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PH.D. THESIS

# *COLORED COMBINATORIAL STRUCTURES:*
## *HOMOMORPHISMS AND COUNTING*

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# Part II

# Ramsey and anti-Ramsey results in finite groups

# Chapter 10

# Preliminaries

In this section we give the concepts and terminology that we will use to develop our work. We also present an historical overview of the arithmetic Ramsey theory and anti–Ramsey theory on the integers. Finally we give some tools from additive combinatorics that we will use in Chapter 11.

## 10.1 Basic definitions

We shall denote the set of integers by $\mathbb{Z}$, the set of positive integers by $\mathbb{Z}^+$, and the set of integers modulo $n$ by $\mathbb{Z}/n\mathbb{Z}$.

Most of the results presented in Sections 10.2 and 10.3 will be confined to the interval of integers $[1, n] := \{1, 2, ..., n\}$, and the set of natural numbers denoted by $\mathbb{N}$.

Given a set $X$, an *$r$–coloring* of $X$ is a function $c : X \to C$ where $|C| = r$. Typically, the elements of $C$ are called the *colors*, and we use $C = \{1, 2, ..., r\}$. We can think of an $r$–coloring $c$ of a set $X$, as a partition of $X$ into $r$ subsets $X_1, X_2, ..., X_3$ by associating the subset $X_i$ with the set $\{x \in X : c(x) = i\}$. The subsets $X_1, X_2, ..., X_3$ are called *color classes*. For 3–colorings we usually use $C = \{R, G, B\}$ in order to have the color classes Red Green and Blue. An *equinumerous* $r$–coloring os a set $X$, is an $r$–coloring in which all color classes have the same cardinality.

Let $Y$ be a subset of $X$ and $c$ be a coloring on $X$, then $Y$ is said to be *monochromatic* under $c$, if $c$ assigns the same color to the elements of $Y$. On the contrary, $Y$ is said to be *rainbow* under $c$, if $c$ assigns pairwise distinct colors to the elements of $Y$.

Most of the structures that we consider in this work can be described as solutions of systems of equations, being the main examples arithmetic progressions and Schur triples.

For a positive integer $t$, a *$t$–term arithmetic progression* $AP(t)$, is a sequence of the form $\{a, a+d.a+2d, ..., a+(t-1)d\}$, where $a \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$. We shall observe that, when $t = 3$, a 3–term arithmetic progression $AP(3)$, is a triple $(x, y, z)$ satisfying the equation $x + y = 2z$. A *Schur triple* is a triple $(x, y, z)$ satisfying the equation $x + y = z$.

Let $X$ be a finite set with cardinality $n$ and let $S$ be a set of vectors in $X^d$. We said that $S$ is an *orthogonal array* of *degree* $d$ and *strength* $k$ if, for any choice of $k$ columns, each $k$-vector of $X^k$ appears in exactly one vector of $S$. In other words, if we specify any set of $k$ entries $a_1, \cdots, a_k$ and any set of subscripts $1 \leq i_1 < i_2 < \cdots < i_k \leq d$, we find exactly one vector $\mathbf{y} = (y_1, y_2, \ldots, y_d)$ in $S$ with $y_{i_1} = a_1, y_{i_2} = a_2, \ldots, y_{i_k} = a_k$. We denote by $OA(d, k)$ the family of orthogonal arrays of degree $d$ and strength $k$ on $X$.

The set of Schur triples in a finite group form an orthogonal array $OA(3, 2)$. The same is true for 3–term arithmetic progressions if the order of the group is relatively prime with 6.

## 10.2 Arithmetic Ramsey theory: three classical results

Ramsey's theorem was not the first theorem in the area now known as Ramsey theory. The results that are generally accepted to be the earliest Ramsey–type theorems are due, in chronological order, to Hilbert, Schur and van der Waerden. All this results which preceded Ramsey's theorem, deal with colorings of the integers.

In this section we present three classical theorems concerning Ramsey theory on the integers, that became the starting point of an area that is still very active today: *arithmetic Ramsey theory.*

### 10.2.1 Van der Waerden's Theorem

We start with the van der Waerden theorem, which was proved in 1927, and its perhaps the most fundamental Ramsey-type theorem on the integers. Loosely, it says that for any given coloring of $\mathbb{Z}^+$, monochromatic arithmetic progressions cannot be avoided.

**Theorem 10.1 (van der Waerden's Theorem [67], 1927)** *For all positive integers $k$ and $t$, if $n$ is sufficiently large, then every $k$-coloring of $[1, n]$ contains a monochromatic $t$-term arithmetic progression.*

The least positive integer $n = w(t; k)$ which satisfies the van der Waerden theorem, is known as the *van der Waerden number*. The van der Waerden numbers $w(2; r)$ are easy to find, as it is shown in the next example.

**Example 10.1** $w(2; r) = r+1$. *To see this, observe that* $w(2; r) > r$ *(since an* $r$*-coloring of* $[1, r]$ *in which every integer get a distinct color does not contains a* 2*–term arithmetic progression), and by a simple application of the pigeonhole principle we get* $w(2; r) \leq r+1$ *(since any* 2*–element set of integers is a* 2*–term arithmetic progression).*

For $k \geq 3$, the evaluation of these numbers becomes much more difficult. In fact, the only known van der Waerden numbers, for $k \geq 3$, are: $w(3; 2) = 9$, $w(3; 3) = 27$, $w(3; 4) = 76$, $w(4; 2) = 35$, and $w(5; 2) = 178$. Thus, besides trying to find exact values of the van der Waerden numbers, to estimate $w(k; r)$ in terms of $k$ and $n$, becomes one of the central problems in Ramsey theory.

There are many other interesting aspects to study in this framework. For instance consider the following question:

- How few arithmetic progression can a subset $S \subset [1, n]$ have, if $S$ has a given density?

In the case of 3–term arithmetic progressions, it is shown in [23] that for $n$ sufficiently large, every 2–coloring of $[1, n]$ admits $\Theta(n^2)$ monochromatic 3–term arithmetic progressions. In 1999, Ron Graham propose the following problem:

- Let $V(n)$ be the minimum number of monochromatic 3–term arithmetic progressions in any 2–coloring of $[1, n]$. Given $V(n) = \beta n^2(1 + o(1))$, determine $\beta$.

It was conjectured that $\beta = 1/16$. In 2008 this conjecture was disproved by Parrilo et.al. [49]. They do not determine the exact value of $\beta$, but provided fairly good upper and lower bounds.

**Theorem 10.2 (Parrilo, Robertson and Saracino [49], 2008)**

$$\frac{1675}{32768} n^2(1 + o(1)) \leq V(n) \leq \frac{117}{2192} n^2(1 + o(1))$$

Now let $p$ be a prime and $S \subset \mathbb{Z}/p\mathbb{Z}$ a set of density $\upsilon p$, with $\upsilon$ a constant. Denote by $t_3(S)$ the number of monochromatic 3–term arithmetic progressions in $S$. From an old result of Varnavides [66] it is know that the normalized count of $t_3(S)$ satisfies:

$$\frac{1}{p^2}\min\{t_3(S) : S \subset \mathbb{Z}/p\mathbb{Z}, |S| = \upsilon p\} \geq c(\upsilon) > 0$$

where $c(\upsilon)$ is a constant depending only on $\upsilon$. Recently Croot [17] answer a question of Green [18] and show that the limit:

$$\lim_{p\to\infty}\frac{1}{p^2}\min\{t_3(S) : S \subset \mathbb{Z}/p\mathbb{Z}, |S| = \upsilon p\},$$

does exist. However the actual value of that limit is still an open question.

### 10.2.2 Schur's Theorem

We now present a theorem proved by Schur in 1916, which is one of the earliest results in Ramsey theory.

**Theorem 10.3 (Schur's Theorem [58], 1916)** *For every $k$, if $n$ is sufficiently large, then every $k$-coloring of $[1, n]$ contains a monochromatic solution of the equation $x+y = z$.*

A solution of the equation $x+y = z$ is named a *Schur triple*. The least positive integer $n = s(k)$ which satisfies the Schur theorem, is known as the *Schur number*.

**Example 10.2** $s(2) = 5$. *By means of the following* $2$*–coloring of* $[1, 4]$ *we get* $s(2) > 4$*:*

1234

*It can be shown that for every* $2$*–coloring of* $[1, 5]$ *(there are 32 of them) there will exist integers* $x, y, z$ *(not necessarily distinct) in the same color class satisfying* $x + y = z$*:*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 12345 | 12345 | 12345 | 12345 | 12345 | 12345 | 12345 | 12345 |
| 12345 | 12345 | 12345 | 12345 | 12345 | 12345 | 12345 | 12345 |
| 12345 | 12345 | 12345 | 12345 | 12345 | 12345 | 12345 | 12345 |
| 12345 | 12345 | 12345 | 12345 | 12345 | 12345 | 12345 | 12345 |

The only Schur numbers that are currently known are: $s(1) = 2$, $s(2) = 5$, $s(3) = 14$ and $s(4) = 45$.

Concerning the problem of determine the minimum number of monochromatic Schur triples in a 2–coloring of $[1, n]$, it has been shown by several authors [19, 56, 54] that this number is:

$$N^2/22 + O(N).$$

The problem of finding the minimal number of monochromatic Schur triples was first proposed by Graham et al. [23] in 1996. It was solved by Robertson and Zeilberger [54] in 1998; their proof relies on discrete calculus and a Maple package. A little later Schoen [56] gave a paper-and-pencil proof of the same result. Finally, in 2003, Datskovsky [19] provide a simple proof of this result, and also shows that the number of monochromatic Schur triples modulo $n$ equals:

$$n^2 - |S_1||S_2|$$

where $[1, n] = S_1 \cup S_2$ is a 2–coloring of $[1, n]$. Thus, the total number of monochromatic Schur triples in every 2–coloring of the group $\mathbb{Z}/n\mathbb{Z}$ depends only on the cardinality of the color clases but not on the distribution on the colors.

### 10.2.3 Rado's Theorem

The third classical theorem we mention is Rado's theorem, which is a generalization of Schur's theorem. Thinking of Schur's theorem as a theorem about the homogeneous linear equation $x + y - z = 0$, we ask the most general question:

- Which systems, $\mathcal{L}$, of homogeneous linear equations with integer coefficients have the following property: for every $r \geq 1$, there exist a least positive integer $n := n(\mathcal{L}; r)$ such that every $r$–coloring of $[1, n]$ yields a monochromatic solution to $\mathcal{L}$?

In a series of articles published in the 1930's, Rado completely answered this question. Since Rado's theorem, in its most general form, is a bit complicate to describe, we will mention here just the special case in which the system consist of only a single equation.

We first need the following definition.

**Definition 10.1** *For $r \geq 1$, a linear equation $\mathcal{E}$ is called $r$–**regular** if there exists $n = n(\mathcal{E}; r)$ such that for every $r$–coloring of $[1, n]$ there is a monochromatic solution to $\mathcal{E}$. It is called regular if it is $r$–regular for all $r \geq 1$.*

Thus, Schur's theorem states that the equation $x + y = z$ is regular.

**Theorem 10.4 (Rado's single equation Theorem, 1916)** *Let $\mathcal{E}$ represent the linear equation $\sum_{i=1}^{n} c_i x_i = 0$, where $c_i \in \mathbb{Z} - 0$ for $1 \leq i \leq n$. Then $\mathcal{E}$ is regular if and only if some nonempty subset of the $c_i$'s sums to $0$.*

**Example 10.3** *It follows by Rados's theorem that the $x + y - z = 0$ (the Schur equation) is regular, since the sum of the first and third coefficients is $0$.*

## 10.3 Rainbow Ramsey theory: historical overview

In this Section we give an overview on the current state in the so called *rainbow Ramsey Theory*. This new trend studies the existence of rainbow (hetero–chromatic) structures in a colored universe, under certain density conditions on the colorings. In particular, the results we present here are related to the existence of rainbow arithmetic progressions in $[1, n]$ and $\mathbb{N}$.

There are previous results regarding the existence of rainbow structures in the context of *canonical Ramsey theory* (canonical theorems prove the existence of *either* a monochromatic structure *or* a rainbow structure). Here our interest is not on "either-or"–type statements. In a sense, the theorems we describe below can be thought of as the first rainbow counterparts of classical theorems in Ramsey theory, such as Schur's and van der Waerden's theorems.

### 10.3.1 Rainbow Schur triples

More than seven decades after Schur's theorem was published, Alekseev and Savchev [3] consider what is now called an *un–Schur* problem. Recall that a solution of the equation $x + y = z$ is named a Schur triple.

**Theorem 10.5 (Alekseev and Savchev [3], 1987)** *Every equinumerous $3$–coloring of $[1, 3n]$ contains a rainbow Schur triple.*

It is natural to ask whether the condition of equal cardinalities on the color classes, can be weakened. In fact, in 1990 Schönhaim [57] answers this question.

**Theorem 10.6 (Schönhaim [57], 1990)** *Every* $3$*–coloring of* $[1, n]$*, such that every color class has cardinality grater than* $n/4$*, contains a rainbow Schur triple, and* $n/4$ *is optimal.*

### 10.3.2 Rainbow arithmetic progressions in $\mathbb{N}$ and $[n]$

Motivated by these results, Jungić and Radoičić [34] proved the analogous result of Theorem 10.5, concerning 3–term arithmetic progressions. Recall that a solution of the equation $x + y = 2z$ is a 3–term arithmetic progression, and is denoted by $AP(3)$.

**Theorem 10.7 (Jungić and Radoičić [34], 2003)** *Every equinumerous* $3$*–coloring of* $[1, 3n]$*, contains a rainbow* $AP(3)$*.*

In 2003, Jungić, Licht, Mahdian, Nešetřil and Radoičić [32] proved an infinite version of Theorem 10.7. Let $c : \mathbb{N} \to \{R, G, B\}$ be a 3–coloring of the set of natural numbers with colors Red, Green, and Blue. We define $R_c(n)$ (resp. $G_c(n)$, $B_c(n)$) to be the number of integers less than or equal to $n$ that are colored with color Red (resp. Green, Blue).

**Theorem 10.8 (Jungić, Licht, Mahdian, Nešetřil and Radoičić [32], 2003)** *If* $c : \mathbb{N} \to \{R, G, B\}$ *satisfies the following density condition:*

$$\limsup(\min(R_c(n), G_c(n), B_c(n)) - n/6) = +\infty$$

*then* $c$ *contains a rainbow* $AP(3)$*.*

Basically Theorem 10.8 states that every 3–coloring of the set of natural numbers with the upper density of each color grater than $1/6$ contains a rainbow $AP(3)$.

The next example given in [7] shows that Theorem 10.8 is the best possible.

**Example 10.4 ([7])** *Consider the following coloring of* $\mathbb{N}$*:*

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, \ldots$$

*that is:*

$$c(i) = \begin{cases} R & \textit{if } i \equiv 1 \pmod 6 \\ G & \textit{if } i \equiv 4 \pmod 6 \\ B & \textit{otherwise} \end{cases}$$

*Observe that c contains no rainbow $AP(3)$ and* $\min\{R_c(n), G_c(n), B_c(n)\} = \lfloor (n+2)/6 \rfloor$.

Based on the computer evidence, the authors in [32] conjectured a value $r(n)$, for which every 3–coloring of $[1, n]$ with the smallest color class grater than $r(n)$, contains a rainbow $AP(3)$. They also showed that, this conjecture, if true, is optimal. In 2004, Axenovich and Fon-Der-Flass [7] proved successfully that conjecture.

**Theorem 10.9 (Axenovich and Fon-Der-Flass [7], 2004)** *For every $n \geq 3$, every partition of $[1, n]$ into three color classes R, G, B, where* $\min |R|, |G|, |B| > r(n)$, *where*

$$r(n) = \begin{cases} \lfloor (n+2)/6 \rfloor & \textit{if mnot} \equiv 2 \pmod 6 \\ (n+4)/6 & \textit{if } m \equiv 2 \pmod 6 \end{cases}$$

*contains a rainbow $AP(3)$.*

For $mnot \equiv 2 \pmod 6$ the coloring described in Example 10.4, shows that Theorem 10.9 is tight in this case. The following coloring shows that Theorem 10.9 is tight as well for the remaining case.

**Example 10.5 ([7])** *Let $n = 6k + 2$, we define a coloring c as follows:*

$$c(i) = \begin{cases} R & \textit{if } i \leq 2k+1 \textit{ and } i \textit{ is odd} \\ G & \textit{if } i \geq 4k+2 \textit{ and } i \textit{ is even} \\ B & \textit{otherwise} \end{cases}$$

*It is not difficult to see that c contains no rainbow $AP(3)$ and:*

$$\min\{R_c(n), G_c(n), B_c(n)\} = k + 1 = (n+4)/6$$

*For $k = 3$ ($n = 20$) it looks like:*

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20

One can think on generalize Theorem 10.9, Theorem 10.8, or even Theorem 10.7 by increasing the number of colors and the length of a rainbow arithmetic progression.

Axenovich and Fon-Der-Flass [7] gave an example, for $k \geq 5$, that no matter how large the smallest color class is, there is a coloring with no rainbow $AP(k)$.

**Example 10.6 ([7])** *Let $n = 2mk$, $k \geq 5$. We subdivide $[1, n]$ into $k$ consecutive intervals of length $2m$ each, say $A_1, ..., A_k$ and let $t = \lfloor k/2 \rfloor$. Then,*

$$c(i) = \begin{cases} j-1 & \text{if } i \in A_j \text{ and } j \neq t, j \neq t+2 \\ t-1 & \text{if } i \in A_t \cup A_{t+2} \text{ and } i \text{ is even} \\ t+1 & \text{if } i \in A_t \cup A_{t+2} \text{ and } i \text{ is odd} \end{cases}$$

*It is not difficult to see that $c$ contains no rainbow $AP(k)$ and the size of each color class is $n/k$.*

*For $k = 4$, $m = 3$ ($n = 24$) it looks like:*

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24,

Concerning the case $k = 4$, it is not even known if for sufficiently large values of $n$, every equinumerous 4–coloring of $[1, 4n]$ contains a rainbow $AP(4)$. Sicherman (see [32]) found equinumerous colorings of $[1, n]$, for $n \leq 60$, without rainbow $AP(4)$. Axenovich and Fon-Der-Flass [7] provide a coloring of $[1, n]$, where $n = 10m + 1$, with the smallest color class of size $(n - 1)/5$ and no rainbow $AP(4)$.

Concerning equinumerous colorings of $\mathbb{N}$, the authors in [32] show examples with no rainbow $AP(k)$, for $k = 5$ and $k = 6$. By using a clever counting argument, C. Sándor (see [33]) generalize those examples and show the existence of equinumerous $k$–colorings of $\mathbb{N}$ with no rainbow $AP(k)$, for every $k \geq 10$. Hence the generalization of Theorem 10.8 is not true for $k = 5, 6$ and $k \geq 10$. If the number of colors is infinite, in the same paper [32] it was shown that there are colorings of $\mathbb{N}$ with infinitely many colors, with each color having positive density such that there is no rainbow $AP(3)$.

### 10.3.3 Rainbow arithmetic progressions in $\mathbb{Z}_n$ and $\mathbb{Z}_p$

An interesting corollary of Theorem 10.8, is the modular version, which states that if $\mathbb{Z}/n\mathbb{Z}$ is colored with 3 colors, in such a way that the size of every color class is greater than $n/6$, then there exist a rainbow $AP(3)$, i.e. a triple $x$,$y$ and $z$, each of different color satisfying $x + y \equiv 2z \pmod{n}$.

It turns out (by Example 10.4) that for $n$ divisible by 6 this condition is tight. However, for other values of $n$ it is possible to obtain weaker assumptions, as we will see later.

**Definition 10.2** *Let $m(n)$ be the largest integer $m$ for which there is a 3–coloring of $\mathbb{Z}/n\mathbb{Z}$, with no rainbow $AP(3)$'s, such that the cardinality of the smaller color class is $m$.*

For $n$ divisible by 6, $m(n) = n/6$. For general $n$, the exact value of $m(n)$ is not known. Jungić, Nešetřil and Radoičić [33] formulated the following Conjecture.

**Definition 10.3** *Let $\mathcal{P}_0$ be the set of primes for which 2 has either multiplicative order $p-1$, or multiplicative order $(p-1)/2$ with $(p-1)/2$ odd. Let $\mathcal{P}_1$ be the set of remaining primes.*

**Example 10.7** *Let us see where, $\mathcal{P}_0$ or $\mathcal{P}_1$, belong some small prime numbers. Observe that for $n = 3, 5, 11,$ and 13, the multiplicative order of 2 in $\mathbb{Z}/n\mathbb{Z}$ is $n-1$; while for $n = 7$, the multiplicative order of 2 in $\mathbb{Z}/n\mathbb{Z}$ is $(n-1)/2$ an odd number. Hence, the fist prime number which does not belong to $\mathcal{P}_0$ is 17:*

$3 \in \mathcal{P}_0$, *sice* $2^2 = 4 \equiv 1 \pmod 3$.

$5 \in \mathcal{P}_0$, *sice* $2^4 = 16 \equiv 1 \pmod 5$.

$7 \in \mathcal{P}_0$, *sice* $2^3 = 8 \equiv 1 \pmod 7$,

$11 \in \mathcal{P}_0$, *sice* $2^{10} = 1024 \equiv 1 \pmod{11}$.

$13 \in \mathcal{P}_0$, *sice* $2^{12} = 4096 \equiv 1 \pmod{13}$.

$17 \in \mathcal{P}_1$, *sice* $2^8 = 256 \equiv 1 \pmod{17}$.

**Conjecture 10.1 (Jungić, Nešetřil and Radoičić [33], 2005)** *Let $n$ be an integer which is not a power of 2. Let $p$ denote the smallest odd prime factor of $n$ in $\mathcal{P}_0$ and let $q$ be the smallest odd prime factor of $n$ in $\mathcal{P}_1$. Then*

$$m(n) = \lfloor \frac{n}{\min\{2p, q\}} \rfloor.$$

For $n = 2^m$ and primes in $\mathcal{P}_0$ it had been already proved in [32] that $m(n) = 0$. Actually these two cases characterize the possible values of $n$ for which every 3–coloring (with nonempty color classes) of $\mathbb{Z}/n\mathbb{Z}$, contains a rainbow $AP(3)$.

**Theorem 10.10 (Jungić, Licht, Mahdian, Nešetřil and Radoičić [32], 2003)** *For every integer $n$, there is a rainbow–free 3–coloring of $\mathbb{Z}/n\mathbb{Z}$ with non–empty color classes, if and only if $n$ does not satisfy any of the following conditions:*

*(a) $n$ is a power of 2.*

*(b) $n \in \mathcal{P}_0$.*

Hence, Theorem 10.10 states that $m(n) = 0$, if and only if, $n$ is a power of 2 or $n$ is a prime in $\mathcal{P}_0$.

**Example 10.8** *For $n = 3, 5, 7, 11$ and 13, every 3–coloring of $\mathbb{Z}/n\mathbb{Z}$ with non–empty color classes contains a rainbow $AP(3)$. For $n = 17$, next we show a 3–coloring of $\mathbb{Z}/n\mathbb{Z}$ with non–empty color classes that contains no rainbow $AP(3)'s$:*

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16$$

For general $n$ the authors of [32] proved the following bounds.

**Theorem 10.11 (Jungić, Licht, Mahdian, Nešetřil and Radoičić [32], 2003)** *Let $n$ be not a power of 2, $q$ be the smallest prime factor of $n$, and $r$ be the smallest odd prime factor of $n$, then:*

$$\lfloor \frac{n}{2r} \rfloor \leq m(n) \leq min(\frac{n}{6}, \frac{n}{q})$$

We shall observe that, concerning cyclic groups of prime order, it follows from Theorem 10.11 that:

$$m(p) \leq 1$$

That is, every 3–coloring of $\mathbb{Z}/p\mathbb{Z}$, with the cardinality of the smallest color class greater than one, contains a rainbow $AP(3)$. In other words, a 3–coloring of $\mathbb{Z}/p\mathbb{Z}$ which is rainbow–free, satisfies that the cardinality of the smallest color class is either one or zero. Actually, from Theorem 10.10, it follows that if $p \in \mathcal{P}_0$ then the size of the smallest color class is zero, and if $p \in \mathcal{P}_1$ then the size of the smallest color class is one.

## 10.4 Some tools from additive combinatorics

Next we give some tools from additive combinatorics that we will use in Chapter 11, to describe 3–colorings on abelian groups of odd order, which has no 3–term arithmetic progression with its members having pairwise distinct colors.

The key idea is to observe that, if $c$ is a 3–coloring of an abelian group of odd order $G$, such that for any pair of color classes $X, Y \in \{R, G, B\}$ we have $|X+Y| \geq |X|+|Y|+1$ then, since $n = |G|$ is odd, we have $|X+Y| > |G| - |2Z|$ and therefore the coloring has a rainbow $AP(3)$. Then, we shall use structural results which provide the structure of pairs of sets $(X, Y)$ with $|X+Y| \leq |X|+|Y|$.

First we need some more definitions. Let $G$ be a finite abelian group and $A, B \subseteq G$ two finite non-empty subsets.

**Definition 10.4** *The* ***sumset*** *of $A$ and $B$ is denoted by $A+B$ and defined as the set of all elements of the group, representable as a sum of an element of $A$ and an element of $B$:*

$$A + B = \{a + b : a \in A, b \in B\}.$$

**Definition 10.5** *The* ***period*** *of a subset $S \subseteq G$, denoted by $\pi(S)$, is the subgroup of $G$ defined by:*

$$\pi(S) = \{g \in G : S + g = S\}$$

Thus, $S$ is a union of cosets of $\pi(S)$, and $\pi(S)$ lies above any subgroup of $G$ such that $S$ is a union of cosets.

**Definition 10.6** *We say that a set $S$ is* ***periodic*** *if $\pi(S)$ is a nontrivial subgroup of $G$ (i.e. $\pi(S) \neq \{0\}$). We say that $S$ is* ***aperiodic*** *otherwise.*

We shall use the following result of Kneser.

**Theorem 10.12 (Kneser)** *Let $(A, B)$ be a pair of finite non-empty subsets of an abelian group $G$. Then, letting $H := \pi(A+B)$, we have:*

$$|A + B| \geq |A + H| + |B + H| - |H|$$

*Moreover, if* $|A+B| \leq |A|+|B|-1$ *then we have equality.*

The next Remarks are direct consequences of Kneser's theorem. If $H$ is a subgroup of $G$, the canonical homomorphism of $G$ onto the quotient group $G/H$ is denoted by $\varphi_{G/H}$, and the full inverse image under $\varphi_{G/H}$ of a subset $\overline{S} \subseteq G/H$ is denoted by $\varphi^{-1}_{G/H}(\overline{S})$.

**Remark 10.1** *If* $A$, $B$ *and* $H$ *are as in Kneser's theorem with* $|A+B| \leq |A|+|B|-1$ *, then there are non cosets of* $H$ *with elements of both subsets* $A$ *and* $B$.

*Proof.* From the conclusion of Kneser's theorem we obtain:

$$(|A+H|-|A|)+(|B+H|-|B|) = |H|+(|A+B|-|A|-|B|)$$

Since $(A, B)$ is a pair with small sumset, it follow that: $|H|+(|A+B|-|A|-|B|) < |H|$, then:

$$(|A+H|-|A|)+(|B+H|-|B|) < |H|$$

which means that $A$ and $B$ are obtained from $A+H = \varphi^{-1}_{G/H}(\overline{A})$ and $B+H = \varphi^{-1}_{G/H}(\overline{B})$ by removing less than $|H|$ elements totally. □

**Remark 10.2** *If* $A$, $B$ *and* $H$ *are as in Kneser's theorem* $|A+B| \leq |A|+|B|-1$, *writing* $\overline{A} = \varphi_{G/H}(A)$ *and* $\overline{B} = \varphi_{G/H}(B)$, *we have:* $|\overline{A}+\overline{B}| = |\overline{A}|+|\overline{B}|-1$.

*Proof.* Observe that $|A+H| = |\overline{A}||H|$, $|B+H| = |\overline{B}||H|$ and $|A+B| = |\overline{A}+\overline{B}||H|$, then the conclusion of Kneser's theorem takes the shape of the claim. □

Thus Knesers's theorem shows that any pair $(A, B)$ with small sumset, can be obtained by "lifting" a pair $(\overline{A}, \overline{B})$ of subsets of a quotient group $G/H$, with $|\overline{A}+\overline{B}| = |\overline{A}|+|\overline{B}|-1$. In other words, Knesers's theorem reduces the problem of classifying all pairs $(A, B)$ with small sumset, to that of describing those pairs for which: $|A+B| = |A|+|B|-1$.

The structure of pairs of sets $(X, Y)$ in an abelian group $G$ verifying $|X+Y| = |X|+|Y|-1$ is given by the Kemperman Structure Theorem (KST). We shall only use the following simplified version of the Theorem which can be easily deduced from Theorem 2 in [38].

**Definition 10.7** *A set* $S \subset G$ *is said to be* ***quasiperiodic*** *if it admits a decomposition* $S = S_0 \cup S_1$, *where each of* $S_0$ *and* $S_1$ *can be empty,* $S_1$ *is* $H$*–periodic (where* $H$ *is nontrivial) and* $S_0$ *is contained in a single coset of* $H$.

Note that every set $S \subset G$ is quasiperiodic with $S_1 = \emptyset$ and $H = G$.

**Theorem 10.13 (Kemperman [35])** *Let $A$ and $B$ be nonempty subsets of an abelian group $G$ verifying*

$$|A+B| = |A| + |B| - 1 \le |G| - 2.$$

*If $A+B$ is aperiodic then one of the following holds:*

(i) $\min\{|A|,|B|\} = 1$.

(ii) *Both $A$ and $B$ are arithmetic progressions with the same common difference.*

(iii) *Both $A$ and $B$ are $H$–quasiperiodic for some nontrivial proper subgroup $H < G$.*

We shall also use the following extension of KST, recently obtained by Grynkiewicz [25], which describes the structure of pairs of sets $(X, Y)$ in an abelian group $G$ verifying $|X + Y| = |X| + |Y|$. Again we only need a simplified version of the full result.

**Theorem 10.14 (Grynkiewicz [25])** *Let $A$ and $B$ be nonempty subsets of an abelian group $G$ of odd order $n$ verifying*

$$|A+B| = |A| + |B| \le |G| - 3.$$

*If $A+B$ is aperiodic then one of the following holds:*

(i) $\min\{|A|,|B|\} = 2$ *or* $|A| = |B| = 3$.

(ii) *Both $A$ and $B$ are $H$–quasiperiodic for some nontrivial proper subgroup $H < G$.*

(iii) *There are $a, b \in G$ such that $|A' + B'| = |A'| + |B'| - 1$ where $A' = A \cup \{a\}$ and $B' = B \cup \{b\}$.*

It is well known that if $A, B$ are subsets of a group $G$ and $|A| + |B| > |G|$ then $A+B = G$. We shall use this simple fact without further reference. The following lemma characterizes the structure of sets $A, B \subset G$ with $|A| + |B| = |G|$ and $A + B \neq G$. We include here a short proof for the benefit of the reader.

**Lemma 10.1** *Let $A, B$ be subsets of a finite abelian group $G$ such that*

$$|A| + |B| = |G|.$$

*If $A + B \neq G$ then there is a subgroup $H$ and $a \in G$ such that*

$$A + H = A,\ B + H = B \ \textit{and}\ A + B = G \setminus (a + H).$$

*Proof.* If $|A+B| = |G|-1$ then the statement holds with $H = \{0\}$. Suppose that $|A+B| \leq |G|-2$ and let $H = P(A+B)$. By Kneser's Theorem $|A+B| = |A+H| + |B+H| - |H| \geq |A|+|B|-|H| = |G|-|H|$. Since $A+B \neq G$ is $H$–periodic we must have $A+B = G \setminus (a+H)$ for some $a \in G$ and $A+H = A$ and $B+H = B$. □

One of the applications of Lemma 10.1 is the following result which will be often used in Chapter 11. Let $H$ be a proper subgroup of $G$. We say that a triple $(X,Y,Z)$ of $H$–cosets is in arithmetic progression if $(X/H)+(Y/H) = 2\cdot(Z/H)$. For $X$ an $H$–coset and $U$ a color class of a coloring we write $X_U := X \cap U$.

**Lemma 10.2 (The 3-cosets Lemma)** *Let $\{A,B,C\}$ be a rainbow–free 3–coloring of an abelian group $G$ with odd order $n$. Let $H < G$ be a subgroup of $G$ and let $(X,Y,Z)$ be a triple of $H$–cosets in arithmetic progression.*

*If each of $X_A$, $Y_B$ and $Z_C$ is non–empty, then*

$$\max\{|X_A|+|Y_B|, |X_A|+|Z_C|, |Z_C|+|Y_B|\} \leq |H|. \tag{10.1}$$

*Moreover, if equality holds then there is a proper subgroup $K < H$ such that two of the sets $X_A, Y_B, Z_C$ are $K$–periodic (the two involved in the equality holding) and the third one is contained in a single coset of $K$.*

*Proof.* Since the coloring is rainbow–free we have $X_A + Y_B \subseteq (2\cdot Z) \setminus (2\cdot Z_C)$. Hence $|X_A+Y_B| < |H|$ which implies $|X_A|+|Y_B| \leq |H|$. Similarly $X_A - (2\cdot Z_C) \subsetneq Y$ and $Y_B - (2\cdot Z_C) \subsetneq X$ imply $|X_A|+|Z_C| \leq |H|$ and $|Y_B|+|Z_C| \leq |H|$ respectively. This proves the first part of the statement.

Suppose that $|X_A|+|Y_B| = |H|$. By Lemma 10.1 there is a subgroup $K < H$ such that both $X_A$ and $Y_B$ are $K$–periodic and $(2\cdot Z)\setminus(X_A+Y_B)$ consists of a single $K$–coset, which contains $2\cdot Z_C$. A symmetric argument applies if $|X_A|+|Z_C| = |H|$ or $|Y_B|+|Z_C| = |H|$. □

# Chapter 11

# Rainbow–free three colorings in abelian groups

## 11.1 Introduction

Recall that a 3–*term arithmetic progression* $AP(3)$, is a triple $(x, y, z)$ with $x + y = 2z$. Given a 3–coloring $c$ of an abelian group $G$, we say that a 3–term arithmetic progression $(x, y, z)$ is *rainbow* under $c$, if the coloring assigns pairwise distinct colors to the elements. Throughout this chapter we will say that a coloring $c$ is *rainbow–free* if there are no rainbow $AP(3)$'s under $c$.

In this chapter we give a structural description of rainbow–free 3–colorings on finite abelian groups of odd order. Our main result, Theorem 11.1 below, identifies the three possible kinds of rainbow-free colorings of an abelian group of odd order $G$ which can be described as follows. There is a proper subgroup $H$ of $G$ such that either the coloring is obtained by lifting a rainbow–free coloring with a color class of size one from a quotient group $G/H$, or there is one coset of $H$ which is bichromatic and $G \setminus H$ is monochromatic, or a combination of the two possibilities above.

Observe that the property of being rainbow–free is invariant up to translation of the coloring:

**Remark 11.1** *If $c$ is a rainbow–free coloring of $G$ then $c'(x) := c(x + g)$ is also rainbow–free.*

We will often use this remark without explicit reference.

The main result can be precisely stated as follows.

**Theorem 11.1** *Let $G$ be a finite abelian group of odd order $n$ and let $c$ be a 3–coloring of $G$ with non–empty color classes $A, B, C$. Then $c$ is rainbow–free if and only if, up to translation, there is a proper subgroup $H < G$ and a color class, say $A$, such that:*

(i) $A \subseteq H$, *and the coloring induced in $H$ is rainbow–free,*

(ii) $\widetilde{B} + H = \widetilde{B}$ *and* $\widetilde{C} + H = \widetilde{C}$,

(iii) $\widetilde{B} = -\widetilde{B} = 2\widetilde{B}$ *and* $\widetilde{C} = -\widetilde{C} = 2\widetilde{C}$.

*where $\widetilde{B} = B \setminus H$ and $\widetilde{C} = C \setminus H$.*

The description of rainbow–free 3–colorings of abelian groups of odd order can be used to prove Conjecture 10.1. Actually the Conjecture holds for general abelian groups of odd order as shown in the next Corollary.

**Corollary 11.1** *Let $G$ be an abelian group of odd order $n$. Let $p$ denote the smallest prime factor of $n$ in $\mathcal{P}_0$ and let $q$ be the smallest prime factor of $n$ in $\mathcal{P}_1$. If $\{A, B, C\}$ is a rainbow–free 3–coloring of $G$ then*

$$\min\{|A|, |B|, |C|\} \leq \left\lfloor \frac{n}{\min\{2p, q\}} \right\rfloor. \tag{11.1}$$

*Moreover, there are rainbow–free 3–colorings of $G$ for which equality holds.*

*Proof.* We first prove (11.1). By Theorem 11.1 (i), one color class, say $A$, is contained in a subgroup $H < G$ and

$$|A| \leq |H| \leq \max\{\frac{n}{p}, \frac{n}{q}\}.$$

Thus we may assume that $|H| = \frac{n}{p}$ and $G/H$ is a cyclic group of prime order $p \in \mathcal{P}_0$, otherwise we are done. By Theorem 11.1 (ii), each of the two sets $\widetilde{B} = B \setminus H$ and $\widetilde{C} = C \setminus H$ is a (possibly empty) union of $H$–cosets.

Suppose that both sets $\widetilde{B}$ and $\widetilde{C}$ are nonempty. Since $p \in \mathcal{P}_0$ it follows from Theorem 10.10 that the 3–coloring of $G/H \simeq \mathbb{Z}/p\mathbb{Z}$ with color classes $A' = \{0\}$, $B' = \widetilde{B}/H$ and $C' = \widetilde{C}/H$ contains a rainbow $AP(3)$ which must be of the form $\{-x, 0, x\}$ for some $x \in G/H$. But this contradicts that $\widetilde{B} = -\widetilde{B}$ and $\widetilde{C} = -\widetilde{C}$, Theorem 11.1 (iii). Hence $G \setminus H$ is monochromatic and thus $H$ contains two colors. It follows that $\min\{|A|, |B|, |C|\} \leq \lfloor \frac{n}{2p} \rfloor$.

Let us show that there are rainbow–free 3–colorings of $G$ for which equality holds in (11.1).

If $2p \leq q$ then choose a subgroup $H < G$ with cardinality $\frac{n}{p}$, consider a partition $A \cup B = H$ where $|A| = \lfloor \frac{n}{2p} \rfloor$, and let $C = G \setminus H$. If $q < 2p$ then choose a subgroup $H < G$ with cardinality $\frac{n}{q}$ and let $B' = \{2, 2^2, \ldots, 2^b = 1\}$ in the cyclic group $G/H \simeq \mathbb{Z}/q\mathbb{Z}$. Since $q \notin \mathcal{P}_0$, the set $C' = G/H \setminus (B' \cup -B' \cup \{0\})$ is nonempty. Define the coloring $A = H$, $B = \pi^{-1}(B' \cup -B')$ and $C = \pi^{-1}(C' \cup -C')$, where $\pi : G \to G/H$ denotes the natural projection. In both cases the coloring satisfies parts (i), (ii) and (iii) of Theorem 11.1 and therefore it is rainbow–free. $\square$

We complete the proof Conjecture 10.1 in Section 11.5 where we show that Theorem 11.1 also implies the conjecture for cyclic groups of even order.

Note however that the conjecture does not hold for general abelian groups of even order as illustrated by the following counterexample.

**Example 11.1** *Let $G := H \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ where $|H|$ is not a power of 2. Consider the following 3–coloring of $G$: let the subgroup $H$ be colored by $A$, color one of the three remaining $H$–cosets of $G$ by $B$ and the remaining two by $C$.*

This coloring is rainbow–free, since a 3–term arithmetic progression in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = G/H$ is of the form $(x, x, y)$. However the smaller color class has cardinality $|H| = |G|/4$ which can be arbitrarily larger than $\min\{\frac{|G|}{2p}, \frac{|G|}{q}\}$ according to the choice of $H$.

The 'if' part of Theorem 11.1 can be easily checked as shown in Section 11.2. For the "only if" part we use the following observation. Let $\{A, B, C\}$ be a 3–coloring of an abelian group $G$ of odd order. If

$$|A + B| \geq |A| + |B| + 1$$

then (since $n = |G|$ is odd) we have $|A + B| > |G| - |2 \cdot C|$, where $A + B = \{a + b,\ a \in A,\ b \in B\}$ denotes the Minkowski sum of $A$ and $B$, and $2 \cdot C = \{2c,\ c \in C\}$. Therefore $A + B \cap 2 \cdot C \neq \emptyset$ and the coloring has a rainbow $AP(3)$. Thus we are only concerned with the situation where $|A + B| \leq |A| + |B|$. For this we use some results in Additive Combinatorics on the structure of sets with small sumset which are recalled in Section 10.4. It follows from Kneser's theorem that the case $|A+B| \leq |A|+|B|-1$ can be reduced to the case of equality. The Kemperman Structure Theorem [35] provides precise information on the structure of sets $A, B$ in an abelian group verifying $|A + B| = |A| + |B| - 1$. This deep structural result has been recently extended by Grynkiewicz [25] to handle the case where $|A + B| = |A| + |B|$. These two structural results are the main tools used to prove Theorem 11.1 in sections 11.2 ( small color classes), 11.3 (structured color classes) and 11.4 (proof of the result).

## 11.2 Small classes and arithmetic progresions

We start by characterizing the structure of a rainbow–free 3–coloring, of an abelian group when there is a color class with just one element.

**Proposition 11.1** *Let $G$ be a finite abelian group and let $c$ be a 3–coloring of $G$ with color classes $A, B, C$ such that $|A| = 1$. Then $c$ is rainbow–free if and only if, up to translation, every $X \in \{A, B, C\}$ satisfies:*

$$2 \cdot X \subset X = -X. \tag{11.2}$$

*Proof.* Suppose first that $c$ is rainbow–free. We may assume that $A = \{0\}$, thus $A$ satisfies (11.2); on the other hand, for every $x \in B$ we must have $-x, 2x \in B$ since otherwise we get a rainbow–free $AP(3)$ of the form $(-x, 0, x)$ or $(0, x, 2x)$. Hence $2 \cdot B \subset B = -B$ and similarly $2 \cdot C \subset C = -C$.

Reciprocally, if all color classes satisfy (11.2) then necessarily $A = \{0\}$ and, any 3–term arithmetic progression containing 0 has its remaining members in the same color class, thus $c$ is rainbow–free. □

Proposition 11.1 proves Theorem 11.1 when one of the colors has cardinality one. Moreover the Proposition also shows the 'if' part of the Theorem.

**Proposition 11.2** *Let $\{A, B, C\}$ be a coloring of an abelian group $G$. If there is a proper subgroup $H$ of $G$ and a color $A$ such that*

(i) *$A \subseteq H$, and the 3–coloring induced in $H$ is rainbow–free,*

(ii) *$\widetilde{B} + H = \widetilde{B}$ and $\widetilde{C} + H = \widetilde{C}$,*

(iii) *$\widetilde{B} = -\widetilde{B} = 2\widetilde{B}$ and $\widetilde{C} = -\widetilde{C} = 2\widetilde{C}$.*

*where $\widetilde{B} = B \setminus H$ and $\widetilde{C} = C \setminus H$, then the 3-coloring is rainbow–free.*

*Proof.* Choose the minimal $H$ for which conditions (i)-(iii) are verified. Then $H$ meets at most two colors, say $A \subset H$ and $B$.

If $C = G \setminus H$ then $A + B$ is contained in $H$, and thus it is disjoint from $2 \cdot C$. Moreover, each of $2 \cdot A$ and $2 \cdot B$ are contained in $H$ and thus disjoint from $A + C = B + C = C$.

Suppose that $C \neq G \setminus H$. Since a rainbow $AP(3)$ in $G$ can not be contained in $H$, it gives rise, by conditions (ii) and (iii), to a rainbow $AP(3)$ in $G/H$ with the coloring

$\{A/H, \widetilde{B}/H, \widetilde{C}/H\}$. However, every color $X$ in this last coloring verifies $X = -X = 2 \cdot X$ and it follows from Proposition 11.1 that it is rainbow–free. □

Observe that the two propositions above do not require that $G$ has odd order.

The 3-cosets Lemma and Proposition 11.1 can also be used to show the structure of a rainbow–free coloring given by the main result when one of the color classes is contained in a single coset.

**Lemma 11.1** *Let $G$ be a finite abelian group with odd order $n$ and let $\{A, B, C\}$ be a 3–coloring of $G$. If one of the three color classes is contained in a single coset of a proper subgroup $H < G$ then Theorem 11.1 holds.*

*In particular, if $B' = B \setminus H$ and $C' = C \setminus H$ are both nonempty, then the 3–coloring $\{A' = A + H, B', C'\}$ is also rainbow–free.*

*Proof.* The 'if' part follows from Proposition 11.2. Suppose that the given coloring is rainbow–free.

We may assume that $A \subset H$ and that $H$ is a minimal subgroup of $G$ which contains $A$.

Suppose that $Y \neq H$ is an $H$–coset which intersects the two remaining color classes $B$ and $C$. Let $Z$ be a third coset such that $(X = H, Y, Z)$ are in arithmetic progression. Since $|Y_B| + |Y_C| + |Z_B| + |Z_C| = 2|H|$, it follows from Lemma 10.2 that $|Y_B| + |Z_C| = |H|$ and $|Y_C| + |Z_B| = |H|$. Moreover, there is a subgroup $K \leq H$ such that $X_A = A$ is contained in a single coset of $K$ and each of $Y_B, Y_C, Z_B, Z_C$ is $K$–periodic. By the minimality of $H$ we have $K = H$ contradicting the existence of the bichromatic coset $Y$. Thus each of $B \setminus H$ and $C \setminus H$ is monochromatic. Then the restriction of the coloring to $H$ must be rainbow–free and, in order that there are no rainbow $AP(3)$ in $H, H + x, H - x$, we must have $X = -X = 2 \cdot X$ for $X = \widetilde{B}, \widetilde{C}$. Note that under these conditions the coloring $\{A', B', C'\}$ is also rainbow–free. □

We next show that the main result also holds if there is a class with two elements or there are two classes with three elements.

**Lemma 11.2** *Let $\{A, B, C\}$ be a coloring of an abelian group $G$ of odd order $n$. If $\min\{|A|, |B|, |C|\} = 2$ then Theorem 11.1 holds.*

*Proof.* By Proposition 11.2 and Lemma 11.1 we only have to show that, if the given coloring $c$ is rainbow–free, then one color is contained in a single coset of a proper subgroup of $G$.

We may assume that $A = \{0, a\}$. Let us show that $a$ generates a proper subgroup $H$ of $G$. Suppose the contrary so that $G = \langle a \rangle \cong \mathbb{Z}/n\mathbb{Z}$.

Since $\{-a, a, 3a\}$ can not be rainbow, we have $c(-a) = c(3a)$. Since $\{-3a, 0, 3a\}$ can not be rainbow we have $c(-3a) = c(3a)$. By iterating this argument, we have

$$c(-a) = c(3a) = c(-3a) = c(5a) = c(-5a) = ... = c((n-2)a) = c(-(n-2)a),$$

so that the color class of $-a$ has $n-2$ elements. But then the third one is empty, a contradiction.

Hence $A \subset H$ and, by Lemma 11.1, Theorem 11.1 holds. □

We next consider the case $|A| = |B| = 3$.

**Lemma 11.3** *Let $\{A, B, C\}$ be a coloring of an abelian group $G$ of odd order $n$. If $|A| = |B| = 3$ and $|A + B| = |A| + |B|$ then Theorem 11.1 holds.*

*Proof.* Let $A = \{0, a, a'\}$. Since $|A + B| = |A| + |B|$ it follows that $B = A + x = \{x, x+a, x+a'\}$ for some $x \in G$. Since $\{-x, 0, x\}$ can not be rainbow we have $-x \in A \cup B$. If $-x \in A$ then $0 \in A \cap B$, a contradiction. Thus $-x \in B$ and $a = -2x$. Since $\{0, x, 2x\}$ can not be rainbow we have $2x \in A \cup B$. If $2x \in B$ then $x \in A \cap B$ (since $A = B - x$). Thus $2x = a$ which implies $B = \{-x, x, 3x\}$ and $A = \{-2x, 0, 2x\}$ and both sets are arithmetic progressions with the same difference contradicting $|A + B| = |A| + |B|$. □

We next prove the case when two color classes are *almost* progressions. An *almost*–progression is an arithmetic progression with one point removed. Observe that, with this definition, the class of almost progressions contains the class of all arithmetic progressions except the ones whose length equals the order of the cyclic group generated by the difference.

**Lemma 11.4** *Let $\{A, B, C\}$ be a coloring of an abelian group $G$ of odd order $n$. Assume that $3 \le |A| \le |B| \le |C|$. If $A$ and $B$ are almost–progressions with the same difference $d$ then Theorem 11.1 holds.*

*Proof.* If $d$ generates a proper subgroup $H$ of $G$ then $A$ is contained in a single coset of $H$ and the result follows by Lemma 11.1.

Thus we may assume that $d$ generates the full group so that $G$ is the cyclic group $\mathbb{Z}/n\mathbb{Z}$ and we may assume $d = 1$. We will show that in this case $c$ contains a rainbow $AP(3)$.

Let $b$ be the minimum circular distance from elements in $A$ to elements in $B$.

If $b = 1$ we may assume that $n-1 \in A$ and $0 \in B$. Since $\max\{|A|,|B|\} \le (n-3)/2$ we have $(n-1)/2 \in C$ giving the rainbow $\{0,(n-1)/2,n-1\}$.

Suppose now that $b > 1$. Since $|A| \ge 3$ we may assume that $\{n-1,0\} \subset A$ and $\{1,2,\ldots,b'\} \subseteq C$ and $b'+1 \in B$ for some $b' \ge b$. If $b'$ is odd then $\{0,(b'+1)/2,b'+1\}$ is rainbow and if $b'$ is even then $\{n-1,(b'+2)/2,b'+1\}$ is rainbow. □

Moreover, if $c$ is rainbow–free then there is a subgroup $H$ and $g \in G$ such that, up to renaming the colors, both $B$ and $C \cup \{0\}$ are $H$–periodic and

$$B + C \supseteq G \setminus (g + H). \tag{11.3}$$

## 11.3 Periodic color classes

In this section we analyze the structure of the color classes when they are close to be periodic. The consideration of these cases arise from the discussion after Theorem 11.1 on sets with small sumset and the KST and Grynkiewicz theorems.

Throughout the section $G$ is an abelian group of odd order $n$ and $c$ is a rainbow–free 3–coloring of $G$ with non–empty color classes $(A,B,C)$. Recall that, for a subset $X \subset G$ and a subgroup $H < G$, we denote by $X/H$ the image of $X$ by the natural projection $\pi : G \to G/H$.

We start with a simple observation.

**Lemma 11.5** *If the three color classes are $K$–periodic for some subgroup $K < G$ then Theorem 11.1 holds in $G$ if and only if it holds in $G/K$.*

*Proof.* Since the three color classes are $K$–periodic, $c$ is rainbow–free if and only if the 3–coloring $\{A/K, B/K, C/K\}$ of $G/K$ is rainbow–free. Moreover, $A/K$ is contained in a single coset of a proper subgroup $H'$ of $G/K$ if and only if $A$ is contained in a single coset of the proper subgroup $H' + K$ in $G$. In view of Proposition 11.2 and Lemma 11.1 this suffices to show the equivalence. □

We next consider the case where two of the color classes are quasiperiodic.

**Lemma 11.6** *Suppose that $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$ are $H$–quasiperiodic decompositions of $A$ and $B$ with $H$ a nontrivial proper subgroup of $G$. If Theorem 11.1 holds in $G/H$ then it holds in $G$.*

*Proof.* By Lemma 11.1 we may assume that none of the color classes is contained in a single coset of a proper subgroup of $G$. By Lemma 11.5, we may also assume that at least two of the color classes are not periodic. Therefore, up to renaming the color classes we may assume that each of the sets $A_0, B_0, A_1$ and $B_1$ are nonempty, and that $|C/H| > 1$. We also assume that $0 \in A_0$.

Let us show that $A_0/H = B_0/H$. Suppose the contrary and let $Z$ be an $H$–coset such that $X = H, Y = B_0 + H$ and $Z$ are in arithmetic progression. Note that both $X$ and $Y$ intersect $C$ and that $Z$ is monochromatic. Thus, up to renaming the color classes, the conditions of Lemma 10.2 are satisfied and $|Z| + |A_0|, |Z| + |B_0| > |H|$, contradicting its conclusion.

Consider the 3–coloring $c_H$ of $G/H$ with color classes $(A', B', C')$ with $A' = A/H$, $B' = B_1/H$ and $C' = G/H \setminus (A' \cup B'))$. Note that $C' = (C \setminus H)/H$.

Observe that $c_H$ is rainbowfree, otherwise we have three $H$–cosets in $G$ in arithmetic progression where at least two of them are monochromatic (since both $B_1$ and $C \setminus H$ are $H$–periodic) and, by the 3–cosets Lemma 10.2, the original coloring is not rainbow–free. Since Theorem 11.1 holds in $G/H$, there is a subgroup $K < G$ containing $H$ such that, up to translation, one of the three chromatic classes of $c_H$ is contained in $K/H$ and the remaining two are $(K/H)$–periodic outside $K/H$.

Suppose that $A' \subset K/H$. Then $A \subset K$ and, by Lemma 11.1, Theorem 11.1 holds for $c$ and $G$ with $K < G$.

Let us show now that $C'$ can not be contained in a single coset of $K/H$ in $G/H$. Suppose the contrary, so that $C$ is contained in a single $K$–coset $X$ of $G$. Let $Z$ be a $K$–coset in arithmetic progression with $X$ and $Y = A_0 + K$. Since $Y$ intersects the two colors, $A$ and $B$, and $Z$ is necessarily monochromatic with color $A$ or $B$, we have $|Z| + |Y_A|, |Z| + |Y_B| > |K|$ contradicting Lemma 10.2.

Suppose now that $B'$ is contained in a single coset of $K/H$ in $G/H$. Consider the coloring $c_K$ of $G/K$ with color classes $(A'', B'', C'')$ with $A'' = A_1/K, B'' = B/K$ and $C'' = G/K \setminus (A'' \cup B'')$. Again $c_K$ is rainbow free since otherwise we have three $K$–cosets in $G$ in arithmetic progression where one of them is monochromatic (since $C''$ is $K$–periodic) and, by Lemma 10.2, the original coloring is not rainbow–free. Moreover, $|B/K| = 2$. It follows from Lemma 11.2 that Theorem 11.1 holds for $c_K$ and $G/K$ with some subgroup $L/K < G/K$. Thus Theorem 11.1 holds for $c$ and $G$ with $L < G$. □

We next consider the case where $A + B$ is $H$–periodic for some subgroup $H$ of $G$. Observe that, since $c$ is rainbow–free, we also have $H \neq G$.

**Lemma 11.7** *Suppose that $A + B$ is $H$–periodic for some proper nontrivial subgroup $H$ of $G$. If Theorem 11.1 holds in $G/H$ then it holds in $G$.*

*Proof.* We show that, under the assumption of the Lemma, both sets $A$ and $B$ admit an $H$–quasiperiodic decomposition and thus Theorem 11.1 follows from Lemma 11.6.

By the Theorem of Kneser we have

$$|A/H + B/H| \geq |A/H| + |B/H| - 1. \tag{11.4}$$

Since $A + B \cap 2 \cdot C = \emptyset$ we have

$$(A + B)/2 \subset G \setminus C = A \cup B, \tag{11.5}$$

where $X/2$ denotes the image of $X \subset G$ by the inverse of the automorphism of $G$ defined as $x \mapsto 2x$. This automorphism leaves all subgroups invariant so that $(A + B)/2$ is also $H$–periodic. Let

$$D = ((A \cup B) + H) \setminus (A + B)/2.$$

Note that the aperiodic parts of $A$ and of $B$ are contained in $D \cup (A \cap B)$. By (11.4) we have

$$|A/H|+|B/H|-|A/H\cap B/H| = |(A\cup B)/H| = |(A+B)/H|+|D/H| \geq |A/H|+|B/H|-1+|D/H|,$$

which implies

$$|D/H| + |A/H \cap B/H| \leq 1.$$

Hence each of $A$ and $B$ admits an $H$–quasiperiodic decomposition. □

Now we prove the case where two of the color classes are *almost* quasiperiodic. A set $X \subset G$ is *almost* $H$–quasiperiodic (resp. almost $H$–periodic) if there is $x \in G$ such that $X \cup \{x\}$ is $H$–quasiperiodic (resp. $H$–periodic).

**Lemma 11.8** *Suppose that $A, B$ are almost $H$–quasiperiodic for some proper nontrivial subgroup $H < G$. If Theorem 11.1 holds in $G/H$ then it holds in $G$.*

*Proof.* We say that a coset $X$ of a subgroup $H < G$ is punctured if all but one of its elements are in the same color class $U \in \{A, B, C\}$. We then say that $X$ is a punctured coset of color $U$.

Since $A$ and $B$ are almost $H$–quasiperiodic, they admit decompositions $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$ where each of $A_0$ and $B_0$ are subsets of some $H$–coset and each of $A_1$ and $B_1$ are almost periodic so that each of them contains at most one punctured coset.

We may assume that at least one of $A_1$ or $B_1$ contains a punctured coset and that $0 < |A_0|, |B_0| < |H|$ since otherwise $A$ and $B$ are quasiperiodic and the result follows from Lemma 11.6. We may also assume that none of $A, B$ and $C$ are periodic since otherwise at least one of $A + B$ or $A + C$ is periodic and the result follows from Lemma 11.7. Finally we may assume that $\min\{|A/H|, |B/H|, |C/H|\} > 1$ since otherwise the result follows from Lemma **??**.

We consider two cases:

*Case 1*: $A_0+H \neq B_0+H$. Let $Z$ be a coset in arithmetic progression with $X = A_0+H$ and $Y = B_0+H$.

We may assume that one of $X, Y$, say $X$, intersects $C$, since otherwise $X$ is the punctured coset of $B_1$ and $Y$ is the punctured coset of $A_1$, which implies that $C$ is periodic. In particular $X \cap B = \emptyset$. Moreover, whatever the colors present in $Z$, the conditions of Lemma 10.2 are satisfied and $Z$ can not be a full coset. Since all $H$–cosets different from $X$ and $Y$ are either monochromatic or punctured, $Z$ is a punctured coset. Moreover it can not be of color $C$ since $Z \cap A_0 = Z \cap B_0 = \emptyset$.

Suppose that $|Z_A| = |Z \cap A| = |H|-1$. Then, again by Lemma 10.2, $|Z_A|+|X_C| = |H|$, which implies $|X_C| = |X \cap C| = 1$ and $|Y_B| = |Y \cap B| = 1$. Thus both $X$ and $Z$ are punctured cosets of color $A$. Since $A$ can not contain more than two partially filled cosets, $Y$ is a punctured coset of color $C$. Finally, the other color in $Z$ must also be $C$ since $Z$ is not the coset containing $B_0$.

Since $|B/H| > 1$ there is a coset $Y' \notin \{X, Y, Z\}$ which intersects $B$. Moreover, $Y'$ is either a full coset or a punctured coset of $B$. Let $Z'$ be a third coset in arithmetic progression with $X$ and $Y'$. Whatever the colors present in $Z'$, the conditions of Lemma 10.2 are satisfied, so that both $Y'$ and $Z'$ must be punctured cosets. Thus $Z'$ must intersect $C$ (there are no punctured cosets with colors $A$ and $B$) and $|X_A|+|Y_B| > |H|$, contradicting Lemma 10.2.

Suppose now that $|Z_B| = |Z \cap B| = |H|-1$. If $Y \cap A \neq \emptyset$ then $Y$ is a punctured coset of color $A$ and $|Y_A|+|Z_B| > |H|$ contradicting Lemma 10.2. Otherwise $Y$ intersects $C$ and application of Lemma 10.2 implies $|Y_C| = |X_A| = 1$. Thus both $Y$ and $Z$ are punctured cosets of $B$ with second color $C$ and $X$ is a punctured coset of $C$ with second color $A$, the same structure as in the case above with colors $A$ and $B$ exchanged.

*Case 2*: $A_0 + H = B_0 + H$. We may assume that at least one of $A_1$ or $B_1$ contains a punctured coset which is not $X$, otherwise $A$ and $B$ are quasiperiodic and the results follows from Lemma 11.6. So let $Y$ be a punctured coset of color $A$ (observe that $Y_B = \emptyset$ since $B_0$ is contained in $X$, thus $|Y_C| = 1$). Let $Z$ be a coset in arithmetic progression with $X$ and $Y$.

We fist prove that $Z$ is not monochromatic. If $Z$ is monochromatic of color $B$ (resp. $C$ or $A$) then $|Z_B|+|Y_C| > |H|$ (resp. $|Z_C|+|Y_A| > |H|$ or $|Z_A|+|Y_C| > |H|$) and we get a contradiction by Lemma 10.2 since $X_A$ (resp. $X_B$) is not empty.

Thus $Z$ must be a punctured coset of color $B$ with $|Z_C| = 1$. Since $|Y_A| + |Z_B| > |H|$

then $X_C = \emptyset$. Since $|Y_A| + |Z_C| = |H|$ then $|B_0| = 1$, but also $|Y_C| + |Z_B| = |H|$ implies $|A_0| = 1$ which is a contradiction. □

## 11.4 Proof of Theorem 11.1

The proof of Theorem 11.1 now follows from the results in sections 11.2 and 11.3 together with the theorems of Kneser, Kemperman and Grynkiewicz. By Proposition 11.2 we only have to prove that if $c$ is a rainbow–free coloring then the color classes verify conditions (i)-(iii) with some proper subgroup $H < G$.

Let $c$ be a rainbow–free 3–coloring with non–empty color classes $\{A, B, C\}$ of the abelian group $G$ of odd order $n$. Recall that (since $|G| = n$ is odd) if $c$ is rainbow free, then for any pair of color classes $X, Y \in \{A, B, C\}$ we have $|X + Y| \leq |X| + |Y|$.

The proof is by induction on the number of (not necessarily distinct) primes dividing $n = |G|$. If $n$ is prime, the statement holds by Theorems 10.10 and 10.11, and by Proposition 11.1. Thus we may assume that the Theorem holds for any proper divisor of $n = |G|$.

By Lemma 11.7 we can assume that $A + B$ is aperiodic. It follows from Kneser's theorem that $|A + B| \geq |A| + |B| - 1$. We consider two cases.

*Case 1:* $|A + B| = |A| + |B| - 1$. It follows from the simplified version of the KST Theorem **??** that one of the following holds:

(i) $\min\{|A|, |B|\} = 1$. In this case the result follows by Proposition 11.1.

(ii) Both $A$ and $B$ are arithmetic progressions with the same common difference $d$. The result follows by Lemma 11.4.

(iii) Both $A$ and $B$ are $H$–quasiperiodic for some nontrivial proper subgroup $H < G$. The result follows by Lemma 11.6.

*Case 2:* $|A + B| = |A| + |B|$. It follows from the simplified version of Grynkiewicz's Theorem 10.14 that one of the following holds:

(i) $\min\{|A|, |B|\} = 2$ or $|A| = |B| = 3$. In this case the result follows by Lemmas 11.2 and 11.3 respectively.

(ii) Both $A$ and $B$ are $H$–quasiperiodic for some nontrivial proper subgroup $H < G$. The result follows by Lemma 11.8.

(iii) There are $a, b \in G$ such that $|A' + B'| = |A'| + |B'| - 1$ where $A' = A \cup \{a\}$ and $B' = B \cup \{b\}$. According to Kemperman's Theorem 10.13, either $A' + B'$ is periodic, in which case $A + B$ is also periodic and the result follows by Lemma 11.7, or $A'$, $B'$ are both quasiperiodic, in which case $A$ and $B$ are almost periodic and we can apply Lemma 11.8, or $A'$, $B'$ are both arithmetic progressions and then $A$ and $B$ are almost arithmetic progressions, a case handled in Lemma 11.4.

## 11.5 The even case

In this Section we shall prove Conjecture 10.1 for cyclic groups of even order. We start with another consequence of Proposition 11.1 which will be useful later on.

**Lemma 11.9** *Let $\{A, B, C\}$ be a rainbow–free 3–coloring of an abelian group. Suppose that there is a subgroup $H$ such that one of the colors, say $A$, is contained in $H$, and each of $\widetilde{B} = B \setminus H$ and $\widetilde{C} = C \setminus H$ are $H$-periodic.*

*There is a proper subgroup $K$ of $G$ containing $H$ such that*

$$B + C \supseteq G \setminus K$$

*and each of $B \setminus K$ and $C \setminus K$ is $K$–periodic.*

*Proof.* We consider two cases.

*Case 1*: $H = \{0\}$.

We have $A = \{0\}$. If $|B + C| = |B| + |C| = |G| - 1$ we can choose $H = \{0\}$.

Suppose that $|B + C| = |B| + |C| - 1$ and let $\{0, x\} = G \setminus (B + C)$. We have $\{0, x\} + B = \{0, x\} - B \subset G \setminus (C \cup \{0\})$. Hence $|\{0, x\} + B| \leq |B| + 1$. By decomposing $B$ into maximal arithmetic progressions with difference $x$ we see that $B$ is a union of cosets of the nontrivial subgroup $K = \langle x \rangle < G$ generated by $x$ and at most one (proper) arithmetic progression with difference $x$. Likewise $\{0, x\} + C \subset G \setminus (B \cup \{0\})$ implies the analogous structure for $C$. Since none of $B$ and $C$ contains the whole subgroup $K$, the only coset where the (proper) arithmetic progressions can sit in is $K$ itself. If both colors meet $K$ then we have a rainbow–free 3–coloring of this cyclic group with all three colors arithmetic progressions. But we can not partition $K \setminus \{0\}$ into two arithmetic progressions $B', C'$ with $B' = -B'$ and $C' = -C'$. Thus only one of the colors, say $C$, meets $K$. Hence the result holds with the subgroup $K$.

Finally suppose that $|B+C| < |B|+|C|-1$. By Kneser's theorem there is a subgroup $K$ such that $B+C$ is $K$–periodic and $|B+C| = |B+K|+|C+K|-|K|$. Since $0 \notin B+C$

we have $B+C \subset G\setminus K$. Hence it follows from $|G|-1 = |B|+|C| \le |B+K|+|C+K| \le |G|$ that one of the sets $B, C$ is $K$–periodic, the second one has one $K$–hole and $B+C = G\setminus K$.

*Case 2*: $H \neq \{0\}$.

Suppose first that $\widetilde{C} = G \setminus H$, so that $\widetilde{B} = \emptyset$ and $B \subset H$. Then $B + C = C$ and the statement holds with $K = H$.

Suppose now that both $\widetilde{B}$ and $\widetilde{C}$ are nonempty. Then $\{A' = A/H, B' = \widetilde{B}/H, C' = \widetilde{C}/H\}$ is a rainbow–free 3–coloring of $G/H$. It follows from Case 1 that there is a proper subgroup $K$ of $G$ containing $H$ such that $B' \setminus (K/H)$ and $C' \setminus (K/H)$ are $K$–periodic and $A' + B' \supseteq (G/H) \setminus (K/H)$. It follows that each of $B \setminus K$ and $C \setminus K$ are $K$–periodic (one of the two may be empty) and $B + C \supseteq G \setminus K$. □

In what follows we use

$$G = L \times \mathbb{Z}/2^m\mathbb{Z}$$

where $L$ has odd order and $m \ge 1$. As usual $\{A, B, C\}$ denotes a rainbow–free 3–coloring of $G$. Let $P_0 = 2 \cdot G$. Since the even factor of $G$ is cyclic we have

$$P_0 \cong L \times \mathbb{Z}/2^{m-1}\mathbb{Z}.$$

For each color $X \in \{A, B, C\}$ we write $X_0 = X \cap P_0$ and $X_1 = X \cap P_1$ where $P_1$ is the second coset of $P_0$ in $G$.

**Lemma 11.10** *None of the two cosets of $P_0$ is monochromatic.*

*Proof.* Suppose the contrary and choose the minimal $m$ for which there is a counterexample to the statement. We may assume that $P_1 = C_1$. Since $A+B \subset P_0 \setminus 2\cdot C_1 = P_0 \setminus 2\cdot P_1$ we have $m \ge 2$ (otherwise $2 \cdot P_1 = P_0$) and $A + B$ is contained in the proper subgroup $2 \cdot P_0$ of $P_0$. Thus $A \cup B$ is contained in one coset of $2 \cdot P_0$ and the second coset of this subgroup in $P_0$ must be colored only with $C$ contradicting the minimality of $m$. □

We next give the structural result analogous to Theorem 11.1 for cyclic groups of even order.

**Theorem 11.2** *Let $G = L \times \mathbb{Z}/2^m\mathbb{Z}$ where $L$ has odd order and $m \ge 0$. There is a proper subgroup $H'$ of $L$ such that one of the colors, say $A$, is contained in one coset of $H = H' \times \mathbb{Z}/2^m\mathbb{Z}$ and each of $B \setminus H$ and $C \setminus H$ is $H$–periodic.*

*Proof.* By Lemma 11.10 we may assume that none of the two cosets of $P_0$ is monochromatic. The proof is by induction on $m$. By Theorem 11.1 the result follows for $m = 0$. Assume $m \ge 1$. We consider three cases.

*Case 1*: One of the two cosets of $P_0$ is bichromatic.

We may assume that $P_1 = B_1 \cup C_1$ and $0 \in A_0 = A$. Thus $|B_1| + |C_1| = |P_0|$ and $B_1 + C_1 \subset P_0$. It follows from Lemma 10.1 and $2 \cdot A \subset P_0 \setminus (B_1 + C_1)$ that there is a proper subgroup $H_1$ of $P_0$ such that $2 \cdot A \subseteq H_1$, and that both $B_1$ and $C_1$ are $H_1$–periodic. Let $H_1 = H_1' \times \mathbb{Z}/2^{m_1}\mathbb{Z}$ where $H_1' < L$ has odd order. We next consider two cases according to $P_0$ being bichromatic or trichromatic.

*Case 1.1*: $P_0$ is bichromatic. We may assume that $P_0 = A_0 \cup B_0$. Since $|A_0| + |B_0| = |P_0|$ and $2 \cdot C_1 \subset P_0 \setminus (A_0 + B_0)$ it follows from Lemma 10.1 that there is a proper subgroup $H_0$ of $P_0$ such that $2 \cdot C_1$ is contained in a single coset of $H_0$, and that both $A_0$ and $B_0$ are $H_0$–periodic. Let $H_0 = H_0' \times \mathbb{Z}/2^{m_0}\mathbb{Z}$ with $H_0' < L$ of odd order.

Now $C_1$ being $H_1$–periodic and $2 \cdot C_1$ contained in a single coset of $H_0$ implies $H_1' \leq H_0'$. By the analogous argument on $A_0$ we get $H_0' \leq H_1'$. Thus $H_0' = H_1'$ and, by symmetry, we may assume $H_0 \leq H_1$. Therefore each color class is $H_0$–periodic.

It follows that $H_0'$ is a proper subgroup of $L$ since otherwise we get the rainbow–free 3–coloring $\{A/L, B/L, C/L\}$ of the cyclic group $G/L$ of order $2^m$, contradicting Theorem 10.10.

Now consider the subgroup $H = H_0' \times \mathbb{Z}/2^m\mathbb{Z}$. Observe that $H$ contains $A_0$ since $2 \cdot A_0 \subset H_1 \leq H$ and the two subgroups $H_1$ and $H$ have the same odd factor. Similarly, since $2 \cdot C_1 \subset H_0$ the color $C_1$ is also contained in $H$. Hence $B$ does not intersect $H$ since otherwise, as all color classes are $H_0'$–periodic, we would get the rainbow-free 3–coloring $\{(A_0 \cap H)/H_0', (B \cap H)/H_0', (C \cap H)/H_0')$ of the cyclic group $G'/H_0'$ of order $2^m$ contradicting again Theorem 10.10. Thus $B = G \setminus H$ and the statement of the Theorem holds with $H' = H_0'$.

*Case 1.2*: $P_0$ is trichromatic. By the induction hypothesis there is a subgroup $H_0 = H_0' \times \mathbb{Z}/2^{m-1}\mathbb{Z}$ such that $A \subseteq H_0$ and $B_0 \setminus H_0$ and $C_0 \setminus H_0$ are $H_0$–periodic. Choose a minimal $H_0'$ with this property. We may assume that $C_0 \setminus H_0$ is nonempty. Since $2 \cdot A \subset H_1' \times \mathbb{Z}/2^{m_1}\mathbb{Z}$ we have $A \subset H' \times \mathbb{Z}/2^{m-1}\mathbb{Z}$ with $H' = H_0' \cap H_1'$. By the minimality of $H_0'$ we have $H' = H_0' \leq H_1'$. Now if for some $x \in L \setminus H_0'$ the coset $X = H_0 + (2x, 0)$ is colored $B$ then $X \subset A_0 + B_0$ disjoint from $2 \cdot C$ implies that the coset $H_0 + (x, 1)$ is also monochromatic of color $B$. By switching the roles of $B$ and $C$ we conclude that $B_1 \setminus (H_0 + (0, 1))$ and $C_1 \setminus (H_0 + (0, 1))$ are also $H_0$–periodic. Let $H = H_0' \times \mathbb{Z}/2^m\mathbb{Z}$. If $C = G \setminus H$ the statement holds with $H$ and we are done. Otherwise the 3–coloring $\{A/H_0, (B \setminus H_0)/H_0, (C \setminus H_0)/H_0\}$ is rainbow–free with a color of cardinality one. By Proposition 11.1 every color $X$ satisfies $2 \cdot X \subset X = -X$. This implies that each of $B \setminus H$ and $C \setminus H$ are not only $H_0$–periodic but in fact $H$–periodic. This concludes this case.

*Case 2*: Both cosets of $P_0$ are trichromatic.

By the induction hypothesis there is a subgroup $H_0 = H_0' \times \mathbb{Z}/2^{m-1}\mathbb{Z}$ of $P_0$ such that $A_0$ is contained in a single coset $H_0$ and $B_0 \setminus H_0$ and $C_0 \setminus H_0$ are $H_0$–periodic. Choose a minimal $H_0'$ with this property.

It follows from Lemma 11.9 that there is a proper subgroup $T_0 = T_0' \times \mathbb{Z}/2^{m-1}\mathbb{Z} < P_0$ containing $H_0$ such that $B_0 + C_0 \supset P_0 \setminus T_0$ and each of $B_0 \setminus T_0$ and $C_0 \setminus T_0$ is $T_0$–periodic.

We have $2 \cdot A_1 \subset P_0 \setminus (B_0 + C_0)$ so that $2 \cdot A_1 \subset T_0$. It follows that $A_1 \subset H$ with $H = T_0' \times \mathbb{Z}/2^m\mathbb{Z}$. Thus $A \subset H$. We now use a similar argument to Case 1.2. For each $x \in L \setminus T_0$ the coset $X = T_0 + (2x, 0) \subset P_0 \setminus T_0$ is monochromatic and, since the coloring is rainbow–free so that $X = A_0 + X$ is disjoint from $T_0 + (2x, 1)$, the coset $T_0 + (x, 1)$ is also monochromatic. Hence each of $B_1 \setminus (T_0 + A_1)$ and $C_1 \setminus (T_0 + A_1)$ is also $T_0$–periodic. Hence, either $G \setminus H$ is monochromatic and we are done, or $\{A/H, (B \setminus H)/H, (C \setminus H)/H\}$ is a rainbow–free 3–coloring of $G/H$ with a color class of cardinality one. By Proposition 11.1 every color $X$ satisfies $2 \cdot X \subset X = -X$. This implies that each of $B \setminus H$ and $C \setminus H$ are not only $T_0$–periodic but in fact $H$–periodic. This completes the proof. □

Theorem 11.2 provides a proof of Conjecture 10.1 for cyclic groups of even order. The proof is completely analogous to the one in Corollary 11.1 for the case of abelian groups of odd order except that we invoke Theorem 11.2 instead of Theorem 11.1.

**Corollary 11.2** *Let $G$ be cyclic group of order $n$. Let $p$ denote the smallest odd prime factor of $n$ in $\mathcal{P}_0$ and let $q$ be the smallest odd prime factor of $n$ in $\mathcal{P}_1$. If $\{A, B, C\}$ is a rainbow–free 3–coloring of $G$ then*

$$\min\{|A|, |B|, |C|\} \leq \left\lfloor \frac{n}{\min\{2p, q\}} \right\rfloor . \tag{11.6}$$

*Moreover, there are rainbow–free 3–colorings of $G$ for which equality holds.*

# Chapter 12

# Colour patterns in orthogonal arrays

## 12.1 Introduction

Some of the phenomena described in past sections on the existence of color patterns in combinatorial structures behave in a particularly nice way when considered in finite groups. A simple example is the fact that the total number of monochromatic Schur triples in every 2–coloring of the group of integers modulo $n$ depends only on the cardinality of the color classes but not on the distribution of the colors, a fact first noticed, as far as we know, by Datskowsky [19] (see Section 10.2.2). The same is true for monochromatic 3–term arithmetic progressions when $n$ is relatively prime with 6, as noted by Croot [17].

In 2005, Cilleruelo, Cameron and Serra [15] gave a combinatorial counting argument which explains the above two results and provides the ground for the following generalizations:

1. Results like the above mentioned ones can be extended to general finite groups. Actually the universe to be colored needs not to be even a group, but simply an orthogonal array. However most of the natural applications seem to be more suited in the context of finite groups.
2. The monochromatic structures include Schur triples, arithmetic progressions, or solutions of more general equations in groups.
3. The counting argument can be applied to colorings with more than two colors and can also be used to study rainbow structures.

Of course there are limitations in such general results, which become less precise with the increasing complexity of the structures we consider.

We give in Section 12.2 a general formulation of the basic counting lemma which is based in a counting argument used in [15]. This basic lemma is stated in the context of orthogonal arrays and its nicest applications give a relationship between the number of monochromatic and rainbow structures depending only on the cardinality of the color classes. Section 12.3 collects some specific applications of the results in Section 12.2, including the study of monochromatic and rainbow Schur triples and arithmetic progressions in finite groups.

## 12.2 A counting argument

Let $X$ be a finite set with cardinality $n$ and let $S$ be a set of vectors in $X^d$. Let $c : X \to [1, r]$ be an $r$–coloring of $X$ with color classes $X_1, \dots, X_r$. A vector $\mathbf{x} = (x_1, x_2, \dots, x_d) \in S$ is *monochromatic* under $c$ if all its coordinates belong to the same color class. When there are either no two coordinates of the same color class or all colors are present, we say that the vector is *rainbow* under $c$. We denote by $M = M(S)$ and $R = R(S)$ the set of monochromatic and rainbow vectors in $S$ respectively.

A set $S$ of $d$-vectors with entries in $X$ is an *orthogonal array* of degree $d$ and strength $k$ if, for any choice of $k$ columns, each $k$-vector of $X^k$ appears in exactly one vector of $S$. In other words, if we specify any set of $k$ entries $a_1, \cdots, a_k$ and any set of subscripts $1 \le i_1 < i_2 < \cdots < i_k \le d$, we find exactly one vector $\mathbf{y} = (y_1, y_2, \dots, y_d)$ in $S$ with $y_{i_1} = a_1, y_{i_2} = a_2, \dots, y_{i_k} = a_k$. We denote by $OA(d, k)$ the family of orthogonal arrays of degree $d$ and strength $k$ on $X$.

Lemma 12.1 below is the basic tool we shall use. It is based on the counting arguments used in [15].

In what follows we use the following notation. The color classes of an $r$–coloring $c$ of $X$ will be denoted by $X_1, X_2, \dots, X_r$, and we denote $x_i = |X_i|/n$. For a vector $\mathbf{u} = (u_1, \dots, u_r)$ with nonnegative integer entries, we denote by $\|\mathbf{u}\| = \sum_i u_i$. The multinomial coefficient $\binom{d}{u_1, u_2, \dots, u_r, d-\|\mathbf{u}\|}$ will be written as $\binom{d}{\mathbf{u}}$. We use the convention $\binom{v}{u} = 0$ if $v < u$ and $\binom{0}{0} = 1$.

**Lemma 12.1** *Let $S$ be an orthogonal array $OA(d, k)$ on $X$ and let $c$ be an $r$–coloring of $X$.*

*For each vector $\mathbf{u} = (u_1, u_2, \dots, u_r)$ with $\|\mathbf{u}\| \le k$ the following equality holds:*

$$\binom{d}{\mathbf{u}} x_1^{u_1} \cdots x_r^{u_r} = \frac{1}{n^k} \sum_{\|\mathbf{v}\|=d} \binom{v_1}{u_1} \binom{v_2}{u_2} \cdots \binom{v_r}{u_r} s(\mathbf{v}), \tag{12.1}$$

*where the sum is extended to all vectors* $\mathbf{v} = (v_1, v_2, \ldots, v_r)$ *with nonnegative integer entries,* $\|\mathbf{v}\| = d$*, and* $s(\mathbf{v})$ *is the number of vectors in* $S$ *with* $v_i$ *coordinates in* $X_i$ *for each* $i = 1, \ldots, r$*.*

*Proof.* Given an ordered partition $V = (V_1, V_2, \ldots, V_r)$ of $[1, d]$, possibly with some empty parts, let us denote by $S(V)$ the set of vectors in $S$ whose entries in $V_i$ belong to $X_i$, $1 \le i \le r$. An $r$-tuple of subsets $(U_1, U_2, \ldots, U_r)$ of $[1, d]$ is of type $\mathbf{u} = (u_1, u_2, \ldots, u_r)$ if $|U_i| = u_i$, $1 \le i \le r$. Denote by $\mathcal{P}^r(\mathbf{u})$ the set of all $r$–tuples of pairwise disjoint subsets of $[1, d]$ of type $\mathbf{u}$. We say that $V$ dominates $U$ and write $V \succeq U$ if $V_i \supset U_i$, $1 \le i \le r$.

Since $S$ is an orthogonal array $OA(d, k)$, there are $k - \|\mathbf{u}\|$ vectors in $S$ which meet a prescribed assignment of $\|\mathbf{u}\|$ coordinates. Hence, for each $r$–tuple of subsets $(U_1, U_2, \ldots U_r)$ in $\mathcal{P}^r(\mathbf{u})$ there are $|X_1|^{u_1}|X_2|^{u_2}\cdots|X_r|^{i_r}|X|^{k-\|\mathbf{u}|}$ vectors in $S$ whose entries in $U_i$ belong to $X_i$, $1 \le i \le r$. Among these vectors we find all vectors in $S(V)$ for each partition $V$ which dominates $U$, that is,

$$\sum_{V \succeq U} S(V) = |X_1|^{u_1}|X_2|^{u_2}\cdots|X_r|^{u_r}|X|^{k-\|\mathbf{u}\|}.$$

Each partition $V$ dominates $\binom{|V_1|}{u_1}\binom{|V_2|}{u_2}\cdots\binom{|V_r|}{u_r}$ $r$–tuples in $\mathcal{P}^r(\mathbf{u})$. Summing up through all $r$–tuples in $\mathcal{P}^r(\mathbf{u})$ we get vectors counted by $s(\mathbf{v})$ for each $\mathbf{v}$ which dominates componentwise the vector $\mathbf{u}$:

$$\begin{aligned}
\binom{d}{\mathbf{u}}|X_1|^{u_1}|X_2|^{u_2}\cdots|X_r|^{u_r}|X|^{k-\|\mathbf{u}|} &= \sum_{U\in\mathcal{P}^r(\mathbf{u})}\sum_{V\succeq U} S(V) \\
&= \sum_{V\succeq U}\sum_{U\in\mathcal{P}^r(\mathbf{u})} S(V) \\
&= \sum_{\|\mathbf{v}\|=d}\binom{v_1}{u_1}\binom{v_2}{u_2}\cdots\binom{v_r}{u_r}\sum_{V\in\mathcal{P}^r(\mathbf{v})} S(V) \\
&= \sum_{\|\mathbf{v}\|=d}\binom{v_1}{u_1}\binom{v_2}{u_2}\cdots\binom{v_r}{u_r} s(\mathbf{v}),
\end{aligned}$$

as claimed. □

Lemma 12.1 gives a relationship between the number of vectors with some specific color patterns and the cardinalities of the color classes. For 2–colorings and orthogonal arrays $OA(d, k)$, Lemma 12.1 gives the following result obtained in [15].

**Theorem 12.1 ([15])** *Let* $S$ *be an orthogonal array* $OA(d, k)$ *on* $X$*. For any* 2*–coloring of* $X$ *we have*

$$\sum_{j=0}^{k}(-1)^j\binom{d}{j}x_1^j = \frac{1}{n^d}\left(s(0,d) + (-1)^k\sum_{j=1}^{d}\binom{j-1}{k}s(i, d-i)\right).$$

□

For orthogonal arrays $OA(d, d-1)$ with arbitrary $d \geq 3$ we get the following relation.

**Theorem 12.2** *Let $S$ be an orthogonal array $OA(d, d-1)$ on $X$. For each $r$–coloring of $X$ we have*

$$\sum_{i=1}^{r} \left((1-x_i)^d - (-1)^d x_i^d\right) = \frac{1}{n^{d-1}} \left( \sum_{i=1}^{r} |S_i| + (-1)^{d-1}|M| \right),$$

*where $S_i$ denotes the set of vectors in $S$ which miss color $i$.*

*Proof.* Consider the alternating sum of the equations (12.1) for vectors of type $(i, 0, \ldots, 0)$; we have

$$\begin{aligned}
\sum_{i=0}^{d-1}(-1)^i \binom{d}{i} |X_1|^i |X|^{d-1-i} &= \sum_{i=0}^{d-1}(-1)^i \sum_{||\mathbf{v}||=d} \binom{v_1}{i} s(\mathbf{v}) \\
&= \sum_{||\mathbf{v}||=d} \left( \sum_{i=0}^{d-1}(-1)^i \binom{v_1}{i} \right) s(\mathbf{v}) \\
&= \sum_{||\mathbf{v}||=d, v_1=0} s(\mathbf{v}) + (-1)^{d+1} s(d, 0, \ldots, 0) \\
&= |S_1| + (-1)^{d+1}|M_1|,
\end{aligned}$$

where $S_1$ denotes the set of vectors which miss color 1 and $M_1$ denotes the set of vectors with all entries of color 1.

By symmetry we get

$$\sum_{i=0}^{d-1}(-1)^i \binom{d}{i} |X_j|^i |X|^{d-1-i} = |S_j| + (-1)^{d+1}|M_j|, \ 1 \leq j \leq r, \tag{12.2}$$

where $S_j$ denotes the set of vectors which miss color $j$ and $M_j$ denotes the set of vectors with all entries of color $j$. We get the result by adding up all the equations in (12.2). □

In particular we get the following Corollary for 3–colorings, which is a slight generalization of a result by Balandraud [8, Corollary 2]. Here a vector is said to be *rainbow* if all colors are present.

**Corollary 12.1** *Let $S$ be an orthogonal array $OA(d, d-1)$ on $X$. For each 3–coloring of $X$ we have*

$$\sum_{i=1}^{3}((1-x_i)^d - (-1)^d x_i^d) - 1 = \frac{1}{n^{d-1}}((1 + (-1)^{d-1})|M| - |R|).$$

*Proof.* With our current notion of rainbow vectors we have

$$|R| = |\cap_{j=1}^{3} \bar{S}_j| = |S| - \sum_{i=1}^{3} |S_i| + |M|.$$

By Theorem 12.2 we have

$$\begin{aligned}\sum_{k=1}^{3}\sum_{i=0}^{d-1}(-1)^i\binom{d}{i}|X_k|^i|X|^{d-1-i} &= \sum_{i=1}^{3}|S_i| + (-1)^{d+1}|M| \\ &= |X|^{d-1} - |R| + (1+(-1)^{d+1})|M|,\end{aligned}$$

as claimed. □

## 12.3 Colour patterns in $OA(3,2)$

Natural extensions of results in Arithmetic Ramsey Theory concern the study of color patterns of structures in groups. The results in Section 12.2 can be directly applied to this setting. The set of Schur triples in a finite group $G$ form an orthogonal array $OA(3,2)$. The same is true for 3–term arithmetic progressions if the order of $G$ is relatively prime with 6. For orthogonal arrays $O(3,2)$ we get a nice relationship between monochromatic and rainbow vectors.

**Theorem 12.3** *Let $S$ be an orthogonal array $OA(3,2)$ on $X$. For any $r$–coloring of $X$ we have*

$$2|M| - |R| = 3\sum_{i=1}^{r}|X_i|^2 - |X|^2.$$

□

*Proof.* By taking $\mathbf{u} = (0,0,0)$ in Lemma 12.1 we get

$$|X|^2 = \sum_{||v||=3} s(\mathbf{v}) = |M| + |R| + |T(2,1)|, \tag{12.3}$$

where $T(2,1) = S \setminus \{M \cup R\}$ denotes the set of vectors in $S$ with exactly two entries of the same colour.

On the other hand, the choice of $\mathbf{u} = (2,0,0)$ in Lemma 12.1 gives

$$3|X_1|^2 = 3s(3,0,0) + s(2,1,0) + s(2,0,1).$$

Adding up similar countings with $(0,2,0)$ and $(0,0,2)$, we have

$$3\sum_{i=1}^{r}|X_i|^2 = 3|M| + |T(2,1)|. \tag{12.4}$$

The result follows by substracting (12.3) from (12.4). □

For 2–colourings we do not have rainbow triples, so that Theorem 12.3 gives a formula for the total number of monochromatic triples in terms of the cardinalities of the color classes. By minimized that formula (with each color class of size $n/2$) we get the minimum number of monochromatic triples in an orthogonal array $OA(3,2)$ for any 2-coloring of $X$. More precisely, we have the next Corolary which is a natural generalization of Corollary 3.1 in [15],

**Corollary 12.2** *Let $S$ be an orthogonal array $OA(3,2)$ on $X$. For any 2-coloring of $X$ we have*

$$|M| = |X_1|^2 - |X_1|\cdot|X_2| + |X_2|^2.$$

*In particular, there are at least $n^2/4$ monochromatic triples in $S$ for any 2–coloring of $X$.*

In the case of 3–colourings Theorem 12.3 has a nice interpretation. Let us call $\sigma_c^2 = \sum_{i=1}^r x_i^2/r - (\sum_{i=1}^r x_i/r)^2$ the *variance* of an $r$–coloring $c$. Then Theorem 12.3 states the following Corollary.

**Corollary 12.3** *Let $S$ be an orthogonal array $OA(3,2)$ on $X$. For any 3-coloring of $X$ we have*

$$2|M| - |R| = 9n^2\sigma_c^2. \tag{12.5}$$

### 12.3.1 Schur triples in finite groups

In this Section we consider some applications for Schur triples in finite groups.

Let $G$ be a finite group of order $n$. We denote by $p(G)$ the smallest prime dividing the order of $G$. A Schur triple in $G$ is a triple of elements $(x,y,z)$ with $xy = z$. The set of Schur triples forms an orthogonal array $OA(3,2)$.

Note that for every coloring of a group $G$ with identity 1 there is always a trivial monochromatic Schur triple, namely $(1,1,1)$. If the color class containing the group identity is not a singleton we also have the trivial Schur triples $(1,x,x)$ for $x$ in the color class of 1. Thus, in dealing with monochromatic Schur triples it makes sense to consider colorings of $G^*$, the nonidentity elements of the group. Note that the original proof of

Schur works fine for finite groups, giving that any $r$–colouring of $G^*$ has a (nontrivial) monochromatic Schur triple provided that the order of the group verifies $|G| \geq er!$. We include here this translation just for completeness.

**Theorem 12.4 (Schur)** *Let $r$ be a positive integer. For every $r$–partition of $G^* = G\backslash\{1\}$ where $G$ is a group of order $|G| \geq er!$ there is a monochromatic Schur triple.*

*Proof.* Let $\{A_1, \dots A_r, \{1\}\}$ a given partition with no nontrivial monochromatic Schur triples. This in particular implies that $A_i g^{-1} \cap A_i \emptyset$ for each $g \in A_i$.

We may assume that $A_1$ is the largest color class, so that $|G| \leq r|A_1| + 1$. For $g_1 \in A_1$ we have $A_1 g_1^{-1} \subset A_2 \cup \dots \cup A_r \cup \{1\}$.

Let $A_2$ be the largest color class in $A_1 g_1^{-1}$, so that $|A_1| \leq (r-1)|A_2| + 1$. Let $g_2 \in A_2$ and note that $A_2 g_2^{-1} \subset A_1 g_1^{-1}$ because $g_2 = a g_1^{-1}$ for some $a \in A_1$. Therefore we have $A_2 g_2^{-1} \subset A_3 \cup \dots \cup A_r \cup \{1\}$. Let $A_3$ be the largest color class in $A_2 g_2^{-1}$ so that $|A_2| \leq (r-2)|A_3| + 1$. Again, for $g_3 \in A_3$ we have $A_3 g_3^{-1} \subset A_2 g_2^{-1} \subset A_1 g_1^{-1}$; hence $A_3 g_3^{-1} \subset A_4 \cup \dots \cup A_r \cup \{1\}$.

By continuing in this way we reach a step $i$ where $|A_{i+1} \cap A_i g_i^{-1}| \leq 1$ (eventually $i = r$.) Hence,

$$|G| \leq r|A_1| + 1 \leq r(r-1)|A_2| + r \leq \dots \leq \sum_{i=0}^{r-1} \frac{r!}{i!} \leq er!.$$

□

Since there are $2|X_i|$ monochromatic Schur triples for $X_i$ the color class containing the identity of the group, we immediately get the following results concerning rainbow Schur–triples.

**Corollary 12.4** *Let $c$ be a 3–coloring of $G$. If $\sigma_c^2 \leq \frac{4\alpha}{9n}$, where $\alpha = \min\{x_1, x_2, x_3\}$ then there is a rainbow Schur–triple under $c$.*

*Proof.* Let $X_i$ be the color class containing the identity of the group, then $2|X_i| \leq |M|$. Since $\alpha \leq x_i$ then $n\alpha \leq |X_i|$ and thus $2n\alpha \leq |M|$. Combining this inequality with equation 12.5 we get,

$$4n\alpha - 9n^2\sigma_c^2 \leq |R|. \tag{12.6}$$

Thus, $|R|$ is positive if $\sigma_c^2 \leq \frac{4\alpha}{9n}$ as claimed. □

**Corollary 12.5** *Let $c$ be an equinumerous 3–coloring of a group $G$ of order $n \equiv 0 \pmod 3$. There is a linear number of rainbow Schur triples under $c$.*

*Proof.* Since the variance of an equinumerous coloring $c$ is $\sigma_c^2 = 0$, from equation 12.6 it follows that $\frac{4}{3}n \leq |R|$. □

Alekseev and Sachev [3] proved that every equinumerous 3-coloring of the integers in $[1, 3n]$ contains a rainbow Schur-triple. The result was later improved by Schönheim [57] who proved that any 3-coloring of the integers in $[1, N]$ such that the smallest color class has more than $N/4$ elements contains a rainbow Schur triple, and this lower bound is best possible. The following example shows that for groups there are also 3–colorings with no rainbow Schur triples such that the smaller color class has cardinality $n/4$:

**Example 12.1** *Let $K < H < G$ be two subgroups of $G$ such that $K$ has index two in $H$ and $H$ has index two in $G$; give color 1 to the elements in $K$, color 2 to the elements in $H \setminus K$ and color 3 the remaining elements of the group. In this example $X_1X_2 = X_2$ and $X_1X_3 = X_2X_3 = X_3$.*

However it is not clear to us that the lower bound $n/4$ for the size of the smaller color class is tight in the case of 3-colorings of groups with no rainbow Schur triples.

It has been shown by several authors [19, 56, 54] that the minimum number of monochromatic Schur triples in a 2–colouring of the integers in $[1, N]$ is $N^2/22 + O(N)$. We just note here that this result can also be obtained from Theorem 12.3 by considering a 3–coloring of the group $\mathbb{Z}/2N\mathbb{Z}$ with chromatic classes $X_1, X_2, X_3$ where $X_3 = [N+1, 2N]$ and $\{X_1, X_2\}$ are the color classes of the given 2–colouring of $[1, N]$.

### 12.3.2 Arithmetic progressions in finite groups

In this Section we consider some applications for arithmetic progressions in finite groups.

A $d$-term arithmetic progression in a group $G$ with $p(G) \geq d$ is a set of the form $\{a, ax, ax^2, \ldots, ax^{d-1}\}$ where $a, x \in G$. The progression is said to be *degenerate* if $x = 1$, the identity of the group. The set of $d$–term arithmetic progressions forms an orthogonal array $OA(d, 2)$. For $d = 3$ we find the nicest applications of the results in Section 12.2. When $G$ is an abelian group the set of 3–term arithmetic progressions correspond to solutions of the equation $x + y = 2z$.

For 3–colorings the following lower bound on the number of monochromatic 3–term arithmetic progressions was given in [15].

**Corollary 12.6 ([15], Corollary 4.3)** *Every 3–coloring of a group $G$ with $p(G) > 17$ has at least $(n^2 + 15n + 32)/48$ monochormatic 3–term arithmetic progressions.*

The authors of [32] prove that, for cyclic groups $G = \mathbb{Z}/n\mathbb{Z}$, every 3–coloring such that the smaller color class has more than $\min\{n/6, n/p(G)\}$ contains a rainbow 3–term arithmetic progression. They also prove that the lower bound on the size of the smaller color class can be decreased to 4 when $n$ is a prime. For general groups the following analogous of Corollaries 12.4 and 12.5 can be derived from Theorem 12.3 and Corollary 12.6.

**Corollary 12.7** *Let $c$ be a 3–coloring of a group $G$ with $p(G) > 17$. If $\alpha = \min\{x_1, x_2, x_3\} \geq 0.26$ then there is a rainbow 3–term arithmetic progression under $c$.*

*Proof.* For a group $G$ with $p(G) > 17$, Corollary 12.6 states $(n^2 + 15n + 32)/48 \leq |M|$. Combining this inequality with Theorem 12.3 we obtain,

$$(25n^2 + 15n + 32)/24 - 3(|X_1|^2 + |X_2|^2 + |X_3|^2) \leq |R|. \tag{12.7}$$

Recall $x_i = |X_i|/n$. Since $\alpha = \min\{x_1, x_2, x_3\}$ then,

$$|X_1|^2 + |X_2|^2 + |X_3|^2 \leq (\alpha n)^2 + (\alpha n)^2 + ((1 - 2\alpha)n)^2. \tag{12.8}$$

From Equations 12.7 and 12.8 it follows that

$$(25n^2 + 15n + 32)/24 - 3(2\alpha^2 + (1 - 2\alpha)^2)n^2 \leq |R|.$$

Thus,

$$\frac{(-432\alpha^2 + 288\alpha - 47)n^2 + 15n + 32}{24} \leq |R|.$$

The coefficient $(-432\alpha^2 + 288\alpha - 47)$ of $n^2$ is positive for $0.26 \leq \alpha \leq 1/3$. Then there are rainbow 3–term arithmetic progression under $c$ if $0.26 \leq \alpha$. □

In our present context, let us say that a 3-coloring is *almost* equinumerous if $x_i = 1/3 + O(1/\sqrt{n})$.

**Corollary 12.8** *Let $c$ be an almost equinumerous 3–coloring of a group $G$ with $p(G) > 17$. Then the number of rainbow 3–term arithmetic progression under $c$ is at least $n^2/24 + O(n)$.*

*Proof.* Since $x_i = 1/3 + O(1/\sqrt{n})$ for every $i$, then

$$|X_i|^2 + |X_2|^2 + |X_3|^2 = 3(1/9 + O(1/n))n^2.$$

Combining this equality with Equation 12.7 we obtain,

$$(25n^2 + 15n + 32)/24 - n^2 + O(n) \leq |R|.$$

Thus, $n^2/24 + O(n) \leq |R|$ as claimed. □

By increasing the number of colors we clearly get rainbow 3–term arithmetic progressions more easily. Another simple application of Theorem 12.3 gives the following Corollary.

**Corollary 12.9** *Every almost equinumerous $r$–coloring, $r > 3$, of a group $G$ of odd order has at least $(1 - 3/r)n^2 + O(n)$ rainbow 3–term arithmetic progressions.*

*Proof.* Since $x_i = 1/r + O(1/\sqrt{n})$ for every $i$, then

$$\sum_{i=1}^{r} |X_i|^2 = r(1/r^2 + O(1/n))n^2.$$

Using this Equality Theorem 12.3 states,

$$2|M| - |R| = 3rn^2(1/r^2 + O(1/n)) - n^2.$$

Thus $(1 - 3/r)n^2 + O(n) \leq |R|$ as claimed. □

By using Theorem 12.3 we can estimate the minimum number of monochromatic 3–term arithmetic progressions in a 2–colouring of the integers $[1, N]$. Consider a 3–coloring of the group $\mathbb{Z}/(2N+1)\mathbb{Z}$ with chromatic classes $X_1, X_2, X_3$ where $X_3 = [N+1, 2N+1]$ and $\{X_1, X_2\}$ are the color classes of the given 2–colouring of $[1, N]$. Denote by $M_i$ the set of monochromatic 3–term arithmetic progressions in $X_i$. Then $|M_1| + |M_2|$ is also the number of monochromatic 3–term arithmetic progressions taken in $[1, N]$.

## 12.4 Sidon equation

The *Sidon* equation, $xy = zt$, has also received much attention. The set of solutions of the Sidon equation forms an orthogonal array $OA(4,3)$. Corollary 12.1 gives the following result.

**Corollary 12.10** *For any 3-coloring of a group $G$, the number of rainbow solutions to the Sidon equation is*

$$|R| = n^3 \left(1 - \sum_{i=1}^{3}((1-x_i)^4 - x_i^4)\right).$$

□

The above results illustrate the application of the counting arguments in Section 12.2 to obtain Ramsey and anti–Ramsey results in colorings of finite groups. These results can be extended to linear equations in abelian groups or, more generally, to equations of the form $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d} = g$, where $\alpha_1, \alpha_2, \ldots, \alpha_d$ are permutations of the elements $G$ and $g \in G$.

# Bibliography


[1] S.D. Adhikari, Aspects of Combinatorics and Combinatorial Number Theory Alpha Sc. International Ltd., 2002

[2] V. Albiero, E. Sopena. Une classification des systèmes d'estampillage berné séquentiels. Reserch report. Bordeaux I University (1995).

[3] V. E. Alekseev and S. Savchev. Problem M. 1040. Kvant, (1987) 4:23.

[4] N. Alon, T.H. Marshall. Homomorphisms of edge-colored graphs and coxeter groups, *J.Algebraic Combin.* **8** (1998),5-13.

[5] K. Appel, W. Haken. Every planar map is for colorable. I. Discharging, *Illinois J. Math.* **21(3)** (1977),429-490.

[6] K. Appel, W. Haken, J. Koch. Every planar map is for colorable. II. Reducibility, *Illinois J. Math.* **21(3)** (1977),491-567.

[7] M. Axenovich, D. Fon-Der-Flaass. *On rainbow arithmetic progressions.* Electronic Journal of Conbinatorics, 11:R1, 2004

[8] E. Balandraud, Coloured Solutions of Equations in Finite Groups, preprint.

[9] B. Bollobs. Extremal graph theory, *London New York Academic Press* 1978.

[10] O.V. Borodin. On acyclic colorings of planar graphs, *Discrete Math.* **25** (1979),211-236.

[11] O. V. Borodin, A. O. Ivanova. An oriented 7–coloring of planar graphs with girth at least 7, *Sib. Electron. Math. Reports* **2** (2005),222-229.

[12] O. V. Borodin, A. O. Ivanova. An oriented coloring of planar graphs with girth at least 4, *Sib. Electron. Math. Reports* **2** (2005),239-249.

[13] O. V. Borodin, A. V. Kostochka, J. Nešetřil, A. Raspaud, E. Sopena. On universal graphs for planar oriented graphs of a given girth, *Discrete Math.* **188** (1998),73–85.

[14] O. V. Borodin, A. V. Kostochka, J. Nešetřil, A. Raspaud, E. Sopena. On the maximum average degree and the oriented chromatic number of a graph, *Discrete Math.* **206** (1999),77–89.

[15] J. Cilleruelo, P. J. Cameron, O. Serra, On monochromatic solutions of equations in groups, *Revista Iberoamericana de Matemàticas* (2005)

[16] B. Courcelle. The monadic second order-logic of graphs. VI: On several representations of graphs by relational structures, *Discret Appl. Math.* **54** (1994)117-149.

[17] E. Croot, The Minimal Number of Three-Term Arithmetic Progressions Modulo a Prime Converges to a Limit, ...

[18] Some Problrms in Additive Combinatorics, AIM ARCC Workshop, compiled by E. Croot and S. Lev.

[19] B.A. Datskovsky, On the number of monochromatic Schur triples, *Advances in Applied Mathematics* **31** (2003) 193–198.

[20] K. Edwars and C. J. H. McDiarmid. The complexity of harmonious colorings for trees, *Discrete Appl. Math.* **57(2-3)** (1995),133-144.

[21] P. Erdős, M. Simonovits, V. Sós, Anti-Ramsey theorems. Infinite and finite sets, *Colloq. Math. Soc. Janos Bolyai*, Vol. **10** (1975), 633–643.

[22] G. Fertin, A. Raspaud, A. Roychowdhury. On the oriented chromatic number of grids, *Inform. Process. Lett.***85** (2003), 261–266.

[23] R. Graham, V. Rödl and A. Ruciński, On Schur properties of random subsets of integers, *J. Numb. Theory* **61** (1996), 388-408.

[24] H. Grötzsch. Ein Dreifarbensatz fur dreikreisfreie Netze auf der Kugel, *Math. Nat. Reihe* **8** (1959) 390-408.

[25] D.J. Grynkiewicz. A step beyond Kemperman's Structure Theorem, *Mathematika* (to appear 2009).

[26] Z. Hedrlin, A. Pultr. Relations (graphs) with given finitely generated semigroups, *Monatsh. Math.***68** (1964) 213-217.

[27] Hell, P.; Kostochka, A. V.; Raspaud, A.; Sopena, E. On nice graphs, *Discrete Math.***234** (2001), 39–51.

[28] P. Hell, J. Nešetřil. Graphs and Homomorphisms, *Oxford Lecture Series in Mathematics and its Applications,* **vol 28**, *Oxford University Press*, (2004).

[29] M. Hosseine Dolma, E. Sopena. On the orientedd chromatic number of halin graphs, *Inform. Process. Lett.* **98(6)** (2006)247-252.

[30] M. Hosseine Dolma, E. Sopena. On the oriented chromatic number of graphs with given excess, *Discrete Mathematics* **306(13)** (2006),1342-1350.

[31] T. R. Jensen, B. Toft. Graph coloring problems, *New York etc. John Wiley Sons cop.* (1995).

[32] V. Jungić, J. Licht, M. Mahdian, J. Nešetřil, and R. Radoičić. Rainbow Arithmetic Progressions and Anti-Ramsey Results. *Combinatorics, Probability and Computing* bf 12 (2003), 599620.

[33] V. Jungić, J. Nešetřil, R. Radoičić, Rainbow Ramsey Theory, *Integers: Electronic Journal of Combinatorial Number Theory* **5 (2)# A09**, (2005).

[34] V. Jungić, R. Radoičić, Rainbow 3-term Arithmetic Progressions. *Integers, the Electronic Journal of Combinatorial Number Theory*, **3:A18**, (2003)

[35] J.H.B. Kemperman. On small sumsets in an abelian group. *Acta Math.* **103** (1960) 63–88.

[36] A. V. Kostochka, E. Sopena, X. Zhu. Acyclic and oriented chromatic numbers of graphs, *J. Graph Theory* **24** (1997), 331–340.

[37] B. Landman, A. Robertson, Ramsey Theory on the Integers, Student Mathematical Library, Vol. 24, AMS , 2003

[38] V.F. Lev. Critical pairs in abelian groups and Kemperman's structure theorem. *International Journal of Number Theory,* **2**, (2006), 379-396.

[39] K. W. Lih, W. F. Wang, X. Zhu. Coloring the square of a $K_4$–minor free graph, *Discrete Math.* 269 (2003) 303-309.

[40] L. Lovás, On chromatic number of finite set-systems. *Acta Math. Acad. Sci. Hungar.* **19**, 59-67

[41] M. Montassier, P. Ochem, A. Pinlou. Strong oriented chromatic number of planar graphs without short cycles, *Discrete Math. Theoret. Comput. Sci.*, **10(1)** (2008), 1-24.

[42] J. Nešetřil, Ramsey classes and homogeneous structures. *Combin. Probab. Comput.* **14. 1-2**, (2005), 171–189.

[43] J. Nešetřil, A. Raspaud. Colored homomorphisms of colored mixed graphs, *J. Combin. Theory Ser. B* **80** (2000), 147-155.

[44] J. Nešetřil, A. Rapaud, E. Sopena. Colorings and girth of oriented planar graphs, *Discrete Math.* **165/166** (1997), 519–530.

[45] J. Nešetřil, E. Sopena, L. Vignal, $T$-preserving homomorphisms of oriented graphs. *Comment. Math. Univ. Carolin.* **38** (1997), 125–136.

[46] P. Ochem. Oriented colorings of triangle-free planar graphs, *Inform Process. Lett.* **92** (2004) 71-76.

[47] P. Ochem. Negative results on acyclic improper colorings, *EuroComb 2005. Berlin, September 5-9 2005. DMTCS Conference* **Volume AE** (2005), 357-362.

[48] P. Ochem, A. Pinlou. Oriented colorings of partial 2-trees, *Inform. Process.Lett.* **108** (2008) 82-86.

[49] P.A. Parrilo, A. Robertson, D. Saracino, On the asymptotic minimum number of monochromatic 3–term arithmetic progressions. *J. Combin. Theory* ser. A **115** (2008) 185-192.

[50] A. Pinlou. An oriented coloring of planar graphs with girth at last five, *Discrete Math*, to appear, 2008.

[51] A. Pinlou. PhD thesis, oriented arc-colorings and oriented vertex-colorings of graphs (in french) (2006).

[52] A. Pinlou, E. Sopena. Oriented vertex and arc colorings of outerplanar graphs, *Inform. Process. Lett.* **100(3)** (2006) 97-104.

[53] A. Raspaud, E. Sopena. Good and semi-strong colorings of oriented planar graphs, *Inform. Process. Lett.* **51(4)** (1994), 171-174.

[54] A. Robertson and D. Zeilberger, A 2-coloring of $[1, N]$ can have $(1/22)N^2 + O(N)$ monochromatic Schur triples, but not less!, *Electron. J. Combin.* **5** (1998) #R19.

[55] G. Sabidussi. Graphs derivatives, *Math. Z.* **76** (1961) 385-401.

[56] T. Schoen, The Number of Monochromatic Schur Triples, *Europ. J. Combinatorics* **20** (1999) 855-866.

[57] J. Schönheim. On partitions of the positive integers with no x, y, z belonging to distinct classes satisfying x + y = z. *In R. A. Mollin, editor, Number Theory: Proceedings of the First Conference of the Canadian Number Theory Association* (1990) 515-528.

[58] I. Schur, Uber die Kongruenz $x^m + y^m \equiv z^m$ mod $p$. *Jahresb. Deutsche Math. Verein,* **25** (1916), 114-117.

[59] E. Sopena. The chromatic number of oriented graphs, *J. Graph Theory* **25(3)** (1997), 191-205.

[60] E. Sopena. Oriented graph coloring, *Discrete Math.* **229** (2001), 359-369.

[61] E. Sopena. There exist oriented planar graphs with oriented chromatic number at least sixteen, *Inform. Process. Lett.* **81(6)** (2002), 309–312.

[62] E. Sopena, *On the oriented chromatic number of graphs with given excess.* Discrete Math. **306** no. **13**, (2006), 1342–1350.

[63] A. Szepietowki, M. Targan. A note on the oriented chromatic number of grids, *Inform. Process. Lett.* **92(2)** (2004) 65-7.

[64] T. Tao, V. Vu. Additive combinatorics. *Cambridge Studies in Advanced Mathematics*, **105**. Cambridge University Press, Cambridge, 2006.

[65] J. Tromp, Unpublished manuscript.

[66] P. Varnavides, On Certain Sets of Positive Density, *J.London Math. Soc.* **34** (1959) 358-360

[67] B.L. van der Warden. Beweis einer Baudetschen Vermutung. *Nieuw Archief voor Wiskunde*, **15** (1927), 212-216.

[68] D.R. Wood. Acyclic, star and oriented colourings of graphs subdivitions, *Discrete Math. Theoret. Comput. Sci.* **7 (1)** (2005) 37-50.