Successions d'interpolació en certs espais de funcions

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Certifico que la present Memòria ha estat
realitzada per en Daniel Blasi Babot,
sota la direcció del Dr. Artur Nicolau Nos.

Bellaterra, desembre de 2007.

Firmat: Dr. Artur Nicolau Nos.

Non gogoa, han zangoa.
(Euskal atsotitza)

On van els teus pensaments, van els teus passos.
(Proverbi basc)

Contents

Introducció

Sigui H^∞ l'àlgebra de les funcions analítiques i acotades al disc unitat obert $\mathbb{D}$. Direm que una successió de punts $\{z_n\} \subset \mathbb{D}$ és d'*interpolació per* H^∞ si per tota successió de valors acotats $\{w_n\}$ existeix una funció f analítica i acotada a $\mathbb{D}$ amb $f(z_n) = w_n$ per tota $n = 1, 2, \cdots$. Als anys 50, R.C. Buck va plantejar el següent problema. Existeixen successions d'interpolació per H^∞ amb infinits punts? R. C. Buck, A. Gleason i D. Newman [New] van obtenir resultats parcials i W. Hayman [Hay] va provar que la condició

$$\inf_k \prod_{n \neq k} \left| \frac{z_n - z_k}{1 - z_n \overline{z}_k} \right| > 0 \tag{0.0.1}$$

era necessària i que una condició lleugerament més forta era també suficient. L'any 1958, L. Carleson [Ca1] va provar que de fet, la condició (0.0.1) era també suficient.

Teorema A. *[Ca1] Si $\{z_n\}$ és una successió de punts al disc unitat $\mathbb{D}$, les següents condicions són equivalents.*

(a) La successió $\{z_n\}$ és d'interpolació per H^∞, és a dir, tot problema d'interpolació

$$f(z_n) = w_n, \quad n = 1, 2, ...$$

amb $\sup_n |w_n| < \infty$ té solució $f \in H^\infty$.

(b) Existeix una constant $\delta > 0$ tal que

$$\prod_{n \neq k} \left| \frac{z_n - z_k}{1 - z_n \overline{z}_k} \right| \geq \delta, \quad k = 1, 2, ...$$

(c) La successió $\{z_n\}$ és separada, és a dir, existeix una constant $c > 0$ amb

$$\rho(z_n, z_k) = \left| \frac{z_n - z_k}{1 - z_n \overline{z}_k} \right| \geq c, \text{ per } n \neq k$$

i existeix una constant $C > 0$ tal que per tot quadrat de Carleson $Q(z)$ es compleix

$$\sum_{z_n \in Q(z)} (1 - |z_n|) \leq C(1 - |z|) \tag{0.0.2}$$

Aquí $Q(z) = \{w \in \mathbb{D} \; : \; |Argz - Argw| \leq \pi(1 - |z|), \; 1 - |w| \leq 1 - |z|\}$ per $z \in \mathbb{D} \setminus \{0\}$ és el *quadrat de Carleson* que es mostra en la figura 1 i $Q(0) = \mathbb{D}$.

Sigui μ una mesura de Borel positiva al disc unitat $\mathbb{D}$. Direm que μ és una *mesura de Carleson* si existeix una constant $C > 0$ de forma que

$$\mu(Q(z)) \leq C(1 - |z|),$$

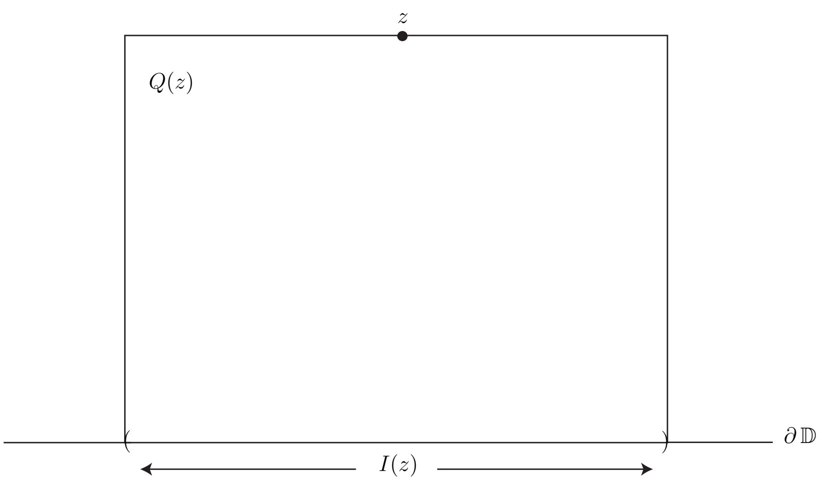

Figure 1: un quadrat de Carleson $Q(z)$

per tot quadrat de Carleson $Q(z)$. Així doncs la condició (0.0.2) es pot reescriure com

$$\mu = \sum_n (1 - |z_n|)\delta_{z_n} \quad \text{és una mesura de Carleson.}$$

L'ampli impacte del teorema de Carleson queda de manifest en el tribut de Peter Jones al treball de Lennart Carleson. *"This result is now understood to be one of the pillars of function theory, and it shows up in areas ranging from the corona problem to operator theory (and many places between)"* [Jo2].

El teorema de Carleson ha estat investigat almenys en dues direccions, que corresponen a intentar entendre:

- El paper que hi juga l'holomorfia
- El paper que hi juga el fet que les funcions siguin acotades

Aquesta memòria consta de dos capítols relacionats amb aquestes dues direccions. Es tracta de capítols independents que poden llegir-se per separat.

En el primer capítol de la memòria estudiem un problema d'interpolació per funcions harmòniques i positives. Abans de plantejar el problema però, farem esment d'alguns dels resultats coneguts en espais de funcions no holomorfes. Als anys 70, L. Carleson i J. Garnett es van plantejar el problema de caracteritzar les successions d'interpolació per h^∞, l'espai de funcions harmòniques i acotades al disc unitat. Una successió $\{z_n\}$ de punts del disc unitat direm que és d'*interpolació per* h^∞ si per qualsevol successió acotada $\{w_n\}$ existeix una funció $f \in h^\infty$ amb $f(z_n) = w_n$ per tota $n = 1, 2, \cdots$.

L'any 1975, L. Carleson i J. Garnett van provar el següent resultat que dóna la descripció de les successions d'interpolació per funcions harmòniques i acotades al disc unitat $\mathbb{D}$. Aquesta descripció resulta ser idèntica a la que caracteritza les successions d'interpolació per l'espai H^∞.

Teorema B. *[CG] Una successió de punts $\{z_n\} \subset \mathbb{D}$ és d'interpolació per $h^\infty(\mathbb{D})$ si i només si $\{z_n\}$ és separada i existeix una constant $C > 0$ tal que*

$$\sum_{z_n \in Q(z)} (1 - |z_n|) \leq C(1 - |z|)$$

per tot quadrat de Carleson $Q(z)$.

Si considerem el problema d'interpolació per funcions harmòniques i acotades al semiplà $\mathbb{R}^2_+$ la situació és anàloga al Teorema B. L. Carleson i J. Garnett van estudiar també el problema de caracteritzar les successions d'interpolació per l'espai $h^\infty(\mathbb{R}^{d+1}_+)$ de les funcions harmòniques i acotades al semiespai $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times \mathbb{R}^+$, per $d > 1$, però el resultat que van obtenir és més feble ja que la prova del cas $d = 1$ utilitza eines de variable complexa. Donat un punt $(x, y) \in \mathbb{R}^{d+1}_+$, definim el *cub de Carleson* Q associat al punt (x, y) com $Q = \{(b, t) \in \mathbb{R}^d \times \mathbb{R}^+ \ : \ |x - b| < l(Q), \ 0 < t < l(Q)\}$, on $l(Q) = y$. Direm també que una *successió* de punts $\{z_n\}$ de $\mathbb{R}^{d+1}_+$ és *separada* si $\inf_{n \neq m} \beta(z_n, z_m) > 0$, on $\beta(z, w)$ és la distància hiperbòlica entre els punts $z, w \in \mathbb{R}^{d+1}_+$. El resultat d'interpolació per $h^\infty(\mathbb{R}^{d+1}_+)$ és el següent:

Teorema C. *[CG] Sigui $d > 1$. Donada una successió de punts $\{z_n = (x_n, y_n)\}$ a $\mathbb{R}^{d+1}_+$, considerem la mesura $\mu = \sum_n y_n^d \delta_{z_n}$. Aleshores,*

(a) Si $\{z_n\}$ és una successió d'interpolació per $h^\infty(\mathbb{R}^{d+1}_+)$ llavors $\{z_n\}$ és separada i la mesura μ compleix $\mu(Q) \leq Cl(Q)^d$ per tot cub de Carleson Q.

(b) Si $\{z_n\}$ és una successió separada i la mesura μ compleix $\mu(Q) \leq Cl(Q)^d$ per tot cub de Carleson Q, llavors podem trencar la successió $\{z_n\}$ en una unió finita de subsuccessions Λ_i, $i = 1, ..., N$,

$$\{z_n\} = \Lambda_1 \cup \cdots \cup \Lambda_N$$

de manera que per qualssevol $i, j = 1, ..., N$, la successió $\Lambda_i \cup \Lambda_j$ és d'interpolació per $h^\infty(\mathbb{R}^{d+1}_+)$.

No se sap si les condicions necessàries de l'apartat (a) del Teorema C són també suficients excepte en dimensió $d = 1$. En cas de tenir condicions més fortes sobre la successió, L. Carleson i J. Garnett van demostrar que la successió és d'interpolació.

Teorema D. *[CG] Donada una successió de punts $\{(x_n, y_n)\}$ de $\mathbb{R}^{d+1}_+$, suposem que existeixen unes constants $C, c > 0$ tals que per tota n, existeix un punt $b_n \in \mathbb{R}^d$, amb $|x_n - b_n| < Cy_n$ de forma que les boles $\{x \in \mathbb{R}^d \ : \ |x - b_n| < c\, y_n\}$ siguin disjuntes. Aleshores la successió $\{(x_n, y_n)\}$ és d'interpolació per $h^\infty(\mathbb{R}^{d+1}_+)$.*

N. Th. Varopoulos [Va] i L. Carleson [Ca3] van obtenir de forma independent condicions necessàries per problemes d'interpolació en contextos més generals quan en lloc de considerar extensions harmòniques de funcions acotades a la frontera del disc $\partial\mathbb{D}$, es considera la convolució contra nuclis més generals que el nucli de Poisson.

El 1994, K. Dyakonov [Dy] va estudiar el problema de caracteritzar les successions d'interpolació en espais de funcions acotades amb nuclis integrables. Si considerem el cas particular del nucli de Poisson el resultat de K. Dyakonov ens diu que si existeixen subconjunts de Borel E_n disjunts dos a dos de $\mathbb{R}^d$ tals que

$$\inf_n \omega(z_n, E_n, \mathbb{R}^{d+1}_+) > \frac{1}{2}$$

aleshores $\{z_n\}$ és una successió d'interpolació per $h^\infty(\mathbb{R}^{d+1}_+)$. Aquí, $\omega(z_n, E_n, \mathbb{R}^{d+1}_+)$ denota la mesura harmònica del subconjunt E_n des del punt z_n.

Donat $1 < p < \infty$, E. Amar [Am] a finals dels anys 70 va plantejar un problema d'interpolació per l'espai $h^p(\mathbb{R}^{d+1}_+)$ de funcions harmòniques al semiespai $\mathbb{R}^{d+1}_+$ obtingudes a partir d'integrals de Poisson de funcions de $L^p(\mathbb{R}^d)$. Donada una successió de punts $\{z_n\} \subset \mathbb{R}^{d+1}_+$, considerem l'operador T_p que assigna a cada funció $u \in h^p$ la successió

$$T_p(u) = \{u(z_n) y_n^{d/p}\}.$$

Direm que la successió $\{z_n\}$ és d'*interpolació per* $h^p(\mathbb{R}^{d+1}_+)$ si l'operador T_p compleix $T_p(h^p) \supset l^p$. Resultats parcials similars al Teorema C van ser obtinguts per J. Garnett [Ga2] i E. Amar [Am] en el context de les funcions de $h^p(\mathbb{R}^{d+1}_+)$ però el problema de si les condicions de separació i Carleson són suficients per caracteritzar les successions d'interpolació per $h^p(\mathbb{R}^{d+1}_+)$ en dimensió $d > 1$ continua sent un problema obert. En el cas $d = 1$ es té una descripció completa anàloga al Teorema B. [Ga2].

En el capítol 1 d'aquesta memòria volem estudiar un problema anàleg d'interpolació al con de funcions harmòniques i positives al disc unitat $\mathbb{D}$ del pla complex que denotarem per $h^+ = h^+(\mathbb{D})$. Si $u \in h^+$, la *desigualtat* clàssica *de Harnack* ens diu que

$$\frac{1-|z|}{1+|z|} \le \frac{u(z)}{u(0)} \le \frac{1+|z|}{1-|z|}$$

per tota $z \in \mathbb{D}$. Recordem que la *distància hiperbòlica* $\beta(z, w)$ entre dos punts $z, w \in \mathbb{D}$ és

$$\beta(z, w) = \log_2 \frac{1 + \left|\frac{z-w}{1-\bar{w}z}\right|}{1 - \left|\frac{z-w}{1-\bar{w}z}\right|}.$$

Aleshores les estimacions anteriors es poden escriure com $|\log_2 u(z) - \log_2 u(0)| \le \beta(z, 0)$. Com que aquestes nocions es preserven per automorfismes del disc, podem deduir que

$$|\log_2 u(z) - \log_2 u(w)| \le \beta(z, w) \tag{0.0.3}$$

per totes $z, w \in \mathbb{D}$. Així doncs, per tota funció $u \in h^+$, una successió de punts $\{z_n\} \subset \mathbb{D}$ i la corresponent successió de valors $w_n = u(z_n)$, $n = 1, 2, \cdots$ estan relacionades per l'estimació

$$|\log_2 w_n - \log_2 w_m| \leq \beta(z_n, z_m), \quad n, m = 1, 2, \cdots . \tag{0.0.4}$$

Malgrat tot, donada una successió de punts $\{z_n\} \subset \mathbb{D}$, no podem esperar poder interpolar per una funció de h^+ tota successió de valors positius $\{w_n\}$ que satisfaci la condició de compatibilitat (0.0.4) a menys que la successió es redueixi a dos únics punts. És ben conegut que si tenim igualtat en (0.0.3) per dos punts diferents $z, w \in \mathbb{D}$, això força que la funció u sigui un nucli de Poisson i per tant no podem esperar interpolar més valors. En altres paraules, l'espai de traça natural donat per la condició (0.0.4) és massa gran, i ens veiem induïts a considerar la següent noció.

Una successió de punts $\{z_n\}$ al disc unitat s'anomenarà d'*interpolació per* h^+ si existeix una constant $\varepsilon = \varepsilon(\{z_n\}) > 0$, tal que per tota successió de valors positius $\{w_n\}$ satisfent

$$|\log_2 w_n - \log_2 w_m| \leq \varepsilon\beta(z_n, z_m), \quad n, m = 1, 2, \cdots \tag{1.0.1}$$

existeix una funció $u \in h^+$ amb $u(z_n) = w_n$, $n = 1, 2, \cdots$

Observem que aquesta noció és conformement invariant, és a dir, si $\{z_n\}$ és una successió d'interpolació per h^+, també ho serà $\{\tau(z_n)\}$ per tot automorfisme τ del disc unitat. A més a més les constants corresponents satisfan que $\varepsilon(\{\tau(z_n)\}) = \varepsilon(\{z_n\})$. El resultat principal del primer capítol és el següent.

Theorem 1.0.1. *Una successió separada $\{z_n\}$ de punts del disc unitat és d'interpolació per h^+ si i només si existeixen constants $M > 0$ i $0 < \alpha < 1$ tals que*

$$\#\{z_j : \beta(z_j, z_n) \leq l\} \leq M 2^{\alpha l} \tag{1.0.2}$$

per $n, l = 1, 2, \cdots$

Restringim l'atenció a successions separades perquè volem considerar un problema d'interpolació per funcions harmòniques i positives i no per les seves derivades. Tot i així cal mencionar que tota successió d'interpolació per h^+ és la unió de com a molt tres successions separades. Anem ara a discutir la condició (1.0.2). Com és habitual en aquests tipus de problemes, la descripció geomètrica de les successions d'interpolació ve donada en termes d'una condició de densitat que diu, en el sentit apropiat, que les successions d'interpolació no són massa denses. El número 2 apareix en (1.0.2) a causa de la normalització de la distància hiperbòlica. Hem escollit aquesta normalització perquè encaixa perfectament amb les descomposicions diàdiques tal i com es pot veure a l'Observació 1.0.2. A la secció 1.3 mostrarem que hi ha una sèrie de condicions que són equivalents a la condició (1.0.2). Per exemple, una successió $\{z_n\}$ satisfà (1.0.2) si i només si existeixen constants $M_1 > 0$ i $0 < \alpha < 1$ tals que

$$\#\left\{z_j : \left|\frac{z_j - z_n}{1 - \bar{z}_n z_j}\right| \leq r\right\} \leq M_1(1 - r)^{-\alpha}$$

per $n = 1, 2, \cdots$ i $0 < r < 1$. També es pot escriure una condició equivalent en termes de mesures de Carleson. Mostrarem en la secció 1.3 que una successió $\{z_n\} \subset \mathbb{D}$ satisfà la condició (1.0.2) si i només si existeixen constants $M_2 > 0$ i $0 < \alpha < 1$ tals que

$$\sum_j (1 - |z_j|)^\alpha \leq M_2 (1 - |z_n|)^\alpha, \quad n = 1, 2, \cdots,$$

on la suma es pren sobre tots els punts $z_j \in \{z_k\}$ tals que $|z_j - z_n| \leq 2(1 - |z_n|)$. Aquesta condició s'assembla a la condició habitual de Carleson amb exponent $\alpha < 1$ pels quadrats de Carleson que contenen un punt en la seva part superior. Anem a discutir el significat geomètric de la condició (1.0.2). Aquesta condició ens diu que, mirades des d'un punt de la successió, les successions que satisfan (1.0.2) són —a escales grans— exponencialment més disperses que les successions separades. De fet, resulta que una successió de punts $\{z_n\} \subset \mathbb{D}$ és una unió finita de successions separades si i només si (1.0.2) es compleix amb $\alpha = 1$. També cal mencionar que en la condició (1.0.2) es compten els punts de la successió que estan a distància hiperbòlica menor que l d'un punt donat z_n de la successió, en lloc de prendre com a punt base un punt qualsevol $z \in \mathbb{D}$ com en [BoNi]. Veure també [S, p. 63–77]. Aquesta última condició és més forta. De fet mostrarem en la secció 1.3 que existeixen dues successions d'interpolació Z_1, Z_2 per h^+ amb $\inf\{\beta(z, \xi) \colon z \in Z_1, \xi \in Z_2\} > 0$ tals que $Z_1 \cup Z_2$ no és una successió d'interpolació per h^+.

És temptador intentar provar del Teorema 1.0.1 utilitzant la condició necessària i suficient de Nevanlinna-Pick per interpolar per funcions analítiques al disc amb part real positiva. En aquesta direcció, P. Koosis [Ko] ha trobat una prova del resultat clàssic de L. Carleson on es descriuen les successions d'interpolació per funcions analítiques i acotades utilitzant la condició de Nevanlinna-Pick. Es pot trobar material relacionat amb el tema a [Bi] i [MS2]. Seria interessant trobar una prova del Teorema 1.0.1 en aquesta línia, però no hem explorat aquesta possibilitat.

Anem a explicar les idees principals de la prova del Teorema 1.0.1. Donat un conjunt $E \subset \mathbb{D}$, sigui E^* la seva projecció radial. És a dir,

$$E^* = \{\, \xi \in \partial\mathbb{D} \colon r\xi \in E, \quad \text{per alguna } 0 \leq r < 1 \,\}.$$

Una aplicació del Lema de Hall dóna que existeix una constant universal $C > 0$ tal que per tota $u \in h^+$ es té que

$$\left| \left\{ z \in \mathbb{D} : \frac{u(z)}{u(0)} > \lambda \right\}^* \right| \leq \frac{C}{\lambda}, \quad \lambda > 0.$$

La necessitat de la condició (1.0.2) es dedueix fàcilment d'aquesta estimació. La prova de la suficiència és més complicada. Donada una successió de punts $\{z_n\} \subset \mathbb{D}$ satisfent (1.0.2) i una successió de valors positius $\{w_n\}$ satisfent la condició de comptabilitat

(1.0.1), hem de trobar una funció $u \in h^+$ tal que $u(z_n) = w_n$, $n = 1, 2, \cdots$ La construcció de la funció $u \in h^+$ es pot dividir en tres passos.

1. Aplicarem un resultat clàssic en Anàlisi Convex anomenat Lema de Farkas que es pot entendre com un anàleg per Cons del Teorema de Hahn-Banach. En lloc de construir directament la funció $u \in h^+$ que interpoli els valors w_n, el *Lema de Farkas* ens diu que n'hi ha prou trobant, per tota partició $\{z_n\} = T \cup S$, una funció $u \in h^+$, que depèn de la partició, amb $u(z_n) \geq w_n$ per $z_n \in T$ i $u(z_n) \leq w_n$ per $z_n \in S$.

2. Sigui $\omega(z, G)$ la mesura harmònica a $\mathbb{D}$ del conjunt $G \subset \partial\mathbb{D}$ des del punt $z \in \mathbb{D}$, és a dir,

$$\omega(z, G) = \frac{1}{2\pi} \int_G \frac{1 - |z|^2}{|\xi - z|^2} \, |d\xi|.$$

Mostrarem que la condició (1.0.2) ens dóna una certa independència de les mesures harmòniques $\{\omega(z_n, \cdot) \colon n = 1, 2, \cdots\}$. De forma imprecisa podríem dir que construirem conjunts $G_n \subset \partial\mathbb{D}$ que compleixen que $\omega(z_n, G_n) \sim 1$ mentre que $\omega(z_n, G_k)$ decau exponencialment amb $\beta(z_n, z_k)$. L'enunciat precís és el Lema 1.2.2 de la secció 1.2. La construcció dels conjunts $\{G_n\}$ utilitza un argument de tipus temps d'atur i constitueix la part més tècnica de la prova.

3. Utilitzant el Teorema B de L. Carleson i J. Garnett on es caracteritzen les successions d'interpolació per l'espai h^∞, és fàcil veure que una successió separada verificant (1.0.2) és d'interpolació per h^∞. Llavors existeix $\gamma > 0$ tal que fixada una partició $\{z_n\} = T \cup S$, existeix una funció harmònica $h = h(T, S)$, amb $\sup\{|h(z)| \colon z \in \mathbb{D}\} < 1$ tal que $h(z_n) = \gamma$ si $z_n \in T$, mentre que $h(z_n) = -\gamma$ si $z_n \in S$. Llavors, utilitzant la condició de comptabilitat (1.0.1) i les estimacions del pas 2, veurem que la funció

$$u(z) = \sum_{z_n \in T} w_n \int_{G_n} \frac{1 - |z|^2}{|\xi - z|^2} \, (1 + h(\xi)) \frac{|d\xi|}{2\pi}, \quad z \in \mathbb{D},$$

verifica $u(z_n) \geq w_n$ si $z_n \in T$ i $u(z_n) \leq w_n$ si $z_n \in S$.

Hom pot considerar també un problema similar en dimensions superiors. Sigui $h^+(\mathbb{R}^{d+1}_+)$ el con de funcions harmòniques i positives al semiespai superior $\mathbb{R}^{d+1}_+$. Direm que una successió de punts $\{z_n\} \subset \mathbb{R}^{d+1}_+$ és d'*interpolació per* $h^+(\mathbb{R}^{d+1}_+)$ si existeix una constant $\varepsilon = \varepsilon(\{z_n\}) > 0$ tal que per tota successió de valors positius $\{w_n\}$ satisfent

$$|\log_2 w_n - \log_2 w_m| \leq \varepsilon \beta(z_n, z_m), \quad n, m = 1, 2, \cdots$$

existeix $u \in h^+(\mathbb{R}^{d+1}_+)$ amb $u(z_n) = w_n$, $n = 1, 2, \cdots$. Quan $d > 1$ no tenim una caracterització geomètrica completa de les successions d'interpolació. En aquesta direcció la situació és anàloga al treball de Carleson i Garnett [CG] per successions d'interpolació a l'espai $h^\infty(\mathbb{R}^{d+1}_+)$ de funcions harmòniques i acotades en $\mathbb{R}^{d+1}_+$ descrit al Teorema C. En la secció 1.3 hi trobem una discussió detallada del problema.

En el segon capítol de la memòria estudiem un problema d'interpolació en certs espais de Banach de funcions analítiques al disc unitat $\mathbb{D}$. Abans d'introduir el problema però, donarem una breu pinzellada d'alguns dels resultats coneguts sobre successions d'interpolació en alguns espais clàssics de funcions analítiques al disc.

Donat un espai de Banach B de funcions analítiques a $\mathbb{D}$ amb avaluacions contínues, denotem el funcional T_z d'avaluació en el punt z com

$$\begin{array}{rccc} T_z: & B & \longrightarrow & \mathbb{C} \\ & f & \mapsto & f(z) \end{array}$$

i la norma del funcional la denotarem per $\|T_z\|$. Sigui B' l'espai dual de B. Llavors existeix $k_z \in B'$ denominat *nucli reproductor* tal que $f(z) =< f, k_z >$ per a tota $f \in B$. A més, $\|k_z\|_{B'} = \|T_z\|$.

Sigui H un espai de Hilbert de funcions analítiques a $\mathbb{D}$ amb avaluacions contínues, amb un producte intern $<,>$ i una norma associada $\|\cdot\|$. Donada una successió de punts $\{z_n\} \subset \mathbb{D}$, considerem els nuclis reproductors k_{z_n} i definim $u_n = k_{z_n}/\|k_{z_n}\|$. Direm que $\{z_n\}$ és una successió d'interpolació per H si l'aplicació

$$x \mapsto \{< x, u_n >\}$$

és exhaustiva de H a l^2. Així doncs, una successió de punts $\{z_n\}$ és d'interpolació per l'espai de Hilbert H si per tota successió de valors $\{w_n\}$ que compleixi $\{w_n/\|k_{z_n}\|\} \in l^2$ existeix una funció $f \in H$ tal que $f(z_n) = w_n$ per tota $n = 1, 2, \cdots$.

Sigui $0 < p < \infty$, denotem per $H^p(\mathbb{D})$ l'espai de Hardy de funcions analítiques f tals que

$$\|f\|_{H^p}^p = \sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt < \infty.$$

Les funcions de H^p tenen límit no tangencial en gairebé tot punt de $\partial\mathbb{D}$ i la norma introduïda coincideix amb la norma L^p d'aquests límits. Considerem també l'espai de Bergman A^p de funcions analítiques a $\mathbb{D}$ amb

$$\|f\|_{A^p}^p = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty.$$

Direm que una successió de punts $\{z_n\}$ és d'*interpolació per H^p (per A^p)* si per tota successió de valors $\{w_n\}$ amb $\{w_n/\|T_{z_n}\|\} \in l^p$, existeix una funció $f \in H^p$ ($f \in A^p$) amb $f(z_n) = w_n$, $n = 1, 2, \cdots$. H. Shapiro i A. Shields [ShSh] l'any 1961 van provar que una successió és d'interpolació per H^p per $1 \leq p < \infty$ si i només si és d'interpolació per H^∞. Dos anys més tard V. Kabaila [Ka] va provar que el mateix resultat era cert per $0 < p < 1$. En el cas dels espais de Bergman A^p, les successions d'interpolació van ser caracteritzades per K. Seip [Se1] l'any 1993 utilitzant una condició de densitat.

Finalment, considerem l'espai de Dirichlet D de funcions analítiques f al disc amb

$$\|f\|_D^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty$$

i de forma més general, els espais de Besov B_p de funcions analítiques al disc amb

$$\|f\|_{B_p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty$$

per $1 < p < \infty$. Observem que quan $p = 2$ coincideix amb l'espai de Dirichlet D.

Donats $1 < p < \infty$ i $0 \leq s < 1$ definim també l'*espai de tipus Besov* $B_p(s)$ com l'espai de funcions analítiques a $\mathbb{D}$ amb

$$\|f\|_{B_p(s)}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z) < \infty.$$

Observem que el cas $s = 0$ correspon a l'espai de Besov clàssic B_p. És fàcil veure que el funcional d'avaluació $T_z : B_p(s) \mapsto \mathbb{C}$ està acotat per tot $z \in \mathbb{D}$ i es pot comprovar que

$$\|T_z\| \approx \begin{cases} \beta(0,z)^{1-1/p} & \text{si} \quad s = 0 \\ (1-|z|^2)^{-s/p} & \text{si} \quad 0 < s \leq 1 \end{cases}.$$

Donada una successió de punts $\{z_n\} \subset \mathbb{D}$, direm que és d'*interpolació per* $B_p(s)$ si l'operador $f \mapsto \{f(z_n)/\|T_{z_n}\|\}$ és acotat i exhaustiu de $B_p(s)$ a l^p. Les successions d'interpolació a l'espai de Dirichlet van ser descrites per D. Marshall - C. Sundberg [MS2] i C. Bishop [Bi] simultàniament als anys 90, i l'any 2000 B. Boe [Bo1] va caracteritzar les successions d'interpolació pels espais de Besov B_p en termes d'una condició de separació i un altra de tipus mesura de Carleson.

Els *multiplicadors de* $B_p(s)$ que denotarem per $\mathcal{M}(B_p(s))$ són les funcions analítiques g tals que gf està a $B_p(s)$ per tota f de $B_p(s)$. És fàcil veure que $\mathcal{M}(B_p(s)) \subset H^\infty$ i definirem les *successions d'interpolació per l'espai de multiplicadors* $\mathcal{M}(B_p(s))$ de la mateixa manera que en l'espai H^∞. Direm que $\{z_n\} \subset \mathbb{D}$ és una successió d'interpolació per $\mathcal{M}(B_p(s))$ si per tota successió de valors $\{w_n\} \in l^\infty$ existeix una funció $g \in \mathcal{M}(B_p(s))$ amb $g(z_n) = w_n$ per tota $n = 1, 2, \cdots$.

Igual que en els espais de Hardy clàssics H^p, direm que una mesura de Borel positiva μ és una (s,p)*-mesura de Carleson* si existeix una constant C positiva tal que

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C \|f\|_{B_p(s)}^p$$

per tota $f \in B_p(s)$. El paper que jugaran les mesures de Carleson en la caracterització de les successions d'interpolació per $B_p(s)$ és similar al que juguen en el cas de H^p. Notem que les (s,p)-mesures de Carleson per $p = 2$, van ser descrites per D. Stegenga [Steg] en termes d'una condició de capacitat. Anys més tard N. Arcozzi, R. Rochberg i E. Sawyer [ARS1], [ARS2] van donar una caracterització geomètrica de les (s,p)-mesures de Carleson que no involucra capacitats.

En el cas de Besov clàssic ($s = 0$) la caracterització de les successions d'interpolació pels espais B_p i $\mathcal{M}(B_p)$ que obtingué Bjarte Böe és la següent.

Theorem E. *[Bo1] Sigui* $1 < p < \infty$*. Donada una successió de punts* $\{z_n\} \subset \mathbb{D} \setminus \{0\}$*, les següents condicions són equivalents:*

1. $\{z_n\}$ *és una successió d'interpolació per* B_p
2. $\{z_n\}$ *és una successió d'interpolació per* $\mathcal{M}(B_p)$
3. *Existeix una constant* $C > 0$ *tal que* $\inf_{m \neq n} \beta(z_n, z_m) \geq C\beta(z_n, 0)$*, per* $n = 1, 2, \cdots$ *i* $\sum_n \frac{1}{\beta(0, z_n)^{p-1}} \delta_{z_n}$ *és una mesura de Carleson per* B_p*.*

Les successions d'interpolació pels espais de Besov $B_p(s)$ amb $s > 0$ ja havien estat caracteritzades per W. S. Cohn.

Theorem F. *[Co] Sigui* $0 < p < \infty$ *i* $s > \max(0, 1-p)$*. Donada una successió de punts* $\{z_n\}$ *del disc unitat* $\mathbb{D}$*, les següents condicions són equivalents:*

1. $\{z_n\}$ *és una successió d'interpolació per* $B_p(s)$*.*
2. $\{z_n\}$ *és separada i la mesura* $\sum_n (1 - |z_n|^2)^s \delta_{z_n}$ *és una* (s, p)*-mesura de Carleson.*

En aquesta memòria estudiem les successions d'interpolació pels espais de tipus Besov $B_p(s)$ i pels corresponents espais de multiplicadors $\mathcal{M}(B_p(s))$ per $1 < p < \infty$ i $0 < s < 1$. Malgrat que les tècniques que utilitzem només funcionen per $0 < s < 1$, tenen l'avantatge, respecte a la prova de W. S. Cohn, que ens permeten caracteritzar també les successions d'interpolació per l'espai de multiplicadors $\mathcal{M}(B_p(s))$. Així doncs el resultat que obtenim no inclou els resultats del Teorema E però sí que trobem certa intersecció amb alguns casos del Teorema F. Observem que la natura dels espais per $s = 0$ i per $s > 0$ és bastant diferent. Per exemple en el cas $s = 0$ no tenim productes de Blaschke infinits a $B_p(s)$ mentre que per $0 < s < 1$ sí. Així doncs les tècniques que utilitzarem són totalment diferents a les utilitzades per B. Boe a [Bo1]. El nostre resultat és el següent.

Theorem 2.5.1. *Siguin* $1 < p < \infty$ *i* $0 < s < 1$*. Donada una successió de punts* $\{z_n\}$ *del disc unitat* $\mathbb{D}$*, les següents condicions són equivalents:*

(UIS) $\{z_n\}$ *és una successió d'interpolació per* $B_p(s)$*.*

(M) $\{z_n\}$ *és una successió d'interpolació per* $\mathcal{M}(B_p(s))$*.*

(CS) $\{z_n\}$ *és separada i la mesura* $\sum_n (1 - |z_n|^2)^s \delta_{z_n}$ *és una* (s, p)*-mesura de Carleson.*

Notem que el cas $p = 2$ d'aquest teorema es dedueix d'un resultat més general de B. Boe [Bo2] que caracteritza les successions d'interpolació per certs espais de Hilbert que tenen nuclis reproductors amb la propietat de Nevanlinna-Pick.

Per provar el Teorema 2.5.1 veurem d'una banda que les condicions (UIS) i (CS) són equivalents i de l'altra que les condicions (M) i (CS) són també equivalents. Anem a explicar els passos que seguirem per veure l'equivalència entre (UIS) i (CS). La necessitat de (CS) és fàcil. Per la suficiència hem de veure que si $\{z_n\}$ compleix (CS) aleshores l'operador

$$\begin{array}{rccc} T: & B_p(s) & \longrightarrow & l^p \\ & f & \longrightarrow & \left\{\frac{f(z_n)}{\|T_{z_n}\|}\right\} \end{array}$$

és acotat i exhaustiu. L'acotació ve donada per la condició de mesura de Carleson mentre que per l'exhaustivitat haurem de treballar una mica més i caldrà introduir l'espai de funcions L^p_s definit a $\partial\mathbb{D}$. Direm que una funció f definida a $\partial\mathbb{D}$ és de L^p_s si $f \in L^p(\partial\mathbb{D})$ i

$$\|f\|^p_{L^p_s} = \int_0^{2\pi}\int_0^{2\pi} \frac{|f(e^{it}) - f(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}}\, du\, dt < \infty.$$

Les funcions de $B_p(s)$ per $1 < p < \infty$ i $0 < s < 1$ tenen límit radial en gairebé tot punt i resulta que si $f \in B_p(s)$, llavors $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}) \in L^p_s(\partial\mathbb{D})$. Això és un resultat clàssic per $p = 2$ i $s = 0$. Recíprocament, si $f \in L^p_s(\partial\mathbb{D})$ i la seva integral de Poisson $f(z)$ és holomorfa a $\mathbb{D}$ llavors $f \in B_p(s)$. Veure la Proposició 2.3.7. De manera anàloga veurem també que una funció analítica i acotada g està a $\mathcal{M}(B_p(s))$ si i només si $g(e^{i\theta}) = \lim_{r\to 1} g(re^{i\theta})$ està a $\mathcal{M}(L^p_s)$, on $\mathcal{M}(L^p_s)$ és l'espai de multiplicadors de L^p_s definit de la manera òbvia.
Per provar l'exhaustivitat de l'operador T, sigui $\{w_n/\|T_{z_n}\|\}$ una successió de l^p. Seguint les tècniques habituals en aquests tipus de problemes, construirem una funció no analítica φ que interpoli els valors $\{w_n\}$ als punts $\{z_n\}$ i que compleixi que

$$\int_{\mathbb{D}} |\overline{\partial}\varphi(z)|^p (1 - |z|^2)^{p-2+s} dA(z) < \infty.$$

Sigui B el producte de Blaschke amb zeros $\{z_n\}$. Resulta que $B \in \mathcal{M}(B_p(s))$. Si Considerem una solució b de

$$\overline{\partial} b = \frac{1}{B}\,\overline{\partial}\varphi,$$

aleshores la funció $f = \varphi - Bb$ és analítica a $\mathbb{D}$ i interpola els valors $\{w_n\}$ en els punts $\{z_n\}$. Per veure que la funció f és de $B_p(s)$ es necessita resoldre el problema $\overline{\partial}$ anterior amb una solució b tal que $b(e^{i\theta}) \in L^p_s$. Veure el Teorema 2.4.1.

Pel què fa a l'equivalència entre (M) i (CS) notem que la necessitat de la condició de separació (S) és elemental. La necessitat de la condició de mesura de Carleson (C) es deduirà d'un argument que combina una fórmula de reproducció per $B_p(s)$ i una idea de N. Varopoulos [Va] que utilitza la desigualtat de Khinchin (2.5.1). Donarem dues demostracions diferents de la suficiència de la condició (CS). Una de les proves utilitza una construcció de J. Earl [Ea] on el problema d'interpolació es resol amb un múltiple d'un producte de Blaschke. L'altra demostració que donem és anàloga a la prova que mostra que (CS) implica (UIS). Donada una successió de valors $\{w_n\} \in l^\infty$, construirem una funció no analítica φ que interpoli els valors $\{w_n\}$ en els punts $\{z_n\}$ i que a més a més compleixi que $|\overline{\partial}\varphi(z)/B(z)|^p(1-|z|^2)^{p-2+s}dA(z)$ sigui una (s,p)-mesura de Carleson. Per aconseguir una solució analítica del problema d'interpolació caldrà trobar una solució del problema

$$\overline{\partial} b = \frac{1}{B}\,\overline{\partial}\varphi.$$

En efecte, llavors la funció $f = \varphi - Bb$ serà analítica i $f(z_n) = w_n$, per $n = 1, 2, \cdots$. Per veure que $f \in \mathcal{M}(B_p(s))$ serà suficient trobar una solució b del problema $\overline{\partial}$ anterior amb $b \in \mathcal{M}(L^p_s)$. L'existència d'una solució del problema $\overline{\partial}$ ens la donarà el Teorema 2.4.2 i aquesta consistirà, de fet, en la solució de P. Jones del problema $\overline{\partial}$ amb estimacions a L^∞ ([Jo1]). La resta de la prova consisteix en mostrar que aquesta solució està a $\mathcal{M}(L^p_s)$. Al Teorema 2.3.11 provarem que una funció b acotada a $\partial\mathbb{D}$ està a $\mathcal{M}(L^p_s)$ si i només si

$$|\nabla P[b](z)|(1-|z|^2)^{p-2+s}dA(z) \tag{0.0.5}$$

és una (s,p)-mesura de Carleson, on $P[b]$ és l'extensió de Poisson al disc unitat de la funció b. Finalment, per comprovar que la mesura (0.0.5) és efectivament una (s,p)-mesura de Carleson, haurem d'estudiar el comportament de l'operador

$$T(\phi) = \int_{\mathbb{D}} \frac{|\phi(w)|}{|1-z\overline{w}|^2}dA(w)$$

sobre funcions ϕ complint que $|\phi(z)|^p(1-|z|^2)^{p-2+s}dA(z)$ sigui una (s,p)-mesura de Carleson. Aquest estudi es resumeix en el Teorema 2.3.2, la prova del qual fa ús d'una caracterització de l'espai $B_p(s)$ que no utilitzi derivades, que és de fet, una generalització d'un resultat de R. Rochberg i Z. Wu [RW] pel cas $p = 2$. Veure el Teorema 2.2.4.

La solució a l'espai $\mathcal{M}(L^p_s)$ del problema $\overline{\partial}$ obtinguda per provar que (CS) implica (M) té interés independent. Efectivament, l'utilitzarem també per donar una nova prova del problema de la corona en l'àlgebra $\mathcal{M}(B_p(s))$ estudiat per Tolokonnikov, veure [Tol] o [Nik, Appendix 3]. Així mateix també utilitzarem la solució del problema $\overline{\partial}$ per donar una descomposició del tipus Fefferman-Stein per funcions de $\mathcal{M}(L^p_s) \subset L^\infty(\partial\mathbb{D})$.

La lletra C denotarà una constant absoluta el valor de la qual pot variar d'una línia a una altra. També $C(M)$ denotarà una constant que depèn de M. La notació $a \approx b$ significa que existeixen constants absolutes $c_1, c_2 > 0$ que satisfan $c_1\, b \leq a \leq c_2\, b$. De manera similar, $a \lesssim b$ significa que la segona desigualtat es compleix.

Esctructura de la Tesis

La tesis consta de dos capítols que corresponen a l'estudi de les successions d'interpolació en dos espais diferents.

El primer capítol està dedicat a l'estudi de les successions d'interpolació per funcions harmòniques i positives. A les dues primeres seccions es planteja i es soluciona el problema i a la tercera secció es mostren condicions equivalents que caracteritzen aquestes successions. A la tercera secció es resol també un problema d'interpolació per funcions holomorfes i acotades sense zeros i s'estudia el problema d'interpolació per funcions harmòniques i positives en dimensions superiors.

Al segon capítol estudiem les successions d'interpolació pels espais de tipus Besov $B_p(s)$. Aquest capítol consta de cinc seccions. A la primera secció s'esmenten diverses desigualtats que utilitzarem al llarg del capítol. La segona secció està dedicada a una caracterització de l'espai de Besov $B_p(s)$ sense utilitzar derivades i també es presenten algunes propietats del nucli reproductor. A la tercera secció s'estudien les mesures de Carleson i s'utilitzen per caracteritzar l'espai de multiplicadors $\mathcal{M}(B_p(s))$. Així mateix s'introdueix l'espai frontera L^p_s i es proven algunes relacions amb l'espai de Besov $B_p(s)$. La secció quatre la dediquem a resoldre problemes $\overline{\partial}$ amb estimacions a L^p_s i $\mathcal{M}(L^p_s)$ i ho apliquem en aquesta mateixa secció a la solució del problema de la corona a l'àlgebra $\mathcal{M}(B_p(s))$ i per provar un resultat de descomposició de tipus Fefferman-Stein. Finalment a la secció cinc caracteritzem les successions d'interpolació per l'espai de Besov $B_p(s)$ i pel corresponent espai de multiplicadors $\mathcal{M}(B_p(s))$.

La tesis està basada en tres articles. El capítol 1 correspon a l'article

[BlNi] D. Blasi & A. Nicolau, *Interpolation by positive harmonic functions*, J. London Math. Soc. (2), **76** (2007), 253–271.

El capítol 2 presenta resultats dels articles

[BP] D. Blasi & J. Pau, *A characterization of Besov type spaces and applications to Hankel type operators*, per aparèixer a Michigan Math. J. **56** (2008).

[ABP] N. Arcozzi, D. Blasi & J. Pau, *Interpolating sequences on Besov type spaces*, en preparació (2008).

Agraïments

> "Al meu poble
> hi havia gent de tres menes:
> els qui creien, els qui no creien
> i els qui tant se'ls en fotia.
> D'aquests darrers, però,
> n'hi havia pocs.
> Al meu poble hi havia,
> sobretot, gent de dues menes:
> els qui creien i els qui no creien.
> I encara val a dir
> que aquests darrers no eren pas molts."
>
> Cròniques canten (Miquel Martí i Pol)

Aquesta memòria vull dedicar-la a tots aquells que creuen en un projecte, que tenen un somni, tots aquells que pertanyen al primer grup i que lluiten per tirar endavant tot el què es proposen. Aquesta tesi va dedicada, en primer lloc, als meus pares pel seu suport incondicional, al meu germà Albert per donar-me un cop de mà sempre que ha fet falta i també al Jordi, el teu record sempre present és com l'aire que empeny els núvols per seguir avançant.

El meu agraïment més sincer a l'Artur Nicolau per haver-me ensenyat a moure'm per la selva de les matemàtiques, pels coneixements que m'ha transmès i per despertar la meva curiositat per intentar entendre els elements que la componen. Per ensenyar-me a explorar els camins que desfan aquest entrellat d'hipòtesis i suposicions tot mostrant un món atractiu i captivador. Diuen que els matemàtics som una gran família. Doncs bé, des d'aquí voldria agrair també al meu germà gran de les matemàtiques, el Jordi Pau, tot el temps que m'ha dedicat i tot el que he après al seu costat.

Però és clar, per poder treballar a gust, el més important és un bon clima de treball i aquest sens dubte ha estat immillorable! Vull donar les gràcies al Departament de matemàtiques de la Universitat Autònoma de Barcelona per les facilitats que m'ha ofert en tot moment, i pel bon ambient que s'hi respira. Així mateix al meu company de despatx, el Ramon, i a la Unitat d'Anàlisi del departament i de la UB pels seminaris conjunts. Una menció especial a la gent de secretaria tot i que molts d'ells ja no hi són; al Toni, la Cristina, l'Ester i la Noelia per la seva eficiència i per estar sempre disposats a donar-me un cop de mà. I què dir dels companys de doctorat? doncs senzillament que no els canviaria pas! Tions, Paca, butifarra, excursions, e-milius, carmanyolades, paelles... són tants els bons moments compartits que no em veuria en cor de llistar-los en aquest paràgraf. I és clar, en els crèdits no hi poden pas faltar el Miquel com a

LaTeX-conseller i la Natàlia com a English-revisora. Tampoc hi podien faltar els amics de la carrera i els qui sempre seran companys de cotxe (les Laures, l'Àlex, el Damià i també la Mireia, el David, la Juani i l'Eduard). Amb el temps, segons diuen, el cotxe ha assolit un altre nivell i les males llengües parlen de l'autobús de Manresa. Una salutació als companys de ruta també!

A finals del 2003 vaig assistir al meu primer congrés de matemàtiques; la *"Conference in honor of Boris Korenblum's 80th birthday"* a Barcelona. Allà vaig conèixer en Milne Anderson i em va dir una frase que recordaré sempre:

"Welcome to the mathematical community. You must know something: doing mathematics you will never be rich but you will meet a lot of people!"

I quanta raó tenia... està clar que fent matemàtiques no et faràs ric, però durant aquest temps a l'univers de les matemàtiques, de gent n'he conegut un munt, d'aquí d'allà i de més enllà i he fet bons amics com el *"Girona de Noruega"*, més senzill de pronunciar que no pas Geir Arne Hjelle, o el Jan Frederick. Una abraçada també pel Ramón, la Lola, la Juana, la Fátima, la Lucy,... i tots aquells que han passat pel departament durant aquests cinc anys.

Tampoc vull oblidar-me d'en Pascal Thomas, en Kristian Seip i en Nicola Arcozzi que tan amablement em van acollir en els seus respectius departaments i em van oferir la possibilitat de treballar conjuntament i conèixer llocs preciosos.

I com diria aquell, fora de les matemàtiques també hi ha vida! Doncs bé, anem a pams. Començaré agraint al meu poble la felicitat que em transmet cada dia en tornar de la universitat, a la seva gent i al nostre país petit que algun dia veurem lliure! Als amics de l'esplai GEPS amb qui he compartit tantes vivències i amb qui junts hem rigut, hem cantat i hem après... molt! Als companys del Centre Excursionista 2x2 per introduir-me al món de la muntanya, perquè puguem muntar moltes històries plegats i en definitiva, perquè el 2x2 és més que un club!

Quan començava el doctorat, amb el Marsi vam anar a parar al Club Alpí Universitari (CAU) que en aquells moments també començava. Allà he conegut gent extraordinària i ells són els culpables que tornés dels caps de setmana més cansat del què els començava, però amb les piles renovades, això sí! Ascensions, travesses, escalades, bivacs, concerts, sopars,... Gràcies Pirineus, gràcies Montserrat, i gràcies als amics que ens uneix una corda, que no són quatre fils; el Guillem, la Laura, el Gerard, el Marsi, l'Alex,... quin valor!!! I també al Martí, la Cristina, el Roger, la Mònica, el Joan, la Núria, el Jordi, la Xus, la Lara, l'Isis, el John... Com diria en Feliu Ventura: que no s'apague la llum!

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I finalment a tu per tenir la santa paciència de llegir tot això!

Però aquesta gran parrafada
no podria pas acabar
sense una que altra corranda
que a uns amics vull dedicar.

A la colla corrandera
que vinguts d'aquí i d'allà
sempre busquen la manera
de deixar anar un disbarat!

Si va bé en unes setmanes
el doctor Blasi seré
però si teniu mal de panxa
no us refieu del meu consell.

I és que els doctors matemàtics
tenen l'especialitat
de fer fora el mal de panxa
fent-te venir mal de cap!

Fent pocions amb conjectures,
lemes i proposicions
hi ha dues opcions: o bé et cures
o tens al-lucinacions!

Com a darrera corranda
amb permís em copiaré,
una que em feu molta gràcia
que va escriure el Mingu Vell.

La de Dani és planta rara,
que té les arrels quadrades,
logaritmes a les fulles
i floretes integrals.

La vida és Rauxa!

Chapter 1

Interpolating sequences for positive harmonic functions

Let $h^+ = h^+(\mathbb{D})$ be the cone of positive harmonic functions in the unit disc $\mathbb{D}$ of the complex plane. As discussed in the introduction, a sequence of points $\{z_n\}$ in the unit disc will be called an *interpolating sequence for* h^+ if there exists a constant $\varepsilon = \varepsilon(\{z_n\}) > 0$, such that for any sequence of positive values $\{w_n\}$ satisfying

$$|\log_2 w_n - \log_2 w_m| \leq \varepsilon\beta(z_n, z_m), \quad n, m = 1, 2, \cdots \tag{1.0.1}$$

there exists a function $u \in h^+$ with $u(z_n) = w_n$, $n = 1, 2, \cdots$, where $\beta(z, w)$ refers to the hyperbolic distance between the points $z, w \in \mathbb{D}$.

The main result of this chapter is the following.

Theorem 1.0.1. *A separated sequence $\{z_n\}$ of points in the unit disc is interpolating for h^+ if and only if there exist constants $M > 0$ and $0 < \alpha < 1$ such that*

$$\#\{z_j : \beta(z_j, z_n) \leq l\} \leq M2^{\alpha l} \tag{1.0.2}$$

for any $n, l = 1, 2, \cdots$

Let us now explain the main ideas of the proof. Let E^* denote the radial projection of a set $E \subset \mathbb{D}$, that is, $E^* = \{\xi \in \partial\mathbb{D} : r\xi \in E \text{ for some } 0 \leq r < 1\}$. An application of Hall's Lemma yields that there exists a universal constant $C > 0$ such that for any $u \in h^+$ one has

$$\left|\left\{z \in \mathbb{D} : \frac{u(z)}{u(0)} > \lambda\right\}^*\right| \leq \frac{C}{\lambda}, \quad \lambda > 0.$$

The necessity of condition (1.0.2) follows easily from this estimate. The proof of the sufficiency is harder. Given a sequence of points $\{z_n\} \subset \mathbb{D}$ satisfying (1.0.2) and a sequence of positive values $\{w_n\}$ satisfying the compatibility condition (1.0.1), one has to find a function $u \in h^+$ such that $u(z_n) = w_n$, $n = 1, 2, \cdots$ The construction of the function $u \in h^+$ may be split into three steps.

1. We will apply a classical result in Convex Analysis called *Farkas' Lemma* which may be understood as an analogue for Cones of the Hahn-Banach Theorem. Actually Farkas' Lemma follows from the Separation Theorem for convex sets in locally convex spaces, but the version we use predates the Separation Theorem. Instead of constructing directly the function $u \in h^+$ which performs the interpolation, *Farkas' Lemma* will tell us that it suffices to prove the following statement. Given any partition of the sequence $\{z_n\}$ into two disjoint subsequences, $\{z_n\} = T \cup S$, there exists a function $u = u(T, S) \in h^+$ such that

$$u(z_n) \geq w_n, \quad \text{if } z_n \in T,$$
$$u(z_n) \leq w_n, \quad \text{if } z_n \in S.$$

2. Let $\omega(z, G)$ denote the harmonic measure in $\mathbb{D}$ of the set $G \subset \partial\mathbb{D}$ from the point $z \in \mathbb{D}$, that is,

$$\omega(z, G) = \frac{1}{2\pi} \int_G \frac{1 - |z|^2}{|\xi - z|^2} |d\xi|.$$

For each point z_n of the sequence $\{z_n\}$ we will construct a set $G_n \subset \partial\mathbb{D}$ and we will show that condition (1.0.2) provides some sort of independence of harmonic measures $\{\omega(z_n, \cdot)\colon n = 1, 2, \cdots\}$. Actually, given $0 < \delta < 1$, there exists $N > 0$ and a collection of pairwise disjoint subsets $\{G_n\}$ of $\partial\mathbb{D}$ such that

$$\omega(z_n, \cup_{k \in A(n)} G_k) \geq 1 - \delta,$$
$$\sum_{k \notin A(n)} 2^{\eta\beta(z_k, z_n)} \omega(z_n, G_k) \leq \delta.$$

Here $A(n)$ denotes the set of indexes k so that $\beta(z_k, z_n) \leq N$. The number $\eta = \eta(\delta, M, \alpha) > 0$ is a constant depending on $\delta > 0$ and on the constants $M > 0$ and $\alpha < 1$ of (1.0.2). The construction of the sets $\{G_n\}$ uses a certain stopping time argument and constitutes the most technical part of the proof.

3. L. Carleson ad J. Garnett found a description of the interpolating sequences for the space h^∞ of bounded harmonic functions in the unit disc (see [Ga1], [Ga3, p. 313] or [CG]). Using their result it is easy to show that a separated sequence verifying (1.0.2) is interpolating for h^∞. Hence there exists $\gamma > 0$ and a harmonic function h, with $\sup\{|h(z)|\colon z \in \mathbb{D}\} < 1$ such that $h(z_n) = \gamma$ if $z_n \in T$, while $h(z_n) = -\gamma$ if $z_n \in S$. Then, for fixed $\varepsilon > 0$ and $\delta > 0$ sufficiently small, using the compatibility condition (1.0.1) and the estimates in step 2, one can show that the function

$$u(z) = \sum_{z_n \in T} w_n \int_{G_n} \frac{1 - |z|^2}{|\xi - z|^2} (1 + h(\xi)) \frac{|d\xi|}{2\pi}, \quad z \in \mathbb{D},$$

verifies $u(z_n) \geq w_n$ if $z_n \in T$ and $u(z_n) \leq w_n$ if $z_n \in S$.

Before proving Theorem 1.0.1 we need some preliminary technical results. Given $z, w \in \mathbb{D}$, let $\rho(z,w) = \left|\dfrac{w-z}{1-\overline{w}z}\right|$ be the pseudohyperbolic distance between those two points. We define the hyperbolic distance between z and w as

$$\beta(z,w) = \log_2 \frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

We choose this normalization of the hyperbolic distance because it fits perfectly well with *dyadic decompositions* of the unit disc. In fact, it is not difficult to prove the following result.

Observation 1.0.2. *Let $k \geq 0$ and let $z, w \in \mathbb{D}$ with $|Arg z - Arg w| < 1 - |z|$ and $1 - |w| = 2^{-k}(1-|z|)$. Then*

$$\beta(z,w) = k + C,$$

where C is a bounded constant not depending on k.

It is well known that for all $z, w \in \mathbb{D} \setminus \{0\}$ with $\beta(z,w) \leq 1$ there exists a constant $C > 0$ not depending on z, w such that

$$\begin{cases} 1-|z| \approx 1-|w| \\ |Arg\,(z) - Arg\,(w)| \leq C(1-|z|) \end{cases} \tag{1.0.3}$$

Observation 1.0.3. *Let $z, w \in \mathbb{D}$ with $\beta(z,w) \leq 1$, then $|1-\overline{w}z| \approx 1-|z|$*

Proof. Given $z, w \in \mathbb{D}$ with $\beta(z,w) \leq 1$, we have that

$$|1-\overline{w}z|^2 = |z-w|^2 + (1-|z|^2)(1-|w|^2) \leq C|1-\overline{w}z|^2 + (1-|z|^2)(1-|w|^2),$$

for a constant $C < 1/2$. Inequality (1.0.3) tells us that $1-|w| \approx 1-|z|$. So we conclude that

$$1-|z| \leq |1-\overline{w}z| \leq C(1-|z|),$$

where C is an absolute constant not depending on the points z, w. □

Given a point $z \in \mathbb{D}$ we define the arc $I(z)$ and the *Carleson square* $Q(z)$ as follows:

$$\begin{aligned} I(z) &= \{e^{i\theta} : \, -\pi(1-|z|) < \theta - \operatorname{Arg} z \leq \pi(1-|z|)\} \\ Q(z) &= \left\{re^{i\theta} : 0 < 1-r \leq 1-|z|, \; e^{i\theta} \in I(z)\right\}. \end{aligned} \tag{1.0.4}$$

Sometimes we will denote $I = I(z)$ and $S(I)$ for the corresponding Carleson square. Given a constant $C > 0$ we denote

$$CI(z) = \left\{e^{i\theta} : \, -\pi\, C(1-|z|) < \theta - \operatorname{Arg} z \leq \pi\, C(1-|z|)\right\}$$

$$CQ(z) = \left\{re^{i\theta} : 0 < 1-r \leq C(1-|z|), \; e^{i\,\theta} \in C\, I(z)\right\}.$$

Observe that if $C(1-|z|) \geq 1$, one has $CI(z) = \partial\mathbb{D}$ and $CQ(z) = \mathbb{D}$. See figure 1. When $z_k \in \{z_n\}$, we simply denote $I_k = I(z_k)$.

1.1 Necessity of condition (1.0.2)

Given a closed set $E \subset \mathbb{D}$, let $\omega(z) := \omega(z, E, \mathbb{D} \setminus E)$ denote the *harmonic measure* from the point $z \in \mathbb{D} \setminus E$ of the set E in the domain $\mathbb{D} \setminus E$. See [MS3] or [Du]. The classical *Hall's Lemma* tells that there exists a universal constant $C > 0$ such that $\omega(0, E, \mathbb{D} \setminus E) \geq C|E^*|$ for any set $E \subset \mathbb{D}$. See [Hal] or [MS1]. Recall that E^* denotes the radial projection of E. If $E \subset \partial\mathbb{D}$, we define

$$\omega(z, E) = \int_E \frac{1-|z|^2}{|\xi - z|^2} \frac{|d\xi|}{2\pi}$$

for any point z in $\mathbb{D}$. The main auxiliary result for the proof of the necessity of condition (1.0.2) is the following.

Lemma 1.1.1. *There exists a constant $C > 0$ such that for any $u \in h^+$ and $\lambda > 0$ one has*

$$\left| \left\{ z \in \mathbb{D} \colon \frac{u(z)}{u(0)} > \lambda \right\}^* \right| \leq \frac{C}{\lambda}$$

Proof. One may assume that $\lambda > 1$. Fix $u \in h^+$, and let $E = \{z \in \mathbb{D} \colon u(z) > \lambda u(0)\}$. The maximum principle shows that

$$u(z) \geq \lambda u(0) \omega(z, E, \mathbb{D} \setminus E), \quad z \in \mathbb{D} \setminus E.$$

Taking $z = 0$, one gets $\omega(0, E, \mathbb{D} \setminus E) \leq \lambda^{-1}$ and applying Hall's Lemma one finishes the proof. □

Proof of the necessity of condition (1.0.2). Let us assume that $\{z_k\}$ is an interpolating sequence for h^+. By conformal invariance it is sufficient to prove (1.0.2) when the base point z_n is the origin. So assume $z_1 = 0$ and take $w_k = 2^{\varepsilon\beta(z_k,0)}$, $k = 1, 2, \cdots$ It is clear that the compatibility condition (1.0.1) holds. So, there exists $u \in h^+$ with $u(z_k) = w_k$, $k = 1, 2, \cdots$ Let D_k be the hyperbolic disc centered at z_k of hyperbolic radius 1. By Harnack's Lemma,

$$u(z) \geq \frac{w_k}{2}, \quad z \in D_k, \quad k = 1, 2, \cdots .$$

Let $A(j)$ denote the set of indexes k corresponding to points z_k with $j-1 \leq \beta(z_k, 0) \leq j$, $j = 1, 2, \cdots$. Then, one deduces

$$u(z) \geq 2^{\varepsilon(j-1)-1}, \quad z \in D_k, \quad k \in A(j).$$

Now, since $u(0)=1$, Lemma 1.1.1 gives

$$\left|\left(\cup_{k\in A(j)} D_k\right)^*\right| \le C2^{\varepsilon(1-j)}.$$

Since the sequence $\{z_k\}$ is separated, the discs $\{D_k\}$ are quasidisjoint and one deduces

$$\sum_{k\in A(j)} 1-|z_k| \le C \sum_{k\in A(j)} |D_k^*| \le C2^{\varepsilon(1-j)}.$$

Since $1-|z_k|$ is comparable to 2^{-j} for any $k\in A(j)$, one deduces

$$\#A(j) \le C2^{(1-\varepsilon)j}.$$

Adding up for $j=1,\cdots,l$, one gets

$$\#\{z_k\colon \beta(z_k,0)\le l\} \le C2^{(1-\varepsilon)l}. \quad \square$$

1.2 Sufficiency of condition (1.0.2)

Given a sequence of points $\{z_n\}\subset\mathbb{D}$ satisfying (1.0.2) and a sequence of positive values $\{w_n\}$ satisfying the compatibility condition (1.0.1), one has to find a function $u\in h^+$ such that $u(z_n)=w_n$, $n=1,2,\cdots$. By a normal families argument, one may assume the sequence $\{z_n\}$ consists of finitely many points. As explained in the introduction of this chapter, the proof consists of three steps.

First Step

Let $e_1,\cdots,e_m$ be a collection of vectors of the euclidian space $\mathbb{R}^d$. *Farkas' Lemma* asserts that a vector $e\in\mathbb{R}^d$ is in the cone generated by $\{e_i\colon i=1,\cdots,m\}$, that is $e=\sum\lambda_i e_i$ for some $\lambda_i\ge 0$, $i=1,\cdots,m$, if and only if $\langle x,e\rangle\le 0$ for any vector $x\in\mathbb{R}^d$ for which $\langle x,e_i\rangle\le 0$, $i=1,\cdots,m$. See [HL]. This classical result will be used in the proof of the next auxiliary result

Lemma 1.2.1. *Let $\{z_n\}$ be a sequence of distinct points in the unit disc and let $\{w_n\}$ be a sequence of positive values. Assume that for every partition of the sequence $\{z_n\}=T\cup S$, into two disjoint subsequences T and S, there exists $u=u(T,S)\in h^+$ such that $u(z_n)\ge w_n$ if $z_n\in T$ and $u(z_n)\le w_n$ if $z_n\in S$. Then, there exists $u\in h^+$ such that $u(z_n)=w_n$, $n=1,2,\cdots$*

Proof of Lemma 1.2.1. By a normal families argument, one may assume that both the sequences of points $\{z_n\}$ and values $\{w_n\}$ consist of finitely many, say d, points. Consider the set of all partitions $\{z_n\}=T_k\cup S_k$, $k=1,\dots,m$ of the sequence $\{z_n\}$. Let

$u_1, \dots, u_m \in h^+$ be the corresponding functions such that $u_k(z_n) \geq w_n$ if $z_n \in T_k$ and $u_k(z_n) \leq w_n$ if $z_n \in S_k$, and consider the vector

$$u_i\colon = (u_i(z_1), \dots, u_i(z_d)), \quad i = 1, \dots, m.$$

If $x \in \mathbb{R}^d$ satisfies $\langle x, u_i \rangle \leq 0$, $i = 1, \dots, m$, that is $\sum_{n=1}^d u_i(z_n) x_n \leq 0$, let $\mathcal{F} = \{z_n \colon x_n \geq 0\}$. Then $\mathcal{F} = T_k$ for some k and let $S_k = \{z_n\} \setminus \mathcal{F}$. Its corresponding function u_k satisfies $x_n w_n \leq x_n u_k(z_n)$ for all $n = 1, \cdots, d$. So,

$$\langle x, w \rangle = \sum_{n=1}^d w_n x_n \leq \sum_{n=1}^d u_k(z_n) w_n \leq 0.$$

Now, by Farkas' Lemma, $w = (w_1, \cdots, w_d)$ is in the cone generated by the vectors $\{u_i, i = 1, \dots, m\}$. So there exist constants $\lambda_i \geq 0$, $i = 1, \cdots, m$ such that $u(z) = \sum_{i=1}^m \lambda_i u_i(z) \in h^+$ and $u(z_n) = w_n$, $n = 1, 2, \dots, d$. □

Second Step

The second step in the proof consists of using condition (1.0.2) to construct a collection of disjoint subsets $\{G_n\}$ of the unit circle which provide a suitable kind of independence of the harmonic measures $\{\omega(z_n, \cdot)\colon n = 1, 2, \cdots\}$. The precise statement is given in the following result which is the main technical part of the proof. Recall that $\omega(z, G)$ denote the harmonic measure in $\mathbb{D}$ of the set $G \subset \partial\mathbb{D}$ from the point $z \in \mathbb{D}$, that is,

$$\omega(z, G) = \frac{1}{2\pi} \int_G \frac{1 - |z|^2}{|\xi - z|^2} |d\xi|.$$

Lemma 1.2.2. *Let $\{z_n\}$ be a sequence of distinct points in the unit disc which satisfies condition (1.0.2). Then for any $\delta > 0$, there exist numbers $N = N(\delta) > 0$, $\eta = \eta(\delta) > 0$ and a collection $\{G_n\}$ of pairwise disjoint subsets of the unit circle such that*

$$\omega\left(z_n, \cup_{k \in A(n)} G_k\right) \geq 1 - \delta, \quad n = 1, 2, \cdots, \tag{1.2.1}$$

and

$$\sum_{k \notin A(n)} 2^{\eta \beta(z_k, z_n)} \omega(z_n, G_k) < \delta, \quad n = 1, 2, \cdots. \tag{1.2.2}$$

Here $A(n) = A(n, N)$ denotes the collection of indexes k such that $\beta(z_k, z_n) \leq N$.

We will use the following two elementary auxiliary results.

Lemma 1.2.3. *Fixed $\delta > 0$, there exists $M_0 = M_0(\delta) > 0$ such that*

$$\omega(z_k, M_0 I_k) \geq 1 - \frac{\delta}{100}, \quad k = 1, 2, \cdots$$

Proof. If $z_k = 0$ one may take $M_0 = 1$. If $z_k \neq 0$ observe that there exists an absolute constant $C_0 > 0$ such that $|e^{it} - z_k| \geq C_0 |t - \operatorname{Arg} z_k|$. Since

$$\omega(z_k, \partial\mathbb{D} \setminus M_0 I_k) = \frac{1 - |z_k|^2}{2\pi} \int_{\partial\mathbb{D}\setminus M_0 I_k} \frac{|d\xi|}{|\xi - z_k|^2},$$

one gets

$$\omega(z_k, \partial\mathbb{D} \setminus M_0 I_k) \leq \frac{1 - |z_k|^2}{2\pi C_0^2} \int_{\pi M_0(1-|z_k|)}^{\infty} \frac{dx}{x^2}.$$

Hence

$$\omega(z_k, \partial\mathbb{D} \setminus M_0 I_k) \leq \frac{1}{\pi^2 C_0^2 M_0}$$

and taking $M_0 = 100/\pi C_0^2 \delta$ the result follows. □

Lemma 1.2.4. *Fixed $M > 0$, there exists a constant $C(M) > 0$ such that for all pairs of points $z, w \in \mathbb{D}$ with $w \in 20MQ(z)$, one has*

$$\left| \beta(z, w) - \log_2 \left(\frac{1 - |z|}{1 - |w|} \right) \right| \leq C(M).$$

Proof. One may assume that $z, w \in \mathbb{D} \setminus \{0\}$. Since

$$|1 - \bar{w}z| \geq (1 - |z||w|) \geq (1 - |z|)$$

and

$$\begin{aligned} |1 - \bar{w}z| &\leq |w| \left| \frac{1}{\bar{w}} - z \right| \\ &\leq |w| \left| \frac{1}{\bar{w}} - e^{i\operatorname{Arg} w} \right| + \left| e^{i\operatorname{Arg} w} - e^{i\operatorname{Arg} z} \right| + \left| e^{i\operatorname{Arg} z} - z \right| \\ &\leq (20M + 20M\pi + 1)(1 - |z|), \end{aligned}$$

we deduce

$$1 - |z| \leq |1 - \bar{w}z| \leq K(M)(1 - |z|),$$

where $K(M) = 20M + 20M\pi + 1$. So,

$$\begin{aligned} \beta(z, w) &= 2\log_2 \left(1 + \left| \frac{z - w}{1 - \bar{w}z} \right| \right) - \log_2 \left(1 - \left| \frac{z - w}{1 - \bar{w}z} \right|^2 \right) \\ &= 2\log_2 \left(1 + \left| \frac{z - w}{1 - \bar{w}z} \right| \right) - \log_2 \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{w}z|^2} \\ &= C + \log_2 \left(\frac{1 - |z|}{1 - |w|} \right) \end{aligned}$$

where $-2 \leq C \leq 2 + 2\log_2 K(M)$. □

Proof of Lemma 1.2.2. The construction of the sets $\{G_n\}$ may be split into three steps.

i) Given a sequence of points $\{z_n\}$ satisfying (1.0.2) and positive constants λ and M_0, let γ be a constant depending on M_0 and on the constant α appearing in (1.0.2). For each $z_k \in \{z_n\}$ we will construct certain points $z_n^\gamma(k) \in \mathbb{D}$ with $I(z_n) \subset I(z_n^{(\gamma)}(k))$ and
$$\sum_{\substack{z_n\in 20M_0Q(z_k)\\ \beta(z_k,z_n)\geq N}} 1-|z_n^\gamma(k)| \leq \lambda(1-|z_k|) \text{ for all } z_k \in \{z_n\}. \tag{1.2.3}$$
Here N is a constant depending on λ, M_0 and on the constants M and α appearing in (1.0.2)

ii) Next, we will construct certain sets $E_k \subset \partial\mathbb{D}$ with $E_k \cap E_j = \emptyset$ if $\beta(z_k, z_j) \geq N$ such that
$$\omega(z_k, E_k) \geq 1 - \frac{\delta}{10}. \tag{1.2.4}$$
In the construction of the sets E_k we will use the points $z_n^\gamma(k)$ of the first step which satisfy the estimate (1.2.3) above for a certain fixed λ sufficiently small.

iii) Finally we will construct the pairwise disjoint sets G_n satisfying conditions (1.2.1) and (1.2.2).

i) Construction of the points $z_n^\gamma(k)$. Fix $\delta > 0$. Applying Lemma 1.2.3, there exists a constant $M_0 = M_0(\delta) > 0$ such that
$$\omega(z_k, M_0 I_k) \geq 1 - \frac{\delta}{100}, \quad k = 1, 2, \cdots \tag{1.2.5}$$
Fix $z_k \in \{z_n\}$. Let $\gamma = \gamma(\alpha) > 0$ be a small number to be fixed later. For any $z_n \in 20M_0Q(z_k)$ with $\beta(z_k, z_n) \geq N$ we define $z_n^\gamma(k)$ as the point in $\mathbb{D}$ satisfying the following three conditions
$$\begin{aligned} &\operatorname{Arg}(z_n) = \operatorname{Arg}(z_n^\gamma(k)),\\ &\beta(z_n^\gamma(k), z_n) = \gamma\beta(z_k, z_n),\\ &|z_n^\gamma(k)| < |z_n|. \end{aligned} \tag{1.2.6}$$
Here $N = N(\gamma, M_0, \lambda)$ is a large number to be fixed later. In particular $N > 0$ will be taken so large that $z_n^\gamma(k) \in 20M_0Q(z_k)$ whenever $z_n \in 20M_0Q(z_k)$ satisfies $\beta(z_n, z_k) > N$. See Figure 1.1.

Using Lemma 1.2.4 and $\beta(z_n^\gamma(k), z_n) = \gamma\beta(z_k, z_n)$ we obtain the following inequalities:
$$\left(\frac{1-|z_k|}{1-|z_n|}\right)^{C^{-1}\gamma} \leq \frac{1-|z_n^\gamma(k)|}{1-|z_n|} \leq \left(\frac{1-|z_k|}{1-|z_n|}\right)^{C\gamma}, \tag{1.2.7}$$

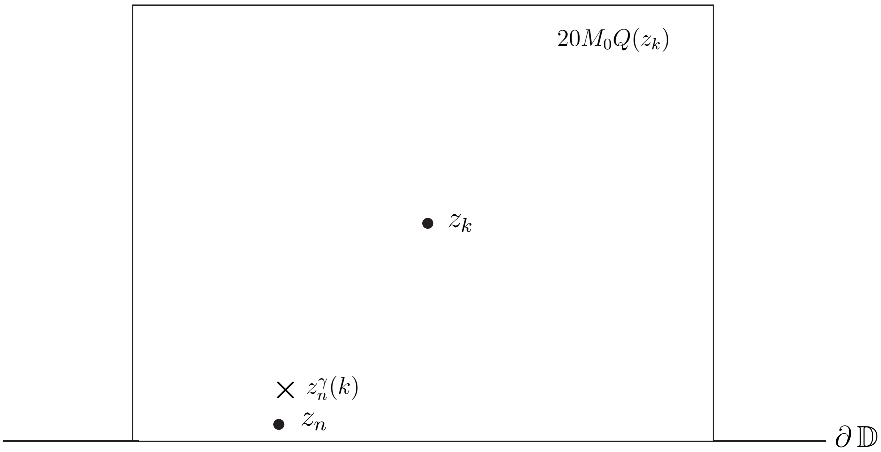

Figure 1.1: definition of $z_n^\gamma(k)$

where C is a constant depending on M_0. So,

$$\sum_{\substack{z_n\in 20M_0Q(z_k)\\ \beta(z_k,z_n)\geq N}} 1-|z_n^\gamma(k)| \leq (1-|z_k|)^{C\gamma}\sum_{j=N}^{\infty}\sum_{\substack{z_n\in 20M_0Q(z_k)\\ j\leq\beta(z_n,z_k)<j+1}}(1-|z_n|)^{1-C\gamma}.$$

Now, if $z_n \in 20M_0Q(z_k)$ and $j \leq \beta(z_n, z_k) < j+1$, Lemma 1.2.4 tells that $1-|z_n| \leq K(M_0)2^{-j}(1-|z_k|)$. So, using (1.0.2), the right hand side term is bounded by

$$K(M_0)^{1-C\gamma}(1-|z_k|)\sum_{j=N}^{\infty} M2^{\alpha j}2^{-j(1-C\gamma)}.$$

Since $\alpha < 1$, taking $\gamma > 0$ so small that $\alpha + C\gamma < 1$, the expression above may be bounded by

$$M\,K(M_0)^{1-C\gamma}\frac{2^{N(\alpha+C\gamma-1)}}{1-2^{\alpha+C\gamma-1}}\,(1-|z_k|).$$

Finally, given $\lambda > 0$ taking N sufficiently large, we obtain

$$\sum_{\substack{z_n\in 20M_0Q(z_k)\\ \beta(z_n,z_k)\geq N}} 1-|z_n^\gamma(k)| \leq \lambda\,(1-|z_k|) \text{ for all } z_k\in\{z_n\}.$$

ii) Construction of the sets $\{E_k\}$. For each $z_n^\gamma(k)$, we define $I_n^\gamma(k) = I(z_n^{(\gamma)}(k))$. Fixed $M_0 > 0$ and $N > 0$, we introduce the notation:

$$B(k) = \{z_n \,:\, |z_n| \geq |z_k|,\ \beta(z_k,z_n)\geq N,\ z_n\in 20M_0Q(z_k)\}.$$

Now we will proof that the sets $E_k = M_0 I_k \setminus \bigcup_{z_n \in B(k)} I_n^\gamma(k)$ satisfy

$$\omega(z_k, E_k) \geq 1 - \frac{\delta}{10}. \tag{1.2.8}$$

Using the elementary estimate of the Poisson Kernel

$$\frac{1-|z_k|^2}{|e^{it}-z_k|^2} \leq \frac{1+|z_k|}{1-|z_k|},$$

one obtains

$$\omega\big(z_k, \bigcup_{z_n \in B(k)} I_n^\gamma(k)\big) \leq \sum_{z_n \in B(k)} \frac{1+|z_k|}{1-|z_k|} \int_{I_n^\gamma(k)} \frac{dt}{2\pi} \leq \frac{2}{1-|z_k|} \sum_{z_n \in B(k)} 1 - |z_n^\gamma(k)|,$$

which by (1.2.3) is smaller than 2λ. Since

$$\omega(z_k, E_k) = \omega(z_k, M_0 I_k) - \omega\left(z_k, \bigcup_{z_n \in B(k)} I_n^\gamma(k)\right),$$

the estimate (1.2.5) tells us that

$$\omega(z_k, E_k) \geq 1 - \frac{\delta}{100} - \lambda.$$

If we take $\lambda > 0$ sufficiently small, we deduce (1.2.8). Since $M_0 I_n \subset I_n^\gamma(k)$, it is clear from the definition that $E_k \cap E_j = \emptyset$ if $\beta(z_k, z_j) > N$.

iii) Construction of the pairwise disjoint sets G_n. We rearrange the sequence $\{z_n\}$ so that $\{1 - |z_n|\}$ decreases. For each point z_n we will construct a set $G_n \subset E_n$ so that the corresponding family $\{G_n\}$ will satisfy (1.2.1), (1.2.2) and $G_n \cap G_m = \emptyset$ if $n \neq m$. The construction will proceed by induction and will ensure that the sets G_n are pairwise disjoint and verify (1.2.1).

Take $G_1 = E_1$. By (1.2.8), the estimate (1.2.1) is satisfied when $n = 1$. Assume that pairwise disjoint subsets $G_1, \cdots, G_{j-1}$ of the unit circle have been defined so that

$$\omega(z_n, \bigcup_{k \leq n, k \in A(n)} G_k) \geq 1 - \delta, \text{ for } n = 1, 2, \cdots, j-1.$$

The set G_j will be constructed according to the following two different situations:

(1) If $\beta(z_j, \{z_1, \cdots, z_{j-1}\}) \geq N$ we define $G_j = E_j$. By (1.2.4) we have

$$\omega(z_j, \bigcup_{k \leq j, k \in A(j)} G_k) \geq \omega(z_j, G_j) \geq 1 - \delta.$$

Now let us show that $G_k \cap G_j = \emptyset$ for any $k = 1, \cdots, j-1$. Since $G_k \subset E_k$ and $G_j \subset M_0 I_j$, it is sufficient to show that $M_0 I_j \cap E_k = \emptyset$ for $k = 1, \cdots, j-1$. Fix $k = 1, \cdots, j-1$ and consider two cases

(a) If $z_j \in 20M_0Q(z_k)$. Since $M_0I_j \subset I_j^\gamma(k)$ and $E_k = M_0I_k \setminus \bigcup I_j^\gamma(k)$, we have $E_k \cap M_0I_j = \emptyset$

(b) If $z_j \notin 20M_0Q(z_k)$. Since $|z_j| > |z_k|$ we have $M_0I_j \cap M_0I_k = \emptyset$. Hence $E_k \cap M_0I_j = \emptyset$.

(2) If $\beta(z_j, \{z_1, \cdots, z_{j-1}\}) \leq N$, consider the set of indexes $\mathcal{F} = \mathcal{F}(j) = \{k \in [1, \cdots, j-1] \colon \beta(z_k, z_j) \leq N\}$. Let us distinguish the following two cases:

(a) If $\omega(z_j, \bigcup_{k\in\mathcal{F}} G_k) \geq 1-\delta$, define $G_j = \emptyset$. It is obvious that

$$\omega(z_j, \bigcup_{k\leq j, k\in A(j)} G_k) \geq 1-\delta.$$

(b) If $\omega(z_j, \bigcup_{k\in\mathcal{F}} G_k) < 1-\delta$, define $G_j = E_j \setminus \bigcup_{k\in\mathcal{F}} G_k$. Arguing as in case 1 one can show that $G_k \cap G_j = \emptyset$ for any $k = 1, \cdots, j-1$. Also, applying (1.2.8), one gets

$$\omega(z_j, \bigcup_{k\leq j, k\in A(j)} G_k) \geq \omega(z_j, E_j) \geq 1-\delta.$$

So, by induction, a family $\{G_n\}$ of pairwise disjoint subsets of the unit circle is constructed so that condition (1.2.1) is satisfied. It just remains to show that the family $\{G_n\}$ verifies (1.2.2), that is, there exists $\eta = \eta(\delta) > 0$ such that

$$\sum_{k\colon \beta(z_k,z_n)\geq N} 2^{\eta\beta(z_k,z_n)}\omega(z_n, G_k) \leq \delta, \quad n = 1, 2, \cdots$$

Fixed $n = 1, 2, \cdots$, consider the following set of indexes:

$$\begin{aligned} \mathcal{A} &= \{k \colon \beta(z_k, z_n) \geq N,\ z_k \in 20M_0Q(z_n)\} \\ \mathcal{B} &= \{k \colon \beta(z_k, z_n) \geq N,\ 2M_0I_k \cap M_0I_n = \emptyset\} \\ \mathcal{C} &= \{k \colon \beta(z_k, z_n) \geq N,\ k \notin \mathcal{A} \cup \mathcal{B}\}. \end{aligned}$$

Now split the sum above into three parts

$$\sum_{k\colon \beta(z_k,z_n)\geq N} 2^{\eta\beta(z_k,z_n)}\omega(z_n, G_k) = (A) + (B) + (C)$$

where

$$(A) = \sum_{k\in\mathcal{A}} 2^{\eta\beta(z_k,z_n)}\omega(z_n, G_k)$$

$$(B) = \sum_{k\in\mathcal{B}} 2^{\eta\beta(z_k,z_n)}\omega(z_n, G_k)$$

$$(C) = \sum_{k\in\mathcal{C}} 2^{\eta\beta(z_k,z_n)}\omega(z_n, G_k)$$

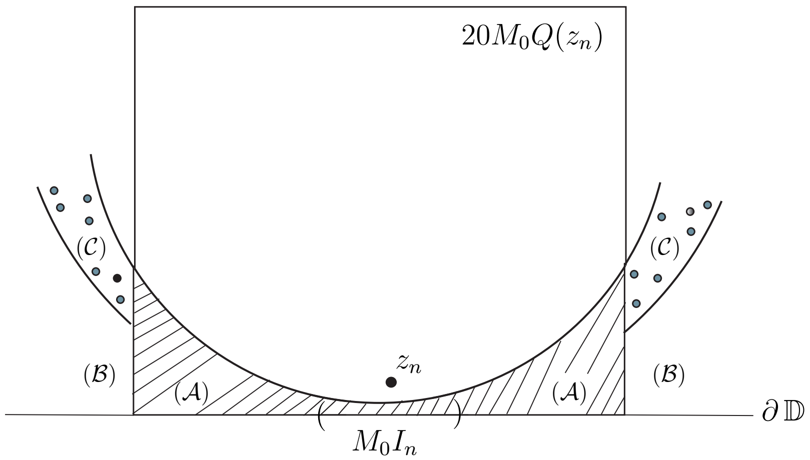

Figure 1.2: The sum is split into three parts corresponding to the location of the points z_k in the regions $(\mathcal{A})$, $(\mathcal{B})$ or $(\mathcal{C})$

See figure 1.2.

In (A) and (B) we will use the estimate $\omega(z_n, G_k) \le C(M_0)2^{-\beta(z_n,z_k)}$ and for (C) we will use the constant $\gamma > 0$ appearing in the construction of the sets E_k.

We first claim that there exists a constant $C = C(M_0) > 0$ such that for points z_k in part (A) or (B), that is those verifying either $z_k \in 20M_0Q(z_n)$ or $2M_0I_k \cap M_0I_n = \emptyset$, one has

$$\omega(z_n, G_k) \le C2^{-\beta(z_k,z_n)}. \tag{1.2.9}$$

For the points z_k in part (A) we have $z_k \in 20M_0Q(z_n)$. Since $G_k \subseteq M_0I_k$, a trivial estimate of the Poisson kernel gives

$$\omega(z_n, G_k) \le \int_{M_0I_k} \frac{1-|z_n|^2}{|e^{it}-z_n|^2}\frac{dt}{2\pi} \le 2M_0\frac{1-|z_k|}{1-|z_n|}.$$

Applying Lemma 1.2.4, since $z_k \in 20M_0Q(z_n)$, one has

$$\log_2\frac{1-|z_k|}{1-|z_n|} \le C(M_0) - \beta(z_k, z_n).$$

Hence, if $z_k \in 20M_0Q(z_n)$ we deduce

$$\omega(z_n, G_k) \le C2^{-\beta(z_k,z_n)}$$

with $C = 2M_02^{C(M_0)}$. For the points z_k in part (B) we have $2M_0I_k \cap M_0I_n = \emptyset$. An easy calculation shows that there exists a constant $C_1 = C_1(M_0)$ such that for any $e^{it} \in I_k$ one has

$$|\, e^{it} - z_n| \ge C_1|1 - z_n\bar{z}_k|.$$

Then

$$\omega(z_n, G_k) \le \int_{M_0 I_k} \frac{1-|z_n|^2}{|e^{it}-z_n|^2}\,\frac{dt}{2\pi} \le C_1^{-2} M_0 \frac{(1-|z_n|^2)(1-|z_k|^2)}{|1-z_n\bar{z}_k|^2}.$$

It is easy to see from the estimates above that there exists a universal constant $C_2 > 0$ such that

$$\beta(z_n, z_k) \le C_2 - \log_2 \frac{(1-|z_n|^2)(1-|z_k|^2)}{|1-z_n\bar{z}_k|^2},$$

and so one deduces

$$\omega(z_n, G_k) \le C 2^{-\beta(z_n,z_k)}$$

with $C = C_1^{-2} M_0 2^{C_2}$. Hence (1.2.9) holds for points z_k in parts (A) and (B). Therefore

$$(\mathrm{A}) + (\mathrm{B}) \le C \sum_{k:\ \beta(z_k,z_n)\ge N} 2^{(\eta-1)\beta(z_n,z_k)}.$$

Observe that condition (1.0.2) gives

$$\sum_{k:\ \beta(z_k,z_n)\le j} 2^{(\eta-1)\beta(z_n,z_k)} \le M 2^{(\eta+\alpha-1)j},$$

for any $j = 1, 2, \cdots$ Since $\alpha < 1$ one may choose $0 < \eta = \eta(\alpha) < 1-\alpha$ so that $\alpha+\eta < 1$. So, adding up for $j \ge N$, one obtains

$$(\mathrm{A}) + (\mathrm{B}) \le CM \frac{2^{(\eta+\alpha-1)N}}{1-2^{\eta+\alpha-1}}.$$

Hence, taking $N > 0$ sufficiently large one deduces

$$(\mathrm{A}) + (\mathrm{B}) \le \frac{\delta}{3}.$$

The estimate of the third term (C) depends on the choice of the constant $\gamma > 0$ appearing in the construction of the sets $\{E_n\}$. For fixed z_n, consider

$$U(n) = \{z_k :\ \beta(z_k, z_n) \ge N,\ z_k \notin 20M_0 Q(z_n),\ 2M_0 I_k \cap M_0 I_n \ne \emptyset\}.$$

So (C) $= \sum_{z_k \in U(n)} 2^{\eta\beta(z_k,z_n)} \omega(z_n, G_k)$.

Observe that if $z_k \in U(n)$, then $|z_k| < |z_n|$ and $z_n \in 3M_0 Q(z_k)$. In particular $z_n \in 20M_0 Q(z_k)$ so, by the construction of the sets $\{G_k\}$, $G_k \subset M_0 I_k \setminus I_n^\gamma(k)$. Hence

$$\omega(z_n, G_k) \le \int\limits_{M_0 I_k \setminus I_n^\gamma(k)} \frac{1-|z_n|^2}{|\xi - z_n|^2}\,\frac{|d\xi|}{2\pi} \le \int\limits_{\partial\mathbb{D}\setminus I_n^\gamma(k)} \frac{1-|z_n|^2}{|\xi - z_n|^2}\,\frac{|d\xi|}{2\pi}$$

and a change of variable gives an absolute constant $C_3 > 0$ such that

$$\omega(z_n, G_k) \le C_3\,(1-|z_n|) \int_{1-|z_n^\gamma(k)|}^{\infty} \frac{dx}{x^2} \le C_3\, \frac{1-|z_n|}{1-|z_n^\gamma(k)|}. \tag{1.2.10}$$

This estimate is worse than (1.2.9) which was used for (A) and (B) but it is good enough for our purposes. The key is that in (C) we sum over "few" terms corresponding to the points $z_k \in U(n)$.

Observe that if $z_k \in U(n)$, z_k belongs to the Stolz angle $\Gamma_n = \Gamma_n(M_0) = \{z \in \mathbb{D}\colon |z - e^{i \operatorname{Arg} z_n}| \leq 11M_0(1-|z|)\}$ with vertex $e^{i\operatorname{Arg} z_n}$ and a certain opening depending on M_0. To see this we only need to observe that $2M_0 I_k \cap M_0 I_n \neq \emptyset$ implies $|\operatorname{Arg} z_k - \operatorname{Arg} z_n| \leq 10M_0(1-|z_k|)$ and use this inequality to get

$$|z_k - e^{i\operatorname{Arg} z_n}| \leq 11M_0(1-|z_k|).$$

Define $V(n) = \{z_k \in \Gamma_n\colon |z_k| < |z_n|, \beta(z_k, z_n) \geq N\}$ and then,

$$(\mathrm{C}) = \sum_{z_k \in U(n)} 2^{\eta\beta(z_k,z_n)} \omega(z_n, G_k) \leq \sum_{z_k \in V(n)} 2^{\eta\beta(z_k,z_n)} \omega(z_n, G_k).$$

Using inequalities (1.2.10) and (1.2.7) we obtain

$$(\mathrm{C}) \leq C_3 \sum_{z_k \in V(n)} 2^{\eta\beta(z_k,z_n)} \frac{1-|z_n|}{1-|z_n^{\gamma}(k)|} \leq C_3 \sum_{z_k \in V(n)} 2^{\eta\beta(z_k,z_n)} \left(\frac{1-|z_n|}{1-|z_k|}\right)^{C^{-1}\gamma}.$$

Since $z_n \in 3M_0 Q(z_k)$, Lemma 1.2.4 gives

$$\left|\beta(z_n, z_k) - \log_2 \frac{1-|z_k|}{1-|z_n|}\right| \leq C(M_0).$$

Hence

$$\frac{1-|z_n|}{1-|z_k|} \leq 2^{C(M_0) - \beta(z_n,z_k)}.$$

Therefore

$$(\mathrm{C}) \leq C_3\, 2^{C(M_0)C^{-1}\gamma} \sum_{z_k \in V(n)} 2^{(\eta - C^{-1}\gamma)\beta(z_n,z_k)}.$$

Since the sequence $\{z_n\}$ is separated, there exists $C_4 = C_4(M_0) > 0$ such that for any $j \geq 0$, the number of points $z_k \in V_n$ with $j \leq \beta(z_k, z_n) \leq j+1$ is at most C_4. Hence

$$(\mathrm{C}) \leq C_3\, C_4 2^{C(M_0)C^{-1}\gamma} \sum_{j=N}^{\infty} 2^{(\eta - C^{-1}\gamma)j}.$$

Taking $\eta > 0$ so small that $\eta - C^{-1}\gamma < 0$ and taking N sufficiently large, we deduce

$$(\mathrm{C}) \leq \frac{\delta}{3}.$$

So condition (1.2.2) is satisfied and the proof of Lemma 1.2.2 is finished. $\square$

Third Step

On the last step given a partition $\{z_n\} = T \cup S$ the sets $\{G_n\}$ constructed on step 3.2 will be used to find a function $u = u(T, S)$ satisfying the conditions stated in Lemma 1.2.1. This will end the proof of the sufficiency of condition (1.0.2).

A sequence of points $\{z_n\}$ in the unit disc is called an interpolating sequence for the space h^∞ of bounded harmonic functions in the unit disc if for any bounded sequence $\{w_n\}$ of real numbers there exists $u \in h^\infty$ with $u(z_n) = w_n$, $n = 1, 2, \cdots$. L. Carleson and J. Garnett characterized the interpolating sequences for h^∞ as those sequences $\{z_n\}$ satisfying $\inf_{n \neq m} \beta(z_n, z_m) > 0$ and

$$\sup \frac{1}{\ell(Q)} \sum_{z_n \in Q} (1 - |z_n|) < \infty, \tag{1.2.11}$$

where the supremum is taken over all Carleson squares of the form

$$Q = \{re^{i\theta} \colon 0 < 1 - r < \ell(Q), \quad |\theta - \theta_0| < \ell(Q)\}$$

for some $\theta_0 \in [0, 2\pi)$. See [CG], [Ga1] or [Ga3, p. 313]. We next show that a separated sequence $\{z_n\}$ satisfying (1.0.2) verifies the condition above. Actually it is sufficient to show (1.2.11) for Carleson squares Q which contain a point of the sequence $\{z_n\}$ in its top part $T(Q) = \{re^{i\theta} \in Q \colon 1 - r > \ell(Q)/2\}$. Let Q be a Carleson square of this type. Let $z_n \in T(Q)$ and $A(j) = \{k \colon z_k \in Q, j - 1 \leq \beta(z_k, z_n) < j\}$. Lemma 1.2.4 tells that for any $k \in A(j)$ the quantity $1 - |z_k|$ is comparable to $2^{-j}\ell(Q)$. Hence condition (1.0.2) yields

$$\sum_{k \in A(j)} 1 - |z_k| \leq C_1 2^{-j} \ell(Q) \# A(j) \leq C_1 M 2^{(\alpha-1)j} \ell(Q).$$

Since $\alpha < 1$, adding up over $j = 1, 2, \cdots$, one obtains (1.2.11). Hence $\{z_n\}$ is an interpolating sequence for h^∞. Then by the Open Mapping Theorem, there exists a constant $\gamma = \gamma(\{z_n\}) > 0$ such that for any partition of the sequence $\{z_n\} = T \cup S$ there exists $h = h(T, S) \in h^\infty$ with $\sup\{|h(z)| \colon z \in \mathbb{D}\} < 1$ and $h(z_n) > \gamma$ for $z_n \in T$ while $h(z_n) < -\gamma$ for $z_n \in S$. Let $\delta > 0$ be a small number to be fixed later and let $N = N(\delta)$, $\eta = \eta(\delta)$ be the positive constants and $\{G_n\}$ the pairwise disjoint collection of subsets of the unit circle given in Lemma 1.2.2. Let $\varepsilon = \varepsilon(\delta)$ be a small number to be fixed later which will satisfy $\varepsilon\delta^{-1} \to 0$ as δ tends to 0. Let $\{w_k\}$ be a sequence of positive numbers satisfying the compatibility condition (1.0.1). Given a partition $\{z_n\} = T \cup S$, consider the function $u = u(T, S) \in h^+$ defined by

$$u(z) = \sum_k w_k \int_{G_k} P_z(\xi)(1 + h(\xi))|d\xi|,$$

where $h = h(T, S)$ and

$$P_z(\xi) = \frac{1}{2\pi} \frac{1 - |z|^2}{|\xi - z|^2}$$

is the Poisson kernel. Our goal is to show that $u(z_n) \geq w_n$ for $z_n \in T$ and $u(z_n) \leq w_n$ for $z_n \in S$. For $n = 1, 2, \cdots$, let $A(n)$ be the set of indexes k such that $\beta(z_k, z_n) \leq N$. Write $u(z_n) = \text{(I)} + \text{(II)}$, where

$$\text{(I)} = \sum_{k \notin A(n)} \omega_k \int_{G_k} P_{z_n}(\xi)(1 + h(\xi))\, |d\xi|,$$

$$\text{(II)} = \sum_{k \in A(n)} \omega_k \int_{G_k} P_{z_n}(\xi)(1 + h(\xi))\, |d\xi|.$$

We first show that

$$\text{(I)} < 2\delta w_n, \quad n = 1, 2, \cdots \tag{1.2.12}$$

Actually if the constant $\varepsilon = \varepsilon(\delta) > 0$ is taken so that $\varepsilon < \eta$, the compatibility condition (1.0.1) tells that (I) can be bounded by

$$w_n \sum_{k \notin A(n)} 2^{\eta\beta(z_k, z_n)} 2\omega(z_n, G_k)$$

which, by (1.2.2), is bounded by $2\delta w_n$. Hence (1.2.12) holds.

For the other term, using that the sets $\{G_n\}$ are pairwise disjoint and the compatibility condition (1.0.1) we have

$$\text{(II)} = \sum_{k \in A(n)} w_k \int_{G_k} P_{z_n}(\xi)(1 + h(\xi))|d\xi| \leq 2^{\varepsilon N} w_n (1 + h(z_n)).$$

Also, since $\sup\{|h(z_n)| \colon z \in \mathbb{D}\} \leq 1$, the compatibility condition (1.0.1) and the estimate (1.2.1) yield

$$\begin{aligned}\text{(II)} &\geq w_n 2^{-\varepsilon N} \left(1 + h(z_n) - \int_{\partial\mathbb{D} \setminus \bigcup_{k \in A(n)} G_k} P_{z_n}(\xi)\,(1 + h(\xi))|d\xi|\right) \\ &\geq 2^{-\varepsilon N} w_n (1 + h(z_n) - 2\delta).\end{aligned}$$

So

$$2^{-\varepsilon N} w_n (1 + h(z_n) - 2\delta) \leq \text{(II)} \leq 2^{\varepsilon N} w_n (1 + h(z_n)).$$

Hence

(a) If $z_n \in T$, $h(z_n) \geq \gamma$ and then $u(z_n) \geq \text{(II)} \geq w_n 2^{-\varepsilon N}(1 + \gamma - 2\delta)$.

(b) If $z_n \in S$, $h(z_n) \leq -\gamma$ and then $u(z_n) = \text{(I)} + \text{(II)} \leq w_n(2\delta + 2^{\varepsilon N}(1 - \gamma))$.

For fixed $\gamma > 0$, taking $\delta = \delta(\gamma) > 0$ and $\varepsilon = \varepsilon(\delta, \eta, N) > 0$ sufficiently small, we deduce that $u(z_n) \geq w_n$ if $z_n \in T$ and $u(z_n) \leq w_n$ if $z_n \in S$. An application of Lemma 1.2.1 concludes the proof of the sufficiency of condition (1.0.2).

□

1.3 Other results

Equivalent conditions

In this section several geometric conditions which are equivalent to (1.0.2) are collected.

Proposition 1.3.1. *Let $\{z_n\}$ be a sequence of distinct points in $\mathbb{D}$. Then the following are equivalent:*

(a) *Condition (1.0.2) holds, that is, there exist constants $M > 0$ and $0 < \alpha < 1$ such that*

$$\#\{z_j \colon \beta(z_j, z_n) \leq l\} \leq M\, 2^{\alpha l}$$

for any $n, l = 1, 2 \cdots$

(b) *There exist constants $M_1 > 0$ and $0 < \alpha < 1$ such that*

$$\#\left\{z_j \colon \left|\frac{z_j - z_n}{1 - \bar{z}_n z_j}\right| \leq r\right\} \leq M_1 (1-r)^{-\alpha},$$

for any $0 < r < 1$ and any $n = 1, 2, \cdots$

(c) *There exist constants $M_2 > 0$ and $0 < \alpha < 1$ such that*

$$\#\{z_j \in Q(z_n) \colon 2^{-l-1}(1-|z_n|) \leq 1 - |z_j| \leq 2^{-l}(1-|z_n|)\} \leq M_2\, 2^{\alpha\, l}$$

for any $n, l = 1, 2, \cdots$

(d) *There exist constants $M_3 > 0$ and $0 < \alpha < 1$ such that*

$$\sum_{z_j \in Q(z_n)} (1-|z_j|)^{\alpha} \leq M_3 (1-|z_n|)^{\alpha},$$

for any $n = 1, 2, \cdots$.

Proof. The equivalence between (a) and (b) follows from the following obvious observation. Let $z, w \in \mathbb{D}$, then $\beta(z,w) \leq l$ if and only if

$$\left|\frac{z-w}{1-\overline{w}z}\right| = \frac{2^{\beta(z,w)} - 1}{2^{\beta(z,w)} + 1} = 1 - \frac{2}{2^{\beta(z,w)} + 1} \leq 1 - \frac{2}{2^l + 1}$$

Assume (a) holds. Fix two positive integers n, l. Let $z_j \in Q(z_n)$ satisfying

$$2^{-l-1}(1-|z_n|) \leq 1 - |z_j| \leq 2^{-l}(1-|z_n|).$$

Applying Lemma 1.2.4 one shows that there exists a universal constant $C > 0$ such that

$$|\beta(z_n, z_j) - l| \leq C.$$

Hence

$$\Big\{z_j \in Q(z_n)\colon 2^{-l-1}(1-|z_n|) \leq 1-|z_j| \leq 2^{-l}(1-|z_n|)\Big\} \subseteq \{z_j\colon \beta(z_j, z_n) \leq l + C\}$$

and condition (1.0.2) gives (c). Adding up over $l = 1, 2, \cdots$ one shows that (c) implies (d). Assume (d) holds and let us show condition (1.0.2). By conformal invariance one may assume $z_n = 0$. So condition (d) tells us that

$$\sum_{j=1}^{\infty}(1-|z_j|)^{\alpha} \leq M_3.$$

Since $\beta(z_j, 0) \leq l$ implies

$$1 - |z_j| \geq 2^{-l},$$

we deduce

$$\#\{z_j\colon \beta(z_j, 0) \leq l\} \leq 2^{\alpha l} \sum_{z_j\colon \beta(z_j,0)\leq l} (1-|z_j|)^{\alpha} \leq M_3\, 2^{\alpha l}$$

which gives (1.0.2). □

As mentioned in the introduction, condition (1.0.2) says how dense is the sequence when one looks at it from a point of the sequence. It is worth mentioning that one can not take as a base point an arbitrary point in the unit disc. This follows from the following example of two separated interpolating sequences for h^+ which will be called Z_1, Z_2 so that $\inf\{\beta(z,\xi)\colon z \in Z_1,\ \xi \in Z_2\} > 0$ but such that the union $Z_1 \cup Z_2$ is not an interpolating sequence for h^+. For instance one may take $Z_1 = \{r_k\}$ where $r_1 = 0$, $r_k \to 1$ and $\beta(r_k, r_{k+1}) \to \infty$ as $k \to \infty$. For each $k = 1, 2, \cdots$, choose points $\{z_1^{(k)}, \cdots, z_{N(k)}^{(k)}\}$, $N(k) = 2^{n_k}$, equally distributed in the hyperbolic cercle centered at r_k of hyperbolic radius n_k. Here $n_k \to \infty$ as $k \to \infty$ in such a way that $n_k < \beta(r_k, r_{k+1})/4$. Let $Z_2 = \{z_i^{(k)}\colon i = 1, \cdots, N(k), k = 1, 2, \cdots\}$. It can be shown that Z_1 and Z_2 satisfy condition (1.0.2) with the exponent $\alpha = 1/2$, while $Z_1 \cup Z_2$ does not fulfill (1.0.2) for any $0 < \alpha < 1$ because the number of points in Z_2 at hyperbolic distance n_k from the point $r_k \in Z_1$ is 2^{n_k}. See figure 1.3.

An interpolation problem for bounded Analytic Functions without zeros

Let H^∞ denote the algebra of bounded analytic functions in the unit disc $\mathbb{D}$. Let $(H^\infty)^*$ be the subalgebra of H^∞ which consists on the functions in H^∞ without zeros in $\mathbb{D}$. If $f \in (H^\infty)^*$ then $\log(\|f\|_\infty/|f(z)|) \in h^+$. So if $\{z_n\}$ is a sequence in $\mathbb{D}$ and $t_n = \log(\|f\|_\infty/|f(z_n)|)$, Harnack's inequality tells us that

$$|\log_2 t_n - \log_2 t_m| \leq \beta(z_n, z_m), \qquad n, m = 1, 2, \cdots.$$

So, as before, we may consider a notion of interpolating sequence.

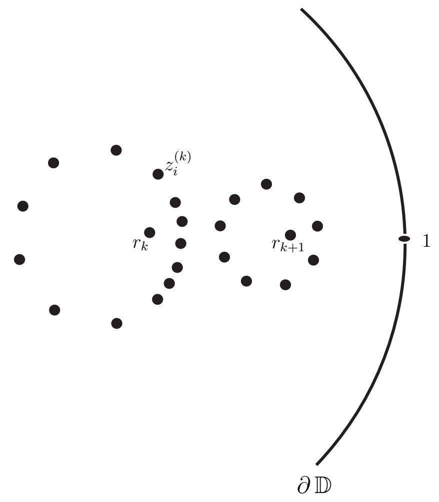

Figure 1.3: Interpolating sequences for h^+

Definition 1.3.2. A sequence of points $\{z_n\}$ in the unit disc is called an *interpolating sequence for* $(H^\infty)^*$ if there exist constants $\varepsilon > 0$ and $0 < C < \infty$ such that for any sequence of non-vanishing complex values $\{w_n\}$, $|w_n| < C, n = 1, 2, \cdots$, satisfying

$$\left|\log_2\left(\log_2\left(\frac{C}{|w_n|}\right)\right) - \log_2\left(\log_2\left(\frac{C}{|w_m|}\right)\right)\right| \leq \varepsilon\beta(z_n, z_m),\ n, m = 1, 2, \cdots \quad (1.3.1)$$

there exists a function $f \in (H^\infty)^*$ with $f(z_n) = w_n$, $n = 1, 2, \cdots$.

The characterization of the interpolating sequences for $(H^\infty)^*$ is given in the following result.

Theorem 1.3.3. *A separated sequence $\{z_n\}$ of points in the unit disc is interpolating for $(H^\infty)^*$ if and only if there exist constants $M > 0$ and $0 < \alpha < 1$ such that*

$$\#\{z_j \colon \beta(z_j, z_n) \leq \ell\} \leq M2^{\alpha\ell} \text{ for any } n, \ell = 1, 2, \cdots \quad (1.3.2)$$

Proof of Theorem 1.3.3. Let us start by showing the necessity of condition (1.3.2). Given a separated interpolating sequence $\{z_n\}$ for $(H^\infty)^*$ consider the constants $\varepsilon > 0$ and $C < \infty$ given in definition 1.3.2. Define the sequence of positive values $t_n = 2^{\varepsilon\beta(0,z_n)}$, $n = 1, 2, \cdots$. It is clear that

$$|\log_2 t_n - \log_2 t_m| \leq \varepsilon\beta(z_n, z_m), \quad n, m = 1, 2, \cdots .$$

Then, if we consider a sequence of complex values $\{w_n\}$ with $t_n = \log(C/|w_n|)$, we have $\sup_n |w_n| \leq C$ and furthermore $\{w_n\}$ satisfies condition (1.3.1). So, there exists a function $f \in H^\infty$ without zeros with $f(z_n) = w_n$, $n = 1, 2, \cdots$. The function $v(z) =$

$\log\left(\frac{C}{|f(z)|}\right)$ is a harmonic function, $v(z) \geq \log(C/\|f\|_\infty) := -k_1$, and interpolates the values $\{t_n\}$ at the points $\{z_n\}$. So, $u(z) = v(z) + k_1 \in h^+(\mathbb{D})$ and $u(z_n) = t_n + k_1 = 2^{\varepsilon\beta(0,z_n)} + k_1$, $n = 1, 2, , \cdots$. Now, arguing as in the proof of the necessity of Theorem 1.0.1, we can conclude that there exist constants $M > 0$ and $0 < \alpha < 1$ such that

$$\#\{z_j : \beta(z_j, z_n) \leq \ell\} \leq M2^{\alpha\ell} \text{ for any } n, \ell = 1, 2, \cdots$$

Let us now show the sufficiency of condition (1.3.2). Given a separated sequence $\{z_n\}$ satisfying (1.3.2) and $\{w_n\}$ satisfying (1.3.1) for some ε, C, consider $t_n = \log \frac{C}{|w_n|}$. We can take $C > \|w_n\|_\infty$. Then obviously $\{t_n\}$ satisfies the compatibility condition (1.0.1). So, there exists a function $u \in h^+(\mathbb{D})$ with $u(z_n) = \log \frac{C}{|w_n|}$, for $n = 1, 2, \cdots$. Consider $u_0(z) = u(z) - \log(C)$ and let $\widetilde{u_0}(z)$ be the harmonic conjugate function of $u_0(z)$. Then $e^{-(u_0 + i\widetilde{u_0})}$ is a bounded analytic function that interpolates de values $\{|w_n|\gamma_n\}$ at the points $\{z_n\}$, where $\gamma_n = e^{-i\widetilde{u}_0(z_n)}$, $n = 1, 2, \cdots$. The sequence $\{z_n\}$ is separated and satisfies condition (1.0.2), so it is an interpolating sequence for H^∞ (see [Ca1] or [Ga3]). So there exists a bounded analytic function $g(z)$ such that $g(z_n) = -\operatorname{Arg}(\gamma_n) + \operatorname{Arg}(w_n)$ and then the function $h(z) = e^{-u_0 - i\widetilde{u_0}} e^{ig}$ is a bounded analytic function without zeros with $h(z_n) = w_n$ for any $n = 1, 2, \cdots$ □

Higher Dimensions

Let $h^\infty(\mathbb{R}^{d+1}_+)$ be the space of bounded harmonic functions in the upper-half space $\mathbb{R}^{d+1}_+ = \{(x, y) : x \in \mathbb{R}^d, y > 0\}$. A sequence of points $\{z_n\} \subset \mathbb{R}^{d+1}_+$ is called an interpolating sequence for $h^\infty(\mathbb{R}^{d+1}_+)$ if for any bounded sequence $\{w_n\}$ of real numbers there exists $u \in h^\infty(\mathbb{R}^{d+1}_+)$ with $u(z_n) = w_n$, $n = 1, 2, \cdots$. When the dimension $d > 1$, there is no complete geometric description of the interpolating sequences for $h^\infty(\mathbb{R}^{d+1}_+)$. In [Ca1] and [CG], L. Carleson and J. Garnett proved Theorem C. Moreover in [CG], the authors present several geometric conditions on the sequence $\{z_n\}$ which imply that $\{z_n\}$ is an interpolating sequence for $h^\infty(\mathbb{R}^{d+1}_+)$. However it is not known if the two necessary conditions in part (a) of Theorem C are also sufficient. Related interpolation problems have been considered in [Am] and [Dy]. The situation for interpolating sequences for the space $h^+(\mathbb{R}^{d+1}_+)$ of positive harmonic functions in $\mathbb{R}^{d+1}_+$ is quite similar. A sequence of points $\{z_n\} \subset \mathbb{R}^{d+1}_+$ will be called an *interpolating sequence for* $h^+(\mathbb{R}^{d+1}_+)$ if there exists a constant $\varepsilon = \varepsilon(\{z_n\}) > 0$ such that for any sequence $\{w_n\}$ of positive values satisfying

$$|\log_2 w_n - \log_2 w_m| \leq \varepsilon\beta(z_n, z_m), \quad n, m = 1, 2, \cdots,$$

there exists a function $u \in h^+(\mathbb{R}^{d+1}_+)$ with $u(z_n) = w_n$, $n = 1, 2, \cdots$. Here $\beta(z, w)$ denotes the hyperbolic distance between the points $z, w \in \mathbb{R}^{d+1}_+$,

$$\beta(z, w) = \log_2 \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

where $\rho(z,w) = |z-w|/|z-\bar{w}|$ and $\bar{w} = (w_1, \cdots, w_d, -w_{d+1})$. As before, a sequence of points $\{z_n\} \subset \mathbb{R}^{d+1}_+$ is called separated if $\inf_{n\neq m} \beta(z_n, z_m) > 0$.

Theorem 1.3.4. *Let $\{z_n\}$ be a separated sequence of points in the upper-half space $\mathbb{R}^{d+1}_+$, $d > 1$.*

(a) *Assume that $\{z_n\}$ is an interpolating sequence for $h^+(\mathbb{R}^{d+1}_+)$. Then there exist constants $M > 0$ and $0 < \alpha < 1$ such that*

$$\#\{z_k \colon \beta(z_k, z_n) \leq l\} \leq M 2^{\alpha d l}, \quad l, n = 1, 2, \cdots \tag{1.3.3}$$

(b) *Assume that $\{z_n\}$ satisfies the condition (1.3.3) above. Then $\{z_n\}$ can be split into a finite number of disjoint subsequences Λ_i, $i = 1, \cdots, N$,*

$$\{z_n\} = \Lambda_1 \cup \cdots \cup \Lambda_n,$$

such that $\Lambda_i \cup \Lambda_j$ is an interpolating sequence for $h^+(\mathbb{R}^{d+1}_+)$ for any $i, j = 1, \cdots, N$

The proof of (a) follows the same lines of the proof of the necessity in Theorem 1.0.1. The first two steps 1.2.1 and 1.2.2 of the proof of the sufficiency in Theorem 1.0.1 can be extended to several variables. However the third step 3.3 can not be fulfilled because we have not been able to show that a separated sequence satisfying condition (1.3.3) is an interpolating sequence for $h^\infty(\mathbb{R}^{d+1}_+)$. Since it is clear that condition (1.3.3) implies that $\mu(Q) \leq C\, l(Q)^d$ for any Carleson cube Q, applying Theorem C of L. Carleson and J. Garnett, the sequence $\{z_n\}$ can be split into a finite number of disjoint subsequences $\Lambda_1, \cdots, \Lambda_N$ such that $\Lambda_i \cup \Lambda_j$ is an interpolating sequence for $h^\infty(\mathbb{R}^{d+1}_+)$, $i, j = 1, \cdots, N$. Arguing as in step 3.3 of the proof of the sufficiency, one can show that for any $i, j = 1, \cdots, N$, the sequence $\Lambda_i \cup \Lambda_j$ is an interpolating sequence for $h^+(\mathbb{R}^{d+1}_+)$.

It is worth mentioning that we have not been able to prove that a separated sequence verifying (1.3.3) is interpolating for $h^+(\mathbb{R}^{d+1}_+)$, when $d > 1$.

Chapter 2

Interpolating sequences for Besov type spaces.

2.1 Preliminaries

Bergman Spaces

Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that the are of $\mathbb{D}$ is 1. In rectangular and polar coordinates,

$$dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi}\, r\, dr\, d\theta.$$

Some spaces will be defined in terms of the measure $dA(z)$. Given a real number β, let

$$dA_\beta(z) = (1+\beta)(1-|z|^2)^\beta\, dA(z).$$

For $\beta > -1$ and $0 < p < \infty$, the Bergman space A^p_β consists of all analytic functions in $L^p(\mathbb{D}, dA_\beta)$ with norm

$$\|f\|^p_{A^p_\beta} = \int_{\mathbb{D}} |f(z)|^p\, dA_\beta(z) < \infty.$$

The following result can be found in [HKZ, Corollary 1.5] and can be referred as a reproducing formula for functions in the Bergman space. When $z = 0$, the proof is a simple application of the mean value property and the rotation invariance of dA_β. The general case follows raplacing f by $f \circ \varphi_z$, with the change of variables $w = \varphi_z(\xi)$ and then replacing the function $f(w)$ by the function $(1-\overline{w}z)^{2+\beta} f(w)$.

Lemma G. *Let $\beta > -1$, if v is a function in A^1_β, then*

$$v(z) = \int_{\mathbb{D}} \frac{v(w)}{(1-\bar{w}z)^{2+\beta}}\, dA_\beta(w)$$

for all $z \in \mathbb{D}$ and the integral converges uniformly for z in every compact subset of $\mathbb{D}$.

The following result is from [HKZ, Theorem 1.9].

Lemma H. *Let a and β be real numbers, and let $T(v)$ be the integral operator defined by*

$$T(v)(z) = \int_{\mathbb{D}} \frac{v(w)}{|1-\bar{w}z|^{2+a}} (1-|w|^2)^a \, dA(w).$$

Let $1 \le p < \infty$. Then $T(v)$ is bounded from $L^p(\mathbb{D}, dA_\beta)$ to itself if and only if

$$0 < 1+\beta < p\,(a+1).$$

Another useful result that can be found in [HKZ, Proposition 1.11].

Lemma I. *Suppose $1 \le p < +\infty$, $-1 < \beta < +\infty$, and that n is a positive integer. Then an analytic function f in $\mathbb{D}$ belongs to A^p_β if and only if the function $(1-|z|^2)^n f^{(n)}(z)$ is in $L^p(\mathbb{D}, dA_\beta)$.*

Some inequalities

We shall summarize now some well-known inequalities that we will use. Further details may be found in [Stein, Appendix A].

Minkowski's inequality for integrals states that the norm of an integral is not greater than the integral of the corresponding norms. In explicit form, for the case of L^p spaces, this can be restated as follows. Let $1 \le p < \infty$, then

$$\left(\int_{\mathscr{Y}} \left(\int_{\mathscr{X}} |F(x,y)| \, dx\right)^p dy\right)^{1/p} \le \int_{\mathscr{X}} \left(\int_{\mathscr{Y}} |F(x,y)|^p \, dy\right)^{1/p} dx. \tag{2.1.1}$$

Here $F(x,y)$ is a measurable function on the σ-finite product measure space $\mathscr{X} \times \mathscr{Y}$; dx and dy are the measures on $\mathscr{X}$ and $\mathscr{Y}$ respectively.

Another useful result is the following pair of inequalities due to *Hardy*:

$$\begin{aligned} \left(\int_0^C \left(\int_0^x f(y)dy\right)^p x^{-r-1}dx\right)^{1/p} &\le p/r \left(\int_0^C (y\,f(y))^p y^{-r-1} dy\right)^{1/p}, \\ \left(\int_0^C \left(\int_x^C f(y)dy\right)^p x^{r-1}dx\right)^{1/p} &\le p/r \left(\int_0^C (y\,f(y))^p y^{-r-1} dy\right)^{1/p}. \end{aligned} \tag{2.1.2}$$

Here $f \ge 0$, $p \ge 1$, $r > 0$, and $0 < C \le \infty$.

The following lemma is standard and a proof can be found in [Zhu, Section 4.2].

Lemma J. *Let $z \in \mathbb{D}$, $t > -1$ and $c > 0$. Then*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-\bar{w}z|^{2+t+c}} \, dA(w) \approx (1-|z|^2)^{-c}.$$

The following useful inequality is from [OF, Lemma 2.5]. A proof of this result can be found in [Zhao].

Lemma K. *Let $a > -1$, $r, t \geq 0$, and $r + t - a > 2$. If $t < a + 2 < r$ then we have*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^a}{|1-\bar{w}z|^r\,|1-\bar{w}\zeta|^t}\,dA(w) \leq C\,\frac{(1-|z|^2)^{2+a-r}}{|1-\bar{\zeta}z|^t}.$$

Note that Lemma K is a generalization of one of the inequalities in Lemma J.

2.2 Some results for $B_p(s)$

Reproducing kernel

For $1 < p < \infty$ and $s \geq 0$, let $B_p(s)$ be the Besov space of those analytic functions on the unit disc $\mathbb{D}$ for which

$$\|f\|^p_{B_p(s)} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s}\,dA(z) < \infty.$$

We will use the notation $dA_{p,s}(z) = (1-|z|^2)^{p-2+s}dA(z)$. Note that the dual space of $B_p(s)$ is isomorphic to $B_q(s)$, where q is the conjugate exponent of p, under the pairing

$$\langle f, h\rangle_s = f(0)\overline{h(0)} + \int_{\mathbb{D}} f'(z)\overline{h'(z)}\,(1-|z|^2)^s\,dA(z), \tag{2.2.1}$$

defined for $f \in B_p(s)$ and $h \in B_q(s)$. Using the inequality $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$ which holds for all $a, b \geq 0$, we obtain

$$|\langle f, h\rangle_s| \leq \|f\|_{B_p(s)}\|h\|_{B_q(s)}, \qquad \text{for all } f \in B_p(s),\ h \in B_q(s). \tag{2.2.2}$$

Next we will use the reproducing formula given in Lemma G to deduce a reproducing formula for functions in $B_p(s)$. Let $f \in B_p(s)$, and let $0 < s < 1$ and $1 < p < \infty$, then by Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|(1-|z|^2)^s dA(z)\\
&\quad\leq \left(\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s}dA(z)\right)^{1/p}\left(\int_{\mathbb{D}}(1-|z|^2)^{q-2+s}dA(z)\right)^{1/q}\\
&\quad= C\left(\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2+s}dA(z)\right)^{1/p} < \infty.
\end{aligned}$$

So $f' \in A^1_s$ and Lemma G gives

$$f'(\xi) = \int_{\mathbb{D}} \frac{f'(w)}{(1-\overline{w}\,\xi)^{2+s}}\,dA_s(w).$$

Now, for $z \in \mathbb{D}$, we have

$$\int_0^z f'(\xi)\,d\xi = \int_{\mathbb{D}} f'(w) \left(\int_0^z \frac{d\xi}{(1-\overline{w}\,\xi)^{2+s}} \right) dA_s(w)$$

and we obtain the following reproducing formula

$$f(z) = f(0) + \int_{\mathbb{D}} f'(w) \left(\frac{1-(1-\overline{w}z)^{1+s}}{\overline{w}\,(1-\overline{w}z)^{1+s}} \right) (1-|w|^2)^s\, dA(w). \tag{2.2.3}$$

Then with the notation of the pairing (2.2.1),

$$f(z) = \langle f, k_z \rangle_s = f(0)\overline{k_z(0)} + \int_{\mathbb{D}} f'(w)\overline{k_z'(w)}(1-|w|^2)^s\, dA(w),$$

with

$$k_z'(w) = \frac{1-(1-w\overline{z})^{1+s}}{w\,(1-w\overline{z})^{1+s}} \quad \text{and} \quad k_z(0) = 1. \tag{2.2.4}$$

We can state some properties of the reproducing kernel k_z.

Lemma 2.2.1. *Let $1 < p < \infty$, $0 < s < 1$ and let $z, w \in \mathbb{D}$ with $\beta(z,w) \leq 1$, then*

$$\|k_z - k_w\|_{B_q(s)} \leq C \frac{|z-w|}{(1-|w|^2)^{1+s/p}}, \qquad \textit{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Observe that

$$|k_z'(\nu) - k_w'(\nu)| = \left| \frac{(1-\overline{w}\nu)^{1+s} - (1-\overline{z}\nu)^{1+s}}{(1-\overline{z}\nu)^{1+s}(1-\overline{w}\nu)^{1+s}} \right| \lesssim \left| \frac{(1-\xi\overline{\nu})^s}{(1-\overline{z}\nu)^{1+s}(1-\overline{w}\nu)^{1+s}} \right| |z-w|$$

for some $\xi \in [z,w]$. Since $\beta(z,w) \leq 1$, we saw in Observation 1.0.3 that

$$|1-\overline{\xi}\nu| \approx |1-\overline{w}\nu| \approx |1-\overline{z}\nu|.$$

So,

$$|k_z'(\nu) - k_w'(\nu)| \leq C \frac{|z-w|}{|1-\overline{w}\nu|^{2+s}}. \tag{2.2.5}$$

Now, using this inequality,

$$\begin{aligned} \|k_z - k_w\|_{B_q(s)} &= \left(\int_{\mathbb{D}} |k_z'(\nu) - k_w'(\nu)|^q (1-|\nu|^2)^{q-2+s} dA(\nu) \right)^{1/q} \\ &\leq C|z-w| \left(\int_{\mathbb{D}} \frac{(1-|\nu|^2)^{q-2+s}}{|1-\overline{w}\nu|^{q(2+s)}}\, dA(\nu) \right)^{1/q} \end{aligned}$$

and applying Lemma J,

$$\|k_z - k_w\|_{B_q(s)} \leq C \frac{|z-w|}{(1-|w|^2)^{\frac{q+(q-1)s}{q}}} = C \frac{|z-w|}{(1-|w|^2)^{1+s/p}}.$$

□

We can now estimate the norm of the reproducing kernel k_z.

Lemma 2.2.2. *Let $1 < p < \infty$, $0 < s < 1$, and let k_z be the reproducing kernel defined in* (2.2.4)*, then*

$$\|k_z\|^p_{B_q(s)} \approx \frac{1}{(1-|z|^2)^s}, \quad \textit{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Using the explicit form (2.2.4) of k'_z and the fact that

$$\sup_{z,\, w \in \mathbb{D}} \left| \frac{(1-\bar{w}z)^t - 1}{\bar{w}} \right| \le C \quad \text{for } t \ge 1, \tag{2.2.6}$$

we have

$$\begin{aligned}
\|k_z\|^q_{B_q(s)} &= 1 + \int_{\mathbb{D}} |k'_z(w)|^q (1-|w|^2)^{q-2+s} dA(w) \\
&= 1 + \int_{\mathbb{D}} \left| \frac{1-(1-w\overline{z})^{1+s}}{w\,(1-w\overline{z})^{1+s}} \right|^q (1-|w|^2)^{q-2+s} dA(w) \\
&\le 1 + C \int_{\mathbb{D}} \frac{(1-|w|^2)^{q-2+s}}{|1-w\overline{z}|^{q(1+s)}}\, dA(w),
\end{aligned}$$

and applying Lemma J we obtain

$$\|k_z\|^q_{B_q(s)} \le 1 + \frac{C}{(1-|z|^2)^{s(q-1)}} \lesssim \frac{1}{(1-|z|^2)^{s(q-1)}}.$$

For the other inequality, let $D_h(z,1)$ be the hyperbolic disc with center z and radius 1. If $w \in D_h(z,1)$ then, by Observation 1.0.3, $|1-\overline{w}z| \approx 1-|z|$. So, there exists a constant $r_0(s) < 1$ such that

$$|1-\overline{w}z|^{1+s} \le 1/2 \quad \text{for all } |z| \ge r_0(s).$$

We will distinguish two cases:
If $|z| \le r_0(s)$ then $(1-|z|)^{s(q-1)} \ge (1-r_0(s))^{s(q-1)} = C$, so

$$\|k_z\|^q_{B_q(s)} \ge |k_z(0)| = 1 \gtrsim \frac{1}{(1-|z|)^{s(q-1)}}.$$

If $|z| \ge r_0(s)$ then $|1-(1-w\overline{z})^{1+s}| \ge 1 - |1-w\overline{z}|^{1+s} \ge 1/2$. So,

$$\begin{aligned}
\|k_z\|^q_{B_q(s)} &= 1 + \int_{\mathbb{D}} \left| \frac{1-(1-w\overline{z})^{1+s}}{w\,(1-w\overline{z})^{1+s}} \right|^q (1-|w|^2)^{q-2+s} dA(w) \\
&\ge \frac{1}{2^q} \int_{D_h(z,1)} \frac{(1-|w|^2)^{q-2+s}}{|1-w\overline{z}|^{(1+s)q}}\, dA(w) \gtrsim \frac{1}{(1-|z|^2)^{s(q-1)}}.
\end{aligned}$$

□

A characterization of $B_p(s)$ without derivatives

We begin this section with a result that can be of independent interest.

Lemma 2.2.3. *Let $1 < p < \infty$, and let $\sigma > -1$, $b \geq 0$ with $b < 2+\sigma$. Let f be analytic on $\mathbb{D}$, then for any $\zeta \in \mathbb{D}$ one has*

$$\int_{\mathbb{D}} |f(z) - f(0)|^p \, \frac{(1-|z|^2)^\sigma}{|1-\bar{\zeta}z|^b} \, dA(z) \leq C \int_{\mathbb{D}} (1-|z|^2)^p |f'(z)|^p \, \frac{(1-|z|^2)^\sigma}{|1-\bar{\zeta}z|^b} \, dA(z).$$

Here C is an absolute constant which does not depend on ζ.

Proof. The case $b = 0$ is proved in [HKZ]. So, assume that $b > 0$. Choose $\varepsilon > 0$ with

$$\sigma - \varepsilon \max(1, p-1) > -1 \quad \text{and} \quad b + \varepsilon(p-1) < 2+\sigma.$$

Without loss of generality we may assume

$$\int_{\mathbb{D}} (1-|z|^2)^p |f'(z)|^p \, \frac{(1-|z|^2)^\sigma}{|1-\bar{\zeta}z|^b} \, dA(z) < \infty.$$

Observe that Hölder's inequality gives

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|(1-|z|^2)^{1+\sigma} dA(z) \\
&\quad \leq \left(\int_{\mathbb{D}} (1-|z|^2)^p |f'(z)|^p \, \frac{(1-|z|^2)^\sigma}{|1-\bar{\zeta}z|^b} \, dA(z) \right)^{1/p} \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^\sigma}{|1-\bar{\zeta}z|^{(1-q)b}} dA(z) \right)^{1/q} \\
&\quad \leq C \left(\int_{\mathbb{D}} (1-|z|^2)^p |f'(z)|^p \, \frac{(1-|z|^2)^\sigma}{|1-\bar{\zeta}z|^b} \, dA(z) \right)^{1/p},
\end{aligned}$$

so $f \in A^1_{1+\sigma}$. Then by the reproducing formula (2.2.3), we have

$$f(z) - f(0) = \int_{\mathbb{D}} \frac{1-(1-\bar{w}z)^{2+\sigma}}{\bar{w}\,(1-\bar{w}z)^{2+\sigma}} \, f'(w)\,(1-|w|^2)^{1+\sigma} \, dA(w).$$

By inequality (2.2.6), Hölder's inequality and Lemma J,

$$\begin{aligned}
|f(z)-f(0)|^p &\leq C \left(\int_{\mathbb{D}} \frac{|f'(w)|(1-|w|^2)^{1+\sigma}}{|1-\bar{w}z|^{2+\sigma}} \, dA(w) \right)^p \\
&\leq C \left(\int_{\mathbb{D}} |f'(w)|^p \, \frac{(1-|w|^2)^{(1+\varepsilon)p+\sigma-\varepsilon}}{|1-\bar{w}z|^{2+\sigma}} \, dA(w) \right) \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^{\sigma-\varepsilon}}{|1-\bar{w}z|^{2+\sigma}} \, dA(w) \right)^{p-1} \\
&\leq C \left(\int_{\mathbb{D}} |f'(w)|^p \, \frac{(1-|w|^2)^{(1+\varepsilon)p+\sigma-\varepsilon}}{|1-\bar{w}z|^{2+\sigma}} \, dA(w) \right) (1-|z|^2)^{-\varepsilon(p-1)}
\end{aligned}$$

since $\sigma - \varepsilon > -1$. Now, by Fubini's theorem we have

$$\int_{\mathbb{D}} |f(z) - f(0)|^p \, \frac{(1-|z|^2)^\sigma}{|1-\bar{\zeta}z|^b} \, dA(z)$$

$$\lesssim \int_{\mathbb{D}} \Big(\int_{\mathbb{D}} |f'(w)|^p \, \frac{(1-|w|^2)^{(1+\varepsilon)p+\sigma-\varepsilon}}{|1-\bar{w}z|^{2+\sigma}} \, dA(w) \Big) \, \frac{(1-|z|^2)^{-\varepsilon(p-1)+\sigma}}{|1-\bar{\zeta}z|^b} \, dA(z)$$

$$= \int_{\mathbb{D}} |f'(w)|^p \, (1-|w|^2)^{(1+\varepsilon)p+\sigma-\varepsilon} \Big(\int_{\mathbb{D}} \frac{(1-|z|^2)^{-\varepsilon(p-1)+\sigma}}{|1-\bar{w}z|^{2+\sigma} \, |1-\bar{\zeta}z|^b} \, dA(z) \Big) \, dA(w),$$

that by Lemma K, is bounded by constant times

$$\int_{\mathbb{D}} (1-|w|^2)^p \, |f'(w)|^p \, \frac{(1-|w|^2)^\sigma}{|1-\bar{\zeta}w|^b} \, dA(w),$$

and this finishes the proof. □

The following Theorem is a generalization of a result in [RW] where a characterization of $B_2(s)$ without derivatives for $s \geq 0$ is given. The case $s = 0$ had already been considered in [AFP]. Our proof is quite different from the one by Rochberg and Wu, since in [RW] they use some Hilbert space techniques.

This result presents a description of the functions in $B_p(s)$ which use oscillation instead of derivatives.

Theorem 2.2.4. *Let* $1 < p < \infty$ *and* $\sigma, \tau > -1$, $s \geq 0$ *such that* $\min(\sigma, \tau) > s - 2$. *Then, for any analytic function* f,

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^p}{|1-\bar{w}z|^{4+\sigma+\tau-s}} \, (1-|w|^2)^\sigma \, (1-|z|^2)^\tau \, dA(z) \, dA(w)$$
$$\approx \int_{\mathbb{D}} |f'(z)|^p \, (1-|z|^2)^{p-2+s} \, dA(z).$$

Proof. Notice that if $\sigma \neq \tau$, say $\sigma \geq \tau$, then by the fact that $(1-|z|), (1-|w|) \leq |1-\overline{z}w|$, for $z, w \in \mathbb{D}$, we have

$$\frac{(1-|z|^2)^\sigma (1-|w|^2)^\sigma}{|1-\overline{w}z|^{4+2\sigma-s}} \leq \frac{(1-|z|^2)^\sigma (1-|w|^2)^\tau}{|1-\overline{w}z|^{4+\sigma+\tau-s}} \leq \frac{(1-|z|^2)^\tau (1-|w|^2)^\tau}{|1-\overline{w}z|^{4+2\tau-s}}.$$

Hence, the case $\sigma \neq \tau$ can be obtained from the case $\sigma = \tau$. Let us consider the case $\sigma = \tau$. We first prove the upper estimate. Let

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z} \qquad \text{for all } z, w \in \mathbb{D},$$

then

$$1-|\varphi_w(z)|^2=\frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{w}z|^2}\quad\text{and}\quad|\varphi_w'(z)|=\frac{1-|\varphi_w(z)|^2}{1-|z|^2}.$$

Observe also that if $\zeta=\varphi_w(z)$ then $z=\varphi_w(\zeta)$,

$$1-|\zeta|^2=\frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{w}z|^2}\qquad\qquad |1-\overline{w}\zeta|=\frac{1-|w|^2}{|1-\overline{w}z|}$$

$$dA(\zeta)=|\varphi_w'(z)|^2dA(z)=\frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\,dA(z)\qquad\qquad |\varphi_w'(\zeta)|=\frac{|1-\overline{w}z|^2}{1-|w|^2}.$$

The change of variables $\zeta=\varphi_w(z)$ gives

$$\begin{aligned}
&\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^{4-s+2\sigma}}\,(1-|w|^2)^\sigma\,(1-|z|^2)^\sigma\,dA(z)\,dA(w)\\
&=\int_{\mathbb{D}}\int_{\mathbb{D}}|f(z)-f(w)|^p\frac{(1-|w|^2)^\sigma(1-|z|^2)^\sigma}{|1-\overline{w}z|^{2\sigma}}\\
&\qquad\times\frac{|1-\overline{w}z|^s}{(1-|w|^2)^s}\left(\frac{1-|w|^2}{|1-\overline{w}z|^2}\right)^2 dA(z)(1-|w|^2)^{s-2}dA(w)\\
&=\int_{\mathbb{D}}\Big(\int_{\mathbb{D}}|(f\circ\varphi_w)(\zeta)-(f\circ\varphi_w)(0)|^p\,\frac{(1-|\zeta|^2)^\sigma}{|1-\bar{w}\zeta|^s}\,dA(\zeta)\Big)(1-|w|^2)^{s-2}\,dA(w).
\end{aligned}$$

Since $\sigma>s-2$, Lemma 2.2.3 tells that this last quantity is smaller than constant times

$$\int_{\mathbb{D}}(1-|w|^2)^{s-2}\int_{\mathbb{D}}|(f\circ\varphi_w)'(\zeta)|^p\,\frac{(1-|\zeta|^2)^{p+\sigma}}{|1-\bar{w}\zeta|^s}\,dA(\zeta)\,dA(w),$$

and, by the change of variables $z=\varphi_w(\zeta)$ and Fubini's theorem, this quantity is

$$\begin{aligned}
&\int_{\mathbb{D}}(1-|w|^2)^{s-2}\int_{\mathbb{D}}|f'(z)|^p\left(\frac{|1-\overline{w}z|^2}{1-|w|^2}\right)^p\left(\frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{w}z|^2}\right)^{p+\sigma}\\
&\qquad\times\left(\frac{|1-\overline{w}z|}{1-|w|^2}\right)^s\frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\,dA(z)\,dA(w)\\
&=\int_{\mathbb{D}}|f'(z)|^p\,(1-|z|^2)^{p+\sigma}\Big(\int_{\mathbb{D}}\frac{(1-|w|^2)^\sigma}{|1-\bar{w}z|^{4-s+2\sigma}}\,dA(w)\Big)\,dA(z)\\
&\le C\int_{\mathbb{D}}|f'(z)|^p\,(1-|z|^2)^{p-2+s}\,dA(z),
\end{aligned}$$

after an application of Lemma J. This proves the upper estimate.

To prove the lower estimate, we use Cauchy's integral formula on the circle and obtain

$$f'(0)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\theta})-f(0)}{(Re^{i\theta})^2}\,d\theta,\quad\text{for all } 0<R<1.$$

Multiplying by R^3 and integrating with respect to R in both sides of the equality we get

$$|f'(0)|^p \le C(p,r)\int_{|w|<r}|f(w)-f(0)|^p dA(w), \qquad \text{for all } 0<r<1 \text{ and } 1<p<\infty.$$

Therefore if we take $r=1/2$, since $(1-|w|^2)\ge 3/4$, we have

$$|f'(0)|^p \le C\int_{\mathbb{D}}|f(w)-f(0)|^p(1-|w|^2)^\beta dA(w), \quad \text{for all } \beta\ge 0.$$

Replacing f by $f\circ\varphi_z$ we get

$$(1-|z|^2)^p\,|f'(z)|^p \lesssim \int_{\mathbb{D}}|f\circ\varphi_z(w)-f(z)|^p\,dA_\beta(w). \tag{2.2.7}$$

Choose $\beta=2+\sigma>0$. Then, by (2.2.7), and the change of variables $w=\varphi_z(\zeta)$ we get

$$\begin{aligned}
&\int_{\mathbb{D}}(1-|z|^2)^p\,|f'(z)|^p\,(1-|z|^2)^{s-2}\,dA(z)\\
&\lesssim \int_{\mathbb{D}}\frac{dA(z)}{(1-|z|^2)^{2-s}}\int_{\mathbb{D}}|f\circ\varphi_z(w)-f(z)|^p\,dA_\beta(w)\\
&=\int_{\mathbb{D}}\frac{dA(z)}{(1-|z|^2)^{2-s}}\int_{\mathbb{D}}|f(\zeta)-f(z)|^p\,|\varphi_z'(\zeta)|^2\,(1-|\varphi_z(\zeta)|^2)^\beta\,dA(\zeta)\\
&=\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{|f(\zeta)-f(z)|^p}{|1-\bar z\zeta|^{4+2\beta}}\,(1-|z|^2)^{\beta+s}\,(1-|\zeta|^2)^\beta\,dA(z)\,dA(\zeta)\\
&=\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{|f(\zeta)-f(z)|^p}{|1-\bar z\zeta|^{4-s+2\sigma}}\,(1-|z|^2)^\sigma\,(1-|\zeta|^2)^\sigma\,\frac{(1-|z|^2)^{2+s}\,(1-|\zeta|^2)^2}{|1-\bar z\zeta|^{4+s}}\,dA(z)\,dA(\zeta)\\
&\lesssim \int_{\mathbb{D}}\int_{\mathbb{D}}\frac{|f(\zeta)-f(z)|^p}{|1-\bar z\zeta|^{4-s+2\sigma}}\,(1-|z|^2)^\sigma\,(1-|\zeta|^2)^\sigma\,dA(z)\,dA(\zeta),
\end{aligned}$$

and this finishes the proof. □

2.3 Carleson measures and multipliers

Let $1<p<\infty$. *Classical Carleson measures* were introduced by L. Carleson [Ca1] as those Borel measures $\mu\ge 0$ supported on the unit disc for which there exists a constant $K=K(\mu)>0$ such that

$$\int_{\mathbb{D}}|f(z)|^p d\mu(z)\le K\int_0^{2\pi}|f(e^{i\theta})|^p d\theta$$

for any analytic function f in $\overline{\mathbb{D}}$. He proved that μ is a Carleson measure if and only if $\mu(S(I))\le C|I|$ for any arc $I\subset\partial\mathbb{D}$ defined as in (1.0.4).

Given $a>0$ and a positive measure μ on $\mathbb{D}$ we say that μ is an *a-Carleson measure* if

$$\mu(S(I))\le C|I|^{\,a} \quad \text{for all arcs } I \text{ in } \partial\mathbb{D}.$$

Carleson measures for $B_p(s)$

A positive measure μ on $\mathbb{D}$ is an (s,p)-*Carleson measure* if

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \le C \, \|f\|^p_{B_p(s)}$$

whenever $f \in B_p(s)$. The best constant C, denoted by $\|\mu\|_{p,s}$, is said to be the (s,p)-Carleson measure norm of μ.

Let $\mu_1(z) = dA(z)$ be the Lebesgue measure in $\mathbb{D}$, then μ_1 is an (s,p)-Carleson measure for $1 < p < \infty$ and $0 \le s \le 2$. Now consider, for any $z_0 \in \mathbb{D}$, the atomic measure $\mu_2(z) = \delta_{z_0}(z)$, then $\mu_2(z)$ is an $(s,p)-$Carleson measure for $1 < p < \infty$ and $s \ge 0$. The (s,p)-Carleson measures are geometrically described in [ARS1] and [ARS2], but for our purposes we only need the following simple result.

Lemma 2.3.1. *Let $1 < p < \infty$, $s > 1-p$ and let μ be an (s,p)-Carleson measure. Then for each $\varepsilon > 0$*

$$\sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|^2)^\varepsilon}{|1-\bar{w}z|^{\varepsilon+s}} \, d\mu(z) < \infty.$$

Proof. Let

$$f_w(z) = \frac{(1-|w|^2)^{\varepsilon/p}}{(1-\bar{w}z)^{(\varepsilon+s)/p}}.$$

We have that $f_w \in B_p(s)$ with $\|f_w\|^p_{B_p(s)} \le C$, where C is a positive constant independent of w. In fact, applying Lemma J in the last inequality,

$$\|f_w\|^p_{B_p(s)} = (1-|w|^2)^\varepsilon + C \int_{\mathbb{D}} \frac{(1-|w|^2)^\varepsilon (1-|z|^2)^{p-2+s}}{|1-\overline{w}z|^{\varepsilon+s+p}} \, dA(z) \le C.$$

Therefore

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^\varepsilon}{|1-\bar{w}z|^{\varepsilon+s}} \, d\mu(z) = \int_{\mathbb{D}} |f_w(z)|^p \, d\mu(z) \le C \, \|f_w\|^p_{B_p(s)} \le C.$$

□

It is worth mentioning that the condition in Lemma 2.3.1 does not characterize $(s,p)-$Carleson measures. The following result which will be used later tells that $(s,p)-$Carleson measures are stable under a certain operator T.

Theorem 2.3.2. *Let $a > -1$, $s \ge 0$ and v an integrable function in $\mathbb{D}$. Consider*

$$T(v)(z) = \int_{\mathbb{D}} \frac{v(w)}{|1-\bar{w}z|^{2+a}} \, (1-|w|^2)^a \, dA(w).$$

(*i*) *Let* $1 < p < 2$, *and suppose*

$$\begin{array}{lll} a\,p & > (-1+s)(p-1) & \text{if } s \le 1 \\ a & > -1+s & \text{if } s > 1. \end{array}$$

Suppose also that

$$\sup_{z\in\mathbb{D}} (1-|z|^2)\,|v(z)| < \infty. \tag{2.3.1}$$

If the measure $|v(z)|^p\,(1-|z|^2)^{p-2+s}\,dA(z)$ *is an* (s,p)*-Carleson measure, then the measure* $|T(v)(z)|^p\,(1-|z|^2)^{p-2+s}\,dA(z)$ *is also an* (s,p)*-Carleson measure.*

(*ii*) *Let* $p \ge 2$, $\beta > max(-1, -2+s)$ *and*

$$a\,p > 1 + \beta - p + \max(0, -1+s).$$

If $|v(z)|^p\,(1-|z|^2)^{\beta}\,dA(z)$ *is an* (s,p)*-Carleson measure, then* $|T(v)(z)|^p\,(1-|z|^2)^{\beta}\,dA(z)$ *is also an* (s,p)*-Carleson measure.*

Remarks: For $p = 2$, this result is proved by Rochberg and Wu in [RW]. The case $s = 0$ and $\beta = p-2$ is proved in [Bo1], but only for the range $a\,p > 1$. Also, when $s \le 1$, the condition on a can be rewritten as

$$\max(p,q)\,a > \beta - (p-1).$$

For $p \ge 2$, we do not need condition (2.3.1). Also, for $1 < p < 2$, the result is only obtained for $\beta = p-2+s$. It could be interesting to extend the result for β in the same range as in the case $p \ge 2$. Observe also that we will only apply Theorem 2.3.2 for the case $a = 0$, $\beta = p-2+s$.

Proof of Theorem 2.3.2. We must show that for all $f \in B_p(s)$ we have

$$\int_{\mathbb{D}} |f(z)|^p\,|T(v)(z)|^p\,(1-|z|^2)^{\beta}\,dA(z) \le C\|f\|^p_{B_p(s)}$$

for some positive constant C, where $\beta = p-2+s$ for $1 < p < 2$.

Put $fT(v) = (fT(v) - T(fv)) + T(fv)$. By hypothesis $0 < 1+\beta < p\,(a+1)$ so, applying Lemma H and the fact that $|v(z)|^p\,(1-|z|^2)^{\beta}$ is an (s,p)-Carleson measure, we have

$$\begin{aligned} \int_{\mathbb{D}} |T(fv)(z)|^p\,(1-|z|^2)^{\beta}\,dA(z) &\lesssim \int_{\mathbb{D}} |f(z)|^p\,|v(z)|^p\,(1-|z|^2)^{\beta}\,dA(z) \\ &\le C\,\|f\|^p_{B_p(s)}. \end{aligned}$$

On the other hand, we have

$$f(z)T(v)(z) - T(fv)(z) = \int_{\mathbb{D}} \frac{(f(z)-f(w))\,v(w)}{|1-\bar{w}z|^{2+a}}\,(1-|w|^2)^a\,dA(w). \tag{2.3.2}$$

Now we consider separately the cases $1 < p < 2$ and $p \ge 2$.

The case $1<p<2$.
In this case $\beta=p-2+s$, and since $(1-|z|^2)\,|v(z)|\le C$ and $p<2$ we have

$$|v(z)|=|v(z)|^{p-1}\,|v(z)|^{2-p}\le C\,|v(z)|^{p-1}\,(1-|z|^2)^{p-2},$$

and this gives

$$|f(z)T(v)(z)-T(fv)(z)|^p\lesssim\Big(\int_{\mathbb{D}}\frac{|f(z)-f(w)|}{|1-\bar{w}z|^{2+a}}\,|v(w)|^{p-1}\,(1-|w|^2)^{p-2+a}\,dA(w)\Big)^p.$$

If $s=0$, then by Hölder's inequality

$$\begin{aligned}\Big(\int_{\mathbb{D}}&\frac{|f(z)-f(w)|}{|1-\bar{w}z|^{2+a}}\,|v(w)|^{p-1}\,(1-|w|^2)^{p-2+a}\,dA(w)\Big)^p\\&\le C\|v\|^{p(p-1)}_{L^p(dA_\beta)}\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^{(2+a)p}}\,(1-|w|^2)^{a\,p+(p-2)}\,dA(w).\end{aligned}$$

Hence, by (2.3.2) and Theorem 2.2.4 with $\sigma=a\,p+(p-2)$ and $\tau=p-2$ (we can apply it since $ap>1-p$),

$$\begin{aligned}&\int_{\mathbb{D}}|f(z)T(v)(z)-T(fv)(z)|^p\,(1-|z|^2)^{p-2}\,dA(z)\\&\quad\le C\,\|v\|^{p(p-1)}_{L^p(dA_\beta)}\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^{(2+a)p}}(1-|z|^2)^\tau\,(1-|w|^2)^\sigma\,dA(z)\,dA(w)\\&\quad\le C\,\|v\|^{p(p-1)}_{L^p(dA_\beta)}\,\|f\|^p_{B_p(s)}.\end{aligned}$$

If $s>0$, we apply Hölder's inequality again, and then Lemma 2.3.1, to obtain

$$\begin{aligned}&\Big(\int_{\mathbb{D}}\frac{|f(z)-f(w)|}{|1-\bar{w}z|^{2+a}}\,|v(w)|^{p-1}\,(1-|w|^2)^{p-2+a}\,dA(w)\Big)^p\\&\le\Big(\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^{p\,(1+a)+1}}\,(1-|w|^2)^{a\,p-1+(1-s)(p-1)}\,dA(w)\Big)\Big(\int_{\mathbb{D}}\frac{|v(w)|^p\,dA_{p,s}(w)}{|1-\bar{w}z|^{1-s+s}}\Big)^{p-1}\\&\lesssim(1-|z|^2)^{(-1+s)(p-1)}\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^{p\,(1+a)+1}}\,(1-|w|^2)^{a\,p-1+(1-s)(p-1)}\,dA(w).\end{aligned}$$

Therefore we have

$$\begin{aligned}\int_{\mathbb{D}}&|f(z)T(v)(z)-T(fv)(z)|^p\,(1-|z|^2)^{p-2+s}\,dA(z)\\&\lesssim\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^{p\,(1+a)+1}}(1-|z|^2)^\sigma\,(1-|w|^2)^\tau\,dA(z)\,dA(w),\end{aligned}$$

where $\sigma=s(p-1)-1+s$ and $\tau=a\,p-1+(1-s)(p-1)$. Since $s>0$ then $\sigma>-1$, and $\sigma>s-2$. Also, the conditions on a ensure that $\tau>-1$ and $\tau>s-2$. Since

$$4+\sigma+\tau-s=p\,(1+a)+1,$$

we can apply Theorem 2.2.4 to obtain

$$\int_{\mathbb{D}} |f(z)T(v)(z) - T(fv)(z)|^p \, (1-|z|^2)^{p-2+s} \, dA(z) \lesssim \|f\|^p_{B_p(s)}.$$

The case $p \geq 2$.
Choose $\varepsilon > 0$ such that $\beta > \max(-1, -2+s) + \varepsilon(p-1)$ and

$$a\,p > 1 + \beta - p + \max(0, -1+s) + \varepsilon(p-2).$$

Let q be the conjugate exponent of p. By Hölder's inequality we have

$$\begin{aligned}|f(z)T(v)(z) - T(fv)(z)|^p &\leq \Big(\int_{\mathbb{D}} \frac{|v(w)|\,|f(z)-f(w)|}{|1-\bar{w}z|^{2+a}} \, (1-|w|^2)^a \, dA(w)\Big)^p \\ &\leq \Big(\int_{\mathbb{D}} |v(w)|^q \frac{(1-|w|^2)^\gamma}{|1-\bar{w}z|^{\frac{A}{(p-1)}}} \, dA(w)\Big)^{p/q} \int_{\mathbb{D}} \frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^B} \, (1-|w|^2)^\tau \, dA(w),\end{aligned}$$

where

$$\begin{aligned}&\gamma = \tfrac{\beta-(1-\varepsilon)(p-2)}{p-1}, && \tau = a\,p - \beta + (1-\varepsilon)(p-2)\\ &A = \varepsilon + s + (1+2\varepsilon)(p-2), && B = (2+a)p - \varepsilon - s - (1+2\varepsilon)(p-2).\end{aligned}$$

Since $p \geq 2$ we can apply Hölder's inequality once again with exponent $p/q \geq 1$, and then Lemma 2.3.1 and Lemma J to obtain

$$\begin{aligned}&\Big(\int_{\mathbb{D}} |v(w)|^q \, \frac{(1-|w|^2)^\gamma}{|1-\bar{w}z|^{\frac{A}{(p-1)}}} \, dA(w)\Big)^{p/q} \\ &\qquad \leq \Big(\int_{\mathbb{D}} \frac{|v(w)|^p \, (1-|w|^2)^\beta}{|1-\bar{w}z|^{\varepsilon+s}} \, dA(w)\Big)\Big(\int_{\mathbb{D}} \frac{(1-|w|^2)^{-1+\varepsilon}}{|1-\bar{w}z|^{1+2\varepsilon}} \, dA(w)\Big)^{p-2} \\ &\qquad \lesssim (1-|z|^2)^{-\varepsilon} \, (1-|z|^2)^{-\varepsilon(p-2)} = (1-|z|^2)^{-\varepsilon(p-1)}.\end{aligned}$$

Therefore, if $\sigma = \beta - \varepsilon(p-1)$, by Theorem 2.2.4 we have

$$\begin{aligned}&\int_{\mathbb{D}} |f(z)T(v)(z) - T(fv)(z)|^p \, (1-|z|^2)^\beta \, dA(z) \\ &\qquad \lesssim \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^B} \, (1-|z|^2)^\sigma \, (1-|w|^2)^\tau \, dA(z) \, dA(w) \\ &\qquad \lesssim \|f\|^p_{B_p(s)},\end{aligned}$$

since $4+\sigma+\tau-s = B$, and the choice of $\varepsilon > 0$ ensures that $\sigma, \tau > -1$ and $\min(\sigma,\tau) > s-2$. This finishes the proof. □

The following is a well known generalization of the classical Möbius invariant version of Carleson measures. See [Ga3, Chapter VI, Lemma 3.3] for the classical case and [ASX, Lemma 2.1] for the general case.

Lemma 2.3.3. *Let $a > 0$. Then the following are equivalent:*

(i) $\mu(S(I)) \leq C|I|^a$ *for all Carleson sector* $S(I)$

(ii) $\displaystyle\sup_{z_0\in\mathbb{D}} \int_{\mathbb{D}} \left(\frac{1-|z_0|^2}{|1-\overline{z}_0 z|^2} \right)^a d\mu(z) \leq C.$

Proof. First we will show that (ii) implies (i). Let $Q = S(I)$ be a Carleson square with $z_0 \in T(Q)$. For all $z \in Q$,

$$|1-\overline{z}_0 z| \leq |z_0| \left| \frac{1}{\overline{z}_0} - e^{iArg\, z_0} \right| + |z_0| \left| e^{i\, Arg z_0} - z \right| \leq C(1-|z_0|).$$

On the other hand $|1-\overline{z_0}z| \geq 1-|z_0|$. Hence

$$\frac{C}{1-|z_0|} \leq \frac{1-|z_0|}{|1-\overline{z_0}z|^2} \leq \frac{1}{1-|z_0|}, \quad \text{for all } z \in Q.$$

Then, by hypothesis (ii),

$$\mu(S(I)) \approx |I|^{\,a} \int_Q \left(\frac{1-|z_0|^2}{|1-\overline{z_0}z|^2} \right)^a d\mu(z) \leq C|I|^a.$$

For the other implication, given $z_0 \in \mathbb{D}$, consider $I = I(z_0)$, $Q = Q(z_0)$ and let $N \in \mathbb{N}$ such that $2^{N-1}(1-|z_0|) < 1 \leq 2^N(1-|z_0|)$. If we define

$$I_n = 2^n I, \quad \text{for } n = 0, ..., N-1 \quad \text{and} \quad I_N = \partial\mathbb{D} \setminus I_{N-1},$$

then

$$\mathbb{D} = \bigcup_{n\geq 0} C_n, \quad \text{where } C_0 = S(I) \text{ and } C_n = S(I_n) \setminus S(I_{n-1}), \text{ for } n = 1, 2, ...$$

By hypothesis (i),

$$\mu(C_n) \leq \mu(S(I_n)) \leq C|I_n|^{\,a},$$

and observe that

$$|1-\overline{z_0}z| \geq C \max(\,1-|z|\,, |Arg\, z - Arg\, z_0|\,) \geq C\,|I_n| \quad \text{for all } z \in C_n.$$

So,

$$\int_{\mathbb{D}} \left(\frac{1-|z_0|^2}{|1-\overline{z}_0 z|^2} \right)^a d\mu(z) = \sum_{n=0}^{N} \int_{C_n} \left(\frac{1-|z_0|^2}{|1-\overline{z}_0 z|^2} \right)^a d\mu(z)$$

$$\leq C \sum_{n=0}^{N} \frac{|I_0|^a |I_n|^a}{|I_n|^{2\,a}} \leq C \sum_{n\geq 0} \left(\frac{1}{2^a} \right)^n \leq C.$$

□

As an immediate consequence of the previous results we obtain the following one.

Lemma 2.3.4. *Let $1 < p < \infty$ and $s \geq 0$. If μ is an (s,p)-Carleson measure, then there exists a constant $C > 0$ such that $\mu(S(I)) \leq C|I|^s$ for all sectors $S(I)$.*

Proof. It follows immediately by choosing $\varepsilon = s$ on Lemma 2.3.1 and then using Lemma 2.3.3. □

Multipliers of $B_p(s)$

Recall that

$$\mathcal{M}(B_p(s)) = \{g \in Hol(\mathbb{D}) : gf \in B_p(s) \text{ whenever } f \in B_p(s)\}.$$

Let $g \in \mathcal{M}(B_p(s))$, then by the Closed graph Theorem, the operator

$$\begin{array}{ccc} B_p(s) & \longrightarrow & B_p(s) \\ f & \longmapsto & fg \end{array}$$

is bounded and we define the multiplier norm of g as

$$\|g\|_{\mathcal{M}(B_p(s))} = \inf_{\|f\|_{B_p(s)}=1} \|fg\|_{B_p(s)}.$$

The following well known result (see [ARS1, p. 472]) describes the multipliers of $B_p(s)$ in terms of (s,p)-Carleson measures.

Lemma L. *Let $1 < p < \infty$ and $0 \leq s < 1$. Then $g \in M(B_p(s))$ if and only if $g \in H^\infty$ and $|g'(z)|^p\,(1-|z|^2)^{p-2+s}\,dA(z)$ is an (s,p)-Carleson measure.*

Proof. Suppose first that $g \in \mathcal{M}(B_p(s))$. By the closed graph theorem $\|fg\|_{B_p(s)} \leq C\|f\|_{B_p(s)}$, whenever $f \in B_p(s)$. Using inequality (2.2.2) and Lemma 2.2.2, we obtain that for all $h \in B_p(s)$,

$$|h(z)| = |\langle h, k_z\rangle| \leq \|h\|_{B_p(s)}\|k_z\|_{B_q(s)} \leq C\|h\|_{B_p(s)}(1-|z|^2)^{-s/p}. \tag{2.3.3}$$

For each $a \in \mathbb{D}$, the function $f_a(z) = \dfrac{(1-|a|^2)^{s/p}}{(1-\bar{a}z)^{2s/p}}$ is in $B_p(s)$ with $\|f_a\|_{B_p(s)} \leq C$ (see Lemma 2.3.1). Since $g \in M(B_p(s))$, we can apply (2.3.3) with the function $h = gf_a$ to get

$$|\,g(z)f_a(z)| \leq C(1-|z|^2)^{-s/p}.$$

Taking $a = z$ we get that $g \in H^\infty$. In order to see that $|g'(z)|^p(1-|z|^2)^{p-2+s}\,dA(z)$ is an (s,p)-Carleson measure just note that

$$\int_{\mathbb{D}} |f(z)|^p|g'(z)|^p(1-|z|^2)^{p-2+s}\,dA(z) \leq C\left(\|g\|_\infty^p\|f\|_{B_p(s)}^p + \|fg\|_{B_p(s)}^p\right) \lesssim \|f\|_{B_p(s)}^p$$

since $|f(z)g'(z)|^p \lesssim |(fg)'(z)|^p + |f'(z)g(z)|^p$. The other implication is easier. Let $f \in B_p(s)$, then

$$\|fg\|_{B_p(s)} \lesssim \|f\|_{B_p(s)}\|g\|_{H^\infty} + \int_{\mathbb{D}} |f(z)|^p |g'(z)|^p (1-|z|^2)^{p-2+s} dA(z)$$

and using the hypothesis on g we obtain that $\|fg\|_{B_p(s)} \leq C\|f\|_{B_p(s)}$, so $g \in \mathcal{M}(B_p(s))$. □

Proposition 2.3.5. *Let $1 < p < \infty$, $0 < s < 1$, and let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $\sum(1-|z_n|^2)^s \delta_{z_n}$ is an (s,p)-Carleson measure. Then the Blaschke product*

$$B(z) = \prod_n \frac{|z_n|}{z_n} \frac{z - z_n}{1 - \bar{z}_n z}$$

is a multiplier of $B_p(s)$.

Proof. Note first that the assumption also implies that the measure

$$\mu = \sum_n (1-|z_n|^2)\, \delta_{z_n} \tag{2.3.4}$$

is a Carleson measure. Indeed, let $S(I)$ be a Carleson box, then by Lemma 2.3.4,

$$\mu(S(I)) = \sum_{z_n \in S(I)} (1-|z_n|^2) \leq |I|^{1-s} \sum_{z_n \in S(I)} (1-|z_n|^2)^s \leq C|I|.$$

In particular, by Lemma 2.3.3 with $a = 1$, one has

$$\sup_{z\in\mathbb{D}} \sum_n \frac{(1-|z_n|^2)(1-|z|^2)}{|1-\overline{z_n}z|^2} \leq C. \tag{2.3.5}$$

Since $\|B\|_\infty \leq 1$ then, by Lemma L, the condition that $B \in \mathcal{M}(B_p(s))$ is equivalent to the fact that $|B'(z)|^p(1-|z|^2)^{p-2+s}\, dA(z)$ is an (s,p)-Carleson measure. Observe that

$$B'(z) = \sum_n \frac{|z_n|}{z_n} \frac{1-|z_n|^2}{(1-\overline{z}_n z)^2} \prod_{m\neq n} \frac{|z_m|}{z_m} \frac{z_m - z}{1-\bar{z}_m z},$$

so

$$|B'(z)| \leq \sum_n \frac{(1-|z_n|^2)}{|1-\bar{z}_n z|^2}, \tag{2.3.6}$$

and together with (2.3.5) we obtain that

$$(1-|z|^2)|B'(z)| \leq C \quad \text{for all } z \in \mathbb{D}. \tag{2.3.7}$$

Let $f \in B_p(s)$, using the estimation (2.3.6) and (2.3.7), we have

$$
\begin{aligned}
&\int_{\mathbb{D}} |f(z)|^p \, |B'(z)|^p \, (1-|z|^2)^{p-2+s} \, dA(z) \\
&\quad = \int_{\mathbb{D}} |f(z)|^p \, |B'(z)| \, ((1-|z|^2)|B'(z)|)^{p-1} (1-|z|^2)^{-1+s} dA(z) \\
&\quad \lesssim \int_{\mathbb{D}} |f(z)|^p \Big(\sum_n \frac{1-|z_n|^2}{|1-\bar{z}_n z|^2} \Big) (1-|z|^2)^{-1+s} \, dA(z) \\
&\quad = \sum_n (1-|z_n|^2) \int_{\mathbb{D}} |f(z)|^p \, \frac{(1-|z|^2)^{-1+s}}{|1-\bar{z}_n z|^2} \, dA(z) \\
&\quad \lesssim \sum_n (1-|z_n|^2) \int_{\mathbb{D}} |f(z_n)|^p \, \frac{(1-|z|^2)^{-1+s}}{|1-\bar{z}_n z|^2} \, dA(z) \\
&\qquad + \sum_n (1-|z_n|^2) \int_{\mathbb{D}} |f(z)-f(z_n)|^p \, \frac{(1-|z|^2)^{-1+s}}{|1-\bar{z}_n z|^2} \, dA(z) \\
&\quad = I_1 + I_2.
\end{aligned}
$$

By Lemma J and the fact that $\sum_n (1-|z_n|^2)^s \delta_{z_n}$ is an (s,p)-Carleson measure, we have

$$
\begin{aligned}
I_1 &= \sum_n \, |f(z_n)|^p \, (1-|z_n|^2) \int_{\mathbb{D}} \frac{(1-|z|^2)^{-1+s}}{|1-\bar{z}_n z|^2} \, dA(z) \\
&\lesssim \sum_n \, |f(z_n)|^p \, (1-|z_n|^2)^s \lesssim \|f\|^p_{B_p(s)}.
\end{aligned}
$$

On the other hand, let

$$
\varphi_n(z) = \frac{z_n - z}{1-\overline{z_n} z}, \qquad \text{for } n = 1, \cdots ,
$$

making the change of variables $w = \varphi_n(z)$ we have

$$
\begin{aligned}
I_2 &= \sum_n \, (1-|z_n|^2) \int_{\mathbb{D}} |f(z)-f(z_n)|^p \, \frac{(1-|z|^2)^{-1+s}}{|1-\bar{z}_n z|^2} \, dA(z) \\
&= \sum_n \, (1-|z_n|^2)^s \int_{\mathbb{D}} |f_n(w)-f_n(0)|^p \, \frac{(1-|w|^2)^{-1+s}}{|1-\bar{z}_n w|^{2s}} \, dA(w),
\end{aligned}
$$

where $f_n = f \circ \varphi_n$. Therefore, since $0 < s < 1$, by Lemma 2.2.3 and the change of variables $z = \varphi_n(w)$, we get

$$\begin{aligned} I_2 &\lesssim \sum_n (1-|z_n|^2)^s \int_{\mathbb{D}} |f_n'(w)|^p \frac{(1-|w|^2)^{p-1+s}}{|1-\bar{z}_n w|^{2s}} \, dA(w) \\ &= \sum_n |f'(\varphi_n(w))|^p (1-|w|^2)(|\varphi_n'(w)|(1-|w|^2))^{p-2} \\ &\qquad \times \left(\frac{(1-|z_n|^2)(1-|w|^2)}{|1-\bar{z}_n w|^2} \right)^s |\varphi_n'(w)|^2 \, dA(w) \\ &= \sum_n \int_{\mathbb{D}} |f'(z)|^p \, (1-|\varphi_n(z)|^2) \, (1-|z|^2)^{p-2+s} \, dA(z) \\ &= \int_{\mathbb{D}} |f'(z)|^p \Big(\sum_n \frac{(1-|z_n|^2)\,(1-|z|^2)}{|1-\bar{z}_n z|^2} \Big) (1-|z|^2)^{p-2+s} \, dA(z), \end{aligned}$$

and finally, applying inequality (2.3.5) to the last term, the proof is finished. □

Boundary values

Let $1 < p < \infty$ and $0 < s < 1$, since $B_p(s) \subset H^p$, it follows that every function f in $B_p(s)$ has nontangential limits a.e. on $\partial\mathbb{D}$. Denote by $f_b \in L^p(\partial\mathbb{D})$ the boundary values of f (taken as a nontangential limit). The purpose of this section is to give a description of the space $B_p(s)$ in terms of its boundary values. Let $f \in L^p(\partial\mathbb{D})$, we say that that the function f is in L^p_s if

$$\|f\|^p_{L^p_s} = \int_0^{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}} \, du \, dt < \infty.$$

The main purpose of this section is to show that the boundary values of functions in $B_p(s)$ are in L^p_s. Our arguments follow closely the proof in [Nic].

Lemma 2.3.6. *Let $1 < p < \infty$ and $0 < s < 1$. Let $f \in L^p(\partial\mathbb{D})$ and let $F \in C^1(\mathbb{D})$ with $\lim_{r\to 1} F(re^{i\theta}) = f(e^{i\theta})$ for a.e. $e^{i\theta} \in \partial\mathbb{D}$. Then*

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}} \, du \, dt \le C \int_{\mathbb{D}} |\nabla F(z)|^p (1-|z|^2)^{p-2+s} dA(z), \qquad (2.3.8)$$

where C is an absolute constant.

Proof. The proof is based on an idea in [Stein, chapter V]. Changing the coordinates $t = u + h$ we obtain

$$\begin{aligned} \int_0^{2\pi} \int_0^{2\pi} \frac{|f(e^{it}) - f(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}} \, du \, dt &= 2 \int_0^{\pi} \left(\int_0^{2\pi} \frac{|f(e^{i(u+h)}) - f(e^{iu})|^p}{|e^{ih} - 1|^{2-s}} \, du \right) dh \\ &\le C \int_0^{\pi} \frac{1}{h^{2-s}} \left(\int_0^{2\pi} |f(e^{i(u+h)}) - f(e^{iu})|^p \, du \right) dh. \end{aligned}$$

Consider

$$
\begin{aligned}
&|f(e^{i(u+h)}) - f(e^{iu})| \\
&\qquad \le |f(e^{i(u+h)}) - F(re^{i(u+h)})| + |F(r(e^{i(u+h)}) - F(re^{iu}))| + |F(re^{iu}) - f(e^{iu})| \\
&\qquad \le \int_r^1 |\nabla F(xe^{i(u+h)})|\,dx + \int_0^h |\nabla F(re^{i(u+t)})|\,dt + \int_r^1 |\nabla F(xe^{iu})|\,dx,
\end{aligned}
$$

for any $r \in (0,1)$. Apply Minkowski's integral inequality (2.1.1), to get

$$
\begin{aligned}
&\int_0^{2\pi} |f(e^{i(u+h)}) - f(e^{iu})|^p\,du \\
&\quad \lesssim \int_0^{2\pi} \left(\int_r^1 |\nabla F(xe^{iu})|\,dx\right)^p du + \int_0^{2\pi} \left(\int_0^h |\nabla F(re^{i(u+t)})|\,dt\right)^p du \\
&\quad \lesssim \left(\int_r^1 \left(\int_0^{2\pi} |\nabla F(xe^{iu})|^p\,du\right)^{1/p} dx\right)^p \\
&\qquad + \left(\int_0^h \left(\int_0^{2\pi} |\nabla F(re^{i(u+t)})|^p\,du\right)^{1/p} dt\right)^p = (I) + (II).
\end{aligned}
$$

Therefore,

$$
\int_0^{2\pi}\int_0^{2\pi} \frac{|f(e^{it}) - f(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}}\,du\,dt \le C \int_0^{\pi} \frac{1}{h^{2-s}}\,(I)\,dh + \int_0^{\pi} \frac{1}{h^{2-s}}\,(II)\,dh.
$$

For the term (I) we make the change of variables $\psi = h/(2\pi)$ and take $r = 1-\psi \in (1/2, 1)$,

$$
\begin{aligned}
\int_0^{\pi} \frac{1}{h^{2-s}}\,(I)\,dh &= \int_0^{\pi} \frac{1}{h^{2-s}} \left(\int_r^1 \left(\int_0^{2\pi} |\nabla F(xe^{iu})|^p du\right)^{1/p} dx\right)^p dh \\
&= C \int_0^{1/2} \frac{1}{\psi^{2-s}} \left(\int_{1-\psi}^1 \left(\int_0^{2\pi} |\nabla F(xe^{iu})|^p du\right)^{1/p} dx\right)^p d\psi.
\end{aligned}
$$

After the change of variables $x = 1-\varphi$, we apply Hardy's inequality (2.1.2) and obtain

$$
\begin{aligned}
\int_0^{\pi} \frac{1}{h^{2-s}}\,(I)\,dh &= C \int_0^{1/2} \frac{1}{\psi^{2-s}} \left(\int_0^{\psi} \left(\int_0^{2\pi} |\nabla F((1-\varphi)e^{iu})|^p du\right)^{1/p} d\varphi\right)^p d\psi \\
&\le C \int_0^{1/2}\int_0^{2\pi} |\nabla F((1-\varphi)e^{iu})|^p\ du\ \varphi^{p-2+s}\,d\varphi \\
&\le C \int_{\mathbb{D}} |\nabla F(z)|^p (1-|z|^2)^{p-2+s} dA(z).
\end{aligned}
$$

For the term (II) we make the change of variables $\psi = h/(2\pi)$, take $r = 1-\psi$ and integrate with respect to t and we obtain

$$\begin{aligned}\int_0^\pi \frac{1}{h^{2-s}}\,(II)\,dh &= \int_0^\pi \frac{1}{h^{2-s}}\Big(\int_0^h \Big(\int_0^{2\pi} |\nabla F(re^{iu})|^p du\Big)^{1/p} dt\Big)^p dh\\ &= C\int_0^{1/2} \frac{1}{\psi^{2-s}}\Big(\int_0^\psi \Big(\int_0^{2\pi} |\nabla F((1-\psi)e^{iu})|^p du\Big)^{1/p} dt\Big)^p d\psi\\ &= C\int_0^{1/2} \psi^{p-2+s}\Big(\int_0^{2\pi} \big|\nabla F((1-h)e^{iu})\big|^p du\Big) d\psi\\ &\le C\int_{\mathbb{D}} |\nabla F(z)|^p (1-|z|^2)^{p-2+s} dA(z).\end{aligned}$$

This finishes the proof. □

Proposition 2.3.7. *Let $1<p<\infty$ and $0<s<1$. Let $f \in L^p(\partial\mathbb{D})$, then $f \in L^p_s$ if and only if*

$$\int_{\mathbb{D}} |\nabla P[f](z)|^p (1-|z|^2)^{p-2+s} dA(z) < \infty,$$

where $P[f]$ is the Poisson integral of f. Moreover, there exists a universal constant C such that

$$C^{-1}\|f\|^p_{L^p_s} \le \int_{\mathbb{D}} |\nabla P[f](z)|^p (1-|z|^2)^{p-2+s} dA(z) \le C\|f\|^p_{L^p_s}.$$

Proof. Consider $f \in L^p(\partial\mathbb{D})$, and let $P[f]$ be the Poisson integral of f. Then, it follows from Lemma 2.3.6 that

$$\|f\|^p_{L^p_s} \le C\int_{\mathbb{D}} |\nabla P[f](z)|^p (1-|z|^2)^{p-2+s} dA(z).$$

For the converse, since

$$P[f](z) = \int_0^{2\pi} f(e^{it})\,\frac{(1-|z|^2)}{|e^{it}-z|^2}\frac{dt}{2\pi},$$

an easy computation gives

$$\begin{aligned}\frac{\partial P[f]}{\partial z}(z) &= \int_0^{2\pi} \frac{f(e^{it})e^{it}}{(e^{it}-z)^2}\frac{dt}{2\pi} = \int_0^{2\pi} \frac{(f(e^{it})-f(e^{iu}))e^{it}}{(e^{it}-z)^2}\frac{dt}{2\pi},\\ \frac{\partial P[f]}{\partial \bar z}(z) &= \int_0^{2\pi} \frac{f(e^{it})}{e^{it}(e^{it}-z)^2}\frac{dt}{2\pi} = \int_0^{2\pi} \frac{(f(e^{it})-f(e^{iu}))}{e^{it}(e^{it}-z)^2}\frac{dt}{2\pi}\end{aligned}$$

and consequently,

$$|\nabla P[f](re^{iu})| \le C\int_0^{2\pi} \frac{|f(e^{it})-f(e^{iu})|}{|e^{it}-re^{iu}|^2}\,dt. \tag{2.3.9}$$

Therefore, using (2.3.9) and Hölder's inequality, we get

$$
\begin{aligned}
&\int_{\mathbb{D}} |\nabla P[f](z)|^p (1-|z|^2)^{p-2+s}\, dA(z) \\
&\le C \int_0^{2\pi}\int_0^1 \left(\int_0^{2\pi} \frac{|f(e^{it})-f(e^{iu})|}{|e^{it}-re^{iu}|^2}\, dt\right)^p (1-r^2)^{p-2+s}\, r\, dr\, du \\
&\le C \int_0^{2\pi}\int_0^1 \left(\int_0^{2\pi} \frac{|f(e^{it})-f(e^{iu})|^p}{|e^{it}-re^{iu}|^2}\, dt\right) \\
&\quad \times \left(\int_0^{2\pi} \frac{1}{|e^{it}-re^{iu}|^2}\, dt\right)^{p-1} (1-r^2)^{p-2+s}\, r\, dr\, du \\
&= C \int_0^{2\pi}\int_0^1 \int_0^{2\pi} \frac{|f(e^{it})-f(e^{iu})|^p}{|e^{it}-re^{iu}|^2}\, dt\, (1-r^2)^{-1+s}\, r\, dr\, du \\
&= C \int_0^{2\pi}\int_0^{2\pi} |f(e^{it})-f(e^{iu})|^p \left(\int_0^1 \frac{r(1-r^2)^{-1+s}}{|e^{it}-re^{iu}|^2}\, dr\right) dt\, du.
\end{aligned}
$$

The proof will be finished if we show that

$$
\int_0^1 \frac{r(1-r^2)^{-1+s}}{|e^{it}-re^{iu}|^2}\, dr \le C \frac{1}{|e^{it}-e^{iu}|^{2-s}} \qquad \text{for all } t,u \in [0,2\pi].
$$

We can assume $|\, e^{it}-e^{iu}| \le 1/2$. Consider

$$
\begin{aligned}
R_0 &= \{r \in [0,1) \text{ such that } 0 < 1-r \le |e^{it}-e^{iu}|\}, \\
R_n &= \{r \in [0,1) \text{ such that } 2^{n-1}|e^{it}-e^{iu}| \le 1-r \le 2^n|e^{it}-e^{iu}|\} \qquad \text{for } n = 1, ..., N,
\end{aligned}
$$

where $N \in \mathbb{N}$ satisfies $2^{N-1}|e^{iu}-e^{it}| < 1 < 2^N|e^{iu}-e^{it}|$. Then, if $n \ge 1$

$$
|e^{it}-re^{iu}| \ge (1-r) - |e^{it}-e^{iu}| \gtrsim 2^n|e^{it}-e^{iu}| \qquad \text{if } r \in R_n.
$$

So,

$$
\begin{aligned}
&\int_0^1 \frac{r(1-r^2)^{-1+s}}{|e^{it}-re^{iu}|^2}\, dr = \sum_{n=0}^N \int_{R_n} \frac{r(1-r^2)^{-1+s}}{|e^{it}-re^{iu}|^2}\, dr \lesssim \sum_{n=0}^N \frac{\big(2^n|e^{it}-e^{iu}|\big)^{-1+s}}{\big(2^n|e^{it}-e^{iu}|\big)^2} 2^n|e^{it}-e^{iu}| \\
&\quad = \sum_{n=0}^N \left(\frac{1}{2^{2-s}}\right)^n \frac{1}{|e^{it}-e^{iu}|^{2-s}} \lesssim \frac{1}{|e^{it}-e^{iu}|^{2-s}},
\end{aligned}
$$

and we conclude that

$$
\int_{\mathbb{D}} |\nabla P[f](z)|^p (1-|z|^2)^{p-2+s} dA(z) \le C \int_0^{2\pi}\int_0^{2\pi} \frac{|f(e^{it})-f(e^{iu})|^p}{|e^{it}-e^{iu}|^{2-s}}\, du\, dt.
$$

□

As an immediate consequence of Proposition 2.3.7 we obtain the following result.

Corollary 2.3.8. *Let f be analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Let f_b be the boundary values of f on $\partial\mathbb{D}$ so that $f = P[f_b]$. Then $f \in B_p(s)$ if and only if $f_b \in L^p_s$. Moreover, there exists a universal constant $C > 0$ such that*

$$C^{-1}\|f\|_{L^p_s} \leq \|f\|_{B_p(s)} \leq C\|f\|_{L^p_s}, \quad \textit{for any } f(0) = 0.$$

Multipliers of L^p_s

Let $\mathcal{M}(L^p_s)$ be the algebra of (pointwise) multipliers of L^p_s, that is,

$$\mathcal{M}(L^p_s) = \{g : \partial\mathbb{D} \to \mathbb{C} : gf \in L^p_s \text{ whenever } f \in L^p_s\}.$$

The main purpose of this section is to present a description of the multipliers of L^p_s in terms of (s,p)-Carleson measures.

Lemma 2.3.9. *Let $1 < p < \infty$ and $0 < s < 1$. Let μ be a finite positive measure in the closed unit disc. Then the following conditions are equivalent:*

(i) μ is an (s,p)-Carleson measure

(ii) $\int_{\mathbb{D}} |P[u](z)|^p \, d\mu(z) \leq C\|u\|^p_{L^p_s}$ for all $u \in L^p_s$ with $\int_0^{2\pi} u(e^{i\theta})\, d\theta = 0$

Proof. First we will proof that $(i) \Rightarrow (ii)$. Let μ be an (s,p)-Carleson measure and $u \in L^p_s$. We can assume u is real valued. Let $\widetilde{P}[u](z)$ be the harmonic conjugate function of $P[u](z)$ with $\widetilde{P}[u](0) = 0$. By the Cauchy-Riemann's equations,

$$|\nabla P[u](z)| = |U'(z)|, \quad \text{where } U(z) = P[u](z) + i\,\widetilde{P}[u](z).$$

Observe that $U(0) = 0$. Now, applying Proposition 2.3.7 with the function $u \in L^p_s$, we deduce that $U \in B_p(s)$. Since μ is an $(s,p)-$Carleson measure,

$$\int_{\mathbb{D}} |P[u](z)|^p d\mu(z) \leq \int_{\mathbb{D}} |U(z)|^p d\mu(z) \leq C \int_{\mathbb{D}} |U'(z)|^p (1-|z|^2)^{p-2+s}\, dA(z),$$

and by Proposition 2.3.7, this last integral is comparable to $\|u\|^p_{L^p_s}$.

Let us show that $(ii) \Rightarrow (i)$. We need to prove that

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C\|f\|^p_{B_p(s)}$$

for any $f \in B_p(s)$. Let f_b be the boundary function of f. By Proposition 2.3.7 and hypothesis (ii), $f_b \in L^p_s$ and

$$\int_{\mathbb{D}} |f(z) - f(0)|^p \, d\mu(z) \leq C\|f_b\|^p_{L^p_s} \leq C \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z).$$

So we can conclude that μ is an $(s,p)-$Carleson measure. □

Lemma 2.3.10. *Let $g \in L^\infty(\partial\mathbb{D})$ and let $G \in L^\infty(\mathbb{D}) \cap C^1(\mathbb{D})$ with $\lim_{r\to 1} G(re^{i\theta}) = g(e^{i\theta})$ for a.e. $e^{i\theta} \in \partial\mathbb{D}$. If*

$$|\nabla G(z)|^p (1-|z|^2)^{p-2+s} dA(z)$$

is an (s,p)-Carleson measure, then $g \in \mathcal{M}(L^p_s)$.

Proof. Observe that Lemma 2.3.6 tells that the function g is in L^p_s. We must show that $gf \in L^p_s$ whenever f is in L^p_s. So, let $f \in L^p_s$. Without loss of generality we may think $P[f](0) = 0$. Using Lemma 2.3.6 with the extension $G\,P[f]$ of gf on $\mathbb{D}$ we get

$$\begin{aligned}\|gf\|^p_{L^p_s} &= \int_0^{2\pi}\int_0^{2\pi} \frac{|gf(e^{it}) - gf(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}}\, du\, dt \le C \int_{\mathbb{D}} |\nabla (G\,P[f])(z)|^p \, dA_{p,s}(z) \\ &\le C\left(\int_{\mathbb{D}} |G(z)|^p |\nabla P[f](z)|^p \, dA_{p,s}(z) + \int_{\mathbb{D}} |P[f](z)|^p |\nabla G(z)|^p \, dA_{p,s}(z)\right).\end{aligned}$$

Since $G \in L^\infty(\mathbb{D})$, for the first term of the sum, using Proposition 2.3.7 with $f \in L^p_s$ we obtain

$$\int_{\mathbb{D}} |G(z)|^p |\nabla P[f](z)|^p (1-|z|^2)^{p-2+s} dA(z) \le C\|f\|^p_{L^p_s}.$$

For the second term we use the fact that $|\nabla G(z)|^p \, dA_{p,s}(z)$ is an (s,p)-Carleson measure and Lemma 2.3.9 to get

$$\int_{\mathbb{D}} |P[f](z)|^p |\nabla G(z)|^p (1-|z|^2)^{p-2+s} dA(z) \le C\|f\|^p_{L^p_s}.$$

Hence $\|gf\|_{L^p_s} \le C\|f\|_{L^p_s}$ for all $f \in L^p_s$ with $P[f](0) = 0$. So $g \in \mathcal{M}(L^p_s)$, which finishes the proof. □

Now we can give a description of the multipliers of L^p_s in terms of (s,p)-Carleson measures.

Theorem 2.3.11. *Let $1 < p < \infty$ and $0 < s < 1$, the following conditions are equivalent:*

(i) $g \in \mathcal{M}(L^p_s)$

(ii) $g \in L^\infty(\partial\mathbb{D})$ *and* $|\nabla P[g](z)|^p (1-|z|^2)^{p-2+s} dA(z)$ *is an (s,p)-Carleson measure.*

Proof. The implication $(ii) \Rightarrow (i)$ follows from Lemma 2.3.10. For the converse, let $g \in \mathcal{M}(L^p_s)$. Without loss of generality we may think that $P[g](0) = 0$. We start by proving that $g \in L^\infty(\partial\mathbb{D})$ using an argument from [Ch]. Consider the operator of multiplication by g, then by the closed-graph theorem we have

$$\|gf\|^p_{L^p_s} \le C\Big(\|f\|^p_{L^p_s} + \Big| \int_0^{2\pi} f(e^{i\theta})\, d\theta \Big|^p \Big), \quad \text{for all } f \in L^p_s.$$

Taking $\varphi(z) = g(z)/C^{1/p}$ we have that

$$\|\varphi f\|^p_{L^p_s} \leq \|f\|^p_{L^p_s} + \Big| \int_0^{2\pi} f(e^{i\theta})\, d\theta \Big|^p, \qquad \text{for all } f \in L^p_s.$$

Since $1 \in L^p_s$, we see that $\varphi \in L^p_s$ and so $\varphi^n \in L^p_s$ with $\|\varphi^n\|_{L^p_s} \leq \|\varphi\|_{L^p_s}$, for $n = 1, 2, \cdots$. Note that

$$\varphi^n(e^{it})(f(e^{it}) - f(e^{iu})) = (\varphi^n f)(e^{it}) - (\varphi^n f)(e^{iu}) - f(e^{iu})(\varphi^n(e^{it}) - \varphi^n(e^{iu})).$$

Hence,

$$\begin{aligned}
&\int_0^{2\pi}\!\!\int_0^{2\pi} |\varphi(e^{it})|^{np} \frac{|f(e^{it}) - f(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}}\, dt\, du \\
&\quad \lesssim \int_0^{2\pi}\!\!\int_0^{2\pi} \frac{|(\varphi^n f)(e^{it}) - (\varphi^n f)(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}}\, dt\, du \\
&\qquad + \int_0^{2\pi}\!\!\int_0^{2\pi} |f(e^{iu})|^p \frac{|\varphi^n(e^{it}) - \varphi^n(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}}\, dt\, du \\
&\quad = \|\varphi^n f\|^p_{L^p_s} + \int_0^{2\pi}\!\!\int_0^{2\pi} |f(e^{iu})|^p \frac{|\varphi^n(e^{it}) - \varphi^n(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}}\, dt\, du \\
&\quad \leq \|f\|^p_{L^p_s} + \int_0^{2\pi}\!\!\int_0^{2\pi} |f(e^{iu})|^p \frac{|\varphi^n(e^{it}) - \varphi^n(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}}\, dt\, du.
\end{aligned}$$

Now, let f be the identity function on $\partial\mathbb{D}$, we then have

$$\begin{aligned}
&\int_0^{2\pi}\!\!\int_0^{2\pi} |\varphi(e^{it})|^{np} |e^{it} - e^{iu}|^{p-2+s}\, dt\, du \\
&\qquad \lesssim 1 + \int_0^{2\pi}\!\!\int_0^{2\pi} \frac{|\varphi^n(e^{it}) - \varphi^n(e^{iu})|^p}{|e^{it} - e^{iu}|^{2-s}}\, dt\, du \leq C.
\end{aligned}$$

Therefore, since n is arbitrary and the constant C is independent of n, we have $|\varphi(\xi)| \leq 1$ for almost every $\xi \in \partial\mathbb{D}$ and so $g \in L^\infty(\partial\mathbb{D})$.

Now we are going to show that the measure $|\nabla P[g](z)|^p(1 - |z|^2)^{p-2+s} dA(z)$ is an (s, p)-Carleson measure, that is,

$$\int_{\mathbb{D}} |f(z)|^p |\nabla P[g](z)|^p (1 - |z|^2)^{p-2+s} dA(z) \leq C\|f\|^p_{B_p(s)},$$

whenever f is in $B_p(s)$. Let $f \in B_p(s)$, since $g \in \mathcal{M}(L^p_s)$, we have that $f_b\, g \in L^p_s$, so by proposition 2.3.7,

$$\begin{aligned}
&\int_{\mathbb{D}} |\nabla(f P[g])(z)|^p (1 - |z|^2)^{p-2+s} dA(z) \\
&\qquad \leq C\|fg\|_{L^p_s} \leq C\big(\, \|f\|^p_{L^p_s} + |f(0)|^p \,\big) \leq C\|f\|^p_{B_p(s)}.
\end{aligned} \tag{2.3.10}$$

Furthermore, since $g \in L^\infty(\partial\mathbb{D})$ then $P[g] \in L^\infty(\mathbb{D})$ and we obtain

$$\int_{\mathbb{D}} |P[g](z)|^p |\nabla f(z)|^p (1-|z|^2)^{p-2+s} dA(z) \leq C\|f\|^p_{B_p(s)}. \tag{2.3.11}$$

Finally, using the inequalities (2.3.10) and (2.3.11), we get

$$\begin{aligned}\int_{\mathbb{D}} &|f(z)|^p |\nabla P[g](z)|^p (1-|z|^2)^{p-2+s} dA(z) \\ &\lesssim \int_{\mathbb{D}} |\nabla(fP[g])(z)|^p (1-|z|^2)^{p-2+s} dA(z) \\ &\quad + \int_{\mathbb{D}} |P[g](z)|^p |\nabla f(z)|^p (1-|z|^2)^{p-2+s} dA(z) \\ &\leq C\|f\|^p_{B_p(s)}.\end{aligned}$$

Therefore, the measure $|\nabla P[g](z)|^p (1-|z|^2)^{p-2+s} dA(z)$ is an (s,p)-Carleson measure and the proof is complete. □

As an immediate consequence of the previous results we can deduce the following relationship between multipliers in $B_p(s)$ and multipliers in L^p_s.

Corollary 2.3.12. *Let g be a bounded analytic function in $\mathbb{D}$ and let g_b be its boundary values so that $g = P[g_b]$. Then $g \in \mathcal{M}(B_p(s))$ if and only if $g_b \in \mathcal{M}(L^p_s)$.*

Proof. By Lemma L, the function $g \in \mathcal{M}(B_p(s))$ if and only if $|g'(z)|^p (1-|z|^2)^{p-2+s} dA(z)$ is an (s,p)-Carleson measure. And using Theorem 2.3.11, this is equivalent to the fact that $g_b \in \mathcal{M}(L^p_s)$. □

2.4 A $\bar{\partial}$-problem

A $\bar{\partial}$-problem with estimates

In the classical setting of bounded analytic functions, the Corona Theorem and the characterization of interpolating sequences can be proved using solutions of $\overline{\partial}$-problems with uniform estimates. In our situation the result we need is Theorem 2.4.2. We start with the following more simple result.

Theorem 2.4.1. *Let $1 < p < \infty$ and $0 \leq s < 1$. If $\phi \in \mathcal{C}(\overline{\mathbb{D}})$, then there is a function $b \in C(\overline{\mathbb{D}})$ such that*

$$\frac{\partial b}{\partial \overline{z}} = \phi$$

in the sense of distributions and such that the boundary value function $b_{|\partial\mathbb{D}}$ belongs to L^p_s. Moreover $\|b_{\,|\partial\mathbb{D}}\|_{L^p_s} \leq C\|\phi\|_{L^p(dA_{p,s})}$, where C is an absolute constant.

Proof. Consider the function

$$u(z) = \int_{\mathbb{D}} \frac{\phi(w)}{(z-w)}\, dA(w).$$

It is easy to check that $u \in \mathcal{C}(\overline{\mathbb{D}})$ and $\overline{\partial} u = \phi$. We will show that $u_{|\partial\mathbb{D}} \in L^p_s$. Consider the function

$$v(z) = \int_{\mathbb{D}} \frac{\phi(w)}{(1-w\overline{z})}\, dA(w).$$

Observe that $v(z) = z\, u(z)$ for all $z \in \partial\mathbb{D}$. So, it is sufficient to show that $v_{|\partial\mathbb{D}} \in L^p_s$. An easy calculation tells

$$|\nabla v(z)| \lesssim \int_{\mathbb{D}} \frac{|\phi(w)|}{|1-\overline{w}z|^2}\, dA(w).$$

If we define

$$T(\phi)(z) = \int_{\mathbb{D}} \frac{\phi(w)}{|1-\overline{w}z|^2}\, dA(w),$$

Lemma H tells that T is a bounded operator from $L^p(dA_{p,s})$ to itself. Since the function ϕ belongs to $L^p(dA_{p,s})$,

$$\int_{\mathbb{D}} |\nabla v(z)|^p (1-|z|^2)^{p-2+s} dA(z) \lesssim \|T(|\phi|)\|_{L^p(dA_{p,s})} \lesssim \|\phi\|_{L^p(dA_{p,s})} < \infty.$$

So, by Lemma 2.3.6, $v_{|\partial\mathbb{D}} \in L^p_s$ and the proof is finished. □

The main result of this section is the following.

Theorem 2.4.2. *Let $p > 1$ and $0 \leq s < 1$. Let ϕ be a continuous function in $\mathbb{D}$ such that $|\phi(z)|^p(1-|z|^2)^{p-2+s} dA(z)$ is an (s,p)-Carleson measure (and $|\phi(z)|(1-|z|) \leq C$ for $1 < p < 2$), then there is a function $b \in C(\overline{\mathbb{D}})$ such that*

$$\frac{\partial b}{\partial \overline{z}} = \phi(z)$$

and such that the boundary value function $b_{|\partial\mathbb{D}}$ belongs to $\mathcal{M}(L^p_s)$.

The proof of Theorem 2.4.2 uses two auxiliary results.

Lemma 2.4.3. *Let $1 < p < \infty$, $0 \leq s < 1$. If $|\phi(z)|^p(1-|z|^2)^{p-2+s}\, dA(z)$ is an (s,p)-Carleson measure, then $|\phi(z)|\, dA(z)$ is a Carleson measure.*

Proof. Given a Carleson sector $S(I)$, applying Hölder's inequality and Lemma 2.3.4, one has

$$\begin{aligned}
\int_{S(I)} |\phi(z)|\, dA(z) &= \int_{S(I)} |\phi(z)|(1-|z|^2)^{\frac{p-2+s}{p}} (1-|z|^2)^{\frac{-(p-2+s)}{p}}\, dA(z) \\
&\leq C \left(\int_{S(I)} |\phi(z)|^p (1-|z|^2)^{p-2+s} dA(z) \right)^{\frac{1}{p}} \\
&\qquad \times \left(\int_{S(I)} (1-|z|^2)^{\frac{-q}{p}(p-2+s)} dA(z) \right)^{\frac{1}{q}} \\
&\leq C\, |I|^{\frac{s}{p}} \left(\int_{S(I)} \frac{dA(z)}{(1-|z|^2)^{\frac{p-2+s}{p-1}}} \right)^{\frac{p-1}{p}} \\
&\leq C\, |I|^{\frac{s}{p}} \left(|I|^{\frac{p-s}{p-1}} \right)^{\frac{p-1}{p}} = C\, |I|
\end{aligned}$$

Here q satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Observe that in last line we have used $s < 1$. □

The proof of Theorem 2.4.2 is based in the following beautiful result of P. Jones [Jo1].

Theorem 2.4.4 (Jones). *Let $d\mu$ be a Borel measure on $\mathbb{D}$ such that $d|\mu|$ is a classical Carleson measure. For $z \in \overline{\mathbb{D}}$ and $\xi \in \mathbb{D}$, consider*

$$K(\mu, z, \xi) = \frac{2i}{\pi} \frac{1-|\xi|^2}{(1-\bar{\xi} z)(z-\xi)} \exp\left[\int_{|w| \geq |\xi|} \left(\frac{1+\overline{w}\,\xi}{1-\overline{w}\,\xi} - \frac{1+\overline{w}z}{1-\overline{w}z} \right) d|\mu|(w) \right]$$

and

$$\|\mu\|_1 = \sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|}.$$

Then

$$S_0(\mu)(z) = \int_{\mathbb{D}} K\left(\frac{\mu}{\|\mu\|_1}, z, \xi \right) d\mu(\xi) \tag{2.4.1}$$

satisfies $S_0(\mu) \in L^1_{loc}(\mathbb{D})$ and $\partial S_0(\mu)/\partial\overline{z} = \mu$ in the sense of distributions. Moreover, if $z \in \partial\mathbb{D}$, then the integral in (2.4.1) *converges absolutely and*

$$\sup_{z \in \partial\mathbb{D}} \int_{\mathbb{D}} \left| K\left(\frac{\mu}{\|\mu\|_1}, z, \xi \right) \right| d|\mu|(\xi) \leq C\|\mu\|_1.$$

In particular, $S_0(\mu) \in L^\infty(\partial\mathbb{D})$ and $\|S_0(\mu)\|_{L^\infty(\partial\mathbb{D})} \leq C\|\mu\|_1$.

Proof of Theorem 2.4.2. By hypothesis, $|\,\phi(z)|^p(1-|z|^2)^{p-2+s}dA(z)$ is an (s,p)-Carleson measure, so applying Lemma 2.4.3, $|\,\phi(z)|\, dA(z)$ is a classical Carleson measure. Without loss of generality we can assume that both Carleson norms are smaller than 1. Let $b = S_0(\mu)$, where $d\mu(z) = \phi(z)\, dA(z)$. Thus, by Theorem 2.4.4, the function b is defined at almost every point of $\overline{\mathbb{D}}$ and satisfies the equation $\partial b/\partial \overline{z} = \phi(z)$ in the sense of distributions. Moreover, the boundary value function $b_{|\partial\mathbb{D}}$ lies in $L^\infty(\partial\mathbb{D})$. Our aim is to verify that the boundary value function $b_{|\partial\mathbb{D}}$ lies in $\mathcal{M}(L^p_s)$. For this purpose, consider

$$\begin{aligned} b^*(z) =& \frac{2i}{\pi}\int_{\mathbb{D}} \frac{1-|\xi|^2}{|1-\bar{\xi}z|^2}\cdot \\ & \exp\left[\int_{|w|\geq|\xi|}\left(\frac{1+\overline{w}\,\xi}{1-\overline{w}\,\xi}-\frac{1+\overline{w}\,z}{1-\overline{w}\,z}\right)|\phi(w)|\,dA(w)\right]\phi(\xi)\,dA(\xi)\end{aligned}$$

which satisfies $b^*(z) = z\, b(z)$, for any $z \in \partial\mathbb{D}$. Hence it is sufficient to prove that $b^* \in \mathcal{M}(L^p_s)$. Now write

$$b^*(z) = \frac{2i}{\pi}\int_{\mathbb{D}} h_1(z)\, h_2(z)\, \exp\left(\int_{|w|\geq|\xi|}\frac{1+\overline{w}\,\xi}{1-\overline{w}\,\xi}|\,\phi(w)|\,dA(w)\right)\phi(\xi)\,dA(\xi),\ z\in\mathbb{D},$$

where

$$h_1(z) = \frac{1-|\xi|^2}{|1-\bar{\xi}z|^2},\quad h_2(z) = \exp\left(-\int_{|w|\geq|\xi|}\frac{1+\overline{w}\,z}{1-\overline{w}\,z}\,|\phi(w)|\,dA(w)\right).$$

Since $d|\,\mu|\,(z) = |\,\phi(z)|\,dA(z)$ is a classical Carleson measure,

$$\begin{aligned} &\operatorname{Re}\left(\int_{|w|\geq|\xi|}\frac{1+\overline{w}\,\xi}{1-\overline{w}\,\xi}|\phi(w)|dA(w)\right) \\ &= \int_{|w|\geq|\xi|}\operatorname{Re}\left(\frac{1+\overline{w}\,\xi}{1-\overline{w}\,\xi}\right)|\phi(w)|dA(w) \\ &\leq 2\int_{\mathbb{D}}\frac{1-|\xi|^2}{|1-\overline{w}\,\xi|^2}\,|\phi(w)|dA(w)\leq C,\end{aligned} \tag{2.4.2}$$

where the constant C is independent of ξ. Hence $e^{-C} \leq |h_2(z)| \leq 1$ for any $z \in \mathbb{D}$. The main technical estimate in the proof is

$$|\nabla b^*(z)| \leq C\int_{\mathbb{D}}\frac{|\phi(w)|}{|1-z\,\overline{w}\,|^2}\,dA(w). \tag{2.4.3}$$

Observe that

$$|\nabla b^*(z)| \leq C\int_{\mathbb{D}}(|h_1(z)\nabla h_2(z)| + |h_2(z)\nabla h_1(z)|)\,|\phi(\xi)|\,dA(\xi).$$

Since $|h_2| \leq 1$, we have

$$|h_2(z)\nabla h_1(z)| \leq \frac{C}{|1-\bar{\xi}z|^2}.$$

We also have

$$|h_1(z)\nabla h_2(z)| \leq C\frac{1-|\,z\,\overline{\xi}\,|}{|1-z\,\overline{\xi}\,|^2}\,|h_2(z)|\int_{\mathbb{D}}\frac{|\phi(w)|}{|1-\overline{w}\,z|^2}dA(w).$$

Lemma 2.1 in [Jo1] states that

$$\int_{\mathbb{D}}\frac{1-|z\,\overline{\xi}\,|}{|1-z\,\overline{\xi}\,|^2}\;\exp\left[-\int_{|w|\geq|\xi|}\frac{1-|z\overline{w}\,|^2}{|1-z\overline{w}\,|^2}\,|\phi(w)|\,dA(w)\right]|\phi(\xi)|\,dA(\xi)\leq 1. \qquad (2.4.4)$$

Therefore

$$\int_{\mathbb{D}}|h_1(z)\nabla h_2(z)|\,|\phi(\xi)|dA(\xi)\leq C\int_{\mathbb{D}}\frac{|\phi(w)|}{|1-z\,\overline{w}\,|^2}\,dA(w),$$

and this finishes the proof of (2.4.3). Since $|\phi(z)|^p(1-|z|^2)^{p-2+s}dA(z)$ is an (s,p)-Carleson measure, we apply Theorem 2.3.2 and obtain that $|\nabla b^*(z)|^p(1-|z|^2)^{p-2+s}dA(z)$ is also an (s,p)-Carleson measure. Given $z\in\mathbb{D}$, by inequalities (2.4.2) and (2.4.4) we have that

$$\begin{aligned}|b^*(z)| &\leq C\int_{\mathbb{D}}\frac{1-|\xi|^2}{|1-\overline{\xi}z|^2}\;\exp\left(-\int_{|w|\geq|\xi|}Re\left(\frac{1+\overline{w}\,z}{1-\overline{w}\,z}\right)|\phi(w)|\,dA(w)\right)|\phi(\xi)|\,dA(\xi)\\ &\leq C\int_{\mathbb{D}}\frac{1-|z\overline{\xi}|^2}{|1-\overline{\xi}z|^2}\;\exp\left(-\int_{|w|\geq|\xi|}\frac{1-|z\,\overline{w}|^2}{|1-z\,\overline{w}|^2}|\phi(w)|\,dA(w)\right)|\phi(\xi)|\,dA(\xi)\leq C.\end{aligned}$$

Thus $b^*\in L^\infty(\mathbb{D})$, and Lemma 2.3.10 tells us that $b^*\in\mathcal{M}(L^p_s)$ which finishes the proof. □

The Corona Problem in $\mathcal{M}(B_p(s))$

Let Y be a subalgebra (nonclosed) of H^∞. If, for an arbitrary finite set of functions $f_1,\cdots,f_n\in Y$ satisfying the condition

$$\inf_{z\in\mathbb{D}}\sum_{j=1}^{n}|f_j(z)|=\sigma>0.$$

there is a set of functions $g_1,\cdots,g_n\in Y$ such that

$$f_1g_1+\cdots+f_ng_n\equiv 1,$$

then we will say that *the corona problem* is solvable in the algebra Y. The corona problem in H^∞ is usually described with the help of solutions of $\overline{\partial}$ equations [Ga3].

It is easy to see that $\mathcal{M}(B_p(s))$ is a Banach algebra with the norm

$$\|f\|_{\mathcal{M}(B_p(s))}=\sup\{\|f\,g\|_{B_p(s)}\;:\;g\in B_p(s),\;\|g\|_{B_p(s)}=1\}.$$

Let $1<p<\infty$ and $0<s<1$, in this section we will consider the corona problem for the algebra $\mathcal{M}(B_p(s))$. This was first proved by V. Tolokonnikov ([Tol]). Here we present an approach based on $\overline{\partial}$ techniques.

Theorem 2.4.5. *Let $1 < p < \infty$, $0 < s < 1$ and $f_1, ..., f_n \in \mathcal{M}(B_p(s))$. Assume*

$$\inf_{z\in\mathbb{D}} \sum_{j=1}^{n} |f_j(z)| = \sigma > 0. \tag{2.4.5}$$

Then there exist $g_1, \cdots, g_n \in \mathcal{M}(B_p(s))$ with

$$f_1 g_1 + \cdots + f_n g_n \equiv 1. \tag{2.4.6}$$

Proof. By normal families, we may assume that $f_1, \cdots, f_n$ are holomorphic in a neighborhood of the closed unit disc. As in the classical case of bounded analytic functions, the proof is based on solutions of $\overline{\partial}$-problems with appropriate estimates. Take

$$\varphi_j(z) = \frac{\overline{f}_j(z)}{\sum_{l=1}^{n} |f_l(z)|^2}, \quad j = 1, ...n.$$

By (2.4.5), the denominator is bounded below. So,

$$\varphi_j \in \mathcal{C}^\infty(\mathbb{D}); \quad \|\varphi_j\|_\infty \le \sigma^{-1}\|f_j\|_\infty < \infty; \quad f_1\varphi_1 + ... + f_n\varphi_n \equiv 1 \text{ on } \overline{\mathbb{D}}.$$

Observe that $|\nabla\varphi_j(z)| \le C(\sigma, n)\sum_{j=1}^{n} |f_j'(z)|$. So, $|\nabla\varphi_j(z)|^p(1-|z|^2)^{p-2+s}dA(z)$ is an $(s,p)-$Carleson measure. Hence Lemma 2.3.10 tells that $\varphi_j \in \mathcal{M}(L^p_s)$. The difficulty is that φ_j may not be analytic on $\mathbb{D}$. To rectify that, we write

$$g_j(z) = \varphi_j(z) + \sum_{k=1}^{n}(b_{j,k}(z) - b_{k,j}(z))f_k(z)$$

where the functions $b_{j,k}$ to be determined will be the solution of a certain $\overline{\partial}-$problem. It is obvious that

$$f_1 g_1 + ... + f_n g_n \equiv 1 \quad \text{on } \mathbb{D}.$$

We want the functions g_j to be analytic and this will follow if we can find solutions $b_{j,k}$ of the $\overline{\partial}-$problems

$$\overline{\partial}\, b_{j,k} = \varphi_j\, \overline{\partial}\, \varphi_k, \qquad j\,, k = 1, ..., n. \tag{2.4.7}$$

Finally, since g_j should lie in $\mathcal{M}(B_p(s))$, we will need $b_{j,k} \in \mathcal{M}(L^p_s)$. A calculation shows

$$\frac{\partial\varphi_k}{\partial\overline{z}} = \frac{\overline{f_k'}}{\sum|f_l|^2} - \frac{\overline{f_k}\sum f_l\overline{f_l'}}{(\sum|f_l|^2)^2} = \frac{\sum f_l(\overline{f_l}\,\overline{f_k'} - \overline{f_k}\,\overline{f_l'})}{(\sum|f_l|^2)^2}.$$

Thus, $|\varphi_j\overline{\partial}\varphi_k| \le C(\sigma)\sum_l |f_l'|$. Hence

$$\sup_{z\in\mathbb{D}} |\varphi_j(z)\overline{\partial}\varphi_k(z)|(1-|z|) \le C \sup_{z\in\mathbb{D}} \sum_{l=1}^{n} |f_l'(z)|(1-|z|) \le C.$$

Observe also that since

$$|f_l'(z)|^p(1-|z|^2)^{p-2+s}dA(z) \text{ is an } (s,p)\text{-Carleson measure}, \quad \text{for } l=1,...,n$$

then $|\varphi_j(z)\,\overline{\partial}\,\varphi_k(z)|^p(1-|z|^2)^{p-2+s}dA(z)$ is also an (s,p)-Carleson measure. Applying Theorem 2.4.2, the equation (2.4.7) has a solution $b_{j,k} \in \mathcal{M}(L_s^p)$. So, $g_j \in \mathcal{M}(L_s^p)$ for $j=1,...,n$ and Corollary 2.3.12 tells that $g_j \in \mathcal{M}(B_p(s))$ for $j=1,...,n$. □

Fefferman-Stein decomposition

As is well known, there is a close relation between $\overline{\partial}$-equations and the *Fefferman-Stein decomposition* asserting that any $f \in BMO(\partial\mathbb{D})$ can be decomposed into $f = u + \widetilde{v}$, where $u, v \in L^\infty(\partial\mathbb{D})$ and $\widetilde{v}$ means the conjugate function of v. So, it is not surprising that solving $\overline{\partial}$-equations with appropriate estimates leads to the following result. Also, it should be recalled that $\mathcal{M}(L_s^p) \subset L^\infty(\partial\mathbb{D})$.

Theorem 2.4.6. *Let $1 < p < \infty$, $0 < s < 1$, $f \in L^p(\partial\mathbb{D})$ and let $P[f]$ denote the Poisson extension of the function f. Assume*

$$|\nabla P[f](z)|^p(1-|z|^2)^{p-2+s}dA(z) \textit{ is an } (s,p)\textit{-Carleson measure,}$$

then $f = u + i\,\widetilde{v}$, where $u, v \in \mathcal{M}(L_s^p)$.

Proof. Without loss of generality we may assume that f is a continuous real valued function. Consider the $\overline{\partial}$-problem

$$\overline{\partial}\, b = \overline{\partial}\, P[f].$$

Since $|\nabla P[f](z)|^p(1-|z|^2)^{p-2+s}dA(z)$ is an (s,p)-Carleson measure then

$$\sup_{z\in\mathbb{D}} |\nabla P[f](z)|(1-|z|) \le C.$$

So, by Theorem 2.4.2, there exists a solution b of the previous $\overline{\partial}$-problem such that $b \in \mathcal{M}(L_s^p)$. Observe that the function $P[f]-b$ is analytic on $\mathbb{D}$. So,

$$P[f]-b = h + i\,\widetilde{h}, \quad \text{where } h \text{ and } \widetilde{h} \text{ are harmonic conjugate functions.}$$

Since f is real valued, we have that a.e. on $\partial\mathbb{D}$, $f = Re\, b + h = u + \widetilde{v}$, where $u = Re\, b$ and $v = \widetilde{h} = -Im\, b$. Thus, since $b \in \mathcal{M}(L_s^p)$, this finishes the proof. □

2.5 Interpolating sequences for $B_p(s)$

Consider the point evaluation functional

$$\begin{array}{cccc} T_n: & B_p(s) & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & f(z_n) \end{array}$$

A sequence of points $\{z_n\} \subset \mathbb{D}$ is called an *interpolating sequence* for the Besov type space $B_p(s)$ if the map $f \mapsto \left\{\frac{f(z_n)}{\|T_n\|}\right\}$ transforms $B_p(s)$ onto and into l^p, where $\|T_n\|$ denotes the norm of the point-evaluation functional at z_n. Using the pairing in section 2.2 and the reproducing kernel, Lemma 2.2.2 tells that the norm of the point evaluation functional is comparable to

$$\|k_{z_n}\|_{B_q(s)} \approx \frac{1}{(1-|z_n|^2)^{s/p}}.$$

The corresponding notion of *interpolating sequences for the multiplier space* $\mathcal{M}(B_p(s))$ consists on the sequences $\{z_n\} \subset \mathbb{D}$ for which the map $g \mapsto \{g(z_n)\}$ transforms the space of multipliers of $B_p(s)$ onto l^∞.

The characterization of the interpolating sequences for both spaces is given in the following theorem which is the main result of this Chapter.

Theorem 2.5.1. *Let $1 < p < \infty$ and $0 < s < 1$. Let $\{z_n\}$ be a sequence of points in the unit disc $\mathbb{D}$. The following conditions are equivalent:*

(M) *Z is an interpolating sequence for $\mathcal{M}(B_p(s))$.*

(UIS) *Z is an interpolating sequence for $B_p(s)$.*

(CS) *Z is a separated sequence and $\mu_Z = \sum_{z_n \in Z}(1-|z_n|^2)^s \delta_{z_n}$ is an (s,p)-Carleson measure.*

We point out that in the case $p = 2$, Theorem 2.5.1 was first proved in [Bo2] and after in [Xi]. When $s = 0$, that is, when working with functions in the classical Besov spaces, the result was proved by C. Bishop [Bi] and Marshall&Sundberg [MS2] for $p = 2$. The general case $1 < p < \infty$ and $s = 0$ is due to B. Boe [Bo1].

2.5.1 Necessity of condition (CS)

(M)⇒(CS)

In this section we will prove that (CS), that is,

- Z is a separated sequence (S)
- $\mu_Z = \sum_{z_n \in Z}(1-|z_n|^2)^s \delta_{z_n}$ is an (s,p)-Carleson measure (C)

are necessary conditions for a sequence $\{z_n\}$ to be interpolating for $\mathcal{M}(B_p(s))$. Since $\{z_n\}$ is an interpolating sequence for $\mathcal{M}(B_p(s)) \subset H^\infty$, in particular $\{z_n\}$ is an interpolating sequence for H^∞. Then, the separation condition follows. For the proof of the necessity of the Carleson measure condition (C) we will need the following Lemma for the case $1 < p \leq 2$.

Lemma 2.5.2. *Let $1 < p \leq 2$, and $0 < s < 1$. If $\{z_n\}$ is an interpolating sequence for $\mathcal{M}(B_p(s))$ then*

$$\sup_{z\in\mathbb{D}} \sum_{k=1}^{\infty} \frac{(1-|z|^2)^s\,(1-|z_k|^2)^s}{|1-\bar{z}_k z|^{2s}} < \infty.$$

A proof of this Lemma can be done using a result in [NX, Theorem 1.3]. Lemma 2.5.2 can also be proven by combining Khinchin's inequality (2.5.1) and a reproducing formula for $B_p(s)$.

The necessity of the Carleson measure condition (C) in Theorem 2.5.1 will follow from an argument which combines Khinchin's inequality and a reproducing formula for $B_p(s)$. The idea goes back to N. Varopoulos [Va]. To be precise, given finitely many complex numbers $w_1, ..., w_n$, consider the 2^n possible sums

$$\sum_{j=1}^{n} \pm w_j$$

obtained as the plus-minus signs vary in the 2^n possible ways. For $r > 0$ we use

$$\mathcal{E}\left(\Big|\sum_{j=1}^{n} \pm w_j\Big|^r\right)$$

to denote the average value of

$$\Big|\sum_{j=1}^{n} \pm w_j\Big|^r$$

over the 2^n choices of sign. *Khinchin's inequality* states an estimate on the expectation below,

$$\mathcal{E}\left(\Big|\sum_{j=1}^{n} \pm w_j\Big|^r\right) \leq C_r \left(\sum_{j=1}^{n} |w_j|^2\right)^{r/2} \tag{2.5.1}$$

where C_r is a constant that does not depend on "n" (see [Ga3, p. 302]). Actually $C_r = 1$ if $r \leq 2$. This inequality will be used in the reproducing formula for $B_p(s)$.

Suppose now that (M) holds. Let $\varepsilon_k^j = \pm 1$ and let $g_j \in \mathcal{M}(B_p(s))$ with $g_j(z_k) = \varepsilon_k^j$, for $k, j = 1, 2, \cdots$. Let $f \in B_p(s)$, without loss of generality we may think $f(z_n) \neq 0$ for all $n = 1, \cdots$. Applying the reproducing formula (2.2.3) we obtain

$$(f\,g_j)(z_k) = (f\,g_j)(0) + \int_{\mathbb{D}} (f\,g_j)'(w)\overline{k'_{z_k}(w)}(1-|w|^2)^s dA(w), \tag{2.5.2}$$

where

$$k'_{z_k}(w) = \frac{1-(1-w\overline{z_k})^{1+s}}{w(1-w\overline{z_k})^{1+s}}.$$

In particular, by (2.2.6),

$$|k'_{z_k}(w)| \lesssim \frac{1}{|1-\overline{w}z_k|^{1+s}}. \tag{2.5.3}$$

Observe that if $0 < r \leq 1$, then

$$\sum |a_k| \leq \Big(\sum |a_k|^r\Big)^{1/r}. \tag{2.5.4}$$

Fix $n > 1$, and $j = 1, \cdots$. By (2.5.2), we have

$$\begin{aligned}
\sum_{k=1}^{n} |f(z_k)|^p (1-|z_k|^2)^s &= \sum_{k=1}^{n} \varepsilon_k^j (f\,g_j)(z_k)\overline{f(z_k)}|f(z_k)|^{p-2}(1-|z_k|^2)^s \\
&= (f\,g_j)(0) \sum_{k=1}^{n} \varepsilon_k^j \overline{f(z_k)}|f(z_k)|^{p-2}(1-|z_k|^2)^s \\
&\quad + \sum_{k=1}^{n} \varepsilon_k^j \left(\int_{\mathbb{D}} (f\,g_j)'(w)\,\overline{k'_{z_k}(w)}\,(1-|w|^2)^s dA(w) \right) \overline{f(z_k)}|f(z_k)|^{p-2}(1-|z_k|^2)^s \\
&= I_1 + I_2
\end{aligned}$$

We will compute the expectation of both sides of this identity. Let q be the conjugate exponent of p. Since $|f(0)| \leq \|f\|_{B_p(s)}$ for all $f \in B_p(s)$, applying Khinchine's inequality with $r = 1$,

$$\begin{aligned}
\mathcal{E}(|I_1|) &\leq C\|fg\|_{B_p(s)}\,\mathcal{E}\left(\Big| \sum_{k=1}^{n} \varepsilon_k^j \overline{f(z_k)}|f(z_k)|^{p-2}(1-|z_k|^2)^s \Big| \right) \\
&\leq C\|f\|_{B_p(s)} \left(\sum_{k=1}^{n} |f(z_k)|^{2(p-1)}(1-|z_k|^2)^{2s} \right)^{1/2}.
\end{aligned}$$

• If $p \geq 2$, then $2(p-1) \geq p$, and therefore, applying (2.5.4) with $r = \frac{p}{2(p-1)} = q/2 \leq 1$ we get

$$\mathcal{E}(|I_1|) \leq C\|f\|_{B_p(s)} \left(\sum_{k=1}^{n} |f(z_k)|^p\,(1-|z_k|^2)^{qs} \right)^{1/q}$$

$$\leq C\|f\|_{B_p(s)} \left(\sum_{k=1}^{n} |f(z_k)|^p\,(1-|z_k|^2)^{s} \right)^{1/q}.$$

• If $1 < p < 2$ we apply Holder's inequality with exponent $q/2 > 1$ to get

$$\begin{aligned}\mathcal{E}(|I_1|) &\leq C\|f\|_{B_p(s)} \left(\sum_{k=1}^{n} |f(z_k)|^{q(p-1)}(1-|z_k|^2)^{2s}\right)^{1/q} \left(\sum_{k=1}^{n}(1-|z_k|^2)^{2s}\right)^{\frac{2-p}{2p}} \\ &\leq C\|f\|_{B_p(s)} \left(\sum_{k=1}^{n} |f(z_k)|^{p}(1-|z_k|^2)^{s}\right)^{1/q},\end{aligned}$$

by Lemma 2.5.2 with $z = 0$. Let us now estimate the expected value of the second term I_2. Applying Fubini and Hölder's inequality, we obtain that $\mathcal{E}(|I_2|)$ equals

$$\mathcal{E}\left(\Big|\int_{\mathbb{D}}(f\,g_j)'(w)\left(\sum_{k=1}^{n}\varepsilon_k^j\,\overline{k'_{z_k}(w)}\;\overline{f(z_k)}|f(z_k)|^{p-2}(1-|z_k|^2)^s\right)(1-|w|^2)^s dA(w)\Big|\right)$$

$$\leq \|f\,g_j\|_{B_p(s)}$$

$$\left(\int_{\mathbb{D}}\mathcal{E}\left(\Big|\sum_{k=1}^{n}\varepsilon_k^j\,\overline{k'_{z_k}(w)}\;\overline{f(z_k)}|f(z_k)|^{p-2}(1-|z_k|^2)^s\Big|^q\right)(1-|w|^2)^{s+(q-2)}dA(w)\right)^{1/q}.$$

Now, by Khinchine's inequality (2.5.1) with $r = q$, the last expression can be bounded above by constant times

$$\|f\|_{B_p(s)}\left(\int_{\mathbb{D}}\left(\sum_{k=1}^{n}|\,k'_{z_k}(w)|^2|f(z_k)|^{2(p-1)}(1-|z_k|^2)^{2s}\right)^{q/2}(1-|w|^2)^{q-2+s}dA(w)\right)^{1/q}.$$

• If $p \geq 2$, we use (2.5.4) with $r = q/2 \leq 1$ and (2.5.3) to obtain

$$\begin{aligned}\mathcal{E}(|I_2|) &\lesssim \|f\|_{B_p(s)} \\ &\left(\int_{\mathbb{D}}\left(\sum_{k=1}^{n}|\,k'_{z_k}(w)|^q\cdot|f(z_k)|^{q(p-1)}(1-|z_k|^2)^{qs}\right)(1-|w|^2)^{q-2+s}dA(w)\right)^{1/q} \\ &\lesssim \|f\|_{B_p(s)}\left(\sum_{k=1}^{n}|f(z_k)|^p(1-|z_k|^2)^{qs}\int_{\mathbb{D}}\frac{(1-|w|^2)^{q-2+s}}{|1-\overline{w}z_k|^{(1+s)q}}dA(w)\right)^{1/q} \\ &\lesssim \|f\|_{B_p(s)}\left(\sum_{k=1}^{n}|f(z_k)|^p(1-|z_k|^2)^{s}\right)^{1/q},\end{aligned}$$

after an application of Lemma J.

• If $1 < p < 2$, we apply estimate (2.5.3) and use Hölder's inequality with exponent $q/2 \geq 1$. Then we apply Lemma 2.5.2 to obtain

$$\begin{aligned}
&\mathcal{E}(|I_2|) \lesssim \|f\|_{B_p(s)} \\
&\left(\int_{\mathbb{D}} \Big(\sum_{k=1}^{n} |f(z_k)|^p \frac{(1-|z_k|^2)^{s+\frac{qs}{2}}}{|1-\bar{w}z_k|^{q+2s}}\Big) \Big(\sum_{k=1}^{n} \frac{(1-|z_k|^2)^s}{|1-\bar{w}z_k|^{2s}}\Big)^{(q-2)/2} (1-|w|^2)^{q-2+s} dA(w)\right)^{1/q} \\
&\lesssim \|f\|_{B_p(s)} \left(\int_{\mathbb{D}} \Big(\sum_{k=1}^{n} |f(z_k)|^p \frac{(1-|z_k|^2)^{s+\frac{qs}{2}}}{|1-\bar{w}z_k|^{q+2s}}\Big) (1-|w|^2)^{\frac{-s(q-2)}{2}} (1-|w|^2)^{q-2+s} dA(w)\right)^{1/q} \\
&= \|f\|_{B_p(s)} \left(\sum_{k=1}^{n} |f(z_k)|^p (1-|z_k|^2)^{s+\frac{qs}{2}} \int_{\mathbb{D}} \frac{(1-|w|^2)^{q+2s-2-s\frac{q}{2}}}{|1-\bar{w}z|^{q+2s}} dA(w)\right)^{1/q} \\
&\lesssim \|f\|_{B_p(s)} \left(\sum_{k=1}^{n} |f(z_k)|^p (1-|z_k|^2)^{s+\frac{qs}{2}} (1-|z_k|^2)^{-s\frac{q}{2}}\right)^{1/q} \\
&= \|f\|_{B_p(s)} \left(\sum_{k=1}^{n} |f(z_k)|^p (1-|z_k|^2)^{s}\right)^{1/q},
\end{aligned}$$

after an application of Lemma J. Then,

$$\sum_{k=1}^{n} |f(z_k)|^p (1-|z_k|^2)^s \leq \mathcal{E}(|I_1|) + \mathcal{E}(|I_2|) \leq C\|f\|_{B_p(s)} \Big(\sum_{k=1}^{n} |f(z_k)|^p (1-|z_k|^2)^s\Big)^{1/q},$$

where the constant C is independent of n. So we can conclude that

$$\sum_{k\geq 1} |f(z_k)|^p (1-|z_k|^2)^s \leq C\|f\|^p_{B_p(s)}.$$

Hence

$$\sum_{z_n \in Z} (1-|z_n|^2)^s \delta_{z_n} \text{ is an } (s,p)\text{-Carleson measure,}$$

and the proof is finished. Note that we only use Lemma 2.5.2 when $1 < p < 2$.

(UIS) ⇒ (CS)

Let k_z be the reproducing kernel associated to the pairing

$$\langle f, h\rangle_s = f(0)\overline{h(0)} + \int_{\mathbb{D}} f'(z)\overline{h'(z)}\,(1-|z|^2)^s\, dA(z),$$

defined for $f \in B_p(s)$ and $h \in B_q(s)$. By Lemma 2.2.2,

$$\|k_z\|^p_{B_q(s)} \approx \frac{1}{(1-|z|^2)^s}.$$

Assume that the map $T: \ f \longrightarrow \left\{\frac{f(z_n)}{\|k_{z_n}\|_{B_q(s)}}\right\}$ from $B_p(s)$ to l^p, is bounded and onto. We want to see that the measure $d\mu = \sum(1-|z_n|^2)^s\delta_{z_n}$ is an (s,p)-Carleson measure, that is,

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C\|f\|^p_{B_p(s)} \quad \text{for all } f \in B_p(s).$$

This follows easily from the boundedness of T. Actually if $f \in B_p(s)$ one has

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) = \sum |f(z_n)|^p(1-|z_n|^2)^s \leq C\sum \frac{|f(z_n)|^p}{\|k_{z_n}\|^p_{B_q(s)}} \leq C\|f\|^p_{B_p(s)}$$

using the fact that the operator T is bounded.

Now we will see that the sequence $\{z_n\}$ is separated. Suppose $z_n, z_m \in Z$ with $\beta(z_n,z_m) \leq 1$. Then, by Lemma 2.2.1,

$$\|k_{z_n} - k_{z_m}\|_{B_q(s)} \leq C\frac{|z_n - z_m|}{(1-|z_n|^2)^{1+s/p}}.$$

Consider the sequence $\{w_k\}$ given by $w_n = 1$ and $w_k = 0$ for $k \neq n$. By hypothesis (UIS), there exists a function $f \in B_p(s)$ with $f(z_k) = w_k$ for $k = 1, \cdots$. Moreover by the open mapping Theorem, we may assume

$$\|f\|^p_{B_p(s)} \lesssim \sum \frac{|f(z_m)|^p}{\|k_{z_m}\|^p_{B_q(s)}} = \frac{1}{\|k_{z_n}\|^p_{B_q(s)}}$$

Hence for $m \neq n$, one has

$$\|f\|_{B_p(s)} \leq C\frac{|f(z_n) - f(z_m)|}{\|k_{z_n}\|_{B_q(s)}} \leq C\frac{|< f, k_{z_n} - k_{z_m} >|}{(1-|z_n|^2)^{-s/p}} \leq C\frac{\|f\|_{B_p(s)}\|k_{z_n} - k_{z_m}\|_{B_q(s)}}{(1-|z_n|^2)^{-s/p}}.$$

So,

$$\|k_{z_n} - k_{z_m}\|_{B_q(s)} \geq C(1-|z_n|^2)^{-s/p}.$$

Since $\beta(z_n,z_m) \leq 1$, we have $1-|z_n|^2 \approx |1-\overline{z}_m z_n|$, and Lemma 2.2.1 gives

$$C \leq \frac{\|k_{z_n} - k_{z_m}\|_{B_q(s)}}{(1-|z_n|^2)^{-s/p}} \leq C\frac{|z_n - z_m|(1-|z_n|^2)^{s/p}}{(1-|z_n|^2)^{1+s/p}} \leq C\frac{|z_n - z_m|}{|1-\overline{z}_m z_n|} = C\rho(z_n,z_m).$$

So, $\rho(z_n,z_m) \geq C$ and the sequence $\{z_n\}$ is separated.

2.5.2 Sufficiency of condition (CS)

(CS) $\Rightarrow$ (UIS)

Given a separated sequence $\{z_n\} \subset \mathbb{D}$ such that $\mu = \sum_n (1-|z_n|^2)^s\delta_{z_n}$ is an (s,p)-Carleson measure, consider the map

$$\begin{array}{rccc} T: & B_p(s) & \longrightarrow & l^p \\ & f & \longrightarrow & \left\{\frac{f(z_n)}{\|k_{z_n}\|_{B_q(s)}}\right\}. \end{array}$$

We want to see that the map T is bounded and onto. Recall that $\|k_z\|^p_{B_q(s)} \approx (1-|z|)^{-s}$. Since μ is an (s,p)-Carleson measure, we deduce

$$\left\| \frac{f(z_n)}{\|k_{z_n}\|_{B_q(s)}} \right\|^p_{l^p} \approx \sum_n |f(z_n)|^p (1-|z_n|)^s = \int_{\mathbb{D}} |f(z)|^p d\mu(z) \lesssim \|f\|^p_{B_p(s)},$$

for all $f \in B_p(s)$. So the map T is bounded.

To see that T is also onto, consider an arbitrary sequence $\{w_n\} \in l^p$. We will find a function $f \in B_p(s)$ with

$$\frac{f(z_n)}{\|k_{z_n}\|_{B_q(s)}} = w_n, \qquad \text{for} \qquad n = 1, 2, \cdots.$$

By a normal family argument we may assume that the sequence $\{z_n\}$ is finite and $0 \notin \{z_n\}$. Let $D(z,r)$ denote the pseudohyperbolic disc of center z and pseudohyperbolic radius r. Since $\{z_n\}$ is separated, there exists $\varepsilon > 0$ such that the pseudohyperbolic discs $\{D(z_n, 2\varepsilon) \ : \ n = 1, 2, \cdots\}$ are pairwise disjoints. With standard arguments, we construct a smooth function φ with the following properties:

(a) $\varphi(z) = w_n \|k_{z_n}\|_{B_q(s)}$ for $z \in D(z_n, \varepsilon)$;

(b) φ vanishes outside $\cup_n D(z_n, 2\varepsilon)$;

(c) $(1-|z|)|\nabla\varphi(z)| \leq C|w_n| \, \|k_{z_n}\|_{B_q(s)}$ for all $z \in D(z_n, 2\varepsilon)$;

Using the above conditions and Lemma 2.2.2,

$$\begin{aligned}&\int_{\mathbb{D}} |\nabla\varphi(z)|^p (1-|z|^2)^{p-2+s} dA(z)\\ &\qquad\lesssim \sum_{n=1}^{N} \int_{D(z_n,2\varepsilon)} |w_n|^p \, \|k_{z_n}\|^p_{B_q(s)} \, (1-|z|)^{s-2} \, dA(z) \leq C \|w_n\|^p_{l^p},\end{aligned}$$

where C is an absolute constant non depending on N, the number of points of the sequence. Let $B(z)$ be the Blaschke product with zeros $\{z_n\}$ and let δ be a positive constant. By (2.3.4), $\sum (1-|z_n|)\delta_{z_n}$ is a Carleson measure, and there exists a constant $C > 0$ such that $|B(z)| \geq C$ for any $z \in \mathbb{D}$ with $\inf \beta(z, z_n) \geq \delta$. Hence $|B(z)| \geq C$ for all $z \in \operatorname{supp}(\overline{\partial}\varphi(z))$, thus

$$\frac{\overline{\partial}\varphi}{B} \in \mathcal{C}(\overline{\mathbb{D}}).$$

So, by Theorem 2.4.1, we can solve the $\bar{\partial}$-equation

$$\overline{\partial} b = \frac{1}{B} \, \overline{\partial}\varphi$$

in the variable function b obtaining a solution $b \in L^p_s$ with

$$\|b\|^p_{L^p_s} \leq C \int_{\mathbb{D}} |\nabla \varphi(z)|^p (1-|z|^2)^{p-2+s} dA(z).$$

Without loss of generality we may assume $b(0) = 0$. Consider now the function

$$f = \varphi - Bb.$$

Hence f is analytic in the unit disc $\mathbb{D}$ and

$$f(z_n) = \varphi(z_n) = w_n, \quad \text{for} \quad n = 1, 2, \cdots.$$

It only remains to prove that the function f is in $B_p(s)$. By Lemma 2.3.5, the Blaschke product $B \in \mathcal{M}(B_p(s))$. Hence $Bb \in L^p_s$ with $\|Bb\|_{L^p_s} \leq C\|b\|_{L^p_s}$ and thus $f_{|\partial\mathbb{D}} \in L^p_s$. By Corollary 2.3.8 we conclude that the function $f \in B_p(s)$ with

$$\|f\|_{B_p(s)} \leq C\|f_{|\partial\mathbb{D}}\|_{L^p_s} \leq C\|b\|_{L^p_s} \leq C\|w_n\|_{l^p},$$

where C is an absolute constant not depending on the number of points N.

(CS) $\Rightarrow$ (M)

Let Z be a separated sequence such that $\mu_Z = \sum_{z_n \in Z} (1-|z_n|^2)^s \delta_{z_n}$ is an (s,p)-Carleson measure. We want to see that Z is an interpolating sequence for $\mathcal{M}(B_p(s))$. Before proving this, we will present some auxiliary results.

Let $W = \{w_n\}$ be a sequence of values with $\sup_n |w_n| \leq 1$. By a normal family argument we may assume that the sequence $\{z_n\}$ is finite and $0 \notin \{z_n\}$. Since $Z = \{z_n\}$ is separated, there exists $\varepsilon > 0$ such that the pseudohyperbolic discs $\{D(z_n, 2\varepsilon) \;:\; n = 1, 2, \cdots\}$ are pairwise disjoints. With standard arguments, we construct a smooth solution φ of the interpolating problem with the following properties:

(a) $\varphi(z) = w_n$ for $z \in D(z_n, \varepsilon)$;

(b) φ vanishes outside $\cup_n D(z_n, 2\varepsilon)$;

(c) $0 \leq |\varphi| \leq 1$ everywhere on $\mathbb{D}$;

(d) $(1-|z|)|\nabla\varphi(z)| \leq C$ for all $z \in \mathbb{D}$;

Claim 2.5.3. *Let φ be a function in $\mathbb{D}$ with the above properties, then the measure $d\mu_\varphi(z) = |\nabla\varphi(z)|^p(1-|z|^2)^{p-2+s}dA(z)$ is an (s,p)-Carleson measure.*

Proof. Let D_n be the pseudohyperbolic disc centered at z_n and pseudohyperbolic radius 2ε, and let $f \in B_p(s)$. Since φ is supported in $\cup D_n$ and $(1-|z|)|\nabla\varphi(z)| \leq C$, we have

$$\int_{\mathbb{D}} |f(z)|^p |\nabla\varphi(z)|^p (1-|z|^2)^{p-2+s} dA(z) \lesssim \sum_n \int_{D_n} |f(z)|^p (1-|z|^2)^{s-2} dA(z)$$
$$\lesssim \sum_n |f(z_n)|^p (1-|z_n|^2)^s + \sum_n \int_{D_n} |f(z)-f(z_n)|^p (1-|z|^2)^{s-2} dA(z).$$

Since $\sum_n (1-|z_n|^2)^s \delta_{z_n}$ is an (s,p)-Carleson measure,

$$\sum_n |f(z_n)|^p (1-|z_n|^2)^s \leq C\|f\|^p_{B_p(s)}.$$

So, if we show that

$$\sum_n \int_{D_n} |f(z)-f(z_n)|^p (1-|z|^2)^{s-2} dA(z) \leq C\|f\|^p_{B_p(s)}$$

we obtain

$$\int_{\mathbb{D}} |f(z)|^p |\nabla\varphi(z)|^p (1-|z|^2)^{p-2+s} dA(z) \leq C\|f\|^p_{B_p(s)},$$

proving that the measure $|\nabla\varphi(z)|^p(1-|z|^2)^{p-2+s}dA(z)$ is an $(s,p)-$Carleson measure. Let $z, z_n \in D_n$, the reproducing formula (2.2.3) for $B_p(s)$ and inequality (2.2.5) tell that

$$|f(z)-f(z_n)| = \left|\int_{\mathbb{D}} f'(w)\left(\overline{k'_z(w)} - \overline{k'_{z_n}(w)}\right)(1-|w^2|)^s dA(w)\right|$$
$$\leq C\int_{\mathbb{D}} |f'(w)| \frac{|z-z_n|}{|1-\overline{w}z|^{2+s}} (1-|w|^2)^s dA(w).$$

Choose $t>0$ with $\max(0,2-p) < tp+1+s\cdot\min(1,p-1)$. Then, by Hölder's inequality and Lemma J,

$$|f(z)-f(z_n)|^p \lesssim (1-|z|^2)^p \left(\int_{\mathbb{D}} |f'(w)|^p \frac{(1-|w|^2)^{tp}}{|1-\overline{w}z|^{2+s}}\, dA_{p,s}(w)\right)$$
$$\cdot \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^{-tq}}{|1-\overline{w}z|^{2+s}} dA_{q,s}(w)\right)^{p-1}$$
$$\lesssim \left(\int_{\mathbb{D}} |f'(w)|^p \frac{(1-|w|^2)^{tp}}{|1-\overline{w}z|^{2+s}} dA_{p,s}(w)\right)(1-|z|^2)^{2-tp}.$$

Hence, by Fubini's theorem and Lemma J we obtain

$$\begin{aligned}
&\sum_n \int_{D_n} |f(z)-f(z_n)|^p\,(1-|z|^2)^{s-2} dA(z) \\
&\quad \lesssim \sum_n \int_{D_n} \left(\int_{\mathbb{D}} |f'(w)|^p \frac{(1-|w|^2)^{tp}}{|1-\overline{w}z|^{2+s}}\, dA_{p,s}(w) \right) (1-|z|^2)^{s-tp}\, dA(z) \\
&\quad \le \int_{\mathbb{D}} |f'(w)|^p \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^{s-tp}}{|1-\overline{w}z|^{2+s}}\, dA(z) \right) (1-|w|^2)^{tp}\, dA_{p,s}(w) \\
&\quad \lesssim \int_{\mathbb{D}} |f'(w)|^p\, dA_{p,s}(w) \le \|f\|^p_{B_p(s)}
\end{aligned}$$

Note that the conditions on t ensures that the application of Lemma J is correct. So, the proof is complete. $\square$

Once we have the function φ, consider the Blashcke product $B(z)$ with zeros at the points $\{z_n\}$. Since as before, $|B(z)| \ge C$ for all $z \in \operatorname{supp}(\overline{\partial}\varphi(z))$, we have

$$\left| \frac{\overline{\partial}\varphi(z)}{B(z)} \right|^p (1-|z|^2)^{p-2+s} \le C|\nabla\varphi(z)|^p (1-|z|^2)^{p-2+s}$$

so, by Claim 2.5.3,

$$\left| \frac{\overline{\partial}\varphi(z)}{B(z)} \right|^p (1-|z|^2)^{p-2+s} dA(z) \text{ is an } (s,p)\text{-Carleson measure.}$$

By Theorem 2.4.2, we can solve the $\bar{\partial}$-equation

$$\overline{\partial} b = \frac{1}{B}\,\overline{\partial}\varphi$$

in the variable function b obtaining a solution $b \in \mathcal{M}(L^p_s)$ with $b(0)=0$ and

$$\|b\|_{L^p_s} \le C \int_{\mathbb{D}} |\nabla\varphi(z)|^p\,(1-|z|^2)^{p-2+s} dA(z).$$

Consider now

$$g = \varphi - Bb.$$

By construction, g is analytic in the unit disc $\mathbb{D}$ and $g(z_n) = \varphi(z_n) = w_n,\ n = 1, 2, \cdots$. It only remains to prove that the function $g \in \mathcal{M}(B_p(s))$. By Lemma 2.3.5, the Blaschke product $B \in \mathcal{M}(B_p(s))$ with $\|Bb\|_{L^p_s} \le C\|b\|_{L^p_s}$, so $g \in \mathcal{M}(L^p_s)$. Since $g \in \mathcal{C}(\overline{\mathbb{D}})$ then $g = P[g_{|\partial\mathbb{D}}]$ and applying Corollary 2.3.12 we get that $g \in \mathcal{M}(B_p(s))$ with

$$\|g\|_{B_p(s)} \le C\|\nabla\varphi\|_{L^p(dA_{p,s})},$$

where C is an absolute constant not depending on N, the number of points of the sequence $\{z_n\}$.

(CS) $\Rightarrow$ (M): A constructive proof

J.P. Earl gave a method to construct a bounded analytic function having prescribed bounded values on a uniformly separated sequence. See [Ea] or [Ga3, VII.5].

Theorem M (Earl). *Let $\{z_n\}$ be a sequence in the unit disc $\mathbb{D}$ such that*

$$\prod_{j\neq k}\left|\frac{z_k - z_j}{1-\overline{z}_j\, z_k}\right| \geq \delta, \qquad k = 1, 2, ... \tag{2.5.5}$$

Then there is a constant $C = C(\{z_n\}) > 0$ such that whenever $\{w_n\} \in l^\infty$, there exist $\{\xi_n\}$ satisfying

$$\rho(\xi_n, z_n) \leq \frac{\delta}{3} \qquad \text{and} \qquad \prod_{j\neq k}\left|\frac{\xi_k - \xi_j}{1-\overline{\xi}_j\, \xi_k}\right| \geq \delta/3, \qquad k = 1, 2, ...$$

such that the function

$$g(z) = C\left(\sup_n |w_n|\right) B(z),$$

where B(z) is the Blaschke product with zeros $\{\xi_n\}$ satisfies

$$g(z_n) = w_n, \qquad n = 1, 2, ...$$

Let $\{z_n\}$ be a separated sequence with $\sum(1-|z_n|)^s\delta_{z_n}$ an (s,p)-Carleson measure, and let $\{w_n\} \in l^\infty$. Using Earl's result we will show that the interpolation problem $g(z_n) = w_n$ can be solved with a multiple of a Blaschke product $B \in \mathcal{M}(B_p(s))$. In fact, by (2.3.4), it follows that $\sum(1-|z_n|^2)\delta_{z_n}$ is a classical Carleson measure. Hence (2.5.5) holds, and applying J.P. Earl's result stated above, we obtain a Blaschke product B with zeros $\{\xi_n\}$ such that a multiple of B solves the interpolation problem. By proposition 2.3.5, if $\sum_n(1-|\xi_n|^2)^s\delta_{\xi_n}$ is an (s,p)-Carleson measure then the Blaschke product B is in $\mathcal{M}(B_p(s))$. So we will be finished if we prove the following result.

Lemma 2.5.4. *Let $\{z_n\}$ be a separated sequence of points in $\mathbb{D}$ with $\rho(z_n, z_m) \geq \delta$ for $m \neq n$, and suppose that $\sum(1-|z_n|)^s\delta_{z_n}$ is an (s,p)-Carleson measure. Let $\{\xi_k\}$ be a sequence of points in the unit disc such that*

$$\rho(\xi_k, z_k) \leq \delta/3, \quad k = 1, 2, ...$$

for some fixed $\delta > 0$. Then $\sum(1-|\xi_k|)^s\delta_{\xi_k}$ is also an (s,p)-Carleson measure.

Proof. Let $f \in B_p(s)$. Since $\rho(z_n, \xi_n) \leq \delta/3$ then $1-|z_n| \approx 1-|\xi_n|$. So,

$$\sum |f(\xi_n)|^p\,(1-|\xi_n|^2)^s \lesssim (I) + (II),$$

where $(I) = \sum |f(z_n)|^p\,(1-|z_n|^2)^s$ and $(II) = \sum |f(z_n)-f(\xi_n)|^p\,(1-|z_n|^2)^s$. Now, since $\sum(1-|z_n|^2)^s\delta_{z_n}$ is an (s,p)-Carleson measure, $(I) \le C\|f\|^p_{B_p(s)}$. For the other term, since $|f'|$ is a subharmonic function,

$$|f(z_n) - f(\xi_n)| \le \int_{\gamma_n} |f'(\sigma)|\,ds(\sigma) \le \int_{\gamma_n} \frac{1}{|D_\sigma|}\int_{D_\sigma} |f'(w)|dA(w)\,ds(\sigma),$$

where D_σ is the pseudohyperbolic disc of center σ and radius $\delta/6$ and γ_n is the hyperbolic geodesic joining z_n and ξ_n. Since $\rho(z_n,\xi_n) \le \delta/3$, one has $D_\sigma \subset \widetilde{D}_{z_n}$ for any $\sigma \in \gamma_n$, where $\widetilde{D}_{z_n}$ is the pseudohyperbolic disc of center z_n and radius $\delta/2$. Observe that $|D_\sigma| \approx (1-|z_n|)^2$ and that $|z_n - \xi_n| \lesssim 1 - |z_n|$. Then,

$$|f(z_n) - f(\xi_n)| \lesssim \int_{\gamma_n} \frac{1}{(1-|z_n|)^2}\int_{\widetilde{D}_{z_n}} |f'(w)|\,dA(w)\,ds(\sigma) \lesssim \frac{1}{(1-|z_n|)}\int_{\widetilde{D}_{z_n}} |f'(w)|\,dA(w).$$

Since $\rho(z_n, z_m) \ge \delta$ for $m \ne n$, the discs $\widetilde{D}_{z_n}$ are pairwise disjoint and finally

$$\begin{aligned}
(II) &= \sum |f(z_n) - f(\xi_n)|^p\,(1-|z_n|^2)^s \\
&\lesssim \sum \frac{1}{(1-|z_n|)^p}\left(\int_{\widetilde{D}_{z_n}} |f'(w)|dA(w)\right)^p (1-|z_n|^2)^s \\
&\lesssim \sum \left(\int_{\widetilde{D}_{z_n}} |f'(w)|(1-|w|^2)^{-1+s/p}dA(w)\right)^p \\
&\le \sum \left(\int_{\widetilde{D}_{z_n}} |f'(w)|^p(1-|w|^2)^{p-2+s}dA(w)\right)\left(\int_{\widetilde{D}_n} (1-|w|^2)^{-2}\right)^{p/q} \\
&\lesssim \sum \int_{\widetilde{D}_{z_n}} |f'(w)|^p(1-|w|^2)^{p-2+s}dA(w) \le \|f\|^p_{B_p(s)}.
\end{aligned}$$

So, $\sum(1-|z_n|^2)\delta_{z_n}$ is an (s,p)-Carleson measure. □

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