Universitat Autònoma de Barcelona

Combinatorial dynamics of strip patterns of quasiperiodic skew products in the cylinder

Leopoldo Morales López

Combinatorial dynamics of strip patterns of quasiperiodic skew products in the cylinder

Thesis submitted by Leopoldo Morales López for the degree of Philosophæ Doctor by the Universitat Autònoma de Barcelona under the supervision of Prof. Lluís Alsedà i Soler and Prof. Francisco Mañosas Capellades.

Dr. Alsedà i Soler Lluís   Dr. Mañosas Capellades Francisco

Leopoldo Morales López

Programa de Doctorat en Matemàtiques
Departament de Matemàtiques, Universitat Autònoma de Barcelona
Bellaterra 2016

"Les coses secretes pertanyen al Senyor, el nostre Déu, però les revelades són per a nosaltres i per als nostres fills per sempre, a fi que posem en pràctica totes les paraules d'aquesta llei."
Deuteronomi 29:29

Agraïments

A Déu,
Si no fos per ell no hauria vingut a aquest meravellós *país petit* Catalunya.

A la Generalitat de Catalunya,
Per la seva generositat.

Especialment a en Lluís i en Topa, els meus directors,
Per acceptar-me com estudiant. Sense el seu temps, consells, discussions i la seva paciencia, res d'això seria possible.

A la Nancy, la meva dona, i companya d'aventures,
M'agafo als teus braços com el nàufrag a las seves taules.

Als meus fills, la Montse i l'Andreu,
Els meus ulls, sense ells no veia bé.

A la Esglesia Baptista Missionera en Cerdanyola,
Per que fan que aquest ofici sigui el millor.

A les esglésies Baptistes missioneres a Mèxic,
Per enviar-me com missioner.

Al meu pare†, La meva mare i els meus germans,
pels seus ànims, rialles i plors.

Finalment a tos els meus amics i germans estimats que he fet a Catalunya dels que no faig la llista sencera, per no en oblidar-me de cap, però tots sou una benedicció de Déu.

Contents

Introducción

En las últimas dos décadas, se han hecho muchos trabajos dedicados a encontrar y estudiar *Atractores Extraños no caoticos* (SNA, por sus siglas en inglés). El término SNA fue introducido y estudiado por C. Grebogi, E. Ott, S. Pelikan, y J. A. Yorke, en el artículo *Strange attractors that are not chaotic* [7]. Cabe mencionar que, antes de que la noción de SNA fuera formalizada, ya existían construcciones de funciones que contenían objetos similares, algunas de ellas, se pueden encontrar en [11], [12] y [16]. Pero, después de [7], el estudio de estos objetos se hizo popular rápidamente y apareció un notable número de artículos estudiando diferentes modelos en los cuales también aparecen dichos SNA. Posteriormente, en [10] fue publicado otro modelo importante, *el modelo de Keller*, el cual es una versión abstracta del modelo contenido en [7].

Estrechamente ligados al estudio de dichos objetos, los autores Roberta Fabbri, Tobias Jäger, Russell Johnson y Gerhard Keller publicaron el artículo *A Sharkovskiĭ-type Theorem for Minimally Forced Interval Maps* [9]. En el mismo, el teorema de Sharkovskiĭ fue extendido a una clase de sistemas que son, esencialmente, funciones del intervalo forzadas cuasiperiodicamente. Antes de describir, brevemente, las herramientas y conjuntos que se definen en [9], haremos un breve resumen del Teorema de Sharkovskiĭ y, mencionaremos algunas de sus consecuencias más importantes.

Sharkovskiĭ enunció y demostró su célebre teorema en el año 1964 en [14]. Este resultado supuso, entre otros aspectos, el inicio del estudio de lo que hoy conocemos como dinámica combinatoria en el intervalo. En dicho teorema se introduce la siguiente ordenación de los números naturales:

$$\begin{aligned}
&3 \succ 5 \succ 7 \succ 9 \succ \ldots \succ 2\cdot 3 \succ 2\cdot 5 \succ 2\cdot 7 \succ 2\cdot 9 \succ \ldots \succ \\
&2^2\cdot 3 \succ 2^2\cdot 5 \succ 2^2\cdot 7 \succ 2^2\cdot 9 \succ \ldots \succ 2^n\cdot 3 \succ 2^n\cdot 5 \succ 2^n\cdot 7 \succ 2^n\cdot 9 \succ \ldots \succ \\
&2^\infty \ldots \succ 2^n \succ \ldots \succ 2^3 \succ 2^2 \succ 2 \succ 1.
\end{aligned}$$

Observemos que el mínimo es 1 y el máximo es 3. Necesitamos incluir el símbolo 2^∞ para asegurar la existencia del supremo de ciertos conjuntos, en particular el supremo de $\{1, 2, 4, \ldots, 2^n, \ldots\}$ es 2^∞.

Dado $\mathbb{I}$ un intervalo en la recta real, definiremos el conjunto $\mathcal{C}^0(\mathbb{I},\mathbb{I}) = \{f : \mathbb{I} \to \mathbb{I} : f \text{ es una función contínua }\}$. Fijada una función $f \in \mathcal{C}^0(\mathbb{I},\mathbb{I})$ y un punto $x \in \mathbb{I}$ diremos que $\{f^n(x) : n \in \mathbb{N}\}$ es la órbita de x. Si existe $m \in \mathbb{N}$ tal que $f^m(x) = x$ diremos que la órbita de x es periódica y si $f^k(x) \neq x$ para toda $k < m$, diremos que x tiene periodo m. Observemos que, particularmente, una órbita $A = \{f^n(x) : n \in \mathbb{N}\}$ es *invariante* pues satisface $f(A) \subset A$.

El teorema de Sharkovskiĭ, para $\mathbb{I}$, afirma: Toda función $f \in C^0(\mathbb{I},\mathbb{I})$ que tiene una órbita periódica de periodo q, también tiene una órbita periódica de periodo $p \in \mathbb{N}$ para cada $p < q$. Recíprocamente, para cada $q \in \mathbb{N} \cup \{2^\infty\}$ existe una función $f_q \in C^0(\mathbb{I},\mathbb{I})$ tal que el conjunto de puntos periódicos de f_q es $\{p \in \mathbb{N} : p < q\}$.

Este resultado establece que la existencia de órbitas periódicas, de un determinado periodo, en una aplicación del intervalo "fuerza" la existencia de órbitas periódicas de otros periodos. Un refinamiento de este teorema es lo que conocemos como teoría del *forcing de órbitas periódicas en el intervalo*.

Fijado un periodo, es inmediato observar que hay distintos tipos combinatorios de órbitas del mismo periodo. Sea $P = \{p_1 < \ldots < p_n\}$ una órbita periódica de período n de una función f del intervalo. Podemos asociar a la órbita periódica una permutación σ, de orden n (a partir de ahora, n-ciclo) dada por $\sigma(i) = j$ si y solo si $f(p_i) = p_j$. Asociamos así a una órbita periódica P de periodo n un n-ciclo σ al que llamamos *pattern* de P.

Diremos que un pattern σ fuerza otro pattern τ si toda función del intervalo que tiene una órbita periódica con el pattern σ tiene también una órbita periódica con el pattern τ. La teoría del forcing en el intervalo prueba que la anterior relación es una relación de orden parcial y describe con exactitud el conjunto de patterns forzados por un pattern prefijado.

Volviendo al artículo [9], en él, el Teorema de Sharkovskiĭ fue extendido a una clase de funciones triangulares en el cilindro. A fin de enunciar las principales propiedades de dicha clase y objetos introducidos en dicho artículo, primero estableceremos un poco de notación.

Dados $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ y $\mathbb{I} = [0,1] \subset \mathbb{R}$, denotamos por Ω al Cilindro $\mathbb{S}^1 \times \mathbb{I}$. Escribiremos un punto en Ω como (θ, x) donde $\theta \in \mathbb{S}^1$ y $x \in \mathbb{I}$. Denotaremos por $\mathcal{S}(\Omega)$ a la clase de funciones forzadas cuasiperiodicamente de Ω en Ω, que son de la forma: $F(\theta,x) = (R(\theta), f(\theta,x))$ donde $R(\theta) = \theta + \omega \pmod 1, \omega \in \mathbb{R}\backslash\mathbb{Q}$ y $f : \Omega \to \mathbb{I}$.

En [9] los autores consideran conjuntos invariantes, que no son órbitas periódicas de puntos. Ni tan solo objetos minimales. Ellos consideran bandas periódicas, objetos que pasamos a definir. Denotamos por A^θ a la fibra de un subconjunto A de Ω en un punto $\theta \in \mathbb{S}^1$. Diremos que una *banda* es un subconjunto cerrado A del cilindro, tal que A^θ es un intervalo para toda θ en un residual de $\mathbb{S}^1$. Recordemos que $G \subseteq \mathbb{S}^1$ es un *subconjunto residual* si contiene la intersección de una familia numerable de subconjuntos abiertos y densos de $\mathbb{S}^1$.

Por otro lado, dos bandas A y B satisfacen $A < B$ (Definición 3.13 en [9]) si existe un conjunto residual $G \subset \mathbb{S}^1$, tal que para toda $(\theta,x) \in A$ y $(\theta,y) \in B$ implica $x < y$ para toda $\theta \in G$. Diremos que las bandas son *ordenadas* si, o bien $A < B$ o bien $A > B$. Finalmente, decimos que una banda

$B \subset \Omega$ es n*-periódica*, para una función $F \in \mathcal{S}(\Omega)$ (Definición 3.15 en [9]), si $F^n(B) = B$ y los conjuntos imagen $B, F^1(B), F^2(B), \ldots, F^{n-1}(B)$ son disjuntos y ordenados en pares.

En el caso trivial en el que f no depende de θ las bandas periódicas son conjuntos de círculos en el cilindro que son obtenidos como productos del círculo $\mathbb{S}^1$ multiplicado por órbitas periódicas P (o órbitas periódicas de intervalos) de f, es decir: $\mathbb{S}^1 \times P$.

El Teorema de Sharkovskiĭ dado en [9] establece que toda función $F \in \mathcal{S}(\Omega)$ que tiene una banda q-periódica tiene también una banda p-periódica, para todo $p \in \mathbb{N}$ tal que $q \succ p$. Al igual que en el caso del intervalo el recíproco de éste teorema también es cierto. Basta tomar funciones en las cuales la función en la segunda componente es desacoplada.

Nuestro primer objetivo, desarrollado en el Capítulo 1, es extender el teorema principal en [9] para obtener una teoría del forcing entre patterns de bandas periódicas. Demostraremos que la relación de forcing en el intervalo y en nuestra clase coinciden. Provaremos que una permutación cíclica τ fuerza ν como pattern en el intervalo si y solo si τ fuerza ν como pattern en el cilindro (en el Teorema A enunciaremos una versión más precisa). Una consecuencia inmediata del forcing entre patterns de bandas periódicas, es que tiene como corolario (Corolario 1.28) el teorema de Sharkovskiĭ para skew-products cuasiperiodicmente forzados en el cilindro provado en [9]. Lo usaremos también en los resultados que mencionamos a continuación.

El Teorema A, nos da herramientas para estudiar la *entropía* de las funciones skew-product forzadas cuasiperiodicamente en el cilindro. Recordemos que la entropía topológica és una medida lo caótico que puede ser un sistema. Para ello definimos la noción de s-herradura para skew-products forzados cuasiperiodicamente en el cilindro y demostramos, como en el caso del intervalo, que si una función skew-product cuasiperiodicamente forzada en el cilindro tiene una s-herradura entonces su entropía topologica es mayor o igual que $\log(s)$ (Teorema B). Observemos que éste teorema es importante, pues nos facilita el cálculo de cotas inferiores para la entropía.

El concepto de s-herradura, es parte fundamental, para demostrar el resultado que establece que si un skew-product forzado cuasiperiódicamente en el cilindro, tiene una órbita periódica, con pattern τ, entonces $h(F) \geq h(f_\tau)$, donde f_τ denota la función *connect-the-dots* en el intervalo sobre una órbita periódica con pattern τ. Esto implica que si el periodo de τ es $2^n q$ con $n \geq 0$ y $q \geq 1$ impar, entonces $h(F) \geq \frac{\log(\lambda_q)}{2^n}$, donde $\lambda_1 = 1$ y, para toda $q \geq 3$, λ_q es la raíz más grande del polinomio $x^q - 2x^{q-2} - 1$. Aún más, para cada $m = 2^n q$ con $n \geq 0$ y $q \geq 1$ impar, existe un skew-product cuasiperiodicamente forzado en el cilindro F_m con una órbita periódica de periodo m tal que $h(F_m) = \frac{\log(\lambda_q)}{2^n}$ (Teorema C). Esto extiende el resultado análogo, para funciones en el intervalo, a skew-products forzados cuasiperiodicamente en el cilindro.

El teorema de Sharkovskiĭ para bandas periódicas remite de manera natural a las siguientes preguntas ¿Es cierto el teorema de Sharkovskiĭ para curvas periódicas? y más generalmente: ¿Es cierto que todo skew product forzado cuasiperiodicamente tiene una curva invariante? El segundo capítulo de la memoría está dedicado a dar una respuesta negativa a ambas cuestiones

(Teorema D). Concretamente construiremos un skew-product forzado cuasiperiódicamente que tiene una curva 2-periódica y no tiene una curva invariante. En está construcción jugará un papel muy relevante unos objetos que llamamos pseudo-curvas (llamadas bandas pinchadas núcleo en [9]). La ventaja de usarlas es que se puede definir correctamente el espacio de pseudo-curvas, que equipado con la métrica adecuada es completo. Éste es un hecho extraordinariamente útil en la demostración del Teorema 2.45.

El capítulo se divide en tres partes. En la primera (Sección 2.2) desarrollamos una *Teoría general de las pseudo-curvas*. Analizamos a las pseudo-curvas como un espacio métrico y demostramos que es un espacio métrico completo. En la segunda parte (Sección 2.3), construimos una pseudo-curva, que no es una curva, que jugará un papel esencial en nuestra construcción. En la tercera parte (Secciones 2.4, 2.5, 2.6, 2.7) construimos la función que nos dejará invariante la pseudo-curva y demostramos el Teorema D. Dada la dificultad técnica de algunos resultados necesarios para la prueba del Teorema D, hemos pospuesto su demostración a las secciones 2.8, 2.9 y 2.10.

Finalmente, el Capítulo 1 ha sido publicado como artículo [2], en la revista *Journal of Mathematical Analysis and Applications*. El Capítulo 2 será enviado como artículo [3] a una revista especializada.

1

Entropy for skew-products in the cylinder

1.1 Introduction

In this chapter we want to study the coexistence and implications between periodic objects of maps on the cylinder $\Omega = \mathbb{S}^1 \times \mathbb{I}$, of the form:

$$F\colon \begin{pmatrix} \theta \\ x \end{pmatrix} \longrightarrow \begin{pmatrix} R_\omega(\theta) \\ f(\theta, x) \end{pmatrix},$$

where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, $\mathbb{I} = [0, 1]$, $R_\omega(\theta) = \theta + \omega \pmod 1$ with $\omega \in \mathbb{R}\backslash\mathbb{Q}$ and $f(\theta, x) = f_\theta(x)$ is continuous on both variables. To study this class of maps, in [9], were developed clever techniques that lead to a theorem of the Sharkovskiĭ type for this class of maps and periodic orbits of appropriate objects.

We aim at extending these results and techniques to study the combinatorial dynamics (*forcing*) and entropy of the skew-products from the class $\mathcal{S}(\Omega)$ consisting on all maps of the above type.

As already remarked in [9], instead of $\mathbb{S}^1$ we could take any compact metric space Θ that admits a minimal homeomorphism $R\colon \Theta \longrightarrow \Theta$ such that R^ℓ is minimal for every $\ell > 1$. However, for simplicity and clarity we will remain in the class $\mathcal{S}(\Omega)$.

Before stating the main results of this chapter, we will recall the extension of Sharkovskiĭ Theorem to $\mathcal{S}(\Omega)$ from [9], together with the necessary notation. We start by clarifying the notion of a periodic orbit for maps from $\mathcal{S}(\Omega)$. To this end we informally introduce some key notions that will be defined more precisely in Section 1.2.

Let X be a compact metric space. A subset $G \subset X$ is *residual* if it contains the intersection of a countable family of open dense subsets in X.

In what follows, $\pi\colon \Omega \longrightarrow \mathbb{S}^1$ will denote the standard projection from Ω to the circle.

Instead of periodic points we use objects that project over the whole $\mathbb{S}^1$, called *strips* in [9, Definition 3.9]. A *strip in Ω* is a closed set $B \subset \Omega$ such that $\pi(B) = \mathbb{S}^1$ (i.e., B projects on the whole $\mathbb{S}^1$) and $\pi^{-1}(\theta) \cap B$ is a closed interval (perhaps degenerate to a point) for every θ in a residual set of $\mathbb{S}^1$.

Given two strips A and B, we will write $A < B$ and $A \leq B$ ([9, Definition 3.13]) if there exists a residual set $G \subset \mathbb{S}^1$, such that for every $(\theta, x) \in A \cap \pi^{-1}(G)$ and $(\theta, y) \in B \cap \pi^{-1}(G)$ it follows that $x < y$ and, respectively, $x \leq y$. We say that the strips A and B are *ordered*[1] (respectively *weakly ordered*) if either $A < B$ or $A > B$ (respectively $A \leq B$ or $A \geq B$).

Given $F \in \mathcal{S}(\Omega)$ and $n \in \mathbb{N}$, a strip $B \subset \Omega$ is called *n-periodic* for F ([9, Definition 3.15]), if $F^n(B) = B$ and the image sets $B,\ F(B),\ F^2(B), \dots, F^{n-1}(B)$ are pairwise disjoint and pairwise ordered.

To state the main theorem of [9] we need to recall the *Sharkovskiĭ Ordering* ([14, 15]). The *Sharkovskiĭ Ordering* is a linear ordering of $\mathbb{N}$ defined as follows:

$$\begin{aligned}
&3 \ {}_{\text{Sh}}\!> 5 \ {}_{\text{Sh}}\!> 7 \ {}_{\text{Sh}}\!> 9 \ {}_{\text{Sh}}\!> \cdots \ {}_{\text{Sh}}\!> \\
&2\cdot 3 \ {}_{\text{Sh}}\!> 2\cdot 5 \ {}_{\text{Sh}}\!> 2\cdot 7 \ {}_{\text{Sh}}\!> 2\cdot 9 \ {}_{\text{Sh}}\!> \cdots \ {}_{\text{Sh}}\!> \\
&4\cdot 3 \ {}_{\text{Sh}}\!> 4\cdot 5 \ {}_{\text{Sh}}\!> 4\cdot 7 \ {}_{\text{Sh}}\!> 4\cdot 9 \ {}_{\text{Sh}}\!> \cdots \ {}_{\text{Sh}}\!> \\
&\qquad\qquad\vdots \\
&2^n\cdot 3 \ {}_{\text{Sh}}\!> 2^n\cdot 5 \ {}_{\text{Sh}}\!> 2^n\cdot 7 \ {}_{\text{Sh}}\!> 2^n\cdot 9 \ {}_{\text{Sh}}\!> \cdots \ {}_{\text{Sh}}\!> \\
&\qquad\qquad\vdots \\
&\cdots \ {}_{\text{Sh}}\!> 2^n \ {}_{\text{Sh}}\!> \cdots \ {}_{\text{Sh}}\!> 16 \ {}_{\text{Sh}}\!> 8 \ {}_{\text{Sh}}\!> 4 \ {}_{\text{Sh}}\!> 2 \ {}_{\text{Sh}}\!> 1.
\end{aligned}$$

In the ordering ${}_{\text{Sh}}\!\geq$ the least element is 1 and the largest one is 3. The supremum of the set $\{1, 2, 4, \dots, 2^n, \dots\}$ does not exist.

Sharkovskiĭ Theorem for maps from $\mathcal{S}(\Omega)$ 1 ([9]) *Assume that the map $F \in \mathcal{S}(\Omega)$ has a p-periodic strip. Then F has a q-periodic strip for every $q <_{\text{Sh}} p$.*

Our first main result (Theorem A) concerns the forcing relation. As we will see in detail, the *strips patterns* of periodic orbits of strips of maps from $\mathcal{S}(\Omega)$ can be formalized in a natural way as cyclic permutations, as in the case of the periodic patterns for interval maps. Our first main result states that a cyclic permutation τ forces a cyclic permutation ν as interval patterns if and only if τ forces ν as strips patterns.

Since the Sharkovskiĭ Theorem in the interval follows from the forcing relation, a corollary of Theorem A is the Sharkovskiĭ Theorem for maps from $\mathcal{S}(\Omega)$.

Next, an s-horseshoe for maps from $\mathcal{S}(\Omega)$ can be defined also in a natural way. Our second main result (Theorem B) states that if a map $F \in \mathcal{S}(\Omega)$ has an s-horseshoe then $h(F)$, the *topological entropy of F*, satisfies $h(F) \geq \log(s)$. This is a generalization of the well known result for the interval.

The third main result of the chapter (Theorem C) states that if a map $F \in \mathcal{S}(\Omega)$ has a periodic orbit of strips with strips pattern τ, then $h(F) \geq h(f_\tau)$, where f_τ denotes the *connect-the-dots*

[1] This notion will be defined with greater detail but equivalently in Definition 1.17. We are giving here this less technical definition just to simplify this general section.

interval map over a periodic orbit with pattern τ. A corollary of this fact and the lower bounds of the topological entropy of interval maps from [4] is that, if the period of τ is $2^n q$ with $n \geq 0$ and $q \geq 1$ odd, then $h(F) \geq \frac{\log(\lambda_q)}{2^n}$, where $\lambda_1 = 1$ and, for each $q \geq 3$, λ_q is the largest root of the polynomial $x^q - 2x^{q-2} - 1$. Moreover, for every $m = 2^n q$ with $n \geq 0$ and $q \geq 1$ odd, there exists a quasiperiodically forced skew-product on the cylinder F_m with a periodic orbit of strips of period m such that $h(F_m) = \frac{\log(\lambda_q)}{2^n}$.

The chapter is organized as follows. In Section 1.2 we introduce the notation and we state the results in detail and in Section 1.3 we prove Theorem A Finally, in Section 1.4 we prove Theorems B and C.

1.2 Definitions and statements of results

We start by recalling the notion of interval pattern and related results. Afterwards we will introduce the natural extension to the class $\mathcal{S}(\Omega)$ by defining the cylinder patterns.

In what follows we will denote the class of continuous maps from the interval $\mathbb{I}$ to itself by $\mathcal{C}^0(\mathbb{I}, \mathbb{I})$.

1.2.1 Interval patterns

Given $f \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$, we say that $p \in \mathbb{I}$ is an n-periodic point of f if $f^n(p) = p$ and $f^j(p) \neq p$ for $j = 1, 2, \ldots, n-1$. The set of points $\{p, f(p), f^2(p), \ldots, f^{n-1}(p)\}$ will be called a *periodic orbit*. A periodic orbit $P = \{p_1, p_2, \ldots, p_n\}$ is said to have the *spatial labelling* if $p_1 < p_2 < \cdots < p_n$. In what follows, every periodic orbit will be assumed to have the spatial labelling unless otherwise stated.

Definition 1.1 (Interval pattern). *Let $P = \{p_1 < p_2 < \cdots < p_n\}$ be a periodic orbit of a map $f \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$ and let τ be a cyclic permutation over $\{1, 2, \ldots, n\}$. The periodic orbit P is said to have the* (periodic) interval pattern τ *if and only if $f(p_i) = p_{\tau(i)}$ for $i = 1, 2, \ldots, n$. The period of P, n, will also be called the* period of τ. ◘

Remark 1.2. Every cyclic permutation can occur as interval pattern. ◘

To study the dynamics of functions from $\mathcal{C}^0(\mathbb{I}, \mathbb{I})$ we introduce the following ordering on the set of interval patterns.

Definition 1.3 (Forcing). *Given two interval patterns τ and ν, we say that τ* forces ν, as interval patterns, *denoted by $\tau \Longrightarrow_{\mathbb{I}} \nu$, if and only if* every $f \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$ *that has a periodic orbit with interval pattern τ also has a periodic orbit with interval pattern ν. By [1, Theorem 2.5], the relation $\Longrightarrow_{\mathbb{I}}$ is a partial ordering.* ◘

Next we define a *canonical map* for an interval pattern as follows.

Definition 1.4 (τ-linear map). *Let $f \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$ and let $P = \{p_1, p_2, \dots, p_n\}$ be an n-periodic orbit of f with the spatial labelling ($p_1 < p_2 < \cdots < p_n$). We define the P-*linear map *f_P as the unique map in $\mathcal{C}^0(\mathbb{I}, \mathbb{I})$ such that $f_P(p_i) = f(p_i)$ for $i = 1, 2, \dots, n$, f_P is affine in each interval of the form $[p_i, p_{i+1}]$ for $i = 1, 2, \dots, n-1$, and f_P is constant on each of the two connected components of $\mathbb{I} \setminus [p_1, p_n]$. The map f_P is also called P-*connect-the-dots map*.*

Observe that the map f_P is uniquely determined by P and $f\big|_P$.

*Let τ be the pattern of the periodic orbit P. The map f_P will also be called a τ-*linear map *and denoted by f_τ. Then the maps f_τ are not unique but all maps $f_\tau\big|_{[\min P, \max P]}$ are topologically conjugate and, thus, they have the same topological entropy and periodic orbits.* ◘

The next result is a useful characterization of the forcing relation of interval patterns in terms of the τ-linear maps.

Theorem 1.5 (Characterization of the forcing relation). *Let τ and ν be two interval patterns. Then, $\tau \Longrightarrow_{\mathbb{I}} \nu$ if and only if f_τ has a periodic orbit with pattern ν.*

1.2.2 Strips Theory

In this subsection we introduce a new (more restrictive) kind of strips with better properties and we study the basic properties that we will need throughout the chapter. To introduce this new kind of strips we first need to introduce the notion of a *core* of a set.

Given a compact metric space (X, d) we denote the set of all closed (compact) subsets of X by 2^X, and we endow this space with the Hausdorff metric

$$\begin{aligned} H_d(B, C) &= \max\{\max_{b \in B} \min_{c \in C} d(c, b), \max_{c \in C} \min_{b \in B} d(c, b)\} \\ &= \max\{\max_{b \in B} d(b, C), \max_{c \in C} d(c, B)\}. \end{aligned}$$

It is well known that $(2^X, H_d)$ is compact. Also, given a set A we will denote the closure of A by $\overline{A}$.

Definition 1.6 ([9]). *Let M be a subset of 2^Ω. We define the *core* of M, denoted M^c, as*

$$\bigcap_{G \in \mathcal{G}(\mathbb{S}^1)} \overline{M \cap \pi^{-1}(G)},$$

where $\mathcal{G}(\mathbb{S}^1)$ denotes the set of all residual subsets of $\mathbb{S}^1$. Observe that if M is compact, then $M^c \subset M$ and, $\pi(M) = \mathbb{S}^1$ implies $\pi(M^c) = \mathbb{S}^1$. ◘

This definition of *core* is rather intrincate. Below we settle an equivalent and more useful definition in the spirit of Lemma 3.2 and Remark 3.3 of [9]. The role of the resial of continuity in this equivalent definition is stated without proof in [9] and, hence, we include the proof for completeness.

Let $M \in 2^{\Omega}$ be such that $\pi(M) = \mathbb{S}^1$. We define the map $\phi_{_M} : \mathbb{S}^1 \longrightarrow 2^{\mathbb{I}}$ by $\phi_{_M}(\theta) := M^{\theta}$, and $G_{_M} := \{\theta \in \mathbb{S}^1 : \phi_{_M}$ is continuous at $\theta\}$. It can be easily seen that $\phi_{_M}$ is upper semicontinuous (i.e. for every $\theta \in \mathbb{S}^1$ and every open $U \subset \mathbb{I}$ such that $\phi_{_M}(\theta) \subset U$, $\{z \in \mathbb{S}^1 : \phi_{_M}(z) \subset U\}$ is open in $\mathbb{S}^1$). Hence, by [8, Theorem 7.10], the set $G_{_M}$ is residual. The set $G_{_M}$ will be called the *residual of continuity of* M.

Given $G \subset \mathbb{S}^1$ and a map $\varphi\colon G \longrightarrow 2^{\mathbb{I}}$, $\operatorname{Graph}(\varphi) := \{(\theta, \varphi(\theta)) : \theta \in G\} \subset \mathbb{S}^1 \times 2^{\mathbb{I}}$ denotes the *graph of* φ. By abuse of notation we will identify $\operatorname{Graph}(\varphi)$ with the set $\bigcup_{\theta\in G}\{\theta\} \times \varphi(\theta)$. Hence, we will consider $\operatorname{Graph}(\varphi)$ as a subset of Ω (or of $G \times \mathbb{I}$), and $\overline{\operatorname{Graph}(\varphi)}$ is a compact subset of Ω.

Lemma 1.7. *Let M be a compact subset of Ω. Then,*

$$M^c = \overline{\operatorname{Graph}\left(\phi_{_M}\big|_G\right)} = \overline{M \cap \pi^{-1}(G)}$$

for every residual set $G \subset G_{_M}$. Moreover, $M \cap \pi^{-1}(G) = M^c \cap \pi^{-1}(G)$ and $(M^c)^c = M^c$.

Proof. We start by proving the first statement of the lemma. Notice that if

$$\overline{M \cap \pi^{-1}(G_{_M})} \subset \overline{M \cap \pi^{-1}(H)} \quad \text{for every } H \in \mathcal{G}, \tag{1.1}$$

then

$$\overline{M \cap \pi^{-1}(G)} \subset \overline{M \cap \pi^{-1}(G_{_M})} \subset M^c = \bigcap_{H\in\mathcal{G}} \overline{M \cap \pi^{-1}(H)} \subset \overline{M \cap \pi^{-1}(G)}.$$

Hence, we only have to prove (1.1).

Let $H \in \mathcal{G}$ and let $(\theta, x) \in M \cap \pi^{-1}(G_{_M})$ (i.e. $\theta \in G_{_M}$ and $(\theta, x) \in M^{\theta} = \phi_{_M}(\theta)$). Since H is residual, it is dense in $\mathbb{S}^1$. Therefore, there exists a sequence $\{\theta_n\}_{n=1}^{\infty} \subset H$ converging to θ. Since $\theta \in G_{_M}$, $\phi_{_M}$ is continuous in θ. So, $\lim \phi_{_M}(\theta_n) = \phi_{_M}(\theta)$ and, for every $\varepsilon > 0$ exists $N \in \mathbb{N}$ such that $d\big((\theta, x), \phi_{_M}(\theta_n)\big) \le H_d(\phi_{_M}(\theta), \phi_{_M}(\theta_n)) < \varepsilon$ for every $n \ge N$. Since the sets $\phi_{_M}(\theta_n)$ are compact, for every $n \in \mathbb{N}$, there exists $(\theta_n, x_n) \in \phi_{_M}(\theta_n) \subset M \cap \pi^{-1}(H)$ such that $d\big((\theta, x), (\theta_n, x_n)\big) = d\big((\theta, x), \phi_{_M}(\theta_n)\big)$. Thus, $\lim(\theta_n, x_n) = (\theta, x)$ and, hence, $(\theta, x) \in \overline{M \cap \pi^{-1}(H)}$. This implies $M \cap \pi^{-1}(G_{_M}) \subset \overline{M \cap \pi^{-1}(H)}$ which, in turn, implies (1.1).

By the first statement,

$$M \cap \pi^{-1}(G) \subset \overline{M \cap \pi^{-1}(G)} \cap \pi^{-1}(G) = M^c \cap \pi^{-1}(G) \subset M \cap \pi^{-1}(G).$$

Now, to end the proof of the lemma, take $\widetilde{G} = G_{_M} \cap G_{_{M^c}}$, which is a residual set. By the part of the lemma already proven we have,

$$(M^c)^c = \overline{M^c \cap \pi^{-1}\big(\widetilde{G}\big)} = \overline{M \cap \pi^{-1}\big(\widetilde{G}\big)} = M^c.$$

Now we are ready to define the notion of band.

Definition 1.8 (Band). *Every strip $A \subset \Omega$ such that $A^c = A$ will be called a* band. ◘

Remark 1.9. In view of Lemma 1.7 a band could be equivalently defined as follows: A *band* is a set of the form $\overline{\mathrm{Graph}(\varphi)}$, where φ is a continuous map from a residual set of $\mathbb{S}^1$ to the connected elements (intervals) of $2^{\mathbb{I}}$. ◘

Given $F \in \mathcal{S}(\Omega)$, a strip A is $F-$*invariant* if $F(A) \subset A$ and $F-$*strongly invariant* if $F(A) = A$. As usual, the intersection of two $F-$invariant strips is either empty or an $F-$invariant strip. An invariant strip is called *minimal* if it does not have a strictly contained invariant strip.

Remark 1.10. From Corollary 3.5 and Lemmas 3.10 and 3.11 of [9] it follows that the bands in Ω have the following properties for every map from $\mathcal{S}(\Omega)$:

(1) The image of a band is a band.
(2) Every invariant strip contains an invariant minimal strip.
(3) Every invariant minimal strip is a strongly invariant band.

◘

Moreover, the Sharkovskiĭ Theorem for maps from $\mathcal{S}(\Omega)$ is indeed,

Sharkovskiĭ Theorem for maps from $\mathcal{S}(\Omega)$ 2 ([9]) *Assume that $F \in \mathcal{S}(\Omega)$ has a p-periodic strip. Then F has a q-periodic band for every $q <_{\mathrm{Sh}} p$.*

Next we introduce a particular kind of bands that play a key role in this theory since they allow us to better study and work with the bands.

Given a set $A \subset \Omega$ and $\theta \in \Omega$ we will denote the set $A \cap \pi^{-1}(\theta)$ by A^θ.

Definition 1.11. *A band A is called* solid *when A^θ is an interval for every $\theta \in \mathbb{S}^1$ and $\delta(A) := \inf\{\mathrm{diam}(A^\theta) : \theta \in \mathbb{S}^1\} > 0$. Also, A is called* pinched *if A^θ is a point for each θ in a residual subset of $\mathbb{S}^1$.* ◘

Remark 1.12. From [9, Theorem 4.11] it follows that there are only two kind of strongly invariant bands: *solid* or *pinched*. ◘

Despite of the fact that the above notion of pinched band is completely natural, for several reasons that will become clear later (see also [3]) we prefer to view the pinched bands as *pseudo-curves* in the spirit of Remark 1.9:

Definition 1.13. *Let G be a residual set of $\mathbb{S}^1$ and let $\varphi\colon G \longrightarrow \mathbb{I}$ be a continuous map from G to $\mathbb{I}$. The set $\overline{\mathrm{Graph}(\varphi)}$ will be called a* pseudo-curve. ◘

The next remark summarizes the basic properties of the pseudo-curves.

Remark 1.14. The following properties of the pseudo-curves are easy to prove:

(1) Every pseudo-curve is a band. In particular $\pi\left(\overline{\mathrm{Graph}(\varphi)}\right) = \mathbb{S}^1$.

(2) The image of a pseudo-curve is a pseudo-curve. Moreover, every invariant pseudo-curve is strongly invariant and minimal.

Now assume that $\overline{\text{Graph}(\varphi)}$ is a pseudo-curve where φ is a map from G to $\mathbb{I}$. Then,

(3) $G_{\overline{\text{Graph}(\varphi)}} \supset G$ (see e.g. [13, Lemma 7.2]).
(4) $\overline{\text{Graph}(\varphi)} \cap \pi^{-1}(G) = \text{Graph}(\varphi)$.

□

Next we want to define a partial ordering in the set of strips. We recall that a map g from $\mathbb{S}^1$ to $\mathbb{I}$ is *lower semicontinuous* (respectively *upper semicontinuous*) at $\theta \in \mathbb{S}^1$ if for every $\lambda < g(\theta)$ (respectively $\lambda > g(\theta)$) there exists a neighbourhood V of θ such that $\lambda < g(V)$ (respectively $\lambda > g(V)$). When this condition holds at every point in $\mathbb{S}^1$ g is said to be *lower semicontinuous on* $\mathbb{S}^1$ (respectively *upper semicontinuous on* $\mathbb{S}^1$).

Definition 1.15 ([9, Definition 4.1(a)]). *Given $A \in 2^{\Omega}$ such that $\pi(A) = \mathbb{S}^1$ we define the functions*

$$M_A(\theta) := \max\{x \in \mathbb{I} : (\theta, x) \in A\}$$
$$m_A(\theta) := \min\{x \in \mathbb{I} : (\theta, x) \in A\}.$$

It can be seen that M_A is an upper semicontinuous function from $\mathbb{S}^1$ to $\mathbb{I}$ and m_A is a lower semicontinuous function from $\mathbb{S}^1$ to $\mathbb{I}$. From [8, Theorem 7.10], each of the functions m_A and M_A is continuous on a residual set of $\mathbb{S}^1$. We denote by G_{m_A} (respectively G_{M_A}) the residual set of continuity of m_A (respectively M_A). □

Remark 1.16. If A is a pseudo-curve, it follows from [13, Lemma 7.2] that $G_A = G_{M_A} = G_{m_A} = \{\theta \in \mathbb{S}^1 : M_A(\theta) = m_A(\theta)\}$ (that is, A is pinched in $G_A = G_{M_A} = G_{m_A}$) and, hence,

$$A = \overline{\text{Graph}\left(M_A\big|_{G_{M_A}}\right)} = \overline{\text{Graph}\left(m_A\big|_{G_{m_A}}\right)}.$$

□

Definition 1.17 ([9, Definition 3.13]). *Given two strips A and B we write $A < B$ (respectively $A \le B$) if there exists a residual set G in $\mathbb{S}^1$ such that $M_A(\theta) < m_B(\theta)$ (respectively $M_A(\theta) \le m_B(\theta)$) for all $\theta \in G$. We say that two strips are* ordered *(respectively* weakly ordered*) if either $A < B$ or $A > B$ (respectively $A \le B$ or $A \ge B$).* □

Remark 1.18. It follows from the definition that two (weakly) ordered strips have pairwise disjoint interiors. □

The above ordering can be better formulated in terms of the *covers* of a strip.

Definition 1.19. *Let* $A \subset \Omega$ *be a strip. We define the* top cover of A *as the pseudo-curve defined by* $M_A|_{G_{M_A}}$*:*

$$A^+ := \overline{\mathrm{Graph}\left(M_A|_{G_{M_A}}\right)},$$

and the bottom cover of A *as the pseudo-curve defined by* $m_A|_{G_{m_A}}$*:*

$$A^- := \overline{\mathrm{Graph}\left(m_A|_{G_{m_A}}\right)}.$$

◻

Remark 1.20. The sets A^+ and A^- are bands but in general do not coincide with $\overline{\mathrm{Graph}\,(M_A)}$ and $\overline{\mathrm{Graph}\,(m_A)}$ respectively. Moreover, if A is a pseudo-curve then, from Remark 1.16, $A^+ = A^- = A$. ◻

Remark 1.21. Let A and B be two strips. By Remark 1.16 we have, $A < B$ if and only if $A^+ < B^-$ and $A \le B$ if and only if $A^+ \le B^-$. ◻

To end this subsection we introduce the useful notion of *band between two pseudo-curves*. Although this definition is inspired in the definition of a *basic strip* from [9] (see Definition 1.39) we follow our approach based in pseudo-curves.

Definition 1.22. *Let* A *and* B *be pseudo-curves such that* $A < B$*. We define the* band between A and B *as:*

$$E_{AB} := \overline{\bigcup_{\theta \in G_{M_A} \cap G_{m_B}} \{\theta\} \times (M_A(\theta), m_B(\theta))}.$$

◻

The properties of the set E_{AB} are summarized by:

Lemma 1.23. *Let* A *and* B *be pseudo-curves such that* $A < B$*. Then,*

(a) $E_{AB}^- = A$ *and* $E_{AB}^+ = B$*. Moreover,* $(E_{AB})^\theta = \{\theta\} \times [M_A(\theta), m_B(\theta)]$ *for every* $\theta \in G_{M_A} \cap G_{m_B}$.
(b) E_{AB} *is a band.*
(c) $E_{AB} := \overline{\mathrm{Int}\,(E_{AB})}$*. In particular,* E_{AB} *has non-empty interior.*

Proof. From the definition of E_{AB} it follows that

$$\mathrm{Graph}\left(M_A|_{G_{M_A} \cap G_{m_B}}\right) \subset E_{AB}.$$

Thus,

$$A = A^c = \overline{\mathrm{Graph}\left(M_A|_{G_{M_A} \cap G_{m_B}}\right)} \subset E_{AB}$$

by Remarks 1.14(1) and 1.16 and Lemma 1.7. Consequently, $m_{E_{AB}} \leq m_A$. Now we will prove that $m_{E_{AB}} \geq m_A$ and, hence, $m_{E_{AB}} = m_A$. To see this note that, for every $\theta \in \mathbb{S}^1$, there exists a sequence

$$\{(\theta_n, x_n)\}_{n\in\mathbb{N}} \subset \bigcup_{\theta \in G_{M_A} \cap G_{m_B}} \{\theta\} \times (M_A(\theta), m_B(\theta))$$

which converges to $(\theta, m_{E_{AB}}(\theta))$. Observe that $x_n \geq M_A(\theta_n) \geq m_A(\theta_n)$ for every n. Therefore, by Remark 1.16 $m_{E_{AB}}(\theta) = \lim_n x_n \geq \liminf_n m_A(\theta_n) \geq m_A(\theta)$.

Since $m_{E_{AB}} = m_A$, from Definition 1.19 and Remark 1.16 it follows that $E^-_{AB} = A$.

In a similar way we get that $M_{E_{AB}} = M_B$ and $E^+_{AB} = B$.

Then, by the part already proven and Remark 1.16,

$$(E_{AB})^\theta = \{\theta\} \times [M_A(\theta), m_B(\theta)] \quad \text{for every } \theta \in G_{M_A} \cap G_{m_B}. \tag{1.2}$$

This ends the proof of (a).

Now we prove (b). From the previous statement it follows that E_{AB} is a strip. Hence, we have to show that $(E_{AB})^c = E_{AB}$ which, by Definition 1.6, it reduces to prove that $E_{AB} \subset (E_{AB})^c$. Moreover, it is enough to show that

$$E^\theta_{AB} \subset (E_{AB})^c \quad \text{for every } \theta \in G_{M_A} \cap G_{m_B} \tag{1.3}$$

because, by (1.2),

$$E_{AB} \subset \overline{\bigcup_{\theta \in G_{M_A} \cap G_{m_B}} E^\theta_{AB}} \subset \overline{(E_{AB})^c} = (E_{AB})^c.$$

To prove (1.3) observe that, since $G_{M_A} \cap G_{m_B} \cap G_{E_{AB}}$ is a residual set (contained in $G_{E_{AB}}$), from Lemma 1.7 we get

$$(E_{AB})^c = \overline{E_{AB} \cap \pi^{-1}\left(G_{M_A} \cap G_{m_B} \cap G_{E_{AB}}\right)} = \overline{\bigcup_{\theta \in G_{M_A} \cap G_{m_B} \cap G_{E_{AB}}} E^\theta_{AB}}. \tag{1.4}$$

In particular,

$$\bigcup_{\theta \in G_{M_A} \cap G_{m_B} \cap G_{E_{AB}}} E^\theta_{AB} \subset (E_{AB})^c.$$

Fix $\theta \in \left(G_{M_A} \cap G_{m_B}\right) \backslash G_{E_{AB}}$. Since $G_{M_A} \cap G_{m_B} \cap G_{E_{AB}}$ is a residual set, there exists a sequence $\{\theta_n\}_{n=1}^\infty \subset G_{M_A} \cap G_{m_B} \cap G_{E_{AB}}$ whose limit is θ. The continuity of the functions M_A and m_B in $G_{M_A} \cap G_{m_B}$ implies that $\lim M_A(\theta_n) = M_A(\theta)$ and $\lim m_B(\theta_n) = m_B(\theta)$. Therefore, again by (1.2), every point of E^θ_{AB} is limit of points in $\{E^{\theta_n}_{AB}\}_{n=1}^\infty$. This implies that $E^\theta_{AB} \subset (E_{AB})^c$ by (1.4). This ends the proof of (b).

To prove (c) observe that $\overline{\operatorname{Int}(E_{AB})} \subset E_{AB}$. So, it is enough to show that

$$\bigcup_{\theta \in G_{M_A} \cap G_{m_B}} \{\theta\} \times (M_A(\theta), m_B(\theta)) \subset \operatorname{Int}(E_{AB}).$$

Take $(\theta, x) \in \{\theta\} \times (M_A(\theta), m_B(\theta))$ with $\theta \in G_{M_A} \cap G_{m_B}$. Since $x \neq M_A(\theta)$ and $x \neq m_B(\theta)$, there exists $\varepsilon > 0$ such that $x > M_A(\theta) + \varepsilon$ and $x < m_B(\theta) - \varepsilon$. On the other hand, the continuity of M_A and m_B on $G_{M_A} \cap G_{m_B}$ implies that there exist $\delta > 0$ such that $\theta' \in G_{M_A} \cap G_{m_B}$ and $|\theta - \theta'| < \delta$ implies $|M_A(\theta) - M_A(\theta')| < \varepsilon$ and $|m_B(\theta) - m_B(\theta')| < \varepsilon$. Now we define

$$r := \min \{\delta, |x - M_A(\theta) - \varepsilon|, |x - m_B(\theta) + \varepsilon|\} > 0.$$

Observe that, with this choice of r, $M_A(\theta) + \varepsilon \leq x - r < x + r \leq m_B(\theta) - \varepsilon$.

Let $U := \{(\theta', y) \in \Omega : |\theta - \theta'| < r \text{ and } |x - y| < r\}$ be an open neighbourhood of (θ, x). We will prove that every $(\theta', y) \in U$ belongs to E_{AB}. If $\theta' \in G_{M_A} \cap G_{m_B}$, from the choice of δ and r, it follows that $(\theta', y) \in \{\theta'\} \times (x - r, x + r) \subset \{\theta'\} \times [M_A(\theta'), m_B(\theta')] \subset E_{AB}$. Now assume that $\theta' \notin G_{M_A} \cap G_{m_B}$ and consider a sequence $\{\theta_n\}_{n \in \mathbb{N}} \subset G_{M_A} \cap G_{m_B} \cap (\theta - r, \theta + r)$ converging to θ'. Clearly, $(\theta_n, y) \in U$ for every $n \in \mathbb{N}$ and, by the part already proven, $(\theta_n, y) \in E_{AB}$. Consequently, since E_{AB} is closed, $(\theta', y) = \lim(\theta_n, y) \in E_{AB}$.

1.2.3 Strip patterns

In this subsection we define the notion of strips pattern and forcing for maps from $\mathcal{S}(\Omega)$ along the lines of Subsection 1.2.1.

Definition 1.24 ([9, Definition 3.15]). *Let $F \in \mathcal{S}(\Omega)$. We say that a strip $A \subseteq \Omega$ is a p-*periodic strip *if $F^p(A) = A$ and the strips $A, F(A), \dots, F^{p-1}(A)$ are pairwise disjoint and ordered. The set $\{A, F(A), \dots, F^{p-1}(A)\}$ is called an n-*periodic orbit of strips*.*

By Remarks 1.10 and 1.12, it follows that we can restrict our attention to two kind of periodic orbit of bands: the solid ones and the pseudo-curves. ◘

A periodic orbit of strips $\{B_1, B_2, \dots, B_p\}$ is said to have the *spatial labelling* if $B_1 < B_2 < \dots < B_p$. In what follows we will assume that every periodic orbit of strips has the spatial labelling.

Definition 1.25 (Strip pattern). *Let $F \in \mathcal{S}(\Omega)$ and let $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$ be a periodic orbit of strips. The* strips pattern *of $\mathcal{B}$ is the permutation τ such that $F(B_i) = B_{\tau(i)}$ for every $i = 1, 2, \dots, n$.*

When a map $F \in \mathcal{S}(\Omega)$ has a periodic orbit of strips with strips pattern τ we say that F exhibits the pattern *τ.* ◘

Remark 1.26. Interval and strips patterns are formally the same algebraic objects; that is cyclic permutations. ◘

Definition 1.27 (Forcing). *Let τ and ν be strips patterns. We say that τ forces ν in Ω, denoted by $\tau \Longrightarrow_\Omega \nu$, if and only if every map $F \in \mathcal{S}(\Omega)$ that exhibits the strips pattern τ also exhibits the quasiperiodic pattern ν.* ◘

The next theorem is the first main result of this chapter. It characterizes the relation $\Longrightarrow_{\Omega}$ by comparison with $\Longrightarrow_{\mathbb{I}}$.

Theorem A *Let τ and ν be patterns (both in $\mathbb{I}$ and Ω). Then,*

$$\tau \Longrightarrow_{\mathbb{I}} \nu \quad \textit{if and only if} \quad \tau \Longrightarrow_{\Omega} \nu.$$

The first important consequence of Theorem A is the next result which follows from the fact that the Sharkovskiĭ theorem is a corollary of the forcing relation for interval maps.

Corollary 1.28. *The Sharkovskiĭ Theorem for maps from $\mathcal{S}(\Omega)$ holds.*

Proof. Assume that $F \in \mathcal{S}(\Omega)$ exhibits a p-periodic strips pattern τ and let $q \in \mathbb{N}$ be such that $p \,{}_{\text{Sh}}\!> q$. By [1, Corollary 2.7.4], $\tau \Longrightarrow_{\mathbb{I}} \nu$ for some strips pattern ν of period q. Then, by Theorem A, $\tau \Longrightarrow_{\Omega} \nu$ and, by definition, F also has a q-periodic orbit of strips (with strips pattern ν). Then the corollary follows from Remark 1.10(2,3).

Next we are going to study the relation between the forcing relation and the topological entropy of maps from $\mathcal{S}(\Omega)$. To this end we introduce the notion of horseshoe in $\mathcal{S}(\Omega)$.

Let $F \in \mathcal{S}(\Omega)$ and let A and B be bands in Ω. We say that A *F-covers* B if either $F(A^-) \leq B^-$ and $F(A^+) \geq B^+$, or $F(A^-) \geq B^+$ and $F(A^+) \leq B^-$.

Definition 1.29 (Horseshoe). *An s-*horseshoe *for a map $F \in \mathcal{S}(\Omega)$ is a pair $(J, \mathcal{D})$ where J is a band and $\mathcal{D}$ is a set of $s \geq 2$ pairwise weakly ordered bands, each of them with non-empty interior, such that L $F-$covers J for every $L \in \mathcal{D}$. Observe that, by Remark 1.18, the elements of $\mathcal{D}$ have pairwise disjoint interiors.* ◘

The next theorem is the second main result of the chapter. It relates the topological entropy of maps from $\mathcal{S}(\Omega)$ with horseshoes.

Theorem B *Assume that $F \in \mathcal{S}(\Omega)$ has an s-horseshoe. Then*

$$h(F) \geq \log(s).$$

Next we want to introduce a class of maps that play the role of the connect-the-dots maps in the interval case and use them to study the topological entropy in relation with the periodic orbits of strips.

Definition 1.30 (Quasiperiodic τ-linear map). *Given a strips pattern τ we define a* quasiperiodic τ-linear map *$F_\tau \in \mathcal{S}(\Omega)$ as:*

$$F_\tau(\theta, x) := (R_\omega(\theta), f_\tau(x))$$

where R_ω is the irrational rotation by angle ω and f_τ is a τ-linear interval map (Definition 1.4 — recall that τ is also an interval pattern). ◘

Remark 1.31. Since, by definition, f_τ has a periodic orbit with interval pattern τ, F_τ has a periodic orbit of bands (in fact curves which are horizontal circles) with strips pattern τ. ◘

The next main result shows that the quasiperiodic τ-linear maps have minimal entropy among all maps from $\mathcal{S}(\Omega)$ which exhibit the strips pattern τ, again as in the interval case.

Theorem C *Assume that $F \in \mathcal{S}(\Omega)$ exhibits the strips pattern τ. Then*

$$h(F) \geq h(F_\tau) = h(f_\tau).$$

Theorem C has an interesting consequence concerning the entropy of strips patterns that we define as follows.

Definition 1.32 (Entropy of strips patterns). *Given a strips pattern τ we define the* entropy of τ *as*

$$h(\tau) := \inf\{h(F) : F \in \mathcal{S}(\Omega) \text{ and } F \text{ exhibits the strips pattern } \tau\}.$$

◘

With this definition, in view of the Remark 1.31, Theorem C can be written as follows:

Theorem C *Assume that $F \in \mathcal{S}(\Omega)$ exhibits the strips pattern τ. Then*

$$h(\tau) = h(F_\tau) = h(f_\tau).$$

By [1, Corollary 4.4.7] and [1, Lemma 4.4.11] we immediately get the following simple but important corollary of Theorem C which will allow us to obtain lower bounds of the topological entropy depending on the set of periods.

Corollary 1.33. *Assume that τ and ν are strips patterns such that $\tau \Longrightarrow_\Omega \nu$. Then $h(\tau) \geq h(\nu)$.*

Corollary 1.34. *Assume that $F \in \mathcal{S}(\Omega)$ has a periodic orbit of strips of period $2^n q$ with $n \geq 0$ and $q \geq 1$ odd. Then,*

$$h(F) \geq \frac{\log(\lambda_q)}{2^n}$$

where $\lambda_1 = 1$ and, for each $q \geq 3$ odd, λ_q is the largest root of the polynomial $x^q - 2x^{q-2} - 1$. Moreover, for every $m = 2^n q$ with $n \geq 0$ and $q \geq 1$ odd, there exists a map $F_m \in \mathcal{S}(\Omega)$ with a periodic orbit of bands of period m such that $h(F_m) = \frac{\log(\lambda_q)}{2^n}$.

Proof. Let τ denote the strips pattern of a periodic orbit of strips of F of period $2^n q$. By Theorem C and [4] (see also Corollaries 4.4.7 and 4.4.18 of [1]) we get that

$$h(F) \geq h(f_\tau) \geq \frac{\log \lambda_q}{2^n}.$$

To prove the second statement we use [1, Theorem 4.4.17]: for every $m = 2^n q$ there exists a primary pattern ν_m of period m such that $h(f_{\nu_m}) = \frac{\log \lambda_q}{2^n}$. Then, from Theorem C, we can take $F_m = F_{\nu_m}$.

1.3 Proof of Theorem A

To prove Theorem A we need some more notation and preliminary results.

An important tool in the study of patterns is the Markov graph. Signed Markov graphs are a specialization of Markov graphs. Next we define them and clarify the relation with our situation.

A a *combinatorial (directed) signed graph* is defined as a pair $G = (V, \mathcal{A})$ where V is a finite set, called the *set of vertices*, and $\mathcal{A} \subset V \times V \times \{+, -\}$ is called the *set of signed arrows*. Given a *signed arrow* $\alpha = (I, J, s) \in \mathcal{A}$, I is the *beginning of* α, J is the *end of* α and s is the *sign of* α. Such an arrow α is denoted by $I \xrightarrow{s} J$.

1.3.1 Signed Markov graphs in the interval

We start by introducing the notion of *signed covering*. In what follows, $\operatorname{Bd}(A)$ will denote the boundary of A.

Definition 1.35. *Let $f \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$ and let $I, J \subset \mathbb{I}$ be two intervals. We say that I* positively F-covers *J, denoted by $I \xrightarrow{+} J$ (or $I \xrightarrow[f]{+} J$ if we need to specify the map), if $f(\min I) \le \min J < \max J \le f(\max I)$ and, analogously, we say that I* negatively F-covers *J, denoted by $I \xrightarrow{-} J$ (or $I \xrightarrow[f]{-} J$), if $f(\max I) \le \min J < \max J \le f(\min I)$. Observe that if $I \xrightarrow{s_1} J_1$ and $I \xrightarrow{s_2} J_2$ then $s_1 = s_2$.*

We will write $I \overset{s_1}{\Longrightarrow} J$ or $I \underset{f}{\overset{s_1}{\Longrightarrow}} J$ to denote that $f(I) = J$ and $I \xrightarrow[f]{s_1} J$ (in particular, $f(\operatorname{Bd}(I)) = \operatorname{Bd}(J)$). ◘

We associate a signed graph to a periodic orbit of an interval map as follows.

Definition 1.36. *Let $f \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$ and let P be a periodic orbit f. A P-*basic interval *is the closure of a connected component of $[\min P, \max P] \backslash P$. The P-*signed Markov graph of *f is the combinatorial signed graph that has the set of all basic intervals as set of vertices V and the signed arrows in $\mathcal{A}$ are the ones given by Definition 1.35.* ◘

Remark 1.37. Observe that the P-signed Markov graph of f depends only on $f\big|_P$ or more precisely on the pattern of P. It does not depend on the concrete choice of the points of P and on the graph of f outside P. Consequently, if P is a periodic orbit of $f \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$ and Q is a periodic orbit of $g \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$ with the same pattern then the P-signed Markov graph of f and the Q-signed Markov graph of g coincide. In particular, the P-signed Markov graph of f and the P-signed Markov graph of f_P coincide. ◘

1.3.2 Signed Markov graphs in *Ω*

Now we also associate a signed graph to a periodic orbit of strips. We start by defining the notion of *signed covering* for bands. It is an improvement of the notion of F-covering introduced before.

Definition 1.38 (Signed covering [9, Definition 4.14]). *Let $F \in \mathcal{S}(\Omega)$ and let A and B be bands in Ω. We say that A* positively F-covers B, *denoted by $A \xrightarrow{+} B$ (or $A \xrightarrow[F]{+} B$ if we need to specify the map), if*[2] *$F(A^-) \le B^-$ and $F(A^+) \ge B^+$ and, analogously, we say that A* negatively F-covers B, *denoted by $A \xrightarrow{-} B$ (or $A \xrightarrow[F]{-} B$), if $F(A^-) \ge B^+$ and $F(A^+) \le B^-$.*

Observe that, as in the interval case (see Definition 1.35), if $A \xrightarrow{s_1} B_1$ and $A \xrightarrow{s_2} B_2$, then $s_1 = s_2$. We will write $A \xRightarrow{s_1} B$ or $A \xRightarrow[F]{s_1} B$ to denote that $F(A) = B$ and $A \xrightarrow[F]{s_1} B$. ◘

Next, by using the notion of band between two pseudo-curves, we will define the analogous of basic interval (*basic band*) and signed Markov graph for maps from $\mathcal{S}(\Omega)$.

Definition 1.39. *Let $F \in \mathcal{S}(\Omega)$ and let $\mathcal{B} = \{B_1, B_2, \dots, B_n\}$ be a periodic orbit of strips of F with the spatial labelling (that is, $B_1 < B_2 < \dots < B_n$). For every $i = 1, 2, \dots, n-1$ the band (see Remark 1.21 and Lemma 1.23)*

$$I_{B_i B_{i+1}} := E_{B_i^+ B_{i+1}^-} = \overline{\operatorname{Int}\left(E_{B_i^+ B_{i+1}^-}\right)}$$

will be called a basic band. *Observe that, from Lemma 1.23(a), $I^-_{B_i B_{i+1}} = B_i^+$ and $I^+_{B_i B_{i+1}} = B^-_{i+1}$.*

The $\mathcal{B}$-signed Markov graph of F *is the combinatorial signed graph that has the set of all basic bands as set of vertices V and the signed arrows in $\mathcal{A}$ are the ones given by Definition 1.38.* ◘

Clearly, all the basic bands are contained in $E_{B_1 B_n}$, $I_{B_i B_{i+1}} \le I_{B_{i+1} B_{i+2}}$ for $i = 1, 2, \dots, n-2$ and if $I_{B_i B_{i+1}} \cap I_{B_j B_{j+1}} \neq \emptyset$ then $|i - j| = 1$.

Remark 1.40. As in the interval case (see Remark 1.37) the P-signed Markov graph of F is a pattern invariant. Moreover, if P is a periodic orbit of $F \in \mathcal{S}(\Omega)$ and Q is a periodic orbit of the interval map $f \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$ with the same pattern, then the P-signed Markov graph of F and the Q-signed Markov graph of f coincide. In particular, the P-signed Markov graph of F and the P-signed Markov graph of f_P coincide. ◘

The following lemma summarizes the properties of basic bands and arrows. We will use it in the proof of Theorem A.

Lemma 1.41. *The following statements hold.*

(a) Let $F \in \mathcal{S}(\Omega)$ and let A and B be bands such that there is a signed arrow $A \xrightarrow{s} B$ from A to B in the signed Markov graph of F. Then,
- *(a.1) $F(A) \supset B$.*
- *(a.2) $A \xrightarrow{s} D$ for every band $D \subset B$.*
- *(a.3) There exists a band $C \subset A$ such that $C \xRightarrow{s} B$. Moreover, $F(C^+) \subset B^+$ and $F(C^-) \subset B^-$ if $s = +$, and $F(C^-) \subset B^+$ and $F(C^+) \subset B^-$ if $s = -$.*

[2] Although these definitions are formally different from [9, Definition 4.14], they are equivalent by [9, Lemma 4.3(c,d)] and the definitions of the weak ordering of strips.

(a.4) Assume that $A \xrightarrow{s} \widetilde{B}$ with $B \le \widetilde{B}$ and let C and $\widetilde{C}$ denote the bands given by (a.3) for B and $\widetilde{B}$ respectively. Then, $C \le \widetilde{C}$ if $s = +$, and $C \ge \widetilde{C}$ if $s = -$.

(b) Let $F \in \mathcal{S}(\Omega)$ and let A be a band such that $A \xrightarrow{\pm} A$. Then there exists a band $A_\infty \subset A$ such that $A_\infty \xRightarrow{\pm} A_\infty$.

Proof. Statement (a.1) is [9, Lemma 4.15] and (a.2) follows directly from the definitions. Statements (a.3,4) are [9, Lemma 4.19] while statement (b) is [9, Lemma 4.21].

1.3.3 Loops of signed Markov graphs

Given a combinatorial signed Markov graph G, a sequence of arrows

$$\alpha = I_0 \xrightarrow{s_0} I_1 \xrightarrow{s_1} \cdots \xrightarrow{s_{m-1}} I_{m-1}$$

will be called a *path of length* m. The length of α will be denoted by $|\alpha|$. When a path begins and ends in the same vertex (i.e. $I_{m-1} = I_0$) it will be called a *loop*. Observe that, then $I_1 \xrightarrow{s_1} I_2 \xrightarrow{s_2} \cdots \xrightarrow{s_{m-2}} I_{m-1} \xrightarrow{s_m} I_1$ is also a loop in G. This loop is called a *shift* of α and denoted by $S(\alpha)$. For $n \ge 0$, we will denote by S^n the n-th iterate of the shift. That is,

$$S^n(\alpha) = I_{j_0} \xrightarrow{s_{j_0}} I_{j_1} \xrightarrow{s_{j_1}} I_{j_2} \xrightarrow{s_{j_2}} \cdots \xrightarrow{s_{j_{m-2}}} I_{j_{m-1}},$$

where $j_r = r + n \pmod m$. Note that $S^{km}(\alpha) = \alpha$ for every $k \ge 0$.

Let $\alpha = I_0 \xrightarrow{s_0} I_1 \xrightarrow{s_1} \cdots \xrightarrow{s_{m-2}} I_{m-1}$ and $\beta = J_0 \xrightarrow{r_0} J_1 \xrightarrow{r_1} \cdots \xrightarrow{r_{l-2}} J_{l-1}$ be two paths such that the last vertex of α coincides with the first vertex of β (i.e. $I_{m-1} = J_0$). The path $I_0 \xrightarrow{s_0} I_1 \xrightarrow{s_1} \cdots \xrightarrow{s_{m-1}} J_0 \xrightarrow{r_0} J_1 \xrightarrow{r_1} \cdots \xrightarrow{r_{l-1}} J_{l-1}$ is the *concatenation* of α and β and is denoted by $\alpha\beta$. In this spirit, for every $n \ge 1$, α^n will denote the concatenation of α with himself n-times. the path α^n will be called the *n-repetition of* α. Also, α^∞ will denote the infinite path $\alpha\alpha\alpha\cdots$.

A loop is called *simple* if it is not a repetition of a shorter loop. Observe that, in that case, the length of the shorter loop divides the length of the long one.

The next lemma translates the non-repetitiveness of a loop to conditions on its liftings. Its proof is folk knowledge.

Lemma 1.42. *Let α be a signed loop of length n in a combinatorial signed Markov graph G. If α is simple, $S^i(\alpha) \ne S^j(\alpha)$ for every $i \ne j$.*

Given a path $\alpha = I_0 \xrightarrow{s_0} I_1 \xrightarrow{s_1} \ldots I_{m-1} \xrightarrow{s_{m-1}} I_m$ we define the *sign* of α, denoted by $\operatorname{Sign}(\alpha)$, as $\prod_{i=1}^m s_i$, where in this expression we use the obvious multiplication rules:

$$+ \cdot + = - \cdot - = +, \text{ and}$$
$$+ \cdot - = - \cdot + = -.$$

Finally we introduce a (lexicographical) *ordering* in the set of paths of signed combinatorial graphs. To this end we start by introducing a linear ordering in the set of vertices. This ordering is arbitrary but fixed.

In the case of Markov graphs, the spatial labelling of orbits induces a natural ordering in the set of basic intervals or basic bands, which is the ordering that we are going to adopt. More precisely, if $P = \{p_0, p_1, \dots, p_{n-1}\}$ is a periodic orbit with the spatial labelling, then we endow the set of vertices (basic intervals) of the associated signed Markov graph with the following ordering:

$$[p_0, p_1] < [p_1, p_2] < \cdots < [p_{n-2}, p_{n-1}].$$

Analogously, if $\mathcal{B} = \{B_0, B_1, \dots, B_{n-1}\}$ is a periodic orbit of strips with the spatial labelling, then we endow the set of vertices (basic intervals) of the associated signed Markov graph with the following ordering:

$$I_{B_0B_1} < I_{B_1B_2} < \cdots < I_{B_{n-2}B_{n-1}}.$$

Then, the above ordering in the set of vertices naturally induces a *lexicographical ordering* in the set of paths of the signed combinatorial graph as follows. Let

$$\alpha = I_0 \xrightarrow{s_0} I_1 \xrightarrow{s_1} \cdots I_{n-1} \xrightarrow{s_{n-1}} I_n \text{ and}$$
$$\beta = J_0 \xrightarrow{r_0} J_1 \xrightarrow{r_1} \cdots J_{m-1} \xrightarrow{r_{m-1}} J_m$$

be paths such that there exists $k \leq \min\{n, m\}$ with $I_k \neq J_k$ and $I_i = J_i$ for $i = 0, 1, \dots, k-1$ (recall that, by Definition 1.35, if $I_i = J_i$ then the signs s_i and r_i of the corresponding arrows coincide). We write $\alpha < \beta$ if and only if

$$\begin{cases} I_k < J_k & \text{when } s = +\text{, or} \\ I_k > J_k & \text{when } s = -\text{,} \end{cases}$$

where $s = \operatorname{Sign}\left(I_0 \xrightarrow{s_0} I_1 \xrightarrow{s_1} \cdots I_{k-1} \xrightarrow{s_{k-1}} I_k\right) = s_0 s_1 \cdots s_{k-1}$.

Next we relate the loops in signed Markov graphs with periodic orbits.

Definition 1.43. *Let $f \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$ and let p be a periodic point of f and let*

$$\alpha = J_0 \xrightarrow{s_0} J_1 \xrightarrow{s_1} \cdots J_{n-1} \xrightarrow{s_{n-1}} J_0$$

be a loop in the P-signed Markov graph of f. We say that α and p are associated *if p has period n and $f^i(p) \in J_i$ for every $i = 0, 1, \dots, n-1$. Observe that in such case $S^m(\alpha)$ and $f^m(p)$ are associated for all $m \geq 1$.* ◘

The next lemma relates the ordering of periodic points with the ordering of the associated loops. Its proof is a simple exercise.

Lemma 1.44. *Let $f \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$ and let f_P be a P-linear map, where P is a periodic orbit. Let x and y be two distinct periodic points of f_P associated respectively to two distinct loops α and β in the P-signed Markov graph of f_P. Then $x < y$ if and only if $\alpha < \beta$. Consequently, for every $n \geq 1$, $f^n(x) < f^n(y)$ if and only if $S^n(\alpha) < S^n(\beta)$.*

The next lemma is folk knowledge but we include the proof because we are not able to provide an explicit reference for it.

Lemma 1.45. *Let τ be a pattern and let $f_\tau = f_P$ be a P-linear map, where P is a periodic orbit of f_P of pattern τ. Assume that $\{q_0, q_1, q_2, \ldots, q_m\}$ is a periodic orbit of f_τ with pattern $\nu \neq \tau$. Then there exists a unique loop α in the P-signed Markov graph of f_P associated to q_0. Moreover, α is simple.*

Proof. The existence and unicity of the loop α follows from [1, Lemma 1.2.12]. We have to show that α is simple. Assume that α is the k repetition of a loop

$$\beta = J_0 \xrightarrow{s_0} J_1 \xrightarrow{s_1} \cdots J_{\ell-1} \xrightarrow{s_{\ell-1}} J_0$$

of length ℓ with $k \geq 2$ and $m = k\ell$. By [1, Lemma 1.2.6], there exist intervals $K_0 \subset J_0, K_1 \subset J_1, \ldots, K_{\ell-1} \subset J_{\ell-1}$ such tat $K_i \overset{s_i}{\Longrightarrow} K_{i+1}$ for $i = 0, 1, \ldots, \ell-2$ and $K_{\ell-1} \overset{s_{\ell-1}}{\Longrightarrow} J_0$. Clearly, since f_P is P-linear, $f_P^\ell\big|_{K_0}$ is an affine map from K_0 onto J_0. On the other hand, since q_0 is associated to $\alpha = \beta^k$ it follows that $f_P^i(q_0), f_P^{i+\ell}(q_0), \ldots, f_P^{i+(k-1)\ell}(q_0) \in J_i$ for $i = 0, 1, \ldots, \ell-1$ and, consequently, $q_0, f_P^\ell(q_0), \ldots, f_P^{(k-1)\ell}(q_0) \in K_0$. Consequently, since $f_P^\ell\left(f_P^{(k-1)\ell}(q_0)\right) = f_p^m(q_0) = q_0$, it follows that $\{q_0, f_P^\ell(q_0), \ldots, f_P^{(k-1)\ell}(q_0)\}$ is a periodic orbit $f_P^\ell\big|_{K_0}$ with period $k \geq 2$. The affinity of $f_P^\ell\big|_{K_0}$ implies that $f_P^\ell\big|_{K_0}$ is decreasing with slope -1 and $k = 2$. The fact that $f_P^\ell\big|_{K_0}(K_0) = J_0$ implies that $K_0 = J_0$ and the endpoints of J_0 are also a periodic orbit of $f_P^\ell\big|_{K_0}$ of period 2. In this situation P and $\{q_0, q_1, q_2, \ldots, q_m\}$ both have the same period and pattern; a contradiction.

Now we want to extend the notion of associated periodic orbit and loop and Lemma 1.44 to periodic orbits of strips.

Definition 1.46. *Let $F \in \mathcal{S}(\Omega)$ and let and let $\mathcal{B}$ be a periodic orbit of strips of F. We say that a loop*

$$\alpha = J_0 \xrightarrow{s_0} J_1 \xrightarrow{s_1} \cdots J_{n-1} \xrightarrow{s_{n-1}} J_0$$

in the $\mathcal{B}$-signed Markov graph of F and a strip A are associated *if A is an $n-$periodic strip of F and $F^i(A) \in J_i$ for every $i = 0, 1, \ldots, n-1$. Observe that in such case $S^m(\alpha)$ and $F^m(A)$ are associated for all $m \geq 1$.* ◘

The next lemma extends Lemma 1.2.7 and Corollary 1.2.8 of [1] to quasiperiodically forced skew products on the cylinder.

Lemma 1.47. *Let $F \in \mathcal{S}(\Omega)$ and let $J_0, J_1, \dots, J_{n-1}$ be basic bands such that*

$$\alpha = J_0 \xrightarrow{s_0} J_1 \xrightarrow{s_1} \cdots J_{n-1} \xrightarrow{s_{n-1}} J_0$$

is a simple loop in a signed Markov graph of F. Then there exists a periodic band $C \subset J_0$ associated to α (and hence of period n). Moreover, for every $i, j \in \{0, 1, \dots, n-1\}$, $F^i(C) < F^j(C)$ if and only if $S^i(\alpha) < S^j(\alpha)$.

Proof. Let A be a basic band and let $B_1 \le B_2 \le \cdots \le B_m$ be all basic bands F-covered by A. By Lemma 1.41(a.3,4) there exist bands $K(A, B_1) \le K(A, B_2) \le \cdots \le K(A, B_m)$ contained in A such that $K(A, B_i) \stackrel{s_A}{\Longrightarrow} B_i$ for $i = 1, 2, \dots, m$, where s_A denotes the sign of all arrows $A \xrightarrow{s_A} B_i$ (see Definition 1.38).

Now we recursively define a family of $2n$ bands in the following way. We set $K_{2n-1} := K(J_{n-1}, J_0) \subset J_{n-1}$ so that $K_{2n-1} \stackrel{s_{n-1}}{\Longrightarrow} J_0$.

Then, assume that $K_j \subset J_{j \pmod n}$ have already been defined for $j = i+1, i+2, \dots, 2n-1$ and $i \in \{0, 1, \dots, 2n-2\}$. Since $J_{\widetilde{\imath}} \xrightarrow{s_{\widetilde{\imath}}} J_{i+1 \pmod n}$ with $\widetilde{\imath} = i \pmod n$, by Lemma 1.41(a.2,3), there exists a band $K_i \subset K\left(J_{\widetilde{\imath}}, J_{i+1 \pmod n}\right) \subset J_{\widetilde{\imath}}$ such that $K_i \stackrel{s_{\widetilde{\imath}}}{\Longrightarrow} K_{i+1}$.

Now we claim that for every $i, j \in \{0, 1, \dots, n-1\}$ $S^i(\alpha) < S^j(\alpha)$ is equivalent to $K_i \le K_j$. If $S^i(\alpha) \ne S^j(\alpha)$ there exists $k \in \{0, 1, \dots, n-1\}$ such that

$$S^i(\alpha) = J_i \xrightarrow{s_i} J_{i+1} \xrightarrow{s_{i+1}} \cdots J_{k+i-1} \xrightarrow{s_{k+i-1}} J_{k+i} \xrightarrow{s_{k+i}} J_{k+i+1} \cdots \text{ and}$$
$$S^j(\alpha) = J_i \xrightarrow{s_i} J_{i+1} \xrightarrow{s_{i+1}} \cdots J_{k+i-1} \xrightarrow{s_{k+i-1}} J_{k+j} \xrightarrow{s_{k+j}} J_{k+j+1} \cdots$$

with $J_{k+i \pmod n} \ne J_{k+j \pmod n}$ (where every sub-index in the above paths must be read modulo n). By construction, $K_{k+i} \subset J_{k+i \pmod n}$ and $K_{k+j} \subset J_{k+j \pmod n}$. Hence, $K_{k+i} \le K_{k+j}$ if and only if $J_{k+i \pmod n} < J_{k+j \pmod n}$. By definition

$$K_{k+i-1} \stackrel{s_{k+i-1 \pmod n}}{\Longrightarrow} K_{k+i} \quad \text{and} \quad K_{k+i-1} \subset K\left(J_{k+i-1 \pmod n}, J_{k+i \pmod n}\right),$$

and

$$K_{k+j-1} \stackrel{s_{k+i-1 \pmod n}}{\Longrightarrow} K_{k+j} \quad \text{and} \quad K_{k+j-1} \subset K\left(J_{k+i-1 \pmod n}, J_{k+j \pmod n}\right).$$

Thus, $K_{k+i-1} \le K_{k+j-1}$ if and only if $K_{k+i} \le K_{k+j}$ and $s_{k+i-1 \pmod n} = +$. So, $K_{k+i-1} \le K_{k+j-1}$ if and only if $J_{k+i \pmod n} < J_{k+j \pmod n}$ and $s_{k+i-1 \pmod n} = +$. By iterating this argument $k-1$ times backwards we get that $K_i \le K_j$ if and only if $J_{k+i \pmod n} < J_{k+j \pmod n}$ and

$$\operatorname{Sign}\left(J_i \xrightarrow{s_i} J_{i+1} \xrightarrow{s_{i+1}} \cdots J_{k+i-1} \xrightarrow{s_{k+i-1}} J_{k+i}\right) = s_i s_{i+1} \cdots s_{k+i-1} = +$$

(where every sub-index in the above formula is modulo n). This ends the proof of the claim.

Observe that, since $K_n \subset J_0$, from the construction of the sets K_n we get that $K_{n-1} \subset K_{2n-1}, K_{n-2} \subset K_{2n-2}, \dots, K_0 \subset K_n$ and $K_0 \underset{F^n}{\stackrel{\operatorname{Sign}(\alpha)}{\Longrightarrow}} K_n$. Then, by Lemma 1.41(a.2,b) there exists a band $C \subset K_0 \subset J_0$ such that $C \underset{F^n}{\stackrel{\operatorname{Sign}(\alpha)}{\Longrightarrow}} C$ and $F^i(C) \subset K_i \subset J_i$ for $i = 0, 1, \dots, n-1$.

Since C is a periodic strip, $F^i(C)$ and $F^j(C)$ are either disjoint or equal. Hence, by the claim, $F^i(C) < F^j(C)$ if and only if $S^i(\alpha) < S^j(\alpha)$. Now, Lemma 1.42 tells us that $S^i(\alpha) \neq S^j(\alpha)$ whenever $i \neq j$. Consequently, $F^i(C) \neq F^j(C)$ whenever $i \neq j$ and C has period n. This ends the proof of the lemma.

Remark 1.48. From the construction in the above proof it follows that if $F \in \mathcal{S}(\Omega)$ and

$$\alpha = J_0 \xrightarrow{s_0} J_1 \xrightarrow{s_1} \cdots J_{n-1} \xrightarrow{s_{n-1}} J_0$$

is a loop in the a signed Markov graph of F by basic bands, then there exist bands $K_0 = K_0(\alpha) \subset J_0,\ K_1 \subset J_1,\ \ldots,\ K_{n-1} \subset J_{n-1}$ such that $K_i \overset{s_i}{\Longrightarrow} K_{i+1}$ for $i = 0, 1, \ldots, n-2$ and $K_{n-1} \overset{s_{n-1}}{\Longrightarrow} J_0$. In particular, $K_0 \underset{F^n}{\overset{\text{Sign}(\alpha)}{\Longrightarrow}} J_0$. Moreover, if β is another loop such that $\alpha^\infty \neq \beta^\infty$, then $K_0(\alpha)$ and $K_0(\beta)$ have pairwise disjoint interiors. ◘

1.3.4 Proof of Theorem A

We start this subsection with a lemma that studies the periodic orbits of the uncoupled quasiperiodically forced skew-products on the cylinder (in particular for the maps F_τ).

Lemma 1.49. *Let $f \in \mathcal{C}^0(\mathbb{I}, \mathbb{I})$ and let F be a map from $\mathcal{S}(\Omega)$ such that $F(\theta, x) = (R_\omega(\theta), f(x))$. Then, the following statements hold.*

(a) Assume that $P = \{p_1, p_2, \ldots, p_n\}$ is a periodic orbit of f with pattern τ. Then $\mathbb{S}^1 \times P$ is a periodic orbit of F with pattern τ.

(b) If B is a periodic orbit of strips of F with pattern τ then there exists a periodic orbit P of f with pattern τ such that $\mathbb{S}^1 \times P$ is a periodic orbit of F with pattern τ and $\mathbb{S}^1 \times P \subset B$. In particular, every cyclic permutation is a pattern of a function of $F \in \mathcal{S}(\Omega)$.

Proof. The first statement follows directly from the definition of a pattern. Now we prove (b). Let $B = \{B_1, B_2, \ldots, B_n\}$ be periodic orbit of strips of F with pattern τ (that is, $F(B_i) = B_{\tau(i)}$ for $i = 1, 2, \ldots, n$). Since $F = (R_\omega, f)$ it follows that $F^k = (R_\omega^k, f^k)$ for every $k \in \mathbb{N}$ (so the iterates of F are also uncoupled quasiperiodically forced skew-products). So, since $F^n(B_i) = B_i$ for every i, it follows that the strips B_i are horizontal. That is, for every i there exists a closed interval $J_i \subset \mathbb{I}$ such that $B_i = \mathbb{S}^1 \times J_i$. Moreover, since the strips are pairwise disjoint, so are the intervals J_i. Clearly, $f(J_i) = J_{\tau(i)}$ for every i and, hence, $f^n(J_1) = J_1$. So, there exists a point $p_1 \in J_1$ such that $f^n(p_1) = p_1$ and $f^k(p_1) \in f^k(J_1) = J_{\tau^k(1)}$ for $k \geq 0$. Since the intervals J_i are pairwise disjoint, the set $P = \{p_1, f(p_1), \ldots, f^{n-1}(p_1)\}$ is a periodic orbit of f of period n such that $\mathbb{S}^1 \times P \subset B$. Moreover, if we set $f^k(p_1) = p_{\tau^k(1)}$ for $k = 1, 2, \ldots, n-1$, then P has the spatial labelling and it follows that the pattern of P is τ.

Proof (Proof of Theorem A). First we prove that $\tau \Longrightarrow_\Omega \nu$ implies $\tau \Longrightarrow_\mathbb{I} \nu$. The assumption $\tau \Longrightarrow_\Omega \nu$ implies that every map $F \in \mathcal{S}(\Omega)$ that exhibits the strips pattern τ also exhibits the strips pattern ν. In particular, the map F_τ has a periodic orbit of strips with pattern ν. By Lemma 1.49, f_τ has a periodic orbit with pattern ν. Therefore, $\tau \Longrightarrow_{\mathbb{I}} \nu$ by the characterization of the forcing relation in the interval (Theorem 1.5).

Now we prove that $\tau \Longrightarrow_{\mathbb{I}} \nu$ implies $\tau \Longrightarrow_{\Omega} \nu$. Clearly, we may assume that $\nu \neq \tau$. We have to show that every $F \in \mathcal{S}(\Omega)$ that has a periodic orbit of strips $B = \{B_0, B_1, \dots, B_{n-1}\}$ with strips pattern τ also has a periodic orbit of strips with strips pattern ν.

We consider the map $f_\tau = f_P$ where P is a periodic orbit with pattern τ. By Theorem 1.5, f_τ has periodic orbit $Q = \{q_0, q_1, \dots, q_{n-1}\}$ with pattern ν. Since Q has the spatial labelling, $q_0 = \min Q$,

Since $\nu \neq \tau$, by Lemma 1.45, there exists a simple loop

$$\alpha = I_0 \xrightarrow{s_0} I_1 \xrightarrow{s_1} \cdots \xrightarrow{s_{n-2}} I_{n-1} \xrightarrow{s_{n-1}} I_0$$

in the P-signed Markov graph of f_τ associated to q_0. Moreover, by Definition 1.43,

$$\begin{array}{ll} q_0 & \sim I_0 \xrightarrow{s_0} I_1 \xrightarrow{s_1} \cdots \xrightarrow{s_{n-2}} I_{n-1} \xrightarrow{s_{n-1}} I_0 \\ f_\tau(q_0) & \sim I_1 \xrightarrow{s_1} I_2 \xrightarrow{s_2} \cdots \xrightarrow{s_{n-1}} I_0 \xrightarrow{s_0} I_1 \\ f_\tau^2(q_0) & \sim I_2 \xrightarrow{s_2} I_3 \xrightarrow{s_3} \cdots \xrightarrow{s_{n-1}} I_0 \xrightarrow{s_0} I_1 \xrightarrow{s_1} I_2 \\ \vdots & \qquad\qquad\vdots \\ f_\tau^{n-1}(q_0) & \sim I_{n-1} \xrightarrow{s_{n-1}} I_0 \xrightarrow{s_0} I_1 \xrightarrow{s_1} I_2 \cdots \xrightarrow{s_{n-2}} I_{n-1}, \end{array}$$

where the symbol $\sim$ means "associated with". By Remark 1.40 (see also Remark 1.37), the above loop α also exists in the B-signed Markov graph of F by replacing the basic intervals $I_i = [q_i, q_{i+1}]$ by the basic bands $I_{B_i B_{i+1}}$:

$$\alpha = I_{B_0 B_1} \xrightarrow{s_0} I_{B_1 B_2} \xrightarrow{s_1} \cdots \xrightarrow{s_{n-2}} I_{B_{n-2} B_{n-1}} \xrightarrow{s_{n-1}} I_{B_0 B_1}.$$

By Lemma 1.47 and Definition 1.46, F has a periodic band Q_0 associated to α (and hence of period n), and

$$\begin{array}{rl} Q_0 \sim & I_{B_0 B_1} \xrightarrow{s_0} I_{B_1 B_2} \xrightarrow{s_1} \cdots \xrightarrow{s_{n-2}} I_{B_{n-2} B_{n-1}} \xrightarrow{s_{n-1}} I_{B_0 B_1} \\ F(Q_0) \sim & I_{B_1 B_2} \xrightarrow{s_1} I_{B_2 B_3} \xrightarrow{s_2} \cdots \xrightarrow{s_{n-2}} I_{B_{n-2} B_{n-1}} \xrightarrow{s_{n-1}} I_{B_0 B_1} \xrightarrow{s_0} I_{B_1 B_2} \\ F^2(Q_0) \sim & I_{B_2 B_3} \xrightarrow{s_2} \cdots \xrightarrow{s_{n-2}} I_{B_{n-2} B_{n-1}} \xrightarrow{s_{n-1}} I_{B_0 B_1} \xrightarrow{s_0} I_{B_1 B_2} \xrightarrow{s_1} I_{B_2 B_3} \\ \vdots \quad & \qquad\qquad\vdots \\ F^{n-1}(Q_0) \sim & I_{B_{n-2} B_{n-1}} \xrightarrow{s_{n-1}} I_{B_0 B_1} \xrightarrow{s_0} I_{B_1 B_2} \xrightarrow{s_1} \cdots \xrightarrow{s_{n-2}} I_{B_{n-2} B_{n-1}}. \end{array}$$

By Lemmas 1.44 and 1.42, the order of the points $f_\tau^i(q_0)$ induces an order on the shifts of the loop $S^i(\alpha)$, with the usual lexicographical ordering and, by Lemma 1.47, the order of the shifts $S^i(\alpha)$ induces the same order on the bands $F^i(Q_0)$. Thus, for every $i, j \in \{0, 1, \dots, n-1\}$, $i \neq j$, $F^i(Q_0) < F^j(Q_0)$ if and only if $f_\tau^i(q_0) < f_\tau^j(q_0)$. So, $\{Q_0, F(Q_0), F^2(Q_0), \dots, F^{n-1}(Q_0)\}$ and $\{q_0, q_1, \dots, q_{n-1}\}$ have the same pattern ν. This concludes the proof.

1.4 Proof of Theorems B and C

We start by proving Theorem B.

The next technical lemma is inspired in [1, Lemma 4.3.1].

Lemma 1.50. *Let $F \in \mathcal{S}(\Omega)$ and let $(J, \mathcal{D})$ be an s-horseshoe of F. Then, there exists $\mathcal{D}_n$, a set of s^n pairwise weakly ordered bands contained in J, each of them with non-empty interior, such that $(J, \mathcal{D}_n)$ is a s^n-horseshoe for F^n.*

Proof. We use induction. For $n = 1$ there is nothing to prove.

Suppose that the induction hypothesis holds for some n and let $D \in \mathcal{D}$ and $C \in \mathcal{D}_n$. Since $C \subset J$ has non-empty interior and $D \xrightarrow{\pm} J$, by Lemma 1.41(a.2,3), there exists a band $B(D, C) \subset D$ with non-empty interior such that $B(D, C) \xRightarrow{\pm} C$. Moreover, given $C' \in \mathcal{D}_n$ with $C' \neq C$, $B(D, C)$ and $B(D, C')$ can be chosen to be weakly ordered because C and C' are weakly ordered by assumption. Since, $C \in \mathcal{D}_n$, $B(D, C) \xrightarrow[F^{n+1}]{\pm} J$. Thus, the family

$$\mathcal{D}_{n+1} = \{B(D, C) : D \in \mathcal{D} \text{ and } C \in \mathcal{D}_n\}$$

consists of s^{n+1} pairwise weakly ordered bands contained in J, each of them with non-empty interior, such that $B(D, C)$ F^{n+1}-covers J. Consequently, $(J, \mathcal{D}_{n+1})$ is an s^{n+1}-horseshoe for F^{n+1}.

Proof (Proof of Theorem B). Fix $n > 0$. By Lemma 1.50, F^n has a s^n-horseshoe $(J, \mathcal{D})$. Remove the smallest and the biggest band of $\mathcal{D}$ and call K the smallest band that contains the remaining elements of $\mathcal{D}$. Clearly, K is contained in the interior of J. Thus, by Lemma 1.41(a.2,3), each element D of $\mathcal{D}$ contains in its interior a band $A(D)$ such that $A(D) \xRightarrow[F^n]{\pm} K$. Then there exists an open cover $\mathcal{B}$ of the strip J (formed by open sets B such B^θ is an open interval for every $\theta \in \mathbb{S}^1$), such that for each $D \in \mathcal{D}\big|_K$, the band $A(D)$ intersects only one element $B(D)$ of $\mathcal{B}$ (then it has to be contained in it) and if $D, D' \in \mathcal{D}\big|_K$ with $D \neq D'$ then $B(D) \neq B(D')$. For $D_0, D_1, \dots, D_{k-1} \in \mathcal{D}\big|_K$ the set $\cap_{i=0}^{k-1} F^{-n}(A(D_i))$ is non-empty and intersects only one element of $\mathcal{B}^k_{F^n}$, namely $\cap_{i=0}^{k-1} F^{-n}(B(D_i))$. Therefore the sets $\cap_{i=0}^{k-1} F^{-n}(A(D_i))$ are different for different sequences $(D_0, D_1, \dots, D_{k-1})$, and thus

$$\mathcal{N}(\mathcal{B}^k_{F^n}) \geq (\operatorname{Card} \mathcal{D} - 2)^k,$$

where $\mathcal{N}(\mathcal{B}^k_{F^n})$ is defined as in [1, Section 4.1]. Hence,

$$h(F) = \frac{1}{n} h(F^n) \geq \frac{1}{n} h(F^n, \mathcal{B}) \geq \frac{1}{n} \log(\operatorname{Card}(\mathcal{D}) - 2) = \frac{1}{n} \log(s^n - 2).$$

Since n is arbitrary, we obtain $h(F) \geq \log(s)$.

Now we aim at proving Theorem C. To this end we have to introduce some more notation and preliminary results concerning the *topological entropy*.

Given a map $f \in \mathcal{S}(\Omega)$, $h(F\big|_{I_\theta})$ is defined for every $I_\theta := \{\theta\} \times \mathbb{I}$ (despite of the fact that it is not F-invariant) by using the Bowen definition of the topological entropy (c.f. [5,6]). Moreover, the Bowen Formula gives

$$\max\{h(R), h_{\text{fib}}(F)\} \leq h(F) \leq h(R) + h_{\text{fib}}(F)$$

where

$$h_{\text{fib}}(F) = \sup_{\theta \in \mathbb{S}^1} h(F\big|_{I_\theta}).$$

Since $h(R) = 0$, it follows that $h(F) = h_{\text{fib}}(F)$.

In the particular case of the uncoupled maps $F_\tau = (R, f_\tau)$ we easily get the following result:

Lemma 1.51. *Let τ be a pattern (both in $\mathbb{I}$ and Ω). Then $h(F_\tau\big|_{I_\theta}) = h(f_\tau)$ for every $\theta \in \mathbb{S}^1$. Consequently,*

$$h(F_\tau) = h_{\text{fib}}(F_\tau) = h(f_\tau).$$

Given a signed Markov graph G with vertices $I_1, I_2, \dots, I_n$ we associate to it a $n \times n$ *transition matrix* $T_G = (t_{ij})$ by setting $t_{ij} = 1$ if and only if there is a signed arrow from the vertex I_i to the vertex I_j in G. Otherwise, t_{ij} is set to 0.

The spectral radius of a matrix T, denoted by $\rho(T)$, is equal to the maximum of the absolute values of the eigenvalues of T.

Lemma 1.52. *Let P be a periodic orbit of strips of $F \in \mathcal{S}(\Omega)$ and let T be the transition matrix of the P-signed Markov graph of F. Then*

$$h(F) \geq \max\{0, \log(\rho(T))\}.$$

Proof. If $\rho(T) \leq 1$ then there is nothing to prove. So, we assume that $\rho(T) > 1$. Let J be the i-th P-basic band and let s be the i-th entry of the diagonal of T^n. By [1, Lemma 4.4.1] there are s loops of length n in the P-signed Markov graph of F beginning and ending at J. Hence, if $s \geq 2$, F^n has an s-horseshoe $(J, \mathcal{D})$ by Remark 1.48. By Theorem B, $h(F) = \frac{1}{n} h(F^n) \geq \frac{1}{n}\log(s)$.

If there are k basic bands, the trace of T^n is not larger than k times the maximal entry on the diagonal of T^n. Hence, $h(F) \geq \frac{1}{n}\log\left(\frac{1}{k}\operatorname{tr}(T^n)\right)$. Therefore, by [1, Lemma 4.4.2],

$$h(F) \geq \limsup_{n\to\infty} \frac{1}{n}\log\left(\frac{1}{k}\operatorname{tr}(T^n)\right) = \limsup_{n\to\infty}\frac{1}{n}\log(\operatorname{tr}(T^n) = \log(\rho(T)).$$

Proof (Proof of Theorem C). Let P be a periodic orbit of strips with pattern τ and let T be the transition matrix of the P-signed Markov graph of F. Let $f_\tau = f_Q$ be a Q-linear map in $\mathcal{C}^0(\mathbb{I}, \mathbb{I})$, where Q is a periodic orbit of f_Q with pattern τ. In view of Remark 1.40 (see also Remark 1.37), T is also the transition matrix of the Q-signed Markov graph of f_τ. Consequently, by [1, Theorem 4.4.5], $h(f_\tau) = \max\{0, \log(\rho(T))\}$. By Lemmas 1.52 and 1.51,

$$h(F) \geq \max\{0, \log(\rho(T))\} = h(f_\tau) = h(F_\tau).$$

2

A skew-product aplication without invariant curves

2.1 Introduction

We consider the coexistence and implications between periodic objects of maps on the cylinder $\Omega = \mathbb{S}^1 \times \mathbb{I}$, of the form:

$$F\colon \begin{pmatrix}\theta\\ x\end{pmatrix} \longrightarrow \begin{pmatrix}R_\omega(\theta)\\ \zeta(\theta,x)\end{pmatrix},$$

where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, $\mathbb{I}$ is an interval of the real line, $R_\omega(\theta) = \theta + \omega \pmod 1$ with $\omega \in \mathbb{R}\backslash\mathbb{Q}$ and $\zeta(\theta,x) = \zeta_\theta(x)$ is continuous on both variables. The class of all maps of the above type will be denoted by $\mathcal{S}(\Omega)$.

In this setting a very basic and natural question is the following: *is it true that any map in the class $\mathcal{S}(\Omega)$ has an invariant curve?*

In [9], the authors created an appropriate topological framework that allowed them to obtain the following extension of the Sharkovskiĭ Theorem to the class $\mathcal{S}(\Omega)$[1].

Let X be a compact metric space. We recall that a subset $G \subset X$ is *residual* if it contains the intersection of a countable family of open dense subsets in X.

In what follows, $\pi\colon \Omega \longrightarrow \mathbb{S}^1$ will denote the standard projection from Ω to the circle. Given a set $B \subset \mathbb{S}^1$, for convenience we will use the following notation:

$$\upuparrows B := \pi^{-1}(B) = B \times \mathbb{I} \subset \Omega$$

In the particular case when $B = \{\theta\}$, instead of $\upuparrows\{\theta\}$ we will simply write $\upuparrows\theta$. Also, given $A \subset \Omega$, we will denote by $A^{\upuparrows B}$ the set

$$A \cap \upuparrows B = \{(\theta,x) \in \Omega : \theta \in B \text{ and } (\theta,x) \in A\}.$$

In the particular case when $B = \{\theta\}$, instead of $A^{\upuparrows\theta}$ we will simply write A^θ.

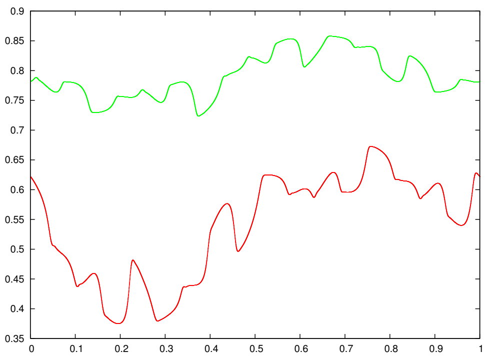
[1] As already remarked in [9], instead of $\mathbb{S}^1$ we could take any compact metric space Θ that admits a minimal homeomorphism $R\colon \Theta \longrightarrow \Theta$ such that R^ℓ is minimal for every $\ell > 1$. However, for simplicity and clarity we will remain in the class $\mathcal{S}(\Omega)$.

Instead of periodic points we use objects that project over the whole $\mathbb{S}^1$, called *strips* in [9, Definition 3.9]. A set $B \subset \Omega$ such that $\pi(B) = \mathbb{S}^1$ (i.e., B projects on the whole $\mathbb{S}^1$) will be called a *circular set*.

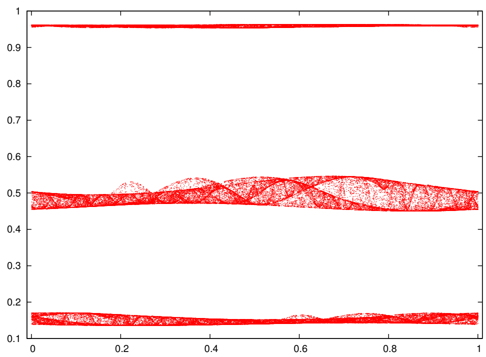
Definition 2.1. *A* strip in Ω *is a compact circular set* $B \subset \Omega$ *such that* B^θ *is a closed interval (perhaps degenerate to a point) for every* θ *in a residual set of* $\mathbb{S}^1$. ◘

Given two strips A and B, we will write $A < B$ and $A \le B$ ([9, Definition 3.13]) if there exists a residual set $G \subset \mathbb{S}^1$, such that for every $(\theta, x) \in A^{\upharpoonleft\upharpoonright G}$ and $(\theta, y) \in B^{\upharpoonleft\upharpoonright G}$ it follows that $x < y$ and, respectively, $x \le y$. We say that the strips A and B are *ordered* (respectively *weakly ordered*) if either $A < B$ or $A > B$ (respectively $A \le B$ or $A \ge B$).

Definition 2.2 ([9, Definition 3.15]). *A strip* $B \subset \Omega$ *is called* n-periodic *for* $F \in \mathcal{S}(\Omega)$ *if* $F^n(B) = B$ *and the image sets* $B,\ F(B),\ F^2(B), \dots, F^{n-1}(B)$ *are pairwise disjoint and pairwise ordered (see Figure 2.1 for examples).* ◘



(a) $3.28x(1-x) + \frac{4}{100}\cos(2\pi\theta)$



(b) $3.85x(1-x)(1 + \frac{111}{10^5}\cos(2\pi\theta))$

Figure 2.1: In the left picture we show an example two periodic orbit of curves, and in the second we show a possible example of a three periodic orbit solid strips.

To state the main theorem of [9] we need to recall the *Sharkovskiĭ Ordering* ([14, 15]). The *Sharkovskiĭ Ordering* is a linear ordering of $\mathbb{N}$ defined as follows:

$$3 \;{}_{\text{Sh}}\!\!> 5 \;{}_{\text{Sh}}\!\!> 7 \;{}_{\text{Sh}}\!\!> 9 \;{}_{\text{Sh}}\!\!> \cdots \;{}_{\text{Sh}}\!\!>$$
$$2\cdot 3 \;{}_{\text{Sh}}\!\!> 2\cdot 5 \;{}_{\text{Sh}}\!\!> 2\cdot 7 \;{}_{\text{Sh}}\!\!> 2\cdot 9 \;{}_{\text{Sh}}\!\!> \cdots \;{}_{\text{Sh}}\!\!>$$
$$4\cdot 3 \;{}_{\text{Sh}}\!\!> 4\cdot 5 \;{}_{\text{Sh}}\!\!> 4\cdot 7 \;{}_{\text{Sh}}\!\!> 4\cdot 9 \;{}_{\text{Sh}}\!\!> \cdots \;{}_{\text{Sh}}\!\!>$$
$$\vdots$$
$$2^n\cdot 3 \;{}_{\text{Sh}}\!\!> 2^n\cdot 5 \;{}_{\text{Sh}}\!\!> 2^n\cdot 7 \;{}_{\text{Sh}}\!\!> 2^n\cdot 9 \;{}_{\text{Sh}}\!\!> \cdots \;{}_{\text{Sh}}\!\!>$$
$$\vdots$$
$$\cdots \;{}_{\text{Sh}}\!\!> 2^n \;{}_{\text{Sh}}\!\!> \cdots \;{}_{\text{Sh}}\!\!> 16 \;{}_{\text{Sh}}\!\!> 8 \;{}_{\text{Sh}}\!\!> 4 \;{}_{\text{Sh}}\!\!> 2 \;{}_{\text{Sh}}\!\!> 1.$$

In the ordering ${}_{\text{Sh}}\!\!\geq$ the least element is 1 and the largest one is 3. The supremum of the set $\{1, 2, 4, \dots, 2^n, \dots\}$ does not exist.

Sharkovskiĭ Theorem for maps from $\mathcal{S}(\Omega)$ 3 ([9]) *Assume that the map $F \in \mathcal{S}(\Omega)$ has a p-periodic strip. Then F has a q-periodic strip for every $q <_{\text{Sh}} p$.*

In view of this result, the new following natural question (that is stronger that the previous one) arises: *Does Theorem 3 holds when restricted to curves?* where a curve is defined as the graph of a continuous map from $\mathbb{S}^1$ to $\mathbb{I}$. More precisely, *is it true that if F has a q-periodic curve and $p \leq_{\text{Sh}} q$ then does there exists a p-periodic curve of F?*

The aim of this chapter is to answer both of the above questions in the negative by constructing a counterexample. This is done by the following result which is the main result of the chapter.

Theorem D *There exists a map $T \in \mathcal{S}(\Omega)$ with $f(\theta, \cdot)$ non-increasing for every $\theta \in \mathbb{S}^1$, such that T permutes the upper and lower circles of Ω (thus having a periodic orbit of period two of curves), and T does not have any invariant curve.*

The construction will be done in two steps. First, in Section 2.3, we construct a strip A which is a pseudo-curve which is not a curve. This strip is obtained as a *limit* of sets defined inductively by using of a collection of *winged boxes* $\mathcal{R}^{\smile}(i^*) \subset \Omega$. Second, we construct a Cauchy sequence $\{T_m\}_{m=0}^{\infty}$ that gives as a limit the function T from Theorem D having A as invariant set. To this end, in Section 2.4 we define a collection of auxiliary functions G_i defined on the winged boxes $\mathcal{R}^{\smile}(i^*)$. Next, in Section 2.5 we introduce a notion of *depth* in the set of winged boxes $\mathcal{R}^{\smile}(i^*)$ which defines a convenient stratification in the set of winged boxes $\mathcal{R}^{\smile}(i^*)$. In Section 2.6 we study the wings of box and its interaction with boxes of higher depth. In Section 2.7, by using the auxiliary functions from Section 2.4, the stratification from Section 2.5 and the technical results from Section 2.6 we construct the Cauchy sequence $\{T_m\}_{m=0}^{\infty} \subset \mathcal{S}(\Omega)$, we define the map $T = \lim_{m\to\infty} T_m$, and we prove Theorem D.

For clarity, we omit the proofs of all results from Section 2.7. These proofs will be provided in Sections 2.8, 2.9 and 2.10. Section 2.2 is devoted to introduce the necessary definitions and, in particular, to introduce the notion of pseudo-curve and some necessary results on the space of pseudo-curves.

2.2 Definitions and preliminary results

The main aim of this section is to introduce the definition and basic results about pseudo-curves.

Given $G \subset \mathbb{S}^1$ and a map $\varphi\colon G \longrightarrow \mathbb{I}$, $\operatorname{Graph}(\varphi)$ denotes the *graph of* φ. Also, given a set A we will denote the closure of A by $\overline{A}$.

Definition 2.3 (Pseudo-curve). *Let G be a residual set of $\mathbb{S}^1$ and let $\varphi\colon G \longrightarrow \mathbb{I}$ be a continuous map from G to $\mathbb{I}$. The set $\overline{\operatorname{Graph}(\varphi)}$, denoted by $\mathfrak{A}_{(\varphi,G)}$, will be called a* pseudo-curve. *Notice that every pseudo-curve is a compact circular set.*

Also, $\boldsymbol{\mathcal{A}}$ will denote the class of all pseudo-curves. ◘

A set $A \subset \Omega$ is F*-invariant* (respectively *strongly F-invariant*) if $F(A) \subset A$ (respectively $F(A) = A$). Observe that if $F \in \mathcal{S}(\Omega)$, every compact F-invariant set is circular. A closed invariant set is called *minimal* if it does not contain any proper closed invariant set.

An *arc of a curve* is the graph of a continuous function from an arc of $\mathbb{S}^1$ to $\mathbb{I}$.

The pseudo-curves have the following properties which are easy to prove:

Lemma 2.4. *Given a pseudo-curve $\mathfrak{A}_{(\varphi,G)} \in \boldsymbol{\mathcal{A}}$ the following statements hold.*

(a) $\mathfrak{A}^{\theta}_{(\varphi,G)}$ consists of a single point for every $\theta \in G$. Consequently,

$$\mathfrak{A}^{\Uparrow G}_{(\varphi,G)} = \operatorname{Graph}(\varphi).$$

(b) Every circular compact set contained in a pseudo-curve coincides with the pseudo-curve.

(c) $\mathfrak{A}_{(\varphi,G)} = \overline{\operatorname{Graph}(\varphi|_{\widetilde{G}})}$ for every $\widetilde{G} \subset G$ dense in $\mathbb{S}^1$.

(d) If $\mathfrak{A}_{(\varphi,G)}$ contains a curve then it is a curve.

Proof. We start by proving (a). By the definition of a pseudo-curve we have $\operatorname{Graph}(\varphi) \subset \mathfrak{A}^{\Uparrow G}_{(\varphi,G)}$. To prove the other inclusion fix $\theta \in G$ and $x \in \mathbb{I}$ such that $(\theta, x) \in \mathfrak{A}_{(\varphi,G)}$. Then, there exists a sequence $\{(\theta_n, \varphi(\theta_n))\}_{n=1}^{\infty} \subset \operatorname{Graph}(\varphi)$ such that $\lim_{n\to\infty}(\theta_n, \varphi(\theta_n)) = (\theta, x)$. The continuity of φ in G (and hence in θ) implies $x = \varphi(\theta)$ and, therefore, $(\theta, x) \in \operatorname{Graph}(\varphi)$.

Now we prove (b). Assume that $B \subset \mathfrak{A}_{(\varphi,G)}$ is a circular compact set. From the assumptions and statement (a) we get $\mathfrak{A}^{\Uparrow G}_{(\varphi,G)} = B^{\Uparrow G}$. Hence,

$$\mathfrak{A}_{(\varphi,G)} = \overline{\operatorname{Graph}(\varphi)} = \overline{\mathfrak{A}^{\Uparrow G}_{(\varphi,G)}} = \overline{B^{\Uparrow G}} \subset B.$$

Now (d) follows directly from (b) and the fact that a curve is compact since it is the graph of a continuous function. Statement (c) also follows from (b) because $\overline{\operatorname{Graph}\left(\varphi|_{\widetilde{G}}\right)} \subset \mathfrak{A}_{(\varphi,G)}$ and $\overline{\operatorname{Graph}\left(\varphi|_{\widetilde{G}}\right)}$ is a circular set (since $\widetilde{G}$ is dense in $\mathbb{S}^1$).

We also will be interested in the pseudo-curves as a possible invariant objects of maps from $\mathcal{S}(\Omega)$. The next lemma studies their properties in this case.

Lemma 2.5. *Let $F \in \mathcal{S}(\Omega)$ and assume that $\mathfrak{A}_{(\varphi,G)} \in \boldsymbol{\mathcal{A}}$ is an F-invariant pseudo-curve. Then,*

(a) $\mathfrak{A}_{(\varphi,G)}$ is strongly F-invariant and minimal.
(b) If $\mathfrak{A}_{(\varphi,G)}$ contains an arc of a curve then it is a curve.

Proof. We start by proving (a). Let $B \subset \mathfrak{A}_{(\varphi,G)}$ be a closed invariant set. We have that B is circular and, by Lemma 2.4(b), $B = \mathfrak{A}_{(\varphi,G)}$. Hence, $\mathfrak{A}_{(\varphi,G)}$ is minimal.

On the other hand, $F(\mathfrak{A}_{(\varphi,G)}) \subset \mathfrak{A}_{(\varphi,G)}$ implies $F^2(\mathfrak{A}_{(\varphi,G)}) \subset F(\mathfrak{A}_{(\varphi,G)})$ and, hence, $F(\mathfrak{A}_{(\varphi,G)})$ is a compact F-invariant set. Therefore, by the part already proven, $F(\mathfrak{A}_{(\varphi,G)}) = \mathfrak{A}_{(\varphi,G)}$.

Now we prove (b). Let S be an (open) arc of $\mathbb{S}^1$ and let $\xi\colon S \longrightarrow \mathbb{I}$ be a continuous map such that $\operatorname{Graph}(\xi) \subset \mathfrak{A}_{(\varphi,G)}$. Clearly, there exists $m \in \mathbb{N}$ such that $\bigcup_{i=0}^{m} R_\omega^i(S) = \mathbb{S}^1$. Now we set $\xi_0 := \xi$ and, for $i = 1, 2, \dots, m$, we define $\xi_i\colon R_\omega^i(S) \longrightarrow \mathbb{I}$ by

$$\xi_i(\theta) := f\left(R_\omega^{-1}(\theta), \xi_{i-1}\left(R_\omega^{-1}(\theta)\right)\right).$$

The continuity of f implies that every ξ_i is an arc of a curve and $\operatorname{Graph}(\xi_i) = F(\operatorname{Graph}(\xi_{i-1}))$. Hence,

$$\bigcup_{i=0}^{m} \operatorname{Graph}(\xi_i) = \bigcup_{i=0}^{m} F^i(\operatorname{Graph}(\xi)) \subset \mathfrak{A}_{(\varphi,G)}$$

because $\mathfrak{A}_{(\varphi,G)}$ is F-invariant.

In view of Lemma 2.4(d) we only have to show that $\bigcup_{i=0}^{m} \operatorname{Graph}(\xi_i)$ is a curve. We will prove prove this by induction.

Assume that $\emptyset \neq M \subsetneq \{0, 1, 2, \dots, m\}$ verifies that $S_M := \bigcup_{i \in M} R_\omega^i(S)$ is an (open) arc of $\mathbb{S}^1$ and $\bigcup_{i \in M} \operatorname{Graph}(\xi_i)$ is an arc of a curve (initially we can take M to be any unitary subset of $\{0, 1, 2, \dots, m\}$). Then, there exists a continuous map $\xi_{_M} : S_M \longrightarrow \mathbb{I}$ such that $\operatorname{Graph}(\xi_{_M}) = \bigcup_{i \in M} \operatorname{Graph}(\xi_i)$.

Clearly, there exists $j \in \{0, 1, 2, \dots, m\} \backslash M$ such that $S_{M,j} := S_M \cap R_\omega^j(S) \neq \emptyset$. The set $S_{M,j}$ is an open arc of $\mathbb{S}^1$ and, by Lemma 2.4(a), $\xi_{_M}\big|_{S_{M,j} \cap G} = \xi_j\big|_{S_{M,j} \cap G}$ because $\operatorname{Graph}(\xi_{_M}), \operatorname{Graph}(\xi_j) \subset \mathfrak{A}_{(\varphi,G)}$. Since $S_{M,j} \cap G$ is dense in $S_{M,j}$, given $\theta \in S_{M,j} \backslash G$, there exists a sequence $\{\theta_n\}_{n=0}^{\infty} \subset S_{M,j} \cap G$ converging to θ. The continuity of $\xi_{_M}$ and ξ_j on $S_{M,j}$ implies that

$$\xi_{_M}(\theta) = \lim_{n\to\infty} \xi_{_M}(\theta_n) = \lim_{n\to\infty} \xi_j(\theta_n) = \xi_j(\theta).$$

Consequently, $\xi_{_M}\big|_{S_{M,j}} = \xi_j\big|_{S_{M,j}}$ and $\operatorname{Graph}(\xi_{_M}) \cup \operatorname{Graph}(\xi_j)$ is an arc of a curve (defined on the open arc $S_M \cup R_\omega^j(S)$). By redefining M as $M \cup \{j\}$ and iterating this procedure until $M \cup \{j\} = \{0, 1, 2, \dots, m\}$ we see that the whole $\bigcup_{i=0}^{m} \operatorname{Graph}(\xi_i)$ is a curve.

Next we will introduce and study the space of pseudo-curves.

Definition 2.6. *We define the* space of pseudo-curve generators *as*

$$\mathfrak{C} := \{(\varphi, G) : G \text{ is a residual set in } \mathbb{S}^1 \text{ and } \varphi\colon G \longrightarrow \mathbb{I} \text{ is a continuous map}\}.$$

On $\mathfrak{C}$ *we also define the* supremum pseudo-metric $\mathsf{d}_\infty : \mathfrak{C} \times \mathfrak{C} \longrightarrow \mathbb{R}^+$ *by:*

$$\mathsf{d}_\infty\big((\varphi, G), (\varphi', G')\big) := \sup_{\theta \in G \cap G'} |\varphi(\theta) - \varphi'(\theta)|.$$

Clearly, $\mathsf{d}_\infty((\varphi, G), (\varphi', G')) = 0$ *if and only if* $\varphi\big|_{G\cap G'} = \varphi'\big|_{G\cap G'}$ *and, hence,* d_∞ *is a pseudo-metric.* ◻

The next lemma will be useful in using the metric d_∞.

Lemma 2.7. *Let* $(\varphi, G), (\varphi', G') \in \mathfrak{C}$. *Then,*

$$\mathsf{d}_\infty\big((\varphi, G), (\varphi', G')\big) = \sup_{\theta \in \widetilde{G}} |\varphi(\theta) - \varphi'(\theta)|$$

for every $\widetilde{G} \subset G \cap G'$ *dense in* $\mathbb{S}^1$.

Proof. Set $\mathsf{d}_{\infty,\widetilde{G}}\big((\varphi, G), (\varphi', G')\big) := \sup_{\theta\in\widetilde{G}}|\varphi(\theta) - \varphi'(\theta)|$. With this notation, we clearly have $\mathsf{d}_{\infty,\widetilde{G}}\big((\varphi, G), (\varphi', G')\big) \le \mathsf{d}_\infty\big((\varphi, G), (\varphi', G')\big)$.

To prove the reverse inequality take $\theta \in (G \cap G')\backslash\widetilde{G}$. Since $\widetilde{G}$ is dense in $\mathbb{S}^1$, there exists a sequence $\{\theta_n\}_{n=0}^\infty \subset \widetilde{G}$ converging to θ. On the other hand, by definition, the maps φ and φ', are continuous in $G \cap G'$ (and, hence, in θ). Consequently, $|\varphi(\theta), \varphi'(\theta)| = \lim_{n\to\infty}|\varphi(\theta_n) - \varphi'(\theta_n)| \le \mathsf{d}_{\infty,\widetilde{G}}\big((\varphi, G), (\varphi', G')\big)$. This ends the proof of the lemma.

As it is customary we will introduce an equivalent relation in the space of pseudo-curve generators so that the quotient space will be a metric space.

Definition 2.8. *Two pseudo-curve generators* $(\varphi, G), (\varphi', G') \in \mathfrak{C}$ *are said to be equivalent, denoted by* $(\varphi, G) \sim (\varphi', G')$ *if and only if* $\mathfrak{A}_{(\varphi,G)} = \mathfrak{A}_{(\varphi',G')}$. *Clearly* $\sim$ *is an equivalence relation in* $\mathfrak{C}$. *The* $\sim$*-equivalence class of* $(\varphi, G) \in \mathfrak{C}$ *will be denoted by* $[\varphi, G]$. ◻

Remark 2.9. From Lemma 2.4(a,c) it follows that $(\varphi, G) \sim (\varphi', G')$ if and only if $\varphi\big|_{\widetilde{G}} = \varphi'\big|_{\widetilde{G}}$ for every $\widetilde{G} \subset G\cap G'$ dense in $\mathbb{S}^1$. In particular, by taking $\widetilde{G} = G\cap G'$, we get that $\mathsf{d}_\infty((\varphi, G), (\varphi', G')) = 0$ if and only if $(\varphi, G) \sim (\varphi', G')$. ◻

Definition 2.10. *The space* $\mathfrak{C}/\sim$ *will be called the* space of pseudo-curves generator classes *and denoted by* $\mathcal{PC}$. *Also, on* $\mathcal{PC}$ *we define the* supremum metric, *also denoted* $\mathsf{d}_\infty : \mathcal{PC} \times \mathcal{PC} \longrightarrow \mathbb{R}^+$ *by abuse of notation, in the following way. Given* $A = [\varphi_A, G_A], B = [\varphi_B, G_B] \in \mathcal{PC}$ *we set*

$$\mathsf{d}_\infty(A, B) := \mathsf{d}_\infty\big((\varphi_A, G_A), (\varphi_B, G_B)\big).$$

Note that d_∞ *is well defined. To see this take* $[\varphi_A, G_A] = [\varphi_{'A}, G_{A'}], [\varphi_B, G_B] \in \mathfrak{C}$. *Then, by Lemma 2.7 and Remark 2.9 applied to* $\widetilde{G} = G_A \cap G_{A'} \cap G_B$ *we get* $\mathsf{d}_\infty\big((\varphi_A, G_A), (\varphi_B, G_B)\big) = \mathsf{d}_\infty\big((\varphi_{A'}, G_{A'}), (\varphi_B, G_B)\big)$. ◻

The next result establishes the basic properties of the space of pseudo-curves generator classes $(\mathcal{PC}, \mathsf{d}_\infty)$.

Proposition 2.11. *The space of pseudo-curves generator classes $\mathcal{PC}$ is a complete metric space.*

Proof. The fact that d_∞ is a metric in $\mathcal{PC}$ follows from Remark 2.9.

Now we prove that $\mathcal{PC}$ is complete. Assume that $\{[\varphi_n, G_n]\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{PC}$. We have to see that $\lim_{n\to\infty}[\varphi_n, G_n] \in \mathcal{PC}$.

Set, $G := \cap_{i=1}^\infty G_n$. Since this intersection is countable, G is still a residual set. The definition of d_∞ implies that the sequence $\{\varphi_n(\theta)\}_{n=1}^\infty \subset \mathbb{I}$ is a Cauchy sequence in $\mathbb{I}$ for every $\theta \in G$. So, it is convergent and we can define a map $\varphi\colon G \longrightarrow \mathbb{I}$ by $\varphi(\theta) := \lim_{n\to\infty} \varphi_n(\theta)$.

If $(\varphi, G) \in \mathfrak{C}$ we have $[\varphi, G] \in \mathcal{PC}$ and, from the definition of φ it follows that

$$\lim_{n\to\infty} \mathsf{d}_\infty([\varphi, G], [\varphi_n, G_n]) = \sup_{\theta\in G\cap G_n} \lim_{n\to\infty} |\varphi(\theta) - \varphi_n(\theta)| = 0.$$

Consequently, $[\varphi, G] = \lim_{n\to\infty}[\varphi_n, G_n]$. Since φ is the uniform limit of a sequence of continuous functions on G, it is continuous on G. That is, $(\varphi, G) \in \mathfrak{C}$.

In what follows we want to look at the space $\boldsymbol{\mathcal{A}}$ as a metric space and relate this metric space with $(\mathcal{PC}, \mathsf{d}_\infty)$.

Let ρ denote the euclidean metric in Ω. Then, the space (Ω, ρ) is a compact metric space. We recall that the *Hausdorff metric* is defined in the space of compact subsets of (Ω, ρ), by

$$H_\rho(\mathfrak{A}, \mathsf{B}) = \max\left\{\max_{(\theta,x)\in\mathfrak{A}} \rho((\theta,x), \mathsf{B}), \max_{(\theta,x)\in\mathsf{B}} \rho((\theta,x), \mathfrak{A})\right\}.$$

Then, $(\boldsymbol{\mathcal{A}}, H_\rho)$ is a metric space. To study the relation between $(\mathcal{PC}, \mathsf{d}_\infty)$ and $(\boldsymbol{\mathcal{A}}, H_\rho)$ we need a couple of simple technical results.

Lemma 2.12. *Let $\mathfrak{A}, \mathsf{B} \subset \Omega$ be compact circular sets. Then,*

$$H_\rho(\mathfrak{A}, \mathsf{B}) \le \max_{\theta\in\mathbb{S}^1} H_\rho\big(\mathfrak{A}^\theta, \mathsf{B}^\theta\big).$$

Proof. It follows directly from the definitions:

$$
\begin{aligned}
H_\rho(\mathfrak{A}, \mathsf{B}) &\le \max\left\{ \sup_{(\theta,x)\in\mathfrak{A}} \rho\Big((\theta,x), \mathsf{B}^\theta\Big), \sup_{(\theta,x)\in\mathsf{B}} \rho\Big((\theta,x), \mathfrak{A}^\theta\Big) \right\} \\
&= \max\left\{ \sup_{\theta\in\mathbb{S}^1} \max_{\{x\in\mathbb{I}\,:\,(\theta,x)\in\mathfrak{A}\}} \rho\Big((\theta,x), \mathsf{B}^\theta\Big), \right. \\
&\qquad\qquad \left. \sup_{\theta\in\mathbb{S}^1} \max_{\{x\in\mathbb{I}\,:\,(\theta,x)\in\mathsf{B}\}} \rho\Big((\theta,x), \mathfrak{A}^\theta\Big) \right\} \\
&= \sup_{\theta\in\mathbb{S}^1} \max\left\{ \max_{\{x\in\mathbb{I}\,:\,(\theta,x)\in\mathfrak{A}\}} \rho\Big((\theta,x), \mathsf{B}^\theta\Big), \max_{\{x\in\mathbb{I}\,:\,(\theta,x)\in\mathsf{B}\}} \rho\Big((\theta,x), \mathfrak{A}^\theta\Big) \right\} \\
&= \sup_{\theta\in\mathbb{S}^1} H_\rho\left(\mathfrak{A}^\theta, \mathsf{B}^\theta\right).
\end{aligned}
$$

Proposition 2.13. *Let* $(\varphi, G), (\widetilde{\varphi}, \widetilde{G}) \in \mathfrak{C}$. *Then,*

$$
H_\rho\left(\mathfrak{A}_{(\varphi,G)}, \mathfrak{A}_{(\widetilde{\varphi},\widetilde{G})}\right) \le \sup_{\theta\in\mathbb{S}^1} H_\rho\left(\mathfrak{A}^\theta_{(\varphi,G)}, \mathfrak{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}\right) = \mathsf{d}_\infty\big((\varphi,G),(\widetilde{\varphi},\widetilde{G})\big).
$$

Proof. The first inequality follows from Lemma 2.12.

Now we prove the second equality. By Lemma 2.4(a),

$$
\mathsf{d}_\infty\big((\varphi,G),(\widetilde{\varphi},\widetilde{G})\big) = \sup_{\theta\in G\cap\widetilde{G}} |\varphi(\theta) - \widetilde{\varphi}(\theta)| = \sup_{\theta\in G\cap\widetilde{G}} H_\rho\left(\mathfrak{A}^\theta_{(\varphi,G)}, \mathfrak{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}\right).
$$

So, to end the proof of the lemma, we have to see that

$$
H_\rho\left(\mathfrak{A}^\theta_{(\varphi,G)}, \mathfrak{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}\right) \le \mathsf{d}_\infty\big((\varphi,G),(\widetilde{\varphi},\widetilde{G})\big) \quad \text{for every} \quad \theta \in \mathbb{S}^1 \backslash (G\cap\widetilde{G}).
$$

Fix $\theta \in \mathbb{S}^1\backslash(G\cap\widetilde{G})$. From the definition of the Hausdorff metric it follows that there exist $x, y \in \mathbb{I}$ such that $H_\rho\left(\mathfrak{A}^\theta_{(\varphi,G)}, \mathfrak{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}\right) = |x-y|$, $(\theta, x) \in \mathfrak{A}^\theta_{(\varphi,G)}$, and $(\theta, y) \in \mathfrak{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}$.

Since $G\cap\widetilde{G}$ is residual (and thus dense) in $\mathbb{S}^1$, from Lemma 2.4(a,c) it follows that there exists sequences $\{(\theta_n, \varphi(\theta_n))\}_{n=0}^\infty$, $\{(\theta_n, \widetilde{\varphi}(\theta_n))\}_{n=0}^\infty \subset \upuparrows(G\cap\widetilde{G})$ such that $\lim_{n\to\infty}(\theta_n, \varphi(\theta_n)) = (\theta, x)$ and $\lim_{n\to\infty}(\theta_n, \widetilde{\varphi}(\theta_n)) = (\theta, y)$. Hence,

$$
H_\rho\left(\mathfrak{A}^\theta_{(\varphi,G)}, \mathfrak{A}^\theta_{(\widetilde{\varphi},\widetilde{G})}\right) = |x-y| = \lim_{n\to\infty} |\varphi(\theta_n) - \widetilde{\varphi}(\theta_n)| \le \mathsf{d}_\infty\big((\varphi,G),(\widetilde{\varphi},\widetilde{G})\big).
$$

Proposition 2.13 tells us that that if $\{[\varphi_n, G_n]\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{PC}$ then $\mathfrak{A}_{(\varphi_n,G_n)}$ is a Cauchy sequence in $(\boldsymbol{\mathcal{A}}, H_\rho)$, and if $[\varphi, G] = \lim_{n\to\infty}[\varphi_n, G_n]$ then $\mathfrak{A}_{(\varphi,G)} = \lim_{n\to\infty} \mathfrak{A}_{(\varphi_n,G_n)}$. Unfortunately the space $(\boldsymbol{\mathcal{A}}, H_\rho)$ is not complete as the following simple example shows.

Example 2.14 (The space $(\boldsymbol{\mathcal{A}}, H_\rho)$ *is not complete).* Consider continuous maps $\xi_n \colon \mathbb{S}^1 \longrightarrow \mathbb{I}$ with $n \in \mathbb{N},\ n \ge 2$, defined by

$$
\xi_n(\theta) = \begin{cases} 2n\theta & \text{if } \theta \in [0, \frac{1}{2n}], \\ 2(1-n\theta) & \text{if } \theta \in [\frac{1}{2n}, \frac{1}{n}], \\ 0 & \text{if } \theta \ge \frac{1}{n}. \end{cases}
$$

Clearly, $(\xi_n, \mathbb{S}^1) \in \mathfrak{C}$ and $H_\rho(\mathfrak{A}_{(\xi_n,\mathbb{S}^1)}, \mathfrak{A}_{(\xi_m,\mathbb{S}^1)}) \le \frac{1}{\min\{n,m\}}$. Hence, $\{\mathfrak{A}_{(\xi_n,\mathbb{S}^1)}\}$ is a Cauchy sequence in $\boldsymbol{\mathcal{A}}$. However, the sequence $\{\mathfrak{A}_{(\xi_n,\mathbb{S}^1)}\}$ has no limit in $\boldsymbol{\mathcal{A}}$. Indeed, $\lim_{n\to\infty} \mathfrak{A}_{(\xi_n,\mathbb{S}^1)} = \mathsf{L} = (\mathbb{S}^1 \times \{0\}) \cup (\{0\} \times [0,1])$, which is not the closure of the graph of a continuous map on a residual set of $\mathbb{S}^1$ (in other words, $\mathsf{L} \notin \boldsymbol{\mathcal{A}}$). This is consistent with the fact that, clearly, $\{[\xi_n, \mathbb{S}^1]\}$ is not a Cauchy sequence in $(\mathcal{PC}, \mathsf{d}_\infty)$. ◘

2.3 Construction of a connected pseudo-curve

The aim of this subsection is to construct a strip $\mathfrak{A} = \mathfrak{A}_{(\gamma,G)}$ as a connected pseudo-curve with certain topological properties that will allow us to define the map $T \in \mathcal{S}(\Omega)$ having this pseudo-curve as the only proper invariant object. The pseudo-curve $\mathfrak{A}_{(\gamma,G)}$ will be obtained as a limit in $\mathcal{PC}$ of a sequence of pseudo-curves that will be constructed recursively.

We will start by introducing the necessary notation.

In what follows, for simplicity, we will take the interval $\mathbb{I}$ as the interval $[-2, 2]$. Also, fix $\omega \in [0,1]\backslash\mathbb{Q}$. For any $\ell \in \mathbb{Z}$ set $\ell^* = \ell\omega \pmod 1$ and $O^*(\omega) = \{\ell^* : \ell \in \mathbb{Z}\}$. That is, $O^*(\omega)$ is the orbit of 0 by the rotation of angle ω.

We will denote by $\mathsf{d}_{\mathbb{S}^1}$ the arc distance on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. That is, for $\theta_1, \theta_2 \in \mathbb{S}^1$, we set

$$\mathsf{d}_{\mathbb{S}^1}(\theta_1, \theta_2) := \begin{cases} \theta_2 - \theta_1 & \text{when } \theta_1 \le \theta_2, \text{ and} \\ (\theta_2 + 1) - \theta_1 & \text{when } \theta_1 > \theta_2. \end{cases}$$

The closed arc of $\mathbb{S}^1$ joining θ_1 and θ_2 in the natural direction will be denoted by $[\theta_1, \theta_2]$. That is,

$$[\theta_1, \theta_2] = \begin{cases} \{t \pmod 1 : \theta_1 \le t \le \theta_2\} & \text{when } \theta_1 \le \theta_2, \text{ and} \\ \{t \pmod 1 : \theta_1 \le t \le \theta_2 + 1\} & \text{when } \theta_1 > \theta_2. \end{cases}$$

The open arc of $\mathbb{S}^1$ joining θ_1 and θ_2 will be denoted by $(\theta_1, \theta_2) = [\theta_1, \theta_2]\backslash\{\theta_1, \theta_2\}$, and is defined analogously with strict inequalities Given an arc $B \subset \mathbb{S}^1$, $\operatorname{Bd}(B)$ will denote the set of endpoints of B.

We will denote the open (respectively closed) ball (in $\mathbb{S}^1$) of radius δ centred at $\theta \in \mathbb{S}^1$ by $B_\delta(\theta)$ (respectively $B_\delta[\theta]$):

$$B_\delta(\theta) = \{\widetilde{\theta} \in \mathbb{S}^1 : \mathsf{d}_{\mathbb{S}^1}(\theta, \widetilde{\theta}) < \delta\} = (\theta - \delta \pmod 1, \theta + \delta \pmod 1), \text{ and}$$
$$B_\delta[\theta] = \overline{B_\delta(\theta)} = \{\widetilde{\theta} \in \mathbb{S}^1 : \mathsf{d}_{\mathbb{S}^1}(\theta, \widetilde{\theta}) \le \delta\} = [\theta - \delta \pmod 1, \theta + \delta \pmod 1].$$

We consider the space Ω endowed the metric induced by the maximum of $\mathsf{d}_{\mathbb{S}^1}$ and the absolute value on $\mathbb{I}$. That is, given $(\theta, x), (\nu, y) \in \Omega$ we set

$$\mathsf{d}_\Omega((\theta, x), (\nu, y)) := \max\left\{\mathsf{d}_{\mathbb{S}^1}(\theta, \nu), |x - y|\right\}.$$

Then, given $A \subset \Omega$ we will denote the *interior of* A by $\operatorname{Int}(A)$ and $\operatorname{diam}(A)$ will denote the *diameter of* A whenever A is compact.

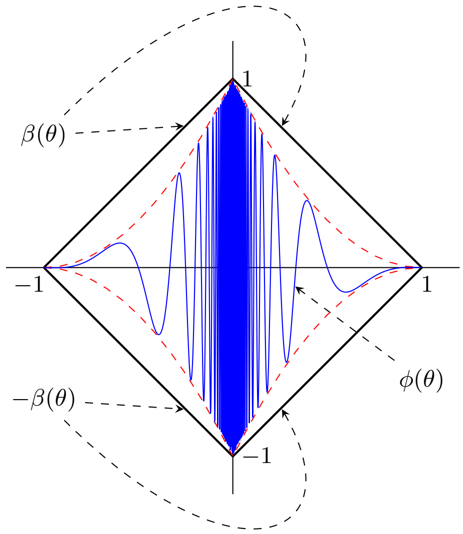


Figure 2.2: The graphs of the functions ϕ (in blue) and $\pm\beta$ in thick black. The red dashed curve is $(1-|x|)^2$.

To define the sequence of pseudo-curves that will converge to $\mathfrak{A}_{(\gamma,G)}$ we first need to construct an auxiliary family $\{\mathcal{R}(\ell^*)\}_{\ell\in\mathbb{Z}}$ of compact regions in Ω and a family of compact sets $\{\Gamma\varphi_{\ell^*}\}_{\ell\in\mathbb{Z}}$ such that, for every $\ell \in \mathbb{Z}$, $\Gamma\varphi_{\ell^*} \subset \mathcal{R}(\ell^*)$ and it is the restriction of a pseudo-curve generator to $\pi(\mathcal{R}(\ell^*))$. To do this we define the auxiliary functions $\beta\colon [-1,1] \longrightarrow [-1,1]$ and $\phi\colon [-1,1]\backslash\{0\} \longrightarrow [-1,1]$ by (see Figure 2.2):

$$\beta(x) := 1-|x| \qquad \text{and} \qquad \phi(x) := (1-|x|)^2 \sin\left(\frac{\pi}{x}\right).$$

Note that $-\beta(x) < \phi(x) < \beta(x)$, for all $x \in [-1,1]\backslash\{0\}$ and the graphs of $-\beta$ and β intersect the closure of the graph of ϕ only at the points $(0,-1)$, $(0,1)$, $(-1,0)$ and $(1,0)$.

To define the families $\{\mathcal{R}(\ell^*)\}_{\ell\in\mathbb{Z}}$ and $\{\Gamma\varphi_{\ell^*}\}_{\ell\in\mathbb{Z}}$ we use the following *generic boxes*.

For every $\theta \in \mathbb{S}^1$ and $\delta < \frac{1}{2}$, $\vartheta_\theta\colon [-\delta,\delta] \longrightarrow \mathbb{S}^1$ denotes the map defined by $\vartheta_\theta(x) = x+\theta \pmod 1$. Clearly ϑ_θ is a homeomorphism between $[-\delta,\delta]$ and $B_\delta[\theta]$. Finally $\vartheta_\theta^{-1}\colon B_\delta[\theta] \longrightarrow [-\delta,\delta]$ denotes the inverse homeomorphism of ϑ_θ.

Definition 2.15 (Generic boxes). *Fix $\ell, n \in \mathbb{Z}$, $n \geq |\ell|$, $\alpha \in (0, 2^{-n})$, $\delta \in (0,\alpha)$, $a \in [-1,1]$ and $a^+, a^- \in B_a(2^{-n}\beta(\delta))$ (see Figure 2.3). Now we consider the Jordan closed curve in Ω, formed by the graphs of the functions*

$$a + 2^{-n}(\beta \circ \vartheta_{\ell^*}^{-1})\big|_{B_\delta[\ell^*]} \quad \text{and} \quad a - 2^{-n}(\beta \circ \vartheta_{\ell^*}^{-1})\big|_{B_\delta[\ell^*]},$$

together with the four segments that join the points:

$$(\ell^* - \alpha, a^-) \text{ with } \big(\ell^* - \delta, a - 2^{-n}\beta(-\delta)\big),$$
$$(\ell^* - \alpha, a^-) \text{ with } \big(\ell^* - \delta, a + 2^{-n}\beta(-\delta)\big),$$
$$(\ell^* + \alpha, a^+) \text{ with } \big(\ell^* + \delta, a - 2^{-n}\beta(\delta)\big), \text{ and}$$
$$(\ell^* + \alpha, a^+) \text{ with } \big(\ell^* + \delta, a + 2^{-n}\beta(\delta)\big).$$

We denote the closure of the connected component of the complement of the above Jordan curve in Ω that contains the point (ℓ^*, a) by $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$ (the coloured region in Figure 2.3). Observe that $\pi\left(\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)\right)$, the projection of $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$ to $\mathbb{S}^1$, is $B_\alpha[\ell^*] = [\ell^* - \alpha, \ell^* + \alpha]$.

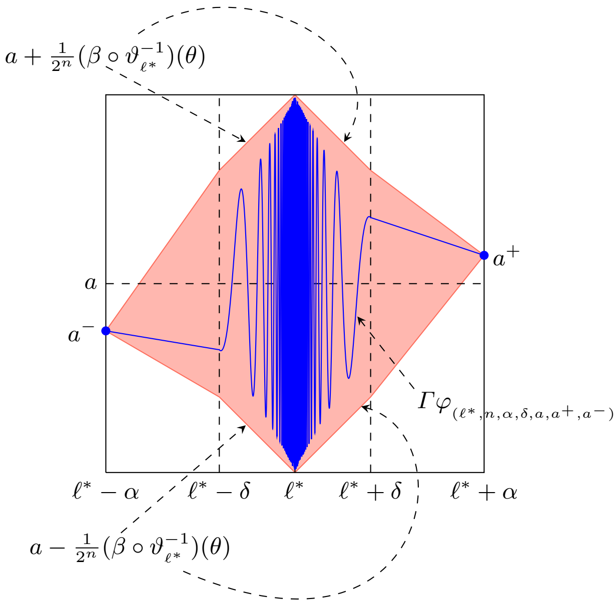


Figure 2.3: The region $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$ is the colored filled area, delimited in the rectangle $\upuparrows B_\delta[\ell^*]$ by the graphs of the functions $a \pm \frac{1}{2^n}(\beta \circ \vartheta_{\ell^*}^{-1})(\theta)$. In blue the set $\Gamma\varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)}$ inductively defining the pseudo-curve.

We denote by

$$\varphi_{\ell^*} = \varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)} \colon B_\alpha[\ell^*] \setminus \{\ell^*\} \longrightarrow \mathbb{I}$$

the continuous map defined as follows:

(i) $\varphi_{\ell^*}\big|_{B_\delta[\ell^*] \setminus \{\ell^*\}} = a + (-1)^\ell 2^{-n}(\phi \circ \vartheta_{\ell^*}^{-1})$.
(ii) $\varphi_{\ell^*}(\ell^* - \alpha) = a^-$ *and* $\varphi_{\ell^*}(\ell^* + \alpha) = a^+$.
(iii) $\varphi_{\ell^*}\big|_{[\ell^* - \alpha, \ell^* - \delta]}$ *and* $\varphi_{\ell^*}\big|_{[\ell^* + \delta, \ell^* + \alpha]}$ *are affine.*

We also denote by $\Gamma\varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)} \subset \mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$ the closure in Ω of the graph of $\varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)}$. ◘

Remark 2.16. The region $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$ and the set $\Gamma\varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)}$ satisfy the following properties:

(1) $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-) \subset B_\alpha[\ell^*] \times [a - 2^{-n}, a + 2^{-n}]$.
(2) $\operatorname{diam}(\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)) = \operatorname{diam}(\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)^{\ell^*}) = 2 \cdot 2^{-n}$.
(3) The sets $\Gamma\varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)}$ and $\partial\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$ only intersect at the points $(\ell^*, a - 2^{-n})$, $(\ell^*, a + 2^{-n})$, $(\ell^* - \alpha, a^-)$ and $(\ell^* + \alpha, a^+)$.
(4) $\left(\Gamma\varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)}\right)^{\ell^*} = \mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)^{\ell^*}$ is an interval.
(5) Let $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$ and $\mathcal{R}(k^*, \widetilde{n}, \widetilde{\alpha}, \widetilde{\delta}, \widetilde{a}, \widetilde{a}^+, \widetilde{a}^-)$ be two regions, then $B_\alpha[\ell^*] \cap B_{\widetilde{\alpha}}[k^*] = \emptyset$ implies

$$\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-) \cap \mathcal{R}(k^*, \widetilde{n}, \widetilde{\alpha}, \widetilde{\delta}, \widetilde{a}, \widetilde{a}^+, \widetilde{a}^-) = \emptyset.$$

◘

For every $j \in \mathbb{Z}^+$, we set

$$Z_j := \{i \in \mathbb{Z} : |i| \leq j\} = \{-j, -j+1, \ldots, -1, 0, 1, \ldots, j-1, j\} \text{ and}$$
$$Z_j^* := \{i^* : i \in Z_j\}.$$

With the help of the sets $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$ and $\Gamma\varphi_{(\ell^*,n,\alpha,\delta,a,a^+,a^-)}$, which are the "bricks" of our construction we are ready to define the sequence of pseudo-curve generators $\{(\gamma_j, \mathbb{S}^1 \backslash Z_j^*)\}_{j=0}^\infty$ that we are looking for.

To do this, for every $j \geq 0$ we define

- a strictly increasing sequence $\{n_j\}_{j=0}^\infty \subset \mathbb{N}$,
- a strictly decreasing sequence $\{\alpha_j\}_{j=0}^\infty$ such that $2^{-n_{j+1}} < \alpha_j < 2^{-n_j}$
- and a sequence $\{\delta_j\}_{j=0}^\infty$ with $2^{-n_{j+1}} < \delta_j < \alpha_j$

verifying some technical properties that we will make explicit below, and we define a sequence of boxes $\mathcal{R}(j^*) := \mathcal{R}(j^*, n_j, \alpha_j, \delta_j, a_j, a_j^+, a_j^-)$ and $\mathcal{R}((-j)^*) := \mathcal{R}((-j)^*, n_j, \alpha_j, \delta_j, a_{-j}, a_{-j}^+, a_{-j}^-)$ (for $j = 0$ both sets coincide) with projections

$$\pi\left(\mathcal{R}(j^*)\right) = B_{\alpha_j}\left[j^*\right] \quad \text{and} \quad \pi\left(\mathcal{R}((-j)^*)\right) = B_{\alpha_j}\left[(-j)^*\right].$$

Finally, with the use of all these sequences and objects we can define our functions $\gamma_j\big|_{\mathbb{S}^1 \backslash Z_j^*}$.

Observe that we are using the intervals of the form $B_{\alpha_{|\ell|}}[\ell^*]$, $B_{\delta_{|\ell|}}[\ell^*]$ and also $B_{\alpha_{|\ell|-1}}[\ell^*]$ when ℓ is negative. To ease the use of these intervals we introduce the following notation:

$$B_\ell^{\frown}[\ell^*] := \begin{cases} B_{\alpha_\ell}[\ell^*] & \text{if } \ell \geq 0, \text{ or} \\ B_{\alpha_{|\ell+1|}}[\ell^*] & \text{if } \ell < 0, \end{cases} \quad \text{and} \quad B_\ell^{\frown}(\ell^*) := \begin{cases} B_{\alpha_\ell}(\ell^*) & \text{if } \ell \geq 0, \text{ or} \\ B_{\alpha_{|\ell+1|}}(\ell^*) & \text{if } \ell < 0. \end{cases}$$

Notice that the ball $B_\ell^{\frown}[\ell^*]$ has diameter α_j for $\ell \in \{j, -(j+1)\}$.

Remark 2.17. With the above notation $B_{\alpha_{|\ell|}}[\ell^*] \subsetneq B_\ell^{\frown}(\ell^*)$ for every $\ell < 0$. Moreover, for $\ell \in \mathbb{Z}$ and $j \in \mathbb{Z}^+$,

$$R_\omega\left(B_{\alpha_j}[\ell^*]\right) = B_{\alpha_j}[(\ell+1)^*], \text{ and}$$
$$R_\omega\left(B_\ell^{\frown}[\ell^*]\right) = \begin{cases} B_{\alpha_\ell}[(\ell+1)^*] & \text{if } \ell \geq 0\text{, or} \\ B_{\alpha_{|\ell+1|}}[(\ell+1)^*] & \text{if } \ell < 0. \end{cases}$$

Also, the same formulae holds with α replaced by δ and for open balls. ◘

The next crucial definition fixes in detail all quantities and objects mentioned above.

Definition 2.18. *We start by defining* $\mathcal{R}(0^*) := \mathcal{R}(0^*, n_0, \alpha_0, \delta_0, 0, 0, 0)$ *and* $\varphi_{0^*} := \varphi_{(0^*, n_0, \alpha_0, \delta_0, 0, 0, 0)}$ *by choosing (Definition 2.15)* $n_0 = 1$, $\alpha_0 < \frac{1}{2} = 2^{-n_0}$ *and* $\delta_0 < \alpha_0$ *small enough so that the intervals* $B_0^{\frown}[0^*] = B_{\alpha_0}[0^*]$, $B_{\alpha_0}[1^*]$ *and* $B_{-1}^{\frown}[(-1)^*] = B_{\alpha_0}[(-1)^*]$ *are pairwise disjoint; and* $(-2)^*, 2^* \notin B_{-1}^{\frown}[(-1)^*]$ *and, additionally,* $\operatorname{Bd}(B_{\alpha_0}[0^*]) \cap O^*(\omega) = \emptyset$.

We also set $a_0^+ = a_0^- = a_0 = 0$, *and we define the map* $\gamma_0 : \mathbb{S}^1 \backslash \{0\} \longrightarrow \mathbb{I}$ *by*

$$\gamma_0(\theta) = \begin{cases} \varphi_{0^*}(\theta) & \textit{if } \theta \in B_{\alpha_0}[0^*] \backslash \{0\}, \\ 0 & \textit{if } \theta \notin B_{\alpha_0}[0^*]. \end{cases}$$

For consistency with the definition of γ_j *in the case* $j \geq 1$, *we define the map* $\gamma_{-1} : \mathbb{S}^1 \backslash \{0\} \longrightarrow \mathbb{I}$ *by* $\gamma_{-1}(\theta) = 0$ *for every* $\theta \in \mathbb{S}^1$. *Then, notice that,* $a_0 = \gamma_{-1}(0^*)$, $a_0^\pm = \varphi_{0^*}(0^* \pm \alpha_0) = \gamma_{-1}(0^* \pm \alpha_0)$, *and* $\gamma_0(\theta) = \gamma_{-1}(\theta)$ *for every* $\theta \notin B_{\alpha_0}[0^*]$.

Next, for every $j \in \mathbb{N}$ *we define* $\mathcal{R}(j^*)$, $\mathcal{R}((-j)^*)$ *and* $(\gamma_j, \mathbb{S}^1 \backslash Z_j^*)$ *from the corresponding boxes* $\mathcal{R}(i^*)$ *and* $B_{\alpha_{|i|}}[i^*] \subset B_i^{\frown}[i^*]$ *for* $i \in Z_{j-1}$, *and* $(\gamma_{j-1}, \mathbb{S}^1 \backslash Z_{j-1}^*)$ *as follows. We take* n_j, δ_j *and* α_j *such that (see Figure 2.4 to fix ideas):*

(R.1) $n_j > n_{j-1}$, $\delta_j < \alpha_j < 2^{-n_j} < \delta_{j-1} < \alpha_{j-1}$ *and*

$$\Big(\operatorname{Bd}\left(B_{\alpha_j}[(-j)^*]\right) \cup \operatorname{Bd}\left(B_{\alpha_j}[j^*]\right)\Big) \cap O^*(\omega) = \emptyset.$$

(R.2) The intervals

$$\begin{aligned} &B_j^{\frown}[j^*] = B_{\alpha_j}[j^*], \\ &R_\omega\left(B_{\alpha_j}[j^*]\right) = B_{\alpha_j}[(j+1)^*], \\ &B_{-j}^{\frown}[(-j)^*] = B_{\alpha_{j-1}}[(-j)^*] \textit{ and} \\ &B_{-(j+1)}^{\frown}[(-(j+1))^*] = B_{\alpha_j}[(-(j+1))^*] \end{aligned}$$

are pairwise disjoint,

$$\gamma_{j-1}\left(B_{\alpha_j}[\ell^*]\right) \subset \left[\gamma_{j-1}(\ell^*) - 2^{-n_j}, \gamma_{j-1}(\ell^*) + 2^{-n_j}\right]$$

for every $\ell \in \{j+1, -(j+1)\}$,

$$B^{\frown}_{\ell}[\ell^*] \cap Z^*_{j+1} = \{\ell^*\} \text{ for } \ell \in \{j, -(j+1)\} \text{ and}$$
$$B_{\alpha_j}[(j+1)^*] \cap Z^*_{j+1} = \{(j+1)^*\},$$

and $(-(j+2))^*, (j+2)^* \notin B^{\frown}_{-(j+1)}[(-(j+1))^*] = B_{\alpha_j}[(-(j+1))^*]$.

(R.3) $\operatorname{Bd}\left(B_{\alpha_{|k|}}[(k+1)^*]\right) \cap \left(B_{\alpha_j}[j^*] \cup B_{\alpha_j}[(-j)^*]\right) = \emptyset$ *for every* $k \in Z_{j-1}$.

(R.4) Assume that there exists $k \in Z_{j-1}$ *such that* $B_{\alpha_j}[(j+1)^*] \cap B^{\frown}_k[k^*] \neq \emptyset$ *and* $|k|$ *is maximal verifying these conditions. Then,* $B_{\alpha_j}[(j+1)^*]$ *is contained in one of the two connected components of* $B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ *when* $B_{\alpha_j}[(j+1)^*] \cap B_{\alpha_{|k|}}[k^*] \neq \emptyset$*, and* $B_{\alpha_j}[(j+1)^*]$ *is contained in one of the two connected components of* $B^{\frown}_k(k^*) \setminus B_{\alpha_{|k|}}[k^*]$ *if* $B_{\alpha_j}[(j+1)^*] \cap B_{\alpha_{|k|}}[k^*] = \emptyset$ *(note that, in this case,* k *must be negative).*

(R.5) Let $\ell \in \{j, -(j+1)\}$ *(recall that the ball* $B^{\frown}_{\ell}[\ell^*]$ *has diameter* α_j *for these two values of* ℓ *and only for them).*

(R.5.i) If $\ell^* \notin \bigcup_{i \in Z_{j-1}} B^{\frown}_i[i^*]$ *then,* $B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_i[i^*] = \emptyset$ *for every* $i \in Z_{j-1}$.

(R.5.ii) If $\ell^* \in B^{\frown}_m[m^*]$ *for some* $m \in Z_{j-1}$ *such that* $|m|$ *is maximal with these properties, then*

(R.5.ii.1) $B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_i[i^*] = \emptyset$ *for every* $i \in Z_{j-1}$ *such that* $|i| \geq |m|,\ i \neq m$*, and*

(R.5.ii.2) $B^{\frown}_{\ell}[\ell^*]$ *is contained in (a connected component of)*

$$\begin{aligned} B^{\frown}_m(m^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right) = \\ \left(m^* - \alpha_{|m|-1}, m^* - \alpha_{|m|}\right) \cup \left(m^* - \alpha_{|m|}, m^*\right) \cup \\ \left(m^*, m^* + \alpha_{|m|}\right) \cup \left(m^* + \alpha_{|m|}, m^* + \alpha_{|m|-1}\right) \end{aligned}$$

(observe that $B^{\frown}_{\ell}[\ell^*] \subset B^{\frown}_m(m^*) \setminus B_{\alpha_{|m|}}[m^*]$ *can only happen when* $m < 0$ *since* $B^{\frown}_m[m^*] = B_{\alpha_{|m|}}[m^*]$ *for* $m \geq 0$*).*

(R.6) Let $\ell \in \{j, -j\}$. *If* $B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_m[m^*] = \emptyset$ *for every* $m \in Z_j$, $m \neq \ell$ *then, to define* $\mathcal{R}(\ell^*)$ *and the map* φ_{ℓ^*}*, we set*

$$a_{\ell} = \gamma_{j-1}(\ell^*) = a^{\pm}_{\ell} = \gamma_{j-1}(\ell^* \pm \alpha_j) = 0.$$

Otherwise, there exists $m \in Z_{j-1}$ *such that* $B^{\frown}_{\ell}[\ell^*]$ *is contained in a connected component of* $B^{\frown}_m(m^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$ *and* $|m|$ *is maximal with these properties. Then, to define* $\mathcal{R}(\ell^*)$ *and the map* φ_{ℓ^*}*, we set*

(R.6.i) $a_{\ell} := \gamma_{|m|}(\ell^*)$, $a^{\pm}_{\ell} := \gamma_{|m|}(\ell^* \pm \alpha_j)$ *and* $\operatorname{Graph}\left(\gamma_{|m|}\big|_{B_{\alpha_j}[\ell^*]}\right) \subset \mathcal{R}(\ell^*)$.

(R.6.ii) Assume that there exists $k \in Z_{|m|} \subset Z_{j-1}$ *such that* $B^{\frown}_{\ell}[\ell^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$. *Then,* $\mathcal{R}(\ell^*)$ *is contained in one of the two connected components of* $\operatorname{Int}\left(\mathcal{R}(k^*) \setminus \upuparrows k^*\right)$.

Finally we define $\gamma_j : \mathbb{S}^1 \setminus Z^*_j \longrightarrow \mathbb{I}$ *by*

$$\gamma_j(\theta) = \begin{cases} \varphi_{j^*}(\theta) & \text{if } \theta \in B_{\alpha_j}[j^*] \setminus \{j^*\}, \\ \varphi_{(-j)^*}(\theta) & \text{if } \theta \in B_{\alpha_j}[(-j)^*] \setminus \{(-j)^*\}, \\ \gamma_{j-1}(\theta) & \text{if } \theta \notin \left(B_{\alpha_j}[j^*] \cup B_{\alpha_j}[(-j)^*] \cup Z^*_{j-1}\right). \end{cases}$$

(notice that $Z^*_j = Z^*_{j-1} \cup \{j^*, (-j)^*\}$*).* ◘

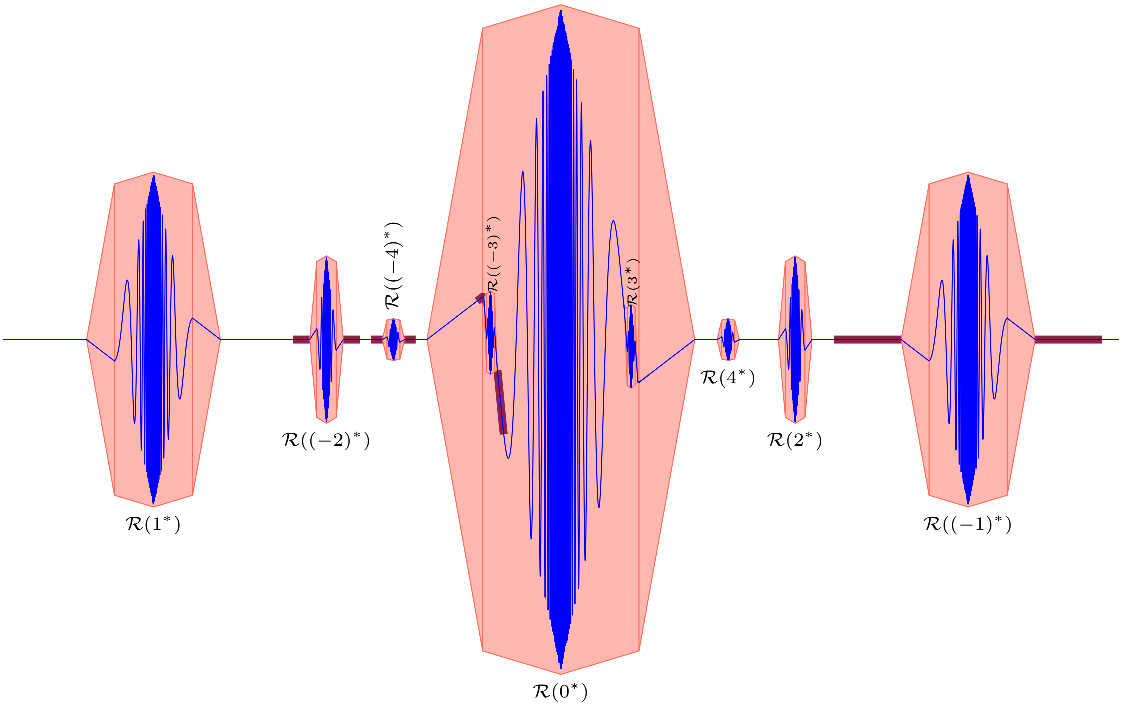


Figure 2.4: The boxes $\mathcal{R}(\ell^*)$ for $\ell \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ and the graph of γ_4. The wings are represented as a thick garnet curve surrounding the graph of γ_4. For clarity the scale and separation between boxes is not preserved. The circle $\mathbb{S}^1$ is parametrized as $[-\frac{1}{2}, \frac{1}{2})$.

For every $\ell \in \mathbb{Z}$ we define the *winged region associated to* ℓ as

$$\mathcal{R}^{\frown}(\ell^*) := \begin{cases} \mathcal{R}(\ell^*) & \text{if } \ell \geq 0, \text{ or} \\ \mathcal{R}(\ell^*) \cup \operatorname{Graph}\Big(\gamma_{|\ell|}\big|_{B^{\frown}_{\ell}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\Big) & \text{if } \ell < 0. \end{cases}$$

The next technical lemma shows that the objects from Definition 2.18 exist (that is, they are well defined), and studies some of the basic properties of the family of pseudo-curve generators $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^{\infty}$.

Remark 2.19 (Explicit consequences of Definition 2.18). The following statements are easy consequences of Definition 2.18. They are stated explicitly for easiness of usage.

(R.1) $n_j > j$. This follows from Definition 2.18(R.1) and the fact that we have set $n_0 = 1$ and $n_j > n_{j-1}$ for $j \in \mathbb{N}$.

(R.2) For every $j \in \mathbb{N}$,

$$B^{\frown}_{-j}[(-j)^*] \cap Z^*_{j+1} = \{(-j)^*\}.$$

This follows from Definition 2.18(R.2) for $j - 1$. We get

$$B^{\frown}_{-j}[(-j)^*] \cap Z_j^* = \{(-j)^*\} \quad \text{and} \quad (-(j+1))^*, (j+1)^* \notin B^{\frown}_{-j}[(-j)^*].$$

which shows the statement.

(R.6) Let $j \in \mathbb{N}$ and $\ell \in \{j, -j\}$, and assume that $B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_{m}[m^*] = \emptyset$ for every $m \in Z_j$, $m \neq \ell$. Then, $\gamma_r\big|_{B^{\frown}_{\ell}[\ell^*]} = \gamma_0\big|_{B^{\frown}_{\ell}[\ell^*]} \equiv 0$ for $r = 1, 2, \ldots, j-1$.

(R.6.i) Assume that here exists $m \in Z_{j-1}$ such that $B^{\frown}_{\ell}[\ell^*]$ is contained in a connected component of $B^{\frown}_{m}(m^*) \setminus \big(\operatorname{Bd}\big(B_{\alpha_{|m|}}[m^*]\big) \cup \{m^*\}\big)$ and $|m|$ is maximal with these properties. Then, $\gamma_r\big|_{B^{\frown}_{\ell}[\ell^*]} = \gamma_{|m|}\big|_{B^{\frown}_{\ell}[\ell^*]}$ for $r = |m|+1, |m|+2, \ldots, j-1$.

(R.6.ii) Assume that there exists $k \in Z_{|m|} \subset Z_{j-1}$ such that $B^{\frown}_{\ell}[\ell^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ and $|k|$ is maximal with these properties. Then, $\gamma_r\big|_{B^{\frown}_{\ell}[\ell^*]} = \gamma_{|k|}\big|_{B^{\frown}_{\ell}[\ell^*]}$ for $r = |k|+1, |k|+2, \ldots, |m|$.

To prove (R.6) notice that when $B^{\frown}_{\ell}[\ell^*] \cap B_{\alpha_{|m|}}[m^*] \subset B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_{m}[m^*] = \emptyset$ for every $m \in Z_j$, $m \neq \ell$, from the definition of γ_r for $0 \leq r < j$ we get that $\gamma_r\big|_{B^{\frown}_{\ell}[\ell^*]} = \gamma_0\big|_{B^{\frown}_{\ell}[\ell^*]} \equiv 0$ for $r = 1, 2, \ldots, j-1$.

(R.6.i) The maximality of $|m|$, together with Definition 2.18(R.2), imply that $B^{\frown}_{\ell}[\ell^*] \cap B_{\alpha_{|i|}}[i^*] \subset B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_{i}[i^*] = \emptyset$ for every $i \in Z_{j-1}$, $|i| \geq |m|$, $i \neq m$. So, by the definition of the functions γ_r,

$$\gamma_r\big|_{B_{\alpha_j}[\ell^*]} = \gamma_{|m|}\big|_{B_{\alpha_j}[\ell^*]} \quad \text{for} \quad r = |m|+1, |m|+2, \ldots, j-1.$$

(R.6.ii) When $|k| = |m|$ (R.6.ii) holds trivially. So, assume that $|k| < |m|$. As in the case (R.6.i), the maximality of $|k|$ and Definition 2.18(R.2) imply that $B^{\frown}_{\ell}[\ell^*] \cap B_{\alpha_{|r|}}[r^*] = \emptyset$ for every $r \in Z_{j-1}$, $|r| \geq |k|$, $r \neq k$. So, (R.6.ii) follows from the definition of the functions γ_r. ◻

Lemma 2.20. *For every $j \in \mathbb{Z}^+$ the regions $\mathcal{R}(j^*)$ and $\mathcal{R}((-j)^*)$ (and hence $\mathcal{R}^{\frown}(j^*)$ and $\mathcal{R}^{\frown}((-j)^*)$), and the maps $(\gamma_j, \mathbb{S}^1 \backslash Z_j^*)$ are well defined. Moreover, the following statements hold:*

(a) $(\gamma_j, \mathbb{S}^1 \backslash Z_j^*) \in \mathfrak{C}$. *Furthermore, for every $\ell \in \{j+1, -(j+1)\}$,*

$$\gamma_j\left(B_{\alpha_j}[\ell^*]\right) \subset \left[\gamma_j(\ell^*) - 2^{-n_j}, \gamma_j(\ell^*) + 2^{-n_j}\right].$$

(b) $\bigcup_{\ell \in \mathbb{Z}} \mathcal{R}^{\frown}(\ell^*) \subset \mathbb{S}^1 \times [-1,1]$ *and* $\operatorname{Graph}\left(\gamma_j\big|_{\mathbb{S}^1 \backslash Z_j^*}\right) \subset \mathbb{S}^1 \times [-1,1]$.

(c) For $\ell \in \{j, -j\}$ we have $\operatorname{Graph}\left(\gamma_{j-1}\big|_{B_{\alpha_j}[\ell^*]}\right) \subset \mathcal{R}(\ell^*)$, $a_\ell = \gamma_{j-1}(\ell^*)$, *and* $a_\ell^{\pm} = \varphi_{\ell^*}(\ell^* \pm \alpha_j) = \gamma_{j-1}(\ell^* \pm \alpha_j)$.

(d) $\operatorname{Graph}\left(\gamma_n\big|_{B_{\alpha_j}[\ell^*] \backslash Z_n^*}\right) \subset \mathcal{R}(\ell^*)$ *for every $n \geq j$ and $\ell \in \{j, -j\}$.*

(e) For every $\ell \in \{j, -j\}$,

$$\gamma_j\big|_{\left(B^{\frown}_{\ell}[\ell^*] \backslash B_{\alpha_j}(\ell^*)\right) \cup R_\omega\left(B^{\frown}_{\ell}[\ell^*] \backslash B_{\alpha_j}(\ell^*)\right)} = \gamma_{j-1}\big|_{\left(B^{\frown}_{\ell}[\ell^*] \backslash B_{\alpha_j}(\ell^*)\right) \cup R_\omega\left(B^{\frown}_{\ell}[\ell^*] \backslash B_{\alpha_j}(\ell^*)\right)}.$$

Moreover, for every $\theta \in \operatorname{Bd}(B^{\frown}_{\ell}[\ell^*] \backslash B_{\alpha_j}(\ell^*)) = \operatorname{Bd}(B_{\alpha_j}[\ell^*]) \cup \operatorname{Bd}(B^{\frown}_{\ell}[\ell^*])$, we have $\theta \notin B^{\frown}_{n}[n^*] \cup B^{\frown}_{-n}[(-n)^*]$ and $\gamma_n(\theta) = \gamma_j(\theta) = \gamma_{j-1}(\theta)$ for every $n > j$, and $R_\omega(\theta) \notin B_{\alpha_n}[n^*] \cup B_{\alpha_n}[(-n)^*]$ and $\gamma_n\left(R_\omega(\theta)\right) = \gamma_{j-1}\left(R_\omega(\theta)\right)$ for every $n \geq j$.

(f) *For every* $\ell \in \mathbb{Z}$, $\mathcal{R}^{\frown}(\ell^*)$ *is a compact connected set such that* $\pi\left(\mathcal{R}^{\frown}(\ell^*)\right) = B^{\frown}_{\ell}[\ell^*]$, $\gamma_{|\ell|}\big|_{B^{\frown}_{\ell}[\ell^*]\setminus B_{\alpha_{|\ell|}}(\ell^*)}$ *is continuous and*

$$\operatorname{diam}\left(\mathcal{R}^{\frown}(\ell^*)\right) = \begin{cases} \operatorname{diam}\left(\mathcal{R}(\ell^*)\right) = \operatorname{diam}\left(\mathcal{R}((-\ell)^*)\right) = 2 \cdot 2^{-n_\ell} \leq 2^{-\ell} & \text{if } \ell \geq 0, \\ 2 \cdot 2^{-n_{|\ell+1|}} \leq 2 \cdot 2^{-|\ell|} & \text{if } \ell < 0. \end{cases}$$

(g) *Given* $\ell, m \in \mathbb{Z}$ *such that* $|\ell| \geq |m|$, $\ell \neq m$ *and* $B^{\frown}_{\ell}[\ell^*] \cap B^{\frown}_{m}[m^*] \neq \emptyset$, *it follows that* $|\ell| > |m|$, *and either* $B^{\frown}_{\ell}[\ell^*] \subset B_{\alpha_{|m|}}(m^*)\setminus\{m^*\}$ *and the region* $\mathcal{R}^{\frown}(\ell^*)$ *is contained in one of the two connected components of* $\operatorname{Int}\left(\mathcal{R}(m^*)\setminus \upuparrows m^*\right)$, *or* $m < 0$ *and* $B^{\frown}_{\ell}[\ell^*]$ *is contained in one of the two connected components of* $B^{\frown}_{m}(m^*)\setminus B_{\alpha_{|m|}}[m^*]$.

Proof. We start by proving the first statement of the lemma and (a) by induction.

Observe that $n_0 = 1$, α_0, δ_0 and γ_0 are defined so that Definition 2.18(R.1–2) for $j = 0$ and $(\gamma_0, \mathbb{S}^1\setminus Z^*_0) \in \mathfrak{C}$ are verified except for the obvious fact that $B^{\frown}_{-j}[(-j)^*] = B^{\frown}_{j}[j^*]$. On the other hand, by construction, $B_{\alpha_0}[0^*]$ is disjoint from $B_{\alpha_0}[1^*]$ and $B_{\alpha_0}[(-1)^*]$. Then, by the definition of γ_0,

$$\gamma_0\left(B_{\alpha_0}[\ell^*]\right) = \{0\} \subset [-\tfrac{1}{2}, \tfrac{1}{2}] = \left[\gamma_0(\ell^*) - 2^{-n_0}, \gamma_0(\ell^*) + 2^{-n_0}\right]$$

for $\ell \in \{1, -1\}$. Hence, (a) holds.

Fix $j > 0$ and assume that we have defined n_ℓ, α_ℓ, δ_ℓ and γ_ℓ such that all Definition 2.18(R.1–6) above and (a) hold for $\ell = 0, 1, \ldots, j-1$.

Since the elements of Z^*_{j+2} are pairwise different, we can choose an integer $n_j > n_{j-1}$ and δ_j and α_j small enough so that

- $0 < \delta_j < \alpha_j < 2^{-n_j} < \delta_{j-1}$,
- $(-(j+2))^*, (j+2)^* \notin B^{\frown}_{-(j+1)}[(-(j+1))^*] = B_{\alpha_j}[(-(j+1))^*]$,
- the three intervals $B^{\frown}_{j}[j^*] = B_{\alpha_j}[j^*]$, $R_\omega\left(B_{\alpha_j}[j^*]\right) = B_{\alpha_j}[(j+1)^*]$ and $B^{\frown}_{-(j+1)}[(-(j+1))^*]$ are pairwise disjoint,
- $B^{\frown}_{\ell}[\ell^*] \cap Z^*_{j+1} = \{\ell^*\}$ for $\ell \in \{j, -(j+1)\}$, $B_{\alpha_j}[(j+1)^*] \cap Z^*_{j+1} = \{(j+1)^*\}$ and, additionally,
- $\left(\operatorname{Bd}\left(B_{\alpha_j}[(-j)^*]\right) \cup \operatorname{Bd}\left(B_{\alpha_j}[j^*]\right)\right) \cap O^*(\omega) = \emptyset$.

Then, Definition 2.18(R.1) is verified. Moreover, from the above conditions it follows that $B_{\alpha_j}[\ell^*] \cap Z^*_{j+1} = \{\ell^*\}$ for every $\ell \in \{j+1, -(j+1)\}$. Thus, by statement (a) for $j-1$, γ_{j-1} is defined and continuous on $\ell^* \in B_{\alpha_j}[\ell^*]$ because this interval is disjoint from Z^*_{j-1}. Hence, we can decrease the value of α_j (and, accordingly, the value of $0 < \delta_j < \alpha_j$), if necessary, to get

- $$\gamma_{j-1}\left(B_{\alpha_j}[\ell^*]\right) \subset \left[\gamma_{j-1}(\ell^*) - 2^{-n_j}, \gamma_{j-1}(\ell^*) + 2^{-n_j}\right]$$

  for every $\ell \in \{j+1, -(j+1)\}$.

To see that Definition 2.18(R.2) is verified it remains to show that the intervals $B^{\frown}_{j}[j^*]$, $B_{\alpha_j}[(j+1)^*]$ and $B^{\frown}_{-(j+1)}[(-(j+1))^*]$ are disjoint from $B^{\frown}_{-j}[(-j)^*]$. By induction, Definition 2.18(R.2) holds for $j-1$. Thus we see, that $(-(j+1))^*, (j+1)^* \notin B^{\frown}_{-j}[(-j)^*]$, and $R_\omega\left(B_{\alpha_{j-1}}[(j-1)^*]\right) = B_{\alpha_{j-1}}[j^*]$ is disjoint from $B^{\frown}_{-j}[(-j)^*]$. Hence, we can decrease the value of α_j (and, accordingly, the value of $0 < \delta_j < \alpha_j$), if necessary, until $B_{\alpha_j}[(j+1)^*]$ and $B^{\frown}_{-(j+1)}[(-(j+1))^*] = B_{\alpha_j}[(-(j+1))^*]$ are disjoint from $B^{\frown}_{-j}[(-j)^*]$. On the other hand we have that $\alpha_j < 2^{-n_j} < \delta_{j-1} < \alpha_{j-1}$. So, $B^{\frown}_j[j^*] = B_{\alpha_j}[j^*] \subset B_{\alpha_{j-1}}[j^*]$ is disjoint from $B^{\frown}_{-j}[(-j)^*]$.

Up to now we have seen that we can choose n_j, δ_j and α_j so that Definition 2.18(R.1–2) hold for j. Let us see that we can choose α_j such that Definition 2.18(R.3) also holds. Observe that for every $\ell, i \in \mathbb{Z}$ and every $m \geq 0$ it follows that $\operatorname{Bd}\left(B_{\alpha_m}[\ell^*]\right) \cap O^*(\omega) \neq \emptyset$ if and only if $\operatorname{Bd}\left(R^i_\omega\left(B_{\alpha_m}[\ell^*]\right)\right) \cap O^*(\omega) = \operatorname{Bd}\left(B_{\alpha_m}[(\ell+i)^*]\right) \cap O^*(\omega) \neq \emptyset$. Therefore, by using Definition 2.18(R.1) inductively, we obtain

$$\bigcup_{k \in Z_{j-1}} \operatorname{Bd}\left(B_{\alpha_{|k|}}[(k+1)^*]\right) \cap \{(-j)^*, j^*\} \subset \bigcup_{k \in Z_{j-1}} \operatorname{Bd}\left(B_{\alpha_{|k|}}[(k+1)^*]\right) \cap O^*(\omega) = \emptyset.$$

Consequently, since $\bigcup_{k \in Z_{j-1}} \operatorname{Bd}\left(B_{\alpha_{|k|}}[(k+1)^*]\right)$ is a finite set, by decreasing again the value of α_j, if necessary, we can achieve that Definition 2.18(R.3) holds for j and Definition 2.18(R.1–2) are still verified.

Next we will take care of Definition 2.18(R.4). If $(j+1)^* \notin \bigcup_{i \in Z_{j-1}} B^{\frown}_i[i^*]$, by decreasing again the value of α_j (and δ_j), if necessary, we can achieve that $B_{\alpha_j}[(j+1)^*] \cap \left(\bigcup_{i \in Z_{j-1}} B^{\frown}_i[i^*]\right) = \emptyset$ while preserving that Definition 2.18(R.1–3) are verified for j. In this case Definition 2.18(R.4) holds trivially.

Conversely, assume that there exists $k \in Z_{j-1}$ such that $(j+1)^* \in B^{\frown}_k[k^*]$ and $|k|$ is maximal verifying these conditions. By Definition 2.18(R.2), k is unique (that is, the condition cannot be verified by k and $-k$ simultaneously). On the other hand, by the Definition 2.18(R.1) for $|k|$ and $|k|-1$ and the comment above, $(j+1)^* \notin \operatorname{Bd}\left(B^{\frown}_k[k^*]\right) \cup \operatorname{Bd}\left(B_{\alpha_{|k|}}[k^*]\right)$. Since $k \in Z_{j-1}$, $|k| \leq j-1$ and, hence, $(j+1)^* \notin Z^*_{|k|}$ (in particular $j^* \neq k^*$). Consequently, $(j+1)^*$ is contained in one of the connected components of $B^{\frown}_k(k^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup Z^*_{|k|}\right)$. Then, by decreasing again the value of α_j, if necessary, we can get that $B_{\alpha_j}[(j+1)^*]$ is contained in the connected component of $B^{\frown}_k(k^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup Z^*_{|k|}\right)$ where $(j+1)^*$ lies, while preserving that Definition 2.18(R.1–3) are verified for j. Consequently, Definition 2.18(R.1–4) hold for j.

Now we will deal with Definition 2.18(R.5). If $\ell^* \notin \bigcup_{i \in Z_{j-1}} B^{\frown}_i[i^*]$, by decreasing again the value of α_j, if necessary, we can get Definition 2.18(R.5.i) while preserving that Definition 2.18(R.1–4) are verified for j.

Assume that there exists $m \in Z_{j-1}$ such that $\ell^* \in B^{\frown}_m[m^*]$ and $|m|$ is maximal with these properties. As in the above construction, by Definition 2.18(R.1–2),

$$\ell^* \in B^{\frown}_m(m^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$$

and m is unique (that is, the condition cannot be verified simultaneously by m and $-m$). Consequently, $\ell^* \notin B^{\frown}_i[i^*]$ for every $i \in Z_{j-1}$ such that $|i| \geq |m|$, $i \neq m$. Thus, by decreasing again the value of α_j, if necessary, we can get that Definition 2.18(R.1–4) still hold, Definition 2.18(R.5.ii.1) is verified and the interval $B_\ell^{\frown}[\ell^*]$ is contained in the connected component of $B_m^{\frown}(m^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$ where ℓ^* lies. So, Definition 2.18(R.5.ii.2) also holds.

We claim that

for every $\ell, m \in \mathbb{Z}$ such that $|m| \leq |\ell| \leq j$, $\ell \neq m$, either $B_\ell^{\frown}[\ell^*] \cap B_m^{\frown}[m^*] = \emptyset$ or $|m| < |\ell|$ and $B_\ell^{\frown}[\ell^*]$ is contained in a connected component of

$$B_m^{\frown}(m^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right).$$

We prove the claim by induction. Observe that the claim holds trivially for $|m| \leq |\ell| \leq 1$ because $B_0^{\frown}[0^*]$, $B_1^{\frown}[1^*] = B_{\alpha_1}[1^*] \subset B_{\alpha_0}[1^*]$ and $B_{-1}^{\frown}[(-1)^*]$ are pairwise disjoint by construction.

Assume that the claim holds for every $|m| \leq |\ell| < j$. So, to prove the claim, we may assume that $\ell \in \{j, -j\}$, $m \in Z_{j-1} \cup \{-\ell\}$ and $B_\ell^{\frown}[\ell^*] \cap B_m^{\frown}[m^*] \neq \emptyset$. By Definition 2.18(R.2), $B_j^{\frown}[j^*] \cap B_{-j}^{\frown}[(-j)^*] = \emptyset$. Consequently, $m \neq -\ell$ (that is, $m \in Z_{j-1}$ and $|\ell| = j > |m|$). On the other hand, if $\ell = -j$, Definition 2.18(R.2) for $j-1$ shows that $B_{j-1}^{\frown}[(j-1)^*]$, $B_{-(j-1)}^{\frown}[(-(j-1))^*]$ and $B_{-j}^{\frown}[(-j)^*]$ are pairwise disjoint. Thus, $m \in Z_{j-2}$ in this case.

Hence, by the Definition 2.18(R.5) for j when $\ell = j$ and for $j-1$ when $\ell = -j$, there exists $k \in Z_{j-1}$ (in fact when $\ell = -j$, $k \in Z_{j-2}$) such that $B_\ell^{\frown}[\ell^*]$ is contained in a connected component of $B_k^{\frown}(k^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup \{k^*\}\right)$ and $|\ell| = j > |k| \geq |m|$.

If $m = k$ then the claim holds. Otherwise, $m \neq k$ and since $j = |\ell| > |k| \geq |m|$, by the induction hypotheses, $|k| > |m|$, and $B_k^{\frown}[k^*]$ is contained in a connected component of $B_m^{\frown}(m^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$. So, the claim holds also in this case. This ends the proof of the claim.

Finally, we consider Definition 2.18(R.6). The fact that either $B_\ell^{\frown}[\ell^*] \cap B_m^{\frown}[m^*] = \emptyset$ for every $m \in Z_j$, $m \neq \ell$ or there exists $m \in Z_{j-1}$ such that $B_\ell^{\frown}[\ell^*]$ is contained in a connected component of $B_m^{\frown}(m^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$ follows from the claim.

To show that Definition 2.18(R.6.i) can be guaranteed, it is enough to decrease again the value of α_j, if necessary, until $B_{\alpha_j}[\ell^*]$ is disjoint from $Z_{|m|}^*$ and Definition 2.18(R.1–5) are still verified. Thus by (a) for $|m|$, $\gamma_{|m|}$ is well defined and continuous on $B_{\alpha_j}[\ell^*]$. So, we can set $a_\ell := \gamma_{|m|}(\ell^*)$ and, by decreasing again α_j (if necessary), we get $\operatorname{Graph}\left(\gamma_{|m|}\big|_{B_{\alpha_j}[\ell^*]}\right) \subset \mathcal{R}(j^*)$.

To show that Definition 2.18(R.6.ii) can be guaranteed we first assume that $k = m$. As before, if necessary, we can increase the value of n_j and, accordingly, decrease the values of $\alpha_j < 2^{-n_j}$ and $0 < \delta_j < \alpha_j$ so that Definition 2.18(R.1–5) and (R.6.i) are still verified for j and in addition,

$$(\ell^*, a_\ell + 2^{-n_j}), (\ell^*, a_\ell - 2^{-n_j}) \in \operatorname{Int}(\mathcal{R}(k^*))$$

and the region $\mathcal{R}(\ell^*)$ is contained in one of the two connected components of $\operatorname{Int}\left(\mathcal{R}(k^*) \setminus \upuparrows k^*\right)$.

Assume now that $k \neq m$ (recall that $|k| \leq |m| < j$). In this case we have $B_\ell^{\frown}[\ell^*] \subset B_m^{\frown}(m^*) \cap B_{\alpha_{|k|}}(k^*)$. In particular, $B_m^{\frown}(m^*) \cap B_{\alpha_{|k|}}(k^*) \neq \emptyset$ and, by the above claim, $|k| < |m|$ and $B_\ell^{\frown}[\ell^*] \subset$

$B^{\frown}_m[m^*]$ is contained in a connected component of $B^{\frown}_k(k^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup \{k^*\}\right)$. The fact that $B^{\frown}_\ell[\ell^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ implies that $B^{\frown}_\ell[\ell^*] \subset B^{\frown}_m[m^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$. Then, as above we can increase the value of n_j and, accordingly, decrease the values of $\alpha_j < 2^{-n_j}$ and $0 < \delta_j < \alpha_j$ so that Definition 2.18(R.1–5) and (R.6.i) are still verified,

$$(\ell^*, a_\ell + 2^{-n_j}), (\ell^*, a_\ell - 2^{-n_j}) \in \operatorname{Int}(\mathcal{R}(k^*))$$

and the region $\mathcal{R}(\ell^*)$ is contained in one of the two connected components of $\operatorname{Int}\left(\mathcal{R}(k^*) \setminus \upuparrows k^*\right)$.

Now assume that $|k|$ is not maximal verifying the assumptions. Then, there exists $r \in Z_{|m|} \subset Z_{j-1}$ such that $B^{\frown}_\ell[\ell^*] \subset B_{\alpha_{|r|}}(r^*) \setminus \{r^*\}$ and $|r|$ is maximal with these properties.

We have $|k| \le |r| \le |m| < j$ and

$$B^{\frown}_r[r^*] \cap B^{\frown}_k[k^*] \supset B_{\alpha_{|r|}}(r^*) \cap B_{\alpha_{|k|}}(k^*) \neq \emptyset$$

because $B^{\frown}_\ell[\ell^*] \subset B_{\alpha_{|r|}}(r^*) \cap B_{\alpha_{|k|}}(k^*)$. Then, by the claim, $|k| < |r|$ and $B^{\frown}_r[r^*]$ is contained in a connected component of $B^{\frown}_k(k^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup \{k^*\}\right)$. The fact that $B^{\frown}_\ell[\ell^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ implies that $B^{\frown}_r[r^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$. By the part already proven and Definition 2.18(R.6.ii) for $|r| < j$ we get that $\mathcal{R}(\ell^*)$ is contained in one of the two connected components of $\operatorname{Int}\left(\mathcal{R}(r^*) \setminus \upuparrows r^*\right)$ and $\mathcal{R}(r^*)$ is contained in one of the two connected components of $\operatorname{Int}\left(\mathcal{R}(k^*) \setminus \upuparrows k^*\right)$. This shows that Definition 2.18(R.6.ii) can be guaranteed.

Let us prove that (a) holds for j. Since the set $\mathbb{S}^1 \setminus Z^*_j$ is residual, to prove that $(\gamma_j, \mathbb{S}^1 \setminus Z^*_j) \in \mathfrak{C}$ we have to show that $\gamma_j\big|_{\mathbb{S}^1 \setminus Z^*_j}$ is continuous. Note that, from Definition 2.18(R.6.ii), $a^{\pm}_\ell = \varphi_{\ell^*}(\ell^* \pm \alpha_j) = \gamma_{j-1}(\ell^* \pm \alpha_j)$. Hence, the continuity of $\gamma_j\big|_{\mathbb{S}^1 \setminus Z^*_j}$ follows from the fact that γ_{j-1} is continuous on $\mathbb{S}^1 \setminus Z^*_{j-1} \supset \mathbb{S}^1 \setminus Z^*_j$ and the continuity of φ_{j^*} and $\varphi_{(-j)^*}$ (Definition 2.15).

This ends the proof of the first statement of the lemma and the first statement of (a). For every $\ell \in \{j+1, -(j+1)\}$, from By Definition 2.18(R.1,2) we get:

$$\gamma_{j-1}\left(B_{\alpha_j}[\ell^*]\right) \subset \left[\gamma_{j-1}(\ell^*) - 2^{-n_j}, \gamma_{j-1}(\ell^*) + 2^{-n_j}\right]$$

$B_{\alpha_j}[\ell^*]$ is disjoint from $B_{\alpha_j}[j^*]$ and $B_{\alpha_{j-1}}[(-j)^*] \supset B_{\alpha_j}[(-j))^*]$, and

$$\{\ell^*\} \notin B_{\alpha_j}[\ell^*] \cap Z^*_{j-1} \subset B_{\alpha_j}[\ell^*] \cap Z^*_{j+1} = \{\ell^*\}.$$

So, from the definition of γ_j it follows that

$$\gamma_j\big|_{B_{\alpha_j}[\ell^*]} = \gamma_{j-1}\big|_{B_{\alpha_j}[\ell^*]}$$

and, thus, (a) holds.

Statement (c) follows immediately from Definition 2.18(R.6) and Remark 2.19(R.6).

Next we prove (b,d,e,f,g).

(d) When $n = j$, we get $B_{\alpha_j}[\ell^*] \setminus Z^*_j = B_{\alpha_j}[\ell^*] \setminus \{\ell^*\}$ from Definition 2.18(R.2). Hence,

$$\operatorname{Graph}\left(\gamma_j\big|_{B_{\alpha_j}[\ell^*] \setminus Z^*_j}\right) \subset \mathcal{R}(\ell^*)$$

by the definition of γ_j (Definition 2.18) and the definition of φ_{ℓ^*} (Definition 2.15).

Now assume that $n > j$ and fix $\theta \in B_{\alpha_j}[\ell^*] \setminus Z_n^*$. We have to show that the point $(\theta, \gamma_n(\theta)) \in \mathcal{R}(\ell^*)$. If $\theta \notin B_{\alpha_{|m|}}[m^*]$ for every m such that $j < |m| \le n$ then, by the iterative use of the definition of γ_i for $i = j+1, j+2, \dots, n$ (Definition 2.18) and Definition 2.15,

$$(\theta, \gamma_n(\theta)) = (\theta, \gamma_{n-1}(\theta)) = \cdots = (\theta, \gamma_{j+1}(\theta)) = (\theta, \gamma_j(\theta)) = (\theta, \varphi_{\ell^*}(\theta)) \in \mathcal{R}(\ell^*).$$

Otherwise, by Definition 2.18(R.2), there exists $m \in \mathbb{Z}$ such that $|\ell| < |m| \le n$, $\theta \in B_{\alpha_{|m|}}[m^*] \setminus Z_n^*$, and $\theta \notin B_{\alpha_{|s|}}[s^*]$ for every s such that $|m| < |s| \le n$. This implies that $B_\ell^{\frown}[\ell^*] \cap B_m^{\frown}[m^*] \supset B_{\alpha_j}[\ell^*] \cap B_{\alpha_{|m|}}[m^*] \neq \emptyset$ and $|m|$ is maximal with these properties. So, by the claim for $j = |m|$, $B_m^{\frown}[m^*]$ is contained in a connected component of $B_\ell^{\frown}(\ell^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|\ell|}}[\ell^*]\right) \cup \{\ell^*\}\right)$. Moreover, since $\theta \in B_m^{\frown}(m^*) \cap B_{\alpha_j}[\ell^*] \neq \emptyset$, $B_m^{\frown}[m^*] \subset B_{\alpha_{|\ell|}}[\ell^*] \setminus \{\ell^*\}$. Thus, by Definition 2.18(R.6.ii) and Remark 2.19(R.6.ii) for $j = |m|$, ℓ replaced by m and k replaced by ℓ, $\mathcal{R}(m^*) \subset \mathcal{R}(\ell^*)$ and (d) follows from the part already proven by replacing ℓ by m and j by $|m|$.

(g) By the claim we have that for every $\ell, m \in \mathbb{Z}$ such that $|\ell| \ge |m|$, $\ell \neq m$ and $B_\ell^{\frown}[\ell^*] \cap B_m^{\frown}[m^*] \neq \emptyset$, it follows that $|\ell| > |m|$, and $B_\ell^{\frown}[\ell^*]$ is contained in a connected component of $B_m^{\frown}(m^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|m|}}[m^*]\right) \cup \{m^*\}\right)$. Only it remains to show that if $B_\ell^{\frown}[\ell^*] \subset B_{\alpha_{|m|}}(m^*) \setminus \{m^*\}$, then the region $\mathcal{R}^{\frown}(\ell^*)$ is contained in one of the two connected components of $\operatorname{Int}\left(\mathcal{R}(m^*) \setminus \upuparrows m^*\right)$. By Definition 2.18(R.6.ii) we know that this holds for $\mathcal{R}(\ell^*)$ instead of $\mathcal{R}^{\frown}(\ell^*)$. Hence, if $\ell \ge 0$, (g) holds because $\mathcal{R}^{\frown}(\ell^*) = \mathcal{R}(\ell^*)$. Assume now that $\ell < 0$. Since $\mathcal{R}^{\frown}(\ell^*) = \mathcal{R}(\ell^*) \cup \operatorname{Graph}\left(\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right)$ is connected, $\mathcal{R}(\ell^*) \subset \mathcal{R}(m^*)$, and $\operatorname{Int}\left(\mathcal{R}(m^*) \setminus \upuparrows m^*\right)$ has two connected components, it is enough to show that

$$\operatorname{Graph}\left(\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right) \subset \mathcal{R}(m^*).$$

Since $B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*) \subset B_\ell^{\frown}[\ell^*] \subset B_{\alpha_{|m|}}(m^*) \setminus \{m^*\}$, statement (g) follows from (d) with ℓ replaced by m, j by $|m|$ and n replaced by $|\ell|$.

(b) With (g) in mind we set

$$\mathsf{D} := \{\ell \in \mathbb{Z} : \mathcal{R}^{\frown}(\ell^*) \not\subset \mathcal{R}(i^*) \text{ for every } i \in \mathbb{Z} \setminus \{\ell\}\}.$$

Clearly,

$$\begin{aligned}\bigcup_{\ell \in \mathbb{Z}} \mathcal{R}^{\frown}(\ell^*) &= \left(\bigcup_{i \in \mathbb{Z} \setminus \mathsf{D}} \mathcal{R}^{\frown}(i^*)\right) \cup \left(\bigcup_{\ell \in \mathsf{D}} \mathcal{R}^{\frown}(\ell^*)\right) \\ &\subset \left(\bigcup_{i \in \mathsf{D}} \mathcal{R}(i^*)\right) \cup \left(\bigcup_{\ell \in \mathsf{D}} \mathcal{R}^{\frown}(\ell^*)\right) = \bigcup_{\ell \in \mathsf{D}} \mathcal{R}^{\frown}(\ell^*)\end{aligned}$$

Case 2.21. Claim: For every $\ell \in \mathsf{D}$, $\gamma_{|\ell|-1}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)} \equiv 0$.

First we prove statement (b) from the above claim and then we will prove the claim. To this end we start by pointing out few elementary facts.

From the definition of $\mathcal{R}^{\frown}(\ell^*)$ we see that $\mathcal{R}^{\frown}(\ell^*)\backslash\mathcal{R}(\ell^*) = \emptyset$ for every $\ell \geq 0$ and $\mathcal{R}^{\frown}(\ell^*)\backslash\mathcal{R}(\ell^*) \subset \operatorname{Graph}\Big(\gamma_{|\ell|}\big|_{B^{\frown}_{\ell}[\ell^*]\backslash B_{\alpha_{|\ell|}}(\ell^*)}\Big)$ for every $\ell < 0$. So, in any case,

$$\mathcal{R}^{\frown}(\ell^*)\backslash\mathcal{R}(\ell^*) \subset \operatorname{Graph}\Big(\gamma_{|\ell|}\big|_{B^{\frown}_{\ell}[\ell^*]\backslash B_{\alpha_{|\ell|}}(\ell^*)}\Big) \quad \text{for every} \quad \ell \in \mathbb{Z}.$$

On the other hand, the arc $B^{\frown}_{\ell}[\ell^*] \supset B^{\frown}_{\ell}[\ell^*]\backslash B_{\alpha_{|\ell|}}(\ell^*)$ is disjoint from the arc $B^{\frown}_{-\ell}[(-\ell)^*] \supset B_{\alpha_{|\ell|}}[(-\ell)^*]$ by Definition 2.18(R.2). Thus, by Definition 2.18 and (a),

$$\gamma_{|\ell|-1}\big|_{B^{\frown}_{\ell}[\ell^*]\backslash B_{\alpha_{|\ell|}}(\ell^*)} = \gamma_{|\ell|}\big|_{B^{\frown}_{\ell}[\ell^*]\backslash B_{\alpha_{|\ell|}}(\ell^*)}.$$

Furthermore, by the Claim and Definition 2.18(R.6), $a^+_\ell = a^-_\ell = a_\ell = 0$ for every $\ell \in D$. So, by Remark 2.16(1),

$$\mathcal{R}(\ell^*) \subset B_{\alpha_{|\ell|}}[\ell^*] \times [-2^{-n_{|\ell|}}, 2^{-n_{|\ell|}}] \subset B_{\alpha_{|\ell|}}[\ell^*] \times [-2^{-|\ell|}, 2^{-|\ell|}] \subset B_{\alpha_{|\ell|}}[\ell^*] \times [-1, 1].$$

Therefore, summarizing and using again by the Claim,

$$\begin{aligned}
\bigcup_{\ell\in\mathbb{Z}} \mathcal{R}^{\frown}(\ell^*) \subset \bigcup_{\ell\in\mathsf{D}} \mathcal{R}^{\frown}(\ell^*) &\subset \bigcup_{\ell\in\mathsf{D}} \left(\mathcal{R}(\ell^*) \cup \operatorname{Graph}\Big(\gamma_{|\ell|}\big|_{B^{\frown}_{\ell}[\ell^*]\backslash B_{\alpha_{|\ell|}}(\ell^*)}\Big)\right) \\
&= \left(\bigcup_{\ell\in\mathsf{D}} \mathcal{R}(\ell^*)\right) \cup \left(\bigcup_{\ell\in\mathsf{D}} \operatorname{Graph}\Big(\gamma_{|\ell|-1}\big|_{B^{\frown}_{\ell}[\ell^*]\backslash B_{\alpha_{|\ell|}}(\ell^*)}\Big)\right) \\
&\subset \left(\bigcup_{\ell\in\mathsf{D}} B_{\alpha_{|\ell|}}[\ell^*]\right) \times [-1, 1] \cup \mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times [-1, 1].
\end{aligned}$$

So, the first part of (b) is proved, provided that the claim holds. Let us prove the second statement of (b). Observe that, since

$$\left(\bigcup_{\ell\in\mathbb{Z}} \mathcal{R}(\ell^*)\right) \cup \mathbb{S}^1 \times \{0\} \subset \left(\bigcup_{\ell\in\mathbb{Z}} \mathcal{R}^{\frown}(\ell^*)\right) \cup \mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times [-1, 1],$$

it is enough to show that

$$\operatorname{Graph}\left(\gamma_j\big|_{\mathbb{S}^1\backslash Z^*_j}\right) \subset \left(\bigcup_{\ell\in\mathbb{Z}} \mathcal{R}(\ell^*)\right) \cup \mathbb{S}^1 \times \{0\}$$

for every $j \in \mathbb{Z}^+$. We will prove this statement by induction on j.

By construction we have

$$\operatorname{Graph}\left(\gamma_0\big|_{\mathbb{S}^1\backslash\{0^*\}}\right) \subset \mathcal{R}(0^*) \cup \mathbb{S}^1 \times \{0\} \subset \left(\bigcup_{\ell\in\mathbb{Z}} \mathcal{R}(\ell^*)\right) \cup \mathbb{S}^1 \times \{0\}.$$

So, the statement holds for $j = 0$. Now assume that it holds for some $j \geq 0$, and prove it for $j + 1$. By Definition 2.18 and (d),

$$
\begin{aligned}
\text{Graph}\left(\gamma_{j+1}\big|_{\mathbb{S}^1\setminus Z^*_{j+1}}\right) &\subset \mathcal{R}(j^*)\cup\mathcal{R}((-j)^*)\cup\text{Graph}\left(\gamma_j\big|_{\mathbb{S}^1\setminus Z^*_j}\right)\\
&\subset \mathcal{R}(j^*)\cup\mathcal{R}((-j)^*)\cup\left(\bigcup_{\ell\in\mathbb{Z}}\mathcal{R}(\ell^*)\right)\cup\mathbb{S}^1\times\{0\}\\
&\subset \left(\bigcup_{\ell\in\mathbb{Z}}\mathcal{R}(\ell^*)\right)\cup\mathbb{S}^1\times\{0\}.
\end{aligned}
$$

To end the proof of (b) it remains to show the Claim.

Let $\ell\in\mathsf{D}$ and $m\in Z_{|\ell|}$, $m\neq\ell$. Then, either

$$
\begin{cases}
B^{\frown}_{\ell}[\ell^*]\cap B^{\frown}_{m}[m^*]=\emptyset \text{ or}\\
|\ell|>|m|,\ m<0 \text{ and } B^{\frown}_{\ell}[\ell^*]\subset B^{\frown}_{m}(m^*)\setminus B_{\alpha_{|m|}}[m^*]\,.
\end{cases} \tag{2.1}
$$

To see this, observe that if $B^{\frown}_{\ell}[\ell^*]\cap B^{\frown}_{m}[m^*]\neq\emptyset$ then, by (g), $|\ell|>|m|$ and either $\mathcal{R}^{\frown}(\ell^*)\subset\mathcal{R}(m^*)$ or $m<0$ and $B^{\frown}_{\ell}[\ell^*]\subset B^{\frown}_{m}(m^*)\setminus B_{\alpha_{|m|}}[m^*]$, and the first possibility is ruled out because $\ell\in\mathsf{D}$.

By using iteratively the dichotomy (2.1) we get that, for every $\ell\in\mathsf{D}$, there exists a sequence $m_0,m_1,\dots,m_k=\ell\in\mathbb{Z}$ with $k\geq 0$ such that $B^{\frown}_{m_0}[(m_0)^*]\cap B^{\frown}_{q}[q^*]=\emptyset$ for every $q\in Z_{|m_0|}$, $q\neq m_0$ and, in the case $k>0$, $|m_0|<|m_1|<\cdots<|m_k|=|\ell|$ and, for every $p=0,1,\dots,k-1$,

- $m_p<0$,
- $B^{\frown}_{m_{p+1}}[(m_{p+1})^*]\subset B^{\frown}_{m_p}((m_p)^*)\setminus B_{\alpha_{|m_p|}}[(m_p)^*]$ and
- $B^{\frown}_{m_{p+1}}[(m_{p+1})^*]\cap B^{\frown}_{q}[q^*]=\emptyset$ for every $q\in Z_{|m_{p+1}|}$, $q\neq m_p,m_{p+1}$ and $|m_p|\leq|q|$.

The condition $B^{\frown}_{m_0}[(m_0)^*]\cap B^{\frown}_{q}[q^*]=\emptyset$ for every $q\in Z_{|m_0|},\ q\neq m_0$ implies

$$
\gamma_{|m_0|-1}\big|_{B^{\frown}_{m_0}[(m_0)^*]}=\gamma_{|m_0|-2}\big|_{B^{\frown}_{m_0}[(m_0)^*]}=\cdots=\gamma_0\big|_{B^{\frown}_{m_0}[(m_0)^*]}\equiv 0
$$

by Definition 2.18(R.6) and Remark 2.19(R.6) (with $\ell=m_0$). This ends the proof of the Claim when $k=0$.

Assume now that $k>0$. As before we have

$$
\gamma_{|m_0|-1}\big|_{B^{\frown}_{m_0}[(m_0)^*]\setminus B_{\alpha_{|m_0|}}((m_0)^*)}=\gamma_{|m_0|}\big|_{B^{\frown}_{m_0}[(m_0)^*]\setminus B_{\alpha_{|m_0|}}((m_0)^*)}.
$$

This, together with the inclusion,

$$
B^{\frown}_{m_1}[(m_1)^*]\subset B^{\frown}_{m_0}((m_0)^*)\setminus B_{\alpha_{|m_0|}}[(m_0)^*]
$$

implies that

$$
\gamma_{|m_0|}\big|_{B^{\frown}_{m_1}[(m_1)^*]}\equiv 0.
$$

Then, by Definition 2.18(R.6.i) and Remark 2.19(R.6.i) with $\ell=m_1$,

$$
0\equiv\gamma_{|m_0|}\big|_{B^{\frown}_{m_1}[(m_1)^*]}=\gamma_{|m_0|+1}\big|_{B^{\frown}_{m_1}[(m_1)^*]}=\cdots=\gamma_{|m_1|-1}\big|_{B^{\frown}_{m_1}[(m_1)^*]}.
$$

If $k=1$ we are done. Otherwise, $k\geq 2$ and, as above,

$$\gamma_{|m_1|}\big|_{B^{\frown}_{m_2}[(m_2)^*]} \equiv 0.$$

By iterating the above arguments at most k times the Claim holds. This ends the proof of (b).

(e) By Definition 2.18(R.2) and Remark 2.19(R.2) it follows that

$$\theta \notin Z^*_{j+1} \cup B_{\alpha_j}(\ell^*) \cup B^{\frown}_{-\ell}((-\ell)^*) \quad \text{for every} \quad \theta \in B^{\frown}_{\ell}[\ell^*] \setminus B_{\alpha_j}(\ell^*).$$

So, by (a), $\gamma_{j-1}(\theta)$ is well defined and γ_{j-1} is continuous at θ. Thus, by the definition of γ_j (Definition 2.18) and the continuity of γ_{j-1} at θ, $\gamma_j(\theta) = \gamma_{j-1}(\theta)$.

Now assume that $\theta \in \operatorname{Bd}(B^{\frown}_{\ell}[\ell^*] \setminus B_{\alpha_j}(\ell^*)) = \operatorname{Bd}(B_{\alpha_j}[\ell^*]) \cup \operatorname{Bd}(B^{\frown}_{\ell}[\ell^*])$. By (g), $\theta \notin B^{\frown}_{n}[n^*] \cup B^{\frown}_{-n}[(-n)^*]$ for every $n > j$. So, by the iterative use of the definition of γ_i for $i = j+1, j+2, \dots, n$ (Definition 2.18) we get

$$\gamma_j(\theta) = \gamma_{j+1}(\theta) = \cdots = \gamma_{n-1}(\theta) = \gamma_n(\theta).$$

Now we prove the part of (e) concerning $R_\omega(B^{\frown}_{\ell}[\ell^*] \setminus B_{\alpha_j}(\ell^*))$. We first assume that $\ell = j \geq 0$. Then,

$$B^{\frown}_{j}[j^*] = B_{\alpha_j}[j^*],\ \theta \in \operatorname{Bd}(B_{\alpha_j}[j^*]) \quad \text{and} \quad R_\omega(\theta) \in \operatorname{Bd}(B_{\alpha_j}[(j+1)^*]).$$

Again by Definition 2.18(R.2), $R_\omega(\theta) \notin Z^*_{j+1} \cup B_{\alpha_j}[j^*] \cup B^{\frown}_{-j}[(-j)^*]$. So, by (a) and the definition of γ_j (Definition 2.18), $\gamma_{j-1}\big(R_\omega(\theta)\big)$ is well defined and $\gamma_j\big(R_\omega(\theta)\big) = \gamma_{j-1}\big(R_\omega(\theta)\big)$. By Definition 2.18(R.3) (with $j = n$ and $k = \ell = j$), $R_\omega(\theta) \notin B_{\alpha_n}[n^*] \cup B_{\alpha_n}[(-n)^*]$ for every $n > j$. So, $\gamma_n\big(R_\omega(\theta)\big) = \gamma_j\big(R_\omega(\theta)\big)$ as above.

Assume now that $\ell = -j < 0$. In this case we have $B^{\frown}_{\ell}[\ell^*] = B_{\alpha_{|\ell+1|}}[\ell^*]$ and, hence, $R_\omega(\theta) \in B_{\alpha_{|\ell+1|}}[(\ell+1)^*] \setminus B_{\alpha_j}((\ell+1)^*)$. By Definition 2.18(R.1) we have

$$B_{\alpha_j}[(\ell+1)^*] \subset B_{\alpha_{|\ell+1|}}[(\ell+1)^*] \subset B^{\frown}_{\ell+1}[(\ell+1)^*].$$

Thus, $R_\omega(\theta) \in B^{\frown}_{\ell+1}[(\ell+1)^*] \setminus \{(\ell+1)^*\}$. Again by Definition 2.18(R.2) and Remark 2.19(R.2) (with j replaced by $-(\ell+1)$),

$$R_\omega(\theta) \notin Z^*_{\ell} \cup B_{\alpha_{-(\ell+1)}}[(-\ell)^*] \cup B^{\frown}_{\ell}[\ell^*] \supset Z^*_j \cup B_{\alpha_j}[j^*] \cup B^{\frown}_{-j}[(-j)^*].$$

So, by (a) and the definition of γ_j (Definition 2.18), $\gamma_{j-1}\big(R_\omega(\theta)\big)$ is well defined and $\gamma_j\big(R_\omega(\theta)\big) = \gamma_{j-1}\big(R_\omega(\theta)\big)$.

To end the proof of (e), assume as above that $\theta \in \operatorname{Bd}(B_{\alpha_j}[\ell^*]) \cup \operatorname{Bd}(B^{\frown}_{\ell}[\ell^*])$ and, hence, $R_\omega(\theta) \in \operatorname{Bd}(B_{\alpha_j}[(\ell+1)^*]) \cup \operatorname{Bd}\big(B_{\alpha_{|\ell+1|}}[(\ell+1)^*]\big)$. We have to show that, in this case, $R_\omega(\theta) \notin B_{\alpha_n}[n^*] \cup B_{\alpha_n}[(-n)^*]$ for every $n > j$ (the fact that $\gamma_n\big(R_\omega(\theta)\big) = \gamma_j\big(R_\omega(\theta)\big)$ follows as above). When $R_\omega(\theta) \in \operatorname{Bd}(B_{\alpha_j}[(\ell+1)^*])$ this follows from Definition 2.18(R.3) as before. Assume now that $R_\omega(\theta) \in \operatorname{Bd}\big(B_{\alpha_{|\ell+1|}}[(\ell+1)^*]\big)$. Then, by (g), $R_\omega(\theta) \notin B^{\frown}_{n}[n^*] \cup B^{\frown}_{-n}[(-n)^*]$ for every $n > j$.

(f) If $\ell \geq 0$ then the first two statements of (f) follow directly from the definitions. Moreover, by Remarks 2.16(2) and 2.19(R.1),

$$\operatorname{diam}\left(\mathcal{R}^{\frown}(\ell^*)\right) = \operatorname{diam}\left(\mathcal{R}(\ell^*)\right) = \operatorname{diam}\left(\mathcal{R}((-\ell)^*)\right) = 2 \cdot 2^{-n_\ell} \le 2 \cdot 2^{-(\ell+1)} = 2^{-\ell}.$$

Assume that $\ell < 0$. From Definition 2.18(R.2) and Remark 2.19(R.2) we get $\left(B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)\right) \cap Z_{|\ell|}^* = \emptyset$ and, hence, $\gamma_{|\ell|}$ is continuous in an open neighbourhood of $B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)$ by (a). On the other hand, by (d), $\left(\theta, \gamma_{|\ell|}(\theta)\right) \in \mathcal{R}(\ell^*)$ for every $\theta \in \operatorname{Bd}\left(B_{\alpha_{|\ell|}}[\ell^*]\right) \subset B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)$. Thus,

$$\mathcal{R}^{\frown}(\ell^*) = \mathcal{R}(\ell^*) \cup \operatorname{Graph}\Big(\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\Big)$$

is closed, connected and projects onto the whole $B_\ell^{\frown}[\ell^*]$.

On the other hand, by (e) and (a) (since $\ell < 0$, $|\ell + 1| = |\ell| - 1$),

$$\begin{aligned}\gamma_{|\ell|}\left(B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)\right) &= \gamma_{|\ell|-1}\left(B_{\alpha_{|\ell+1|}}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)\right) \\ &\subset \left[\gamma_{|\ell|-1}(\ell^*) - 2^{-n_{|\ell|-1}}, \gamma_{|\ell|-1}(\ell^*) + 2^{-n_{|\ell|-1}}\right].\end{aligned}$$

Thus, by Remark 2.16(1), (c) and Definition 2.18(R.1),

$$\begin{aligned}\mathcal{R}^{\frown}(\ell^*) &= \mathcal{R}(\ell^*) \cup \operatorname{Graph}\Big(\gamma_{|\ell|}\big|_{B_\ell^{\frown}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\Big) \\ &\subset B_{\alpha_{|\ell|}}[\ell^*] \times \left[\gamma_{|\ell|-1}(\ell^*) - 2^{-n_{|\ell|}}, \gamma_{|\ell|-1}(\ell^*) + 2^{-n_{|\ell|}}\right] \cup \\ &\qquad \left(B_{\alpha_{|\ell+1|}}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)\right) \times \left[\gamma_{|\ell|-1}(\ell^*) - 2^{-n_{|\ell|-1}}, \gamma_{|\ell|-1}(\ell^*) + 2^{-n_{|\ell|-1}}\right] \\ &\subset B_{\alpha_{|\ell+1|}}[\ell^*] \times \left[\gamma_{|\ell|-1}(\ell^*) - 2^{-n_{|\ell|-1}}, \gamma_{|\ell|-1}(\ell^*) + 2^{-n_{|\ell|-1}}\right].\end{aligned}$$

Hence, by Definition 2.18(R.1) and Remark 2.19(R.1),

$$\operatorname{diam}\left(\mathcal{R}^{\frown}(\ell^*)\right) \le 2 \cdot \max\{\alpha_{|\ell+1|}, 2^{-n_{|\ell|-1}}\} = 2 \cdot 2^{-n_{|\ell|-1}} \le 2 \cdot 2^{-|\ell|}.$$

The next results allow us to define the limit pseudo-curve generated by the sequence $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty$.

Lemma 2.22. *The sequence $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty \subset \mathfrak{C}$ is convergent in $\mathfrak{C}$.*

Proof. By Proposition 2.11 it suffices to show that $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty$ is a Cauchy sequence in $\mathfrak{C}$. By the definition of γ_i (Definition 2.18) we have

$$\begin{aligned}\mathsf{d}_\infty\left(\gamma_{i-1}, \gamma_i\right) &= \sup_{\theta \in \mathbb{S}^1 \setminus Z_i^*} |\gamma_{i-1}(\theta) - \gamma_i(\theta)| \\ &= \sup_{\theta \in \left(B_{\alpha_i}[i^*] \setminus \{i^*\}\right) \cup \left(B_{\alpha_i}[(-i)^*] \setminus \{(-i)^*\}\right)} |\gamma_{i-1}(\theta) - \gamma_i(\theta)|.\end{aligned}$$

By Lemmas 2.20(c,d), and Definition 2.18(R.2) and Remark 2.19(R.2),

$$(\theta, \gamma_{i-1}(\theta)), (\theta, \gamma_i(\theta)) \in \mathcal{R}(\ell^*) \quad \text{for} \quad \theta \in B_{\alpha_i}[\ell^*] \setminus \{\ell^*\} \text{ and } \ell \in \{i, -i\}.$$

Hence, by Lemma 2.20(f),

$$\mathsf{d}_\infty(\gamma_{i-1}, \gamma_i) \le \operatorname{diam}(\mathcal{R}(i^*)) = \operatorname{diam}(\mathcal{R}((-i)^*)) \le 2^{-i}.$$

Since n_i is a strictly increasing sequence, for every $m \geq 0$,

$$\mathsf{d}_{\infty}(\gamma_{i+m}, \gamma_i) \leq \sum_{k=i+1}^{i+m} 2^{-k} < 2^{-(i+1)} \sum_{k=0}^{\infty} \tfrac{1}{2^k} = 2 \cdot 2^{-(i+1)},$$

and consequently $\{(\gamma_i, \mathbb{S}^1 \backslash Z_i^*)\}_{i=0}^{\infty}$ is a Cauchy sequence in $\mathfrak{C}$.

Lemma 2.22 allows us to define the following limit pseudo-curve generator of the sequence $\{(\gamma_i, \mathbb{S}^1 \backslash Z_i^*)\}_{i=0}^{\infty}$.

Definition 2.23. *There exists $(\gamma, \mathbb{S}^1 \backslash O^*(\omega)) \in \mathfrak{C}$ such that*

$$(\gamma, \mathbb{S}^1 \backslash O^*(\omega)) = \lim_{i \to \infty} (\gamma_i, \mathbb{S}^1 \backslash Z_i^*)$$

(that is, $\gamma(\theta) = \lim_{i \to \infty} \gamma_i(\theta)$ for every $\theta \in \mathbb{S}^1 \backslash O^*(\omega)$). Observe that

$$\mathbb{S}^1 \backslash O^*(\omega) = \bigcap_{i=1}^{\infty} (\mathbb{S}^1 \backslash Z_i^*)$$

is a residual set in $\mathbb{S}^1$. ◘

Now, we are ready to define the sequence of pseudo-curves associated to the sequence $\{(\gamma_i, \mathbb{S}^1 \backslash Z_i^*)\}_{i=0}^{\infty}$, and to the limit pseudo-curve generator $(\gamma, \mathbb{S}^1 \backslash O^*(\omega))$. This will finally define the pseudo-curve $\mathfrak{A}$ that we want to construct.

Definition 2.24. *We denote by*

$$\mathfrak{A}_j := \mathfrak{A}_{(\gamma_j, \mathbb{S}^1 \backslash Z_j^*)} = \overline{\operatorname{Graph}(\gamma_j, \mathbb{S}^1 \backslash Z_j^*)}$$

the pseudo-curve defined by $(\gamma_j, \mathbb{S}^1 \backslash Z_j^*) \in \mathfrak{C}$, and

$$\mathfrak{A} = \mathfrak{A}_{(\gamma, \mathbb{S}^1 \backslash O^*(\omega))} := \overline{\operatorname{Graph}(\gamma, \mathbb{S}^1 \backslash O^*(\omega))}.$$

By Definition 2.23 and Proposition 2.13, $\mathfrak{A} = \lim_{j \to \infty} \mathfrak{A}_{(\gamma_j, \mathbb{S}^1 \backslash Z_j^*)}$. ◘

The next lemmas study the properties the pseudo-curves $\mathfrak{A}_j$ and $\mathfrak{A}$.

Lemma 2.25. *The following statements hold for every $\ell \in \mathbb{Z}$:*

(a) $\mathfrak{A}_n^{\theta} \subset \mathcal{R}(\ell^*)^{\theta}$ for every $n \geq |\ell| - 1$ and $\theta \in B_{\alpha_{|\ell|}}[\ell^*]$.
(b) $\mathfrak{A}_n^{\ell^*} = \mathfrak{A}_{|\ell|}^{\ell^*} \subset \mathcal{R}(\ell^*)^{\ell^*}$ for every $n \geq |\ell|$. Moreover, $\mathfrak{A}_{|\ell|}^{\ell^*} = \mathcal{R}(\ell^*)^{\ell^*}$ is a non-degenerate interval.
(c) $\mathfrak{A}_{\ell}^{\theta} = \{(\theta, \gamma_{\ell}(\theta))\}$ for every $\theta \in \mathbb{S}^1 \backslash Z_{\ell}^*$.
(d) $\mathfrak{A}_{|\ell|} \subset \mathbb{S}^1 \times [-1, 1]$.

Proof. **(a)** By Lemma 2.20(c,d), $\operatorname{Graph}\left(\gamma_n\big|_{B_{\alpha_{|\ell|}}[\ell^*] \backslash Z_n^*}\right) \subset \mathcal{R}(\ell^*)$. Then, the statement follows from the compacity of $\mathcal{R}(\ell^*)$.

(b) From the definition of γ_i (Definition 2.18) and Definition 2.18(R.2), for every $n > |\ell|$ there exists an $\varepsilon(n) > 0$ such that $\gamma_n(\theta) = \gamma_{|\ell|}(\theta)$ for every $\theta \in B_{\varepsilon(n)}(\ell^*) \setminus \{\ell^*\}$. Hence $\mathfrak{A}_n^{\ell^*} = \mathfrak{A}_{|\ell|}^{\ell^*}$. Moreover, $\gamma_{|\ell|}$ coincides with φ_{ℓ^*} in a neighbourhood of ℓ^*. Thus, $\mathfrak{A}_{|\ell|}^{\ell^*} = \mathcal{R}(\ell^*)^{\ell^*}$ and it is an interval by Definition 2.15 and Remark 2.16(4).

Finally statement (c) follows from Lemma 2.4(a) and Definition 2.24, and (d) from Lemma 2.20(b).

Lemma 2.26. *The following statements hold.*

(a) $\mathfrak{A}^\theta \subset \mathcal{R}(\ell^*)^\theta$ *for every* $\ell \in \mathbb{Z}$ *and* $\theta \in B_{\alpha_{|\ell|}}[\ell^*]$.
(b) $\mathfrak{A}^{\ell^*} = \mathfrak{A}_{|\ell|}^{\ell^*}$ *for every* $\ell \in \mathbb{Z}$. *In particular* $\mathfrak{A}^{\ell^*}$ *is a non-degenerate interval.*
(c) If $\theta \notin O^*(\omega)$, *then* $\mathfrak{A}^\theta = \{(\theta, \gamma(\theta))\}$.
(d) $\mathfrak{A} \subset \mathbb{S}^1 \times [-1, 1]$.

Proof. Statement (c) follows directly from Lemma 2.4(a).

Now we prove (a). From Lemma 2.25(a), $\mathfrak{A}_n^\theta \subset \mathcal{R}(\ell^*)$ for every $\ell \in \mathbb{Z}$ and $n \geq |\ell|$. On the other hand, by Definition 2.23 and Proposition 2.13, $\mathfrak{A}^\theta = \lim_{n\to\infty} \mathfrak{A}_n^\theta$. Hence the result follows from the compacity of $\mathcal{R}(\ell^*)$.

By Lemma 2.25(b) and the part of the lemma already proved we have

$$\mathfrak{A}^{\ell^*} = \lim_{n\to\infty} \mathfrak{A}_n^{\ell^*} = \mathfrak{A}_{|\ell|}^{\ell^*}.$$

Statement (d) follows from Lemma 2.25(d), the compacity of $\mathbb{S}^1 \times [-1, 1]$ and the fact that $\mathfrak{A} = \lim_{j\to\infty} \mathfrak{A}_j$.

The next proposition, summarizes the main properties of the set $\mathfrak{A}$.

Proposition 2.27. *The set* $\mathfrak{A}$ *is a connected, does not contain any arc of curve and* $\Omega \setminus \mathfrak{A}$ *has two connected components.*

Proof. From statements (b) and (c) of the previous lemma, we know that $\mathfrak{A}^\theta$ is connected for every $\theta \in \mathbb{S}^1$.

If $\mathfrak{A}$ is not connected there exist closed (in $\mathfrak{A}$) sets U and V such that $U \cap V = \emptyset$ and $U \cup V = \mathfrak{A}$. Observe that $\pi(U) \cup \pi(V) = \pi(\mathfrak{A}) = \mathbb{S}^1$ because every pseudo-curve is a circular set. Moreover, since $\mathfrak{A}$ is compact, U and V are also compact sets of Ω. Hence, $\pi(U)$ are $\pi(V)$ compact in $\mathbb{S}^1$. Since $\mathbb{S}^1$ is connected, $\pi(U) \cap \pi(V) \neq \emptyset$. For every $\theta \in \pi(U) \cap \pi(V)$ we have,

$$\mathfrak{A}^\theta = (U \cup V)^\theta = U^\theta \cup V^\theta.$$

The sets U^θ and V^θ are closed, non-empty and disjoint. Consequently, $\mathfrak{A}^\theta$ is not connected; a contradiction. This proves that $\mathfrak{A}$ is connected.

By Lemma 2.26(b), $\mathfrak{A}^{\ell^*}$ is a non-degenerate interval for every $\ell \in O^*(\omega)$. Then, since $O^*(\omega)$ is dense in $\mathbb{S}^1$, $\mathfrak{A}$ does not contain any arc of curve by Lemma 2.5(b).

To prove that $\Omega \setminus \mathfrak{A}$ has two connected components we define

$$\Omega_- := \{(\theta, y) \in \Omega : y < \min\{x \in \mathbb{I} : (\theta, x) \in \mathfrak{A}\}\}, \text{ and}$$
$$\Omega_+ := \{(\theta, y) \in \Omega : y > \max\{x \in \mathbb{I} : (\theta, x) \in \mathfrak{A}\}\}.$$

By Lemma 2.26(d) we know that

$$-1 \leq \min\{x \in \mathbb{I} : (\theta, x) \in \mathfrak{A}\} \leq \max\{x \in \mathbb{I} : (\theta, x) \in \mathfrak{A}\} \leq 1.$$

Hence, $\Omega \backslash \mathfrak{A} = \Omega_- \cup \Omega_+$, Ω_+ and Ω_- are disjoint open circular subsets of Ω and $\Omega_- \supset \mathbb{S}^1 \times [-2, -1]$ and $\Omega_+ \supset \mathbb{S}^1 \times [1, 2]$ (in particular, for every $\theta \in \mathbb{S}^1$, Ω_+^θ and Ω_-^θ are non-degenerate intervals). Thus, Ω_+ and Ω_- are arc-wise connected and, hence, connected.

2.4 A collection of auxiliary functions G_i defined on the boxes $\mathcal{R}^{\smile}(i^*)$

In this section we define a family of auxiliary functions $G_i \colon \mathcal{R}(i^*) \longrightarrow \Omega$ with $i \in \mathbb{Z}$ and study their properties.

In what follows we consider the supremum metric d_∞ on the class of all functions $F \colon A \longrightarrow \Omega$ with $A \subset \Omega$. That is, given $F, G \colon A \longrightarrow \Omega$ we set

$$\mathsf{d}_\infty(F, G) := \sup_{(\theta,x) \in A} \mathsf{d}_\Omega(F(\theta, x), G(\theta, x)).$$

In the special case when F and G are skew products with the same base, that is when $F(\theta, x) = (R(\theta), f(\theta, x))$ and $G(\theta, x) = (R(\theta), g(\theta, x))$, then

$$\mathsf{d}_\infty(F, G) := \sup_{(\theta,x) \in A} |f(\theta, x) - g(\theta, x)|.$$

Observe that $(\mathcal{S}(\Omega), \mathsf{d}_\infty)$ is a complete metric space.

Before defining the maps G_i we need to introduce the necessary notation, and recall and collect some basic facts that we will use in this definition and to study their properties.

For every $i \in \mathbb{Z}$, we define

$$M_i \colon B_i^{\smile}[i^*] \longrightarrow \mathbb{I} \quad \text{by} \quad M_i(\theta) := \max\{x \in \mathbb{I} : (\theta, x) \in \mathcal{R}^{\smile}(i^*)\}, \text{ and}$$
$$m_i \colon B_i^{\smile}[i^*] \longrightarrow \mathbb{I} \quad \text{by} \quad m_i(\theta) := \min\{x \in \mathbb{I} : (\theta, x) \in \mathcal{R}^{\smile}(i^*)\}.$$

The next simple lemma states the basic properties of the maps m_i and M_i.

Lemma 2.28. *The following statements hold for every $i \in \mathbb{Z}$*

(a) $-1 \leq m_i(\theta) \leq M_i(\theta) \leq 1$ for every $\theta \in B_i^{\smile}[i^*]$.
(b) m_i and M_i are continuous.
(c) $m_i\big|_{B_{\alpha_{|i|}}[i^*]}$ and $M_i\big|_{B_{\alpha_{|i|}}[i^*]}$ are piecewise linear.
(d) $m_i(\theta) = M_i(\theta) = \gamma_{|i|}(\theta)$ if and only if $\theta \in B_i^{\smile}[i^*] \backslash B_{\alpha_{|i|}}(i^*)$.

Proof. It follows easily from Definition 2.15, the definition of a winged region and Lemma 2.20(b,f).

Notice that, for every $i \in \mathbb{Z}$,

$$\mathcal{R}^{\smallfrown}(i^*) = \bigcup_{\theta \in B_i^{\smallfrown}[i^*]} \mathcal{R}^{\smallfrown}(i^*)^\theta = \bigcup_{\theta \in B_i^{\smallfrown}[i^*]} \{\theta\} \times [m_i(\theta), M_i(\theta)].$$

In what follows the interval $[m_i(\theta), M_i(\theta)] \subset \mathbb{I}$, defined for every $\theta \in B_i^{\smallfrown}[i^*]$, will be denoted by $\mathbb{I}_{i,\theta}$. Clearly, for every $\theta \in B_i^{\smallfrown}[i^*]$, $\mathcal{R}^{\smallfrown}(i^*)^\theta = \{\theta\} \times \mathbb{I}_{i,\theta}$.

By Definition 2.18(R.2) and Remark 2.19(R.2),

$$B_i^{\smallfrown}[i^*] \setminus \{i^*\} \quad \text{is disjoint from} \quad Z^*_{|i|}.$$

Hence, Lemmas 2.20(a,d) and 2.25(c) can be summarized as:

$$\begin{cases} \gamma_{|\ell|}\big|_{B_\ell^{\smallfrown}[\ell^*]\setminus\{\ell^*\}} \quad \text{is continuous,} \\ \gamma_{|\ell|}(\theta) \in \mathbb{I}_{\ell,\theta} \quad \text{for every} \quad \theta \in B_\ell^{\smallfrown}[\ell^*] \setminus \{\ell^*\}, \text{ and} \\ \mathfrak{A}^\theta_{|\ell|} = \{(\theta, \gamma_{|\ell|}(\theta))\} \quad \text{for every} \quad \theta \in B_\ell^{\smallfrown}[\ell^*] \setminus \{\ell^*\} \end{cases} \tag{2.1}$$

for $\ell \in \{i, i+1\}$.

Now we define a family of continuous maps $G_i \colon \mathcal{R}^{\smallfrown}(i^*) \longrightarrow \Omega$ with $i \in \mathbb{Z}$, by

$$G_i(\theta, x) = \big(R_\omega(\theta), g_i(\theta, x)\big)$$

Also, for every $\theta \in B_i^{\smallfrown}[i^*]$, we will denote the map $g_i(\theta, \cdot) \colon \mathbb{I}_{i,\theta} \longrightarrow \mathbb{I}$ by $g_{i,\theta}$.

To define the functions $g_{i,\theta}$, for clarity, we will consider separately two different situations:

- $i \geq 0$, when $\mathcal{R}^{\smallfrown}(i^*) = \mathcal{R}(i^*)$, $B_i^{\smallfrown}[i^*] = B_{\alpha_{|i|}}[i^*]$ and $G_i(\mathcal{R}(i^*))$ strictly contains the smaller box $\mathcal{R}((i+1)^*)$, and
- $i \leq -1$, when $G_i(\mathcal{R}^{\smallfrown}(i^*))$ is strictly contained in the bigger box $\mathcal{R}((i+1)^*)$.

We start by defining $g_{i,\theta}$ for $i \geq 0$ in three different ways, depending on the base point $\theta \in B_{\alpha_i}[i^*]$. In this definition, for simplicity we will use $\mathcal{R}(i^*)$ instead of $\mathcal{R}^{\smallfrown}(i^*)$ and $B_{\alpha_{|i|}}[i^*]$ instead of $B_i^{\smallfrown}[i^*]$.

Notice that, by Definition 2.18(R.1) and Lemma 2.20(c),

$$\begin{aligned} &\text{for every } i \geq 0 \\ &B_{\delta_{i+1}}[i^*] \subset B_{\alpha_{i+1}}(i^*) \quad \text{and} \quad B_{\alpha_{i+1}}[i^*] \subset B_{\delta_i}(i^*) \subset B_{\alpha_i}(i^*), \text{ and} \\ &\gamma_{i-1}(i^*) = a_i \quad \text{and} \quad \gamma_i((i+1)^*) = a_{i+1}. \end{aligned} \tag{2.2}$$

Definition 2.29 (Definition of g_i for $i \geq 0$).

$\boldsymbol{\theta \in B_{\delta_{i+1}}[i^*]}$ $\qquad g_{i,\theta}(x) := \gamma_i((i+1)^*) + \frac{2^{n_i}}{2^{n_{i+1}}}\left(\gamma_{i-1}(i^*) - x\right).$

$\boldsymbol{\theta \in B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)}$	*we define* $g_{i,\theta}$ *to be the unique piecewise affine map with two affine pieces, defined on* $\mathbb{I}_{i,\theta}$*, whose graph joins* $\big(m_i(\theta), M_{i+1}\big(R_\omega(\theta)\big)\big)$ *with* $\big(\gamma_i(\theta), \gamma_{i+1}\big(R_\omega(\theta)\big)\big)$*, and this with the point* $\big(M_i(\theta), m_{i+1}\big(R_\omega(\theta)\big)\big)$ *(in particular,* $g_{i,\theta}\big(\gamma_i(\theta)\big) = \gamma_{i+1}\big(R_\omega(\theta)\big)$*),*
$\boldsymbol{\theta \in B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*)}$	$g_{i,\theta}(x) := \gamma_{i+1}\big(R_\omega(\theta)\big)$ *(that is,* $g_{i,\theta}$ *is constant).*

□

The next lemma states the basic properties of the functions G_i for $i \geq 0$.

Lemma 2.30. *The following statements hold for every* $i \geq 0$ *:*

(a) The map $g_{i,\theta}$ *is well defined and non-increasing for every* $\theta \in B_{\alpha_i}[i^*]$*. Moreover,* $-1 \leq g_{i,\theta}(x) \leq 1$ *for every* $\theta \in B_{\alpha_i}[i^*]$ *and* $x \in \mathbb{I}_{i,\theta}$*. Furthermore, the function* G_i *is continuous.*

(b) $G_i\big|_{\mathcal{R}(i^*)^\theta}$ *is affine and* $G_i\big(\mathcal{R}(i^*)^\theta\big) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ *for every* $\theta \in B_{\delta_{i+1}}[i^*]$*;* $G_i\big|_{\mathcal{R}(i^*)^\theta}$ *is piecewise affine with two pieces and* $G_i\big(\mathcal{R}(i^*)^\theta\big) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ *for every* $\theta \in B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)$*; and*
$G_i\big(\mathcal{R}(i^*)^\theta\big) = \mathfrak{A}_{i+1}^{R_\omega(\theta)}$ *for every* $\theta \in B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*)$*.*

(c) $G_i(\mathfrak{A}_i^\theta) = \mathfrak{A}_{i+1}^{R_\omega(\theta)}$ *for every* $\theta \in B_{\alpha_i}[i^*]$*.*

Proof. We will prove all statements of the lemma simultaneously and according to the regions in the definition of the map g_i.

- We start with the region $\mathcal{R}(i^*)^{\upuparrows B_{\delta_{i+1}}[i^*]}$.

Let $z \in [-\delta_i, \delta_i] \subset \mathbb{R}$ and let $\theta = i^* + z \in B_{\delta_i}[i^*]$. From Definition 2.15 and (2.2) we get

$$\begin{aligned} m_i(\theta) &= a_i - 2^{-n_i}(1-z) = \gamma_{i-1}(i^*) - 2^{-n_i}(1-z), \text{ and} \\ M_i(\theta) &= a_i + 2^{-n_i}(1-z) = \gamma_{i-1}(i^*) + 2^{-n_i}(1-z). \end{aligned} \tag{2.3}$$

In a similar way, for every $\theta \in B_{\delta_{i+1}}[i^*]$ (that is, $z \in [-\delta_{i+1}, \delta_{i+1}]$), we have $R_\omega(\theta) = (i+1)^* + z \in B_{\delta_{i+1}}[(i+1)^*]$, and

$$\begin{aligned} m_{i+1}(R_\omega(\theta)) &= a_{i+1} - 2^{-n_{i+1}}(1-z) = \gamma_i((i+1)^*) - 2^{-n_{i+1}}(1-z), \text{ and} \\ M_{i+1}(R_\omega(\theta)) &= a_{i+1} + 2^{-n_{i+1}}(1-z) = \gamma_i((i+1)^*) + 2^{-n_{i+1}}(1-z). \end{aligned} \tag{2.4}$$

Hence, for every $\theta \in B_{\delta_{i+1}}[i^*]$,

$$\begin{aligned} g_{i,\theta}(m_i(\theta)) &= \gamma_i((i+1)^*) + \tfrac{2^{n_i}}{2^{n_{i+1}}} 2^{-n_i}(1-z) = \gamma_i((i+1)^*) + 2^{-n_{i+1}}(1-z) \\ &= M_{i+1}(R_\omega(\theta)), \\ g_{i,\theta}(M_i(\theta)) &= \gamma_i((i+1)^*) - \tfrac{2^{n_i}}{2^{n_{i+1}}} 2^{-n_i}(1-z) = \gamma_i((i+1)^*) - 2^{-n_{i+1}}(1-z) \\ &= m_{i+1}(R_\omega(\theta)). \end{aligned} \tag{2.5}$$

So, $g_{i,\theta}\big|_{\mathbb{I}_{i,\theta}}$ is the affine map whose graph joins the point

$$\big(m_i(\theta), M_{i+1}(R_\omega(\theta))\big) \text{ with } \big(M_i(\theta), m_{i+1}(R_\omega(\theta))\big).$$

In particular, $g_{i,\theta}$ sends the interval $\mathbb{I}_{i,\theta}$ affinely onto $\mathbb{I}_{i+1,R_\omega(\theta)}$ or, equivalently, G_i sends the interval $\mathcal{R}(i^*)^\theta$ affinely onto $\mathcal{R}((i+1)^*)^{R_\omega(\theta)}$. Then, by Lemma 2.20(b), this implies that $-1 \leq g_{i,\theta}(x) \leq 1$ for every $x \in \mathbb{I}_{i,\theta}$. Moreover, the continuity of the maps $m_i,\ M_i,\ m_{i+1} \circ R_\omega$ and $M_{i+1} \circ R_\omega$ imply that g_i is well defined and continuous on $\mathcal{R}(i^*)^{\upuparrows B_{\delta_{i+1}}[i^*]}$

Next we will prove that $G_i\big(\mathfrak{A}_i^\theta\big) = \mathfrak{A}_{i+1}^{R_\omega(\theta)}$ for every $\theta \in B_{\delta_{i+1}}[i^*]$. We take $\theta = i^* + z \in B_{\delta_{i+1}}[i^*] \setminus \{i^*\}$. Then, clearly, $z \in [-\delta_{i+1}, \delta_{i+1}] \setminus \{0\} \subset \mathbb{R}$. By Definitions 2.18 and 2.15 and statement (2.2),

$$\begin{aligned}
\gamma_i(\theta) = \varphi_{i^*}(\theta) = a_i + 2^{-n_i} d = \gamma_{i-1}(i^*) + 2^{-n_i} d \in \mathbb{I}_{i,\theta}, \text{ and}\\
\gamma_{i+1}(R_\omega(\theta)) = \varphi_{(i+1)^*}(\theta) = a_{i+1} - 2^{-n_{i+1}} d = \gamma_{i-1}(i^*) - 2^{-n_{i+1}} d \in \mathbb{I}_{i+1,R_\omega(\theta)},
\end{aligned}$$

where $d = (-1)^i \phi(z)$. So, for every $\theta \in B_{\delta_{i+1}}[i^*] \setminus \{i^*\}$,

$$g_{i,\theta}(\gamma_i(\theta)) = \gamma_i((i+1)^*) - \frac{2^{n_i}}{2^{n_{i+1}}} 2^{-n_i} d = \gamma_{i+1}(R_\omega(\theta)). \tag{2.6}$$

Thus, from (2.2) and (2.1) we get

$$\begin{aligned}
G_i\big(\mathfrak{A}_i^\theta\big) &= G_i\big(\{(\theta, \gamma_i(\theta))\}\big) = \{(R_\omega(\theta), g_{i,\theta}(\gamma_i(\theta)))\}\\
&= \{(R_\omega(\theta), \gamma_{i+1}(R_\omega(\theta)))\} = \mathfrak{A}_{i+1}^{R_\omega(\theta)}
\end{aligned}$$

for every $\theta \in B_{\delta_{i+1}}[i^*] \setminus \{i^*\}$. On the other hand, by the part already proven, g_{i,i^*} sends the interval $\mathbb{I}_{i,i^*}$ affinely to $\mathbb{I}_{i+1,(i+1)^*}$ or, equivalently, G_i sends the interval $\mathcal{R}(i^*)^{i^*} = \{i^*\} \times \mathbb{I}_{i,i^*}$ affinely onto $\mathcal{R}((i+1)^*)^{(i+1)^*} = \{(i+1)^*\} \times \mathbb{I}_{i,(i+1)^*}$. This implies that $G_i\big(\mathfrak{A}_i^{i^*}\big) = \mathfrak{A}_{i+1}^{(i+1)^*}$ by Lemma 2.25(b). Hence, $G_i\big(\mathfrak{A}_i^\theta\big) = \mathfrak{A}_{i+1}^{R_\omega(\theta)}$ for every $\theta \in B_{\delta_{i+1}}[i^*]$.

- Now we study $\mathcal{R}(i^*)^{\upuparrows\left(B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)\right)}$.

Observe that $R_\omega(B_\alpha[i^*] \setminus \{i^*\}) = B_\alpha[(i+1)^*] \setminus \{(i+1)^*\}$ for $\alpha \in \{\alpha_i, \alpha_{i+1}\}$. Then, by (2.1)

$$\begin{aligned}
&\gamma_{i+1} \circ R_\omega\big|_{B_{\alpha_i}[i^*] \setminus \{i^*\}} \quad \text{is continuous, and}\\
&\gamma_{i+1}(R_\omega(\theta)) \in \mathbb{I}_{i+1,R_\omega(\theta)} \quad \text{for every} \quad \theta \in B_{\alpha_{i+1}}[i^*] \setminus \{i^*\}.
\end{aligned} \tag{2.7}$$

So, the continuity of the maps $m_i,\ M_i,\ m_{i+1} \circ R_\omega$ and $M_{i+1} \circ R_\omega$ imply that g_i is well defined and continuous on $\mathcal{R}(i^*)^{\upuparrows\left(B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)\right)}$, and

$$\big(\gamma_i(\theta), \gamma_{i+1}(R_\omega(\theta))\big) \in \mathbb{I}_{i,\theta} \times \mathbb{I}_{i+1,R_\omega(\theta)}$$

for every $\theta \in B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)$. Consequently, $g_{i,\theta}$ maps $\mathbb{I}_{i,\theta}$ piecewise affinely with two pieces onto $\mathbb{I}_{i+1,R_\omega(\theta)}$ or, equivalently, G_i sends the interval $\mathcal{R}(i^*)^\theta$ piecewise affinely with two pieces onto $\mathcal{R}((i+1)^*)^{R_\omega(\theta)}$. Again, by Lemma 2.20(b), this implies that $-1 \leq g_{i,\theta}(x) \leq 1$ for every $x \in \mathbb{I}_{i,\theta}$. On the other hand, from (2.2) and (2.1) we have

$$G_i\big(\mathfrak{A}_i^{\theta}\big) = G_i\big(\{(\theta, \gamma_{_i}(\theta))\}\big) = \{(R_\omega(\theta), g_{_{i,\theta}}(\gamma_{_i}(\theta)))\}$$
$$= \{(R_\omega(\theta), \gamma_{_{i+1}}(R_\omega(\theta)))\} = \mathfrak{A}_{i+1}^{R_\omega(\theta)}$$

for every $\theta \in B_{\alpha_{i+1}}[i^*] \setminus B_{\delta_{i+1}}(i^*)$.

- Finally, we study the region $\mathcal{R}(i^*)^{\upuparrows\left(B_{\alpha_i}[i^*]\setminus B_{\alpha_{i+1}}(i^*)\right)}$.

In this case, by definition and Lemma 2.20(b) we have $-1 \leq g_{_{i,\theta}}(x) \leq 1$ for every $x \in \mathbb{I}_{i,\theta}$. By (2.7), $g_i(\cdot, x) = \gamma_{_{i+1}} \circ R_\omega$ is well defined and continuous in both variables on $\mathcal{R}(i^*)^{\upuparrows\left(B_{\alpha_i}[i^*]\setminus B_{\alpha_{i+1}}(i^*)\right)}$ because m_i and M_i are continuous. Moreover, for every $\theta \in B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*)$ and x such that $(\theta, x) \in \mathcal{R}(i^*)^{\theta}$, we have

$$\{G_i(\theta, x)\} = \{(R_\omega(\theta), g_i(\theta, x))\} = \{(R_\omega(\theta), \gamma_{_{i+1}}(R_\omega(\theta))\} = \mathfrak{A}_{i+1}^{R_\omega(\theta)}$$

by Definition 2.24 and Lemma 2.4(a). Thus, by Lemma 2.25(a),

$$G_i\big(\mathfrak{A}_i^{\theta}\big) = G_i\big(\mathcal{R}(i^*)^{\theta}\big) = \mathfrak{A}_{i+1}^{R_\omega(\theta)}.$$

From all the previous arguments (b) and (c) follow. To end the proof of (a) we have to see that G_i is well defined and globally continuous. This amounts to show that it is well defined on the fibres

$$\mathcal{R}(i^*)^{(i^*\pm\delta_{i+1})} = \{i^* \pm \delta_{i+1}\} \times \mathbb{I}_{i,i^*\pm\delta_{i+1}} \text{ and}$$
$$\mathcal{R}(i^*)^{(i^*\pm\alpha_{i+1})} = \{i^* \pm \alpha_{i+1}\} \times \mathbb{I}_{i,i^*\pm\alpha_{i+1}}.$$

We will only show that the two definitions of g_i coincide on $\{\theta\} \times \mathbb{I}_{i,\theta}$ with $\theta \in \{i^* + \delta_{i+1}, i^* + \alpha_{i+1}\}$. The case $\theta \in \{i^* - \delta_{i+1}, i^* - \alpha_{i+1}\}$ follows analogously.

We start with $\theta = i^* + \alpha_{i+1} \in B_{\delta_i}(i^*)$. In this case, $R_\omega(\theta) = (i+1)^* + \alpha_{i+1} \in \operatorname{Bd}(B_{\alpha_{i+1}}[(i+1)^*])$ and, by Definition 2.15 and Lemma 2.20(c),

$$M_{i+1}(R_\omega(\theta)) = m_{i+1}(R_\omega(\theta)) = a_{i+1}^{+} = \gamma_{_{i+1}}(R_\omega(\theta)).$$

Thus, the piecewise affine map whose graph joins the points

$$\big(m_i(\theta), M_{i+1}(R_\omega(\theta))\big), \big(\gamma_{_i}(\theta), \gamma_{_{i+1}}(R_\omega(\theta))\big), \text{ and } \big(M_i(\theta), m_{i+1}(R_\omega(\theta))\big)$$

is the constant map $\gamma_{_{i+1}}(R_\omega(\theta))$. Hence, $g_{_{i,\theta}}$ is well defined for $\theta = i^* + \alpha_{i+1}$.

Now we deal with the case $\theta = i^* + \delta_{i+1} \in B_{\delta_i}[i^*]$. By (2.5) and (2.6) we know that the points $\big(m_i(\theta), M_{i+1}(R_\omega(\theta))\big)$, $\big(\gamma_{_i}(\theta), \gamma_{_{i+1}}(R_\omega(\theta))\big)$ and $\big(M_i(\theta), m_{i+1}(R_\omega(\theta))\big)$ belong to

$$\operatorname{Graph}\left(x \mapsto \gamma_{_i}((i+1)^*) + \frac{2^{n_i}}{2^{n_{i+1}}}\big(\gamma_{_{i-1}}(i^*) - x\big)\right).$$

Consequently, the map $\gamma_{_i}((i+1)^*) + \frac{2^{n_i}}{2^{n_{i+1}}}\big(\gamma_{_{i-1}}(i^*) - x\big)$ coincides with the piecewise affine map whose graph joins $\big(m_i(\theta), M_{i+1}(R_\omega(\theta))\big)$, $\big(\gamma_{_i}(\theta), \gamma_{_{i+1}}(R_\omega(\theta))\big)$ and $\big(M_i(\theta), m_{i+1}(R_\omega(\theta))\big)$. This ends the proof of (a).

Now we define $g_{i,\theta}$ for $i < 0$. In this case, since we are going from a smaller box $\mathcal{R}^{\smile}(i^*)$ to a bigger one, we only need to define $g_{i,\theta}$ in two different ways, depending on the base point $\theta \in B_i^{\smile}[i^*]$.

As in the previous case we need to fix some facts about the elements that we will use in the definition.

By Definition 2.18(R.1) and Lemma 2.20(c),

$$\begin{aligned}
&\text{for every } i < 0 \\
&B_{\delta_{|i|}}\left[(i+1)^*\right] \subset\ B_{\alpha_{|i|}}\left[(i+1)^*\right] \subset B_{\delta_{|i+1|}}\left((i+1)^*\right) \subset B_{\alpha_{|i+1|}}\left((i+1)^*\right), \\
&R_\omega\left(B_i^{\smile}\left[i^*\right]\right) = B_{\alpha_{|i+1|}}\left[(i+1)^*\right], \quad B_{\delta_{|i|}}\left[i^*\right] \subset B_{\alpha_{|i|}}\left(i^*\right), \text{ and} \\
&\gamma_{|i+1|}(i^*) = a_i \quad \text{and} \quad \gamma_{|i+2|}((i+1)^*) = a_{i+1}.
\end{aligned} \tag{2.8}$$

Consequently, from (2.1) and Definitions 2.15 and 2.18 we get

$$\begin{aligned}
&m_i(\theta) < \gamma_{|i|}(\theta) < M_i(\theta) \text{ and} \\
&m_{i+1}(R_\omega(\theta)) < \gamma_{|i+1|}(R_\omega(\theta)) < M_{i+1}(R_\omega(\theta))
\end{aligned}$$

for every $\theta \in B_{\alpha_{|i|}}(i^*) \setminus \{i^*\}$ (and $R_\omega(\theta) \in B_{\alpha_{|i|}}((i+1)^*) \setminus \{(i+1)^*\}$). Then,

$$\widetilde{\kappa}_i(\theta) = \min\left\{1, \frac{m_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}(\gamma_{|i|}(\theta) - M_i(\theta))}, \frac{M_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}(\gamma_{|i|}(\theta) - m_i(\theta))}\right\} > 0$$

defines a continuous function $\widetilde{\kappa}_i \colon B_{\alpha_{|i|}}(i^*) \setminus B_{\delta_{|i|}}(i^*) \longrightarrow (0,1]$. To define the map g_i we need an auxiliary function

$$\kappa_i \colon B_{\alpha_{|i|}}[i^*] \setminus B_{\delta_{|i|}}(i^*) \longrightarrow [0,1]$$

such that κ_i is non-decreasing and continuous, $\kappa_i(i^* \pm \delta_{|i|}) = \widetilde{\kappa}_i(i^* \pm \delta_{|i|})$, and $\kappa_i(\theta) \le \widetilde{\kappa}_i(\theta)$ for every $\theta \in B_{\alpha_{|i|}}(i^*) \setminus B_{\delta_{|i|}}(i^*)$. In principle any such function would do, but for definiteness, and to show that such function exists, we note that we can take, for instance,

$$\kappa_i(\theta) = \begin{cases} \inf_{t \in [\theta, i^* - \delta_{|i|}] \cap B_{\alpha_{|i|}}(i^*)} \widetilde{\kappa}_i(t) & \text{if } \theta \le i^* - \delta_{|i|}, \\ \inf_{t \in [i^* + \delta_{|i|}, \theta] \cap B_{\alpha_{|i|}}(i^*)} \widetilde{\kappa}_i(t) & \text{if } \theta \ge i^* + \delta_{|i|}. \end{cases}$$

It is easy to check that this map verifies the desired properties.

Definition 2.31 (Definition of g_i for $i < 0$). *For every $(\theta, x) \in \mathcal{R}^{\smile}(i^*)$ we set*

$$g_{i,\theta}(x) := \begin{cases} \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\left(\gamma_{|i+1|}(i^*) - x\right) + \gamma_{|i+2|}((i+1)^*) & \textit{if } \theta \in B_{\delta_{|i|}}[i^*], \\ \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\kappa_i(\theta)\left(\gamma_{|i|}(\theta) - x\right) + \gamma_{|i+1|}(R_\omega(\theta)) & \textit{if } \theta \in B_{\alpha_{|i|}}[i^*] \setminus B_{\delta_{|i|}}(i^*) \\ \gamma_{|i+1|}(R_\omega(\theta)) & \textit{if } \theta \in B_i^{\smile}[i^*] \setminus B_{\alpha_{|i|}}(i^*). \end{cases}$$

∎

The next lemma states the basic properties of the functions G_i for $i < 0$.

Lemma 2.32. *The following statements hold for every $i < 0$:*

(a) The map $g_{i,\theta}$ is well defined and non-increasing for every $\theta \in B_{\alpha_i}[i^*]$. Moreover, $-1 \leq g_{i,\theta}(x) \leq 1$ for every $\theta \in B_{\alpha_i}[i^*]$ and $x \in \mathbb{I}_{i,\theta}$. Furthermore, the function G_i is continuous.

(b) $G_i\big|_{\mathcal{R}^{\frown}(i^*)^\theta}$ is affine, $G_i(\mathcal{R}^{\frown}(i^*)^\theta) \subset \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ for every $\theta \in B_i^{\frown}[i^*]$ and $G_i(\mathcal{R}^{\frown}(i^*)^\theta) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ for every $\theta \in B_{\delta_{|i|}}[i^*]$.

(c) $G_i(\mathfrak{A}^\theta_{|i|}) = \mathfrak{A}^{R_\omega(\theta)}_{|i+1|}$ for every $\theta \in B_i^{\frown}[i^*]$.

Proof. First we will prove that the map G_i is continuous and that $G_i\big|_{\mathcal{R}^{\frown}(i^*)^\theta}$ is affine, according to the three regions in the definition.

- As in the previous lemma we start with $\mathcal{R}^{\frown}(i^*)^{\upuparrows B_{\delta_{|i|}}[i^*]} = \mathcal{R}(i^*)^{\upuparrows B_{\delta_{|i|}}[i^*]}$.

As in the same case of Lemma 2.30, by using (2.8) instead of (2.2), it follows that $g_{i,\theta}\big|_{\mathbb{I}_{i,\theta}}$ is the affine map whose graph joins the points $\big(m_i(\theta), M_{i+1}(R_\omega(\theta))\big)$ and $\big(M_i(\theta), m_{i+1}(R_\omega(\theta))\big)$, g_i is well defined and continuous on $\mathcal{R}(i^*)^{\upuparrows B_{\delta_{|i|}}[i^*]}$,

$$g_{i,\theta}(\gamma_{|i|}(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)) \text{ for every } \theta \in B_{\delta_{|i|}}[i^*] \setminus \{i^*\},$$
G_i sends the interval $\mathcal{R}(i^*)^\theta$ affinely onto $\mathcal{R}((i+1)^*)^\theta$, and
$$G_i(\mathfrak{A}^\theta_{|i|}) = \mathfrak{A}^{R_\omega(\theta)}_{|i+1|} \quad \text{for every } \theta \in B_{\delta_{|i|}}[i^*].$$

- $\mathcal{R}^{\frown}(i^*)^{\upuparrows\left(B_{\alpha_{|i|}}[i^*]\setminus B_{\delta_{|i|}}(i^*)\right)} = \mathcal{R}(i^*)^{\upuparrows\left(B_{\alpha_{|i|}}[i^*]\setminus B_{\delta_{|i|}}(i^*)\right)}$.

From (2.1) we know that the maps $\gamma_{|i|}$ and $\gamma_{|i+1|} \circ R_\omega$ are continuous on the domain $B_{\alpha_{|i|}}[i^*] \setminus B_{\delta_{|i|}}(i^*)$. Hence, the continuity of g_i follows from the continuity of the maps κ_i, m_i, M_i, $m_{i+1} \circ R_\omega$ and $M_{i+1} \circ R_\omega$.

Notice that, from the definition of g_i in this region we clearly have that

$$g_{i,\theta}(\gamma_{|i|}(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)), \text{ and}$$

$G_i\big|_{\mathcal{R}^{\frown}(i^*)^\theta} = g_i(\theta, \cdot)$ is affine.

- $\mathcal{R}^{\frown}(i^*)^{\upuparrows\left(B_i^{\frown}[i^*]\setminus B_{\alpha_{|i|}}(i^*)\right)}$.

In this case we have $m_i(\theta) = \gamma_{|i|}(\theta) = M_i(\theta)$ by definition. Then, the map $G_i\big|_{\mathcal{R}^{\frown}(i^*)^\theta} = g_i(\theta, \cdot)$ is affine because it is constant, and g_i is continuous because $\gamma_{|i|}$ and $\gamma_{|i+1|} \circ R_\omega$ are continuous on the domain $B_i^{\frown}[i^*] \setminus \{i^*\}$ by (2.1).

To end the proof of (a) we have to see that G_i is well defined and globally continuous. This amounts to show that it is well defined on the fibres

$$\mathcal{R}(i^*)^{(i^* \pm \delta_{|i|})} \quad \text{and} \quad \mathcal{R}(i^*)^{(i^* \pm \alpha_{|i|})}$$

We start by showing that the two definitions of g_i coincide on the fibres $\mathcal{R}(i^*)^\theta$ for $\theta \in \{i^* \pm \alpha_{|i|}\}$. In this case we have $m_i(\theta) = \gamma_{|i|}(\theta) = M_i(\theta)$. Consequently, $\mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\}$ and

$$\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\kappa_i(\theta)\left(\gamma_{|i|}(\theta) - x\right) + \gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{|i+1|}(R_\omega(\theta))$$

for $x \in \mathbb{I}_{i,\theta}$.

Next we consider $\mathcal{R}(i^*)^\theta = \{\theta\} \times \mathbb{I}_{i,\theta}$ with $\theta = i^* + \delta_{|i|}$. We will show that the two definitions of g_i coincide on this set. The case $\theta = i^* - \delta_{|i|}$ follows analogously.

For simplicity we will denote

$$\begin{aligned} g_{i,\theta}^{\delta_{|i|}}(x) &:= \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\left(\gamma_{|i+1|}(i^*) - x\right) + \gamma_{|i+2|}((i+1)^*), \text{ and} \\ \xi_{i,\theta}(x) &:= \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\left(\gamma_{|i|}(\theta) - x\right) + \gamma_{|i+1|}(R_\omega(\theta)). \end{aligned}$$

Notice that $g_{i,\theta}^{\delta_{|i|}}$ is the map $g_{i,\theta}$ as defined in the first region while

$$\kappa_i(\theta)\left(\xi_{i,\theta} - \gamma_{|i+1|}(R_\omega(\theta))\right) + \gamma_{|i+1|}(R_\omega(\theta))$$

is the map $g_{i,\theta}$ as defined in the second region. In a similar way to the previous lemma we have that $\left(\gamma_{|i|}(\theta), \gamma_{|i+1|}(R_\omega(\theta))\right) \in \operatorname{Graph}\left(g_{i,\theta}^{\delta_{|i|}}\right)$. Hence, since $g_{i,\theta}^{\delta_{|i|}}$ is affine with slope $-\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}$, it follows that $g_{i,\theta}^{\delta_{|i|}} = \xi_{i,\theta}$. So, to end the proof of the lemma, we only have to see that $\kappa_i(i^* + \delta_{|i|}) = \widetilde{\kappa}_i(i^* + \delta_{|i|}) = 1$.

Since the points $\left(m_i(\theta), M_{i+1}(R_\omega(\theta))\right)$ and $\left(M_i(\theta), m_{i+1}(R_\omega(\theta))\right)$ also belong to $\operatorname{Graph}\left(g_{i,\theta}^{\delta_{|i|}}\right) = \operatorname{Graph}\left(\xi_{i,\theta}\right)$, it follows that

$$\begin{aligned} m_{i+1}(R_\omega(\theta)) &= \xi_{i,\theta}(M_i(\theta)) = \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\left(\gamma_{|i|}(\theta) - M_i(\theta)\right) + \gamma_{|i+1|}(R_\omega(\theta)), \text{ and} \\ M_{i+1}(R_\omega(\theta)) &= \xi_{i,\theta}(m_i(\theta)) = \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\left(\gamma_{|i|}(\theta) - m_i(\theta)\right) + \gamma_{|i+1|}(R_\omega(\theta)). \end{aligned}$$

This shows that $\widetilde{\kappa}_i(i^* + \delta_{|i|}) = \widetilde{\kappa}_i(\theta) = 1$ and ends the proof of (a).

Now we prove (b) according to the three regions in the definition. From the part of the lemma already proven we already know that $G_i\big|_{\mathcal{R}^{\frown}(i^*)^\theta}$ is affine, and $G_i\left(\mathcal{R}^{\frown}(i^*)^\theta\right) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ for every $\theta \in B_{\delta_{|i|}}[i^*]$. So, to end the proof of (b) we have to see that

$$g_{i,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{i+1,R_\omega(\theta)} \tag{2.9}$$

for every $\theta \in B_i^{\frown}[i^*] \setminus B_{\delta_{|i|}}[i^*]$ (by definition, since $i < 0$, $B_i^{\frown}[i^*] = B_{\alpha_{|i+1|}}[i^*]$; therefore, $R_\omega(\theta) \in B_{\alpha_{|i+1|}}[(i+1)^*]$ and $\mathbb{I}_{i+1,R_\omega(\theta)} = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$).

For $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$, by (2.1), we have

$$g_{i,\theta}(\mathbb{I}_{i,\theta}) = \{\gamma_{|i+1|}(R_\omega(\theta))\} \subset \mathbb{I}_{i+1,R_\omega(\theta)}.$$

Now we consider $\theta \in B_{\alpha_{|i|}}(i^*) \setminus B_{\delta_{|i|}}[i^*]$. Since

$$\kappa_i(\theta) \le \widetilde{\kappa}_i(\theta) \le \frac{M_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\left(\gamma_{|i|}(\theta) - m_i(\theta)\right)},$$

we have

$$\begin{aligned} g_{i,\theta}(m_i(\theta)) &\le \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} \frac{M_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}\left(\gamma_{|i|}(\theta) - m_i(\theta)\right)} \big(\gamma_{|i|}(\theta) - m_i(\theta)\big) + \gamma_{|i+1|}(R_\omega(\theta)) \\ &= M_{i+1}(R_\omega(\theta)). \end{aligned}$$

An analogous computation shows that $g_{i,\theta}(M_i(\theta)) \ge m_{i+1}(R_\omega(\theta))$. Hence, (2.9) holds because $g_{i,\theta}$ is affine. This ends the proof of (b).

Then, by Lemma 2.20(b), Statement (b) of the lemma implies that $-1 \le g_{i,\theta}(x) \le 1$ for every $x \in \mathbb{I}_{i,\theta}$.

By the part of the lemma already proved we know that $G_i\big(\mathfrak{A}^{\theta}_{|i|}\big) = \mathfrak{A}^{R_\omega(\theta)}_{|i+1|}$ for every $\theta \in B_{\delta_{|i|}}[i^*]$. On the other hand, as in the previous lemma, from (2.8) and (2.1) we get

$$\begin{aligned} G_i\big(\mathfrak{A}^{\theta}_{|i|}\big) &= G_i\big(\{(\theta, \gamma_{|i|}(\theta))\}\big) = \{(R_\omega(\theta), g_{i,\theta}(\gamma_{|i|}(\theta)))\} \\ &= \{(R_\omega(\theta), \gamma_{|i+1|}(R_\omega(\theta)))\} = \mathfrak{A}^{R_\omega(\theta)}_{|i+1|} \end{aligned}$$

for every $\theta \in B_i^{\frown}[i^*] \setminus B_{\delta_{|i|}}[i^*]$. So, (c) holds.

Up to now we have defined the family of auxiliary functions $G_i \colon \mathcal{R}^{\frown}(i^*) \longrightarrow \Omega$ with $i \in \mathbb{Z}$. The next step before being able to define the family $\{T_m\} \subset \mathcal{S}(\Omega)$ is to fix some stratification in the set of boxes $\mathcal{R}^{\frown}(i^*)$.

2.5 A stratification in the set of boxes $\mathcal{R}^{\frown}(i^*)$

In this section we introduce a notion of *depth* in the set of arcs $B_i^{\frown}[i^*]$ defined earlier. This notion introduce a stratification in the set of boxes $\mathcal{R}^{\frown}(i^*)$ that we study below.

Definition 2.33. *For every $\ell \in \mathbb{Z}$ we define the* depth of ℓ, *which will be denoted by* $\mathsf{depth}(\ell)$, *as the cardinality of the set (see Lemma 2.20(g))*

$$\begin{aligned} &\{i \in \mathbb{Z} : B_\ell^{\frown}[\ell^*] \subsetneq B_i^{\frown}[i^*]\} = \{i \in \mathbb{Z} : B_\ell^{\frown}[\ell^*] \cap B_i^{\frown}[i^*] \neq \emptyset\} = \\ &\{i \in \mathbb{Z} : \mathcal{R}^{\frown}(\ell^*) \subsetneq \mathcal{R}^{\frown}(i^*)\} = \{i \in \mathbb{Z} : \mathcal{R}^{\frown}(\ell^*) \cap \mathcal{R}^{\frown}(i^*) \neq \emptyset\}. \end{aligned}$$

Also, for every $m \in \mathbb{Z}^+$, we denote

$$\begin{aligned} \mathfrak{D}_m &:= \{\ell \in \mathbb{Z} : \ \mathsf{depth}(\ell) = m\}, \\ \mathfrak{D}^*_m &:= \{i^* : i \in \mathfrak{D}_m\}, \text{ and} \\ \mu_m &:= \min\{|i| : i \in \mathfrak{D}_m\}. \end{aligned}$$

◘

The next lemma studies the stratification on $\mathbb{Z}$ created by the notion of *depth*.

Lemma 2.34. *The following statements hold:*

(a) $\mathfrak{D}_{m+1} \subset \{\ell \in \mathbb{Z} : \exists\, i \in \mathfrak{D}_m$ *such that* $B_\ell^{\frown}[\ell^*] \subsetneq B_i^{\frown}[i^*]\}$.

(b) For every $\ell, k \in \mathfrak{D}_m$ *it follows that* $B_\ell^{\frown}[\ell^*] \cap B_k^{\frown}[k^*] = \emptyset$.

Proof. Observe that if $B_\ell^{\frown}[\ell^*] \subsetneq B_i^{\frown}[i^*]$ then $\mathsf{depth}(\ell) \geq \mathsf{depth}(i) + 1$. Hence, (a) holds.

Statement (b) follows from Lemma 2.20(g).

In what follows, for every $m \in \mathbb{Z}^+$ we set

$$\mathbb{B}_m^{\frown} := \bigcup_{i \in \mathfrak{D}_m} B_i^{\frown}[i^*] \supset \mathfrak{D}_m^*.$$

Note that, by Lemma 2.34(b), $\mathbb{B}_m^{\frown}$ is a disjoint union of closed arcs. Therefore, for every $\theta \in \mathbb{B}_m^{\frown}$, there exists a unique $i \in \mathfrak{D}_m$ such that $\theta \in B_i^{\frown}[i^*]$. We will denote such integer i by $\mathsf{b}^{\frown}(\theta, m) \in \mathfrak{D}_m$.

The next two lemmas study the properties of the winged boxes $B_i^{\frown}[i^*]$ and $\mathcal{R}^{\frown}(i^*)$ according to the depth stratification. Lemma 2.36 is the real motivation to introduce the winged boxes.

Lemma 2.35. *The following statements hold:*

(a) The sequence $\{\mu_m\}_{m=0}^{\infty}$ *is strictly increasing. In particular* $\lim_{m\to\infty} \mu_m = \infty$.

(b) For every $m \in \mathbb{Z}^+$, $\mathbb{B}_m^{\frown}$ *is dense in* $\mathbb{S}^1$, $\mathbb{B}_{m+1}^{\frown} \subset \mathbb{B}_m^{\frown}$ *and* $\mathfrak{D}_m^* \cap \mathbb{B}_{m+1}^{\frown} = \emptyset$.

(c) $O^*(\omega) \subset \mathbb{B}_0^{\frown}$, *and* $\mathfrak{A}^\theta = \{(\theta, 0)\}$ *for every* $\theta \in \mathbb{S}^1 \backslash \mathbb{B}_0^{\frown}$.

(d) Let $i \in \mathbb{Z}$ *and* $\theta \in B_i^{\frown}[i^*] \backslash \mathbb{B}_{\mathsf{depth}(i)+1}^{\frown}$. *Then,* $\theta \notin O^*(\omega)$ *unless* $\theta = i^*$, *and* $\mathfrak{A}_n^\theta = \mathfrak{A}_{|i|}^\theta$ *for every* $n \geq |i|$. *In particular* $\mathfrak{A}^\theta = \mathfrak{A}_{|i|}^\theta$.

Proof. By Lemmas 2.34(a) and 2.20(g) it follows that for every $m \in \mathbb{Z}+$ and $\ell \in \mathfrak{D}_{m+1}$ there exists $i \in \mathfrak{D}_m$ such that $B_\ell^{\frown}[\ell^*] \subsetneq B_i^{\frown}[i^*]$ and $|i| < |\ell|$. Thus, $\mathbb{B}_{m+1}^{\frown} \subset \mathbb{B}_m^{\frown}$ and $\mu_m < \mu_{m+1}$. This proves (a) and the second statement of (b).

Next we will show that $i^* \notin \mathbb{B}_{m+1}^{\frown}$ for every $i \in \mathfrak{D}_m$. Assume by way of contradiction that there exists $i \in \mathfrak{D}_m$ such that $i^* \in \mathbb{B}_{m+1}^{\frown}$. Let $k = \mathsf{b}^{\frown}(i^*, m+1) \in \mathfrak{D}_{m+1}$. Clearly, $i \neq k$ and $i^* \in B_k^{\frown}[k^*]$. Then, by Lemma 2.20(g), $|k| < |i|$ and $B_i^{\frown}[i^*] \subsetneq B_k^{\frown}[k^*]$. Thus,

$$m = \mathsf{depth}(i) \geq \mathsf{depth}(k) + 1 = m + 2;$$

a contradiction.

Now we prove the first statement of (c). From the definitions and the part of (b) already proven we have

$$O^*(\omega) \subset \bigcup_{i \in \mathbb{Z}} B_i^{\frown}[i^*] \subset \bigcup_{m=0}^{\infty} \mathbb{B}_m^{\frown} = \mathbb{B}_0^{\frown}.$$

To end the proof of (b) it remains to show the density of $\mathbb{B}^{\frown}_{m}$. We will do it by induction on m. Clearly $\mathbb{B}^{\frown}_{0} \supset O^*(\omega)$ is dense in $\mathbb{S}^1$ because so is $O^*(\omega)$. Suppose that (b) holds for $\mathbb{B}^{\frown}_{m}$. We will show that (b) also holds for $\mathbb{B}^{\frown}_{m+1}$. Choose $\theta \in \mathbb{B}^{\frown}_{m}$ and set $i = \mathsf{b}^{\frown}(\theta, m)$. Since $O^*(\omega)$ is dense in $\mathbb{S}^1$, there exists a sequence $\{s_n\}_{n=0}^{\infty} \subset \mathbb{Z}$ such that $s_n^* \in B_i^{\frown}(i^*)$ and $\lim_{n\to\infty} s_n^* = \theta$. As above, we get that $\mathsf{depth}(s_n) \geq \mathsf{depth}(i) + 1 = m + 1$. Moreover, $s_n^* \in \mathbb{B}^{\frown}_{\mathsf{depth}(s_n)} \subset \mathbb{B}^{\frown}_{m+1}$ for every n. Consequently, $\mathbb{B}^{\frown}_{m} \subset \overline{\mathbb{B}^{\frown}_{m+1}}$, and the density of $\mathbb{B}^{\frown}_{m+1}$ follows from the density of $\mathbb{B}^{\frown}_{m}$.

Next we prove the second statement of (c). From above it follows that

$$\bigcup_{i\in\mathbb{Z}} B_{\alpha_{|i|}}[i^*] \subset \bigcup_{i\in\mathbb{Z}} B_i^{\frown}[i^*] \subset \mathbb{B}^{\frown}_{0}.$$

Hence, by the definition of the maps γ_m (Definition 2.18) it follows that $\gamma_m(\theta) = \gamma_0(\theta) = 0$ for every $\theta \notin \mathbb{B}^{\frown}_{0}$ and $m \in \mathbb{Z}^+$. So, $\gamma(\theta) = \lim_{m\to\infty} \gamma_m(\theta) = 0$, and $\mathfrak{A}^\theta = \{(\theta, \gamma(\theta))\} = \{(\theta, 0)\}$ by Lemma 2.26(c). This ends the proof of (c).

(d) If $\theta = i^*$ then the statement follows from Lemmas 2.25(b) and 2.26(b). So, we assume that $\theta \neq i^*$.

By Definition 2.18(R.2) and Remark 2.19(R.2) we get that $\theta \notin Z^*_{|i|+1}$. Hence, if $\theta \in O^*(\omega)$, it follows that $\theta = k^* \in \mathbb{B}^{\frown}_{\mathsf{depth}(k)}$ with $|k| > |i| + 1$ and $B_k^{\frown}[k^*] \cap B_i^{\frown}[i^*] \neq \emptyset$. Thus, by Lemma 2.20(g), $\mathsf{depth}(k) \geq \mathsf{depth}(i) + 1$. By (b), this implies that $\theta = k^* \in \mathbb{B}^{\frown}_{\mathsf{depth}(i)+1}$; a contradiction. Therefore, $\theta \notin O^*(\omega)$. On the other hand, $\theta \notin B_{-i}^{\frown}[(-i)^*]$ by Definition 2.18(R.2).

If $\theta \notin B_{\alpha_{|k|}}[k^*]$ for every $k \in \mathbb{Z}$ such that $|k| > |i|$, then $\gamma_n(\theta) = \gamma_{|i|}(\theta)$ and $\mathfrak{A}^\theta_n = \mathfrak{A}^\theta_{|i|}$ for every $n \geq |i|$, by Definition 2.18 and Lemma 2.25(c).

Now assume that $\theta \in B_{\alpha_{|k|}}[k^*]$ for some $k \in \mathbb{Z}$ such that $|k| > |i|$ and $|k|$ is minimal with these properties. If $\theta \in B_k^{\frown}(k^*)$, as above we get that $\mathsf{depth}(k) \geq \mathsf{depth}(i) + 1$ and $\theta \in \mathbb{B}^{\frown}_{\mathsf{depth}(k)} \subset \mathbb{B}^{\frown}_{\mathsf{depth}(i)+1}$. Thus, $\theta \in \operatorname{Bd}(B_k^{\frown}[k^*]) = \operatorname{Bd}(B_{\alpha_{|k|}}[k^*])$ and $k \geq 0$. So, by Lemma 2.20(c) and the definition of the maps γ_j (Definition 2.18), $\gamma_{|k|}(\theta) = \gamma_{|k|-1}(\theta)$. Moreover, by Lemma 2.20(e), $\gamma_j(\theta) = \gamma_{|k|}(\theta)$ for every $j > |k|$. On the other hand, the minimality of $|k|$ implies that $\theta \notin B_{\alpha_{|\ell|}}[\ell^*]$ for every $\ell \in \mathbb{Z}$ such that $|k| > |\ell| > |i|$. Hence, by the definition of the maps γ_j (Definition 2.18), $\gamma_j(\theta) = \gamma_{|i|}(\theta)$ for every $|k| > j > |i|$. In short, we have proved that $\gamma_j(\theta) = \gamma_{|i|}(\theta)$ for every $j \geq |i|$. Thus, as above, $\mathfrak{A}^\theta_n = \mathfrak{A}^\theta_{|i|}$ for every $n \geq |i|$. This ends the proof of the lemma.

Lemma 2.36. *Assume that $B_i^{\frown}[i^*] \subset B_k^{\frown}[k^*]$ for some $i \in \mathfrak{D}_m$, $k \in \mathfrak{D}_{m-1}$ and $m \in \mathbb{N}$. Then, $|k| < |i|$ and $|k+1| < |i+1|$ unless $k \geq 0$ and $i = -(k+2)$ (whence $|k+1| = |i+1|$). Moreover, the following statements hold:*

(a) For every $\theta \in B_i^{\frown}[i^*]$,

$$\gamma_{|k|}(\theta) = \gamma_{|k|+1}(\theta) = \cdots = \gamma_{|i|-1}(\theta) \in \mathbb{I}_{i,\theta}$$

and, when $|k+1| < |i+1|$,

$$\gamma_{|k+1|}(R_\omega(\theta)) = \gamma_{|k+1|+1}(R_\omega(\theta)) = \cdots = \gamma_{|i+1|-1}(R_\omega(\theta))$$

(b) For every $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$,

$$\gamma_{|i|}(\theta) = \gamma_{|i|-1}(\theta) \quad \textit{and} \quad \mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}.$$

Proof. The fact that $|k| < |i|$ follows from Lemma 2.20(g). Therefore, either $|k+1| < |i+1|$ or $k \geq 0$, $i = -(k+2)$ and $|k+1| = |i+1|$ or $k \geq 0$, $i = -(k+1)$ and $|k+1| > |i+1|$. In the last case, $B_i^{\frown}[i^*] = B_{-(k+1)}^{\frown}[(-(k+1))^*]$ and $B_k^{\frown}[k^*]$ must be disjoint by Definition 2.18(R.2) (with $j = k$); which is a contradiction. Thus $|k+1| < |i+1|$ unless $k \geq 0$ and $i = -(k+2)$ ($|k+1| = |i+1|$).

By Definition 2.18(R.2) and Remark 2.19(R.2), $B_i^{\frown}[i^*] \cap Z_{|i|-1}^* = \emptyset$. Hence, from the definition of the maps γ_j (Definition 2.18), to prove that

$$\gamma_{|k|}\big|_{B_i^{\frown}[i^*]} = \gamma_{|k|+1}\big|_{B_i^{\frown}[i^*]} = \cdots = \gamma_{|i|-2}\big|_{B_i^{\frown}[i^*]} = \gamma_{|i|-1}\big|_{B_i^{\frown}[i^*]},$$

it is enough to show that $B_{\alpha_{|\ell|}}[\ell^*] \cap B_i^{\frown}[i^*] = \emptyset$ for every ℓ such that $|k| < |\ell| < |i|$. Assume that $B_{\alpha_{|\ell|}}[\ell^*] \cap B_i^{\frown}[i^*] \neq \emptyset$ for some ℓ such that $|k| < |\ell| < |i|$. Then,

$$\emptyset \neq B_{\alpha_{|\ell|}}[\ell^*] \cap B_i^{\frown}[i^*] \subset B_\ell^{\frown}[\ell^*] \cap B_i^{\frown}[i^*] \subset B_\ell^{\frown}[\ell^*] \cap B_k^{\frown}[k^*]$$

and, by Lemma 2.20(g),

$$B_i^{\frown}[i^*] \subsetneq B_\ell^{\frown}[\ell^*] \subsetneq B_k^{\frown}[k^*].$$

So, in a similar way as before,

$$m = \mathsf{depth}(i) \geq \mathsf{depth}(\ell) + 1 \geq \mathsf{depth}(k) + 2 = m + 1;$$

a contradiction. This ends the proof of the first statement of (a).

Now we show that if $|k+1| < |i+1| - 1$, then

$$\gamma_{|k+1|}(R_\omega(\theta)) = \gamma_{|k+1|+1}(R_\omega(\theta)) = \cdots = \gamma_{|i+1|-1}(R_\omega(\theta)),$$

and are well defined.

First we prove that $\gamma_\ell(R_\omega(\theta))$ is well defined for every $\ell = 0, 1, \ldots, |i+1| - 1$. For every $\theta \in B_i^{\frown}[i^*]$ we have

$$R_\omega(\theta) \in R_\omega\left(B_i^{\frown}[i^*]\right) = \begin{cases} B_{\alpha_i}[(i+1)^*] & \text{when } i \geq 0 \text{, and} \\ B_{\alpha_{|i+1|}}[(i+1)^*] \subset B_{i+1}^{\frown}[(i+1)^*] & \text{when } i < 0. \end{cases}$$

In any case, by Definition 2.18(R.2) and Remark 2.19(R.2) with $j = i$ when $i \geq 0$ and $\ell = -(j+1) = i+1$ when $i < 0$, and Lemma 2.20(a),

$$R_\omega(\theta) \notin \begin{cases} Z_i^* & \text{when } i \geq 0 \text{, and} \\ Z_{|i+1|-1}^* & \text{when } i < 0, \end{cases}$$

and $\gamma_\ell(R_\omega(\theta))$ is well defined for $\ell = 0, 1, \ldots, |i+1|-1$ (recall that $Z_m^* \subset Z_{m+1}^*$ for every $m \geq 0$).

Now, assume by way of contradiction that

$$\gamma_{\ell}\left(R_{\omega}(\theta)\right) \neq \gamma_{\ell-1}\left(R_{\omega}(\theta)\right) \text{ for some } \ell \in \{|k+1|+1, |k+1|+2, \dots, |i+1|-1\},$$

and ℓ is minimal with this property (observe that $\ell \geq 1$). By the definition of the map γ_{ℓ} (Definition 2.18),

$$R_{\omega}(\theta) \in B_{\alpha_{\ell}}\left((q+1)^*\right) \quad \text{with} \quad q \in \{\ell-1, -(\ell+1)\}$$

and, hence, $\theta \in B_{\alpha_{\ell}}(q^*)$.

Since $|k+1|+1 \leq \ell < |i+1|$, when $q = -(\ell+1) \leq -2$,

$$|k+1|+2 \leq -q \leq |i+1| \text{ and } B_{\alpha_{\ell}}\left(q^*\right) = B^{\frown}_{-(\ell+1)}\left((-(\ell+1))^*\right) = B^{\frown}_{q}\left(q^*\right).$$

Otherwise, when $q = \ell - 1 \geq 0$, $|k+1| \leq q \leq |i+1| - 2$ and

$$B_{\alpha_{\ell}}\left(q^*\right) \subset B_{\alpha_{\ell-1}}\left((\ell-1)^*\right) = B^{\frown}_{\ell-1}\left((\ell-1)^*\right) = B^{\frown}_{q}\left(q^*\right),$$

by Definition 2.18(R.1).

Next we want to use Lemma 2.20(g) to show that $B^{\frown}_{i}[i^*] \subsetneq B^{\frown}_{q}[q^*] \subsetneq B^{\frown}_{k}[k^*]$. To this end we have to compare $|q|$ with $|i|$ and $|k|$.

Notice $B^{\frown}_{q}[q^*] \cap B^{\frown}_{k}[k^*] \neq \emptyset$ because

$$\theta \in B^{\frown}_{q}\left(q^*\right) \cap B^{\frown}_{i}\left[i^*\right] \subset B^{\frown}_{q}\left(q^*\right) \cap B^{\frown}_{k}\left[k^*\right].$$

If $k \geq 0$, $|q| \geq |k+1| > |k|$. When $k, q < 0$, $|q| \geq |k+1| + 2 = |k| + 1 > |k|$. If $k < 0$ and $q \geq 0$, $|q| = q \geq |k+1| = |k| - 1$. If $q = |k| - 1$ (that is, $k = -(q+1)$), as above, by Definition 2.18(R.2) with $j = q$ we get $B^{\frown}_{k}[k^*] \cap B^{\frown}_{q}[q^*] = \emptyset$; a contradiction. So, $|q| > |k|$ unless $|q| = |k|$ and $k < 0 \leq q$. Summarizing, we have shown that $|q| \geq |k|$ and $q \neq k$. Then, from Lemma 2.20(g) we get that $|q| > |k|$ and $B^{\frown}_{q}[q^*] \subsetneq B^{\frown}_{k}[k^*]$.

Now we will study the relation of $B^{\frown}_{q}[q^*]$ with the box $B^{\frown}_{i}[i^*]$. From above we get that $B^{\frown}_{q}[q^*] \cap B^{\frown}_{i}[i^*] \neq \emptyset$. If $i < 0$, $|q| \leq |i+1| = |i| - 1$. When $q, i \geq 0$, we have $|q| = q \leq |i+1| - 2 = |i| - 1$. If $i \geq 0$ and $q < 0$, $|q| \leq |i+1| = |i| + 1$.

Assume that $i \geq 0$ and $q = -(i+1) < 0$. In this case, additionally, $q = -(\ell+1)$ and, thus, $i = \ell \geq 1$. Then,

$$\begin{aligned}
R_{\omega}(\theta) &\in R_{\omega}\left(B^{\frown}_{i}\left[i^*\right]\right) = R_{\omega}\left(B_{\alpha_i}\left[i^*\right]\right) = B_{\alpha_i}\left[(i+1)^*\right], \text{ and} \\
R_{\omega}(\theta) &\in B_{\alpha_{\ell}}\left((q+1)^*\right) = B_{\alpha_i}\left((-i)^*\right) \subset B^{\frown}_{-i}\left((-i)^*\right),
\end{aligned}$$

which is a contradiction by Definition 2.18(R.2). Summarizing, $|q| < |i|$ unless $|q| = |i|$ and $q < 0 \leq i$ (that is, $|q| \leq |i|$ and $q \neq i$). Then, again by Lemma 2.20(g), $|q| < |i|$ and $B^{\frown}_{i}[i^*] \subsetneq B^{\frown}_{q}[q^*] \subsetneq B^{\frown}_{k}[k^*]$. So, as before,

$$m = \mathsf{depth}(i) \geq \mathsf{depth}(q) + 1 \geq \mathsf{depth}(k) + 2 = m + 1;$$

a contradiction. This ends the proof of (a).

Now we assume that $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$. By Lemmas 2.20(e) and 2.28(d),

$$\gamma_{|i|}(\theta) = \gamma_{|i|-1}(\theta) \quad \text{and} \quad \mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|i|-1}(\theta)\} = \{\gamma_{|k|}(\theta)\}.$$

On the other hand, by Lemma 2.35(b), $\mathfrak{D}^*_{m-1} \cap \mathbb{B}^{\frown}_m = \emptyset$ which implies that $\theta \neq k^*$ because $k^* \in \mathfrak{D}^*_{m-1}$ and $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset \mathbb{B}^{\frown}_m$. So, by (2.1),

$$\mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}.$$

Now we prove that $\gamma_{|i|-1}(\theta) \in \mathbb{I}_{i,\theta}$ for every $\theta \in B_i^{\frown}[i^*]$. From above, we have $\mathbb{I}_{i,\theta} = \{\gamma_{|i|-1}(\theta)\}$ for every $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$. Moreover, when $\theta \in B_{\alpha_{|i|}}(i^*)$ the statement follows directly from Lemma 2.20(c). Thus, (b) is proved.

2.6 Boxes in the wings

To prove Theorem D we will inductively construct a Cauchy sequence $\{T_m\}_{m=0}^{\infty} \subset \mathcal{S}(\Omega)$ that gives the function T from Theorem D as a limit.

This section is devoted to study the points in the wings of boxes in the circle and its interaction with boxes of higher depth. The resulting technology is necessary to be able to construct the sequence $\{T_m\}_{m=0}^{\infty}$ so that it is Cauchy sequence. Unfortunately this will complicate even more the definition of the functions T_m and the proof of its continuity.

We start by introducing some more notation. For every $m \in \mathbb{Z}^+$ we set

$$\mathbb{B}_m := \bigcup_{i \in \mathfrak{D}_m} B_{\alpha_{|i|}}[i^*] \subset \mathbb{B}^{\frown}_m, \text{ and}$$
$$\mathbb{WDB}_m := \Big\{\theta \in \mathbb{B}^{\frown}_m \setminus \mathbb{B}_m : \theta \in \mathbb{B}_j \text{ for some } j > m\Big\}.$$

On the other hand, the smallest number j from the above definition will be called the *least essential depth of θ below m,* and will be denoted by $\mathsf{led}(\theta, m)$. That is, $\mathsf{led}(\theta, m)$ denotes the positive integer larger than m such that

$$\theta \in \mathbb{B}^{\frown}_j \setminus \mathbb{B}_j \text{ for } j = m, m+1, \dots, \mathsf{led}(\theta, m) - 1 \quad \text{and} \quad \theta \in \mathbb{B}_{\mathsf{led}(\theta,m)}.$$

The following simple lemmas are useful to better understand and use the above definitions. The next lemma establishes the relation between boxes in the wings of increasing depth.

Lemma 2.37. *Assume that $\theta \in \mathbb{WDB}_m$ for some $m \in \mathbb{Z}^+$ and set $\ell = \mathsf{led}(\theta, m)$. Then, the following statements hold.*

(a) For every $j = m, m+1, \dots, \ell$ the numbers $\mathsf{i}_j = \mathsf{b}^{\frown}(\theta, j) \in \mathfrak{D}_j$ are well defined and are all of them negative except, perhaps, $\mathsf{i}_\ell = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta, m))$.

(b)

$$|\mathsf{i}_m| < |\mathsf{i}_{m+1}| < \cdots < |\mathsf{i}_{\ell-1}| < |\mathsf{i}_\ell|, \textit{and}$$
$$\theta \in B_{\alpha_{|\mathsf{i}_\ell|}}[(\mathsf{i}_\ell)^*] \subset B^{\frown}_{\mathsf{i}_{\ell-1}}((\mathsf{i}_{\ell-1})^*) \setminus B_{\alpha_{|\mathsf{i}_{\ell-1}|}}[(\mathsf{i}_{\ell-1})^*]$$
$$\subset B^{\frown}_{\mathsf{i}_{\ell-2}}((\mathsf{i}_{\ell-2})^*) \setminus B_{\alpha_{|\mathsf{i}_{\ell-2}|}}[(\mathsf{i}_{\ell-2})^*] \subset \cdots \subset B^{\frown}_{\mathsf{i}_m}((\mathsf{i}_m)^*) \setminus B_{\alpha_{|\mathsf{i}_m|}}[(\mathsf{i}_m)^*].$$

(c) For every $j = m, m+1, \ldots, \ell-1$, $B_{\alpha_{|\mathsf{i}_\ell|}}[(\mathsf{i}_\ell)^*] \subset \mathbb{WDB}_j$, $\mathsf{led}(\nu, j) = \mathsf{led}(\theta, m)$ *and* $\mathsf{b}^{\frown}(\nu, \mathsf{led}(\nu, j)) = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta, m)) = \mathsf{i}_\ell$ *for every* $\nu \in B_{\alpha_{|\mathsf{i}_\ell|}}[(\mathsf{i}_\ell)^*]$.

(d) $\mathbb{I}_{\mathsf{i}_m,\nu} = \{\gamma_{|\mathsf{i}_m|}(\nu)\} \subset \mathbb{I}_{\mathsf{i}_\ell,\nu}$ *for every* $\nu \in B_{\alpha_{|\mathsf{i}_\ell|}}((\mathsf{i}_\ell)^*)$ *and*

$$\mathbb{I}_{\mathsf{i}_m,\nu} = \{\gamma_{|\mathsf{i}_m|}(\nu)\} = \{m_{\mathsf{i}_\ell}(\nu)\} = \{M_{\mathsf{i}_\ell}(\nu)\} = \{\gamma_{|\mathsf{i}_\ell|}(\nu)\} = \mathbb{I}_{\mathsf{i}_\ell,\nu}$$

for every $\nu \in \operatorname{Bd}\left(B_{\alpha_{|\mathsf{i}_\ell|}}[(\mathsf{i}_\ell)^*]\right)$.

Proof. Since $B^{\frown}_i[i^*] = B_{\alpha_i}[i^*]$ for every $i \geq 0$,

$$\mathbb{B}^{\frown}_m \setminus \mathbb{B}_m = \bigcup_{\substack{i \in \mathfrak{D}_m \\ i<0}} \left(B^{\frown}_i[i^*] \setminus B_{\alpha_{|i|}}[i^*]\right) \tag{2.1}$$

for every $m \in \mathbb{Z}^+$.

Statement (a) follows from Lemma 2.35(b) and (2.1). Then, (b) follows from Lemma 2.20(g). Statement (c) is an easy consequence of (b) and the definitions.

Now we prove (d) iteratively. Fix $\nu \in B_{\alpha_{|\mathsf{i}_\ell|}}((\mathsf{i}_\ell)^*)$. By (b)

$$\nu \in B^{\frown}_{\mathsf{i}_{m+1}}((\mathsf{i}_{m+1})^*) \setminus B_{\alpha_{|\mathsf{i}_{m+1}|}}[(\mathsf{i}_{m+1})^*] \subset B^{\frown}_{\mathsf{i}_m}((\mathsf{i}_m)^*) \setminus B_{\alpha_{|\mathsf{i}_m|}}[(\mathsf{i}_m)^*]$$

provided that $\ell = \mathsf{led}(\theta, m) > m+1$. Hence, by Lemmas 2.28(d) and 2.36,

$$\gamma_{|\mathsf{i}_m|}(\nu) = \gamma_{|\mathsf{i}_m|+1}(\nu) = \cdots = \gamma_{|\mathsf{i}_{m+1}|}(\nu), \text{ and}$$
$$\mathbb{I}_{\mathsf{i}_m,\nu} = \{\gamma_{|\mathsf{i}_m|}(\nu)\} = \{\gamma_{|\mathsf{i}_{m+1}|}(\nu)\} = \mathbb{I}_{\mathsf{i}_{m+1},\nu}.$$

By iterating this argument we get,

$$\gamma_{|\mathsf{i}_m|}(\nu) = \gamma_{|\mathsf{i}_m|+1}(\nu) = \cdots = \gamma_{|\mathsf{i}_{\ell-1}|}(\nu) \quad \text{and} \quad \mathbb{I}_{\mathsf{i}_m,\nu} = \mathbb{I}_{\mathsf{i}_{\ell-1},\nu}.$$

Again by (b) and Lemmas 2.28(d) and 2.36,

$$\gamma_{|\mathsf{i}_m|}(\nu) = \gamma_{|\mathsf{i}_m|+1}(\nu) = \cdots = \gamma_{|\mathsf{i}_\ell|}(\nu) \quad \text{and} \quad \mathbb{I}_{\mathsf{i}_m,\nu} = \mathbb{I}_{\mathsf{i}_\ell,\nu}$$

when $\nu \in \operatorname{Bd}\left(B_{\alpha_{|\mathsf{i}_\ell|}}[(\mathsf{i}_\ell)^*]\right)$ and, otherwise,

$$\gamma_{|\mathsf{i}_m|}(\nu) = \gamma_{|\mathsf{i}_m|+1}(\nu) = \cdots = \gamma_{|\mathsf{i}_{\ell-1}|}(\nu) \quad \text{and} \quad \mathbb{I}_{\mathsf{i}_m,\nu} \subset \mathbb{I}_{\mathsf{i}_\ell,\nu}.$$

Equipped with above results and definition we are going to define two maps, analogous to the maps m_i and M_i, on the wings of the negative boxes.

Definition 2.38. *For every $m \in \mathbb{Z}^+$ we define*

$$\mathfrak{WFD}_m := \{\mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta, m)) : \theta \in \mathbb{WDB}_m\} \subset \mathbb{Z},$$
$$\mathbb{WIB}_m := \operatorname{Int}(\mathbb{WDB}_m) = \bigcup_{i \in \mathfrak{WFD}_m} B_{\alpha_{|i|}}(i^*),$$
$$\mathbb{WB}^{\frown}_m := \bigcup_{\substack{i \in \mathfrak{D}_m \\ i<0}} \left(B^{\frown}_i[i^*] \setminus B_{\alpha_{|i|}}(i^*)\right), \text{ and}$$
$$\mathbb{EB}^{\frown}_m := \bigcup_{i \in \mathfrak{D}_m} \operatorname{Bd}\left(B^{\frown}_i[i^*]\right) \subset \mathbb{B}^{\frown}_m.$$

By Lemma 2.37(a,c), $\mathfrak{WFD}_m$ is well defined and

$$\mathbb{WIB}_m \subset \mathbb{WDB}_m \subset \mathbb{B}^{\frown}_m \setminus \mathbb{B}_m \subset \mathbb{WB}^{\frown}_m.$$

Consequently,

$$\mathbb{B}^{\frown}_m = \mathbb{B}_m \cup \mathbb{WB}^{\frown}_m.$$

Then, we can define functions $\tau_m \colon \mathbb{WB}^{\frown}_m \longrightarrow \mathbb{I}$ and $\lambda_m \colon \mathbb{WB}^{\frown}_m \longrightarrow \mathbb{I}$ as follows:

$$\tau_m(\theta) := \begin{cases} M_{\mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta,m))}(\theta) & \text{if } \theta \in \mathbb{WIB}_m, \\ \gamma_{|\mathsf{b}^{\frown}(\theta,m)|}(\theta) & \text{otherwise,} \end{cases}$$
$$\lambda_m(\theta) := \begin{cases} m_{\mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta,m))}(\theta) & \text{if } \theta \in \mathbb{WIB}_m, \\ \gamma_{|\mathsf{b}^{\frown}(\theta,m)|}(\theta) & \text{otherwise.} \end{cases}$$

Clearly, by Lemmas 2.28(a) and 2.20(b),

$$-1 \le \lambda_m(\theta) \le \tau_m(\theta) \le 1$$

for every $\theta \in \mathbb{WB}^{\frown}_m$. So, we can define

$$\mathbb{IW}_{m,\theta} := [\lambda_m(\theta), \tau_m(\theta)] \subset [0,1].$$

□

The next lemmas will help us in the definition and study of the maps T_m.

Lemma 2.39. *The following statements hold for every $m \in \mathbb{Z}^+$.*

(a) $\mathbb{WIB}_m \cap \mathbb{B}_m = \mathbb{WIB}_m \cap \mathbb{EB}^{\frown}_m = \emptyset$.
(b) Let $\theta \in \mathbb{WB}^{\frown}_m$. Then, $\mathbb{I}_{\mathsf{b}^{\frown}(\theta,m),\theta} = \left\{\gamma_{|\mathsf{b}^{\frown}(\theta,m)|}(\theta)\right\}$,

$$\mathbb{I}_{\mathsf{b}^{\frown}(\theta,m),\theta} = \mathbb{IW}_{m,\theta} \qquad \text{when } \theta \notin \mathbb{WIB}_m, \text{ and}$$
$$\mathbb{I}_{\mathsf{b}^{\frown}(\theta,m),\theta} \subset \mathbb{IW}_{m,\theta} \qquad \text{when } \theta \in \mathbb{WIB}_m.$$

(c) Assume that $m \in \mathbb{N}$ and let U be a connected component of $\mathbb{WB}^{\smile}_m$ such that $U \subset \mathbb{WB}^{\smile}_{m-1}$. Then, $\mathbb{WDB}_m \cap U \subset \mathbb{WDB}_{m-1}$, $\mathbb{WIB}_m \cap U = \mathbb{WIB}_{m-1} \cap U$ and $\mathbb{IW}_{m,\theta} = \mathbb{IW}_{m-1,\theta}$ for every $\theta \in U$.

Proof. **(a)** By Lemma 2.37(b),

$$\theta \in B^{\smile}_{\mathsf{b}^{\smile}(\theta,m)}\left((\mathsf{b}^{\smile}(\theta,m))^*\right) \setminus B_{\alpha_{|\mathsf{b}^{\smile}(\theta,m)|}}\left[(\mathsf{b}^{\smile}(\theta,m))^*\right]$$

and $\mathsf{b}^{\smile}(\theta,m) < 0$ for every $\theta \in \mathbb{WIB}_m \subset \mathbb{WDB}_m$. So, by Lemma 2.34(b), we get $\theta \notin \mathbb{B}_m \cup \mathbb{EB}^{\smile}_m$.

(b) The fact that $\mathbb{I}_{\mathsf{b}^{\smile}(\theta,m),\theta} = \left\{\gamma_{|\mathsf{b}^{\smile}(\theta,m)|}(\theta)\right\}$ follows from Lemma 2.28(d). The other two statements follow from Definition 2.38 and Lemma 2.37(d).

(c) The assumption that U is a connected component of $\mathbb{WB}^{\smile}_m$ and $U \subset \mathbb{WB}^{\smile}_{m-1}$ implies by Lemmas 2.34(b) and 2.20(g) that there exist $i \in \mathfrak{D}_m$ and $k \in \mathfrak{D}_{m-1}$, $i, k < 0$, such that U is a connected component of

$$B^{\smile}_i[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset B^{\smile}_k(k^*) \setminus B_{\alpha_{|k|}}[k^*] \subset \mathbb{WB}^{\smile}_{m-1}.$$

Again by Lemma 2.34(b) this implies that $U \subset \mathbb{B}^{\smile}_{m-1} \setminus \mathbb{B}_{m-1}$. Moreover, by definition, $\mathbb{WDB}_m \subset \mathbb{B}^{\smile}_m \setminus \mathbb{B}_m$. Consequently, $\mathbb{WDB}_m \cap U \subset \mathbb{WDB}_{m-1}$.

Let $\theta \in \mathbb{WIB}_m \cap U \subset \mathbb{WDB}_m \cap U \subset \mathbb{WDB}_{m-1} \cap U$. By Definition 2.38 and Lemma 2.37(a,b), $i = \mathsf{b}^{\smile}(\theta,m)$ and there exists $\ell = \mathsf{b}^{\smile}(\theta, \mathsf{led}(\theta,m)) \in \mathfrak{WFD}_m$ such that

$$\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset B^{\smile}_i[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset B^{\smile}_k(k^*) \setminus B_{\alpha_{|k|}}[k^*].$$

Therefore, again by Lemma 2.37(a–c) and Definition 2.38, $\mathsf{led}(\theta, m-1) = \mathsf{led}(\theta,m)$,

$$\ell = \mathsf{b}^{\smile}(\theta, \mathsf{led}(\theta,m)) = \mathsf{b}^{\smile}(\theta, \mathsf{led}(\theta,m-1)) \in \mathfrak{WFD}_{m-1}$$

and $\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset \mathbb{WIB}_{m-1}$. Hence, $\mathbb{WIB}_m \cap U \subset \mathbb{WIB}_{m-1}$.

Now assume that $\theta \in \mathbb{WIB}_{m-1} \cap U$. As above, there exist $r = \mathsf{b}^{\smile}(\theta,m) \in \mathfrak{D}_m$ and $\ell = \mathsf{b}^{\smile}(\theta, \mathsf{led}(\theta,m-1)) \in \mathfrak{WFD}_{m-1}$ such that

$$\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset B^{\smile}_r(r^*) \setminus B_{\alpha_{|r|}}[r^*] \subset B^{\smile}_k(k^*) \setminus B_{\alpha_{|k|}}[k^*].$$

Since $\theta \in U \subset B^{\smile}_i[i^*]$, Lemma 2.34(b) gives $i = r$ and $\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset U$. Moreover, by Lemma 2.37(c), $\ell = \mathsf{b}^{\smile}(\theta, \mathsf{led}(\theta,m-1)) = \mathsf{b}^{\smile}(\theta, \mathsf{led}(\theta,m)) \in \mathfrak{WFD}_m$ and, so, $\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset \mathbb{WIB}_m$. Thus, $\mathbb{WIB}_m \cap U = \mathbb{WIB}_{m-1} \cap U$.

To end the proof of the lemma we have to show that $\mathbb{IW}_{m,\theta} = \mathbb{IW}_{m-1,\theta}$ for every $\theta \in U$. Assume first that $\theta \in U \setminus \mathbb{WIB}_m \subset \mathbb{WB}^{\smile}_m \setminus \mathbb{WIB}_m$. Then,

$$\theta \in U \setminus \mathbb{WIB}_m = U \setminus \mathbb{WIB}_{m-1} \subset \mathbb{WB}^{\smile}_{m-1} \setminus \mathbb{WIB}_{m-1}$$

and, by (b) and Lemmas 2.28(d) and 2.36,

$$\mathbb{IW}_{m,\theta} = \mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} = \mathbb{I}_{k,\theta} = \mathbb{IW}_{m-1,\theta}.$$

If $\theta \in U \cap \mathbb{WIB}_m = U \cap \mathbb{WIB}_{m-1}$ then we get

$$\begin{aligned}\mathbb{IW}_{m,\theta} &= \left[m_{\mathsf{b}^{\frown}(\theta,\mathsf{led}(\theta,m))}(\theta), M_{\mathsf{b}^{\frown}(\theta,\mathsf{led}(\theta,m))}(\theta)\right] \\ &= \left[m_{\mathsf{b}^{\frown}(\theta,\mathsf{led}(\theta,m-1))}(\theta), M_{\mathsf{b}^{\frown}(\theta,\mathsf{led}(\theta,m-1))}(\theta)\right] = \mathbb{IW}_{m-1,\theta}\end{aligned}$$

from Definition 2.38 and Lemma 2.37(c).

Lemma 2.40. *Let $m \in \mathbb{Z}^+$ and let U be a connected component of $\mathbb{WB}^{\frown}_m$. Then, the functions $\lambda_m\big|_U$ and $\tau_m\big|_U$ are continuous.*

Proof. We will prove only the continuity of $\lambda_m\big|_U$. The proof of the continuity of $\tau_m\big|_U$ is analogous.

By Lemmas 2.37(c) and 2.28(b) we get

for every $\ell \in \mathfrak{WFD}_m$, $\ell = \mathsf{b}^{\frown}(\nu, \mathsf{led}(\nu, m))$ for every $\nu \in B_{\alpha_{|\ell|}}[\ell^*]$, and the function m_ℓ is continuous on $B_{\alpha_{|\ell|}}[\ell^*]$. (2.2)

Let $\ell \in \mathfrak{WFD}_m$ be such that $B_{\alpha_{|\ell|}}(\ell^*) \subset \mathbb{WIB}_m \cap U$. Thus, by (2.2), the function $\lambda_m = m_\ell$ is continuous on $B_{\alpha_{|\ell|}}(\ell^*)$.

So, we have to show that λ_m is continuous at every $\theta \in U \setminus \mathbb{WIB}_m$. To show this we will use a simple usual ε–δ game. Fix $\varepsilon > 0$.

By Lemma 2.34(b) it follows that U is a connected component of $B^{\frown}_i[i^*] \setminus B_{\alpha_{|i|}}(i^*)$ for some $i \in \mathfrak{D}_m$, $i < 0$, and

$$\mathsf{b}^{\frown}(\nu, m) = i \quad \text{for every} \quad \nu \in U. \tag{2.3}$$

By Lemma 2.20(a) and Definition 2.18(R.2) and Remark 2.19(R.2), the function $\gamma_{|i|}\big|_U$ is continuous. So,

there exists $\overline{\delta}_{|i|} = \overline{\delta}_{|i|}(\theta) > 0$ such that $|\gamma_{|i|}(\theta), \gamma_{|i|}(\nu)| < \varepsilon/2$ provided that $\mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \overline{\delta}_{|i|}$. (2.4)

On the other hand, by (2.2),

for every $\ell \in \mathfrak{WFD}_m$, there exists $\delta_\ell > 0$ such that $|m_\ell(\widetilde{\theta}), m_\ell(\nu)| < \varepsilon/2$ for every $\widetilde{\theta} \in \operatorname{Bd}\left(B_{\alpha_{|\ell|}}[\ell^*]\right)$ and $\nu \in \operatorname{Bd} B_{\alpha_{|\ell|}}[\ell^*]$ such that $\mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta_\ell$. (2.5)

Now we will define δ. Note that there exists $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon/2$. Then we set:

$$\delta = \delta(\theta) := \min\left\{\overline{\delta}_{|i|}(\theta), \min\{\delta_\ell : \ell \in \mathfrak{WFD}_m \text{ and } |\ell| < N\}\right\}.$$

Clearly, $\delta > 0$ because the set $\{\ell \in \mathfrak{WFD}_m : |\ell| < N\}$ is finite.

To end the proof of the lemma we have to show that

$$|\lambda_m(\theta) - \lambda_m(\nu)| < \varepsilon$$

whenever $\nu \in U$ and $\mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta$.

Assume that $\nu \in U$ and $\mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta$ (recall that we have the assumption that $\theta \notin \mathbb{WIB}_m$). If $\nu \notin \mathbb{WIB}_m$, then $\mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta \le \overline{\delta}_{|i|}(\theta)$ and, by (2.3) and (2.4),

$$|\lambda_m(\theta) - \lambda_m(\nu)| = |\gamma_{|i|}(\theta) - \gamma_{|i|}(\nu)| < \varepsilon/2 < \varepsilon.$$

Now assume that there exists $\ell \in \mathfrak{WFD}_m$ such that $\nu \in B_{\alpha_{|\ell|}}(\ell^*) \subset \mathbb{WIB}_m$. Clearly, there exists $\widetilde{\theta} \in \operatorname{Bd}\left(B_{\alpha_{|\ell|}}[\ell^*]\right)$ such that

$$\mathsf{d}_{\mathbb{S}^1}(\theta, \widetilde{\theta}) < \mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta \le \overline{\delta}_{|i|}(\theta) \text{ and}$$
$$\mathsf{d}_{\mathbb{S}^1}(\widetilde{\theta}, \nu) < \mathsf{d}_{\mathbb{S}^1}(\theta, \nu) < \delta.$$

Observe that, by Lemma 2.34(b), $\widetilde{\theta} \notin \mathbb{WIB}_m$. Hence, by (2.3) and Lemma 2.37(c,d),

$$\lambda_m(\widetilde{\theta}) = \gamma_{|i|}(\widetilde{\theta}) = m_\ell(\widetilde{\theta}).$$

If $|\ell| < N$, then $\mathsf{d}_{\mathbb{S}^1}(\widetilde{\theta}, \nu) < \delta \le \delta_\ell$ and, by (2.5), $|m_\ell(\widetilde{\theta}) - m_\ell(\nu)| < \varepsilon/2$. Otherwise, by Lemma 2.20(f),

$$|m_\ell(\widetilde{\theta}) - m_\ell(\nu)| < \operatorname{diam}(\mathcal{R}(\ell^*)) \le 2^{-|\ell|} \le 2^{-N} < \varepsilon/2.$$

In any case, $|m_\ell(\widetilde{\theta}) - m_\ell(\nu)| < \varepsilon/2$. Thus, again by (2.3) and (2.4),

$$\begin{aligned}|\lambda_m(\theta) - \lambda_m(\nu)| &\le |\lambda_m(\theta) - \lambda_m(\widetilde{\theta})| + |\lambda_m(\widetilde{\theta}) - \lambda_m(\nu)| \\ &= |\gamma_{|i|}(\theta) - \gamma_{|i|}(\widetilde{\theta})| + |m_\ell(\widetilde{\theta}) - m_\ell(\nu)| < \varepsilon.\end{aligned}$$

2.7 A Cauchy sequence of skew products. Proof of Theorem D

In this section prove Theorem D. To do this we inductively construct a Cauchy sequence $\{T_m\}_{m=0}^\infty \subset \mathcal{S}(\Omega)$ that gives the function T from Theorem D as a limit.

The sequence $\{T_m\}_{m=0}^\infty \subset \mathcal{S}(\Omega)$ is defined so that

$$T_m(\theta, x) = (R_\omega(\theta), f_m(\theta, x))$$

and $f_m \colon \Omega \longrightarrow \mathbb{I}$ is continuous in both variables. To build these functions we will use the auxiliary functions $G_i \colon \mathcal{R}(i^*) \longrightarrow \Omega$ with $i \in \mathbb{Z}$ from Section 2.4. The maps $f_m(\theta, \cdot)$ will also be denoted as $f_{m,\theta}$, and will be defined non-increasing, and such that $f_{m,\theta}(2) = -2$ and $f_{m,\theta}(-2) = 2$ for every $\theta \in \mathbb{S}^1$.

To make more evident the strategy of the construction of this sequence of maps we will separate several cases, and we will state without proofs the results that study these maps. After establishing all the definitions and results related to the construction of the sequence $\{T_m\}_{m=0}^\infty$ without having been distracted by the technicalities involving the proofs, we will proceed to provide the missing proofs. More precisely, we will start by defining the map T_0 and stating without proof the proposition that summarizes the necessary properties of this map. Next we will inductively define the maps $\{T_m\}_{m=1}^{\infty} \subset \mathcal{S}(\Omega)$ and state without proof the proposition that establishes the properties of the whole sequence $\{T_m\}_{m=0}^{\infty}$.

Then, as we have said, we prove Theorem D and in the next three sections we will provide all pending proofs.

In what follows $\mathcal{C}(\mathbb{I}, \mathbb{I})$ will denote the class of all continuous maps from $\mathbb{I}$ to itself. We endow $\mathcal{C}(\mathbb{I}, \mathbb{I})$ with the supremum metric denoted by $\|\cdot\|$ so that $(\mathcal{C}(\mathbb{I}, \mathbb{I}), \|\cdot\|)$ is a complete metric space.

Next we define the map T_0.

Definition 2.41 (The map T_0). *Assume first that $\theta \in \mathbb{B}_0^{\frown}$ and let $\mathsf{i} = \mathsf{b}^{\frown}(\theta, 0)$ (that is $\theta \in B_{\mathsf{i}}^{\frown}[\mathsf{i}^*]$). In this case we set:*

$$f_{0,\theta}(x) = \begin{cases} g_{\mathsf{i},\theta}(x) & \text{if } x \in \mathbb{I}_{\mathsf{i},\theta}, \\ \frac{g_{\mathsf{i},\theta}(m_{\mathsf{i}}(\theta))-2}{m_{\mathsf{i}}(\theta)+2}(x+2)+2 & \text{if } x \in [-2, m_{\mathsf{i}}(\theta)], \\ \frac{g_{\mathsf{i},\theta}(M_{\mathsf{i}}(\theta))+2}{M_{\mathsf{i}}(\theta)-2}(x-2)-2 & \text{if } x \in [M_{\mathsf{i}}(\theta), 2]. \end{cases}$$

If $\theta \in \mathbb{S}^1 \setminus \mathbb{B}_0^{\frown}$ then we define $f_{0,\theta}$ to be the unique piecewise affine map with two affine pieces whose graph joins the point $(-2, 2)$ with $(0, \gamma(R_\omega(\theta)))$, and this with the point $(2, -2)$. ◘

Next we introduce some more notation to be able to define the maps $\{T_m\}_{m=1}^{\infty}$. For every $k \in \mathbb{Z}$ we set

$$\mathsf{V}_{k^*}^{\frown} := \upuparrows B_k^{\frown}[k^*] = B_k^{\frown}[k^*] \times \mathbb{I}$$

and, for every $m \in \mathbb{Z}^+$,

$$\mathbb{V}_m^{\frown} := \upuparrows \mathbb{B}_m^{\frown} = \mathbb{B}_m^{\frown} \times \mathbb{I} = \bigcup_{i \in \mathfrak{D}_m} \mathsf{V}_{i^*}^{\frown}.$$

Definition 2.42 (The maps T_m with $m > 0$). *Now we assume that we have defined the function T_{m-1} for some $m \geq 1$ and we define*

$$T_m(\theta, x) = (R_\omega(\theta), f_m(\theta, x))$$

as follows. By Lemma 2.34(b), for every $(\theta, x) \in \mathbb{V}_m^{\frown}$, we have

$$\theta \in B_{\mathsf{i}}^{\frown}[\mathsf{i}^*] \subset \mathbb{B}_m^{\frown} \quad \textit{with} \quad \mathsf{i} = \mathsf{b}^{\frown}(\theta, m) \in \mathfrak{D}_m$$

(and, of course, $x \in \mathbb{I}$). Then we define:

$$f_{m,\theta}(x) = \begin{cases} f_{m-1,\theta}(x) & \textit{if } \theta \in \mathbb{S}^1 \setminus \mathbb{B}^{\frown}_{m};\ x \in \mathbb{I}, \\ g_{\mathsf{i},\theta}(x) & \textit{if } \theta \in \mathbb{B}_{m};\ x \in \mathbb{I}_{\mathsf{i},\theta}, \\ \frac{2-g_{\mathsf{i},\theta}(m_{\mathsf{i}}(\theta))}{2-f_{m-1,\theta}(m_{\mathsf{i}}(\theta))}(f_{m-1,\theta}(x)-2)+2 & \textit{if } \theta \in \mathbb{B}_{m};\ x \in [-2, m_{\mathsf{i}}(\theta)], \\ \frac{2+g_{\mathsf{i},\theta}(M_{\mathsf{i}}(\theta))}{2+f_{m-1,\theta}(M_{\mathsf{i}}(\theta))}(f_{m-1,\theta}(x)+2)-2 & \textit{if } \theta \in \mathbb{B}_{m};\ x \in [M_{\mathsf{i}}(\theta), 2], \\ \gamma_{|\mathsf{i}+1|}\left(R_{\omega}(\theta)\right) & \textit{if } \theta \in \mathbb{WB}^{\frown}_{m};\ x \in \mathbb{IW}_{m,\theta}, \\ \frac{2-\gamma_{|\mathsf{i}+1|}(R_{\omega}(\theta))}{2-f_{m-1,\theta}(\lambda_m(\theta))}(f_{m-1,\theta}(x)-2)+2 & \textit{if } \theta \in \mathbb{WB}^{\frown}_{m};\ x \in [-2, \lambda_m(\theta)], \\ \frac{2+\gamma_{|\mathsf{i}+1|}(R_{\omega}(\theta))}{2+f_{m-1,\theta}(\tau_m(\theta))}(f_{m-1,\theta}(x)+2)-2 & \textit{if } \theta \in \mathbb{WB}^{\frown}_{m};\ x \in [\tau_m(\theta), 2]. \end{cases}$$

Since $\mathbb{V}^{\frown}_{m} \subset \mathbb{V}^{\frown}_{m-1}$, $f_{m-1,\theta}$ is defined on $\mathbb{V}^{\frown}_{m}$. Moreover, the above formula defines $f_{m,\theta}$ for every $\theta \in \mathbb{B}^{\frown}_{m}$ since, by Definition 2.38, $\mathbb{B}^{\frown}_{m} = \mathbb{B}_{m} \cup \mathbb{WB}^{\frown}_{m}$. We also remark that $f_{m,\theta}$ formally is defined in two different ways when $\theta \in \mathbb{WB}^{\frown}_{m} \cap \mathbb{B}_{m}$. Later on we will show that $f_{m,\theta}$ is well defined. ◘

The next proposition studies the maps $\{T_m\}_{m=0}^{\infty}$ and describes their properties.

Proposition 2.43. *The following statements hold for every $m \in \mathbb{Z}^+$.*

(a) The map T_m is well defined, continuous and belongs to $\mathcal{S}(\Omega)$.

(b) For every $\theta \in \mathbb{S}^1$, $f_{m,\theta}$ is non-increasing, and $f_{m,\theta}(2) = -2$, $f_{m,\theta}(-2) = 2$. Moreover, $-1 \leq f_{0,\theta}\left(M_{\mathfrak{b}^{\frown}(\theta,m)}(\theta)\right) \leq f_{0,\theta}\left(m_{\mathfrak{b}^{\frown}(\theta,m)}(\theta)\right) \leq 1$ for every $\theta \in \mathbb{B}^{\frown}_{m}$.

(c) For every $i \in \mathfrak{D}_{m}$, $T_m\big|_{\mathcal{R}^{\frown}(i^*)} = G_i$, $T_m\left(\mathfrak{A}^{i^*}_{|i|}\right) = \mathfrak{A}^{(i+1)^*}_{|i+1|}$, and $T_k\big|_{\{i^*\}\times\mathbb{I}} = T_m\big|_{\{i^*\}\times\mathbb{I}}$ (that is, $f_{k,i^*} = f_{m,i^*}$) for every $k > m$.

The next result shows that the sequence $\{T_m\}_{m=0}^{\infty}$ has a limit in $\mathcal{S}(\Omega)$.

Proposition 2.44. *For every $m \geq 2$ and $\theta \in \mathbb{S}^1$,*

$$\|f_{m,\theta} - f_{m-1,\theta}\| \leq 2 \cdot 2^{-|\mathfrak{b}^{\frown}(\theta,m-1)|}. \tag{2.1}$$

Moreover, the sequence $\{T_m\}_{k=0}^{\infty}$ is a Cauchy sequence.

Finally we are ready to prove the main result of the chapter. It follows from the next result which gives a more concrete version of Theorem D.

Theorem 2.45. *There exists a map $T \in \mathcal{S}(\Omega)$ with $f(\theta, \cdot)$ non-increasing for every $\theta \in \mathbb{S}^1$, such that T permutes the upper and lower circles of Ω (thus having a periodic orbit of period two of curves), and there exists a connected pseudo-curve $\mathfrak{A} \subset \Omega$ which does not contain any arc of a curve such that $T(\mathfrak{A}) = \mathfrak{A}$ and there does not exist any T-invariant curve.*

Proof. By Propositions 2.43 and 2.44, there exists a map

$$T(\theta, x) = (R_{\omega}(\theta), f(\theta, x)) = (R_{\omega}(\theta), \lim_{m\to\infty} f_m(\theta, x)) \in \mathcal{S}(\Omega)$$

with $f(\theta, \cdot)$ non-increasing for every $\theta \in \mathbb{S}^1$ such that T permutes the upper and lower circles of Ω (that is, $f(\theta, 2) = -2$ and $f(\theta, -2) = 2$). As the connected set $\mathfrak{A}$ we take the one given by Proposition 2.27 (and Definition 2.24).

To end the proof of the theorem we need to show that $T(\mathfrak{A}) = \mathfrak{A}$, since this already implies that there does not exist any T-invariant curve. To see it, assume by way of contradiction that there exists an invariant curve and denote its graph by B. Since B is the graph of a (continuous) curve, it is compact and connected. On the other hand, let Ω_+ and Ω_- be the two connected components of $\Omega \backslash \mathfrak{A}$ from the proof of Proposition 2.27. The facts that $T(\mathfrak{A}) = \mathfrak{A}$, $f(\theta, \cdot)$ is decreasing for every $\theta \in \mathbb{S}^1$, and T permutes the upper and lower circles of Ω imply that $T(\Omega_+) = \Omega_-$ and $T(\Omega_-) = \Omega_+$. Hence, by the invariance of B, $B \not\subseteq \Omega_+$ and $B \not\subseteq \Omega_-$. The connectivity of $\mathfrak{A}$ and B imply that there exists $(\theta, x) \in \mathfrak{A} \cap B$. Consequently,

$$B = \overline{\{T^n(\theta, x) : n \in \mathbb{Z}^+\}} \subset \mathfrak{A};$$

a contradiction because $\mathfrak{A}$ does not contain any arc of a curve.

So, only it remains to prove that $T(\mathfrak{A}) = \mathfrak{A}$. By using Proposition 2.43(c) and Lemma 2.26(b) we get that $T_m\left(\mathfrak{A}^{i^*}\right) = \mathfrak{A}^{(i+1)^*}$, and $T_k\big|_{\mathfrak{A}^{i^*}} = T_m\big|_{\mathfrak{A}^{i^*}}$ for every $k, m \in \mathbb{Z}^+,\ k \geq m$ and $i \in \mathfrak{D}_m$. Consequently, by the definition of the map T we have, $T(\mathfrak{A}^{i^*}) = \mathfrak{A}^{(i+1)^*}$ for every $i \in \mathbb{Z}$ or, equivalently, $T\big(\mathfrak{A}^{\upuparrows O^*(\omega)}\big) = \mathfrak{A}^{\upuparrows O^*(\omega)}$.

Now we consider $\mathfrak{A}^\theta$ with $\theta \in \mathbb{S}^1 \backslash O^*(\omega)$. Since $O^*(\omega)$ is dense in $\mathbb{S}^1$, there exists a sequence $\{(\theta_n, x_n)\}_{n=0}^\infty \subset \mathfrak{A}^{\upuparrows O^*(\omega)}$ such that $\lim_{n\to\infty} \theta_n = \theta$. By the compacity of $\mathfrak{A}$ we can assume (by taking a convergent subsequence, if necessary) that $\{(\theta_n, x_n)\}_{n=0}^\infty$ is convergent to a point $(\theta, x) \in \mathfrak{A}$. By Lemma 2.26(c), $\mathfrak{A}^\theta = (\theta, x)$ (and $x = \gamma(\theta)$). On the other hand, by the part of the statement already proven, $T(\theta_n, x_n) \in \mathfrak{A}$ for every n. Hence, by the continuity of T and the compacity of $\mathfrak{A}$,

$$T(\theta, x) = (R_\omega(\theta), f(\theta, x)) = \lim_{n\to\infty} T(\theta_n, x_n) \in \mathfrak{A}^{R_\omega(\theta)}.$$

Since $\theta \notin O^*(\omega)$ we have that $R_\omega(\theta) \notin O^*(\omega)$ and, again by Lemma 2.26(c), $\mathfrak{A}^{R_\omega(\theta)}$ consists of a unique point. Hence, $T(\mathfrak{A}^\theta) = \mathfrak{A}^{R_\omega(\theta)}$ for every $\theta \in \mathbb{S}^1 \backslash O^*(\omega)$. Equivalently, $T\left(\mathfrak{A}^{\upuparrows\left(\mathbb{S}^1 \backslash O^*(\omega)\right)}\right) = \mathfrak{A}^{\upuparrows\left(\mathbb{S}^1 \backslash O^*(\omega)\right)}$. This ends the proof of the theorem.

2.8 Proof of Proposition 2.43 in the case $m = 0$

This section is devoted to prove Proposition 2.43 for $m = 0$; that is, to study the map T_0. It is the first technical counterpart of Section 2.7.

To prove Proposition 2.43 for T_0 we will need some more notation and a technical lemma.

Given a skew product $F(\theta, x) = (R_\omega(\theta), \zeta(\theta, x)$ from $\Omega = \mathbb{S}^1 \times \mathbb{I}$ to itself we define the *fibre map function of* F, $\mathsf{fib}(F) \colon \mathbb{S}^1 \longrightarrow \mathcal{C}(\mathbb{I}, \mathbb{I})$ by $\mathsf{fib}(F)(\theta) := \zeta(\theta, \cdot)$. A simple exercise shows that F is continuous if and only if $\zeta(\theta, \cdot)$ is continuous for every $\theta \in \mathbb{S}^1$, and $\mathsf{fib}(F)$ is continuous.

Lemma 2.46. *Let $\theta \in \operatorname{Bd}(B_i^{\frown}[i^*])$ for some $i \in \mathfrak{D}_0$. Then, $m_i(\theta) = M_i(\theta) = 0$, $g_i(\theta, m_i(\theta)) = \gamma(R_\omega(\theta))$, and $f_{0,\theta}$ is the unique piecewise affine map with two affine pieces whose graph joins the point $(-2, 2)$ with $(0, \gamma(R_\omega(\theta)))$, and this with the point $(2, -2)$.*

Proof. By Lemma 2.28(d) and Definition 2.41, we have $m_i(\theta) = M_i(\theta)$. Hence, $f_{0,\theta}$ is the piecewise affine map with two affine pieces whose graph joins the point $(-2, 2)$ with $\left(m_i(\theta), g_{i,\theta}\left(m_i(\theta)\right)\right)$, and this with the point $(2, -2)$. So, we need to show that $m_i(\theta) = 0$, and $g_{i,\theta}\left(m_i(\theta)\right) = \gamma(R_\omega(\theta))$.

Lemma 2.20(g) and the fact that $\mathsf{depth}\, i = 0$, $B_i^{\frown}[i^*] \cap B_\ell^{\frown}[\ell^*] = \emptyset$ for every $\ell \in Z_{|i|}$, $i \neq \ell$. Consequently, by Definition 2.18(R.6), $m_i(\theta) = M_i(\theta) = a_i^- = 0$.

Now we show that $g_{i,\theta}(m_i(\theta)) = \gamma(R_\omega(\theta))$. From the definition of the map g_i (Definitions 2.29 and 2.31), Lemma 2.20(e) and Definitions 2.23 and 2.18(R.1), we get

$$g_{i,\theta}(m_i(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)) = \gamma(R_\omega(\theta)).$$

This ends the proof of the lemma.

Proof (Proof of Proposition 2.43 for $m = 0$). By Lemma 2.20(b),

$$-1 \leq m_{\mathfrak{b}^{\frown}(\theta,0)}(\theta) \leq M_{\mathfrak{b}^{\frown}(\theta,0)}(\theta) \leq 1$$

for every $\theta \in \mathbb{B}_0^{\frown}$. So, T_0 is well defined.

(b) If $\theta \in \mathbb{S}^1 \backslash \mathbb{B}_0^{\frown}$, then the statement follows directly from Definition 2.41. Now assume that $\theta \in \mathbb{B}_0^{\frown}$ and let $i = \mathfrak{b}^{\frown}(\theta, 0)$. From the definition of the maps $g_{i,\theta}$ (Definitions 2.29 and 2.31) and Definition 2.41, it follows that $f_{0,\theta}\big|_{\mathbb{I}_{i,\theta}}$ is piecewise affine and non-increasing. On the other hand, again by Definition 2.41, $f_{0,\theta}\big|_{[-2,m_i(\theta)]}$ and $f_{0,\theta}\big|_{[M_i(\theta),2]}$ are affine with negative slope and $f_{0,\theta}(2) = -2$ and $f_{0,\theta}(-2) = 2$. The fact that

$$-1 \leq f_{0,\theta}\left(M_{\mathfrak{b}^{\frown}(\theta,0)}(\theta)\right) \leq f_{0,\theta}\left(m_{\mathfrak{b}^{\frown}(\theta,0)}(\theta)\right) \leq 1$$

for every $\theta \in \mathbb{B}_0^{\frown}$ follows from Definition 2.41 and Lemmas 2.30(a) and 2.32(a). This ends the proof of (b).

(c) Recall that

$$\mathcal{R}^{\frown}(i^*) = \bigcup_{\theta \in B_i^{\frown}[i^*]} \{\theta\} \times \mathbb{I}_{i,\theta}.$$

Hence, from Definition 2.41 and the definition of G_i (Definitions 2.29 and 2.31) it follows that

$$T_m(\theta, x) = \left(R_\omega(\theta), f_m(\theta, x)\right) = \left(R_\omega(\theta), g_{i,\theta}(x)\right) = G_i(\theta, x),$$

for every $(\theta, x) \in \mathcal{R}^{\frown}(i^*)$. Thus, $T_0\left(\mathfrak{A}_{|i|}^{i^*}\right) = \mathfrak{A}_{|i+1|}^{(i+1)^*}$ from Lemmas 2.25(b), 2.30(c) and 2.32(c). On the other hand, Lemma 2.35(b) implies that $i^* \in \mathbb{B}_0^{\frown}$ but $i^* \notin \mathbb{B}_k^{\frown}$ for every $k \in \mathbb{N}$. Then, we get $f_{k,i^*} = f_{0,i^*}$ from Definition 2.42.

(a) Since T_0 is a skew product with base R_ω we only have to prove that f_0 is continuous.

By Definition 2.41, for every $\theta \in \mathbb{S}^1$, the map $f_{0,\theta}$ is continuous. So we have to prove that the map $\mathsf{fib}(T_0)$ (that is, the map $s \mapsto f_{0,s}$) is continuous.

In the rest of the proof we will denote

$$\mathbb{IB}^{\frown}_{0} := \bigcup_{i \in \mathfrak{D}_0} B^{\frown}_i(i^*) \subset \mathbb{B}^{\frown}_{0}.$$

Clearly, since for every $i \in \mathbb{Z}$, the maps m_i and M_i are continuous on $B^{\frown}_i[i^*]$, it follows that the map $s \mapsto f_{0,s}$ is continuous on $\mathbb{IB}^{\frown}_{0}$. Thus, we have to see that the fibre map function is continuous at every $\theta \in \mathbb{S}^1 \backslash \mathbb{IB}^{\frown}_{0}$; that is, $\lim_{j\to\infty} f_{0,\theta_j} = f_{0,\theta}$ for every $\{\theta_j\}_{j=1}^{\infty} \subset \mathbb{S}^1$ converging to θ. Given $\alpha > 0$, we can consider four sets associated to such a sequence:

$$\begin{aligned}
&\{j \in \mathbb{N} : \theta_j \in \mathbb{S}^1 \backslash \mathbb{IB}^{\frown}_{0}\}, \quad \{j \in \mathbb{N} : \theta_j \in \mathbb{IB}^{\frown}_{0} \backslash B_\alpha(\theta)\},\\
&\{j \in \mathbb{N} : \theta_j \in (\theta, \theta+\alpha) \cap \mathbb{IB}^{\frown}_{0}\} \quad \text{and} \quad \{j \in \mathbb{N} : \theta_j \in (\theta-\alpha, \theta) \cap \mathbb{IB}^{\frown}_{0}\}.
\end{aligned}$$

Observe that the second set $\{j \in \mathbb{N} : \theta_j \in \mathbb{IB}^{\frown}_{0} \backslash B_\alpha(\theta)\}$ is always finite and that any of the other three sets gives rise to a subsequence of $\{\theta_j\}_{j=1}^{\infty}$ converging to θ, when it is infinite. Consequently, the continuity of the fibre map function $s \mapsto f_{0,s}$ at θ is equivalent to the fact that $\lim_{j\to\infty} f_{0,\theta_j} = f_{0,\theta}$ for every $\{\theta_j\}_{j=1}^{\infty}$ converging to θ and such that, for some $\alpha > 0$, $\{\theta_j\}_{j=1}^{\infty}$ is contained either in $\mathbb{S}^1 \backslash \mathbb{IB}^{\frown}_{0}$, or $(\theta, \theta+\alpha) \cap \mathbb{IB}^{\frown}_{0}$, or $(\theta-\alpha, \theta) \cap \mathbb{IB}^{\frown}_{0}$. We will only deal with the first two cases since the proof in the last case (for $(\theta-\alpha, \theta)$) can be done symmetrically.

Case 2.47. Case 1: $\lim_{j\to\infty} \theta_j = \theta$ and $\{\theta_j\}_{j=1}^{\infty} \subset \mathbb{S}^1 \backslash \mathbb{IB}^{\frown}_{0}$.

By Definition 2.41 and Lemma 2.46, f_{0,θ_j} (respectively $f_{0,\theta}$) is the unique piecewise affine map with two affine pieces whose graph joins the point $(-2, 2)$ with $(0, \gamma(R_\omega(\theta_j)))$ (respectively $(0, \gamma(R_\omega(\theta)))$), and this with the point $(2, -2)$. By Lemma 2.35(c) and Definition 2.23 the function γ is continuous at $R_\omega(\theta) \notin O^*(\omega)$. Hence, $\lim_{j\to\infty} \gamma(R_\omega(\theta_j)) = \gamma(R_\omega(\theta))$ and, thus, $\lim_{j\to\infty} f_{0,\theta_j} = f_{0,\theta}$.

Case 2.48. Case 2: $\lim_{j\to\infty} \theta_j = \theta$ and $\{\theta_j\}_{j=1}^{\infty} \subset (\theta, \theta+\alpha) \cap \mathbb{IB}^{\frown}_{0}$.

If there exists $i \in \mathfrak{D}_0$ such that θ is the left endpoint of $B^{\frown}_i[i^*] \subset \mathbb{B}^{\frown}_0$ then the result follows from Definition 2.41, the continuity of the maps m_i and M_i and the continuity of the maps g_i (Lemmas 2.30(a) and 2.32(a)).

Assume now that θ is not the left endpoint of $B^{\frown}_i(i^*)$ for every $i \in \mathfrak{D}_0$. For every $j \in \mathbb{N}$ we set $\mathsf{i}_j := \mathsf{b}^{\frown}(\theta_j, 0) \in \mathfrak{D}_0$ (that is, $\theta_j \in B^{\frown}_{\mathsf{i}_j}((\mathsf{i}_j)^*)$).

We claim that $\lim_{j\to\infty} |\mathsf{i}_j| = \infty$ and consequently, by Definition 2.18(R.1),

$$\lim_{j\to\infty} 2^{-n_{|\mathsf{i}_j+1|}} = \lim_{j\to\infty} 2^{-n_{|\mathsf{i}_j|}} = 0. \tag{2.1}$$

To prove this claim, assume by way of contradiction that there exists L such that for every $k \in \mathbb{N}$ there exists $j_k \geq k$ such that $|\mathsf{i}_{j_k}| \leq L$. Then,

$$\{\theta_{j_k}\}_{k=1}^{\infty} \subset \bigcup_{k=1}^{\infty} B_{\mathsf{i}_{j_k}}^{\frown}((\mathsf{i}_{j_k})^*)$$

and, since $\{\mathsf{i}_{j_k} : k \in \mathbb{N}\}$ is finite, it follows that there exists $i \in \{\mathsf{i}_{j_k} : k \in \mathbb{N}\} \subset \mathfrak{D}_0$ and a subsequence of $\{\theta_{j_k}\}_{k=1}^{\infty}$, that by abuse of notation will also be called $\{\theta_{j_k}\}$, such that $\{\theta_{j_k}\}_{k=1}^{\infty} \subset B_i^{\frown}(i^*)$. So,

$$\theta = \lim_{k\to\infty} \theta_{j_k} \in B_i^{\frown}[i^*]\,;$$

a contradiction. So, the claim (and hence (2.1)) holds.

Next we claim that the conditions

$$\lim_{j\to\infty} M_{\mathsf{i}_j}(\theta_j) = \lim_{j\to\infty} m_{\mathsf{i}_j}(\theta_j) = 0, \text{ and} \tag{2.2}$$

there exists a sequence $\{x_j\}_{j=1}^{\infty}$ with $x_j \in \mathbb{I}_{\mathsf{i}_j,\theta_j} = [m_{\mathsf{i}_j}(\theta_j), M_{\mathsf{i}_j}(\theta_j)]$ for every j, such that $\lim_{j\to\infty} f_{0,\theta_j}(x_j) = \gamma(R_\omega(\theta))$ (2.3)

imply

$$\lim_{j\to\infty} f_{0,\theta_j} = f_{0,\theta}.$$

To prove the claim notice that, by Definition 2.41 and Lemma 2.46, $f_{0,\theta}$ is the unique piecewise affine map with two affine pieces whose graph joins the point $(-2, 2)$ with $(0, \gamma(R_\omega(\theta)))$, and this with the point $(2, -2)$. On the other hand, for every j,

- $f_{0,\theta_j}\big|_{[-2,m_{\mathsf{i}_j}(\theta_j)]}$ is the affine map joining the point $(-2, 2)$ with the point $(m_{\mathsf{i}_j}(\theta_j), g_{\mathsf{i}_j}(\theta_j, m_{\mathsf{i}_j}(\theta_j)))$, and
- $f_{0,\theta_j}\big|_{[M_{\mathsf{i}_j}(\theta_j),2]}$ is the affine map joining the point $(M_{\mathsf{i}_j}(\theta_j), g_{\mathsf{i}_j}(\theta_j, M_{\mathsf{i}_j}(\theta_j)))$ with the point$(2, -2)$

(see Figure 2.5). Moreover, from the part of the proposition already proven we know that f_{0,θ_j} is non-increasing and continuous. Therefore, the claim holds provided that

$$\lim_{j\to\infty} \operatorname{diam}\left(f_{0,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right)\right) = 0$$

(see again Figure 2.5).

When $\theta_j \in B_{\alpha_{\mathsf{i}_j}}[(\mathsf{i}_j)^*] \setminus B_{\alpha_{\mathsf{i}_j+1}}((\mathsf{i}_j)^*)$ and $\mathsf{i}_j \geq 0$, by Definitions 2.41 and 2.29,

$$\operatorname{diam}\left(f_{0,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right)\right) = \operatorname{diam}\left(g_{\mathsf{i}_j,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right)\right) = \operatorname{diam}\left(\{\gamma_{\mathsf{i}_j+1}(R_\omega(\theta_j))\}\right) = 0.$$

Otherwise, by Definition 2.41, and Lemmas 2.30(b) and 2.32(b),

$$\begin{aligned}\{R_\omega(\theta_j)\} \times f_{0,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right) = \{R_\omega(\theta_j)\} \times g_{\mathsf{i}_j,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right) = G_{\mathsf{i}_j}\left(\mathcal{R}((\mathsf{i}_j)^*)^{\theta_j}\right)\\ \subset \mathcal{R}((\mathsf{i}_j+1)^*)^{R_\omega(\theta_j)}.\end{aligned}$$

So, by Remark 2.16(2),

$$\operatorname{diam}\left(f_{0,\theta_j}\left(\mathbb{I}_{\mathsf{i}_j,\theta_j}\right)\right) \leq \operatorname{diam}\left(\mathcal{R}((\mathsf{i}_j+1)^*)\right) \leq 2\cdot 2^{-n_{|\mathsf{i}_j+1|}}.$$

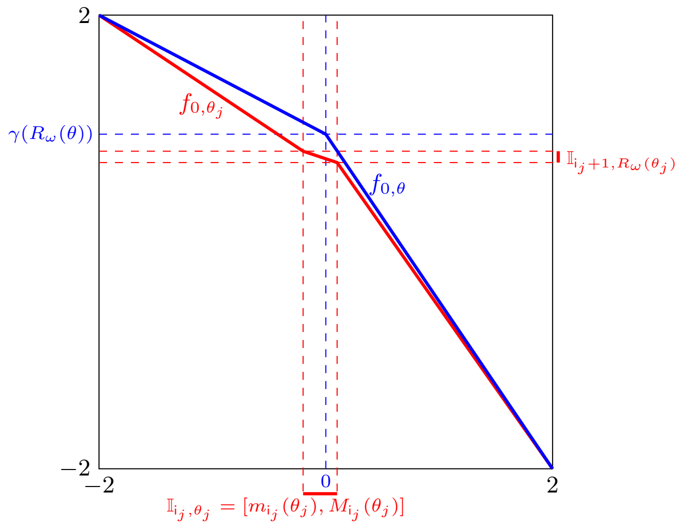


Figure 2.5: A symbolic representation of the maps $f_{0,\theta}$ and f_{0,θ_j} in Case 2 of the proof of Proof of Proposition 2.43 for $m = 0$. The map $f_{0,\theta}$ and the points 0 and $\gamma(R_\omega(\theta))$ are drawn in blue. The map f_{0,θ_j} and the corresponding intervals $\mathbb{I}_{\mathrm{i}_j,\theta_j}$ and $\mathbb{I}_{\mathrm{i}_j+1,R_\omega(\theta_j)}$ are drawn in red.

In any case,

$$0 \leq \operatorname{diam}\left(f_{0,\theta_j}\left(\mathbb{I}_{\mathrm{i}_j,\theta_j}\right)\right) \leq 2 \cdot 2^{-n_{|\mathrm{i}_j+1|}} \quad \text{for every} \quad j \in \mathbb{N}$$

and, by (2.1), $\lim_{j\to\infty} \operatorname{diam}\left(f_{0,\theta_j}\left(\mathbb{I}_{\mathrm{i}_j,\theta_j}\right)\right) = 0$. This ends the proof of the claim.

By the last claim, to end the proof of the proposition in the case $m = 0$ it is enough to show that (2.2–2.3) hold. We start by proving (2.2). By Lemma 2.46,

$$m_{\mathrm{i}_j}(\operatorname{Bd}(B_{\mathrm{i}_j}^{\frown}[(\mathrm{i}_j)^*])) = M_{\mathrm{i}_j}(\operatorname{Bd}(B_{\mathrm{i}_j}^{\frown}[(\mathrm{i}_j)^*])) = 0,$$

and from the definition of the maps m_{i_j} and M_{i_j}, Definition 2.15 (or Lemma 2.28) and Remark 2.16(2), for every $s \in B_{\mathrm{i}_j}^{\frown}((\mathrm{i}_j)^*)$ we get

$$\begin{aligned} &-1 \leq m_{\mathrm{i}_j}(s) < 0 < M_{\mathrm{i}_j}(s) \leq 1, \text{ and} \\ &M_{\mathrm{i}_j}(s) - m_{\mathrm{i}_j}(s) = \operatorname{diam}\left(\mathbb{I}_{\mathrm{i}_j,s}\right) \leq 2 \cdot 2^{-n_{|\mathrm{i}_j|}}. \end{aligned} \tag{2.4}$$

So, (2.2) holds by (2.1). Now we prove (2.3).

By (2.1), (2.2) and (2.8), it follows that

$$\begin{aligned} &m_{\mathrm{i}_j}(\theta_j) < \gamma_{|\mathrm{i}_j|}(\theta_j) < M_{\mathrm{i}_j}(\theta_j) && \text{if } \theta_j \neq (\mathrm{i}_j)^*, \text{ and} \\ &m_{\mathrm{i}_j}(\theta_j) < \gamma_{|\mathrm{i}_j|-1}(\theta_j) = 0 < M_{\mathrm{i}_j}(\theta_j) && \text{if } \theta_j = (\mathrm{i}_j)^*. \end{aligned}$$

Also, from Definition 2.41, the definitions of G_i and $g_{i,\theta}$ (Definitions 2.29 and 2.31), and Lemmas 2.30(c) and 2.32(c) we get

$$f_{0,\theta_j}(\gamma_{|\mathsf{i}_j|}(\theta_j)) = g_{\mathsf{i}_j,\theta_j}(\gamma_{|\mathsf{i}_j|}(\theta_j)) = \gamma_{|\mathsf{i}_j+1|}(R_\omega(\theta_j)) \qquad \text{if } \theta_j \neq (\mathsf{i}_j)^*,$$
$$f_{0,\theta_j}(\gamma_{\mathsf{i}_j-1}(\theta_j)) = g_{\mathsf{i}_j,\theta_j}(\gamma_{\mathsf{i}_j-1}(\theta_j)) = \gamma_{\mathsf{i}_j}(R_\omega(\theta_j)) \qquad \text{if } \theta_j = (\mathsf{i}_j)^* \text{ and } \mathsf{i}_j \geq 0, \text{ and}$$
$$f_{0,\theta_j}(\gamma_{|\mathsf{i}_j|-1}(\theta_j)) = g_{\mathsf{i}_j,\theta_j}(\gamma_{|\mathsf{i}_j+1|}(\theta_j)) = \gamma_{|\mathsf{i}_j+2|}(R_\omega(\theta_j)) \qquad \text{if } \theta_j = (\mathsf{i}_j)^* \text{ and } \mathsf{i}_j < 0.$$

Thus, to prove (2.3), we have to show that

$$\begin{cases} \lim_{j\to\infty} \gamma_{|\mathsf{i}_j+1|}(R_\omega(\theta_j)) = \gamma(R_\omega(\theta)) & \text{if } \theta_j \neq (\mathsf{i}_j)^*, \\ \lim_{j\to\infty} \gamma_{\mathsf{i}_j}(R_\omega(\theta_j)) = \gamma(R_\omega(\theta)) & \text{if } \theta_j = (\mathsf{i}_j)^* \text{ and } \mathsf{i}_j \geq 0, \text{ and} \\ \lim_{j\to\infty} \gamma_{|\mathsf{i}_j+2|}(R_\omega(\theta_j)) = \gamma(R_\omega(\theta)) & \text{if } \theta_j = (\mathsf{i}_j)^* \text{ and } \mathsf{i}_j < 0 \end{cases} \tag{2.5}$$

(that is, we take $x_j := \gamma_{|\mathsf{i}_j|}(\theta_j)$ if $\theta_j \neq (\mathsf{i}_j)^*$, $x_j := \gamma_{\mathsf{i}_j-1}(\theta_j)$ if $\theta_j = (\mathsf{i}_j)^*$ and $\mathsf{i}_j \geq 0$, and $x_j := \gamma_{|\mathsf{i}_j|-1}(\theta_j)$ if $\theta_j = (\mathsf{i}_j)^*$ and $\mathsf{i}_j < 0$).

Let $\varepsilon > 0$. By Lemma 2.35(c) and Definition 2.18(R.1) we have that $\theta \notin O^*(\omega)$ and, hence, $R_\omega(\theta) \notin O^*(\omega)$. By the continuity of γ on $\mathbb{S}^1 \backslash O^*(\omega)$ and the fact that $\lim_{i\to\infty} \gamma_i = \gamma$, there exist $\delta > 0$ and $L \in \mathbb{N}$ such that

$$|\gamma(R_\omega(\theta)) - \gamma(\widehat{\theta})| < \varepsilon/2 \quad \text{for every } \widehat{\theta} \in B_\delta\left(R_\omega(\theta)\right) \backslash O^*(\omega), \text{ and}$$
$$d_\infty\left(\gamma, \gamma_i\right) < \varepsilon/2 \quad \text{for every } i \geq L.$$

Then, since $\lim_{j\to\infty} \theta_j = \theta$ and $\lim_{j\to\infty} |\mathsf{i}_j| = \infty$, there exists $N \in \mathbb{N}$ such that $|\theta - \theta_j| < \delta/2$, and $|\mathsf{i}_j| \geq L + 2$ for every $j \geq N$.

First we will show that

$$|\gamma(R_\omega(\theta)) - \gamma_{|\mathsf{i}_j+1|}(R_\omega(\theta_j))| \leq \varepsilon$$

for every $j \geq N$ such that $\theta_j \neq (\mathsf{i}_j)^*$. To see it observe that, by Definition 2.18(R.2) and Remark 2.19(R.2), $\theta_j, R_\omega(\theta_j) \notin Z^*_{|\mathsf{i}_j+1|}$ whenever $\theta_j \neq (\mathsf{i}_j)^*$. Thus, $\gamma_{|\mathsf{i}_j+1|}$ is continuous at $R_\omega(\theta_j)$ by Lemma 2.20(a).

Also, there exists a sequence $\{\widehat{\theta}_{j\ell}\}_{\ell=1}^\infty \subset \left(B_{\delta/2}(\theta_j) \cap B^{\frown}_{\mathsf{i}_j}((\mathsf{i}_j)^*)\right) \backslash O^*(\omega)$ converging to θ_j, because $\mathbb{S}^1 \backslash O^*(\omega)$ is dense in $\mathbb{S}^1$. Clearly, for every $j \geq N$, we have $\{R_\omega(\widehat{\theta}_{j\ell})\}_{\ell=1}^\infty \subset B_\delta(R_\omega(\theta)) \backslash O^*(\omega)$ and $\lim_{\ell\to\infty} R_\omega(\widehat{\theta}_{j\ell}) = R_\omega(\theta_j)$. Moreover, since $\{R_\omega(\widehat{\theta}_{j\ell})\}_{\ell=1}^\infty \subset \mathbb{S}^1 \backslash O^*(\omega) \subset \mathbb{S}^1 \backslash Z^*_{|\mathsf{i}_j+1|}$, $\gamma_{|\mathsf{i}_j+1|}$ is defined for every $R_\omega(\widehat{\theta}_{j\ell})$. Then, for every $j \geq N$ and $\ell \in \mathbb{N}$, we have

$$\begin{aligned} |\gamma(R_\omega(\theta)) - \gamma_{|\mathsf{i}_j+1|}(R_\omega(\widehat{\theta}_{j\ell}))| &\leq |\gamma(R_\omega(\theta)) - \gamma(R_\omega(\widehat{\theta}_{j\ell}))| + \\ &\qquad |\gamma(R_\omega(\widehat{\theta}_{j\ell})) - \gamma_{|\mathsf{i}_j+1|}(R_\omega(\widehat{\theta}_{j\ell}))| \\ &< \tfrac{\varepsilon}{2} + d_\infty\Big(\gamma, \gamma_{|\mathsf{i}_j+1|}\Big) < \varepsilon. \end{aligned}$$

Consequently,

$$|\gamma(R_\omega(\theta)) - \gamma_{|\mathsf{i}_j+1|}(R_\omega(\theta_j))| = \lim_{\ell\to\infty} |\gamma(R_\omega(\theta)) - \gamma_{|\mathsf{i}_j+1|}(R_\omega(\widehat{\theta}_{j\ell}))| \leq \varepsilon$$

This ends the proof of the first equality of (2.5). The second and third equalities of (2.5) follow as above by replacing $\gamma_{|\mathsf{i}_j+1|}$ by γ_{i_j} (respectively $\gamma_{|\mathsf{i}_j+2|}$), and noting that

$$R_\omega(\theta_j) = R_\omega((\mathsf{i}_j)^*) = \begin{cases} ((\mathsf{i}_j+1))^* \notin Z^*_{\mathsf{i}_j} & \text{if } \mathsf{i}_j \geq 0, \text{ and} \\ ((-(|\mathsf{i}_j|-1)))^* \notin Z^*_{|\mathsf{i}_j|-2} & \text{if } \mathsf{i}_j < 0. \end{cases}$$

This ends the proof of the continuity of T_0, and the proposition for the case $m = 0$.

2.9 Proof of Proposition 2.43 for $m > 0$

This section is the second technical counterpart of Section 2.7 and is devoted to prove Proposition 2.43 for every map T_m with $m > 0$. To do this we will need some more technical results. Also we will use the notion of fibre map function introduced in the previous section.

The next two lemmas establish some basic properties of the maps $T_m\big|_{\mathbb{V}^{\frown}_m}$ and clarify some aspects of Definition 2.42.

Lemma 2.49. *For every $m \in \mathbb{N}$ and for every $\theta \in \mathbb{B}^{\frown}_m$,*

$$f_{m,\theta}\big|_{\mathbb{I}_{\mathsf{i},\theta}} = g_{\mathsf{i},\theta}\big|_{\mathbb{I}_{\mathsf{i},\theta}},$$

where $\mathsf{i} = \mathsf{b}^{\frown}(\theta, m)$. Moreover, assume that $\theta \in \mathbb{WB}^{\frown}_m \setminus \mathbb{WIB}_m$. Then,

$$f_{m,\theta}(x) = \begin{cases} g_{\mathsf{i},\theta}(x) & \text{if } x \in \mathbb{I}_{\mathsf{i},\theta}, \\ \frac{2-g_{\mathsf{i},\theta}(m_\mathsf{i}(\theta))}{2-f_{m-1,\theta}(m_\mathsf{i}(\theta))}(f_{m-1,\theta}(x)-2)+2 & \text{if } x \in [-2, m_\mathsf{i}(\theta)], \\ \frac{2+g_{\mathsf{i},\theta}(M_\mathsf{i}(\theta))}{2+f_{m-1,\theta}(M_\mathsf{i}(\theta))}(f_{m-1,\theta}(x)+2)-2 & \text{if } x \in [M_\mathsf{i}(\theta), 2]. \end{cases}$$

Proof. We start by proving the first statement. When $\theta \in \mathbb{B}_m$ there is nothing to prove. So, assume that $\theta \in \mathbb{B}^{\frown}_m \setminus \mathbb{B}_m$. By Definition 2.38, $\theta \in \mathbb{WB}^{\frown}_m$, $\mathsf{i} < 0$ and $\theta \in B^{\frown}_\mathsf{i}[\mathsf{i}^*] \setminus B_{\alpha_{|\mathsf{i}|}}(\mathsf{i}^*)$. By Lemma 2.39(b),

$$\mathbb{I}_{\mathsf{i},\theta} = \{\gamma_{|\mathsf{i}|}(\theta)\} \subset \mathbb{IW}_{m,\theta}.$$

Consequently, by Definition 2.42 and the definition of the maps $g_{\mathsf{i},\theta}$ for $\mathsf{i} < 0$ (Definition 2.31 — notice that $\mathbb{I}_{\mathsf{i},\theta} \subset \mathcal{R}^{\frown}(\mathsf{i}^*)$ by definition),

$$f_{m,\theta}\left(\gamma_{|\mathsf{i}|}(\theta)\right) = \gamma_{|\mathsf{i}+1|}\left(R_\omega(\theta)\right) = g_{\mathsf{i},\theta}\left(\gamma_{|\mathsf{i}|}(\theta)\right).$$

So, the first statement holds. Now we prove the second one. By Lemma 2.39(b),

$$\mathbb{I}_{\mathsf{i},\theta} = \{m_\mathsf{i}(\theta)\} = \{M_\mathsf{i}(\theta)\} = \{\gamma_{|\mathsf{i}|}(\theta)\} = \{\lambda_m(\theta)\} = \{\tau_m(\theta)\} = \mathbb{IW}_{m,\theta}.$$

Thus, by the part already proven, the formulas

$$\begin{cases} g_{\mathsf{i},\theta}(x) & \text{if } x \in \mathbb{I}_{\mathsf{i},\theta}, \\ \frac{2-g_{\mathsf{i},\theta}(m_\mathsf{i}(\theta))}{2-f_{m-1,\theta}(m_\mathsf{i}(\theta))}(f_{m-1,\theta}(x)-2)+2 & \text{if } x \in [-2, m_\mathsf{i}(\theta)], \\ \frac{2+g_{\mathsf{i},\theta}(M_\mathsf{i}(\theta))}{2+f_{m-1,\theta}(M_\mathsf{i}(\theta))}(f_{m-1,\theta}(x)+2)-2 & \text{if } x \in [M_\mathsf{i}(\theta), 2], \end{cases}$$

and

$$\begin{cases} \gamma_{|i+1|}\left(R_\omega(\theta)\right) & \text{if } x \in \mathbb{IW}_{m,\theta}, \\ \frac{2-\gamma_{|i+1|}(R_\omega(\theta))}{2-f_{m-1,\theta}(\lambda_m(\theta))}(f_{m-1,\theta}(x)-2)+2 & \text{if } x \in [-2, \lambda_m(\theta)], \\ \frac{2+\gamma_{|i+1|}(R_\omega(\theta))}{2+f_{m-1,\theta}(\tau_m(\theta))}(f_{m-1,\theta}(x)+2)-2 & \text{if } x \in [\tau_m(\theta), 2], \end{cases}$$

coincide.

Lemma 2.50. *The following statements hold for every $m \in \mathbb{N}$ and $i \in \mathfrak{D}_m$:*

(a) The map $T_m\big|_{\mathsf{V}_{i^*}^{\frown}}$ is well defined and continuous.
(b) For every $\theta \in B_i^{\frown}[i^*]$,

(b.i) $f_{m,\theta}(2) = -2$ and $f_{m,\theta}(-2) = 2$,

(b.ii) $f_{m,\theta}$ is piecewise affine and non-increasing, and

(b.iii) $-1 \leq f_{m,\theta}\left(M_i(\theta)\right) \leq f_{m,\theta}\left(m_i(\theta)\right) \leq 1$.

(c) $T_m\big|_{\mathcal{R}^{\frown}(i^*)} = G_i$ and $T_m\left(\mathfrak{A}_{|i|}^{i^*}\right) = \mathfrak{A}_{|i+1|}^{(i+1)^*}$.

Proof. Clearly, $T_m\big|_{\mathsf{V}_{i^*}^{\frown}}$ is well defined and continuous if and only if so is $f_m\big|_{\mathsf{V}_{i^*}^{\frown}}$.

We will prove by induction on $m \in \mathbb{Z}^+$ that, (a), (b) and

(b.iv) $f_{m,\theta}\big|_{[-2,-1]}$ and $f_{m,\theta}\big|_{[1,2]}$ are affine, $f_{m,\theta}(-1) < 2$ and $f_{m,\theta}(1) > -2$

hold for every $\theta \in B_i^{\frown}[i^*]$.

First we will show that (a), (b) and (b.iv) hold for $m = 0$ and $i \in \mathfrak{D}_0$ (we are including the map f_0 studied earlier to correctly start the induction process). By Proposition 2.43(a,b) for $m = 0$ we have that $T_0\big|_{\mathsf{V}_{i^*}^{\frown}}$ is well defined and continuous and (b) holds. By Definition 2.41, we also know that $f_{m,\theta}\big|_{[-2,m_i(\theta)]}$ and $f_{m,\theta}\big|_{[M_i(\theta),2]}$ are affine. Then, (b.iv) follows from $-1 \leq m_i(\theta) \leq M_i(\theta) \leq 1$ (see Lemma 2.28(a)) and (b.iii).

Assume now that (a), (b) and (b.iv) hold for some $m-1 \in \mathbb{Z}+$ and prove it for m and $i \in \mathfrak{D}_m$. By Lemma 2.34(a), $\theta \in B_i^{\frown}[i^*] \subsetneq B_k^{\frown}[k^*]$ for some $k \in \mathfrak{D}_{m-1}$. Consequently, $\mathsf{V}_{i^*}^{\frown} \subset \mathsf{V}_{k^*}^{\frown}$ and $f_{m-1}\big|_{\mathsf{V}_{i^*}^{\frown}}$ is well defined and continuous.

By Lemma 2.28(a) and Definition 2.38,

$$\begin{aligned} &-1 \leq m_i(\theta) \leq M_i(\theta) \leq 1 && \text{for } \theta \in B_i^{\frown}[i^*], \text{ and} \\ &-1 \leq \lambda_m(\theta) \leq \tau_m(\theta) \leq 1 && \text{for } \theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset \mathbb{WB}_m^{\frown}\ (i<0). \end{aligned} \tag{2.1}$$

Consequently, by (b.ii) and (b.iv) for $m-1$,

$$-2 < f_{m-1,\theta}(1) \leq f_{m-1,\theta}\left(M_i(\theta)\right) \leq f_{m-1,\theta}\left(m_i(\theta)\right) \leq f_{m,\theta}(-1) < 2$$

for every $\theta \in B_i^{\frown}[i^*]$, and

$$-2 < f_{m-1,\theta}(1) \leq f_{m-1,\theta}\left(\tau_m(\theta)\right) \leq f_{m-1,\theta}\left(\lambda_m(\theta)\right) \leq f_{m,\theta}(-1) < 2$$

for $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset \mathbb{WB}_m^{\frown}$ when $i < 0$.

On the other hand, as it was observed in Definition 2.42, $f_{m,\theta}$ is defined in two different ways when $\theta \in \mathbb{WB}_m^{\frown} \cap \mathbb{B}_m$. In such a case, by Lemmas 2.39(a,b) and 2.49, $\theta \notin \mathbb{WIB}_m$ and both definitions for $f_{m,\theta}$ coincide. Hence, $f_m\big|_{\mathsf{V}_{i^*}^{\frown}}$ is well defined.

Now we prove that $f_m\big|_{\mathsf{V}_{i^*}^{\frown}}$ is continuous by using the continuity of $f_{m-1}\big|_{\mathsf{V}_{i^*}^{\frown}}$. Since $B_{\alpha_{|i|}}[i^*] \subset \mathbb{B}_m$, by Definition 2.42, the continuity of the maps m_i and M_i (see Lemma 2.28(b)), and the continuity of the maps g_i (Lemmas 2.30(a) and 2.32 (a)), $f_m\big|_{\Uparrow B_{\alpha_{|i|}}[i^*]}$ is continuous. Now we assume that $i < 0$ and we study the continuity of $f_m\big|_{\Uparrow U}$ on a connected component U of $B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$. Observe that, by Definition 2.38 and Lemma 2.34(b), U is a connected component of $\mathbb{WB}_m^{\frown}$. Then, again by Definition 2.42, the continuity of the maps $\lambda_m\big|_U$ and $\tau_m\big|_U$ (Lemma 2.40), and the continuity of the map $\gamma_{|i|}\big|_U$ (Lemma 2.20(a) and Definition 2.18(R.2) and Remark 2.19(R.2)), $f_m\big|_{\Uparrow U}$ is continuous. Therefore, $f_m\big|_{\mathsf{V}_{i^*}^{\frown}}$ is continuous because it is well defined on $\Uparrow\left(\left(B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*)\right) \cap B_{\alpha_{|i|}}[i^*]\right)$.

Let $\theta \in B_{\alpha_{|i|}}[i^*] \subset \mathbb{B}_m$. By Definition 2.42, and the definition of the maps $g_{i,\theta}$ (Definitions 2.29 and 2.31), $f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}}$ is piecewise affine and non-increasing. So, by Lemma 2.49 for $m-1$ and Definition 2.42, $f_{m,\theta}(2) = -2$, $f_{m,\theta}(-2) = 2$, and $f_{m,\theta}\big|_{[-2,m_i(\theta)]}$ and $f_{m,\theta}\big|_{[M_i(\theta),2]}$ are affine transformations of the map $f_{m-1,\theta}$ with positive slope. Hence, (b.i,ii) hold for $f_{m,\theta}$ in this case. Moreover, (b.iv) is verified by (2.1) and (b.iv) for $m-1$.

Consider $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset \mathbb{WB}_m^{\frown}$. Again by Definition 2.42, $f_{m,\theta}\big|_{\mathbb{IW}_{m,\theta}}$ is constant. Then, (b.i,ii) and (b.iv) hold for $f_{m,\theta}$ as above by replacing $m_i(\theta)$ and $M_i(\theta)$ by $\lambda_m(\theta)$ and $\tau_m(\theta)$, respectively.

By (b.ii) and (2.1) we have $f_{m,\theta}\left(M_i(\theta)\right) \leq f_{m,\theta}\left(m_i(\theta)\right)$. Hence, (b.iii) follows from Lemma 2.49, Definition 2.42, Lemmas 2.30(b) and 2.25(c), Definition 2.18(R.2) and Remark 2.19(R.2), Lemma 2.32(b) and Lemma 2.20(b).

(c) In a similar way to the proof of Proposition 2.43 for the case $m = 0$,

$$\mathcal{R}^{\frown}(i^*) = \bigcup_{\theta \in B_i^{\frown}[i^*]} \{\theta\} \times \mathbb{I}_{i,\theta} \subset \mathsf{V}_{i^*}^{\frown} \subset \mathbb{V}_m^{\frown}$$

and, by Definition 2.42, Lemma 2.49 and the definition of G_i (Definitions 2.29 and 2.31) it follows that

$$T_m(\theta, x) = \left(R_\omega(\theta), f_m(\theta, x)\right) = \left(R_\omega(\theta), g_{i,\theta}(x)\right) = G_i(\theta, x),$$

for every $(\theta, x) \in \mathcal{R}^{\frown}(i^*)$. Thus, $T_m\left(\mathfrak{A}_{|i|}^{i^*}\right) = \mathfrak{A}_{|i+1|}^{(i+1)^*}$ from Lemmas 2.25(b), 2.30(c) and 2.32(c).

The next technical lemma compares the images of $f_{m,\theta}$ and $f_{m-1,\theta}$ on a point. It is an extension of Lemma 2.36.

Lemma 2.51. *Assume that $B_i^{\frown}[i^*] \subset B_k^{\frown}[k^*]$ for some $i \in \mathfrak{D}_m$, $k \in \mathfrak{D}_{m-1}$ and $m \in \mathbb{N}$. Then, for every $\theta \in B_i^{\frown}[i^*] \setminus B_{\alpha_{|i|}}(i^*)$, $m_i(\theta) = M_i(\theta) = \gamma_i(\theta)$ and*

$$f_{m,\theta}(m_i(\theta)) = g_{i,\theta}(m_i(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)), \text{ and}$$
$$f_{m-1,\theta}(m_i(\theta)) = g_{k,\theta}(m_i(\theta)) = \gamma_{|k+1|}(R_\omega(\theta)).$$

Proof. The fact that $m_i(\theta) = M_i(\theta) = \gamma_i(\theta)$ follows directly from the definitions. The first equation follows from Lemma 2.49, and the definition of the map $g_{i,\theta}$ (Definitions 2.29 and 2.31).

By Lemma 2.36, $\mathbb{I}_{i,\theta} = \{m_i(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}$. Moreover, as in the proof of Lemma 2.36, $\theta \neq k^*$. Consequently, by Definition 2.41, Lemma 2.49, Lemmas 2.30(c) and 2.32(c) and (2.1) (alternatively, for the last equality check directly the proofs of the Lemmas 2.30(c) and 2.32(c)),

$$f_{m-1,\theta}(m_i(\theta)) = g_{k,\theta}(m_i(\theta)) = g_{k,\theta}\left(\gamma_{|k|}(\theta)\right) = \gamma_{|k+1|}(R_\omega(\theta)).$$

The following lemma is the analogue of Lemma 2.46 for $m \geq 1$. To state it we will use the set

$$\upuparrows\mathbb{EB}_m^{\frown} = \mathbb{EB}_m^{\frown} \times \mathbb{I} \subset \mathbb{V}_m^{\frown}.$$

Lemma 2.52. $T_m\big|_{\upuparrows\mathbb{EB}_m^{\frown}} = T_{m-1}\big|_{\upuparrows\mathbb{EB}_m^{\frown}}$ *for every $m \in \mathbb{N}$. Equivalently, $f_{m,\theta} = f_{m-1,\theta}$ for every $m \in \mathbb{N}$ and $\theta \in \mathbb{EB}_m^{\frown}$.*

Proof. Fix $m \in \mathbb{N}$ and $\theta \in \mathbb{EB}_m^{\frown} \subset \mathbb{B}_m^{\frown}$. By Lemma 2.34(a,b), there exist $i \in \mathfrak{D}_m$ and $k \in \mathfrak{D}_{m-1}$ such that $\theta \in \operatorname{Bd}(B_i^{\frown}[i^*]) \subset B_i^{\frown}[i^*] \subsetneq B_k^{\frown}[k^*]$. So, we are in the assumptions of Lemmas 2.36 and 2.51 and, hence,

$$\mathbb{I}_{i,\theta} = \{m_i(\theta)\} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta},$$
$$f_{m,\theta}(m_i(\theta)) = g_{i,\theta}(m_i(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)), \text{ and}$$
$$f_{m-1,\theta}(m_i(\theta)) = g_{k,\theta}(m_i(\theta)) = \gamma_{|k+1|}(R_\omega(\theta)).$$

Thus, if $i \geq 0$, $\theta \in \mathbb{B}_m$ and, by Definition 2.42 and Lemma 2.50(a), to prove that $f_{m,\theta} = f_{m-1,\theta}$ we only have to show that

$$g_{i,\theta}(m_i(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{|k+1|}(R_\omega(\theta)) = f_{m-1,\theta}(m_i(\theta)).$$

When $i < 0$, $\theta \in \mathbb{WB}_m^{\frown} \cap \mathbb{EB}_m^{\frown}$ and, by Lemma 2.39(a), $\theta \notin \mathbb{WIB}_m$. Then, by Lemma 2.49, we get again that

$$g_{i,\theta}(m_i(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{|k+1|}(R_\omega(\theta)) = f_{m-1,\theta}(m_i(\theta)).$$

implies $f_{m,\theta} = f_{m-1,\theta}$.

If $|k+1| = |i+1|$ there is nothing to prove. So, by Lemma 2.36, we can assume that $|k+1| < |i+1|$ and we have

$$\gamma_{|k+1|}(R_\omega(\theta)) = \gamma_{|k+1|+1}(R_\omega(\theta)) = \cdots = \gamma_{|i+1|-1}(R_\omega(\theta)).$$

Hence, we have to show that $\gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{|i+1|-1}(R_\omega(\theta))$. If $i \geq 0$ we get

$$\gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{i+1}(R_\omega(\theta)) = \gamma_i(R_\omega(\theta)) = \gamma_{|i+1|-1}(R_\omega(\theta))$$

by Lemma 2.20(e). Otherwise we have $i < 0$, $\theta \in \operatorname{Bd}(B_i^{\frown}[i^*]) = \operatorname{Bd}\left(B_{\alpha_{|i+1|}}[i^*]\right)$ and, consequently, $R_\omega(\theta) \in \operatorname{Bd}\left(B_{\alpha_{|i+1|}}[(i+1)^*]\right)$. Again by Lemma 2.20(e) for $j = |i+1|$,

$$\gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{|i+1|-1}(R_\omega(\theta)).$$

This ends the proof of the lemma.

Now we aim at computing two different kind of upper bounds for $\|f_{m,\theta} - f_{m-1,\theta}\|$ (Lemma 2.54 and Proposition 2.44). This will be a key tool in the proof of Propositions 2.43 for $m > 0$ and 2.44. The next two lemmas and remark will be useful to automate and simplify the proofs of these two results.

Lemma 2.53.

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \begin{cases} \left\| f_{m,\theta}\big|_{\mathbb{I}_{\mathfrak{b}^{\frown}(\theta,m),\theta}} - f_{m-1,\theta}\big|_{\mathbb{I}_{\mathfrak{b}^{\frown}(\theta,m),\theta}} \right\| & \textit{when } \theta \in \mathbb{B}_m^{\frown} \setminus \mathbb{WIB}_m\textit{, and} \\ \left\| f_{m,\theta}\big|_{\mathbb{IW}_{m,\theta}} - f_{m-1,\theta}\big|_{\mathbb{IW}_{m,\theta}} \right\| & \textit{when } \theta \in \mathbb{WIB}_m, \end{cases}$$

for every $m \geq 2$ and $\theta \in \mathbb{B}_m^{\frown}$.

Proof. Set $i = \mathfrak{b}^{\frown}(\theta, m) \in \mathfrak{D}_m$, so that $\theta \in B_i^{\frown}[i^*]$.

When $\theta \in \mathbb{B}_m^{\frown} \setminus \mathbb{WIB}_m = \mathbb{B}_m \cup \mathbb{WB}_m^{\frown} \setminus \mathbb{WIB}_m$, by Definition 2.42 and Lemma 2.49, it is enough to show that

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))|$$

for every $x \in [-2, m_i(\theta)]$, and

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(M_i(\theta)) - f_{m-1,\theta}(M_i(\theta))|$$

for every $x \in [M_i(\theta), 2]$. We will prove the first statement. The second one follows similarly.

Definition 2.42 and Lemma 2.49 give

$$\begin{aligned} f_{m,\theta}(x) - f_{m-1,\theta}(x) &= \frac{2 - g_{i,\theta}(m_i(\theta))}{2 - f_{m-1,\theta}(m_i(\theta))}(f_{m-1,\theta}(x) - 2) + 2 - f_{m-1,\theta}(x) \\ &= \frac{2 - f_{m,\theta}(m_i(\theta))}{2 - f_{m-1,\theta}(m_i(\theta))}(f_{m-1,\theta}(x) - 2) - (f_{m-1,\theta}(x) - 2) \\ &= (f_{m-1,\theta}(x) - 2)\left(\frac{2 - f_{m,\theta}(m_i(\theta))}{2 - f_{m-1,\theta}(m_i(\theta))} - 1\right) \\ &= (2 - f_{m-1,\theta}(x))\frac{f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))}{2 - f_{m-1,\theta}(m_i(\theta))}. \end{aligned}$$

By Lemma 2.50(b), $2 \geq f_{m-1,\theta}(x) \geq f_{m-1,\theta}(m_i(\theta))$ and $1 \geq f_{m-1,\theta}(m_i(\theta))$. Hence,

$$
\begin{aligned}
|f_{m,\theta}(x) - f_{m-1,\theta}(x)| &= (2 - f_{m-1,\theta}(x)) \frac{|f_{m,\theta}\,(m_i(\theta)) - f_{m-1,\theta}\,(m_i(\theta))|}{2 - f_{m-1,\theta}\,(m_i(\theta))} \\
&\leq |f_{m,\theta}\,(m_i(\theta)) - f_{m-1,\theta}\,(m_i(\theta))|.
\end{aligned}
$$

Now assume that $\theta \in \mathbb{WIB}_m \subset \mathbb{WB}_m^{\frown}$. By Definition 2.42 it is enough to show that

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(\lambda_m(\theta)) - f_{m-1,\theta}(\lambda_m(\theta))|$$

for every $x \in [-2, \lambda_m(\theta)]$, and

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(\tau_m(\theta)) - f_{m-1,\theta}(\tau_m(\theta))|$$

for every $x \in [\tau_m(\theta), 2]$. As before, we will prove the first statement. The second one follows similarly. We have

$$f_{m,\theta}(x) - f_{m-1,\theta}(x) = (2 - f_{m-1,\theta}(x)) \frac{f_{m,\theta}\,(\lambda_m(\theta)) - f_{m-1,\theta}\,(\lambda_m(\theta))}{2 - f_{m-1,\theta}\,(\lambda_m(\theta))}.$$

By Lemma 2.50(b), $2 \geq f_{m-1,\theta}(x) \geq f_{m-1,\theta}\,(\lambda_m(\theta))$ and hence,

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}\,(m_i(\theta)) - f_{m-1,\theta}\,(m_i(\theta))|$$

provided that $2 - f_{m-1,\theta}\,(\lambda_m(\theta)) \neq 0$. Assume by way of contradiction that we have $f_{m-1,\theta}\,(\lambda_m(\theta)) = 2$. Then, by Definition 2.38 and Lemma 2.50(b), $-1 \leq \lambda_m(\theta)$ and

$$2 \geq f_{m-1,\theta}(-1) \geq f_{m-1,\theta}\,(\lambda_m(\theta)) = 2;$$

which contradicts statement (b.iv) from the proof of Lemma 2.50.

Next we compute an upper bound for $\|f_{m,\theta} - f_{m-1,\theta}\|$ for every $\theta \in B_i^{\frown}[i^*]$ and $i \in \mathfrak{D}_m$ such that $\operatorname{diam}(B_i^{\frown}[i^*])$ is small enough.

Lemma 2.54. *Assume that T_{m-1} is continuous for some $m \geq 2$ and let ε be positive. Then, there exist $\varrho_m(\varepsilon) \in \mathbb{N}$ such that*

$$\|f_{m,\theta} - f_{m-1,\theta}\| \leq \varepsilon$$

for every $\theta \in B_i^{\frown}[i^*]$ and $i \in \mathfrak{D}_m$ (that is, $B_i^{\frown}[i^*] \subset \mathbb{B}_m^{\frown}$) such that $|i| \geq \varrho_m(\varepsilon)$.

Proof. Since T_{m-1} is uniformly continuous, there exists $\delta_{m-1} = \delta_{m-1}(\varepsilon) > 0$ such that

$$\mathsf{d}_{\Omega}(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) < \varepsilon$$

provided that $\mathsf{d}_{\Omega}((\theta, x), (\nu, y)) < \delta_{m-1}$. We choose $\varrho_m = \varrho_m(\varepsilon) \in \mathbb{N}$ such that

$$3 \cdot 2^{-\varrho_m} < \min\{\delta_{m-1}(\varepsilon/2), \varepsilon/2\}.$$

Assume that $i \in \mathfrak{D}_m$ verifies $|i| \geq \varrho_m(\varepsilon)$ and let $(\theta, x) \in \mathsf{V}_{i^*}^{\frown} = B_i^{\frown}[i^*] \times \mathbb{I}$. When $\theta \in B_i^{\frown}[i^*] \setminus \mathbb{WIB}_m$ we can use Lemma 2.53 with $\mathbb{I}_{i,\theta}$ to compute $\|f_{m,\theta} - f_{m-1,\theta}\|$. We have to show that $|f_{m,\theta}(x) - f_{m-1,\theta}(x)| < \varepsilon$ for every $x \in \mathbb{I}_{i,\theta}$.

Let $\nu \in \operatorname{Bd}(B_i^{\smile}[i^*]) \subset \mathbb{EB}_m^{\smile}$. We have $(\theta,x),(\nu,m_i(\nu)) \in \mathcal{R}^{\smile}(i^*)$ and, by Lemmas 2.50(c) and 2.20(f),

$$\begin{aligned}
\mathsf{d}_{\Omega}(T_m(\theta,x),T_m(\nu,m_i(\nu))) &= \mathsf{d}_{\Omega}(G_i(\theta,x),G_i(\nu,m_i(\nu)))\\
&\le \operatorname{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right), \text{ and}\\
\mathsf{d}_{\Omega}((\theta,x),(\nu,m_i(\nu))) &\le \operatorname{diam}\left(\mathcal{R}^{\smile}(i^*)\right) \le 2\cdot 2^{-|i|} < 3\cdot 2^{-\varrho_m} < \delta_{m-1}(\varepsilon/2).
\end{aligned}$$

Thus,

$$\mathsf{d}_{\Omega}(T_{m-1}(\theta,x),T_{m-1}(\nu,m_i(\nu)) < \varepsilon/2.$$

Consequently, by Lemma 2.52,

$$\begin{aligned}
|f_{m,\theta}(x)-f_{m-1,\theta}(x)| &= \mathsf{d}_{\Omega}(T_m(\theta,x),T_{m-1}(\theta,x))\\
&\le \mathsf{d}_{\Omega}(T_m(\theta,x),T_{m-1}(\nu,m_i(\nu))) +\\
&\qquad \mathsf{d}_{\Omega}(T_{m-1}(\nu,m_i(\nu)),T_{m-1}(\theta,x))\\
&< \mathsf{d}_{\Omega}(T_m(\theta,x),T_m(\nu,m_i(\nu))) + \varepsilon/2\\
&< \operatorname{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) + \varepsilon/2.
\end{aligned}$$

Now we look at the size of $G_i\left(\mathcal{R}^{\smile}(i^*)\right)$. When $i<0$, from Lemmas 2.32(b) and 2.20(f), we obtain

$$\operatorname{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) \le \operatorname{diam}\left(\mathcal{R}((i+1)^*)\right) \le 2^{-(|i|-1)} < 2\cdot 2^{-|i|}. \tag{2.2}$$

When $i \ge 0$, from Lemma 2.30(b) we get

$$G_i\left(\mathcal{R}^{\smile}(i^*)\right) = G_i\left(\mathcal{R}(i^*)\right) \subset \mathcal{R}((i+1)^*) \cup \mathfrak{A}_{i+1}^{\upuparrows\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}.$$

Moreover, as in the proof of Lemma 2.20(f) for $\ell<0$, the set

$$\mathcal{R}((i+1)^*) \cup \mathfrak{A}_{i+1}^{\upuparrows\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}$$

is connected. So, by Lemma 2.20(f),

$$\begin{aligned}
\operatorname{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) &\le \operatorname{diam}\left(\mathcal{R}((i+1)^*) \cup \mathfrak{A}_{i+1}^{\upuparrows\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}\right)\\
&\le \operatorname{diam}\left(\mathcal{R}((i+1)^*)\right) + \operatorname{diam}\left(\mathfrak{A}_{i+1}^{\upuparrows\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}\right)\\
&\le 2^{-(i+1)} + \operatorname{diam}\left(\mathfrak{A}_{i+1}^{\upuparrows\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}\right).
\end{aligned}$$

As noticed earlier, $B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*)$ is disjoint from

$$B_{\alpha_{i+1}}\left((i+1)^*\right) \cup B_{-(i+1)}^{\smile}\left[(-(i+1))^*\right] \cup Z_{i+1}^*$$

by Definition 2.18(R.2) and Remark 2.19(R.2). So, by Lemma 2.25(c), Definition 2.18 and Lemma 2.20(a),

$$\mathfrak{A}_{i+1}^{\nu} = \{(\nu, \gamma_{i+1}(\nu)\} = \{(\nu, \gamma_i(\nu)\} \\ \in \{\nu\} \times \left[\gamma_i((i+1)^*) - 2^{-n_i}, \gamma_i((i+1)^*) + 2^{-n_i}\right].$$

for every $\nu \in B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*)$. On the other hand, $\gamma_i((i+1)^*) \in \mathbb{I}_{i+1,(i+1)^*}$ by Lemma 2.20(c). Hence, by Remark 2.16(2), Definition 2.18(R.1) and Remark 2.19(R.1),

$$\begin{aligned} &\operatorname{diam}\left(\mathfrak{A}_{i+1}^{\Uparrow\left(B_{\alpha_i}[(i+1)^*]\setminus B_{\alpha_{i+1}}((i+1)^*)\right)}\right) \\ &\quad \le \max\left\{\operatorname{diam}\left(B_{\alpha_i}\left[(i+1)^*\right]\setminus B_{\alpha_{i+1}}\left((i+1)^*\right)\right), 2\cdot\left(2^{-n_i}+2^{-n_{i+1}}\right)\right\} \\ &\quad \le 2\cdot\max\left\{\alpha_i, 2^{-n_i}+2^{-n_{i+1}}\right\} = 2\cdot\left(2^{-n_i}+2^{-n_{i+1}}\right) \\ &\quad < 4\cdot 2^{-n_i} \le 2\cdot 2^{-i}. \end{aligned}$$

Summarizing, when $i \ge 0$,

$$\operatorname{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) \le 2^{-(i+1)} + 2\cdot 2^{-i} < 3\cdot 2^{-i}$$

and, from (2.2),

$$\operatorname{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) < 3\cdot 2^{-|i|} \le 3\cdot 2^{-\varrho_m} < \varepsilon/2$$

for every $i \in \mathbb{Z}^+$. Thus, for every $x \in \mathbb{I}_{i,\theta}$,

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| < \operatorname{diam}\left(G_i\left(\mathcal{R}^{\smile}(i^*)\right)\right) + \varepsilon/2 < \varepsilon.$$

Now assume that $\theta \in B_i^{\smile}[i^*] \cap \mathbb{WIB}_m$. We can use Lemma 2.53 with $\mathbb{IW}_{m,\theta}$ to compute $\|f_{m,\theta} - f_{m-1,\theta}\|$. We have to show that $|f_{m,\theta}(x) - f_{m-1,\theta}(x)| < \varepsilon$ for every $x \in \mathbb{IW}_{m,\theta}$. Since $\theta \in \mathbb{WIB}_m$, by Definition 2.38 and Lemma 2.39(b), $i < 0$, $\theta \in \mathbb{WB}_m^{\smile}$ and

$$\mathbb{I}_{i,\theta} = \left\{\gamma_{|i|}(\theta)\right\} \subset \mathbb{IW}_{m,\theta} = \mathbb{I}_{\ell,\theta} \ni x$$

with $\ell = \mathsf{b}^{\smile}(\theta, \mathsf{led}(\theta, m)) \in \mathfrak{WFD}_m$. In this case we will consider the points $(\theta, x) \in \mathcal{R}(\ell^*)$ and $(\nu, m_i(\nu)), (\theta, \gamma_{|i|}(\theta)) \in \mathcal{R}^{\smile}(i^*)$ with $\nu \in \operatorname{Bd}(B_i^{\smile}[i^*]) \subset \mathbb{EB}_m^{\smile}$. By Lemma 2.37(b), Remark 2.16(2) and Lemma 2.20(f), $|i| < |\ell|$ and

$$\begin{aligned} \mathsf{d}_{\Omega}((\theta, x), (\nu, m_i(\nu)) &\le \mathsf{d}_{\Omega}((\theta, x), (\theta, \gamma_{|i|}(\theta)) + \mathsf{d}_{\Omega}((\theta, \gamma_{|i|}(\theta)), (\nu, m_i(\nu)) \\ &\le |x - \gamma_{|i|}(\theta)| + \operatorname{diam}\left(\mathcal{R}^{\smile}(i^*)\right) \\ &\le \operatorname{diam}\left(\mathcal{R}(\ell^*)\right) + \operatorname{diam}\left(\mathcal{R}^{\smile}(i^*)\right) \\ &\le 2^{-|\ell|} + 2\cdot 2^{-|i|} < 3\cdot 2^{-|i|} \le 3\cdot 2^{-\varrho_m} < \delta_{m-1}(\varepsilon/2). \end{aligned}$$

Thus,

$$\mathsf{d}_{\Omega}(T_{m-1}(\theta, x), T_{m-1}(\nu, m_i(\nu)) < \varepsilon/2.$$

On the other hand, by Lemma 2.50(c), Definition 2.42 and (2.2),

$$\begin{aligned}
&\mathsf{d}_\Omega(T_m(\theta,x),T_m(\nu,m_i(\nu)))\\
&\qquad\le \mathsf{d}_\Omega(T_m(\theta,x),T_m(\theta,\gamma_{|i|}(\theta)))+\mathsf{d}_\Omega(T_m(\theta,\gamma_{|i|}(\theta)),T_m(\nu,m_i(\nu)))\\
&\qquad\le |f_{m,\theta}(x)-f_{m,\theta}(\gamma_{|i|}(\theta))|+\mathsf{d}_\Omega(G_i(\theta,\gamma_{|i|}(\theta)),G_i(\nu,m_i(\nu)))\\
&\qquad= \mathsf{d}_\Omega(G_i(\theta,x),G_i(\nu,m_i(\nu)))\le \operatorname{diam}\left(G_i\left(\mathcal{R}^{\frown}(i^*)\right)\right)<2\cdot 2^{-|i|}\\
&\qquad\le 3\cdot 2^{-\varrho_m}<\varepsilon/2.
\end{aligned}$$

So, in a similar way as before, Lemma 2.52 gives

$$\begin{aligned}
|f_{m,\theta}(x)-f_{m-1,\theta}(x)| &= \mathsf{d}_\Omega(T_m(\theta,x),T_{m-1}(\theta,x))\\
&\le \mathsf{d}_\Omega(T_m(\theta,x),T_{m-1}(\nu,m_i(\nu)))+\\
&\qquad\qquad \mathsf{d}_\Omega(T_{m-1}(\nu,m_i(\nu)),T_{m-1}(\theta,x))\\
&<\varepsilon.
\end{aligned}$$

Proof (Proof of Proposition 2.43 for $m>0$).

(a) We start by proving by induction on m that T_m is continuous for every $m\in\mathbb{Z}^+$.

By Proposition 2.43(a) for $m=0$, T_0 is continuous. So, we may assume that T_{m-1} is continuous for some $m\in\mathbb{N}$ and prove that T_m is continuous.

Let $\varepsilon>0$ be fixed but arbitrary, and let $(\theta,x),(\nu,y)\in\Omega$. We have to show that there exists $\delta(\varepsilon)>0$ such that

$$\mathsf{d}_\Omega(T_m(\theta,x),T_m(\nu,y))<\varepsilon\quad\text{when}\quad \mathsf{d}_\Omega((\theta,x),(\nu,y))<\delta.$$

We start by defining $\delta(\varepsilon)$. To this end we need to introduce some more notation and establish some facts about the maps T_m and T_{m-1}.

Since T_{m-1} is uniformly continuous, we know that

there exists $\delta_{m-1}=\delta_{m-1}(\varepsilon)>0$ such that $\mathsf{d}_\Omega(T_{m-1}(\theta,x),T_{m-1}(\nu,y))<\varepsilon$ provided that $\mathsf{d}_\Omega((\theta,x),(\nu,y))<\delta_{m-1}$. (2.3)

On the other hand, Lemma 2.50(a) tells us that $T_m\big|_{\mathsf{V}^{\frown}_{i^*}}$ is uniformly continuous for every $i\in\mathfrak{D}_m$. So, for every $i\in\mathfrak{D}_m$,

there exists $\delta_{m,i}=\delta_{m,i}(\varepsilon)>0$ such that $\mathsf{d}_\Omega(T_m(\theta,x),T_m(\nu,y))<\varepsilon$ for every $(\theta,x),(\nu,y)\in\mathsf{V}^{\frown}_{i^*}\subset\mathbb{V}^{\frown}_m$ verifying $\mathsf{d}_\Omega((\theta,x),(\nu,y))<\delta_{m,i}(\varepsilon)$. (2.4)

Then, by using the numbers $\delta_{m-1}(\varepsilon/7)$ given by (2.3), $\delta_{m,i}(\varepsilon/7)$ given by (2.4) and $\varrho_m(\varepsilon/7)$ given by Lemma 2.54, we set

$$\delta=\delta(\varepsilon):=\min\left\{\delta_{m-1}(\varepsilon/7),\min\{\delta_{m,i}(\varepsilon/7): i\in\mathfrak{D}_m\cap Z_{\varrho_m(\varepsilon/7)}\}\right\}.$$

Clearly, $\delta>0$ because the set $\mathfrak{D}_m\cap Z_{\varrho_m(\varepsilon/7)}$ is finite.

Now we will show that if $\mathsf{d}_\Omega((\theta,x),(\nu,y))<\delta$, then $\mathsf{d}_\Omega(T_m(\theta,x),T_m(\nu,y))<\varepsilon$.

Assume first that $(\theta,x),(\nu,y) \in \mathsf{V}^{\frown}_{\ell^*}$ for some $\ell \in \mathfrak{D}_m \cap Z_{\varrho_m(\varepsilon/7)}$. We have

$$\mathsf{d}_\Omega((\theta,x),(\nu,y)) < \delta \leq \min\{\delta_{m,i}(\varepsilon/7) : i \in \mathfrak{D}_m \cap Z_{\varrho_m(\varepsilon/7)}\} \leq \delta_{m,\ell}(\varepsilon/7).$$

Hence, by (2.4),

$$\mathsf{d}_\Omega(T_m(\theta,x),T_m(\nu,y)) < \varepsilon/7 < \varepsilon.$$

Next we assume that $(\theta,x),(\nu,y) \in \mathsf{V}^{\frown}_{\ell^*}$ for some $\ell \in \mathfrak{D}_m$ such that $|\ell| > \varrho_m(\varepsilon/7)$ (in particular, $\theta,\nu \in B^{\frown}_\ell[\ell^*]$). In this situation we have

$$\mathsf{d}_\Omega((\theta,x),(\nu,y)) < \delta \leq \delta_{m-1}(\varepsilon/7)$$

and, by (2.3) and Lemma 2.54,

$$\begin{aligned}
\mathsf{d}_\Omega(T_m(\theta,x),T_m(\nu,y)) &\leq \mathsf{d}_\Omega(T_m(\theta,x),T_{m-1}(\theta,x)) + \mathsf{d}_\Omega(T_{m-1}(\theta,x),T_{m-1}(\nu,y)) + \\
&\qquad \mathsf{d}_\Omega(T_{m-1}(\nu,y),T_m(\nu,y)) \\
&= |f_{m,\theta}(x) - f_{m-1,\theta}(x)| + \mathsf{d}_\Omega(T_{m-1}(\theta,x),T_{m-1}(\nu,y)) + \\
&\qquad |f_{m,\nu}(y) - f_{m-1,\nu}(y)| \\
&\leq \|f_{m,\theta} - f_{m-1,\theta}\| + \mathsf{d}_\Omega(T_{m-1}(\theta,x),T_{m-1}(\nu,y)) + \\
&\qquad \|f_{m,\nu} - f_{m-1,\nu}\| \\
&< \tfrac{3}{7}\varepsilon < \varepsilon.
\end{aligned}$$

In summary, we have proved that

$$\mathsf{d}_\Omega(T_m(\theta,x),T_m(\nu,y)) < \tfrac{3}{7}\varepsilon$$

when $\mathsf{d}_\Omega((\theta,x),(\nu,y)) < \delta$ and $(\theta,x),(\nu,y) \in \mathsf{V}^{\frown}_{\ell^*}$ for some $\ell \in \mathfrak{D}_m$.

Next we assume that $(\theta,x),(\nu,y) \in \mathbb{V}^{\frown}_m$ but $(\theta,x),(\nu,y) \notin \mathsf{V}^{\frown}_{\ell^*}$ for every $\ell \in \mathfrak{D}_m$. By Lemma 2.34(a,b), there exist $i = \mathsf{b}^{\frown}(\theta,m)$, $k = \mathsf{b}^{\frown}(\nu,m) \in \mathfrak{D}_m$, $i \neq k$, such that $\theta \in B^{\frown}_i[i^*]$, $(\theta,x) \in \mathsf{V}^{\frown}_{i^*}$, $\nu \in B^{\frown}_k[k^*]$ and $(\nu,y) \in \mathsf{V}^{\frown}_{k^*}$. Then, there exist

$$\widetilde{\theta} \in A \cap \operatorname{Bd}\left(B^{\frown}_i[i^*]\right) \subset \mathbb{EB}^{\frown}_m \quad \text{and} \quad \widetilde{\nu} \in A \cap \operatorname{Bd}\left(B^{\frown}_k[k^*]\right) \subset \mathbb{EB}^{\frown}_m,$$

where A denotes the closed arc of $\mathbb{S}^1$ such that

$$\operatorname{diam}(A) = \mathsf{d}_{\mathbb{S}^1}(\theta,\nu) \quad \text{and} \quad \operatorname{Bd}(A) = \{\theta,\nu\}.$$

Clearly we have, $(\theta,x),(\widetilde{\theta},x) \in \mathsf{V}^{\frown}_{i^*}$, $(\nu,y),(\widetilde{\nu},y) \in \mathsf{V}^{\frown}_{k^*}$ and, by the previous case,

$$\begin{gathered}
\mathsf{d}_\Omega\left((\theta,x),(\widetilde{\theta},x)\right) = \mathsf{d}_{\mathbb{S}^1}\left(\theta,\widetilde{\theta}\right) \leq \mathsf{d}_{\mathbb{S}^1}(\theta,\nu) \leq \mathsf{d}_\Omega((\theta,x),(\nu,y)) < \delta, \\
\mathsf{d}_\Omega\left(T_m(\theta,x),T_m(\widetilde{\theta},x)\right) < \tfrac{3}{7}\varepsilon \\
\mathsf{d}_\Omega\left((\nu,y),(\widetilde{\nu},y)\right) = \mathsf{d}_{\mathbb{S}^1}(\nu,\widetilde{\nu}) \leq \mathsf{d}_{\mathbb{S}^1}(\theta,\nu) \leq \mathsf{d}_\Omega((\theta,x),(\nu,y)) < \delta, \text{ and} \\
\mathsf{d}_\Omega\left(T_m(\nu,y),T_m(\widetilde{\nu},y)\right) < \tfrac{3}{7}\varepsilon.
\end{gathered}$$

On the other hand, $(\widetilde{\theta}, x), (\widetilde{\nu}, y) \in \upuparrows\mathbb{E}\mathbb{B}^{\frown}_{m} \subset \mathbb{V}^{\frown}_{m} \subset \mathbb{V}^{\frown}_{m-1}$ and, by Lemma 2.52 and (2.3),

$$\begin{aligned}
\mathsf{d}_{\Omega}\left((\widetilde{\theta}, x), (\widetilde{\nu}, y)\right) &= \max\left\{\mathsf{d}_{\mathbb{S}^1}(\widetilde{\theta}, \widetilde{\nu}), |x-y|\right\} \le \max\left\{\mathsf{d}_{\mathbb{S}^1}(\theta, \nu), |x-y|\right\} \\
&= \mathsf{d}_{\Omega}((\theta, x), (\nu, y)) < \delta \le \delta_{m,i}(\varepsilon/7), \text{ and} \\
\mathsf{d}_{\Omega}(T_m(\theta, x), T_m(\nu, y)) &\le \mathsf{d}_{\Omega}\left(T_m(\theta, x), T_m(\widetilde{\theta}, x)\right) + \mathsf{d}_{\Omega}\left(T_m(\widetilde{\theta}, x), T_m(\widetilde{\nu}, y)\right) + \\
&\qquad \mathsf{d}_{\Omega}\left(T_m(\widetilde{\nu}, y), T_m(\nu, y)\right) \\
&< \tfrac{3}{7}\varepsilon + \mathsf{d}_{\Omega}\left(T_{m-1}(\widetilde{\theta}, x), T_{m-1}(\widetilde{\nu}, y)\right) + \tfrac{3}{7}\varepsilon = \varepsilon.
\end{aligned}$$

If $(\theta, x), (\nu, y) \notin \mathbb{V}^{\frown}_{m}$ then, by Definition 2.42 and (2.3) ,

$$\mathsf{d}_{\Omega}(T_m(\theta, x), T_m(\nu, y)) = \mathsf{d}_{\Omega}(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) < \varepsilon/7 < \varepsilon$$

because $\mathsf{d}_{\Omega}((\theta, x), (\nu, y)) < \delta \le \delta_{m-1}(\varepsilon/7)$.

Lastly, assume that $(\nu, y) \notin \mathbb{V}^{\frown}_{m}$ but $(\theta, x) \in \mathsf{V}^{\frown}_{i^*} \subset \mathbb{V}^{\frown}_{m}$, for some $i \in \mathfrak{D}_m$ (that is, $\theta \in B^{\frown}_{i}[i^*]$). In this situation, as before, there exists $\widetilde{\theta} \in \operatorname{Bd}(B^{\frown}_{i}[i^*]) \subset \mathbb{E}\mathbb{B}^{\frown}_{m}$ such that, by Lemma 2.52 and Definition 2.42 ($(\widetilde{\theta}, x) \in \upuparrows\mathbb{E}\mathbb{B}^{\frown}_{m} \subset \mathbb{V}^{\frown}_{m} \subset \mathbb{V}^{\frown}_{m-1}$), and (2.3),

$$\begin{aligned}
\mathsf{d}_{\Omega}\left((\theta, x), (\widetilde{\theta}, x)\right) &< \delta, \\
\mathsf{d}_{\Omega}\left((\widetilde{\theta}, x), (\nu, y)\right) &< \delta \le \delta_{m-1}(\varepsilon/7), \\
\mathsf{d}_{\Omega}\left(T_m(\theta, x), T_m(\widetilde{\theta}, x)\right) &< \tfrac{3}{7}\varepsilon, \text{ and} \\
\mathsf{d}_{\Omega}(T_m(\theta, x), T_m(\nu, y)) &\le \mathsf{d}_{\Omega}\left(T_m(\theta, x), T_m(\widetilde{\theta}, x)\right) + \mathsf{d}_{\Omega}\left(T_m(\widetilde{\theta}, x), T_m(\nu, y)\right) \\
&< \tfrac{3}{7}\varepsilon + \mathsf{d}_{\Omega}\left(T_{m-1}(\widetilde{\theta}, x), T_{m-1}(\nu, y)\right) < \varepsilon.
\end{aligned}$$

This ends the proof of the continuity of T_m and, hence, of (a).

(b) When $\theta \in \mathbb{B}^{\frown}_{m}$ the statement follows from Lemma 2.50(b). When $\theta \in \mathbb{S}^1 \setminus \mathbb{B}^{\frown}_{m}$, it follows from the part already proven and the continuity of T_m.

(c) The first two statements follow from Lemma 2.50(c) and statement (a). On the other hand, as in the proof of Proposition 2.43(c) for $m = 0$, Lemma 2.35(b) implies that $i^* \in \mathbb{B}^{\frown}_{m}$ but $i^* \notin \mathbb{B}^{\frown}_{k}$ for every $k > m$. Then, we get $f_{k,i^*} = f_{m,i^*}$ from Definition 2.42.

2.10 Proof of Proposition 2.44

This section is devoted to prove Proposition 2.44. It is the third technical counterpart of Section 2.7. In contrast to Lemma 2.54 the bound given by Proposition 2.44. is valid for every $\theta \in \mathbb{B}^{\frown}_{m}$.

Before starting the proof of this proposition we will state and prove a number of very simple lemmas that will help in automating the proof of Proposition 2.44.

Lemma 2.55. *Assume that $B_i^{\frown}[i^*] \subset B_k^{\frown}[k^*]$ for some $i \in \mathfrak{D}_m,\ k \in \mathfrak{D}_{m-1}$ and $m \geq 2$, and assume that either*

$$i < 0 \text{ and } \theta \in B_i^{\frown}[i^*] \setminus \{i^*\} \text{ or } i \geq 0 \text{ and } \theta \in B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*)\,.$$

Then,

$$|\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))| \leq 2^{-|k|}.$$

Proof. The lemma holds trivially when $|k+1| = |i+1|$. Thus, we may assume that $|k+1| \neq |i+1|$. Then by Lemma 2.36, $|k| < |i|$, $|k+1| < |i+1|$ and

$$\gamma_{|k+1|}(R_\omega(\theta)) = \gamma_{|k+1|+1}(R_\omega(\theta)) = \cdots = \gamma_{|i+1|-1}(R_\omega(\theta))\,.$$

By assumption we have

$$\theta \in \begin{cases} B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*) & \text{when } i \geq 0, \text{ and} \\ B_i^{\frown}[i^*] \setminus \{i^*\} = B_{\alpha_{|i+1|}}[i^*] \setminus \{i^*\} & \text{when } i < 0, \end{cases}$$

and, hence,

$$R_\omega(\theta) \in \begin{cases} B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*) & \text{when } i \geq 0, \text{ and} \\ B_{\alpha_{|i+1|}}[(i+1)^*] \setminus \{(i+1)^*\} & \text{when } i < 0. \end{cases}$$

Thus, in the case $i \geq 0$ we have

$$R_\omega(\theta) \notin B_{\alpha_{i+1}}((i+1)^*) \cup B_{-(i+1)}^{\frown}[(-(i+1))^*] \cup Z_{i+1}^*$$

by Definition 2.18(R.2) and Remark 2.19(R.2). So, by Definition 2.18,

$$\gamma_{i+1}(R_\omega(\theta)) = \gamma_i(R_\omega(\theta)) = \gamma_{|k+1|}(R_\omega(\theta))\,.$$

This ends the proof of the lemma in this case.

Assume now that $i < 0$. By Lemma 2.20(c,d,f) and Definition 2.18(R.2) and Remark 2.19(R.2),

$$\begin{aligned} |\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))| &= |\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|i+1|-1}(R_\omega(\theta))| \\ &\leq \operatorname{diam}(\mathcal{R}((i+1)^*)) \leq 2^{-|i+1|} \leq 2^{-|k|} \end{aligned}$$

(observe that $|i+1| > |k+1| \geq |k| - 1$).

Lemma 2.56. *Let $s, t \in \mathbb{Z}$, $s \neq t$ be such that $\theta \in B_s^{\frown}(s^*) \setminus B_{\alpha_{|s|}}(s^*)$, and either $t < 0$ and $\theta \in B_{\alpha_{|t|}}(t^*)$ or $t \geq 0$ and $\theta \in B_{\alpha_{t+1}}(t^*)$. Then, the following statements hold:*

(a) $R_\omega(\theta) \in B_{\alpha_{|s+1|}}((s+1)^*) \cap B_{\alpha_{|t+1|}}((t+1)^*)\,.$

(b) Let $u, v \in \mathbb{Z}$ be such that $\{u, v\} = \{s, t\}$ and $|u+1| \leq |v+1|$.

Then, $\mathbb{I}_{v+1, R_\omega(\theta)} \subset \mathbb{I}_{u+1, R_\omega(\theta)}$.

(c)

$$|x - y| \leq 2 \cdot 2^{-|u|}$$

for every $x \in \mathbb{I}_{t+1, R_\omega(\theta)}$ and $y \in \mathbb{I}_{s+1, R_\omega(\theta)}$.

Proof. By assumption we have

$$\theta \in \begin{cases} B_{\alpha_{t+1}}(t^*) & \text{when } t \geq 0, \text{ and} \\ B_{\alpha_{|t|}}(t^*) \subset B_t^{\frown}(t^*) = B_{\alpha_{|t+1|}}(t^*) & \text{when } t < 0. \end{cases}$$

Hence, $R_\omega(\theta) \in B_{\alpha_{|t+1|}}((t+1)^*)$. Moreover, as in the proof of Lemma 2.55, $s < 0$ and $R_\omega(\theta) \in B_{\alpha_{|s+1|}}((s+1)^*)$. This proves (a).

Now we prove (b). From (a) we have

$$\begin{aligned} R_\omega(\theta) &\in B_{\alpha_{|u+1|}}\left((u+1)^*\right) \cap B_{\alpha_{|v+1|}}\left((v+1)^*\right) \\ &\subset B_{\alpha_{|u+1|}}\left((u+1)^*\right) \cap B_{v+1}^{\frown}\left[(v+1)^*\right]. \end{aligned}$$

Moreover, $s \neq t$ implies $u+1 \neq v+1$ and we have $|u+1| \leq |v+1|$ by assumption. Consequently, by Lemma 2.20(g,d) and Definition 2.18(R.2) and Remark 2.19(R.2), $|u+1| < |v+1|$ and

$$\mathcal{R}((v+1)^*) \subset \operatorname{Int}\Big(\mathcal{R}((u+1)^*) \setminus \upuparrows\{(u+1)^*\}\Big)$$

which implies (b).

Thus, $x, y \in \mathbb{I}_{u+1, R_\omega(\theta)}$ and, by Lemma 2.20(f),

$$|x-y| \leq \operatorname{diam}\left(\mathcal{R}((u+1)^*)\right) \leq 2^{-|u+1|} \leq 2^{-(|u|-1)} = 2 \cdot 2^{-|u|}.$$

Now we are ready to start the proof of Proposition 2.44.

Proof (Proof of Proposition 2.44). We start by showing that $\{T_m\}_{k=0}^{\infty}$ is a Cauchy sequence, assuming that the bound (2.1) holds for every $m \geq 2$ and $\theta \in \mathbb{S}^1$.

We start by estimating $\mathsf{d}_\infty(T_m, T_{m+1})$ for every $m \in \mathbb{N}$. From (2.1) and the definition of μ_m

$$\mathsf{d}_\infty(T_m, T_{m+1}) = \sup_{\theta \in \mathbb{S}^1} \|f_{m,\theta} - f_{m+1,\theta}\| \leq 2 \cdot \sup_{\theta \in \mathbb{S}^1} 2^{-|\mathsf{b}^{\frown}(\theta, m)|} \leq 2 \cdot 2^{-\mu_m}.$$

By Lemma 2.35(a) $\{\mu_m\}_{m=0}^{\infty}$ is strictly increasing (and $\lim_{m\to\infty} \mu_m = \infty$). Therefore, for every $\varepsilon > 0$, there exists $N \geq 2$, such that $4 \cdot 2^{-\mu_m} < \varepsilon$ for every $m \geq N$. Hence,

$$\begin{aligned} \mathsf{d}_\infty(T_m, T_{m+i}) &\leq \sum_{\ell=m}^{m+i-1} \mathsf{d}_\infty(T_\ell, T_{\ell+1}) \leq 2 \cdot \sum_{\ell=m}^{m+i-1} 2^{-\mu_\ell} \\ &\leq 2 \cdot 2^{-\mu_m} \sum_{\ell=0}^{\infty} 2^{-\ell} = 4 \cdot 2^{-\mu_m} \leq 4 \cdot 2^{-\mu_N} < \varepsilon \end{aligned}$$

for every $m \geq N$ and $i \in \mathbb{N}$. So, $\{T_m\}_{k=0}^{\infty}$ is a Cauchy sequence.

Now we prove (2.1). That is,

$$\|f_{m,\theta} - f_{m-1,\theta}\| \leq 2 \cdot 2^{-|\mathsf{b}^{\frown}(\theta, m-1)|}$$

for every $m \geq 2$ and $\theta \in \mathbb{S}^1$.

From Definition 2.42 and Lemma 2.52 we know that $f_{m,\theta} = f_{m-1,\theta}$ for every $\theta \in \left(\mathbb{S}^1 \setminus \mathbb{B}^{\frown}_{m}\right) \cup \mathbb{EB}^{\frown}_{m}$. Then, (2.1) holds in this case.

In the rest of the involved proof we assume that $\theta \in \mathbb{B}^{\frown}_{m} \setminus \mathbb{EB}^{\frown}_{m}$. Thus, by Lemmas 2.34(a,b), 2.20(g) and 2.36,

$$\begin{aligned}
&\theta \in B^{\frown}_{i}(i^*) \subset B^{\frown}_{k}(k^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup \{k^*\}\right) \textit{ where}\\
&i = \mathsf{b}^{\frown}(\theta, m) \in \mathfrak{D}_{m},\ k = \mathsf{b}^{\frown}(\theta, m-1) \in \mathfrak{D}_{m-1},\\
&|k| < |i|, \textit{ and } |k+1| \leq |i+1|.
\end{aligned}$$

Moreover, $\mathsf{V}^{\frown}_{i^*} \subset \mathsf{V}^{\frown}_{k^*} \subset \mathbb{V}^{\frown}_{m-1}$. Consequently, by Lemma 2.50(a,b), the maps $f_{m,\theta}$ and $f_{m-1,\theta}$ are well defined, continuous, piecewise affine and non-increasing, and $f_{m,\theta}(2) = f_{m-1,\theta}(2) = -2$ and $f_{m,\theta}(-2) = f_{m-1,\theta}(-2) = 2$ (see Figures 2.6, 2.7 and 2.8 for some examples in generic cases).

We split the proof into three cases according to whether θ belongs to

$$B^{\frown}_{i}(i^*) \setminus B_{\alpha_{|i|}}(i^*),\ B_{\alpha_{|i|}}(i^*) \subset B^{\frown}_{k}(k^*) \setminus B_{\alpha_{|k|}}[k^*] \text{ or } B_{\alpha_{|i|}}(i^*) \subset B_{\alpha_{|k|}}(k^*).$$

Case 2.57. Case 1. $\theta \in B^{\frown}_{i}(i^*) \setminus B_{\alpha_{|i|}}(i^*)$.

We have $i < 0$ because $B^{\frown}_{i}(i^*) = B_{\alpha_i}(i^*)$ for $i \geq 0$. Moreover, by Definition 2.38, $\theta \in \mathbb{WB}^{\frown}_{m}$.

To deal with this case we consider three subcases.

Case 2.58. Subcase 1.1. $\theta \in \left(B^{\frown}_{i}(i^*) \setminus B_{\alpha_{|i|}}(i^*)\right) \setminus \mathbb{WIB}_{m}$.

By Lemmas 2.36, 2.51, 2.53 and 2.55,

$$\begin{aligned}
\mathbb{I}_{i,\theta} &= \{m_i(\theta)\} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta},\\
f_{m,\theta}\left(m_i(\theta)\right) &= \gamma_{|i+1|}\left(R_\omega(\theta)\right),\\
f_{m-1,\theta}\left(m_i(\theta)\right) &= \gamma_{|k+1|}\left(R_\omega(\theta)\right), \text{ and}\\
\|f_{m,\theta} - f_{m-1,\theta}\| &= \left\| f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}} - f_{m-1,\theta}\big|_{\mathbb{I}_{i,\theta}} \right\| = \left|f_{m,\theta}\left(m_i(\theta)\right) - f_{m-1,\theta}\left(m_i(\theta)\right)\right|\\
&= \left|\gamma_{|i+1|}\left(R_\omega(\theta)\right) - \gamma_{|k+1|}\left(R_\omega(\theta)\right)\right| \leq 2^{-|\mathsf{b}^{\frown}(\theta,m-1)|}.
\end{aligned}$$

Case 2.59. Subcase 1.2. $\theta \in \left(B^{\frown}_{i}(i^*) \setminus B_{\alpha_{|i|}}(i^*)\right) \cap \mathbb{WIB}_{m}$ and $B^{\frown}_{i}(i^*) \subset B^{\frown}_{k}(k^*) \setminus B_{\alpha_{|k|}}[k^*]$.

In this subcase, by Definition 2.38 we have

$$\theta \in B^{\frown}_{k}(k^*) \setminus B_{\alpha_{|k|}}[k^*] \subset \mathbb{WB}^{\frown}_{m-1}$$

(recall that $i < 0$). Then, by Lemmas 2.36 and 2.39(b,c), Definition 2.42 and Lemmas 2.53 and 2.55,

$$\begin{aligned}
\mathbb{I}_{i,\theta} &= \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{IW}_{m,\theta} = \mathbb{IW}_{m-1,\theta},\\
f_{m,\theta}(x) &= \gamma_{|i+1|}\left(R_\omega(\theta)\right) \text{ for every } x \in \mathbb{IW}_{m,\theta},\\
f_{m-1,\theta}(x) &= \gamma_{|k+1|}\left(R_\omega(\theta)\right) \text{ for every } x \in \mathbb{IW}_{m-1,\theta}, \text{ and}\\
\|f_{m,\theta} - f_{m-1,\theta}\| &= \left\| f_{m,\theta}\big|_{\mathbb{IW}_{m,\theta}} - f_{m-1,\theta}\big|_{\mathbb{IW}_{m,\theta}} \right\|\\
&= \left|\gamma_{|i+1|}\left(R_\omega(\theta)\right) - \gamma_{|k+1|}\left(R_\omega(\theta)\right)\right| \leq 2^{-|\mathsf{b}^{\frown}(\theta,m-1)|}.
\end{aligned}$$

Observe that since $B_i^{\frown}(i^*)$ is connected and

$$B_i^{\frown}(i^*) \subset B_k^{\frown}(k^*) \setminus \left(\operatorname{Bd}\left(B_{\alpha_{|k|}}[k^*]\right) \cup \{k^*\}\right),$$

$B_i^{\frown}(i^*) \not\subset B_k^{\frown}(k^*) \setminus B_{\alpha_{|k|}}[k^*]$ implies $B_i^{\frown}(i^*) \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$.

Case 2.60. Subcase 1.3. $\theta \in \left(B_i^{\frown}(i^*) \setminus B_{\alpha_{|i|}}(i^*)\right) \cap \mathbb{WIB}_m$ and $B_i^{\frown}(i^*) \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ (see Figure 2.6 for a symbolic representation of this case).

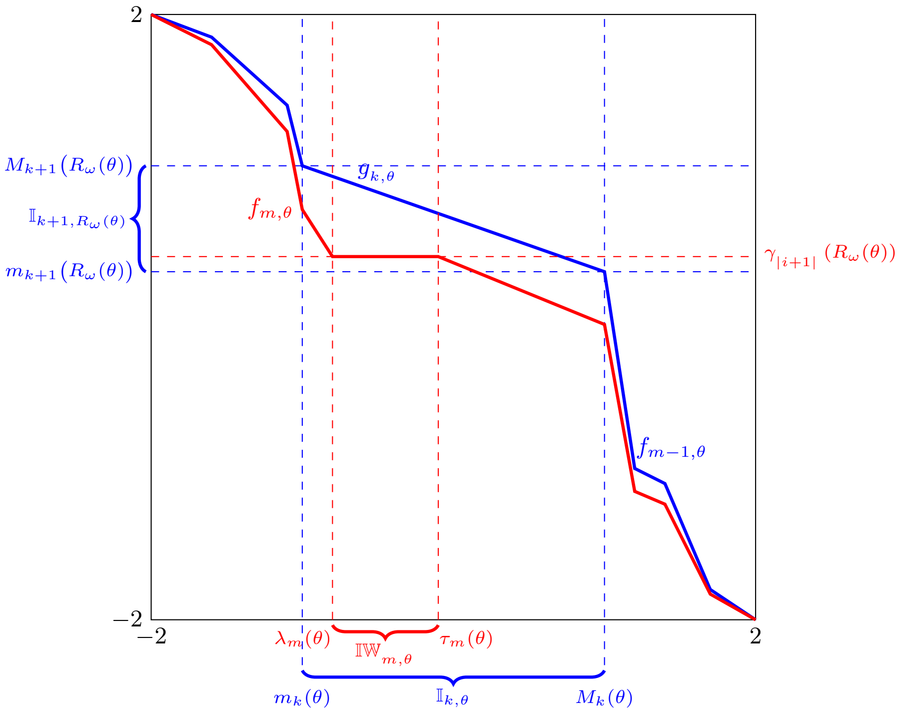


Figure 2.6: A symbolic representation of the maps $f_{m,\theta}$ and $f_{m-1,\theta}$ in Subcase 1.3 of Proposition 2.44 ($\theta \in \left(B_i^{\frown}(i^*) \setminus B_{\alpha_{|i|}}(i^*)\right) \cap \mathbb{WIB}_m$ and $B_i^{\frown}(i^*) \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$). The map $f_{m-1,\theta}$ and the corresponding intervals $\mathbb{I}_{k,\theta}$ and $\mathbb{I}_{k+1,R_\omega(\theta)}$ are drawn in blue. The map $f_{m,\theta}$, the interval $\mathbb{IW}_{m,\theta}$ and the point $\gamma_{|i+1|}(R_\omega(\theta))$ are drawn in red.

By Lemmas 2.36 and 2.39(b) and Definition 2.42,

$$\mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{IW}_{m,\theta}, \text{ and}$$
$$f_{m,\theta}(x) = \gamma_{|i+1|}(R_\omega(\theta)) \text{ for every } x \in \mathbb{IW}_{m,\theta}.$$

On the other hand, by Definition 2.38 and Lemma 2.37(a,b), $\theta \in \mathbb{WIB}_m \subset \mathbb{WDB}_m$, and

$$\theta \in B_{\alpha_{|\ell|}}[\ell^*] \subset B_i^{\frown}(i^*) \setminus B_{\alpha_{|i|}}[i^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$$

with $\ell = \mathsf{b}^{\frown}(\theta, \mathsf{led}\,(\theta, m)) \in \mathfrak{WFD}_m$ and $|\ell| > |i| > |k|$. Then, by Lemma 2.20(g) and Definition 2.38, $\mathcal{R}(\ell^*) \subset \operatorname{Int}\left(\mathcal{R}(k^*) \setminus \upuparrows k^*\right)$ and

$$\mathbb{IW}_{m,\theta} = \mathbb{I}_{\ell,\theta} \subset \mathbb{I}_{k,\theta}.$$

Moreover, since $\theta \in B_{\alpha_{|k|}}(k^*) \subset \mathbb{B}_m$, Definition 2.42, Lemmas 2.30(b) and 2.32(b), and the definition of the maps $g_{i,\theta}$ for $i \geq 0$ (Definition 2.29) give

$$\begin{aligned} f_{m-1,\theta}\left(\mathbb{IW}_{m,\theta}\right) &\subset f_{m-1,\theta}\left(\mathbb{I}_{k,\theta}\right) \\ &\subset \begin{cases} \mathbb{I}_{k+1,R_\omega(\theta)} & \text{if } k < 0 \text{ or } k \geq 0 \text{ and } \theta \in B_{\alpha_{k+1}}(k^*)\,, \\ \{\gamma_{k+1}\left(R_\omega(\theta)\right)\} & \text{if } k \geq 0 \text{ and } \theta \in B_{\alpha_k}[k^*] \setminus B_{\alpha_{k+1}}(k^*). \end{cases} \end{aligned}$$

Now, as before, we will use Lemma 2.53 to bound $\|f_{m,\theta} - f_{m-1,\theta}\|$. We start with the simplest case: $k \geq 0$ and $\theta \in B_{\alpha_k}[k^*] \setminus B_{\alpha_{k+1}}(k^*)$. By Lemma 2.55,

$$\begin{aligned} \|f_{m,\theta} - f_{m-1,\theta}\| &= \left\| f_{m,\theta}\big|_{\mathbb{IW}_{m,\theta}} - f_{m-1,\theta}\big|_{\mathbb{IW}_{m,\theta}} \right\| \\ &= |\gamma_{|i+1|}\left(R_\omega(\theta)\right) - \gamma_{|k+1|}\left(R_\omega(\theta)\right)| \leq 2^{-|\mathsf{b}^{\frown}(\theta, m-1)|}. \end{aligned}$$

Now we assume that $k < 0$ or $k \geq 0$ and $\theta \in B_{\alpha_{k+1}}(k^*)$. In this case Lemma 2.56 applies. By Lemmas 2.56, 2.20(d) and Definition 2.18(R.2) and Remark 2.19(R.2), and Lemma 2.53 we have

$$\begin{aligned} \gamma_{|i+1|}\left(R_\omega(\theta)\right) &\in \mathbb{I}_{i+1,R_\omega(\theta)} \subset \mathbb{I}_{k+1,R_\omega(\theta)}, \\ f_{m-1,\theta}(x) &\in \mathbb{I}_{k+1,R_\omega(\theta)} \quad \text{for every } x \in \mathbb{IW}_{m,\theta}. \end{aligned}$$

and

$$\begin{aligned} \|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x \in \mathbb{IW}_{m,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| \\ &= \sup_{x \in \mathbb{IW}_{m,\theta}} |\gamma_{|i+1|}\left(R_\omega(\theta)\right) - f_{m-1,\theta}(x)| \\ &\leq 2 \cdot 2^{-|k|} = 2 \cdot 2^{-|\mathsf{b}^{\frown}(\theta, m-1)|}. \end{aligned}$$

This ends the proof of the proposition in this case.

Case 2.61. Case 2. $\theta \in B_{\alpha_{|i|}}(i^*) \subset B_{\widetilde{k}}(k^*) \setminus B_{\alpha_{|k|}}[k^*]$ (see Figure 2.7 for a symbolic representation of this case).

In this case we will use Lemma 2.53 with $\mathbb{I}_{i,\theta}$. Thus, we need to compare the maps $f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}}$ and $f_{m-1,\theta}\big|_{\mathbb{I}_{i,\theta}}$.

Directly from the definitions we get $k < 0$, $B_{\alpha_{|i|}}[i^*] \subset \mathbb{B}_m$ and $B_{\alpha_{|k|}}[k^*] \subset \mathbb{B}_{m-1}$. Consequently, by Lemma 2.34(b) and Definition 2.38,

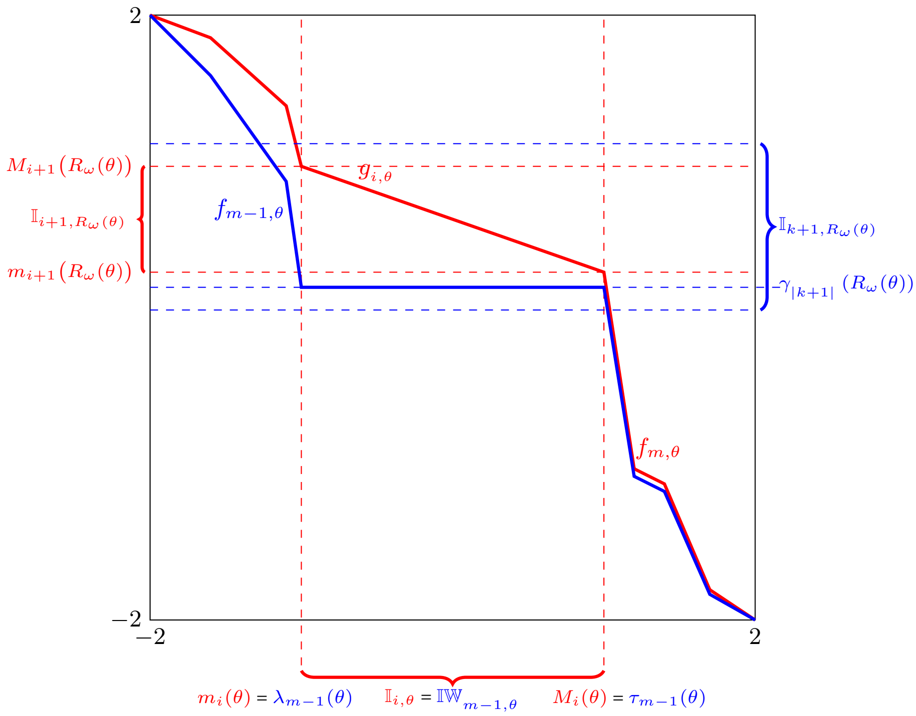


Figure 2.7: A symbolic representation of the maps $f_{m,\theta}$ and $f_{m-1,\theta}$ in Case 2 ($\theta \in B_{\alpha_{|i|}}(i^*) \subset B^{\frown}_{k}(k^*) \setminus B_{\alpha_{|k|}}[k^*]$) of Proposition 2.44. The map $f_{m-1,\theta}$ and the corresponding intervals $\mathbb{IW}_{m-1,\theta}$ and $\mathbb{I}_{k+1,R_\omega(\theta)}$ are drawn in blue. The map $f_{m,\theta}$ and the corresponding intervals $\mathbb{I}_{i,\theta} = \mathbb{IW}_{m-1,\theta}$ and $\mathbb{I}_{i+1,R_\omega(\theta)}$ are drawn in red.

$$\theta \in \mathbb{B}_m \quad \text{and} \quad \theta \in \mathbb{B}^{\frown}_{m-1} \setminus \mathbb{B}_{m-1} \subset \mathbb{WDB}_{m-1} \subset \mathbb{WB}^{\frown}_{m-1}.$$

Moreover, $\mathsf{led}(\theta, m-1) = m$, $i = \mathsf{b}^{\frown}(\theta, m) = \mathsf{b}^{\frown}(\theta, \mathsf{led}(\theta, m-1)) \in \mathfrak{WFD}_{m-1}$ and, by Definition 2.38, $\theta \in \mathbb{WIB}_{m-1}$, and

$$\mathbb{IW}_{m-1,\theta} = \mathbb{I}_{i,\theta}.$$

Furthermore, since $k < 0$, as in the proof of Lemma 2.55, $R_\omega(\theta) \in B_{\alpha_{|k+1|}}((k+1)^*)$. Thus, Definition 2.42, Lemma 2.20(d) and Definition 2.18(R.2) and Remark 2.19(R.2), give

$$f_{m-1,\theta}(x) = \gamma_{|k+1|}(R_\omega(\theta)) \in \mathbb{I}_{k+1,R_\omega(\theta)}$$

for every $x \in \mathbb{I}_{i,\theta} = \mathbb{IW}_{m-1,\theta}$.

Now we will use Lemma 2.53 to bound the norm $\|f_{m,\theta} - f_{m-1,\theta}\|$. By Definition 2.38 and Lemma 2.53, $\theta \in \mathbb{B}_m \subset \mathbb{B}^{\frown}_{m} \setminus \mathbb{WIB}_m$, and

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \sup_{x\in\mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| = \sup_{x\in\mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - \gamma_{|k+1|}(R_\omega(\theta))|.$$

Next we will compute $f_{m,\theta}(\mathbb{I}_{i,\theta})$. We start with the simplest case: $i \geq 0$ and $\theta \in B_{\alpha_i}(i^*) \setminus B_{\alpha_{i+1}}(i^*)$. By Definition 2.42, the definition of the maps $g_{i,\theta}$ for $i \geq 0$ (Definition 2.29) and Lemma 2.55,

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \sup_{x\in\mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - \gamma_{|k+1|}(R_\omega(\theta))| = |\gamma_{i+1}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))| \leq 2^{-|\widetilde{\mathsf{b}}(\theta,m-1)|}.$$

Assume that $i < 0$ or $i \geq 0$ and $\theta \in B_{\alpha_{i+1}}(i^*)$. Then, again by Definition 2.42 and Lemmas 2.30(b), 2.32(b) and 2.56,

$$f_{m,\theta}(x) \in \mathbb{I}_{i+1,R_\omega(\theta)} \subset \mathbb{I}_{k+1,R_\omega(\theta)} \quad \text{for every} \quad x \in \mathbb{I}_{i,\theta},$$

and

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \sup_{x\in\mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - \gamma_{|k+1|}(R_\omega(\theta))| \leq 2 \cdot 2^{-|k|} = 2 \cdot 2^{-|\widetilde{\mathsf{b}}(\theta,m-1)|}.$$

This ends the proof of the proposition in Case 2.

Case 2.62. Case 3. $\theta \in B_{\alpha_{|i|}}(i^*) \subset B_{\alpha_{|k|}}(k^*)$.

In this case we have $B_{\alpha_{|i|}}(i^*) \subset \mathbb{B}_m$ and $B_{\alpha_{|k|}}(k^*) \subset \mathbb{B}_{m-1}$ so that, $\theta \in \mathbb{B}_m \cap \mathbb{B}_{m-1}$. Moreover, by Lemma 2.20(g), $\mathcal{R}(i^*) \subset \operatorname{Int}\left(\mathcal{R}(k^*) \setminus \upuparrows k^*\right)$ and, hence,

$$\mathbb{I}_{i,\theta} \subset \mathbb{I}_{k,\theta}.$$

Since $\theta \in \mathbb{B}_m$, by Definition 2.38 and Lemma 2.53, $\theta \in \widetilde{\mathbb{B}_m} \setminus \mathbb{WIB}_m$, and

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \left\| f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}} - f_{m-1,\theta}\big|_{\mathbb{I}_{i,\theta}} \right\| = \sup_{x\in\mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)|.$$

Thus, we need to compare the maps $f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}}$ and $f_{m-1,\theta}\big|_{\mathbb{I}_{i,\theta}}$. To do this we consider two subcases.

Case 2.63. Subcase 3.1. Either $k < 0$ or $k \geq 0$ and $\theta \in B_{\alpha_{k+1}}(k^*)$
(see Figure 2.8 for a symbolic representation of this case).

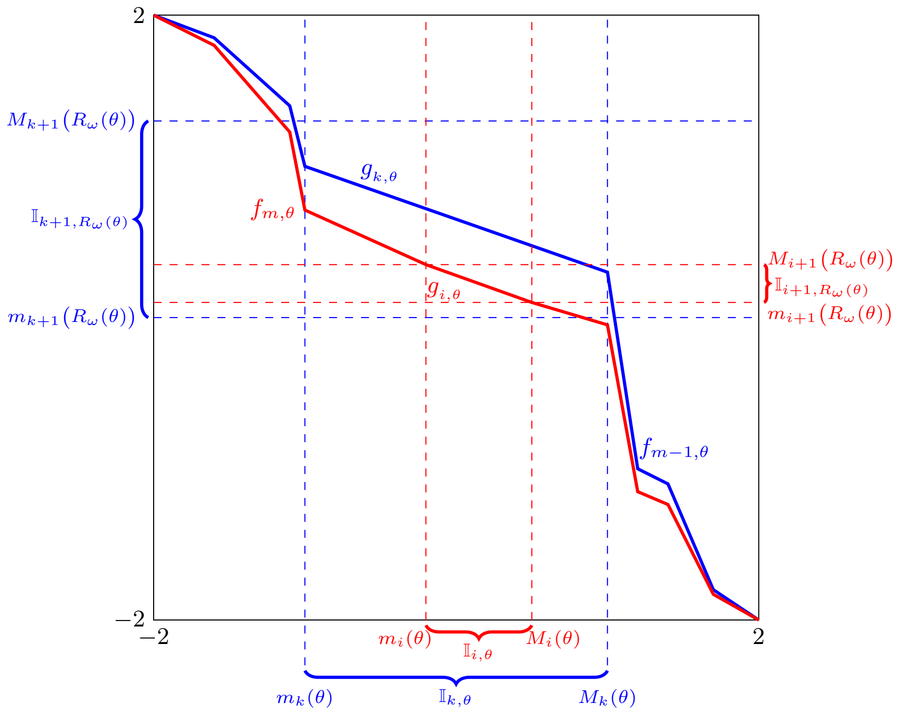


Figure 2.8: A symbolic representation of the maps $f_{m,\theta}$ and $f_{m-1,\theta}$ in Subcase 3.1 from the proof of Proposition 2.44 ($\theta \in B_{\alpha_{|i|}}(i^*)$ and $\mathbb{I}_{i,\theta} \subset \mathbb{I}_{k,\theta}$ and either $k < 0$ or $k \geq 0$ and $i^* \in B_{\alpha_{k+1}}[k^*]$). The map $f_{m-1,\theta}$ and the corresponding intervals $\mathbb{I}_{k,\theta}$ and $\mathbb{I}_{k+1,R_\omega(\theta)}$ are drawn in blue. The map $f_{m,\theta}$ and the corresponding intervals $\mathbb{I}_{i,\theta}$ and $\mathbb{I}_{i+1,R_\omega(\theta)}$ are drawn in red.

In this situation we aim at proving that

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}), f_{m,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

We start with $f_{m-1,\theta}(\mathbb{I}_{i,\theta})$. By Definition 2.42 and Lemmas 2.30(b) and 2.32(b) we obtain

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}) \subset f_{m-1,\theta}(\mathbb{I}_{k,\theta}) = g_{k,\theta}(\mathbb{I}_{k,\theta}) \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

Next we show that $f_{m,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{k+1,R_\omega(\theta)}$.
Since $k < 0$ or $k \geq 0$ and $\theta \in B_{\alpha_{k+1}}(k^*)$, by Definition 2.18(R.1) we obtain

$$R_\omega(\theta) \in \begin{cases} R_\omega\left(B_{\alpha_{|k|}}(k^*)\right) = B_{\alpha_{|k|}}((k+1)^*) \subset B_{\alpha_{|k+1|}}((k+1)^*) & \text{if } k < 0, \\ R_\omega\left(B_{\alpha_{k+1}}(k^*)\right) = B_{\alpha_{k+1}}((k+1)^*) & \text{if } k \geq 0 \text{ and } \theta \in B_{\alpha_{k+1}}(k^*). \end{cases} \tag{2.1}$$

Assume that $i < 0$ or $i \geq 0$ and $\theta \in B_{\alpha_{i+1}}(i^*)$. By (2.1) with k replaced by i,

$$R_\omega(\theta) \in B_{\alpha_{|i+1|}}((i+1)^*) \cap B_{\alpha_{k+1}}((k+1)^*) \subset B_{\widetilde{i+1}}[(i+1)^*] \cap B_{\widetilde{k+1}}[(k+1)^*].$$

Therefore, since $|k+1| \leq |i+1|$ and $k+1 \neq i+1$, from Lemma 2.20(g) we obtain $|k+1| < |i+1|$,

$$B_{\alpha_{|i+1|}}[(i+1)^*] \subset B_{\alpha_{|k+1|}}((k+1)^*) \setminus \{(k+1)^*\}, \text{ and}$$
$$\mathcal{R}((i+1)^*) \subset \operatorname{Int}\Big(\mathcal{R}((k+1)^*) \setminus \upuparrows (k+1)^*\Big).$$

Thus, by Definition 2.42 and Lemmas 2.30(b) and 2.32(b),

$$f_{m,\theta}(\mathbb{I}_{i,\theta}) = g_{i,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{i+1,R_\omega(\theta)} \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

Now we will consider the case $i \geq 0$ and $\theta \in B_{\alpha_i}(i^*) \setminus B_{\alpha_{i+1}}(i^*)$. The fact that $|k| < |i| = i$ implies $|k+1| \leq |k| + 1 \leq i$. We claim that

$$B_{\alpha_i}((i+1)^*) \subset B_{\alpha_{|k+1|}}((k+1)^*) \setminus \{(k+1)^*\}.$$

To prove the claim note that, by (2.1),

$$R_\omega(\theta) \in R_\omega(B_{\alpha_i}(i^*)) \cap B_{\alpha_{|k+1|}}((k+1)^*) \subset B_{\alpha_i}((i+1)^*) \cap B^{\frown}_{k+1}[(k+1)^*].$$

Moreover, the interval $B_{\alpha_i}((i+1)^*)$ is disjoint from $B^{\frown}_{i}[i^*]$ and $B^{\frown}_{-i}[(-i)^*]$ by Definition 2.18(R.2). Thus, $i \neq k+1, -(k+1)$ and, hence, $|k+1| < i$ (that is, $k+1 \in Z_{i-1}$). So, there exists $q \in Z_{i-1}$ such that $B_{\alpha_i}[(i+1)^*] \cap B^{\frown}_{q}[q^*] \neq \emptyset$ and $|q| \geq |k+1|$ is maximal verifying these conditions. By Definition 2.18(R.4),

$$B_{\alpha_i}((i+1)^*) \subset B^{\frown}_{q}(q^*) \setminus \big(\operatorname{Bd}\big(B_{\alpha_{|q|}}[q^*]\big) \cup \{q^*\}\big).$$

So, the claim holds when $q = k+1$. Assume that $q \neq k+1$. Then,

$$R_\omega(\theta) \in B_{\alpha_i}((i+1)^*) \cap B_{\alpha_{|k+1|}}((k+1)^*) \subset B^{\frown}_{q}(q^*) \cap B_{\alpha_{|k+1|}}((k+1)^*).$$

Hence, by Lemma 2.20(g), $|q| > |k+1|$ and

$$B_{\alpha_i}((i+1)^*) \subset B^{\frown}_{q}[q^*] \subset B_{\alpha_{|k+1|}}((k+1)^*) \setminus \{(k+1)^*\}.$$

This ends the proof of the claim.

On the other hand, by Definition 2.18(R.2) and Remark 2.19(R.2),

$$\big(B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*)\big) \cap Z_{i+1} = \emptyset.$$

Thus, by the claim,

$$\begin{aligned} R_\omega(\theta) \in R_\omega\big(B_{\alpha_i}(i^*) \setminus B_{\alpha_{i+1}}(i^*)\big) &= B_{\alpha_i}((i+1)^*) \setminus B_{\alpha_{i+1}}((i+1)^*) \\ &\subset B_{\alpha_{|k+1|}}((k+1)^*) \setminus Z_{i+1}. \end{aligned}$$

By Definition 2.42, the definition of the maps $g_{i,\theta}$ for $i \geq 0$ (Definition 2.29) and Lemma 2.20(d) (with $\ell = k+1$ and $n = i+1$),

$$f_{m,\theta}(\mathbb{I}_{i,\theta}) = g_{i,\theta}(\mathbb{I}_{i,\theta}) = \big\{\gamma_{i+1}(R_\omega(\theta))\big\} \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

Summarizing, we have proved that

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}), f_{m,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

So, by Lemma 2.20(f) (and the fact that $|k+1| \geq |k| - 1$),

$$\begin{aligned}\|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x\in\mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq \operatorname{diam}\left(\mathbb{I}_{k+1,R_\omega(\theta)}\right) \\ &\leq \operatorname{diam}\left(\mathcal{R}((k+1)^*)\right) \leq 2^{-|k+1|} \leq 2 \cdot 2^{-|k|} = 2 \cdot 2^{-|\tilde{\mathfrak{b}}(\theta,m-1)|}.\end{aligned}$$

This ends the proof of the proposition in this subcase.

Case 2.64. Subcase 3.2. $k \geq 0$ and $\theta \in B_{\alpha_k}(k^*) \setminus B_{\alpha_{k+1}}(k^*)$.

We start by computing $f_{m-1,\theta}(\mathbb{I}_{i,\theta})$. By Definition 2.42 and the definition of the maps $g_{k,\theta}$ for $k \geq 0$ (Definition 2.29),

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}) \subset f_{m-1,\theta}(\mathbb{I}_{k,\theta}) = g_{k,\theta}(\mathbb{I}_{k,\theta}) = \{\gamma_{k+1}(R_\omega(\theta))\}.$$

Analogously, if $i \geq 0$ and $\theta \in B_{\alpha_i}(i^*) \setminus B_{\alpha_{i+1}}(i^*)$,

$$f_{m,\theta}(\mathbb{I}_{i,\theta}) = g_{i,\theta}(\mathbb{I}_{i,\theta}) = \{\gamma_{i+1}(R_\omega(\theta))\}.$$

Then, by Lemma 2.55,

$$\begin{aligned}\|f_{m,\theta} - f_{m-1,\theta}\| &= \left\| f_{m,\theta}\big|_{\mathbb{I}_{i,\theta}} - f_{m-1,\theta}\big|_{\mathbb{I}_{i,\theta}} \right\| \\ &= |\gamma_{i+1}(R_\omega(\theta)) - \gamma_{k+1}(R_\omega(\theta))| \leq 2^{-|\tilde{\mathfrak{b}}(\theta,m-1)|}.\end{aligned}$$

Assume now that $i < 0$ or $i \geq 0$ and $\theta \in B_{\alpha_{i+1}}(i^*)$. By (2.1), Definition 2.42 and Lemmas 2.30(b) and 2.32(b)

$$\begin{aligned} R_\omega(\theta) &\in B_{\alpha_{|i+1|}}((i+1)^*), \text{ and} \\ f_{m,\theta}(\mathbb{I}_{i,\theta}) &= g_{i,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{i+1,R_\omega(\theta)}.\end{aligned}$$

Moreover, if $k+1 < |i+1|$, by Lemmas 2.36(a) and 2.20(c), we have

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}) = \left\{\gamma_{k+1}(R_\omega(\theta))\right\} = \left\{\gamma_{|i+1|-1}(R_\omega(\theta))\right\} \subset \mathbb{I}_{i+1,R_\omega(\theta)}.$$

Therefore, by Lemma 2.20(f),

$$\begin{aligned}\|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x\in\mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| \\ &= \sup_{x\in\mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - \gamma_{|i+1|-1}(R_\omega(\theta))| \\ &\leq \operatorname{diam}\left(\mathbb{I}_{i+1,R_\omega(\theta)}\right) \leq \operatorname{diam}\left(\mathcal{R}((i+1)^*)\right) \leq 2^{-|i+1|} \\ &< 2^{-(k+1)} < 2^{-|\tilde{\mathfrak{b}}(\theta,m-1)|}.\end{aligned}$$

So, to end the proof of the proposition we have to show that, in this subcase, $k+1 < |i+1|$. To prove this, notice that when $i \geq 0$, $k+1 = |k|+1 < |i|+1 = |i+1|$. So, assume by way of contradiction that $i < 0$ and $k+1 = |i+1|$ (recall that $k+1 \leq |i+1|$). Then, $k+1 = -(i+1)$ and, hence,

$$
\begin{aligned}
&R_\omega(\theta) \in R_\omega\left(B_{\alpha_k}\left(k^*\right)\right) = B_{\alpha_k}\left((k+1)^*\right), \text{ and} \\
&R_\omega(\theta) \in B_{\alpha_{|i+1|}}\left((i+1)^*\right) = B_{\alpha_{k+1}}\left((-(k+1))^*\right) \subset B^{\smile}_{-(k+1)}\left((-(k+1))^*\right),
\end{aligned}
$$

which is a contradiction by Definition 2.18(R.2).

References

1. Lluís Alsedà, Jaume Llibre, and Michał Misiurewicz. *Combinatorial dynamics and entropy in dimension one*, volume 5 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co. Inc., River Edge, NJ, second edition, 2000.
2. Lluís Alsedà, Francesc Mañosas, and Leopoldo Morales. Forcing and entropy of strip patterns of quasiperiodic skew products in the cylinder. *J. Math. Anal. Appl.*, 429:542–561, 2015.
3. Lluís Alsedà, Francesc Mañosas, and Leopoldo Morales. A skew-product map on the cylinder without fixed-curves. preprint, 2016.
4. Louis Block, John Guckenheimer, Michał Misiurewicz, and Lai Sang Young. Periodic points and topological entropy of one-dimensional maps. In *Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)*, volume 819 of *Lecture Notes in Math.*, pages 18–34. Springer, Berlin, 1980.
5. R. Bowen. Erratum to "Entropy for group endomorphisms and homogeneous spaces". *Trans. Amer. Math. Soc.*, 181:509–510, 1973.
6. Rufus Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.*, 153:401–414, 1971.
7. S. Pelikan C. Grebogi, E. Ott and J. A. Yorke. Strange attractors that are not chaotic. *Phys. D*, 13(1-2)(1):261–268, 1984.
8. Gustave Choquet. *Topology*. Translated from the French by Amiel Feinstein. Pure and Applied Mathematics, Vol. XIX. Academic Press, New York-London, 1966.
9. Roberta Fabbri, Tobias Jäger, Russel Johnson, and Gerhard Keller. A Sharkovskii-type theorem for minimally forced interval maps. *Topol. Methods Nonlinear Anal.*, 26(1):163–188, 2005.
10. G. Keller. A note on strange nonchaotic attractors. *Fund. Math.*, 151(2)(1):139–148, 1996.
11. V.M. Million. Proof of the existence of irregular systems of linear differential equations with almost periodic coefficients. *Differ. Uravn.*, 4 (3)(1):391–396, 1968.
12. V.M. Million. Proof of the existence of irregular systems of linear differential equations with quasi periodic coefficients. *Differ. Uravn.*, 5 (11)(1):1979–1983, 1969.
13. Sam B. Nadler, Jr. *Continuum theory*, volume 158 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1992. An introduction.
14. O. M. Šarkovs′kiĭ. Co-existence of cycles of a continuous mapping of the line into itself. *Ukrain. Mat. Ž.*, 16:61–71, 1964.

15. A. N. Sharkovskiĭ. Coexistence of cycles of a continuous map of the line into itself. In *Thirty years after Sharkovskiĭ's theorem: new perspectives (Murcia, 1994)*, volume 8 of *World Sci. Ser. Nonlinear Sci. Ser. B Spec. Theme Issues Proc.*, pages 1–11. World Sci. Publ., River Edge, NJ, 1995. Translated by J. Tolosa, Reprint of the paper reviewed in MR1361914 (96j:58058).
16. R.E. Vinograd. A problem suggested by n.p. erugin. *Differ. Uravn.*, 11 (4)(1):632–638, 1975.