Essays on Banking, Financial Fragility and CEO Compensation

Zhao Li

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DIRECTOR DE LA TESI

Prof. Xavier Freixas

Departament Departament d'Economia i Empresa

Universitat Pompeu Fabra Barcelona

This thesis is dedicated to Yunyi and Muyin.

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Abstract

This dissertation investigates two issues: (1) banks' funding liquidity risk and its implications, and (2) the optimal design of CEO compensation contracts. The first chapter analyzes the interaction between banks' funding liquidity risk and asset market illiquidity. We emphasize how the lack of information distorts asset prices. In the model, asset fire sales, bank runs, and financial contagion, are all self-fulfilling, and emerge in a rational expectations equilibrium. The model also delivers new policy insights on bank capital holding and asset purchase programs. The second chapter also relates bank liquidity issues. It emphasizes how credit information sharing schemes can contribute to asset market liquidity, and therefore provides a novel exposition for the existence of such schemes. The last chapter is devoted to the design of CEO compensation contracts, where the coexistence of stock and option grants is rationalized as an optimal contract in a multiplicative model.

Resum

Aquesta tesi investiga dues qüestions: (1) la del risc de liquiditat dels bancs i les seves implicacions, i (2) el disseny òptim dels contractes de compensació CEO. En el primer capítol s'analitza la interacció entre el risc de liquiditat de finançament dels bancs i la falta de liquiditat del mercat d'actius. El model fa èmfasi en que la manca d'informació distorsiona els preus dels actius. Per aquesta raó, les vendes forçades d'actius, les corregudes bancàries, i el contagi financer, són formes de profecies autosatisfetes, i resulten en un equilibri d'expectatives racionals. El model també ofereix noves perspectives de política a la bateria de programes de compra d'actius i de regulació del capital. El segon capítol es refereix també a problemes de liquiditat del banc. Es posa l'accent en com els sistemes d'intercanvi d'informació creditícia poden contribuir a la liquiditat del mercat d'actius, i, per tant, proporciona una nova justificació de l'existència d'aquests intercanvis d'informació. L'últim capítol es dedica al disseny de contractes de compensació del CEO, on la coexistència d'accions i opcions subvencions es racionalitza com un contracte òptim en un model multiplicador.

Preface

The recent banking crisis highlights the risk of illiquidity. On the one hand, market liquidity evaporated and asset prices dropped sharply. On the other hand, as funding liquidity dried up, even well capitalized banks found it difficult to rollover their short-term debts and had to resort to central banks. The first and second chapters of this dissertation, while using different modeling frameworks and dealing with different topics, have a common emphasis on the issue of bank liquidity.

In the first chapter, I analyze the interaction between market liquidity and funding liquidity and its policy implications. It is a joint paper with Kebin Ma, a very close friend of mine. In a global-games framework, we endogenize bank asset fire sales by emphasizing asymmetric information: asset prices collapse because in banking crises, assets sold by illiquid banks can hardly be distinguished from those by insolvent banks. The lack of information makes runs and fire sales self-fulfilling and mutually reinforcing; it also generates financial contagion when banks have common risk exposures. The theoretical framework delivers several policy insights. (1) High capital holding can have unintended consequences on bank liquidity, because a run on a well-capitalized bank signals unusually high risk and exacerbates fire sales. (2) A regulator can improve financial stability by purchasing assets at a committed price. Such intervention resemebles an asset purchase program and can break down the vicious cycle fueled by beliefs. Finally, (3) regulatory transparency involves a trade-off: while a favorable disclosure saves banks from illiquidity, acknowledging a crisis aggravates financial instability.

While treating bank's funding liquidity risk as exogenous, the second chapter proposes a novel rationale for the existence of bank information sharing schemes. We suggest that banks can voluntarily disclose borrowers' credit history in order to maintain asset market liquidity. By entering an information sharing scheme, banks will face less adverse selection when selling their loans in secondary markets. This reduces the cost of asset liquidation in case of liquidity shocks. The benefit, however, has to be weighed against higher competition and lower profitability in prime loan markets. Information sharing can arise endogenously as banks tradeoff between asset liquidity and rent extraction. Different from the literature, we allow for non-verifiable credit history of borrowers', and show that banks still have incentives to truthfully disclose such information in competitive credit markets.

The last chapter is devoted to the design of optimal CEO compensation contracts. I offer a model to contribute to the open debate about whether stock options should be a part of the optimal CEO compensation contracts. The insights delivered in this model can shed some light on the discussion of bank CEO risk taking and compensation regulation, even if the modeling framework is not specific to banking.

In the third chapter, I analyze the design of compensation contracts to motivate a risk neutral CEO's effort of developing risky project opportunity, then to induce his best project choice for the firm. Restricted stock induces the CEO to select the project that maximizes the firm's expected value while stock options are superior in motivating the managerial effort. When the risky project has sufficient "upside" value than the firm's existing safe project, it is optimal to pay the CEO solely in restricted stock. Otherwise, the firm faces a trade-off between motivating the CEO's effort and mitigating his excessive risk taking. The second best contract is a combination of restricted stock and stock options. I extend the model to consider a competitive CEO market and find out that there could be circumstances where larger firms hire lower ability CEOs in the market equilibrium.

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Chapter 1

Self-fulfilling Fire-sales, Bank Runs and Contagion: Implications for Bank Capital and Regulatory Transparency

1.1 Introduction

The recent banking crisis highlights the risk of illiquidity. On the one hand, market liquidity evaporated and asset prices dropped sharply. On the other hand, as funding liquidity dried up, even well capitalized banks found it difficult to rollover their short-term debts and had to resort to central banks.

The two types of illiquidity closely link to each other. First, it has been well acknowledged that market illiquidity contributes to funding illiquidity. As market liquidity diminishes, potential fire-sale losses from early liquidation make creditors panic. Creditors can have coordination failures in rolling over their short-term debts and thus deprive a healthy financial institution of its funding. A bank can be solvent but illiquid: being able to repay in full its debts if no run happens, but being liquidated early if its creditors do not roll over their debts. The point has been emphasized by work like (Morris and Shin, 2000), (Rochet and Vives, 2004) and (Goldstein and Pauzner, 2005). However, the literature has ignored the feedback from bank runs to asset prices, treating separately two interconnected issues that in our opinion should be integrated.

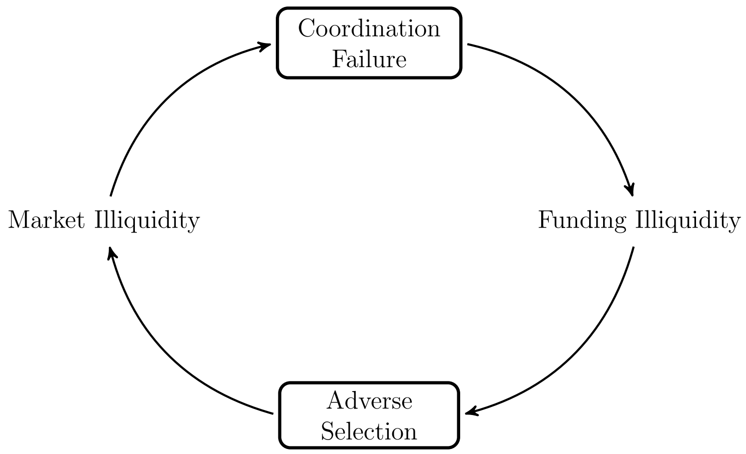
Indeed, funding illiquidity also feeds market illiquidity. Bank runs can lead to fire-sales, depress asset prices, and in extreme cases, freeze up markets. As narrated by (Acharya and Roubini, 2009):

> *...the collapses on June 20, 2007, of two highly levered Bear Stearns-managed hedge funds that invested in subprime asset-backed securities (ABSs)...as the prices of collateralized debt obligations (CDOs) began to fall...lenders to the funds demanded more collateral...Merrill Lynch, seized $800 million of their assets and tried to auction them off. When only $100 million worth could be sold, the illiquid nature and declining value of the assets became quite evident.*

The mutual reinforcement between market and funding illiquidity with the emergence of "liquidity spirals", is first outlined by (Brunnermeier and Pedersen, 2009). Our model provides new insights into this issue by emphasizing coordination failures among creditors and asymmetric information in the secondary asset market.

We present a theoretical framework where asset fire-sales and bank runs/contagion happen in a self-fulfilling manner. When buyers of a bank's assets are uninformed of a bank's asset quality, observing a run will imply low asset values from buyers' perspective. As the uninformed buyers cannot distinguish assets sold by solvent-but-illiquid banks from those by insolvent ones, such adverse selection will distort downwards their willingness to pay. As a result, a solvent bank will not recoup a fair value for its assets on sale. The friction leads to a vicious circle. First, low asset prices fuel self-fulfilling bank runs: To avoid fire-sale losses caused by other creditors' early withdrawals, a creditor has the incentive to withdraw funds from a solvent bank. Strategic complementarities can create successful runs and illiquid banks. Second, fire sales are self-fulfilling too. Out of the fear of low fire-sale prices, creditors run on a solvent bank and force early liquidation. Yet it is the run and liquidation, by pooling the solvent with the insolvent, that leads to the low fire-sale prices in the first place. In this sense, the creditors' pessimistic expectation realizes itself. The self-fulfilling bank runs and fire sales intertwine and feed back into each other. Driven by the adverse selection, the whole banking crisis of “twin” illiquidity rises as a self-fulfilling prophecy. (See Figure 1.1 for an illustration.)

Figure 1.1: A banking crisis of “twin”illiquidity



For the financial system, contagion happens in a similar self-fulfilling manner except with one more ingredient—a common risk factor. As the uninformed buyers form rational expectations, they revise their expectation downwards of the common risk factor upon observing a bank run. The reduced expectation lowers their willingness to pay for other banks' assets, which in turn precipitates runs in all other banks. It should be noted that such contagion (the run to other banks) again reflects the mutually reinforcing interaction between fire-sales and runs. Anticipating the declining asset prices due to buyers' lower expectation of common risk factor, the creditors of other banks panic and run, and the run confirms the worsening expectation and leads to further distressed asset prices.

As a defining feature that distinguishes the current model from the literature, we have buyers' beliefs, asset prices, bank runs, and contagion, all endogenous and jointly determined in a rational expectation equilibrium. We prove that the equilibrium exists and is unique. These features of our model allow us to deliver several policy insights. In particular, we show that increasing bank capital and regulatory transparency can have unintended consequences, and refore challenge some conventional wisdom.

First, while our paper confirms that well capitalized banks have larger buffers against fire-sale losses, our analysis also reveals that once asset prices are endogenous, the situation is more complex and increasing capital also has unintended consequences on illiquidity and total credit risk. In particular, increasing bank capital can negatively affect asset prices via buyers' beliefs. For an individual bank, buyers' posterior beliefs on the bank's asset value deteriorate when a run happens. And the deterioration is particularly strong when the bank maintains a high capital ratio. Because well capitalized banks are able to sustain large losses, if a run happens to such a bank, the bank's losses must be unusually high. Therefore, given that a bank faces a run, buyers' valuation of its assets decreases in its capital level. The low willingness to pay contributes to creditors' coordination failure and makes the run more likely to happen in the first place. We show that in some extreme cases, increasing bank capital cannot reduce the risk of bank runs at all.

Second, our theoretical model confirms the effectiveness of asset purchase programs in promoting financial stability. In an asset purchase program where a regulator purchases bank assets at a committed price, the vicious cycle fueled by beliefs can be broken down. We argue that regulators hold more commitment power than other market participants, and the lack of commitment in ordinary asset buyers is at the very root of financial instability in this model. In particular, an ordinary asset buyer would behave according to her rational beliefs, and would avoid losses in every realized state. This can generate the vicious cycle discussed above because the buyer's pessimistic belief can lead to negative market outcomes (e.g., more bank runs) which in turn justify itself. A regulator with commitment power, on the other hand, can resist such pessimistic belief updating. We show that even if the regulator has no better information than ordinary asset buyers, he can still break even and promote financial stability.

Finally, the information-based run and contagion links to the debate on regulatory transparency.[1] Our paper considers such transparency to be a double-edged sword. If the disclosed information reassures the asset buyers, illiquid banks will be saved. However, if the assistance program adds to pessimistic market inference, e.g., its size greater than expected, the assistance program itself will be contagious. Once the severity of the problem is acknowledged, market participants further revise down the expected performance

[1] For instance, whether or not regulators should disclose information concerning their assistance programs.

of all financial institutions', leading to greater fire-sale losses and triggering illiquidity, even for healthy institutions. The fall of Bear Stearns was an interesting case in this aspect. As documented in (Brunnermeier, 2009):

> *...March 11, 2008, when the Federal Reserve announced its $200 billion Term Securities Lending Facility. ... However, some market participants might have (mistakenly) interpreted this move as a sign that the Fed knew that some investment bank might be in difficulty. Naturally, they pointed to the smallest, most leveraged investment bank with large mortgage exposure: Bear Stearns.*

It was unclear whether Bear Sterns was truly insolvent or not. Yet because market participants believed the Fed was better informed and the action of Fed reflected that superior information, the attack began.

Our theoretical framework is related to the literature on bank runs and financial contagion. Since (Diamond and Dybvig, 1983) the literature is concerned with the financial fragility caused by runs.[2] Following their seminal contribution there was a debate as to whether bank runs are due to pure panic or unfavorable information on banks' fundamentals.[3] The gap between the panic and fundamental view is bridged by the application of global games. Using the concept, papers such as (Morris and Shin, 2000), (Rochet and Vives, 2004) and (Goldstein and Pauzner, 2005) refine the multiple equilibria in (Diamond and Dybvig, 1983) and emphasize the role of early liquidation loss in causing bank runs: An extra buffer of cash flow is needed to reassure creditors and to prevent runs. Weak banks that fail to provide the extra buffer become "solvent but illiquid". A limitation of the existing models is that they build on the simplifying assumption of exogenous fire-sale losses,[4] so that the models ignore the reinforcing effects of runs on fire-sales. In contrast, the current paper explores the relationship: As it is difficult to

[2]It should be mentioned that some papers also consider the positive role of bank run as disciplinary device: (Calomiris and Kahn, 1991) and (Diamond and Rajan, 2001).

[3]The papers emphasizing banks' weak fundamentals in causing runs are (Chari and Jagannathan, 1988), (Jacklin and Bhattacharya, 1988) and (Allen and Gale, 1998). (Friedman and Schwartz, 1963) provides empirical support for the panic view. Contradicting evidence in favor of the fundamental view is present in (Gorton, 1988), (Calomiris and Gorton, 1991) and (Calomiris and Mason, 2003).

[4]For example, (Rochet and Vives, 2004) assumes an exogenous fire-sale discount and (Morris and Shin, 2009) assumes exogenous haircut.

distinguish the illiquid banks from those insolvent ones, the adverse selection causes the low asset prices and fire-sale losses.[5]

A natural corollary of assuming an exogenous fire-sale price is that funding liquidity risk will be always reduced by higher capital, because the returns generated on capital add to the buffer against fire-sale losses. With endogenous fire-sale prices the current paper takes a broader view: while acknowledging the buffer effect of capital, we point out that greater capital can also contribute to illiquidity via buyers' pessimistic inference.

Predicting an interaction between market liquidity and funding liquidity, our model is most closely related to (Brunnermeier and Pedersen, 2009), who emphasize a haircut constraint on a speculator that supplies liquidity to a financial market with limited participation. In their model, asset prices are volatile because there is an asynchronization between selling and buying. This paper differs from theirs in two aspects. First, the funding liquity risk rises as a result of equilibrium bank runs caused by the wholesale creditors' coordination failures. Second, this paper emphasizes the asymmetric information on asset qualities, and how such adverse selection causes asset illiquidity.

In our paper, contagion is generated not only by the actual realization of common risk factor but also by its perception: A bank failure casts shadow on the perceived common risk factor; and the created negative informational externalities affect all the other banks. This observation is mostly related to the literature of information contagion, as exemplified by (Acharya and Thakor, 2011) and (Oh, 2012).[6] Compared to the existing work, the current paper emphasizes the self-fulfilling nature of such contagion and the two-way feedback between runs and fire-sales.

On the application to capital requirements, the paper relates to a few papers that show increased capital requirements can increase bank risk. (Martinez-Miera, 2009) argues that equity increases banks' cost of funding, which leads to higher loan rates and spurs

[5]While the current paper justifies the low asset price by informational frictions, low asset prices can also be explained by fixed short-term cash supply—the cash-in-the-market argument pioneered by (Shleifer and Vishny, 1992) and (Allen and Gale, 1994).

[6]There are other approaches to model contagion. For instance, (Freixas and Parigi, 1998) and (Freixas, Parigi, and Rochet, 2000) model the direct linkages of banks through payment system. (Allen and Gale, 2000) emphasizes the role of interbank market. (Gorton, 1988) studies banks' common risk exposures directly contribute to systemic risk.

risk-taking by borrowers. As a result, banks' portfolio risk rises passively. (Hakenes and Schnabel, 2007) argue that a higher capital requirement erodes charter value and induces banks' active risk taking; when the higher capital requirement decreases credit supply, it also leads to borrower risk-taking via a hike in loan rate. What all these papers have in common is that they all focus on solvency risk. To the best of our knowledge, the current study is the first to show capital can contribute to illiquidity, contagion and systemic risk.

The discussion on disclosure policy is most related to several recent papers on the instability consequences of public signals: (Morrison and White, 2010) is concerned that a public bailout can reveal regulatory deficiency and make market participants lose their confidence in all other banks under the same regulation. (Dang, Gorton, and Holmström, 2010) shows that a public signal makes debt-like securities information sensitive, could otherwise increase adverse selection. (Wang, 2013) empirically documents that after individual banks were identified in Trouble Asset Relief Program (TARP), bank run probabilities, as reflected in CDS spread and stock market abnormal returns, rose dramatically, an outcome the author attributes to the bad news nature of public bailout. Our paper abstracts from specific policy announcements and shows that as long as market participants believe the regulator is better informed, any regulatory action and announcement concerning banks' common risk exposure may generate financial contagion.

The paper proceeds as follows. Section 2 lays out the model. Section 3 presents the baseline bank-run model under asset market adverse selection and endogenous fire-sale price. With only one bank and one state, the baseline model allows us to discuss the first policy issue that whether higher capital can lead to greater illiquidity risk and total credit risk. In Section 4, we analyze contagion in the full fledged model with two banks and two states. We are able to address the second policy issue that whether regulators should disclose information on aggregate states. In Section 5, we discuss briefly the implications for other related policy issues such as liquidity requirements and lender of last resort policies. Section 6 concludes.

1.2 Model setup

We consider a three-date ($t = 0, 1, 2$) economy with two banks.[7] At $t = 0$, banks are identical. Each of them holds a unit portfolio of long-term assets, and finances them with equity E, retail deposits F, and short-term wholesale debts $1-E-F$. There are two groups of active players: banks' wholesale creditors and uninformed buyers of banks' assets. Both groups of players are risk neutral. We assume that retail deposits are fully insured so that depositors act only passively. Since their claims are risk free, the depositors will always hold their claims to maturity, and demand only a gross risk-free rate which we normalize to 1. We also assume that the financial safety net is provided to banks free of charge. We consider banks as contractual arrangements among claim holders, designed to fulfil the function of liquidity and maturity transformation.[8] Therefore, banks in our model are passive, with given loan portfolios and liability structures.

Banks' wholesale debts are risky, demandable, and raised from a continuum of creditors. Provided that a bank does not fail, a wholesale debt contract promises a gross interest rate $r_D > 1$ at $t = 2$, and qr_D if a wholesale creditor withdraws early at $t = 1$. Here $q < 1$ reflects the penalty for the early withdrawal. A bank run occurs if a positive mass of wholesale creditors withdraw funds from their bank at $t = 1$. For the ease of future exposition, we denote by D_1 the total amount of debts a bank needs to repay at $t = 1$ if *all* wholesale creditors withdraw early, and by D_2 the total amount of debts a bank needs to repay at $t = 2$ if *no* wholesale creditor withdraws early.

$$D_1 \equiv (1 - E - F)qr_D$$

$$D_2 \equiv (1 - E - F)r_D + F$$

A bank's portfolio generates a random cash flow $\tilde{\theta}$ at $t = 2$. For simplicity, we assume that $\tilde{\theta}$ follows a uniform distribution on $[\underline{\theta}_s, \overline{\theta}]$, and the random cash flows of the two banks are independent and identically distributed. Subscript s denotes the realization of an aggregate state that affects both banks. There are two possible states, G and B

[7]It should be emphasized that all results of the current paper can be generalized to a N-bank case.
[8]This view can be traced back to (Diamond and Dybvig, 1983).

(e.g., housing market boom or bust), and the two states occur with an equal probability. With $\underline{\theta}_G > \underline{\theta}_B$, State G is more favorable than State B. Therefore, the value of a bank's assets is not only affected by its idiosyncratic risk (the realizations of $\tilde{\theta}$) but also by the aggregate risk s. On the other hand, $\overline{\theta}$ is assumed to be the same across states. This reflects the fact that banks hold mostly debt claims whose highest payoffs are capped by their face values. We further make the following three assumptions on parameters.

$$D_2 > \underline{\theta}_s \tag{1.1}$$

$$\left(\underline{\theta}_B + \overline{\theta}\right)/2 > D_2 \tag{1.2}$$

$$F > D_1 \tag{1.3}$$

As D_2 denotes a bank's total debt obligation at $t = 2$, inequality (1.1) states that there is a positive probability of bankruptcy in both states. Inequality (1.2) states that, in the absence of bankruptcy cost, even if the realization of the state is unfavorable, the expected cash flow of a bank's asset is still greater than its debt obligations, so that bank lending is viable. Finally, inequality (1.3) states that a bank's retail debts exceed its wholesale debts, which is a realistic scenario and helps to simplify the analysis of bank run games.[9] We also assume that bankruptcy costs are sufficiently high such that once a bank fails, its residual value drops to zero.

Banks' assets are long-term, taking two periods to mature. In particular, we assume that at $t = 1$ the assets cannot be physically liquidated. Therefore, if a wholesale run happens, to meet the liquidity demand, a bank has to financially liquidate its assets in a secondary asset market, and sell them to outside asset buyers. As early liquidation is costly in this model, a bank will sell its assets if and only if it faces a bank run.

1.2.1 Secondary asset market

Potential buyers in the secondary asset market are uninformed: they are unable to observe either the aggregate state s or any bank's cash flow θ. Yet, they can observe the

[9] It should be emphasized that the condition is more than a technical assumption. It is realistic in the sense that despite of the rapid growth of wholesale funding, most of commercial banks and bank holding companies are still financed more by retail deposits than wholesale debts.

number of bank runs, and based on the observable outcome, form rational expectations about the quality of assets on sale. In this two-bank setup, there are three distinctive outcomes from the buyers' perspective, i.e., the number of bank runs $N = 0$, 1, or 2.

We assume the following sequential moves between asset buyers and wholesale creditors. Asset buyers first post a price scheme $\mathbb{P} = (P_1 \ P_2)$, and offer to purchase bank assets on sale at price P_1 when the number of bank runs $N = 1$, and P_2 when $N = 2$. Having observed the price scheme, wholesale creditors play a bank run game, making their individual decisions simultaneously on whether to withdraw their funds early. In case that any bank run happens, transactions take place at the offered price, and assets are transferred to buyers.

The price scheme $\mathbb{P}$ is complete in the sense that an asset price is specified for each distinctive outcome of bank run games where bank assets are on sale. Depending on the number of runs observed, the prices that buyers offer can differ. In fact, in the absence of commitment power, the asset buyers' decisions need to be time consistent so that they will not revoke their posted price after the outcomes of bank run games are revealed. As a result, the price P_1 and P_2 will have to reflect buyers' posterior beliefs on asset qualities. As buyers form different posterior beliefs when observing different numbers of bank runs, their offered prices will vary with the number of bank runs.

The asset market is assumed to be perfectly competitive, and the buyers compete in the price schemes that they offer. In equilibrium, based on their posterior beliefs, the asset buyers should perceive themselves breaking even in expectation when purchasing bank assets at their posted prices. As the buyers make time-consistent decisions and do not revoke their offers, they must make no loss for any realized number of bank runs.

1.2.2 Bank run game

The demandable nature of wholesale debts allows creditors to withdraw their funds before a bank's assets mature, which will force the bank to liquidate its assets prematurely. When assets are sold for less than their fundamental values, there will be an early liquidation loss, or an asset fire sale. While the creditors who withdraw early can avoid suffering from the fire sale, those who do not withdraw will receive zero payoffs if the bank fails. As a result, creditors' actions to withdraw display strategic complementarities, and it can be in the interest of all creditors to run on a bank that is otherwise solvent.

A bank run game of complete information can have two strict equilibria that all creditors withdraw from the bank, and that nobody withdraws. To refine the equilibria, we take the global-games approach pioneered by (Carlsson and Van Damme, 1993) and study games with incomplete information, where common knowledge on θ does not exist among creditors. We assume that at the beginning of $t = 1$, both aggregate risk (State s) and idiosyncratic risk (cash flow θ) have been realized, but the information is not fully revealed to players. For a given bank, each individual creditor only privately observes a noisy signal $x_i = \theta + \epsilon_i$. The noise ϵ_i is drawn from a uniform distribution with a support $[-\epsilon, \epsilon]$, where ϵ can be arbitrarily small. Based on their private signals, the creditors play a bank-run game with each other. Each of the creditors has two possible actions: to wait until maturity or to withdraw early, and follows a threshold strategy: to withdraw early if and only if their individual private signal is lower than a critical level $\hat{x}$. In this two-bank setup, we also assume that each creditor holds claims in both banks, and observes independent noisy signals for both banks' cash flows.

The maturity mismatch between banks' liabilities and assets, together with potential asset fire sales, exposes banks to the risk of runs. In particular, a run and premature liquidation at $t = 1$ can cause failure to a bank that is otherwise solvent at $t = 2$. In order to reassure its creditors not to withdraw early, a bank has to be more than merely solvent, and should be able to absorb potential fire-sale losses. This implies a critical cash flow $\hat{\theta} > D_2$ for a bank to survive a run. The distance between $\hat{\theta}$ and D_2 provides a measure of financial instability. Moreover, a lower asset price implies greater fire-sale losses, and a higher critical cash flow $\hat{\theta}$ for a bank to survive a run.

Given our assumption that bankruptcy costs result in zero residual value, if a bank is to fail at $t = 1$, a wholesale creditor will receive zero payoff whether he withdraws early or not. In this case of indifference, we assume that the creditor will always withdraw. One justification can be that wholesale creditors receive arbitrarily small reputational benefits by running on a bank that is doomed to failed.[10]

1.2.3 Asymmetric information on cash flow θ

As asset buyers are intelligent, they can solve creditors' bank run game and form rational beliefs on the qualities of assets on sale. In particular, they know that a bank will be forced into an asset sale if and only if its cash flow is below $\hat{\theta}$. However, the lack of more detailed information makes solvent banks (those with $D_2 \leq \theta < \hat{\theta}$) indistinguishable from the insolvent ones (those with $\theta < D_2$). As an equilibrium asset price reflects only the average quality of assets on sale, a bank with cash flow θ greater than the price but less than $\hat{\theta}$ will face an asset fire sale.

As a lower asset price pushes $\hat{\theta}$ upwards, there will be two-way feedback between asset fire sales and bank runs. When asset buyers offer a low price for a bank's assets, a run is triggered, which generates the pooling of assets, and thus fully justifies the low asset price offered in the first place. As a result, both fire sales and bank runs occur in a self-fulfilling manner.

1.2.4 Belief updating on State s

While asset buyers hold a prior belief that State B and G occur with an equal probability, after observing any bank runs, they update their beliefs according to Bayes' rule and consider State B to be more likely. The pessimistic belief updating can lead to financial contagion. In particular, a bank may face no runs if the other bank does not face a run, but will if the other one does. This defines financial contagion in our model.

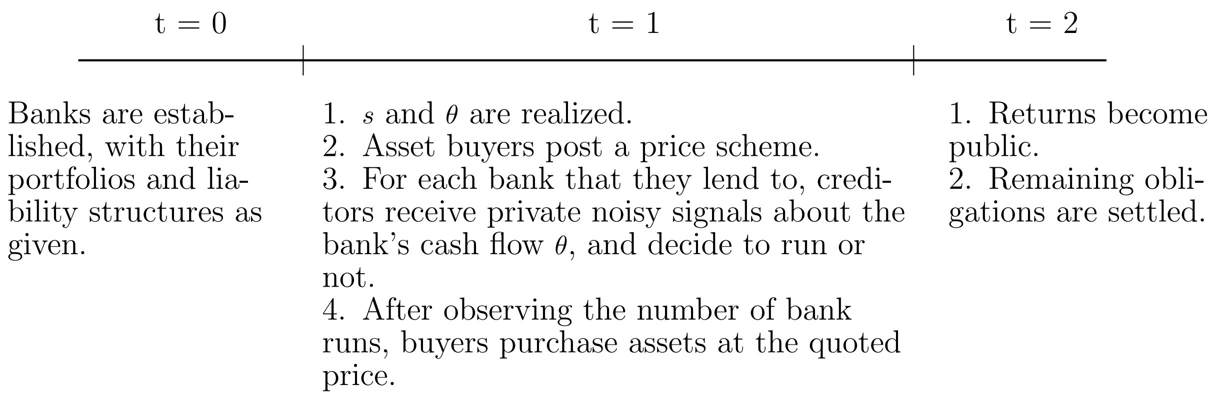
In the current model, financial contagion is self-fulfilling too. When observing more bank runs, asset buyers infer State B to be increasingly likely and reduce their offered asset prices accordingly. The fear of increased liquidation losses makes wholesale creditors panic even more, and leads to simultaneous bank runs in the first place.

[10] For more detailed discussion on this assumption, please see (Rochet and Vives, 2004).

1.2.5 Timing

The timing of the model is summarized in Figure 1.2. Events at $t = 1$ take place sequentially.

Figure 1.2: Timing of the game



1.3 Self-fulfilling bank runs and fire sales

Depending on the realization of $\tilde{\theta}$, the model can have two types of equilibria: one type with bank runs, and the other without. The market equilibrium with bank runs consists of two parts. First, the bank run games feature threshold equilibria. That is, when N runs happen and bank assets are sold for an equilibrium price P_N^*, a bank will experience a run if and only if the bank's cash flow is lower than a unique threshold $\theta_N^* \equiv \hat{\theta}(P_N^*)$, $N \in \{1, 2\}$. Second, the competitive asset market is in a rational expectations equilibrium. That is, asset buyers form a rational belief about the quality of assets on sale based on the number of bank runs N. In particular, they anticipate $\theta < \hat{\theta}(P_N^*)$, and Bayesian update their beliefs on State s. According to such posterior beliefs, asset buyers who purchase bank assets at an equilibrium price P_N^* should perceive themselves breaking even in expectation. Moreover, the buyers should find themselves unable to profitably deviate from bidding P_N^*.

Definition. *Denote $\hat{\theta}(P_N)$ the threshold equilibrium of the bank run game for a given* *asset price* P_N*; and* $P_N\left(\hat{\theta}\right)$ *the price scheme by which asset buyers break even in expectation for a given* $\hat{\theta}$ *and their rational beliefs about* θ *and* s*. The equilibrium of the model is defined by equilibrium critical cash flows* $\theta_N^* \equiv \hat{\theta}\left(P_N^*\right)$*,* $N \in \{1, 2\}$*, and an equilibrium asset price scheme* $\mathbb{P}^* = \left(P_1^* \; P_2^*\right)$ *with* $P_N^* \equiv P_N\left(\theta_N^*\right)$*. The combination of* θ_N^* *and* P_N^* *is such that: when there are* N *bank runs in the economy, (1) a successful bank run happens if and only if the bank's cash flow is lower than* θ_N^**; (2) the competitive asset market is in a rational expectations equilibrium, where asset buyers form rational beliefs about State* s *and the quality of assets on sale. Based on their posterior beliefs, the buyers perceive themselves making zero profit in expectation by purchasing bank assets at* P_N^* *and cannot make profitable deviation.*

It takes four steps to obtain the equilibrium.

- First, we show that equilibrium asset prices P_N^* cannot be lower than D_1 or higher than D_2 (subsection 3.1). This restricts the set of candidate equilibria and will facilitate the solution of bank run games.

- Second, solving the model using backward induction, we start with creditors who move last and solve the bank run game using the concept of global games. For a *given* asset price $P_N \in (D_1, D_2)$, we derive a unique critical cash flow $\hat{\theta}\left(P_N\right)$, so that a bank run will happen if and only if the bank's cash flow $\theta < \left(P_N\right)$ (subsection 3.2).

- Third, we characterize asset buyers' posterior beliefs on asset qualities when N bank runs occur. In particular, they expect only those assets with quality $\theta < \hat{\theta}(P_N)$ to be on sale, and update their beliefs about State s using Bayes' rule. It should be emphasised that the buyers' rational beliefs are functions of asset prices that they offer (subsection 3.3).

- Finally, we solve for the equilibrium of the model by examining equilibrium asset price schemes. As asset buyers offer different prices given different numbers of bank runs, we solve for equilibrium prices P_N^* for each $N \in \{1, 2\}$. For N observed bank runs, in a competitive equilibrium, P_N^* should be equal to the expected asset quality based buyers' posterior beliefs (subsection 3.3).

To illustrate the main intuition behind the feedback between bank runs and fire sales, we present in subsection 3.4 a simplified version of the model where there is only one state so that asset buyers cannot update their beliefs on State s. This simplification allows us to derive a closed-form solution to our model, and is sufficient to generate some interesting result such as unintended liquidity consequences of bank capital. The full-fledged model with different states and asset buyers' belief updating on s is analyzed in section 4.

1.3.1 Restricting the set of candidate equilibria

For an equilibrium price cannot be negative, a candidate equilibrium price P_N^* can only fall into one of three regions, $0 \leq P_N^* \leq D_1$, $D_1 < P_N^* < D_2$, and $P_N^* \geq D_2$. We discuss the existence and uniqueness of equilibrium for each of the three regions, and show that any equilibrium price P_N^* cannot be lower than D_1, nor greater than D_2 provided that $F > D_1$.

Suppose $P_N^* \geq D_2$. Then, for any bank with $\theta \in [D_2, \overline{\theta}]$, it is suboptimal for its wholesale creditors to withdraw early. This is because with $P_N^* \geq D_2$, an asset sale at $t = 1$ will not hurt the bank's capability to repay its liabilities at either $t = 1$ or $t = 2$. As a result, by running on the bank, a creditor will only incur the penalty for early withdrawal. This implies that whenever a run happens, it must be the case that the bank is fundamentally insolvent with $\theta < D_2$. Therefore, the highest asset quality that buyers can expect is D_2, with the expected quality strictly lower than that. As asset buyers break even and pay a price equal to the expected quality, the price that the buyers are willing to pay must be strictly smaller D_2. This contradicts the presumption $P_N^* \geq D_2$.

Now, suppose $P_N^* \leq D_1$. Then, a bank with $\theta \in [\underline{\theta}_s, D_2]$ will for sure fail, either because sufficiently many creditors run at $t = 1$, or because of fundamental insolvency at $t = 2$. Under the assumption that wholesale creditors run on banks that are doomed to fail, we know that successful runs must happen to those banks with $\theta \in [\underline{\theta}_s, D_2]$. This implies that the expected quality of assets on sale is at least $(\underline{\theta}_B + D_2)/2$. As asset buyers only break even in equilibrium, the price they offer must be greater than that. Therefore, we have $P_N^* > (\underline{\theta}_B + D_2)/2 > D_2/2$. By the definitions of D_1 and D_2, we further have $D_2/2 = [(1 - E - F)r_D + F]/2 > [(1 - E - F)qr_D + F]/2 = (D_1 + F)/2$, which is in turn greater than D_1, provided $F > D_1$. Again, this contradicts the presumption $P_N^* \leq D_1$. We summarize these results in Lemma 1.

Lemma 1. *An equilibrium asset price cannot less than or equal to D_2. And an equilibrium asset price cannot be greater than or equal to D_1 either, provided $F > D_1$.*

1.3.2 Threshold equilibrium for bank run games

We solve the model by backward induction, and start with the subgame of bank runs. We show that for a *given* price $P_N \in (D_1, D_2)$ the bank run game has a unique threshold equilibrium characterized by a critical cash flow $\hat{\theta}(P_N)$. A successful bank run happens if and only if the bank's cash flow is lower than $\hat{\theta}(P_N)$.

To solve for the optimal strategy of creditors, we first derive their payoffs for action "wait" and "withdraw" as functions of the number of other creditors who withdraw from the bank. Denote by $L \in [0, 1]$ the fraction of creditors who withdraw from the bank at $t = 1$. A bank that faces a total withdrawal of LD_1 can meet the demand for liquidity with a partial liquidation by selling a f fraction of its assets.[11]

$$f = \frac{LD_1}{P_N} < 1 \tag{1.4}$$

After liquidating f fraction of its assets, the bank will fail at $t = 2$ if and only if the value of its remaining assets $(1 - f)\theta$ is lower thanits remaining liabilities $F + (1 - L)(1 - E - F)r_D$. That is,

$$(1 - f)\theta \leq F + (1 - L)(1 - E - F)r_D. \tag{1.5}$$

Thus, a bank will fail at $t = 2$ if and only if the fraction of creditors' withdrawal exceeds a threshold L^c.

$$L \geq \frac{P_N[\theta - F - (1 - E - F)r_D)]}{(q\theta - P_N)(1 - E - F)r_D} = \frac{P_N(\theta - D_2)}{[\theta - P_N/q]D_1} \equiv L^c. \tag{1.6}$$

Such a $t = 2$ failure happens because the partial early liquidation incurs a cost of fire sale. When a sufficiently large number of creditors withdraw and the bank is forced to

[11]Here $f < 1$ is guaranteed by $P_N > D_1$ and $L \leq 1$. Note that three factors contribute to a high fraction of asset liquidation: (i) a large number of early withdrawals, (ii) low market price P for assets on sale, and (iii) a high level of wholesale debts.

liquidate a significant share of assets prematurely, the remaining assets will not generate sufficient cash flows to meet the remaining liabilities. The creditors who withdraw early at $t = 1$ therefore can impose negative externalities on creditors who choose to wait.

Depending on the amount of early withdrawals L, a creditor's payoffs of playing withdraw or stay are tabulated as follows.

	$L \in [0, L^c)$	$L \in [L^c, 1]$
withdraw	qr_D	qr_D
stay	r_D	0

Note that if a creditor withdraws, his payoff will always be $W_{run}(L) = qr_D$. Instead, if he waits, his payoff depends on the action of other creditors.

$$W_{wait}(L) = \begin{cases} r_D & L \in [0, L^c] \\ 0 & L \in [L^c, 1] \end{cases}$$

Defining the difference between the creditor's payoffs of withdraw and stay as $DW(L) \equiv W_{run}(L) - W_{wait}(L)$, one has

$$DW(L) = \begin{cases} -(1-q)r_D & L \in [0, L^c] \\ qr_D & L \in [L^c, 1] \end{cases}$$

The strategic complementarity is clear: when a sufficient large number of other creditors choose to withdraw ($L > L^C$), a wholesale creditor receives better payoff is better by withdrawal than to wait. In fact, when there is complete information on θ, the bank run game has two equilibria in which either all creditors withdraw or all creditors wait. We refine the multiple equilibria using the technique of global games.

The analysis follows a standard global games approach. We give here the outline of the proof, and interested readers can refer to 1.A for full details. First, we establish the existence of a lower dominance region $[\underline{\theta}_s, \theta^L]$, where and it is a dominant strategy for all wholesale creditors to withdraw early, independent of the private signal that they receive. Similarly, we show there exists an upper dominance region $[\theta^U(P_N), \overline{\theta}]$, where it is a dominant strategy for all creditors to wait.[12] For the intermediate range $\theta^L < \theta < \theta^U(P_N)$, a creditor's payoff depends on the actions of other creditors. So, as a second step, we characterize a creditor i's ex-post belief about the other creditors' actions, conditional on his private signal $x_i = \theta + \epsilon_i$. The belief is a conditional distribution of L. The creditor then choose his optimal action based on the ex-post belief and payoff function $DW(L)$. Finally, for the limiting case where the noise of the signal approaches zero, we obtain a unique threshold

$$\hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - qD_1/P_N} \tag{1.7}$$

such that a successful bank run will happen if and only if the bank's cash flow $\theta < \hat{\theta}(P_N)$. The results are summarized in Proposition 1.

Proposition 1. *For a secondary market asset price $P_N \in (D_1, D_2)$, the bank run game has a unique threshold equilibrium: a successful run occurs to a bank if the bank's cash flow fall below a critical level $\hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - qD_1/P_N}$.*

Proof. See 1.A. *Q.E.D.*

Expression (1.7) establishes a one-to-one correspondence between the asset price P_N and the critical cash flow $\hat{\theta}(P_N)$. Note that the critical cash flow $\hat{\theta}(P_N)$ is decreasing in P_N. A lower asset price makes successful bank runs more likely.

1.3.3 Asset market equilibrium

The uninformed asset buyers observe neither θ nor State s, but they can form rational beliefs about the quality of asset on sale. First of all, they anticipate the threshold equilibrium for the bank run game to be characterized by $\hat{\theta}(P_N)$. Therefore, when N bank runs happen, the asset buyers form a rational belief that only those assets of quality $\theta < \hat{\theta}(P_N)$ will be on sale. Second, the asset buyers also update their beliefs about State s using Bayes' rule. We denote $\omega_N^G\left(\hat{\theta}(P_N)\right)$ the buyers' posterior belief that $s = G$ when the observed number of bank runs equals N, and $\omega_N^B\left(\hat{\theta}(P_N)\right)$ the posterior belief

[12] One can derive the upper and lower bounds explicitly and show that $\theta^L = D_2$ and $\theta^U(P_N) = \frac{F}{1-D_1/P_N}$.

for $s = B$. It should be emphasized that the posterior beliefs depend on buyers' offered price P_N.

Note that two factors can contribute to asset fire sales. First, conditional on a bank run has happened, the cash flow of the bank must be lower than $\hat{\theta}(P_N)$. The buyers face an adversely selected asset pool in the sense that only those banks with low cash flow will be forced into asset sales. Second, any observed bank runs also indicate that $s = B$ is more likely. This further reduces the expected quality of assets on sale, which in turn reduces buyers' willingness to pay.

When the asset market is perfectly competitive, an equilibrium asset price must satisfy two conditions. First, based on their rational expectations about θ and s, the buyers should make zero expected profit by purchasing bank assets at the posted price. In other words, when there are N bank runs, an equilibrium asset price P_N^* equals the expected asset quality.

$$P_N^* = E\left[\theta|\theta < \hat{\theta}(P_N^*)\right] = \omega_N^G\left(\hat{\theta}(P_N^*)\right)\frac{\underline{\theta}_G + \hat{\theta}(P_N^*)}{2} + \omega_N^B\left(\hat{\theta}(P_N^*)\right)\frac{\underline{\theta}_B + \hat{\theta}(P_N^*)}{2} \tag{1.8}$$

Second, a buyer should not be able to make profitable deviation by unilaterally bidding a higher price. Therefore, their expected net payoff, $E\left[\theta|\theta < \hat{\theta}(P_N), N\right] - P_N$, should not increase in P_N.

The equilibrium has a fixed-point representation: P_N^* should be a fixed point for function $E\left[\theta|\theta < \hat{\theta}(P_N), N\right]$. We show that for *each* $N \in \{1, 2\}$, the fixed-point equilibrium exists and is unique. We also verify that the equilibrium is stable in the sense that a buyer cannot profitably deviate by unilaterally bidding a higher price.

1.3.4 A baseline model

The feedback between a bank run and an asset fire sale can be examined without different aggregate states. Therefore, to illustrate the main intuition, we analyze a baseline case of our model with $\underline{\theta}_B = \underline{\theta}_G = \underline{\theta}$. As buyers do not update their beliefs about State s, their posted price scheme will consist of only one unified price P. For this baseline model, we denote market equilibrium by $\{\theta_e, P_e\}$, and obtain closed-form solutions.

As discussed, intelligent asset buyers can solve the subgame of bank runs and anticipate only those assets of quality $\theta < \hat{\theta}(P)$ to be on sale. On the other hand, when the asset market is in a competitive equilibrium, asset buyers who purchase banks' asset at the posted price should break even in expectation. Given their belief $\theta \sim U\left(\underline{\theta}, \hat{\theta}(P)\right)$, a candidate equilibrium price P_e must satisfy the following zero-profit condition.

$$P_e = \frac{\hat{\theta}(P_e) + \underline{\theta}}{2} \tag{1.9}$$

With $\hat{\theta}(P)$ derived in equation (1.7), we can write the condition explicitly.

$$P_e = \frac{1}{2}\left(\frac{D_2 - D_1}{1 - qD_1/P_e} + \underline{\theta}\right) \tag{1.10}$$

Equation (1.10) has one and only one root in interval (D_1, D_2). We obtain the following closed-form solution of equilibrium asset price P_e.[13]

$$P_e = \frac{(D_2 - D_1) + 2qD_1 + \underline{\theta} + \sqrt{[(D_2 - D_1) + 2qD_1 + \underline{\theta}]^2 - 8qD_1\underline{\theta}}}{4} \tag{1.11}$$

For P_e to be an equilibrium, asset buyers should not have profitable deviation by unilaterally bidding a higher price than P_e. That is, a buyer's expected payoff, $E[\theta|\theta < \hat{\theta}] - P$, should not increase in P. In the baseline model, the asset buyers' expected payoff takes the form

$$\pi(P) = \frac{1}{2}\left(\frac{D_2 - D_1}{1 - qD_1/P} + \underline{\theta}\right) - P.$$

For $P > D_1$, the expected payoff monotonically decreases in P.

$$\frac{d\pi(P)}{dP} = -\frac{qD_1(D_2 - D_1)}{2(1 - \frac{q}{P}D_1)^2 P^2} - 1 < 0$$

From (1.10), the equilibrium asset price is such that $\pi(P_e) = 0$, an asset buyer will earn negative profit if unilaterally bidding a higher price $P > P_e$. Intuitively, by bidding a higher price P, a buyer decreases her expected payoff in two ways. First, a higher bid

[13] Details can be found in 1.B.1.

increases the cost for acquiring a piece of asset, and directly reduces the her payoff. Second, a higher price P also alleviates the bank run risks, making fewer banks sell for liquidity reasons. As a result, the buyer faces a pool of assets with deteriorating qualities where more banks are selling assets because of fundamental insolvency. This again reduces her expected payoff.

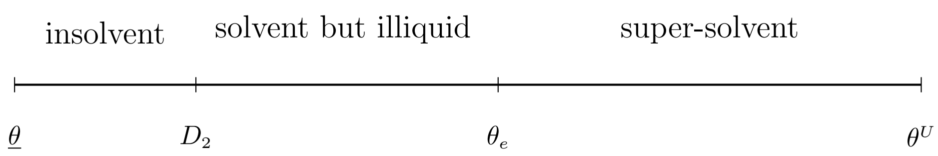
Having solved P_e, we can obtain the corresponding equilibrium critical cash flow $\theta_e \equiv \hat{\theta}(P_e)$ from expression (1.9). One can also verify $\theta_e \in (\theta^L, \theta^U)$.

$$\theta_e = \frac{(D_2 - D_1) + 2qD_1 - \underline{\theta} + \sqrt{[(D_2 - D_1) + 2qD_1 - \underline{\theta}]^2 + 4(D_2 - D_1)\underline{\theta}}}{2} \tag{1.12}$$

The market equilibrium $\{\theta_e, P_e\}$ reflects asymmetric information on asset qualities. By offering P_e, an uninformed buyer makes a loss when the bank is insolvent, and a profit when the bank is only illiquid. Furthermore, as a lower $\underline{\theta}$ aggravates the information asymmetry, it reduces the buyers' willingness to pay, and makes banks more likely to be illiquid. Mathematically, we have θ_e decreasing in $\underline{\theta}$.

Figure 1.3 illustrates the equilibrium funding liquidity risk. A bank with $\theta \in (D_2, \theta_e]$ may not fail and can fully repay its debt obligations if no bank run happens, yet it will fail because of premature asset liquidation caused by the run of its wholesale creditors.

Figure 1.3: Illustration



Proposition 2. *The baseline model has an unique equilibrium, with equilibrium asset price P_e and equilibrium critical cash flow θ_e specified in (1.11) and (1.12) respectively. A bank with cash flow $\theta \in (D_2, \theta_e)$ is solvent but illiquid: it will fail because of a wholesale debt run, even though its assets can generate a cash flow greater than its liabilities D_2.*

Proof. See 1.B.1. *Q.E.D.*

1.3.5 Application I: bank capital and bank run risk

It is an entrenched belief that capital helps reduces bank run risk. An application of the current framework, however, shows that the relationship is more subtle. We show that once asset prices are endogenous, capital also contributes to bank runs via stressed asset prices.

We model an increase of bank capital in its most simplistic form. We assume that a bank maintains its unit portfolio size, increasing its equity from E to $E + \Delta$, and at the same time decreasing its retail deposits from F to $F - \Delta$. In other words, an increase in capital reduces D_2 to $D_2 - \Delta$ but does not affect D_1. We then examine how increasing bank capital affects the risk of bank runs. To measure bank run risks, we follow (Morris and Shin, 2009) and define the illiquidity risk as $IL \equiv \hat{\theta}(P) - D_2$, with IL standing for illiquidity.[14]

Under exogenous asset prices, a natural corollary of Proposition 1 is that a higher capital always reduces funding liquidity risks, because the cash flow generated by capital serves as an extra buffer against fire-sale losses. The value of wholesale debts is better protected and wholesale creditors have less incentive to run, a channel that we call "buffer effect". Recall that $\hat{\theta}(P) = \frac{D_2 - D_1}{1 - \frac{q}{P} D_1}$, we can write IL explicitly as

$$IL = \frac{D_2 - D_1}{1 - \frac{q}{P} D_1} - D_2. \tag{1.13}$$

With price P exogenous and not a function of Δ, it is straightforward to verify that increasing bank capital unambiguously reduces illiquidity.

$$\frac{\partial IL}{\partial \Delta} = -\frac{qD_1}{P - qD_1} < 0 \tag{1.14}$$

With endogenous asset prices, the situation is more complicated. Once investors rationally update their beliefs of a bank's asset qualities, a higher capital level also contributes to bank runs by reducing endogenous fire-sale prices. The intuition is as follows. In terms

[14] Strictly speaking, the illiquidity risk should be measured as the probability $Prob(D_2 < \theta < \hat{\theta}(P)) = \frac{\hat{\theta}(P) - D_2}{\bar{\theta} - \underline{\theta}}$. We drop the the denominator because it is a constant and does not affect comparative statistics.

of inferring the realization of θ, a bank run presents more negative news when it happens to a well-capitalized bank than when it happens to a poorly capitalized bank. Because a well-capitalized bank is able to sustain large losses, the fundamental of the bank must be unusually poor for a run to happen. With such pessimistic inference about θ, buyers' willingness to pay for the bank's asset decreases with the observed capital level. Therefore, a change in bank capital affects illiquidity not only via D_2 but also via endogenous asset price P_e.

$$\frac{\partial IL}{\partial \Delta} = \frac{\partial IL}{\partial D_2}\frac{\partial D_2}{\partial \Delta} + \frac{\partial IL}{\partial P_e}\frac{\partial P_e}{\partial \Delta} \tag{1.15}$$

The first term captures the traditional "buffer effect" as in the case where the asset price is exogenous. Captured by the second term is a new channel that we want to emphasize: increasing capital also affects banks' funding liquidity risk via endogenous asset price.

To see that higher capital leads to lower secondary market asset prices. One can simply take the first order derivative of the closed-form solution of P_e, which gives

$$\frac{\partial P_e}{\partial \Delta} = -\frac{1}{4} - \frac{1}{4}\frac{D_1 + D_2 + \underline{\theta}}{\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}} < 0.$$

Increasing capital decreases asset buyers' willingness to pay for a bank's assets on sale, which in turn makes creditors panic and bank runs more likely. And this is captured by

$$\frac{\partial IL}{\partial P_e}\frac{\partial P_e}{\partial \Delta} > 0.$$

Hence, capital can contribute to funding liquidity risk by reducing endogenous asset prices, a mechanism we dub "inference effect". Comparing expression (1.14) with (1.15), it should be clear that with endogenous asset price and the "inference effect", capital is less able to contain bank run risks as compared to the case where asset price is exogenous. Buyers' rational beliefs limit the role of capital in containing funding liquidity risk.

The overall impact of capital on funding liquidity risk depends on the relative strength of the "buffer effect" and the "inference effect". Using the closed form solution of P_e and θ_e, one can write the overall impact of an increase in capital explicitly.

$$\frac{\partial IL}{\partial \Delta} = -\frac{qD_1}{P_e - qD_1} + \frac{q(D_2 - D_1)D_1}{4(P_e - qD_1)^2}\left[\frac{1}{4} + \frac{D_1 + D_2 + \underline{\theta}}{4\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}}\right] \tag{1.16}$$

It can be shown that in an extreme case where $\underline{\theta} = 0$, $\partial IL/\partial\Delta = 0$ and increasing capital cannot reduce funding liquidity risk at all. Intuitively, a lower $\underline{\theta}$ reduces the expected quality of assets on sale, and therefore reduces buyers' willingness to pay. That is,

$$\frac{\partial}{\partial \underline{\theta}}\left(\frac{\partial P_e}{\partial \Delta}\right) > 0.$$

Such drop of price is most pronounced when $\underline{\theta} = 0$. In that case, the "inference effect" reaches its maximum and completely offsets the "buffer effect" of capital. We summarize the results in the following proposition.

Proposition 3. *In equilibrium, higher bank capital leads to a lower fire-sale asset price. Compared to the case where the price is exogenous, capital is less able to reduce the risk of illiquidity. And in an extreme case where $\underline{\theta} = 0$, higher capital does not reduce bank illiquidity at all.*

Proof. See 1.B.2 *Q.E.D.*

The result suggests that the design of prudential regulations has to take into account the responses of market participants. Compared to the situation wehre regulations are lax, market participants' interpretion of the same piece of negative news can be more pessimistic under strigent regulations. When they panic according to their pessimitic beliefs, the effectiveness of stringent prudential regulations will be reduced, or even completely wiped out.

1.4 Self-fulfilling bank runs and financial contagion

In this section, we extend the baseline case to include two banks and two states. Asset buyers will be able to update their beliefs about State s based on different numbers of bank runs. They perceive $s = B$ to be more likely when more bank runs are observed. In the absence of commitment power, equilibrium prices that buyers offer must reflect the posterior beliefs, and therefore vary with the number of observed bank runs. We characterize market equilibrium with a single bank run and that with two bank runs respectively. We show that for a given $N \in \{1,2\}$, there exists a unique market equilibrium characterized by $\{P_N^*, \theta_N^*\}$ (section 4.1 and 4.2). We further establish that financial contagion can rise as a multiple-equilibria phenomenon, highlighting how pessmistic beliefs can drive financial instability (section 4.3). Finally, we discuss, how an asset purchase program committed by a regulator can improve financial stability over the market equilibria (section 4.4).

1.4.1 Market equilibrium with a single bank run

We start with characterizing the equilibrium with a single bank run, $\{P_1^*, \theta_1^*\}$. For a *given* asset price P_1 that corresponds to the one-bank run outcome, the bank run game has a unique threshold equilibrium characterized by $\hat{\theta}(P_1)$. So asset buyers know that a bank run happens if and only if bank's cash flow is lower than $\hat{\theta}(P_1)$, and update their beliefs about the aggregate state according to Bayes' rule. Recall that $\omega_1^s\left(\hat{\theta}(P_1)\right)$ denotes buyers' posterior belieft for State s when they observe a single bank run. We formulate their posterior belifs as follows.

$$
\begin{aligned}
\omega_1^B\left(\hat{\theta}(P_1)\right) \equiv Prob(s=B|N=1) &= \frac{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)}{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)+\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)} \\
&= \frac{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)}{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)+\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)} \\
\omega_1^G\left(\hat{\theta}(P_1)\right) \equiv Prob(s=G|N=1) &= \frac{\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)}{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)+\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)\left(\overline{\theta}-\hat{\theta}(P_1)\right)} \\
&= \frac{\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)}{\left(\hat{\theta}(P_1)-\underline{\theta}_B\right)+\left(\hat{\theta}(P_1)-\underline{\theta}_G\right)}
\end{aligned}
$$

When the competitive asset market is in a rational expectations equilibrium, based on their posterior beliefs, asset buyers should perceive themselves breaking even when purchasing bank assets for price P_1^*. Their ex-post zero-profit condition (1.8) can now write as the following.

$$P_1^* = E\left[\theta|\theta < \hat{\theta}(P_1^*), N = 1\right] = \omega_1^B\left(\hat{\theta}(P_1^*)\right)\frac{\theta_B + \hat{\theta}(P_1^*)}{2} + \omega_1^G\left(\hat{\theta}(P_1^*)\right)\frac{\theta_G + \hat{\theta}(P_1^*)}{2} \tag{1.17}$$

A candidate equilibrium price P_1^* must lie between D_1 and D_2, and should be a fixed point to function $E\left[\theta|\theta < \hat{\theta}(P_1^*), N = 1\right]$. With $\theta_1^* \equiv \hat{\theta}(P_1^*)$, we can re-write the zero-profit condition (1.17) as a function of θ_1^*.

$$F_1(\theta_1^*) \equiv \omega_1^B(\theta_1^*)\frac{\theta_B + \theta_1^*}{2} + \omega_1^G(\theta_1^*)\frac{\theta_G + \theta_1^*}{2} - \frac{qD_1\theta_1^*}{\theta_1^* - (D_2 - D_1)} = 0 \tag{1.18}$$

The expression implies that asset buyers' net payoff equals zero in expectation. And finding a fixed point P_1^* is equivalent to finding a solution for equation (1.18).

For P_1^* to be an equilibrium, an asset buyer must not profit by unilaterally rising her bid above P_1^*. That is function F_1 should not increase in P_1 (or equivalently, not decrease in θ_1). Such monotonicity holds and the intuition is as follows. First of all, as discussed in section 3.3, increasing price rises the cost for acquiring bank assets and also leads to an deteriorting quality in the asset pool. Second, when P increases, the bank run risks are mitigated, and a bank selling its assets is more likely to be fundamental insolvent than facing a pure liquidity problem. For a *given* number of bank runs observed, this suggests that $s = B$ is more likely. Mathematically, one can verify $\partial\omega_1^B\left(\hat{\theta}(P_1)\right)/\partial P > 0$. The result is summarized in Lemma 2 below.

Lemma 2. *$F_1(\theta)$ monotonically increases in θ, meaning that given a single bank run observed, a buyer's expected payoff monotonically decreases in her bid P.*

Proof. See 1.B.3. *Q.E.D.*

With extra complications introduced by the posterior beliefs on s, we can no longer obtain closed-form solution for P_1^* and θ_1^*. Instead, we prove that there exists a θ_1^* that satisfies equation (1.18) in interval $(\theta^L, \theta^U(P_1^*))$, and a corresponding P_1^* that satisfies equation (1.17) in interval (D_1, D_2). The proof is based on the continuity of $F_1(\theta_1)$. In particular, we show that $F_1(\theta_1)$ is negative at θ^L and positive at θ^U. Furthermore, given the monotonicity of $F_1(\theta)$, once such an equilibrium exists, it is also unique. As a result, the market equilibrium with one bank run can be characterized by a unique pair $\{P_1^*, \theta_1^*\}$. The result is summarized in the proposition below.

Proposition 4. *There exists a unique equilibrium critical cash flow* $\theta_1^* \in (\theta^L, \theta^U(P_1^*))$ *and a unique equilibrium asset price* $P_1^* \in (D_1, D_2)$ *corresponding when the asset buyers observe only one bank run has happened. The equilibrium asset price* P_1^* *and the equilibrium threshold* θ_1^* *are specified in* (1.17) *and* (1.18). *A bank with cash flow* $\theta \in (D_2, \theta_1^*]$ *is solvent but illiquid.*

Proof. See 1.B.4. *Q.E.D.*

1.4.2 Market equilibrium with two bank runs

Following the same approach as the last section, we now characterize the equilibrium with two bank runs. For a *given* asset price P_2 that corresponds to a two-bank-run outcome, a bank will face a run if and only if its cash flow $\theta < \hat{\theta}(P_2)$. Then, we formulate asset buyers' posterior beliefs about State s according to Bayes' rule.

$$\omega_2^B\left(\hat{\theta}(P_2)\right) \equiv Prob(s = B|N = 2) = \frac{\left(\hat{\theta}(P_2) - \underline{\theta}_B\right)^2}{\left(\hat{\theta}(P_2) - \underline{\theta}_B\right)^2 + \left(\hat{\theta}(P_2) - \underline{\theta}_G\right)^2}$$

$$\omega_2^G\left(\hat{\theta}(P_2)\right) \equiv Prob(s = G|N = 2) = \frac{\left(\hat{\theta}(P_2) - \underline{\theta}_G\right)^2}{\left(\hat{\theta}(P_2) - \underline{\theta}_B\right)^2 + \left(\hat{\theta}(P_2) - \underline{\theta}_G\right)^2}.$$

Based on the posterior beliefs, the asset buyers' break-even condition can be written as follows.

$$P_2^* = E\left[\theta|\theta < \hat{\theta}(P_2^*), N = 2\right] = \omega_2^B\left(\hat{\theta}(P_2^*)\right)\frac{\underline{\theta}_B + \hat{\theta}(P_2^*)}{2} + \omega_2^G\left(\hat{\theta}(P_2^*)\right)\frac{\underline{\theta}_G + \hat{\theta}(P_2^*)}{2} \tag{1.19}$$

And the equilibrium threshold $\theta_2^* \equiv \hat{\theta}(P_2^*)$ makes the following equation $F_2(\theta_2^*) = 0$.

$$F_2(\theta_2^*) \equiv \omega_2^B(\theta_2^*)\frac{\underline{\theta}_B + \theta_2^*}{2} + \omega_2^G(\theta_2^*)\frac{\underline{\theta}_G + \theta_2^*}{2} - \frac{qD_1\theta_2^*}{\theta_2^* - (D_2 - D_1)} = 0 \tag{1.20}$$

Lemma 3 shows that buyers' expected payoff monotonically decreases in P_2, so that they have no profitable deviation. Thus, any solution to equation (1.20) is indeed a market equilibrium.

Lemma 3. *$F_2(\theta)$ monotonically increases in θ, meaning that given two bank runs observed, a buyer's expected payoff monotonically decreases in her bid P.*

Proof. See 1.B.5. *Q.E.D.*

To prove the existence of and uniqueness of the equilibrium, we again use the monotonicity and continuity of function $F_2(\theta_2)$. We show that $F_2(\theta_2)$ is negative at θ^L and positive at θ^U, so that the market equilibrium with one bank run can be characterized by a unique pair $\{P_2^*, \theta_2^*\}$. The result is summarized in Proposition 5.

Proposition 5. *There exists an unique equilibrium critical cash flow $\{\theta_2^* \in (\theta^L, \theta^U(P_2^*))$ and a unique equilibrium asset price $P_2^* \in (D_1, D_2)$ when the asset buyers observe that two bank runs have happened. The equilibrium asset price P_2^* and the equilibrium threshold θ_2^* are specified in (1.19) and (1.20). A bank with cash flow $\theta \in (D_2, \theta_2^*]$ is solvent but illiquid.*

Proof. See 1.B.6. *Q.E.D.*

1.4.3 Financial contagion and multiple equilibria

$\theta_2^* > \theta_1^*$ would imply potential contagion. In particular, when a bank's cash flow lies between θ_1^* and θ_2^*, the bank will face no run if the other bank does not face a run, and will fail in a wholesale run if the other bank does. We prove with Lemma 4 that $\theta_2^* > \theta_1^*$ is indeed the case. Intuitively, the asset buyers form more pessimistic beliefs about State s when having observed more bank runs. Their willingness to pay for banks' assets decreases as banks' expected asset qualities are lower in State B. This in turn reduces equilibrium asset price pushes up the equilibrium critical cash flow that a bank has to meet to survive a run.

Lemma 4. *When more runs are observed, the equilibrium market asset price is lower $P_2^* < P_1^*$ and the risk of bank runs is higher $\theta_2^* > \theta_1^*$.*

Proof. See 1.B.7. *Q.E.D.*

Financial contagion emerge as a multiple-equilibrium phenomenon in the current model. In fact, when a bank's cash flow $\theta \in (\theta_1^*, \theta_2^*)$ and the other bank's cash flow $\theta < \theta_2^*$, the equilibrium number of bank runs depends on creditors' beliefs about each others' strategies. As only two threshold strategies can be be rationalized as part of a market equilibrium, i.e., an optimistic threshold strategy, 'to run if and only if $x < \theta_1^*$', and a pessimistic threshold strategy, 'to run if and only if $x < \theta_2^*$', we can focus on those two threshold strategies only. We show that financial contagion can happen purely because of creditors' pessimistic beliefs.

For the ease of exposition, we label the two banks as Bank i and j, and discuss the following two cases respectively. (1) Bank i has a cash flow $\theta \in (\theta_1^*, \theta_2^*)$ and Bank j has a cash flow $\theta < \theta_1^*$.[15] And (2) Bank i and j both have cash flows between θ_1^* and θ_2^*.

In the first case, the equilibrium number of bank runs can be either 1 or 2, depending on creditors' belief about each others' strategies. With a cash flow $\theta < \theta_1^*$, Bank j will fail in a run whether creditors follow the optimistic or pessimistic strategy. Therefore, there will be at least one bank run in the economy. Whether Bank i will have a run, however, depends on creditors' beliefs. If creditors believe that a positive mass among them follow the pessimistic strategy, they will expect a run on Bank i and an asset price P_2^*, so that it is optimal to join the run. As a result, that all creditors withdraw early from Bank i can emerge as an equilibrium. On contrast, if all creditors believe that none of them follow the pessimistic strategy, they would expect the asset price to be P_1^*, and only Bank j to fail, which justifies their optimistic belief/strategy in the first place.

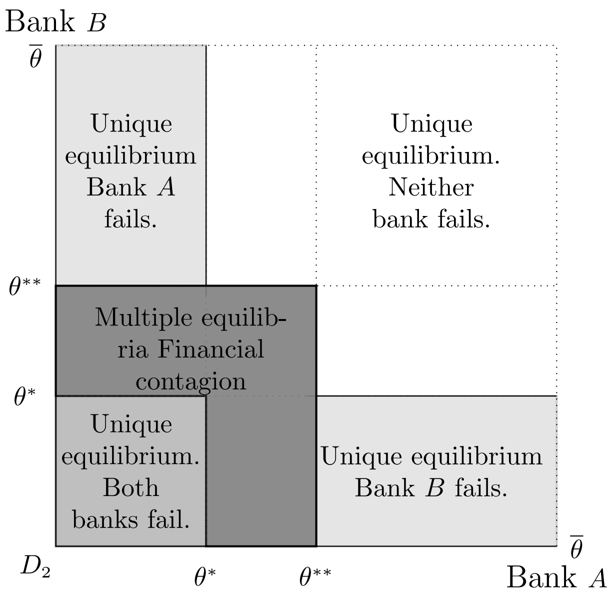
In the second case, the equilibrium number of bank runs can be either 0 or 2, depending again on creditors' beliefs. If all creditors believe that none of them follow the pes-

[15]The symmetric case where Bank i has a cash flow $\theta < \theta_1^*$, and Bank j has $\theta \in (\theta_1^*, \theta_2^*)$ can be analyzed with the same reasoning.

simistic strategy, no run will happen, because both banks' cash flows are higher than θ_1^*. Therefore, $N = 0$ can be an equilibrium. On contrast, if a creditor believes that a positive mass among them follow the pessimistic strategy, he will expect two bank runs and assets sold for price P_2^*, so that it is optimal for him to join the run. Therefore, $N = 0$ can emerge as an equilibrium. The creditor's belief must be that a positive mass of creditors will run both banks. This is because if the pessimistic creditors are present in one bank, then those creditors' strategy cannot be rationalized. Therfore, $N = 1$ cannot be an equilibrium.

In sum, multiple equilibria can emerge when a bank's cash flow in $[\theta_1^*, \theta_2^*]$ and the other bank's cash flow below θ_2^*. The contagion is self-fulfilling and can be fuelled completely by creditors' beliefs. In Figure 1.4, we plot the possible equilibrium outcomes for different combinations of bank cash flows, and summarize the results in Proposition 6.

Figure 1.4: Summary



Proposition 6. *When one bank's cash flow belongs to $[\theta_1^*, \theta_2^*]$ and the other bank's cash flow is lower than θ_2^*, multiple market equilibria exist, and financial contagion can happen because of creditors' pessimistic beliefs.*

1.4.4 Application II: Asset purchase program

We show in this section that a regulator with commitment power can promote financial stability even if he is not better informed than the asset buyers. The welfare-improving policy intervention that we propose resembles asset purchase programs such as Term Asset-Backed Securities Loan Facility (TALF).

We consider the following policy intervention: the regulator makes a promise to purchase bank assets at a *unified* price P_A in case any bank run happens. In particular, the unified price P_A does not depend on the number of bank runs in the economy. The regulator is assumed to have full commitment power and will not revoke his offer after having observed the actual number of bank runs. Under the policy intervention, the model has the following revised timeline.

Figure 1.5: Timing of the game: with announced price

t = 0	t = 0.5	t = 1	t = 2
Banks are established, with their portfolios and liability structures as given.	The regulator makes a promises to buy assets at a unified price P_A, in case any bank run happens.	1. s and θ are realized. 2. Asset buyers post a price scheme. 3. For each bank that they lend to, creditors receive private noisy signals about the bank's cash flow θ, and decide to run or not. 4. Assets are sold to the party that offers the highest price.	1. Returns are realized. 2. Remaining obligations are settled

The regulator is risk-neutral, and subject to an ex-ante budget constraint: he should not make any loss in expectation. To maximise social welfare, he will choose an optimal price P_A^* so that he only breaks even. This is because any higher price that leads to a positive expected profit will come at a cost of letting more solvent banks fail in runs.

The regulator is different from the ordinary asset buyers in the market because he holds full commitment power. In particular, he does not require to break even for each observed number of bank runs, but only to break even ex ante. The commitment power allows the regulator to disregard new information such as the number of bank runs, and can therefore avoid the vicious cycle fuelled by pessimistic belief updating.

We now derive the ex-ante break-even price P_A^*. As a first step, we solve for price P_A^*, under the assumption that banks will sell their assets to the regulator instead of to the asset buyers in the secondary market. When wholesale creditors expect bank assets to be sold at price P_A, we know from Section 3.3 that the critical cash flow of the bank run game is $\hat{\theta}(P_A)$. So the regulator understands that only those assets with $\theta < \hat{\theta}(P_A)$ will be on sale, with $\hat{\theta}(P_A)$ again defined by expression (1.7).

$$\hat{\theta}(P_A) = \frac{D_2 - D_1}{1 - qD_1/P_A} \tag{1.21}$$

As the regulator commits to price P_A before observing any number of bank runs, he holds the prior belief $Prob(s = G) = Prob(s = B) = 1/2$. From this ex-ante perspective, the regulator's break-even condition can be written as follows.

$$P_A^* = \frac{1}{2}\frac{\underline{\theta}_B + \hat{\theta}(P_A^*)}{2} + \frac{1}{2}\frac{\underline{\theta}_G + \hat{\theta}(P_A^*)}{2} \tag{1.22}$$

Using expression (1.21), we can rewrite equation the ex-ante break-even condition into a quadratic function of P_A^*, which has the following root between D_1 and D_2.

$$P_A^* = \frac{[2(D_2 - D_1) + 4qD_1 + (\underline{\theta}_B + \underline{\theta}_G)] + \sqrt{[2(D_2 - D_1) + 4qD_1 + (\underline{\theta}_B + \underline{\theta}_G)]^2 - 16qD_1(\underline{\theta}_B + \underline{\theta}_G)}}{8} \tag{1.23}$$

Having obtained P_A^*, we can derive the corresponding $\hat{\theta}(P_A^*)$ using equation (1.21). Following the same procedure in the proof of Proposition 4, we can prove that $\hat{\theta}(P_A^*) \in (\theta^L, \theta^U(P_A^*))$, so that the policy intervention cannot completely eliminate inefficient bank runs.

Lemma 5. *Suppose that facing runs, banks can sell their assets at a unified price committed by the regulator. The regulator can break even ex ante by offering price P_A^* as in (1.23). And the bank run game has a threshold equilibrium where a run happens if and only if $\theta < \hat{\theta}(P_A^*)$.*

Proof. See 1.B.8. *Q.E.D.*

Now one can verify that P_A^* is higher than what market offers (i.e., $P_A^* > P_1^* > P_2^*$), so that banks will indeed sell their assets to the regulator. As $\hat{\theta}(P)$ decreases with P, the policy intervention improves financial stability as compared to the market equilibria. In particular, the asset purchase commited by the regulator reduces (though does not eliminate) the risk of bank runs, and completely rules out financial contagion. The result is summarised in Proposition 7.

Proposition 7. *The regulator's ex-ante break-even price P_A^* is higher than the prices in market equilibria, so that banks will sell their assets to the regulator when they face runs. With $P_A^* > P_1^* > P_2^*$ and $\hat{\theta}(P_A^*) < \hat{\theta}(P_1^*) < \hat{\theta}(P_2^*)$, the regulator can reduce bank run risks and eliminate financial contagion.*

Proof. See 1.B.9. *Q.E.D.*

With his commitment power, the regulator can disregard the outcome of bank run games and stick to a unified asset price. The commitment power allows the regulator to avoid the viscous cycle between bank runs and fire sales that is fueled by pessimistic beliefs in market. As the regulator only needs to break even ex ante given his prior belief about State s, he can use the profit from State G to compensate the loss in State B.

The ordinary buyers in market, on the other hand, are unable to do so. Without commitment power, they must not make expected loss given any realized number of bank runs. In other words, they are constrained by ex-post break-even conditions. In fact, if an asset buyer offers the same price P_A^*, she will revoke the offer when a bank run actually happens, because in that case she will form a posterior belief that $s = B$ is more likely and will no longer consider herself breaking even by purchasing bank assets at P_A^*. To break even from this ex-post perspective, the asset buyer has to lower her offered price, so as to decrease the loss from purchasing assets with $\theta \in [\underline{\theta}_s, P_N^*)$, and to increase the profit from purchasing assets with $\theta \in \left[P_N^*, \hat{\theta}(P_N^*)\right)$. The lack of commitment power therefore leads to lower asset prices, which in turn result in more bank runs, and justify the pessimistic beliefs in the first place.

1.5 More policy discussion

It should be noticed that the model is sufficiently rich for other policy analyses. We present here one more application on regulatory transparency.[16]

From the discussion in the above sections, the lack of information on aggregate state contributes to the financial instability. It seems that the promotion of information transparency on aggregate risk can serve as an remedy to restore financial safety. In fact, this seemingly natural solution can lead to even greater market distress. To shed some light on this discussion, we consider a situation where a regulator has superior information about aggregate state s and can credibly convey the information to the market. While it could reduce the illiquidity caused by the buyers' pessimistic belief updating in the good state, regulatory transparency exacerbates the situation when the state is bad. Thus, the regulator faces a tradeoff when making the ex ante disclosure decision.

1.5.1 Trade-offs for regulatory transparency

We assume that legislation allows the regulator to perfectly commit to disclose the true information. To concentrate on the effects of disclosure, we consider a simplistic case where the regulator observes the aggregate state s perfectly. Once the regulator commits to disclose information, the information concerning the aggregate state will be released before market trading. The information set of creditors and asset buyers changes correspondingly. In contrast to the belief updating about state upon observing the number of bank runs, now buyers know with certainty the aggregate state. Therefore the price conditions on the true state that the regulator discloses.

For brevity, we omit the derivation of the rational expectations equilibrium. Let θ_e^G, P_e^G and θ_e^B, P_e^B denote the equilibrium critical cash flow and asset price in the good state and in the bad state respectively. The following Lemma 5 shows the effect of disclosing aggregate state on the banks' illiquidity.

[16] In reality, the examples of regulatory transparency are such as the establishment of an early warning system, the release of stress testing parameters or the announcement of the size of assistance program.

Lemma 6. *Under regulatory disclosure there exist a unique critical cash flows θ_e^s ($s = G, B$) corresponding to the true state s ($s = G, B$). In the true state s, banks with cash flows $\theta \in (D_2, \theta_e^s]$ are solvent but illiquid. The regulatory announcements eliminates the multiple equilibria caused by the asset buyers' beliefs about the aggregate state.*

Proof. See 1.B.10. *Q.E.D.*

The asset buyers now know the aggregate state after hearing the announcement. There is no needs to form beliefs about the realization of s based on the observation of bank runs. The regulatory disclosure eliminates different inferences as a source of multiple equilibria: Instead of two rational expectations equilibria depending on buyers' the beliefs, there is a single equilibrium depending on the announced (realized) state.

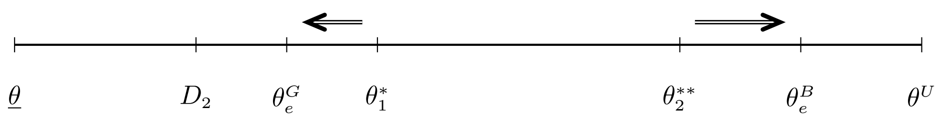
Intuitively, $\theta_e^G < \theta_1^*$ and $\theta_e^B > \theta_2^{**}$. Even if observing two bank runs, buyers cannot be certain that $s = B$. But as as long as the regulatory disclosure is accurate, buyers will lower their valuation of assets further if $s = B$ is communicated. Similarly, making a favorable disclosure will save certain banks from illiquidity, as market participants are reassured. We now show this rigorously.

Proposition 8. *$\theta_e^G < \theta_1^*$ and $\theta_e^B > \theta_2^{**}$. The regulatory announcement reduces illiquidity if $s = G$ but increases it if $s = B$.*

Proof. See 1.B.11. *Q.E.D.*

Graphically, we have Figure 1.6.

Figure 1.6: Regulatory Transparency



The disclosed information, when favorable, reduces adverse selection: banks with $\theta \in (\theta_e^G, \theta_1^*]$ are saved from bank runs. However, acknowledge a crisis in the bad state will exacerbate liquidity problems: a solvent bank is more likely to suffer from illiquidity when market participants are more aware that the whole economy is in the bad state. In particular, banks with $\theta \in (\theta_2^{**}, \theta_e^B]$ will for sure be confronted with runs. Therefore, in determining whether to commit to disclose information, the regulator faces a trade-off: if the state is good, it saves banks from illiquidity; if the state is bad, transparency will create even more panic and runs by pushing asset prices further down. Intuitively, when social cost of bank failure in the crisis state, i.e., $s = B$ is sufficiently large, it is suboptimal for the regulator to disclose the information to market.

1.5.2 Regulatory transparency vs. Asset purchase

Now we run a horse racing between different policy interventions, examining whether regulatory transparency can outperform an asset purchase program as modelled in section 4.4. We show that an asset purchase program, which does not require superior information on the aggregate state, can actually achieve a higher level of financial stability.

Indeed, asset purchase program and regulatory transparency are mutually exclusive. Once credibly disclosing the true state to be G, the equilibrium market price will be P_e^G higher than the announced price P_A^* derived in section 4.4. The reason is again, the market participants acknowledge the state is indeed good, the creditors are reassured and the asset buyers' willingness to pay is highest. Suppose asset purchase program and regulatory disclosure coexist, it will be impossible for the regulator to purchase any assets in the good state unless his announced price is even higher than P_e^G. However, it is never credible for the regulator to commit to purchase assets at such a price as he makes losses even from an ex ante perspective.

We now conduct a cost and benefit analysis to evaluate the regulator's optimal policy choice between an asset purchase program and regulatory transparency. For simplicity, we concentrate on the social cost of bank failure. We denote this social cost to be C and assume it is independent of the number of bank runs (failures) and the residual cash flows. The regulator's objective, then is to choose the policy intervention, which minimizes the expected social cost of bank failure. We denote SC^{AP} and SC^{RT} as the expected social costs when implementing the asset purchase program and the regulatory transparency, respectively.

From section 4.4, SC^{AP} can be formulated as

$$SC^{AP} = \frac{1}{2} \cdot \frac{\hat{\theta}(P_A^*) - \underline{\theta}_B}{\overline{\theta} - \underline{\theta}_B} C + \frac{1}{2} \cdot \frac{\hat{\theta}(P_A^*) - \underline{\theta}_G}{\overline{\theta} - \underline{\theta}_G} C. \quad (1.24)$$

Recall that from section 4.4, $\hat{\theta}(P_A^*)$ is the critical cash flow when the regulator makes the ex ante break even announcement P_A^*.

Note that $\theta_e^s = \hat{\theta}(P_e^s)$ $(s = G, B)$. On the other hand, the regulator's objective when implementing the regulatory transparency policy is

$$SC^{RT} = \frac{1}{2} \cdot \frac{\hat{\theta}(P_e^B) - \underline{\theta}_B}{\overline{\theta} - \underline{\theta}_B} C + \frac{1}{2} \cdot \frac{\hat{\theta}(P_e^G) - \underline{\theta}_G}{\overline{\theta} - \underline{\theta}_G} C. \quad (1.25)$$

Proposition 9 shows that the social cost when the regulator implements asset purchase program is strictly lower.

Corollary 1. *Social cost due to bank failure is lower when the regulator implements the asset purchase program $SC^{AP} < SC^{RT}$. The economy is better off when the regulator suppress his superior information regarding to the true state.*

Proof. See 1.B.12. *Q.E.D.*

Intuitively, the result depends on the critical cash flow as a function of P, $\theta(\hat{P})$ is decreasing and convex in P.

1.6 Concluding remarks

In this paper, we investigate the relationship between fire-sales and bank runs. We present a model where fire-sale prices and bank runs, driven by asymmetric information and buyers' belief updating, are endogenously determined in rational expectations equilibrium. Furthermore, we extend the model to incorporate contagion when there is a common risk exposure. We draw several results from our analysis. First, fire-sales and bank runs are self-fulfilling and mutually reinforcing: when creditors anticipate low prices for a bank's asset sales, a run will be triggered, which generates fire-sales and the corresponding collapse in prices, thus fully justifying creditors' strategies. Second, as a bank fails, asset buyers lower their expectation of common risk factor and perceive banks' asset to be less valuable: the declining asset price will precipitate runs at all other banks.

Based on the model, we draw policy implications regarding capital and regulatory transparency. We show that high capital overall makes the banking industry more resilient against systemic crises. Also, complementary to conventional wisdom, capital can also have side effects on both illiquidity and contagion because buyers' inference via endogenous fire sale prices. A run presents more negative news, both for idiosyncratic and common risk factors, when it happens to a well-capitalized bank. Asset buyers' perceived asset quality will deteriorate further compared to the case where runs are on poorly capitalized banks. As buyers' willingness to pay drops more sharply, it is more likely that creditors panic such that funding liquidity dries up and contagion starts. We also show that regulatory transparency is a double-edged sword: On the one hand, it eliminates the multiple equilibria due to the buyers' beliefs about the aggregate state. On the other hand, it saves illiquid banks when the disclosure is favorable. However, it amplifies illiquidity and contagion problem when the disclosure deteriorates market beliefs. When systemic crisis is more costly than individual bank failures, the desirability of revealing aggregate risk is open to question.

Appendix 1.A Bank run game for $D_1 < P_N < D_2$

In this section, we solve the creditors' bank run game for a given secondary market price P_N belongs to the interval (D_1, D_2).

1.A.1 Lower and upper dominance regions

Following the standard procedure of global games, we start with the lower dominance region denoting as $[\underline{\theta}, \theta^L]$. Suppose all other creditors stay untill $t = 2$ when the bank's cash flow realizes in this region, then the fraction of creditors who withdraw is $L = 0$. Under this case, there is no bank run. In this circumstance, a creditor i still withdraws at $t = 1$ if and only if the inequality (1.5) holds for $L = 0$, that is $\theta \leq F + (1 - E - F)r_D = D_2$. The bank's fundamental is so poor that the bank still fails at $t = 2$ even if there is no premature liquidation of its assets. A creditor i who waits will get zero because of the bankruptcy. Instead, he will get qr_D if withdrawing early. Thus, we define $\theta^L = D_2$. In our analysis, the support of noise ϵ is taken to be arbitrarily close to zero, so creditors are sure when the bank's cash flow realizes in $[\underline{\theta}, D_2)$. Thus, a creditor's dominant strategy is to withdraw at $t = 1$ to get qr_D in this circumstance.

Second, we choose $\theta^U(P_N) = \frac{F}{1 - D_1/P_N}$ given the asset price P_N. Then the upper dominance region is $[\theta^U(P_N), \overline{\theta}]$. Note that we can always have

$$\overline{\theta} > \frac{F}{1 - D_1/P_N}$$

by assuming $\overline{\theta}$ is sufficiently large to keep the existence of the upper dominance region. Now suppose all other creditors withdraw early when the bank's fundamental realizes in the upper dominance region, then $L = 1$. Under this case, a successful bank run always occurs irrespective of the bank's cash flow. Yet, the bank still survives at $t = 2$ if the inequality (1.5) does not hold, $(1 - D_1/P_N)\theta > F$. In other words, the bank always survives if its realized cash flow is sufficiently large $\theta > \frac{F}{1 - D_1/P_N}$. Again, the creditors' signals are arbitrarily accurate, they are sure when the bank's cash flow realizes in $(\theta^U(P_N), \overline{\theta}]$. A creditor's dominant strategy is to stay until $t = 2$ to avoid the penalty for early withdrawal $(r_D > qr_D)$.

1.A.2 Beliefs of creditors outside the dominance regions

In this subsection, we characterize creditors' beliefs when the bank's cash flow realizes in the intermediate region $(\theta^L, \theta^U(P_N))$. Now creditors' actions depend on their beliefs about the actions of other creditors. The signals regarding to the bank's realized cash flow form their beliefs.

To proceed, we first determine the fraction of creditors who withdraw at $t = 1$ as a function of a bank's realized cash flow and the threshold. Formally, when a bank's cash flow θ realizes in the region $(\theta^L, \theta^U(P_N))$, a creditor i receives a signal $x_i = \theta + \epsilon_i$, with $\epsilon_i \sim U(-\epsilon, \epsilon)$ as the noise about the realized fundamental. We suppose each creditor acts according to a threshold strategy and set the threshold signal as $\hat{x}$, i.e., a creditor i withdraws at $t = 1$ if $x_i < \hat{x}$, stays until $t = 2$ if $x_i > \hat{s}$. The fraction of creditors who withdraw at $t = 1$ should be a function of the realized cash flow θ and the threshold of signals $\hat{x}$, that is $L = L(\theta, \hat{x})$. This is because the decision to withdraw or stay depends on both the realization of the cash flow and the strategy of other players. To achieve model tractability, we follow the classic approach in global games by assuming that the creditors' signal about the realized cash flow is sufficiently accurate. The noise ϵ_i is distributed on an arbitrarily small interval, $\epsilon \to 0$. As a result, we can consider the threshold of signal $\hat{x}$ approximately to be a threshold of bank's cash flow $\hat{\theta}$, as $\hat{x}$ and $\hat{\theta}$ are arbitrarily close. Then a representative creditor i withdraws at $t = 1$ if $x_i < \hat{\theta}$, stays till $t = 2$ if $x_i > \hat{\theta}$ and the fraction of early withdrawals is $L(\theta, \hat{\theta})$.

Our second step is to determine the functional form of $L(\theta, \hat{\theta})$. For a realized θ, we have three cases: (i) When $\theta + \epsilon < \hat{\theta}$, even the highest possible signal is below the threshold $\hat{\theta}$. According to the definition of threshold strategy, all creditors withdraw at $t = 1$, and $L(\theta, \hat{\theta}) = 1$. (ii) When $\theta - \epsilon > \hat{\theta}$, even the lowest possible signal exceeds the threshold $\hat{\theta}$. Then all creditors stay till $t = 2$. (iii) When θ falls into the intermediate range $[\hat{\theta} - \epsilon, \hat{\theta} + \epsilon]$, the fraction of creditors who withdraw at $t = 1$ is determined as

$$L(\theta, \hat{\theta}) = Prob(x_i < \hat{\theta}|\theta) = Prob(\epsilon_i < \hat{\theta} - \theta|\theta) = \frac{\hat{\theta} - \theta - (-\epsilon)}{2\epsilon} = \frac{\hat{\theta} - \theta + \epsilon}{2\epsilon}. \tag{A.1}$$

A creditor who receives a signal x_i holds a posterior belief that the fundamental follows a uniform distribution on $[x_i - \epsilon, x_i + \epsilon]$ because the noise ϵ_i is uniformly distributed on $[-\epsilon, \epsilon]$. As the proportion of creditors who withdraw is a function of the fundamental, each creditor forms a posterior belief about the proportion.

The third step is to derive those posterior beliefs. To begin with, we show that the distribution is uniform on $[0, 1]$ for the marginal creditor who happens to observe $s_i = \hat{\theta}$. Indeed, we have

$$Prob\left(L(\theta, \hat{\theta}) \leq \hat{L} \middle| x_i = \hat{\theta}\right) = Prob\left(\frac{\hat{\theta} - \theta + \epsilon}{2\epsilon} \leq \hat{L} \middle| x_i = \hat{\theta}\right) = Prob\left(\theta \geq \hat{\theta} + \epsilon - 2\epsilon\hat{L} \middle| x_i = \hat{\theta}\right).$$

On the other hand, we know that, conditional on $x_i = \hat{\theta}$, the marginal creditor has a posterior belief that θ is uniformly distributed on $[\hat{\theta} - \epsilon, \hat{\theta} + \epsilon]$, which implies $Prob\left(L(\theta, \hat{\theta}) \leq \hat{L} \middle| x_i\right) = \hat{L}$. Therefore, the marginal creditor holds a posterior belief that the fraction of creditors who withdraw at $t = 1$ forms a uniform distribution on $[0, 1]$, that is $L(\theta, \hat{\theta} | x_i = \hat{\theta}) \sim U(0, 1)$.

We then move onto the slightly more complicated cases for the non marginal creditor, $x_i \neq \hat{\theta}$. Without loss of generality, we start with the case $x_i > \hat{\theta}$. Remember that a creditor who receives a signal x_i holds a posterior belief that the fundamental follows a uniform distribution on $[x_i - \epsilon, x_i + \epsilon]$. Given $x_i > \hat{\theta}$, the upper bound of the support is greater than $\hat{\theta} + \epsilon$. And we know that when $\theta > \hat{\theta} + \epsilon$, all creditors stay and $L = 0$. In fact, we can divide the support of θ into two sections: $[x_i - \epsilon, \hat{\theta} + \epsilon]$ and $[\hat{\theta} + \epsilon, x_i + \epsilon]$. As we have discussed, the second section corresponds to a posterior belief $L(\theta, \hat{\theta} | x_i) = 0$. Therefore in the eyes of a creditor i who receives $x_i > \hat{\theta}$, there will be a positive probability mass on $L = 0$. On the other hand, we can show that the posterior belief of θ continues to be a uniform distribution on $[x_i - \epsilon, \hat{\theta} + \epsilon] \subset [\hat{\theta} - \epsilon, \hat{\theta} + \epsilon]$. Since θ is again within the intermediate range $[\hat{\theta} - \epsilon, \hat{\theta} + \epsilon]$, the expression of $L(\theta, \hat{\theta})$ will follow expression (A.1), and we can derive the posterior belief on L as follows.

$$Prob\left(L(\theta, \hat{\theta}) \leq \hat{L} | x_i\right) = Prob\left(\frac{\hat{\theta} - \theta + \epsilon}{2\epsilon} \leq \hat{L} \middle| x_i\right) = Prob(\theta \geq \hat{\theta} + \epsilon - 2\epsilon\hat{L} | x_i)$$

Because the player perceives a uniform distribution of θ on $[x_i - \epsilon, \hat{\theta} + \epsilon]$, the probability above can be calculated as $\frac{\hat{L}}{1 - (s_i - \hat{\theta})/2\epsilon}$, and this is a uniform distribution on $\left[0, 1 - \frac{x_i - \hat{\theta}}{2\epsilon}\right]$. Notice that the density function on this interval is 1, thus the probability uniformly allocated on this interval is $1-\frac{x_i-\hat{\theta}}{2\epsilon}$, and the probability mass at $L(\theta,\hat{\theta})=0$ is $Prob\left(L(\theta,\hat{\theta})=0\right)=\frac{x_i-\hat{\theta}}{2\epsilon}$. A creditor who observes $x_i>\hat{\theta}$ holds a more optimistic belief that a smaller proportion of creditors will withdraw (reflected by the positive probability mass on $L=0$ where no one withdraws). As the marginal creditor who observes $x_i=\hat{\theta}$ is indifferent between withdrawing or not, the player who observes $x_i>\hat{\theta}$ will prefer to stay. Moreover, the higher the signal s_i received, the more optimistic belief a creditor i holds ($Prob\left(L(\theta,\hat{\theta})=0\right)=\frac{x_i-\hat{\theta}}{2\epsilon}$ increases in x_i).

The case $x_i<\hat{\theta}$ follows exactly the same procedure. We can show that from the perspective of a creditor who observes $x_i<\hat{\theta}$, L has a mixed distribution: it is uniformly distributed on $\left[\frac{\hat{\theta}-x_i}{2\epsilon},1\right]$ with density function 1, and has with a positive probability mass at $L(\theta,\hat{\theta})=1$. The probability mass at $L=1$ is $Prob\left(L(\theta,\hat{\theta})=1\right)=\frac{\hat{\theta}-x_i}{2\epsilon}$, where creditor i believes every one withdraws. Thus, a creditor who observes $x_i<\hat{\theta}$ will be more pessimistic and prefer to withdraw. Moreover, the lower the signal s_i received, more pessimistic belief a creditor i holds ($Prob\left(L(\theta,\hat{\theta})=1\right)=\frac{\hat{\theta}-x_i}{2\epsilon}$ increases in x_i).

1.A.3 Threshold Equilibrium

The previous subsections show that upper and lower dominance regions are existent and any creditor whose signal is higher (lower) than $\hat{\theta}(P_N)$ is more prone to stay (withdraw). Now we formally derive the value of this critical cash flow by the indifference condition of the marginal creditor. Remember that the marginal creditor, observing exactly $\hat{\theta}$, is indifferent between stay and withdraw. We have derived that his belief is $L\sim U(0,1)$ and formulated the difference $DW(L)$ in the section 3.2. The creditor's indifference condition then can be expressed as

$$\int_0^1 DW(L)dL=0,$$

or

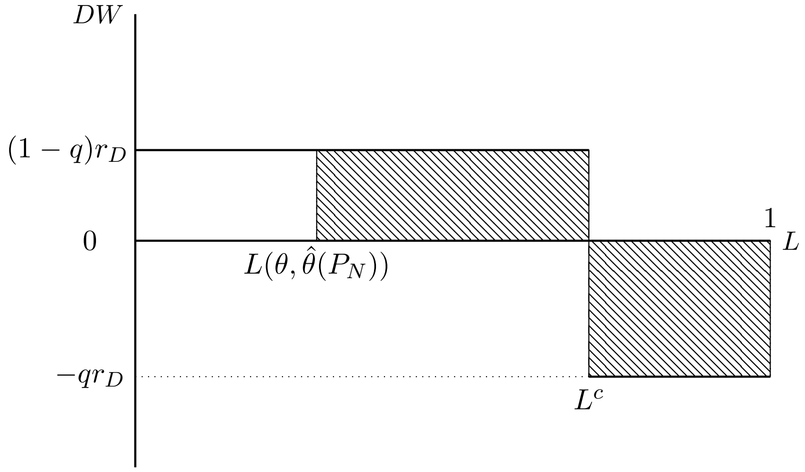
$$\int_{L^c}^1 qr_D dL-\int_0^{L^C}(1-q)r_D dL=qr_D(1-L^c)-(1-q)r_D L^c=0.$$

Recall the definition of L^c, $L^c = \frac{P_N(\theta - D_2)}{(\theta - P_N/q)D_1}$. The indifference condition implies a unique critical cash flow $\hat{\theta}$ for a given asset price $P_N \in (D_1, D_2)$.

$$\hat{\theta}(P_N) = \frac{D_2 - D_1}{1 - qD_1/P_N},$$

For a given asset price $P_N \in (D_1, D_2)$, a run happens to banks with $\theta < \hat{\theta}(P_N)$. Geometrically, we present the indifference condition in Figure 3.

Figure A.7: Payoff differences and the decision to withdraw



Appendix 1.B Proofs to Lemmas and Propositions

1.B.1 Proposition 2. Solution to the baseline model

Proof. To solve the equilibrium critical cash flow θ_e, note that (**??**) is actually a quadratic function of $\hat{\theta}$

$$\hat{\theta}^2 - [(D_2 - D_1) + 2qD_1 - \underline{\theta}]\,\hat{\theta} - (D_2 - D_1)\underline{\theta} = 0.$$

Using the quadratic formula, we can obtain two solutions and retain the positive one

$$\theta_e = \frac{(D_2 - D_1) + 2qD_1 - \underline{\theta} + \sqrt{[(D_2 - D_1) + 2qD_1 - \underline{\theta}]^2 + 4(D_2 - D_1)\underline{\theta}}}{2}.$$

The equilibrium asset price P_e can be obtained by solving (1.10) or directly from the zero profit condition $P_e = \frac{\theta_e + \underline{\theta}}{2}$. We have

$$P_e = \frac{(D_2 - D_1) + 2qD_1 + \underline{\theta} + \sqrt{[(D_2 - D_1) + 2qD_1 + \underline{\theta}]^2 - 8qD_1\underline{\theta}}}{4},$$

Note that $[(D_2 - D_1) + 2qD_1 - \underline{\theta}]^2 + 4(D_2 - D_1)\underline{\theta} = [(D_2 - D_1) + 2qD_1 + \underline{\theta}]^2 - 8qD_1\underline{\theta}$.

To prove $\theta_e > D_2$ and $P_e \in (D_1, D_2)$, we let q be sufficiently close to 1 to simplify the calculation. Note that this assumption is innocuous as q is the penalty for early withdrawal, in reality such penalty is small for demandable debts, i.e., $q \to 1$. So θ_e and P_e turn into

$$\theta_e = \frac{(D_1 + D_2 - \underline{\theta}) + \sqrt{[(D_1 + D_2 - \underline{\theta})^2 + 4F\underline{\theta}}}{2}, \quad P_e = \frac{(D_1 + D_2 + \underline{\theta}) + \sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8qD_1\underline{\theta}}}{4}.$$

With the analytical solution, it can be verified easily that $D_1 < P_e < D_2$ and $\theta_e > D_2$. To prove $\theta_e < \theta^U(P_e)$, note that $\theta_e = \frac{D_2 - D_1}{1 - qD_1/P_e}$ and $\theta^U(P_e) = \frac{F}{1 - D_1/P_e}$. With $P_e > D_1$, $\theta^U(P_e)$ is finite, thus can be assumed to be less than $\overline{\theta}$. By the definition of D_2 and D_1, we have $\theta_e = \theta^U(P_e)$ when $q \to 1$. Then consider the following derivative

$$\begin{aligned} lim_{q \to 1^-} \frac{d}{dq}\left[\theta^U(P_e) - \theta_e\right] =& lim_{q \to 1^-} \frac{d}{dq}\left[\frac{F}{1 - \frac{q(1-E-F)r_D}{P_e}} - \frac{F + (1-q)(1-E-F)r_D}{1 - \frac{q^2(1-E-F)r_D}{P_e}}\right] \\ =& \frac{P_e - D_2}{P_e}. \end{aligned}$$

Thus, there exists an interval for q such that when $q \in (1 - \epsilon, 1)$, $\frac{d}{dq}(\theta^U(P_e) - \theta_e) < 0$ if and only if $P_e < D_2$. Combining with $\theta_e = \theta^U(P_e)$ when $q = 1$, we obtain $\theta^U(P_e) > \theta_e$ when $q \in (1 - \epsilon, 1)$. That is $\theta_e \in [D_2, \theta^U(P_e)] \subset [D_2, \overline{\theta}]$.

To conclude, θ_e and P_e derived above is the unique equilibrium critical cash flow and asset price in the baseline model. Thus, the creditors' beliefs and asset buyers' beliefs are consistent. *Q.E.D.*

1.B.2 Proposition 3. Bank capital and illiquidity

Proof. We show that increasing capital is less able, or even has no effect in reducing a bank's illiquidity risk when asset price is endogenous.

From (1.16), we obtain

$$\frac{\partial IL}{\partial \Delta} = -\frac{qD_1}{P_e - qD_1} \frac{(P_e - qD_1)\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}} - (D_2 - D_1)P_e}{(P_e - qD_1)\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}}$$

Provided that $P_e > D_1$, we have $sgn\left(\frac{\partial IL}{\partial \Delta}\right) = -sgn\left[(P_e - qD_1)\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}} - (D_2 - D_1)P_e\right]$. With the analytical form of P_e from Appendix B.2, we have

$$sgn\left[(P_e - qD_1)\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}} - (D_2 - D_1)P_e\right] = sgn\left[D_1(D_2 - P_e) + (\underline{\theta} - D_1)(P_e - D_1)\right]$$

When $\underline{\theta} = 0$, it can be further verified that $P_e = \frac{D_1+D_2}{2}$ and $D_1(D_2 - P_e) + (\underline{\theta} - D_1)(P_e - D_1) = 0$. Thus, we obtain $sgn\left(\frac{\partial IL}{\partial \Delta}\right) = 0$. In this case, increasing capital (increase Δ) has no effect on a bank's illiquidity risk.

When $\underline{\theta} > 0$, we take the derivative $\frac{\partial}{\partial \underline{\theta}}\left[D_1(D_2 - P_e) + (\underline{\theta} - D_1)(P_e - D_1)\right] = (\underline{\theta} - 2D_1)\frac{\partial P_e}{\partial \underline{\theta}} + (P_e - D_1)$. Recall again $P_e = \frac{(D_1+D_2+\underline{\theta})+\sqrt{(D_1+D_2+\underline{\theta})^2-8D_1\underline{\theta}}}{4}$, we can calculate $\frac{\partial P_e}{\partial \underline{\theta}} = \frac{P_e - D_1}{\sqrt{(D_1+D_2+\underline{\theta})^2-8D_1\underline{\theta}}}$. Thus, we have

$$(\underline{\theta} - 2D_1)\frac{\partial P_e}{\partial \underline{\theta}} + (P_e - D_1) = (P_e - D_1)\frac{\underline{\theta} - 2D_1 + \sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}}{\sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}}.$$

As $P_e > D_1$, the sign of this term depends on $\underline{\theta} - 2D_1 + \sqrt{(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta}}$. When $2D_1 < \underline{\theta}$, this term is of course larger than zero. When $2D_1 > \underline{\theta}$, it can be verified that $(D_1 + D_2 + \underline{\theta})^2 - 8D_1\underline{\theta} > (2D_1 - \underline{\theta})^2$. Again, we have the term is larger than zero. When $\underline{\theta} > 0$, we proved that

$$\frac{\partial}{\partial \underline{\theta}}\left[D_1(D_2 - P_e) + (\underline{\theta} - D_1)(P_e - D_1)\right] > 0.$$

Notice that $D_1(D_2 - P_e) + (\underline{\theta} - D_1)(P_e - D_1) = 0$ when $\underline{\theta} = 0$. As a result, when $\underline{\theta} > 0$, this term is larger than zero. In the end, we have

$$sgn\left(\frac{\partial IL}{\partial \Delta}\right) = -sgn\,(D_1(D_2 - P_e) + (\underline{\theta} - D_1)(P_e - D_1)) < 0.$$

Increasing capital reduces illiquidity risk in the cases where $\underline{\theta} > 0$.

To summarize, when $\underline{\theta} = 0$, increasing capital has no effect on illiquidity risk. When $\underline{\theta} > 0$, Increasing capital reduces illiquidity risk. But one thing should be emphasized is that increasing capital is less able to reduce illiquidity because of the "inferencing effect".

Q.E.D.

1.B.3 Lemma 2. The monotonicity of $F_1(\theta)$

Proof. We start with $F_1(\theta)$, the buyers' expected payoff when they expecting one bank run. With the ex post beliefs about state estabilished, $F_1(\theta)$ can be explicitly expressed as:

$$\begin{aligned} F_1(\theta) &= \frac{\theta - \underline{\theta}_B}{(\theta - \underline{\theta}_B) + (\theta - \underline{\theta}_G)} \frac{\theta + \underline{\theta}_B}{2} + \frac{\theta - \underline{\theta}_G}{(\theta - \underline{\theta}_B) + (\theta - \underline{\theta}_G)} \frac{\theta + \underline{\theta}_G}{2} - \frac{qD_1}{1 - \frac{D_2 - D_1}{\theta}} \\ &= \frac{1}{2} \frac{2\theta^2 - (\underline{\theta}_B^2 + \underline{\theta}_G^2)}{2\theta - (\underline{\theta}_B + \underline{\theta}_G)} - \frac{qD_1\theta}{\theta - (D_2 - D_1)} \end{aligned}$$

To check the monotonicity of $F_1(\theta)$, we take the derivative:

$$\frac{dF_1(\theta)}{d\theta} = \frac{1}{2} \frac{[2\theta - (\underline{\theta}_B + \underline{\theta}_G)]^2 + (\underline{\theta}_B - \underline{\theta}_G)^2}{[2\theta - (\underline{\theta}_B + \underline{\theta}_G)]^2} + \frac{qD_1(D_2 - D_1)}{[\theta - (D_2 - D_1)]^2} > 0$$

Q.E.D.

1.B.4 Proposition 4. The existence and uniqueness of θ_1^*

Proof. It takes two steps to prove Proposition 4. First, we prove the existence and the uniqueness of θ_1^* in the interval $[D_2, \overline{\theta}]$. Second, we prove the equilibrium cash flow $\theta_1^* \in [\theta^L, \theta^U(P_1^*)]$ and the equilibrium price $P_1^* \in (D_1, D_2)$. Note that $F_1(\theta)$ can be rewritten as

$$F_1(\theta) = \omega_1^B(\theta, 1)\pi^B(\theta) + \omega_1^G(\theta, 1)\pi^G(\theta),$$

where $\pi^s(\theta) = \frac{\underline{\theta}_s+\theta}{2} - \frac{qD_1\theta}{\theta-(D_2-D_1)}$. Thus, the equilibrium condition can be also rewritten as

$$\omega_1^B(\theta_1^*)\pi^B(\theta_1^*) + \omega_1^G(\theta_1^*)\pi^G(\theta_1^*) = 0 \tag{A.2}$$

Step 1: We prove by continuity that there exists $\theta_1^* \in [D_2, \overline{\theta}]$ such that $F_1(\theta_1^*) = 0$.

We value the function $F_1(\theta)$ at $\theta = D_2$. Notice that

$$\omega_1^B(D_2) = \frac{D_2 - \underline{\theta}_B}{(D_2 - \underline{\theta}_B) + (D_2 - \underline{\theta}_G)} > 0 \quad \text{and} \quad \omega_1^G(D_2) = \frac{D_2 - \underline{\theta}_G}{(D_2 - \underline{\theta}_B) + (D_2 - \underline{\theta}_G)} > 0.$$

Moreover, as q sufficiently close to 1, it holds that

$$\pi^B(D_2) = \frac{D_2 + \underline{\theta}_B}{2} - qD_2 < 0 \quad \text{and} \quad \pi^G(D_2) = \frac{D_2 + \underline{\theta}_G}{2} - qD_2 < 0,$$

by the parameter assumption 1.1. Therefore, we have $F_1(D_2) < 0$.

Now we examine $F_1(\theta)$ at $\theta = \overline{\theta}$. Similarly, we have

$$\omega_1^B(\overline{\theta}) = \frac{\overline{\theta} - \underline{\theta}_B}{(\overline{\theta} - \underline{\theta}_B) + (\overline{\theta} - \underline{\theta}_G)} > 0, \quad \text{and} \quad \omega_1^G(\overline{\theta}) = \frac{\overline{\theta} - \underline{\theta}_G}{(\overline{\theta} - \underline{\theta}_B) + (\overline{\theta} - \underline{\theta}_G)} > 0.$$

And under our assumption 1.2, it holds that

$$\pi^B(\overline{\theta}) = \frac{\overline{\theta} + \underline{\theta}_B}{2} - \frac{qD_1\overline{\theta}}{\overline{\theta} - (D_2 - D_1)} > 0 \quad \text{and} \quad \pi^G(\overline{\theta}) = \frac{\overline{\theta} + \underline{\theta}_G}{2} - \frac{qD_1\overline{\theta}}{\overline{\theta} - (D_2 - D_1)} > 0.$$

These inequalities hold when q is sufficiently close to 1 as

$$\lim_{q\to1^-} \pi^B(\overline{\theta}) = \frac{(\overline{\theta} + \underline{\theta}_B)(\overline{\theta} - F) - 2D_1\overline{\theta}}{\overline{\theta} - F} > \frac{2D_2(\overline{\theta} - F) - 2D_1\overline{\theta}}{\overline{\theta} - F} > \frac{2F(\overline{\theta} - D_2)}{\overline{\theta} - F}$$

Notice that $D_2 - D_1$ tends to F when $q \to 1$[1]. The first inequality is by the efficiency assumption 1.2, $\frac{\overline{\theta}+\underline{\theta}_B}{2} > D_2$. And $\overline{\theta} > D_2$ follows the efficiency assumption 1.2 as well. The proof $\pi^G(\overline{\theta}) > 0$ of course holds. Therefore, we have: $F_1(\overline{\theta}) > 0$. By the continuity of function $F_1(\theta)$, there exists $\theta_1^* \in (D_2, \overline{\theta})$ such that $F_1(\theta_1^*) = 0$.

Step 2: We prove $P_1^* \in (D_1, D_2)$ and $\theta_1^* < \theta^U(P_1^*)$.

Note that $\theta_1^* = \frac{D_2 - D_1}{1 - qD_1/P_1^*}$ holds in equilibrium. This P_1^* is unique for $\theta_1^* \in [D_2, \overline{\theta}]$, as such θ_1^* is unique. Moreover, P_1^* can not be equal to D_1. Otherwise, θ_1^* can never belong to a finite region $[D_2, \overline{\theta}]$.

We then consider the case when $q \to 1$: $\lim_{q \to 1} \frac{D_2 - D_1}{1 - \frac{q}{P_1^*} D_1} = \frac{D_2 - D_1}{1 - \frac{D_1}{P_1^*}}$. It can be seen that $\frac{D_2 - D_1}{1 - \frac{D_1}{P_1^*}} > 0$ only if $P_1^* > D_1$ and $\frac{D_2 - D_1}{1 - \frac{D_1}{P_1^*}} > \theta^L = D_2$ only if $P_1^* < D_2$. Thus, we prove P_1^* belongs to (D_1, D_2). With $\theta_1^* = \frac{D_2 - D_1}{1 - qD_1/P_e^*}$, $F_1(\theta_1^*) = 0$ can be also rewritten as $\omega_1^B(P_1^*) \cdot \pi^B(P_1^*) + \omega_1^G(P_1^*) \cdot \pi^G(P_1^*) = 0$. Hence, such P_1^* indeed makes the asset buyers earn zero profit when one bank run is observed.

Similar as in 1.B.1, the derivative $lim_{q \to 1^-} \frac{d}{dq} \left[\theta^U(P_1^*) - \hat{\theta}(P_1^*) \right] = \frac{P_1^* - D_2}{P_1^*}$. Having proved $P_1^* < D_2$, $\theta^U(P_1^*) > \hat{\theta}(P_1^*)$ when $q \in (1 - \epsilon, 1)$. That is $\theta_1^* \in [D_2, \theta^U(P_1^*)] \subset [D_2, \overline{\theta}]$. Recall 1.A, such $\theta_1^* = \frac{D_2 - D_1}{1 - qD_1/P_1^*}$ is indeed a threshold equilibrium given asset price P_1^*.

To summarize, the unique price $P_1^* \in (D_1, D_2)$ indeed makes the asset buyers make zero profit, and no incentive to deviate. And the unqiue $\theta_1^* \in [D_2, \overline{\theta}]$ is indeed an equilibrium critical cash flow. Combine those two, the equilibrium $\{\theta_1^*, P_1^*\}$ exists and is unique when one bank run is observed. *Q.E.D.*

1.B.5 Lemma 3. The monotonicity of $F_2(\theta)$

Proof. We show the monotonicity of $F_2(\theta)$, the buyers' expected payoff when they expecting two bank runs. We write explicitly function $F_2(\theta)$ as:

$$F_2(\theta) = \frac{1}{2} \left[\frac{(\theta - \underline{\theta}_B)^2 (\theta + \underline{\theta}_B)}{(\theta - \underline{\theta}_B)^2 + (\theta - \underline{\theta}_G)^2} + \frac{(\theta - \underline{\theta}_G)^2 (\theta + \underline{\theta}_G)}{(\theta - \underline{\theta}_B)^2 + (\theta - \underline{\theta}_G)^2} \right] - \frac{qD_1\theta}{\theta - (D_2 - D_1)}$$

Again, we take the derivative of $F_2(\theta)$ respect to θ. The derivative to θ of the first term in the parenthesis is:

$$\frac{2(\theta+\underline{\theta}_B)[(\theta-\underline{\theta}_G)^2(\theta-\underline{\theta}_B)-(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)]+(\theta-\underline{\theta}_B)^4+(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)^2}{[(\theta-\underline{\theta}_B)^2+(\theta-\underline{\theta}_G)^2]^2}$$

The derivative to θ of the second term in the parenthesis is:

$$\frac{2(\theta+\underline{\theta}_G)[(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)-(\theta-\underline{\theta}_G)^2(\theta-\underline{\theta}_B)]+(\theta-\underline{\theta}_G)^4+(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)^2}{[(\theta-\underline{\theta}_B)^2+(\theta-\underline{\theta}_G)^2]^2}$$

Notice that

$$\begin{aligned}&2(\theta+\underline{\theta}_B)[(\theta-\underline{\theta}_G)^2(\theta-\underline{\theta}_B)-(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)]+2(\theta+\underline{\theta}_G)[(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)-(\theta-\underline{\theta}_G)^2(\theta-\underline{\theta}_B)]\\&=2(\underline{\theta}_G-\underline{\theta}_B)^2(\theta-\underline{\theta}_B)(\theta-\underline{\theta}_G)>0\end{aligned}$$

And the derivative for the last term is again, $-\frac{dP(\theta)}{d\theta}$, positive. Put these discussions altogether, we obtain

$$\frac{dF_2(\theta)}{d\theta}=\frac{2(\underline{\theta}_G-\underline{\theta}_B)^2(\theta-\underline{\theta}_B)(\theta-\underline{\theta}_G)+(\theta-\underline{\theta}_B)^4+(\theta-\underline{\theta}_G)^4+2(\theta-\underline{\theta}_B)^2(\theta-\underline{\theta}_G)^2}{2[(\theta-\underline{\theta}_B)^2+(\theta-\underline{\theta}_G)^2]^2}+\frac{qD_1(D_2-D_1)}{[\theta-(D_2-D_1)]^2}>0$$

Q.E.D.

1.B.6 Proposition 5. The existence and uniqueness of θ_2^*

Proof. We follow the same argument as the proof in 1.B.4. Similar;y, the equilibrium condition can be expressed as

$$F_2(\theta_2^*)=\omega_2^B(\theta_2^*)\pi^B(\theta_2^*)+\omega_2^G(\theta_2^*)\pi^G(\theta_2^*)=0 \tag{A.3}$$

To check the step 1. Notice that:

$$\omega_2^B(D_2)=\frac{(D_2-\underline{\theta}_1)^2}{(D_2-\underline{\theta}_B)^2+(D_2-\underline{\theta}_G)^2}>0,\qquad \omega_2^G(D_2)=\frac{(D_2-\underline{\theta}_G)^2}{(D_2-\underline{\theta}_B)^2+(D_2-\underline{\theta}_G)^2}>0.$$

Moreover,

$$\omega_2^B(\bar{\theta}) = \frac{(\bar{\theta} - \underline{\theta}_B)^2}{(\bar{\theta} - \underline{\theta}_B)^2 + (\bar{\theta} - \underline{\theta}_G)^2} > 0, \qquad \omega_2^G(\bar{\theta}) = \frac{(\bar{\theta} - \underline{\theta}_G)^2}{(\bar{\theta} - \underline{\theta}_B)^2 + (\bar{\theta} - \underline{\theta}_G)^2} > 0$$

The sign of function $F_2(\theta)$ depends on $\pi^B(\theta)$ and $\pi^G(\theta)$, which have the same definitions as in 1.B.4. We have already showed that: $\pi^B(D_2) < 0, \pi^B(\bar{\theta}) > 0$ and $\pi^G(D_2) < 0, \pi^G(\bar{\theta}) > 0$. Thus we can again claim:

$$F_2(D_2) < 0 \qquad \text{and} \qquad F_2(\bar{\theta}) > 0.$$

By the continuity of $F_2(\theta)$, there exists a $\theta_2^* \in (D_2, \bar{\theta})$ satisfying $F_2(\theta_2^*) = 0$. Then by Lemma 1, θ_2^* necessarily belongs to $(D_2, \theta^U(P_2^*))$ with $P_2^* = \frac{qD_1\theta_2^*}{\bar{\theta}-(D_2-D_1)}$.

Since F_2 is monotonically increasing in θ, the uniqueness of this θ_2^* is again guaranteed. The equilibrium $\{\theta_2^*, P_2^*\}$ exists and is unique.

Then step 2 follows exactly the procedure as in 1.B.4, we thus omit it. *Q.E.D.*

1.B.7 Proposition 6. Financial contagion

Proof. The proof hinges on the monotonicity of two ratios

$$\frac{\omega_2^B(\theta)}{\omega_2^G(\theta)} = \frac{(\theta - \underline{\theta}_B)^2}{(\theta - \underline{\theta}_G)^2} \qquad \text{and} \qquad \frac{\pi^G(\theta)}{\pi^B(\theta)} = \frac{\frac{\theta+\underline{\theta}_G}{2} - P(\theta)}{\frac{\theta+\underline{\theta}_B}{2} - P(\theta)}.$$

The first is a conditional likelihood ratio and the second is a payoff ratio. It can be shown both ratios are strictly monotonically decreasing in θ, that is

$$\frac{d}{d\theta}\left(\frac{\omega_2^B(\theta)}{\omega_2^G(\theta)}\right) = -\frac{2(\theta - \underline{\theta}_B)(\underline{\theta}_G - \underline{\theta}_B)}{(\theta - \underline{\theta}_G)} < 0$$
$$\frac{d}{d\theta}\left(\frac{\pi^G(\theta)}{\pi^B(\theta)}\right) = -\frac{[\frac{1}{2} - P'(\theta)][\frac{\underline{\theta}_G - \underline{\theta}_B}{2}]}{[\frac{\theta+\underline{\theta}_B}{2} - P(\theta)]} < 0$$

We focus on the interior realization of cash flow, then $\theta > \underline{\theta}_G$. And remember $P'(\theta) < 0$ from the Appendix B.4.

Furthermore, notice that for $\omega_1^B(\theta)/\omega_1^G(\theta) > 1$, we have

$$\frac{\omega_1^B(\theta)}{\omega_1^G(\theta)} < \left[\frac{\omega_1^B(\theta)}{\omega_1^G(\theta)}\right]^2 = \frac{\omega_2^B(\theta)}{\omega_2^G(\theta)} \tag{A.4}$$

Now we prove by contradiction. Suppose $\theta_1^* > \theta_2^{**}$. By the monotonicity of $\pi^G(\theta)/\pi^B(\theta)$, we will have

$$\frac{\pi^G(\theta_1^*)}{\pi^B(\theta_1^*)} < \frac{\pi^G(\theta_2^{**})}{\pi^B(\theta_2^{**})}.$$

By the equilibrium conditions (A.2) and (A.3), we have

$$\frac{\pi^G(\theta_1^*)}{\pi^B(\theta_1^*)} = -\frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)} \qquad \text{and} \qquad \frac{\pi^G(\theta_2^{**})}{\pi^B(\theta_2^{**})} = -\frac{\omega_2^B(\theta_2^{**})}{\omega_2^G(\theta_2^{**})},$$

which implies

$$\frac{\omega_2^B(\theta_2^{**})}{\omega_2^G(\theta_2^{**})} < \frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)}.$$

By condition (A.4), we know

$$\frac{\omega_2^B(\theta_2^{**})}{\omega_2^G(\theta_2^{**})} < \frac{\omega_1^B(\theta_1^*)}{\omega_1^G(\theta_1^*)} < \frac{\omega_2^B(\theta_1^*)}{\omega_2^G(\theta_1^*)}.$$

But this contradicts the monotonicity of $\omega_2^B(\theta)/\omega_2^G(\theta)$. Therefore, we prove $\theta_2^{**} > \theta_1^*$. *Q.E.D.*

1.B.8 Lemma 4. Regulator's break even price P_A^*

Proof. By inserting (1.21) into (1.22), one can obtain the following equation.

$$4(P_A)^2 - [2(D_2 - D_1) + 4qD_1 + (\underline{\theta}_B + \underline{\theta}_G)]P_A + qD_1(\underline{\theta}_B + \underline{\theta}_G) = 0$$

The positive solution of this quadratic function is

$$P_A^* = \frac{[2(D_2 - D_1) + 4qD_1 + (\underline{\theta}_B + \underline{\theta}_G)] + \sqrt{[2(D_2 - D_1) + 4qD_1 + (\underline{\theta}_B + \underline{\theta}_G)]^2 - 16qD_1(\underline{\theta}_B + \underline{\theta}_G)}}{8}$$

Following the proof in 1.B.1, we can check that $P_A^* \in (D_1, D_2)$. Moreover, we can also check that the regulator does not have profitable deviation by unilaterally bid higher price than P_A^*. *Q.E.D.*

1.B.9 Proposition 7. Asset purchase

Proof. Recall that θ_1^* solves $F_1(\theta_1^*) = 0$. $F_1(\theta)$ can be rewritten as

$$F_1(\theta) = \frac{1}{2}\frac{\underline{\theta}_B + \theta}{2} + \frac{1}{2}\frac{\underline{\theta}_G + \theta}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)} - \frac{(\underline{\theta}_G - \underline{\theta}_B)^2}{4[(\theta - \underline{\theta}_G) + (\theta - \underline{\theta}_B)]}$$

While, we can define

$$F_A(\theta) = \frac{1}{2}\frac{\underline{\theta}_B + \theta}{2} + \frac{1}{2}\frac{\underline{\theta}_G + \theta}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)},$$

where $F_A(\theta_A^*) = 0$. Insert θ_A^* into it, we have

$$F_1(\theta_A^*) = F_A(\theta_A^*) - \frac{(\underline{\theta}_G - \underline{\theta}_B)^2}{4[(\theta - \underline{\theta}_G) + (\theta - \underline{\theta}_B)]} = -\frac{(\underline{\theta}_G - \underline{\theta}_B)^2}{4[(\theta - \underline{\theta}_G) + (\theta - \underline{\theta}_B)]} < 0$$

Recall again $F_1(\theta)$ is increasing in θ. We have $\theta_A^* < \theta_1^*$. Then $P_A^* > P_1^*$ immediately follows. *Q.E.D.*

1.B.10 Lemma 5. Regulatory transparency and bank runs

Proof. We solve here only for the equilibrium in state $s = G$. The equilibrium under $s = 1$ can be solved with the same procedure. The equilibrium is determined by a system of two equations:

$$\begin{cases} \theta_e^G = \frac{D_2 - D_1}{1 - \frac{q}{P_e^G} D_1} \\ P_e^G = \frac{\theta_e^G + \underline{\theta}_G}{2} \end{cases}$$

Solving the system of equations as in the Appendix B, we have the equilibrium critical cash flow and the endogenous fire-sale price:

$$\theta_e^G = \frac{(D_2 - D_1) + 2qD_1 - \underline{\theta}_G + \sqrt{[(D_2 - D_1) + 2qD_1 - \underline{\theta}_G]^2 + 4(D_2 - D_1)\underline{\theta}_G}}{2}$$
$$P_e^G = \frac{(D_2 - D_1) + 2qD_1 + \underline{\theta}_G \pm \sqrt{[(D_2 - D_1) + 2qD_1 + \underline{\theta}_G]^2 - 8qD_1\underline{\theta}_G}}{4}$$

When q is sufficiently close to 1, we have

$$\theta_e^G = \frac{(D_1 + D_2 - \underline{\theta}_G) + \sqrt{[D_1 + D_2 - \underline{\theta}_G]^2 + 4F\underline{\theta}_G}}{2}$$
$$P_e^G = \frac{(D_1 + D_2 + \underline{\theta}_G) + \sqrt{[D_1 + D_2 + \underline{\theta}_G]^2 - 8D_1\underline{\theta}_G}}{4}$$

It is straightforward to check that $\theta_G^* \in (D_2, \underline{\theta}]$ as in Appendix B.2. *Q.E.D.*

1.B.11 Proposition 8. Regulatory transparency and illiquidity

Proof. We start by proving $\theta_e^G < \theta_1^*$. Recall that $F_1(\theta_1^*) = 0$ and $F_1(\theta)$ is monotonically increasing. So $\theta_e^G < \theta_1^*$ will hold if and only if $F_1(\theta_e^G) < 0$. To proceed, we write $F_1(\theta)$ explicitly

$$F_1(\theta) = \frac{\theta - \underline{\theta}_B}{(\theta - \underline{\theta}_B) + (\theta - \underline{\theta}_G)} \frac{\theta + \underline{\theta}_B}{2} + \frac{\theta - \underline{\theta}_G}{(\theta - \underline{\theta}_B) + (\theta - \underline{\theta}_G)} \frac{\theta + \underline{\theta}_G}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)}.$$

We can rewrite $F_1(\theta)$ as follows

$$\begin{aligned} F_1(\theta) &= \frac{\theta - \underline{\theta}_B}{(\theta - \underline{\theta}_B) + (\theta - \underline{\theta}_G)} \left[\frac{\theta + \underline{\theta}_B}{2} - \frac{\theta + \underline{\theta}_G}{2}\right] + \frac{\theta + \underline{\theta}_G}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)}. \\ &= -\frac{\theta - \underline{\theta}_B}{(\theta - \underline{\theta}_B) + (\theta - \underline{\theta}_G)} \frac{\underline{\theta}_G - \underline{\theta}_B}{2} + \frac{\theta + \underline{\theta}_G}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)}. \end{aligned}$$

We then evaluation $F_1(\theta)$ at θ_e^G, that is

$$F_1(\theta_e^G) = -\frac{\theta_e^G - \underline{\theta}_B}{(\theta_e^G - \underline{\theta}_B) + (\theta_e^G - \underline{\theta}_G)} \frac{\underline{\theta}_G - \underline{\theta}_B}{2} < 0.$$

Remember that the term $\frac{\theta_e^G + \underline{\theta}_G}{2} - \frac{qD_1\theta_e^G}{\theta_e^G - (D_2 - D_1)} = \frac{\theta_e^G + \underline{\theta}_G}{2} - P_e^G = 0$. Then we have $\theta_e^G < \theta_e^*$.

We then prove $\theta_e^B > \theta_2^{**}$. Recall that $F_2(\theta_e^{**}) = 0$ and $F_2(\theta)$ is monotonically increasing. So $\theta_e^B > \theta_e^{**}$ will hold if and only if $F_2(\theta_e^B) > 0$. Similarly, we can write $F_2(\theta)$ as

$$F_2(\theta) = \frac{(\theta - \underline{\theta}_G)^2}{(\theta - \underline{\theta}_B)^2 + (\theta - \underline{\theta}_G)^2} \frac{\underline{\theta}_G - \underline{\theta}_B}{2} + \frac{\theta + \underline{\theta}_B}{2} - \frac{qD_1\theta}{\theta - (D_2 - D_1)}.$$

We evaluation $F_2(\theta)$ at θ_e^B, and for the similar argument

$$F_2(\theta_e^B) = \frac{(\theta_e^B - \underline{\theta}_G)^2}{(\theta_e^B - \underline{\theta}_B)^2 + (\theta_e^B - \underline{\theta}_G)^2} \frac{\underline{\theta}_G - \underline{\theta}_B}{2} > 0$$

Then we have $\theta_e^B > \theta_2^{**}$.

Q.E.D.

1.B.12 Proposition 9. Socially undesirable disclosure

Proof. It can be seen easily $SC^{AP} < SC^{RT}$ if and only if $\theta^a < \frac{\theta_e^G + \theta_e^B}{2}$. Consider the auxiliary function

$$G(\theta) = \frac{2qD_1\theta}{\theta - (D_2 - D_1)} - \theta.$$

Then θ^a satisfies $G(\theta^a) = \frac{\underline{\theta}_B + \underline{\theta}_G}{2}$. θ_e^G and θ_e^B combined satisfy $\frac{1}{2}G(\theta_e^G) + \frac{1}{2}G(\theta_e^B) = \frac{\underline{\theta}_B + \underline{\theta}_G}{2}$. Together we obtain

$$G(\theta^a) = \frac{1}{2}G(\theta_e^G) + \frac{1}{2}G(\theta_e^B)$$

It is fairly easy to check that $G' = -\frac{2qD_1(D_2-D_1)}{[\theta-(D_2-D_1)]^2} < 0$ and $G''(\theta) = \frac{4qD_1(D_2-D_1)[\theta-(D_2-D_1)]}{[\theta-(D_2-D_1)]^4} > 0$, thus G is a decreasing convex function. We further have

$$G(\theta^a) = \frac{1}{2}G(\theta_e^G) + \frac{1}{2}G(\theta_e^B) > G(\frac{\theta_e^G + \theta_e^B}{2})$$

Lastly, because the function G is decreasing, we obtain $\theta_A^* < \frac{\theta_e^G + \theta_e^B}{2}$. The social cost due to illiquidity is lower than the regulator chooses to implement the asset purchase program.

Q.E.D.

Chapter 2

Bank Information Sharing and Liquidity Risk

2.1 Introduction

One of the rationales for the existence of banks is their roles in liquidity transformation. Borrowing short-term and lending long-term, banks face funding liquidity risk which is an innate characteristic of financial intermediation (Diamond and Dybvig, 1983). This paper argues that such funding risk can be at the root of the existence of information sharing agreements among banks. The need of information sharing arises because banks in need of liquidity have to sell their assets in secondary markets. Information asymmetry in such markets can make the cost of asset liquidation particularly high (i.e., fire-sales). In order to mitigate adverse selection problems, banks could find it convenient to share information about the quality of assets that they hold. This reduces the cost of asset liquidation when liquidity needs materialize. Information sharing allows banks to reduce adverse selection in secondary loan markets, which in turn reduces the damage of asset fire sales in case of liquidity needs.[17]

The benefit of information sharing, however, has to be traded off with its potential cost.

[17] A similar argument can be made for collateralized borrowing and securitization, where the reduced adverse selection will lead to lower haircut and higher prices for securitized assets.

Letting other banks know the credit worthiness of its own borrowers, an incumbent bank sacrifices its market power. Likely its competitors will forcefully compete for the good borrowers. The intensified competition will reduce the incumbent bank's profitability. We develop a simple model to analyze this trade-off.

We consider an economy made of two banks, one borrower, and many asset buyers. One of the banks is a relationship bank that has a long standing lending relationship with the borrower. It knows both the credit worthiness (i.e., the type) and the credit history (i.e., the repayments) of the borrower's. While the information on borrower credit worthiness cannot be communicated, the credit history can be shared. The second bank is a distant bank, and it has no lending relationship with the borrower so it does not have any information about the borrower's credit worthiness or history. This bank can however compete for the borrower by offering competitive loan rates. The borrower can be risky or safe. While both types have projects of positive NPV, the safe borrower surely brings the project to maturity while the risky one does so only with a certain probability. The distant bank can lose from lending if it cannot price the loan correctly.

The relationship bank is subject to liquidity risk, which we model as a possibility of an (idiosyncratic) bank run. When liquidity need arises, the relationship bank can sell in a secondary market the loan it has granted to the borrower. Since the quality of the loan is unknown to third parties, the secondary market for asset is characterized by adverse selection. Even if the relationship bank holds a safe loan, to sell that at a discount can incur the risk of bankruptcy. Sharing information ex-ante is beneficial because it reduces the adverse selection problem and boosts the liquidation value. The relationship bank trades off higher asset liquidity against rent extraction, and it will voluntarily share information when the benefit outweighs the cost.

The analysis unfolds in three steps. First, we provide an existence result, pinning down the conditions under which information sharing can save the relationship bank from illiquidity. This happens when the participation in an information sharing scheme actually boosts the asset price in the secondary market. This result is not trivial to obtain because information sharing has two countervailing effects on the asset price. On the one hand, observing a good credit history, the asset buyers are willing to pay more for the bank's loan on sale. As the quality of the loan (i.e., the borrower's type) is more likely to be high quality. The adverse selection on the high quality borrower reduces as a result of information sharing. On the other hand, the distant bank competes more aggressively for this loan exactly for the same reason. This drives down the loan rate charged by the relationship bank on the high quality borrower. Since the loan is less profitable, its price in the secondary market decreases. We show that the first effect always dominates.

Second, we look at the equilibrium and characterize the conditions when the relationship bank actually chooses to share information. These conditions coincide with the existence conditions if the relationship bank's probability of becoming illiquid (bank run) is sufficiently high. Indeed, the benefit of information sharing is high and the relationship bank finds it optimal to share information whenever is feasible. Otherwise, when the probability of a run is low, the parameter constellation in which the relationship bank chooses to share information is smaller than the one in which information sharing saves the relationship bank from illiquidity. This occurs because the reduction in expected profits due to more intense competition overcomes the expected benefit of the higher asset's liquidation value.

Lastly, we relax the common assumption in the existing literature that the shared credit history is verifiable. The relationship bank can lie about the borrower's credit history when it shares this information. There are both theoretical and practical reasons to think that such assumption is quite restrictive. From a theoretical point of view, a natural way to sustain truth telling would be to employ a dynamic setting where banks have some reputation at stake. This would induce them to say the truth. We use instead a static game to show that truth telling under information sharing can be indeed a perfect Bayesian equilibrium. From a practical point of view, the verifiability assumption can be rationalized in certain contexts, but it maybe be quite unrealistic in other circumstances. For example, (Giannetti, Liberti, and Sturgess, 2015) show that banks manipulate the credit ratings of their borrowers in the Argentinian credit registry. On a more casual level, information manipulation can take place in the form of 'zombie' lending, like it occurred in Japan with the ever-greening phenomenon or in Spain where banks kept on lending to real estate firms likely to be in distress after housing market crash. We allow for the possibility that banks can manipulate credit reporting and overstate past loan performance. We show under which conditions the relationship bank has an incentive to truthfully disclose the information on the borrower's credit history. It turns out that it exists a narrower parameter constellation than the one in which information sharing is chosen in equilibrium under the assumption of verifiable credit history. In particular, banks have the incentive to truthfully communicate borrower' credit history when credit market is competitive. In fact, one necessary condition for information sharing to be sustained as a truth-telling equilibrium is that the relationship bank can increase the loan rate charged on borrower with bad credit history.

The conjecture that information sharing is driven by market liquidity is novel and complementary to existing rationales. Previous literature has mostly rationalized the presence of information sharing by focusing on the loan market. Sharing information can either reduce adverse selection (Pagano and Jappelli, 1993) or mitigate moral hazard (Padilla and Pagano, 1997) and (Padilla and Pagano, 2000). In their seminal paper, (Pagano and Jappelli, 1993) rationalizes the existence of information sharing as a mechanism to have more accurate information about borrowers that change location and therefore the bank from which they borrow. Sharing ex-ante information about borrowers reduces their riskiness and increases banks' expected profits. This beneficial role is traded off against the cost of losing the information advantage over the competitors. We see information sharing as stemming also from frictions on the secondary market for asset sale instead only on the prime loan market. The two explanations are in principle not mutually exclusive but complementary.

Another strand of the literature argues that information sharing allows the incumbent bank to extract more monopolistic rent. When competition for borrowers occurs in two periods, inviting the competitor to enter in the second period by sharing information actually dampens the competition in the first period (Bouckaert and Degryse, 2006) and (Gehrig and Stenbacka, 2007). Sharing information about the past defaulted borrowers deters the entry of competitor, which allows the incumbent bank to capture those unlucky but still good borrowers (Bouckaert and Degryse, 2006). This mechanism is also present in our model, and it is related to our analysis with unverifiable credit history. The incumbent (relationship) bank has an incentive to report the true credit history if it can charge higher loan rates to a good borrower with bad credit history.

Finally, a couple of papers link information sharing to other banking activities. For example, information sharing can complement collateral requirement since the bank is able to charge high collateral requirement only after the high risk borrowers are identified via information sharing (Karapetyan and Stacescu, 2014a). Information sharing can also complement information acquisition. After hard information is communicated, collecting soft information to boost profit becomes a more urgent task for the bank (Karapetyan and Stacescu, 2014b). In those papers, the goal is not to provide a rationale of why banks voluntary choose to share information but how information sharing affects other dimensions of bank lending decisions.

Our novel theoretical exposition also opens road for future empirical research. The model generates complementary empirical implications that information sharing will facilitate banks' liquidity management and loan securitization. The model also suggests that information sharing system can be more easily established, and can work more effectively, in countries with competitive banking sector, and in credit market segments where competition is strong. These empirical predictions would complement the existing empirical literature which has mostly focused on the impact of information sharing on bank risks and firms' access to bank financing.[18]

The remainder of this paper is organized as follows. In the next section we present the model. In Section 2.3 we show under which conditions information sharing arises endogenously when borrower's credit history is verifiable. Section 2.4 shows when information sharing is still chosen in equilibrium when credit history is not verifiable. Section 2.5 analyzes welfare and policy implication. Section 2.6 concludes.

2.2 Model Setup

The economy consists of banks, a relationship bank and a distant bank, one borrower and many depositors and asset buyers. All agents are risk neutral. The gross return of

[18] (Doblas-Madrid and Minetti, 2013) finds that information sharing reduces contract delinquencies. (Houston, Lin, Lin, and Ma, 2010) finds that information sharing is correlated with lower bank insolvency risk and likelihood of financial crisis. (Brown, Jappelli, and Pagano, 2009) finds that information sharing improves credit availability and lower cost of credit to firms in transition countries.

the risk-free asset is indicated as r_0.

For simplicity, we assume that a bank has one loan on its balance sheet. The loan requires 1 unit of initial funding, and its returns depend on the type of the borrower. The borrower can be either safe (H-type) or risky (L-type). The ex-ante probability of the safe type $\Pr(H)$ is equal to α, and for the risky type $\Pr(L)$ is equal to $1-\alpha$. A safe borrower always generates a payoff $R > r_0$, and a risky borrower has a payoff that depends on an aggregate state $s = \{G, B\}$. In the good state G, the payoff is the same as a safe borrower R, but in the bad state B the payoff is 0. The ex-ante probabilities of the two states are $\Pr(G) = \pi$ and $\Pr(B) = 1 - \pi$, respectively. Throughout the paper, we assume no credit rationing. Even a risky loan has a positive NPV, that is, $\pi R > r_0$.[19]

The relationship bank has an ongoing lending relationship with a borrower. It privately observes both the credit worthiness (i.e., the type) and the payment history of the relationship borrower. The distant bank, on the other hand, has no lending relationship with the borrower and observes no information about the borrower's type. It does not know the credit history either, unless the relationship bank shares such information. The distant bank can compete for the borrower by offering lower loan rates, but to initiate the new lending relationship it bears a fixed cost c. Such cost instead represents a sunk cost for the relationship bank.[20]

We make a distinction between soft and hard information. While borrower's credit worthiness (type) is assumed to be soft information and cannot be communicated to the others, credit history is assumed to be hard information and can be shared with third parties. We model information sharing as a unilateral decision of the relationship bank. If the bank chooses to share the credit history of its borrower, it makes announcement about whether the borrower had defaulted or not. We label a credit history with previous defaults as D, and a credit history without defaults as $\overline{D}$. A safe borrower has a credit history $\overline{D}$ with probability 1, and a risky borrower has a credit history $\overline{D}$ with probability

[19]One potential interpretation is to consider the H-type being prime mortgage borrowers, and L-type being subprime borrowers. While both can pay back their loans in a housing boom, the subprime borrowers will default once housing price drops. However, the probability of a housing market boom is sufficiently large that it is still profitable to lend to both types.

[20]One possible interpretation of the fixed cost c can be the fixed cost paid by the bank to establish new branches, hire and train new staffs, etc. Alternatively it can represents the borrower's switching cost that is paid by the bank.

π and a credit history D with probability $1 - \pi$.[21]

The relationship bank and the distant bank compete for the borrower by offering loan rates. The banks are financed solely by deposits. We abstract from risk-shifting induced by limited liability, and assume that there is perfect market discipline so that deposit rates are determined based on bank's risk. Depositors are assumed to have the same information about the borrower as the distant bank. In a competitive deposit market, the depositors demand to earn the risk-free rate r_0 in expectation.

To capture funding liquidity risk, we assume the probability that the relationship bank faces a run equals to $\Pr(\text{run}) = \rho$. In such a case all depositors withdraw their funds. Otherwise, we have no bank run with probability $\Pr(\text{no run}) = 1-\rho$. When a run happens, the relationship bank needs to raise liquidity to meet the depositors' withdrawals. We assume that physical liquidation of the bank's loan is not feasible, and only financial liquidation—a loan sale to asset buyers—is possible. We also assume that the loan is indivisible and the bank has to sell it as a whole. The state $s = \{G, B\}$ realizes after the loan competition, and it becomes public information. Asset buyers observe the true state, but are uninformed of the credit worthiness of the relationship borrower's. They can nevertheless condition their bids on the borrower's credit history if the relationship bank shares the information. We assume that the secondary asset market is competitive, and risk neutral asset buyers only require to break even in expectation.

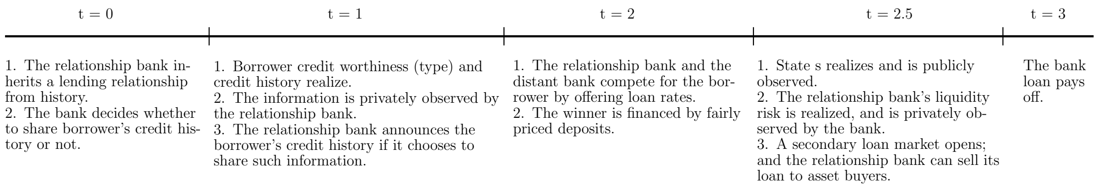
Notice that bank asset can be on sale for two reason: either due to funding liquidity need, in which case H-type loans can be on sale, or due to strategic sale for arbitrage reason, in which case only L-type loans will be sold. The possibility of strategic asset sale leads to adverse selection in the secondary asset market. Therefore, H-type loans are underpriced during asset sale and even a solvent relationship bank owning an H-type loan can fail due to illiquidity. In case of a bank failure, we assume that bankruptcy costs result in zero salvage value. Such liquidity risk and costly liquidation gives the relationship bank the incentive to disclose the credit history of its borrower, in the hope

[21] It is equivalent to assume there was a first round of lending before the current model. If the borrower is safe, it generated a non default credit history. If the borrower is risky, its payoff depended on the state when the first round of lending occurred. If the state was good, the risky borrower did not default as well, instead if the state was bad, the risky borrower had a credit history of D.

that such information sharing can boost asset market liquidity.

The sequence of events is summarized in Figure 2.1.

Figure 2.1: Time line of the model



The timing captures the fact that information sharing is a long-term decision (commitment), while competition in the loan market and the liquidity risk faced by the bank are shorter-term concerns.

At $t = 0$ the relationship bank inherits a lending relationship and decides to participate in the information sharing scheme or not. At $t = 1$, the borrower's type and credit history realizes. The relationship bank privately observes these information and announces the borrower's credit history if it chose to participate in information sharing scheme in the previous stage. At $t = 2$, the two banks compete in loan rates for the opportunity to lend to the borrower again. The winning bank is financed by competitive depositors. At $t = 2.5$, the aggregate state realizes and is publicly observed. The relationship bank's liquidity risk realizes and is only privately known. The relationship bank raises liquidity by selling its loan on the secondary asset market. Finally, at $t = 3$ the loan pays off.

2.3 Verifiable Credit History

We solve the decentralized solution by backward induction. Therefore we proceed as follows: i) determine the prices at which loans are traded in the secondary asset market; ii) compute the deposit rates at which depositors supply their fund to the bank; iii) determine the loan rates at which the bank offers credit to the borrower; iv) decide if the relationship bank wants to share or not the information on the borrower's credit history.

Depending on whether banks share information or not, the game has different information structures. Without information sharing, asset prices, loan rates and deposit rates cannot be conditional on the borrower's credit history. On the contrary, such variables will depend on credit history if information is shared. Through this section we follow the literature and assume that credit history, once shared, is perfectly verifiable. In Section 2.4 we allow for the possibility that the relationship bank can manipulate the credit history and overstate past loan performance.

2.3.1 Asset Prices

We determine at which price loans are traded in the secondary market taking as given loan rates and deposit rates. We indicate with P_i^s the asset price in state $s = \{G, B\}$ and with information-sharing regime $i = \{N, S\}$, where N is no information sharing in place, and S refers to the presence of information sharing. Like all other agents, asset buyers can perfectly observe state s, but they cannot observe whether the loan sale is for liquidity reason or for arbitrage. Accordingly, the pricing of loans is state-contingent and takes into account the relationship bank's strategic behaviors.

Without information sharing, if the aggregate state is good, the borrower will generate the same payoff, and therefore $P_N^G = R_N$ independently of the borrower's type. That is, asset buyers are competitive so they bid until zero profit. If the state is bad, the L-type borrower will generate a zero payoff. Asset buyers cannot update their prior beliefs since the relationship bank does not share any information on borrower's credit history. For any positive price, L-type loan will be on sale even if the relationship bank faces no bank run. Due to the presence of L-type loan, H-type loan will be sold at a discount. Consequently, it is sold by the relationship bank only if there is urgent liquidity needs to meet the depositors' withdrawals. The market is characterized by adverse selection. The price P_N^B is determined by the following break-even condition of asset buyers

$$\Pr(L)(0 - P_N^B) + \Pr(H)\Pr(\text{run})(R_N - P_N^B) = 0,$$

which implies

$$P_N^B = \frac{\alpha\rho}{(1-\alpha)+\alpha\rho} R_N. \tag{2.1}$$

It follows immediately that the H-type loan is underpriced (lower than the fundamental value R_N) because of adverse selection in the secondary asset market.

With information sharing, asset prices can be conditional on the borrower's credit history D and $\overline{D}$ too. If the state is good no loan will default, the prices equal to the face value of loans. We have

$$P_S^G(D) = R_S(D)$$

and

$$P_S^G(\overline{D}) = R_S(\overline{D}),$$

where $R_S(D)$ and $R_S(\overline{D})$ denote the loan rates for a borrower with and without default history, respectively. Notice that asset prices are different because the loans rate are different, conditional on the information released. When the state is bad, asset buyers can update their beliefs accordingly. When the relationship bank announce a previous default then the borrower is perceived as a L-type for sure, therefore posterior beliefs are $\Pr(H \mid D) = 0$ and $\Pr(L \mid D) = 1$. Since a L-type loan defaults in state B with certainty, we have $P_S^B(D) = 0$. When the announced credit history is $\overline{D}$ (no default), then posterior beliefs, according to Bayesian rule, are

$$\Pr(H \mid \overline{D}) = \frac{\Pr(H, \overline{D})}{\Pr(\overline{D})} = \frac{\alpha}{\alpha + (1-\alpha)\pi} > \alpha$$

and

$$\Pr(L \mid \overline{D}) = \frac{\Pr(L, \overline{D})}{\Pr(\overline{D})} = \frac{(1-\alpha)\pi}{\alpha + (1-\alpha)\pi} < 1 - \alpha.$$

Intuitively, asset buyers uses the credit history as a noisy signal of the loan quality. A loan with a good credit history $\overline{D}$ is more likely to be of H-type, thus $\Pr(H \mid \overline{D}) > \alpha$.

Given the posterior beliefs, asset buyers anticipate that the relationship bank always sells L-type loan and withholds the H-type loan to maturity if no bank run occurs, therefore the price $P_S^B(\overline{D})$ they are willing to pay is given by the following break even condition

$$\Pr(L \mid \overline{D})[0 - P_S^B(\overline{D})] + \Pr(H \mid \overline{D}) \Pr(\text{run})[R_s(\overline{D}) - P_S^B(\overline{D})] = 0,$$

which implies

$$P_S^B(\overline{D}) = \frac{\alpha\rho}{(1-\alpha)\pi + \alpha\rho} R_s(\overline{D}). \tag{2.2}$$

Comparing (2.1) with (2.2), conditional on $\overline{D}$-history, the perceived chance that a loan is H-type is higher under information sharing. This is because a L-type borrower with bad credit history D can no longer be pooled with a H-type in asset sales. Information sharing therefore mitigates the adverse selection problem. However, we cannot yet draw a final conclusion on the relationship between the asset prices until we determine the equilibrium loan rates R_N and $R_s(\overline{D})$.

2.3.2 Deposit Rates

We assume that deposits are fairly priced for the risk and that depositors have the same information on the credit worthiness of loan applicants as the distant bank. Consequently, the pricing of deposit rates can be conditional on the riskiness of bank's loan as well as the past credit information of the loan applicants if the relationship bank shared this piece of information. We determine equilibrium deposit rates r_i, with $i = \{N, S\}$, taking as given the loan rates.

On the equilibrium path, it will be the relationship bank that finances the loan. We first discuss the deposit rates charged to the relationship bank, i.e. the deposit rates on equilibrium path. Besides the fundamental asset risk, the liquidity risk faced by the relationship bank is endogenized in pricing the deposit rates. A *necessary* condition for a candidate deposit rate to be an equilibrium one is that the depositors break even by earning zero expected payoff under this rate. The break-even condition is only necessary because we have to check the depositors do not have a profitable deviation by charging a lower rate than the break-even one. Since deposits can be either risky or safe, a break-even deposit rate can be so high that the relationship bank cannot survive a run. In this case, lowering the deposit rate can save the relationship bank and it can guarantee to the depositors a positive payoff.

Consider the situation where relationship bank does not participate in the information sharing program, and denote r_N as the equilibrium deposit rate. When the loan opportunity is risky, define $\widehat{r}_N$ as the break-even rate for risky deposit, we have

$$[\Pr(G) + \Pr(H)\Pr(B)\Pr(\text{no run})]\widehat{r}_N = r_0,$$

which implies

$$\widehat{r}_N = \frac{r_0}{\pi + \alpha(1-\pi)(1-\rho)} > r_0.$$

Notice that deposit rate is charged before the realization of the state s and of the (possible) bank run. Facing a bank run, the relationship bank will be bankrupt. We implicitly assume that the parameter values are such that $P_N^B < \widehat{r_N}$ in the case of risky deposits. Recall that there is zero salvage value when bankruptcy occurs, then a candidate equilibrium rate is $\widehat{r}_N$ in case of risky deposits. On the other hand, if the parameter values are such that $P_N^B > \widehat{r_N}$, the relationship bank will survive a bank run. The deposits are safe, then a candidate equilibrium deposit rate is simply r_0 in case of safe deposits.

The following Lemma characterizes the equilibrium deposit rates in case information sharing is not in place.

Lemma 2.1. *Assume there is no information sharing, then deposit rates are as follows: (i) If $P_N^B \geq r_0$ then $r_N = r_0$; (ii) If $P_N^B < r_0$ then $r_N = \widehat{r}_N$.*

The proof is in the Appendix. The intuition is that when the price of the asset to liquidate is greater than or equal to the risk-free rate, then deposits are not risky and depositors can be remunerated with the risk-free rate. Otherwise, if the price of the asset is less than the risk-free rate, bankruptcy occurs and deposits become risky. Depositors anticipate this possibility, and they have to be remunerated with the interest $\widehat{r}_N$ higher than the risk-free rate.

We now characterize deposit rates when the relationship bank adopts the information sharing regime. The deposit rates are now conditional on the credit history of the borrower. If the borrower has a credit history with default (i.e., a D-history) then depositors knows the borrower is surely L-type and $P_S^B(D) = 0$. Therefore depositors are paid only if the state is G. This leads depositors to ask a deposit rate $r_S(D)$ that satisfies the break-even condition $\Pr(G) r_S(D) = r_0$. Accordingly we have

$$r_S(D) = \frac{r_0}{\pi} > r_0. \tag{2.3}$$

When the borrower has a $\overline{D}$-history (i.e., no default) the analysis is similar to the no information sharing, and the candidate equilibrium deposit rates depend on parameter values. Defining the break-even deposit rate for risky deposits by $\widehat{r}_S(\overline{D})$, we have

$$[\Pr(G) + \Pr(B) \Pr(H \mid \overline{D}) \Pr(\text{no run})] \widehat{r}_S(\overline{D}) = r_0$$

that implies

$$\widehat{r}_S(\overline{D}) = \frac{\alpha + (1-\alpha)\pi}{\alpha + (1-\alpha)\pi^2 - (1-\pi)\alpha\rho} r_0 > r_0.$$

Again, if the parameter values are such that $P_S^B(\overline{D}) > r_0$, a candidate equilibrium deposit rate is $\widehat{r}_S(\overline{D})$. Instead, if the parameter values are such that $P_S^B(\overline{D}) > r_0$, a candidate equilibrium deposit rate is simply the risk-free rate r_0. The following Lemma characterizes the equilibrium deposit rates when the no default history $\overline{D}$ is reported.

Lemma 2.2. *Assume information sharing is in place and the borrower has a $\overline{D}$-history, then deposits rates are as follows: (i) If $P_S^B(\overline{D}) \geq r_0$ then $r_S(\overline{D}) = r_0$; (ii) If $P_S^B(\overline{D}) < r_0$ then $r_S(\overline{D}) = \widehat{r}_S(\overline{D})$.*

The proof is provided in the Appendix, and the intuition is similar to Lemma 2.1. When the price of the asset in the secondary market is sufficiently high, the equilibrium deposit rate is equal to the risk-free rate. Otherwise, deposits are risky and consequently the equilibrium deposit rate is higher than the risk-free rate.

We now compute the break-even deposit rates r_i^E with $i = \{N, S\}$ charged to the distant bank. These deposit rates are off-equilibrium rates since it is the relationship bank that finances the loan in equilibrium. Remember that the distant bank only faces the fundamental asset risk.[22] Without information sharing, the deposit rate r_N^E is determined by depositors' break-even condition as follows

$$\Pr(H)r_N^E + \Pr(L)\Pr(G)r_N^E = r_0,$$

which implies

$$r_N^E = \frac{r_0}{\alpha + (1-\alpha)\pi} > r_0. \tag{2.4}$$

Under the information sharing regime, the deposit rate $r_S^E(\overline{D})$ charged when the borrower has no previous default is determined by depositors' break even condition

$$\Pr(H \mid \overline{D})r_S^E(\overline{D}) + \Pr(L \mid \overline{D})\Pr(G)r_S^E(\overline{D}) = r_0,$$

which implies

$$r_S^E(\overline{D}) = \frac{\alpha + (1-\alpha)\pi}{\alpha + (1-\alpha)\pi^2} r_0 > r_0. \tag{2.5}$$

Finally, the deposit rate $r_S^E(D)$ charged when the borrower has a default history is given by the depositors' break even condition $\Pr(G)r_S^E(D) = r_0$, which implies $r_S^E(D) = r_0/\pi$.

2.3.3 Loans Rates

We assume the credit market is contestable, then the loan rates charged to the borrower are determined by the break-even condition of the distant bank that tries to enter the loan market. We call R_i^E the loan rate offered by the distant (entrant) bank to the borrower under information-sharing regime $i = \{N, S\}$. As noticed, we assume that the distant bank does not face liquidity risk but only fundamental asset risk.

Without information sharing, the distant bank holds the prior belief on the borrower's type. The break-even condition for the distant bank is

$$\Pr(H)(R_N^E - r_N^E) + \Pr(L)\Pr(G)(R_N^E - r_N^E) = c,$$

[22] While the relationship bank faces the liquidity risk, that the distant bank does not face, the relationship bank has an extra tool (information sharing decision) to manage that risk. Our set up is symmetric in this respect.

where c is the fix entry cost and r_N^E is the deposit rate paid by the distant bank to its depositors determined in (2.4). Combining the two expressions, we get

$$R_N^E = \frac{c + r_0}{Pr(H) + Pr(L)Pr(G)} = \frac{c + r_0}{\alpha + (1-\alpha)\pi}.$$

With information sharing in place, loan rates are contingent on credit history. If the distant bank observes a previous default, then the borrower is surely an L-type. The distant bank's break-even condition is

$$\Pr(G)[R_S^E(D) - r_S^E(D)] = c,$$

where $r_S^E(D) = r_0/\pi$. Combining these two expressions, we get

$$R_S^E(D) = \frac{c + r_0}{\pi}.$$

When the credit history of the borrower is $\overline{D}$, the distant bank updates its belief and its break-even condition is

$$\Pr(H \mid \overline{D})[R_S^E(\overline{D}) - r_S^E(\overline{D})] + \Pr(L \mid \overline{D}) \Pr(G)[R_S^E(\overline{D}) - r_S^E(\overline{D})] = c,$$

where $r_S^E(\overline{D})$ is given by (2.5). Combining the two expressions, we get

$$R_S^E(\overline{D}) = \frac{c + r_0}{Pr(H|\overline{D}) + Pr(L|\overline{D})Pr(G)} = \frac{\alpha + (1-\alpha)\pi}{\alpha + (1-\alpha)\pi^2}(c + r_0).$$

A simple comparison of the loan rates makes it possible to rank them as follows.

Lemma 2.3. *The ranking of the loan rates charged by the distant bank is* $R_S^E(\overline{D}) < R_N^E < R_S^E(D)$.

When information sharing is in place, and the borrower has the no-default history $\overline{D}$, the distant bank offers the lowest loan rate since it is more likely that the borrower is H-type. On the contrary, if the credit history presents defaults the distant bank charges the highest loan rate since the borrower is surely an L-type. Without information sharing, the distant bank offers an average loan rate (reflecting the prior belief about borrower's type).

The equilibrium loan rate also depends on the contestability of the loan market. Suppose $R_i^E > R$, then the payoff R from the project (loan) is too low and entry into such loan market is never profitable for the distant bank. Then the relationship bank can charge the monopolistic loan rate taking the entire payoff from the project. Suppose, otherwise, $R_i^E \leq R$. In this case the payoff R is high enough to induce the distant bank to enter the loan market. The relationship bank in this case can only undercut the loan rate to R_i^E. The equilibrium loan rate is determined by the break-even loan rate charged by the distant bank. Let us indicate the equilibrium loan rate as R_i^* under information-sharing regime $i = \{N, S\}$. The following lemma characterizes the equilibrium loan rates.

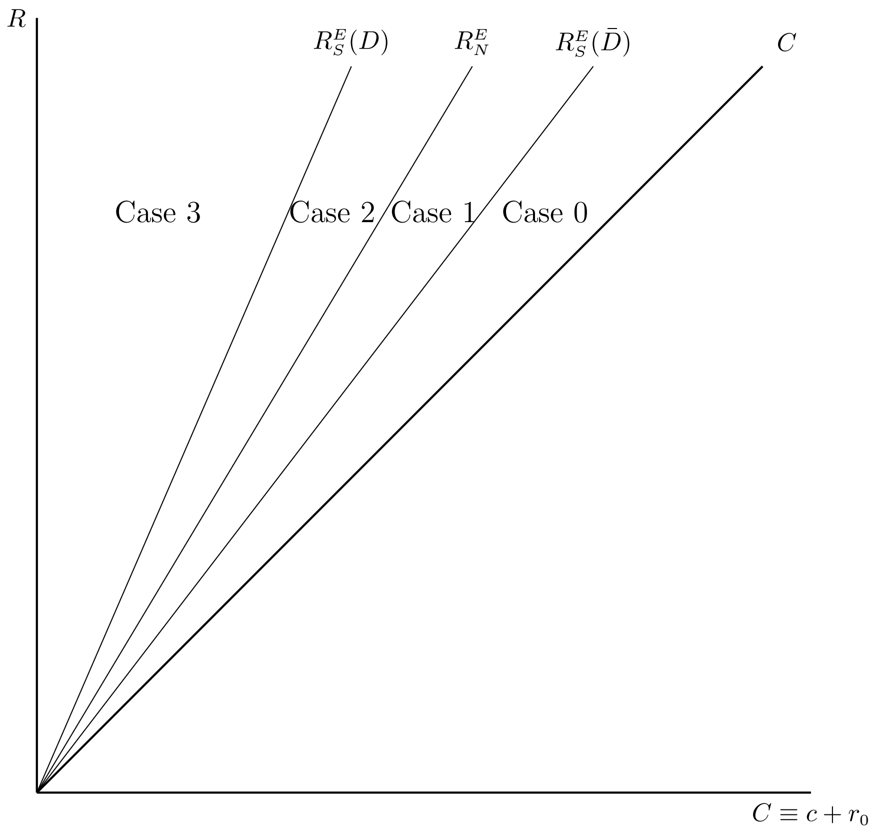
Lemma 2.4. *In equilibrium the loan is financed by the relationship bank. The equilibrium loan rates depend on the relationship between the distant bank's break-even loan rates and the project's return R. We have the following four cases:*

- *Case 0: If $R \in R_0 = (c + r_0, R_S^E(\overline{D})]$ then $R_S^*(\overline{D}) = R_N^* = R_S^*(D) = R$*
- *Case 1: If $R \in R_1 = (R_S^E(\overline{D}), R_N^E]$ then $R_S^*(\overline{D}) = R_S^E(\overline{D})$ and $R_N^* = R_S^*(D) = R$*
- *Case 2: If $R \in R_2 = (R_N^E, R_S^E(D)]$ then $R_S^*(\overline{D}) = R_S^E(\overline{D})$, $R_N^* = R_N^E$ and $R_S^*(D) = R$*
- *Case 3: If $R \in R_3 = (R_S^E(D), \infty)$ then $R_S^*(\overline{D}) = R_S^E(\overline{D})$, $R_N^* = R_N^E$ and $R_S^*(D) = R_S^E(D)$.*

Where R_j, with $j = \{0, 1, 2, 3\}$, denotes the set of payoffs of the project's return R for each case j. Consider Case 0, the payoff R is so low that distant bank does find convenient to enter the loan market. In this case, the loan market is least contestable, and the relationship bank charges the monopolistic loan rate R irrespective of the borrower's credit history. The higher R, and the more contestable the loan market becomes. In Case 3, the loan market is the most contestable since R is so high that the distant bank competes for a loan even when the borrower shows the defaulted D-history. The four cases are mutually exclusive, as it is clear from Figure 2.2 that represent them graphically.

Recall expressions (2.1) and (2.2), and the fact that the perceived loan quality is higher for a $\overline{D}$-loan with information sharing than for a loan with unknown credit history. The

Figure 2.2: Equilibrium loan rates: Interior and corner solutions



benefit of information sharing is to mitigate the adverse selection. However, we noticed that there is also a second effect that goes through the equilibrium loan rates R_N^* and $R_s^*(\overline{D})$. As $R_S^*(\overline{D}) \leq R_N^*$, it seems that information sharing *may* result in $P_S^B(\overline{D}) < P_N^B$ as it decreases loan rate from R_N^* to $R_S^*(\overline{D})$. We establish in Proposition 2.1 in the next section that the effect of reduced adverse selection is of the first order importance, and it is always true that $P_S^B(\overline{D}) > P_N^B$.

2.3.4 The Benefit of Information Sharing

We now show that in each of the cases $j = \{0, 1, 2, 3\}$, corresponding to different degree of loan market contestability, there exists a set of parameter values that guarantees the existence of a region where information sharing is indeed beneficial to the relationship bank. To be more specific, we show that there exists a parameter region where the relationship bank owning an H-type loan will survive from bank run when sharing information but will fail otherwise in the bad state. To understand intuitively when this can be the case, recall the analysis in Section 2.3.1 about the asset prices in the secondary loan market.

When the state is bad (B) only an H-type loan generates positive payoff, and there is adverse selection in the secondary market. If asset buyers do not know the exact type of a loan, it results in the underpricing of an H-type loan. Relationship bank may fail from a bank run even if it hold a safe H-type loan. Sharing information on credit history could therefore boost the asset price in the secondary market by mitigating the adverse selection. However, the distant bank also competes more fiercely with the relationship bank in the prime loan market for a borrower with good credit history. Accordingly, the relationship bank's profitability of financing an H-type of loan decreases. This in turn negatively affects the asset price in the secondary market.

The following result establishes the existence of a set of parameter values that guarantees that information sharing indeed promotes market liquidity in the bad state. Under such parameter values, the positive effect of mitigating adverse selection dominates the negative effect of lower profitability. We have

Proposition 2.1. *In the bad state, the equilibrium asset price is $P_S^B(\overline{D}) > P_N^B$. Whenever $P_S^B(\overline{D}) > r_0 > P_N^B$ information sharing can save the relationship bank from illiquidity.*

The proof is in the Appendix. The result can be easily verified with Case 0, where equilibrium loan rates are equal to R regardless of the information sharing regime. Indeed with $R_S^*(\overline{D}) = R_N^* = R$, the comparison between expression (2.1) and (2.2) is straightforward and we have $P_S^B(\overline{D}) > P_N^B$.

The result also hold for all other cases because of the presence of adverse selection both in the prime loan market and in the secondary asset market. We discuss Case 2 to give some the intuition. The best way to examine the relationship between $P_S^B(\overline{D})$ and P_N^B is to consider their ratio, which can be decomposed into a product of two elements

$$\frac{P_N^B}{P_S^B(\overline{D})} = \underbrace{\left(\frac{\Pr(L,\overline{D}) + \Pr(H)\Pr(\text{run})}{\Pr(L) + \Pr(H)\Pr(\text{run})}\right)}_{(1)} \underbrace{\left(\frac{\Pr(H) + \Pr(L,\overline{D})\Pr(G)}{\Pr(H) + \Pr(L)\Pr(G)}\frac{1}{\Pr(\overline{D})}\right)}_{(2)}.$$

Part (1) represents an increase in asset quality in the secondary market due to information sharing. It is a ratio of the expected average asset quality of $\overline{D}$-type loan under information sharing and the average asset quality under no information sharing regime in the secondary market. This ratio has a upper bound because of adverse selection in the secondary market

$$\frac{\Pr(L,\overline{D}) + \Pr(H)\Pr(\text{run})}{\Pr(L) + \Pr(H)\Pr(\text{run})} \leq \Pr(\overline{D}).$$

When the probability of a run increases, it becomes less likely that assets are on sale for strategic reason. As a result, the adverse selection in the secondary market decreases, and the gap in asset qualities under the two information regimes diminishes. However, it reaches a limit when $\Pr(\text{run}) \to 1$. Indeed, the adverse selection in the secondary market completely disappear when $\Pr(\text{run}) = 1$, and Part (1) reaches its upper bound $\Pr(\overline{D})$.[23]

Part (2) represents the extra rent that the relationship bank can extract from a $\overline{D}$-type borrower by not sharing information, and this rent diminishes when the adverse selection is mitigated in the prime loan market. Suppose a L-type borrower always generates a default credit history D in the previous lending relationship, the adverse selection would disappear in the prime loan market. Since under this assumption, the non default credit history (default credit history) must be generated by a H-type (L-type) borrower. With $\Pr(L,\overline{D}) \to \Pr(L)$, Part (2) reaches its upper bound $1/\Pr(\overline{D})$[24]

$$1 < \frac{\Pr(H) + \Pr(L,\overline{D})\Pr(G)}{\Pr(H) + \Pr(L)\Pr(G)} \frac{1}{\Pr(\overline{D})} \leq \frac{1}{\Pr(\overline{D})}.$$

The stronger the adverse selection in the prime loan market is, or the bigger the gap between $\Pr(L,\overline{D})$ and $\Pr(L)$ is, and the smaller Part (2) becomes. When adverse selection is mitigated for $\overline{D}$-type loan, the relationship bank extracts less profitability from financing $\overline{D}$-type of loan because the distant bank undercuts more for this type of loan.

Since both Part (1) and Part (2) are bounded from above, and the upper bounds are $\Pr(\overline{D})$ and $1/\Pr(\overline{D})$ respectively, we can conclude that $P_N^B < P_S^B(\overline{D})$ always holds. The

[23] Without information sharing, the average loan quality $\Pr(L) + \Pr(H)\Pr(\text{run})$ tends to 1, loan of any type will be sold for liquidity when $\Pr(\text{run}) = 1$. Similarly, with information sharing, any $\overline{D}$-type loan has to be sold for liquidity when $\Pr(\text{run}) = 1$, the average loan quality $\Pr(L,\overline{D}) + \Pr(H)\Pr(\text{run})$ tends to $\Pr(\overline{D})$.

[24] This is true because $\Pr(L,\overline{D}) \leq \Pr(L)$.

benefit of information sharing from the increase in average asset quality dominates the losses of information sharing from the reduction in rent extraction on the $\overline{D}$-type of borrower. Once this result is established, there must exist a set of parameters where the risk-free rate r_0 lies between the two prices and information sharing can save the bank from illiquidity. We will focus on those cases throughout the paper.

A corollary of Proposition 2.1 regards the equilibrium deposit rates that make information sharing valuable. We have

Corollary 2.1. *If* $r_N = \widehat{r}_N$ *and* $r_S(\overline{D}) = r_0$ *then information sharing can save the relationship bank from illiquidity.*

The intuition is as follows. For information sharing to be valuable, it must be able to prevent bank illiquidity. On the one hand, without information sharing, the relationship bank must face liquidity risk and it fails because of the run when the state is bad, even if it holds the safe H-type loan. This implies that the equilibrium deposit rate without information sharing r_N has to be risky. On the other hand, with information sharing, the relationship bank must never fail because of the run when the state is bad, even if it lends to the L-type borrower (in that case it would sell the asset for arbitrage which is the source of adverse selection). This implies that the equilibrium deposit rate with information sharing $r_S(\overline{D})$ has to be equal to the risk-free rate r_0.

Information sharing can endogenously emerge only inside the set of parameters specified in Proposition 2.1. Under this parameters restriction, the equilibrium deposit rates are those specified in Corollary 1. All other combinations of parameter values would not allow information sharing to be an equilibrium outcome. For example, assume $r_N = r_S(\overline{D}) = r_0$, then the relationship bank does not face any liquidity risk, therefore it will always survive with and without information sharing. Given that information sharing does not reduce liquidity risk, but it only intensify competition on the loan rates, the relationship bank will not choose to share its information on borrower's credit history. Similarly, assume $r_N = \widehat{r}_N$ and $r_S(\overline{D}) = \widehat{r}_S(\overline{D})$. The relationship bank faces liquidity risk and it would fail in case of a run both with and without information sharing. The bank again does not gain anything to disclose its information on the borrower. Finally, consider the case $r_N = r_0$ and $r_S(\overline{D}) = \widehat{r}_S(\overline{D})$. The relationship bank would fail in case of a run with information sharing and it survives without information sharing. The choice about sharing information is again clear. Notice however that the last case cannot exist since the parameter restrictions generate an empty set.

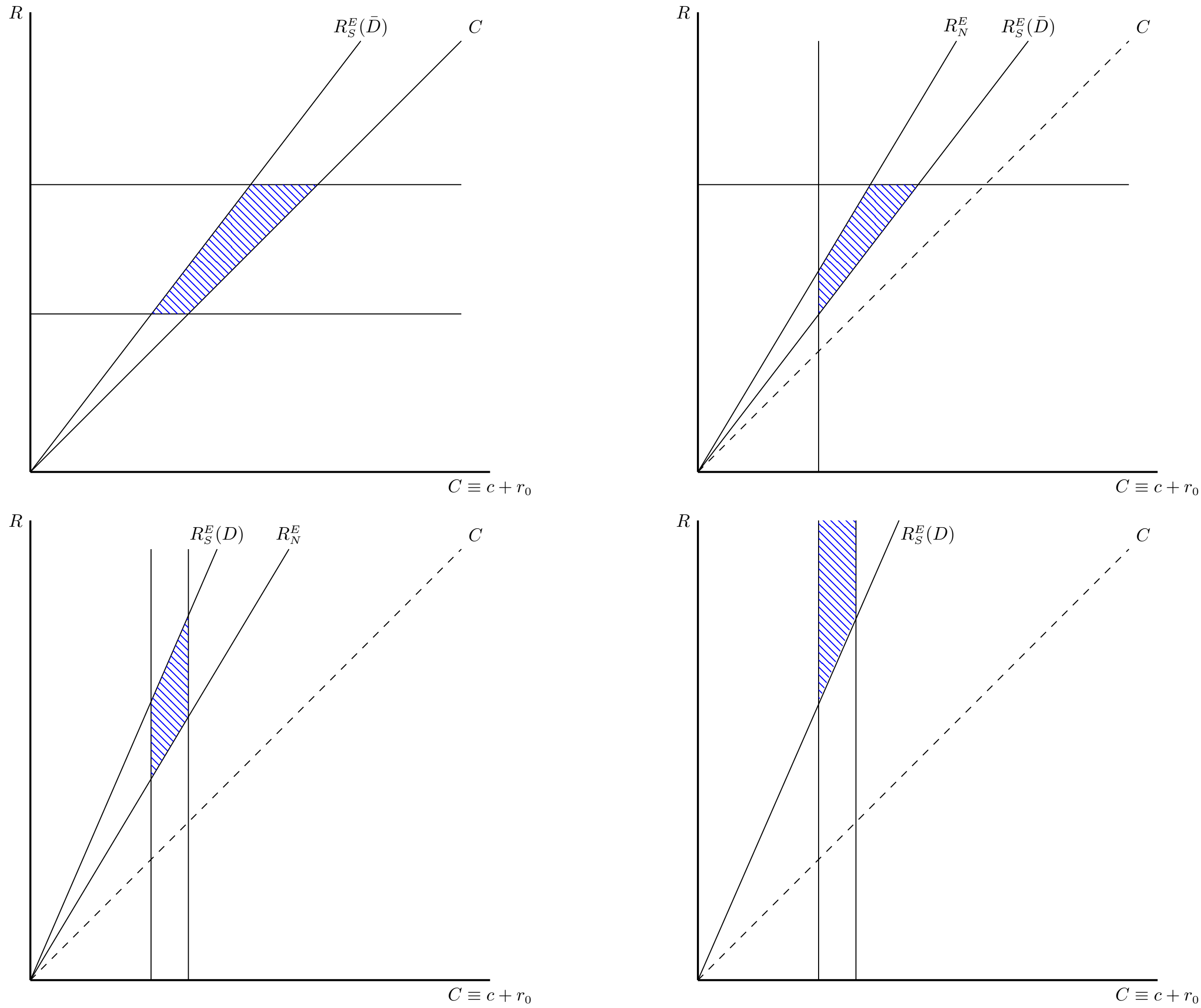
Given the result in Proposition 2.1, we define the set F_j with $j = \{0, 1, 2, 3\}$ such that the condition $P_S^B(\overline{D}) > r_0 > P_N^B$ holds. This is the set of parameters in each case j such that the relationship bank with $\overline{D}$-history loan survives from bank run in bad state when sharing information and fails because illiquidity in the bad state without information sharing. Recall that R_j, with $j = \{0, 1, 2, 3\}$, gives the set of payoffs R that defines different levels of contestability in the prime loan market. We define the intersection set $\Psi_j = R_j \bigcap F_j$ with $j = \{0, 1, 2, 3\}$. We have:

- $\Psi_0 = R_0 \bigcap F_0$ with $F_0 = \{R | \frac{(1-\alpha)\pi+\alpha\rho}{\alpha\rho} r_0 < R < \frac{(1-\alpha)+\alpha\rho}{\alpha\rho} r_0\}$.
- $\Psi_1 = R_1 \bigcap F_1$ with $F_1 = \{R | R < \frac{\alpha\rho+(1-\alpha)}{\alpha} r_0 \quad \text{and} \quad c + r_0 > \frac{\alpha\rho+(1-\alpha)\pi}{\alpha\rho} \frac{\alpha+(1-\alpha)\pi^2}{\alpha+(1-\alpha)\pi} r_0\}$.
- $\Psi_2 = R_2 \bigcap F_2$ with $F_2 = \{R | \frac{(1-\alpha)\pi+\alpha\rho}{\alpha\rho} \frac{\alpha+(1-\alpha)\pi^2}{\alpha+(1-\alpha)\pi} r_0 < c + r_0 < \frac{(1-\alpha)+\alpha\rho}{\alpha\rho} [\alpha + (1-\alpha)\pi] r_0\}$.
- $\Psi_3 = R_3 \bigcap F_3$ with $F_3 = F_2$.

Notice that the prices P_N^B and $P_S^B(\overline{D})$ are the same under Case 2 and Case 3. This is because the prime loan market is more contestable under these two cases. The distant bank competes with the relationship bank for a loan without knowing the credit history as well as for a loan with good credit history. Therefore we have $F_3 = F_2$. Figure 2.3 presents Cases 0, 1, 2 and 3 each with its respective blue shaded area in which the condition in Proposition 2.1 holds. In each of the four cases the relevant area exists, and we indicate this area as Ψ_j with $j = \{0, 1, 2, 3\}$. The non-shaded areas in Figure 2.3 correspond to the set of parameters in which information sharing is not beneficial in saving the relationship bank from illiquidity and then it cannot emerge in equilibrium.[25] We therefore do not further consider in our analysis such parameter values.

[25] To guarantee that the area where information sharing is beneficial exists in all four cases, we impose a further parameter restricion ($\frac{(1-\alpha)\pi+\alpha\rho}{\alpha\rho} > \frac{1}{\pi}$). The analysis of the relevant areas would be the same without such restriction.

Figure 2.3: Regions where information sharing can save the relationship bank from illiquidity



2.3.5 Equilibrium Information Sharing

We are now in a position to determine when information sharing emerges as an equilibrium of our game. We focus on the regions Ψ_j with $j = \{0, 1, 2, 3\}$. At $t = 0$, the relationship bank decides whether to choose the information sharing regime or the no information sharing regime by comparing the expected profits in those two regimes. Let us call the relationship bank's expected profits at $t = 0$ with V_i, where like before $i = \{N, S\}$.

The relationship bank's expected profits under no information sharing regime is

$$V_N = [\Pr(G) + \Pr(B)\Pr(H)\Pr(\text{no run})](R_N^* - r_N).$$

In the good state, the relationship bank will always survive irrespective of the type of its loan. However, in the bad state the relationship bank holding an H-type loan will survive only if there is no bank run.[26] Without information sharing scheme, the relationship bank cannot charge discriminative prices conditional on the borrower's type. Otherwise, it will reveal the borrower's type to the distant bank. Recall that the equilibrium deposit rate r_N under no information sharing regime is risky. That is, $r_N = \widehat{r}_N$ which is determined by $[\Pr(G) + \Pr(B)\Pr(H)\Pr(\text{no run})]\widehat{r}_N = r_0$. Therefore, we obtain

$$V_N = [\alpha + (1-\alpha)\pi^2]R_N^* + (1-\alpha)(1-\pi)\pi R_N^* - \alpha(1-\pi)\rho R_N^* - r_0.$$

When the relationship bank participates in the information sharing regime, its expected profits V_S are

$$V_S = \Pr(\overline{D})[\Pr(H|\overline{D})V_S^H(\overline{D}) + \Pr(L|\overline{D})V_S^L(\overline{D})] + \Pr(D)V_S^L(D), \tag{2.6}$$

where $V_S^H(\overline{D})$ and $V_S^L(\overline{D})$ are the expected profits of financing an H-type and an L-type borrower, respectively, when they generate the non default credit history $\overline{D}$. While $V_S^L(D)$ is the expected profit of financing an L-type borrower with default credit history D. Notice that when a loan has a credit history $\overline{D}$, with posterior probability $\Pr(H|\overline{D})$ it is an H-type loan. Moreover, $\Pr(D) = \Pr(L)\Pr(B) = (1-\alpha)(1-\pi)$ and $\Pr(\overline{D}) = 1 - \Pr(D) = \alpha + (1-\alpha)\pi$.

The expected profit of financing an H-type borrower with credit history $\overline{D}$ is

$$V_S^H(\overline{D}) = [\Pr(G) + \Pr(B)\Pr(\text{no run})]R_S^*(\overline{D}) + \Pr(B)\Pr(\text{run})P_S^B(\overline{D}) - r_0.$$

Notice that, given that we focus on the case in which information sharing saves the relationship bank from illiquidity, we have $r_S(\overline{D}) = r_0$. Moreover, the relationship bank will withhold H-type loan to maturity if no bank run occurs because $P_S^B(\overline{D}) = \frac{\alpha\rho}{(1-\alpha)\pi+\alpha\rho}R_S^*(\overline{D}) <$

[26]Recall that we focus on the case where the relationship bank with an H-type loan will survive from bank run when sharing information but will fail otherwise.

$R_S^*(\overline{D})$. Similarly, the expected profit of financing a L-type borrower with credit history $\overline{D}$ is given by

$$V_S^L(\overline{D}) = \Pr(G)R_S^*(\overline{D}) + \Pr(B)P_S^B(\overline{D}) - r_0.$$

When the relationship bank holds an L-type loan, in the bad state B the bank will sell it on the secondary market even without facing a run. Finally, a borrower that generates a default credit history D must be an L-type borrower. The equilibrium deposit rate is risky, that is $r_S(D) = r_0/\pi$. The expected profit of financing such a loan is

$$V_S^L(D) = \Pr(G)[R_S^*(D) - r_0/\pi] = \Pr(G)R_S^*(D) - r_0.$$

Insert the expressions of $V_S^H(\overline{D})$, $V_S^L(\overline{D})$ and $V_S^L(D)$ into equation (2.6), and we get after rearranging

$$V_S = [\alpha + (1-\alpha)\pi^2]R_S^*(\overline{D}) + (1-\alpha)(1-\pi)\pi R_S^*(D) - r_0.$$

Information sharing is preferred by the relationship bank if and only if $V_S - V_N > 0$. The difference between the expected profits in the two regimes can be rewritten as

$$V_S - V_N = \underbrace{[\alpha + (1-\alpha)\pi^2](R_S^*(\overline{D}) - R_N^*)}_{(1)} + \underbrace{(1-\alpha)(1-\pi)\pi(R_S^*(D) - R_N^*)}_{(2)} + \underbrace{\alpha(1-\pi)\rho R_N^*}_{(3)}.$$

The interpretation of the three terms is quite intuitive. Term (1) represents the *competition* effect, and it has a negative consequence on the adoption of the information sharing regime since $R_S^*(\overline{D}) \leq R_N^*$. Sharing information about the credit history encourages the distant bank to compete for the borrower with good credit history, i.e. $\overline{D}$-history. The expected profits of the relationship bank is reduced due to this effect because the entrant bank undercuts the loan rate when $\overline{D}$-history is observed. Term (2) is understood as the *capturing* effect, and it has positive impact on sharing information since $R_S^*(D) \geq R_N^*$. Sharing information about the borrower with bad credit history, i.e. D-history, deters the entry of distant bank. Thus the relationship bank can discriminate the borrower with D-history by charging higher loan rate. The expected profits of the relationship bank increases due to this effect. Finally, Term (3) denotes the *liquidity* effect, which is always positive. Sharing credit information of a borrower with good credit history reduces the adverse selection in the secondary credit market. In the bad state of nature, the relationship bank will be saved from potential bank run. This effect increases the expected profits of the relationship bank by avoiding costly asset liquidation.

The overall effect crucially depends if the capturing effect together with the liquidity effect dominate the competition effect. In that case the relationship bank chooses information sharing regime to maximize its expected profits. Denote with φ_j where $j = \{0, 1, 2, 3\}$ the set of parameters in which $V_S > V_N$ holds, then we have

Proposition 2.2. *The relationship bank chooses voluntarily to share information on* $\varphi_j = \Psi_j$ *with* $j = \{0, 3\}$ *and on* $\varphi_j \subseteq \Psi_j$ *with* $j = \{1, 2\}$. *Moreover, if* $\rho > (1 - \alpha)(1 - \pi)$ *then information sharing is chosen on* $\varphi_j = \Psi_j$ $\forall j$.

The proof is in the Appendix. The intuition is the following. In Cases 0 and 3 the set of parameters φ_j in which the relationship bank decide to share information coincides with the set Ψ_j in which information sharing saves the relationship bank from illiquidity. The reason is that there is no cost for the relationship bank to share information in both cases. In Case 0 because the distant bank never compete for the borrower, and in Case 3 because the distant bank always compete for the borrower. This is not true in Cases 1 and 2. In those two cases, the competition effect could overcome the sum of the capturing and the liquidity effects and the relationship bank would find it profitable to not sharing information. This reduces the set of parameters φ_j in which sharing information is actually chosen versus the set of parameters Ψ_j in which is actually beneficial. However, when the probability of bank run is sufficiently high, the benefit from sharing information becomes sufficiently high that the relationship bank find it convenient to share information whenever is beneficial to do so also in Cases 1 and 2 .

Figure 2.4: Regions where information sharing leads to greater value for the relationship bank(for $\rho < (1-\alpha)(1-\pi)$)

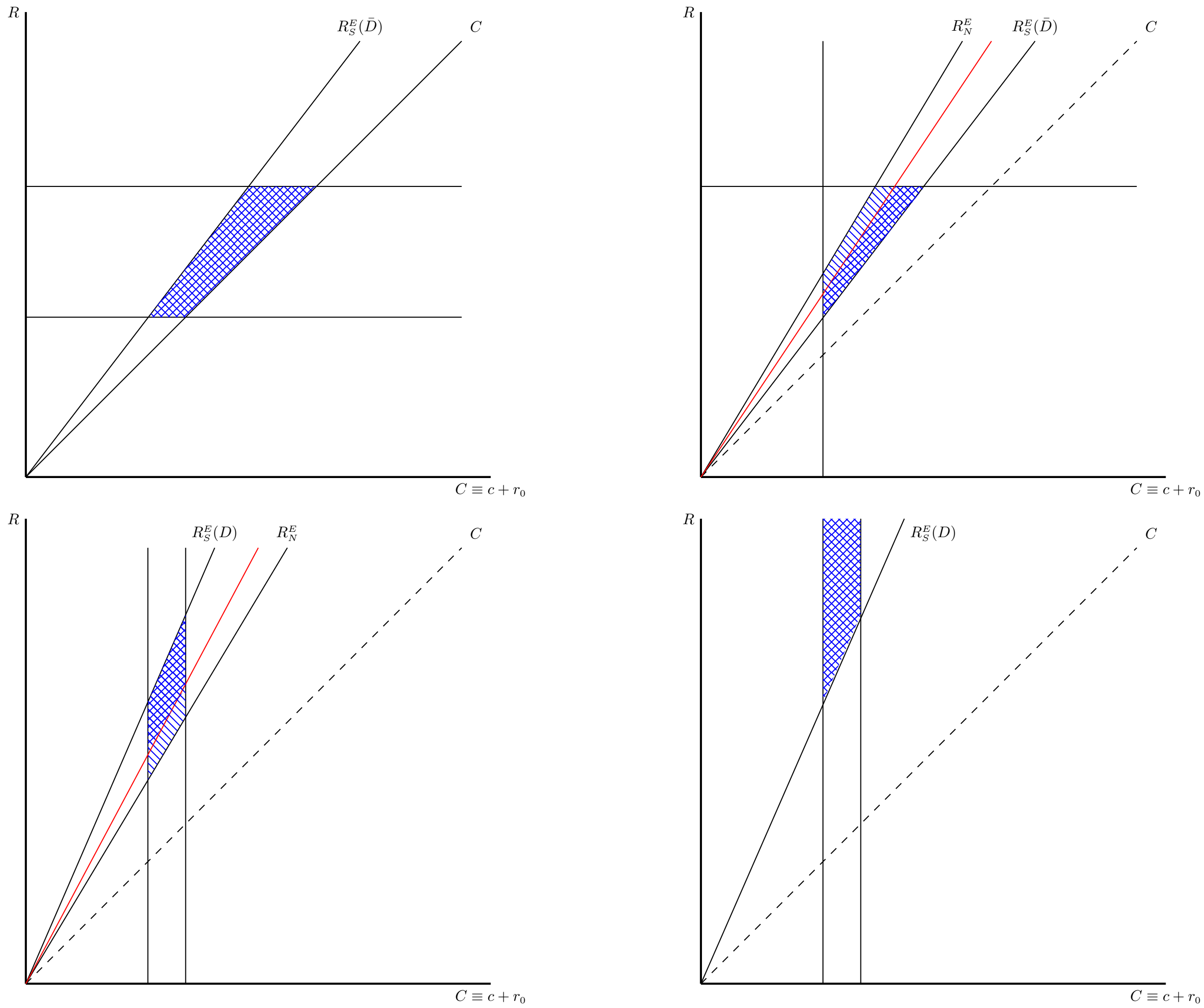


Figure 2.4 shows Cases 0, 1, 2 and 3 corresponding to different degree of loan market contestability. In each graph, the double-shaded blue area corresponds to the set of parameters φ_j. Clearly the double-shaded areas in Cases 0 and 3 correspond to the shaded areas in Figure 2.3. When ρ is lower than $(1-\alpha)(1-\pi)$, the double-shaded areas in the graphs of Cases 1 and 2 are smaller than the corresponding areas in Figure 2.3 (the red line is the boundary of the double-shaded area in which the relationship bank voluntarily chooses to share information). When ρ is higher than $(1-\alpha)(1-\pi)$ Figure 2.3 and 2.4 coincide.

2.4 Unverifiable Credit History

In this section we relax the assumption of verifiable credit history. If the reported borrower's credit history is not verifiable, the relationship bank that chooses to share such information may have an incentive to misreport the borrower's credit history after observing it. In particular, the relationship bank may have an incentive to overstate the borrower's credit history, that is to report a default credit history D as a non default credit history $\overline{D}$.[27] We have the following

Proposition 2.3. *The relationship bank truthfully discloses the borrower's credit history only if it leads to an increase in the loan rate for borrowers who have a default history* D*. This does not occur on* φ_j *with* $j = \{0, 1\}$*, and it does occur on* φ_2*, for sufficiently low* ρ*, and always on* φ_3*.*

The proof in the Appendix. In order to sustain truthfully reporting the credit history as an equilibrium, a necessary condition is that the relationship bank must suffer a loss when deviating from the equilibrium strategy. Consider the case in which the relationship bank lends to an L-type of borrower, which generated a default credit history D. If the relationship bank truthfully reveals the credit history, it is able to charge the loan rate $R_S^*(D)$. Yet, the relationship bank will not survive if the state is bad (i.e., with probability $1 - \pi$), because the asset buyers know that a loan with a credit history D is L-type and will generate zero payoff in state B. If the relationship bank lies about the credit history, the asset buyers as well as the distant bank will perceive the borrower to be more likely an H-type. Accordingly, the loan rate charged by the relationship bank is $R_S^*(\overline{D})$, which could be lower than $R_S^*(D)$ due to the intensified competition. However, cheating gives the relationship bank more resilience against the future liquidity shock since it can sell the loan in the secondary market at the price $P_S^B(\overline{D}) > r_0$ when the state is bad. The relationship bank always survives when cheating. Thus, the relationship bank trades off the benefit of market liquidity (surviving in state B) versus the loss in profitability

[27] We assume misreporting $\overline{D}$ as D to be impossible, that is the relationship bank cannot claim non-defaulted borrower as defaulted. This is because borrowers have means and incentive to correct it or act against it (e.g., FCA in US). Moreover, according to the documentations in www.doingbusiness.com, borrowers can access their own credit record. A false report about defaulting can result in a legal dispute.

(potential decrease in loan rate from $R^*_S(D)$ to $R^*_S(\overline{D})$) when deciding to tell the truth about the reported credit history. Notice that a pre-requisite for the relationship bank to manipulate the reported credit history is that it must choose the information sharing regime in the first place. Thus, we focus our discussion on the intuition in each case, on the parameter sets φ_j, with $j = \{0, 1, 2, 3\}$, defined in Section 2.3.5.

Consider Case 0. We have $R^*_S(D) = R^*_S(\overline{D}) = R$ therefore the relationship bank always has incentive to misreport the true D-history as $\overline{D}$-history in the parameters space φ_0. The loan market is least contestable and we have $R^*_S(D) = R^*_S(\overline{D}) = R$. Assuming truthfully reporting, ex-ante participating in information sharing is more profitable for the relationship bank in the parameter set φ_0. However, when the relationship bank observes a credit history D ex-post, it will incur no loss in profit to misreport the credit history as $\overline{D}$ because $R^*_S(D) = R^*_S(\overline{D})$. Consequently, the relationship bank will always misreport the true D-history as $\overline{D}$-history in the parameters set φ_0. Truthfully reporting the credit history can never be an equilibrium in Case 0.

Since in the other cases we have $R^*_S(D) > R^*_S(\overline{D})$, there is hope for the relationship bank to report the true credit history. However, as noticed, this is only a necessary condition. Even if ex-post the relationship bank has an incentive to tell the truth, it is possible that ex-ante it is not willing to share information. The parameters that guarantee the ex-post truth telling have to be consistent with those that induce ex-ante information sharing.

Consider Case 1. On the one hand, assuming truthfully reporting, the relationship bank ex-ante prefers to participate in information sharing scheme when R is low. This is because its expected profit without sharing information is increasing in R $(R^*_N = R)$, while the expected profit with information sharing is increasing in R only if the relationship bank lend to an L-type borrower. On the other hand, in order to make the relationship bank report the true credit history ex-post, R must be high. This is because the deviating penalty increases with R, that is $R^*_S(D) = R$ while $R^*_S(\overline{D})$ is an internal solution thus it does not depend on R. It turns out that the ex-ante and ex-post conditions on R determine an empty set and therefore truthfully reporting can not be sustained as an equilibrium in the parameter space φ_1.

Consider Case 2. On the one hand, assuming truthfully reporting, the relationship bank ex-ante prefers to participate information sharing scheme when R is high. This is because the loan market becomes more contestable, the expected profit without information sharing does not depend on R any more (R_N^* becomes an internal solution), while the expected profit with information sharing is increasing in R (with L-type borrower, the loan rate is $R_S^*(D) = R$). On the other hand, in order to make the relationship bank report the true credit history ex-post, the return R must be high since, as in Case 1, $R_S^*(D) = R$ and $R_S^*(\overline{D})$ is an internal solution. It turns out that the ex-ante condition on R is more restrictive than the ex-post condition only if ρ is lower than a critical value $\hat{\rho}$.[28]. Under this condition, whenever the relationship bank finds ex-ante optimal to share information it also will report ex-post the true credit history, and truthful reporting can be sustained as an equilibrium in the parameter space φ_2.

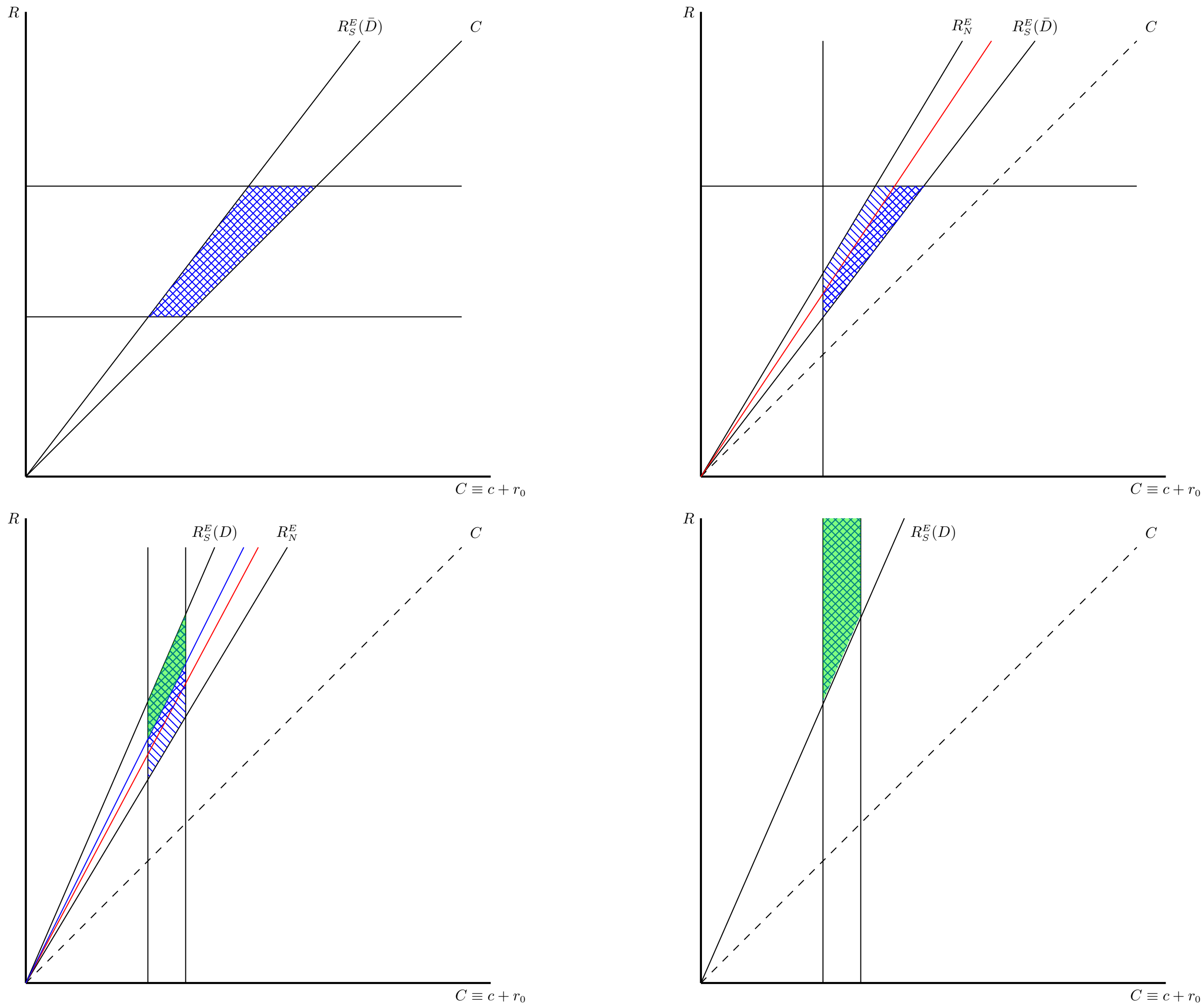
Finally, consider Case 3. Assuming truthfully reporting, the relationship bank ex-ante always prefer information sharing (irrespective of R). Moreover, the prime loan market is most contestable, $R_S^*(D) = \frac{c+r_0}{\pi} > R_S^*(\overline{D})$. It turns out that the relationship bank earns a strictly negative profit by ex-post misreporting D history with $\overline{D}$. This is because, $R_S^*(D)$ is substantially higher than $R_S^*(\overline{D})$, so the relationship bank's expected loss in profit overcomes its expected gain from market liquidity by misreporting the credit history. As a result, truthful reporting is sustained as an equilibrium in the parameter space φ_3.

To sum up, by truthfully reporting the credit history, the relationship bank can discriminate the L-type borrower by charging higher loan rate. When misreport the credit history, the relationship bank has to charge a lower loan rate but benefits from the higher market liquidity to survive potential runs. If the market is less contestable, the profit from the discriminative loan pricing is bounded above by the loan's return R. Thus, in Case 0 and 1, the benefit from the higher market liquidity to save the bank from run in state B, dominates the loss in profit. The relationship bank will lie in those Cases. However, in Case 2 and 3, the return R is sufficiently large and the profit from discriminative loan pricing tends to dominate the benefit from market liquidity. Truthfully reporting the credit history can be sustained as equilibrium in those two Cases.

[28] $\hat{\rho}$ is the value under which the relationship bank is indifferent between reporting the true credit history about D and lying D history to $\overline{D}$

Figure 2.5 shows Cases 0, 1, 2 and 3 each with its respective dark-blue area corresponding to the set of parameters in which truth-telling is an equilibrium. In Cases 0 and 1 such area is empty since truth-telling is not possible under these Cases. In Case 2 we show a situation where truth-telling can be sustained in a subsect of φ_2, which occurs when $\rho < \min\{\hat{\rho}, (1-\alpha)(1-\rho)\}$. In case 3, since truth-telling is always sustained in the entire regions φ_3, the dark-blue area coincide with the area in Figure 2.4.

Figure 2.5: Regions where truthful information sharing can be sustained in a perfect Bayesian equilibrium (for $\rho < \min\{\hat{\rho}, (1-\alpha)(1-\pi)\}$)



2.5 Welfare and Policy Implication

We first notice what is the socially efficient level of information sharing. Suppose a benevolent social planner knows borrower's type, then the planner would always invest (all positive NPV projects). Moreover, there are two sources of frictions: i) information power of the relationship bank over the borrower; ii) adverse selection in the secondary market for loan sale. Since both frictions are reduced by information sharing, from a social perspective maximum information sharing is preferred. Indeed, the planner does not care about friction i), but reducing friction ii) is better for everybody.

From a private perspective, relationship bank values information sharing since it reduces the adverse selection problem in the secondary asset market enhancing asset market liquidity. But it also reduces market power vis a vis the borrower. This can generates a private level of information sharing that is less than the efficient one.

This is seen comparing the shaded areas in Figure 2.3 and the double-shaded areas in Figure 2.4. In Cases 0 and 3 the two areas coincide so there is no inefficient choice. However in Cases 1 and 2 the relationship bank chooses a level of information sharing that is less than what would be (socially) optimal. In this Cases sharing information is costly, and the private cost of the relationship bank is higher than the social cost.

The endogenous arise of private registries is rational from the bank's point of view, but can be inefficiently low in some circumstances. A public registry can increase welfare in Cases 1 and 2, without harming in Cases 0 and 3.

2.6 Conclusion

This paper formally analyzes the conjecture according to which banks' decision to share information about the credit history of their borrowers is driven by the needs for market liquidity. To meet urgent liquidity needs, banks have to make loan sale in the secondary market. However, the information friction in loan markets makes this sale costly and good loans can be priced below their fundamental value. This concern became very evident during the financial crisis started in the summer of 2007. Several potentially solvent banks risk to fail because they could not raise enough short term liquidity.

This basic observation implies that banks could find convenient to share information on their loans in order to reduce the information asymmetry about their quality in case they have to sell them in the secondary market. Information sharing can be a solution to reduce the cost of urgent liquidity needs so to make banks more resilient to funding risk. Clearly, sharing information makes banks to lose the rent they extract if credit information were not communicated. Banks may be no longer able to lock in their loan applicants because competing banks also know about the quality of those loans. Eventually, the benefit of a greater secondary market liquidity has to be traded off with the loss in information rent. We show that it possible to rationalize information sharing as such device. We show under which conditions information sharing is feasible, and when is actually chosen by the banks in equilibrium.

We also show that our rationale for information sharing is robust to truth telling. A common assumption in the literature is that when banks communicate the credit information, they share it truthfully. We allow banks to manipulate the information they release by reporting bad loans as good ones. The reason is for the banks to increase the liquidation value in the secondary market. We show that when banks lose too much in information rent from good borrowers with bad credit history, then information sharing is a truth telling device.

Coherently with previous theoretical model of information sharing, the existing empirical literature has mostly focused on the impact of information sharing on bank risks and firms' access to bank financing. Our theoretical contribution generates new empirical implications. In particular, information sharing should facilitate banks liquidity management and loan securitization. The model also suggests that information sharing can be more easily established, and work more effectively, in countries with competitive banking sector, and in credit market segments where competition is strong.

Proof of Lemma 2.1. Recall that the depositors' break even rates are r_0 when deposits are safe and $\widehat{r}_N$ ($> r_0$) when deposits are risky. Depositors are competitive so they bid against each other in determining the equilibrium deposit rate r_N to finance the bank. Depositors take the asset price P_N^B and the break even rates as given. Under the assumption of perfect competition, a necessary condition for the equilibrium deposit rate is that it has to guarantee zero expected profits to depositors.

We prove statement (i) by contradiction. Let us consider three cases.

Case (a), the parameters are such that $P_N^B > \widehat{r}_N$. Assume the equilibrium deposit rate is $r_N > P_N^B > \widehat{r}_N$. If this rate were indeed the equilibrium rate, then the deposits were risky because the asset price P_N^B is not enough to repay r_N in equilibrium. Their break even rate is $\widehat{r}_N$. However, the depositors could make positive profit if this were the case, since $r_N > \widehat{r}_N$. A deposit rate higher than P_N^B cannot be an equilibrium. Assume the equilibrium deposit rate is $P_N^B \geq r_N > \widehat{r}_N$. If this were the case, the deposits are safe, depositors' break even rate is r_0. But if this rate were the equilibrium rate, again the depositors could make positive profit since $\widehat{r}_N > r_0$. The equilibrium deposit rate can not be higher than $\widehat{r}_N$. Assume the equilibrium rate is $\widehat{r}_N \geq r_N > r_0$. Deposits are again safe, and depositors can make positive profit since r_N is larger than r_0. Lastly, assume the equilibrium rate is $r_0 > r_N$. If this were the case, the depositors make negative profit. As a result, the only candidate equilibrium deposit rate is $r_N = r_0$. Under this rate, the deposits are safe and depositors make zero expected profit. Each depositor does not have incentive to undercut below $r_N = r_0$. Thus $r_N = r_0$ is the unique equilibrium deposit rate.

Case (b), the parameters are such that $P_N^B = \widehat{r}_N$. Assume the equilibrium deposit rate is $r_N > P_N^B = \widehat{r}_N$, then the deposits are risky. But if r_N were the equilibrium rate, the depositors would earn positive profit because $r_N > \widehat{r}_N$. Assume the equilibrium rate is $P_N^B = \widehat{r}_N \geq r_N > r_0$, then the deposits are again safe but depositors would earn positive profit since $r_N > r_0$. If $r_0 > r_N$ depositors make negative profit. Thus, the unique equilibrium deposit rate is again $r_N = r_0$, under which the depositors have no incentive to undercut.

Case (c), the parameters are such that $r_0 \leq P_N^B < \widehat{r}_N$. Assume $r_N > \widehat{r}_N > P_N^B$, then deposits are risky. The rate r_N is making depositors earn positive profit. Assume $r_N =$

$\widehat{r}_N > P_N^B$, then deposits are risky and depositors earn zero profit. But if this rate were the equilibrium rate, then the depositors can offer an alternative rate as $r_N = P_N^B - \epsilon$. Under this new rate $r_N < P_N^B$, the deposits become safe and the depositors can instead make positive profit as $r_N = P_N^B - \epsilon \geq r_0$. There exists a profitable deviation. Assume $\widehat{r}_N > r_N > P_N^B$, the deposits are risky and the depositors will never finance the bank as they make negative profit. Assume $\widehat{r}_N > P_N^B \geq r_N > r_0$, the deposits are risk-free but the depositors could make positive profit. Lastly, assume $r_0 > r_N$, the depositors again make negative profit. We have the unique equilibrium deposit rate is $r_N = r_0$. Under this rate, the deposits are safe and the depositors make zero profit. The depositors have no incentive to undercut further otherwise they make negative profit.

In sum, the unique equilibrium deposit rate is $r_N = r_0$, and deposits are safe.

To prove statement (ii), notice that the only case to consider is $\widehat{r}_N > r_0 > P_N^B$. Assume $r_N > \widehat{r}_N > r_0 > P_N^B$, the deposits are risky yet under this rate the depositors could make positive profit. Assume $\widehat{r}_N > r_N > r_0 > P_N^B$ or $\widehat{r}_N > r_0 \geq r_N > P_N^B$, the deposits are also risky but the depositors make negative profit. Assume $\widehat{r}_N > r_0 > P_N^B \geq r_N$, the deposits are safe but the depositors make negative profit. Lastly, assume $r_N = \widehat{r}_N > r_0 > P_N^B$, then the deposits are risky but make zero expected profit. They have no incentive to undercut further since otherwise they will make negative profit. Thus, the unique equilibrium deposit rate is $r_N = \widehat{r}_N$ and deposits are risky. *Q.E.D.*

Proof of Lemma 2.2. The logic of the proof is similar to the one provided in Lemma 1, with the only difference that we focus on the loan with a past non-defaulted history $\overline{D}$. The depositors' break even rates are r_0 when their deposits are safe and $\widehat{r}_S(\overline{D})$ when the deposits are risky. Depositors are competitive, so they bid against each other in determining the equilibrium deposit rate $r_S(\overline{D})$.

We prove statement (i) by contradiction, and we consider three cases.

Case (a), the parameters are such that $P_S^B(\overline{D}) > \widehat{r}_S(\overline{D})$. Assume $r_S(\overline{D}) > P_S^B(\overline{D}) > \widehat{r}_S(\overline{D})$, we have risky deposits but positive profit. Assume $P_S^B(\overline{D}) > r_S(\overline{D}) > \widehat{r}_S(\overline{D}) > r_0$ and $\widehat{r}_S(\overline{D}) > r_S(\overline{D}) > r_0$, we have safe deposits but positive profit. Assume $r_0 > r_S(\overline{D})$, we have negative profit. The unique equilibrium rate is $r_S(\overline{D}) = r_0$.

Case (b), the parameters are such that $P_S^B(\overline{D}) = \widehat{r_S}(\overline{D}) > r_0$. Assume $r_S(\overline{D}) > P_S^B(\overline{D}) = \widehat{r_S}(\overline{D})$, we have risky deposits but positive profit. Assume $\widehat{r_S}(\overline{D}) > r_S(\overline{D}) > r_0$, we again have safe deposits but positive profit. Assume $r_0 > r_S(\overline{D})$, we have negative profit. The unique equilibrium rate is $r_S(\overline{D}) = r_0$.

Case (c), the parameters are such that $P_S^B(\overline{D}) < \widehat{r_S}(\overline{D})$. Assume $r_S(\overline{D}) > \widehat{r_S}(\overline{D}) > P_S^B(\overline{D})$, we have the deposits are risky but depositors are making positive profit. Assume $r_S(\overline{D}) = \widehat{r_S}(\overline{D}) > P_S^B(\overline{D})$, the deposits are again risky and the depositors earn zero profit. But the depositors can undercut to offer $r_S(\overline{D}) = P_S^B(\overline{D}) - \epsilon$ to make the deposits safe and earn positive profit. Assume $\widehat{r_S}(\overline{D}) > r_S(\overline{D}) > P_S^B(\overline{D})$, the deposits are risky and the depositors make negative profit. Assume $\widehat{r_S}(\overline{D}) > P_S^B(\overline{D}) \geq r_S(\overline{D}) > r_0$, the deposits are risk-free but the depositors could make positive profit. Last, assume $r_0 > r_S(\overline{D})$ and depositors get negative profit. We have the unique equilibrium deposit rate is $r_S(\overline{D}) = r_0$. Under this rate, the deposits for the bank with a loan of past history $\overline{D}$ are safe, the depositors make zero expected profit. The depositors have no incentive to undercut otherwise they make negative profit.

To prove statement (ii), notice that we have to consider the case in which $\widehat{r_S}(\overline{D}) > r_0 > P_S^B(\overline{D})$. Assume $r_S(\overline{D}) > \widehat{r_S}(\overline{D}) > r_0 > P_S^B(\overline{D})$, then deposits are risky yet the depositors make positive profit. Assume $\widehat{r_S}(\overline{D}) > r_S(\overline{D}) > r_0 > P_S^B(\overline{D})$, then deposits are risky and the depositors make negative profit. Assume $r_S(\overline{D}) = \widehat{r_S}(\overline{D}) > r_0 > P_S^B(\overline{D})$, then deposits are risky but the depositors make zero expected profit. They have no incentive to undercut as well since otherwise they will make negative profit. Thus, the unique equilibrium deposit rate is $r_S(\overline{D}) = \widehat{r_S}(\overline{D})$ and we have risky deposits for a bank with a loan of past history $\overline{D}$. *Q.E.D.*

Proof of Proposition 2.1. Recall expressions (2.1) and (2.2) that determines equilibrium asset prices in the secondary market. They are

$$P_N^B = \frac{\alpha\rho}{(1-\alpha)+\alpha\rho} R_N^*$$

and

$$P^B_S(\overline{D}) = \frac{\alpha\rho}{(1-\alpha)\pi + \alpha\rho} R^*_S(\overline{D}),$$

where R^*_N and $R^*_S(\overline{D})$ are the equilibrium loan rates under no information sharing and information sharing regime, respectively. Notice that the average loan quality in the secondary market without information sharing ($\frac{\alpha\rho}{(1-\alpha)+\alpha\rho}$) is lower than the average loan quality with information sharing ($\frac{\alpha\rho}{(1-\alpha)\pi+\alpha\rho}$).

Consider Case 0. The distant bank does not compete for any loan even if the relationship bank shared the credit history of the borrower. The relationship bank extracts the entire payoff of the loan irrespective of the information sharing regime, that is $R^*_S(\overline{D}) = R^*_N = R$. Information sharing solely brings in the benefit from boosting asset liquidity for loan with $\overline{D}$ history. Consequently, $P^B_S(\overline{D}) > P^B_N$.

Consider Case 2 (for the easy of exposition it is convenient to analyze this case first). Distant bank competes both under information sharing (and the borrower has no default history $\overline{D}$) and when there is no information sharing. The equilibrium loan rates are therefore

$$R^*_N = \frac{c + r_0}{\alpha + (1-\alpha)\pi} > \frac{\alpha + (1-\alpha)\pi}{\alpha + (1-\alpha)\pi^2}(c + r_0) = R^*_S(\overline{D}).$$

We want to show that

$$P^B_N = \frac{\alpha\rho}{(1-\alpha) + \alpha\rho}\frac{c + r_0}{\alpha + (1-\alpha)\pi} < \frac{\alpha\rho}{(1-\alpha)\pi + \alpha\rho}\frac{\alpha + (1-\alpha)\pi}{\alpha + (1-\alpha)\pi^2}(c + r_0) = P^B_S(\overline{D}),$$

which can be rewritten as

$$\frac{(1-\alpha)\pi + \alpha\rho}{(1-\alpha) + \alpha\rho}\frac{\alpha + (1-\alpha)\pi^2}{[\alpha + (1-\alpha)\pi]^2} < 1.$$

To show that the last inequality holds, we notice that the ratio $\frac{(1-\alpha)\pi+\alpha\rho}{(1-\alpha)+\alpha\rho}$ is increasing in ρ, so its maximum value is reached when $\rho = 1$ and it equal to $(1-\alpha)\pi + \alpha$ $\left(= Pr(\overline{D})\right)$. Therefore, the maximum value of the LHS of the last inequality can written as

$$[(1-\alpha)\pi + \alpha]\frac{\alpha + (1-\alpha)\pi^2}{[\alpha + (1-\alpha)\pi]^2} = \frac{\alpha + (1-\alpha)\pi^2}{\alpha + (1-\alpha)\pi},$$

which is smaller than 1 since $\pi \in (0, 1)$. Thus, $P^B_S(\overline{D}) > P^B_N$.

Consider Case 1. The distant bank only competes for the loan with past non-defaulted history $\overline{D}$. The equilibrium loan rate $R_S^*(\overline{D})$ is determined by the distant bank. Without information sharing, the relationship bank can discriminate the borrower by charging $R_N^* = R > R_S^*(\overline{D})$. The competition effect is clearly smaller than under Case 2. Since $P_S^B(\overline{D}) > P_N^B$ always holds in Case 2, then it necessarily holds also in Case 1.

Consider Case 3. The distant bank competes no matter the past history of the borrower. The relevant equilibrium loan rates R_N^* and $R_S^*(\overline{D})$ do not change with respect Case 2. The relationship between the prices $P_S^B(\overline{D})$ and P_N^B is the same as the one analyzed in Case 2. Thus, $P_S^B(\overline{D}) > P_N^B$.

Since we have that is all cases $P_N^B < P_S^B(\overline{D})$, by continuity when r_0 is located in between these two prices the relationship bank survives from illiquidity under information sharing regime and fails under no information sharing regime. *Q.E.D.*

Proof of Proposition 2.2. For each Case $j = \{0, 1, 2, 3\}$ we consider the parameter set Ψ_j defined in Proposition 1.

Consider Case 0. We have: $V_S = [\alpha + (1-\alpha)\pi]R - r_0$ and $V_N = [\alpha - \alpha(1-\pi)\rho + (1-\alpha)\pi]R - r_0$. Then $V_S > V_N$ for the entire region Ψ_0. Thus $\varphi_0 = \Psi_0$.

Consider Case 1. We have: $V_S = [\alpha + (1-\alpha)\pi](c + r_0) + (1-\alpha)(1-\pi)\pi R - r_0$ and $V_N = [\alpha - \alpha(1-\pi)\rho + (1-\alpha)\pi]R - r_0$. Therefore,

$$V_S - V_N = [\alpha + (1-\alpha)\pi](c + r_0) - [(1-\alpha)\pi^2 + \alpha - \alpha(1-\pi)\rho]R.$$

Notice that $(1-\alpha)\pi^2 + \alpha - \alpha(1-\pi)\rho > 0$. We have that $V_S - V_N > 0$ if and only if

$$R < \frac{\alpha + (1-\alpha)\pi}{\alpha - \alpha(1-\pi)\rho + (1-\alpha)\pi^2}(c + r_0) \equiv R_1.$$

We define the region φ_1 as follows

$$\varphi_1 = \Psi_1 \bigcap \{R | R < R_1\} \subseteq \Psi_1.$$

If R_1 is greater than the upper bound R_N^E of R defining Case 1 then information sharing is preferred for the entire region Ψ_1. That is, if

$$R_1 = \frac{\alpha + (1-\alpha)\pi}{\alpha - \alpha(1-\pi)\rho + (1-\alpha)\pi^2}(c + r_0) > \frac{\alpha + (1-\alpha)\pi}{\alpha + (1-\alpha)\pi^2}(c + r_0) = R_N^E$$

the set φ_1 coincides with Ψ_1. We can simplify the last inequality as

$$\rho > (1-\alpha)(1-\pi).$$

Otherwise, when $\rho < (1-\alpha)(1-\pi)$, we have $\varphi_1 \subset \Psi_1$. Indeed, notice that R_1 is increasing in ρ. When $\rho \to 0$, we have $R_1 \to \frac{\alpha+(1-\alpha)\pi}{\alpha+(1-\alpha)\pi^2}(c + r_0) = R_S^E(\overline{D})$. Recall the definition of region Ψ_1, we always have such $\varphi_1 = \Psi_1 \bigcap \{R | R < R_1\}$ non-empty for any value of $\rho \in (0, 1)$ and $\varphi_1 \subset \Psi_1$ when $\rho < (1-\alpha)(1-\pi)$.

Consider Case 2. We have $V_S = [\alpha + (1-\alpha)\pi](c + r_0) + (1-\alpha)(1-\pi)\pi R - r_0$ and $V_N = [\alpha - \alpha(1-\pi)\rho + (1-\alpha)\pi]\frac{c+r_0}{\alpha+(1-\alpha)\pi} - r_0$. Therefore,

$$V_S - V_N = [\alpha + (1-\alpha)\pi](c + r_0) + (1-\alpha)(1-\pi)\pi R - [1 - \frac{\alpha(1-\pi)\rho}{\alpha + (1-\alpha)\pi}](c + r_0).$$

We have $V_S - V_N > 0$ if and only if

$$R > [1 - \frac{\alpha\rho}{(1-\alpha)[\alpha + (1-\alpha)\pi]}]\frac{c + r_0}{\pi} \equiv R_2.$$

We define the set φ_2 as follows

$$\varphi_2 = \Psi_2 \bigcap \{R | R > R_2\} \subseteq \Psi_2.$$

If R_2 is lower than the lower bound of R defining Case 2 then information sharing is preferred for the entire region Ψ_2. That is, if

$$[1 - \frac{\alpha\rho}{(1-\alpha)[\alpha + (1-\alpha)\pi]}]\frac{c + r_0}{\pi} < \frac{c + r_0}{\alpha + (1-\alpha)\pi}$$

the set $\varphi_2 = \Psi_2$. We can simplify the last inequality again as

$$\rho > (1-\alpha)(1-\pi).$$

Otherwise, when $\rho < (1-\alpha)(1-\pi)$ we have $\varphi_2 \subset \Psi_2$. Indeed, also R_2 is decreasing in ρ. When $\rho \to 0$, we have $R_2 \to \frac{c+r_0}{\pi} = R_S^E(D)$. Recall the definition of region Ψ_2, we always have such φ_2 non-empty for all $\rho \in (0,1)$ and $\varphi_2 \subset \Psi_2$ when $\rho < (1-\alpha)(1-\pi)$.

Consider Case 3. We have $V_S = c$ and $V_N = c - \alpha(1-\pi)\rho\frac{c+r_0}{\alpha+(1-\alpha)\pi}$, therefore

$$V_S - V_N = \alpha(1-\pi)\rho\frac{c+r_0}{\alpha+(1-\alpha)\pi} > 0.$$

In this case we have $\varphi_3 = \Psi_3$ and information sharing is preferred by the relationship bank. *Q.E.D.*

Proof of Proposition 2.3. Suppose the distant bank, depositors and asset buyers all hold the belief that the relationship bank will tell the truth about the credit history of the borrower. We analyze the profitable deviation of the relationship bank to announce truthfully a defaulted D-history under such belief. We focus our discussion on the parameter set φ_j with $j = \{0,1,2,3\}$ defined in Proposition 2.

Consider Case 0. We first compute the relationship bank's expected profit at $t = 1$ of truthfully reporting a loan with default credit history D. Recalling that $R_S^*(D) = R$ in this case, we have

$$V_S(D) = \pi R_S^*(D) - r_0 = \pi R - r_0. \tag{A.1}$$

The expected profit of misreporting the borrower's true credit history (i.e., reporting the false $\overline{D}$-history) is

$$V_S(D,\overline{D}) = \Pr(G)R_S^*(\overline{D}) + \Pr(B)P_S^B(\overline{D}) - r_0 = \pi R + (1-\pi)\frac{\alpha\rho}{\alpha\rho+(1-\alpha)\pi}R - r_0.$$

Notice the relationship bank does not fail by misreporting the credit history. Clearly we have $V_S(D) - V_S(D,\overline{D}) < 0$. The relationship bank finds it profitable to misreport the borrower's credit history. The benefit from the deviation $(1-\pi)\frac{\alpha\rho}{\alpha\rho+(1-\alpha)\pi}R$ is the expected liquidation loss in case of bank run. Under this case, the belief of outsiders can not be rationalized, and truthful information sharing can not be sustained as a Perfect Bayesian Equilibrium in the set of parameter φ_0.

Consider Case 1. Like in Case 0, the relevant equilibrium loan rate is $R_S^*(D) = R$. Then reporting the true default history gives the same expected profit as in (A.1). The expected profit of misreporting the true credit history with the false $\overline{D}$-history can be expressed as

$$\begin{aligned} V_S(D,\overline{D}) &= \Pr(G)R_S^*(\overline{D}) + \Pr(B)P_S^B(\overline{D}) - r_0 \\ &= \pi\frac{\alpha+(1-\alpha)\pi}{\alpha+(1-\alpha)\pi^2}(c+r_0) + (1-\pi)\frac{\alpha\rho}{\alpha\rho+(1-\alpha)\pi}\frac{\alpha+(1-\alpha)\pi}{\alpha+(1-\alpha)\pi^2}(c+r_0) - r_0, \end{aligned}$$

since $R_S^*(\overline{D}) = \frac{\alpha+(1-\alpha)\pi}{\alpha+(1-\alpha)\pi^2}(c+r_0)$ in this Case. Then we have

$$V_S(D,\overline{D}) = \frac{\alpha\rho+(1-\alpha)\pi^2}{\alpha\rho+(1-\alpha)\pi}\frac{\alpha+(1-\alpha)\pi}{\alpha+(1-\alpha)\pi^2}(c+r_0) - r_0. \tag{A.2}$$

Then the ex-post incentive compatibility constraint to tell the truth is

$$V_S(D) - V_S(D,\overline{D}) = \pi R - [\pi + (1-\pi)\frac{\alpha\rho}{(1-\alpha)\pi+\alpha\rho}]\frac{\alpha+(1-\alpha)\pi}{\alpha+(1-\alpha)\pi^2}(c+r_0) > 0,$$

which can be simplified as

$$R > \left[\frac{\alpha\rho+(1-\alpha)\pi^2}{\alpha\rho+(1-\alpha)\pi}\frac{\alpha+(1-\alpha)\pi}{\alpha+(1-\alpha)\pi^2}\right]\frac{c+r_0}{\pi} \equiv \underline{R}. \tag{A.3}$$

Information sharing is ex-ante chosen in Case 1 when (recall the definition of R_1 in the proof of Proposition 2.2)

$$R < \frac{\alpha+(1-\alpha)\pi}{\alpha-\alpha(1-\alpha)\rho+(1-\alpha)\pi^2}(c+r_0) \equiv R_1.$$

It can be shown that $R_1 - \underline{R} = -\alpha^2(1-\rho)\rho(1-\pi) < 0$. Consequently, there exists no R such that the relationship bank will ex-ante participate in information sharing scheme and ex-post report the true default credit history of a borrower. The belief of outsiders can not be rationalized and truthful information sharing can not be sustained as a Perfect Bayesian Equilibrium in the set of parameter φ_1.

Consider Case 2. We again have $R_S^*(D) = R$. Reporting the true default history gives the same expected profit as in (A.1). The expected profit of misreporting the true credit history is the same as in expression (A.2), since $R_S^*(\overline{D}) = \frac{\alpha+(1-\alpha)\pi}{\alpha+(1-\alpha)\pi^2}(c+r_0)$ also in this Case.

Therefore the condition on R to ensure ex-post the relationship bank tells the truth is the same as in (A.3). Information sharing is ex-ante chosen in Case 2 when (recall the definition of R_2 in the proof of Proposition 2.2)

$$R > \left[1 - \frac{\alpha\rho}{(1-\alpha)[\alpha + (1-\alpha)\pi]}\right] \frac{c + r_0}{\pi} \equiv R_2.$$

Information sharing can be sustained as a Perfect Bayesian Equilibrium only if both the inequality $R > R_2$ and the condition (A.3) are satisfied. In particular, we find a region of parameters in which whenever is ex-ante optimal for the relationship bank to share information is also ex-post convenient for it to tell the true credit history. This implies to impose the following restriction

$$1 - \frac{\alpha\rho}{(1-\alpha)[\alpha + (1-\alpha)\pi]} > \frac{\alpha\rho + (1-\alpha)\pi^2}{\alpha\rho + (1-\alpha)\pi} \frac{\alpha + (1-\alpha)\pi}{\alpha + (1-\alpha)\pi^2}. \tag{A.4}$$

Note that the expression (A.4) can be rewritten as

$$1 - \frac{\alpha\rho}{(1-\alpha)[\alpha + (1-\alpha)\pi]} - \frac{\alpha\rho + (1-\alpha)\pi^2}{\alpha\rho + (1-\alpha)\pi} \frac{\alpha + (1-\alpha)\pi}{\alpha + (1-\alpha)\pi^2} = 0.$$

We define a function $F(\rho) = 1 - \frac{\alpha\rho}{(1-\alpha)[\alpha+(1-\alpha)\pi]} - \frac{\alpha\rho+(1-\alpha)\pi^2}{\alpha\rho+(1-\alpha)\pi} \frac{\alpha+(1-\alpha)\pi}{\alpha+(1-\alpha)\pi^2}$. It can be checked that

$$F'(\rho) = -\frac{\alpha}{(1-\alpha)[\alpha + (1-\alpha)\pi]} - \frac{\alpha(1-\alpha)\pi(1-\pi))}{[\alpha\rho + (1-\alpha)\pi]^2} \frac{\alpha + (1-\alpha)\pi}{\alpha + (1-\alpha)\pi^2} < 0.$$

Moreover, we can take the limits

$$\lim_{\rho \to 0} F(\rho) = 1 - \frac{\alpha\pi + (1-\alpha)\pi^2}{\alpha + (1-\alpha)\pi^2} > 0$$
$$\lim_{\rho \to 1} F(\rho) = -\frac{\alpha}{(1-\alpha)[\alpha + (1-\alpha)\pi]} < 0.$$

Thus, there exists a unique $\hat{\rho}$ such that $F(\hat{\rho}) = 0$. Whenever $0 < \rho < \hat{\rho}$, we have $F(\rho) > 0$ and expression (A.4) holds. Then truth telling can be sustained as a Perfect Bayesian Equilibrium in the set of parameter φ_2. Recall that we established in Proposition 2.2 that φ_2 is non-empty for all $\rho \in (0, 1)$.

Consider Case 3. In this Case we have $R_S^*(D) = (c + r_0)/\pi$ since the distant bank competes also for the defaulted borrower. Reporting the true default history gives an expected profit equal to

$$V_S(D) = \pi R_S^*(D) - r_0 = c.$$

The expected profit of misreporting the credit history is the same as in (A.2), and since

$$\frac{\alpha\rho + (1-\alpha)\pi^2}{\alpha\rho + (1-\alpha)\pi} \frac{\alpha + (1-\alpha)\pi}{\alpha + (1-\alpha)\pi^2} < 1,$$

we have $V_S(D, \overline{D}) - V_S(D) < 0$. The belief of outsiders can be rationalized, and truthful information sharing can be sustained as a Perfect Bayesian Equilibrium in the set of parameter φ_3. *Q.E.D.*

Chapter 3

CEO Compensation Design in a Multiplicative Model

3.1 Introduction

Both restricted stock and stock options are designed to link CEOs' future wealth to firms' stock price performance, therefore, to mitigate agency problems between the firms' shareholders and CEOs outlined in (Jensen and Meckling, 1976). But there is an open debate about whether stock options should be a part of the optimal CEO compensation contracts. (Hall and Murphy, 2002) and (Jenter, 2002) demonstrate that stock options are inefficient either because their economic value to risk averse CEOs or their created incentives are overstated. Yet, (Lambert and Larcker, 2004) shows that stock options dominate restricted stock in providing incentives in a model where effort affects the distribution of firm's stock price, i.e., more incentives are conveyed through stock options in the range where stock prices are higher. (Dittmann and Maug, 2007) calibrates the traditional principal-agent model in (Hölmstrom, 1979) with constant relative risk aversion and log-normal stock prices and find that the optimal contract should not include any stock options. Motivated by these conflicting findings, alternative models and theories to explain the observed CEO compensation contracts are developed.[29] In this paper, we rationalize the mix of restricted stock and stock options in the optimal CEO compensation contracts in a model where the CEO's preference and the firm's production functions are both multiplicative.

It is well recognized that non-linearities in the pay structure affect CEO's risk taking behavior. (Guay, 1999) suggests that boards add stock options to CEO compensation contracts to induce risk averse CEOs to adopt risky but value enhancing projects. However, in the aftermath of the 2008-2009 financial crisis, there emerges another concern that stock options provide CEOs with incentives to engage in excessive risk taking. [30] We show that stock options indeed induce the CEOs to follow a project choice rule that is both excessive risky and shareholders' value decreasing compared to the optimal one.

We present a model where the CEO first exerts effort to develop a new project for the firm. The new project is risky, yet increases the expected firm value compared to a safe project originally owned by the firm. However, the project implementation is not made based on the ex ante information. Instead, we allow the CEO to collect private information about the risky project after exerting effort and make a second choice about which project to implement based on that piece of information. The CEO's effort is assumed to have a multiplicative effect on the firm value. Thus, we consider that CEO's actions, such as exerting effort and making project choices, are "rolled out" across the entire firm and thus have a greater effect in a larger firm. Moreover, CEO's preference is also assumed to be multiplicative. Both of the assumptions are made because they are better fit to the empirical data compared to the ones in the traditional agency model, i.e., effort has an additive impact on firm value and CEO has an additive cost of effort that is independent of his wealth.[31]

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[29] Papers such as (Hemmer, Kim, and Verrecchia, 1999), (Dittmann, Maug, and Spalt, 2010) and (Chaigneau, 2013) postulate the specific forms of CEO's preferences while papers such as (Feltham and Wu, 2001) and (Dittmann and Yu, 2011) justify the optimality of the observed mixed contracts with shareholders' risk-taking incentive. There is a paralleled inefficient contracting view as well. (Yermack, 1995) challenges the explanatory power of efficient contracting for the pattern of stock option awards and (Bebchuk and Fried, 2004) argues that CEOs of large corporations design their own pays to extract more rents.

[30] Regulatory actions have already been taken to limit the use of stock options. For instance, in the American Recovery and Reinvestment act of 2009, stock options are explicitly prohibited in the executive compensation plans for the TARP recipients. Only two types of compensation, base salaries and restricted stock limited no more than half of base salaries are allowed.

[31] For further justifications of the multiplicative specifications, readers can refer to (Edmans and

In the paper, we first demonstrate that when stock options should be a part of the optimal compensation contract and derive the optimal mix of stock and options taking into account both effort and project choice problems. Different from the previous papers, risk taking in our model is excessive from the shareholders' perspective and is a side effect of the provision of effort incentives. The CEO, after privately observing the quality of new project generated from his effort, makes the project implementation choice between the new risky one and the old safe one to maximize his expected compensation. Paying the CEO in restricted stock perfectly aligns his interests with that of the firm's shareholders in selecting the project. On the other hand, the CEO makes excessively risky project choices from the shareholders' perspective when options are included in his pay contract. The reason is straightforward that options make the CEO's pay function convex in the firm's value. As a result of motivating effort and curbing excessive risk taking, stock should always be a part of the optimal contract. Moreover, options should not be included in the pay contract if stock perform perfectly the role of motivating effort. However, if the CEO's effort is not extremely productive, or the CEO's private benefit from shirking is quite high, restricted stock cannot motivate effort even if effort is still worth exerting. The described circumstances occur as a special feature of the model where CEO's utility is multiplicative between monetary compensation and private benefit. When the CEO shirks, the monetary compensation from restricted stock is lower as the firm value is lower, yet the CEO's total utility could be higher because the private benefit is higher. Thus, to motivate effort, the pay structure needs to be convex in the firm value, i.e., to be insensitive in the lower value to further reduce the monetary compensation when the CEO shirks. Under the circumstances described above, stock options as a part of optimal compensation can be rationalized.[32] The shareholders face a tradeoff of motivating effort and mitigating excessively risky project choice induced by options when deciding the optimal mix of restricted stock and stock options in the compensation package.

Second, we extend the multiplicative model with CEO's effort issue in (Edmans, Gabaix, and Landier, 2009) to consider a second agency problem: CEO's project choice. In our

Gabaix, 2009).

[32]Notice that in this paper, we focus on the choice between restricted stock and stock options. Including options is not the only means to increase the convexity of the pay function. Cash bonuses with a performance target, for instance, could have the similar effect. But we could show as in (Innes, 1990) and (Inderst and Mueller, 2010), options are optimal in providing convexity.

model, the CEO's incentive compatibility constraint has the following form: The ratio of the CEO's expected pay in case of exerting effort to his expected pay in case of shirking must exceed the ratio of the CEO's private benefit in case of shirking to his private benefit in case of exerting effort. The optimal contract is no longer detail-neutral as in (Edmans, Gabaix, and Landier, 2009). It still predicts the amount of restricted stock or stock options to be proportional to the CEO's total pay, yet a proper measure of the firm risk is necessary to provide the right "amount" of risk taking incentives. This is a direct implication that the CEO makes project choice for the firm. When the CEO's action concerns firm's risk choice, the optimal mix of stock and options is a function of the optimal project choice.

The third objective of the paper is to analyze the general market equilibrium by embedding the above double agency problem into a labor market where firms with different asset sizes compete for CEOs with different ability. Our departure from the traditional work[33] is that the managerial contracting affects the general equilibrium matching between firms and CEOs. In our model, high ability CEOs are with high chances to discover risky project with better quality. We show that, in the circumstance where firms are able to contract for CEOs' effort and the optimal project choices, a firm's expected value will be higher if it is matched with a higher ability CEO. Thus, there is positive assortative matching in the market equilibrium: Larger firms hire higher ability CEOs. However, in the circumstance where the optimal project choice can not be attained, larger firms may tend to hire lower ability CEOs. In the second best contracting, CEOs make excessively risky project choices, in which case firms' expected value will be decreased. A higher ability CEO is with higher chances to generate the projects that are considered as excessively risky from the perspective of a firm's shareholders. To motivate effort in the second best setting, a firm has to pay its CEO in stock options, thus makes concession in curbing the CEO's excessive risk taking. Consequently, when a higher ability CEO is hired, the likelihood that this CEO is able to implement the excessively risky project is higher. Once the value losses due to the excessive risk taking is large, a firm tends to hire a lower ability CEO to reduce the probability of excessive risk taking. Positive assortative matching between firm size and CEO ability may fail to hold because the

[33] In (Edmans, Gabaix, and Landier, 2009), managerial contracting is separated from the equilibrium assignment.

friction generated from the managerial contracting.

Our paper is related to the traditional literature in the following aspects. First, our analysis of the optimal structure of compensation contract is obviously linked to the debate about the optimality of stock options. Our paper contributes to the debate by focusing on the different role of restricted stock and stock options in providing effort and risk taking incentives. Stock always provide CEOs with the "right" incentives in taking the project risk, while options dominate stock in providing the effort incentives. Thus, options and stock are not always perfect substitutes in the optimal compensation contract.

Our double agency framework is built on the earlier work of (Lambert, 1986) where firm contracts with CEO for both the managerial effort and best project choice. (Hirshleifer and Suh, 1992) further analyzes a model where the manager first selects the project risk then exerts effort and shows that the optimal curvature of the manager's compensation contract depends on the trade-off between controlling project risk and motivating effort. The focus of these models is to motivate the risk averse CEO to take value enhancing risk. Instead, our model analyzes the CEO's excessive risky project choice as a side effect of providing effort incentives and focus on the choice between restricted stock and stock options.

The agency issues are studied in a model of multiplicative specifications for CEO preference and production functions. The academic researches of CEO compensation have drawn attention to the multiplicative model primarily because its fit for the empirical data. (Edmans, Gabaix, and Landier, 2009) studies a multiplicative model in presence of CEO's moral hazard problem and obtains a parsimonious form of optimal incentive contract that rationalizes the CEO's low fractional ownership and its negative relationship with firm size. (Edmans and Gabaix, 2011) further introduces CEO's risk aversion into (Edmans, Gabaix, and Landier, 2009). Our model follows the multiplicative specifications but focuses on the structure of contract and the interaction between the double agency issues: effort and project choice. (Thanassoulis, 2013) applies a multiplicative model to analyze CEO's short termism behavior and its interaction with industry structure. (Peng and Röell, 2014) also analyzes a multiplicative model and finds convex contracts for some parameterizations but their main focus is on price manipulation.

The works as (Edmans, Gabaix, and Landier, 2009), (Edmans and Gabaix, 2011), (Thanassoulis, 2012) and (Thanassoulis, 2013) consider the optimal contracting problem with a labor market where firms act competitively in hiring CEOs to endogenize not only the pay structure and but also the pay level. In most papers, the general equilibrium prescribes a positive assorting matching assignment between firms with different sizes and CEOs with different ability. There is one distinction, (Edmans and Gabaix, 2011) shows taking in account the CEO's risk aversion, equilibrium assignment is distorted from the positive assortative matching. Our model also considers the equilibrium assignment in the CEO labor market and challenges the positive assortative matching from another angle. We allow CEOs to make project choice decision and show the circumstance where large firms prefer to hire CEOs with lower ability occurs when reducing the excess in the CEOs' risk taking choice is the firms' major concern.

Section 2 presents the setup of basic model. We analyze the optimal contracting where the board of a representative firm contracts with a representative CEO in fixed salary, restricted stock or stock options for effort and project choice in Section 3. Section 4 extends the baseline model to incorporate a competitive labor market where the competitive assignment between firms and CEOs is analyzed. Section 4 discusses the exercise price of options and the relationship between firm size and structure of optimal contract. Section 5 concludes.

3.2 The Basic Model

The basic model considers the case where a risk neutral female principal contracts with a risk neutral male agent to elect effort and induce best project choice in a firm. The principal can be the board whose objective is to maximize the expected net firm' value. The agent is the CEO hired by the firm whose objective is to maximize his expected utility. The model has one period and 3 dates $t = 0, 0.5, 1$. At $t = 0$, the board offers a contract to the CEO, the CEO chooses whether to exert effort to expand the firm's project opportunity. At $t = 0.5$, if effort is exerted, the CEO makes a second choice regarding the project implementation based on his private information. At $t = 1$, the return realizes.

3.2.1 Projects, effort and project choice

The initial ($t = 0$) value of the firm's asset is I. The firm has already owned a safe project. Investing asset I in the safe project, the firm's value at $t = 1$ is $\underline{y}I$, where the unit value of the safe project is certain $\underline{y} > 1$. Alternatively, the CEO can expend effort to innovate a new risky project for the firm. Conditional on the risky project being implemented, the firm's value at $t = 1$ is $\tilde{y}I$ if the project succeeds and zero if it fails. The success probability of the risky project is φ with the random unit value $\tilde{y}$ distributed on the interval $[\underline{y}, \overline{y}]$. If the CEO shirks, the only available project is the safe project. We assume that the risky project has higher expected value than the safe project at $t = 0$ and focus on the case that the firm always wants to induce the CEO's effort.

Denote the CEO's unobservable action to exert effort (shirk) as $e = 1$ ($e = 0$). Once effort is exerted, the CEO can privately observe a signal θ about the risky project at $t = 0.5$. The conditional distribution function of the unit value $\tilde{y}$ is $F(y|\theta)$ on $[\underline{y}, \overline{y}]$. We assume that the signal is informative in the sense that the project with higher θ has better distribution of value: $F(y|\theta_2)$ first order stochastically dominates $F(y|\theta_1)$ for any $\theta_2 > \theta_1$. The signal can be understood as the project quality, higher quality risky project has better value distribution. On the other hand, the board only knows that θ is distributed on the interval $[\underline{\theta}, \overline{\theta}]$ with distribution function $G(\theta)$.

After receiving the private signal about the quality of the new risky project, the CEO decides which project to implement at $t = 0.5$. We assume the project implementation is mutually exclusive between the safe and the risky project. The unit value of safe project is certain while the expected value of risky project increases in its realized quality θ[34], we assume the CEO's project choice follows a threshold rule, i.e., there exists a threshold

[34]We assumed that quality θ improves the distribution $F(y|\theta)$ in the sense of FOSD. A necessary condition of FOSD is that for any $\overline{\theta} \geq \theta_2 > \theta_1 \geq \underline{\theta}$, the expectation $E[y|\theta_2] > E[y|\theta_1]$.

$\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that he chooses the risky project once the quality is larger than the threshold $\theta > \hat{\theta}$ and the safe project otherwise.

3.2.2 CEO Compensation

The firm's value at $t = 1$ is observable and verifiable, yet, the action of exerting effort and project quality θ are the CEO's private information. The board faces a double agency problem. Judging from the realized firm value, she neither knows for sure whether a choice of risky project is because of its higher expected value nor a choice of safe project is not because the CEO shirks. To tackle with this problem, the board can design a CEO compensation contract with three components, fixed salary, restricted stock and stock options. Fixed salary α is the amount of cash granted to the CEO on a non-performance basis. Normalize the number of the bank's outstanding shares to be one, $\beta \in [0, 1]$ is the number of the shares granted to the CEO at $t = 0$. The CEO is allowed to sell these shares at $t = 1$. $\gamma \in [0, 1]$ is the number of the stock options granted to the CEO at $t = 0$ with an exercise price $\hat{y} \in (\underline{y}, \overline{y})$. To summarize, the CEO's total realized compensation at $t = 1$ is

$$c(yI) = \alpha + \beta yI + \gamma \max(y - \hat{y}, 0)I$$

Lastly, there is the normal limited liability constraint $\alpha \geq 0$.

3.2.3 Utility

The board maximizes the expected net firm value (the total value net the compensation cost) from the investment. The CEO has a multiplicative utility function:

$$U = E[c\phi(e)].$$

c is denoted as the CEO's monetary compensation. We let $\phi(e)$ characterizes the cost of effort with $\phi(1) = 1 < \phi = \phi(0)$ and $\phi > 1$. Thus, $\phi(e)$ can be understood as the CEO's equivalent monetary utility from leisure, with more utility derived from leisure when the CEO shirks. The CEO's reservation utility is u. Notice that the CEO's utility is a multiplication of the monetary compensation and utility from leisure. This type of multiplicative preference is commonly used in Macroeconomics and Labor Economics. In (Edmans, Gabaix, and Landier, 2009), the authors find this type of multiplicative preference fits the empirical evidences regarding to the incentives and firm sizes quite well. Throughout the paper, we will assume the optimal level of effort is always to be $e = 1$.

3.3 Optimal Contracting in Partial Equilibrium

In this section, we discuss the contracting in the basic model treating the CEO's reservation utility u as exogenous. In the next section, we will extend the basic model to incorporate a competitive labor market. We first present the benchmark case where the contract could be directly written on the CEO's effort and best project choice maximizing the firm's value. We then move to the case where compensation contract needs to be designed to trade off the provision of effort and controlling excessive risk taking. We characterize the IC constraint in a multiplicative model, which is slightly different from the one in additive model. We show in the first best contracting, the optimal contract can use only stock to provide incentives. While the second best contract prescribes an optimal mix of shares and options trading off the needs to motivate effort and control excessive risk taking.

3.3.1 Benchmark:Contractible Effort and Project Choice

To start, suppose the contract can directly designate the CEO to exert effort and choose the project desired by the board. Then the board simply pays the CEO to participate $c = u$, demands the CEO's effort and chooses the project choice to maximize the expected net firm value

$$\Pi(\hat{\theta}) = \int_{\hat{\theta}}^{\overline{\theta}} E[y|\theta] I dG(\theta) + \int_{\underline{\theta}}^{\hat{\theta}} y_s I dG(\theta). \tag{3.1}$$

We assume this function is strictly concave to $\hat{\theta}$. Thus, the first order condition with respect to project choice $\hat{\theta}$ gives $E[y|\hat{\theta}] = \underline{y}$, where $E[y|\hat{\theta}] = \varphi \int_{\underline{y}}^{\overline{y}} y dF(y|\hat{\theta})$. The first best project choice is the solution of this first order condition. As we have already discussed, the expected value of risky project conditional on a project choice $\hat{\theta}$ is increasing in the project choice. We focus on the interior solution by assuming $E[y|\underline{\theta}] < E[y|\theta^{FB}] < E[y|\overline{\theta}]$. Then there exists a unique first best project choice $\theta^{FB} \in (\underline{\theta}, \overline{\theta})$ defined as the solution

$$E[y|\theta^{FB}] = \underline{y} \tag{3.2}$$

To maximize the expected firm value, The risky project is chosen if and only if its quality is higher than θ^{FB}. We then turn to the case that the compensation contract can only be contingent on the realization of the firm value.

3.3.2 Managerial Contracting in the Partial Equilibrium

In this subsection, we analyze the contracting problem between the board and the CEO. Under the equilibrium approach that effort is exerted, the CEO's expected compensation from a risky project with quality θ at $t = 0.5$ is

$$\alpha + \beta E[y|\theta]I + \gamma E[\max(y - \hat{y}, 0)|\theta]I$$

at $t = 0.5$. The component $E[\max(y - \hat{y}, 0)|\theta] = \varphi \int_{\underline{y}}^{\overline{y}} \max(y - \hat{y}, 0) dF(y|\theta)$ is the unit value of the option conditional on the project quality θ and exercise price $\hat{y}$. On the other hand, the CEO's compensation from the safe project is simply $\alpha + \beta \underline{y} I$. The risk neutral CEO chooses to implement the project with higher expected compensation, thus he chooses the risky project if and only if its quality is higher than the threshold, which is define as

$$\beta E[y|\hat{\theta}] + \gamma E[\max(y - \hat{y}, 0)|\hat{\theta}] = \beta \underline{y}. \tag{3.3}$$

For the similar reason, the unit option value increases in θ as well. We assume $\hat{\theta}$ defined in this equation is interior as well. Thus, $\hat{\theta}$ defined in (3.3) is existent and unqiue.

Anticipating the project choice rule $\hat{\theta}$, the expected compensation when the CEO exerts effort at $t = 0$ is

$$E[c\phi(e)|e = 1, \hat{\theta}] = \alpha + \int_{\hat{\theta}}^{\overline{\theta}} \{\beta E[y|\theta] + \gamma E[\max(y - \hat{y}, 0)|\theta]\} IdG(\theta) + \int_{\underline{\theta}}^{\hat{\theta}} \beta \underline{y} IdG(\theta).$$

This expression can be reformulated to

$$E[c\phi(e)|e = 1, \hat{\theta}] = \alpha + \beta[\underline{y} + \int_{\hat{\theta}}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta)]I + \gamma \int_{\hat{\theta}}^{\overline{\theta}} V_o(\hat{\theta}) dG(\theta))I$$

where the integrand $\Delta V_y(\theta) = E[y|\theta] - \underline{y}$ is the difference in expected value between implementing a risky project and a safe project conditional on the quality θ. Then, the term $\int_{\hat{\theta}}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta)$ is the unit "upside" value of implementing the risky project conditional on that the CEO follows project choice rule $\hat{\theta}$. The integrand $V_o(\theta) = E[\max(y - \hat{y}, 0)|\theta]$ is the expected unit value of option conditional on the quality θ. Similarly, the term $\int_{\hat{\theta}}^{\overline{\theta}} V_o(\hat{\theta}) dG(\theta))$ represents the unit option value conditional on that the CEO follows project choice rule $\hat{\theta}$. And the term in the parenthesis $\underline{y} + \int_{\hat{\theta}}^{\overline{\theta}} \Delta V_y dG(\theta)$ is the expected unit firm value conditional on the project choice $\hat{\theta}$. On the other hand, the CEO's compensation is $E[c\phi(e)|e = 0] = \phi[\alpha + \beta \underline{y} I]$ when he shirks. The utility is a multiplication of the monetary compensation $\alpha + \beta \underline{y} I$ and ϕ the private benefit from leisure. Remember that the private benefit is 1 when the CEO works. Thus it is incentive compatible for the CEO to exert effort if $E[c\phi(e)|e = 1] \geq E[c\phi(e)|e = 0]$, that is

$$\frac{\alpha + \beta[\underline{y} + \int_{\hat{\theta}}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta)]I + \gamma \int_{\hat{\theta}}^{\overline{\theta}} V_o(\theta) dG(\theta))I}{\alpha + \beta \underline{y} I} \geq \phi \tag{3.4}$$

Notice that the IC constraint in a multiplicative preference model differs from the one in an additive model. The incentives to exert effort prescribes a ratio between the CEO's expected pay when he works to the expected pay when he shirks conditional on the CEO's ex post project choice. Motivating effort requires this ratio between the CEO's expected pays conditional on a certain project choice $\hat{\theta}$ outweighs the ratio between the CEO's private benefit from leisure when he shirks to the private benefit when he works $\frac{\phi}{1}$.

The CEO's participation constraint $E[c\phi(e)|e = 1] \geq u$ can be expresses as

$$\alpha + \beta[\underline{y} + \int_{\hat{\theta}}^{\overline{\theta}} \Delta V_y(\theta)dG(\theta)]I + \gamma \int_{\hat{\theta}}^{\overline{\theta}} V_o(\theta)dG(\theta))I \geq u. \tag{3.5}$$

Then there is the limited liability constraints $\alpha \geq 0$. The expected end-of-period return of the firm can be expressed as:

$$\Pi = [\underline{y} + \int_{\hat{\theta}}^{\overline{\theta}} \Delta V_y(\theta)dG(\theta)]I - \left\{\alpha + \beta[\underline{y} + \int_{\hat{\theta}}^{\overline{\theta}} \Delta V_y(\theta)dG(\theta)]I + \gamma \int_{\hat{\theta}}^{\overline{\theta}} V_o(\theta)dG(\theta))I\right\}. \tag{3.6}$$

The firm maximizes the expected return (3.6) subjects to the project choice condition (3.3), the IC constraint for effort (3.4), the participation constraint (3.5) and the limited liability constraints.

We assume that firm's objective function Π is strictly concave to the project choice rule $\hat{\theta}$. To solve this program, we construct the Lagrange function, the firm's program can be write as:

$$\max_{\alpha,\beta,\gamma,\hat{\theta}} \quad \mathbf{L} = \Pi + \lambda_1\{E[cg(e)|e = 1, \hat{\theta}] - E[cg(e)|e = 0]\} + \lambda_2\{E[cg(e)|e = 1, \hat{\theta}] - u\}$$
$$+ \eta\alpha + \mu\{\beta\underline{y} - \beta E[y|\hat{\theta}] - \gamma V_o(\hat{\theta})\}I.$$

λ_1, λ_2 and η are non-negative lagrange multipliers associated with the IC constraint, IR constraint and the limited liability constraint. We solve this lagrange problem in the appendix. The solution is presented in the following Lemma.

Lemma 1. *The solution of the board's program satisfies*

$$(\lambda_1 + \lambda_2 - 1) + \eta = \phi\lambda_1 \tag{3.7}$$

$$(\lambda_1 + \lambda_2 - 1)[\underline{y} + \int_{\hat{\theta}}^{\overline{\theta}} \Delta V_y(\theta)dG(\theta)] = \phi\lambda_1\underline{y} + \mu\Delta V_y(\hat{\theta}) \tag{3.8}$$

$$(\lambda_1 + \lambda_2 - 1)\int_{\hat{\theta}}^{\overline{\theta}} V_o(\theta)dG(\theta) = \mu V_o(\hat{\theta}) \tag{3.9}$$

$$-\Delta V_y(\hat{\theta}) + (\lambda_1 + \lambda_2 - 1)\left[-\beta\Delta V_y(\hat{\theta}) - \gamma V_o(\hat{\theta})\right] - \mu\left[\frac{dE[y|\hat{\theta}]}{d\hat{\theta}} + \frac{dV_o(\hat{\theta})}{d\hat{\theta}}\right] = 0 \tag{3.10}$$

There are two sets of solution depending on parameter value. (1) If the parameters satisfy *the condition*

$$1 + \int_{\theta^{FB}}^{\bar{\theta}} \frac{\Delta V_y(\theta)}{\underline{y}} dG(\theta) \geq \phi \tag{3.11}$$

the solution is $\mu = 0$, $\lambda_1 = 0$, $\lambda_2 > 0$ *and* $\eta = 0$. *(2) If the parameters are such that (3.11) does not hold, the solution is* $\mu > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$ *and* $\eta > 0$. *There exists no solution for* $\mu < 0$.

Proof. See Appendix A *Q.E.D.*

The four first order conditions are derived with respect to the amount of fixed salary α, the number of shares β, the number of options γ and the project choice rule $\hat{\theta}$. We first consider the FOC of the board's program with respect to the project choice rule $\hat{\theta}$

$$-\Delta V_y(\hat{\theta}) + (\lambda_1 + \lambda_2 - 1)\left[-\beta \Delta V_y(\hat{\theta}) - \gamma V_o(\hat{\theta})\right] - \mu\left(\frac{dE[y|\hat{\theta}]}{d\hat{\theta}} + \frac{dV_o(\hat{\theta})}{d\hat{\theta}}\right) = 0$$

From the CEO's ex post project choice condition (3.3), the part

$$-\beta \Delta V_y(\hat{\theta}) - \gamma V_o(\hat{\theta}) = \beta \underline{y} - \beta E[y|\hat{\theta}] - \gamma E[\max(y - \hat{y}, 0)|\hat{\theta}] = 0.$$

Moreover, we know that $\frac{dE[y|\hat{\theta}]}{d\hat{\theta}} > 0$ and $\frac{dV_O(\hat{\theta})}{d\hat{\theta}} > 0$ because the distribution function $F(y|\theta)$ satisfies FOSD with respect to θ and both the integrands y and $\max(y-\hat{y}, 0)$ are increasing function of y. Thus, the sign of μ is determined by

$$sgn[\mu] = sgn[-\Delta V_y(\hat{\theta})] = sgn[\underline{y} - E[y|\hat{\theta}]].$$

When $\mu = 0$, the CEO maximizing the expected compensation based on the ex post information of quality follows the project choice rule $\underline{y} = E[y|\hat{\theta}]$. From the discussion in the last subsection, this equation defines the unique project choice rule that maximizes the expected value of the firm, i.e. $\hat{\theta} = \theta^{FB}$. In order this equation to be held, from (3.3), it must be $\gamma = 0$ and $\beta > 0$. The firm has to grant the CEO with positive amount of stock but no options. Intuitively, notice first the amount of fixed salary is irrelevant to the CEO's project choice decision as it is paid across all states. Second, options motivate the CEO to make more risky project choice. By applying implicit function theorem on

(3.3), we obtain

$$\frac{\partial\hat{\theta}}{\partial\gamma} = -\frac{V_o(\hat{\theta})}{\frac{dE[y|\hat{\theta}]}{d\hat{\theta}} + \frac{dV_O(\hat{\theta})}{d\hat{\theta}}} < 0.$$

As to the stock, the impact on project choice decision depends on the range of project choice

$$\frac{\partial\hat{\theta}}{\partial\beta} = -\frac{\Delta V_y(\hat{\theta})}{\frac{dE[y|\hat{\theta}]}{d\hat{\theta}} + \frac{dV_O(\hat{\theta})}{d\hat{\theta}}} = \begin{cases} > 0 & \hat{\theta} < \theta^{FB} \\ = 0 & \hat{\theta} = \theta^{FB} \\ < 0 & \hat{\theta} > \theta^{FB} \end{cases}$$

Thus, at the first best project choice rule θ^{FB}, changing the amount of the stock has no marginal effect on the CEO's project choice decision. In order to induce the CEO to follow the first best choice rule, it is necessary that the firm uses solely restricted stock. But this is only possible only if (3.11) satisfies. To understand this condition, notice that it actually comes from (3.4)

$$\frac{\beta[\underline{y} + \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta)dG(\theta)]I}{\beta\underline{y}I} \geq \phi$$

with $\alpha = 0$ and $\gamma = 0$. Thus, this condition is, in fact, the necessary condition for the firm can use only stock to motivate the managerial effort. Notice that in the multiplicative model, the condition (3.11) prescribes a ratio between the expected compensation from stock when the CEO works and the expected compensation from stock when the CEO shirks. This ratio has to be higher than the private benefit from shirking ϕ. Actually, this condition is sufficient to induce the effort and first best project choice rule as well. This explains why in the Lemma 1, the IC constraint is slack $\lambda_1 = 0$. Notice that when (3.11) holds, the firm simply chooses the amount of fixed salary to make the CEO just participate to minimize the compensation cost, so $\lambda_2 > 0$ and $\eta = 0$, IR constraint is binding while LL constraint is slack. This leads to our Proposition 1 that characterizes the first best contract.

Proposition 1. *The first best CEO compensation contract is as follows. When condition (3.11) holds, effort and the project choice* θ^* *can be implemented through a contract with shares and fixed salary. It prescribes* $\left\{\alpha^{FB} = [1 - \frac{(\phi-1)\underline{y}}{\int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta)dG(\theta)}]\frac{u}{\phi}, \beta^{FB}I = \frac{\phi-1}{\int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta)dG(\theta)}\frac{u}{\phi}, \gamma^{FB} = 0\right\}$. *The expected firm value is*

$$\Pi(\theta^{FB}) - u = [\underline{y} + \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta)dG(\theta)]I - u \tag{3.12}$$

Proof. See Apendix B. *Q.E.D.*

To get a more intuitive understanding of this contracts, now (3.11) holds, the board use stock alone to motivate effort and implement the project choice θ^{FB}. The benefit of exerting effort for the firm is the "upside" value (expected increase in firm value) from the risky project $\int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta) I$. The cost of motivating effort for the board is the increase in the amount of total monetary compensation $\frac{\phi-1}{\phi}u$. Notice that when the CEO exerts no effort, the board offers a total pay $\frac{u}{\phi}$ to induce the CEO just to participate. To motivate effort, the compensation contract prescribes that the minimum amount of shares β^{FB} must be at least to maintain the equality between the benefit and cost $\beta^{FB} I \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta) = \frac{\phi-1}{\phi}u$. And remember that under the first solution the CEO's IC is slack.

Consider the second solution in Lemma 1. $\mu > 0$ corresponds to the case (3.11) does not hold. Effort and the first best project choice can not be induced together when the compensation contract is only consist of restricted stock and fixed salary. Under this case, the board can include options, which realize only if the high values occur $y > \hat{y}$ into the compensation contract to motivate the CEO's effort. In our model, the high firm values occur only if the CEO exerts effort to develop the risky project, but the argument is more general. According to (Innes, 1990), as long as effort increases the occurrence of the higher values in the sense that the distribution function of firm value satisfies the monotonic likelihood ratio property, option-like instrument is superior to motivate effort. In our current model, this advantage of options to motivate effort has to be balanced with the board's needs of controlling excessive risky project choice, which will be presented soon. Before presenting our main tradeoff regarding the restricted stock and stock options, we briefly analyze the case where only options are used to highlight the CEO's excessive risk taking induced by options.

Consider the case where the compensation contract is only consist of fixed salary α and γ units of options. Along the equilibrium approach that effort has already been exerted, the CEO's expected compensation from a risky project with quality θ is $\alpha + \gamma V_o(\hat{\theta})$. On the other hand, the compensation from the safe project now results in only the fixed salary α. Options make the CEO's compensation non responsive to the project values that are below the exercise price $\hat{y}$. A direct implication is that the CEO always selects the risky project no matter its quality, as $\gamma V_o(\hat{\theta})I > 0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. In this case, we write the project choice rule $\hat{\theta} = \underline{\theta}$ denoting that any risky project is chosen by the CEO ex post.

Knowing the CEO always chooses the risky project, the option contract designed to motivate effort and ensure participation solves $\alpha+\gamma\int_{\underline{\theta}}^{\overline{\theta}} V_o(\theta)dG(\theta)I \geq \phi\alpha$ and $\alpha+\gamma\int_{\underline{\theta}}^{\overline{\theta}} V_o(\theta)dG(\theta)I \geq u$. Motivating managerial effort is not an issue when the compensation contract containing only fixed salary and options. There exists a continuum of contracts that results the same compensation cost for the firm. For instance, the board can simply choose the amount of fixed salary as $\alpha = 0$ and the unit of options as $\gamma I = \frac{u}{\int_{\underline{\theta}}^{\overline{\theta}} V_o(\theta)dG(\theta)}$. Under this contract, effort is induced while participation constraint is just binding.

The firm's expected value decreases as the project choice deviates from the first best rule θ^{FB}. The problem of options is that the CEO always makes the most dangerous project choice, i.e. $\hat{\theta} = \underline{\theta}$ under pure option contracts (the contracts without restricted stock). Thus, in the following, we discuss the tradeoff of granting options faced by the board.

Following the discussion of the case $\mu > 0$, we consider the compensation contract is consist of both stock and options. Stocks, even if they can not motivate effort when (3.11) does not hold, are granted to balance the excessive risk taking incentives of the CEO from the options. The board now faces a tradeoff in the designing of compensation contract: motivating effort against controlling the excessive risk taking. It can be seen from the derivatives $\frac{\partial\hat{\theta}}{\partial\beta} > 0$ and $\frac{\partial\hat{\theta}}{\partial\beta} > 0$ when $\hat{\theta} < \theta^{FB}$. Increasing marginally the amount of stock increases the project choice rule while increasing marginally the amount of options decreases the project choice conditional on the project choice rule lies in the range $[\underline{\theta}, \theta^{FB})$. The second best contract balancing this tradeoff prescribes the following: Given the amount of shares, the amount of options is chosen to be just enough to motivate the CEO's effort. This provides the CEO with the least incentives to make risky project choices that are considered as excessive risky according to the first best choice rule. Further reduces risk taking incentives by decreasing the amount of options is not feasible, as the CEO's IC constraint is binding, $\lambda_1 > 0$. Moreover, fixed salary is paid across all states, the CEO derives utility α when he works and $\phi\alpha > \alpha$ when he shirks. Thus, fixed salary actually disincentivizes the effort if the CEO has multiplicative preference. When the board needs to limit the amount of options to control excessive risk choice, there is no doubt she first chooses $\alpha = 0$. This is why the limited liability constraint is binding, $\eta > 0$ in the second solution.

Then the binding IC constraint can be expressed as

$$\frac{\beta[\underline{y} + \int_{\hat{\theta}}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta)]I + \gamma \int_{\hat{\theta}}^{\overline{\theta}} V_o(\theta) dG(\theta))I}{\beta \underline{y} I} = \phi$$

Combine with the project choice rule (3.3), this equation can be expressed as

$$1 + \int_{\hat{\theta}}^{\overline{\theta}} \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\hat{\theta})}{\underline{y}} \frac{V_o(\theta)}{V_o(\hat{\theta})} \right] dG(\theta) = \phi$$

Remember that $V_o(\theta) = E[\max(y - \hat{y}, 0)|\theta]$. We can decompose this equation into more intuitive form

$$\underbrace{1 + \int_{\hat{\theta}}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta)}_{\text{Incentives provided by 1 stock}} + \frac{-\Delta V_y(\hat{\theta})}{V_o(\hat{\theta})} \underbrace{\int_{\hat{\theta}}^{\overline{y}} V_o(\theta) dG(\theta)}_{\text{Incentives provided by 1 option}} = \phi \underline{y}.$$

Normalize the amount of restricted stock to be one, the term $\frac{-\Delta V_y(\hat{\theta})}{V_o(\hat{\theta})} > 0$ represents the amount of options required to maintain the project choice rule to be $\hat{\theta}$. The optimal contract prescribes that the incentives (expected compensation) provided by stock and options conditional on the project choice rule to be $\hat{\theta}$ just offset the CEO's private benefit of shirking, i.e., higher private benefit from leisure ϕ multiply the monetary compensation $\underline{y}$ from one stock.

Motivating the managerial effort in the second best setting in a multiplicative preference model is different from that in an additive model. In an additive model, if the limited liability constraint is binding, the CEO receives limited liability rent and the project choice is excessive risky compared to the first best level.[35] However, in a multiplicative model, when the limited liability constraint binds, the CEO receives no rent. The reason is that the board can scale up or down the level of stock and options while maintaining

[35]For the moral hazard problem when effort and risk choice are both concerned in an additive model, readers can refer to the paper (Inderst and Mueller, 2010).

the optimal mix of the two to induce the CEO's effort. Then to minimize the compensation cost, the board chooses the level of stock and options to just bind the CEO's IR constraint, thus $\lambda_2 > 0$ again.

The following Proposition 2 characterizes the second best project choice θ^{**}, proves its existence and uniqueness and derives the optimal contract.

Proposition 2. *When the condition (3.11) does not hold, effort can not be motivated solely by stock and fixed salary. The board chooses the amount of restricted stock and stock options trading off the needs to motivate effort against the needs to curb the excessive risky project choice. The second best contract prescribes* $\{\alpha^* = 0, \beta^* I = \frac{u}{\phi \underline{y}}, \gamma^* I = \frac{-\Delta V_y(\theta^*)}{V_o(\theta^*)} \frac{u}{\phi \underline{y}}\}$. *The second best project choice* θ^* *solving*

$$1 + \int_{\theta^*}^{\overline{\theta}} \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\theta^*)}{\underline{y}} \frac{V_o(\theta)}{V_o(\theta^*)} \right] dG(\theta) = \phi \tag{3.13}$$

is existent and unique in $(\underline{\theta}, \theta^{FB})$, *thus it is more risky than the first best choice rule* θ^{FB}. *The expected firm value is*

$$\Pi(\theta^*) - u = [\underline{y} + \int_{\theta^*}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta)] I - u \tag{3.14}$$

Proof. See Appendix C *Q.E.D.*

Now (3.11) does not hold, managerial effort can not be solely motivated by restricted stock. The total amount of options granted is $\gamma^* I = \frac{-\Delta V_y(\theta^*)}{V_o(\theta^*)} \frac{u}{\phi} > 0$ inducing the CEO to follow a more risky project choice rule than the first best one $\theta^* < \theta^{FB}$. Notice that $\Delta V_y(\theta^*) = E[y|\theta^*] - \underline{y} < 0$ under this case. This is due to the fact that $E[y|\theta]$ increases in θ. The optimal contract prescribes a mix of shares and options trading off the benefit of using options to motivate effort against its shortcoming of leading to excessive risky project choice.

Lastly, there exist no solution such that $\mu < 0$, i.e. $\hat{\theta} > \theta^{FB}$. This is due to our risk neutrality assumption of the CEO's preference. In our model, the distortion highlights the binding limited liability constraint and the private benefit is a proportion to the monetary compensation. As a result, there exists no risk and efficiency tradeoff as in the case where the CEO has a risk averse preference, he does not seek an insurance for bearing the risk from the uncertainty in the compensation. If $\mu < 0$ were a solution, it actually required that the board granted the CEO with negative amount of options $\gamma < 0$. This is because in the (3.3), $E[y|\hat{\theta}] > \underline{y}$ when $\hat{\theta} > \theta^{FB}$.

3.4 Optimal Contract in General Market Equilibrium

In this section, we incorporate a competitive labor market between firms and CEOs. In most models with the competitive labor market, the equilibrium between firms and CEOs results in a positive assortative matching assignment, i.e., larger firms are assigned to higher ability CEOs. (Edmans, Gabaix, and Landier, 2009) shows that this assignment maximizes the total surplus of production in the economy and can be separated from the provision of effort incentives. In our model, we present a different situation where provision of incentives matters for the equilibrium assignment. We show in this section, if the firms can always provide enough effort incentives while maintaining the first best project choice, then positive assortative matching holds. Otherwise, there could be circumstances where larger firms hire CEOs with lower ability to reduce the chance that CEOs make excessive project choices.

We first extend the basic model to consider a labor market that opens at $t = 0$. The market participants are N firms ranking by their asset sizes $I^1 < I^2 < ... < I^N$ and N CEOs ranking by their monitoring ability $G_1 < G_2 < ... < G_N$. The monitoring ability of a CEO n is characterized by the conditional distribution function $G_n(\theta)$ of the quality θ of the risky project generated by the CEO. We assume that the higher ability CEO has higher chances to generate the better quality projects. The ability improves the distribution of θ in the sense of first order stochastic dominance, i.e., for any two CEOs m, n with ranking $m < n$, $G_n(\theta)$ FOSD $G_m(\theta)$ [36]. Larger firms and higher ability CEOs are labeled in higher indices. We assume that CEOs' ability and firms' asset sizes are observable

[36]Remember that the expected value of risky project $E[y|\theta]$ increases in the project quality θ because the conditional distribution function of project value $F(y|\theta)$ is assumed to be FOSD in the quality.

by all the market participants at $t = 0$. The firms make competitive bids in the form of compensation to hire the CEOs. One firm hires exactly one CEO to maximize expected firm value and each CEO accepts the contract with the highest expected total utility. Finally, we normalize the outside opportunity of the CEO with lowest ability to be $\underline{u}$.

We first consider the case where all CEOs can be hired in the first best contracts. Following the discussion in the last section, we can denote a firm's expected value in the partial equilibrium as a function of its asset size, the ability of CEO being hired and the optimal project choice. A firm indexed by i hires a CEO indexed by m in the first best contracting, its expected value is

$$\Pi(\theta^{FB}, i, m) - u_m = [\underline{y} + \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG_m(\theta)] I^i - u_m$$

Notice that in the first best contracting, the CEO's ability does not affect the optimal project choice θ^{FB}. The reason is that the first best project choice follows a "first order" rule, the CEO chooses the project to maximize compensation after the quality θ is realized. On the other hand, the ability of CEO m determines the distribution $G_m(\theta)$ of quality θ, thus affects the provision of the ex ante effort of CEO to develop risky project instead of the ex post project choice after θ is realized. To attain the first best contracting requires a condition similar to (3.11) holds for all CEOs.

The following Lemma 2 establishes the complementary of CEO ability and firm asset in the first best contracting in order to generate the productive efficiency.

Lemma 2. *Once the sufficient and necessary condition for all N CEOs are hired in the first best contracts $1 + \int_{\theta^{FB}}^{\overline{\theta}} \frac{\Delta V_y(\theta)}{\underline{y}} dG_1(\theta) \geq \phi$ is satisfied, the total production Π of any firm-CEO pair exhibits complementarity between CEO's ability and firm's asset size. That is, for CEO $m < n$ and firm $i < j$*

$$\Pi(\theta^{FB}, j, n) + \Pi(\theta^{FB}, i, m) > \Pi(\theta^{FB}, j, m) + \Pi(\theta^{FB}, i, n)$$

Proof. See Appendix D *Q.E.D.*

If the firms can secure the first best contracting with the CEOs, there is complementarity between CEOs' ability and firms' asset in production. Consequently, firms with larger asset size hires higher ability CEOs produces larger economic surplus. The following Proposition 3 characterizes the general market equilibrium in the case of first best contracting.

Proposition 3. *In the circumstance where all CEOs can be induced to choose the optimal project choice θ^{FB}. The equilibrium in the competitive labor market results in positive assortative matching: a firm with kth largest asset size exactly hires a CEO with the same ranking in ability.*

Proof. See Appendix E. *Q.E.D.*

Remember that $G_i(\theta) < G_{i-1}(\theta)$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$. The complementarity property holds, a match between larger firm and higher ability CEO creates higher (expected) surplus. In the simplest case when two firms compete for two CEOs with different ability, the larger firm is able to offer the higher ability CEO with higher expected compensation. In equilibrium, the larger firm actually makes a bid equaling to the smaller firm's maximum willingness to pay for the higher ability CEO. Thus, the equilibrium results in a positive assortative matching. Apply this logic to the competitive bidding between N firms and N CEOs, we obtain the result in Proposition 3. In the equilibrium, the effort of CEOs can always be induced, a CEO's total utility equals to the expected pay. The total pay of CEO k matched with bank k is such that if this CEO were hired by firm $k-1$, firm $k-1$ would be indifference from hiring the lower ability CEO $k-1$. In a other words, the total pay u_k received by CEO k matched with firm k equals to bank $k-1$'s maximum willingness to pay for this CEO, that is

$$\begin{aligned} u_{k,k-1} &= \left[\int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG_k - \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG_{k-1} \right] I^{k-1} + u_{k-1} \\ &= \left[\int_{\theta^{FB}}^{\overline{\theta}} [G_{k-1}(\theta) - G_k(\theta)] \frac{d\Delta V_y(\theta)}{d\theta} d\theta \right] I^{k-1} + u_{k-1}. \end{aligned}$$

The second equation is the result of integrating by parts. As a result, the bank k simply offers $u_k = u_{k,k-1}$ to win the competition because it has more resources. In the proof of Proposition 4, we also derive each CEO's total pay in equilibrium by iteration.

We then turn to the case of second best contracting. We suppose now a firm indexed by i can only hire a CEO indexed m in the second best contracting. From Proposition 2, the firm i's expected value is

$$\Pi(\theta_m^*, i, m) - u_m = [\underline{y} + \int_{\theta_m^*}^{\overline{\theta}} \Delta V_y(\theta) dG_m(\theta)] I^i - u_m$$

Different from the case in the first best contracting, now the second best project choice θ^* is affected by the CEO's ability. θ^* is distorted downwards from θ^{FB} because options are included in the contract to motivate managerial effort. From the condition (3.13), we know that the distribution of θ affects the second best optimal threshold θ^*. As a result, we denote θ_m^* as the second best project choice rule made by the CEO m whose ability is characterized by $G_m(\theta)$.

The following Lemma 3 characterizes the relationship between the CEO ability and the second best project choice.

Lemma 3. *The sufficient and necessary condition for all N CEOs are hired in the second best contracts is $1 + \int_{\theta^{FB}}^{\overline{\theta}} \frac{\Delta V_y(\theta)}{\underline{y}} dG_N(\theta) < \phi$. Under this condition, the higher ability CEO makes safer second best project choice, i.e., for any two CEOs m, n with G_n FOSD G_m, $\theta_m^* < \theta_n^* < \theta^{FB}$.*

Proof. See Appendix F *Q.E.D.*

The intuition for this result has the root that higher ability CEO generates better distribution of risky project. Consider the equation determining the second best project choice (3.13). Normalize the number of restricted stock to be one, the higher ability CEO receives more compensation (in expectation) from restricted stock because the quality of risky project generated by his effort is better. On the other hand, ability does not affect the CEO's utility when he shirks. This means a firm's (unit) restricted stock deliver more incentives when higher ability CEO is hired. The firm thus use relatively less options when the higher ability CEO is hired. This leads to a less risky second best project choice rule made by the higher ability CEO.

Given the Lemma 4, the following Proposition 5 characterizes the hiring for CEOs in the second best contracting. Different from the first best case, we show that large firms can end up in hiring low ability CEOs.

Proposition 4. *Suppose N CEOs are hired in the second best contracting. There exists cases that the expected value of a firm is lower when a higher ability CEO is hired. The positive assortative matching does not always hold.*

Now we are in the second best world. Consider a firm's decision between hiring a two CEOs m and n with $G_n(\theta)$ FOSD $G_m(\theta)$, the CEO n has higher ability in generating new risky project. The firm is willing to pay the higher ability CEO with higher compensation only if the firm value increases under his management. Then we consider the difference in the expected unit value when the firm hires the higher ability CEO n other than the CEO m

$$\int_{\theta_n^*}^{\overline{\theta}} \Delta V_y(\theta) dG_n(\theta) - \int_{\theta_m^*}^{\overline{\theta}} \Delta V_y(\theta) dG_m(\theta).$$

It can be further expressed into the two terms

$$\int_{\theta_n^*}^{\overline{\theta}} \Delta V_y(\theta) d[G_n(\theta) - G_m(\theta)] - \int_{\theta_m^*}^{\theta_n^*} \Delta V_y(\theta) dG_m(\theta)$$

The second term in this expression $-\int_{\theta_m^*}^{\theta_n^*} \Delta V_y(\theta) dG_m(\theta)$ is positive. This is because $\theta_m^* < \theta_n^*$ and the risky project has a lower value than the safe project when its quality is lower than θ^{FB}. Thus the difference in expected value between the risky project and the safe project is negative $\Delta V_y(\theta) < 0$ under this case. This term represents a positive effect of the CEO ability on the firm value: when hiring a higher ability CEO, the firm is able to design contract to induce the CEO to take although still excessive but less risk. The firm's expected value increases accordingly due to this reason. We call this as the "risk reduction" effect.

Then we look at the difference $\int_{\theta_n^*}^{\overline{\theta}} \Delta V_y(\theta) d[G_n(\theta) - G_m(\theta)]$. We integrate it by parts and obtain

$$\int_{\theta_n^*}^{\overline{\theta}} \Delta V_y(\theta) d[G_n(\theta) - G_m(\theta)] = \Delta V_y(\theta)[G_n(\theta) - G_m(\theta)] \mid_{\theta_n^*}^{\overline{\theta}} - \int_{\theta_n^*}^{\overline{\theta}} [G_n(\theta) - G_m(\theta)] \frac{dV_y(\theta)}{d\theta} d\theta$$

The term $-\int_{\theta_n^*}^{\overline{\theta}}[G_n(\theta)-G_m(\theta)]\frac{dV_y(\theta)}{d\theta}d\theta$ is positive. This is because hiring higher ability CEO n than lower ability CEO m increases the distribution of quality of the risky project: the chance that θ is high increases. Notice that $G_n(\theta) - G_m(\theta)$ is negative for any $\theta \in [\underline{\theta}, \overline{\theta}]$. This effect is labeled as "distribution improving" effect, it increases the expected firm as well.

Lastly, the term

$$\Delta V_y(\theta)[G_n(\theta) - G_m(\theta)] \mid_{\theta_n^*}^{\overline{\theta}} = -\Delta V_y(\theta_n^*)[G_n(\theta_n^*) - G_m(\theta_n^*)]$$

is negative. Notice again, $\Delta V_y(\theta_n^*) < 0$ for $\theta_n^* < \theta^{FB}$ and $G_n(\theta_n^*) - G_m(\theta_n^*) < 0$. In the second best contracting, the CEO makes value reducing project choice for the quality lies in the interval $[\theta^*, \theta^{FB})$. Depending on the distribution function G, it could be possible that when a lower ability CEO m is hired, the chance that the quality of the risky project realizes in this interval is lower the one when a higher ability CEO n is hired. The reason is again due to the FOSD assumption. Then from ex ante ($t = 0$) point of view, the low ability CEO could have lower chance to choose the value reducing project even if he acts according to a more risky project choice rule $\theta_m^* < \theta_n^*$. This effect is the side effect of the fact that CEO ability improves distribution of quality of risky project. Due to this adverse effect, the firm value decreases when hiring higher ability CEO.

In sum, it is difficult to identify the final effect of CEO ability on firm's value in the second best world. It depends on the specific forms of the distribution functions G and F. When the adverse effect dominates the "risk reduction" and "distribution improving" effects, the firm value is actually lower when hiring a higher ability CEO. In case of first best contracting, there is only "distribution improving" effect. So in the first best contracting the unit firm value increases when hiring higher ability CEO.

3.5 Discussions

In this section, we discuss the issues that have not been covered so far. The first issue is the optimal exercise price of the options. The second issue is the relationship between firm size, risk and the structure of optimal contract.

3.5.1 Optimal Exercise Price y^*.

In this subsection, we analyze the role of exercise price $\hat{y}$ in the compensation design. In the previous analysis, $\hat{y} \in (\underline{y}, \overline{y})$ is treated as an exogenous variable. Now we discuss the optimal choice of exercise price $\hat{y}$ of the options.

We consider the optimal choice of exercise price as a two step program. First, the firm chooses the optimal mix of stock and options to maximize the expected firm value for any exercise price $\hat{y} \in (\underline{y}, \overline{y})$. Then firm chooses the optimal exercise price y^* from the feasible set $(\underline{y}, \overline{y})$. The first step has already been done in the section 3. We then consider the board's choice of exercise price y^*. To determine the optimal exercise price y^*, we impose another constraint on the total amount of shares and options:

$$\beta + \gamma < 1.$$

It means under any state of world, the CEO never owns the entire firm (remember that we normalize the firm's outstanding shares to be one).

The optimal choice of exercise price is irrelevant in the first best contracting as options are never used. We focus on the case of second best contracting. From Proposition 2, we know that the firm can always induce effort without giving any rent to the CEO. Thus, the exercise price only affects the optimal project choice defined as the solution of (3.13). We now write this condition as a function of exercise price as well

$$\underline{y} + \int_{\theta^*}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta) + \frac{-\Delta V_y(\theta^*)}{V_o(\theta^*, \hat{y})} \int_{\theta^*}^{\overline{y}} V_o(\theta, \hat{y}) dG(\theta) = \phi \underline{y}. \tag{3.15}$$

Notice that exercise price $\hat{y}$ affects the optimal project choice indirectly through its influence on the expected value of option $V_o(\theta, \hat{y}) = \varphi \int_{\underline{y}}^{\overline{y}} \max(y - \hat{y}, 0) dF(y|\theta)$.

An increasing in price $\hat{y}$ imposes an adverse "incentive" effect on project choice. It can be calculated that

$$\frac{\partial}{\partial \hat{y}} V_o(\theta, \hat{y}) = \frac{\partial}{\partial \hat{y}} \varphi \int_{\underline{y}}^{\overline{y}} \max(y - \hat{y}, 0) dF(y|\theta) = -\varphi \int_{\underline{y}}^{\overline{y}} dF(y|\theta) < 0$$

Increase $\hat{y}$ decreases the expected value of option $V_o(\theta)$ conditional on any quality θ of the risky project. The CEO's expected compensation from option $\int_{\theta^*}^{\overline{\theta}} V_o(\theta, \hat{y}) dG(\theta)$ conditional on the optimal project choice rule θ^* decreases as the expected value of option for all the risky project $\theta \geq \theta^*$ decreases. The firm has to increase the amount of options $(\frac{-\Delta V_y(\theta^*)}{V_o(\theta^*, \hat{y})})$ to induce effort for each unit of stock granted. As a result, the CEO's optimal project choice becomes more risky, θ^* further decreases from θ^{FB}.

On the other hand, there is positive "project choice" effect. The project choice rule (3.3) prescribes for one unit stock granted, the amount of options required to maintain the project choice to be θ^* is $\frac{-\Delta V_y(\theta^*)}{V_o(\theta^*, \hat{y})}$. Exactly because the same reason, increase $\hat{y}$ decreases the expected value of option, the relative amount of options decreases for one unit of stock granted. Thus, the CEO is induced to make safer project choice ex post.

The two effects have opposite implications on the optimal project choice. The overall effect can be seen by taking partial derivative of equation (3.15) with respect to $\hat{y}$

$$\frac{-\Delta V_y(\theta^*)}{V_o(\theta^*, \hat{y})} \frac{\partial}{\partial \hat{y}} \left(\int_{\theta^*}^{\overline{y}} V_o(\theta, \hat{y}) dG(\theta) \right) + \frac{\partial}{\partial \hat{y}} \left(\frac{-\Delta V_y(\theta^*)}{V_o(\theta^*, \hat{y})} \right) \int_{\theta^*}^{\overline{y}} V_o(\theta, \hat{y}) dG(\theta)$$

The first term represents the negative "incentive" effect, while the second term represents the positive "project choice" effect. The overall effect is presented in the following lemma 2.

Lemma 4. *When $F(y|\theta)$ satisfies $F''_{y\theta} > 0$, the positive "project choice" effect dominates the negative "incentive" effect.*

Proof. See Appendix G *Q.E.D.*

Once the Lemma 2 holds, the following Proposition 3 summarizes the effect of exercise price on the optimal project choice.

Proposition 5. *When the condition in Lemma 4 holds, increase the exercise price of options decreases the CEO's incentives to make excessive projec choice,* $\frac{\partial \theta^*}{\partial \hat{y}} > 0$.

In the proof of Proposition 2, we have established the fact that the partial derivative of the equation (3.15) with respect to θ^* is negative. It ensures the uniqueness of optimal project choice. Under the condition presented in Lemma 2, the partial derivative of (3.15) with respect to $\hat{y}$ is positive. From the implicit function theorem, we have $\frac{\partial \theta^*}{\partial \hat{y}} > 0$. Increase the exercise price $\hat{y}$ actually induces the CEO to make the less risky optimal project choice.

Recall that in Proposition 2, the optimal amount of options is $\gamma^* I = \frac{-\Delta V_y(\theta^*)}{V_O(\theta^*, \hat{y})} \frac{u}{\phi \underline{y}}$, while the amount of stock is irrelevant to the exercise price. Due to the dominating "project choice" effect, the amount of options increases when exercise price $\hat{y}$ increase. To achieve the safest project choice, the firm optimally chooses the highest possible $\hat{y}$ such that the amount of option γ^* is maximized, that is $1 - \beta^*$. The optimal project choice and optimal exercise price are jointly determined by (3.15) and the binding constraint $\beta^* + \gamma^* = 1$.

Option induces the CEO to make excessive risky project choice compared the first best level. However, when the exercise price is endogenized, we obtain a result that the firm use the highest possible amount of options. This result is actually quite intuitive: as high exercise price reduces the chance that the option be finally exercised (the realized firm value is less likely above the exercise price), the CEO would rather select the safe project to enjoy the certain compensation from stock. Thus, a risky project is chosen only if its quality is high enough. A high exercise price of options decreases the CEO's incentive to make excessive project choice.

3.5.2 Firm size, risk and the structure of optimal contract

In this subsection, in order to analyze the relation between firm size and structure of optimal contract as well as risk, we further make two assumptions. First, we focus on an opposite situation to the one described in Proposition 4 by assuming away the ambiguity on the CEO ability. The benefit from "risk reduction" and "distribution improving" effects dominates the cost from the adverse effect. Second, we assume there will be different structure of optimal contract in the general equilibrium, thus the parameters are such that $1+\int_{\theta^{FB}}^{\overline{\theta}} \frac{\Delta V_y(\theta)}{\underline{y}} dG_N(\theta) > \phi$ and $1+\int_{\theta^{FB}}^{\overline{\theta}} \frac{\Delta V_y(\theta)}{\underline{y}} dG_1(\theta) < \phi$.

Under these assumptions, the partial equilibrium firm value could be either $\Pi(\hat{\theta}, j, n)$ or $\Pi(\theta_n^*, j, n)$ depending on whether the first best contracting is attained, that is (3.11) holds or not. The following lemma shows that under these assumptions, a firm always wants to hire higher ability CEO.

Lemma 5. *The expected (gross) firm value is higher when higher ability CEO is hired.*

Proof. See Appendix H *Q.E.D.*

Consider a firm j has the opportunity between hiring CEO m and n, with G_n FOSD G_m. Because ability improves the distribution of risky project in the sense of FOSD, the gross firm value is higher when hiring the higher ability CEO if first best contracting is attained when hiring both CEOs. Then by the assumption in this subsection that the expected firm value is also higher when hiring the higher ability CEO if only second best contracting is attained. The rest case is that the firm can attain first best contracting only when hiring CEO n, the higher ability one. Then it follows that θ^{FB} is first best project choice when hiring any CEOs, the firm still has higher expected value when hiring CEO n.

Follow the Lemma 5, we introduce the following Proposition 6. It characterizes a general market equilibrium where both contracting structures are presented.

Proposition 6. *There exists a critical CEO such that first best contracting can be attained for the CEOs with ability higher than (or equal to) him. The equilibrium results in PAM. There is seperation in structure of optimal contracts and project choice rule: firms hire CEOs with ability higher (lower) than the critical CEO in stock (stock and options) to implement the first (second) best project choice rule. CEOs in smaller firms follows more risky project choice rule.*

The intuition for this result is straightforward. The critical CEO is defined such that for

CEOs who have higher ability than this CEO, (3.11) holds; while for CEOs who have lower ability than this CEO, (3.11) does not hold. Then there is a partition of the set of CEOs according to their ability. This partition is possible since the term $\int_{\theta^{FB}}^{\overline{\theta}} \frac{\Delta V_y(\theta)}{\underline{y}} dG(\theta)$ in (3.11) increases with the CEO's ability, i.e. for any G_n and G_m such that G_n FOSD G_m, the term is higher when the distribution is G_n. Follow the Lemma 5, we can establish the similar complementarity result when expected firm value increases when higher ability CEO is hired. Thus, the equilibrium assignment results in PAM. For the CEOs with ability higher than the critical CEO, the firms offer first best contracts to implement the same project choice rule θ^{FB}. While for the CEOs with ability lower than the critical CEO, the firms offer second best contracts to implement the project choice rule θ^*. Then from the Lemma 3, the CEOs with lower ability make more risky project choices under second best contracting. And firms with smaller asset sizes hire those CEOs with lower ability in positive assortative matching.

3.6 Conclusion

In this paper, we present a theoretical model where firms optimally design the compensation contracts to their CEOs. In the model, a CEO exerts privately observable effort as well as makes the project choice based on his own private information. Stocks perfectly incentivize the CEO to select the best project for the firm, while options are superior in motivating the CEO's effort. When effort is the prior concern of the firm, we find that both stock and options must be part of the second best optimal contract. The CEO and the firm in our model has multiplicative preference and production function. The incentive compatibility constraint prescribes a ratio between the expected pay in case of exerting effort and the pay in case of shirking must exceed the ratio of the CEO's private benefit in case of shirking and in case of exerting effort. We show that information about the project risk should be a part of the optimal compensation contract. We further analyze the productive efficiency in general equilibrium. We find that in the real world with distortion, there may be circumstance that larger firms would rather hire the CEOs with low ability to reduce the cost of excessive risky project choice.

Appendix 3.A Proof of Lemma 1

Proof. Take the first order derivatives with respect to the four arguments α, β, γ and $\hat{\theta}$, we have the system of solution presented in the main text. We discuss different sets of solution according to the value of μ.

First, we look at the first order derivative with respective to $\hat{\theta}$ 3.10. From 3.6, we know that in 3.10, the term $-\beta\Delta V_y(\hat{\theta}) - \gamma V_o(\hat{\theta})$ equals to zero representing that the CEO makes the ex post project choice to maximize his expected compensation. Moreover, we know that from the FOSD assumption of distribution function $F(\theta)$, the two expectations $E[y|\hat{\theta}]$ and $V_o(\hat{\theta})$ are both increasing in $\hat{\theta}$. So the sign of the lagrange multiplier μ is determined by

$$sgn[\mu] = sgn[-\Delta V_y(\hat{\theta})].$$

Suppose $\mu = 0$, we have $E[y|\hat{\theta}] = \underline{y}$. It follows that γ must be zero from 3.3, as the expected option value $V_o(\hat{\theta})$ is strictly positive for all $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$. From our discussion in the last subsection, θ^{FB} is the unique value satisfing this equation, we have $\hat{\theta} = \theta^{FB}$. Then from 3.9, we have $\lambda_1 + \lambda_2 = 1$ when $\mu = 0$. Inserting $\lambda_1 + \lambda_2 = 1$ and $\mu = 0$ to 3.8, we have $\lambda_1 = 0$ and $\lambda_2 = 1$. Insert $\lambda_1 = 0$ and $\lambda_1 + \lambda_2 - 1 = 0$ into 3.7, we finally get $\eta = 0$. Thus, we find the first set of solution corresponding to the case that $\mu = 0$. Under this set of solution, IC and LL constraints are slack, IR constraint is binding. We insert $\hat{\theta} = \theta^{FB}$ and $\gamma = 0$ into the slack IC constraint 3.4 to get

$$\beta[\underline{y} + \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta) - \phi \underline{y}] I > (\phi - 1)\alpha$$

From the binding IR constraint, we solve $\alpha = u - \beta[\underline{y} + \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta)] I$. Insert it into the first inequality, we get

$$\beta I > \frac{1}{\int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG(\theta)} \frac{\phi - 1}{\phi} u.$$

Then from the binding IR constraint again, we have

$$0 < \alpha < \left[1 - \frac{(\phi-1)\underline{y}}{\int_{\theta^{FB}}^{\bar{\theta}} \Delta V_y(\theta)dG(\theta)}\right]\frac{u}{\phi}.$$

When $\mu = 0$, in order for the solution to be exist, a necessary condition is

$$1 + \int_{\theta^{FB}}^{\bar{\theta}} \frac{\Delta V_y(\theta)}{\underline{y}} dG(\theta) > \phi.$$

Otherwise, the LL constraint will be violated. On the other hand, once this condition satisfies, the board can always choose a combination of fixed salary and shares that satisfies the above two inequalities to implement θ^{FB} and induce effort while making the CEO break even. Thus, it is a sufficient condition as well.

Suppose $\mu > 0$, then $E[y|\hat{\theta}] < \underline{y}$. It follows that γ must be positive. We have $\hat{\theta} < \theta^{FB}$, the CEO makes more risky project choice than the one maximizing the expected firm value. From 3.9, we have $\lambda_1 + \lambda_2 > 1$. Then 3.8 becomes

$$\phi\lambda_1\underline{y} = (\lambda_1 + \lambda_2 - 1)[\underline{y} + \int_{\hat{\theta}}^{\bar{\theta}} \Delta V_y(\theta)dG(\theta)] - \mu[E[y|\hat{\theta}] - \underline{y}] > 0$$

This shows $\lambda_1 > 0$. Combining 3.7 and 3.8, we can solve

$$\eta = (\lambda_1 + \lambda_2 - 1)\int_{\hat{\theta}}^{\bar{\theta}} \frac{\Delta V_y(\theta)}{\underline{y}} dG(\theta) - \mu\frac{E[y|\hat{\theta}] - \underline{y}}{\underline{y}} > 0.$$

When $\mu > 0$, there exists a second set of solution. It entails both IC and LL is binding. Insert the project choice rule 3.3 and $\alpha = 0$ into the binding IC, we have the project choice $\hat{\theta}$ is the solution of

$$1 + \int_{\hat{\theta}}^{\bar{\theta}} \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\hat{\theta})}{\underline{y}}\frac{V_o(\theta)}{V_o(\hat{\theta})}\right] dG(\theta) = \phi \tag{A.1}$$

When $\mu < 0$, the CEO makes conservative project choice $\hat{\theta} > \theta^*$. From project choice rule 3.3, this requires $\gamma < 0$, which is impossible in our risk neutral setup.

To summarize, we found two sets of solutions corresponding to the different parameters value characterized in 3.11,

Q.E.D.

Appendix 3.B Proof of Proposition 1

Proof. The proof of Proposition 1 follows directly the proof of Lemma 1. Notice that Lemma 1 predicts a continuum of contracts that induce effort and implement θ^{FB} when 3.11 holds. We choose the optimal contract as the one with maximum amount of fixed salary as in (Edmans, Gabaix, and Landier, 2009). *Q.E.D.*

Appendix 3.C Proof of Proposition 2

Proof. When $\mu > 0$, the optimal project $\hat{\theta} < \theta^{FB}$ satisfies the equation A.1 from Lemma 1. Notice that now the limited liability constraint is binding, thus 3.11 does not hold.

We start by proving the existence and uniqueness of $\theta^* \in (\underline{\theta}, \theta^{FB})$ as a solution of A.1. For simplicity, we denote an auxiliary function as

$$\Psi(\hat{\theta}) = 1 + \int_{\hat{\theta}}^{\overline{\theta}} \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\hat{\theta})}{\underline{y}} \frac{V_o(\theta)}{V_o(\hat{\theta})} \right] dG(\theta) - \phi$$

We assume this function is continuous in $\hat{\theta}$. We first take the limit of $\hat{\theta}$ to θ^{FB}

$$\lim_{\hat{\theta} \to \theta^{FB}} \Psi(\hat{\theta}) = 1 + \int_{\theta^{FB}}^{\overline{\theta}} \frac{\Delta V_y(\theta)}{\underline{y}} dG(\theta) - \phi < 0$$

This is because now 3.11 does not hold. Then we assume that

$$\lim_{\hat{\theta} \to \underline{\theta}} \Psi(\hat{\theta}) = 1 + \int_{\underline{\theta}}^{\overline{\theta}} \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\underline{\theta})}{\underline{y}} \frac{V_o(\theta)}{V_o(\underline{\theta})} \right] dG(\theta) - \phi > 0$$

As the board can always choose the exercise price $\hat{y}$ of the option. Especially, the board can choose $\hat{y}$ high enough to make $-\frac{\Delta V_y(\hat{\theta})}{V_o(\hat{\theta})}$ sufficiently large when $\hat{\theta}$ is close to $\underline{\theta}$. By the continuity, there exists $\hat{\theta} \in (\underline{\theta}, \theta^{FB})$ such that $\Psi(\hat{\theta}) = 0$.

Then it can be calculated that

$$\frac{\partial \Psi(\hat{\theta})}{\partial \hat{\theta}} = \int_{\hat{\theta}}^{\overline{\theta}} \frac{V_o(\theta)}{\underline{y}} \frac{\partial}{\partial \hat{\theta}} \left(\frac{-\Delta V_y(\hat{\theta})}{V_o(\hat{\theta})} \right) dG(\theta) < 0$$

where

$$\frac{\partial}{\partial \hat{\theta}} \left(\frac{-\Delta V_y(\hat{\theta})}{V_o(\hat{\theta})} \right) = \frac{-\frac{\partial}{\partial \hat{\theta}} E[y|\hat{\theta}] V_o(\hat{\theta}) + \Delta V_y(\hat{\theta}) \frac{\partial}{\partial \hat{\theta}} V_o(\hat{\theta})}{[V_o(\hat{\theta})]^2} < 0$$

Remember that $\Delta V_y(\hat{\theta}) = E[y|\hat{\theta}] - \underline{y} < 0$ when $\hat{\theta} < \theta^{FB}$. And the two expectation $E[y|\hat{\theta}]$ and $V_o(\hat{\theta})$ increases in $\hat{\theta}$ because the FOSD assumption.

So the solution is existent and unique. We define it as θ^* such that $\Psi(\theta^*) = 0$.

Then knowing the value of θ^*, the board can always choose the amount of shares and options to make IR constraint binding. The reason is stated in the main text. With the binding LL constraint, the second best contract is consist of $\beta^* I$ units of shares and $\gamma^* I$ units of options. Combining the binding IC and IR constraints, we have $\beta^* I = \frac{u}{\phi \underline{y}}$. Then use the project choice condition 3.3, $\gamma^* I = \frac{-\Delta V_y(\theta^*)}{V_o(\theta^*)} \beta^* I = \frac{-\Delta V_y(\theta^*)}{V_o(\theta^*)} \frac{u}{\phi \underline{y}}$. Notice that now $\Delta V_y(\theta^*) < 0$ for $\theta^* < \theta^{FB}$.

Q.E.D.

Appendix 3.D Proof of Lemma 2

Proof. We first show when $1 + \int_{\theta^{FB}}^{\overline{\theta}} \frac{\Delta V_y(\theta)}{\underline{y}} dG_1(\theta) \geq \phi$ satisfies, then a firm can secure first best contracting when hiring any CEOs in the labor market. For any CEO n other than the CEO with the lowest ability $G_1(\theta)$, we can show

$$\begin{aligned}
&\int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG_n(\theta) - \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG_1(\theta) \\
&= \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) d[G_n(\theta) - G_1(\theta)] \\
&= \Delta V_y(\theta)[G_n(\theta) - G_1(\theta)] \mid |_{\theta^{FB}}^{\overline{\theta}} - \int_{\theta^{FB}}^{\overline{\theta}} [G_n(\theta) - G_1(\theta)] \frac{dV_y(\theta)}{d\theta} d\theta \\
&= \int_{\theta^{FB}}^{\overline{\theta}} [G_m(\theta) - G_1(\theta)] \frac{dV_y(\theta)}{d\theta} d\theta > 0
\end{aligned}$$

Notice that $\Delta V_y(\theta^{FB}) = 0$ from the project choice rule 3.3. G_n FOSD G_1 for all higher ability CEOs than the CEO with the lowest ranking in ability. Then for $\theta \in [\theta^{FB}, \overline{\theta}]$, we have $G_m(\theta) > G_1(\theta)$. And $\frac{dV_y(\theta)}{d\theta} > 0$ showed in the last section. Then, **??** is the sufficient and necessary condition for firms to offer the first best contracts to CEO 1, it is the sufficient and necessary condition to attain first best contracting for any higher ability CEO.

By the definition of Π, we have

$$\begin{aligned}
&[\Pi(\theta^{FB}, j, n) - \Pi(\theta^{FB}, j, m)] - [\Pi(\theta^{FB}, i, n) - \Pi(\theta^{FB}, i, m)] \\
&= \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) d[G_n(\theta) - G_m(\theta)](I^j - I^i) \\
&= \left[\int_{\theta^{FB}}^{\overline{\theta}} [G_m(\theta) - G_n(\theta)] \frac{dV_y(\theta)}{d\theta} d\theta \right] (I^j - I^i) > 0
\end{aligned}$$

Again, G_n FOSD G_m, thus for all $\theta \in [\theta^{FB}, \overline{\theta}]$, we have $G_m(\theta) > G_n(\theta)$.

Q.E.D.

Appendix 3.E Proof of Proposition 3

Proof. Now we are in the first best world. The proof follows (Edmans, Gabaix, and Landier, 2009) and (Thanassoulis, 2013). Consider the case that two firms i and j with $I^j > I^i$ compete for hiring two CEOs m and n with G_n FOSD G_m. Let's consider the bids offered by the firms are in the form of expected total utility in the eyes of CEOs. A

CEO accepts the bid that delivers him the highest expected total utility. As as result, the firms choose the structure of the compensation contract to attain the optimal contracting described as in last section.

Let's analyze firm i's decision between hiring CEO m or CEO n. Notice that we assume the firm can induce the first best project choice. Fixed the outside utility of the CEO m as u_m, the firm's expected value of hiring this CEO is:

$$\Pi(\theta^{FB}, i, m) = [\underline{y} + \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG_m(\theta)] I^i - u_m.$$

On the other hand, the firm i's highest possible bid $u_{i,n}$ for hiring higher ability CEO n is:

$$u_{i,n} = u_m + \left[\int_{\theta^{FB}}^{\overline{\theta}} [G_m(\theta) - G_n(\theta)] \frac{dV_y(\theta)}{d\theta} d\theta \right] I^i.$$

The term in the parenthesis is the expected value increasing from hiring the higher ability CEO n, which we have calculated in the proof of Lemma 3.

Similarly, we have the firm j's highest bid for the CEO n is:

$$u_{j,n} = u_m + \left[\int_{\theta^{FB}}^{\overline{\theta}} [G_m(\theta) - G_n(\theta)] \frac{dV_y(\theta)}{d\theta} d\theta \right] I^j.$$

Immediately we have the firm with larger asset size I^j can always bids higher for higher ability CEO

$$u_{j,n} - u_{i,n} = \left[\int_{\theta^{FB}}^{\overline{\theta}} [G_m(\theta) - G_n(\theta)] \frac{dV_y(\theta)}{d\theta} d\theta \right] (I^j - I^i).$$

Thus, in the equilibrium the larger firm matches the bids of the smaller firm $u_{j,n} = u_{i,n}$ to make the CEO just indifferent from accepting the two bids.

Iterate this argument for the hiring between the highest ability and second highest ability CEO N and $N-1$ and for the CEOs with lower ranks in their ability. We always have the firm with larger asset size bids higher for the more talented CEO. Because we assume the number of banks and CEOs are equal, the firm ranking in the k's position in asset size exactly hires the CEO ranking in the same position in ability.

Moreover, we can derive the total pay for CEO u_k by iteration

$$u_k = \sum_{i=2}^{k} \int_{\theta^{FB}}^{\overline{\theta}} [G_{i-1}(\theta) - G_i(\theta)] \frac{d\Delta V_y(\theta)}{d\theta} d\theta I^{i-1} + \underline{u}.$$

Q.E.D.

Appendix 3.F Proof of Lemma 3

Proof. The condition that all CEOs are hired in the second best contracting is similar to the one in the first best contracting world. If the CEO with the highest ability can not be hired in first best contract then the result applies to all other CEOs with lower ability.

Define two auxiliary functions as

$$\Psi_m(\hat{\theta}) = 1 + \int_{\hat{\theta}}^{\overline{\theta}} \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\hat{\theta})}{\underline{y}} \frac{V_o(\theta)}{V_o(\hat{\theta})} \right] dG_m(\theta) - \phi$$

It is a function of the CEO's ability as well. Similarly, a CEO n's project choice θ_n^* is given by

$$\Psi_n(\hat{\theta}) = 1 + \int_{\hat{\theta}}^{\overline{\theta}} \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\hat{\theta})}{\underline{y}} \frac{V_o(\theta)}{V_o(\hat{\theta})} \right] dG_n(\theta) - \phi$$

Then the optimal project made by CEO m and n in the second best contracting are given by $\Psi_m(\theta_m^*) = 0$ and $\Psi_n(\theta_n^*) = 0$. We consider the following difference

$$\begin{aligned}
&\Psi_n(\theta_n^*) - \Psi_m(\theta_n^*) \\
&= \int_{\theta_n^*}^{\overline{\theta}} \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\theta_n^*)}{\underline{y}} \frac{V_o(\theta)}{V_o(\theta_n^*)} \right] dG_n(\theta) - \int_{\theta_n^*}^{\overline{\theta}} \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\theta_n^*)}{\underline{y}} \frac{V_o(\theta)}{V_o(\theta_n^*)} \right] dG_m(\theta) \\
&= \int_{\theta_n^*}^{\overline{\theta}} \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\theta_n^*)}{\underline{y}} \frac{V_o(\theta)}{V_o(\theta_n^*)} \right] d[G_n(\theta) - G_m(\theta)] \\
&= \left\{ \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\theta_n^*)}{\underline{y}} \frac{V_o(\theta)}{V_o(\theta_n^*)} \right] [G_n(\theta) - G_m(\theta)] \right\} |_{\theta_n^*}^{\overline{\theta}} - \int_{\theta_n^*}^{\overline{\theta}} [G_n(\theta) - G_m(\theta)] d \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\theta_n^*)}{\underline{y}} \frac{V_o(\theta)}{V_o(\theta_n^*)} \right] \\
&= - \int_{\theta_n^*}^{\overline{\theta}} [G_n(\theta) - G_m(\theta)] d \left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\theta_n^*)}{\underline{y}} \frac{V_o(\theta)}{V_o(\theta_n^*)} \right] > 0.
\end{aligned}$$

The last term is larger than zero because first $G_n(\theta) < G_n(\theta)$ for any $\theta \in [\underline{\theta}, \overline{\theta}]$. Second, the derivative is

$$d\left[\frac{\Delta V_y(\theta)}{\underline{y}} - \frac{\Delta V_y(\theta_n^*)}{\underline{y}}\frac{V_o(\theta)}{V_o(\theta_n^*)}\right] = \frac{d\Delta V_y(\theta)/d\theta}{\underline{y}} - \frac{\Delta V_y(\theta_n^*)}{\underline{y}}\frac{dV_o(\theta)/d\theta}{V_o(\theta_n^*)}$$
$$= \frac{dE[y|\theta]/d\theta}{\underline{y}} - \frac{\Delta V_y(\theta_n^*)}{\underline{y}}\frac{dV_o(\theta)/d\theta}{V_o(\theta_n^*)} > 0.$$

This is because $\frac{dE[y|\theta]}{d\theta} = \frac{d\int_{\underline{y}}^{\overline{y}} y dF(y|\theta)}{d\theta} > 0$ and $\frac{dV_o(\theta)}{d\theta} = \frac{d\int_{\underline{y}}^{\overline{y}} \max(y-\hat{y},0)dF(y|\theta)}{d\theta} > 0$ as $F(y|\theta)$ satisfies FOSD and the two integrands are increasing in y. And in the second best contracting $\theta_n^* < \theta^{FB}$, thus $\Delta V_y(\theta_n^*) = E[y|\theta_n^*] - \underline{y} < 0$. So we have $\Psi_m(\theta_n^*) < 0$.

From the Proof of Proposition 2, we know that $\frac{d\Psi_m(\hat{\theta})}{d\hat{\theta}} < 0$. Thus $\theta_n^* > \theta_m^*$ as θ_m^* is the value such that $\Psi_m(\theta_m^*) = 0$.

Q.E.D.

Appendix 3.G Proof of Lemma 4

Proof. We use an auxiliary function similar to the one defined in the proof of Proposition 2 to take into account of the exercise price $\hat{y}$

$$\Psi(\theta^*, \hat{y}) = \overline{y} + \int_{\theta^*}^{\overline{\theta}} \Delta V_y(\theta)dG(\theta) + \frac{-\Delta V_y(\theta^*)}{V_o(\theta^*, \hat{y})}\int_{\theta^*}^{\overline{y}} V_o(\theta, \hat{y})dG(\theta) - \phi\underline{y}.$$

From the proof of Proposition 2, we know that $\frac{\partial\Psi(\hat{\theta},\hat{y})}{\partial\hat{\theta}} < 0$.

Then we consider exercise price $\hat{y}$. In the main text, we consider two effects of increasing the exercise price on the CEO's optimal project choice, a negative "incentive" effect and a positive "project choice" effect. The total effect of increasing $\hat{y}$ thus can be obtained as

$$\begin{aligned}
&\frac{\partial}{\partial \hat{y}}\left(\frac{-\Delta V_y(\theta^*)}{V_o(\theta^*,\hat{y})}\int_{\theta^*}^{\overline{y}} V_o(\theta,\hat{y})dG(\theta)\right)\\
&=\frac{-\Delta V_y(\theta^*)[1-F(\hat{y}|\theta^*)]}{[V_o(\theta^*,\hat{y})]^2}\int_{\theta^*}^{\overline{\theta}} V_o(\theta,\hat{y})dG(\theta)+\frac{\Delta V_y(\theta^*)}{V_o(\theta^*,\hat{y})}\int_{\theta^*}^{\overline{\theta}}[1-F(\hat{y}|\theta)]dG(\theta)\\
&=\frac{-\Delta V_y(\theta^*)[1-F(\hat{y}|\theta^*)]}{[V_o(\theta^*,\hat{y})]^2}\left(\int_{\theta^*}^{\overline{\theta}} V_o(\theta,\hat{y})dG(\theta)-\int_{\theta^*}^{\overline{\theta}}\frac{1-F(\hat{y}|\theta)}{1-F(\hat{y}|\theta^*)}V_o(\theta^*,\hat{y})dG(\theta)\right)\\
&=\frac{-\Delta V_y(\theta^*)[1-F(\hat{y}|\theta^*)]}{[V_o(\theta^*,\hat{y})]^2}\int_{\theta^*}^{\overline{\theta}}\int_{\hat{y}}^{\overline{y}}[1-F(\hat{y}|\theta)](y-\hat{y})\left(\frac{f(y|\theta)}{1-F(\hat{y}|\theta)}-\frac{f(y|\theta^*)}{1-F(\hat{y}|\theta^*)}\right)dydG(\theta)
\end{aligned}$$

Consider the following derivative

$$\frac{\partial}{\partial\theta}\frac{f(y|\theta)}{1-F(\hat{y}|\theta)}=\frac{f'_\theta(y|\theta)[1-F(\hat{y}|\theta)]+f(\hat{y}|\theta)F'_\theta(\hat{y}|\theta)}{[1-F(\hat{y}|\theta)]^2}>0$$

Notice that we have assumed that the board's objective is strictly concave to $\hat{\theta}$. Thus $f'_\theta(y|\theta)=F''_{y\theta}(y|\theta)>0$. The last term $F'_\theta(y|\theta)$ is positive as well under this assumption. So the term in the parenthesis is positive because the integration is on the interval $\theta\in[\theta^*,\overline{\theta}]$. Eventually, we have

$$\frac{\partial}{\partial \hat{y}}\left(\frac{-\Delta V_y(\theta^*)}{V_o(\theta^*,\hat{y})}\int_{\theta^*}^{\overline{y}} V_o(\theta,\hat{y})dG(\theta)\right)>0.$$

Then $\frac{\partial\Psi(\theta^*,\hat{y})}{\partial\hat{y}}>0$. And we have already proven in Proposition 2 that $\frac{\partial\Psi(\theta^*,\hat{y})}{\partial\theta^*}<0$. By the implicit function theorem, $\frac{\partial\theta^*}{\partial\hat{y}}>0$. The "project choice" effect dominates.

Q.E.D.

Appendix 3.H Proof of Lemma 5

Proof. Suppose a firm j has the opportunity of hiring two CEOs m,n with G_n FOSD G_m. We have already proved in Lemma 2 that if first best contracting is attained when hiring both CEOs, $\Pi(\theta^{FB},j,n)>\Pi(\theta^{FB},j,m)$.

Then by the assumption in this subsection that if only second best contracting is attained when hiring both CEOs, we have $\Pi(\theta^*_n,j,n)>\Pi(\theta^*_m,j,m)$.

We now focus on the case that if first best contracting is attained for only one CEO. Recall in Lemma 2, first best contracting must be attained under the higher ability CEO G_n. Thus, we have the expected firm value when hiring these two CEOs are respectively

$$\Pi(\theta^{FB}, j, n) = [\underline{y} + \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG_n(\theta)] I^j \quad \Pi(\theta^*, j, m) = [\underline{y} + \int_{\theta^*_m}^{\overline{\theta}} \Delta V_y(\theta) dG_m(\theta)] I^j.$$

Because θ^{FB} is the global maximizer of $\int_{\hat{\theta}}^{\overline{\theta}} \Delta V_y(\theta) dG_m(\theta)$. Then we have $\int_{\theta^*_m}^{\overline{\theta}} \Delta V_y(\theta) dG_m(\theta) < \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG_m(\theta)$. The latter expression is the firm's expected unit value when hiring CEO m without any agency problem. We have it is in turn less than the expected unit value when hiring CEO n, $\int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG_m(\theta) < \int_{\theta^{FB}}^{\overline{\theta}} \Delta V_y(\theta) dG_n(\theta)$, again from Lemma 2.

Thus, we conclude for a given firm it always wants to hire the higher ability CEO. *Q.E.D.*

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