UNIVERSITAT ROVIRA I VIRGILI

THE SIMULTANEOUS (STRONG) METRIC DIMENSION OF GRAPH FAMILIES

Yunior Ramírez Cruz

DOCTORAL THESIS

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WE STATE that the present study, entitled "The Simultaneous (Strong) Metric Dimension of Graph Families", presented by Yunior Ramírez-Cruz for the award of the degree of Doctor, has been carried out under our supervision at the Department of Computer Engineering and Mathematics of this university.

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Contents

Introduction

Graphs may be used to model a large variety of network structures. For instance, in computer networks, servers, hosts or hubs can be represented as vertices in a graph and edges can represent connections between them. Likewise, the Web, social networks or transportation infrastructures can be modelled as graphs, where the vertices represent webpages, users and population centres, respectively; and the edges represent hyperlinks, personal relations, and roads, in that order.

In the aforementioned graph-based representation of a computer network, each vertex may be seen as a possible location for an intruder (fault in the network, spoiled device, unauthorized connection) and, in this sense, a correct surveillance of each vertex of the graph to control such a possible intruder would be worthwhile. According to this fact, it would be desirable to uniquely recognize each vertex of the graph. In order to solve this problem, Slater [78, 80] brought in the notion of locating sets and locating number of graphs. Also, Harary and Melter [36] independently introduced the same concept, but using the terms resolving sets and metric dimension to refer to locating sets and locating number, respectively. Moreover, in a more recent article, by Sebö and Tannier [76], the terminology of metric generators and metric dimension for the concepts mentioned above, began to be used. These terms arose from the notion of metric generators of metric spaces, introduced by Blumenthal in [4]. In this thesis we follow this terminology, as well as the notation introduced in [76].

Informally, a metric generator is an ordered subset S of vertices in a graph G, such that every vertex of G is uniquely determined by its vector of distances to the vertices in S. The cardinality of a minimum metric generator for G is called the metric dimension of G.

After the first papers on this topic were published, some authors developed diverse theoretical works on the subject including, for example, [8, 9, 11, 12, 15, 19, 37, 38, 43, 48, 52, 64, 68, 71, 77, 81, 86]. Several applications of the metric generators have also been presented. In Chemistry, a usual representation for the structure of a chemical compound is a labeled graph where the vertex and edge labels specify the atom and bond types, respectively. As described in [12, 15], metric generators allow to obtain unique representations for chemical substances. In particular, they were used in pharmaceutical research for discovering patterns common to a variety of drugs, as described in [44, 45]. Furthermore, this topic has some applications to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [64]. Other applications to navigation of robots in networks and other areas appear in [12, 39, 48]. Some interesting connections between metric generators in graphs and the Mastermind game or coin weighing have been presented in [9]. Moreover, we refer the reader to the work [1], where some historical evolution, non-standard terminologies and more references to this topic can be found.

Apart from the initial concept of metric generator, numerous variations of the concept have been studied. In general, these variations can be classified into five types. Notice that we do not mention all of them, but just some of the most remarkable ones, according to our point of view.

1. Metric generators which also satisfy other properties of the graph:
    - resolving dominating set [6], when the metric generator is also a dominating set;
    - independent resolving set [16], when the metric generator is also an independent set;
    - connected resolving set [74, 75], when the subgraph induced by the metric generator is connected.
2. Metric generators which have a modified condition of resolvability:
    - adjacency resolving set [43], a set such that any two different vertices not belonging to the set have different neighborhood in this set;
    - strong metric generator [67, 76], metric generators where a stronger condition is set for a vertex to distinguish a vertex pair, namely that this vertex and the two vertices of the pair (in either order) lie in a minimum-length path;

- local metric generator [70], a set such that every two adjacent vertices of the graph have distinct vectors of distances to the vertices in this set;

3. Metric generators featuring a combination of criteria 1 and 2:

- locating-dominating set [79, 80], locating set (any two different vertices not belonging to the set have different neighbors in this set) which is a dominating set;
- identifying code [34, 47], a set such that any two different vertices of the graph have different closed neighborhoods in this set and is also a dominating set.

4. Partitions of the vertex set of a graph having some metric properties:

- resolving partitions [17, 32, 72], a partition such that every two different vertices of the graph have distinct vectors of distances to the sets of the partition;
- strong resolving partition [85], a partition where every two different vertices of the graph belonging to the same set of the partition are strongly resolved by some set of the partition;
- metric coloring [14], a partition such that every two adjacent vertices of the graph have distinct vectors of distances to the set of the partition.

5. Variants which are extensions of the metric generators:

- k-metric generator [24, 22], a set such that any pair of vertices of the graph is distinguished by at least k vertices of this set.

Consider the following problem proposed in [48], which deals with the movement of a point-robot in a "graph space". The robot can locate itself by the presence of distinctively labeled "landmarks" in the graph space. On a graph, there is neither the concept of direction nor that of visibility. Instead, it was assumed in [48] that the robot can sense the distances to a set of landmarks. If the robot knows its distances to a sufficiently large number of landmarks, its position on the graph can be uniquely determined. This suggests the following question: *given a graph G, what is the smallest number of landmarks needed, and where should they be located, so that the distances* *to the landmarks uniquely determine the robot's position on G?* This problem can be solved by determining the metric dimension and a metric basis of G.

In this thesis, we consider the following extension of the robot navigation problem. Suppose that the topology of the navigation network may change within a range of possible graphs, say $G_1, G_2, ..., G_k$. This scenario may reflect, for example, the use of a dynamic network whose links change over time. In this case, the problem mentioned above becomes that of determining the minimum cardinality of a set S of vertices which is simultaneously a metric generator for each graph G_i, $i \in \{1, ..., k\}$. So, if S is a solution to this problem, then the position of a robot can be uniquely determined by the distance to the elements of S, regardless of the graph G_i that models the network along whose edges the robot moves at each moment.

To handle situations as the one described above, we introduce the notion of simultaneous metric generator, which naturally leads to that of simultaneous metric basis and simultaneous metric dimension. Throughout the thesis, we study the behaviour of these parameters on a wide variety of graph families and introduce analogous simultaneity notions to other variants of resolvability, namely adjacency generators and strong metric generators. Our study involves both the combinatorial properties of these parameters and complexity issues regarding their computation.

The study of simultaneous parameters in graph families was introduced by Brigham and Dutton in [7], where they studied simultaneous domination. This idea should not be confused with studies on families sharing a constant value on a parameter, for instance the study presented in [40], where several graph families such that all of its members have the same metric dimension are studied.

The thesis is organized as follows. In Chapter 1, we recall some basic definitions on graph theory and present the main concepts regarding resolvability, focusing on the three variants of interest for the thesis: metric, adjacency and strong metric generators. Chapter 2 introduces the main topic of the thesis, the simultaneous metric dimension of graph families, and presents a number of important results on this parameter. The study of the simultaneous metric dimension is continued in Chapter 3, which focuses in families composed by product graphs. In this chapter, a second notion of simultaneous resolvability is introduced, namely the simultaneous adjacency dimension, which is shown to be a valuable tool for studying the simultaneous metric dimension of these families. We further explore into the extensibility of the notion of simultaneity in Chapter 4, where we define and study the simultaneous strong metric dimension. Finally, Chapter 5 discusses the issues related to the computability of the simultaneous resolvability parameters presented throughout the thesis. To conclude, we briefly discuss the most important results presented in the thesis, the associated scientific production and the most promising directions of future work.

Chapter 1

Basic concepts and tools

We begin by establishing the basic terminology and notation used throughout the thesis. For the sake of completeness we refer the reader to the books [20, 82]. Graphs considered herein are undirected, finite and contain neither loops nor multiple edges. Let $G = (V, E)$ be a graph of order $n = |V(G)|$. A graph is nontrivial if $n \geq 2$. We use the notation $u \sim v$ (negated as $u \nsim v$) for two adjacent vertices u and v of G, and the notation $G \cong H$ for two isomorphic graphs G and H. For a vertex v of G, $N_G(v)$ denotes the set of neighbours of v in G, *i.e.*, $N_G(v) = \{u \in V(G) : \; u \sim v\}$. The set $N_G(v)$ is called the *open neighbourhood* of the vertex v in G and $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighbourhood* of v in G. The *degree* of a vertex v of G is denoted by $\delta_G(v)$, *i.e.*, $\delta_G(v) = |N_G(v)|$. The open neighbourhood of a set $S \subseteq V(G)$ of vertices of G is $N_G(S) = \bigcup_{v \in S} N_G(v)$ and the closed neighbourhood of S is $N_G[S] = N_G(S) \cup S$. A *dominating set* of a graph G is a set $M \subseteq V(G)$ such that $N_G[M] = V(G)$. The minimum cardinality of a dominating set of G is its domination number, denoted by $\gamma(G)$. If there is no ambiguity, we will simply write $N(v)$, $N[v]$, $\delta(v)$, $N(S)$ or $N[S]$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *girth* of a graph G is the length of a shortest cycle contained in G, and is defined as $\mathtt{g}(G)$.

We use the notation K_n, C_n, P_n, and N_n for the *complete graph*, *cycle*, *path*, and *empty graph*, respectively, of order n. Moreover, we write $K_{s,t}$ for the *complete bipartite graph* of order $s + t$ and, in particular, we write $K_{1,n}$ for the *star graph* of order $n + 1$. Let T be a *tree*, a vertex of degree one in T is called a *leaf* and the set of leaves in T is denoted by $\Omega(T)$.

The *distance* between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path between u and v in G. The *diameter* of a graph G, denoted by $D(G)$, is the longest distance between any two vertices in G. If G is not connected, then we assume that the distance between any two vertices belonging to different connected components of G is infinity and, thus, its diameter is $D(G) = \infty$.

We recall that the *complement* of a graph G is a graph $G^c = (V(G), E^c)$ such that $uv \in E^c$ if and only if $uv \notin E(G)$. For a set $X \subseteq V(G)$, the *subgraph induced* by X is denoted by $\langle X\rangle_G$. If there is no ambiguity, we will simply write $\langle X\rangle$, and if $X = \{v\}$ we will write $\langle v\rangle$. A vertex of a graph is a *simplicial vertex* if the subgraph induced by its neighbours is a complete graph. Given a graph G, we denote by $\sigma(G)$ the set of simplicial vertices of G. Note that for a tree T, $\sigma(T) = |\Omega(T)|$. We recall that a *clique* in a graph G is a set of pairwise adjacent vertices. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a maximum clique in G. Two distinct vertices u, v are called *true twins* if $N[u] = N[v]$. Likewise, two distinct vertices u, v are called *false twins* if $N(u) = N(v)$. In general, two distinct vertices u, v are called *twins* if they are true twins or they are false twins. In this sense, a vertex x is a *twin* if there exists $y \neq x$ such that they are twins. We say that $X \subset V(G)$ is a *twins-free clique* in G if the subgraph induced by X is a clique and every $u, v \in X$ satisfy $N_G[u] \neq N_G[v]$, *i.e.*, the subgraph induced by X is a clique and it contains no true twins. Note that, by definition, cliques do not contain false twins. We say that the *twins-free clique number* of G, denoted by $\varpi(G)$, is the maximum cardinality among all twins-free cliques in G. Clearly, $\omega(G) \geq \varpi(G)$. We refer to a twins-free clique of a graph G of cardinality $\varpi(G)$ as a $\varpi(G)$-set of G. Finally, recall that an *independent set* is a set of pairwise non-adjacent vertices and that the *independence number* of a graph G, denoted by $\alpha(G)$, is the number of vertices in a maximum independent set of G. Figure 1.1 shows examples of basic concepts such as twins and twins-free cliques.

The *Cartesian product* $G \Box H$ of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph whose vertex set is $V(G \Box H) = V_1 \times V_2$ and any two distinct vertices $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ are adjacent in $G \Box H$ if and only if either $x_1 = y_1$ and $x_2 \sim y_2$, or $x_1 \sim y_1$ and $x_2 = y_2$. The *hypercube* of order 2^r, $r \geq 0$, denoted by Q_r, is defined recursively as

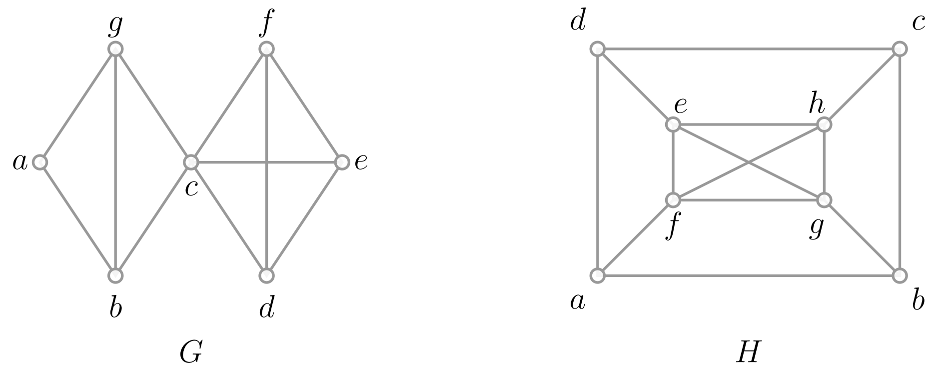

Figure 1.1: The set $\{d, e, f\} \subset V(G)$ is composed by true twin vertices in G. Notice that b and g are true twin vertices in G which are not simplicial, while f and d are true twin and simplicial vertices. The set $\{e, f, g, h\} \subset V(H)$ is a twins-free clique in H.

$$Q_r = \begin{cases} K_1 & \text{if } r = 0 \\ K_2 \Box Q_{r-1} & \text{otherwise.} \end{cases}$$

A graph G is 2-*antipodal* if for each vertex $x \in V(G)$ there exists exactly one vertex $y \in V(G)$ such that $d_G(x, y) = D(G)$. For example, even cycles are 2-antipodal graphs. Other definitions not defined herein will be given the first time that the concept appears in the text.

1.1 Resolvability

A *metric space* is a pair of the form (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function, referred to as a *metric* , such that for any $x, y, z \in X$,

(i) $d(x, y) \geq 0$,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) $d(x, y) = d(y, x)$, and

(iv) $d(x, y) \leq d(x, z) + d(z, y)$.

A *generator* for a metric space is a set $S \subseteq X$ with the property that every element of X is uniquely determined by its distances from the elements of S. Given a simple and connected graph G, we consider the metric d_G :

$V(G) \times V(G) \to \mathbb{N} \cup \{0\}$, where $d_G(x, y)$ is the length of a shortest path between x and y. The pair $(V(G), d_G)$ is readily seen to be a metric space. A vertex $v \in V(G)$ is said to *distinguish* two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subset V(G)$ is said to be a *metric generator* for G if any pair of vertices of G is distinguished by some element of S. Assume that an order is imposed on the elements of a set $S = \{w_1, w_2, \ldots, w_k\}$. Then, the *metric vector*, or *metric representation*, of a vertex $v \in V(G)$ relative to S is the vector $(d_G(v, w_1), d_G(v, w_2), \ldots, d_G(v, w_k))$. Thus, S is a metric generator if distinct vertices have distinct metric vectors relative to S. A minimum cardinality metric generator is called a *metric basis* and its cardinality, the *metric dimension* of G, is denoted by $\dim(G)$.

A related parameter was introduced in [43] for studying the metric dimension of lexicographic product graphs. A set $S \subset V(G)$ is said to be an *adjacency generator* for G if for any pair of vertices $u, v \in V(G)$ there exists some $x \in S$ such that x is adjacent to exactly one of u and v. A minimum cardinality adjacency generator is called an *adjacency basis* of G, and its cardinality the *adjacency dimension* of G, denoted by $\dim_A(G)$. Since any adjacency basis is a metric generator, $\dim(G) \leq \dim_A(G)$. Besides, for any connected graph G of diameter at most two, $\dim_A(G) = \dim(G)$ [43]. As pointed out in [26, 27], any adjacency generator of a graph $G = (V, E)$ is also a metric generator in a suitably chosen metric space. Given a positive integer t, we define the distance function $d_{G,t} : V \times V \to \mathbb{N} \cup \{0\}$, where

$$d_{G,t}(x, y) = \min\{d_G(x, y), t\}.$$

Then any metric generator for $(V, d_{G,t})$ is a metric generator for $(V, d_{G,t+1})$ and, as a consequence, the metric dimension of $(V, d_{G,t+1})$ is less than or equal to the metric dimension of $(V, d_{G,t})$. In particular, the metric dimension of $(V, d_{G,1})$ is equal to $|V| - 1$, the metric dimension of $(V, d_{G,2})$ is equal to $\dim_A(G)$ and, if G has diameter $D(G)$, then $d_{G,D(G)} = d_G$ and so the metric dimension of $(V, d_{G,D(G)})$ is equal to $\dim(G)$. Notice that when using the metric $d_{G,t}$ the concept of metric generator needs not be restricted to the case of connected graphs[1]. Moreover, we have that S is an adjacency generator for G if and only if it is an adjacency generator for its complement G^c. This is

[1]For any pair of vertices x, y belonging to different connected components of G we can assume that $d_G(x, y) = \infty$ and so $d_{G,t}(x, y) = t$ for any t greater than or equal to the maximum diameter of a connected component of G.

justified by the fact that, given an adjacency generator S for G, it holds that for every $x, y \in V - S$ there exists $s \in S$ such that s is adjacent to exactly one of x and y, and this property also holds in G^c. Thus, $\dim_A(G) = \dim_A(G^c)$.

The metric dimension has been studied for a wide variety of graphs, e.g. trees [12, 36, 78], unicyclic graphs [12, 68], wheel graphs [37, 77], fan graphs [37], lexicographic product graphs [43], strong product graphs [71], Cartesian product graphs [37, 48] and corona product graphs [86]. Moreover, integer programming models and metaheuristic approaches have been presented for computing or approximating this parameter [12, 19, 52]. As we mentioned previously, the adjacency dimension was introduced as an auxiliary tool for the study of the metric dimension of lexicographic product graphs [43]. Moreover, the adjacency dimension of corona product graphs, as well as its relation to the simultaneous metric dimension of such products, is studied in [26, 27].

A vertex $w \in V(G)$ *strongly distinguishes* two different vertices $u, v \in V(G)$ if $d_G(w, u) = d_G(w, v) + d_G(v, u)$ or $d_G(w, v) = d_G(w, u) + d_G(u, v)$, *i.e.*, there exists some shortest $w - u$ path containing v or some shortest $w - v$ path containing u. A set S of vertices in a connected graph G is a *strong metric generator* for G if every pair of vertices of G is strongly distinguished by some vertex of S. A minimum cardinality strong metric generator for G is called a *strong metric basis* of G, and its cardinality is the *strong metric dimension* of G, denoted by $\dim_s(G)$.

One can immediately see that a strong metric generator is also a metric generator, which leads to $\dim(G) \leq \dim_s(G)$. It was shown in [12] that $\dim(G) = 1$ if and only if G is a path. It now readily follows that $\dim_s(G) = 1$ if and only if G is a path. At the other extreme we see that $\dim_s(G) = n - 1$ if and only if G is the complete graph of order n. For the cycle C_n of order n, the strong metric dimension is $\dim_s(C_n) = \lceil n/2 \rceil$, and if T is a tree, then its strong metric dimension equals $|\Omega(T)| - 1$ (see [76]).

A number of results have been presented regarding the strong metric dimension of Cartesian product graphs [54, 67, 73], Cayley graphs [67], distance-hereditary graphs [63], convex polytopes [50], strong product graphs [61, 62], corona product graphs [57], rooted product graphs [58], lexicographic product graphs [59], Cartesian sum graphs [60] and direct product graphs [73]. Also, some Nordhaus-Gaddum type results for the strong metric dimension of a graph and its complement are known [88]. Beside the theoretical results related to the strong metric dimension, a mathematical programming model [50] and metaheuristic approaches [51, 65] for computing or estimating this parameter have been developed. For more information we refer the reader to [53] as a short survey on the strong metric dimension.

A set S of vertices of G is a *vertex cover* of G if every edge of G is incident with at least one vertex of S. The *vertex cover number* of G, denoted by $\beta(G)$, is the smallest cardinality of a vertex cover of G. We refer to a $\beta(G)$-set in a graph G as a vertex cover of cardinality $\beta(G)$. Oellermann and Peters-Fransen [67] showed that the problem of finding the strong metric dimension of a connected graph G can be transformed into the problem of finding the vertex cover number of another related graph, which they called the *strong resolving graph*. We now describe this approach in detail.

A vertex u of G is *maximally distant* from v if for every vertex $w \in N_G(u)$, $d_G(v,w) \leq d_G(u,v)$. We denote by $M_G(v)$ the set of vertices of G which are maximally distant from v. The collection of all vertices of G that are maximally distant from some vertex of the graph is called the *boundary* of the graph, see [5, 10], and is denoted by $\partial(G)$[2]. If u is maximally distant from v and v is maximally distant from u, then we say that u and v are *mutually maximally distant*. If u is maximally distant from v, and v is not maximally distant from u, then v has a neighbour v_1, such that $d_G(v_1,u) > d_G(v,u)$, *i.e.*, $d_G(v_1,u) = d_G(v,u)+1$. It is easily seen that u is maximally distant from v_1. If v_1 is not maximally distant from u, then v_1 has a neighbour v_2, such that $d_G(v_2,u) > d_G(v_1,u)$. Continuing in this manner we construct a sequence of vertices $v_1, v_2, \ldots$ such that $d_G(v_{i+1},u) > d_G(v_i,u)$ for every i. Since G is finite this sequence terminates with some v_k. Thus for all neighbours x of v_k we have $d_G(v_k,u) \geq d_G(x,u)$, and so v_k is maximally distant from u and u is maximally distant from v_k. Hence every boundary vertex belongs to the set $S = \{u \in V(G) :$ there exists $v \in V(G)$ such that u, v are mutually maximally distant$\}$. Moreover, every vertex of S is a boundary vertex.

For some basic graph classes, such as complete graphs, complete bipartite graphs, cycle graphs and hypercubes, the boundary is simply the whole vertex set. It is not difficult to see that this property also holds for all 2-antipodal graphs. Notice that the boundary of a tree consists of its leaves. Also, it

[2]In fact, the boundary $\partial(G)$ of a graph was defined first in [13] as the subgraph of G induced by the set mentioned in our work with the same notation. We follow the approach of [5, 10] where the boundary of the graph is just the subset of the boundary vertices defined in this article.

is readily seen that $\sigma(G) \subseteq \partial(G)$. As a direct consequence of the definition of mutually maximally distant vertices, we have that every pair of mutually maximally distant vertices x, y of a connected graph G and every strong metric basis S of G satisfy $x \in S$ or $y \in S$.

Based on the previous definitions, the strong resolving graph of a graph $G = (V, E)$, was defined in [67] as the graph $G_{SR} = (V, E')$ where two vertices u, v are adjacent if and only if u and v are mutually maximally distant in G. To illustrate these notions, Figure 1.2 shows examples of basic concepts such as maximally distant vertices, mutually maximally distant vertices and boundary, whereas Figure 1.3 shows the strong resolving graph G_{SR} of the graph G depicted in Figure 1.2.

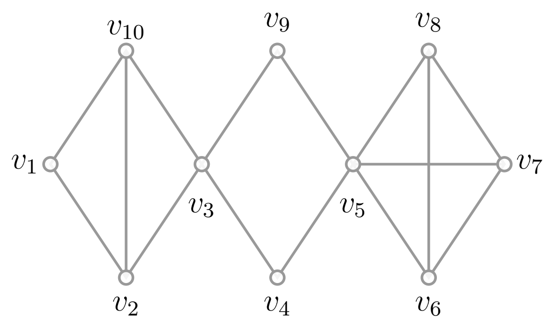

Figure 1.2: All vertices of the set $\{v_1, v_6, v_7, v_8\}$ are pairwise mutually maximally distant. Also, v_2 and v_{10} (v_4 and v_9) are mutually maximally distant. Thus, the boundary of G is $\partial(G) = \{v_1, v_2, v_4, v_6, v_7, v_8, v_9, v_{10}\}$. Now, $M_G(d) = \{v_1, v_6, v_7, v_8, v_9\}$ is the set of vertices which are maximally distant from v_4. Nevertheless, the vertex v_4 is maximally distant only from the vertex v_9.

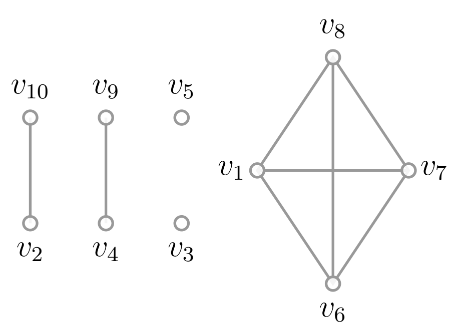

Figure 1.3: Strong resolving graph of the graph G shown in Figure 1.2.

The following result provides a powerful tool for finding the strong metric dimension of a graph.

Theorem 1.1. [67] *For any connected graph G,*

$$\dim_s(G) = \beta(G_{SR}).$$

For some types of graphs, the strong resolving graphs can be obtained relatively easily, as the next result exemplifies, so applying Theorem 1.1 allows to determine their strong metric dimensions.

Remark 1.2.

(a) *If $\partial(G) = \sigma(G)$, then $G_{SR} \cong K_{\partial(G)}$. In particular, $(K_n)_{SR} \cong K_n$ and for any tree T, $(T)_{SR} \cong K_{|\Omega(T)|}$.*

(b) *For any 2-antipodal graph G of order n, $G_{SR} \cong \bigcup_{i=1}^{\frac{n}{2}} K_2$. Even cycles are 2-antipodal. Thus, $(C_{2k})_{SR} \cong \bigcup_{i=1}^{k} K_2$.*

(c) *For odd cycles $(C_{2k+1})_{SR} \cong C_{2k+1}$.*

Chapter 2

The simultaneous metric dimension of graph families

In this chapter, we introduce the concept of simultaneous metric dimension and investigate its core properties, namely its bounds, extreme values and its relations to the metric dimensions of individual graphs composing the families. We also analyse the behaviour of this parameter on several families for which interesting facts may be pointed out.

Given a family $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ of (not necessarily edge-disjoint) connected graphs $G_i = (V, E_i)$ with common vertex set[1] V (the union of whose edge sets is not necessarily the complete graph), we define a *simultaneous metric generator* for $\mathcal{G}$ to be a set $S \subseteq V$ such that S is simultaneously a metric generator for each G_i. We say that a minimum cardinality simultaneous metric generator for $\mathcal{G}$ is a *simultaneous metric basis* of $\mathcal{G}$, and its cardinality the *simultaneous metric dimension* of $\mathcal{G}$, denoted by $\mathrm{Sd}(\mathcal{G})$ or explicitly by $\mathrm{Sd}(G_1, G_2, ..., G_k)$. An example is shown in Figure 2.1, where the set $\{v_3, v_4\}$ is a simultaneous metric basis of the family $\{G_1, G_2, G_3\}$.

2.1 General bounds

The following result is a direct consequence of the definition of simultaneous metric generators and bases.

[1]Although, in general, we will denote the common vertex set simply as V, when necessary we will use the notation $V(\mathcal{G})$ to avoid ambiguities.

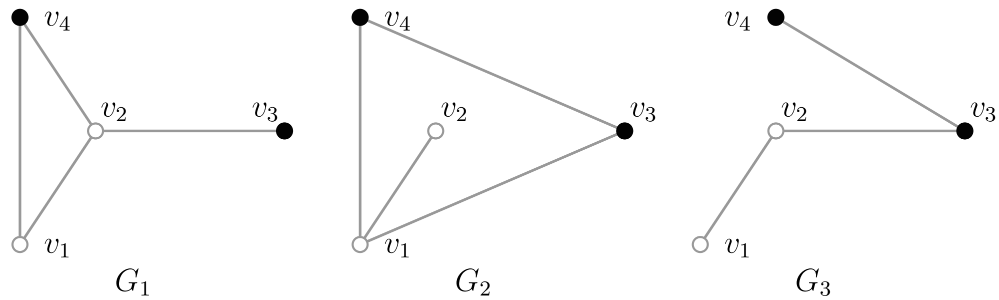

Figure 2.1: The set $\{v_3, v_4\}$ is a simultaneous metric basis of $\{G_1, G_2, G_3\}$. Thus, $\mathrm{Sd}(G_1, G_2, G_3) = 2$.

Remark 2.1. *For any family $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ of connected graphs with common vertex set V and any subfamily $\mathcal{H}$ of $\mathcal{G}$,*

$$\mathrm{Sd}(\mathcal{H}) \leq \mathrm{Sd}(\mathcal{G}) \leq \min\left\{|V| - 1, \sum_{i=1}^{k} \dim(G_i)\right\}.$$

In particular,

$$\max_{i \in \{1,...,k\}} \{\dim(G_i)\} \leq \mathrm{Sd}(\mathcal{G}).$$

The inequalities above are sharp. For instance, for the family of graphs shown in Figure 2.1 we have $\mathrm{Sd}(G_1, G_2, G_3) = 2 = \dim(G_1) = \dim(G_2) = \max_{i \in \{1,2,3\}} \{\dim(G_i)\}$, while for the family of graphs shown in Figure 2.2 we have that $\mathrm{Sd}(G_1, G_2, G_3) = 3 = |V| - 1$.

The following result is a direct consequence of Remark 2.1.

Corollary 2.2. *Let $\mathcal{G}$ be a family of connected graphs on a common vertex set. If $K_n \in \mathcal{G}$, then*

$$\mathrm{Sd}(\mathcal{G}) = n - 1.$$

As shown in Figure 2.2, the converse of Corollary 2.2 does not hold.

Theorem 2.3. *Let $\mathcal{G}$ be a family of connected graphs with the same vertex set V. Then $\mathrm{Sd}(\mathcal{G}) = |V| - 1$ if and only if for every pair $u, v \in V$, there exists a graph $G_{uv} \in \mathcal{G}$ such that u and v are twin vertices in G_{uv}.*

Proof. We first note that for any connected graph $G = (V, E)$ and any vertex $v \in V$ the set $V - \{v\}$ is a metric generator for G. So, if $\mathrm{Sd}(\mathcal{G}) = |V| - 1$, then for every $v \in V$, the set $V - \{v\}$ is a simultaneous metric basis of $\mathcal{G}$ and, as a consequence, for every $u \in V - \{v\}$ there exists a graph $G_{uv} \in \mathcal{G}$

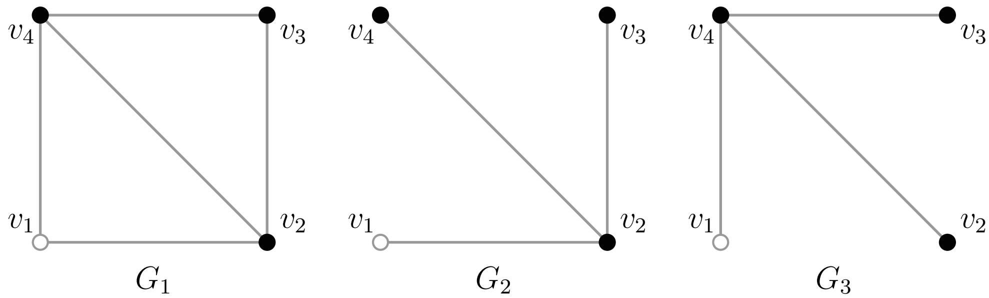

Figure 2.2: The set $\{v_2, v_3, v_4\}$ is a simultaneous metric basis of the family $\{G_1, G_2, G_3\}$. Thus, $\mathrm{Sd}(G_1, G_2, G_3) = 3 = |V| - 1$.

such that the set $V - \{u, v\}$ is not a metric generator for G_{uv}, *i.e.*, for every $x \in V - \{u, v\}$ we have $d_{G_{u,v}}(u, x) = d_{G_{u,v}}(v, x)$. So u and v must be twin vertices in $G_{u,v}$.

Conversely, if for every $u, v \in V$ there exists a graph $G_{uv} \in \mathcal{G}$ such that u and v are twin vertices in G_{uv}, then for any simultaneous metric basis B of $\mathcal{G}$ either $u \in B$ or $v \in B$. Hence, all but one element of V must belong to B. Therefore $|B| \geq |V| - 1$ and, by Remark 2.1, we conclude that $\mathrm{Sd}(\mathcal{G}) = |V| - 1$. □

Notice that Corollary 2.2 is also a consequence of Theorem 2.3 as is the next result. We recall that the centre of a star graph $K_{1,t}$ is the vertex of degree t.

Corollary 2.4. *Let $\mathcal{G}$ be a family of connected graphs with the same vertex set V. If $\mathcal{G}$ contains three star graphs having different centers, then* $\mathrm{Sd}(\mathcal{G}) = |V| - 1$.

It was shown in [12] that for any connected graph G of order n and diameter $D(G)$,

$$\dim(G) \leq n - D(G). \tag{2.1}$$

Our next result is an extension of (2.1) to the case of the simultaneous metric dimension.

Theorem 2.5. *Let $\mathcal{G}$ be a family of graphs with common vertex set V that have a shortest path of length d in common. Then*

$$\mathrm{Sd}(\mathcal{G}) \leq |V| - d.$$

Proof. Let $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ be a family of graphs with common vertex set V having a shortest path $v_0, v_1, ..., v_d$ in common. Let $W = V - \{v_1, ..., v_d\}$. Since $d_{G_j}(v_0, v_i) = i$, for $i \in \{1, ..., d\}$, we conclude that W is a metric generator for each G_j. Hence, $\mathrm{Sd}(\mathcal{G}) \leq |W| = |V| - d$. □

Let $r \geq 3$ be an integer. Label the vertices of K_r and $K_{1,r-1}$ with the same set of labels and suppose c is the label of the centre of the star $K_{1,r-1}$. Let P_d, $d \geq 2$, be an a–b path of order d whose vertex set is disjoint from that of K_r. Let G_1 be the graph obtained from the complete graph $K_r = (V', E')$, $r \geq 3$, and the path graph P_d, $d \geq 2$, by identifying the leaf a of P_d, with the vertex c of K_r and calling it c, and let G_2 be the graph obtained by identifying the leaf a of P_d with the center c of the star $K_{1,r-1}$ and also calling it c. Figure 2.3 illustrates this construction. In this case, G_1 and G_2 have the same vertex set V (where $|V| = d+r-1$). For any $v \in V(K_r) - \{c\}$ we have $d_{G_1}(b, v) = d_{G_2}(b, v) = d$ and $V(P_d) \cup \{v\}$ is a shortest path of length d in both graphs G_1 and G_2. Moreover, $W = (V' - \{v, c\}) \cup \{b\}$ is a simultaneous metric basis of $\{G_1, G_2\}$ and so $\mathrm{Sd}(G_1, G_2) = |V| - d$. Therefore, the bound described above is sharp.

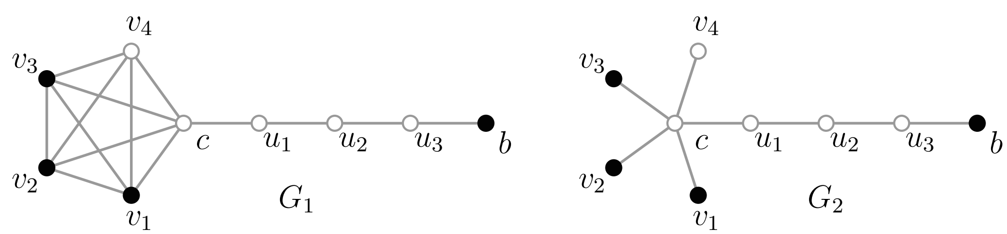

Figure 2.3: The family $\mathcal{G} = \{G_1, G_2\}$ satisfies $\mathrm{Sd}(\mathcal{G}) = |V| - d$.

2.2 Families of graphs with small metric dimension

In this section we focus on families of graphs on the same vertex set each of which have dimension 1 or 2. As we mentioned previously, it was shown in [12] that $\dim(G) = 1$ if and only if G is a path. The first result in this section deals with families of graphs for which the simultaneous metric dimension is as small as possible.

Theorem 2.6. *Let $\mathcal{G}$ be a family of connected graphs on a common vertex set. Then*

(i) $\mathrm{Sd}(\mathcal{G}) = 1$ *if and only if* $\mathcal{G}$ *is a collection of paths that share a common leaf.*

(ii) *If* $\mathcal{G}$ *is a collection of paths, then* $1 \leq \mathrm{Sd}(\mathcal{G}) \leq 2$.

Proof. If $\mathrm{Sd}(\mathcal{G}) = 1$, then the family $\mathcal{G}$ is a collection of paths. Moreover, if v is a vertex of degree 2 in a path P, then v does not distinguish its neighbours and, as a consequence, $\{v\}$ is a metric basis of P if and only if v is a leaf of P. Therefore, (i) follows.

Since any path has metric dimension 1, and any pair of distinct vertices of a path P is a metric generator for P, we conclude that (ii) follows. □

Theorem 2.7. *Let $\mathcal{G}$ be a family of graphs on a common vertex set V such that $\mathcal{G}$ does not only consist of paths. Let $\mathcal{H}$ be the collection of elements of $\mathcal{G}$ which are not paths. Then*

$$\mathrm{Sd}(\mathcal{G}) = \mathrm{Sd}(\mathcal{H}).$$

Proof. Since $\mathcal{H}$ is a non-empty subfamily of $\mathcal{G}$ we conclude that $\mathrm{Sd}(\mathcal{G}) \geq \mathrm{Sd}(\mathcal{H})$. From Theorem 2.6 (i), it follows that $\mathrm{Sd}(\mathcal{H}) \geq 2$. Moreover, as any pair of vertices of a path P is a metric generator for P, it follows that if $B \subseteq V$ is a simultaneous metric basis of $\mathcal{H}$, then B is a simultaneous metric generator for $\mathcal{G}$ and, as a result, $\mathrm{Sd}(\mathcal{G}) \leq |B| = \mathrm{Sd}(\mathcal{H})$. □

Theorem 2.8. *Let $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ be a family of paths and cycles on a common vertex set V, which contains at least one cycle. Then the following assertions hold:*

(i) *If* $|V|$ *is odd, then* $\mathrm{Sd}(\mathcal{G}) = 2$.

(ii) *If* $|V|$ *is even, then* $2 \leq \mathrm{Sd}(\mathcal{G}) \leq 3$. *Moreover,* $\mathrm{Sd}(\mathcal{G}) = 2$ *if and only if there exist two vertices* $u, v \in V$ *which are not mutually antipodal in any cycle* $G_i \in \mathcal{G}$.

(iii) *If* $|V|$ *is even and* $\mathcal{G}$ *contains fewer than* $n-1$ *cycles, then* $\mathrm{Sd}(\mathcal{G}) = 2$. *Moreover, this result is the best possible in the sense that there exists a family of* $(n-1)$ *cycles of order* n *on the same vertex set whose simultaneous metric dimension is* 3.

Proof. By Theorem 2.7, we have that $\mathrm{Sd}(\mathcal{G}) = \mathrm{Sd}(\mathcal{C})$, where $\mathcal{C}$ is the subfamily of $\mathcal{G}$ containing all cycles. With this fact in mind, for the remainder of the proof we will assume that $\mathcal{G}$ is composed only by cycles.

The result is clear for $|V| = 3$. Let C_n be a cycle of order $|V| = n \geq 4$. We first assume that n is odd. In this case, given four different vertices $u, v, x, y \in V(C_n)$ we have $d_{C_n}(u, x) \neq d_{C_n}(u, y)$ or $d_{C_n}(v, x) \neq d_{C_n}(v, y)$. Hence, we conclude that $\{u, v\}$ is a metric generator for C_n and, since $\dim(C_n) > 1$, we conclude that $\{u, v\}$ is a metric basis for C_n. Thus, $\{u, v\}$ is a simultaneous metric basis for $\mathcal{G}$. Therefore, in this case $\mathrm{Sd}(\mathcal{G}) = 2$. Thus (i) holds.

From now on we assume that $|V| = n$ is even. Note that in this case every G_i is a 2-antipodal graph. Let $u, v \in V(C_n)$ be two vertices which are not mutually antipodal in C_n. Since for every pair of distinct vertices $x, y \in V(C_n)$, we have $d_{C_n}(u, x) \neq d_{C_n}(u, y)$ or $d_{C_n}(v, x) \neq d_{C_n}(v, y)$, we conclude that $\{u, v\}$ is a metric generator for C_n and, since $\dim(C_n) > 1$, we conclude that $\{u, v\}$ is a metric basis. Clearly, no pair of mutually antipodal vertices form a metric basis for C_n. Therefore, $\mathrm{Sd}(\mathcal{G}) = 2$ if and only if there are two vertices $u, v \in V$ which are not mutually antipodal in G_i for every $i \in \{1, ..., k\}$. Suppose that, for every pair of distinct vertices $u, v \in V$, there exists $G_i \in \mathcal{G}$ such that u and v are mutually antipodal in G_i. In this case we have $\mathrm{Sd}(\mathcal{G}) \geq 3$. Now, since for three different vertices $u, v, w \in V$, only two of them may be mutually antipodal in G_i, we conclude that $\{u, v, w\}$ is a simultaneous metric generator for $\mathcal{G}$. Therefore, in this case, $\mathrm{Sd}(\mathcal{G}) = 3$. This completes the proof of (ii).

Since each of the k cycles in $\mathcal{G}$ has $n/2$ antipodal pairs it follows that if $k < n - 1$ or equivalently $\frac{nk}{2} < \binom{n}{2}$, then $\mathrm{Sd}(\mathcal{G}) = 2$. This inequality is best possible in the sense that there is a collection of $(n - 1)$ cycles $\mathcal{G} = \{C'_1, C'_2, \ldots, C'_{n-1}\}$ with vertex set $\{1, 2, \ldots, n\}$ such that each of the $\binom{n}{2}$ possible pairs from $\{1, 2, \ldots, n\}$ is an antipodal pair on exactly one of these cycles and hence $\mathrm{Sd}(\mathcal{G}) = 3$. We construct the labeling of these cycles by assigning pairs of labels to antipodal pairs in such a way that a given pair is assigned to exactly one of these $(n-1)$ cycles. Consider the upper triangular array whose $(i, j)^{th}$ entry is (i, j) for $1 \leq i < j \leq n$. Select the first nonempty entry in row 1. This entry is the ordered pair $(1, 2)$. Begin by assigning the labels 1 and 2 to the vertices in positions 1 and $n/2$ on C'_1. Now mark rows and columns 1 and 2 used and mark the pair $(1, 2)$ as unavailable. Find the first unused row and subject to this the first unused column and let the corresponding entry in the array be say (i_{1_2}, j_{1_2}). Assign i_{1_2} and j_{1_2} to vertices in positions 2 and $1+n/2$ on C'_1 and mark both rows and columns i_{1_2} and j_{1_2} as used and the pair (i_{1_2}, j_{1_2}) as unavailable. Next find the first available pair in the first unused row and subject to this in an unused column, say (i_{1_3}, j_{1_3}). Assign the labels i_{1_3} and j_{1_3} to the vertices in C'_1 in positions 3 and $2+n/2$, respectively. We continue this process until all rows and columns of the array have been marked used. Moreover, whenever the entries of an ordered pair are used as labels of vertices in C'_1 we mark that pair as unavailable. Now reset the labels on all rows and columns in the array as unused but do not reset the labels on the ordered pairs. Next find the first available entry say (i_{2_1}, j_{2_1}) in row 1 and assign i_{2_1} and j_{2_1} to the vertices in positions 1 and $n/2$, respectively, of C'_2. Mark rows and columns i_{2_1} and j_{2_1} as used and mark the pair (i_{2_1}, j_{2_1}) as unavailable. Now find the first non-empty available entry in the first unmarked row and subject to this in the first unmarked column, say (i_{2_2}, j_{2_2}), and assign i_{2_2} and j_{2_2} to vertices in positions 2 and $1+n/2$ in C'_2. Continue in this manner until entries of each ordered pair in the triangular array have been assigned as labels to antipodal vertices in one of the cycles in $\mathcal{G}$. Then $\mathrm{Sd}(\mathcal{G}) = 3$. This completes the proof of (iii). □

2.3 Bounds for the simultaneous metric dimension of families of trees

We first introduce some necessary definitions. A vertex of degree at least 2 in a graph G is called an *interior vertex*. The set of interior vertices of graph G is denoted by $\mathcal{I}(G)$. A vertex of degree at least 3 is called a *major vertex* of G. Any leaf u of G is said to be a *terminal vertex of a major vertex* v of G if $d(u,v) < d(u,w)$ for every other major vertex w of G. The *terminal degree* $ter_G(v)$ of a major vertex v in G is the number of terminal vertices of v in G, i.e., the number of paths in $G-v$, while $TER_G(v)$ represents the set of terminal vertices of v in G. If there is no ambiguity, we will simply write $ter(v)$ and $TER(v)$. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. The set of exterior major vertices of graph G is denoted by $\mathcal{M}(G)$. It was shown in [12] that a metric generator W of a tree T may be constructed as follows: for each exterior major vertex of T select a vertex from each of the paths of $T-v$ except from exactly one such path and place it in W. So $\dim(T) = \sum_{w\in\mathcal{M}(T)}(ter(w)-1)$.

The following result shows an upper bound on the simultaneous metric dimension of families composed by trees.

Proposition 2.9. *Let $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ be a family of trees, which are different from paths, defined on a common vertex set V, and let $S_{\mathcal{I}} = \bigcap_{i=1}^{k} \mathcal{I}(T_i)$ be the set of vertices that are simultaneously interior vertices of every tree $T_i \in \mathcal{T}$. Then*

$$\mathrm{Sd}(\mathcal{T}) \leq |V| - |S_{\mathcal{I}}| - 1.$$

Proof. Using the ideas that underly the validity of the algorithm for constructing a (minimum) resolving set of a tree described in [12], it is possible to construct a set S, which is simultaneously a metric generator for every tree $T_i \in \mathcal{T}$ by constructing metric generators W_i for every tree T_i as described and letting $S = \bigcup_{i=1}^{k} W_i$. Any such set S will not contain a vertex that is not in $S_{\mathcal{I}}$, so

$$\mathrm{Sd}(\mathcal{T}) \leq |S| \leq |V| - |S_{\mathcal{I}}|$$

Moreover, for every vertex $u \in V - S_{\mathcal{I}}$ and every tree $T_i \in \mathcal{T}$, either:

(i) u is a terminal vertex of an exterior major vertex x of T_i, in which case every other terminal vertex of x, other than u, may be selected when constructing W_i, and hence W_i may be constructed in such a way that $u \notin W_i$; or

(ii) u is not a terminal vertex of any exterior major vertex of T_i, in which case W_i may be constructed in such a way that $u \notin W_i$.

Thus, for every vertex $u \in V - S_{\mathcal{I}}$, the set S may be constructed in such a way that $u \notin S$ and, as a result, $\mathrm{Sd}(\mathcal{T}) \leq |S| \leq |V| - |S_{\mathcal{I}}| - 1$. □

The bound presented above is sharp. For instance, equality is achieved for the graph family shown in Figure 2.4, where $S_{\mathcal{I}} = \{m_1, m_2, i_1\}$, any triple of leaves is a simultaneous metric generator, e.g. $\{l_1, l_2, l_3\}$, whereas no pair of vertices is a simultaneous metric generator. Thus $\mathrm{Sd}(\mathcal{T}) = 3 = |V| - |S_{\mathcal{I}}| - 1$.

However, there are families $\mathcal{T}$ of trees on the same vertex set for which the ratio $\frac{\mathrm{Sd}(\mathcal{T})}{|V|-|S_{\mathcal{I}}|-1}$ can be made arbitrarily small. To see this let $r, s \geq 3$ be integers and let $V = \{(i,j) | 1 \leq i \leq r,\ 1 \leq j \leq s\} \cup \{x\}$. So $|V| = rs + 1$. Let T_1 be the tree obtained from the paths $Q_i = (i,1)(i,2)\ldots(i,s)x$ for $1 \leq i \leq r$ by identifying the vertex x from each of the paths. So T_1 is isomorphic to

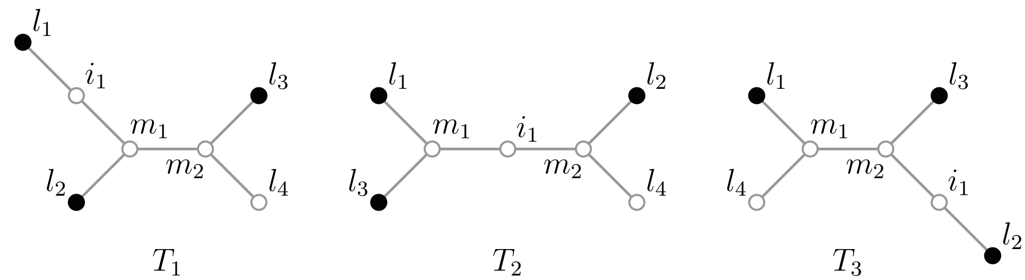

Figure 2.4: A family of trees $\mathcal{T} = \{T_1, T_2, T_3\}$ such that $\mathrm{Sd}(\mathcal{T}) = 3 = |V| - |S_{\mathcal{I}}| - 1$.

the tree obtained from the star $K_{1,r}$ by subdividing each edge $s-1$ times. For $2 \le j < s$ let T_j be obtained from T_1 by adding the edge $(i,1)(i,j+1)$ and deleting the edge $(i,j)(i,j+1)$ for $1 \le i \le r$. Finally let T_s be obtained from T_1 by adding the edge $(i,1)x$ and deleting the edge $(i,s)x$ for $1 \le i \le r$. Let $\mathcal{T} = \{T_j | 1 \le j \le s\}$. Then $S_{\mathcal{I}} = \{x\}$. So $|V| - |S_{\mathcal{I}}| - 1 = rs - 1$. It is not difficult to see that $\{(i,1) | 1 \le i \le r-1\}$ is a minimum resolving set for each T_j. Hence $\mathrm{Sd}(\mathcal{T}) = r - 1$. So $\frac{\mathrm{Sd}(\mathcal{T})}{|V|-|S_{\mathcal{I}}|-1} = \frac{r-1}{rs-1}$. By choosing s large enough this can be made as small as we wish. Note also that this family of trees achieves the lower bound given in Remark 2.1.

2.4 Families composed by a graph and a minimally differing variation

Here, we focus on the following question: given a graph G whose metric dimension is known, if a small modification is performed on G, thus obtaining a new graph G', what is the behaviour of $\mathrm{Sd}(G, G')$ with respect to $\dim(G)$? Answering this question in the general case is hard. Here, we will analyse a number of particular cases. We say that a graph G_2 is obtained from a graph G_1 by an *edge exchange* if there is an edge e not in G_1 and an edge f in G_1 such that $G_2 = G_1 + e - f$. Throughout this section, we will study families composed by two graphs such that each one of them is obtained from the other by an edge exchange.

For any tree T we shall denote by $\mathcal{B}(T)$ the set of its metric bases constructed as described in Section 2.3.

Remark 2.10. *Let T be a tree obtained from a path graph by an edge exchange. If T is not a path, then*

$$\dim(T) = 2.$$

Proof. We assume that T is a tree different from a path. In that case, either T has exactly one exterior major vertex having exactly three terminal vertices, or it has exactly two exterior major vertices having exactly two terminal vertices each. In both cases we obtain $\dim(T) = \sigma(T) - ex(T) = 2$. □

Remark 2.11. *Let T be a tree obtained from a path graph P by an edge exchange. If T is a path graph having a leaf in common with P, then*

$$\mathrm{Sd}(P, T) = 1,$$

otherwise

$$\mathrm{Sd}(P, T) = 2.$$

Proof. If T is a path graph having a leaf in common with P, then $\mathrm{Sd}(P, T) = 1$ by Theorem 2.6 (i). Now, if T is a path graph which has no common leaves with P, then by Theorem 2.6 (ii) it holds that $\mathrm{Sd}(P, T) = 2$.

Finally, suppose that T is a tree different from a path. In that case, by Remark 2.10, $\dim(T) = 2$ and so Theorem 2.7 leads to $\mathrm{Sd}(P, T) = 2$. □

Let $G = (V, E)$ be a graph and let e_1, e_2 be two different edges of its complement. Let $G_1 = G + e_1 = (V, E_1)$ and $G_2 = G + e_2 = (V, E_2)$ be the graphs whose edge sets are $E_1 = E \cup \{e_1\}$ and $E_2 = E \cup \{e_2\}$, respectively. Clearly, G_2 is obtained from G_1 by an edge exchange and vice versa.

Remark 2.12. *Let P be a path graph of order at least four and let e_1, e_2 be two different edges of its complement. Then,*

$$\mathrm{Sd}(P + e_1, P + e_2) = 2.$$

Proof. Since $P + e_1$ and $P + e_2$ are not path graphs, $\mathrm{Sd}(P + e_1, P + e_2) \geq 2$ and so we only need to show that $\mathrm{Sd}(P + e_1, P + e_2) \leq 2$. To this end, we denote by $V = \{v_1, ..., v_n\}$ the vertex set of P, where v_i is adjacent to v_{i+1}, for every $i \in \{1, ..., n-1\}$. Also, let $e_1 = v_p v_q$, $1 \leq p < q \leq n$, and $e_2 = v_r v_s$, $1 \leq r < s \leq n$. In order to show that $\{v_1, v_n\}$ is a metric generator for $P + e_1$, we differentiate the following four cases:

(1) $e_1 = v_1v_n$. In this case, $P + e_1$ is a cycle graph where v_1 and v_n are adjacent, so $\{v_1, v_n\}$ is a metric generator.

(2) $1 < p < q = n$. In this case, $P + e_1$ is a unicyclic graph where v_p has degree three, v_1 has degree one and the remaining vertices have degree two. Consider two different vertices $u, v \in V - \{v_1, v_n\}$. If u or v belong to the path from v_1 to v_p, then v_1 distinguishes them. If both, u and v, belong to the cycle of $P + e_1$, then $d(u, v_1) = d(u, v_p) + d(v_p, v_1)$ and $d(v, v_1) = d(v, v_p) + d(v_p, v_1)$. Thus, if v_p distinguishes u and v so does v_1, otherwise v_n does.

(3) $1 = p < q < n$. This case is analogous to case 2.

(4) $1 < p < q < n$. In this case, $P + e_1$ is a unicyclic graph where v_p and v_q have degree three, v_1 and v_n have degree one and the remaining vertices have degree two. Consider two different vertices $u, v \in V - \{v_1, v_n\}$. If u or v belong to the path from v_1 to v_p (or to the path from v_q to v_n), then v_1 (or v_n) distinguishes them. If both u and v belong to the cycle, then $d(u, v_1) = d(u, v_p) + d(v_p, v_1)$, $d(v, v_1) = d(v, v_p) + d(v_p, v_1)$, $d(u, v_n) = d(u, v_q) + d(v_q, v_n)$ and $d(v, v_n) = d(v, v_q) + d(v_q, v_n)$. Thus, if v_p distinguishes u and v so does v_1, otherwise v_q distinguishes them, which means that v_n also does.

According to the four cases above, we conclude that $\{v_1, v_n\}$ is a metric generator for $P+e_1$ and, by analogy, we deduce that $\{v_1, v_n\}$ is also a metric generator for $P + e_2$. Thus, $\mathrm{Sd}(P + e_1, P + e_2) \leq 2$ and, as a consequence, the result follows. $\square$

We now present results analogous to those of Remarks 2.11 and 2.12 for the case of cycles.

Remark 2.13. *For any graph G obtained from a cycle graph C by an edge exchange,*

$$\mathrm{Sd}(G, C) = 2.$$

Proof. Since G and C are not path graphs, $\mathrm{Sd}(G, C) \geq 2$ and so it remains to show that $\mathrm{Sd}(G, C) \leq 2$. Assume that $G = C + e - f$ and $f = v_iv_j$. As v_i and v_j are adjacent in C, they are not antipodal vertices and so $\{v_i, v_j\}$ is a metric generator for C. Now, since G is isomorphic to the graphs of the form $P + e_1$, as described in Remark 2.12, by analogy to the proof of Remark 2.12

(cases 2, 3 and 4) we deduce that $\{v_i, v_j\}$ is also a metric generator for G. Consequently, $\mathrm{Sd}(G, \mathcal{C}) \leq 2$. □

Remark 2.14. *Let C be a cycle graph of order at least four and let e be an edge of its complement. Then,*

$$\dim(C + e) = 2.$$

Proof. Since $C + e$ is not a path graph, $\dim(C + e) \geq 2$, so we only need to show that $\dim(C + e) \leq 2$.

If C has order four, then there is only one graph of the form $C + e$, for which it is straightforward to verify that $\dim(C + e) = 2$.

Now, suppose C has order $n \geq 5$ and take $e = v_i v_j$. Note that $C + e$ is a bicyclic graph where v_i and v_j are vertices of degree three and the remaining vertices have degree two. We denote by C_{n_1} and C_{n-n_1+2} the two graphs obtained as induced subgraphs of $C + e$ which are isomorphic to a cycle of order n_1 and a cycle of order $n - n_1 + 2$, respectively. Since $n \geq 5$, we have that $n_1 > 3$ or $n - n_1 + 2 > 3$. We assume, without loss of generality, that $n_1 > 3$. Let $a, b \in V(C_{n_1})$ be two vertices such that:

- if n_1 is even, $a \sim b$ and $d(v_i, a) = d(v_j, b)$,
- if n_1 is odd, $a \sim x \sim b$, where $x \in V(C_{n_1})$ is the only vertex such that $d(x, v_i) = d(x, v_j)$.

We claim that $\{a, b\}$ is a metric generator for $C + e$. Consider two different vertices $u, v \in V(C + e) - \{a, b\}$. We differentiate the following cases, where the distances are taken in $C + e$:

(1) $u, v \in V(C_{n_1})$. It may be verified that $\{a, b\}$ is a metric generator for C_{n_1}, hence $d(u, a) \neq d(v, a)$ or $d(u, b) \neq d(v, b)$.

(2) $u \in V(C_{n_1})$ and $v \in V(C_{n-n_1+2}) - \{v_i, v_j\}$. In this case, $d(u, a) < d(v, a)$ or $d(u, b) < d(v, b)$.

(3) $u, v \in V(C_{n-n_1+2}) - \{v_i, v_j\}$. In this case, if $d(u, a) = d(v, a)$, then $d(u, v_i) = d(v, v_i)$, so $d(u, v_j) \neq d(v, v_j)$ and, consequently, $d(u, b) \neq d(v, b)$.

According to the three cases above, $\{a, b\}$ is a metric generator for $C + e$ and, as a result, the proof is complete. □

Corollary 2.15. *Let C be a cycle graph of order $n \geq 4$ and let e_1, e_2 be two different edges of its complement. Then,*

$$2 \leq \mathrm{Sd}(C + e_1, C + e_2) = \mathrm{Sd}(C, C + e_1, C + e_2) \leq 4.$$

To illustrate the different cases of Corollary 2.15, consider the cycle C_{10} where $V(C_{10}) = \{v_1, ..., v_{10}\}$, v_i is adjacent to v_{i+1} for every $i \in \{1, ..., 9\}$ and v_1 is adjacent to v_{10}. If we make $e_1 = v_4v_9$ and $e_2 = v_5v_8$, it may be verified that the sets $\{v_1, v_2\}$ and $\{v_6, v_7\}$ are the simultaneous metric bases of $\mathcal{G} = \{C_{10}+e_1, C_{10}+e_2\}$, so $\mathrm{Sd}(\mathcal{G}) = 2$. Alternatively, if we make $e_1 = v_4v_9$ and $e_2 = v_3v_8$, it may be verified that the sets $\{v_1, v_2, v_{10}\}$ and $\{v_5, v_6, v_7\}$ are the simultaneous metric bases of $\mathcal{G}$, so $\mathrm{Sd}(\mathcal{G}) = 3$. Finally, by making $e_1 = v_4v_9$ and $e_2 = v_1v_8$, we have that the sets $\{v_1, v_2, v_4, v_5\}$, $\{v_1, v_2, v_9, v_{10}\}$, $\{v_4, v_5, v_6, v_7\}$ and $\{v_6, v_7, v_9, v_{10}\}$ are the simultaneous metric bases of $\mathcal{G}$, so $\mathrm{Sd}(\mathcal{G}) = 4$.

We now study the case of families composed by two trees, both different from a path, one of which is obtained from the other by an edge exchange.

Theorem 2.16. *Let T_1 be a tree of order $n \geq 4$ and let T_2 be a tree obtained from T_1 by an edge exchange. Then,*

$$\dim(T_1) \leq \mathrm{Sd}(T_1, T_2) \leq \dim(T_1) + 2.$$

Proof. The lower bound is a direct consequence of Remark 2.1. Consider that $T_2 = T_1 + e - f$, where $e = v_rv_s$ and $f = v_iv_j$. To deduce the upper bound, we will show that for any metric basis $B \in \mathcal{B}(T_1)$, the set $S = B \cup \{v_i, v_j\}$ is a metric generator for T_2, and thus it is a simultaneous metric generator for $\{T_1, T_2\}$. First of all, notice that $\Omega(T_2) \subseteq \Omega(T_1) \cup \{v_i, v_j\}$. Depending on the positions of v_i and v_j in T_1, we differentiate the following cases:

(1) v_i and v_j lie on the path L that connects $v_p \in \mathcal{M}(T_1)$ to $v_x \in TER_{T_1}(v_p)$. We consider, without loss of generality, that v_i is closer to v_p than v_j. In this case, we have that $TER_{T_2}(v_p) - TER_{T_1}(v_p) \in \{\emptyset, \{v_i\}, \{v_j\}, \{v_i, v_j\}\}$. Due to the connectivity of T_2, either v_r or v_s lies on the path L' connecting v_j to v_x, so we assume, without loss of generality, that v_r lies on L'.

On one hand, if $v_r \in \mathcal{M}(T_2)$, then $TER_{T_2}(v_r) = \{v_j, v_x\}$ and, for every $v \in (\mathcal{M}(T_2) - \{v_r\}) - \mathcal{M}(T_1)$, $ter_{T_2}(v) = 1$. Furthermore, under this assumptions, for every $v \in (\mathcal{M}(T_1) - \{v_p\}) \cap \mathcal{M}(T_2)$, we have that $TER_{T_2}(v) \subseteq TER_{T_1}(v)$.

Alternatively, if $v_r \notin \mathcal{M}(T_2)$ and $v_s \in \mathcal{M}(T_2)$, then either $v_j \in TER_{T_2}(v_s)$ or v_j is a vertex of degree 2 lying on the path that connects v_s to v_x in T_2. Furthermore, for every $v \in (\mathcal{M}(T_2) - \{v_s\}) - \mathcal{M}(T_1)$, we have that $ter_{T_2}(v) = 1$, and for every $v \in (\mathcal{M}(T_1) - \{v_p, v_s\}) \cap \mathcal{M}(T_2)$, we have that $TER_{T_2}(v) \subseteq TER_{T_1}(v)$.

Finally, if $v_r \notin \mathcal{M}(T_2)$ and $v_s \notin \mathcal{M}(T_2)$, then $v_s \in TER_{T_1}(v_p) \cup \{v_i\}$ or $v_s \in TER_{T_1}(w)$, where $w \in \mathcal{M}(T_1) - \{v_p\}$. In the first case, $v_j \in TER_{T_2}(v_p)$ or $v_x \in TER_{T_2}(v_p)$ and v_j is a vertex of degree 2 lying on the path that connects v_p to v_x in T_2, whereas in the second case either $v_j \in TER_{T_2}(w)$ or $v_x \in TER_{T_2}(w)$ and v_j is a vertex of degree 2 lying on the path that connects w to v_x in T_2. Furthermore, $\mathcal{M}(T_2) = \mathcal{M}(T_1)$ and for every $v \in \mathcal{M}(T_2) - \{v_p, w\}$, we have that $TER_{T_2}(v) = TER_{T_1}(v)$.

In consequence, for any metric basis $B \in \mathcal{B}(T_1)$, the set $B \cup \{v_i, v_j\}$ is a metric generator for T_2, and thus a simultaneous metric generator for $\{T_1, T_2\}$.

(2) v_i and v_j lie on the path L which connects two major vertices v_p and v_q of T_1 and contains no other major vertex. Here we assume, without loss of generality, that v_i is closer to v_p than v_j. In this case, if $v_r \in \mathcal{M}(T_2) - \mathcal{M}(T_1)$, then $ter_{T_2}(v_r) = 1$. Likewise, if $v_s \in \mathcal{M}(T_2) - \mathcal{M}(T_1)$, we have that $ter_{T_2}(v_s) = 1$. Furthermore, $TER_{T_2}(v_p) - TER_{T_1}(v_p) \in \{\emptyset, \{v_i\}\}$ and $TER_{T_2}(v_q) - TER_{T_1}(v_q) \in \{\emptyset, \{v_j\}\}$. Finally, for every $v \in (\mathcal{M}(T_2) - \{v_r, v_s\}) - \mathcal{M}(T_1)$, we have that $ter_{T_2}(v) = 1$, and for every $v \in (\mathcal{M}(T_1) - \{v_p, v_q\}) \cap \mathcal{M}(T_2)$, we have that $TER_{T_2}(v) \subseteq TER_{T_1}(v)$.

In consequence, for any metric basis $B \in \mathcal{B}(T_1)$, the set $B \cup \{v_i, v_j\}$ is a metric generator for T_2, and thus a simultaneous metric generator for $\{T_1, T_2\}$.

Summing up the cases discussed above, we may conclude that for any metric basis B of T_1, the set $S = B \cup \{v_i, v_j\}$ is a simultaneous metric generator for $\{T_1, T_2\}$, so $\mathrm{Sd}(T_1, T_2) \leq |S| \leq |B| + 2 = \dim(T_1) + 2$. □

Corollary 2.17. *Let T_1 be a tree of order $n \geq 4$ and let T_2 be a tree obtained from T_1 by an edge exchange. Then,*

$$\dim(T_1) - 2 \leq \dim(T_2) \leq \dim(T_1) + 2.$$

Proof. Let f be an edge of T_1 and let e be an edge of its complement. Then $T_2 = T_1 + e - f$ if and only if $T_1 = T_2 + f - e$. Hence, the result is a direct consequence of Theorem 2.16, according to which $\dim(T_2) \leq \mathrm{Sd}(T_1, T_2) \leq \dim(T_1) + 2$ and $\dim(T_1) \leq \mathrm{Sd}(T_1, T_2) \leq \dim(T_2) + 2$. □

Finally, we address other type of families composed by two graphs featuring larger differences from one another. The notation $A \triangledown B$ represents the symmetric difference of the sets A and B.

Remark 2.18. *Let $V = \{v_1, v_2, \ldots, v_n\}$ and $V' = V \cup \{v_{n+1}\}$. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two connected graphs on the common vertex set V and let $G'_1 = (V', E'_1)$ and $G'_2 = (V', E'_2)$ be two graphs whose edge sets are $E'_1 = E_1 \cup \{v_i v_{n+1}\}$ and $E'_2 = E_2 \cup \{v_j v_{n+1}\}$, for some $v_i, v_j \in V$. If there exist two simultaneous metric bases B_1 and B_2 of $\{G_1, G_2\}$ such that $B_1 \triangledown B_2 = \{v_i, v_j\}$, then*

$$\mathrm{Sd}(G'_1, G'_2) = \mathrm{Sd}(G_1, G_2),$$

otherwise,

$$\mathrm{Sd}(G_1, G_2) \leq \mathrm{Sd}(G'_1, G'_2) \leq \mathrm{Sd}(G_1, G_2) + 1.$$

Proof. Any pair of different vertices $u, v \in V$ distinguished in G'_1 or G'_2 by v_{n+1} is also distinguished in G_1 by v_i or by v_j in G_2, so a simultaneous metric basis of $\{G'_1, G'_2\}$ must contain at least as many vertices as a simultaneous metric basis of $\{G_1, G_2\}$. Thus, $\mathrm{Sd}(G'_1, G'_2) \geq \mathrm{Sd}(G_1, G_2)$.

First assume that there exist two simultaneous metric bases B_1 and B_2 of $\{G_1, G_2\}$ such that $B_1 \triangledown B_2 = \{v_i, v_j\}$. Let $S = (B_1 \cap B_2) \cup \{v_{n+1}\}$. We claim that S is a simultaneous metric generator for $\{G'_1, G'_2\}$. We assume, without loss of generality, that $v_i \in B_1$. If a pair of different vertices is distinguished in G_1 by v_i, it is also distinguished in G'_1 by v_{n+1}, otherwise it is distinguished by some $x \in B_1 - \{v_i\} \subseteq S$. The same reasoning is valid for v_j on G_2, so S is simultaneously a metric generator for G'_1 and G'_2. Thus, $\mathrm{Sd}(G'_1, G'_2) \leq |S| = \mathrm{Sd}(G_1, G_2)$, so the equality holds.

For the general case, let B be a simultaneous metric basis of $\{G_1, G_2\}$. Clearly, $B \cup \{v_{n+1}\}$ is simultaneously a metric generator for G'_1 and G'_2, so $\mathrm{Sd}(G'_1, G'_2) \leq \mathrm{Sd}(G_1, G_2) + 1$. □

A particular case of Remark 2.18 deals with another case of a family of graphs $\{G_1, G_2\}$ where G_2 is obtained from G_1 by an edge exchange and *vice versa*.

Corollary 2.19. *Let $G = (V, E)$ be a connected graph of order $n \geq 2$ and let $V' = V \cup \{v_{n+1}\}$. Let $G_1 = (V', E_1)$ and $G_2 = (V', E_2)$ be two graphs whose edge sets are $E_1 = E \cup \{v_i v_{n+1}\}$ and $E_2 = E \cup \{v_j v_{n+1}\}$, for some $v_i, v_j \in V$, $i \neq j$. If there exist two metric bases B_1 and B_2 of G such that $B_1 \triangledown B_2 = \{v_i, v_j\}$, then*

$$\mathrm{Sd}(G_1, G_2) = \dim(G),$$

otherwise,

$$\dim(G) \leq \mathrm{Sd}(G_1, G_2) \leq \dim(G) + 1.$$

2.5 Large families of graphs with a fixed simultaneous metric basis and a large common induced subgraph

Intuitively, it is expectable that the simultaneous metric dimension of large families is considerably larger than the metric dimension of any of its individual member graphs. However, as we will show in this section, there exist large families of graphs where this difference is as small as desired. We accomplish this by describing a general approach for constructing large graph families for which the simultaneous metric dimension attains the lower bound given in Remark 2.1. Moreover, we show that the graphs in such families contain large isomorphic common induced subgraphs.

Let $G = (V, E)$ be a graph and let $\mathrm{Perm}(V)$ be the set of all permutations of V. Given a subset $X \subseteq V$, the *stabilizer* of X is the set of permutations $\mathcal{S}(X) = \{f \in \mathrm{Perm}(V) : f(x) = x, \text{ for every } x \in X\}$. As usual, we denote by $f(X)$ the image of a subset X under f, *i.e.*, $f(X) = \{f(x) : x \in X\}$.

Let B be metric basis of a graph $G = (V, E)$ of diameter $D(G)$. For any $r \in \{0, 1, ..., D(G)\}$ we define the set

$$\mathbf{B}_r(B) = \bigcup_{x \in B} \{y \in V : d_G(x, y) \leq r\}.$$

In particular, $\mathbf{B}_0(B) = B$ and $\mathbf{B}_1(B) = \bigcup_{x \in B} N_G[x]$. Moreover, since B is a metric basis of G, $|\mathbf{B}_{D(G)-1}(B)| \geq |V| - 1$.

Let G be a connected graph that is not complete. Given a permutation $f \in \mathcal{S}(B)$ of V we say that a graph $G' = (V, E')$ belongs to the family

$\mathcal{G}_{B,f}(G)$ if and only if $N_{G'}(f(v)) = f(N_G(v))$, for every $v \in \mathbf{B}_{D(G)-2}(B)$. In particular, if $D(G) = 2$ and $f \in \mathcal{S}(B)$, then $G' = (V, E')$ belongs to the family $\mathcal{G}_{B,f}(G)$ if and only if $N_{G'}(x) = f(N_G(x))$, for every $x \in B$. Moreover, if G is a complete graph, we define $\mathcal{G}_{B,f}(G) = \{G\}$.

Remark 2.20. *Let B be a metric basis of a connected non-complete graph G, let $f \in \mathcal{S}(B)$ and $G' \in \mathcal{G}_{B,f}(G)$. Then for any $b \in B$ and $k \in \{1, ..., D(G) - 1\}$, a sequence $b = v_0, v_1, ..., v_{k-1}, v_k = v$ is a path in G if and only if the sequence $b = f(v_0), f(v_1), ..., f(v_{k-1}), f(v_k) = f(v)$ is a path in G'.*

Proof. Let $b \in \mathbf{B}$. Since $G' \in \mathcal{G}_{B,f}(G)$ and $b = v_0 \in \mathbf{B}_{D(G)-2}(B)$, we have that $f(v_1) \in N_{G'}(f(v_0))$ if and only if $v_1 \in N_G(v_0)$ and, in general, if $v_i \in \mathbf{B}_{D(G)-2}(B)$, then $f(v_{i+1}) \in N_{G'}(f(v_i))$ if and only if $v_{i+1} \in N_G(v_i)$. Therefore, for any $k \in \{1, ..., D(G) - 1\}$, a sequence $(b =)f(v_0), f(v_1), ..., f(v_{k-1}), f(v_k)(= f(v))$ is a path in G' if and only if $(b =)v_0, v_1, ..., v_{k-1}, v_k(= v)$ is a path in G. □

Corollary 2.21. *Let B be a metric basis of a connected graph G, let $f \in \mathcal{S}(B)$ and $G' \in \mathcal{G}_{B,f}(G)$. Then for any $b \in B$ and $v \in \mathbf{B}_{D(G)-1}(B)$, $d_G(b, v) = k$ if and only if $d_{G'}(b, f(v)) = k$.*

Corollary 2.22. *Let B be a metric basis of a connected graph G, let $f \in \mathcal{S}(B)$ and $G' \in \mathcal{G}_{B,f}(G)$. Then $\langle \mathbf{B}_{D(G)-2}(B)\rangle \cong \langle \mathbf{B}_{D(G')-2}(B)\rangle$.*

Proof. Since $G' \in \mathcal{G}_{B,f}(G)$, the function f is a bijection from $V(G)$ onto $V(G')$. It remains to show that the restriction of f to $\langle \mathbf{B}_{D(G)-2}(B)\rangle$ is an isomorphism, i.e., we need to show that uv is an edge of $\langle \mathbf{B}_{D(G)-2}(B)\rangle$ if and only if $f(u)f(v)$ is an edge of $\langle \mathbf{B}_{D(G')-2}(B)\rangle$. Let $u, v \in \mathbf{B}_{D(G)-2}(B)$. Let k be the length of a shortest path from the set $\{u, v\}$ to the set B. Then there is a $b \in B$ such that $k = \min\{d_G(b, u), d_G(b, v)\} \leq D(G) - 2$. We may assume that $d_G(b, u) = k$. So there is a path $(b =)v_0, v_1, ..., v_{k-1}, v_k(= u)$ in $\langle \mathbf{B}_{D(G)-2}(B)\rangle$. By Remark 2.20 $(b =)v_0, v_1, ..., v_{k-1}, v_k(= u), v$ is a path in G if and only if $(b =)f(v_0), f(v_1), ..., f(v_{k-1}), f(v_k)(= f(u)), f(v)$ is a path in G'. So $uv \in E(\langle \mathbf{B}_{D(G)-2}(B)\rangle)$ if and only if $f(u)f(v) \in E(\langle \mathbf{B}_{D(G')-2}(B)\rangle)$. □

Now we define a family of graphs $\mathcal{G}_B(G)$, associated to B in G, as follows:

$$\mathcal{G}_B(G) = \bigcup_{f \in \mathcal{S}(B)} \mathcal{G}_{B,f}(G).$$

Notice that if $\mathbf{B}_{D(G)-2}(B) \subsetneq V$, then any graph $G' \in \mathcal{G}_B(G)$ is isomorphic to a graph $G^* = (V, E^*)$ whose edge set E^* can be partitioned into two sets E_1^*, E_2^*, where E_1^* consists of all edges of G having at least one vertex in $\mathbf{B}_{D(G)-2}(B)$ and E_2^* is a subset of edges of a complete graph whose vertex set is $V - \mathbf{B}_{D(G)-2}(B)$. Hence, $\mathcal{G}_B(G)$ contains $2^{\frac{l(l-1)}{2}}|V - B|!$ different labeled graphs, where $l = |V - \mathbf{B}_{D(G)-2}(B)|$. Clearly, if $|\mathbf{B}_{D(G)-1}(B)| = |V|$, then all these graphs are connected and if $|\mathbf{B}_{D(G)-1}(B)| = |V| - 1$, then $2^{\frac{(l-1)(l-2)}{2}}(2^{l-1} - 1)|V - B|!$ of these graphs are connected.

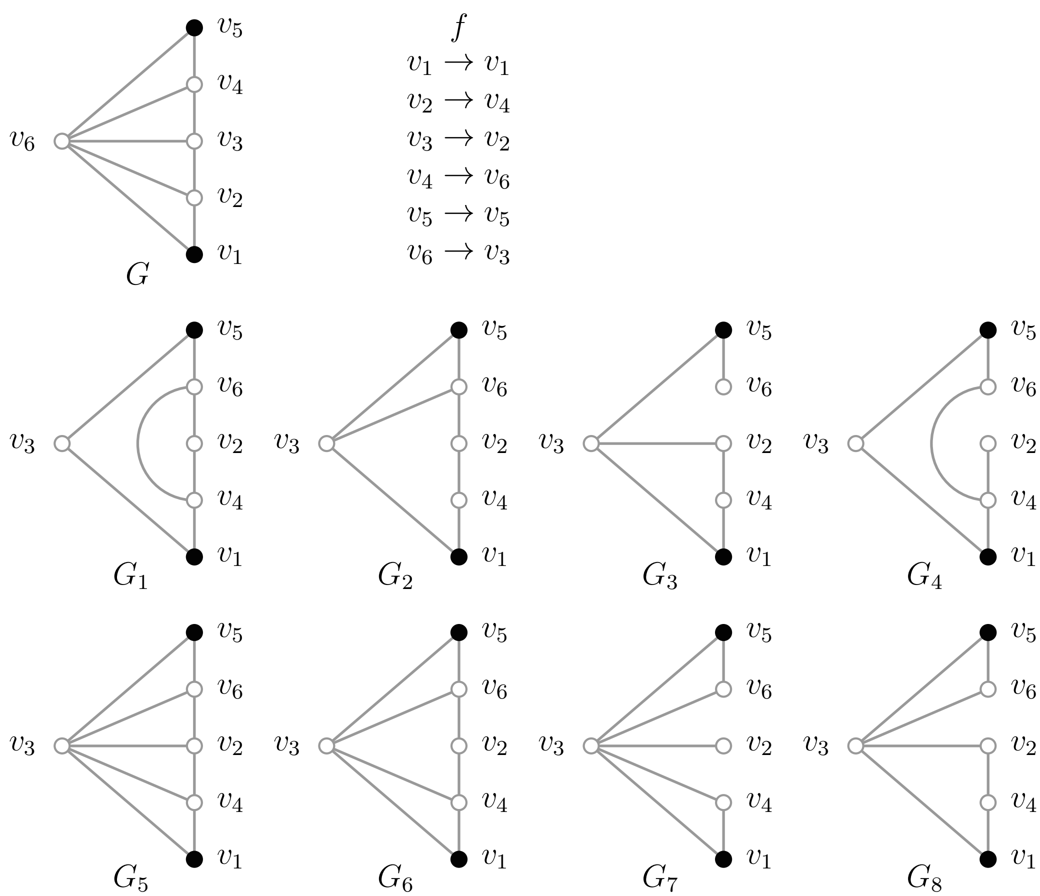

Figure 2.5: $B = \{1, 5\}$ is a metric basis of G, $f \in \mathcal{S}(B)$ and $\{G_1, ..., G_8\} \subseteq \mathcal{G}_f$

Now, if $\mathbf{B}_{D(G)-2}(B) = V$, then $\mathcal{G}_B(G)$ consists of graphs isomorphic to each other, having the basis B in common and, as a consequence, for any non-empty subfamily $\mathcal{H} \subseteq \mathcal{G}_B(G)$ we have $\mathrm{Sd}(\mathcal{H}) = \dim(G)$. As the next result shows, this conclusion on $\mathrm{Sd}(\mathcal{H})$ need not be restricted to the case $\mathbf{B}_{D(G)-2}(B) = V$.

Theorem 2.23. *Any metric basis B of a connected graph G is a simultaneous metric generator for any family of connected graphs $\mathcal{H} \subseteq \mathcal{G}_B(G)$. Moreover, if $G \in \mathcal{H}$, then*

$$\mathrm{Sd}(\mathcal{H}) = \dim(G).$$

Proof. Assume that B is a metric basis of a connected graph $G = (V, E)$, $f \in \mathcal{S}(B)$ and $G' \in \mathcal{G}_{B,f}(G)$. We shall show that B is a metric generator for G'. To this end, we take two different vertices $u', v' \in V - B$ of G' and the corresponding vertices $u, v \in V$ of G such that $f(u) = u'$ and $f(v) = v'$. Since $u \neq v$ and $u, v \notin B$, there exists $b \in B$ such that $d_G(u, b) \neq d_G(v, b)$. Now, consider the following two cases for u, v:

(1) $u, v \in \mathbf{B}_{D(G)-1}(B)$. In this case, since $d_G(u, b) \neq d_G(v, b)$, Corollary 2.21 leads to $d_{G'}(u', b) \neq d_{G'}(v', b)$.

(2) $u \in \mathbf{B}_{D(G)-1}(B)$ and $v \notin \mathbf{B}_{D(G)-1}(B)$. By Corollary 2.21, $d_{G'}(u', b) \leq D(G) - 1$ and, if $d_{G'}(v', b) \leq D(G) - 1$, then $d_G(v, b) \leq D(G) - 1$, which is not possible since $v \notin \mathbf{B}_{D(G)-1}(B)$. Hence, $d_{G'}(v', b) \geq D(G)$ and so $d_{G'}(u', b) \neq d_{G'}(v', b)$.

Notice that since B is a metric basis of G, the case $u, v \notin \mathbf{B}_{D(G)-1}(B)$ is not possible.

According to the two cases above, B is a metric generator for G' and, as a consequence, B is also a simultaneous metric generator for any family of connected graphs $\mathcal{H} \subseteq \mathcal{G}_B(G)$. Thus $\mathrm{Sd}(\mathcal{H}) \leq |B| = \dim(G)$ and, if $G \in \mathcal{H}$, then $\mathrm{Sd}(\mathcal{H}) \geq \dim(G)$. Therefore, the result follows. □

Figure 2.5 shows a graph G for which $B = \{v_1, v_5\}$ is a metric basis. The map f belongs to the stabilizer of B and $\{G_1, ..., G_8\}$ is a subfamily of $\mathcal{G}_{B,f}(G)$. In this case, the family $\mathcal{G}_B(G)$ contains 1344 different connected graphs; 48 of them are paths and B is a metric basis of the remaining 1296 connected graphs.

Chapter 3

Families composed by product graphs

In this chapter, we study the simultaneous metric dimension of families composed by product graphs. In particular, we focus on families composed by lexicographic and corona product graphs. Within the first case, we study the particular subcase of families composed by join graphs. Throughout the chapter, a second notion of simultaneous resolvability, namely the simultaneous adjacency dimension, is used as a tool for characterizing the simultaneous metric dimension of the studied families. The chapter is organized as follows. Section 3.1 gives an overview of the graph products we treat. Then, Section 3.2 introduces the simultaneous adjacency dimension and studies its properties. Finally, we introduce our results on families composed by join graphs, standard lexicographic product graphs, and corona product graphs in Sections 3.3, 3.4 and 3.5, respectively.

3.1 Overview

Let G be a graph of order n, and let $(H_1, H_2, \dots, H_n)$ be an ordered n-tuple of graphs of orders n'_1, n'_2, $\dots$, n'_n, respectively. The *lexicographic product* of G and $(H_1, H_2, \dots, H_n)$ is the graph $G \circ (H_1, H_2, \dots, H_n)$, such that $V(G \circ (H_1, H_2, \dots, H_n)) = \bigcup_{u_i \in V(G)} (\{u_i\} \times V(H_i))$ and $(u_i, v_r)(u_j, v_s) \in E(G \circ (H_1, H_2, \dots, H_n))$ if and only if $u_i u_j \in E(G)$ or $i = j$ and $v_r v_s \in E(H_i)$. As we mentioned previously, we will restrict our study to two particular cases. First, given two vertex-disjoint graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, the *join* of G and H, denoted as $G + H$, is the graph with vertex set $V(G + H) =$

$V_1 \cup V_2$ and edge set $E(G+H) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. Join graphs are lexicographic product graphs, as $G + H \cong P_2 \circ (G, H)$. The other particular case we will focus on is the most traditionally studied standard lexicographic product graph, where $H_i \cong H$ for every $i \in \{1, \dots, n\}$, which is denoted as $G \circ H$ for simplicity.

In the literature we can also find the names the *composition* or the *substitution* for the lexicographic product. The lexicographic product is clearly not commutative, while it is associative [35, 41]. Moreover, a lexicographic product graph $G \circ H$ is connected if and only if G is connected. Figure 3.1 illustrates two examples of lexicographic products and at the same time emphasizes the fact that the lexicographic product is not commutative.

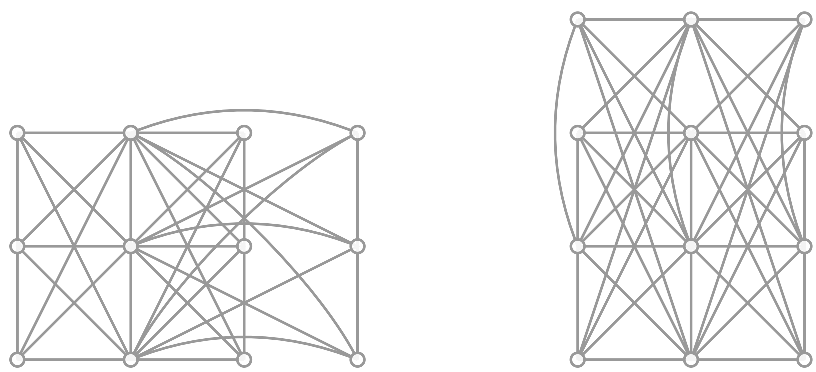

Figure 3.1: Lexicographic products $K_{1,3} \circ P_3$ and $P_3 \circ K_{1,3}$.

The lexicographic product of graphs has been studied from several points of view. The investigation includes, for instance, the metric and strong metric dimensions [43, 56], independence number [31], domination number [66], chromatic number [18, 31], connectivity [83], and hamiltonicity [2, 55]. For more details see [35, 41].

Let G and H be two graphs of order n and n', respectively. The *corona product* of G and H, denoted $G \odot H$, is defined as the graph obtained from G and H by taking one copy of G and n copies of H and joining by an edge each vertex from the i-th copy of H with the i-th vertex of G. Notice that the corona product graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$.

Observe that $G \odot H$ is connected if and only if G is connected. Moreover, it is readily seen from the definition that this product is neither an associative nor a commutative operation. Figure 3.2 shows some examples of corona products and also underscores the fact that the corona product is not commutative.

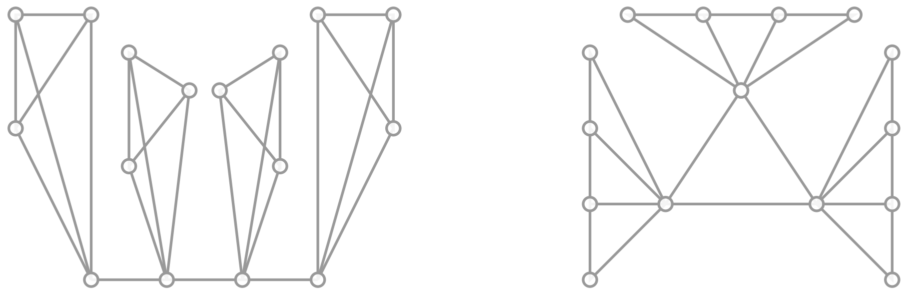

Figure 3.2: Corona products $P_4 \odot C_3$ and $C_3 \odot P_4$.

The concept of corona product of two graphs was first introduced by Frucht and Harary [28]. Despite the fact that the corona product is a simple operation on two graphs and some mathematical properties are merely direct consequences of its factors, it is interesting to study metric dimension-related parameters on this product, as those presented in [3, 22, 23, 25, 26, 27, 33, 42, 56, 57, 69, 72, 86]. Besides, several studies have been presented on domination [33], some topological indices [84, 87] and the equitable chromatic number [29] of corona product graphs.

3.2 The simultaneous adjacency dimension of graph families

Let $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ be a family of (not necessarily edge-disjoint) connected graphs $G_i = (V, E_i)$ with common vertex set V (the union of whose edge sets is not necessarily the complete graph). By analogy to the definitions of simultaneous metric generator, basis and dimension presented in Chapter 2, we define a *simultaneous adjacency generator* for $\mathcal{G}$ to be a set $S \subset V$ such that S is simultaneously an adjacency generator for each G_i. We say that a minimum cardinality simultaneous adjacency generator for $\mathcal{G}$ is a *simultaneous adjacency basis* of $\mathcal{G}$, and its cardinality the *simultaneous adjacency dimension* of $\mathcal{G}$, denoted by $\mathrm{Sd}_A(\mathcal{G})$ or explicitly by $\mathrm{Sd}_A(G_1, G_2, ..., G_k)$. For instance, the set $\{v_1, v_3, v_6, v_7, v_8\}$ is a simultaneous adjacency basis of the family $\mathcal{G} = \{G_1, G_2, G_3\}$ shown in Figure 3.3, while the set $\{v_1, v_6, v_7, v_8\}$ is a simultaneous metric basis, so $\mathrm{Sd}_A(\mathcal{G}) = 5$ and $\mathrm{Sd}(\mathcal{G}) = 4$.

We now analyse the main properties of the simultaneous adjacency dimension and, in a manner analogous as we did for the simultaneous metric

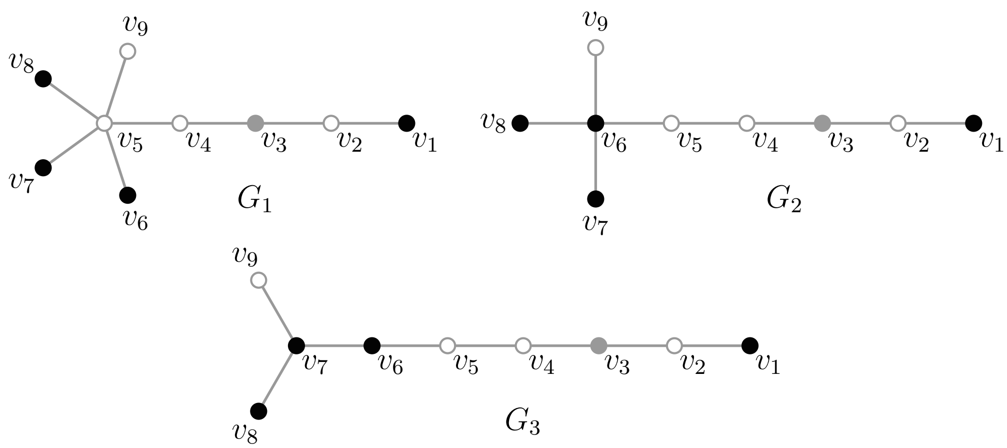

Figure 3.3: The set $\{v_1, v_3, v_6, v_7, v_8\}$ is a simultaneous adjacency basis of $\{G_1, G_2, G_3\}$, whereas $\{v_1, v_6, v_7, v_8\}$ is a simultaneous metric basis.

dimension, we analyse how it is possible to obtain large families of graphs having a fixed adjacency basis and a large common induced subgraph.

Remark 3.1. *For any family $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ of connected graphs on a common vertex set V, the following results hold:*

(i) $\mathrm{Sd}_A(\mathcal{G}) \geq \max_{i\in\{1,\dots,k\}}\{\dim_A(G_i)\}$.

(ii) $\mathrm{Sd}_A(\mathcal{G}) \geq \mathrm{Sd}(\mathcal{G})$.

(iii) $\mathrm{Sd}_A(\mathcal{G}) \leq |V| - 1$.

Proof. (i) is deduced directly from the definition of simultaneous adjacency dimension, while (iii) is obtained from the fact that for any non-trivial graph $G = (V, E)$ it holds that for any $v \in V$ the set $V - \{v\}$ is an adjacency generator. Let B be a simultaneous adjacency basis of $\mathcal{G}$ and let $u, v \in V - B$, be two different vertices. For every graph G_i, there exists $x \in B$ such that $d_{G_i,2}(u, x) \neq d_{G_i,2}(v, x)$, so $d_{G_i}(u, x) \neq d_{G_i}(v, x)$. Thus, B is a simultaneous metric generator for $\mathcal{G}$ and, as a consequence, (ii) follows. $\square$

As pointed out in [43], $\dim_A(G) = n-1$ if and only if $G = K_n$ or $G = N_n$. The following result follows directly from Remark 3.1.

Corollary 3.2. *Let $\mathcal{G}$ be a graph family on a common vertex set V. If $K_{|V|} \in \mathcal{G}$ or $N_{|V|} \in \mathcal{G}$, then* $\mathrm{Sd}_A(\mathcal{G}) = |V| - 1$.

The converse of Corollary 3.2 does not hold, as we will exemplify in Corollary 3.4. We first note the following result, which is a direct consequence of Theorem 2.3 and Remark 3.1 (ii), (iii) and characterizes a large number of cases where the upper bound of (iii) is reached.

Remark 3.3. *Let $\mathcal{G}$ be a graph family on a common vertex set V. If for every pair $u, v \in V$ there exists a graph $G_{uv} \in \mathcal{G}$ such that u and v are twins in G_{uv}, then* $\mathrm{Sd}_A(\mathcal{G}) = |V| - 1$.

For a star graph $K_{1,r}$, $r \geq 3$, it is known that $\dim_A(K_{1,r}) = r - 1$ and every adjacency basis is composed by all but one of its leaves. For a finite set $V = \{v_1, v_2, \ldots, v_n\}$, $n \geq 4$, let $K^i_{1,n-1}$ be the star graph having v_i as its central vertex and $V - \{v_i\}$ as its leaves. We define the family $\mathcal{K}(V) = \{K^i_{1,n-1} : v_i \in V\}$. Any pair of vertices $v_p, v_q \in V$ are twins in every $K^i_{1,n-1} \in \mathcal{K}(V) - \{K^p_{1,n-1}, K^q_{1,n-1}\}$, so the following result is a direct consequence of Remark 3.3.

Corollary 3.4. *For every finite set V of size $|V| \geq 4$,* $\mathrm{Sd}_A(\mathcal{K}(V)) = |V| - 1$.

Let $P_3^{(1)} = (V, E_1)$, $P_3^{(2)} = (V, E_2)$ and $P_3^{(3)} = (V, E_3)$ be the three different path graphs defined on the common vertex set $V = \{v_1, v_2, v_3\}$, where v_i is the vertex of degree two in $P_3^{(i)}$, for $i \in \{1, 2, 3\}$. It was shown in [43] that $\dim_A(G) = 1$ if and only if $G \in \{P_1, P_2, P_3, P_2^c, P_3^c\}$. The following result follows directly from this fact.

Remark 3.5. *The following statements hold:*

(i) $\mathrm{Sd}_A(\mathcal{G}) = 1$ *if and only if* $\mathcal{G} \subseteq \{P_2, P_2^c\}$, $\mathcal{G} \subseteq \left\{P_3^{(1)}, P_3^{(2)}, \left(P_3^{(1)}\right)^c, \left(P_3^{(2)}\right)^c\right\}$, $\mathcal{G} \subseteq \left\{P_3^{(1)}, P_3^{(3)}, \left(P_3^{(1)}\right)^c, \left(P_3^{(3)}\right)^c\right\}$ *or* $\mathcal{G} \subseteq \left\{P_3^{(2)}, P_3^{(3)}, \left(P_3^{(2)}\right)^c, \left(P_3^{(3)}\right)^c\right\}$.

(ii) $\mathrm{Sd}_A\left(P_3^{(1)}, P_3^{(2)}, P_3^{(3)}, \left(P_3^{(1)}\right)^c, \left(P_3^{(2)}\right)^c, \left(P_3^{(3)}\right)^c\right) = 2$.

The following result is derived from the fact that any graph and its complement have the same set of adjacency bases.

Remark 3.6. *Let $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ be a family of graphs with the same vertex set V, and let $\mathcal{G}^c = \{G_1^c, G_2^c, \ldots, G_k^c\}$ be the family composed by the complements of every graph in $\mathcal{G}$. The following assertions hold:*

(i) $\mathrm{Sd}_A(\mathcal{G}) = \mathrm{Sd}_A(\mathcal{G}^c) = \mathrm{Sd}_A(\mathcal{G} \cup \mathcal{G}^c)$. *Moreover, the simultaneous adjacency bases of $\mathcal{G}$ and $\mathcal{G}^c$ coincide.*

(ii) *For any subfamily of graphs* $\mathcal{G}' \subseteq \mathcal{G}^c$, $\mathrm{Sd}_A(\mathcal{G}) = \mathrm{Sd}_A(\mathcal{G} \cup \mathcal{G}')$.

In Section 2.5, we described an approach for, given a graph G and a metric basis B of G, constructing the family $\mathcal{G}_B(G)$, composed by graphs having a large common induced subgraph, which satisfies $\mathrm{Sd}(\mathcal{G}_B(G)) = \dim(G)$. Now, we will present an analogous approach for, given a graph G and an adjacency basis B of G, constructing the family $\widetilde{\mathcal{G}}_B(G)$, composed by graphs that have a large common induced subgraph, which satisfies $\mathrm{Sd}_A(\widetilde{\mathcal{G}}_B(G)) = \dim_A(G)$.

To begin with, recall that for a graph $G = (V, E)$ and a set $X \subseteq V$, $\mathcal{S}(X)$ denotes the stabilizer of X and $f(X)$ denotes the image of X under f.

Let $G = (V, E)$ be a graph and let $B \subset V$ be a non-empty set. For any permutation $f \in \mathcal{S}(B)$ of V we say that a graph $G' = (V, E')$ belongs to the family $\widetilde{\mathcal{G}}_{B,f}(G)$ if and only if $N_{G'}(x) = f(N_G(x))$, for every $x \in B$. We define the subgraph $\langle B_G \rangle_w = (N_G[B], E_w)$ of G, weakly induced by B, where $N_G[B] = \cup_{x\in B} N_G[x]$ and E_w is the set of all edges having at least one vertex in B. See Figure 3.4 for an example of this construction.

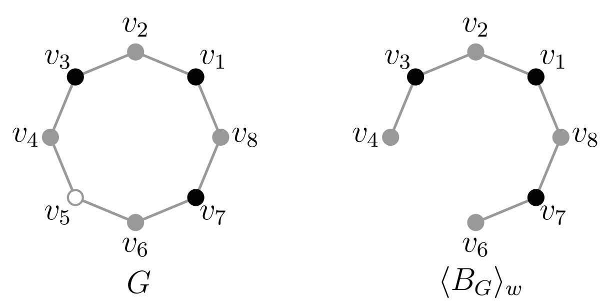

Figure 3.4: The graph $G = C_8$, and the subgraph $\langle B_G \rangle_w$ of G, weakly induced by the adjacency basis $B = \{v_1, v_3, v_7\}$. In this case, $N_G[B] = \{v_1, v_2, v_3, v_4, v_6, v_7, v_8\}$.

Remark 3.7. *Let* $G = (V, E)$ *be a graph and let* $B \subset V$ *be a non-empty set. For any* $f \in \mathcal{S}(B)$ *and any graph* $G' \in \widetilde{\mathcal{G}}_{B,f}(G)$,

$$\langle B_G \rangle_w \cong \langle B_{G'} \rangle_w.$$

Proof. Since $G' \in \widetilde{\mathcal{G}}_{B,f}(G)$, the function f is a bijection from $V(G)$ onto $V(G')$. Now, since $N_{G'}(x) = f(N_G(x))$, for every $x \in B$, we conclude that uv is an edge of $\langle B_G \rangle_w$ if and only if $f(u)f(v)$ is an edge of $\langle B_{G'} \rangle_w$. Therefore, the restriction of f to $\langle B_G \rangle_w$ is an isomorphism. $\square$

Now we define the family $\widetilde{\mathcal{G}}_B(G)$, associated to B, as follows:

$$\widetilde{\mathcal{G}}_B(G) = \bigcup_{f \in \mathcal{S}(B)} \widetilde{\mathcal{G}}_{B,f}(G).$$

With this notation in mind we can state our next result.

Theorem 3.8. *Any adjacency basis B of a graph G is a simultaneous adjacency generator for any family of graphs $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(G)$. Moreover, if $G \in \mathcal{H}$, then*

$$\mathrm{Sd}_A(\mathcal{H}) = \dim_A(G).$$

Proof. Assume that B is an adjacency basis of a graph $G = (V, E)$. Let $f \in \mathcal{S}(B)$ and let $G' = (V, E')$ such that $N_{G'}(x) = f(N_G(x))$, for every $x \in B$. We will show that B is an adjacency generator for any graph G'. To this end, we take two different vertices $u', v' \in V - B$ of G' and the corresponding vertices $u, v \in V$ of G such that $f(u) = u'$ and $f(v) = v'$. Since $u \neq v$ and $u, v \notin B$, there exists $x \in B$ such that $d_{G,2}(u, x) \neq d_{G,2}(v, x)$. Now, since $N_{G'}(x) = f(N_G(x)) = \{f(w) : \ w \in N_G(x)\}$, we obtain that $d_{G',2}(u', x) = d_{G,2}(u, x) \neq d_{G,2}(u, x) = d_{G',2}(v', x)$. Hence, B is an adjacency generator for G' and, in consequence, is also a simultaneous adjacency generator for $\mathcal{H}$. Then we conclude that $\mathrm{Sd}_A(\mathcal{H}) \leq |B| = \dim_A(G)$ and, if $G \in \mathcal{H}$, then $\mathrm{Sd}_A(\mathcal{H}) \geq \dim_A(G)$. Therefore, the result follows. □

Notice that if $G \notin \{K_n, N_n\}$, then the edge set of any graph $G' \in \widetilde{\mathcal{G}}_B(G)$ can be partitioned into two sets E_1, E_2, where E_1 consists of all edges of G having at least one vertex in B and E_2 is a subset of edges of a complete graph whose vertex set is $V - B$. Hence, $\widetilde{\mathcal{G}}_B(G)$ contains $2^{\frac{|V-B|(|V-B|-1)}{2}}|V-B|!$ different labelled graphs. As an example of large graph families that may be obtained according to this procedure, consider the cycle graph C_8, where $\dim_A(C_8) = 3$. For each adjacency basis B of C_8, we have that $|\widetilde{\mathcal{G}}_B(C_8)| = 122880$. To illustrate this, Figure 3.5 shows a graph family $\mathcal{H} = \{H_1, \ldots, H_8\} \subseteq \widetilde{\mathcal{G}}_B(C_8)$, where $B = \{v_1, v_3, v_7\}$, $\{H_1, H_2, H_3, H_4\} \subseteq \widetilde{\mathcal{G}}_{B,f_1}(C_8)$ and $\{H_5, H_6, H_7, H_8\} \subseteq \widetilde{\mathcal{G}}_{B,f_2}(C_8)$.

The next result follows directly from Theorem 3.8 and the fact that $\dim_A(G) = 1$ if and only if $G \in \{P_2, P_3, P_2^c, P_3^c\}$.

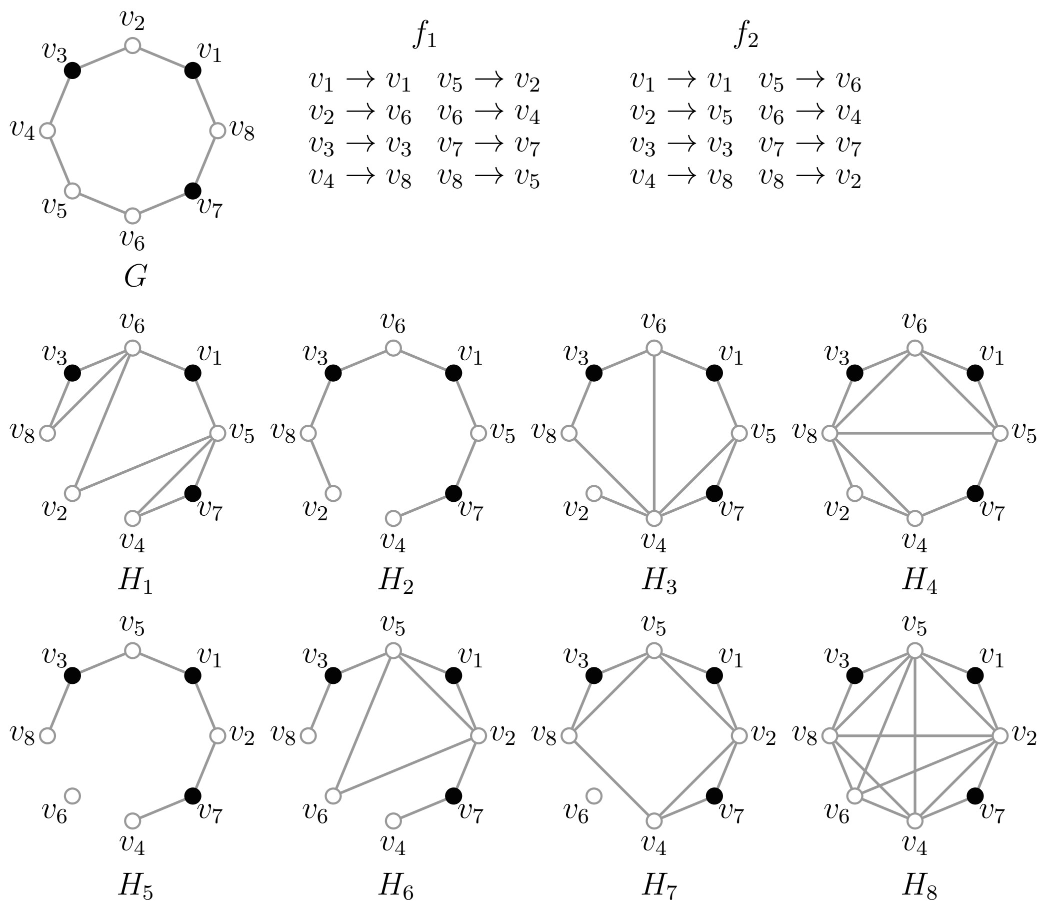

Figure 3.5: A subfamily $\mathcal{H}$ of $\widetilde{\mathcal{G}}_B(C_8)$ for $B = \{v_1, v_3, v_7\}$, where $\{H_1, H_2, H_3, H_4\} \subseteq \widetilde{\mathcal{G}}_{B,f_1}(C_8)$ and $\{H_5, H_6, H_7, H_8\} \subseteq \widetilde{\mathcal{G}}_{B,f_2}(C_8)$. For every $H \in \mathcal{H}$, $\dim_A(H) = \dim_A(C_8) = 3$. Moreover, B is a simultaneous adjacency basis of $\mathcal{H}$, so $\mathrm{Sd}_A(\mathcal{H}) = 3$.

Corollary 3.9. *Let G be a graph of order $n \geq 4$. If $\dim_A(G) = 2$, then for any adjacency basis B of G and any non-empty subfamily $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(G)$,*

$$\mathrm{Sd}_A(\mathcal{H}) = 2.$$

The following result, obtained in [21], shows that Corollary 3.9 is only applicable to families of graphs of order $4, 5$ or 6.

Remark 3.10. [21] *If G is a graph of order $n \geq 7$, then* $\dim_A(G) \geq 3$.

Theorem 3.8 and Remark 3.10 immediately lead to the next result.

Theorem 3.11. *Let B be an adjacency basis of a graph G of order $n \geq 7$. If $\dim_A(G) = 3$, then for any family $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(G)$,*

$$\mathrm{Sd}_A(\mathcal{H}) = 3.$$

The family $\mathcal{H}$ shown in Figure 3.5 is an example of Theorem 3.11.

3.3 Families of join graphs

For a graph family $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$, defined on common vertex set V, and the graph $K_1 = \langle v \rangle$, $v \notin V$, we define the family

$$K_1 + \mathcal{H} = \{K_1 + H : \; H \in \mathcal{H}\}.$$

Notice that, since for any $H \in \mathcal{H}$ the graph $K_1 + H$ has diameter two,

$$\mathrm{Sd}(K_1 + \mathcal{H}) = \mathrm{Sd}_A(K_1 + \mathcal{H}).$$

Theorem 3.12. *Let $\mathcal{G}$ be a family of non-trivial graphs on a common vertex set V. If for every simultaneous adjacency basis B of $\mathcal{G}$ there exist $G \in \mathcal{G}$ and $x \in V$ such that $B \subseteq N_G(x)$, then*

$$\mathrm{Sd}(K_1 + \mathcal{G}) = \mathrm{Sd}_A(\mathcal{G}) + 1.$$

Otherwise,

$$\mathrm{Sd}(K_1 + \mathcal{G}) = \mathrm{Sd}_A(\mathcal{G}).$$

Proof. Let $V(K_1) = \{v_0\}$. Suppose that for every simultaneous adjacency basis B of $\mathcal{G}$ there exist $G \in \mathcal{G}$ and $x \in V$ such that $B \subseteq N_G(x)$. In this case, first notice that for every pair of different vertices $u, v \in V$ we have that $d_{K_1+G,2}(u, v_0) = d_{K_1+G,2}(v, v_0) = 1$, so v_0 does not distinguish any pair of vertices. In consequence, a simultaneous metric basis of $K_1 + \mathcal{G}$ must contain at least as many vertices as a simultaneous adjacency basis of $\mathcal{G}$. Secondly, since $B \subseteq N_{K_1+G}(v_0)$ and $B \subseteq N_{K_1+G}(x)$, a simultaneous metric basis of $K_1 + \mathcal{G}$ must additionally contain some vertex $v \in (V - N_G(x)) \cup \{v_0\}$, so $\mathrm{Sd}(K_1 + \mathcal{G}) \geq \mathrm{Sd}_A(\mathcal{G}) + 1$. Let B be a simultaneous adjacency basis of $\mathcal{G}$ and let $B' = B \cup \{v_0\}$ and $G' \in \mathcal{G}$. For every pair of different vertices $u, v \in V(K_1 + G') - B'$, there exists a vertex $z \in B \subset B'$ such that $d_{K_1+G',2}(u, z) = d_{G',2}(u, z) \neq d_{G',2}(v, z) = d_{K_1+G',2}(v, z)$, so B' is a simultaneous metric generator for $K_1 + \mathcal{G}$ and, as a result, $\mathrm{Sd}(K_1 + \mathcal{G}) \leq |B'| = |B| + 1 = \mathrm{Sd}_A(\mathcal{G}) + 1$. Consequently, $\mathrm{Sd}(K_1 + \mathcal{G}) = \mathrm{Sd}_A(\mathcal{G}) + 1$.

Now suppose that there exists a simultaneous adjacency basis B of $\mathcal{G}$ such that $B \nsubseteq N_G(x)$ for every $G \in \mathcal{G}$ and every $x \in V$. In this case, first recall that a simultaneous metric basis of $K_1 + \mathcal{G}$ must contain as many vertices as a simultaneous adjacency basis of $\mathcal{G}$, so $\mathrm{Sd}(K_1 + \mathcal{G}) \geq \mathrm{Sd}_A(\mathcal{G})$. As above, for every pair of different vertices $u, v \in V - B$, there exists a vertex $z \in B$ such that $d_{K_1+G,2}(u, z) = d_{G,2}(u, z) \neq d_{G,2}(v, z) = d_{K_1+G,2}(v, z)$. Now, for any $u \in V - B$ there exists $u' \in B - N_G(u)$ such that $d_{K_1+G,2}(u, u') = 2 \neq 1 = d_{K_1+G,2}(v_0, u')$. Hence, B is also a simultaneous metric generator for $K_1 + \mathcal{G}$ and, consequently $\mathrm{Sd}(K_1 + \mathcal{G}) \leq |B| = \mathrm{Sd}_A(\mathcal{G})$. Therefore, $\mathrm{Sd}(K_1 + \mathcal{G}) = \mathrm{Sd}_A(\mathcal{G})$. □

Since $K_t + G = K_1 + (K_{t-1} + G)$ for any $t \geq 2$, the previous result can be generalized as follows.

Corollary 3.13. *Let $\mathcal{G}$ be a family of non-trivial graphs on a common vertex set V and let K_t be a complete graph of order $t \geq 1$. If for every simultaneous adjacency basis B of $\mathcal{G}$ there exist $G \in \mathcal{G}$ and $x \in V$ such that $B \subseteq N_G(x)$, then*

$$\mathrm{Sd}(K_t + \mathcal{G}) = \mathrm{Sd}_A(\mathcal{G}) + t.$$

Otherwise,

$$\mathrm{Sd}(K_t + \mathcal{G}) = \mathrm{Sd}_A(\mathcal{G}) + t - 1.$$

By Remark 3.7 and Theorems 3.8 and 3.12 we deduce the following result.

Theorem 3.14. *Let B be an adjacency basis of a graph G and let $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(G)$ such that $G \in \mathcal{H}$. The following assertions hold:*

(i) *If for any adjacency basis B' of G, there exists $v \in V(G)$ such that $B' \subseteq N_G(v)$, then*

$$\mathrm{Sd}(K_1 + \mathcal{H}) = \dim_A(G) + 1.$$

(ii) *If $B \nsubseteq N_G(v)$ for all $v \in V(G)$, then*

$$\mathrm{Sd}(K_1 + \mathcal{H}) = \dim_A(G).$$

Proof. First of all, by Theorem 3.8, $\mathrm{Sd}_A(\mathcal{H}) = \dim_A(G)$ and, as a consequence, every simultaneous adjacency basis of $\mathcal{H}$, which is also a simultaneous metric basis, is an adjacency basis of G. Now, if for any adjacency basis B' of G, there exists $v \in V(G)$ such that $B' \subseteq N_G(v)$, then by Theorem 3.12, $\mathrm{Sd}(K_1 + \mathcal{H}) = \mathrm{Sd}_A(\mathcal{H}) + 1 = \dim_A(G) + 1$. Therefore, (i) follows. On the other hand, if $B \not\subseteq N_G(v)$ for all $v \in V(G)$, then by Remark 3.7 we have that, for every $G' \in \widetilde{\mathcal{G}}_B(G)$ and every $v \in V(G)$, $B \not\subseteq N_{G'}(v)$. Hence, by Theorem 3.12, $\mathrm{Sd}(K_1 + \mathcal{H}) = \mathrm{Sd}_A(\mathcal{H}) = \dim_A(G)$. Therefore, the proof of (ii) is complete. □

To show some particular cases of the results above, we will state the following two results.

Remark 3.15. [43] *For any integer $n \geq 4$,*

$$\dim_A(P_n) = \dim_A(C_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Lemma 3.16. *Let G be a connected graph. If $D(G) \geq 6$, or $G = C_n$ with $n \geq 7$, or G is a graph of girth $\mathtt{g}(G) \geq 5$ and minimum degree $\delta(G) \geq 3$, then for every adjacency generator B for G and every $v \in V(G)$, $B \not\subseteq N_G(v)$.*

Proof. Let B be an adjacency generator for G. First, suppose that there exists $v \in V(G)$ such that $B \subseteq N_G(v)$. Since B is an adjacency generator for G, either B is a dominating set or there exists exactly one vertex $u \in V(G) - B$ which is not dominated by B. In the first case, $D(G) \leq 4$ and in the second one, either $D(G) \leq 5$ or u is an isolated vertex. Hence, if $D(G) \geq 6$, then $B \not\subseteq N_G(v)$.

Now, assume that $\delta(G) \geq 3$. Let $v \in V(G)$, $u \in N_G(v)$ and $x, y \in N_G(u) - \{v\}$. If $\mathtt{g}(G) \geq 5$, then no vertex $z \in N_G[v]$ distinguishes x from y and, since B is an adjacency generator for G, there exists $z' \in B - N_G[v]$ which distinguishes them. Thus, $B \not\subseteq N_G(v)$.

Finally, if $G = C_n$ with $n \geq 7$, then by Remark 3.15 we have $|B| \geq \dim_A(G) = \left\lfloor \frac{2n+2}{5} \right\rfloor \geq 3$ and, since G has maximum degree two, the result follows. □

According to Lemma 3.16, Theorem 3.12 immediately leads to the following result.

Proposition 3.17. *Let $\mathcal{G}$ be a family of graphs on a common vertex set V of cardinality $|V| \geq 7$. If every $G \in \mathcal{G}$ satisfies $D(G) \geq 6$, or $\mathsf{g}(G) \geq 5$ and $\delta(G) \geq 3$, or it is a cycle graph, then*

$$\mathrm{Sd}(K_1 + \mathcal{G}) = \mathrm{Sd}_A(\mathcal{G}).$$

Theorem 3.14 and Lemma 3.16 immediately lead to the following result.

Proposition 3.18. *Let G be a graph of order n and let B be an adjacency basis of G. If G is a cycle graph with $n \geq 7$, or $D(G) \geq 6$, or $\mathsf{g}(G) \geq 5$ and $\delta(G) \geq 3$, then for any family $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(G)$ such that $G \in \mathcal{H}$,*

$$\mathrm{Sd}(K_1 + \mathcal{H}) = \dim_A(G).$$

We now discuss particular cases where $\mathrm{Sd}(K_1 + \mathcal{G}) = \mathrm{Sd}_A(\mathcal{G}) + 1$. First, consider a graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$, defined on a common vertex set of cardinality n, such that $G_i \cong K_n$ for some $i \in \{1, \ldots, k\}$. Since $K_1 + K_n = K_{n+1}$, we have that $\mathrm{Sd}(K_1 + \mathcal{G}) = n = \mathrm{Sd}_A(\mathcal{G}) + 1$. Now recall the families $\mathcal{K}(V)$ of star graphs defined in Section 2.1. The following result holds.

Proposition 3.19. *For every finite set V of cardinality $|V| \geq 4$,*

$$\mathrm{Sd}(K_1 + \mathcal{K}(V)) = \mathrm{Sd}_A(\mathcal{K}(V)) + 1.$$

Proof. Every simultaneous adjacency basis B of $\mathcal{K}(V)$ has the form $V - \{v_i\}$, $i \in \{1, \ldots, n\}$. In $K_{1,n-1}^i$, we have that $B \subseteq N_{K_{1,n-1}^i}(v_i)$, so the result is deduced by Theorem 3.12. □

For two graph families $\mathcal{G} = \{G_1, G_2, \ldots, G_{k_1}\}$ and $\mathcal{H} = \{H_1, H_2, \ldots, H_{k_2}\}$, defined on common vertex sets V_1 and V_2, respectively, such that $V_1 \cap V_2 = \emptyset$, we define the family

$$\mathcal{G} + \mathcal{H} = \{G + H : \; G \in \mathcal{G}, H \in \mathcal{H}\}.$$

Notice that, since for any $G \in \mathcal{G}$ and any $H \in \mathcal{H}$ the graph $G + H$ has diameter two,

$$\mathrm{Sd}(\mathcal{G} + \mathcal{H}) = \mathrm{Sd}_A(\mathcal{G} + \mathcal{H}).$$

Theorem 3.20. *Let $\mathcal{G}$ and $\mathcal{H}$ be two families of non-trivial graphs on common vertex sets V_1 and V_2, respectively. If there exists a simultaneous adjacency basis B of $\mathcal{G}$ such that for every $G \in \mathcal{G}$ and every $g \in V_1$, $B \not\subseteq N_G(g)$, then*

$$\mathrm{Sd}(\mathcal{G}+\mathcal{H}) = \mathrm{Sd}_A(\mathcal{G}) + \mathrm{Sd}_A(\mathcal{H}).$$

Proof. Let B be a simultaneous adjacency basis of $\mathcal{G}$ such that $B \not\subseteq N_G(u)$ for every $u \in V_1$, and let B' be a simultaneous adjacency basis of $\mathcal{H}$. We claim that the set $S = B \cup B'$ is a simultaneous metric generator for $\mathcal{G}+\mathcal{H}$. Consider a pair of different vertices $u, v \in (V_1 \cup V_2) - S$. If $u, v \in V_1$, then there exists $x \in B$ that distinguishes them in every $G \in \mathcal{G}$. An analogous situation occurs for $u, v \in V_2$. If $u \in V_1$ and $v \in V_2$, since $B \not\subseteq N_G(u)$, there exists $x \in B$ such that $d_{G+H,2}(u,x) = 2 \neq 1 = d_{G+H,2}(v,x)$ for every $G \in \mathcal{G}$ and $H \in \mathcal{H}$. Thus, S is a simultaneous metric generator for $\mathcal{G}+\mathcal{H}$ and, as a consequence, $\mathrm{Sd}(\mathcal{G}+\mathcal{H}) \leq |S| = |B| + |B'| = \mathrm{Sd}_A(\mathcal{G}) + \mathrm{Sd}_A(\mathcal{H})$.

To prove that $\mathrm{Sd}(\mathcal{G}+\mathcal{H}) \geq \mathrm{Sd}_A(\mathcal{G}) + \mathrm{Sd}_A(\mathcal{H})$, consider a simultaneous metric basis W of $\mathcal{G}+\mathcal{H}$. Let $W_1 = W \cap V_1$ and let $W_2 = W \cap V_2$. Let $G \in \mathcal{G}$ and $H \in \mathcal{H}$. No pair of different vertices $u, v \in V_2 - W_2$ is distinguished in $G+H$ by any vertex from W_1, whereas no pair of different vertices $u, v \in V_1 - W_1$ is distinguished in $G+H$ by any vertex from W_2, so W_1 is a simultaneous adjacency generator for $\mathcal{G}$ and W_2 is a simultaneous adjacency generator for $\mathcal{H}$. Thus, $\mathrm{Sd}(\mathcal{G}+\mathcal{H}) = |W| = |W_1| + |W_2| \geq \mathrm{Sd}_A(\mathcal{G}) + \mathrm{Sd}_A(\mathcal{H})$. □

By Lemma 3.16 we deduce the following consequence of Theorem 3.20.

Corollary 3.21. *Let $\mathcal{G}$ be a family of graphs on a common vertex set V of cardinality $|V| \geq 7$. If every $G \in \mathcal{G}$ satisfies $D(G) \geq 6$, or $\mathsf{g}(G) \geq 5$ and $\delta(G) \geq 3$, or it is a cycle graph, then for any family $\mathcal{H}$ of non-trivial graphs on a common vertex set,*

$$\mathrm{Sd}(\mathcal{G}+\mathcal{H}) = \mathrm{Sd}_A(\mathcal{G}) + \mathrm{Sd}_A(\mathcal{H}).$$

Theorems 3.8 and 3.20 and Lemma 3.16 lead to the next result.

Theorem 3.22. *Let G be a graph of order n and let B be an adjacency basis of G. If G is a cycle graph with $n \geq 7$, or $D(G) \geq 6$, or $\mathsf{g}(G) \geq 5$ and $\delta(G) \geq 3$, then for any family $\mathcal{G}' \subseteq \widetilde{\mathcal{G}}_B(G)$ such that $G \in \mathcal{G}'$ and any family $\mathcal{H}$ of non-trivial graphs on a common vertex set,*

$$\mathrm{Sd}(\mathcal{G}'+\mathcal{H}) = \dim_A(G) + \mathrm{Sd}_A(\mathcal{H}).$$

The ideas introduced in Theorem 3.8 allow us to define large families composed by subgraphs of a join graph $G+H$, which may be seen as the result of a relaxation of the join operation, in the sense that not every pair of nodes $u \in V(G)$, $v \in V(H)$, must be linked by an edge, yet any adjacency basis of $G+H$ is a simultaneous adjacency generator for the family, and thus a simultaneous metric generator. Since for any adjacency basis B of $G+H$, the family $\mathcal{R}_B$ defined in the next result is a subfamily of $\widetilde{\mathcal{G}}_B(G+H)$, the result follows directly from Theorem 3.8.

Corollary 3.23. *Let G and H be two non-trivial graphs and let B be an adjacency basis of $G+H$. Let $E' = \{uv \in E(G+H) : \ u \in V(G)-B,\ v \in V(H)-B\}$ and let $\mathcal{R}_B = \{R_1, R_2, \ldots, R_k\}$ be a graph family, defined on the common vertex set $V(G+H)$, such that, for every $i \in \{1, \ldots, k\}$, $E(R_i) = E(G+H) - E_i$, for some edge subset $E_i \subseteq E'$. Then*

$$\mathrm{Sd}(\mathcal{R}_B) \leq \dim(G+H).$$

As the next result shows, it is possible to obtain families composed by join graphs of the form $G'+H'$, where G' and H' are the result of applying modifications to G and H, respectively, in such a way that any adjacency basis of $G+H$ is a simultaneous adjacency generator for the family, and thus a simultaneous metric generator.

Corollary 3.24. *Let G and H be two non-trivial graphs and let B be an adjacency basis of $G+H$. Let $B_1 = B \cap V(G)$ and $B_2 = B \cap V(H)$. Then for any family $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_{B_1}(G) + \widetilde{\mathcal{G}}_{B_2}(H)$,*

$$\mathrm{Sd}(\mathcal{H}) \leq \dim(G+H).$$

Moreover, if $G+H \in \mathcal{H}$, then

$$\mathrm{Sd}(\mathcal{H}) = \dim(G+H).$$

Proof. The result is a direct consequence of Theorem 3.8, as $\widetilde{\mathcal{G}}_{B_1}(G)+\widetilde{\mathcal{G}}_{B_2}(H) \subseteq \widetilde{\mathcal{G}}_B(G+H)$. □

Given two families $\mathcal{G}$ and $\mathcal{H}$ of non-trivial graphs on common vertex sets V_1 and V_2, respectively, we define $\mathcal{B}(\mathcal{G})$ and $\mathcal{B}(\mathcal{H})$ as the sets composed by all simultaneous adjacency bases of $\mathcal{G}$ and $\mathcal{H}$, respectively. For a simultaneous adjacency basis $B \in \mathcal{B}(\mathcal{G})$, consider the set

$$P(B) = \{u \in V_1 : \ B \subseteq N_G(u) \text{ for some } G \in \mathcal{G}\}.$$

Similarly, for a simultaneous adjacency basis $B' \in \mathcal{B}(\mathcal{H})$, consider the set

$$Q(B') = \{v \in V_2 : \ B' \subseteq N_H(v) \text{ for some } H \in \mathcal{H}\}.$$

Based on the definitions of $P(B)$ and $Q(B')$, we define the parameter $\psi(\mathcal{G}, \mathcal{H})$ as

$$\psi(\mathcal{G}, \mathcal{H}) = \min_{\substack{B \in \mathcal{B}(\mathcal{G}),\\ B' \in \mathcal{B}(\mathcal{H})}} \{|P(B)|, |Q(B')|\}.$$

The following result holds.

Theorem 3.25. *Let $\mathcal{G}$ and $\mathcal{H}$ be two families of non-trivial graphs on common vertex sets V_1 and V_2, respectively. If for every simultaneous adjacency basis B_1 of $\mathcal{G}$ there exists $G \in \mathcal{G}$ and $g \in V_1$ such that $B_1 \subseteq N_G(g)$ and for every simultaneous adjacency basis B_2 of $\mathcal{H}$ there exists $H \in \mathcal{H}$ and $h \in V_2$ such that $B_2 \subseteq N_H(h)$, then*

$$\mathrm{Sd}_A(\mathcal{G}) + \mathrm{Sd}_A(\mathcal{H}) + 1 \leq \mathrm{Sd}(\mathcal{G} + \mathcal{H}) \leq \mathrm{Sd}_A(\mathcal{G}) + \mathrm{Sd}_A(\mathcal{H}) + \psi(\mathcal{G}, \mathcal{H}).$$

Proof. We first address the proof of the lower bound. Let W be a simultaneous metric basis of $\mathcal{G}+\mathcal{H}$ and let $W_1 = W \cap V_1$ and $W_2 = W \cap V_2$. Let $G \in \mathcal{G}$ and $H \in \mathcal{H}$. Since no pair of different vertices $u, v \in V_2 - W_2$ is distinguished by any vertex in W_1, whereas no pair of different vertices $u, v \in V_1 - W_1$ is distinguished by any vertex in W_2, we conclude that W_1 is an adjacency generator for G and W_2 is an adjacency generator for H. Hence, W_1 is a simultaneous adjacency generator for $\mathcal{G}$ and W_2 is a simultaneous adjacency generator for $\mathcal{H}$. If W_1 is a simultaneous adjacency basis of $\mathcal{G}$ and W_2 is a simultaneous adjacency basis of $\mathcal{H}$, then under the assumptions of this theorem, for at least one graph $G+H \in \mathcal{G}+\mathcal{H}$ there exist $x \in V_1 - W_1$ and $y \in V_2 - W_2$, such that $W \subseteq N_{G+H}(x)$ and $W \subseteq N_{G+H}(y)$, which is a contradiction. Thus, W_1 is not a simultaneous adjacency basis of $\mathcal{G}$ or W_2 is not a simultaneous adjacency basis of $\mathcal{H}$. Hence, $|W_1| \geq \mathrm{Sd}_A(\mathcal{G})+1$ or $|W_2| \geq \mathrm{Sd}_A(\mathcal{H})+1$. In consequence, we have that $\mathrm{Sd}(\mathcal{G} + \mathcal{H}) = |W| = |W_1| + |W_2| \geq \mathrm{Sd}_A(\mathcal{G}) + \mathrm{Sd}_A(\mathcal{H}) + 1$.

We now address the proof of the upper bound. Let B_1 and B_2 be simultaneous adjacency bases of $\mathcal{G}$ and $\mathcal{H}$, respectively, for which $\psi(\mathcal{G}, \mathcal{H})$ is obtained. Assume, without loss of generality, that $|P(B_1)| \leq |Q(B_2)|$. Let $S = B_1 \cup B_2 \cup P(B_1)$. We claim that S is a simultaneous metric generator for $\mathcal{G} + \mathcal{H}$. To show this, we differentiate two cases for any $G \in \mathcal{G}$ and $H \in \mathcal{H}$:

(1) There exists $g \in V_1$ such that $B_1 \subseteq N_G(g)$. We claim that the set $S' = B_1 \cup B_2 \cup \{g\} \subseteq S$ is a metric generator for $G + H$. To see this, we only need to check that for any $u \in V_1 - (B_1 \cup \{g\})$ and $v \in V_2 - B_2$ there exists $s \in S'$ which distinguishes them, as B_1 and B_2 are adjacency generators for G and H, respectively. That is, since g is the sole vertex in V_1 satisfying $N_G(g) \supseteq B_1$, for any $u \in V_1 - (B_1 \cup \{g\})$ and $v \in V_2 - B_2$ there exists $s \in B_1 \subset S'$ such that $d_{G+H,2}(u, s) = 2 \neq 1 = d_{G+H,2}(v, s)$. Hence, the set $S' \subseteq S$ is a metric generator for $G + H$.

(2) No vertex $g \in V_1$ satisfies $B_1 \subseteq N_G(g)$. In this case, the set $S' = B_1 \cup B_2 \subseteq S$ is a metric generator for $G+H$, as B_1 and B_2 are adjacency generators for G and H, respectively, and for any $u \in V_1 - B_1$ and $v \in V_2 - B_2$ there exists $s \in B_1 \subset S'$ such that $d_{G+H,2}(u, s) = 2 \neq 1 = d_{G+H,2}(v, s)$.

Therefore, S is a simultaneous metric generator for $\mathcal{G}+\mathcal{H}$, so $\mathrm{Sd}(\mathcal{G}+\mathcal{H}) \leq |S| = |B_1| + |B_2| + |P(B_1)| = \mathrm{Sd}_A(\mathcal{G}) + \mathrm{Sd}_A(\mathcal{H}) + \psi(\mathcal{G}, \mathcal{H})$. □

As the following corollary shows, the inequalities above are tight.

Corollary 3.26. *Let $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ and $\mathcal{G}' = \{G'_1, G'_2, \ldots, G'_{k'}\}$ be families composed by paths and/or cycle graphs on common vertex sets V and V' of sizes $n \geq 7$ and $n' \geq 7$, respectively. Let $u, v \notin V \cup V'$, $u \neq v$, and let $\mathcal{H} = \{\langle u\rangle + G_1, \langle u\rangle + G_2, \ldots, \langle u\rangle + G_k\}$ and $\mathcal{H}' = \{\langle v\rangle + G'_1, \langle v\rangle + G'_2, \ldots, \langle v\rangle + G'_{k'}\}$. Then,*

$$\mathrm{Sd}(\mathcal{H} + \mathcal{H}') = \mathrm{Sd}_A(\mathcal{H}) + \mathrm{Sd}_A(\mathcal{H}') + 1.$$

Proof. By Lemma 3.16 we have that for every simultaneous adjacency generator B for $G \in \mathcal{G}$ and every $v \in V(G)$, $B \nsubseteq N_G(v)$. Hence, as we have shown in the proof of Theorem 3.12, any simultaneous adjacency basis of $\mathcal{G}$ is a simultaneous adjacency basis of $K_1 + \mathcal{G} \cong \langle u\rangle + \mathcal{G} = \mathcal{H}$ and vice versa. So, for any simultaneous adjacency basis B of $\mathcal{H}$ we have that $P(B) = \{u\}$. Analogously, for any simultaneous adjacency basis B' of $\mathcal{H}'$, we have $Q(B') = \{v\}$ and so $\psi(\mathcal{H}, \mathcal{H}') = 1$. □

Notice that the result above can be extended to any pair of graph families $\mathcal{G}$ and $\mathcal{G}'$ satisfying the premises of Lemma 3.16.

3.4 Families of standard lexicographic product graphs

We begin by stating the following known result.

Claim 3.27. [35] *Let G and H be two non-trivial graphs such that G is connected. Then the following assertions hold for any $a, c \in V(G)$ and $b, d \in V(H)$ such that $a \neq c$.*

(i) $N_{G \circ H}(a, b) = (\{a\} \times N_H(b)) \cup (N_G(a) \times V(H))$.

(ii) $d_{G \circ H}((a, b), (c, d)) = d_G(a, c)$

(iii) $d_{G \circ H}((a, b), (a, d)) = d_{H,2}(b, d)$.

Several results on the metric dimension of the lexicographic product $G \circ H$ of two graphs G and H, and its relation to the adjacency dimension of H, are presented in [43]. In this section, we study the simultaneous metric dimension of several families composed by lexicographic product graphs, exploiting the simultaneous adjacency dimension as an important tool.

First, we introduce some necessary notation. Let S be a subset of $V(G \circ H)$. The *projection* of S onto $V(G)$ is the set $\{u : (u, v) \in S\}$, whereas the projection of S onto $V(H)$ is the set $\{v : (u, v) \in S\}$. We define the *twins equivalence relation* $\mathcal{T}$ on $V(G)$ as follows:

$$x \mathcal{T} y \iff N_G[x] = N_G[y] \text{ or } N_G(x) = N_G(y).$$

In what follows, we will denote the equivalence class of vertex x by $x^* = \{y \in V(G) : y \mathcal{T} x\}$. Notice that every equivalence class may be a singleton set, a clique of size at least two of G or an independent set of size at least two of G. We will refer to equivalence classes which are non-singleton cliques as *true-twins equivalence classes* and to equivalence classes which are non-singleton independent sets as *false-twins equivalence classes*. From now on, $T(G)$ denotes the set of all true-twins equivalence classes in $V(G)$, whereas $F(G)$ denotes the set of all false-twins equivalence classes in $V(G)$. Finally, $V_T(G)$ and $V_F(G)$ denote the sets of vertices belonging to true- and false-twins equivalence classes, respectively.

For two graph families $\mathcal{G} = \{G_1, G_2, \ldots, G_{k_1}\}$ and $\mathcal{H} = \{H_1, H_2, \ldots, H_{k_2}\}$, defined on common vertex sets V_1 and V_2, respectively, we define the family

$$\mathcal{G} \circ \mathcal{H} = \{G \circ H : G \in \mathcal{G}, H \in \mathcal{H}\}.$$

In particular, if $\mathcal{G} = \{G\}$ we will use the notation $G \circ \mathcal{H}$.

Our first result allows to extend any result on the simultaneous adjacency dimension of $\mathcal{G} \circ \mathcal{H}$ to the simultaneous metric dimension, and *vice versa*.

Theorem 3.28. *Let G be a connected graph and let H be a non-trivial graph. Then, every metric generator for $G \circ H$ is also an adjacency generator, and vice versa.*

Proof. By definition, every adjacency generator for $G \circ H$ is also a metric generator, so we only need to prove that any metric generator for $G \circ H$ is also an adjacency generator. Let S be a metric generator for $G \circ H$. For a vertex $u_i \in V(G)$, let $R_i = \{u_i\} \times V(H)$. Notice that $R_i \cap S \neq \emptyset$, for every $u_i \in V(G)$, as no vertex outside of $\{u_i\} \times V(H)$ distinguishes pairs of vertices in $\{u_i\} \times V(H)$. We differentiate the following cases for two different vertices $(u_i, v_r), (u_j, v_s) \in V(G \circ H) - S$:

(1) $i = j$. In this case, no vertex from $R_x \cap S$, $x \neq i$, distinguishes (u_i, v_r) and (u_j, v_s), so there exists $(u_i, v) \in R_i \cap S$ such that $d_{G\circ H,2}((u_i, v_r), (u_i, v)) = d_{G\circ H}((u_i, v_r), (u_i, v)) \neq d_{G\circ H}((u_j, v_s), (u_i, v)) = d_{G\circ H,2}((u_j, v_s), (u_i, v))$.

(2) u_i and u_j are true twins ($i \neq j$). Here, no vertex from $R_x \cap S$, $x \notin \{i, j\}$, distinguishes (u_i, v_r) and (u_j, v_s), so there exists $(u_i, v) \in R_i \cap S$ such that $d_{G\circ H,2}((u_i, v_r), (u_i, v)) = d_{G\circ H}((u_i, v_r), (u_i, v)) = 2 \neq 1 = d_{G\circ H}((u_j, v_s), (u_i, v)) = d_{G\circ H,2}((u_j, v_s), (u_i, v))$, or there exists $(u_j, v) \in R_j \cap S$ such that $d_{G\circ H,2}((u_i, v_r), (u_j, v)) = d_{G\circ H}((u_i, v_r), (u_j, v)) = 1 \neq 2 = d_{G\circ H}((u_j, v_s), (u_j, v)) = d_{G\circ H,2}((u_j, v_s), (u_j, v))$.

(3) u_i and u_j are false twins ($i \neq j$). As in the previous case, no vertex from $R_x \cap S$, $x \notin \{i, j\}$, distinguishes (u_i, v_r) and (u_j, v_s), so there exists $(u_i, v) \in R_i \cap S$ such that $d_{G\circ H,2}((u_i, v_r), (u_i, v)) = d_{G\circ H}((u_i, v_r), (u_i, v)) = 1 \neq 2 = d_{G\circ H}((u_j, v_s), (u_i, v)) = d_{G\circ H,2}((u_j, v_s), (u_i, v))$, or there exists $(u_j, v) \in R_j \cap S$ such that $d_{G\circ H,2}((u_i, v_r), (u_j, v)) = d_{G\circ H}((u_i, v_r), (u_j, v)) = 2 \neq 1 = d_{G\circ H}((u_j, v_s), (u_j, v)) = d_{G\circ H,2}((u_j, v_s), (u_j, v))$.

(4) u_i and u_j are not twins. In this case, there exists $u_x \in V(G) - \{u_i, u_j\}$ such that $d_{G,2}(u_i, u_x) \neq d_{G,2}(u_j, u_x)$. Hence, for any $(u_x, v) \in R_x \cap S$ we have that $d_{G\circ H,2}((u_i, v_r), (u_x, v)) = d_{G,2}(u_i, u_x) \neq d_{G,2}(u_j, u_x) = d_{G\circ H,2}((u_j, v_s), (u_x, v))$.

In conclusion, S is an adjacency generator for $G \circ H$. The proof is complete. □

Corollary 3.29. *For any connected graph and any non-trivial graph H,*

$$\dim(G \circ H) = \dim_A(G \circ H).$$

In general, for every family $\mathcal{G}$ composed by connected graphs on a common vertex set, and every family $\mathcal{H}$ composed by non-trivial graphs on a common vertex set,

$$\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) = \mathrm{Sd}_A(\mathcal{G} \circ \mathcal{H}).$$

We would point out that the equalities above hold, even for lexicographic product graphs of diameter greater than two.

The following result, presented in [43], gives a lower bound on $\dim(G \circ H)$, which depends on the order of G and $\dim_A(H)$.

Theorem 3.30. [43] *Let G be a connected graph of order n and let H be a non-trivial graph. Then* $\dim(G \circ H) \geq n \cdot \dim_A(H)$.

We now generalise the previous result for families composed by lexicographic product graphs.

Theorem 3.31. *Let $\mathcal{G}$ be a family of connected graphs on a common vertex set V_1 and let $\mathcal{H}$ be a family of non-trivial graphs on a common vertex set V_2. Then*

$$\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) \geq |V_1| \cdot \mathrm{Sd}_A(\mathcal{H}).$$

Proof. It was shown in [43] that if S' is a metric generator for $G \circ H$, and $R_i = \{u_i\} \times V(H)$ for some $u_i \in V(G)$, then $S' \cap R_i$ resolves all vertex pairs in R_i, and the projection of $S' \cap R_i$ onto $V(H)$ is an adjacency generator for H. Following an analogous reasoning, consider a simultaneous metric generator S for $\mathcal{G} \circ \mathcal{H}$, and let $R_i = \{u_i\} \times V_2$ for some $u_i \in V_1$. We have that the projection of $S \cap R_i$ onto V_2 is an adjacency generator for every $H \in \mathcal{H}$ and, in consequence, a simultaneous adjacency generator for $\mathcal{H}$, so $|R_i \cap S| \geq \mathrm{Sd}_A(\mathcal{H})$. Thus, $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) = |S| = \sum_{u_i \in V_1} |R_i \cap S| \geq |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$. □

In order to present our next results, we introduce some additional definitions. For a graph family $\mathcal{G}$, defined on a common vertex set V, let $V_M(\mathcal{G}) = \{u : \ u \in V_T(G), u \in V_F(G') \text{ for some } G, G' \in \mathcal{G}\}$. Moreover, for a family $\mathcal{H}$ composed by k_2 non-trivial graphs on a common vertex set V', let $\mathcal{B}_1(\mathcal{H})$ be the set of simultaneous adjacency bases B of $\mathcal{H}$ satisfying $B \nsubseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V'$, and let $\mathcal{B}_2(\mathcal{H})$ be the set of simultaneous adjacency bases of $\mathcal{H}$ that are also dominating sets of every $H \in \mathcal{H}$. Finally, we define the parameter

$$\zeta(\mathcal{H}) = \min\left\{k_2, \min_{\substack{B_1 \in \mathcal{B}_1(\mathcal{H}) \\ B_2 \in \mathcal{B}_2(\mathcal{H})}} \{|B_2 - B_1|\}\right\}.$$

With these definitions in mind, we give the next result.

Theorem 3.32. *Let $\mathcal{G} = \{G_1, G_2, \ldots, G_{k_1}\}$ be a family of connected graphs on a common vertex set V_1, let $\mathcal{H} = \{H_1, H_2, \ldots, H_{k_2}\}$ be a family of non-trivial graphs, defined on a common vertex set V_2, such that $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}_2(\mathcal{H})$ are not empty, and let $\mathcal{H}^c = \{H_1^c, H_2^c, \ldots, H_{k_2}^c\}$. If $V_M(\mathcal{G}) = \emptyset$ or $\mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_2(\mathcal{H}) \neq \emptyset$, then*

$$\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) = \mathrm{Sd}(\mathcal{G} \circ \mathcal{H}^c) = |V_1| \cdot \mathrm{Sd}_A(\mathcal{H}). \tag{3.1}$$

Otherwise,

$$\begin{aligned} |V_1| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_M(\mathcal{G})| &\leq \mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) = \mathrm{Sd}(\mathcal{G} \circ \mathcal{H}^c) \leq \\ &\leq |V_1| \cdot \mathrm{Sd}_A(\mathcal{H}) + \zeta(\mathcal{H}) \cdot |V_M(\mathcal{G})|. \end{aligned} \tag{3.2}$$

Proof. We first assume that $V_M(\mathcal{G}) = \emptyset$. By Theorem 3.31, we have that $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) \geq |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$. Thus, it only remains to prove that $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) \leq |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$. To this end, consider the partition $\{V_1', V_1''\}$ of V_1, where $V_1' = \{u :\ u \in V_T(G) \text{ for some } G \in \mathcal{G}\}$, and a pair of simultaneous adjacency bases $B_1 \in \mathcal{B}_1(\mathcal{H})$ and $B_2 \in \mathcal{B}_2(\mathcal{H})$. Consider the set

$$S = (V_1' \times B_1) \cup (V_1'' \times B_2).$$

It was shown in [43] that a set constructed in this manner, considering $\mathcal{G} = \{G\}$ and $\mathcal{H} = \{H\}$, is a metric generator for $G \circ H$. Following an analogous reasoning, we shall deduce that S is also a metric generator for every $G \circ H \in \mathcal{G} \circ \mathcal{H}$, and thus a simultaneous metric generator for $\mathcal{G} \circ \mathcal{H}$. For the sake of thoroughness of our discussion, we elaborate the four cases for two different vertices $(u_i, v_r), (u_j, v_s) \in V(G \circ H) - S$:

(1) $i = j$. In this case, $r \neq s$. Let $R_i = \{u_i\} \times V_2$. Since $S \cap R_i = \{u_i\} \times B_1$ or $S \cap R_i = \{u_i\} \times B_2$ and both B_1 and B_2 are adjacency

generators for H, there exists $v \in B_1$ such that $d_{H,2}(v, v_r) \neq d_{H,2}(v, v_s)$, or there exists $v \in B_2$ such that $d_{H,2}(v, v_r) \neq d_{H,2}(v, v_s)$. Since for every $(u_i, v_r), (u_i, v_s) \in R_i$ we have that $d_{G \circ H,2}((u_i, v_r), (u_i, v_s)) = d_{H,2}(v_r, v_s)$, we conclude that at least one element from S distinguishes (u_i, v_r) and (u_i, v_s).

(2) $i \neq j$ and u_i, u_j are true twins. Here, since $B_1 \not\subseteq N_H(v_r)$, there exists $v \in B_1$ such that $d_{H,2}(v_r, v) = 2$. Thus, $d_{G \circ H,2}((u_i, v_r), (u_i, v)) = d_{H,2}(v_r, v) = 2 \neq 1 = d_{G,2}(u_j, u_i) = d_{G \circ H,2}((u_j, v_s), (u_i, v))$.

(3) $i \neq j$ and u_i, u_j are false twins. Here, since B_2 is a dominating set of H, there exists $v \in B_2$ such that $d_{H,2}(v_r, v) = 1$. Thus, $d_{G \circ H,2}((u_i, v_r), (u_i, v)) = d_{H,2}(v_r, v) = 1 \neq 2 = d_{G,2}(u_j, u_i) = d_{G \circ H,2}((u_j, v_s), (u_i, v))$.

(4) $i \neq j$ and u_i, u_j are not twins. Here, there exists $u_z \in V_1$ such that $d_{G,2}(u_i, u_z) \neq d_{G,2}(u_j, u_z)$. Since $S \cap R_z \neq \emptyset$, we have that $d_{G \circ H,2}((u_i, v_r), (u_z, v)) = d_{G,2}(u_i, u_z) \neq d_{G,2}(u_j, u_z) = d_{G \circ H,2}((u_j, v_s), (u_z, v))$ for every $(u_z, v) \in S$.

Therefore, S is a metric generator for every $G \circ H \in \mathcal{G} \circ \mathcal{H}$ and, in consequence, a simultaneous metric generator for $\mathcal{G} \circ \mathcal{H}$. Hence, $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) \leq |S| = |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$ and the equality holds.

We now address the proof of $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}^c) = |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$. As pointed out in [43], B_1 is a dominating set of every $H^c \in \mathcal{H}^c$ and B_2 satisfies $B_2 \not\subseteq N_{H^c}(v)$ for every $H^c \in \mathcal{H}^c$ and every $v \in V_2$. Since $\mathrm{Sd}_A(\mathcal{H}) = \mathrm{Sd}_A(\mathcal{H}^c)$, by exchanging the roles of B_1 and B_2 and proceeding in a manner analogous to the one used for proving that $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) \leq |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$, we obtain that $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}^c) \leq |V_1| \cdot \mathrm{Sd}_A(\mathcal{H}^c) = |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$. Since $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}^c) \geq |V_1| \cdot \mathrm{Sd}_A(\mathcal{H}^c) = |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$ by Theorem 3.31, the equality holds.

From now on, we assume that $V_M(\mathcal{G}) \neq \emptyset$ and $\mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_2(\mathcal{H}) \neq \emptyset$. Consider a simultaneous adjacency basis $B \in \mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_2(\mathcal{H})$. By a reasoning analogous to the one previously shown, we have that the set $S = V_1 \times B$ is a metric generator for every $G \circ H \in \mathcal{G} \circ \mathcal{H}$ and every $G \circ H^c \in \mathcal{G} \circ \mathcal{H}^c$. Consequently, S is a simultaneous metric generator for $\mathcal{G} \circ \mathcal{H}$ and $\mathcal{G} \circ \mathcal{H}^c$, so $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) \leq |S| = |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$ and $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}^c) \leq |S| = |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$. By Theorem 3.31, $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) \geq |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$ and $\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}^c) \geq |V_1| \cdot \mathrm{Sd}_A(\mathcal{H})$, so the equalities hold.

From now on, we assume that $V_M(\mathcal{G}) \neq \emptyset$ and $\mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_2(\mathcal{H}) = \emptyset$. Let B be a simultaneous metric basis of $\mathcal{G} \circ \mathcal{H}$ and let $B_p = B \cap (\{u_p\} \times V_2)$

for some $u_p \in V_1$. Recall that, as shown in the proof of Theorem 3.31, the projection of B_p onto V_2 is a simultaneous adjacency generator for $\mathcal{H}$. Let B'_p be the projection onto V_2 of some B_p such that $u_p \in V_M(\mathcal{G})$. Suppose, for the purpose of contradiction, that $|B'_p| = \mathrm{Sd}_A(\mathcal{H})$. Let $G \in \mathcal{G}$ be a graph where $u_p \in V_T(G)$ and let $G' \in \mathcal{G}$ be a graph where $u_p \in V_F(G')$. We have that there exists $v \in V_2 - B'_p$ such that either $B'_p \subseteq N_{H'}(v)$ for some $H' \in \mathcal{H}$ or $B'_p \cap N_{H''}(v) = \emptyset$ for some $H'' \in \mathcal{H}$. In the first case, no vertex $(x, y) \in B$ distinguishes in $G \circ H'$ the vertex (u_p, v) from any vertex (u_t, w) such that u_p and u_t are true twins in G, whereas in the second case, no vertex $(x, y) \in B$ distinguishes in $G' \circ H''$ the vertex (u_p, v) from any vertex (u_f, w) such that u_p and u_f are false twins in G'. In either case, we have a contradiction with the fact that B is a simultaneous metric basis of $\mathcal{G} \circ \mathcal{H}$. Thus, for every $u_p \in V_M(\mathcal{G})$, we have that $|B_p| = |B'_p| \geq \mathrm{Sd}_A(\mathcal{H}) + 1$. In conclusion,

$$\begin{aligned}
\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) &= |B| = \sum_{u_p \in V_1 - V_M(\mathcal{G})} |B_p| + \sum_{u_p \in V_M(\mathcal{G})} |B_p| \geq \\
&\geq \sum_{u_p \in V_1 - V_M(\mathcal{G})} \mathrm{Sd}_A(\mathcal{H}) + \sum_{u_p \in V_M(\mathcal{G})} (\mathrm{Sd}_A(\mathcal{H}) + 1) = \\
&= |V_1 - V_M(\mathcal{G})| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_M(\mathcal{G})| \cdot (\mathrm{Sd}_A(\mathcal{H}) + 1) = \\
&= |V_1| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_M(\mathcal{G})|.
\end{aligned}$$

In order to prove the upper bound, consider the partition $\{V_M(\mathcal{G}), V'_1, V''_1\}$ of V_1, where $V'_1 = \{u : u \in V_T(G) \text{ for some } G \in \mathcal{G}\}$. Since $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}_2(\mathcal{H})$ are disjoint, for any $B_1 \in \mathcal{B}_1(\mathcal{H})$ and $B_2 \in \mathcal{B}_2(\mathcal{H})$, there exist up to k_2 vertices $v_{p_1}, v_{p_2}, \ldots, v_{p_r} \in V_2 - B_1$ such that $B_1 \cap N_H(v_{p_i}) = \emptyset$ for some $H \in \mathcal{H}$ and up to k_2 vertices $v_{q_1}, v_{q_2}, \ldots, v_{q_s} \in V_2 - B_2$ such that $B_2 \subseteq N_H(v_{q_i})$ for some $H \in \mathcal{H}$. We define the sets $B'_1 = B_1 \cup \{v_{p_1}, v_{p_2}, \ldots, v_{p_r}\}$ and $B'_2 = B_2 \cup \{v_{q_1}, v_{q_2}, \ldots, v_{q_s}\}$, which are simultaneous adjacency generators for $\mathcal{H}$ that are also dominating sets of every $H \in \mathcal{H}$ and satisfy $B'_1 \nsubseteq N_H(w)$ and $B'_2 \nsubseteq N_H(w)$ for every $w \in V_2$ and every $H \in \mathcal{H}$.

Consider one $B_1 \in \mathcal{B}_1(\mathcal{H})$ such that $|B'_1|$ is minimum and any $B_2 \in \mathcal{B}_2(\mathcal{H})$. We define the set $S_1 = (V'_1 \times B_1) \cup (V''_1 \times B_2) \cup (V_M(\mathcal{G}) \times B'_1)$. Likewise, consider one $B_2 \in \mathcal{B}_2(\mathcal{H})$ such that $|B'_2|$ is minimum and any $B_1 \in \mathcal{B}_1(\mathcal{H})$. We define the set $S_2 = (V'_1 \times B_1) \cup (V''_1 \times B_2) \cup (V_M(\mathcal{G}) \times B'_2)$. Finally, consider a pair of simultaneous adjacency bases $B_1 \in \mathcal{B}_1(\mathcal{H})$ and $B_2 \in \mathcal{B}_2(\mathcal{H})$ such that $|B_1 \cup B_2|$ is minimum. As $|B_1| = |B_2|$, we have that $|B_2 - B_1| = |B_1 - B_2|$ and is also minimum. We define the set $S_3 =$

$(V_1' \times B_1) \cup (V_1'' \times B_2) \cup (V_M(\mathcal{G}) \times (B_1 \cup B_2))$. Now, recall that for every $G \in \mathcal{G}$ the sets $S = (V_T(G) \times B_1) \cup ((V_1 - V_T(G)) \times B_2)$ and $S' = ((V_1 - V_F(G)) \times B_1) \cup (V_F(G) \times B_2)$ are metric generators for every $G \circ H \in \mathcal{G} \circ \mathcal{H}$. Clearly, $S \subseteq S_1$ or $S' \subseteq S_1$, whereas $S \subseteq S_2$ or $S' \subseteq S_2$, and $S \subseteq S_3$ or $S' \subseteq S_3$, so we have that S_1, S_2 and S_3 are simultaneous metric generators for $\mathcal{G} \circ \mathcal{H}$. Thus,

$$
\begin{aligned}
\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) &\leq \min\{|S_1|, |S_2|, |S_3|\} = \\
&= |V_1 - V_M(\mathcal{G})| \cdot \mathrm{Sd}_A(\mathcal{H}) + \\
&\quad + |V_M(\mathcal{G})| \cdot \min \left\{ \min_{B_1 \in \mathcal{B}_1(\mathcal{H})} \{|B_1'|\}, \min_{B_2 \in \mathcal{B}_2(\mathcal{H})} \{|B_2'|\}, \right. \\
&\quad \left. \min_{\substack{B_1 \in \mathcal{B}_1(\mathcal{H}) \\ B_2 \in \mathcal{B}_2(\mathcal{H})}} \{|B_1 \cup B_2|\} \right\} \leq \\
&\leq |V_1 - V_M(\mathcal{G})| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_M(\mathcal{G})| \cdot (\mathrm{Sd}_A(\mathcal{H}) + \zeta(\mathcal{H})) = \\
&= |V_1| \cdot \mathrm{Sd}_A(\mathcal{H}) + \zeta(\mathcal{H}) \cdot |V_M(\mathcal{G})|.
\end{aligned}
$$

As in the previous cases, by exchanging the roles of $\mathcal{B}_1$ and $\mathcal{B}_2$ for $\mathcal{H}^c$ and proceeding in an analogous manner as above, we obtain that

$$|V_1| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_M(\mathcal{G})| \leq \mathrm{Sd}(\mathcal{G} \circ \mathcal{H}^c) \leq |V_1| \cdot \mathrm{Sd}_A(\mathcal{H}) + \zeta(\mathcal{H}) \cdot |V_M(\mathcal{G})|.$$

The proof is thus complete. □

We now analyse the different cases described in Theorem 3.32. First, note that if $\zeta(\mathcal{H}) = 1$, then Equation (3.2) becomes an equality. In particular, $\zeta(\mathcal{H}) = 1$ for every $\mathcal{H} = \{H\}$. Additionally, if there exists a simultaneous adjacency basis $B_1 \in \mathcal{B}_1(\mathcal{H})$ such that one vertex $v \in V_2 - B_1$ satisfies $B_1 \cap N_H(v) = \emptyset$ for every $H \in \mathcal{H}$, then $\zeta(\mathcal{H}) = 1$. In an analogous manner, if there exists a simultaneous adjacency basis $B_2 \in \mathcal{B}_2(\mathcal{H})$ such that one vertex $v \in V_2 - B_2$ satisfies $B_2 \subseteq N_H(v)$ for every $H \in \mathcal{H}$, then $\zeta(\mathcal{H}) = 1$. Finally, if there exist two simultaneous adjacency bases $B_1 \in \mathcal{B}_1(\mathcal{H})$ and $B_2 \in \mathcal{B}_2(\mathcal{H})$ such that $|B_1 \cup B_2| = \mathrm{Sd}_A(\mathcal{H}) + 1$, then $\zeta(\mathcal{H}) = 1$.

Next, we discuss Equation (3.1). First, note that $V_M(\{G\}) = \emptyset$ for every graph G. Now, we analyse several non-trivial conditions under which a graph family $\mathcal{G}$ composed by connected graphs on a common vertex set satisfies $V_M(\mathcal{G}) = \emptyset$. Consider two vertices u and v that are true twins in some graph G, and a vertex $x \in V(G) - \{u, v\}$ such that $x \sim u$ and $x \sim v$. We have that $\langle \{u, v, x\} \rangle_G \cong C_3$. This fact allows us to characterize a large number of families composed by true-twins-free graphs, for which $V_M(\mathcal{G}) = \emptyset$.

Remark 3.33. *Let $\mathcal{G}$ be a graph family on a common vertex set, such that every $G \in \mathcal{G}$ is a tree or satisfies $\mathsf{g}(G) \geq 4$. Then, $V_M(\mathcal{G}) = \emptyset$.*

In particular, for families composed by path or cycle graphs of order greater than or equal to four, not only all members are true-twins-free, but they are also false-twins-free. Moreover, families composed by hypercubes of order 2^r, $r \geq 2$, satisfy that all their members have girth four.

We now study the behaviour of $V_M(\mathcal{H})$ for $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(G)$, where B is an adjacency basis of G.

Remark 3.34. *For every adjacency basis B of a graph G, and every family $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(G)$,*

$$V_M(\mathcal{H}) = \emptyset.$$

Proof. Let B be an adjacency basis of G. Consider a pair of vertices $x, y \in B$. By the construction of $\widetilde{\mathcal{G}}_B(G)$, we have that in every $H \in \mathcal{H}$ either x and y are true twins, or they are false twins, or they are not twins. Moreover, since B is a simultaneous adjacency generator for $\mathcal{H}$, no pair of vertices $x, y \in V(G) - B$ are twins in any $H \in \mathcal{H}$. Finally, consider two vertices $x \in B$ and $y \in V(G) - B$. If there exist graphs $H_1, H_2, \ldots, H_k \in \mathcal{H}$ where $N_{H_i}(x) = N_{H_i}(y)$, $i \in \{1, \ldots, k\}$, we have that, by the construction of $\widetilde{\mathcal{G}}_B(G)$, either $x \sim y$ in every H_i, $i \in \{1, \ldots, k\}$, or $x \not\sim y$ in every H_i, $i \in \{1, \ldots, k\}$. Hence, x and y are true twins in every H_i, $i \in \{1, \ldots, k\}$, or they are false twins in every H_i, $i \in \{1, \ldots, k\}$. In consequence, $V_M(\mathcal{H}) = \emptyset$. □

We now discuss several cases where a graph family $\mathcal{H}$ satisfies $\mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_2(\mathcal{H}) \neq \emptyset$. First, we introduce an auxiliary result.

Lemma 3.35. *Let P_n and C_n be a path and a cycle graph of order $n \geq 7$. If $n \equiv 1(5)$ or $n \equiv 3(5)$, then no adjacency basis of P_n or C_n is a dominating set. Otherwise, there exist adjacency bases of P_n and C_n that are dominating sets.*

Proof. In C_n, consider the path $v_i v_{i+1} v_{i+2} v_{i+3} v_{i+4}$, where the subscripts are taken modulo n, and an adjacency basis B. If $v_i, v_{i+2} \in B$ and $v_{i+1} \notin B$, then $\{v_{i+1}\}$ is said to be a 1-gap of B. Likewise, if $v_i, v_{i+3} \in B$ and $v_{i+1}, v_{i+2} \notin B$, then $\{v_{i+1}, v_{i+2}\}$ is said to be a 2-gap of B and if $v_i, v_{i+4} \in B$ and $v_{i+1}, v_{i+2}, v_{i+3} \notin B$, then $\{v_{i+1}, v_{i+2}, v_{i+3}\}$ is said to be a 3-gap of B. Since B is an adjacency basis of C_n, it has no gaps of size 4 or larger and it has at most one 3-gap. Moreover, every 2- or 3-gap must be neighboured by two 1-gaps and the number of gaps of either size is at most $\dim_A(C_n)$. We now differentiate the following cases for C_n:

(1) $n = 5k$, $k \geq 2$. In this case, $\dim_A(C_n) = 2k$ and $n - \dim_A(C_n) = 3k$. Since any 2-gap must be neighboured by two 1-gaps, any adjacency basis B has at most k 2-gaps. For any adjacency basis B having exactly k 2-gaps and exactly k 1-gaps, the number of vertices of $V(C_n)-B$ belonging to a 1- or 2-gap is $3k = n-|B|$, so B has no 3-gaps, *i.e.* it is a dominating set.

(2) $n = 5k+1$, $k \geq 2$. In this case, $\dim_A(C_n) = 2k$ and $n - \dim_A(C_n) = 3k+1$. As in the previous case, any adjacency basis B has at most k 2-gaps. Now, assume that B has no 3-gaps. Then $|V(C_n) - B| = 3k < 3k+1 = n-|B|$, which is a contradiction. Thus, any B has a 3-gap, *i.e.* it is not dominating.

(3) $n = 5k+2$, $k \geq 1$. In this case, $\dim_A(C_n) = 2k+1$ and $n - \dim_A(C_n) = 3k+1$. As in the previous cases, any adjacency basis B has at most k 2-gaps. For any adjacency basis B having exactly k 2-gaps and exactly $k+1$ 1-gaps, the number of vertices of $V(C_n) - B$ belonging to a 1- or 2-gap is $3k+1 = n-|B|$, so B has no 3-gaps, *i.e.* it is a dominating set.

(4) $n = 5k+3$, $k \geq 1$. In this case, $\dim_A(C_n) = 2k+1$ and $n - \dim_A(C_n) = 3k+2$. As in the previous cases, any adjacency basis B has at most k 2-gaps. Now assume that B has no 3-gaps. Then $|V(C_n)-B| = 3k+1 < 3k+2 = n-|B|$, which is a contradiction. Thus, any B has a 3-gap, *i.e.* it is not dominating.

(5) $n = 5k+4$, $k \geq 1$. In this case, $\dim_A(C_n) = 2k+2$ and $n - \dim_A(C_n) = 3k+2$. Assume that some adjacency basis B has $k+1$ 2-gaps. Then, B would have at least $k+1$ 1-gaps, making $|V(C_n) - B| \geq 3k+3$, which is a contradiction. So, any adjacency basis B has at most k 2-gaps. For any adjacency basis B having exactly k 2-gaps and exactly $k+2$ 1-gaps, the number of vertices of $V(C_n) - B$ belonging to a 1- or 2-gap is $3k+2 = n-|B|$, so B has no 3-gaps, *i.e.* it is a dominating set.

By the set of cases above, the result holds for C_n.

Now consider the path P_n, $n \mod 5 \in \{0,2,4\}$, and let C'_n be the cycle obtained from P_n by joining its leaves v_1 and v_n by an edge. Let B be an adjacency basis of C'_n which is also a dominating set and satisfies $v_1, v_n \notin B$ (at least one such B exists). Since the only value of $d_{C'_n,2}$ that differs from $d_{P_n,2}$ is $d_{C'_n,2}(v_1,v_n) = 1 \neq 2 = d_{P_n,2}(v_1,v_n)$, it is simple to see that every $v \in V(P_n) - B$ has the same adjacency representation in P_n with respect to B as in C'_n, so B is also an adjacency basis and a dominating set of P_n.

To conclude, consider the path P_n, $n \mod 5 \in \{1,3\}$, and let C'_n be the cycle obtained from P_n by joining its leaves v_1 and v_n by an edge. Consider $V = V(P_n) = V(C_n)$, and let B be an adjacency basis of P_n. Since for two different vertices $x,y \in V$, $d_{C'_n,2}(x,y) \neq d_{P_n,2}(x,y)$ if and only if $x,y \in \{v_1,v_n\}$, we have that if $v_1,v_n \in B$ or $v_1,v_n \notin B$, then B is an adjacency basis of C_n. Moreover, some vertex $w \in V - B$ satisfies $B \cap N_{P_n}(w) = B \cap N_{C'_n}(w) = \emptyset$, so B is not a dominating set of P_n. We now treat the case where $v_1 \in B$ and $v_n \notin B$. If $v_{n-1} \notin B$ then B is not a dominating set of P_n. If $v_{n-1} \in B$ and $v_2 \notin B$, we have that $d_{C'_n,2}(v_2,v_{n-1}) = d_{P_n,2}(v_2,v_{n-1}) = 2 \neq 1 = d_{P_n,2}(v_n,v_{n-1}) = d_{C'_n,2}(v_n,v_{n-1})$, whereas for any other pair of different vertices $x,y \in V - B$ there exists $z \in B$ such that $d_{C'_n,2}(x,z) = d_{P_n,2}(x,z) \neq d_{P_n,2}(y,z) = d_{C'_n,2}(y,z)$, so B is an adjacency basis of C'_n where $\{v_n\}$ is a 1-gap. In consequence, some vertex $w \in V - (B \cup \{v_n\})$ satisfies $B \cap N_{P_n}(w) = B \cap N_{C'_n}(w) = \emptyset$, so B is not a dominating set of P_n. Finally, if $v_2, v_{n-1} \in B$, then for any pair of different vertices $x,y \in V - B$ there exists $z \in B - \{v_1\}$ such that $d_{C'_n,2}(x,z) = d_{P_n,2}(x,z) \neq d_{P_n,2}(y,z) = d_{C'_n,2}(y,z)$, so B is an adjacency basis of C'_n where $\{v_n\}$ is a 1-gap. As in the previous case, some vertex $w \in V - (B \cup \{v_n\})$ satisfies $B \cap N_{P_n}(w) = B \cap N_{C'_n}(w) = \emptyset$, so B is not a dominating set of P_n. The proof is complete. □

The following results hold.

Remark 3.36. *Let P_n be a path graph of order $n \geq 7$, where $n \mod 5 \in \{0,2,4\}$, and let C_n be the cycle graph obtained from P_n by joining its leaves by an edge. Let B be an adjacency basis of P_n and C_n which is also a dominating set of both. Then, every $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(P_n) \cup \widetilde{\mathcal{G}}_B(C_n)$ such that $P_n \in \mathcal{H}$ or $C_n \in \mathcal{H}$ satisfies $\mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_2(\mathcal{H}) \neq \emptyset$.*

Proof. The existence of B is a consequence of Lemma 3.35. Since $P_n \in \mathcal{H}$ or $C_n \in \mathcal{H}$, we have that B is a simultaneous adjacency basis of $\mathcal{H}$. Let $V = V(P_n) = V(C_n)$. By the definition of $\widetilde{\mathcal{G}}_B(G)$, we have that $\bigcup_{v \in B} N_H(v) =$

$\bigcup_{v\in B} N_{P_n}(v) = V$ or $\bigcup_{v\in B} N_H(v) = \bigcup_{v\in B} N_{C_n}(v) = V$ for every $H \in \mathcal{H}$, so B is a dominating set of every $H \in \mathcal{H}$. Moreover, by Lemma 3.16, we have that $B \not\subseteq N_{P_n}(v)$ and $B \not\subseteq N_{C_n}(v)$ for every $v \in V$. Furthermore, by the definition of $\widetilde{\mathcal{G}}_B(G)$, we have that $B \cap N_H(v) = B \cap N_{P_n}(v)$ or $B \cap N_H(v) = B \cap N_{C_n}(v)$ for every $H \in \mathcal{H}$ and every $v \in V$, so $B \not\subseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V$. In consequence, $B \in \mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_2(\mathcal{H})$, so the result holds. □

The following result is a direct consequence of Theorem 3.32 and Remark 3.36.

Proposition 3.37. *Let $\mathcal{G}$ be a family of connected graphs on a common vertex set V, let P_n be a path graph of order $n \geq 7$, where $n \mod 5 \in \{0, 2, 4\}$, and let C_n be the cycle graph obtained from P_n by joining its leaves by an edge. Let B be an adjacency basis of P_n and C_n which is also a dominating set of both. Then, for every $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(P_n) \cup \widetilde{\mathcal{G}}_B(C_n)$ such that $P_n \in \mathcal{H}$ or $C_n \in \mathcal{H}$,*

$$\mathrm{Sd}(\mathcal{G} \circ \mathcal{H}) = |V| \cdot \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Remark 3.38. *Let $\mathcal{H}$ be a graph family on a common vertex set V of cardinality $|V| \geq 7$ such that every $H \in \mathcal{H}$ satisfies $D(H) \geq 6$, or $\mathsf{g}(H) \geq 5$ and $\delta(H) \geq 3$, or it is a cycle graph. Let $\mathcal{H}'$ be a graph family on a common vertex set V' of cardinality $|V'| \geq 7$ satisfying the same conditions as $\mathcal{H}$. Then, $\mathcal{B}_1(\mathcal{H} + \mathcal{H}') \cap \mathcal{B}_2(\mathcal{H} + \mathcal{H}') \neq \emptyset$.*

Proof. As we discussed in the proof of Theorem 3.20, there exists a simultaneous metric basis B of $\mathcal{H} + \mathcal{H}'$, which is also a simultaneous adjacency basis, such that the sets $W = B \cap V$ and $W' = B \cap V'$ satisfy $W \not\subseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V$, and $W' \not\subseteq N_{H'}(w)$ for every $H' \in \mathcal{H}'$ and every $w \in V'$. In consequence, we have that $B \not\subseteq N_{H+H'}(v)$ for every $H + H' \in \mathcal{H} + \mathcal{H}'$ and every $v \in V \cup V'$. Moreover, every vertex in V is dominated by every vertex in W', whereas every vertex in V' is dominated by every vertex in W, so B is a dominating set for every $H + H' \in \mathcal{H} + \mathcal{H}'$. In consequence, $B \in \mathcal{B}_1(\mathcal{H} + \mathcal{H}') \cap \mathcal{B}_2(\mathcal{H} + \mathcal{H}')$, so the result holds. □

By an analogous reasoning, Theorems 3.8 and 3.20 lead to the next result.

Remark 3.39. *Let H be a graph of order n which satisfies $D(H) \geq 6$, or $\mathsf{g}(H) \geq 5$ and $\delta(H) \geq 3$, or it is a cycle graph with $n \geq 7$. Let H' be a graph satisfying the same conditions as H. Let B and B' be adjacency bases of H and H', respectively. Then, any pair of families $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(H)$ and $\mathcal{H}' \subseteq \widetilde{\mathcal{G}}_{B'}(H')$ such that $H \in \mathcal{H}$ and $H' \in \mathcal{H}'$ satisfies $\mathcal{B}_1(\mathcal{H}+\mathcal{H}') \cap \mathcal{B}_2(\mathcal{H}+\mathcal{H}') \neq \emptyset$.*

The two following results are direct consequences of Theorem 3.32 and Remarks 3.38 and 3.39.

Proposition 3.40. *Let $\mathcal{G}$ be a family of connected graphs on a common vertex set V_1. Let $\mathcal{H}$ be a graph family on a common vertex set V_2 of cardinality $|V_2| \geq 7$ such that every $H \in \mathcal{H}$ satisfies $D(H) \geq 6$, or $\mathsf{g}(H) \geq 5$ and $\delta(H) \geq 3$, or it is a cycle graph. Let $\mathcal{H}'$ be a graph family on a common vertex set V_2' of cardinality $|V_2'| \geq 7$ satisfying the same conditions as $\mathcal{H}$. Then,*

$$\mathrm{Sd}(\mathcal{G} \circ (\mathcal{H}+\mathcal{H}')) = |V_1| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_1| \cdot \mathrm{Sd}_A(\mathcal{H}').$$

Proposition 3.41. *Let $\mathcal{G}$ be a family of connected graphs on a common vertex set V. Let H be a graph of order n which satisfies $D(H) \geq 6$, or $\mathsf{g}(H) \geq 5$ and $\delta(H) \geq 3$, or it is a cycle graph with $n \geq 7$. Let H' be a graph satisfying the same conditions as H. Let B and B' be adjacency bases of H and H', respectively. Then, for any pair of families $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(H)$ and $\mathcal{H}' \subseteq \widetilde{\mathcal{G}}_{B'}(H')$ such that $H \in \mathcal{H}$ and $H' \in \mathcal{H}'$,*

$$\mathrm{Sd}(\mathcal{G} \circ (\mathcal{H}+\mathcal{H}')) = |V| \cdot \dim_A(H) + |V| \cdot \dim_A(H').$$

We now analyse several conditions under which a graph family $\mathcal{G}$ composed by connected graphs on a common vertex set satisfies $V_M(\mathcal{G}) \neq \emptyset$ and, in some cases, we exactly determine the value of $V_M(\mathcal{G})$. It is simple to see that any graph of the form $K_t + G$, $t \geq 2$, satisfies $V(K_t) \subseteq v^*$ for some $v^* \in T(K_t + G)$. Likewise, any graph of the form $N_t + G$, $t \geq 2$, satisfies $V(N_t) \subseteq v^*$ for some $v^* \in F(N_t + G)$. Moreover, any complete graph K_n, $n \geq 2$, satisfies $T(G) = \{V(K_n)\}$. The next results are direct consequences of these facts.

Remark 3.42. *Let $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ be a family of connected graphs on a common vertex set V such that, for some $i \in \{1, \ldots, k\}$, $G_i = N_t + G'$, where N_t is an empty graph on the vertex set $V' \subset V$, $|V'| \geq 2$, and $G' = (V - V', E')$. If, for some $j \in \{1, \ldots, k\} - \{i\}$, $G_j = K_t + G''$, where K_t is a complete graph on the vertex set V' and $G'' = (V - V', E'')$, then $V_M(\mathcal{G}) \neq \emptyset$.*

Corollary 3.43. *Let $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ be a family composed by path or cycle graphs on a common vertex set V_1 of size $n \geq 4$, and let $\{K_t, N_t\}$ be a family composed by a complete and an empty graph on a common vertex set V_2 of size $t \geq 2$. Then every non-empty family $\mathcal{H} \subseteq \{N_t + G_1, N_t + G_2, \ldots, N_t + G_k\}$ and every non-empty family $\mathcal{H}' \subseteq \{K_{n+t}, K_t + G_1, K_t + G_2, \ldots, K_t + G_k\}$ satisfy $V_M(\mathcal{H} \cup \mathcal{H}') = V_2$.*

We now analyse cases of families containing a graph and its complement.

Remark 3.44. *Let G be a connected graph such that $|T(G)| \geq 1$ or $|F(G)| \geq 1$, and G^c is connected. Then any family $\mathcal{G}$ composed by connected graphs on a common vertex set such that $G \in \mathcal{G}$ and $G^c \in \mathcal{G}$ satisfies $V_M(\mathcal{G}) \neq \emptyset$.*

Proof. First assume that $|T(G)| \geq 1$. Consider a true-twins equivalence class $v_1^* = \{v_1, v_2, \ldots, v_t\} \in T(G)$. For every pair of vertices $v_i, v_j \in v_1^*$, we have that $N_{G^c}(v_i) = N_{G^c}(v_j)$ and $v_i \nsim_{G^c} v_j$. In consequence, v_1^* is a false-twins equivalence class of G^c. Now assume that $|F(G)| \geq 1$ and consider a false-twins equivalence class $w_1^* = \{w_1, w_2, \ldots, w_f\} \in F(G)$. For every pair of vertices $w_i, w_j \in w_1^*$, we have that $N_{G^c}[w_i] = N_{G^c}[w_j]$, so w_1^* is a true-twins equivalence class of G^c. In consequence, $V_T(G) \cup V_F(G) \subseteq V_M(\mathcal{G})$, so the result follows. □

Corollary 3.45. *For every connected graph G such that G^c is connected, $V_M(\{G, G^c\}) = V_T(G) \cup V_F(G)$.*

Finally, we analyse some examples of families $\mathcal{H}$ satisfying $\mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_2(\mathcal{H}) = \emptyset$. Consider the family $\mathcal{H}_5 = \{P_5, C_5\}$, where $V(P_5) = V(C_5) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(P_5) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}$ and $E(C_5) = E(P_5) \cup \{v_1v_5\}$. We have that $\mathcal{B}_1(\mathcal{H}_5) = \{\{v_1, v_5\}, \{v_2, v_3\}, \{v_3, v_4\}\}$ and $\mathcal{B}_2(\mathcal{H}_5) = \{\{v_2, v_4\}\}$, that is $\mathcal{B}_1(\mathcal{H}_5) \cap \mathcal{B}_2(\mathcal{H}_5) = \emptyset$. Likewise, $\mathcal{B}_1(\{P_5\}) = \{\{v_1, v_5\}, \{v_2, v_3\}, \{v_3, v_4\}\}$ and $\mathcal{B}_2(\{P_5\}) = \{\{v_2, v_4\}\}$, *i.e.* $\mathcal{B}_1(\{P_5\}) \cap \mathcal{B}_2(\{P_5\}) = \emptyset$; whereas $\mathcal{B}_1(\{C_5\}) = \{\{v_1, v_2\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}\}$ and $\mathcal{B}_2(\{C_5\}) = \{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_5\}\}$, *i.e.* $\mathcal{B}_1(\{C_5\}) \cap \mathcal{B}_2(\{C_5\}) = \emptyset$. Moreover, the vertex v_3 satisfies $\{v_2, v_4\} \subseteq N_{P_5}(v_3)$ and $\{v_2, v_4\} \subseteq N_{C_5}(v_3)$, so $\zeta(\mathcal{H}) = 1$ for every non-empty subfamily $\mathcal{H} \subseteq \mathcal{H}_5$.

Additionally, consider the family $\mathcal{H}_{ex}^{(n)} = \{H_1, H_2, H_3, H_4\}$ depicted in Figure 3.6. $\mathcal{H}_{ex}^{(n)}$ is defined on the common vertex set $V = \{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+6}\}$, $n \geq 7$, $n \mod 5 \in \{0, 2, 4\}$, and the dashed lines in the figure indicate that H_i differs from H_j in the fact of containing, or not, each one of the edges v_1v_n and $v_{n+2}v_{n+4}$. Let $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{v_{n+1}, \ldots, v_{n+6}\}$. We have that, for every $H \in \mathcal{H}_{ex}^{(n)}$, $\langle V_1 \rangle_H \cong P_n$ or $\langle V_1 \rangle_H \cong C_n$. In consequence, for every non-empty subfamily $\mathcal{H} \subseteq \mathcal{H}_{ex}^{(n)}$, we have that $\mathrm{Sd}_A(\mathcal{H}) = \dim_A(P_n)+2 = \dim_A(C_n)+2$, and every simultaneous adjacency basis B has the form $B = B' \cup X$, where $X \subset V_2$ and B' is a simultaneous adjacency basis of $\mathcal{H}' = \{\langle V_1 \rangle_H : \ H \in \mathcal{H}\}$. Moreover, we have that $\mathcal{B}_1(\mathcal{H}) = \{B' \cup X\}$, where B' is a simultaneous adjacency basis of $\mathcal{H}'$ that is also a dominating set of every $H' \in \mathcal{H}'$ (Lemma 3.35, and the fact that two graphs in $\mathcal{H}'$ differ at most in the fact of containing, or not, the edge v_1v_n, guarantee the existence of such B') and $X \in \{\{v_{n+2}, v_{n+3}\}, \{v_{n+3}, v_{n+4}\}, \{v_{n+3}, v_{n+5}\}, \{v_{n+3}, v_{n+6}\}, \{v_{n+5}, v_{n+6}\}\}$. Likewise, $\mathcal{B}_2(\mathcal{H}) = \{B' \cup \{v_{n+2}, v_{n+4}\}\}$, where B' is a simultaneous adjacency basis of $\mathcal{H}'$ that is also a dominating set of every $H' \in \mathcal{H}'$. Clearly, $\mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_2(\mathcal{H}) = \emptyset$. Moreover, for every $B \in \mathcal{B}_2(\mathcal{H})$, the vertex v_{n+1} satisfies $B \subseteq N_H(v_{n+1})$ for every $H \in \mathcal{H}$, so $\zeta(\mathcal{H}) = 1$.

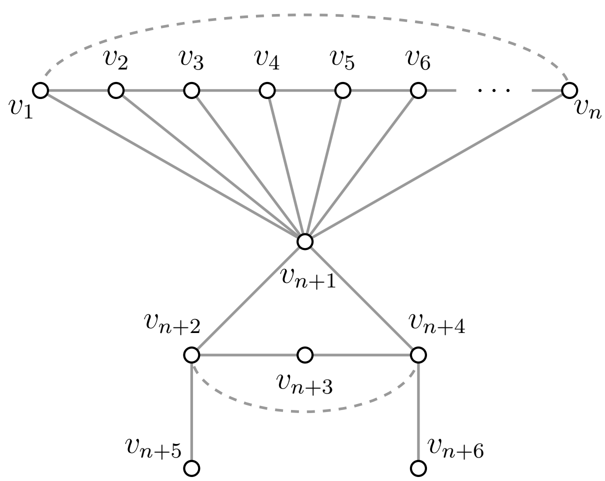

Figure 3.6: For $n \geq 7$, $n \mod 5 \in \{0, 2, 4\}$, every non-empty subfamily $\mathcal{H}$ of the family $\mathcal{H}_{ex}^{(n)} = \{H_1, H_2, H_3, H_4\}$ satisfies $\mathcal{B}_1(\mathcal{H}) \cap \mathcal{B}_2(\mathcal{H}) = \emptyset$ and $\zeta(\mathcal{H}) = 1$.

The aforementioned facts, along with Corollaries 3.43 and 3.45, allows us to obtain examples where Equation (3.2) becomes an equality.

Proposition 3.46. *Let $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ be a family composed by path or cycle graphs on a common vertex set V_1 of size $p \geq 4$, and let $\{K_t, N_t\}$ be a family composed by a complete and an empty graph on a common vertex set V_2 of size $t \geq 2$. Let $\mathcal{G}' \subseteq \{N_t + G_1, N_t + G_2, \ldots, N_t + G_k\}$, $\mathcal{G}' \neq \emptyset$, and* *let $\mathcal{G}'' \subseteq \{K_{n+t}, K_t+G_1, K_t+G_2, \ldots, K_t+G_k\}$, $\mathcal{G}'' \neq \emptyset$. Then, the following assertions hold:*

(i) *For every non-empty subfamily $\mathcal{H} \subseteq \mathcal{H}_5$,*

$$\mathrm{Sd}((\mathcal{G}' \cup \mathcal{G}'') \circ \mathcal{H}) = |V_1 \cup V_2| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_M(\mathcal{G}' \cup \mathcal{G}'')| = 2p + 3t.$$

(ii) *For every $n \geq 7$, where $n \mod 5 \in \{0, 2, 4\}$, and every non-empty subfamily $\mathcal{H} \subseteq \mathcal{H}_{ex}^{(n)}$,*

$$\begin{aligned} \mathrm{Sd}((\mathcal{G}' \cup \mathcal{G}'') \circ \mathcal{H}) &= |V_1 \cup V_2| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_M(\mathcal{G}' \cup \mathcal{G}'')| = \\ &= (p+t) \cdot \left(\left\lfloor \tfrac{2n+2}{5} \right\rfloor + 2\right) + t. \end{aligned}$$

Proposition 3.47. *Let G be a connected graph of order q such that G^c is connected. Then, the following assertions hold:*

(i) *For every non-empty subfamily $\mathcal{H} \subseteq \mathcal{H}_5$,*

$$\mathrm{Sd}(\{G, G^c\} \circ \mathcal{H}) = q \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_M(\{G, G^c\})| = 2q + |V_T(G)| + |V_F(G)|.$$

(ii) *For every $n \geq 7$, where $n \mod 5 \in \{0, 2, 4\}$, and every non-empty subfamily $\mathcal{H} \subseteq \mathcal{H}_{ex}^{(n)}$,*

$$\begin{aligned} \mathrm{Sd}(\{G, G^c\} \circ \mathcal{H}) &= q \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_M(\{G, G^c\})| = \\ &= q \cdot \left(\left\lfloor \tfrac{2n+2}{5} \right\rfloor + 2\right) + |V_T(G)| + |V_F(G)|. \end{aligned}$$

The previous examples additionally show that the bounds of Equation (3.2) are tight. In general, the upper bound is reached when $\min\{|S_1|, |S_2|, |S_3|\} = |S_3|$ or when for every $B_1 \in \mathcal{B}_1(\mathcal{H})$ there exist exactly k_2 vertices $v_{p_1}, v_{p_2}, \ldots, v_{p_r} \in V_2 - B_1$ such that $B_1 \cap N_H(v_{p_i}) = \emptyset$ for some $H \in \mathcal{H}$ and for every $B_2 \in \mathcal{B}_2(\mathcal{H})$ there exist exactly k_2 vertices $v_{q_1}, v_{q_2}, \ldots, v_{q_s} \in V_2 - B_2$ such that $B_2 \subseteq N_H(v_{q_i})$ for some $H \in \mathcal{H}$.

In order to present our next results, we introduce some additional definitions. For a family $\mathcal{H}$ of non-trivial graphs on a common vertex set V, and a simultaneous adjacency basis $B \in \mathcal{B}(\mathcal{H})$, consider the sets

$$P(B) = \{v \in V : \ B \subseteq N_H(v) \text{ for some } H \in \mathcal{H}\}$$

and

$$Q(B) = \{v \in V : \ B \cap N_H(v) = \emptyset \text{ for some } H \in \mathcal{H}\}.$$

Based on the definitions of $P(B)$ and $Q(B)$, we define the parameter

$$\xi(G,\mathcal{H}) = \min_{B\in\mathcal{B}(\mathcal{H})} \{|P(B)|\,(|V_T(G)| - |T(G)|) + |Q(B)|\,(|V_F(G)| - |F(G)|)\}\,.$$

Finally, for a graph G, let $V_T'(G) = \bigcup_{v^*\in T(G)}(v^* - \{v\})$ be the set composed by all vertices, except one, from every true-twins equivalence class of G. Likewise, let $V_F'(G) = \bigcup_{v^*\in F(G)}(v^* - \{v\})$ be the set composed by all vertices, except one, from every false-twins equivalence class of G. For convenience, we will assume without loss of generality that for every graph G a fixed vertex will always be the one excluded from every true or false-twins equivalence class when constructing $V_T'(G)$ or $V_F'(G)$, respectively. With these definitions in mind, we give our next result.

Theorem 3.48. *Let G be a connected graph of order n and let $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ be a family of non-trivial graphs on a common vertex set V_2. If for every simultaneous adjacency basis B of $\mathcal{H}$ there exists $H \in \mathcal{H}$ where one vertex v satisfies $B \subseteq N_H(v)$, or there exists $H' \in \mathcal{H}$ for which B is not a dominating set, then*

$$n \cdot \mathrm{Sd}_A(\mathcal{H}) \le \mathrm{Sd}(G \circ \mathcal{H}) \le n \cdot \mathrm{Sd}_A(\mathcal{H}) + \xi(G,\mathcal{H}).$$

Proof. $\mathrm{Sd}(G\circ\mathcal{H}) \ge n\cdot \mathrm{Sd}_A(\mathcal{H})$ by Theorem 3.31, so we only need to prove that $\mathrm{Sd}(G\circ\mathcal{H}) \le n\cdot\mathrm{Sd}_A(\mathcal{H}) + \xi(G,\mathcal{H})$. Let B be a simultaneous adjacency basis of $\mathcal{H}$ for which $\xi(G,\mathcal{H})$ is obtained. We differentiate the following cases for every graph $H_i \in \mathcal{H}$:

(1) There exist $w_1, w_2 \in V_2$ such that $B \subseteq N_{H_i}(w_1)$ and $B \cap N_{H_i}(w_2) = \emptyset$. In this case, we define the set $S_i = (V(G)\times B) \cup (V_T'(G)\times\{w_1\}) \cup (V_F'(G)\times \{w_2\})$.

(2) There exists $w_1 \in V_2$ such that $B \subseteq N_{H_i}(w_1)$ and there exists no vertex $x \in V_2$ such that $B \cap N_{H_i}(x) = \emptyset$. In this case, we define the set $S_i = (V(G)\times B) \cup (V_T'(G)\times\{w_1\})$.

(3) There exists $w_2 \in V_2$ such that $B \cap N_{H_i}(w_2) = \emptyset$ and there exists no vertex $x \in V_2$ such that $B \subseteq N_{H_i}(x)$. In this case, we define the set $S_i = (V(G)\times B) \cup (V_F'(G)\times\{w_2\})$.

(4) There exists no vertex $x \in V_2$ such that $B \subseteq N_{H_i}(x)$ or $B\cap N_{H_i}(x) = \emptyset$. In this case, we define the set $S_i = V(G)\times B$.

For cases 1, 2 and 3, it is shown in [43] that the corresponding set S_i is a metric generator for $G \circ H_i$. Moreover, as we discussed in the proof of Theorem 3.32, in case 4 the corresponding set S_i is a metric generator for $G \circ H_i$. In consequence, the set $S = \bigcup_{1 \le i \le k} S_i$ is a simultaneous metric generator for $G \circ \mathcal{H}$. Therefore, $\mathrm{Sd}(G \circ \mathcal{H}) \le |S| = n \cdot \mathrm{Sd}_A(\mathcal{H}) + \xi(G, \mathcal{H})$, so the result holds. □

The bounds of the inequalities in Theorem 3.48 are tight. As pointed out in [43], a twins-free graph G satisfies $T(G) = V_T(G) = F(G) = V_F(G) = \emptyset$. In consequence, $\xi(G, \mathcal{H}) = 0$ for any twins-free graph G and any graph family $\mathcal{H}$, so Theorem 3.48 leads to the next result.

Proposition 3.49. *Let G be a twins-free connected graph of order n, and let $\mathcal{H}$ be a family of non-trivial graphs on a common vertex set. Then,*

$$\mathrm{Sd}(G \circ \mathcal{H}) = n \cdot \mathrm{Sd}_A(\mathcal{H}).$$

Recall the families $\mathcal{K}(V)$ of star graphs defined in Section 2.1. The following result is an example of a family for which the upper bound of the inequalities of Theorem 3.48 is reached.

Proposition 3.50. *For every finite set V of size $|V| \ge 4$,*

$$\mathrm{Sd}(P_2 \circ \mathcal{K}(V)) = 2 \cdot |V| - 1.$$

Proof. By Corollary 3.4, every simultaneous adjacency basis B of $\mathcal{K}(V)$ has the form $V - \{v_i\}$, $i \in \{1, \dots, |V|\}$. In $K^i_{1,n-1}$, we have that $B \subseteq N_{K^i_{1,n-1}}(v_i)$, so $\xi(P_2, \mathcal{K}(V)) = 1$. Thus, $\mathrm{Sd}(P_2 \circ \mathcal{K}(V)) \le 2 \cdot \mathrm{Sd}_A(\mathcal{K}(V)) + 1 = 2 \cdot |V| - 1$. Additionally, since $P_2 \circ H \cong H + H$ for any graph H, we have that $\mathrm{Sd}(P_2 \circ \mathcal{K}(V)) = \mathrm{Sd}(\mathcal{K}(V) + \mathcal{K}(V)) \ge 2 \cdot \mathrm{Sd}_A(\mathcal{K}(V)) + 1 = 2 \cdot |V| - 1$ by Theorem 3.25, so the equality holds. □

As we did for join graphs, now we define large families composed by subgraphs of a lexicographic product graph $G \circ H$, which may be seen as the result of a relaxation of the lexicographic product operation, in the sense that not every pair of nodes from two copies of the second factor corresponding to adjacent vertices of the first factor must be linked by an edge. Since for any adjacency basis B of $G \circ H$, the family $\mathcal{R}_B$ defined in the next result is a subfamily of $\widetilde{\mathcal{G}}_B(G \circ H)$, the result follows directly from Theorem 3.8.

Corollary 3.51. *Let G be a connected graph of order n, let H be a non-trivial graph and let B be an adjacency basis of $G \circ H$. Let $E' = \{(u_i, u_j)(u_r, u_s) \in E(G \circ H) :\ i \neq r, (u_i, u_j) \notin B, (u_r, u_s) \notin B\}$ and let $\mathcal{R}_B = \{R_1, R_2, \ldots, R_k\}$ be a graph family, defined on the common vertex set $V(G \circ H)$, such that, for every $l \in \{1, \ldots, k\}$, $E(R_l) = E(G \circ H) - E_l$, for some edge subset $E_l \subseteq E'$. Then*

$$\mathrm{Sd}(\mathcal{R}_B) \leq \dim(G \circ H).$$

3.5 Families of corona product graphs

For two graph families $\mathcal{G} = \{G_1, G_2, \ldots, G_{k_1}\}$ and $\mathcal{H} = \{H_1, H_2, \ldots, H_{k_2}\}$, defined on common vertex sets V and V', respectively, we define the family

$$\mathcal{G} \odot \mathcal{H} = \{G \odot H :\ G \in \mathcal{G}, H \in \mathcal{H}\}.$$

In particular, if $\mathcal{G} = \{G\}$, we will use the notation $G \odot \mathcal{H}$.

Given $G \in \mathcal{G}$ and $H \in \mathcal{H}$, we denote by $H_i = (V_i', E_i)$ the subgraph of $G \odot H$ corresponding to the i-th copy of H. Notice that for any $i \in V$ the graph H_i, which is isomorphic to H, does not depend on G. Hence, the graphs in $\mathcal{G} \odot \mathcal{H}$ are defined on the vertex set $V \cup \left(\bigcup_{i \in V} V_i' \right)$. Analogously, for every $i \in V$ we define the graph family

$$\mathcal{H}_i = \{H_i = (V_i', E_i) :\ H \in \mathcal{H}\}.$$

Also, given a set $W \subset V'$ and $i \in V$, we denote by W_i the subset of V_i' corresponding to W. To clarify this notation, Figure 3.7 shows the graph $C_4 \odot (K_1 \cup K_2)$. In the figure, $V = \{1, 2, 3, 4\}$ and $V' = \{a, b, c\}$, whereas $V_i' = \{a_i, b_i, c_i\}$ for $i \in \{1, 2, 3, 4\}$.

3.5.1 Results on the simultaneous metric dimension

We first introduce a useful relation between the metric generators of two corona product graphs with a common second factor, which allows to determine the simultaneous metric dimension of several families of corona product graphs through the study of the metric dimension of a specific corona product graph.

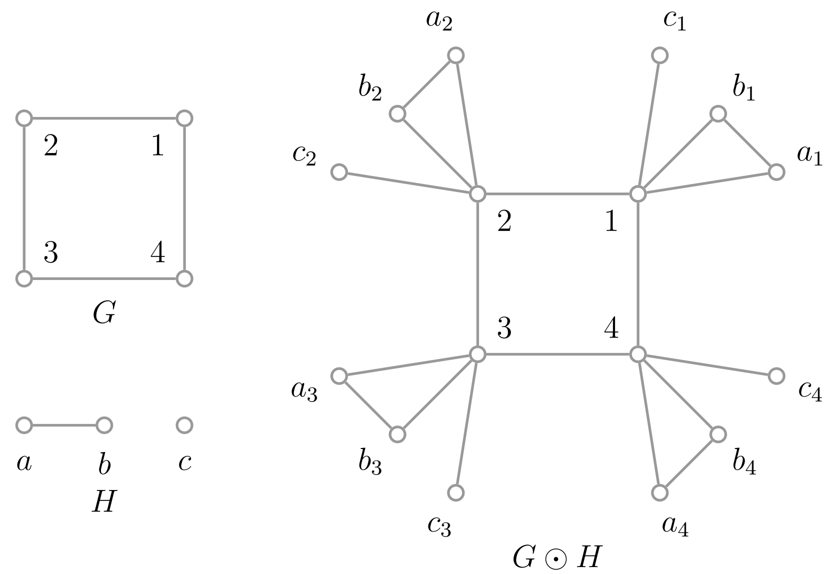

Figure 3.7: The graph $G \odot H$, where $G \cong C_4$ and $H \cong K_1 \cup K_2$.

Theorem 3.52. *Let G_1 and G_2 be two connected non-trivial graphs on a common vertex set and let H be a non-trivial graph. Then any metric generator for $G_1 \odot H$ is a metric generator for $G_2 \odot H$.*

Proof. Let V be the vertex set of G_1 and G_2 and let V' be the vertex set of H. We claim that any metric generator B for $G_1 \odot H$ is a metric generator for $G_2 \odot H$. To see this, we differentiate the following three cases for two different vertices $x, y \in V(G_2 \odot H) - B$.

(1) $x, y \in V_i'$. Since no vertex belonging to $B - V_i'$ distinguishes the pair x, y in $G_1 \odot H$, there must exist $u \in V_i' \cap B$ which distinguishes them. This vertex u also distinguishes x and y in $G_2 \odot H$.

(2) Either $x \in V_i'$ and $y \in V_j'$ or $x = i$ and $y \in V_j'$, where $i \neq j$. For these two possibilities we take $u \in B \cap V_i'$ and we conclude that $d_{G_2 \odot H}(x, u) \leq 2 \neq 3 \leq d_{G_2 \odot H}(y, u)$.

(3) $x = i$ and $y = j$. In this case, for $u \in B \cap V_i'$, we have $d_{G_2 \odot H}(x, u) = 1 \neq 2 \leq d_{G_2 \odot H}(y, u)$.

In conclusion, B is a metric generator for $G_2 \odot H$. □

The following result is a direct consequence of Theorem 3.52.

Corollary 3.53. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set and let $\mathcal{H}$ be a family of non-trivial graphs on a common vertex set. Then, for any $G \in \mathcal{G}$,*

$$\mathrm{Sd}(\mathcal{G} \odot \mathcal{H}) = \mathrm{Sd}(G \odot \mathcal{H}).$$

The following result, obtained in [26], provides a strong link between the metric dimension of the corona product of two graphs and the adjacency dimension of the second graph involved in the product operation.

Theorem 3.54. [26] *For any connected graph G of order $n \geq 2$ and any non-trivial graph H,*

$$\dim(G \odot H) = n \cdot \dim_A(H).$$

We now present a generalisation of Theorem 3.54 to deal with graph families.

Theorem 3.55. *For any family $\mathcal{G}$ composed by connected non-trivial graphs on a common vertex set V and any family $\mathcal{H}$ composed by non-trivial graphs on a common vertex set,*

$$\mathrm{Sd}(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}).$$

Proof. Throughout the proof we consider two arbitrary graphs $G \in \mathcal{G}$ and $H \in \mathcal{H}$. Let B be a simultaneous metric basis of $\mathcal{G} \odot \mathcal{H}$ and let $B_i = B \cap V'_i$. Clearly, $B_i \cap B_j = \emptyset$ for every $i \neq j$. Since no pair of vertices $x, y \in H_i$ is distinguished by any vertex $v \in B_j$, $i \neq j$, we have that B_i is an adjacency generator for H_i. Hence, the set $B' \subset V'$ corresponding to $B_i \subset V'_i$ is an adjacency generator for H and, since B' does not depend on the election of H, it is a simultaneous adjacency generator for $\mathcal{H}$ and, as a result,

$$\mathrm{Sd}(\mathcal{G} \odot \mathcal{H}) = |B| \geq \sum_{i \in V} |B_i| = |V||B'| \geq |V| \cdot \mathrm{Sd}_A(\mathcal{H}).$$

Now, let W be a simultaneous adjacency basis of $\mathcal{H}$ and let $W_i = W \cap V'_i$. By analogy to the proof of Theorem 3.54 we see that $S = \bigcup_{i \in V} W_i$ is a metric generator for $G \odot H$. Since S does not depend on the election of G and H, it is a simultaneous metric generator for $\mathcal{G} \odot \mathcal{H}$ and so

$$\mathrm{Sd}(\mathcal{G} \odot \mathcal{H}) \leq |S| = \sum_{i \in V} |W_i| = |V| \cdot \mathrm{Sd}_A(\mathcal{H}).$$

Therefore, the equality holds. □

The following result is a direct consequence of Theorems 3.8 and 3.55.

Proposition 3.56. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V. Let H be a non-trivial graph and let B be an adjacency basis of H. Then, for every $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(H)$ such that $H \in \mathcal{H}$,*

$$\mathrm{Sd}(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \dim_A(H).$$

3.5.2 Results on the simultaneous adjacency dimension

Given a family $\mathcal{G}$ of connected non-trivial graphs on a common vertex set V and a family $\mathcal{H}$ of non-trivial graphs on a common vertex set, Remark 3.1 and Theorem 3.55 lead to

$$\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H}) \geq \mathrm{Sd}(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}). \tag{3.3}$$

Therefore, there exists an integer $f(\mathcal{G}, \mathcal{H}) \geq 0$ such that

$$\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + f(\mathcal{G}, \mathcal{H}). \tag{3.4}$$

It is easy to check that for any simultaneous adjacency basis W of $\mathcal{H}$ and any $i \in V$, the set $(V - \{i\}) \cup \left(\bigcup_{j \in V} W_j \right)$ is a simultaneous adjacency generator for $\mathcal{G} \odot \mathcal{H}$, where W_j is the subset of V'_j corresponding to $W \subset V'$. Hence,

$$0 \leq f(\mathcal{G}, \mathcal{H}) \leq |V| - 1. \tag{3.5}$$

From now on, our goal is to determine the value of $f(\mathcal{G}, \mathcal{H})$ under different sets of conditions. We begin by pointing out a useful fact which we will use throughout the remainder of this section. Let B be a simultaneous adjacency basis of $\mathcal{G} \odot \mathcal{H}$, and let $B_i = B \cap V'_i$. The following observation is a consequence of the fact that for any graph $G \odot H \in \mathcal{G} \odot \mathcal{H}$ and $i \in V$, no vertex in $B - B_i$ is able to distinguish two vertices in V'_i.

Remark 3.57. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V and let $\mathcal{H}$ be a family of non-trivial graphs on a common vertex set V'. Let B be a simultaneous adjacency basis of $\mathcal{G} \odot \mathcal{H}$ and let $B_i = B \cap V'_i$ for every $i \in V$. Then, B_i is a simultaneous adjacency generator for $\mathcal{H}_i$.*

Now, consider the following known result where $f(G, H) = 0$.

Theorem 3.58. [26] *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If there exists an adjacency basis S of H, which is also a dominating set, and if for every $v \in V(H) - S$, it is satisfied that $S \not\subseteq N_H(v)$, then*

$$\dim_A(G \odot H) = n \cdot \dim_A(H).$$

As the next result shows, Theorem 3.58 can be generalised to the case of families of the form $\mathcal{G} \odot \mathcal{H}$. To this end, recall the notion of simultaneous domination which, as we mentioned previously, was introduced in [7]. On a graph family $\mathcal{G}$, defined on a common vertex set V, a set $M \subseteq V$ is a *simultaneous dominating set* if it is a dominating set of every graph $G \in \mathcal{G}$.

Theorem 3.59. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V and let $\mathcal{H}$ be a family of non-trivial graphs on a common vertex set V'. If there exists a simultaneous adjacency basis B of $\mathcal{H}$ which is also a simultaneous dominating set and satisfies $B \not\subseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V'$, then*

$$\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}).$$

Proof. By (3.3) we only need to show that $\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H}) \leq |V| \cdot \mathrm{Sd}_A(\mathcal{H})$. To this end, assume that B is a simultaneous adjacency basis of $\mathcal{H}$ which is a simultaneous dominating set of $\mathcal{H}$ and satisfies $B \not\subseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V'$. Consider an arbitrary graph $G \odot H \in \mathcal{G} \odot \mathcal{H}$ and let $B_i = B \cap V'_i$, for every $i \in V$. By analogy to the proof of Theorem 3.58 we see that $S = \bigcup_{i \in V} B_i$ is an adjacency generator for $G \odot H$ and, since S does not depend on the election of G and H, it is a simultaneous adjacency generator for $\mathcal{G} \odot \mathcal{H}$. Thus, $\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H}) \leq |S| = |V| \cdot \mathrm{Sd}_A(\mathcal{H})$, and the equality holds. □

The following result is an example of a case where Theorem 3.59 allows to exactly determine the value of $\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H})$ for a large number of graph families.

Proposition 3.60. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V. Let P_n be a path graph of order $n \geq 7$ such that $n \not\equiv 1 \bmod 5$ and $n \not\equiv 3 \bmod 5$, and let C_n be the cycle graph obtained from*

P_n by joining its leaves by an edge. Let B be an adjacency basis of P_n and C_n which is also a dominating set of both. Then, for every $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(P_n) \cup \widetilde{\mathcal{G}}_B(C_n)$ such that $P_n \in \mathcal{H}$ or $C_n \in \mathcal{H}$,

$$\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Proof. The existence of B is a consequence of Lemma 3.35. Since $P_n \in \mathcal{H}$ or $C_n \in \mathcal{H}$, by Theorem 3.8 we deduce that B is a simultaneous adjacency basis of $\mathcal{H}$. Let $V' = V(P_n) = V(C_n)$. By the definition of $\widetilde{\mathcal{G}}_B(G)$, we have that $\bigcup_{v \in B} N_H(v) = \bigcup_{v \in B} N_{P_n}(v) = V'$ or $\bigcup_{v \in B} N_H(v) = \bigcup_{v \in B} N_{C_n}(v) = V'$ for every $H \in \mathcal{H}$, so B is a dominating set of every $H \in \mathcal{H}$. Moreover, by Lemma 3.16, we have that $B \nsubseteq N_{P_n}(v)$ and $B \nsubseteq N_{C_n}(v)$ for every $v \in V'$. Furthermore, by the definition of $\widetilde{\mathcal{G}}_B(G)$, we have that $B \cap N_H(v) = B \cap N_{P_n}(v)$ or $B \cap N_H(v) = B \cap N_{C_n}(v)$ for every $H \in \mathcal{H}$ and every $v \in V'$, so $B \nsubseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V'$. In consequence, the result follows from Remark 3.15 and Theorems 3.8 and 3.59. □

In order to show some cases where $f(\mathcal{G}, \mathcal{H}) = |V| - 1$, we present the following result.

Theorem 3.61. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V and let $\mathcal{H}$ be a family of non-trivial graphs on a common vertex set. If for every simultaneous adjacency basis B of $\mathcal{H}$ there exists $H \in \mathcal{H}$ where B is not a dominating set, then*

$$\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V| - 1.$$

Proof. By (3.4) and (3.5) we have that $\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H}) \leq |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V| - 1$. It remains to prove that $\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H}) \geq |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V| - 1$.

Let U be a simultaneous adjacency basis of $\mathcal{G} \odot \mathcal{H}$, let $U_i = U \cap V_i'$ and let $U_0 = U \cap V$. By Remark 3.57, U_i is a simultaneous adjacency generator for $\mathcal{H}_i$ for every $i \in V$. Consider the partition $\{V_1, V_2\}$ of V defined as

$$V_1 = \{i \in V : \ |U_i| = \mathrm{Sd}_A(\mathcal{H})\} \text{ and } V_2 = \{i \in V : \ |U_i| \geq \mathrm{Sd}_A(\mathcal{H}) + 1\}.$$

For any $i, j \in V_1$, $i \neq j$, we have that there exist a graph $H \in \mathcal{H}$ and two vertices $x \in V_i' - U_i$ and $y \in V_j' - U_j$ such that $U_i \cap N_H(x) = \emptyset$ and $U_j \cap N_H(y) = \emptyset$. Thus, $i \in U$ or $j \in U$ and so $|U_0| \geq |V_1| - 1$. In conclusion,

$$\begin{aligned}\mathrm{Sd}_A(\mathcal{G}\odot\mathcal{H}) &= |U_0| + \sum_{i\in V_1}|U_i| + \sum_{i\in V_2}|U_i| \\ &\geq (|V_1|-1) + |V_1|\cdot \mathrm{Sd}_A(\mathcal{H}) + |V_2|\cdot(\mathrm{Sd}_A(\mathcal{H})+1) \\ &= |V|\cdot \mathrm{Sd}_A(\mathcal{H}) + |V| - 1.\end{aligned}$$

Therefore, the result follows. □

Now we treat some specific families that satisfy the conditions of Theorem 3.61. Lemma 3.35 allows us to give the following result.

Proposition 3.62. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V. Let P_n be a path graph of order $n \geq 7$, $n \equiv 1 \mod 5$ or $n \equiv 3 \mod 5$, and let C_n be the cycle graph obtained from P_n by joining its leaves by an edge. Let B be a simultaneous adjacency basis of $\{P_n, C_n\}$. Then, for every family $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ such that $\mathcal{H}_1$ is composed by paths, $\mathcal{H}_1 \subseteq \widetilde{\mathcal{G}}_B(P_n)$, $P_n \in \mathcal{H}_1$, $\mathcal{H}_2$ is composed by cycles, $\mathcal{H}_2 \subseteq \widetilde{\mathcal{G}}_B(C_n)$, and $C_n \in \mathcal{H}_2$,*

$$\mathrm{Sd}_A(\mathcal{G}\odot\mathcal{H}) = |V|\cdot\left(\left\lfloor\frac{2n+2}{5}\right\rfloor + 1\right) - 1.$$

Proof. Note that B is an adjacency basis of both P_n and C_n. Since $P_n \in \mathcal{H}_1$ and $C_n \in \mathcal{H}_2$, we have that B is a simultaneous adjacency basis of $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ by Theorem 3.8. Moreover, since every $H \in \mathcal{H}_1$ is a path graph and every $H \in \mathcal{H}_2$ is a cycle, we have that $\dim_A(H) = \mathrm{Sd}_A(\mathcal{H})$ for every $H \in \mathcal{H}$, so every simultaneous adjacency basis of $\mathcal{H}$ is an adjacency basis of every $H \in \mathcal{H}$ and, by Lemma 3.35, is not a dominating set of H. Thus, the result follows from Theorem 3.61. □

It is worth noting that for a path graph P_n and a cycle graph C_n, $n \geq 7$, $n \equiv 1 \mod 5$ or $n \equiv 3 \mod 5$, and an adjacency basis B of both, the family $\widetilde{\mathcal{G}}_B(P_n)$ contains $\left(n - \left\lfloor\frac{2n+2}{5}\right\rfloor\right)!$ path graphs, whereas the family $\widetilde{\mathcal{G}}_B(C_n)$ contains $\left(n - \left\lfloor\frac{2n+2}{5}\right\rfloor\right)!$ cycle graphs.

Proposition 3.63. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V and let $\mathcal{H} = \{N_t \cup H_1, N_t \cup H_2, \ldots, N_t \cup H_k\}$, where N_t is an empty graph of order $t \geq 1$ and $H_1, H_2, \ldots, H_k$ are connected non-trivial graphs on a common vertex set. Then,*

$$\mathrm{Sd}_A(\mathcal{G}\odot\mathcal{H}) = |V|\cdot \mathrm{Sd}_A(\mathcal{H}) + |V| - 1.$$

Proof. Consider that the common vertex set of $\mathcal{H}$ has the form $V' = V(N_t) \cup V''$, where $V(N_t)$ and V'' are disjoint. Let B be a simultaneous adjacency basis of $\mathcal{H}$, and let $B'' = B \cap V''$. Consider an arbitrary graph $N_t \cup H \in \mathcal{H}$. The vertices of N_t are false twins, so $V(N_t) \subseteq B$ if and only if there exists $v \in V''$ such that $B \cap N_H(v) = \emptyset$. If such v exists, it is not dominated by B, so the result follows from Theorem 3.61. Otherwise, $V(N_t) - B = \{v'\}$ and $B \cap N_H(v') = \emptyset$, so the result follows from Theorem 3.61. □

Recall that $\gamma(G)$ denotes the domination number of a graph G.

Theorem 3.64. [26] *Let G be a connected graph of order $n \geq 2$ and let H be a non-trivial graph. If there exists an adjacency basis of H, which is also a dominating set and if, for any adjacency basis S of H, there exists $v \in V(H) - S$ such that $S \subseteq N_H(v)$, then*

$$\dim_A(G \odot H) = n \cdot \dim_A(H) + \gamma(G).$$

The *simultaneous domination number* of a family $\mathcal{G}$, which we will denote as $\mathrm{S}\,\gamma(\mathcal{G})$, is the minimum cardinality of a simultaneous dominating set. The next result is a generalisation of Theorem 3.64 to the case of $\mathcal{G} \odot \mathcal{H}$.

Theorem 3.65. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V and let $\mathcal{H}$ be a family of non-trivial graphs on a common vertex set V'. If there exists a simultaneous adjacency basis of $\mathcal{H}$ which is also a simultaneous dominating set, and for every simultaneous adjacency basis B of $\mathcal{H}$ there exist $H \in \mathcal{H}$ and $v \in V' - B$ such that $B \subseteq N_H(v)$, then*

$$\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + \mathrm{S}\,\gamma(\mathcal{G}).$$

Proof. We first address the proof of $\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H}) \geq |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + \mathrm{S}\,\gamma(\mathcal{G})$. Let U be a simultaneous adjacency basis of $\mathcal{G} \odot \mathcal{H}$, let $U_i = U \cap V'_i$, and let $U_0 = U \cap V$. By Remark 3.57, U_i is a simultaneous adjacency generator for $\mathcal{H}_i$ for every $i \in V$. Consider the partition $\{V_1, V_2\}$ of V defined as

$$V_1 = \{i \in V : \ |U_i| = \mathrm{Sd}_A(\mathcal{H})\} \text{ and } V_2 = \{i \in V : \ |U_i| \geq \mathrm{Sd}_A(\mathcal{H}) + 1\}.$$

For every $i \in V_1$, the set U_i is a simultaneous adjacency basis of $\mathcal{H}_i$, so there exist $H \in \mathcal{H}$ and $x \in V'_i$ such that $U_i \subseteq N_H(x)$, causing i and x not to be distinguished by any $y \in U_i$ in any graph belonging to $\mathcal{G} \odot H$. Thus, either $i \in U_0$ or for every $G \in \mathcal{G}$ there exists $z \in U_0$ such that

$d_{G\odot H,2}(i,z) = 1 \neq 2 = d_{G\odot H,2}(x,z)$. In consequence, $V_2 \cup U_0$ must be a simultaneous dominating set of $\mathcal{G}$, so $|V_2 \cup U_0| \geq \mathrm{S}\,\gamma(\mathcal{G})$. Finally,

$$\begin{aligned}
\mathrm{Sd}_A(\mathcal{G}\odot\mathcal{H}) &= \sum_{i\in V_1}|U_i| + \sum_{i\in V_2}|U_i| + |U_0| \\
&\geq \sum_{i\in V_1}\mathrm{Sd}_A(\mathcal{H}) + \sum_{i\in V_2}(\mathrm{Sd}_A(\mathcal{H})+1) + |U_0| \\
&= |V|\cdot\mathrm{Sd}_A(\mathcal{H}) + |V_2| + |U_0| \\
&\geq |V|\cdot\mathrm{Sd}_A(\mathcal{H}) + |V_2\cup U_0| \\
&\geq |V|\cdot\mathrm{Sd}_A(\mathcal{H}) + \mathrm{S}\,\gamma(\mathcal{G}).
\end{aligned}$$

Now, let W be a simultaneous adjacency basis of $\mathcal{H}$ which is also a simultaneous dominating set of $\mathcal{H}$. Consider an arbitrary graph $G\odot H \in \mathcal{G}\odot\mathcal{H}$, and let $W_i = W\cap V_i'$. By analogy to the proof of Theorem 3.64, we have that $S = M\bigcup\left(\bigcup_{i\in V} W_i\right)$, where M is a minimum simultaneous dominating set of $\mathcal{G}$, is an adjacency generator for $G\odot H$. Since S does not depend on the election of G and H, it is a simultaneous adjacency generator for $\mathcal{G}\odot\mathcal{H}$. Thus, $\mathrm{Sd}_A(\mathcal{G}\odot\mathcal{H}) \leq |S| = |V|\cdot\mathrm{Sd}_A(\mathcal{H}) + \mathrm{S}\,\gamma(\mathcal{G})$, so the equality holds. □

Several specific families for which the previous result holds will be described in Theorem 3.72 and Propositions 3.73 and 3.74. Now, in order to present our next result, we need some additional definitions. Let $v\in V(G)$ be a vertex of a graph G and let $G-v$ be the graph obtained by removing from G the vertex v and all its incident edges. Consider the following auxiliary domination parameter, which is defined in [26]:

$$\gamma'(G) = \min_{v\in V(G)}\{\gamma(G-v)\}$$

Theorem 3.66. [26] *Let H be a non-trivial graph such that some of its adjacency bases are also dominating sets, and some are not. If there exists an adjacency basis S' of H such that for every $v\in V(H)-S'$ it is satisfied that $S'\not\subseteq N_H(v)$, and for any adjacency basis S of H which is also a dominating set, there exists some $v\in V(H)-S$ such that $S\subseteq N_H(v)$, then for any connected non-trivial graph G,*

$$\dim_A(G\odot H) = n\cdot\dim_A(H) + \gamma'(G).$$

The following result is a generalisation of Theorem 3.66 to the case of $G \odot \mathcal{H}$.

Theorem 3.67. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family of non-trivial graphs on a common vertex set V' such that some of its simultaneous adjacency bases are also simultaneous dominating sets, and some are not. If there exists a simultaneous adjacency basis B' of $\mathcal{H}$ such that $B' \nsubseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V' - B'$, and for every simultaneous adjacency basis B of $\mathcal{H}$ which is also a simultaneous dominating set there exist $H' \in \mathcal{H}$ and $w \in V' - B$ such that $B \subseteq N_{H'}(w)$, then*

$$\mathrm{Sd}_A(G \odot \mathcal{H}) = n \cdot \mathrm{Sd}_A(\mathcal{H}) + \gamma'(G).$$

Proof. In the family $G \odot \mathcal{H}$, we have that $V = V(G)$. We first address the proof of $\mathrm{Sd}_A(G \odot \mathcal{H}) \geq n \cdot \mathrm{Sd}_A(\mathcal{H}) + \gamma'(G)$. Let U be a simultaneous adjacency basis of $G \odot \mathcal{H}$, let $U_i = U \cap V_i'$, and let $U_0 = B \cap V$. By Remark 3.57, U_i is a simultaneous adjacency generator for $\mathcal{H}_i$ for every $i \in V$. Consider the partition $\{V_1, V_2, V_3\}$ of V, where V_1 contains the vertices $i \in V$ such that U_i is a simultaneous adjacency basis of $\mathcal{H}_i$ but is not a simultaneous dominating set, V_2 contains the vertices $i \in V$ such that U_i is a simultaneous adjacency basis and a simultaneous dominating set of $\mathcal{H}_i$, and V_3 is composed by the vertices $i \in V$ such that U_i is not a simultaneous adjacency basis of $\mathcal{H}_i$.

If $i, j \in V_1$, then there exist a graph $H \in \mathcal{H}$ and two vertices $v_i \in V_i' - U_i$ and $v_j \in V_j' - U_j$ such that $U_i \cap N_H(v_i) = \emptyset$ and $U_j \cap N_H(v_j) = \emptyset$. Thus, $i \in U_0$ or $j \in U_0$, so $|U_0 \cap V_1| \geq |V_1| - 1$. If $i \in V_2$, then there exist $H \in \mathcal{H}$ and $x \in V_i'$ such that $U_i \subseteq N_H(x)$. In consequence, the pair i, x is not distinguished by any $y \in U_i$, so either $i \in U_0$ or there exists $z \in U_0$ such that $d_{G \odot H,2}(i, z) = 1 \neq 2 = d_{G \odot H,2}(x, z)$. Therefore, at most one vertex of G is not dominated by $U_0 \cup V_3$, so $|U_0 \cup V_3| \geq \gamma'(G)$. Finally,

$$\begin{aligned}
\mathrm{Sd}_A(G \odot \mathcal{H}) &= \sum_{i \in V_1 \cup V_2} |U_i| + \sum_{i \in V_3} |U_i| + |U_0| \\
&\geq \sum_{i \in V_1 \cup V_2} \mathrm{Sd}_A(\mathcal{H}) + \sum_{i \in V_3} (\mathrm{Sd}_A(\mathcal{H}) + 1) + |U_0| \\
&= n \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_3| + |U_0| \\
&\geq n \cdot \mathrm{Sd}_A(\mathcal{H}) + |V_3 \cup U_0| \\
&\geq n \cdot \mathrm{Sd}_A(\mathcal{H}) + \gamma'(G).
\end{aligned}$$

Now, let W' be a simultaneous adjacency basis of $\mathcal{H}$ such that $W' \not\subseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V - W'$, and assume that for any simultaneous adjacency basis W of $\mathcal{H}$ which is also a simultaneous dominating set there exist $H' \in \mathcal{H}$ and $w \in V - W$ such that $W \subseteq N_{H'}(w)$. Let W'' be one of such simultaneous adjacency bases of $\mathcal{H}$. Consider an arbitrary graph $G \odot H \in G \odot \mathcal{H}$, let $W'_i = W' \cap V'_i$ and $W''_i = W'' \cap V'_i$. Additionally, let M be a minimum dominating set of $G - n$, assuming without loss of generality that $\gamma'(G) = \gamma(G-n)$, and let $S = M \bigcup W'_n \bigcup \left(\bigcup_{i \in V - \{n\}} W''_i \right)$. By analogy to the proof of Theorem 3.66, we have that S is an adjacency generator for $G \odot H$. Since S does not depend on the election of G and H, it is a simultaneous adjacency generator for $G \odot \mathcal{H}$. Thus, $\mathrm{Sd}_A(G \odot \mathcal{H}) \leq |S| = n \cdot \mathrm{Sd}_A(\mathcal{H}) + \gamma'(G)$, so the equality holds. $\square$

Consider the family $\{P_5, C_5\}$, where C_5 is obtained from P_5 by joining its leaves by an edge. Assume that $V(P_5) = V(C_5) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(P_5) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}$ and $E(C_5) = E(P_5) \cup \{v_1v_5\}$. We have that the set $\{v_2, v_4\}$ is the sole simultaneous adjacency basis which is also a simultaneous dominating set and v_3 satisfies $\{v_2, v_4\} \subseteq N_{P_5}(v_3)$ and $\{v_2, v_4\} \subseteq N_{C_5}(v_3)$. Moreover, the set $\{v_1, v_5\}$ (as well as $\{v_2, v_3\}$ and $\{v_3, v_4\}$) is a simultaneous adjacency basis such that every vertex v_x satisfies $N_{P_5}(v_x) \not\subseteq \{v_1, v_5\}$ and $N_{C_5}(v_x) \not\subseteq \{v_1, v_5\}$. These facts allow us to obtain examples where Theorem 3.67 applies. For instance, for any connected graph G of order $n \geq 2$, we have that $\mathrm{Sd}_A(G \odot \{P_5, C_5\}) = 2n + \gamma'(G)$.

The case where the second factor is a family of join graphs

To begin our presentation, we introduce the following auxiliary result.

Lemma 3.68. *Let $\mathcal{G}$ and $\mathcal{H}$ be two families of non-trivial graphs on common vertex sets V_1 and V_2, respectively. Then, every simultaneous adjacency basis of $\mathcal{G} + \mathcal{H}$ is a simultaneous dominating set of $\mathcal{G} + \mathcal{H}$.*

Proof. Let B be a simultaneous adjacency basis of $\mathcal{G} + \mathcal{H}$, let $W_1 = B \cap V_1$ and $W_2 = B \cap V_2$. Since no pair of different vertices $u, v \in V_2 - W_2$ is distinguished in any $G + H \in \mathcal{G} + \mathcal{H}$ by any vertex from W_1, we have that W_2 is a simultaneous adjacency generator for $\mathcal{H}$ and, in consequence, $W_2 \neq \emptyset$.

By an analogous reasoning we can see that W_1 is a simultaneous adjacency generator for $\mathcal{G}$ and, in consequence, $W_1 \neq \emptyset$. Moreover, every vertex in V_1 is dominated by every vertex in W_2, whereas every vertex in V_2 is dominated by every vertex in W_1, so B is a dominating set for every $G+H \in \mathcal{G}+\mathcal{H}$. □

Recall that Theorem 3.20 characterizes a large number of families of the form $\mathcal{G}+\mathcal{H}$ whose simultaneous adjacency bases are formed by the union of an arbitrary simultaneous adjacency basis of $\mathcal{H}$ and a simultaneous adjacency basis B of $\mathcal{G}$ such that $B \not\subseteq N_G(v)$ for every $G \in \mathcal{G}$ and every $v \in V_1$. With this fact in mind, we present our next result.

Theorem 3.69. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V, and let $\mathcal{H}$ and $\mathcal{H}'$ be families of non-trivial graphs on common vertex sets V_1' and V_2', respectively. If there exist a simultaneous adjacency basis B of $\mathcal{H}$ that satisfies $B \not\subseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V_1'$, and a simultaneous adjacency basis B' of $\mathcal{H}'$ that satisfies $B' \not\subseteq N_{H'}(v')$ for every $H' \in \mathcal{H}'$ and every $v' \in V_2'$, then*

$$\mathrm{Sd}_A(\mathcal{G} \odot (\mathcal{H}+\mathcal{H}')) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V| \cdot \mathrm{Sd}_A(\mathcal{H}').$$

Proof. Let B and B' be simultaneous adjacency bases of $\mathcal{H}$ and $\mathcal{H}'$, respectively, that satisfy the premises of the theorem, and let $S = B \cup B'$. As shown in the proof of Theorem 3.20, S is a simultaneous adjacency basis of $\mathcal{H}+\mathcal{H}'$. Moreover, since $B \not\subseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V_1'$, and $B' \not\subseteq N_{H'}(v')$ for every $H' \in \mathcal{H}'$ and every $v' \in V_2'$, we have that $S \not\subseteq N_{H+H'}(x)$ for every $H+H' \in \mathcal{H}+\mathcal{H}'$ and every $x \in V_1' \cup V_2'$. Finally, by Lemma 3.68, we have that S is a simultaneous dominating set of $\mathcal{H}+\mathcal{H}'$, so $\mathrm{Sd}_A(\mathcal{G} \odot (\mathcal{H}+\mathcal{H}')) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}+\mathcal{H}') = |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V| \cdot \mathrm{Sd}_A(\mathcal{H}')$ by Theorems 3.59 and 3.20. □

The following result is a direct consequence of Lemma 3.16 and Theorem 3.69.

Proposition 3.70. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V. Let $\mathcal{H}$ be a graph family on a common vertex set V_1' of cardinality $|V_1'| \geq 7$ such that every $H \in \mathcal{H}$ is a path graph, a cycle graph, $D(H) \geq 6$, or $\mathsf{g}(H) \geq 5$ and $\delta(H) \geq 3$. Let $\mathcal{H}'$ be a graph family on a common vertex set V_2' of cardinality $|V_2'| \geq 7$ satisfying the same conditions as $\mathcal{H}$. Then,*

$$\mathrm{Sd}_A(\mathcal{G} \odot (\mathcal{H}+\mathcal{H}')) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V| \cdot \mathrm{Sd}_A(\mathcal{H}').$$

In addition, following a reasoning analogous to that of the proofs of Propositions 3.60 and 3.62, we obtain the following result as a consequence of Lemma 3.16 and Theorems 3.8 and 3.69.

Proposition 3.71. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V. Let H be a graph of order $n \geq 7$ which is a path graph, or a cycle graph, or satisfies $D(H) \geq 6$, or $\mathsf{g}(H) \geq 5$ and $\delta(H) \geq 3$. Let H' be a graph of order $n' \geq 7$ that satisfies the same conditions as H. Let B and B' be adjacency bases of H and H', respectively. Then, for any pair of families $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(H)$ and $\mathcal{H}' \subseteq \widetilde{\mathcal{G}}_{B'}(H')$ such that $H \in \mathcal{H}$ and $H' \in \mathcal{H}'$,*

$$\mathrm{Sd}_A(\mathcal{G} \odot (\mathcal{H} + \mathcal{H}')) = |V| \cdot \dim_A(H) + |V| \cdot \dim_A(H').$$

By analogy to the manner in which Theorem 3.69 can be deduced from Theorems 3.59 and 3.20, we present the following result which can be deduced from Theorems 3.65 and 3.20.

Theorem 3.72. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V, and let $\mathcal{H}$ and $\mathcal{H}'$ be families of non-trivial graphs on common vertex sets V_1' and V_2', respectively. If there exists a simultaneous adjacency basis B of $\mathcal{H}$ that satisfies $B \nsubseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V_1'$, and for every simultaneous adjacency basis B' of $\mathcal{H}'$ there exist $H' \in \mathcal{H}$ and $v' \in V_2'$ such that $B' \subseteq N_{H'}(v')$, then*

$$\mathrm{Sd}_A(\mathcal{G} \odot (\mathcal{H} + \mathcal{H}')) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V| \cdot \mathrm{Sd}_A(\mathcal{H}') + \mathrm{S}\,\gamma(\mathcal{G}).$$

Proof. Let S be a simultaneous adjacency basis of $\mathcal{H} + \mathcal{H}'$, let $W = S \cap V_1'$ and let $W' = S \cap V_2'$. As discussed in the proof of Theorem 3.20, W and W' are simultaneous adjacency bases of $\mathcal{H}$ and $\mathcal{H}'$, respectively. Since there exist $H' \in \mathcal{H}$ and $v' \in V_2'$ such that $W' \subseteq N_{H'}(v')$, we have that $S \subseteq N_{H+H'}(v')$ for any $H \in \mathcal{H}$ by the definition of the join operation. Moreover, by Lemma 3.68, S is a simultaneous dominating set of $\mathcal{H} + \mathcal{H}'$, so $\mathrm{Sd}_A(\mathcal{G} \odot (\mathcal{H} + \mathcal{H}')) = |V| \cdot \mathrm{Sd}_A(\mathcal{H} + \mathcal{H}') + \mathrm{S}\,\gamma(\mathcal{G}) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V| \cdot \mathrm{Sd}_A(\mathcal{H}') + \mathrm{S}\,\gamma(\mathcal{G})$ by Theorems 3.65 and 3.20. $\square$

The following results are particular cases of Theorem 3.72.

Proposition 3.73. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V. Let $\mathcal{H}$ be a graph family on a common vertex set V' of cardinality $|V'| \geq 7$ such that every $H \in \mathcal{H}$ is a path graph, a cycle graph,*

$D(H) \geq 6$, or $\mathsf{g}(H) \geq 5$ and $\delta(H) \geq 3$. Let K_t be a complete graph of order $t \geq 2$. Then,

$$\mathrm{Sd}_A(\mathcal{G} \odot (K_t + \mathcal{H})) = |V| \cdot \mathrm{Sd}_A(\mathcal{H}) + |V| \cdot (t-1) + \mathrm{S}\,\gamma(\mathcal{G}).$$

Proof. By Theorem 3.20, $\mathrm{Sd}_A(K_t + \mathcal{H}) = \mathrm{Sd}_A(\mathcal{H}) + t - 1$. Moreover, by Lemma 3.16, every simultaneous adjacency basis B of $\mathcal{H}$ satisfies $B \nsubseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V'$. Furthermore, every adjacency basis of K_t has the form $B' = V(K_t) - \{v\}$, where v is an arbitrary vertex of K_t. Clearly, $B' \subseteq N_{K_t}(v)$, so the result follows from Theorem 3.72. □

Following a reasoning analogous to that of the proofs of Propositions 3.60 and 3.62, we obtain the following result as a consequence of Lemma 3.16 and Theorems 3.8, 3.20 and 3.72.

Proposition 3.74. *Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set V. Let H be a graph of order $n \geq 7$ which is a path graph, or a cycle graph, or satisfies $D(H) \geq 6$, or $\mathsf{g}(H) \geq 5$ and $\delta(H) \geq 3$. Let K_t be a complete graph of order $t \geq 1$. Let B be an adjacency basis of H. Then, for any family $\mathcal{H} \subseteq \widetilde{\mathcal{G}}_B(H)$ such that $H \in \mathcal{H}$,*

$$\mathrm{Sd}_A(\mathcal{G} \odot (K_t + \mathcal{H})) = |V| \cdot \dim_A(H) + |V| \cdot (t-1) + \mathrm{S}\,\gamma(\mathcal{G}).$$

As an example of the previous result, consider an arbitrary family $\mathcal{G}$ composed by connected non-trivial graphs on a common vertex set V, a complete graph K_t of order $t \geq 2$, a path graph P_n of order $n \geq 7$, and the cycle graph C_n obtained from P_n by joining its leaves by an edge. For any simultaneous adjacency basis B of $\{P_n, C_n\}$ and any family $\mathcal{H} \in \widetilde{\mathcal{G}}_B(P_n) \cup \widetilde{\mathcal{G}}_B(C_n)$ such that $P_n \in \mathcal{H}$ or $C_n \in \mathcal{H}$, we have that

$$\mathrm{Sd}_A(\mathcal{G} \odot (K_t + \mathcal{H})) = |V| \cdot \left(\left\lfloor \frac{2n+2}{5} \right\rfloor + t - 1 \right) + \mathrm{S}\,\gamma(\mathcal{G}).$$

Chapter 4

The simultaneous strong metric dimension of graph families

After extensively studying the simultaneous metric dimension, and the related simultaneous adjacency dimension, this chapter explores into the extensibility of the notion of simultaneity to other forms of resolvability. Here, we introduce the simultaneous strong metric dimension. As in Chapter 2, we investigate the core properties of this parameter, including its bounds, extreme values and relations to the individual strong metric dimensions of the graphs composing the families, as well as several families on which interesting facts may be pointed out, namely those composed by a graph and its complement.

Let $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ be a family of (not necessarily edge-disjoint) connected graphs $G_i = (V, E_i)$ with common vertex set V (the union of whose edge sets is not necessarily the complete graph). By analogy to the definitions of simultaneous metric/adjacency generator, basis and dimension, we define a *simultaneous strong metric generator* for $\mathcal{G}$ to be a set $S \subseteq V$ such that S is simultaneously a strong metric generator for each G_i. We say that a minimum cardinality simultaneous strong metric generator for $\mathcal{G}$ is a *simultaneous strong metric basis* of $\mathcal{G}$, and its cardinality the *simultaneous strong metric dimension* of $\mathcal{G}$, denoted by $\mathrm{Sd}_s(\mathcal{G})$ or explicitly by $\mathrm{Sd}_s(G_1, G_2, ..., G_t)$. To illustrate these definitions, Figure 4.1, shows the family $\mathcal{G} = \{G_1, G_2, G_3\}$, for which the set $\{v_1, v_2, v_5, v_7\}$ is a simultaneous strong metric basis, whereas the set $\{v_1, v_5, v_7\}$ is a simultaneous metric basis, so $\mathrm{Sd}_s(\mathcal{G}) = 4$ and $\mathrm{Sd}(\mathcal{G}) = 3$.

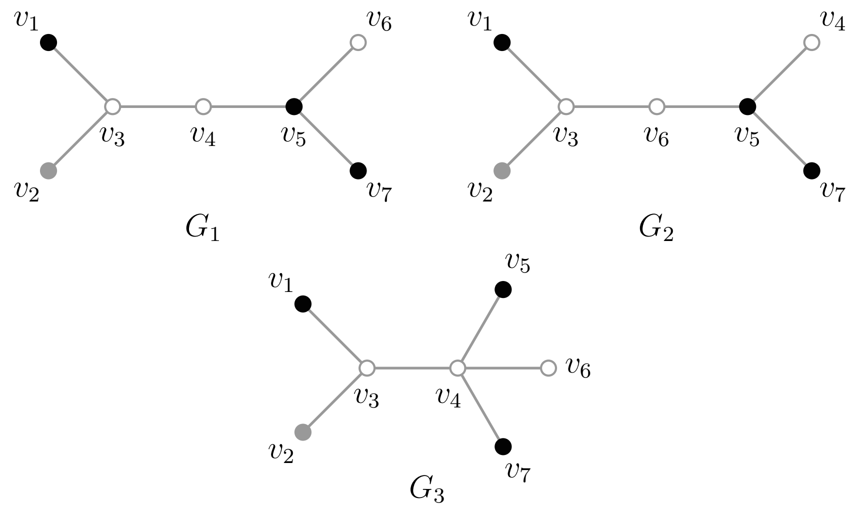

Figure 4.1: The set $\{v_1, v_2, v_5, v_7\}$ is a simultaneous strong metric basis of the family $\mathcal{G} = \{G_1, G_2, G_3\}$, whereas the set $\{v_1, v_5, v_7\}$ is a simultaneous metric basis of $\mathcal{G}$. Thus, $\mathrm{Sd}_s(\mathcal{G}) = 4$ and $\mathrm{Sd}(\mathcal{G}) = 3$.

4.1 General bounds

The following remark is a direct consequence of the fact that every strong metric generator for a graph G is also a metric generator for G.

Remark 4.1. *For any family $\mathcal{G}$ of connected graphs defined on a common vertex set V,*

$$1 \leq \mathrm{Sd}(\mathcal{G}) \leq \mathrm{Sd}_s(\mathcal{G}) \leq |V| - 1.$$

It was shown in [12] that $\dim(G) = 1$ if and only if G is a path. It now readily follows that $\dim_s(G) = 1$ if and only if G is a path. Since any strong metric basis of a path is composed by a leaf, we can state the following remark.

Remark 4.2. *Let $\mathcal{G}$ be a family of connected graphs defined on a common vertex set. Then $\mathrm{Sd}_s(\mathcal{G}) = 1$ if and only if $\mathcal{G}$ is a collection of paths that share a common leaf.*

At the other extreme we see that $\dim_s(G) = n - 1$ if and only if G is the complete graph of order n. Thus, for a family of graphs we have the following straightforward remark.

Remark 4.3. *Let $\mathcal{G}$ be a family of connected graphs defined on a common vertex set. If $K_n \in \mathcal{G}$, then*

$$\mathrm{Sd}_s(\mathcal{G}) = n - 1.$$

A characterization of the graph families for which $\mathrm{Sd}_s(\mathcal{G}) = |V| - 1$ is given in the following result.

Theorem 4.4. *Let $\mathcal{G}$ be a family of connected graphs defined on a common vertex set V. Then $\mathrm{Sd}_s(\mathcal{G}) = |V|-1$ if and only if for every pair $u, v \in V$, there exists a graph $G_{uv} \in \mathcal{G}$ such that u and v are mutually maximally distant in G_{uv}.*

Proof. If $\mathrm{Sd}_s(\mathcal{G}) = |V| - 1$, then for every $v \in V$, the set $V - \{v\}$ is a simultaneous strong metric basis of $\mathcal{G}$ and, as a consequence, for every $u \in V - \{v\}$ there exists a graph $G_{uv} \in \mathcal{G}$ such that the set $V - \{u, v\}$ is not a strong metric generator for G_{uv}. This means that the set $V - \{u, v\}$ is not a vertex cover of $(G_{uv})_{SR}$ and then u and v must be adjacent in $(G_{uv})_{SR}$ or, equivalently, they are mutually maximally distant in G_{uv}.

Conversely, if for every $u, v \in V$ there exists a graph $G_{uv} \in \mathcal{G}$ such that u and v are mutually maximally distant in G_{uv}, then for any strong simultaneous metric basis B of $\mathcal{G}$ either $u \in B$ or $v \in B$. Hence, all but one element of V must belong to B. Therefore $|B| \geq |V|-1$ and we can conclude that $\mathrm{Sd}_s(\mathcal{G}) = |V| - 1$. □

As a non-trivial example of the previous result, recall the family $\mathcal{K}(V)$, defined in Section 3.2, which is composed by $r + 1$ star graphs of the form $K_{1,r}$, defined on a common vertex set V, all of them having different centres. In this case, every pair of vertices is maximally mutually distant in $r - 1$ graphs of the family, so $\mathrm{Sd}_s(\mathcal{G}) = |V| - 1$.

Given a family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ of connected graphs defined on a common vertex set V, we define $\partial(\mathcal{G}) = \bigcup_{G\in\mathcal{G}} \partial(G)$. The following general considerations are true.

Remark 4.5. *For any familly $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ of connected graphs defined on a common vertex set V and any subfamily $\mathcal{H} \subset \mathcal{G}$.*

$$\mathrm{Sd}_s(\mathcal{H}) \leq \mathrm{Sd}_s(\mathcal{G}) \leq \min\left\{|\partial(\mathcal{G})| - 1, \sum_{i=1}^{k} \dim_s(G_i)\right\}.$$

In particular,

$$\max_{i\in\{1,\ldots,k\}}\{\dim_s(G_i)\} \leq \mathrm{Sd}_s(\mathcal{G}).$$

The inequalities above are sharp. For instance, consider a family $\mathcal{H}_1$ of graphs defined on a vertex set V, where some particular vertex $u \in V$ belongs to a simultaneous strong metric basis B. Consider also a family of paths $\mathcal{H}_2$, defined on V, sharing all of them this particular vertex u as one of their leaves. Then B is a simultaneous strong metric basis of the family $\mathcal{H}_1 \cup \mathcal{H}_2$, so that $\mathrm{Sd}_s(\mathcal{H}_1 \cup \mathcal{H}_2) = \mathrm{Sd}_s(\mathcal{H}_1)$.

On the other hand, Remark 1.2 allows to easily construct several families of graphs $\mathcal{G}$ satisfying $\mathrm{Sd}_s(\mathcal{G}) = \dim_s(G)$ for some $G \in \mathcal{G}$. We introduce the following remarks as straightforward examples.

Remark 4.6. *Let $\mathcal{G}$ be a family of trees defined on a common vertex set and let $G \in \mathcal{G}$. If $\sigma(G) \supseteq \sigma(G')$, for all $G' \in \mathcal{G}$, then $\mathrm{Sd}_s(\mathcal{G}) = \dim_s(G)$.*

Notice that a family of trees as the one described above, where the set of leaves of one tree contains the sets of leaves of every other tree in the family, satisfies $\mathrm{Sd}_s(\mathcal{G}) = |\partial(\mathcal{G})| - 1$.

Remark 4.7. *Let $\mathcal{G}$ be a family of 2-antipodal graphs defined on a common vertex set V. If there exits a partition $\{V_1, V_2\}$ of V such that for every $u \in V_1$ and every $G \in \mathcal{G}$, the only vertex diametral to v in G belongs to V_2, then $\mathrm{Sd}_s(\mathcal{G}) = \dim_s(G) = \frac{|V|}{2}$, for all $G \in \mathcal{G}$.*

The next result is a direct consequence of the fact that, in a corona product graph $G \odot H$, no vertex of G is mutually maximally distant with any vertex of $G \odot H$.

Remark 4.8. *Let $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ be a family composed by connected non-trivial graphs, defined on a common vertex set, and let H be a non-trivial graph. Then, for any $i \in \{1, \ldots, k\}$,*

$$\mathrm{Sd}_s(G_1 \odot H, G_2 \odot H, \ldots, G_k \odot H) = \dim_s(G_i \odot H).$$

Finally, consider the family $\mathcal{G} = \{G_1, G_2\}$ shown in Figure 4.2. It is easy to see that $\mathrm{Sd}_s(\mathcal{G}) = \dim_s(G_1) + \dim_s(G_2) = |\partial(\mathcal{G})| - 2 < |\partial(\mathcal{G})| - 1$.

Next, we recall an upper bound for $\dim_s(G)$ obtained in [57]. Recall that $X \subseteq V(G)$ is a *twins-free clique* in G if X is a clique containing no true twins. The *twins-free clique number* of G, denoted by $\varpi(G)$, is the maximum cardinality among all twins-free cliques in G.

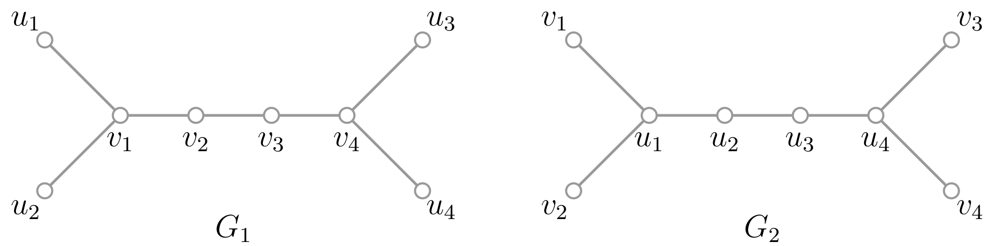

Figure 4.2: The family $\mathcal{G} = \{G_1, G_2\}$ satisfies $\mathrm{Sd}_s(\mathcal{G}) = \dim_s(G_1) + \dim_s(G_2) = 6$.

Theorem 4.9. [57] *For any connected graph G of order $n \geq 2$,*

$$\dim_s(G) \leq n - \varpi(G).$$

Moreover, if $D(G) = 2$, then the equality holds.

Our next result is an extension of Theorem 4.9 to the case of the simultaneous strong metric dimension. We define a *simultaneous twins-free clique* of a family $\mathcal{G}$ of graphs as a set which is a twins-free clique in every $G \in \mathcal{G}$. The *simultaneous twins-free clique number* of $\mathcal{G}$, denoted by $\mathrm{S}\varpi(\mathcal{G})$, is the maximum cardinality among all simultaneous twins-free cliques of $\mathcal{G}$.

Theorem 4.10. *Let $\mathcal{G}$ be a family of connected graphs of order $n \geq 2$ defined on a common vertex set. Then*

$$\mathrm{Sd}_s(\mathcal{G}) \leq n - \mathrm{S}\varpi(\mathcal{G}).$$

Moreover, if every graph belonging to $\mathcal{G}$ has diameter two, then

$$\mathrm{Sd}_s(\mathcal{G}) = n - \mathrm{S}\varpi(\mathcal{G}).$$

Proof. Let W be a simultaneous twins-free clique in $\mathcal{G}$ of maximum cardinality and let $G = (V, E)$ be a graph belonging to $\mathcal{G}$. We will show that $V - W$ is a strong metric generator for G. Since W is a twins-free clique, for any two distinct vertices $u, v \in W$ there exists $s \in V - W$ such that either $s \in N_G(u)$ and $s \notin N_G(v)$ or $s \in N_G(v)$ and $s \notin N_G(u)$. Without loss of generality, we consider $s \in N_G(u)$ and $s \notin N_G(v)$. Thus, $u \in I_G[v, s]$ and, as a consequence, s strongly resolves u and v. Therefore, $\mathrm{Sd}_s(\mathcal{G}) \leq |V - W| = n - \mathrm{S}\varpi(\mathcal{G})$.

Now, suppose that every graph $G = (V, E)$ belonging to $\mathcal{G}$ has diameter two. Let $X \subset V$ be a simultaneous strong metric basis of $\mathcal{G}$ and let $u, v \in V$,

$u \neq v$. If $d_G(u,v) = 2$ or $N_G[u] = N_G[v]$, for some $G \in \mathcal{G}$, then u and v are mutually maximally distant vertices of G, so $u \in X$ or $v \in X$. Hence, for any two distinct vertices $x, y \in V - X$ and any $G \in \mathcal{G}$ we have $d_G(x,y) = 1$ and $N_G[x] \neq N_G[y]$. As a consequence, $V - X$ is a simultaneous twins-free clique of $\mathcal{G}$ and so $n - \mathrm{Sd}_s(\mathcal{G}) = |V - X| \leq \mathrm{S}\varpi(\mathcal{G})$. Therefore, $\mathrm{Sd}_s(\mathcal{G}) \geq n - \mathrm{S}\varpi(\mathcal{G})$ and the result follows. □

Corollary 4.11. *Let $\mathcal{G}$ be a family of graphs of diameter two and order $n \geq 2$ defined on a common vertex set. If $\mathcal{G}$ contains a triangle-free graph, then*

$$n - 2 \leq \mathrm{Sd}_s(\mathcal{G}) \leq n - 1.$$

Finally, we recall the following upper bound on $\dim_s(G)$, obtained in [88].

Theorem 4.12. [88] *For any connected graph G of order n,*

$$\dim_s(G) \leq n - D(G).$$

Given a graph family $\mathcal{G}$ defined on a common vertex set V, we define the parameter $\rho(\mathcal{G}) = |W| - 1$, where $W \subseteq V$ is a maximum cardinality set such that for every $G \in \mathcal{G}$ the subgraph $\langle W \rangle_G$ induced by W in G is a path and there exists $w \in W$ which is a common leaf of all these paths.

Theorem 4.13. *Let $\mathcal{G}$ be a family of graphs defined on a common vertex set V. Then,*

$$\mathrm{Sd}_s(\mathcal{G}) \leq |V| - \rho(\mathcal{G}).$$

Proof. Let $W = \{v_0, v_1, \ldots, v_{\rho(\mathcal{G})}\} \subseteq V$ be a set for which $\rho(\mathcal{G})$ is obtained. Assume, without loss of generality, that v_0 is a common leaf of $\langle W \rangle_G$, for every $G \in \mathcal{G}$, and let $W' = W - \{v_0\}$. Since no pair of vertices $u, v \in W'$ are mutually maximally distant in any $G \in \mathcal{G}$, the set $S = V - W'$ is a simultaneous strong metric generator for $\mathcal{G}$. Thus, $\mathrm{Sd}_s(\mathcal{G}) \leq |S| = |V| - \rho(\mathcal{G})$. □

The inequality above is sharp. A family of graphs $\mathcal{G}$ composed by paths having a common leaf is a trivial example where the inequality is reached. In this case, $\rho(\mathcal{G}) = |V| - 1$, so that $\mathrm{Sd}_s(\mathcal{G}) = 1 = |V| - \rho(\mathcal{G})$. This is not the only circumstance where this occurs. For instance, consider a graph family $\mathcal{G}$ constructed as follows. Consider a star graph $K_{1,r}$ of center u and a complete graph K_{r+1} defined on a common vertex set V'. Let V'' be a set such that $V' \cap V'' = \emptyset$ and let $\{G'_1, G'_2, \ldots, G'_k\}$ be a family composed by paths defined on V'', having a common leaf, say v, and let $\mathcal{G} = \{G_1, H_1, G_2, H_2, \ldots, G_k, H_k\}$ be a graph family such that every G_i is constructed from G'_i and $K_{1,r}$ by identifying u and v, and every H_i is constructed from G'_i and K_{r+1} by identifying u and v. For every $w \in V' - \{u\}$, the set $W = V'' \cup \{w\}$ is a maximum cardinality set such that, for every graph in $\mathcal{G}$, the subgraph induced by W is a path and there exists $w \in W$ which is a common leaf of all these paths, so that $\rho(\mathcal{G}) = |V''|$. Furthermore, the set $V' - \{u\}$ is a simultaneous strong metric basis of $\mathcal{G}$ and, as a result, $\mathrm{Sd}_s(\mathcal{G}) = r = |V| - \rho(\mathcal{G})$.

4.2 Families of the form $\{G, G^c\}$

We first consider the following direct consequence of Theorem 4.4.

Corollary 4.14. *Let G be a graph of order n. Then the following assertions are equivalent:*

(i) $\mathrm{Sd}_s(G, G^c) = n - 1$.

(ii) $D(G) = D(G^c) = 2$.

Proof. Let $x, y \in V(G)$. If $D(G) = D(G^c) = 2$, then either x and y are diametral in G or they are diametral in G^c. Hence, by Theorem 4.4 we obtain $\mathrm{Sd}_s(G, G^c) = n - 1$.

Now, assume that $D(G) \geq 3$. If x, u, v, y is a shortest path from x to y in G, then x and v are not mutually maximally distant in G and, since they are adjacent in G^c and they are not twins, they are not mutually maximally distant in G^c. Thus, by Theorem 4.4 we deduce that $\mathrm{Sd}_s(G, G^c) \leq n - 2$. □

The Petersen graph is an example of graphs where $\mathrm{Sd}_s(G, G^c) = n-1$ and the graphs shown in Figure 4.3 are examples of graphs where $\mathrm{Sd}_s(G, G^c) = n - 2$.

From Theorem 4.9 and Corollary 4.14 we derive the next result.

Theorem 4.15. *For any graph G of order n and $D(G) = 2$ such that G^c is connected,*

$$\mathrm{Sd}_s(G, G^c) \geq n - \varpi(G).$$

Moreover, if $D(G^c) \geq 3$ and $\varpi(G) = 2$, then

$$\mathrm{Sd}_s(G, G^c) = n - 2.$$

Given a graph $G = (V, E)$, we say that a set $S \subset V$ is a *strong resolving cover* for G if S is a vertex cover and a strong metric generator for G.

Theorem 4.16. *If G is a connected graph such that G^c is connected, then any strong resolving cover of G is a simultaneous strong metric generator for $\{G, G^c\}$.*

Proof. Let W be a strong resolving cover of G. We shall show that W is a strong metric generator for G^c. We differentiate two cases for any pair x, y of mutually maximally distant vertices in G^c:

(1) x and y are adjacent in G^c. In this case, x and y are false twins in G (true twins in G^c) and so they are mutually maximally distant in G. Since W is a strong metric generator for G, we conclude that $x \in W$ or $y \in W$.

(2) x and y are not adjacent in G^c. In this case x and y are adjacent in G and, since W is a vertex cover of G, we have that $x \in W$ or $y \in W$.

According to the two cases above, W is a vertex cover of $(G^c)_{SR}$ and, as a consequence, W is a strong metric generator for G^c. Therefore, W is a simultaneous strong metric generator for $\{G, G^c\}$. □

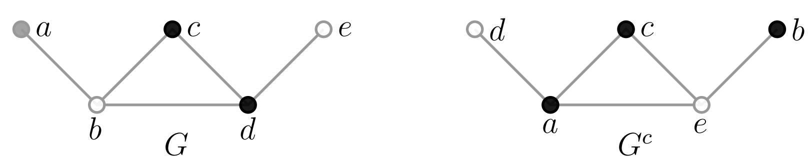

Figure 4.3: $X_1 = \{a, c, d\}$ is a strong resolving cover for G and $X_2 = \{a, c, b\}$ is a strong resolving cover for G^c. Both X_1 and X_2 are simultaneous strong metric bases of $\{G, G^c\}$.

The *strong resolving cover number* of a graph G, denoted by $\beta_s(G)$, is the minimum cardinality among all the strong resolving covers for G. Obviously, for any connected graph of order n,

$$n - 1 \geq \beta_s(G) \geq \max\{\dim_s(G), \beta(G)\}. \tag{4.1}$$

Corollary 4.17. *For any connected graph G such that G^c is connected,*

$$\mathrm{Sd}_s(G, G^c) \leq \min\{\beta_s(G), \beta_s(G^c)\}.$$

Figure 4.3 shows a graph G and its complement G^c. In this case, $\mathrm{Sd}_s(G, G^c) = \beta_s(G) = \beta_s(G^c) = 3 > 2 = \dim_s(G) = \dim_s(G^c) = \beta(G) = \beta(G^c)$. The graph G shown in Figure 4.4 satisfies that $\dim_s(G^c) = 2 < 3 = \beta_s(G^c) = \mathrm{Sd}_s(G, G^c) = \dim_s(G) < 4 = \beta_s(G)$. In this case, $\{2, 4\}$ is a strong metric basis of G^c, $\{2, 3, 4\}$ is a $\beta_s(G^c)$-set which is a simultaneous strong metric basis of $\{G, G^c\}$ and, at the same time, it is a strong metric basis of G, while $\{2, 4, 5, 6\}$ is a $\beta_s(G)$-set.

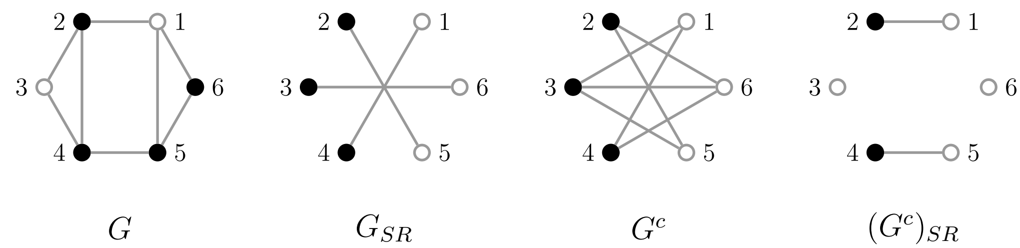

Figure 4.4: The $\beta_s(G^c)$-set $\{2, 3, 4\}$ is a simultaneous strong metric basis of $\{G, G^c\}$.

Theorem 4.18. *Let G be a connected graph such that $D(G^c) = 2$ and let $S \subset V(G)$. Then the following assertions are equivalent.*

(i) *S is a simultaneous strong metric generator for $\{G, G^c\}$.*

(ii) *S is a strong resolving cover for G.*

Proof. Let $G = (V, E)$. Since $D(G^c) = 2$, two vertices $x, y \in V$ are mutually maximally distant in G^c if and only if $d_{G^c}(x, y) = 2$ or $N_{G^c}[x] = N_{G^c}[y]$. Hence, $(G^c)_{SR} = (V, E \cup E')$, where $E' = \{\{x, y\} : N_G(x) = N_G(y)\}$.

Let S be a simultaneous strong metric generator for $\{G, G^c\}$. Since S is a strong metric generator for G^c, we deduce that S is a vertex cover of $(G^c)_{SR} = (V, E \cup E')$, and as a consequence, for any edge $\{x, y\} \in E$, we have that $x \in S$ or $y \in S$. Hence, S is a strong metric generator for G and a vertex cover of G. By Theorem 4.16 we conclude the proof. $\square$

From Theorem 4.18 we deduce the following result.

Corollary 4.19. *For any connected graph G such that $D(G^c) = 2$,*

$$\mathrm{Sd}_s(G, G^c) = \beta_s(G).$$

In order to present the next result, we need to introduce some new notation and terminology. Given a graph G such that $V(G) \neq \partial(G)$, we define the *interior subgraph* of G as the subgraph $\mathring{G}$ induced by $V(G) - \partial(G)$. The parameter $\mathring{\beta}(G)$ is defined as follows.

$$\mathring{\beta}(G) = \begin{cases} 0 & \text{if } V(G) = \partial(G) \\ \\ \beta(\mathring{G}) & \text{otherwise.} \end{cases}$$

Corollary 4.20. *For any connected graph G such that $D(G^c) = 2$,*

$$\mathrm{Sd}_s(G, G^c) \geq \max\{\dim_s(G) + \mathring{\beta}(G), \beta(G)\}.$$

Proof. By Theorem 4.18 and Equation (4.1) we have that $\mathrm{Sd}_s(G, G^c) \geq \beta(G)$. It only remains to prove that $\mathrm{Sd}_s(G, G^c) \geq \dim_s(G) + \mathring{\beta}(G)$. If $V(G) = \partial(G)$, then $\mathring{\beta}(G) = 0$, and by Theorem 4.18 and Equation (4.1) we have $\mathrm{Sd}_s(G, G^c) \geq \dim_s(G) = \dim_s(G) + \mathring{\beta}(G)$. Assume that $V(G) \neq \partial(G)$. Let B be a simultaneous strong metric basis of $\{G, G^c\}$, and let $B_1 = B \cap \partial(G)$ and $B_2 = B - B_1$. Clearly, $|B_1| \geq \dim_s(G)$. Moreover, since no vertex of B_1 covers edges of $\mathring{G}$, by Theorem 4.18 we conclude that B_2 is a vertex cover of $\mathring{G}$, so that $|B_2| \geq \beta(\mathring{G})$. Therefore, $\mathrm{Sd}_s(G, G^c) = |B| = |B_1| + |B_2| \geq \dim_s(G) + \mathring{\beta}(G)$. $\square$

To illustrate this result we take the graph G shown in Figure 4.5. In this case $\mathrm{Sd}_s(G, G^c) = \beta(G) = 5 > 4 = \dim_s(G) + \mathring{\beta}(G)$. In contrast, the equality $\mathrm{Sd}_s(G, G^c) = \dim_s(G) + \mathring{\beta}(G)$ is satisfied for any graph constructed as follows. Let $r, s \geq 2$ and $t \geq 3$ be three integers and let G be the graph constructed from K_r, K_s and P_t by identifying one vertex of K_r with one leaf of P_t and one vertex of K_s with the other leaf of P_t. In this case $\mathrm{Sd}_s(G, G^c) = r + s + \lfloor \frac{t}{2} \rfloor - 1$, $\dim_s(G) = r + s - 1$, $\beta(G) = r + s + \lfloor \frac{t}{2} \rfloor - 2$ and $\mathring{\beta}(G) = \beta(\mathring{G}) = \lfloor \frac{t}{2} \rfloor$. Hence, $\mathrm{Sd}_s(G, G^c) = \dim_s(G) + \mathring{\beta}(G) > \beta(G)$.

Corollary 4.21. *Let G be a connected graph such that $D(G^c) = 2$. Then the following assertions hold.*

(i) $\mathrm{Sd}_s(G, G^c) = \dim_s(G)$ *if and only if there exists a strong metric basis of G which is a vertex cover of G.*

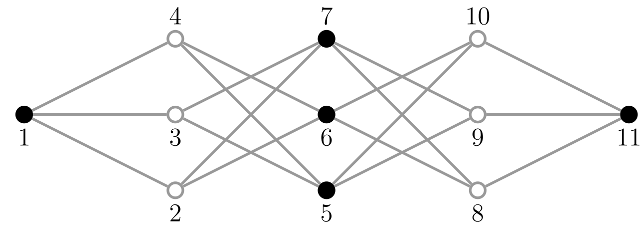

Figure 4.5: The sets $\{1,5,6,7\}$ and $\{5,6,7,11\}$ are the only strong metric bases of G, while $\{1,5,6,7,11\}$ is the only $\beta(G)$-set which is a strong metric generator of G.

(ii) $\mathrm{Sd}_s(G,G^c) = \beta(G)$ *if and only if there exists a* $\beta(G)$*-set which is a strong metric generator of* G.

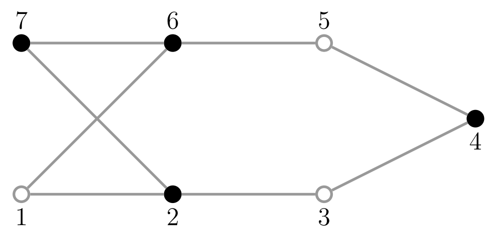

Figure 4.6: The graph G satisfies $\mathrm{Sd}_s(G,G^c) = \dim_s(G) = 4 > 3 = \beta(G)$.

To illustrate the result above we take the graphs shown in Figures 4.5 and 4.6. In both cases $D(G^c) = 2$. Now, in the case of Figure 4.5, the sets $\{1,5,6,7\}$ and $\{5,6,7,11\}$ are the only strong metric bases of G. At the same time, the set $\{1,5,6,7,11\}$ is the only $\beta(G)$-set which is a strong metric generator of G, and so it is the only $\beta_s(G)$-set. Therefore, $\mathrm{Sd}_s(G,G^c) = \beta_s(G) = \beta(G) = 5 > 4 = \dim_s(G)$. In the case of Figure 4.6, $\mathrm{Sd}_s(G,G^c) = \beta_s(G) = \dim_s(G) = 4 > 3 = \beta(G)$, as $\{2,4,6,7\}$ is a strong metric basis of G which is a vertex cover of G and $\{2,4,6\}$ is a $\beta(G)$-set.

The hypercube Q_r, $r \geq 3$, is a 2-antipodal graph, so $\dim_s(Q_r) = 2^{r-1}$. Also, Q_r is a bipartite graph and, for r odd, any colour class forms a strong metric basis which is a vertex cover of minimum cardinality. Since $D((Q_r)^c) = 2$, we conclude that for any odd integer $r \geq 3$,

$$\mathrm{Sd}_s(Q_r,(Q_r)^c) = \dim_s(Q_r) = \beta(Q_r) = 2^{r-1}. \tag{4.2}$$

This is an example where $\mathrm{Sd}_s(G,G^c) = \dim_s(G) = \beta(G)$ and it is a particular case of the next result.

Proposition 4.22. *For any bipartite 2-antipodal graph G of odd diameter and order $n > 2$,*

$$\mathrm{Sd}_s(G, G^c) = \frac{n}{2}.$$

Proof. Let $G = (V_1 \cup V_2, E)$. Since the subgraph of G^c induced by V_i, $i \in \{1, 2\}$, is complete and G is not a complete bipartite graph, we conclude that G^c is connected. Furthermore, since G is 2-antipodal of odd diameter, each vertex $x \in V_1$ is adjacent to a vertex $x' \in V_2$ in G^c and, as a result, $D(G^c) = 2$.

On the other hand, V_1 is a vertex cover of G and since G is a 2-antipodal graph and $D(G)$ is odd, for any $x \in V_1$ there exists exactly one vertex $x' \in V_2$ which is antipodal to x, which implies that V_1 is a strong metric basis of G. Therefore, by Corollary 4.21 we conclude the proof. □

An even-order cycle C_{2k} has odd diameter if k is odd. In this case, $\mathrm{Sd}_s(C_{2k}, (C_{2k})^c) = k$. Note that for k even, $\mathrm{Sd}_s(C_{2k}, (C_{2k})^c) = k + 1$. If G is a bipartite 2-antipodal graph, then the Cartesian product graph $G \Box K_2$ is bipartite and 2-antipodal. Moreover, $D(G \Box K_2) = D(G) + 1$. Therefore, Proposition 4.22 immediately leads to the following result.

Corollary 4.23. *For any bipartite 2-antipodal graph G of even diameter and order n,*

$$\mathrm{Sd}_s(G \Box K_2, (G \Box K_2)^c) = n.$$

Theorem 4.24. *Let G be a connected graph. Then $G_{SR} = G^c$ if and only if $D(G) = 2$ and G is a true-twins-free graph.*

Proof. (Necessity) Assume that $G_{SR} = G^c = (V, E)$, and let $u, v \in V$ be two mutually maximally distant vertices in G.

First consider that u and v are diametral vertices in G. Since u and v are mutually maximally distant in G and $G_{SR} = G^c$, we obtain that u and v are adjacent in G^c and, as a result, $D(G) = d_G(u, v) \geq 2$. Now, suppose that $d_G(u, v) > 2$. Then there exists $w \in N_G(v) - N_G(u)$ such that $d_G(u, w) = D(G) - 1 \geq 2$. Hence, w and u are not mutually maximally distant in G and $w \in N_G(u)$, which contradicts the fact that $G_{SR} = G^c$. Therefore, $D(G) = 2$.

Now assume that u and v are true twins in G. We have that u and v are false twins in G^c and, as a result, they are not adjacent in G^c and they are mutually maximally distant in G, which contradicts the fact that $G_{SR} = G^c$. Therefore, G is a true-twins-free graph.

(Sufficiency) If $G = (V, E)$ is a true-twins-free graph and $D(G) = 2$, then two vertices u, v are mutually maximally distant in G if and only if $d_G(u, v) = 2$. Therefore, $G_{SR} = G^c$. □

Odd-order cycles are an example of the previous result, as $[(C_{2k+1})^c]_{SR} = C_{2k+1}$. Moreover, it is not difficult to show that a simultaneous strong metric basis of $\{C_{2k+1}, (C_{2k+1})^c\}$ is the minimum union of a strong metric basis and a minimum vertex cover of C_{2k+1}, so

$$\mathrm{Sd}_s(C_{2k+1}, (C_{2k+1})^c) = k + \left\lfloor \frac{k}{2} \right\rfloor + 1.$$

Corollary 4.25. *Let G be a true-twins-free graph such that $D(G) = 2$. Then the following assertions hold.*

(i) $\mathrm{Sd}_s(G, G^c) = \dim_s(G)$ *if and only if there exists a $\beta(G^c)$-set which is a strong metric generator for G^c.*

(ii) $\mathrm{Sd}_s(G, G^c) = \dim_s(G) = \dim_s(G^c)$ *if and only if there exists a $\beta(G^c)$-set which is a strong metric basis of G^c.*

The complement of the graph shown in Figure 4.5 has diameter two and $\{1, 5, 6, 7, 11\}$ is a $\beta(G)$-set which is a strong metric generator for G, so that $\mathrm{Sd}_s(G, G^c) = \dim_s(G)$.

Given a graph G, it is well-known that $D(G) \geq 4$ leads to $D(G^c) = 2$. Hence, $D(G) \neq 2$ and $D(G^c) \neq 2$ if and only if $D(G) = D(G^c) = 3$. In particular, for the case of trees we have that $D(T) = 3$ if and only if $D(T^c) = 3$.

Proposition 4.26. *Let T be a tree of order n. If $D(T) = 3$, then*

$$\mathrm{Sd}_s(T, T^c) = n - 2.$$

Proof. Notice that T has $|\Omega(T)| = n - 2$ leaves. Let u and v be the two interior vertices of T. We have that $D(T^c) = 3$ and $d_{T^c}(u, v) = 3$. Any simultaneous strong metric basis of $\{T, T^c\}$ must contain all leaves of T, except one, and one of u and v, so $\mathrm{Sd}_s(T, T^c) \geq |\Omega(T)| - 1 + 1 = n - 2$. Moreover, by Corollary 4.14 we have that $\mathrm{Sd}_s(T, T^c) \leq n - 2$ and so the equality holds. □

Proposition 4.27. *Let T be a tree of order n such that $D(T) \geq 4$, let u be a leaf of T, and let T'_u be the tree obtained from T by removing all leaves, except u. Then,*

$$\beta(\mathring{T}) + |\Omega(T)| - 1 \leq \mathrm{Sd}_s(T, T^c) \leq \beta(T'_u) + |\Omega(T)| - 1.$$

Proof. Note that $\dim_s(T) = |\Omega(T)| - 1$ and $\mathring{\beta}(T) = \beta(\mathring{T})$. Thus, by Corollary 4.20, $\mathrm{Sd}_s(T, T^c) \geq \max\{|\Omega(T)| - 1 + \beta(\mathring{T}), \beta(T)\}$, and as a consequence, $\mathrm{Sd}_s(T, T^c) \geq \beta(\mathring{T}) + |\Omega(T)| - 1$.

To prove the upper bound, let X be a $\beta(T'_u)$-set and let $Y \subset V(T)$ be the set composed by all leaves of T, except u. Notice that $X \cup Y$ is a strong resolving cover of T and $X \cap Y = \emptyset$. Now, since $D(T^c) = 2$, by Theorem 4.18 we conclude that $\mathrm{Sd}_s(T, T^c) = \beta_s(T) \leq |X| + |Y| = \beta(T'_u) + |\Omega(T)| - 1$. □

A particular case of the previous result is that of caterpillar trees T such that $T'_u \cong P_{n-|\Omega(T)|+1}$ for every leaf u of T. In this case, we have that $\mathrm{Sd}_s(T, T^c) = |\Omega(T)| + \left\lceil \frac{n-|\Omega(T)|}{2} \right\rceil - 1$. Moreover, if $D(T) = 4$, then $\mathring{T}$ is a star graph. On the other hand, if $D(T) = 5$, then $\mathring{T}$ is composed by exactly two interior vertices and $|\Omega(\mathring{T})| = n - |\Omega(T)| - 2$ leaves. With these facts in mind, the following two results are straightforward consequences of Proposition 4.27.

Corollary 4.28. *Let T be a tree of order n such that $D(T) = 4$. If the central vertex of $\mathring{T}$ is a support vertex of T, then*

$$\mathrm{Sd}_s(T, T^c) = |\Omega(T)|.$$

Otherwise,

$$\mathrm{Sd}_s(T, T^c) = |\Omega(T)| + 1.$$

Corollary 4.29. *Let T be a tree of order n such that $D(T) = 5$. If an interior vertex of $\mathring{T}$ is a support vertex of T, then*

$$\mathrm{Sd}_s(T, T^c) = |\Omega(T)| + 1.$$

Otherwise,

$$\mathrm{Sd}_s(T, T^c) = |\Omega(T)| + 2.$$

Chapter 5

Computability of simultaneous resolvability parameters

In previous chapters, we have discussed a number of cases where the simultaneous metric, adjacency and strong metric dimensions may be exactly determined or sharply bounded in terms of several parameters of the families and/or their composing graphs. Moreover, some authors have shown methods to efficiently compute some standard resolvability parameters in particular types of graphs, even though it is known that computing these standard resolvability parameters is difficult in the general case. In this chapter, we address the computability of the simultaneous resolvability parameters studied in previous chapters. First, we show that the requirement of simultaneity adds on the complexity of the original problems, making the computation hard even for families composed by graphs whose individual resolvability parameters are easy to compute. Next, in light of this circumstance, we propose several methods for estimating these parameters and study their accuracy on several collections of graph families.

5.1 Overview

It is proven in [48] that the problem of finding the metric dimension of a graph, when stated as a decision problem, is NP-complete. Moreover, the NP-completeness of finding the adjacency dimension and the strong metric dimension of a graph is proven in [26] and [67], respectively. These problems are formally stated as decision problems as follows:

Metric Dimension (DIM)
INSTANCE: A graph $G = (V, E)$ and an integer p, $1 \leq p \leq |V(G)| - 1$.
QUESTION: Is $\dim(G) \leq p$?

Adjacency Dimension (ADIM)
INSTANCE: A graph $G = (V, E)$ and an integer p, $1 \leq p \leq |V(G)| - 1$.
QUESTION: Is $\dim_A(G) \leq p$?

Strong Metric Dimension (SDIM)
INSTANCE: A graph $G = (V, E)$ and an integer p, $1 \leq p \leq |V(G)| - 1$.
QUESTION: Is $\dim_s(G) \leq p$?

In an analogous manner, we define the decision problems associated to finding the simultaneous metric dimension, the simultaneous adjacency dimension, and the simultaneous strong metric dimension of a graph family.

Simultaneous Metric Dimension (SD)
INSTANCE: A graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ on a common vertex set V and an integer p, $1 \leq p \leq |V| - 1$.
QUESTION: Is $\mathrm{Sd}(\mathcal{G}) \leq p$?

Simultaneous Adjacency Dimension (SAD)
INSTANCE: A graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ on a common vertex set V and an integer p, $1 \leq p \leq |V| - 1$.
QUESTION: Is $\mathrm{Sd}_A(\mathcal{G}) \leq p$?

Simultaneous Strong Metric Dimension (SSD)
INSTANCE: A graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ on a common vertex set V and an integer p, $1 \leq p \leq |V| - 1$.
QUESTION: Is $\mathrm{Sd}_s(\mathcal{G}) \leq p$?

With these definitions in mind, it is straightforward to see that SD, SAD and SSD are NP-complete.

Remark 5.1. *The Simultaneous Metric Dimension Problem (SD), the Simultaneous Adjacency Dimension Problem (SAD) and the Simultaneous Strong Metric Dimension Problem (SSD) are NP-complete.*

Proof. It is simple to see that determining whether a vertex set $S \subset V$, $|S| \leq p$, is a simultaneous metric, adjacency or strong metric generator can be done in polynomial time, so SD, SAD and SSD are in NP. Moreover, for any graph $G = (V, E)$ and any integer $1 \leq p \leq |V(G)| - 1$, the corresponding instance of DIM, ADIM or SDIM can be transformed into an instance of SD, SAD or SSD, respectively, in polynomial time by making $\mathcal{G} = \{G\}$, so SD, SAD and SSD are NP-complete. □

5.2 Computational difficulty added by the simultaneity requirement

In the previous section, we saw that the computation of simultaneous resolvability parameters is difficult in the general case, as a direct consequence of the fact that computing the individual parameters of the graphs composing the families is also difficult. However, as we will show, the requirement of simultaneity adds on the difficulty of calculating the individual parameters, making it hard to compute simultaneous resolvability parameters even for families composed by graphs whose individual resolvability parameters are easy to compute.

To begin with, recall that for a tree T, every set composed by all terminal vertices, except one, of every exterior major vertex, is a metric basis of T. Likewise, recall that every set composed by all but one of its leaves is a strong metric basis of T. In consequence, a simple traversal (e.g. a post-order traversal) allows us to compute $\dim(T)$ and $\dim_s(T)$ in polynomial time. Here, we will show that the requirement of simultaneity makes it difficult to compute $\mathrm{Sd}(\mathcal{T})$ and $\mathrm{Sd}_s(\mathcal{T})$ for a family $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ composed by trees on a common vertex set. To this end, we will prove that the decision problems associated to the computation of $\mathrm{Sd}(\mathcal{T})$ and $\mathrm{Sd}_s(\mathcal{T})$ are NP-complete for these families. We do so by showing a transformation from a subcase of the **Hitting set Problem**, which is defined as follows:

Hitting Set Problem (HSP)
INSTANCE: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ of non-empty subsets of a finite set S and a positive integer $p \leq |S|$.
QUESTION: Is there a subset $S' \subseteq S$ with $|S'| \leq p$ such that S' contains at least one element from each subset in $\mathcal{C}$?

The Hitting Set Problem was shown to be NP-complete by Karp [46], as shows the next result.

Lemma 5.2. [30, 46] *The Hitting Set Problem (HSP) is NP-complete, even if $|C_i| \leq 2$ for every $C_i \in \mathcal{C}$.*

In what follows, we will refer to the subcase of HSP where $|C_i| \leq 2$ for every $C_i \in \mathcal{C}$ as HSP2, and will use polynomial time transformations of HSP2 into SD and SSD for families of trees to show their NP-completeness.

Theorem 5.3. *The Simultaneous Metric Dimension Problem (SD) and the Simultaneous Strong Metric Dimension Problem (SSD) are NP-complete for families of trees.*

Proof. As we discussed previously, determining whether a vertex set $S \subset V$, $|S| \leq p$, is a simultaneous (strong) metric generator for a graph family $\mathcal{G}$ can be done in polynomial time, so SD and SSD are in NP.

Now, we will show a polynomial time transformation of HSP2 into SD and SSD. Let $S = \{v_1, v_2, \dots, v_n\}$ be a finite set and let $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$, where every $C_i \in \mathcal{C}$ satisfies $1 \leq |C_i| \leq 2$ and $C_i \subseteq S$. Let p be a positive integer such that $p \leq |S|$, and let $S' = \{w_1, w_2, \dots, w_n\}$ such that $S \cap S' = \emptyset$. We construct the family $\mathcal{T} = \{T_1, T_2, \dots, T_k\}$ composed by trees on the common vertex set $V = S \cup S' \cup \{u\}$, $u \notin S \cup S'$, as follows. For every $r \in \{1, \dots, k\}$, if $C_r = \{v_{i_r}\}$, let P_r be a path on the vertices of $(S - \{v_{i_r}\}) \cup (S' - \{w_{i_r}\})$, and let T_r be the tree obtained from P_r by joining by an edge the vertex u to one end of P_r, and joining the other end of P_r to the vertices v_{i_r} and w_{i_r}. On the other hand, if $C_r = \{v_{i_r}, v_{j_r}\}$, P_r is a path on the vertices of $(S - \{v_{i_r}, v_{j_r}\}) \cup S'$, and T_r is the tree obtained from P_r by joining by an edge the vertex u to one end of P_r, and the other end of P_r to the vertices v_{i_r} and v_{j_r}. Figure 5.1 shows an example of this construction.

In order to prove the validity of this transformation, we claim that there exists a subset $S'' \subseteq S$ of cardinality $|S''| \leq p$ that contains at least one element from each $C_i \in \mathcal{C}$ if and only if $\mathrm{Sd}(\mathcal{T}) = \mathrm{Sd}_s(\mathcal{T}) \leq p + 1$.

To prove this claim, first note that every $T_r \in \mathcal{T}$ satisfies $\mathcal{M}(T_r) = \{x\}$ and $TER_{T_r}(x) = \Omega(T_r)$, so every simultaneous metric basis of $\mathcal{T}$ is a simultaneous strong metric basis, and *vice versa*.

Now, assume that there exists a set $S'' \subseteq S$ which contains at least one element from each $C_i \in \mathcal{C}$ and satisfies $|S''| \leq p$. Since the set $S'' \cup \{u\}$ satisfies $|(S'' \cup \{u\}) \cap \Omega(T_r)| \geq |\Omega(T_r)| - 1$ for every $T_r \in \mathcal{T}$, it is a simultaneous (strong) metric generator for $\mathcal{T}$. Thus, $\mathrm{Sd}(\mathcal{T}) = \mathrm{Sd}_s(\mathcal{T}) \leq p + 1$.

Now, assume that $\mathrm{Sd}(\mathcal{T}) = \mathrm{Sd}_s(\mathcal{T}) \leq p+1$ and let W be a simultaneous (strong) metric generator for $\mathcal{T}$ such that $|W| = p + 1$. Since u is a common leaf of all trees in $\mathcal{T}$, we can assume that $u \in W$, *i.e.*, if $u \notin W$, then for any $T_i \in \mathcal{T}$ and any leaf $x \in W \cap \Omega(T_i)$, the set $(W - \{x\}) \cup \{u\}$ is also a simultaneous (strong) metric generator for $\mathcal{T}$, and so we can replace W by $(W - \{x\}) \cup \{u\}$. Moreover, for every set $C_r \in \mathcal{C}$ such that $W \cap C_r = \emptyset$, we have that $C_r = \{v_{i_r}\}$ and $w_{i_r} \in W$. Hence, the set

$$W' = \bigcup_{W \cap C_r = \emptyset} ((W - \{w_{i_r}\}) \cup \{v_{i_r}\})$$

is also a simultaneous (strong) metric generator for $\mathcal{T}$ of cardinality $|W'| = p + 1$ such that $u \in W'$ and $(W' - \{u\}) \cap C_i \neq \emptyset$ for every $C_i \in \mathcal{C}$. Thus the set $S'' = W' - \{u\}$ satisfies $|S''| \leq p$ and contains at least one element from each $C_i \in \mathcal{C}$.

To conclude our proof, it is simple to verify that the transformation of HSP2 into SD and SSD described above can be done in polynomial time. $\square$

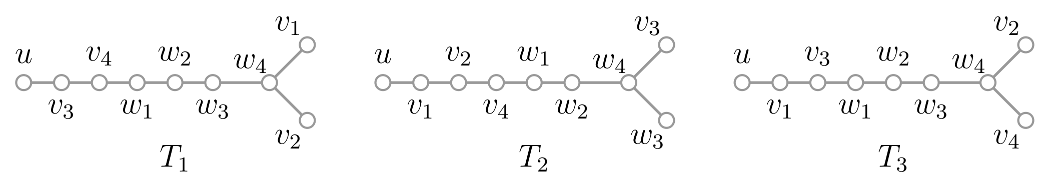

Figure 5.1: The family $\mathcal{T} = \{T_1, T_2, T_3\}$ is constructed for transforming an instance of HSP2, where $S = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{C} = \{\{v_1, v_2\}, \{v_3\}, \{v_2, v_4\}\}$, into instances of SD and SSD for families of trees.

Results analogous to that of Theorem 5.3 can be verified for other classes of graph families. In particular, as an extreme case, we would point out that there exist families composed by graphs whose individual metric dimensions are constant, and small, yet their simultaneous metric dimensions may span a wide range of values and are difficult to compute. For example, consider the so-called *tadpole graphs*[49], unicyclic graphs obtained by taking a path graph P_n and a cycle graph $C_{n'}$, and identifying a leaf of P_n and an arbitrary vertex of $C_{n'}$. These are particular cases of the graphs of the forms $P + e$ and $C + e - f$ described in Section 2.4. As discussed in the proof of Remark 2.12 (cases 2 and 3), any graph G constructed in this manner satisfies $\dim(G) = 2$. However, by Remark 2.1 and Theorem 2.3, we have that a family $\mathcal{G}$ composed by tadpole graphs satisfies $2 \leq \mathrm{Sd}(\mathcal{G}) \leq |V| - 1$, being both bounds tight[1]. Moreover, as illustrated in Figure 5.2, a polynomial-time procedure, similar

[1]The lower bound is trivially satisfied, whereas the upper bound is reached, for instance, by the family composed by all different labelled graphs isomorphic to $K_1 + (K_1 \cup K_2)$.

to that described in the proof of Theorem 5.3, allows to transform an instance of HSP2 into an instance of SD for families of tadpole graphs, in such a way that a solution S'', $|S''| \leq p$, for HSP2 exists if and only if the family $\mathcal{G}$ constructed by this transformation satisfies $\mathrm{Sd}(\mathcal{G}) \leq p+1$, so SD is NP-complete for these families.

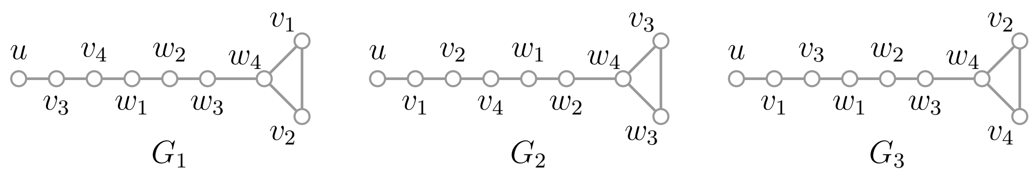

Figure 5.2: The family $\mathcal{G} = \{G_1, G_2, G_3\}$ is constructed for transforming an instance of HSP2, where $S = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{C} = \{\{v_1, v_2\}, \{v_3\}, \{v_2, v_4\}\}$, into an instance of SD for families of tadpole graphs.

5.3 Algorithms for estimating simultaneous resolvability parameters

Here, we present several approaches for obtaining approximate values for simultaneous resolvability parameters. A common idea lies on the conception of all methods, namely that of computing a permutation $S = (v_{i_1}, v_{i_2}, \ldots, v_{i_n})$ of the vertex set V, which imposes an ordering on V, and finding the minimum value θ_S such that the set $W = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{\theta_S}}\}$, composed by the first θ_S vertices according to this ordering, is a simultaneous generator of the desired type. We will refer to this value as the *resolvability threshold* of the given permutation. We will describe two greedy algorithms and a randomized local search procedure for finding a permutation whose resolvability threshold is as close as possible to the exact value of the desired simultaneous resolvability parameter.

5.3.1 Preliminaries

The data structure used for representing one graph is the upper triangular half of the distance matrix, excluding the diagonal. Explicit labels are not used for vertices. Instead, the structure refers to each vertex by its ordinal position in the vector. Thus, the i-th row refers to vertex v_i and contains the distances to the vertices $v_{i+1}, v_{i+2}, \ldots, v_{|V|}$. A graph family is represented as a vector of graph representations. Note that, for a graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ defined on a common vertex set V, the space complexity of this data structure is $O(k \cdot |V|^2)$.

Algorithm 1 Function DIST

function DIST(dt,i,x,u,v)
    if $dt = \mathrm{Sd}$ **then**
        return $d_{G_i}(u,x) \neq d_{G_i}(v,x)$
    else if $dt = \mathrm{Sd}_A$ **then**
        return $d_{G_i,2}(u,x) \neq d_{G_i,2}(v,x)$
    else
        return $d_{G_i}(u,x) = d_{G_i}(u,v) + d_{G_i}(v,x)$ **or** $d_{G_i}(v,x) = d_{G_i}(u,v) + d_{G_i}(u,x)$
    end if
end function

A number of subroutines are common to all methods. We will briefly describe those that are not trivial or simply auxiliary[2]. Boolean function DIST(dt, i, x, u, v) verifies whether the vertex x distinguishes the pair u, v in the graph G_i according to the criterion of the dimension type dt, as described in Algorithm 1. As all the distances are kept in the data structure representing the graph family, the time complexity of function DIST is $O(1)$.

At some point, all the algorithms proposed need to verify whether a vertex set S is a simultaneous generator of a given type for a graph family. This verification is performed by the Boolean function CHECKSIMGEN(dt,S), which is described in Algorithm 2. Note that function CHECKSIMGEN is likely to run faster when the output is **false**. The worst case time complexity of the function is $O(k \cdot |S| \cdot |V|^2)$

5.3.2 Description of the algorithms

Two of the proposed methods are greedy algorithms that rely on the assumption that the likelihood of a vertex belonging to a simultaneous basis

[2]The C++ implementations of the data structures and algorithms described in this chapter are available at `https://github.com/yramirezc/sim-dim-graph-families`.

Algorithm 2 Function checkSimGen

```
function checkSimGen(dt,S)
    for i ← 1 ... |𝒢| do
        for p ← 1 ... |V| − 1 do
            for q ← 1 ... |V| − 1 do
                if v_p ∉ S and v_q ∉ S then
                    foundDistinguisher ← false
                    for x ∈ S do
                        if dist(dt, i, x, v_p, v_q) then
                            foundDistinguisher ← true
                            break for
                        end if
                    end for
                    if not foundDistinguisher then
                        return false
                    end if
                end if
            end for
        end for
    end for
    return true
end function
```

of any type is directly proportional to the number of vertex pairs that it distinguishes.

The first method, *greedy aggregation*, consists on iteratively adding vertices to a set W until a generator is obtained. The method consists on an initialization phase, where the set of vertex pairs distinguished by each vertex is computed, and vertices are decrementally sorted by the sizes of these sets, and a greedy computation phase. In this second phase, a simultaneous generator of the desired type is constructed by iteratively performing two steps. First, a new vertex is added to the generator, and then the remaining vertices are re-sorted according to the number of vertex pairs that are distinguished by them but not by the already added vertices. Algorithm 3 describes greedy aggregation in detail.

Algorithm 3 Greedy aggregation

Require: A graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ on a common vertex set V

$\triangleright$ Initialization
for $v_i \in V$ **do**
  $D_i \leftarrow \emptyset$
  for $v_j \in V - \{v_i\}$ **do**
    for $v_k \in V - \{v_i, v_j\}$ **do**
      for $G_l \in \mathcal{G}$ **do**
        if $\textsc{dist}(dt, l, v_k, v_i, v_j)$ **then**
          $D_i \leftarrow D_i \cup \{\{v_j, v_k\}\}$
        end if
      end for
    end for
  end for
end for
$\textsc{sort}((v_1, D_1), (v_2, D_2), \ldots, (v_n, D_n))$ $\triangleright$ decrementally by $|D_i|$
$\triangleright$ Greedy computation
$j \leftarrow 1$
$W \leftarrow \{v_{i_1}\}$
while $\textsc{checkSimGen}(dt, W) = \textsc{false}$ **do**
  for $l \in \{j+1, j+2, \ldots, n\}$ **do**
    $D_{i_l} \leftarrow D_{i_l} - D_{i_j}$
  end for
  $j \leftarrow j + 1$
  $\textsc{sort}((v_{i_j}, D_{i_j}), (v_{i_{j+1}}, D_{i_{j+1}}), \ldots, (v_{i_n}, D_{i_n}))$ $\triangleright$ decrementally by $|D_{i_l}|$
  $W \leftarrow W \cup \{v_{i_j}\}$
end while
return $|W|$

Remark 5.4. *The time complexity of the greedy aggregation algorithm for a family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ defined on a common vertex set V is $O(k \cdot |V|^4)$.*

Proof. It is simple to see that the time complexity of the initialization phase is $O(k \cdot |V|^3)$. Moreover, function $\textsc{checkSimGen}$, as well as the inverted index update and re-sorting, are called as much as $|V| - 2$ times. Taking into account that the worst case time complexity of updating one entry of the inverted index of distinguished pairs per vertex is $O(k \cdot |V|^2)$ and that the time complexity of efficient sorting algorithms is $O(n \log n)$ on lists composed by n objects, we have that the time complexity of the greedy aggregation algorithm is

$$\begin{aligned}
O(k \cdot |V|^3 & +k \cdot |V|^2 \cdot 1 + (|V|-1) \cdot k \cdot |V|^2 + (|V|-1) \cdot \log(|V|-1)+ \\
& +k \cdot |V|^2 \cdot 2 + (|V|-2) \cdot k \cdot |V|^2 + (|V|-2) \cdot \log(|V|-2)+ \\
& \dots \\
& +k \cdot |V|^2 \cdot (|V|-2) + 2k \cdot |V|^2 + 2 \cdot \log 2) = \\
= O(k \cdot |V|^3 & +k \cdot |V|^2 \cdot \sum_{i=1}^{|V|-2} [i] + k \cdot |V|^2 \cdot \sum_{i=2}^{|V|-1} [i] + \sum_{i=2}^{|V|-1} [i \cdot \log i]) = \\
= O(k \cdot |V|^3 & +k \cdot |V|^2 \cdot \sum_{i=1}^{|V|-2} [i] + k \cdot |V|^2 \cdot \sum_{i=2}^{|V|-1} [i] + \log(|V|) \cdot \sum_{i=2}^{|V|-1} [i]) = \\
& = O(k \cdot |V|^3 + k \cdot |V|^4 + k \cdot |V|^4 + |V|^2 \cdot \log(|V|)) = \\
& = O(k \cdot |V|^4).
\end{aligned}$$

□

Moreover, the space complexity of the inverted index of distinguished vertex pairs per vertex is $O(k \cdot |V|^3)$, which dominates that of the graph family data structure, so the overall space complexity of greedy aggregation is $O(k \cdot |V|^3 + k \cdot |V|^2) = O(k \cdot |V|^3)$.

The second method, *greedy pruning*, consists on iteratively removing vertices from a set W, which is initialized as the entire vertex set, until it stops being a generator. Algorithm 4 describes greedy pruning in detail.

Greedy pruning sorts the vertices only once, so its effective running times are lower than those of greedy aggregation. Note, however, that the asymptotic time complexity of greedy pruning is the same as that of greedy aggregation, *i.e.* $O(k \cdot |V|^4)$, as it is also dominated by the calls of CHECKSIMGEN, which can also be as many as $|V| - 2$. Regarding space complexity, greedy pruning only needs to store counts of the number of distinguished vertices, which makes its space complexity dominated by that of the graph family data structure, *i.e.* $O(k \cdot |V|^2)$. As a final remark, note that the simultaneous generator computed by greedy pruning coincides with the one that would be computed by greedy aggregation if the re-sorting step were not performed. Whether following one constructive strategy or the other is more efficient depends on how probable it is for graph families to have a value of the simultaneous resolvability parameter to compute which is closer to 1 or

Algorithm 4 Greedy pruning

```
Require: A graph family 𝒢 = {G_1, G_2, ..., G_k} on a common vertex set V
▷ Initialization
for v_i ∈ V do
    C_i ← 0
    for v_j ∈ V − {v_i} do
        for v_k ∈ V − {v_i, v_j} do
            for G_l ∈ 𝒢 do
                if DIST(dt, l, v_k, v_i, v_j) then
                    C_i ← C_i + 1
                end if
            end for
        end for
    end for
end for
SORT((v_1, C_1), (v_2, C_2), ..., (v_n, C_n))            ▷ incrementally by C_i
▷ Greedy computation
j ← 1
W ← V
while CHECKSIMGEN(dt, W) = TRUE do
    W ← W − {v_{i_j}}
    j ← j + 1
end while
return |W| + 1
```

to $|V|-1$. Intuitively, we consider that the latter is more likely be the case, hence the choice of pruning rather than aggregation without re-sorting.

The third proposed method is a randomized local search procedure, which consists on running a number of local searches starting in random initial solutions, and selecting the one that obtains the best final solution. Each local search consists on an iterative process where, at every step, given the current solution S, a number of similar solutions are generated by switching the positions of two vertices, one of which is among the first θ_S vertices in S, and the candidate solution that better improves on S (if any) is selected as the new solution. The choice of the pair of vertices to switch is due to the fact that, clearly, switching the positions of two vertices beyond the

Algorithm 5 Randomized local search for simultaneous resolvability parameters

Require: A graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ on a common vertex set V, $runCount$: the number of local searches to run, $maxIters$ the maximum number of iterations to perform in a local search if no convergence is reached, and $candCount$: the number of new candidate solutions to generate in each iteration

$bestResult \leftarrow |V| - 1$
for $r \leftarrow 1 \ldots runCount$ **do**
  $S \leftarrow$ RANDOMPERM(V)
  $resThr \leftarrow$ RESTHRESHOLD(dt, S)
  if $resThr < bestResult$ **then**
    $bestResult \leftarrow resThr$
  end if
  $i \leftarrow 1$
  $notConverged \leftarrow$ **true**
  while $i \leq maxIters$ **and** $notConverged$ **do**
    $notConverged \leftarrow$ **false**
    $\mathcal{S} \leftarrow$ NEWCANDSOLUTIONS$(S, bestResult, candCount)$
    for $S' \in \mathcal{S}$ **do**
      $resThr \leftarrow$ RESTHRESHOLD(dt, S')
      if $resThr < bestResult$ **then**
        $bestResult \leftarrow resThr$
        $S \leftarrow S'$
        $notConverged \leftarrow$ **true**
      end if
    end for
    $i \leftarrow i + 1$
  end while
end for
return $bestResult$

resolvability threshold does not generate a better solution. This method is described in detail in Algorithm 5.

The worst case running time of this randomized local search method occurs when all $runCount$ local searches run up to $maxIters$ times due to non-convergence. Thus, the asymptotic time complexity of the method is determined by the $runCount \cdot maxIters \cdot candCount$ calls of function RESTHRESHOLD, each call of which may in turn call CHECKSIMGEN up to $|V|-1$ times, and is

Algorithm 6 Function RESTHRESHOLD

```
function RESTHRESHOLD(dt, S = (v_{i_1}, v_{i_2}, ..., v_{i_{|V|}}))
    j ← 1
    W ← {v_{i_1}}
    while CHECKSIMGEN(dt, W) = FALSE do
        j ← j + 1
        W ← W ∪ {v_{i_j}}
    end while
    return |W|
end function
```

$$O(runCount \cdot maxIters \cdot candCount \cdot \sum_{i=1}^{|V-1|}(k \cdot i \cdot |V|^2)) =$$
$$= O(runCount \cdot maxIters \cdot candCount \cdot k \cdot |V|^2 \cdot \sum_{i=1}^{|V-1|}(i)) =$$
$$= O(runCount \cdot maxIters \cdot candCount \cdot k \cdot |V|^4).$$

Clearly, the relation between effective running times of the randomized local search method versus that of greedy aggregation and greedy pruning depends on the relations between the values of the parameters $runCount$, $maxIters$ and $candCount$ and those of the implicit constants affecting the running times of both greedy methods. The randomized local search method needs to store at every iteration one list of candidate solutions, which is discarded from one iteration to the next. Thus, its space complexity is dominated by that of storing the graph family data structure and is $O(k \cdot |V|^2 + c \cdot |V|) = O(k \cdot |V|^2)$.

5.3.3 Experiments

In order to assess the accuracy of the proposed methods, we constructed an evaluation benchmark composed by three collections of graph families, each one containing 50 families. The first collection is composed by arbitrary

Algorithm 7 Function NEWCANDSOLUTIONS

function NEWCANDSOLUTIONS($S = (v_{i_1}, v_{i_2}, \ldots, v_{i_{|V|}})$, $resThr$, $candCount$)
  $\mathcal{S} \leftarrow \emptyset$
  for $i \leftarrow 1 \ldots candCount$ **do**
    $x \leftarrow$ RANDOMINT($[1, resThr]$)
    $y \leftarrow$ RANDOMINT($[1, |V|]$)
    if $x < y$ **then**
      $S' \leftarrow (v_{i_1}, \ldots, v_{i_{x-1}}, v_{i_y}, v_{i_{x+1}}, \ldots, v_{i_{y-1}}, v_{i_x}, v_{i_{y+1}}, \ldots, v_{i_{|V|}})$
    else
      $S' \leftarrow (v_{i_1}, \ldots, v_{i_{y-1}}, v_{i_x}, v_{i_{y+1}}, \ldots, v_{i_{x-1}}, v_{i_y}, v_{i_{x+1}}, \ldots, v_{i_{|V|}})$
    end if
    $\mathcal{S} \leftarrow \mathcal{S} \cup \{S'\}$
  end for
  return $\mathcal{S}$
end function

graphs, whereas the second and third collections are composed by families of corona product graphs and trees, respectively. Table 5.1 summarizes the most relevant statistical information of these collections.

For building each family of the first collection, the number of graphs in the family and the size of the common vertex set were randomly set. Then, each graph was constructed by randomly deciding whether each pair of vertices was to be joined by an edge or not. Connectedness was enforced by adding as many extra edges as necessary. Once the families had been constructed, the exact values of their simultaneous metric, adjacency and strong metric dimensions were computed using exhaustive breadth-first search. The need for this exhaustive search imposed a practical constraint on the families of the first collection, namely that of having small simultaneous metric, adjacency and strong metric dimensions.

Families in the second collection were obtained by generating two families $\mathcal{G}$ and $\mathcal{H}$ by the previously described process and computing the family $\mathcal{G} \odot \mathcal{H}$. In this case, exhaustive breadth-first search was used for computing $\mathrm{Sd}_A(\mathcal{H})$, which allowed us to analytically determine $\mathrm{Sd}(\mathcal{G} \odot \mathcal{H})$ applying Theorem 3.55 and $\mathrm{Sd}_A(\mathcal{G} \odot \mathcal{H})$ applying Theorems 3.59, 3.61, 3.65 and 3.67. Thus, although the second factors were constrained to have small simultaneous adjacency dimensions, the graphs of the family themselves were not subject to such constraint. Beyond the fact that the aforementioned results allowed us to analytically determine the exact values of the simultaneous metric and adjacency dimensions of the families composing the collection, we chose to make Collection 2 be composed by families of corona product graphs because such families are particularly difficult for greedy aggregation and greedy pruning. In a corona product graph $G \odot H$, every vertex $u_i \in V(G)$ distinguishes a large number of vertex pairs, including all pairs v, w where $v \in V(H_i)$ and $w \in V(H_j)$, $i \neq j$. Thus, for corona product graphs having large $|V(G)|$, the heuristic that drives both greedy methods is likely to prioritize vertices of G, even though (simultaneous) metric bases must necessarily be composed by vertices from the copies of H and (simultaneous) adjacency bases in most cases only need to contain several vertices of G. This feature of Collection 2 makes it a good example of extreme cases to be handled by the algorithms that we intend to evaluate.

	Coll. 1			Coll. 2			Coll. 3		
	min	mean	max	min	mean	max	min	mean	max
$\|\mathcal{G}\|$	5	17.26	25	2	11.36	25	2	13.62	24
$\|V\|$	12	19.18	25	84	185.38	260	43	139.94	236
$\mathrm{Sd}(\mathcal{G})$	6	7.84	14	39	94.3	147	13	88.08	161
$\mathrm{Sd}_A(\mathcal{G})$	6	8.68	14	51	110.9	167	–	–	–
$\mathrm{Sd}_s(\mathcal{G})$	10	17.66	24	–	–	–	17	104.2	189

Table 5.1: Statistics of the benchmark collections used for the experiments.

As we mentioned previously, families in the third collection are composed by trees. Moreover, such families were constructed in such a way that all trees have a common set of exterior major vertices, all of which, at the same time, have common sets of terminal vertices. In consequence, every set composed by all terminal vertices, except one, of every exterior major vertex, is a metric basis of every tree in the family, so it is also a simultaneous metric basis. Moreover, every set composed by all leaves, except one, is a simultaneous strong metric basis of the family. Consequently, the simultaneous (strong) metric dimensions of all families of this collection were easily determined analytically, so no constraint needed to be posed on their values.

For building each family of the collection, we first set, randomly, the number of trees in the family, the number of exterior major vertices, as well as the terminal degree of every exterior major vertex, and the number of additional vertices. The process for randomly constructing each tree is depicted in Figure 5.3. Initially, a "seed" tree is constructed. This tree is composed by a non-exterior major vertex, joined by edges to every vertex in the defined set of exterior major vertices, which in turn are joined by edges to their associated terminal vertices. The seed tree is then randomly modified as many times as the number of additional vertices, minus one, to obtain each final tree, which is added to the family. Each modification consists on either adding a vertex in a randomly chosen path that joins an exterior major vertex and some of its terminal vertices, or adding a vertex in a randomly chosen path that joins two exterior major vertices.

Summing up, for Collection 1 we determined the exact values of the simultaneous metric, adjacency, and strong metric dimensions; for Collection 2 we determined the exact values of the simultaneous metric and adjacency dimensions; and for Collection 3 we determined the exact values of the simultaneous metric and strong metric dimensions. Afterwards, we obtained the estimated values of these parameters by each algorithm. In the case of randomized local search, we set the values of *maxIters* and *candCount* to 1000 and 100, respectively, and computed partial estimates after 50, 100, 500 and 1000 runs. For a family $\mathcal{G}$, defined on the common vertex set V, let $\mathrm{Sd}^*(\mathcal{G})$ denote an estimate of $\mathrm{Sd}(\mathcal{G})$. The quality of $\mathrm{Sd}^*(\mathcal{G})$ is assessed through the relative error measure

$$\epsilon(\mathrm{Sd}^*(\mathcal{G})) = \frac{\mathrm{Sd}^*(\mathcal{G}) - \mathrm{Sd}(\mathcal{G})}{|V|}.$$

Note that, since all evaluated algorithms compute as their final output the size of a simultaneous metric generator, we have that $\mathrm{Sd}(\mathcal{G}) \le \mathrm{Sd}^*(\mathcal{G}) \le |V|-1$ and so $0 \le \epsilon(\mathrm{Sd}^*(\mathcal{G})) \le \frac{|V|-2}{|V|}$. Also note that we do not use the standard definition of *relative error*, which would be $\epsilon(\mathrm{Sd}^*(\mathcal{G})) = \left|\frac{\mathrm{Sd}(\mathcal{G})-\mathrm{Sd}^*(\mathcal{G})}{\mathrm{Sd}(\mathcal{G})}\right|$, as we consider that it fails to differentiate cases where $|V|$ is relevant to assess the seriousness of errors. For instance, consider two graph families $\mathcal{G}$ and $\mathcal{G}'$ defined on common vertex sets V and V', respectively, such that $|V| \ll |V'|$ and $\mathrm{Sd}(\mathcal{G}) = \mathrm{Sd}(\mathcal{G}')$, e.g. most pairs of families composed by paths and/or cycles characterized in Theorem 2.8. In these cases, we consider that equal absolute errors should not be considered as equally serious, yet the stan-

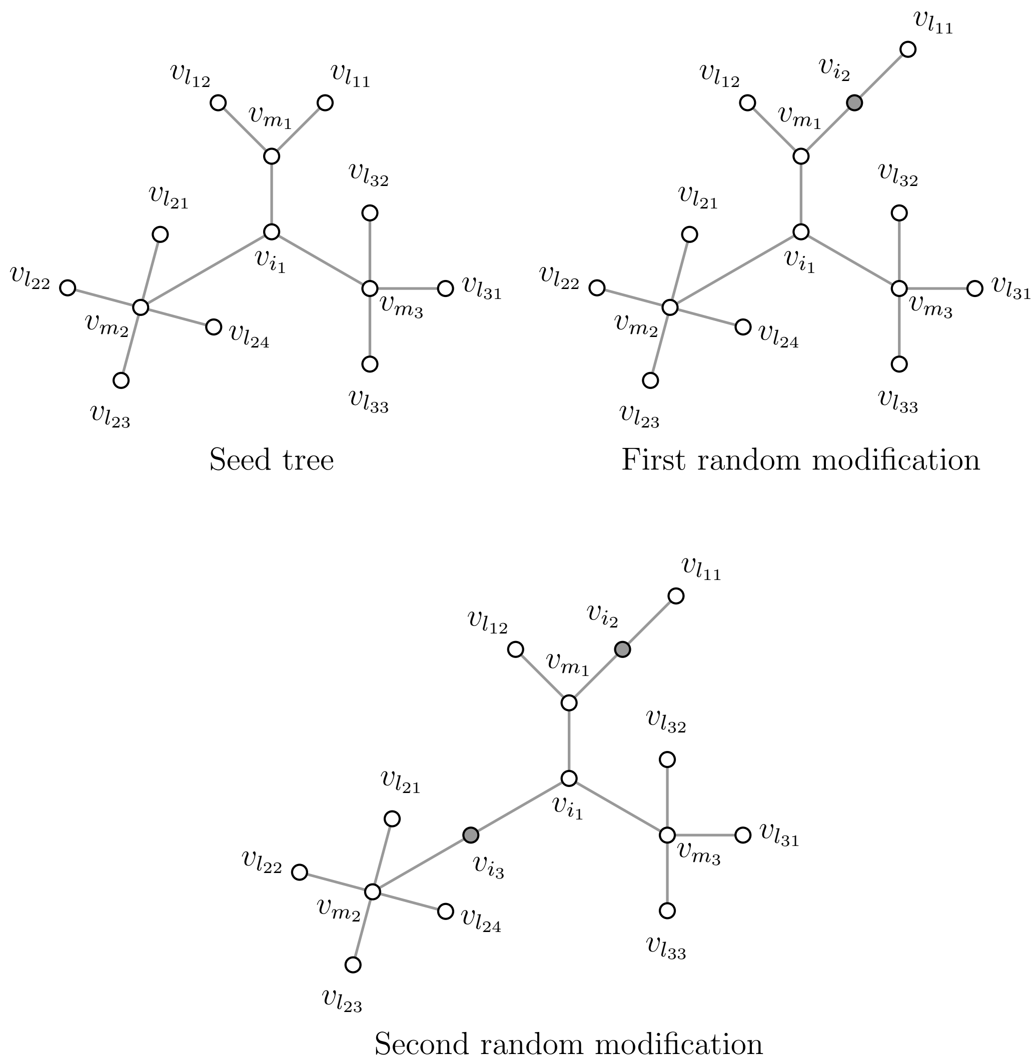

Figure 5.3: Initial steps of the process for randomly constructing a tree with three exterior major vertices having terminal degrees two, three and four.

dard relative errors would be the same. The measure $\epsilon(\mathrm{Sd}(\mathcal{G}))$ handles this situation more adequately.

For the simultaneous adjacency dimension and the simultaneous strong metric dimension, the measures $\epsilon(\mathrm{Sd}^*_A(\mathcal{G}))$ and $\epsilon(\mathrm{Sd}^*_s(\mathcal{G}))$, respectively, are computed in a manner analogous to $\epsilon(\mathrm{Sd}^*(\mathcal{G}))$.

Figures 5.4, 5.5 and 5.6 show the results obtained for the simultaneous metric dimension on the first, second and third collections, respectively. In the figures, each plot represents the values of $\epsilon(\mathrm{Sd}^*(\mathcal{G}))$ for every algorithm on every family. Moreover, dashed horizontal lines represent the mean values for each algorithm on the entire collection. In the x axes, families are arranged in incremental order of $|V|$. In a similar manner, Figures 5.7 and 5.8 show the results obtained for the simultaneous adjacency dimension on the first and second collections, respectively, whereas Figures 5.9 and 5.10 show the results obtained for the simultaneous strong metric dimension on the first and third collections, respectively.

The analysis of these results allowed us to extract a number of conclusions. First, note that the only cases where randomized local search substantially outperforms greedy aggregation are those where the cardinality of the vertex set is considerably small, as can be verified on Collection 1 (clearly for the simultaneous metric and adjacency dimensions and to a lesser extent for the simultaneous strong metric dimension) and a few of the families of Collection 3 having smallest $|V|$. This result comes as no surprise, as performing enough local searches on a small search space is likely to be (almost) equivalent to an exhaustive search. Moreover, from the results on Collections 2 and 3, it is clear that as $|V|$ increases, the results for randomized local search degrade. Besides, the effect of the number of runs on the accuracy of randomized local search is more discrete than we expected.

Secondly, even though Collection 2 was conceived to show the greedy methods at their worst, greedy aggregation actually suffered the lowest error (almost at tie with randomized local search for the simultaneous adjacency dimension). An interesting aspect of the results on Collection 2 are a few families for which greedy aggregation obtained the exact values of the simultaneous metric and adjacency dimensions, in contrast with the generally poorer results. Those cases correspond to families $\mathcal{G} \odot \mathcal{H}$ where $|V(\mathcal{G})|$ is considerably small and $|V(\mathcal{H})|$ is considerably large, so vertices from $|V(\mathcal{G})|$ are not unfairly prioritized. In general, on Collection 2 the results of all methods tend to degrade as $|V(\mathcal{G} \odot \mathcal{H})|$ increases.

The results on Collection 3, whose families are defined on larger vertex sets than Collection 1 and, unlike Collection 2, have no features obviously contradicting the assumptions behind any of the proposed algorithms, allow us to see that greedy aggregation is much more stable as $|V|$ grows, while randomized local search tends to degrade and greedy aggregation tends to slightly improve (more noticeable for the simultaneous strong metric dimension than for the simultaneous metric dimension).

In our opinion, the most important fact highlighted by these results is that, despite its computational cost, the re-sorting stage of greedy aggregation is critical, as it allows it to obtain the overall best results, in contrast to the overall worst results obtained by greedy pruning.

To conclude our discussion, we point out some rules-of-thumb, based on the computational cost of the methods and the observed experimental results, to aid in the selection of one of the proposed algorithms for real-life computations. First, if enough memory is available, greedy aggregation should be the method of choice, as it showed overall best results and higher stability as the cardinality of the vertex set grows. Now, if memory is limited, an extra circumstance should be considered. Up to some value of $|V|$, randomized local search would be the second option, provided that enough computation time is available. However, extrapolating the observed fact that randomized local search tends to degrade as $|V|$ increases while greedy pruning tends to improve (although at a slower pace) we conjecture that for very large instances greedy pruning may be the most appropriate second choice. For instance, we can see that for the simultaneous strong metric dimension, the results of greedy pruning on families of Collection 3 having the largest values of $|V|$ were better than those of randomized local search. Even though this situation did not occur for the simultaneous metric dimension, a trend towards convergence can also be observed.

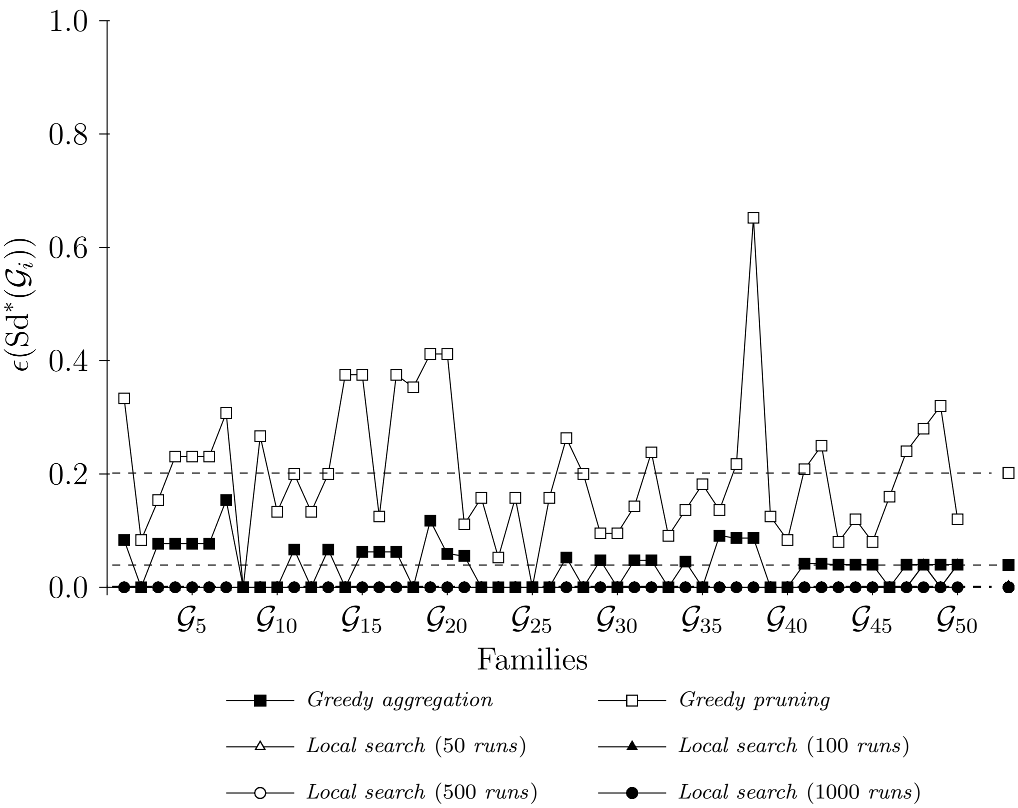

Figure 5.4: Experimental results for the simultaneous metric dimension on Collection 1.

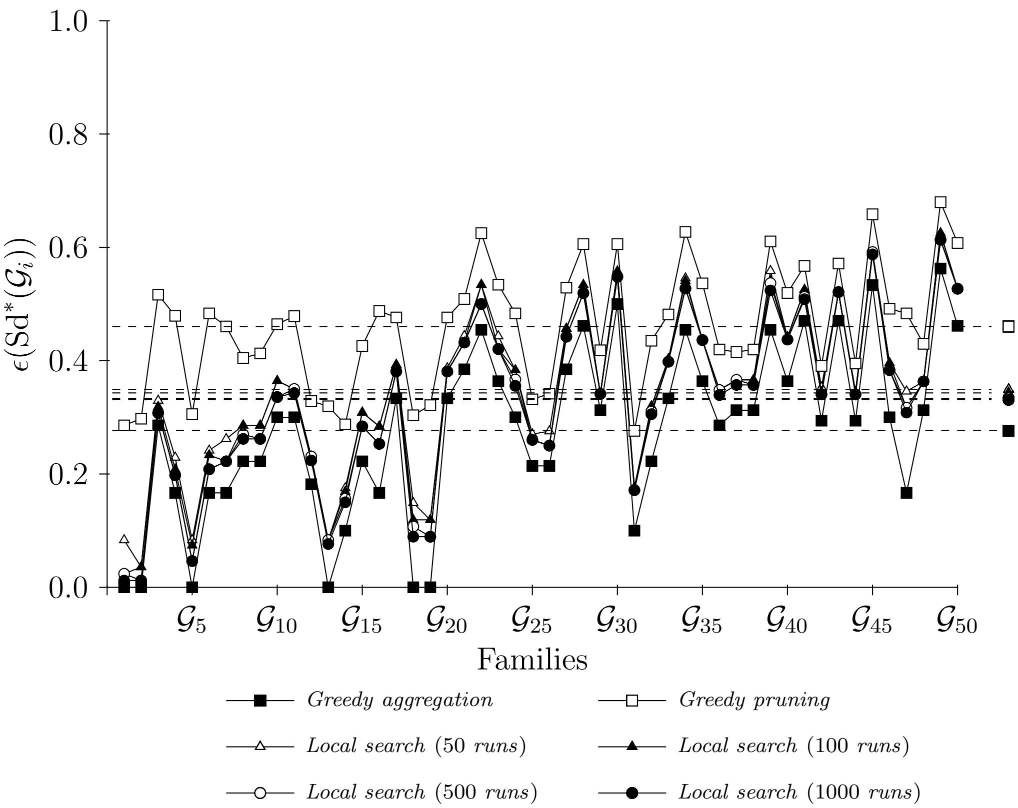

Figure 5.5: Experimental results for the simultaneous metric dimension on Collection 2.

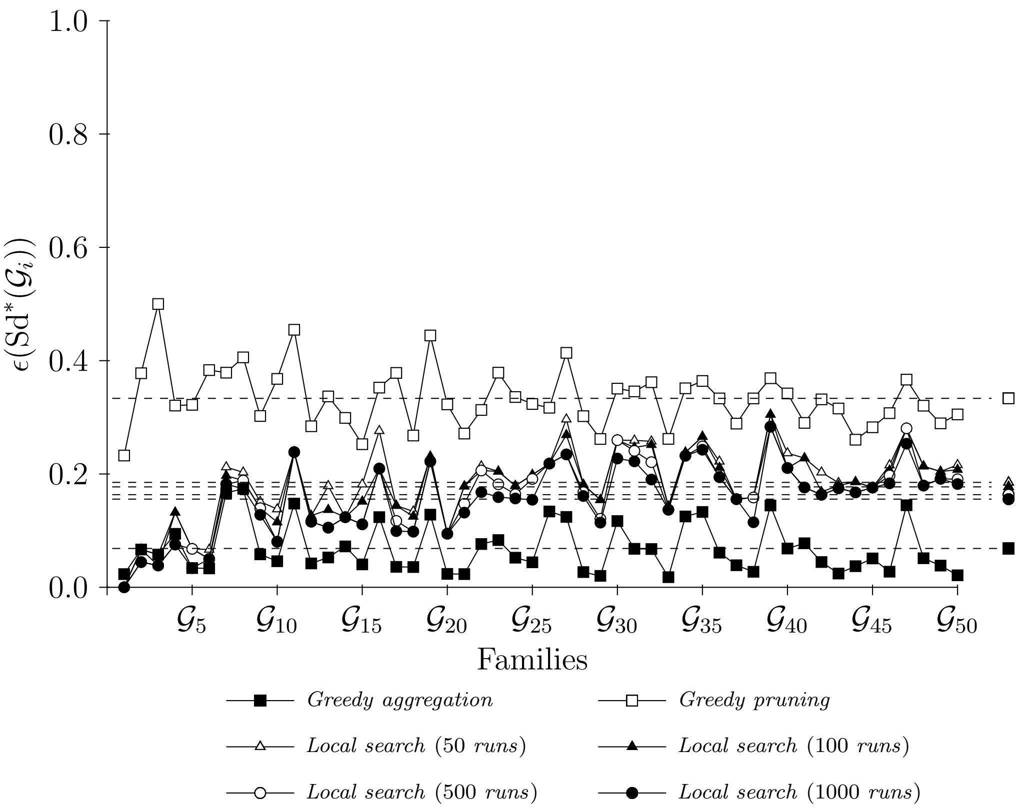

Figure 5.6: Experimental results for the simultaneous metric dimension on Collection 3.

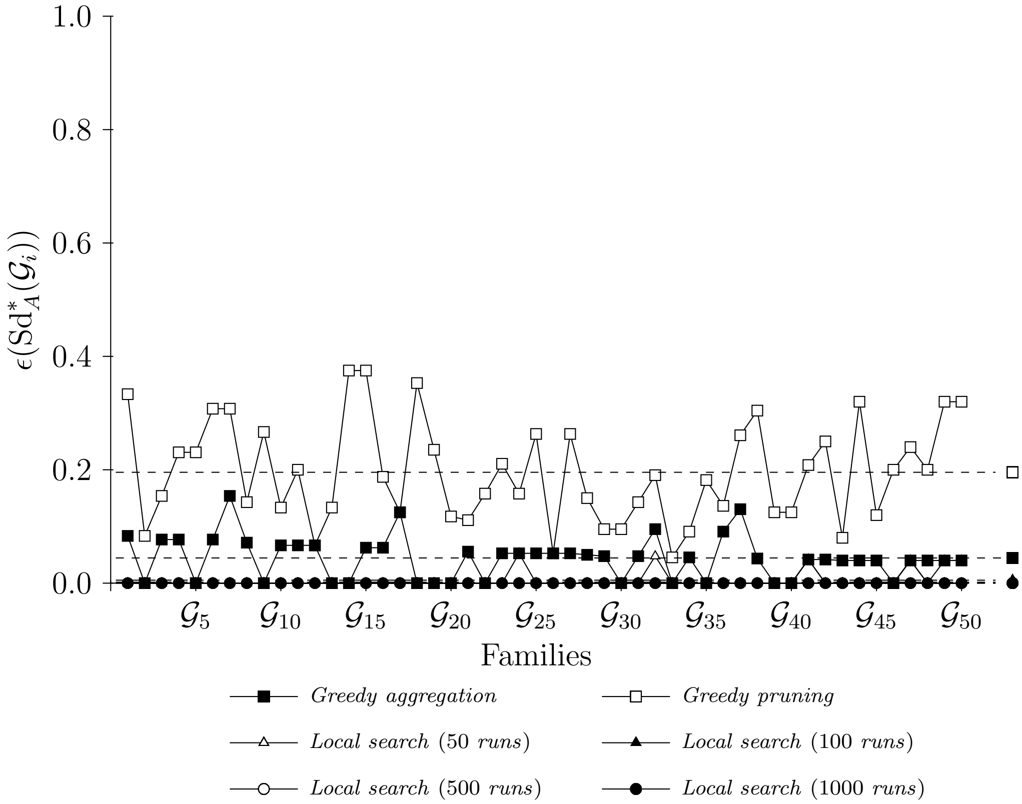

Figure 5.7: Experimental results for the simultaneous adjacency dimension on Collection 1.

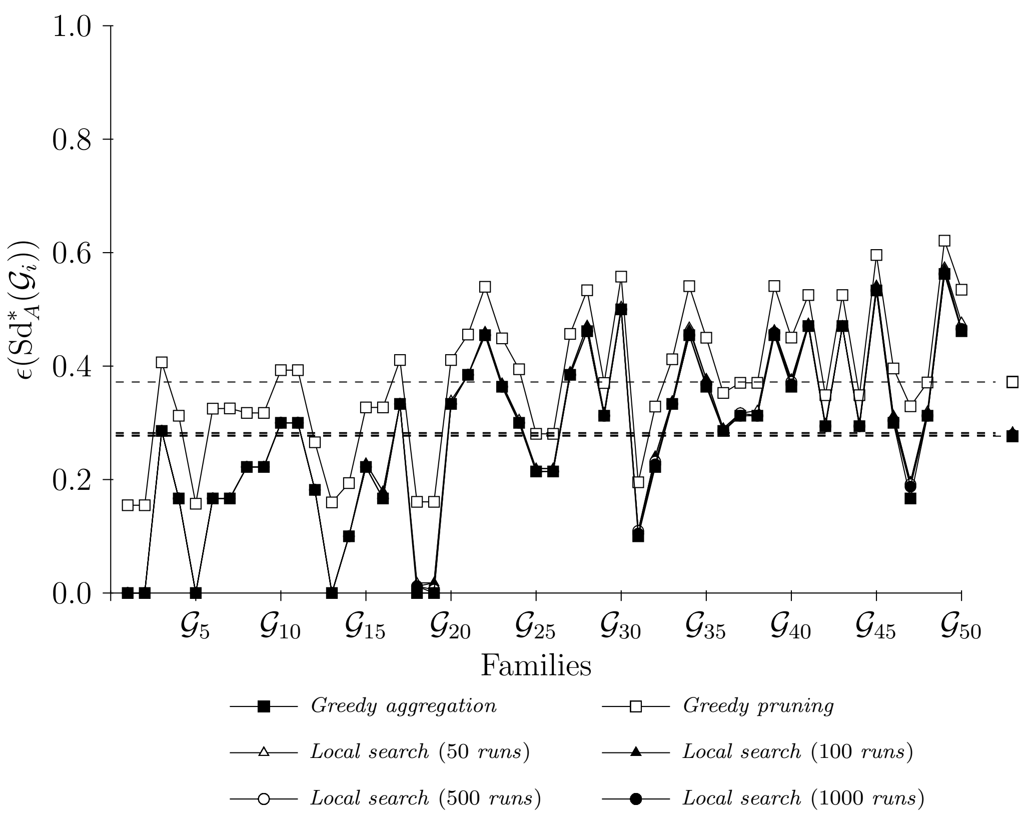

Figure 5.8: Experimental results for the simultaneous adjacency dimension on Collection 2.

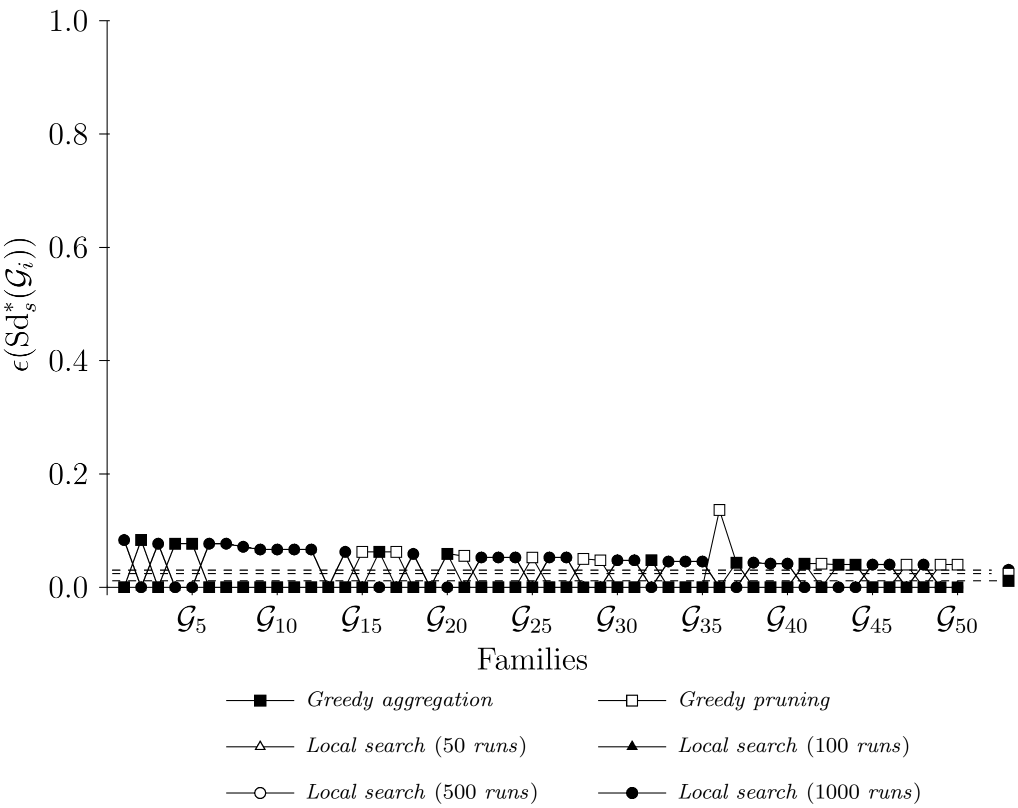

Figure 5.9: Experimental results for the simultaneous strong metric dimension on Collection 1.

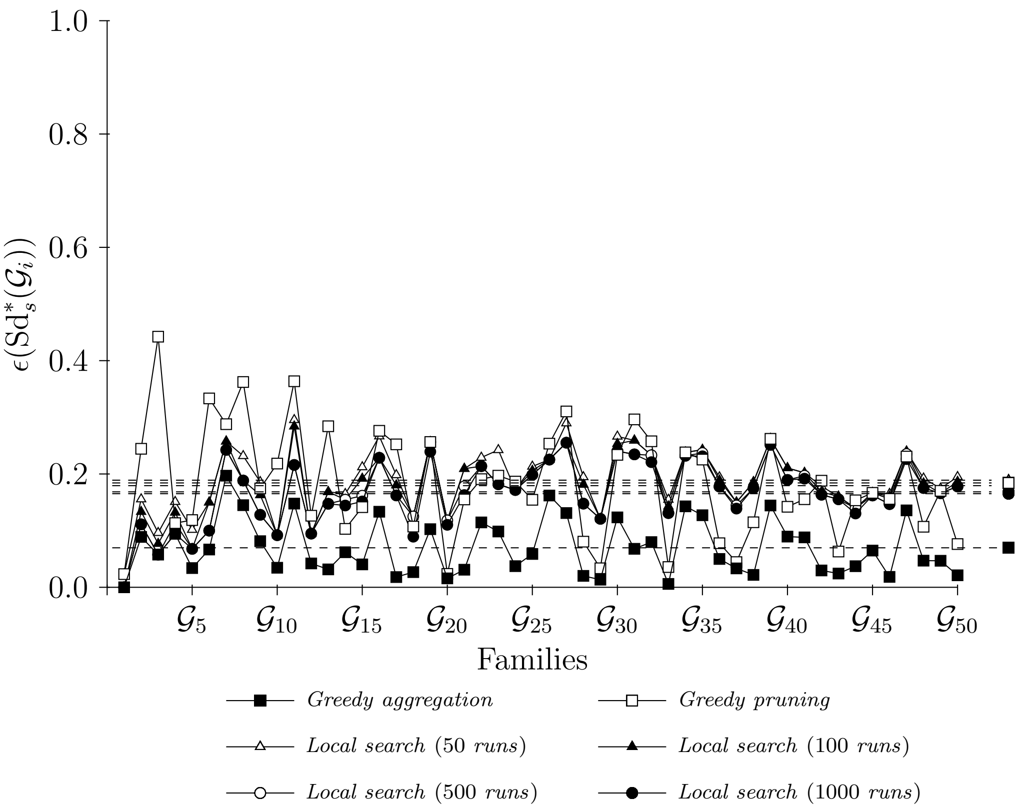

Figure 5.10: Experimental results for the simultaneous strong metric dimension on Collection 3.

Conclusions

In this thesis we have introduced the notion of simultaneous resolvability for graph families defined on a common vertex set. The main results of the thesis have dealt with simultaneous metric generators and bases, as well as the simultaneous metric dimension of such families. Additionally, we have covered two related forms of simultaneous resolvability. Firstly, we treated the simultaneous adjacency dimension, which proved useful for characterizing the simultaneous metric dimension of families composed by lexicographic and corona product graphs. Secondly, we studied the main properties of the simultaneous strong metric dimension. In all cases, our focus was on determining the general bounds for these parameters, their relations to the standard resolvability parameters of the individual graphs and, when possible, giving exact values or sharp bounds for a number of specific families.

Computationally, these problems are far from solved for the general case, as we were able to verify that the requirement of simultaneity adds on the complexity of the calculations involving these resolvability parameters, which had already been proven to be NP-hard for their standard counterparts. In particular, we characterized families composed by graphs for which some standard resolvability parameters can be efficiently computed, while computing the associated simultaneous parameters is NP-hard. To alleviate this problem, we proposed several methods for approximately estimating these parameters and conducted an experimental evaluation to study their behaviour on randomly generated collections of graph families.

Contributions of the thesis

The results presented in this work have been published, or are in the process of been published, in several venues. Several papers have been published, accepted or submitted to ISI-JCR journals, while some of the principal results have been presented in international conferences.

Publications in ISI-JCR journals

- Y. Ramírez-Cruz, O. R. Oellermann, J. A. Rodríguez-Velázquez. The Simultaneous Metric Dimension of Graph Families. *Discrete Applied Mathematics* **198**, 241–250, 2016. DOI: 10.1016/j.dam.2015.06.012.
- Y. Ramírez-Cruz, A. Estrada-Moreno, J. A. Rodríguez-Velázquez. The Simultaneous Metric Dimension of Families Composed by Lexicographic Product Graphs. *Graphs and Combinatorics*, in press, available online Jan. 13, 2016. DOI: 10.1007/s00373-016-1675-1.
- A. Estrada-Moreno, C. García-Gómez, Y. Ramírez-Cruz, J. A. Rodríguez-Velázquez. The Simultaneous Strong Metric Dimension of Graph Families. *Bulletin of the Malaysian Mathematical Sciences Society*, in press, available online Nov. 5, 2015. DOI:10.1007/s40840-015-0268-0.

Papers currently submitted to journals

- Y. Ramírez-Cruz, A. Estrada-Moreno, J. A. Rodríguez-Velázquez. Simultaneous Resolvability in Families of Corona Product Graphs. Submitted to the *Bulletin of the Malaysian Mathematical Sciences Society*.

Publications in conference proceedings

- Y. Ramírez-Cruz, O. R. Oellermann, J. A. Rodríguez-Velázquez. Simultaneous Resolvability in Graph Families. Proceedings of "IX Jornadas de Matemática Discreta y Algorítmica". *Electronic Notes in Discrete Mathematics* **46**, 241–248, 2014.
- A. Estrada-Moreno, C. García-Gómez, Y. Ramírez-Cruz, J. A. Rodríguez-Velázquez. On Simultaneous Strong Metric Generators of Graph Families. Proceedings of "IX Encuentro Andaluz de Matemática Discreta". J. Cáceres and M. L. Puertas (Eds.), *Avances en Matemática Discreta en Andalucía* **IV**, 109–116, 2015.

Contributions to conferences

- Y. Ramírez-Cruz, O. R. Oellermann, J. A. Rodríguez-Velázquez. Simultaneous Resolvability in Graph Families. IX Jornadas de Matemática Discreta y Algorítmica, Tarragona, Spain (2014).

- O. R. Oellermann, Y. Ramírez-Cruz, J. A. Rodríguez-Velázquez. The Simultaneous Metric Dimension of Graph Families. 8^{th} Slovenian Conference on Graph Theory, Kranjska Gora, Slovenia (2015).

- Y. Ramírez-Cruz, O. R. Oellermann, A. Estrada-Moreno, C. García-Gómez, J. A. Rodríguez-Velázquez. The Simultaneous (Strong) Metric Dimension of Graph Families. III Congreso de Jóvenes Investigadores de la Real Sociedad Matemática Española, Murcia, Spain (2015).

- A. Estrada-Moreno, C. García-Gómez, Y. Ramírez-Cruz, J. A. Rodríguez-Velázquez. On Simultaneous Strong Metric Generators of Graph Families. IX Encuentro Andaluz de Matemática Discreta, Almería, Spain (2015).

Other contributions

- Y. Ramírez-Cruz. Notions of Simultaneous Resolvability in Graph Families. A. Valls-Mateu and J. A. Rodríguez-Velázquez (Eds.), *1^{st} URV Doctoral Workshop in Computer Science and Mathematics*, Llibres URV, Tarragona, Spain, 2014, 45–48.

- Y. Ramírez-Cruz. Computability of the Simultaneous (Strong) Metric Dimension of a Graph Family. M. Sánchez-Artigas and A. Valls-Mateu (Eds.), *2^{nd} URV Doctoral Workshop in Computer Science and Mathematics*, Llibres URV, Tarragona, Spain, 2015, 51–55.

Future work

- A vast number of variations of the metric dimension have been presented, as we discussed in Section 1.1. In principle, simultaneous counterparts of all of these parameters can be defined on graph families, which would lead to a wide range of studies.

- Remark 4.8 shows a result on the simultaneous strong metric dimension of some specific families composed by corona product graphs. While this result turned out to be straightforward, it illustrates the interestingness of conducting a deeper study on the simultaneous strong metric dimension of families composed by product graphs. Such study may be based on the results presented in [56, 57, 71].

- A natural extension of the results presented in Section 5.3 is to apply popular metaheuristics to the approximation of simultaneous resolvability parameters, e.g. genetic algorithms, ant-colony optimization, particle swarm optimization, etc.

- Following the line of computing approximate solutions, an alternative approach may be that of defining relaxed notions of resolvability. While the combinatorial study of such variations may be challenging, they may pave the way for the use of a wide range of computational techniques borrowed from other areas, such as data mining and pattern recognition[3], thus enlarging their field of practical applications. To illustrate our point, here we define two intuitively interesting relaxations:

  - *Quasi-simultaneous generators*: For a graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$, defined on a common vertex set V, and a real number $\varsigma \in [0, 1]$, a set $S \subseteq V$ is a ς*-quasi-simultaneous metric / adjacency / strong metric generator* for $\mathcal{G}$ if the number $R_S(\mathcal{G})$ of graphs $G_i \in \mathcal{G}$ for which S is a metric / adjacency / strong metric generator satisfies $\frac{R_S(\mathcal{G})}{k} \geq \varsigma$.
  - *Simultaneous quasi-generators*: For a graph $G = (V, E)$ and a real number $\varsigma \in [0, 1]$, a set $S \subseteq V$ is a *metric / adjacency / strong metric* ς*-quasi-generator* for G if the number $R_S(G)$ of different vertex pairs that are distinguished by some element of S satisfies $\frac{2 \cdot R_S(G)}{|V| \cdot (|V|-1)} \geq \varsigma$.[4] By analogy, for a graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$, defined on a common vertex set V, and a real number $\varsigma \in [0, 1]$, a set $S \subseteq V$ is a *simultaneous metric/adjacency/strong metric* ς*-quasi-generator* for $\mathcal{G}$ if it is a metric / adjacency / strong metric ς-quasi-generator for every $G_i \in \mathcal{G}$.

  Note that simultaneous generators are a particular case of both relaxed variants for $\varsigma = 1$.

[3] For instance, feature selection, frequent itemset mining, clustering, etc.

[4] Note that the total number of different vertex pairs $u, v \in V$ is $\frac{|V| \cdot (|V|-1)}{2}$.

Bibliography

[1] R. F. Bailey, P. J. Cameron, Base size, metric dimension and other invariants of groups and graphs, Bulletin of the London Mathematical Society 43 (2) (2011) 209–242.
URL http://blms.oxfordjournals.org/content/43/2/209.1.abstract

[2] Z. Baranyai, G. R. Szász, Hamiltonian decomposition of lexicographic product, Journal of Combinatorial Theory. Series B 31 (3) (1981) 253–261.
URL http://dx.doi.org/10.1016/0095-8956(81)90028-9

[3] G. Barragán-Ramírez, C. G. Gómez, J. A. Rodríguez-Velázquez, Closed formulae for the local metric dimension of corona product graphs, Electronic Notes in Discrete Mathematics 46 (2014) 27–34.
URL http://www.sciencedirect.com/science/article/pii/S1571065314000067

[4] L. M. Blumenthal, Theory and applications of distance geometry, Oxford University Press, 1953.

[5] B. Brešar, S. Klavžar, A. T. Horvat, On the geodetic number and related metric sets in cartesian product graphs, Discrete Applied Mathematics 308 (23) (2008) 5555–5561.
URL http://www.sciencedirect.com/science/article/pii/S0012365X07008266

[6] R. C. Brigham, G. Chartrand, R. D. Dutton, P. Zhang, Resolving domination in graphs, Mathematica Bohemica 128 (1) (2003) 25–36.
URL http://mb.math.cas.cz/mb128-1/3.html

[7] R. C. Brigham, R. D. Dutton, Factor domination in graphs, Discrete Mathematics 86 (1–3) (1990) 127–136.
URL http://www.sciencedirect.com/science/article/pii/0012365X9090355L

[8] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, On the metric dimension of infinite graphs, Discrete Applied Mathematics 160 (18) (2012) 2618–2626.
URL http://dx.doi.org/10.1016/j.dam.2011.12.009

[9] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of cartesian product of graphs, SIAM Journal on Discrete Mathematics 21 (2) (2007) 423–441.
URL http://epubs.siam.org/doi/abs/10.1137/050641867

[10] J. Cáceres, M. L. Puertas, C. Hernando, M. Mora, I. M. Pelayo, C. Seara, Searching for geodetic boundary vertex sets, Electronic Notes in Discrete Mathematics 19 (2005) 25–31.
URL http://www.sciencedirect.com/science/article/pii/S1571065305050043

[11] G. G. Chappell, J. Gimbel, C. Hartman, Bounds on the metric and partition dimensions of a graph, Ars Combinatoria 88 (2008) 349–366.
URL http://www.cs.uaf.edu/~chappell/papers/metric/metric.pdf

[12] G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Applied Mathematics 105 (1-3) (2000) 99–113.
URL http://dx.doi.org/10.1016/S0166-218X(00)00198-0

[13] G. Chartrand, D. Erwin, G. L. Johns, P. Zhang, Boundary vertices in graphs, Discrete Mathematics 263 (1-3) (2003) 25–34.
URL http://www.sciencedirect.com/science/article/pii/S0012365X02005678#

[14] G. Chartrand, F. Okamoto, P. Zhang, The metric chromatic number of a graph, The Australasian Journal of Combinatorics 44 (2009) 273–286.
URL http://ajc.maths.uq.edu.au/pdf/44/ajc_v44_p273.pdf

[15] G. Chartrand, C. Poisson, P. Zhang, Resolvability and the upper dimension of graphs, Computers & Mathematics with Applications 39 (12) (2000) 19–28.
URL `http://dx.doi.org/10.1016/S0898-1221(00)00126-7`

[16] G. Chartrand, V. Saenpholphat, P. Zhang, The independent resolving number of a graph, Mathematica Bohemica 128 (4) (2003) 379–393.
URL `http://mb.math.cas.cz/mb128-4/4.html`

[17] G. Chartrand, E. Salehi, P. Zhang, The partition dimension of a graph, Aequationes Mathematicae 59 (1-2) (2000) 45–54.
URL `http://dx.doi.org/10.1007/PL00000127`

[18] N. Čižek, S. Klavžar, On the chromatic number of the lexicographic product and the Cartesian sum of graphs, Discrete Mathematics 134 (1-3) (1994) 17–24.
URL `http://www.sciencedirect.com/science/article/pii/0012365X93E0056A`

[19] J. D. Currie, O. R. Oellermann, The metric dimension and metric independence of a graph, Journal of Combinatorial Mathematics and Combinatorial Computing 39 (2001) 157–167.

[20] R. Diestel, Graph Theory, vol. 173 of Graduate Texts in Mathematics, 3rd ed., Springer-Verlag Heidelber, New York, 2005.

[21] A. Estrada-Moreno, Y. Ramírez-Cruz, J. A. Rodríguez-Velázquez, On the adjacency dimension of graphs, Applicable Analysis and Discrete Mathematics. To appear.
URL `http://www.doiserbia.nb.rs/Article.aspx?ID=1452-86301500022E#.VkGwLb-gp-I`

[22] A. Estrada-Moreno, I. Yero, J. Rodríguez-Velázquez, The k-metric dimension of corona product graphs, Bulletin of the Malaysian Mathematical Sciences Society (2015) 1–22.
URL `http://dx.doi.org/10.1007/s40840-015-0282-2`

[23] A. Estrada-Moreno, I. G. Yero, J. A. Rodríguez-Velázquez, The k-metric dimension of corona product graphs II, In progress.

[24] A. Estrada-Moreno, I. G. Yero, J. A. Rodríguez-Velázquez, k-metric resolvability in graphs, Electronic Notes in Discrete Mathematics 46 (2014) 121–128.
URL `http://www.sciencedirect.com/science/article/pii/S1571065314000183`

[25] M. Feng, K. Wang, On the fractional metric dimension of corona product graphs and lexicographic product graphs, arXiv:1206.1906 [math.CO].
URL `http://arxiv.org/abs/1206.1906`

[26] H. Fernau, J. A. Rodríguez-Velázquez, On the (adjacency) metric dimension of corona and strong product graphs and their local variants: combinatorial and computational results, arXiv:1309.2275 [math.CO].
URL `http://arxiv-web3.library.cornell.edu/abs/1309.2275`

[27] H. Fernau, J. A. Rodríguez-Velázquez, Notions of metric dimension of corona products: combinatorial and computational results, in: Computer science—theory and applications, vol. 8476 of Lecture Notes in Comput. Sci., Springer, Cham, 2014, pp. 153–166.

[28] R. Frucht, F. Harary, On the corona of two graphs, Aequationes Mathematicae 4 (3) (1970) 322–325.
URL `http://dx.doi.org/10.1007/BF01844162`

[29] H. Furmańczyk, K. Kaliraj, M. Kubale, J. V. Vivin, Equitable coloring of corona products of graphs, Advances and Applications in Discrete Mathematics 11 (2) (2013) 103–120.
URL `http://www.eti.pg.gda.pl/katedry/kams/wwwkams/pdf/PPH-1210047-DM%20-%20Final%20Version.pdf`

[30] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman & Co., New York, NY, USA, 1979.
URL `http://dl.acm.org/citation.cfm?id=578533`

[31] D. Geller, S. Stahl, The chromatic number and other functions of the lexicographic product, Journal of Combinatorial Theory, Series B 19 (1) (1975) 87–95.
URL `http://www.sciencedirect.com/science/article/pii/0095895675900763`

[32] I. González Yero, M. Jakovac, D. Kuziak, A. Taranenko, The partition dimension of strong product graphs and cartesian product graphs, Discrete Mathematics 331 (2014) 43–52.
URL http://dx.doi.org/10.1016/j.disc.2014.04.026

[33] I. González Yero, D. Kuziak, A. Rondón Aguilar, Coloring, location and domination of corona graphs, Aequationes Mathematicae 86 (1-2) (2013) 1–21.
URL http://dx.doi.org/10.1007/s00010-013-0207-9

[34] S. Gravier, J. Moncel, On graphs having a $v \setminus \{x\}$ set as an identifying code, Discrete Mathematics 307 (3-5) (2007) 432–434.
URL http://dx.doi.org/10.1016/j.disc.2005.09.035

[35] R. Hammack, W. Imrich, S. Klavžar, Handbook of product graphs, Discrete Mathematics and its Applications, 2nd ed., CRC Press, 2011.
URL http://www.crcpress.com/product/isbn/9781439813041

[36] F. Harary, R. A. Melter, On the metric dimension of a graph, Ars Combinatoria 2 (1976) 191–195.
URL http://www.ams.org/mathscinet-getitem?mr=0457289

[37] C. Hernando, M. Mora, I. M. Pelayo, C. Seara, J. Cáceres, M. L. Puertas, On the metric dimension of some families of graphs, Electronic Notes in Discrete Mathematics 22 (2005) 129–133.
URL http://www.sciencedirect.com/science/article/pii/S1571065305051929

[38] C. Hernando, M. Mora, I. M. Pelayo, C. Seara, D. R. Wood, Extremal graph theory for metric dimension and diameter, The Electronic Journal of Combinatorics 17 (R30) (2010) 28.
URL http://www.combinatorics.org/ojs/index.php/eljc/article/view/v17i1r30

[39] B. L. Hulme, A. W. Shiver, P. J. Slater, A boolean algebraic analysis of fire protection, in: R. C.-G. R.E. Burkard, U. Zimmermann (eds.), Algebraic and Combinatorial Methods in Operations Research, vol. 95 of North-Holland Mathematics Studies, North-Holland, 1984, pp. 215–227.

URL http://www.sciencedirect.com/science/article/pii/S0304020808729645

[40] M. Imran, S. A. ul Haq Bokhary, A. Ahmad, A. Semaničová-Feňovčíková, On classes of regular graphs with constant metric dimension, Acta Mathematica Scientia 33 (1) (2013) 187–206.
URL http://www.sciencedirect.com/science/article/pii/S0252960212602045

[41] W. Imrich, S. Klavžar, Product graphs, structure and recognition, Wiley-Interscience series in discrete mathematics and optimization, Wiley, 2000.
URL http://books.google.es/books?id=EOnuAAAAMAAJ

[42] H. Iswadi, E. T. Baskoro, R. Simanjuntak, On the metric dimension of corona product of graphs, Far East Journal of Mathematical Sciences 52 (2) (2011) 155–170.
URL http://www.pphmj.com/abstract/5882.htm

[43] M. Jannesari, B. Omoomi, The metric dimension of the lexicographic product of graphs, Discrete Mathematics 312 (22) (2012) 3349–3356.
URL http://www.sciencedirect.com/science/article/pii/S0012365X12003317

[44] M. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, Journal of Biopharmaceutical Statistics 3 (2) (1993) 203–236, pMID: 8220404.
URL http://www.tandfonline.com/doi/abs/10.1080/10543409308835060

[45] M. Johnson, Browsable structure-activity datasets, in: R. Carbó-Dorca, P. Mezey (eds.), Advances in Molecular Similarity, chap. 8, JAI Press Inc, Stamford, Connecticut, 1998, pp. 153–170.
URL http://books.google.es/books?id=1vvMsHXd2AsC

[46] R. Karp, Reducibility among combinatorial problems, in: R. Miller, J. Thatcher (eds.), Complexity of Computer Computations, Plenum Press, New York, 1972, pp. 85–103.

[47] M. G. Karpovsky, K. Chakrabarty, L. B. Levitin, On a new class of codes for identifying vertices in graphs, IEEE Transactions on Information Theory 44 (1998) 599–611.
URL `http://ieeexplore.ieee.org/iel3/18/14460/00661507.pdf?arnumber=661507`

[48] S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, Discrete Applied Mathematics 70 (3) (1996) 217–229.
URL `http://www.sciencedirect.com/science/article/pii/0166218X95001062`

[49] K. M. Koh, D. G. Rogers, H. K. Teo and K. Y. Yap, Graceful graphs: some further results and problems, Congr. Numerantium 29 (1980) 559–571.

[50] J. Kratica, V. Kovačević-Vujčić, M. Čangalović, M. Stojanović, Minimal doubly resolving sets and the strong metric dimension of some convex polytopes, Applied Mathematics and Computation 218 (19) (2012) 9790–9801.
URL `http://dx.doi.org/10.1016/j.amc.2012.03.047`

[51] J. Kratica, V. Kovačević-Vujčić, M. Čangalović, Computing strong metric dimension of some special classes of graphs by genetic algorithms, Yugoslav Journal of Operations Research 18 (2) (2008) 143–151.
URL `http://www.doiserbia.nb.rs/Article.aspx?id=0354-02430802143K`

[52] J. Kratica, V. Kovačević-Vujčić, M. Čangalović, Computing the metric dimension of graphs by genetic algorithms, Computational Optimization and Applications 44 (2) (2009) 343–361.
URL `http://dx.doi.org/10.1007/s10589-007-9154-5`

[53] J. Kratica, V. Kovačević-Vujčić, M. Čangalović, N. Mladenović, Strong metric dimension: a survey, Yugoslav Journal of Operations Research 24 (2) (2014) 187–198.
URL `http://dx.doi.org/10.2298/YJOR130520042K`

[54] J. Kratica, V. Kovačević-Vujčić, M. Čangalović, M. Stojanović, Minimal doubly resolving sets and the strong metric dimension of Hamming graphs, Applicable Analysis and Discrete Mathematics 6 (1) (2012)

63–71.
URL http://www.doiserbia.nb.rs/Article.aspx?ID=1452-86301100023K

[55] M. Kriesell, A note on hamiltonian cycles in lexicographical products, Journal of Automata, Languages and Combinatorics 2 (2) (1997) 135–138.
URL http://dl.acm.org/citation.cfm?id=273220

[56] D. Kuziak, Strong resolvability in product graphs, Ph.D. thesis, Universitat Rovira i Virgili (2014).

[57] D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, On the strong metric dimension of corona product graphs and join graphs, Discrete Applied Mathematics 161 (7–8) (2013) 1022–1027.
URL http://www.sciencedirect.com/science/article/pii/S0166218X12003897

[58] D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, Strong metric dimension of rooted product graphs, International Journal of Computer Mathematics. To appear.
URL http://www.tandfonline.com/doi/abs/10.1080/00207160.2015.1061656?journalCode=gcom20

[59] D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, Closed formulae for the strong metric dimension of lexicographic product graphs, arXiv:1402.2663v1 [math.CO].
URL http://arxiv.org/abs/1402.2663

[60] D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, On the strong metric dimension of cartesian sum graphs, Fundamenta Informaticae 141(1) (2015), 57–69.
URL http://content.iospress.com/articles/fundamenta-informaticae/fi1263

[61] D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, Erratum to “On the strong metric dimension of the strong products of graphs”, Open Math. 13 (2015) 209–210.
URL http://dx.doi.org/10.1515/math-2015-0020

[62] D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, On the strong metric dimension of the strong products of graphs, Open Math. 13 (2015) 64–74. URL http://dx.doi.org/10.1515/math-2015-0007

[63] T. R. May, O. R. Oellermann, The strong dimension of distance-hereditary graphs, Journal of Combinatorial Mathematics and Combinatorial Computing 76 (2011) 59–73.
URL http://www.combinatorialmath.ca/jcmcc/jcmcc76.html

[64] R. A. Melter, I. Tomescu, Metric bases in digital geometry, Computer Vision, Graphics, and Image Processing 25 (1) (1984) 113–121.
URL http://www.sciencedirect.com/science/article/pii/0734189X84900513

[65] N. Mladenović, J. Kratica, V. Kovačević-Vujčić, M. Čangalović, Variable neighborhood search for the strong metric dimension problem, Electronic Notes in Discrete Mathematics 39 (2012), 51–57.
URL http://dx.doi.org/10.1016/j.endm.2012.10.008

[66] R. J. Nowakowski, D. F. Rall, Associative graph products and their independence, domination and coloring numbers, Discussiones Mathematicae Graph Theory 16 (1996) 53–79.
URL http://www.discuss.wmie.uz.zgora.pl/php/discuss3.php?ip=&url=pdf&nIdA=3544&nIdSesji=-1

[67] O. R. Oellermann, J. Peters-Fransen, The strong metric dimension of graphs and digraphs, Discrete Applied Mathematics 155 (3) (2007) 356–364.
URL http://www.sciencedirect.com/science/article/pii/S0166218X06003015

[68] C. Poisson, P. Zhang, The metric dimension of unicyclic graphs, Journal of Combinatorial Mathematics and Combinatorial Computing 40 (2002) 17–32.

[69] J. A. Rodríguez-Velázquez, G. A. Barragán-Ramírez, C. García Gómez, On the local metric dimension of corona product graphs, Bulletin of the Malaysian Mathematical Sciences Society (2015) 1–17.
URL http://dx.doi.org/10.1007/s40840-015-0283-1

[70] J. A. Rodríguez-Velázquez, C. García Gómez, G. A. Barragán-Ramírez, Computing the local metric dimension of a graph from the local metric dimension of primary subgraphs, International Journal of Computer Mathematics 92 (4) (2015) 686–693.
URL http://dx.doi.org/10.1080/00207160.2014.918608

[71] J. A. Rodríguez-Velázquez, D. Kuziak, I. G. Yero, J. M. Sigarreta, The metric dimension of strong product graphs., Carpathian Journal of Mathematics 31 (2) (2015) 261–268.

[72] J. A. Rodríguez-Velázquez, I. G. Yero, D. Kuziak, The partition dimension of corona product graphs, Ars Combinatoria. To appear.

[73] J. A. Rodríguez-Velázquez, I. G. Yero, D. Kuziak, O. R. Oellermann, On the strong metric dimension of cartesian and direct products of graphs, Discrete Mathematics 335 (0) (2014) 8 – 19.
URL http://www.sciencedirect.com/science/article/pii/S0012365X14002507

[74] V. Saenpholphat, P. Zhang, Connected resolvability of graphs, Czechoslovak Mathematical Journal 53 (4) (2003) 827–840.
URL http://dx.doi.org/10.1023/B%3ACMAJ.0000024524.43125.cd

[75] V. Saenpholphat, P. Zhang, Connected resolving sets in graphs, Ars Combinatoria 68 (2003) 3–16.
URL http://www.combinatorialmath.ca/arscombinatoria/vol68.html

[76] A. Sebö, E. Tannier, On metric generators of graphs, Mathematics of Operations Research 29 (2) (2004) 383–393.
URL http://dx.doi.org/10.1287/moor.1030.0070

[77] B. Shanmukha, B. Sooryanarayana, K. Harinath, Metric dimension of wheels, Far East Journal of Applied Mathematics 8 (3) (2002) 217–229.
URL http://www.pphmj.com/abstract/1365.htm

[78] P. J. Slater, Leaves of trees, Congr. Numerantium 14 (1975) 549–559.

[79] P. J. Slater, Domination and location in acyclic graphs, Networks 17 (1) (1987) 55–64.
URL http://dx.doi.org/10.1002/net.3230170105

[80] P. J. Slater, Dominating and reference sets in a graph, Journal of Mathematical and Physical Sciences 22 (4) (1988) 445–455.
URL `http://www.ams.org/mathscinet-getitem?mr=0966610`

[81] I. Tomescu, Discrepancies between metric dimension and partition dimension of a connected graph, Discrete Applied Mathematics 308 (22) (2008) 5026–5031.
URL `http://www.sciencedirect.com/science/article/pii/S0012365X07007200`

[82] D. B. West, Introduction to graph theory, Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

[83] C. Yang, J.-M. Xu, Connectivity of lexicographic product and direct product of graphs, Ars Combinatoria 111 (2013) 3–12.
URL `http://staff.ustc.edu.cn/~xujm/x126-01.pdf`

[84] Z. Yarahmadi, A. R. Ashrafi, The Szeged, vertex PI, first and second Zagreb indices of corona product of graphs, Filomat 26 (3) (2012) 467–472.
URL `http://www.doiserbia.nb.rs/img/doi/0354-5180/2012/0354-51801203467Y.pdf`

[85] I. G. Yero, On the strong partition dimension of graphs, The Electronic Journal of Combinatorics 21 (3) (2014) #P3.14.

[86] I. G. Yero, D. Kuziak, J. A. Rodríguez-Velázquez, On the metric dimension of corona product graphs, Computers & Mathematics with Applications 61 (9) (2011) 2793–2798.
URL `http://www.sciencedirect.com/science/article/pii/S0898122111002094`

[87] I. G. Yero, J. A. Rodríguez-Velázquez, On the Randić index of corona product graphs, ISRN Discrete Mathematics 2011 (2011) 7, article ID 262183.
URL `http://www.hindawi.com/isrn/dm/2011/262183/`

[88] E. Yi, On strong metric dimension of graphs and their complements, Acta Mathematica Sinica (English Series) 29 (8) (2013) 1479–1492.
URL `http://dx.doi.org/10.1007/s10114-013-2365-z`

Symbol List

The symbols are arranged in the order of the first appearance in the work. Page numbers refer to definitions.

G	simple graph, 7
$V(G)$	set of vertices of G, 7
$E(G)$	set of edges of G, 7
n	order of a graph, 7
$u \sim v$	vertex u is adjacent to v, 7
$u \nsim v$	vertex u is not adjacent to v, 7
$G \cong H$	graphs G and H are isomorphic, 7
$N_G(v)$	open neighbourhood of a vertex v in G, 7
$N_G[v]$	closed neighbourhood of a vertex v in G, 7
$N_G(S)$	open neighbourhood of a subset of $V(G)$, 7
$N_G[S]$	closed neighbourhood of a subset of $V(G)$, 7
$\gamma(G)$	domination number of G, 7
$\delta_G(v)$	degree of a vertex v of G, 7
$N_S(v)$	open neighborhood of a vertex v in the set S, 7
$N_S[v]$	closed neighborhood of a vertex v in the set S, 7
$\delta(G)$	minimum degree of the graph G, 7
$\Delta(G)$	maximum degree of the graph G, 7
$\mathsf{g}(G)$	girth of the graph G, 7
K_n	complete graph of order n, 7
C_n	cycle of order n, 7
P_n	path of order n, 7
N_n	empty graph of order n, 7
$K_{s,t}$	complete bipartite graph of order $s+t$, 7

$K_{1,n}$	star of order $n+1$, 7
T	tree, 7
$\Omega(T)$	set of leaves in the tree T, 7
$d_G(u,v)$	distance between two vertices u and v in G, 8
$D(G)$	diameter of the graph G, 8
G^c	complement of the graph G, 8
$\langle X\rangle_G$	subgraph of G induced by the set X, 8
$\sigma(G)$	set of simplicial vertices of G, 8
$\omega(G)$	clique number of G, 8
$\varpi(G)$	twins-free clique number of G, 8
$\alpha(G)$	independence number of G, 8
$G\Box H$	Cartesian product of two graphs G and H, 8
Q_r	hypercube of order 2^r, 8
d	metric, 9
(X,d)	metric space, 9
$\dim(G)$	metric dimension of G, 10
$\dim_A(G)$	adjacency dimension of G, 10
$d_{G,t}(u,v)$	distance between two vertices u and v in G, bounded by t, 10
$\dim_s(G)$	strong metric dimension of G, 11
$\beta(G)$	vertex cover number of G, 12
$M_G(v)$	set of vertices of G which are maximally distant from v, 12
$\partial(G)$	boundary of the graph G, 12
G_{SR}	strong resolving graph of G, 13
$\mathcal{G}$	graph family on a common vertex set, 15
$\mathrm{Sd}(\mathcal{G})$	simultaneous metric dimension of $\mathcal{G}$, 15
$\mathcal{I}(G)$	set of interior vertices of G, 21
$ter_G(v)$	terminal degree of v in G, 21
$TER_G(v)$	set of terminal vertices of v in G, 21
$\mathcal{M}(G)$	set of exterior major vertices of G, 21
$\mathcal{S}(B)$	stabilizer of B, 30
$\mathcal{G}_B(G)$	family associated to G having B as a simultaneous metric generator, 31
$G+H$	join graph of two graphs G and H, 35

$G \circ H$	lexicographic product of two graphs G and H, 36
$G \odot H$	corona product of two graphs G and H, 36
$\mathrm{Sd}_A(\mathcal{G})$	simultaneous adjacency dimension of $\mathcal{G}$, 37
$\mathcal{K}(V)$	family of star graphs on the common vertex set V, composed by $\lvert V\rvert$ graphs having each a different center, 39
$\mathcal{G}^c$	family composed by the complements of the graphs in $\mathcal{G}$, 39
$\langle B_G \rangle_w$	subgraph of G weakly induced by B, 40
$\widetilde{\mathcal{G}}_B(G)$	family associated to G having B as a simultaneous adjacency generator, 41
$\mathcal{G} + \mathcal{H}$	family composed by join graphs, 46
x^*	equivalence class of x by the twins equivalence relation, 51
$\mathcal{G} \circ \mathcal{H}$	family composed by lexicographic product graphs, 52
$\mathcal{G} \odot \mathcal{H}$	family composed by corona product graphs, 68
$\mathrm{S}\,\gamma(\mathcal{G})$	simultaneous domination number of $\mathcal{G}$, 75
$\mathrm{Sd}_s(\mathcal{G})$	simultaneous strong metric dimension of $\mathcal{G}$, 83
$\mathrm{S}\varpi(\mathcal{G})$	simultaneous twins-free clique number of $\mathcal{G}$, 87
$\beta_s(G)$	strong resolving number of G, 90
θ_S	resolvability threshold of a permutation of $V(\mathcal{G})$, 102
$\mathrm{Sd}^*(\mathcal{G})$	estimate of $\mathrm{Sd}(\mathcal{G})$, 112
$\epsilon(\mathrm{Sd}^*(\mathcal{G}))$	relative error of $\mathrm{Sd}^*(\mathcal{G})$ with respect to $\mathrm{Sd}(\mathcal{G})$, 112
$\mathrm{Sd}^*_A(\mathcal{G})$	estimate of $\mathrm{Sd}_A(\mathcal{G})$, 112
$\epsilon(\mathrm{Sd}^*_A(\mathcal{G}))$	relative error of $\mathrm{Sd}^*_A(\mathcal{G})$ with respect to $\mathrm{Sd}_A(\mathcal{G})$, 112
$\mathrm{Sd}^*_s(\mathcal{G})$	estimate of $\mathrm{Sd}_s(\mathcal{G})$, 112
$\epsilon(\mathrm{Sd}^*_s(\mathcal{G}))$	relative error of $\mathrm{Sd}^*_s(\mathcal{G})$ with respect to $\mathrm{Sd}_s(\mathcal{G})$, 112

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