x, y)) \, \frac{\partial}{\partial x} \; + \; (x + \lambda y + Q_2(x, y)) \, \frac{\partial}{\partial y}.$$

In order to simplify notation, in this chapter we will call a *center* an analytic system of the form (6.1) with $\lambda = 0$ and where the origin O is a center, that is, an isolated singular point with a punctured neighborhood filled of periodic orbits. Isochronicity has been widely studied for centers, see for instance [49] and the references therein. We remark that the period function of a center does not depend on the chosen section Σ . The main methods used in order to study isochronicity of centers can be roughly classified in two categories, linearization and commutation.

Finding a *linearization* for a center $\mathcal{X}$ means finding a transformation $\phi : \mathcal{U} \to \mathcal{U}$ analytic in a neighborhood of the origin such that $\phi(O) = O$, $D\phi(O) = \mathrm{I}$, where I denotes the 2×2 identity matrix, and the transformed system is a linear center, that is $\phi_*(\mathcal{X}) = -y\,\partial/\partial x + x\,\partial/\partial y$. If such a transformation exists, then all the orbits have the same period, coinciding with the period of the linear center. So, a center is isochronous if and only if a linearization can be found.

Finding a *commutator* for a center $\mathcal{X}$ means finding a second vector field $\mathcal{Y}$ analytic in a neighborhood of the origin and of the form

$$\mathcal{Y} = (x + A(x,y))\frac{\partial}{\partial x} + (y + B(x,y))\frac{\partial}{\partial y}, \tag{6.2}$$

with A and B analytic functions of order ≥ 2, such that the Lie bracket $[\mathcal{X}, \mathcal{Y}]$ of the center $\mathcal{X}$ and $\mathcal{Y}$ identically vanishes.

An isolated singular point of a real planar analytic autonomous system is called a *star node* if the linear part of the vector field at the singular point has equal non-zero eigenvalues and it is diagonalizable. Clearly, the origin is a star node for (6.2). By an affine change of coordinates any vector field with a star node can be brought to the form (6.2).

Given two analytic vector fields defined in an open set $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$, we say that they are *transversal in* $\mathcal{U}$ at noncritical points when $\mathcal{X}$ and $\mathcal{Y}$ have isolated singular points, they both have the same critical points in $\mathcal{U}$, and if $p \in \mathcal{U}$ is such that $\mathcal{X}(p) \neq 0$ then the function given by the wedge product of $\mathcal{X}$ and $\mathcal{Y}$ is not zero at p. From now on, we always assume that $\mathcal{X}$ and $\mathcal{Y}$ are analytic vector fields defined in a neighborhood $\mathcal{U}$ of the origin and transversal at non critical points.

The following result, proved in [2], characterizes centers in terms of Lie brackets.

Theorem 6.1 [2] *System* (6.1) *with* $\lambda = 0$ *has a center at the origin if, and only if, there exists an analytic vector field* U *of the form* (6.2) *and an analytic function* $\nu(x,y)$ *with* $\nu(0,0) = 0$ *such that* $[\mathcal{X}, U] = \nu\mathcal{X}$.

The most important result on characterization of isochronous centers appears in [80, 90]. A further study can be found in, for instance, [2, 37, 47] and the references therein. See [18] for a constructive method of U and ν in special cases for polynomial vector fields.

The following theorem, which is stated and proved in [80], gives the equivalence between commutation and isochronicity. It is applied only for centers.

Theorem 6.2 [80] *System* (6.1)*, with* $\lambda = 0$*, has an isochronous center at the origin if, and only if, there exists an analytic vector field* $\mathcal{Y}$ *of the form* (6.2)*, transversal to* $\mathcal{X}$ *and such that* $[\mathcal{X}, \mathcal{Y}] \equiv 0$.

Another work on commuting systems is [81], where M. Sabatini discusses the local and global behavior of the orbits of a pair of commuting systems and he gives several illustrative examples. A wide collection of commutators and linearizations can be found in [29].

When a center is isochronous, it is possible to construct an isochronous section Σ, see [82]. However, the existence of an isochronous section is not strictly dependent on the existence of a center. A system can have a singular point of focus type with an isochronous section. This implies the existence of a neighborhood covered with solutions spiralling towards the singular point, all meeting Σ at equal time intervals. Such a behavior may occur, for instance, in a pendulum with friction, or in an electric circuit with dissipation, see also [82]. Our main result, Theorem 6.8, characterizes when the origin of system (6.1) is isochronous, even when the origin is a center, a weak focus or a strong focus. In this chapter, we adapt the two different techniques usually used for isochronous centers, in order to study isochronous foci.

In Section 6.1 we summarize the known results on isochronicity for foci. It is shown that a strong focus of an analytic system is always isochronous. All the results described in Section 6.1 only apply for systems of the form (6.1) with $\lambda \neq 0$ or for centers.

We will always consider analytic vector fields although many of the stated results apply also for vector fields with weaker differentiability restrictions. The results of Sabatini [82] go on this direction.

Section 6.2 contains the main theorem of this chapter which characterizes isochronicity for the origin of a system (6.1). This result is original when applied to weak foci and gives known results when applied to strong foci or to centers. We modify the commutators' method to study isochronous points. We prove that system (6.1) has a transversal vector field $\mathcal{Y}$ such that the vector field $[\mathcal{X}, \mathcal{Y}]$ is proportional to $\mathcal{Y}$ if, and only if, system (6.1) has an isochronous point at the origin.

We give two examples of weak isochronous foci and we give an example of a family of quadratic systems depending on a parameter $w \in \mathbb{R}$ which never has an isochronous point at the origin in Section 6.3. When $w = 0$ the system is a center and when $w \neq 0$ the system is a weak focus (stable if $w < 0$ and unstable if $w > 0$). Hence, we show that there is no isochronous section for any system of this family.

6.1 Summary of known results

We denote by $\mathcal{U}$ any open neighborhood of the origin and by $\rho : \mathcal{U} \to \mathbb{R}^+ \times \mathbb{R}$ the change to polar coordinates, that is, $\rho(x, y) = (r, \theta)$ with $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. As usual, ρ_* is the push-forward defined by ρ and ρ^* is the corresponding pull-back.

In order to give the definition of isochronous point, we consider the form of (6.1) in polar coordinates, that is, $\rho_*(\mathcal{X}) = rf(r,\theta)\frac{\partial}{\partial r} + g(r,\theta)\frac{\partial}{\partial \theta}$, where f and g are analytic functions in a neighborhood of $\rho(O)$.

Definition 6.3 *The point O of* (6.1) *is said to be isochronous if there exists a local analytic change of variables ϕ with $\phi(O) = O$, $D\phi(O) = \mathrm{I}$ and such that $\rho_*\phi_*(\mathcal{X}) = rf(r,\theta)\frac{\partial}{\partial r} + g(\theta)\frac{\partial}{\partial \theta}$.*

A system (6.1) with an isochronous point at the origin is more easily written using the *arc–length* φ, defined by $\varphi = \int_0^\theta d\theta/g(\theta)$, as new angular variable. In this formulation we end up to the following definition.

Definition 6.4 *The point O of* (6.1) *is said to be isochronous if there exists a local analytic change of variables ϕ with $\phi(O) = O$, $D\phi(O) = \mathrm{I}$ and such that $\rho_*\phi_*(\mathcal{X}) = rf(r,\theta)\frac{\partial}{\partial r} + k\frac{\partial}{\partial \theta}$, $k \in \mathbb{R}$, $k \neq 0$.*

The existence of an isochronous section is equivalent to the existence of the local analytic change of variables ϕ, as we will show in Theorem 6.8. We state the definition of isochronous point by means of ϕ since this is its classical definition which let us give the summary of known results.

Linear foci, $(\lambda x - y)\,\partial/\partial x + (x + \lambda y)\,\partial/\partial y$, are isochronous since their angular speed is constant along rays through the origin. For a linear focus, every ray through the origin is an isochronous section. We say that a vector field $\mathcal{X}$ of the form (6.1) is *linearizable* when there exists a local change of variables ϕ with $\phi(O) = O$, $D\phi(O) = I$ such that $\phi_*(\mathcal{X})$ is a linear focus.

By the above definition, every analytic linearizable focus is isochronous. If ϕ is the linearizing transformation and Σ is a ray, then $\phi^{-1}(\Sigma)$ is an isochronous section of the analytic linearizable focus. Next theorem, which is a special case of classical Poincaré's Theorem, shows that every strong focus of an analytic system is linearizable and therefore isochronous. For a proof, see [31, 78].

Theorem 6.5 [78] *Let us consider the planar real analytic system*

$$\dot{x} = \alpha x - \beta y + g_1(x, y), \qquad \dot{y} = \beta x + \alpha y + g_2(x, y), \tag{6.3}$$

with $\alpha\beta \neq 0$*, and* g_1 *and* g_2 *are of second order in* x *and* y*. Then there exists a real local analytic change of variables* $\phi(x, y) = (u, v)$ *with* $\phi(O) = O$ *and* $D\phi(O) = \mathrm{I}$ *which transforms system* (6.3) *into* $\dot{u} = \alpha u - \beta v$*,* $\dot{v} = \beta u + \alpha v$*.*

This result can also be stated for a system of the form (6.3) satisfying weaker differentiability restrictions. Since we are only concerned with analytic vector fields, we state the result only for the analytic case.

We have seen that every analytic linearizable focus is isochronous, but finding the linearization, and hence the isochronous sections, is usually too difficult. Next theorem proved in [82] shows that it is not necessary to find the explicit form of the linearization, since the orbits of a suitable commutator are isochronous sections of $\mathcal{X}$.

Theorem 6.6 [82] *If the vector field* $\mathcal{X}$ *given by* (6.1) *has a focus* O *and a nontrivial commutator* $\mathcal{Y}$ *with a star node at* O*, then every orbit of* $\mathcal{Y}$ *contained in a neighborhood of* O *is an isochronous section of* $\mathcal{X}$*.*

These result only apply when the vector field $\mathcal{X}$ has a strong focus at the origin or has a center because if the vector field $\mathcal{X}$ has a weak focus at the origin with a nontrivial commutator $\mathcal{Y}$ with a star node at O, then by Theorem 6.1 the vector field has a center at the origin. Next corollary proved in [82] shows that every system with a strong focus and a nontrivial commutator has a commutator with a star node.

Corollary 6.7 [82] *If the vector field* $\mathcal{X}$ *has eigenvalues with non-zero real part at a focus* O *and a nontrivial commutator* $\mathcal{Y}$*, then it has infinitely many isochronous sections.*

In [55] and [82], different sufficient conditions for an analytic vector field to have an isochronous weak focus at O are given. In [82], the particular case of a differential system equivalent to a Liénard equation is taken into account.

6.2 Characterization of isochronous points

The following theorem characterizes when the origin O of a system (6.1) has a section Σ such that the period function $\tau : \Sigma \to \mathbb{R}^+$ is constant, that is, it does not depend on the point $p \in \Sigma$ considered. We will see that if such section exists, then there are an infinite number of them. In particular next theorem characterizes the existence of isochronous points.

Theorem 6.8 *Let us consider an analytic system* (6.1)*. The following statements are equivalent:*

(i) There exists an analytic change of variables $\phi : \mathcal{U} \to \mathcal{U}$, where $\mathcal{U}$ is a neighborhood of the origin, with $\phi(O) = O$ and $D\phi(O) = \mathrm{I}$, such that the transformed system reads for $\rho_* \phi_*(\mathcal{X}) = r f(r,\theta) \frac{\partial}{\partial r} + g(\theta) \frac{\partial}{\partial \theta}$.

(ii) There exists an analytic vector field $\mathcal{Y}$ defined in a neighborhood of the origin of the form

$$\mathcal{Y} = (x + A(x,y)) \frac{\partial}{\partial x} + (y + B(x,y)) \frac{\partial}{\partial y}, \tag{6.4}$$

with A and B analytic functions of order ≥ 2, such that $[\mathcal{X}, \mathcal{Y}] = \mu(x,y)\, \mathcal{Y}$, where $\mu(x,y)$ is a scalar function with $\mu(0,0) = 0$.

(iii) There exists a section Σ such that the period function $\tau : \Sigma \to \mathbb{R}^+$ is constant.

Proof of Theorem 6.8. In order to prove the equivalence of the three statements, it suffices to show (**i**) $\Rightarrow$ (**ii**), (**ii**) $\Rightarrow$ (**iii**) and (**iii**) $\Rightarrow$ (**i**). We also include the proof of (**ii**) $\Rightarrow$ (**i**) and (**iii**) $\Rightarrow$ (**ii**), due to their simplicity and completeness.

In the subsequent, we will denote by a subindex a partial derivative, for instance, if $f(r,\theta)$ is a function of (r,θ), $\frac{\partial f}{\partial r}$ is replaced by f_r.

(**i**) $\Rightarrow$ (**ii**) We define

$$\mathcal{Y} = \phi^* \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right),$$

and we have that $\mathcal{Y}$ has the form described since ϕ is an analytic change such that $\phi(O) = O$ and $D\phi(O) = \mathrm{I}$. Moreover,

$$\begin{aligned}[\mathcal{X},\mathcal{Y}] &= \left[\phi^*\phi_*\mathcal{X}, \phi^*\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\right] = \phi^*\rho^*\left(\left[rf(r,\theta)\frac{\partial}{\partial r} + g(\theta)\frac{\partial}{\partial \theta}, r\frac{\partial}{\partial r}\right]\right)\\ &= \phi^*\rho^*\left(-r^2 f_r(r,\theta)\frac{\partial}{\partial r}\right).\end{aligned}$$

We denote by $\mu(x,y) = \phi^*\rho^*(-rf_r(r,\theta))$. It is obvious that is an analytic scalar function with $\mu(0,0) = 0$. We have

$$[\mathcal{X},\mathcal{Y}] = \mu(x,y)\,\phi^*\rho^*\left(r\frac{\partial}{\partial r}\right) = \mu(x,y)\,\phi^*\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = \mu(x,y)\,\mathcal{Y}.$$

(**ii**) $\Rightarrow$ (**i**) From normal form theory, see [31, 78], we have that there exists an analytic change of variables ϕ, defined in a neighborhood $\mathcal{U}$ of the origin and with $\phi(O) = O$ and $D\phi(O) = \mathrm{I}$, such that $\phi_*(\mathcal{Y}) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

Since $[\mathcal{X},\mathcal{Y}] = \mu\,\mathcal{Y}$, we have $[\phi_*(\mathcal{X}), \phi_*(\mathcal{Y})] = \phi_*(\mu)\,\phi_*(\mathcal{Y})$. We introduce the following notation $\tilde{\mu}(r,\theta) := \rho_*\phi_*(\mu(x,y))$ and $\rho_*\phi_*(\mathcal{X}) := rf(r,\theta)\frac{\partial}{\partial r} + g(r,\theta)\frac{\partial}{\partial \theta}$. Hence,

$$\left[rf(r,\theta)\frac{\partial}{\partial r} + g(r,\theta)\frac{\partial}{\partial \theta},\, r\frac{\partial}{\partial r}\right] = \tilde{\mu}(r,\theta)\,r\frac{\partial}{\partial r}.$$

We compute the Lie bracket and we have the following equality

$$-r^2 f_r(r,\theta)\frac{\partial}{\partial r} - rg_r(r,\theta)\frac{\partial}{\partial \theta} = \tilde{\mu}(r,\theta)\,r\frac{\partial}{\partial r},$$

which implies $g_r(r,\theta) \equiv 0$ and, therefore, $g(r,\theta) = g(\theta)$. We remark that since the origin of the system defined by $\mathcal{X}$ is a monodromic point, we have that $g(\theta) > 0$ or $g(\theta) < 0$ for all $\theta \in \mathbb{R}$. Moreover, as before, we may consider the arc–length $\varphi = \int_0^\theta d\theta/g(\theta)$. This integral is well defined and it gives a change of variable since $g(\theta)$ has a definite sign for all $\theta \in \mathbb{R}$. Then, after this change, the angular speed of the corresponding system is constant.

(**ii**) $\Rightarrow$ (**iii**) This statement is a clear corollary of Theorem 6.6. However, a geometric outline of its proof is easy enough to be given here.

Let, for any $p \in \mathcal{U}$, be $\Phi_t(p)$ the flow of $\mathcal{X}$ and $\Psi_s(p)$ that of $\mathcal{Y}$, with the initial condition $\Phi_0(p) = \Psi_0(p) = p$. Without lack of generality, we can assume that O is the unique singular point for $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{U}$. Let $p, q \in \mathcal{U}$, $p, q \neq O$. By classical Lie theory, we have that the relation $[\mathcal{X}, \mathcal{Y}] = \mu(x, y)\,\mathcal{Y}$ implies that if $\Sigma = \{\Psi_s(p) \,|\, s \in \mathbb{R}\}$ is a solution of $\mathcal{Y}$, then for any $t \in \mathbb{R}$, $\Phi_t(\Sigma)$ is another solution for $\mathcal{Y}$.

It is clear that Σ is a transversal section for $\mathcal{X}$. Let $\tau, \mathcal{P}$ be the corresponding period function and Poincaré map defined on it. We will show that any two points $p, q \in \Sigma$ have the same period function. We have that $\mathcal{P}(p) = \Phi_{\tau(p)}(p)$. The time $\tau(p)$ leaves Σ invariant: $\Phi_{\tau(p)}(\Sigma) \subseteq \Sigma$. Let $q \in \Sigma$, then there exists $s \in \mathbb{R}$ such that $q = \Psi_s(p)$. The minimal time to meet Σ again, that is $\tau(q)$, must coincide with $\tau(p)$ since the time $\tau(p)$ brings the solution Σ into itself. Then $\tau(p) = \tau(q)$.

(**iii**) $\Rightarrow$ (**ii**) We consider $\Phi_t(p)$ the flow of system $\mathcal{X}$ defined in the neighborhood $\mathcal{U}$ of the origin and with the initial condition $\Phi_0(p) = p$.

Given a section through the origin, $\Sigma \subset \mathbb{R}^2$, we consider its parameterization by its arc-parameter σ, that is, there exists a map $c : \mathbb{R} \to \Sigma$ such that $\Sigma = \{c(\sigma) \,|\, \sigma \in \mathbb{R}\}$. We can assume without loss of generality that $\lim_{\sigma \to -\infty} c(\sigma) = O$ and that $\lim_{\sigma \to -\infty} c'(\sigma) \neq (0, 0)$. As usual, $c'(\sigma)$ denotes the derivative of the parameterization of the curve $c : \sigma \mapsto c(\sigma)$ at the value σ. We define the following set of transformations $\Psi : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ in the following way, see Figure 6.1.

If $p \in \Sigma$, that is $p = c(\sigma_0)$ for a certain $\sigma_0 \in \mathbb{R}$, and $s \in \mathbb{R}$ then $\Psi_s(p) := c(\sigma_0 + s)$.

If $p \notin \Sigma$, there exists $t_0(p) \in \mathbb{R}$ such that $\Phi_{t_0(p)}(p) \in \Sigma$, that is, there exists $\sigma_0 \in \mathbb{R}$ such that $c(\sigma_0) = \Phi_{t_0(p)}(p)$. Assume that $t_0(p) > 0$ is the lowest positive real with this property. For any $s \in \mathbb{R}$ we define $\Psi_s(p) = \Phi_{-t_0(p)}(c(\sigma_0 + s))$.

In the subsequent, for any $p \in \mathcal{U}$ we denote by $t_0(p)$ as the lowest positive real such that $\Phi_{t_0(p)}(p) \in \Sigma$. It is clear that $t_0 : \mathcal{U} \to [0, T)$ where $T > 0$ is the period defined by the section Σ. We denote by $\sigma_0(p) \in \mathbb{R}$ the value of the parameter such that $\Phi_{t_0(p)}(p) = c(\sigma_0(p))$.

We are going to prove that the set of transformations defined by Ψ_s is a one–parameter Lie group of point transformations. We need to show the following statements:

(a) For all $s \in \mathbb{R}$, $\Psi_s : \mathcal{U} \to \mathcal{U}$ is bijective.

(b) Ψ_0 is the identity map.

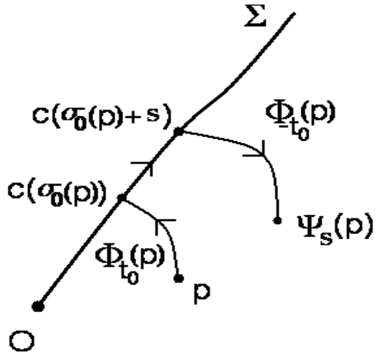

Figure 6.1: Definition of $\Psi_s(p)$.

(c) For any $s_1, s_2 \in \mathbb{R}$, $\Psi_{s_1} \circ \Psi_{s_2} = \Psi_{s_1+s_2}$.

(d) $\Psi \in \mathcal{C}^\omega(\mathbb{R}) \times \mathcal{C}^\omega(\mathcal{U})$.

(a) Fixed $s \in \mathbb{R}$, let us consider any $p \in \mathcal{U}$ and we have $\Psi_s(p) = \Phi_{-t_0(p)}(c(\sigma_0(p) + s))$. Let $p_1, p_2 \in \mathcal{U}$. If $\Psi_s(p_1) = \Psi_s(p_2)$, let q be this point $q = \Psi_s(p_i)$. Then, the points $\Phi_{t_0(p_1)}(q) = c(\sigma_0(p_1) + s)$ and $\Phi_{t_0(p_2)}(q) = c(\sigma(p_2) + s)$ belong to Σ. Both, $t_0(p_1)$ and $t_0(p_2)$ are defined as the minimum positive time with this property so, $t_0(p_1) = t_0(p_2)$. Therefore, $c(\sigma_0(p_2)+s) = c(\sigma_0(p_1) + s)$ and this gives $\sigma_0(p_1) = \sigma_0(p_2)$ which implies $p_1 = p_2$. Then, Ψ_s is injective.

We are going to see that it is exhaustive. Given $q \in \mathcal{U}$ we define the point $p = \Phi_{-t_0(q)}(c(\sigma_0(q) - s))$. Then, $t_0(p) = t_0(q)$, $\sigma_0(p) = \sigma_0(q) - s$ and $\Psi_s(p) = \Phi_{-t_0(q)}(c(\sigma_0(q))) = q$. The fact that the section Σ is isochronous ensures the well-definition of this p.

(b) Given $p \in \mathcal{U}$ we have that $\Psi_0(p) = \Phi_{-t_0(p)}(c(\sigma_0(p)))$ where $c(\sigma_0(p)) = \Phi_{t_0(p)}(p)$. Then, clearly, $\Psi_0(p) = p$.

(c) Given $p \in \mathcal{U}$, it is clear that $t_0(\Phi_{-t_0(p)}(c(\sigma_0 + s_1))) = t_0(\Phi_{-t_0(p)}(c(\sigma_0 + s_1 + s_2))) = t_0(p)$. We have $\Psi_{s_1} \circ \Psi_{s_2}(p) = \Psi_{s_1}(\Phi_{-t_0(p)}(c(\sigma_0 + s_1))) = \Phi_{-t_0(p)}(c(\sigma_0 + s_1 + s_2)) = \Psi_{s_1+s_2}(p)$.

(d) The regularity of Ψ is clear due to the regularity of Φ and c.

Once we have that Ψ is a one–parameter Lie group of point transformations, we apply the first fundamental theorem of Lie, see [70], and we have that there exists an analytic vector field $\mathcal{Y}$ whose flow coincides with

$\Psi_s(p)$. Moreover, $\mathcal{Y}$ is given by $\frac{\partial \Psi_s}{\partial s}(p)_{|s=0}$. By the definition $\Psi_s(p) = \Phi_{t_0(p)}(c(\sigma_0(p)+s))$ we have that $\mathcal{Y}(p) = \mathrm{T}\Phi_{t_0(p)}(c(\sigma_0(p)+s)) \cdot c'(\sigma_0(p)+s)_{|s=0} = \mathrm{T}\Phi_{t_0(p)}(c(\sigma_0(p))) \cdot c'(\sigma_0(p))$, where $\mathrm{T}\Phi_t(q)$ denotes the jacobian matrix of the analytic change of variables Φ_t at the point q and, as before, $c'(\sigma)$ denotes the derivative of the parameterization of the curve $c : \sigma \mapsto c(\sigma)$ at the value σ.

Moreover, by construction, $\mathcal{Y}$ has a star node at the origin. This is clear by the fact that each of its orbits $\Phi_t(\Sigma)$, $t \in [0,T)$, has a different tangent at the origin. Let $\mathcal{Y} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$. Since $\mathcal{Y}$ has a star node at the origin, by a classical result stated in [96], page 63, we have that $\xi(x,y) = xh(x,y) + h.o.t.$ and $\eta(x,y) = yh(x,y) + h.o.t.$, where $h(x,y)$ is a homogeneous polynomial and $h.o.t.$ denotes higher order terms. Therefore, in order to see that $\mathcal{Y}$ is of the form (6.2), we only need to show that the divergence of the vector field $\mathcal{Y}$, that is div$\mathcal{Y}$, is different from zero at the origin, where $\mathrm{div}\mathcal{Y}(x,y) = \frac{\partial \xi}{\partial x}(x,y) + \frac{\partial \eta}{\partial y}(x,y)$. The divergence of the vector field $\mathcal{Y}$ is related to the inverse integrating factor of $\mathcal{Y}$. The inverse integrating factor of $\mathcal{Y}$ is given by $V(x,y) = (\lambda x - y + P_2(x,y))\eta(x,y) - (x + \lambda y + Q_2(x,y))\xi(x,y)$ which is defined in the neighborhood $\mathcal{U}$ of the origin. An easy computation shows that

$$V(\Psi_s(x_0,y_0)) = V(x_0,y_0)\ \exp\left\{\int_0^s \mathrm{div}\mathcal{Y}(\Psi_u(x_0,y_0))du\right\} \qquad (6.5)$$

for any $(x_0,y_0) \in \mathcal{U}$. It is clear that $V(0,0) = 0$. Let $p_0 := (x_0,y_0) \in \mathcal{U} - \{(0,0)\}$ and assume that $V(p_0) = 0$. This implies that the vectors $\mathcal{Y}(p_0)$ and $\mathcal{X}(p_0)$ are parallel. By the definition of $\mathcal{Y}$, we have that

$$\mathcal{Y}(p_0) = \mathrm{T}\Phi_{-t_0(p_0)}(c(\sigma_0(p_0)))\cdot c'(\sigma_0(p_0)) = \mathrm{T}\Phi_{-t_0(p_0)}(\Phi_{t_0(p_0)}(p_0))\cdot\mathcal{Y}(\Phi_{t_0(p_0)}(p_0)).$$

We denote by $q_0 = \Phi_{t_0(p_0)}(p_0)$ and we have $\mathrm{T}\Phi_{t_0(p_0)}(q_0) \cdot \mathcal{Y}(p_0) = \mathcal{Y}(q_0)$. Since Φ is the flow of $\mathcal{X}$, we have $\mathrm{T}\Phi_{t_0(p_0)}(q_0) \cdot \mathcal{X}(p_0) = \mathcal{X}(q_0)$. Therefore, if $\mathcal{Y}(p_0)$ and $\mathcal{X}(p_0)$ are parallel, then $\mathcal{Y}(q_0)$ and $\mathcal{X}(q_0)$ are parallel. However, $q_0 \in \Sigma$ and the vector $\mathcal{Y}(q_0)$ is tangent to Σ at q_0, so the parallelism between $\mathcal{Y}(q_0)$ and $\mathcal{X}(q_0)$ is a contradiction with Σ being a transversal section for $\mathcal{X}$. Therefore, we conclude that $V(x_0,y_0) \neq 0$ for any $(x_0,y_0) \in \mathcal{U} - \{(0,0)\}$.

By using this fact, we prove that $\mathrm{div}\mathcal{Y}(0,0) \neq 0$. Let us consider a point $(x_0,y_0) \in \mathcal{U} - \{(0,0)\}$ and we have that $\lim_{s\to -\infty}\Psi_s(x_0,y_0) = (0,0)$. By continuity and the identity (6.5), we have that the integral $I(x_0,y_0) := \int_{-\infty}^0 \mathrm{div}\mathcal{Y}(\Psi_u(x_0,y_0))du$ diverges. $I(x_0,y_0)$ is continuous, so $I(0,0)$ also diverges. Hence, if $\mathrm{div}\mathcal{Y}(0,0) = 0$, then $I(0,0) = \int_{-\infty}^0 \mathrm{div}\mathcal{Y}(\Psi_u(0,0))du =$

$\int_{-\infty}^{0} \mathrm{div}\mathcal{Y}(0,0)du = 0$, in contradiction with being divergent. Therefore, $\mathrm{div}\mathcal{Y}(0,0) \neq 0$.

Moreover, by definition it is clear that the flow of $\mathcal{X}$ takes solutions of $\mathcal{Y}$ to solutions of $\mathcal{Y}$. Another classical result on Lie symmetries gives that $\mathcal{X}$ is a Lie symmetry for $\mathcal{Y}$ and therefore, there exists an analytic scalar function $\mu : \mathcal{U} \to \mathbb{R}$ such that $[\mathcal{X}, \mathcal{Y}] = \mu\, \mathcal{Y}$. Moreover, $\mu(0,0) = 0$ since both functions defining the vector field $[\mathcal{X}, \mathcal{Y}]$ have order two at the origin and the vector field $\mathcal{Y}$ has order one at the origin.

(i) $\Rightarrow$ **(iii)** The ray $\tilde{\Sigma} = \{(x,0)\,|\, x > 0\}$ is an isochronous section for the system $\phi_*(\mathcal{X})$, where $\rho_* \phi_*(\mathcal{X}) = r f(r,\theta)\frac{\partial}{\partial r} + g(\theta)\frac{\partial}{\partial \theta}$, since $\tilde{\tau} : \tilde{\Sigma} \to \mathbb{R}^+$ is given by $\tilde{\tau}(x) = \int_0^{2\pi} d\theta / g(\theta)$, which is constant for every $x \in \tilde{\Sigma}$. Then, $\Sigma := \phi^{-1}(\tilde{\Sigma})$ is an isochronous section for system (6.1) and the period function is given by $\tau := \phi^*(\tilde{\tau})$. ■

Using Theorem 6.1 and Theorem 6.8, we reencounter the following result which characterizes isochronous centers and which is first stated and proved in [80].

Theorem 6.9 *System* (6.1) *with* $\lambda = 0$ *has an isochronous center at the origin if, and only if, there exists an analytic vector field* $\mathcal{Z}$ *of the form* (6.2) *such that* $[\mathcal{X}, \mathcal{Z}] = 0$.

Proof. Assume that system (6.1), with $\lambda = 0$, has an isochronous center at the origin. Then by Theorems 6.1 and 6.8 there exist analytic vector fields $\mathcal{U}, \mathcal{Y}$ of the form (6.2) and analytic functions ν, μ satisfying $[\mathcal{X}, \mathcal{U}] = \nu \mathcal{X}$ and $[\mathcal{X}, \mathcal{Y}] = \mu \mathcal{Y}$. Since $\mathcal{X}$ and $\mathcal{Y}$ are transversal in a neighborhood of the origin, they define a basis in this neighborhood and therefore, there exist two analytic functions α, β such that $\mathcal{U} = \alpha \mathcal{X} + \beta \mathcal{Y}$. Since both $\mathcal{U}$ and $\mathcal{Y}$ have the form (6.2), we have that $\beta = 1 + \beta_1$ where β_1 is an analytic function of order ≥ 1. We compute

$$
\begin{aligned}
[\mathcal{X}, \mathcal{U}] &= [\mathcal{X}, \alpha\, \mathcal{X} + \beta\, \mathcal{Y}] = \mathcal{X}(\alpha)\, \mathcal{X} + \alpha\, [\mathcal{X}, \mathcal{X}] + \mathcal{X}(\beta)\, \mathcal{Y} + \beta\, [\mathcal{X}, \mathcal{Y}] \\
&= \mathcal{X}(\alpha)\, \mathcal{X} + (\mathcal{X}(\beta) + \beta\mu)\, \mathcal{Y}.
\end{aligned}
$$

Since $[\mathcal{X}, \mathcal{Y}] = \nu \mathcal{X}$, we deduce $\mathcal{X}(\beta) = -\mu\beta$.

We define $\mathcal{Z} = \beta \mathcal{Y}$ which is an analytic vector field with the form (6.2) since $\beta = 1 + \beta_1$ where β_1 is an analytic function of order ≥ 1. Then, $[\mathcal{X}, \mathcal{Z}] = [\mathcal{X}, \beta \mathcal{Y}] = \beta\, [\mathcal{X}, \mathcal{Y}] + \mathcal{X}(\beta)\, \mathcal{Y} = \beta\mu\mathcal{Y} - \mu\beta\mathcal{Y} = 0$. ■

6.3 Examples of application

The methods developed in this chapter can be used to classify isochronous points for polynomial systems. We give some examples of systems of the form (6.1) with an isochronous point at the origin. The determination of the origin being a focus is straightforward by computing Liapunov constants, see for instance [55]. When the origin is a center, a first integral defined on a neighborhood of it is provided. We also give an example of a family of quadratic systems depending on a real parameter $w \neq 0$ which never has an isochronous point at the origin. When $w = 0$, the system has a center, and when $w \neq 0$ the system has a weak focus at the origin.

Example 1. The following system has an isochronous point at the origin.

$$\begin{array}{rcl} \dot{x} & = & -y + \lambda_2 x^3 + \lambda_3 x^2 y + \lambda_4 x y^2, \\ \dot{y} & = & x + \lambda_2 x^2 y + \lambda_3 x y^2 + \lambda_4 y^3, \end{array} \tag{6.6}$$

with $\lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$. In polar coordinates, this system reads for

$$\dot{r} \;=\; \frac{r^3}{2}(\lambda_2 + \lambda_4 + (\lambda_2 - \lambda_4)\cos(2\theta) + \lambda_3 \sin(2\theta)), \quad \dot{\theta} \;=\; 1.$$

Then, by definition, the origin is an isochronous point. A first integral for system (6.6) is given by

$$H(x,y) \;=\; \frac{x^2 + y^2}{1 - \lambda_3 x^2 + (\lambda_2 - \lambda_4)xy + (\lambda_2 + \lambda_4)(x^2 + y^2)\arctan(\frac{y}{x})}.$$

When $\lambda_2 + \lambda_4 \neq 0$, the origin is a focus point and when $\lambda_2 + \lambda_4 = 0$, the origin is a center point. Let us consider $\mathcal{X}$ the corresponding vector field and $\mathcal{Y} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. We have $[\mathcal{X}, \mathcal{Y}] = -2(\lambda_2 x^2 + \lambda_3 xy + \lambda_4 y^2)\mathcal{Y}$.

Example 2. The following system has an isochronous focus at the origin.

$$\begin{array}{rcl} \dot{x} & = & -y - 2xy + xy^2 - 2y^3 + \mu_2(x^3 - xy^2) + \mu_3 x^2 y - y^4 + \\ & & \mu_2(x^2y^2 + y^4) - \mu_2 x y^4 - \mu_3 y^5 - \mu_2 y^6, \\ \dot{y} & = & x + y^2 + y^3 + \mu_2(x^2 y - y^3) + \mu_3 x y^2 + 2\mu_2 x y^3 + \mu_3 y^4 + \mu_2 y^5, \end{array} \tag{6.7}$$

where μ_i are arbitrary real constants for $i = 2, 3$. This system has no constant angular speed. An easy computation shows that the first Liapunov constant equals $-1/2$, so the origin of (6.7) is a stable weak focus point. We use Theorem 6.8 to ensure the property of isochronicity.

Let us consider $\mathcal{X}$ the corresponding vector field and $\mathcal{Y}$ the following analytic vector field

$$\mathcal{Y} = (x - y^2)\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y}.$$

The Lie bracket $[\mathcal{X}, \mathcal{Y}]$ gives $[\mathcal{X}, \mathcal{Y}] = -2(y^2 + \mu_2(x^2 - y^2 + 2xy^2 + y^4) + \mu_3(xy + y^3))\,\mathcal{Y}$. Therefore, the hypothesis of Theorem 6.8 are satisfied and the origin of system (6.7) is an isochronous focus.

Example 3. The following family of quadratic systems depending on the parameter $w \in \mathbb{R}$

$$\dot{x} = -y, \quad \dot{y} = x - 4wxy + 2y^2, \tag{6.8}$$

never has an isochronous point at the origin.

It can be shown that w is the first Liapunov constant for this family of quadratic systems. Hence, when $w > 0$ the origin is an unstable weak focus and when $w < 0$ the origin is a stable weak focus. When $w = 0$, we have that $H(x, y) = (4x + 8y^2 - 1)e^{4x}$ defines a first integral which is analytic in a neighborhood of the origin. So, the origin is a center for $w = 0$.

We will try to construct a vector field $\mathcal{Y}$ and a function μ satisfying Theorem 6.8 and we will get a contradiction. Assume that there exists a vector field $\mathcal{Y}$ with a star node at the origin such that the Lie bracket between the vector field $\mathcal{X}_w$ defined by (6.8) and $\mathcal{Y}$ is equal to $\mu(x, y)\mathcal{Y}$ for a certain scalar analytic function $\mu(x, y)$ with $\mu(0, 0) = 0$. We can write $\mathcal{Y} = (x + \sum_{i>1} A_i(x, y))\,\partial/\partial x + (y + \sum_{i>1} B_i(x, y))\,\partial/\partial y$, where $A_i(x, y), B_i(x, y)$ are homogeneous polynomials of degree i and $\mu(x, y) = \sum_{i>0} m_i(x, y)$ where $m_i(x, y)$ is a homogeneous polynomial of degree i.

Equating the terms of order 2 in the equation $[\mathcal{X}_w, \mathcal{Y}] = \mu\,\mathcal{Y}$ we get the two following equations:

$$\begin{aligned} -y\frac{\partial A_2}{\partial x} + x\frac{\partial A_2}{\partial y} + B_2 &= m_1 x, \\ -y\frac{\partial B_2}{\partial x} + x\frac{\partial B_2}{\partial y} + 4xwy - 2y^2 - A_2 &= m_1 y. \end{aligned}$$

The solution of these two equations is $A_2(x, y) = ax^2 + bxy - (2/3)y^2$,

$B_2(x,y) = (4w/3)x^2 + axy + by^2$ and $m_1(x,y) = (b + (4w/3))x - ((4/3)+a)y$, where a, b are any two real numbers.

Equating the terms of order 3 in the equation $[\mathcal{X}_w, \mathcal{Y}] = \mu\,\mathcal{Y}$, we get the two following equations:

$$-y\frac{\partial A_3}{\partial x} + x\frac{\partial A_3}{\partial y} + (2y^2 - 4wxy)\frac{\partial A_2}{\partial y} + B_3 = m_2 x + m_1 A_2,$$
$$-y\frac{\partial B_3}{\partial x} + x\frac{\partial B_3}{\partial y} + (2y^2 - 4wxy)\frac{\partial B_2}{\partial y} + 4wyA_2 - A_3 - 4(y - wx)B_2 =$$
$$= m_2 y + m_1 B_2.$$

Let us write $m_2(x,y) = \sum_{i+j=2} m_{ij} x^i y^j$, $A_3(x,y) = \sum_{i+j=3} a_{ij} x^i y^j$ and $B_3(x,y) = \sum_{i+j=3} b_{ij} x^i y^j$. We consider the vector of unknowns

$$\mathbf{v} = \{m_{20}, m_{11}, m_{02}, a_{30}, a_{21}, a_{12}, a_{03}, b_{30}, b_{21}, b_{12}, b_{03}\}$$

and we can write the previous two equations as a linear system of eight equations in these eleven unknowns : $\mathbf{M}\,\mathbf{v} = \mathbf{k}$. The matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -3 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -2 & 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -3 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & 0 & 3 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \end{pmatrix}$$

which can be seen that it is of rank 7. The vector $\mathbf{k}$ is

$$\begin{aligned} \mathbf{k} \;=\; & \Big\{\frac{a}{3}(3b+4w), \frac{1}{3}(-4a - 3a^2 + 3b^2 + 16bw), \frac{1}{9}(-36b - 9ab - 56w), \\ & \frac{2}{9}(16+3a), \frac{4}{9}(3b - 8w)w, \frac{1}{9}(9ab + 32w - 36aw), \\ & \frac{1}{3}(2a - 3a^2 + 3b^2 + 4bw), \frac{1}{3}(-4b - 3ab + 8w)\Big\}. \end{aligned}$$

The matrix $(\mathbf{M}|\mathbf{k})$ has rank 8 as the determinant of one of its 8×8 minors equals $1 + w^2$. So, the linear system does not satisfy the compatibility condition and, hence, no such $\mathcal{Y}$ nor μ can exist.

The contents of this chapter belong to the paper entitled *Characterization of isochronous foci for planar analytic differential systems*, authored by J. Giné and M. Grau and which is a preprint, 2003.

Abstract. Consider the two-dimensional autonomous systems of differential equations of the form

$$\dot{x} = \lambda x - y + P(x,y) \ , \quad \dot{y} = x + \lambda y + Q(x,y) \ ,$$

where $P(x,y)$ and $Q(x,y)$ are analytic functions of order ≥ 2. These systems have a focus at the origin if $\lambda \neq 0$, and have either a center or a weak focus if $\lambda = 0$. In this work we study necessary and sufficient conditions for the existence of an isochronous point at the origin. Our result is original when applied to weak foci and gives known results when applied to strong foci or to centers.

Bibliography

[1] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Reprint of the 1972 edition. Dover Publications, Inc., New York, 1992.

[2] A. Algaba, E. Freire and E. Gamero, *Isochronicity via normal form*, Qual. Theory Dyn. Syst. **1** (2000), no. 2, 133–156.

[3] A. Andreev, *Investigation on the behaviour of the integral curves of a system of two differential equations in the neighborhood of a singular point*, Translation Amer. Math. Soc. **8** (1958) 187–207.

[4] A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maĭer, *Qualitative theory of second-order dynamic systems.* Translated from the Russian by D. Louvish. Halsted Press (A division of John Wiley & Sons), New York-Toronto, Ont.; Israel Program for Scientific Translations, Jerusalem-London, 1973.

[5] A.A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maĭer, *Theory of bifurcations of dynamic systems on a plane.* Translated from the Russian. Halsted Press (A division of John Wiley & Sons), New York-Toronto, Ont.; Israel Program for Scientific Translations, Jerusalem-London, 1973.

[6] V.I. Arnol'd, *Chapitres supplémentaires de la théorie des équations différentielles ordinaires.* "Mir", Moscow, 1980.

[7] V.I. Arnol'd, S.M. Guseĭn-Zade and A.N. Varchenko, *Singularities of differentiable maps. Vol. I. The classification of critical points, caustics and wave fronts.* Translated from the Russian by Ian Porteous and Mark Reynolds. Monographs in Mathematics, **82**. Birkhäuser Boston, Inc., Boston, MA, 1985.

[8] V.I. Arnol'd, S.M. Guseĭn-Zade and A.N. Varchenko, *Singularities of differentiable maps. Vol. II. Monodromy and asymptotics of integrals.* Translated from the Russian by Hugh Porteous. Translation revised by the authors and James Montaldi. Monographs in Mathematics, **83**. Birkhäuser Boston, Inc., Boston, MA, 1988.

[9] L. Autonne, *Sur la théorie des équations différentielles du premier ordre et du premier degré.* Journal de l'École Polytechnique, **61**(1891), 35-122; **62**(1892), 47-180.

[10] N.N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or centre type.* Trans. Amer. Math. Soc. **100** (1954), 397–413.

[11] M. Berthier, D. Cerveau and A. Lins Neto, *Sur les feuilletages analytiques réels et le problème du centre*, J. Differential Equations **131** (1996), 244–266.

[12] L. Cairó, H. Giacomini and J. Llibre, *Liouvillian first integrals for the planar Lotka-Volterra system.*, Rend. Circ. Mat. Palermo (2) **52** (2003), no. 3, 389–418.

[13] M. Carnicer, *The Poincaré problem in the nondicritical case*, Ann. Math. **140** (1994), 289–294.

[14] E. Casas-Alvero, *Singularities of Plane Curves.*, London Mathematical Society Lecture Note Series, **276**. Cambridge University Press, Cambridge, 2000.

[15] D. Cerveau and A. Lins Neto, *Holomorphic foliations in* $\mathbb{CP}(2)$ *having an invariant algebraic curve*, Ann. Inst. Fourier **41** (1991), 883–903.

[16] J. Chavarriga, *A new example of quartic algebraic limit cycle for a quadratic system*, Preprint, Universitat de Lleida (1998).

[17] J. Chavarriga and I.A. García, *On the existence of limit cycles for real quadratic differential systems with an invariant cubic.*, Pacific J. Math., to appear.

[18] J. Chavarriga, I.A. García and J. Giné, *On Lie's symmetries for planar polynomial differential systems*, Nonlinearity **14** (2001), no. 4, 863–880.

[19] J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, *On the integrability of two-dimensional flows.* J. Differential Equations **157** (1999), 163–182.

[20] J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, *Darboux integrability and the inverse integrating factor.* J. Differential Equations **194** (2003), no. 1, 116–139.

[21] J. Chavarriga, H. Giacomini and M. Grau, *Necessary conditions for the existence of invariant algebraic curves for planar polynomial systems*, Bull. Sci. Math., to appear.

[22] J. Chavarriga, H. Giacomini and J. Llibre, *Uniqueness of algebraic limit cycles for quadratic systems.*, J. Math. Anal. Appl., **261**, (2001), 85–99.

[23] J. Chavarriga and M. Grau, *Invariant algebraic curves linear in one variable for planar real quadratic systems.*, Appl. Math. Comput. **138** (2003), 291–308.

[24] J. Chavarriga and M. Grau, *A family of quadratic polynomial differential systems non-integrables Darboux and with algebraic solutions of arbitrarily high degree*, Appl. Math. Lett. **16** (2003), 833–837.

[25] J. Chavarriga and J. Llibre, *Invariant algebraic curves and rational first integrals for planar polynomial vector fields.* J. Differential Equations **169** (2001), 1–16.

[26] J. Chavarriga, J. Llibre and J. Moulin-Ollagnier, *On a result of Darboux.* LMS J. Comput. Math. **4** (2001), 197–210 (electronic).

[27] J. Chavarriga, J. Llibre and J. Sorolla, *Algebraic limit cycles of degree* 4 *for quadratic systems.* J. Differential Equations **200** (2004), 206–244.

[28] J. Chavarriga, J. Llibre and J. Sotomayor, *Algebraic solutions for polynomial systems with emphasis in the quadratic case.* Exposition. Math. **15** (1997), no. 2, 161–173.

[29] J. Chavarriga and M. Sabatini, *A survey of isochronous centers*, Qual. Theory Dyn. Syst. **1** (1999), no. 1, 1–70.

[30] Yuan-shün Ch'in, *On algebraic limit cycles of degree* 2 *of the differential equation*

$$\frac{dy}{dx} = \frac{\sum_{0 \leqq i+j \leqq 2} a_{ij} x^i y^j}{\sum_{0 \leqq i+j \leqq 2} b_{ij} x^i y^j}.$$

Sci. Sinica **7** (1958), 934–945, and Acta Math. Sinica **8** (1958), 23–35.

[31] S.N. Chow, C. Li and D. Wang, *Normal forms and bifurcation of planar vector fields*, Cambrige Univ. Press, 1994.

[32] C. Christopher, *Invariant algebraic curves and conditions for a centre.* Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), 1209–1229.

[33] C. Christopher, *Liouvillian first integrals of second order polynomial differential systems.*, Electron. J. Differential Equations, Vol. **1999**(1999), 1–7.

[34] C. Christopher and J. Llibre, *A family of quadratic polynomial differential systems with invariant algebraic curves of arbitrarily high degree without rational first integrals.* Proc. Amer. Math. Soc. **130** (2002), 2025–2030 (electronic).

[35] C. Christopher, J. Llibre and J.V. Pereira, *Multiplicity of invariant algebraic curves in polynomial vector fields.* Preprint, 2004.

[36] C. Christopher, J. Llibre and G. Świrszcz, *Invariant algebraic curves of large degree for quadratic systems.* Preprint, 2003.

[37] R. Conti, *Uniformly isochronous centers of polynomial systems in* $\mathbb{R}^2$, Lecture Notes in Pure Appl. Math., vol. **152**, pp. 21–31, M. Dekker, New York, (1994).

[38] G. Darboux, *Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré (Mélanges)*, Bull. Sci. Math. **32** (1878), 60–96; 123–144; 151–200.

[39] V. A. Dobrovol'skii, N. V. Lokot' and J.-M. Strelcyn, *Mikhail Nikolaevich Lagutinskii* (1871 – 1915)*: un mathématicien méconnu.*

[Mikhail Nikolaevich Lagutinskii (1871–1915): an unrecognized mathematician] Historia Math. **25** (1998), no. 3, 245–264.

[40] H. Dulac, *Détermination et intégration d'une certaine classe d'équations différentielles ayant pour point singulier un centre,* Bull. Sci. Math. Sér. (2) **32**(1) (1908), 230–252.

[41] F. Dumortier, *Singularities of vector fields on the plane*, J. Differential Equations **23**, 53–106 (1977).

[42] R.M. Evdokimenco, *Construction of algebraic paths and the qualitative investigation in the large of the properties of integral curves of a system of differential equations*, Differential Equations **6** (1970), 1349–1358.

[43] R.M. Evdokimenco, *Behavior of integral curves of a dynamic system*, Differential Equations **9** (1974), 1095–1103.

[44] R.M. Evdokimenco, *Investigation in the large of a dynamic system with a given integral curve*, Differential Equations **15** (1979), 215–221.

[45] A. Ferragut, *Polynomial inverse integrating factors of quadratic differential systems.* Master thesis, Universitat Autònoma de Barcelona, 2003.

[46] V.F. Filiptsov, *Algebraic limit cycles*, Differential Equations **9** (1973), 983–988.

[47] E. Freire, A. Gasull and A. Guillamon, *A characterization of isochronous centres in terms of symmetries*, Rev. Mat. Iberoamericana, **20** (2004), no. 1, 205-222.

[48] W. Fulton, *Algebraic curves. An introduction to algebraic geometry.* Notes written with the collaboration of Richard Weiss. Mathematics Lecture Notes Series. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[49] A. Gasull, A. Guillamon and J. Villadelprat, *The period function for second-order quadratic ODEs is monotone*, Qual. Theory Dyn. Syst. **4** (2004), no. 2, 329–352.

[50] I.A. García and J. Giné, *Generalized cofactors and nonlinear superposition principles,* Appl. Math. Lett. **16** (2003), no. 7, 1137–1141.

[51] I.A. García and J. Giné, *Non-algebraic invariant curves for polynomial planar vector fields.* Discrete Contin. Dyn. Syst. **10** (2004), no. 3, 755–768.

[52] H. Giacomini, J. Giné and M. Grau, *Integrability of planar polynomial differential systems through linear differential equations*, Rocky Mountain J. Math., to appear.

[53] H. Giacomini and M. Grau, *On the stability of limit cycles for planar differential systems*, Preprint, 2004.

[54] H. Giacomini, J. Llibre and M. Viano, *On the nonexistence, existence and uniqueness of limit cycles.* Nonlinearity **9** (1996), no. 2, 501–516.

[55] J. Giné, *Isochronous foci for analytic differential systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., **13** (2003), no. 6, 1–7.

[56] J. Giné and M. Grau, *Characterization of isochronous foci for planar analytic differential systems*, Proc. Roy. Soc. Edinburgh Sect. A, submitted to.

[57] J. Giné, X. Santallusia, *On the Poincaré-Lyapunov constants and the Poincaré series*, Appl. Math. (Warsaw) **28** (2001), no. 1, 17–30.

[58] E.A. González Velasco, *Generic properties of polynomial vector fields at infinity.* Trans. Amer. Math. Soc. **143** (1969), 201-222.

[59] D. Hilbert, *Mathematische Problem (lecture)*, Second Internat. Congress Math. Paris 1900, Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. 1900, 253–297.

[60] M.W. Hirsch and S. Smale, *Differential equations, dynamical systems, and linear algebra.* Pure and Applied Mathematics, Vol. **60**. Academic Press, New York-London, 1974.

[61] J.P. Jouanolou, *Équations de Pfaff algébriques.* Lecture Notes in Mathematics, **708**. Springer, Berlin, 1979.

[62] A. Lins Neto, *Folheações em* $\mathbb{CP}(2)$ *com curva algébrica invariante e singularidades radiais*, Instituto de Estudios con Iberoamerica y Portugal, Seminarios Temáticos, Vol **2**, Fasc. III, Septiembre 1998.

[63] A. Lins Neto, *Some examples for the Poincaré and Painlevé problems.*, Ann. Scient. Éc. Norm. Sup., 4^e série, t. **35** (2002), p.231–266.

[64] J. Llibre, *Integrability of polynomial differential systems.* Preprint, 2003.

[65] J. Llibre and G. Rodríguez, *Configurations of limit cycles and planar polynomial vector fields.* J. Differential Equations **198** (2004), no. 2, 374–380.

[66] V.A. Lunkevich and K.S. Sibirskiĭ, *Integrals of a general quadratic differential system in cases of a center.* Diff. Equations **18** (1982), 563–568.

[67] J. Moulin Ollagnier, *About a conjecture on quadratic vector fields.* J. Pure Appl. Algebra **165** (2001), 227–234.

[68] J. Moulin Ollagnier, *Liouvillian integration of the Lotka-Volterra system.*, Qual. Theory Dyn. Syst. **2** (2001), no. 2, 307–358.

[69] R. Moussu, *Une démonstration d'un théorème de Lyapunov-Poincaré*, Astérisque **98-99** (1982), 216–223.

[70] P.J. Olver, *Applications of Lie groups to differential equations*, Second edition. Graduate Texts in Mathematics, **107**. Springer-Verlag, New York, 1993.

[71] P. Painlevé, *Mémoire sur les équations différentielles du premier ordre.* Ann. Sci. École Norm. Sup., **3 serie**, 8(1891), 9-59, 103-140, 201-226 et 276-284; 9(1892), 9-30, 101-144 et 283-308.

[72] C. Pantazi, *Inverse problems of the Darboux theory of integrability for planar polynomial differential systems.* Doctoral thesis, Universitat Autònoma de Barcelona, July 2004.

[73] J. V. Pereira, *Vector fields, invariant varieties and linear systems.* Ann. Inst. Fourier (Grenoble) **51** (2001), no. 5, 1385–1405.

[74] L. Perko, *Differential equations and dynamical systems.* Third edition. Texts in Applied Mathematics, **7**. Springer-Verlag, New York, 2001.

[75] H. POINCARÉ, *Mémoire sur les courbes définies par les équations différentielles*, J. Math. Pures Appl. (3) **7** (1881), 375-422; **8** (1882), 251-296; Oeuvres de Henri Poincaré, vol. I, Gauthier-Villars, Paris 1951, 3–84.

[76] H. POINCARÉ, *Sur l'intégration algébrique des équations différentielles,* C.R. Acad. Sci. Paris **112**(1891), 761-764.

[77] H. POINCARÉ, *Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré,* Rend. Circ. Mat. Palermo **5** (1891), 161–191.

[78] H. POINCARÉ, "Thesis", 1879; also "Oeuvres I", 59-129, Gauthier Villars, Paris, 1928.

[79] M.J. PRELLE AND M.F. SINGER, *Elementary first integrals of differential equations.*, Trans. Amer. Math. Soc. **279** (1983), 215–229.

[80] M. SABATINI, *Characterizing isochronous centres by Lie brackets*, Differential Equations Dynam. Systems **5** (1997), no. 1, 91–99.

[81] M. SABATINI, *Dynamics of commuting systems on two-dimensional manifolds*, Ann. Mat. Pura Appl. (4) **173** (1997), 213–232.

[82] M. SABATINI, *Non periodic isochronous oscillations in plane differential systems*, Ann. Mat. Pura Appl. (4) **182** (2003), 487–501.

[83] D. SCHLOMIUK, *Algebraic particular integrals, integrability and the problem of center.*, Trans. Amer. Math. Soc. **338** (1993), 799–841.

[84] D. SCHLOMIUK, *Algebraic and geometric aspects of the theory of polynomial vector fields.* Bifurcations and Periodic Orbits of Vector Fields, D. Schlomiuk (ed.), (1993), 429–467.

[85] A. SEIDENBERG, *Reduction of singularities of the differential equation* $Ady = Bdx$, Amer. J. Math. **90** (1968) 248–269.

[86] M.F. SINGER, *Liouvillian first integrals of differential equations.*, Trans. Amer. Math. Soc. **333** (1992), 673–688.

[87] SHI SONGLING, *A method of constructing cycles without contact around a weak focus.* J. Differential Equations **41** (1981), 301–312.

[88] J. Sotomayor, *Liçoes de equaçoes diferenciais ordinárias.* Projeto Euclides, **11**. Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1979.

[89] S. Smale, *Mathematical problems for the next century.* Math. Intelligencer **20** (1998), no. 2, 7–15.

[90] M. Villarini, *Regularity properties of the period function near a center of a planar vector field,* Nonlinear Anal. **19** (1992), 787–803.

[91] S. Walcher, *On the Poincaré Problem*, J. Differential Equations **166** (2000), 51–78.

[92] R.J. Walker, *Algebraic curves.* Reprint of the 1950 edition. Springer-Verlag, New York-Heidelberg, 1978.

[93] Eric W. Weisstein, *The Wolfram Functions Site.* `http://functions.wolfram.com`, 2000-2003 Wolfram Research, Inc.

[94] A.I. Yablonskiĭ, *On limit cycles of certain differential equations*, Differential Equations **2** (1966), 164–168.

[95] Ye Yan Qian *and others*, *Theory of limit cycles.* Translations of Mathematical Monographs, **66**. American Mathematical Society, Providence, RI, 1986.

[96] Zhi Fen Zhang *and others* *Qualitative theory of differential equations.* Translations of Mathematical Monographs, **101**. American Mathematical Society, Providence, RI, 1992.

[97] H. Żołądek, *The classification of reversible cubic systems with center.* Topol. Methods Nonlinear Anal. **4** (1994), 79–136.

[98] H. Żołądek, *Remarks on: "The classification of reversible cubic systems with center."* Topol. Methods Nonlinear Anal. **8** (1996), 335–342.

Other publications

The author also has the following publications which are not part of this memory.

J. Chavarriga, J. Giné and M. Grau, *Integrable systems via polynomial inverse integrating factors*, Bull. Sci. Math., **126** (2002), 315–331.

Abstract. We study the integrability of two–dimensional autonomous systems in the plane of the form $\dot{x} = -y + X_s(x, y)$, $\dot{y} = x + Y_s(x, y)$, where $X_s(x, y)$ and $Y_s(x, y)$ are homogeneous polynomials of degree s with $s \geq 2$. Writing this system in polar coordinates, we study the existence of polynomial inverse integrating factors and we give some related invariants, from which we can compute a formal first integral for the system. Finally, we give a family of systems with $s = 4$ and with a center at the origin, via inverse integrating factors, in which radial and angular coefficients do not independently vanish in Liapunov constants.

J. Chavarriga and M. Grau, *Invariant algebraic curves linear in one variable for planar real quadratic systems*, Appl. Math. Comput., **138** (2003), no.2-3, 291–308.

Abstract. In this paper we study algebraic solutions of the form $h(x, y) = p_1(x)y + p_2(x)$, where $p_1(x)$ and $p_2(x)$ are real coprime polynomials and $p_1(x) \neq 0$, for a planar real algebraic quadratic system. We analyze the existence of this kind of solutions depending on the coefficients of the system.

J. Chavarriga and M. Grau, *A family of quadratic polynomial differential systems non-integrables Darboux and with algebraic solutions of arbitrarily high degree*, Appl. Math. Lett., **16** (2003), 833–837.

Abstract. We show that the system $\dot{x} = 1$, $\dot{y} = 2n + 2xy + y^2$ has the algebraic solution $h(x, y) = H_n(x)y + 2nH_{n-1}(x)$, where $H_n(x)$ is the Hermite polynomial of degree n, and the system is not Darboux integrable and has no Darboux integrating factor for any $n \in \mathbb{N}$.

J. Chavarriga, E. Sáez, I. Szántó and M. Grau, *Coexistence of limit cycles and invariant algebraic curves on a Kukles system*, accepted for publication in Nonlinear Anal.

Abstract. We consider a Kukles system of the form $\dot{x} = -y$, $\dot{y} = f(x, y)$ where $f(x, y)$ is a polynomial with real coefficients of degree d without y as a divisor. We study the maximum number of small-amplitude limit cycles for these kind of systems which can coexist with invariant algebraic curves. We give all the possible distributions of invariant straight lines for a Kukles system and we give some bounds for the number of limit cycles. We also give some necessary conditions for the existence of an invariant algebraic curve of degree ≥ 2 and we study the possible coexistence of this curve and a limit cycle.
Finally, we give two examples of cubic Kukles systems both with an invariant hyperbola. In the first example the hyperbola coexists with a center and in the second one it coexists with two small-amplitude limit cycles. These two examples contradict a previous result given in:
Xin An Yang, *A survey of cubic systems.* Ann. Differential Equations **7** (1991), no. 3, 323–363.

The contents of the aforementioned publications are all contained in the author's Master Ph thesis entitled *On the center problem, the integrability and the 16 Hilbert problem for planar polynomial differential systems* which was passed on June of 2002 in the Universitat Autònoma de Barcelona.

J. Chavarriga and M. Grau, *Some open problems related to* 16*b Hilbert problem*, Sci. Ser. A Math. Sci. (N.S.), **9** (2003), 1–26.

The author has the following preprint, 2003.
I.A. García, J. Giné and M. Grau, *A necessary condition in the monodromy problem for analytic differential equations on the plane.*

Abstract. In this paper we give a very easy to compute necessary condition in the monodromy problem for all singular point of analytic differential systems in the real plane. Our main tool is considering the analytic function, *angular speed*, and studying its limit through straight lines to the singular point.

A copy of this article may be downloaded in:
`http://www.udl.es/dept/matematica/ssd/`

The following preprint is a work still in process.

H. Giacomini, M. Grau and J. Llibre, *Cyclicity of the homoclinic orbit in the family of Liénard systems of degree* 4.

Abstract. We consider the following family of Liénard system in $\mathbb{R}^2$ of degree 4:

$$\dot{x} = y - (a_2 x^2 + x^4) - \varepsilon(P_1(\varepsilon)x + P_3(\varepsilon)x^3), \quad \dot{y} = -x,$$

where $P_1(\varepsilon)$ and $P_3(\varepsilon)$ are analytic real functions in ε near $\varepsilon = 0$, $P_1(0)^2 + P_3(0)^2 \neq 0$, and $|\varepsilon| << 1$. When $\varepsilon = 0$, the system has a center at the origin whose period annulus is bounded by a homoclinic orbit Γ. The saddle of Γ is situated in the point $(0, 1, 0)$ of the equator of the Poincaré sphere once the system for $\varepsilon = 0$ has been compactified. We show that one, and only one, limit cycle bifurcates from Γ when perturbing ε.
Our crucial tool consists in a new technique in the study of bifurcations of limit cycles from graphs which considers an approximation of an inverse integrating factor for the system.

Notation

$\mathbb{R}$ denotes the field of real numbers.

$\mathbb{C}$ denotes the field of complex numbers.

i denotes $\sqrt{-1}$.

If $\mathbb{K}$ is any field, we denote by:

$\mathbb{K}[x_1, x_2, \ldots, x_n]$ the ring of **polynomials** in the variables $x_1, x_2, \ldots, x_n$ and coefficients in $\mathbb{K}$,

$\mathbb{K}(x_1, x_2, \ldots, x_n)$ the quotient field of the ring $\mathbb{K}[x_1, x_2, \ldots, x_n]$, that is

$$\mathbb{K}(x_1, x_2, \ldots, x_n) = \left\{ \frac{A}{B} \;\middle|\; A, B \in \mathbb{K}[x_1, x_2, \ldots, x_n],\ B \neq 0 \right\}$$

under the equivalence relation $\dfrac{A}{B} \sim \dfrac{\tilde{A}}{\tilde{B}} \Leftrightarrow A\,\tilde{B} = B\,\tilde{A}$,

$\mathbb{K}[[x_1, x_2, \ldots, x_n]]$ the ring of **formal power series** in the variables $x_1, x_2, \ldots, x_n$ and coefficients in $\mathbb{K}$,

$\mathbb{K}((x_1, x_2, \ldots, x_n))$ the quotient field of the ring $\mathbb{K}[[x_1, x_2, \ldots, x_n]]$, i.e.,

$$\mathbb{K}((x_1, x_2, \ldots, x_n)) = \left\{ \frac{\phi}{\psi} \;\middle|\; \phi, \psi \in \mathbb{K}[[x_1, x_2, \ldots, x_n]],\ \psi \neq 0 \right\}$$

under the equivalence relation $\dfrac{\phi}{\psi} \sim \dfrac{\tilde{\phi}}{\tilde{\psi}} \Leftrightarrow \phi\,\tilde{\psi} = \psi\,\tilde{\phi}$, and

$\mathbb{K}\{x_1, x_2, \ldots, x_n\}$ the ring of **convergent power series** in the variables $x_1, x_2, \ldots, x_n$ and coefficients in $\mathbb{K}$.

$\mathbb{CP}(2)$ denotes the complex projective plane.

ode is the acronym for *ordinary differential equation*.

$\Phi_t(p)$ is the *flow* of a differential system with initial condition $\Phi_0(p) = p$.

Σ denotes a transversal *section* to the flow.

$\mathcal{P}$ denotes the Poincaré map.

$\mathbb{S}^2$ denotes the Poincaré sphere.

$\mathbf{v}$ denotes a non-null vector.

$\mathbf{F}(x, y)$ denotes the vector field at each point defined by a system (1.1), $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$.

∇ denotes the gradient.

div denotes the divergence of a differential system.

$\mathrm{K}(\omega)$ denotes the elliptic integral of first kind.

$\mathrm{E}(\omega)$ denotes the elliptic integral of second kind.

$\Pi(\kappa, \omega)$ denotes the elliptic integral of third kind.

Index

Erratum

Thesis title:	Contribution to the qualitative study of planar differential systems.
Author:	Maria Teresa Grau Montaña
Advisors:	Dr. Javier Chavarriga
	Dr. Héctor Giacomini
Tutor:	Dr. Jaume Llibre
Date of defense:	17^{th} December 2004.

p. 18, lines 3,4 ...where $\Sigma = \{(x,y) \in \mathbb{R}^2 \mid (f_1 \cdot f_2 \cdot \ldots \cdot f_r)(x,y) = 0\}$. We remark that, particularly, if $\lambda_i \in \mathbb{Z}$, $\forall i = 1,2,\ldots,r$, ... *must be substituted with* ...where $\Sigma = \{(x,y) \in \mathbb{R}^2 \mid (f_1 \cdot f_2 \cdot \ldots \cdot f_s)(x,y) = 0\}$. We remark that, particularly, if $\lambda_i \in \mathbb{Z}$, $\forall i = 1,2,\ldots,s$, ...

p. 19, line -12 ...in $\mathbb{C}[x,y]$, λ_i $(1 < i < r)$ and $\mu_j = 0$ $(1 < j < \ell)$ are complex numbers ... *must be substituted with* ...in $\mathbb{C}[x,y]$, λ_i $(1 < i < r)$ and μ_j $(1 < j < \ell)$ are complex numbers ...

p. 21, line -15 An elementary first for this system ... *must be substituted with* An elementary first integral for this system ...

p. 21, line -10 The reciprocals of the statements of Theorems 1.3 and 1.4 are not true. *must be substituted with* The reciprocals of the statement of Theorem 1.3 is not necessarily true. The reciprocal of the statement of Theorem 1.4 is proved in [33].

p. 21, line -4 ...Theorem 1.6 ensures that given an algebraic inverse integrating factor ... *must be substituted with* ...Theorem 1.6 ensures that given a (generalized) Darboux inverse integrating factor ...

p. 24, line 15 As is it proved ... *must be substituted with* As it is proved ...

p. 48, line -12 ... the Lotka-Volterra case and the reversible case. *must be substituted with* ... the Lotka-Volterra case and the critical case.

p. 63, line -2 *Before Lemma* 3.7 *it must be written:* The following lemma is also proved in the article:

A. J. Maciejewski, J. Moulin Ollagnier and A. Nowicki, *Generic polynomial vector fields are not integrable.* Indag. Math. **15** (2004), no. 1, 55–72.

p. 75, line 4 ... planar polynomial differentials systems defined by $\mathcal{C}^1$ functions ... *must be substituted with* ... planar differential systems defined by $\mathcal{C}^1$ functions ...

p. 99, line -10 ... where this integral is done over the oval γ of the curve $v^2 - g(a, \tau) = 0$. *must be substituted with* ... where this integral is done over an homotopic representant of the real oval γ of the curve $v^2 - g(a, \tau) = 0$.

p. 103, line -3 ... where this integral is done over the oval γ of the curve $v^2 - g = 0$. *must be substituted with* ... where this integral is done over an homotopic representant of the real oval γ of the curve $v^2 - g = 0$.

p. 111, line 2 *After the first sentence it must be written:* We only consider, in this chapter, non–degenerate singular points.