Chapter 5. Filter design on the three dimensional spectrum

In this chapter we study the extension of Fourier domain filtering to the color Fourier spectrum and the three dimensional spectrum of color images. We interpret the filtering operations of the color Fourier spectrum as transformations of the color space, and give an interpretation of their effect on correlation operations.

5.1. Color space transformations based on color Fourier spectrum filtering

Linear operations on the color space can be interpreted in terms of a transformation of the color Fourier spectrum. Equivalently, transformations of the color Fourier spectrum involve a transformation of the color space.

Let us consider a linear transformation, noted **T**, of the color space. Each color **c** is transformed onto a new color **c'**=**Tc**. The transform **T** can be represented by a $N \times N$ matrix with matrix elements $T_{n'n}$. Because the color Fourier transform is a linear operator, there is a linear relation, **R**, between the spectrum of a color **c** and the spectrum of its image through **T** (**c**'=**Tc**). Furthermore **R** and **T** satisfy the following relation:

$$\mathbf{R} = \mathbf{F}\mathbf{T}\mathbf{F}^{-1}. \tag{5.1}$$

Here **F** is the matrix that represents the color Fourier transform, defined in Equation 4.12. The explicit form of the matrix elements is written as follows.

$$R_{m'm} = \frac{1}{N}\sum_{n',n=0}^{N-1} T_{n'n} \exp\left[-i\frac{2\pi}{N}(m'n' - nm)\right]. \tag{5.2}$$

$R_{m'm}$ can be identified with the two dimensional Fourier transform of the matrix elements, with the exception of a sign

$$R_{m'm} = FT_{2D}\left[T_{n',N-n}\right]. \tag{5.3}$$

Similarly, the matrix elements, $T_{n'n}$, of the transformation in the color space can be written as a two dimensional Fourier transform of the matrix elements of the transformation in the color Fourier space:

$$T_{n'n} = \frac{1}{N}\sum_{m',m=0}^{N-1} R_{m'm} \exp\left[+i\frac{2\pi}{N}(m'n' - nm)\right]. \tag{5.4}$$

An interesting set of linear transformations are the Fourier spectrum filtering operations. These are operations in which the Fourier spectrum is multiplied by a filter function $H(m)$, and therefore they can be written as diagonal matrices in the color Fourier domain, as follows:

$$R_{m'm} = H(m)\delta_{m'm}. \tag{5.5}$$

Filtering of the color Fourier spectrum introduces a linear transformation of the color space, as follows:

$$T_{n'n} = \frac{1}{N}\sum_{m',m=0}^{N-1} H(m)\delta_{m'm}\exp\left[+i\frac{2\pi}{N}(m'n'-nm)\right]. \tag{5.6}$$

And by developing this we can obtain

$$T_{n'n} = \frac{1}{N}\sum_{m=0}^{N-1} H(m)\exp\left[+i\frac{2\pi}{N}m(n'-n)\right]. \tag{5.7}$$

That is, the operator **T** can be written in terms of the inverse Fourier transform of the filter, $h(n)$ known as the *impulse response function* of the filter.

$$T_{n'n} = h(n'-n). \tag{5.8}$$

Of course, the application of this operator to a color **c** with components $c(n)$ is the convolution of the color with the impulse response function of the filter, let us say:

$$\begin{aligned}[\mathbf{Tc}]_{n'} &= \sum_{n=0}^{N-1} T_{n'n}c(n) \\ &= \sum_{n=0}^{N-1} h(n'-n)c(n)\end{aligned}. \tag{5.9}$$

5.2. Classification of color Fourier spectrum filtering operations

Linear operations associated to color Fourier spectrum filters can always be written as a convolution, i.e. as an operation invariant to shifts along the color axis. Note that the cyclic convolution of any function with a uniform distribution leads to a uniform distribution. This implies that the result of applying any color Fourier spectrum filtering operation to the white color produces the white (or a gray color). Therefore, one can consider that the vector $\mathbf{W}=\zeta\mathbf{R}+\zeta\mathbf{G}+\zeta\mathbf{B}$ is an eigenvector for any color space transformation induced by a color Fourier spectrum filtering operation.

Let us consider the case for N=3, the transformation of the color space corresponding to a filtering operation can be written in terms of the filter impulse response function $h(n)$ as follows:

$$\mathbf{T} = \begin{bmatrix} h(0) & h(1) & h(2) \\ h(2) & h(0) & h(1) \\ h(1) & h(2) & h(0) \end{bmatrix}. \tag{5.10}$$

Note that each row-vector corresponds to a different cyclic translation of the impulse response function. This way, the product of the matrix with a vector gives the convolution of that vector with the impulse response function

The convolution operator can be written in the basis (ζ,η,ξ) associated to the color Fourier transform defined in Equation 4.23, as follows.

$$\mathbf{T}' = \mathbf{A}\mathbf{T}\mathbf{A}^{-1}. \tag{5.11}$$

The explicit expression can be obtained by replacing the explicit expressions of **T** and **A**, then we obtain the following expression:

$$\mathbf{T}' = \begin{bmatrix} \frac{h(0)+h(1)+h(2)}{3} & 0 & 0 \\ 0 & h(0) - \frac{h(1)+h(2)}{2} & \sqrt{3}\frac{h(1)-h(2)}{2} \\ 0 & -\sqrt{3}\frac{h(1)-h(2)}{2} & h(0) - \frac{h(1)+h(2)}{2} \end{bmatrix}. \tag{5.12}$$

In addition, one can write the explicit dependency of the color space transformation on the Fourier domain filter. To do this it is enough to realize that

$$H(0) = \frac{h(0)+h(1)+h(2)}{3}, \tag{5.13a}$$

$$\operatorname{Re} H(1) = h(0) - \frac{h(1)+h(2)}{2} \tag{13b}$$

and

$$\operatorname{Im} H(1) = \sqrt{3}\frac{h(1)-h(2)}{2}. \tag{13c}$$

Then, one can write the transformation expression as follows.

$$\mathbf{T}' = \begin{bmatrix} H(0) & 0 & 0 \\ 0 & \operatorname{Re} H(1) & \operatorname{Im} H(1) \\ 0 & -\operatorname{Im} H(1) & \operatorname{Re} H(1) \end{bmatrix}. \tag{5.14}$$

Here we assume that $h(n)$ is a real valued distribution, because we only consider real transformations of the color space. Therefore $H(2)=H^*(1)$. One can observe that the only nonzero matrix element of the first column is the first row, independently of $h(n)$. That indicates that the white color, that in the (ζ,η,ξ) basis has components $(\zeta,0,0)$, is an eigenvector for any color space transformation generated by a color Fourier spectrum filter.

It results convenient to write the filter in exponential notation to express **T'** as the composition of two operators as follows

$$\mathbf{T}' = |H(1)| \begin{bmatrix} \frac{H(0)}{|H(1)|} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi(1) & \sin\Phi(1) \\ 0 & -\sin\Phi(1) & \cos\Phi(1) \end{bmatrix}. \qquad (5.15)$$

Here $\Phi(1)$ represents the argument of the filter at the m=1 channel. This way, the transformation induced by a color Fourier spectrum filtering operation can be decomposed in three operators. A rescaling of the color space, a dilatation (or contraction) along the ζ axis, which is related to the amplitude modulation of the filter. And a rotation about the same ζ axis, related to the phase modulation of the filter.

5.3. Low pass and High pass filter

Let us consider the N=3 color space transformation induced by a real-valued filter. The filter is also symmetric so as to have a real-valued impulse response function. In this case, the transformation matrix is diagonal in the basis defined by (ζ,η,ξ), and it can be written as follows:

$$\mathbf{T}' = |H(1)| \begin{bmatrix} \frac{H(0)}{|H(1)|} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (5.16)$$

We neglect the factor $|H(1)|$ that indicates a rescaling of the color space because it does not involve any transformation of the color space. The remaining matrix indicates that the color space is dilated in the ζ-direction by a factor of $H(0)/H(1)$. We consider two different situations: when $H(0)/H(1)<1$, known as *high pass filter*, and when $H(0)/H(1)>1$, known as *low pass filter*. We illustrate these transformations in Figure 5.1. We have represented the transformation of a cube (drawn in black line) with its sides normal to the ζ, η and ξ axes. The low pass filter transformation produces a dilatation of the ζ axis, while η and ξ remain constant, therefore the cube is longed along the ζ direction (see the red-lined cube in Figure 5.1a). In other hand, the high pass filter produces a contraction of the ζ axis, therefore the cube is contracted in this direction (Figure 5.1b).

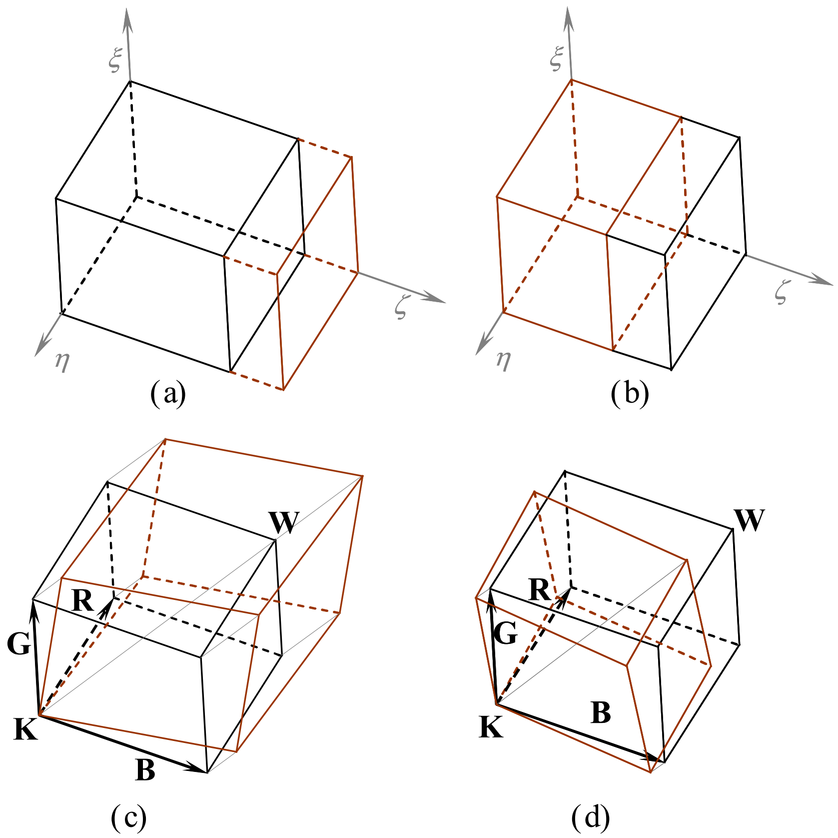


Figure 5.1. Transformation of the color space induced by the low pass and high pass filtering. (a) Low pass in the (ζ,η,ξ) basis. (b) High pass in the (ζ,η,ξ) basis. (c) Low pass in the (r,g,b) basis. (d) High pass in the (r,g,b) basis.

The effect of the same transformations on the *RGB* color cube† has been represented in Figure 5.1c, for the low pass filter, and Figure 5.1d for the high pass filter. One can observe that the dilatation and contraction is in the direction of the diagonal of the cube that links the black (**K**) with the white (**W**), and therefore all the aristae of the cube are changed, what is

† the *RGB* color cube is the cube whose vertices are the black (**K**), the primary colors (**R**, **G**, **B**), the secondary colors (**C**, **M**, **Y**) and the white (**W**).

indicating that none of the *RGB* colors is an eigenvector of the transformation.

The above described transformations of the color space introduced by the high-pass and low-pass filter operations produce a change in the color distribution of the images. The dilatation or contraction of the ζ coordinate of the colors involves a change in their saturation, as can be deduced from Equation 4.25b. When a dilatation of ζ is produced (for a low pass filter), the saturation of the colors is reduced. In the other hand, for a high pass filter, a contraction of the ζ coordinate is produced, and the saturation of the colors is enhanced. Nevertheless, the hue information is not altered by these filters, as can be deduced from the fact that both directions, η and ξ are not altered by the filtering operation.

This is illustrated in Figure 5.2 for the low pass filters. The image in Figure 5.2a has been taken as the input signal for the a low pass filtering operation. We have considered the filter in Figure 5.2b, that enhances the DC term of the color Fourier spectrum of the image with respect to the other channels. The image resulting from this filtering operation is represented in Figure 5.2c. One can observe that the colors of the output image are less saturated than in the original image. The limit case is when the m≠0 channels of the color spectrum are completely removed from the color Fourier spectrum (The filter is shown in Figure 5.2d). In this case, the *RGB* color cube is collapsed onto its diagonal and only the intensity information is preserved by the transformation, leading to a black and white version of the original image, as can be seen in Figure 5.2e.

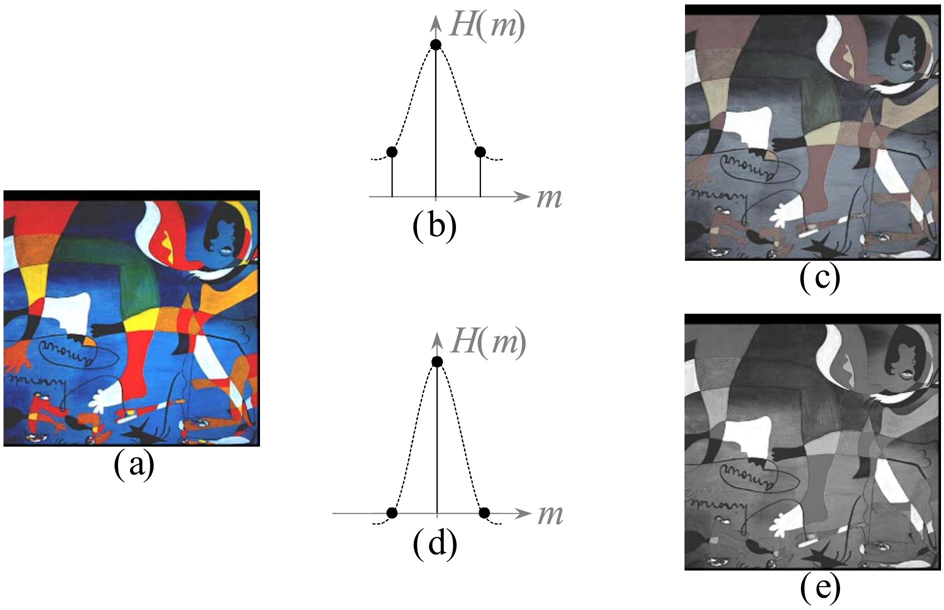
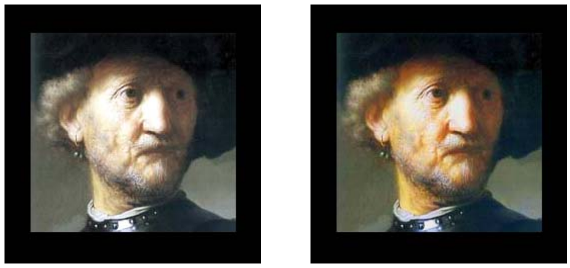


Figure 5.2. Example of Low pass filter. (a) Original image, (b) Low pass filter with a ratio of 25% (c) Corresponding filtered image. (d) Low pass filter with extinction of the nonzero frequencies. (e) Resulting filtered image.

The opposite case, that is, the case of the high pass filter is represented in Figure 5.3. We have considered the input image represented in Figure 5.3a, and in this case we have applied a high pass filter that produces a contraction of the ζ coordinate of the color space. This produces an increase of the saturation of the colors without altering their hue, as can be seen in Figure 5.3b.



(a) (b)

Figure 5.3. Color enhancement by high pass filtering of the color spectrum of an image. (a) Sample image. (b) Image reconstructed from the filtered spectrum.

The effect of the low pass and the high pass filter on the color distribution of images involves a rescaling of their color distribution in the chromaticity histogram. The chromaticity histogram of an image is a two dimensional representation of the color distribution of the image in which the value at each point (x,y) is proportional to the number of pixels of the image with a color that satisfies $R/I=x$, and $G/I=y$. The chromaticity histogram for the image in Figure 5.4a. has been represented in Figure 5.4b. The histogram has been overlapped to the Maxwell triangle. In the histogram, the gray level represents the value, so dark gray levels correspond to low occurrence rates, and light gray levels correspond a high occurrence rates. The zero gray level, corresponding to the null occurrence rate, has been set transparent, to make easier the location of colors. The response of a high pass filter to the signal in Figure 5.4a, whose chromaticity histogram is represented in Figure 5.4b has been represented in Figure 5.4c. The chromaticity histogram of the response has been represented in Figure 5.4d. One can observe that the transformation induced by the high pass filter operation involves a magnification of the distribution of the colors on the chromaticity histogram. The opposite case, that is, the response of a low pass filter to the input image has been represented in Figure 5.4e, and its chromaticity histogram in Figure 5.4f. Now, the colors have lost saturation, and consequently the histogram has been reduced to a small area about the center of the Maxwell's triangle.

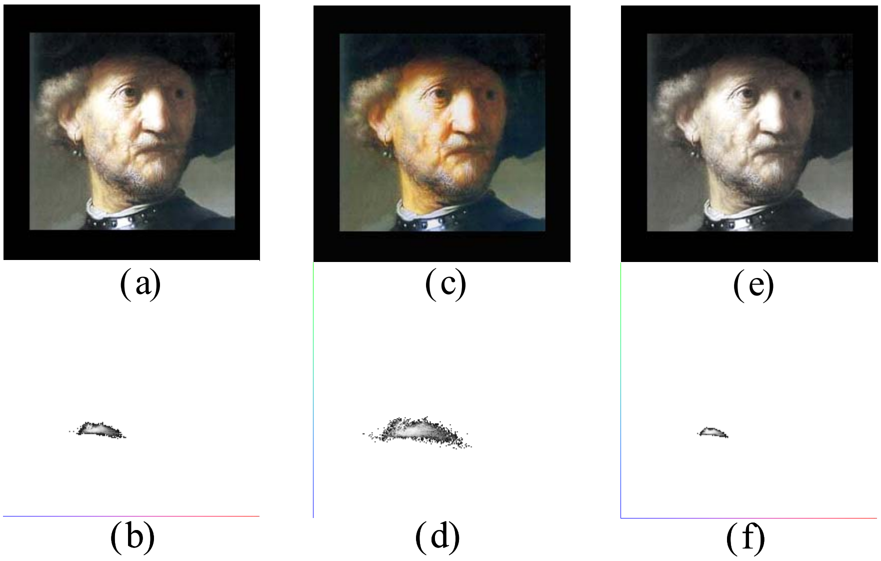


Figure 5.4. Re scaling of the histogram by color Fourier spectrum filtering. (a) Original image. (b) Chromaticity histogram. (c). High pass filter response and its chromaticity histogram (d). Low pass filter response (e) and its chromaticity histogram (f).

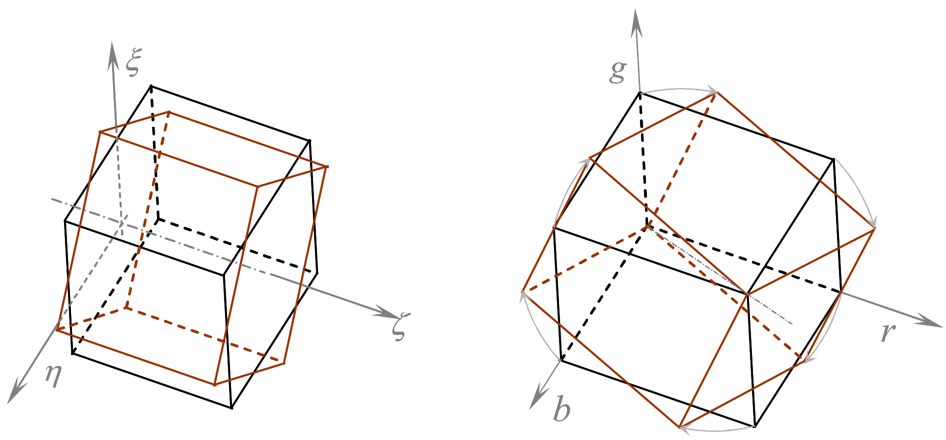
Note that the chromaticity distribution of the image is rescaled but the "shape" of the color distribution in the histogram is preserved. This way, low pass and high pass filtering operations are revealed to be a useful tool to enhance or soften the color distribution of images.

5.4.Linear Phase filtering

Let us consider now that we apply a phase filter to the color Fourier spectrum of a color image. In this case equation 15 can be written as follows:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi(1) & \sin\Phi(1) \\ 0 & -\sin\Phi(1) & \cos\Phi(1) \end{bmatrix}. \tag{5.17}$$

One finds that this matrix corresponds to a rotation of an angle $\Phi(1)$ about the ζ axis of the color space. We have represented this rotation in Figure 5.5a. While the ζ axis is not changed, the η and ξ axes are rotated. This transformation has been represented in the (r,g,b) frame in Figure 5.5b. One can observe that the *RGB* color cube is rotated about its diagonal.



(a) (b)

Figure 5.5. Transformation of the color space induced by a phase filtering. (a) representation in the ζ,η,ξ frame. (b) representation in the r,g,b frame.

Because the phase modulation of the color Fourier spectrum does not change the ζ axis, the intensity of the color is not altered by the induced transformation. In addition, because the transformation can be considered as a rotation, the distance of the colors to the ζ axis is unchanged, and that involves that the saturation (As defined in Equation 4.25b) of the colors is also preserved. This way, only the hue of the colors is affected by the phase filtering of their color Fourier spectra. We illustrate this in Figure 5.6. We have applied a phase filter to the image in Figure 5.6a, whose chromaticity histogram is Figure 5.6b. The resulting response image is represented in Figure 5.6c, one can observe that the colors of the scene have been changed, but their intensity and saturation is kept. That is, only the hue has been altered. In the histogram of the response image, represented in Figure 5.6d, one can observe that the color distribution is rotated with respect to original image.

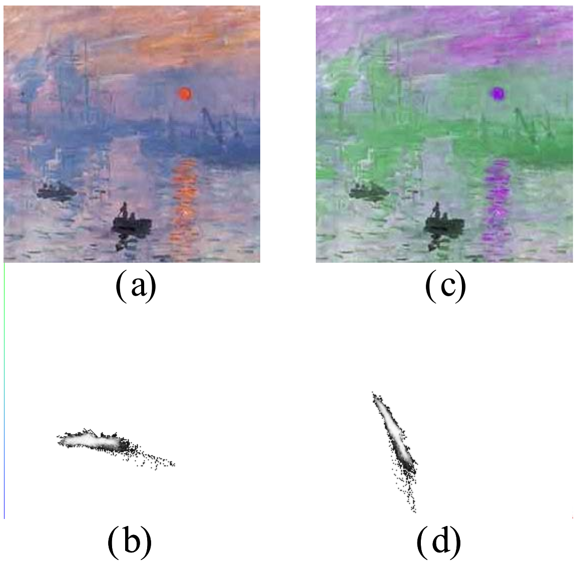


Figure 5.6. Effect of the phase filtering for the color Fourier spectrum of a color image. (a) Sample image (b) Chromaticity histogram of the sample image. (c) Transformed image. (d) Chromaticity histogram of the transformed image.

5.4.1. *Channel sinc-interpolation*

In last section we have given an interpretation of the linear phase filtering of the color Fourier spectrum in terms of rotations of the color space. However, linear phase filtering of the color Fourier spectrum, can also be interpreted in terms of a shift of the signal along the color axis. The shift of the signal can be used for channel interpolation, as we explain next.

Let us consider a phase filter that introduces a shift smaller than one channel on the signal (see Figure 5.7a). Because the sampling points do not change, a new sample of the function is obtained, as represented in Figure 5.7b. The new set of points can also be interpreted as a translation of the sampling points, this way, a new set of values of the function can be obtained, and the interpolation of the function can be performed at these new points, as is represented in Figure 5.7c.

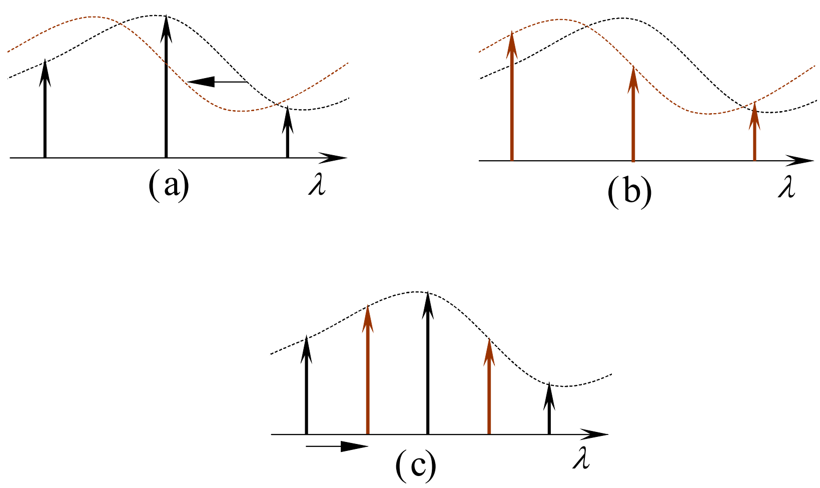


Figure 5.7. Illustration of the sinc-interpolation. (a) The original image is shifted by less than one pixel. (b) A sample of the shifted function is obtained. (c) The new sampling corresponds to the values of the function at the intermediate points of the function.

This way, the interpolation by linear phase filtering (sinc-interpolation) permits to obtain the response of a system with N' detectors from the response of a system with N detectors, where $N'>N$. As the sampling theorem states, aside from edge effects, the result obtained by this sinc-interpolation technique is exact for the spectra whose maximum chromatic frequency is under the Nyquist frequency for the N channel system, that is $N\Delta\lambda/2$.

This is illustrated in Figure 5.8. We have considered the color image in Figure 5.8a, whose (r,g,b) decomposition is presented in Figure 5.8b. The response of a 5-channel system obtained by a phase filtering of the color Fourier spectrum is represented in Figure 5.8c. The n=1 and n=3 channels of the image have been obtained by sinc-interpolation, that is, by means of a phase filtering of the color Fourier spectrum of the (r,g,b) channels.

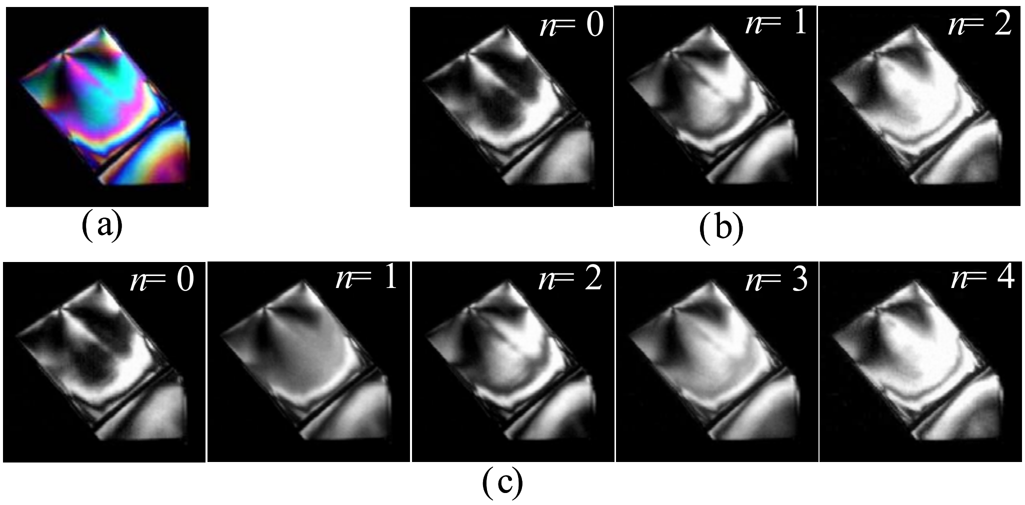


Figure 5.8. (a) Color image. (b) RGB channels. (c) Response for a 5 channel system obtained by sinc-interpolation.

5.5. Whitening of the color Fourier spectrum

Spectrum whitening is a usual operation in signal processing. It consists of a nonlinear transformation in which the magnitude information of the color Fourier spectrum of the signal is removed. The filter is signal dependent, and can be written as

$$H_f(m) = \frac{1}{A(m)}, \tag{5.18}$$

where $A(m)$ is the magnitude of the color Fourier spectrum of the signal. Therefore, the spectrum $F_C(m)=A(m)\exp[i\Phi(m)]$ is transformed onto a new spectrum $F_C'(m)=\exp[i\Phi(m)]$.

An scheme of the color spectrum whitening operation is represented in Figure 5.9 The color Fourier spectrum of the input signal is obtained. Then, the whitening operation is performed, and finally the inverse color Fourier transform is applied to the filtered spectrum and a new color is obtained. A direct interpretation of the color space transformation induced by the spectrum whitening can be given in terms of the *HSI* representation from the relations in Equations 4.25.

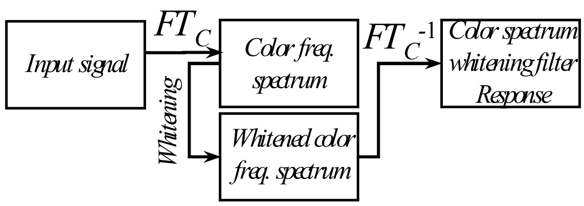


Figure 5.9. Scheme for the color spectrum whitening filter transformation of the color space.

That whitening of the color Fourier spectrum involves the normalization of the DC channel, therefore the intensity of the transformed color has unit intensity. In addition, the saturation of the transformed color is one (following the definition of saturation given in Equation 4.25b) because also the magnitude of the $m\neq0$ channels is normalized to one. However, the hue of the transformed color is the same as the hue of the original signal because the whitening operation does not transform the phase distribution of the color spectrum.

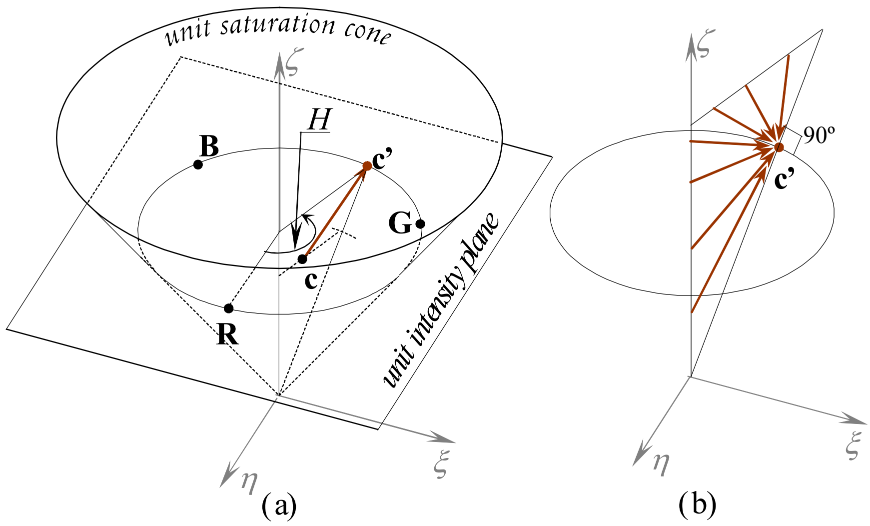


Figure 5.10. Representation of the transformation of the color space induced by the color spectrum whitening operation.

This way, all the colors with a given hue are transformed by the whitening operation onto the color with unit intensity and unit saturation that has that value of hue (see Figure 5.10a), i.e. a color **c** is transformed in the color **c'**, whose saturation and intensity are one, and whose hue is the same as the hue for **c**. Color spectrum whitening operation can be considered as a locally orthogonal projection of the colors onto the circumference defined by the intersection of the unit intensity plane and the unit saturation cone. This is represented in Figure 5.10b. The circumference of colors with unit intensity and unit saturation is represented. We have also represented the plane constituted of all the colors that are transformed onto a color **c'**. This plane is orthogonal to the circumference at the point defined by **c'**.

The axis of the unit saturation cone is the ζ axis and the *r*, *g* and *b* axes are contained in its surface. Therefore it's easy to realize that the cone surface is out of the *RGB* cube except for the primary colors (see Figure 5.11a). That means that the only saturated colors that have three non-negative components are the primary colors **R**, **G** and **B**, and that all the other saturated colors have one negative component (see Figure 5.11b). According to Grassman's color mixture laws, the negative component represents the amount of the primary colors that has to be added to a saturated color to produce a color that is equalized by additive mixture of the remaining primary colors.

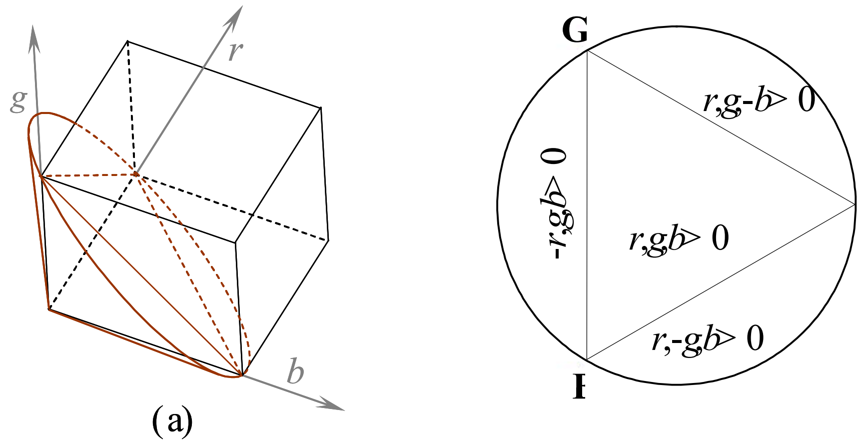


Figure 5.11. (a) Representation of the unit saturation cone in relation to the *RGB* cube. (b) Representation of colors with positive and negative components. Aside from the primary colors the saturated colors have one negative component.

We illustrate the saturation and intensity normalization effect on the colors produced by the color spectrum whitening in Figure 5.12. We have considered the sample image in Figure 5.12a. It contains a number of squares in different colors, with different hue, saturation and intensity values. The squares in each column have the same hue. From top to down, the squares in a column are sorted from less saturated to more saturated, and from more intense to less intense. The response of the color spectrum whitening filter applied to this scene is represented in Figure 5.12b. Because saturated colors have one negative component, we have added a uniform gray background to the response so as to have non-negative components. One can observe that all the colors with the same hue are transformed onto the same color.

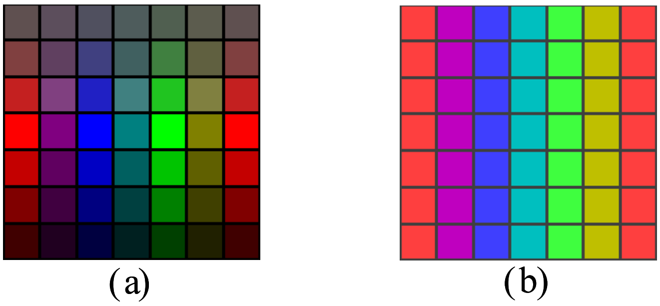


Figure 5.12. (a) Sample scene: palette of colors with different saturation, hue and intensity. (b) Response of the color spectrum whitening filter: The colors are classified by hue.

Because the color spectrum whitening filtering represents a normalization of the saturation, the distribution in a chromaticity histogram of the filter response to any color image is restricted to be in the circumference of maximum saturation. This is illustrated in Figure 5.13. We have applied a whitening operation to the sample scene in Figure 5.13a, whose chromaticity histogram is represented in Figure 5.13b. The response of the filter is represented in Figure 5.13c. It presents colors with negative components, therefore, to be able to represent them, we have subtracted from the image the minimum value between all the components of all the pixels, what is equivalent to add a uniform gray background. One can observe that the colors of the response image are saturated, and that the intensity is normalized. The corresponding chromaticity histogram is represented in Figure 5.13d. One can observe that all the colors are distributed in the circumference tangent to the Maxwell triangle (It looks like an ellipse because of the asymmetrical projection).The circumference, that is originally out of the Maxwell triangle, is inscribed in the

Maxwell triangle because of the addition of the background, that is interpreted as a shift along the ζ axis, as shown in Figure 5.13e. This way, the original circumference Σ is transformed onto the circumference Σ', which is inside the pyramid whose axes are defined by the three primary colors.

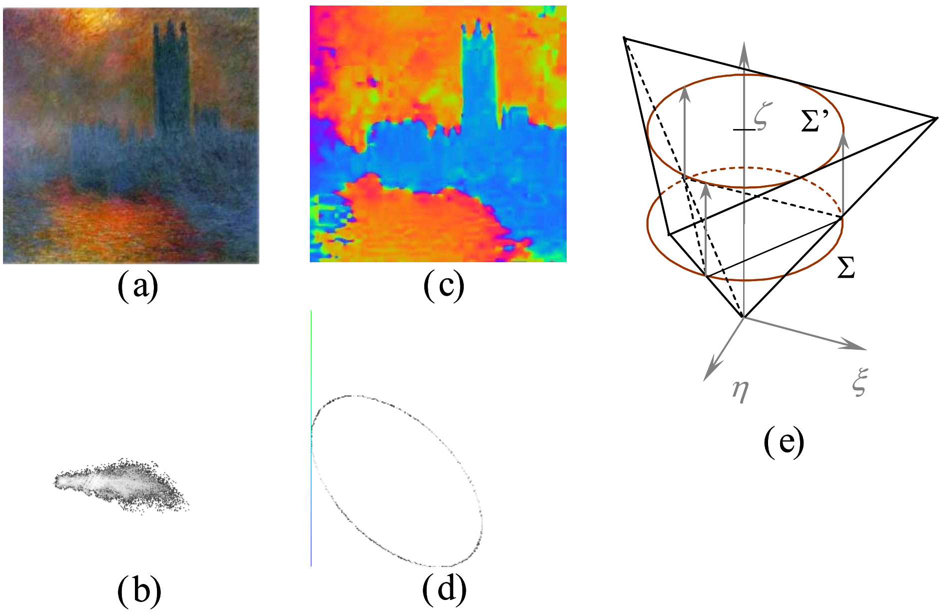


Figure 5.13. (a) Sample scene. (b) Corresponding chromaticity histogram. (c) Color whitened spectrum filter response. (d) Corresponding chromaticity histogram.(e) Color desaturation by addition of white.

5.6.Color-wise correlation

A particular case of filtering of the color Fourier spectrum is the *color-wise correlation*. This is the case when the color spectrum filter consists of the complex conjugated color Fourier spectrum of a reference signal. This way, by applying the correlation theorem one finds that the response of the filter is the correlation between the input and the reference signals along the color axis. One writes the color-wise correlation as follows:

$$[\mathbf{c} \otimes \mathbf{d}](n) = \sum_{n'=0}^{N-1} c(n')d(n'-n), \tag{5.19}$$

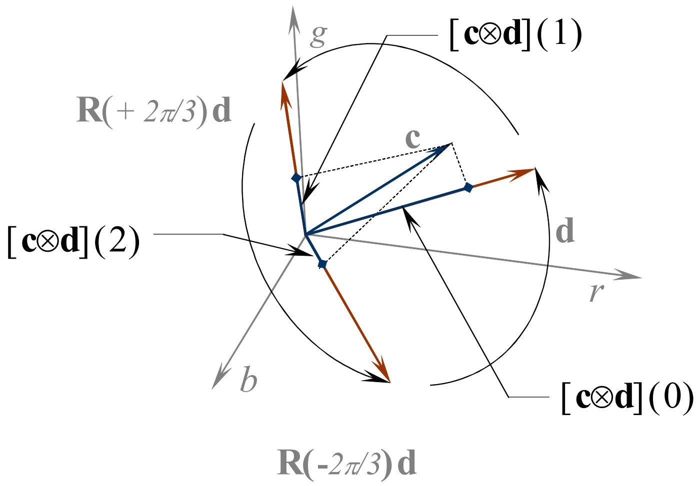


Figure 5.14. Interpretation of the color wise correlation in terms of projections in the color space.

Here, we consider cyclic translations of the reference image. The color-wise correlation can be considered as the scalar product of the vector that represents the color **c** with the vector that represents the reference signal **d**, shifted by n channels. We have demonstrated in Section 5.4 that a cyclic translation of the color distribution, represented by a linear phase filtering can be understood as a rotation of the color space about the ζ axis, noted as $\mathbf{R}_\zeta(\ )$. This way, we can write the channels of the color-wise correlation as the scalar product of the input color with a rotation of the reference color, as follows.

$$[\mathbf{c} \otimes \mathbf{d}](n) = \mathbf{c}\mathbf{R}_\zeta\left(\frac{2\pi}{N}n\right)\mathbf{d}, \tag{5.20}$$

This is illustrated in Figure 5.14: the vector **c** is projected orthogonally onto the reference vector **d**, and onto the vectors that come from the rotation of **d** about the ζ axis. The magnitudes of these components are the values of the color-wise correlation in the different channels. When the input and the reference color are the same color, the projection is maximal in the n=0 channel, Therefore, color-wise correlation can be used as a color recognition technique.

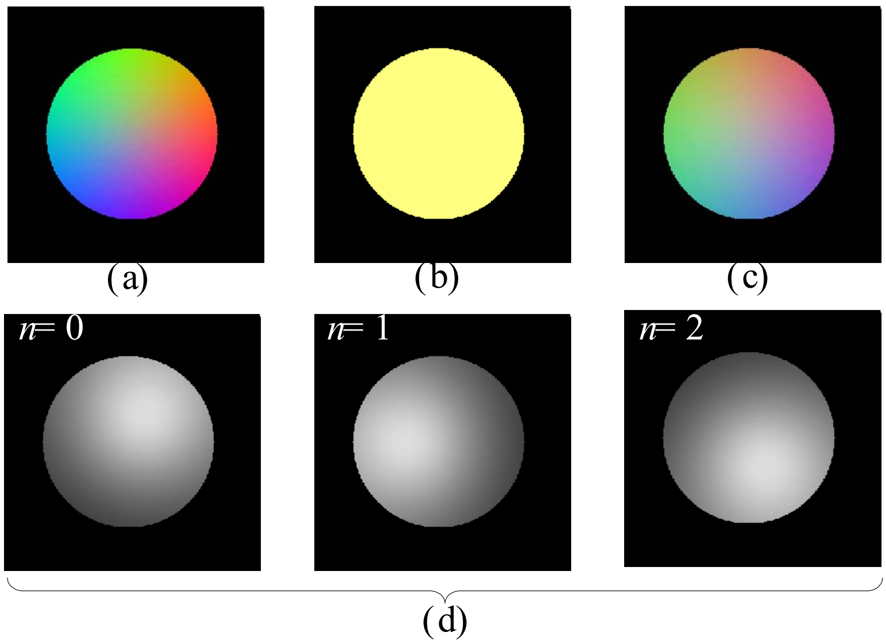


Figure 5.15. (a) Input scene. (b) Reference. (c) Color-wise correlation. (d) Channels of the color-wise correlation.

An example of color-wise correlation is presented in Figure 5.15, We have considered the images in Figure 5.15a and b as the input scene and the reference respectively. We have performed the color-wise correlation, and presented the result in Figure 5.15c, as a color image. The channels of the color-wise correlation are represented in Figure 5.15d. The n=0 channel (colored in red) of the color-wise correlation has its maximum at the pixels where the scene and the reference have more similar colors. For the other channels, the maxima recognize the colors that result from the rotation of the yellow color of the reference by $2\pi/3$ and $-2\pi/3$ respectively about the ζ axis, that is, a certain value of cyan and of magenta respectively.

5.6.1. *Color-wise correlation phase only filter.*

Color recognition by color-wise correlation can be improved by means of modifications of the color Fourier spectrum filter. Usual correlation filters such as the phase only filter and the pure phase correlation (not strictly a filter but a phase only filter combined with a preprocessing of the input), involve a whitening of the color Fourier spectrum. In the case of color-wise correlation the whitening of the color spectrum has an interpretation in terms of the intensity and saturation of the signal color, that allows to give a geometrical interpretation of the phase only filter in the color space..

Let us consider the color-wise correlation using a phase only filter. An scheme of it is presented in Figure 5.16. The color Fourier spectrum is whitened, and the complex conjugate of the resulting filter is multiplied by the color Fourier spectrum of the input signal. Finally the inverse color Fourier transform is applied to the obtained product so as to get the color-wise correlation with a phase only filter matched to the reference.

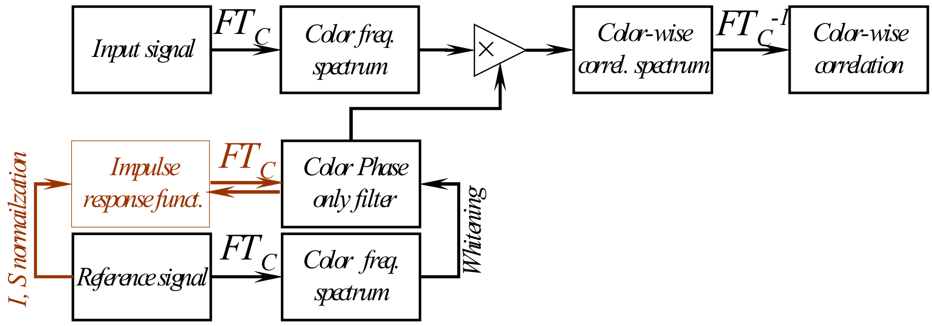


Figure 5.16. Scheme of the color-wise correlation with phase only filter.

We have shown in Section 5.5 that the whitening of the color Fourier spectrum is equivalent to maximize the saturation and to normalize the intensity of the colors. Therefore, the impulse response function of a phase only filter can be obtained directly from the reference scene by modifying the saturation and the intensity of the color (shown in red in Figure 5.16).

This way, the color-wise correlation of a color $\mathbf{c}$ with a phase only filter matched to a color $\mathbf{d}$ can be interpreted as the classic color-wise correlation of $\mathbf{c}$ and $\mathbf{d}^W$, defined as the input response function of the phase only filter matched to $\mathbf{d}$. That is, the n-th channel color-wise correlation is interpreted as the orthogonal projection of $\mathbf{c}$ onto $\mathbf{d}^{W,n}$. Where $\mathbf{d}^{W,n}$ is obtained by applying a rotation to $\mathbf{d}^W$ as follows:

$$\mathbf{d}^{W,n} = \mathbf{R}_{\zeta}\left(\frac{2\pi}{N}n\right)\mathbf{d}^W . \tag{5.21}$$

For the phase only filter, the vectors $\mathbf{d}^{W,n}$ constitute an orthonormal basis of the color space. This is demonstrated as follows: Let us consider the color-wise autocorrelation of $\mathbf{d}^W$. The color-wise autocorrelation is obtained by applying the inverse Fourier transform to the squared magnitude of the color Fourier spectrum of the signal. Because of the whitening operation, the squared magnitude of the color Fourier spectrum of $\mathbf{d}^W$ is uniform, and therefore the autocorrelation is a Kronecker delta function.

$$\left[\mathbf{d}^W \otimes \mathbf{d}^W\right](n) = \delta_K(n), \tag{5.22}$$

Where δ_K takes the following values:

$$\delta_K(n) = \begin{cases} 1 & if \;\; n = 0 \\ 0 & if \;\; n \neq 0 \end{cases}, \tag{5.23}$$

In other hand, the channels of the color-wise correlation can be written as the scalar products of vectors:

$$\left[\mathbf{d}^W \otimes \mathbf{d}^W\right](n) = \mathbf{d}^W \mathbf{R}_\varsigma\left(\frac{2\pi}{N} n\right)\mathbf{d}^W. \tag{5.24}$$

So, we can rename n=p–q and write the rotation as the composition of two rotations, as follows:

$$\left[\mathbf{d}^W \otimes \mathbf{d}^W\right](p-q) = \mathbf{d}^W \mathbf{R}_\varsigma\left(-\frac{2\pi}{N} q\right)\mathbf{R}_\varsigma\left(\frac{2\pi}{N} p\right)\mathbf{d}^W, \tag{5.25}$$

And by applying the definition of $\mathbf{d}^{W,n}$ we get

$$\left[\mathbf{d}^W \otimes \mathbf{d}^W\right](p-q) = \mathbf{d}^{W,q} \cdot \mathbf{d}^{W,p} = \delta_K(p-q). \tag{5.26}$$

That is, the set of colors $\mathbf{d}^{W,n}$ are represented by N unit and orthogonal vectors in the color space, and therefore constitute an ortho-normal basis of the color space..

The color-wise correlation using a phase only filter can be interpreted as the orthogonal projection of the signal color onto the ortho-normal basis of the color space determined by the hue of the reference color. This way, the channels of the color-wise correlation correspond to the components of the signal color in that basis.

This is illustrated for the N=3 color space in Figure 5.17. A phase only filter is matched to the color $\mathbf{d}$. This is equivalent to perform the classic color-wise correlation of the signal color $\mathbf{c}$ with the impulse response of the filter, that is $\mathbf{d}^{W,0}$, that has the same hue as $\mathbf{d}$ but has unit intensity and unit saturation. $\mathbf{d}^{W,0}$ and its translations $\mathbf{d}^{W,1}$ and $\mathbf{d}^{W,2}$ constitute an ortho-normal basis. The different channels of the color-wise correlation correspond to the orthogonal projection of the input color onto the different basis elements, and consequently they correspond to the components of the signal in the basis determined by $\mathbf{d}^{W,n}$.

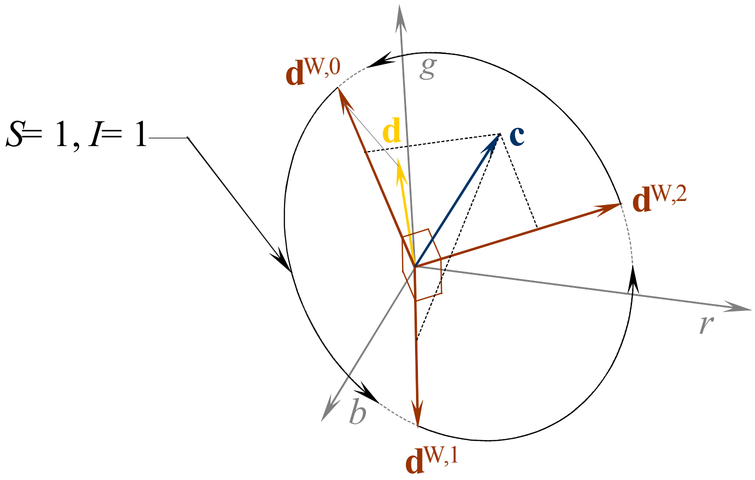


Figure 5.17. Interpretation of the phase only filter color-wise correlation as the projection of the color onto an ortho-normal basis of the color space.

We give an example of color-wise correlation using a phase only filter in Figure 5.18. The image in Figure 5.18a is used as the input signal, it consists in a circle in which each point is colored in a color whose saturation and hue are proportional to the radial and polar coordinate of the point respectively. All the colors have the same intensity. This way, the circle can be considered as a section of the color space normal to the ζ axis. The color-wise correlation of the scene using a phase only filter matched to Figure 5.18b is represented in Figure 5.18c, as a color image, and in Figure 5.18d, where the different channels have been represented separately. One can observe that the n=0 channel of the correlation is maximum at the side of the circle determined by the hue of the reference color. This corresponds to the more saturated color with the hue of the reference signal. The values at the different channels of the correlation indicate the coordinates of the color at each pixel of the input signal in the ortho-normal basis defined by the reference. Because the colors of the input scene are located in a plane of the color space, their coordinates increase linearly along the direction defined by the axes of the new basis. We have drawn in Figure 5.18d the axes of the new basis.

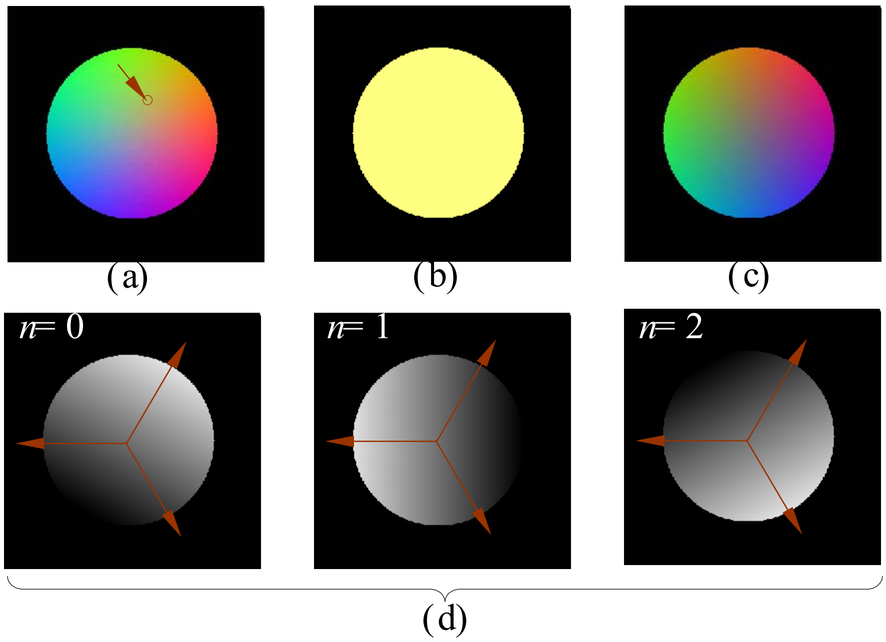


Figure 5.18. Phase only filter for color-wise correlation. (a) Input signal. (b) Reference. (c) Color-wise correlation. (d) Channels of the color-wise correlation.

5.6.2. *Pure phase color-wise correlation*

Whitening of the color Fourier spectrum can be performed for both the scene and the reference signal. One refers to this case as the pure phase color-wise correlation. We have represented an scheme of the pure phase correlation in Figure 5.19. In this case one can consider that the scene is filtered by a color spectrum whitening filter before entering the correlation process. This way, an input signal with the same hue but normalized saturation and intensity is correlated instead of the original input signal. Therefore, the resulting correlation is invariant to intensity changes or saturation changes.

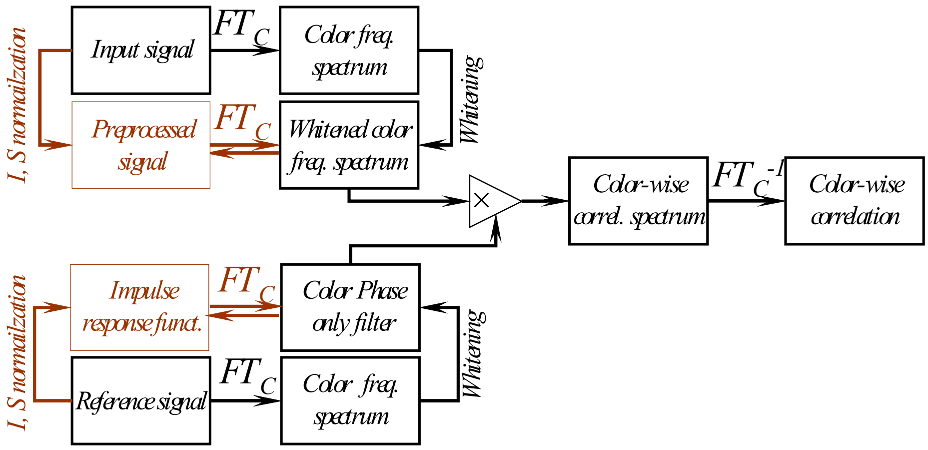


Figure 5.19. Scheme of the pure phase color-wise correlation.

This is illustrated in Figure 5.20. We have considered the input signal represented in Figure 5.20a, that is equal to the input signal in Figure 5.18a. The image obtained after the preprocessing is shown in Figure 5.20b. Note that all the colors are transformed onto colors with unit saturation and intensity, but the hue is not affected. This way, for the considered image, the color distribution is uniform along each radius of the circle.

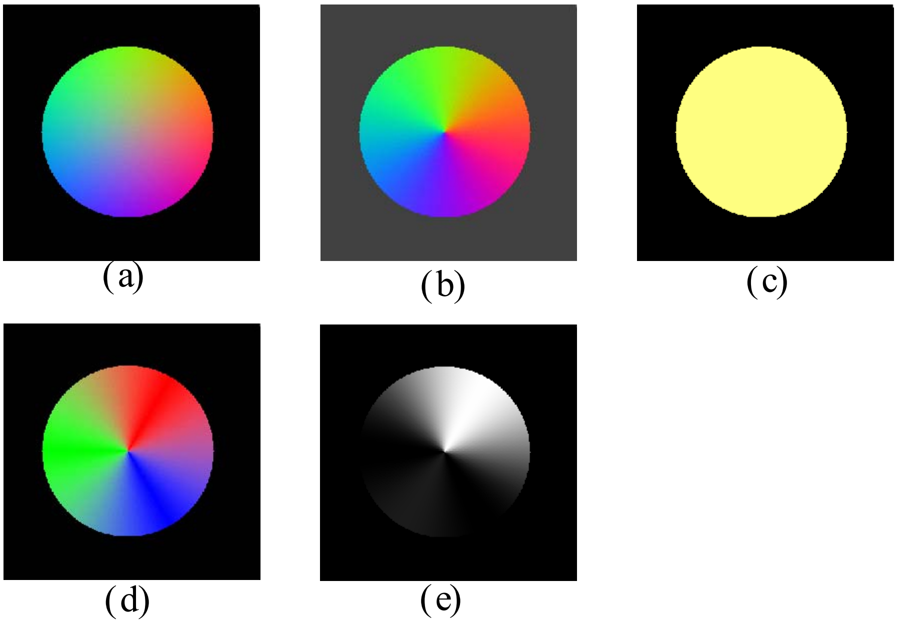


Figure 5.20. (a) Input signal. (b) Preprocessed signal. (c) Reference. (d) Pure phase correlation. (e) n=0 channel of the pure phase correlation.

The reference signal is the yellowish circle in Figure 5.20c, and the color-wise correlation is represented as a color image in Figure 5.20d. One can observe that, the value of the correlation does not change along the radii of the circle. In addition, one can observe the n=0 channel, identified to the red channel (also represented alone in Figure 5.20e) takes its maximum value in the radius of colors with the same hue as the reference signal.

5.7. Whitening operations on the three dimensional spectrum of color images

Whitening operations along the color axis and along the spatial axes can be combined to produce three dimensional filters with different properties. We present in Figure 5.21 the scheme for the design of different filters that involve whitening operations.

Figure 5.21a corresponds to the three dimensional version of the classic matched filter. In this case there is no whitening of the Fourier spectrum. Because there is not any nonlinear operation, one can consider the transformation between the reference signal and the filter as the composition of the color Fourier transform with the spatial two dimensional Fourier transform of the channels of the image. Both operations commute in this case. In addition, the impulse response function of the filter is identified to the reference signal itself.

The scheme in Figure 5.21b corresponds to the whitening of the color Fourier spectrum of the reference signal. We call it the color-whitened filter. In this case, the color Fourier spectrum is whitened before the spatial Fourier transform is performed. This operation is represented by the solid red one-way arrow. This is a non-linear operation, therefore in this case the color Fourier transform and the spatial Fourier transform do not commute. Because the color Fourier spectrum whitening is equivalent to a normalization of the intensity and the saturation of the colors, we can establish a relation between the signal and the impulse response function in the direct domain. This is represented by the red dashed arrow.

Figure 5.21c represents the reciprocal case. In this case the two dimensional Fourier spectrum of the channels of the reference signal are whitened independently. And then the color spectrum is obtained by applying the color Fourier transform. We call this filter the three dimensional spatial-whitened filter. In this case there is not any simple relation in the direct domain between the reference signal and the impulse response function of the filter.

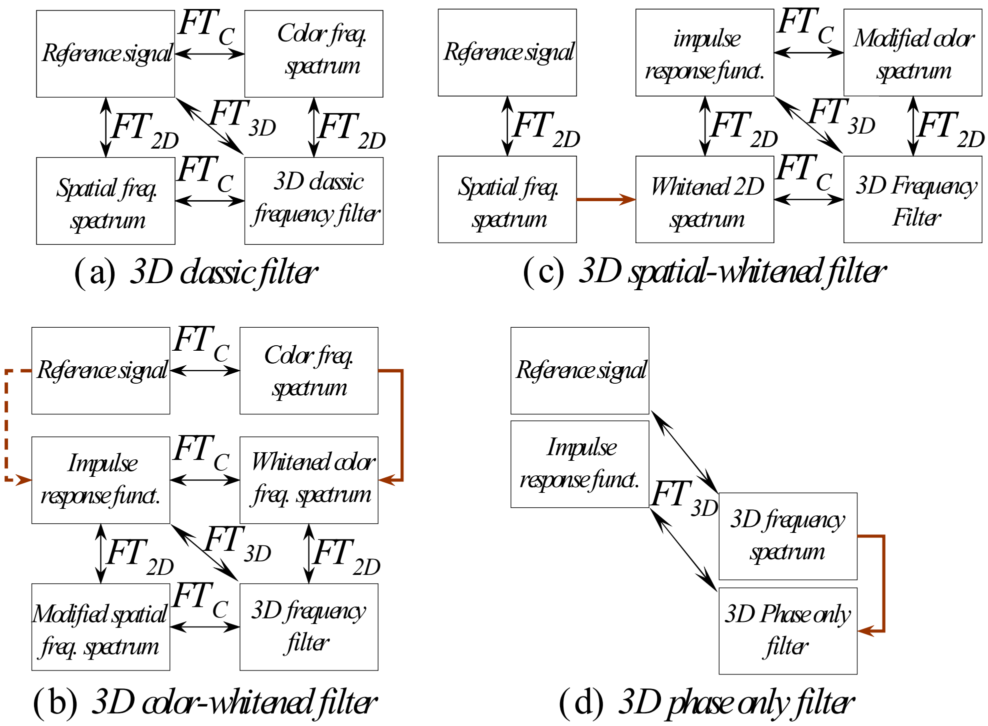


Figure 5.21. Scheme for different whitening operations (a) 3D classic filter. (b) 3D color-whitened filter. (c) 3D space-whitened filter (d) 3D phase only filter.

Finally, the filter corresponding to the whitening of the three dimensional spectrum is represented in Figure 5.21d. It is the natural generalization of the usual phase only filter to three dimensional images in which the color is placed in the third axis.

As we have explained in Section 5.6.1, the color spectrum whitening operation produces an ortho-normalization of the channels. That means that the color-wise cross-correlation between the channels of the filter response is null. That is, the color spectrum whitening involves a decorrelation of the different channels of the signal. Analogously, one can understand that the two dimensional spectrum whitening filter generates a signal in which the different pixels of a channel of the scene are decorrelated each other, but not decorrelated to the pixels of the different channels. Finally, the three dimensional spectrum whitening filtering involves a decorrelation between different pixels, also in different channels.

This is illustrated in Figure 5.22. We have considered the three dimensional autocorrelation for the responses of the following whitening filters: Three dimensional classic filter (Figure 5.22a), Three dimensional color-whitened filter (Figure 5.22b), three dimensional spatial-whitened filter (Figure 5.22c) and finally three dimensional phase only filter. We have represented the impulse response function of the filters (in the left column) and the magnitude of the three channels of the three dimensional autocorrelation (in the remaining columns). In addition, for the two first cases, we have also represented the three dimensional correlation encoded as a color image).

The considered scene consists of a butterfly shape painted in low saturation colors (Figure 5.22a left). One observes that the three dimensional autocorrelation of the original scene presents a wide peak in the three channels of the correlation (Figure

5.22a). This is interpreted as the autocorrelation peak is wide also along the color axis.

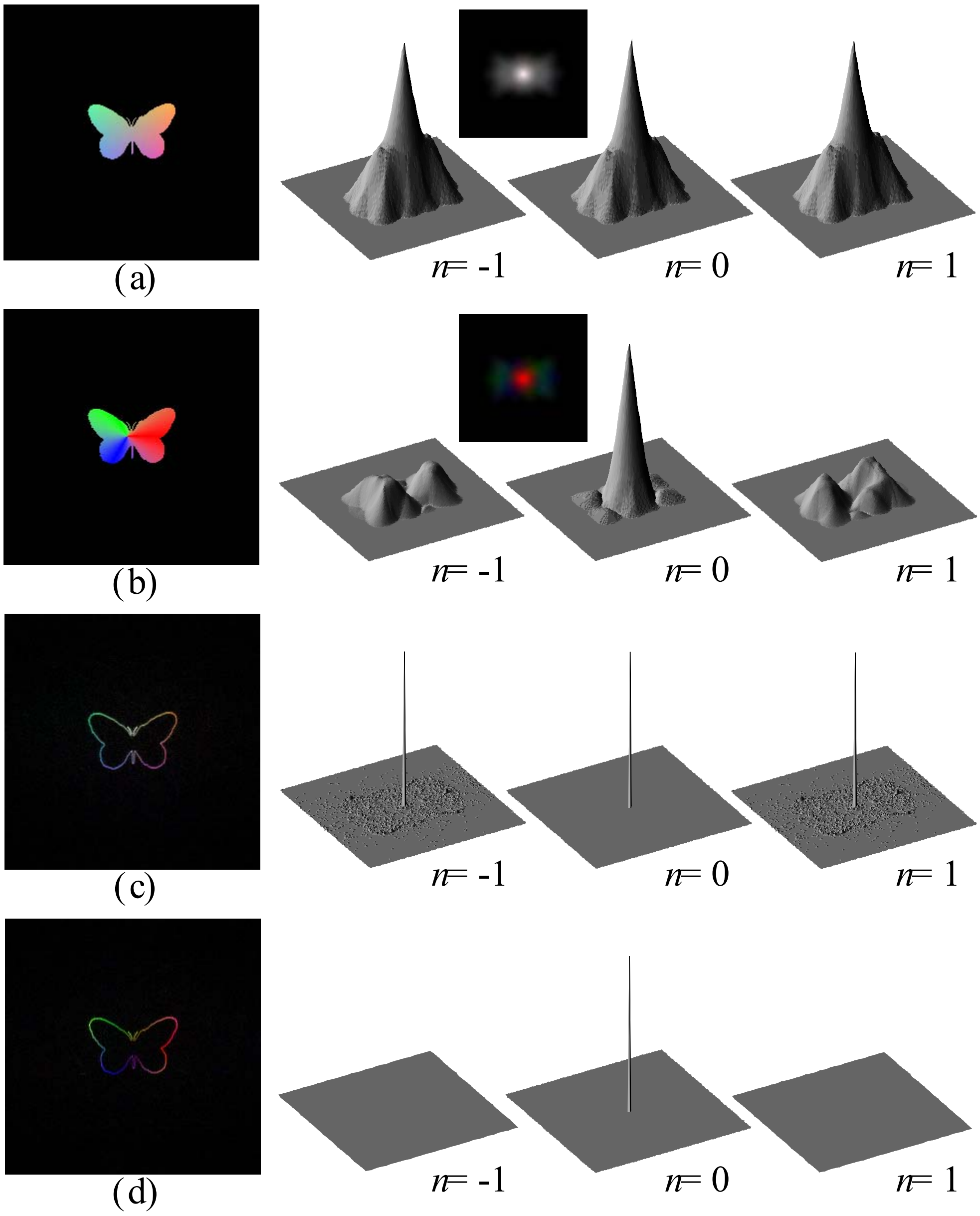


Figure 5.22. (a) Original scene and its three dimensional autocorrelation. (b) Impulse response of the color-spectrum whitening filter and its three dimensional autocorrelation. (c) Idem for spatial-spectrum whitening. (d) Idem for three dimensional spectrum whitening.

The response of the color-spectrum whitening is represented in Figure 5.22b. The shape of the butterfly is not changed, but the colors of the pixels have unit saturation and unit intensity. The autocorrelation presents a peak only in the n=0 channel. That means that the peak is narrower along the color axis than for the classic filter. Note that the value at the center of the n≠0 channels of the correlation is zero. This is a consequence of the decorrelation of the channels of the response image, as we demonstrate next. At the center of the spatial coordinates, the three dimensional correlation can be written as the addition of the color-wise correlation of all the pixels, as follows:

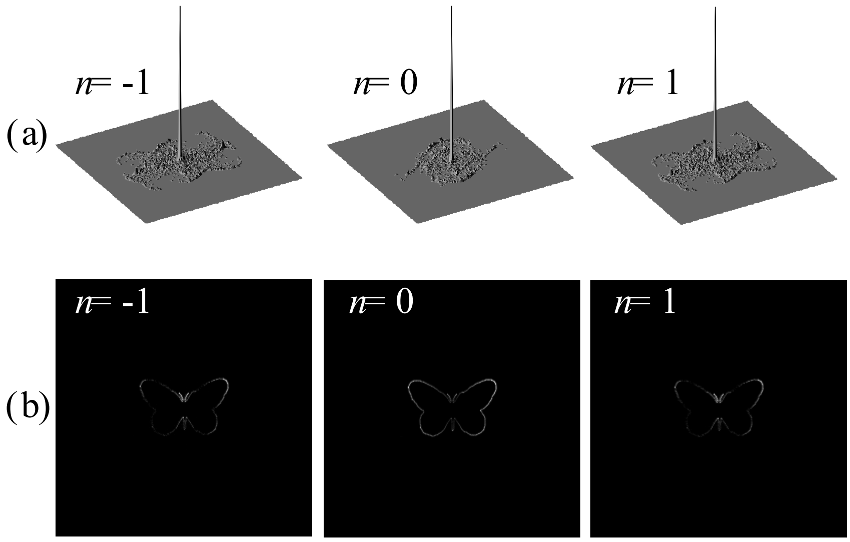
$$\left. s \overset{3D}{\otimes} r \right|_{(0,0,n)} = \sum_{x=0}^{D_x-1} \sum_{y=0}^{D_y-1} \left. s \overset{Color}{\otimes} r \right|_{(x,y,n)} . \qquad (5.27)$$

Because the color components of any pixel are decorrelated with the other color components of the same pixel, all the terms in the series are zero for n≠0. However, the n≠0 channels of the three dimensional correlation can be different from zero at the pixels that are not at the origin because the color components of a pixel are not decorrelated with the color components of different pixels.

When the spatial-frequency spectrum whitening is performed, the corresponding impulse response function is the one presented in Figure 5.22c. The usual edge enhancement is produced, but the hue and saturation of the colors are not changed. In this case, the three dimensional autocorrelation presents a peak that is spatially sharp but wide along the color axis, therefore it is present in the three channels of the three dimensional autocorrelation. Analogously to the case of the color whitening, the fact that the pixels are spatially decorrelated for all the channels, implies that the n=0 channel is zero everywhere except at the origin. However, there is still cross correlation between the different channels of each pixel, and that produces the sharp peaks in the n≠0 channels. Moreover, there is also correlation between the different pixels of different channels of the color image, and this produces the nonzero background in the n≠0 channels.

Finally, when the three dimensional Fourier spectrum of the color image is whitened, the impulse response function is decorrelated both along the spatial and the color dimensions. One can observe in Figure 5.22d that the edges of the butterfly are enhanced, and that the color saturation and intensity are normalized. In this case, the three dimensional correlation has only one sharp peak in the center of the n=0 channel, what means that the image is decorrelated both spatially and also along the color axis.

We have shown that the three dimensional phase only filter impulse response function is decorrelated both along the color axis and along the spatial axes. However, this does not mean that the channels, treated separately are spatially decorrelated, nor that the color distribution of each pixel is decorrelated along the color axis. This is illustrated in Figure 5.23. The profiles of the magnitude of the two dimensional correlation obtained separately for each channel of the impulse response function of the three dimensional phase only filter are represented in Figure 5.23a. One can observe that for the three channels there is a high peak in the center, but the background is not zero, what means that the channels are not spatially decorrelated. We also present in Figure 5.23b (as gray level images) the magnitude of the color-wise correlation for the same response image, The fact that the $n \neq 0$ channels are not null indicates that the image is not decorrelated along the color axis. That is, that the colors of the impulse response function are not on the unit intensity and saturation circle. This is shown in Figure 5.23c, where a chromaticity histogram of the impulse response function of the three dimensional phase only filter is presented.



(c)

Figure 5.23. (a) Spatial two dimensional autocorrelation of the channels of the impulse response function (*IRF*) of the three dimensional phase only filter (*3DPOF*). (b) Channels of the color-wise autocorrelation of the *IRF* of the *3DPOF*. (c) Chromaticity spectrum of the *IRF* of the *3DPOF*

5.8.Whitening filters for color pattern recognition

Whitening operations can be used for color pattern recognition because the decorrelation of the channels implies an increase of the discrimination capability of the filters. We illustrate this in Figure 5.24. We consider the input scene shown in Figure 5.24a. It is composed of two butterflies, the upper one is the butterfly studied in Figure 5.22, and it is the one to be recognized. The other one is a butterfly with the same shape but different color distribution. For both objects the colors have saturation close to zero. We have considered the three dimensional correlation of the scene using the filters that come from whitening the color spectrum, the spatial spectrum and the three dimensional spectrum of the reference object. The three dimensional correlation, and a profile of its n=0 channel are represented in Figure 5.24b, c, d and e for the cases of classic matched filtering, color spectrum whitening, spatial spectrum whitening and three dimensional whitening, respectively.

In the case of the classic matched filter (Figure 5.24b) there are two wide peaks of similar intensities. In addition they are almost white, what indicates that the intensity of the peaks is similar in the three channels of the three dimensional correlation. Because the two butterflies have the same shape and very unsaturated colors, the two peaks have similar intensities, and the discrimination is very poor.

The three dimensional correlation using a 3D color-whitened filter is represented in Figure 5.24c. In this case the two peaks are spatially wide, but their distribution along the color axis is different. This way, the components in the n≠0 channels of the peak corresponding to the target object is lower than for the three dimensional classic matched filter, and it is encoded in a reddish color. However, the peak corresponding to the non-target object is encoded in a greenish color, what indicates that the peak has its maximum at the n=1 channel. This way, in the n=0 channel (that is the channel useful for the recognition) the peak for the target butterfly is clearly more intense than the peak for the non-target object.

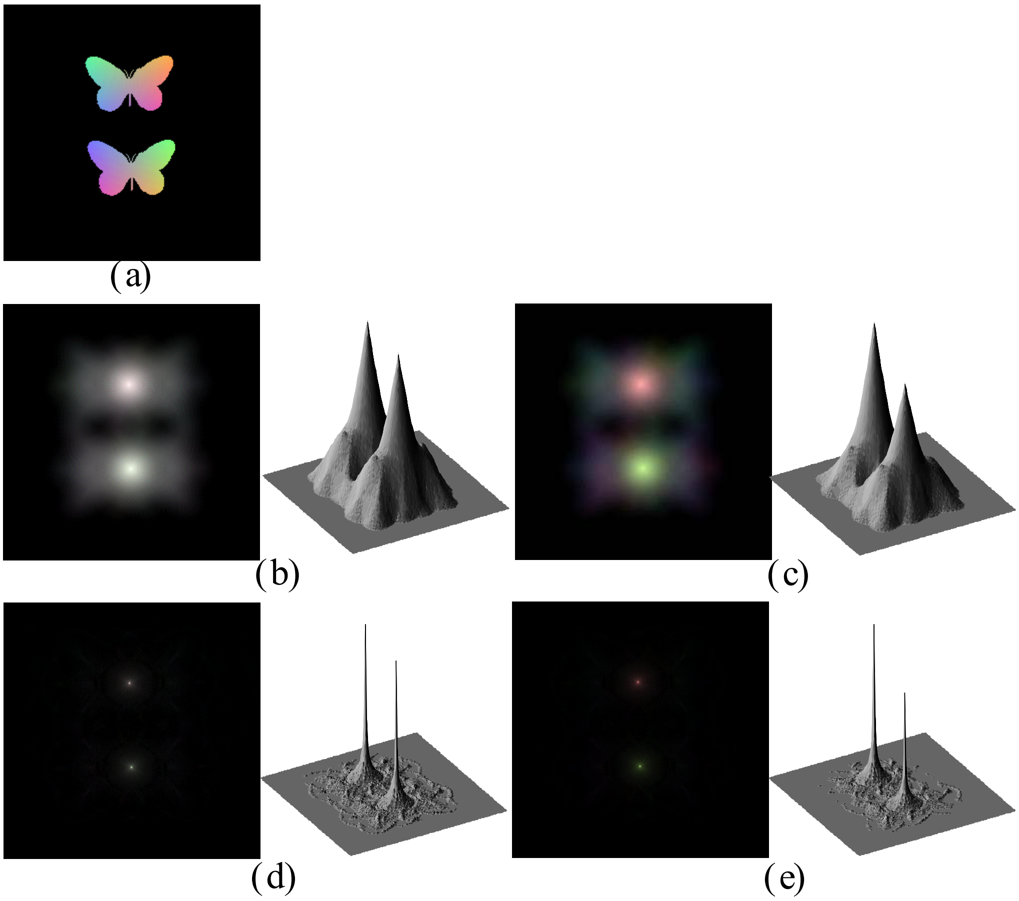


Figure 5.24. (a)Sample input scene. (b) Classic three dimensional correlation. (c) Color spectrum whitening correlation. (d) Spatial spectrum whitening correlation . (e)Three dimensional phase only filter correlation.

Figure 5.24d represents the correlation for the three dimensional spatial-whitened filter. In this case the peaks are spatially sharp but wide along the color axis. This way, because the two butterflies have the same shape and non-saturated colors, in the n=0 channel both peaks have almost the same intensity.

The correlation using the three dimensional phase only filter is represented in Figure 5.24e. One can observe that the three dimensional correlation, encoded as a color image presents two sharp spots colored in two different colors. The spot corresponding to the target object is colored in a reddish color because the n=0 channel is encoded in red. In this case, as an effect of the three dimensional spectrum whitening, that represents whitening along both the color axis and the spatial axes, the n=0 channel of the three dimensional correlation presents two sharp peaks, and the discrimination is improved.

5.8.1. *Input scene filtering operations*

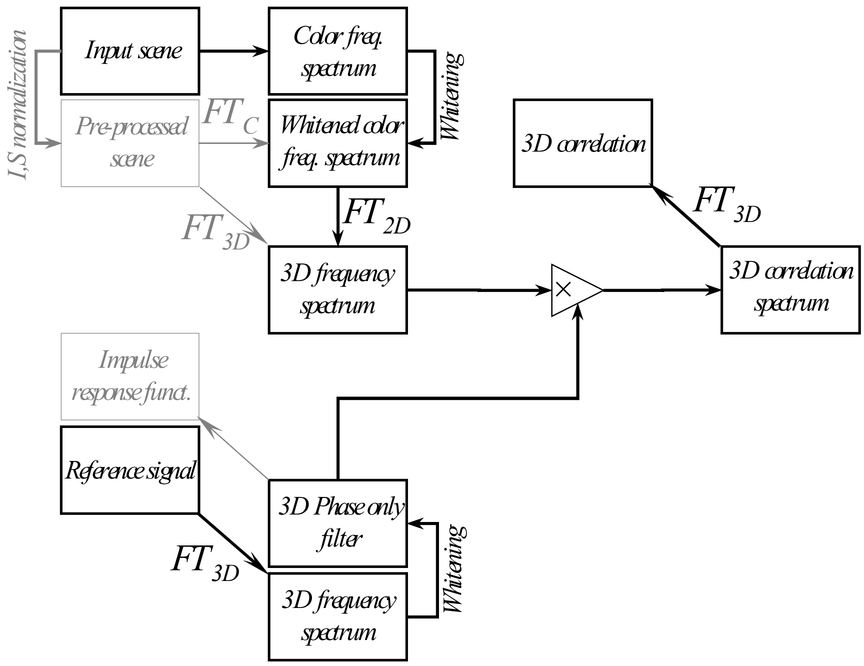


Figure 5.25. Scheme of the three dimensional phase only correlation with scene color spectrum whitening.

We have demonstrated that color spectrum whitening can also be performed in the direct domain by means of a normalization of the intensity and saturation of the colors of the image, so color spectrum whitening can be efficiently performed, without necessity of obtaining the color Fourier spectrum. This enables to do a pre-processing of the scene in which intensity and saturation normalizations are performed so as to increase the discrimination capability of the correlation process. An scheme of this correlation operation is given in Figure 5.25.

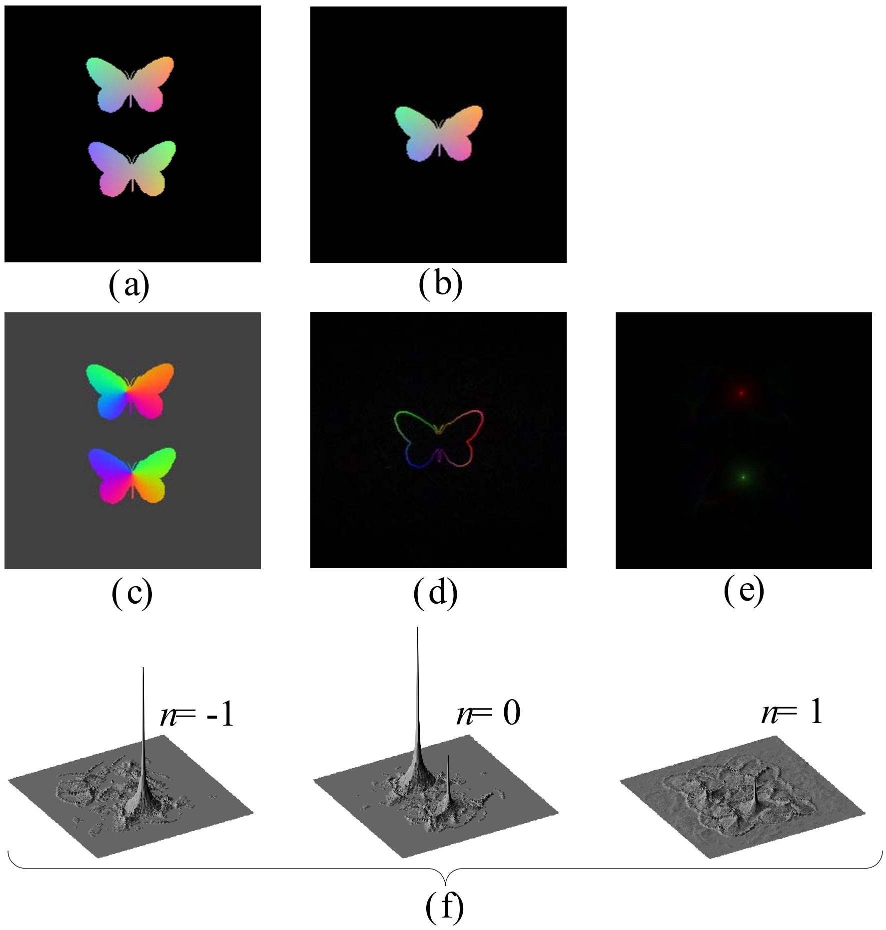


Figure 5.26. Color whitened scene phase only filter correlation. (a) input scene. (b) reference. (c) Preprocessed scene. (d) Filter impulse response function. (e) Color encoded three dimensional correlation. (f) Profile of the channels of the correlation.

The input scene color Fourier spectrum is obtained and whitened, and then the two dimensional Fourier transform is applied to obtain a modified three dimensional spectrum of the input scene. This modified three dimensional spectrum can also be obtained by normalizing the intensity and the saturation of the scene in the direct domain and by applying the three dimensional Fourier transform. By multiplying the three dimensional spectrum of the preprocessed scene by a phase only filter matched to a reference scene the spectrum of a correlation function is obtained. Finally the inverse three dimensional Fourier transform is applied and the correlation function is obtained.

This correlation scheme is applied to the same butterflies scene in Figure 5.24. The input scene and the reference are represented in Figure 5.26a and b respectively. After the preprocessing the scene is transformed onto the image in Figure 5.26c. The magnitude of the impulse response function of the phase only filter is represented in Figure 5.26d. The resulting three dimensional correlation is represented as a color image in Figure 5.26e, and a profile of the magnitude of the channels is given in Figure 5.26f.

The correlation function presents two sharp peaks because the three dimensional phase only filter is used. In addition the two peaks are mostly in different channels, and therefore encoded in different colors. So, while the autocorrelation peak is located at the n=0 channel, the cross correlation is centered at the n=1 channel. This is because the color distribution of the non-target object corresponds to a shift, (by 0.75 of a channel) along the color axis of the color distribution of the target object. In addition, at the n=0 channel, the cross-correlation is lower than for the case without pre-processing. We illustrate it in Figure 5.27. We have represented the magnitude at the n=0 channel of the three dimensional correlation for different whitening operations. Figure 5.27a represents the case for whitening of the spatial-whitening filter. It presents a poor discrimination because both butterflies have the same shape, and almost unsaturated color distributions. Figure 5.27b represents the case for three dimensional phase only filter. The cross correlation peak is lower than for the spatial-whitening case.

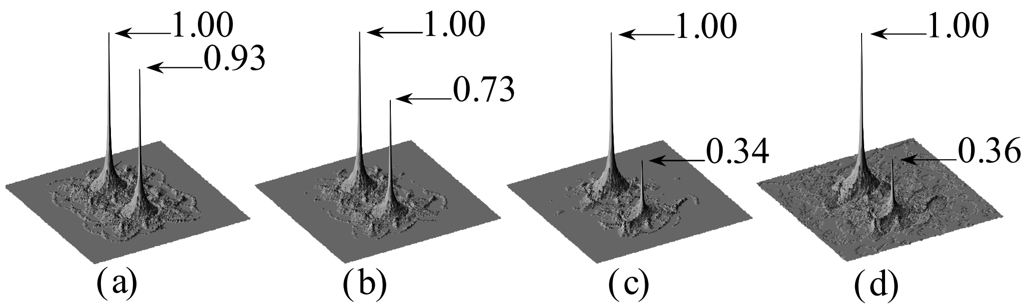


Figure 5.27. Magnitude at the n=0 channel of the three dimensional correlation for the two butterflies scene with different whitening operations:(a) Space-whitened filter correlation. (b) three dimensional phase only filter. (c) scene color-whitening, and *3D* phase only filter. (d) scene color-whitening plus a constant, and *3D* phase only filter.

In addition this filter does not require partial transformations and therefore can be optically implemented as we demonstrate in next chapter.

Both Figure 5.27c and d represent the same case of scene color-whitening and three dimensional phase only filter, however, because the pre-processed scene contains negative values, a constant has been added to the preprocessed scene for in Figure 5.27d so as to remove the negative values and to allow its implementation in an optical correlator. In both cases a great improvement of the discrimination is produced with respect to the cases without scene preprocessing. However, because of adding a constant to the input scene, the contrast of the resulting correlation is affected and the discrimination is a bit smaller.

Note that the correlation schemes in Figure 5.27b, c and d do not require partial Fourier transforms, along the spatial axes or the color axis, only three dimensional Fourier transforms are involved in the correlation. In addition, for Figure 5.27b and d it is not necessary to represent negative valued images. However, to perform these operations optically it is still necessary to represent three dimensional functions and to obtain their three dimensional Fourier spectra transform. The way how to do this is explained in next chapter.