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On the exoticness of some new p-local compact groups

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PhD Thesis

On the exoticness of some new p-local compact groups

Thesis submitted in partial fulfillment of the requirements
for the degree of Philosophiæ Doctor in Mathematics

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2016

To my family

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CHAPTER 1

Introduction

1.1. Motivation

The groups are one of the most interesting mathematical objects since, although they have a very simple definition, they generate a very complex theory.

Since Galois introduced the notion of group to study the solutions of polynomial equations, many mathematicians have worked to understand them. In fact, for almost two hundred years the mathematicians have been trying to classify all finite groups by using normal subgroups and extensions. This led to the notion of finite simple group, as those groups having no proper nontrivial normal subgroups. Therefore, by considering extensions of groups, we can see the finite simple groups as the building blocks for all finite groups.

All the work seems to culminate with the classification of finite simple groups in the mid 2000s but, although the classification statement is pretty simple, the proof of the classification is one of most complex proofs in the history of mathematics. In fact, the complete proof consist of over 12.000 pages spread across over 500 papers. This is why most group theorists are not completely satisfied and still work in a better understanding of the finite groups, by means of a simpler and self-contained proof.

One interesting method to understand any mathematical concept is to look it from a different point of view. For the case of finite groups, one possibility is to look at it from the algebraic topology point of view. In particular, for any finite group G we have a topological space, BG, whose fundamental group is again G. In this sense, the topological space BG contains all the algebraic information of the group G, and we can try to better understand finite groups by doing homotopy theory with BG.

If we want to specialize, a good choice would be to restrict ourselves to finite p-groups, since they satisfy many good properties. This suggests that we could try to isolate the information of the groups at some prime number p, and then use the information at all primes to recover information of the group G. From the topological point of view, this can be achieved by p-completing the topological space BG to obtain another topological space, denoted $BG^{\wedge}_p$, where only the homotopy information at the prime p is preserved. Is in this sense that we talk about p-local properties.

Then, an interesting question to study from the point of view of algebraic topology is: can we characterize those topological spaces which are the p-completed classifying space of some finite group G?

Working to answer this question, Broto-Levi-Oliver introduced in 2003 the notion of p-local finite group, trying to better understand, using algebraic topology tools, how a Sylow p-subgroup of a group G behaves with respect to the conjugation morphisms induced by elements of G.

More precisely, a p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a finite p-group and $\mathcal{F}$ and $\mathcal{L}$ are categories known as saturated fusion system and centric linking system, respectively. A saturated fusion system $\mathcal{F}$ over a finite p-group S consists of a category whose objects are the subgroups of S and the morphisms emulate the behavior of the conjugation morphisms when S is a Sylow p-subgroup of a bigger group G. A centric linking system $\mathcal{L}$ associated to $\mathcal{F}$ is also a category, with objects a certain class of subgroups of S, and whose morphisms contain just enough information to be able to construct a topological space with properties similar to those of the classifying space of a finite group.

Moreover, they were able to characterize those topological spaces which appear as classifying spaces for p-local finite groups. The problem is that, although any finite group G with $S \in \mathrm{Syl}_p(G)$ produces a p-local finite group denoted as $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$, not all p-local finite groups come from finite groups. This last kind of examples are known as exotic p-local finite groups.

Therefore, it turns out to be necessary to understand the exotic examples in order to completely characterize the topological spaces which appear as p-completed classifying spaces of finite groups.

In this thesis we make our contribution to this achievement by constructing new examples of exotic p-local finite groups for $p \geq 5$.

More precisely, consider the action of the cycle $(1, 2, \ldots, p)$ on $(\mathbb{Z}/p^k)^p$ by permutation. This actions leaves invariant a subgroup isomorphic to $(\mathbb{Z}/p^k)^{p-1}$ consisting of the tuples such that the sum of its coordinates is zero. Therefore, we have an action of C_p on $(\mathbb{Z}/p^k)^{p-1}$ and we can consider the group $S_{p,k} = (\mathbb{Z}/p^k)^{p-1} \rtimes C_p$. The center of $S_{p,k}$ is cyclic of order p, so we can consider a generator $\zeta \in Z(S_{p,k})$. Finally, denote by s a generator of the subgroup C_p and consider the elementary abelian subgroup $V = \langle \zeta, s \rangle$. Then, if we write by $T_{p,k}$ the subgroup $(\mathbb{Z}/p^k)^{p-1}$, we have the following result.

Theorem A. *For each $p \geq 5$ and $k \geq 2$, there exists an exotic p-local finite group, denoted by $(S_{p,k}, \mathcal{F}_{p,k}, \mathcal{L}_{p,k})$, such that the outer automorphisms groups in $\mathcal{F}_{p,k}$ satisfy the following:*

- $\mathrm{Out}_{\mathcal{F}_{p,k}}(S_{p,k}) = C_{\frac{p-1}{2}} \times C_{p-1}$
- $\mathrm{Out}_{\mathcal{F}_{p,k}}(T_{p,k}) = A_p \rtimes C_{p-1}$
- $\mathrm{Out}_{\mathcal{F}_{p,k}}(V) = \mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}}$

Moreover, the p-local finite groups $(S_{p,k}, \mathcal{F}_{p,k}, \mathcal{L}_{p,k})$ are simple, in the sense that they have no proper nontrivial normal subsystems.

Another usual method to understand a theory is to generalize the objects we are interested in, trying to strip away their structure to the bare minims, hoping in this way to become able to understand the more intrinsic properties.

One important way to generalize finite groups is to consider compact Lie groups, and if we only want to deal with p-information, the p-compact groups defined by Dwyer and Wilkerson in 1994 are a pure homotopic generalization of the compact Lie group theory.

Following this direction, Broto-Levi-Oliver generalized the notion of p-local finite group to the notion of p-local compact group, motivated by the p-completion of classifying spaces of compact Lie groups and p-compact groups. If we want to make a construction parallel to the p-local finite groups but looking at compact Lie groups and p-compact groups, we need first to consider the Sylow p-subgroups of these kind of groups. These Sylow p-subgroups are discrete p-toral groups and have a well known structure: they are an extension of a finite product of Prüfer groups $\mathbb{Z}/p^\infty$ by a finite p-group π. Then, a p-local compact group is again a triple $(S, \mathcal{F}, \mathcal{L})$, where S is in this case a discrete p-toral group, $\mathcal{F}$ is a saturated fusion system over S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$.

We have, similarly to the finite case, that any compact Lie group and any p-compact group give rise to a p-local compact group. In general we know that, if the group of components of a compact Lie group G is a p-group, then, by p-completing its classifying space BG we obtain a p-compact group $(G^\wedge_p, BG^\wedge_p, \mathrm{Id})$. However, we cannot construct a p-compact group from every compact Lie group. On the other side, nor is it true that every p-compact group is the p-completion of a compact Lie group.

Therefore, one motivation to study p-local compact groups is that they provide a way to study properties of the p-completion of compact Lie groups and p-compact groups both at the same time.

In order to move from the finite case to the compact case, we develop the theory of limit fusion systems. Aschbacher defined the concept of morphism of fusion systems in

2008, and we prove here that, for any direct system of fusion systems, we can consider a limit fusion system which coincides with the categorical direct limit under saturation hypothesis.

Then, using this limit construction, we can define new examples of fusion systems as limits of the new examples of exotic p-local finite groups we described in Theorem A, as well as the limit of a family discovered Broto-Levi-Oliver, denoted by $\widetilde{\mathcal{F}}_{p,k}$, and a family discovered by Díaz-Ruiz-Viruel for $p = 3$, denoted by $\mathcal{F}_{3,k}$.

Since we want to work with p-local compact groups, and not only with fusion systems over discrete p-toral groups, we need to prove that the new fusion systems we construct as the limits of $\mathcal{F}_{3,k}$, $\mathcal{F}_{p,k}$ and $\widetilde{\mathcal{F}}_{p,k}$ are saturated. This will suffice to obtain new p-local compact groups by a result of Levi-Libman in 2015, stating that there exists a unique centric linking system associated to any saturated fusion system over a discrete p-toral group.

To prove the saturation conditions on the new examples of fusion systems we generalize a saturation criterion for finite fusion systems given by Levi-Oliver in 2002 to the case of fusion systems over discrete p-toral groups. Then, we have the next result.

Theorem B. *Let S_p be the direct limit of $S_{p,k}$ under the obvious inclusions. Then, there exists a 3-local compact group and two p-local compact groups for any $p \geq 5$, denoted by $\mathcal{G}_3 = (S_3, \mathcal{F}_3, \mathcal{L}_3)$, $\mathcal{G}_p = (S_p, \mathcal{F}_p, \mathcal{L}_p)$ and $\widetilde{\mathcal{G}}_p = (S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$ such that the outer automorphisms groups in $\mathcal{F}_3$, $\mathcal{F}_p$ and $\widetilde{\mathcal{F}}_p$ satisfy the following:*

$\mathcal{F}$	$\mathrm{Out}_{\mathcal{F}}(S_p)$	$\mathrm{Out}_{\mathcal{F}}(T_p)$	$\mathrm{Out}_{\mathcal{F}}(V)$	*prime*
$\mathcal{F}_3$	$C_2 \times C_2$	$\mathrm{GL}_2(\mathbb{F}_3)$	$\mathrm{GL}_2(\mathbb{F}_3)$	$p = 3$
$\mathcal{F}_p$	$C_{\frac{p-1}{2}} \times C_{p-1}$	$A_p \rtimes C_{p-1}$	$\mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}}$	$p \geq 5$
$\widetilde{\mathcal{F}}_p$	$C_{p-1} \times C_{p-1}$	$\Sigma_p \times C_{p-1}$	$\mathrm{GL}_2(\mathbb{F}_p)$	

where T_p is the direct limit of $T_{p,k}$ and $V = \langle \zeta, s \rangle$. Moreover, the p-local compact groups $\mathcal{G}_3$ and $\mathcal{G}_p$ are simple, in the sense that they have no proper nontrivial normal subsystems.

One important construction we have for any p-local compact group $(S, \mathcal{F}, \mathcal{L})$ is, given any subgroup $P \leq S$ fully centralized in $\mathcal{F}$, the centralizer p-local compact group $(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P))$. Note that, if the p-local compact group $(S, \mathcal{F}, \mathcal{L})$ is constructed from a p-compact group (X, BX, e), we can also consider the centralizer of P in X, denoted by $C_X(P)$, which turns out to be again a p-compact group, so we can use it to construct a p-local compact group.

Then, to study if we can realize the new examples of p-local compact groups by any p-compact group, it is useful to prove a coherence result stating that these two centralizer p-local compact groups we obtain from any p-compact group are in fact the same. More precisely, we have the following result.

Theorem C. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group realized by a p-compact group (X, BX, e). That is, we have $f\colon S \to X$ a Sylow subgroup of X and $(S, \mathcal{F}, \mathcal{L}) \cong (S, \mathcal{F}_{S,f}(X), \mathcal{L}^c_{S,f}(X))$. Then, for any subgroup $P \leq S$ fully centralized in $\mathcal{F}$, the group $C_S(P)$ is a Sylow subgroup of $C_X(P)$ and*

$$(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P)) \cong (C_S(P), \mathcal{F}_{C_S(P),g}(C_X(P)), \mathcal{L}^c_{C_S(P),g}(C_X(P)))$$

With this important result and other lemmas we are able to prove that there is no p-compact group realizing any of the p-local compact groups $\mathcal{G}_3$, $\mathcal{G}_p$ or $\widetilde{\mathcal{G}}_p$. Moreover, using the simplicity property of the new p-local compact groups we also prove that they cannot be realized either by any compact Lie group, giving rise to the next result.

Theorem D. *There does not exist any compact Lie group nor any p-compact group realizing the p-local compact groups $\mathcal{G}_3$, $\mathcal{G}_p$ or $\widetilde{\mathcal{G}}_p$.*

This proves that the class of p-local compact groups is strictly larger than the class formed by compact Lie groups and p-compact groups.

At the end of the thesis, we review some examples of p-local compact groups that can be constructed from limit fusion systems and we study the relation between the exoticness of the saturated fusion systems in the direct system, and the possibility to realize the limit fusion system by compact Lie groups or p-compact groups.

We find examples to almost completely fill this comparison grid. In fact, we only have one unknown cell, leaving an open problem whose solution could lead to a proof for the exoticness of some p-local finite groups without using the classification of finite simple groups. This would, indeed, be an interesting result since we could obtain information about the classification of the p-local finite groups, doing as if the classification of finite simple groups did not exist.

1.2. Organization of this document

This thesis consists of two interconnected parts. In Part I, consisting of Chapter 2 and Chapter 3, we deal with the finite case, while in Part II, consisting of Chapter 4 to Chapter 7, we deal with the compact case.

More precisely, in Chapter 2 we make a review of saturated fusion systems over finite p-groups and the construction of p-local finite groups, including all the necessary notions as the geometric realization of a category or the Bousfield-Kan p-completion functor.

In Chapter 3 we recall first the construction of the exotic examples for $p = 3$ of [17] in Theorem 3.1.3. Then, we present the new examples of exotic p-local finite groups for $p \geq 5$ and we prove Theorem A (restated as Theorem 3.2.7).

Moving to the compact case, in Chapter 4 we review the definition of saturated fusion systems over discrete p-toral groups, joint with the generalization of the p-local finite groups to the p-local compact groups. We also show in Chapter 4 some properties about compact Lie groups and p-compact groups, and how to construct p-local compact groups from them.

The connection between the finite case and the compact case is made precise in Chapter 5, where we define the notion of limit fusion system and we use it to construct the new fusion systems described in Theorem B.

The first section of Chapter 6 is devoted to prove a saturation criterion for fusion systems over discrete p-toral groups. Then, we prove the saturation conditions for the new examples in the second section. Therefore, Theorem B is a consequence of Proposition 5.4.1, Proposition 5.4.2, and Theorem 6.2.3.

Finally, in Chapter 7 we prove Theorem C in the first section, restated as Theorem 7.1.1, and we use it to show the exoticness as p-compact groups for the new examples of p-local compact groups. In the second section we prove that the new examples of p-local compact groups are also exotic as compact Lie groups. Hence, Theorem D is a combination of Theorem 7.1.4 and Theorem 7.2.4.

1.3. Acknowledgements

This thesis wouldn't have been possible without the help of my advisor, Albert Ruiz, and I feel very grateful for his guidance through all of these years. He has not only been an advisor for me, but also a colleague and a friend.

I'm also thankful to all the Mathematics Department in the UAB and, in particular, to the Algebraic Topology Group and its members. More especially, if possible, I appreciate the help of Carles Broto, for all the time we have spent together and the good references he gave to me.

During my PhD, I've been able to spent three months in Copenhagen. This stay was very enjoyable and fruitful, mostly thanks to Jesper Møller and the Centre for Symmetry and Deformation at the University of Copenhagen. Also, I had the pleasure to meet Bob Oliver several times, and I thank him for very good suggestions and ideas. I'm also grateful to Àlex González for the nice hard work days during his visit to Barcelona and for his commitment when we were writing our paper with Albert.

On the personal level, I thank my parents, Antonio and Julia, and my sister, Sonia, for always being there whenever I need them.

And finally, and most importantly, I wouldn't be where I am now without the help and support from my wife, Anabel. She has been by my side since we started our university education and I am really grateful to her.

This thesis has been realized partially under grants BES-2011-044403 and EEBB-I-15-09265. It has also been partially supported by projects MTM2010-20692 and MTM2013-42293-P.

Part I

The finite case

CHAPTER 2

Fusion on finite groups

In this chapter we will make a quick review of fusion on finite groups. In the first section we show the algebraic part of the subject, with the definition of saturated fusion system and some of its properties. The second section deals with the topological part, and we define all the necessary concepts to get to the definition of p-local finite group, such as the geometric realization of a category, the classifying space of a topological group and the p-completion functor.

All the definitions and results in this chapter are in the literature and we provide references for all of them. In some cases there are different names or notations for the same properties and might be the case that the one used here is not exactly the same as in the given reference. This has been done to maintain the coherence of the text, and in all cases the definitions are equivalent. In the case that some well known result is not stated explicitly in the literature, a short proof pointing to the necessary references is given.

2.1. Saturated fusion systems

The fusion system of a finite group G over a p-Sylow subgroup $S \in \mathrm{Syl}_p(G)$ is a category, $\mathcal{F}_S(G)$, which encodes all the information about the conjugacy morphisms between subgroups of S induced by elements of G. More precisely, the objects of $\mathcal{F}_S(G)$ are all subgroups $P \leq S$ and, for $P, Q \leq S$, the morphisms between P and Q are

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \{\varphi \in \mathrm{Hom}(P,Q) \mid \varphi = c_g \text{ for some } g \in G \text{ such that } gPg^{-1} \leq Q\}$$

In general, for two groups P and Q, we denote by $\mathrm{Inj}(P,Q)$ the set of group monomorphisms from P to Q and, if P and Q are subgroups of a group G, we denote by $\mathrm{Hom}_G(P,Q)$ the set of morphisms induced by conjugation for some $g \in G$ such that $gPg^{-1} \leq Q$.

In order to study the fusion of a finite p-group S, we want to consider a category $\mathcal{F}$ encoding information about morphisms between subgroups of S in a similar way as $\mathcal{F}_S(G)$ does when there is a bigger group G such that $S \in \mathrm{Syl}_p(G)$.

Definition 2.1.1 ([11, Definition 1.1])**.** A *fusion system* $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ satisfy the following conditions:

(a) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ for all $P, Q \leq S$.

(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

If $\mathcal{F}$ is a fusion system over a finite p-group S and $P, Q \leq S$ are two subgroups which are isomorphic as objects in $\mathcal{F}$, we say that they are $\mathcal{F}$-conjugated. In that case, we denote $\mathrm{Hom}_{\mathcal{F}}(P, Q) = \mathrm{Iso}_{\mathcal{F}}(P, Q)$. In the particular case when $Q = P$ we denote $\mathrm{Hom}_{\mathcal{F}}(P, P) = \mathrm{Aut}_{\mathcal{F}}(P)$. For a subgroup $P \leq S$, we denote by $P^{\mathcal{F}}$ the set of subgroups of S which are $\mathcal{F}$-conjugated to P.

Remark 2.1.2. Note that, in Definition 2.1.1, the first condition implies that $\mathcal{F}$ has at least the morphisms induced by conjugation for elements of S and the second condition ensures that two subgroups $P, Q \leq S$ which are isomorphic as groups are also $\mathcal{F}$-conjugated if there is a morphism in $\mathcal{F}$ from P to Q, or vice versa. This makes the $\mathcal{F}$-conjugacy relation an equivalence relation.

It turns out that the definition of fusion system is too general for most purposes, so we need to add extra conditions to a fusion system in order to better emulate the behavior of fusion systems of finite groups. For this, we need first to define some conditions about maximality of centralizers and normalizers.

Definition 2.1.3 ([11, Definition 1.2], [5, Definition I.2.2])**.** Let $\mathcal{F}$ be a fusion system over a finite p-group S.

- A subgroup $P \leq S$ is *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.
- A subgroup $P \leq S$ is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.
- A subgroup $P \leq S$ is *fully automized* in $\mathcal{F}$ if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
- A subgroup $P \leq S$ is *receptive* in $\mathcal{F}$ if for each $Q \leq S$ and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$, if we set
$$N_\varphi = \{g \in N_S(Q) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(P)\}$$
then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_Q = \varphi$.

In local group theory it is usual to study the normalizers and centralizers of Sylow subgroups and its subgroups, so it seems natural to ask conditions about these local subgroups to the fusion systems, in order to simulate the existence of an environment group G.

Definition 2.1.4 ([11, Definition 1.2]). Let $\mathcal{F}$ be a fusion system over a finite p-group S. Then $\mathcal{F}$ is a *saturated fusion system* if the following two conditions hold:

(I) Each subgroup $P \leq S$ which is fully normalized in $\mathcal{F}$ is also fully centralized and fully automized in $\mathcal{F}$.

(II) Each subgroup $P \leq S$ which is fully centralized in $\mathcal{F}$ is also receptive in $\mathcal{F}$.

One way to see that this definition indeed shrinks the definition of fusion system to better emulate the behavior of fusion systems of finite groups is shown in the next two propositions.

If we look at the local subgroups of a fusion system constructed from a finite group, we can observe that the properties described on Definition 2.1.3 are strongly related.

Proposition 2.1.5. *Let G be a finite group and $S \in \mathrm{Syl}_p(G)$. Then, $\mathcal{F}_S(G)$ satisfies the following:*

(a) A subgroup $P \leq S$ is fully normalized if and only if it is fully centralized and fully automized.

(b) A subgroup $P \leq S$ is fully centralized if and only if it is fully centralized.

Proof. It is a direct consequence of [34, Proposition 3.7], [34, Proposition 4.4] and [11, Proposition 1.3]. □

In comparison with this result, general fusion systems don't verify these equivalences, but only one of the implications in each case.

Proposition 2.1.6 ([34, Proposition 3.7 and Proposition 4.4]). *Let $\mathcal{F}$ be a fusion system over a finite p-group S. Then,*

(a) Every subgroup of S which is fully automized and receptive is fully normalized.

(b) Every receptive subgroup of S is fully centralized.

We can see then that imposing a fusion system to be saturated completes the equivalences of Proposition 2.1.5. In this way, the subgroups of an abstract saturated fusion system verify the same key properties as the subgroups of fusion systems of finite groups.

As a consequence of Proposition 2.1.5, we have that any finite group G gives rise to a saturated fusion system over any of its Sylows subgroups. In general, not all saturated fusion systems can be constructed in this way.

Definition 2.1.7 ([11, Section 9])**.** A saturated fusion system $\mathcal{F}$ over a finite p-group S will be called *realizable* if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$, and will be called *exotic* otherwise.

Remark 2.1.8. There are several examples of exotic saturated fusion systems over finite p-groups: for $p = 2$, there is only one known family of examples ([28]); for $p \geq 3$ we can find different examples in [11, Section 9], [35] or [17], as well as in Chapter 3.

Given a saturated fusion system $\mathcal{F}$, it is obvious that in every $\mathcal{F}$-conjugacy class of subgroups there is at least one which is fully normalized, fully centralized and receptive. Besides this, there are some of this fully normalized, fully centralized and receptive subgroups which are even more important, in the sense that we can reconstruct all the fusion system $\mathcal{F}$ from the information given by these subgroups. These kind of subgroups will be called $\mathcal{F}$-essential, and we will make precise this notion of control of fusion in Theorem 2.1.13. Before we get to this important theorem we need some other definitions and results.

For any finite group G, let $O_p(G)$ denote the largest normal p-subgroup of G and $O^p(G)$ the smallest normal subgroup H of G such that G/H is a p-group. Also, let $O^{p'}$ denote the smallest normal subgroup H of G such that $|G/H|$ is coprime to p. A proper subgroup $H < G$ is strongly p-embedded if p divides the order of H and, for each $x \in G \setminus H$, the subgroup $H \cap xHx^{-1}$ has order prime to p.

Definition 2.1.9 ([5, Definition I.3.1])**.** Let $\mathcal{F}$ be a fusion system over a finite p-group S.

- For each subgroup $P \leq S$, set $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ and $\mathrm{Out}_S(P) = \mathrm{Aut}_S(P)/\mathrm{Inn}(P)$. Thus $\mathrm{Out}_S(P) \leq \mathrm{Out}_{\mathcal{F}}(P) \leq \mathrm{Out}(P)$.
- A subgroup P of S is $\mathcal{F}$-*centric* if $C_S(Q) = Z(Q)$ for all $Q \in P^{\mathcal{F}}$. Equivalently, P is $\mathcal{F}$-centric if P is fully centralized in $\mathcal{F}$ and $C_S(P) = Z(P)$. Let $\mathcal{F}^c$ denote the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups of S.
- A subgroup P of S is $\mathcal{F}$-*radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ is p-reduced, i.e., if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$.

In [11, Theorem A.10] it is proved that all morphisms in a saturated fusion system $\mathcal{F}$ can be recovered as compositions of restrictions of automorphisms of $\mathcal{F}$-centric, $\mathcal{F}$-radical, fully normalized subgroups. We can restrict a bit more this set of generator subgroups by defining the concept of being $\mathcal{F}$-essential.

Definition 2.1.10 ([5, Definition I.3.2]). Let $\mathcal{F}$ be a fusion system over a finite p-group S. A subgroup P of S is *$\mathcal{F}$-essential* if P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup.

The next proposition shows that this definition in fact restricts the set of generator subgroups and describes the key property of essential subgroups.

Proposition 2.1.11 ([5, Proposition I.3.3]). *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.*

(*a*) *Each $\mathcal{F}$-essential subgroup of S is $\mathcal{F}$-centric, $\mathcal{F}$-radical and fully normalized in $\mathcal{F}$.*

(*b*) *Fix a proper subgroup P of S which is fully normalized, and let $H_P \leq \mathrm{Aut}_{\mathcal{F}}(P)$ be the subgroup generated by those $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ which extend to $\mathcal{F}$-isomorphisms between strictly larger subgroups of S. Then either P is not $\mathcal{F}$-essential and $H_P = \mathrm{Aut}_{\mathcal{F}}(P)$; or P is $\mathcal{F}$-essential and $H_P/\,\mathrm{Inn}(P)$ is strongly p-embedded in $\mathrm{Out}_{\mathcal{F}}(P)$.*

Point (*b*) of previous proposition is saying that, for a non $\mathcal{F}$-essential subgroup, we can extend all its automorphisms to morphisms between larger subgroups, so we can consider all these automorphisms as restrictions of other morphisms. This indicates that we can disregard the non $\mathcal{F}$-essential subgroups while trying to reconstruct the whole fusion system.

In general, given a finite p-group S, we can select a set of monomorphisms between its subgroups and consider the minimal fusion system with these morphisms.

Definition 2.1.12 ([5, Definition I.3.4]). For any set $\mathfrak{X}$ of monomorphisms between subgroups of S, the *fusion system generated by* $\mathfrak{X}$, denoted $\langle \mathfrak{X} \rangle_S$, is the smallest fusion system over S (not necessarily saturated) which contains $\mathfrak{X}$. Thus, the morphisms in $\langle \mathfrak{X} \rangle_S$ are the composites of restrictions of homomorphisms in the set $\mathfrak{X} \cup \mathrm{Inn}(S)$ and their inverses. We write $\langle \mathfrak{X} \rangle = \langle \mathfrak{X} \rangle_S$ when the choice of S is clear.

Now we can make precise the statement about control of fusion by $\mathcal{F}$-essential subgroups mentioned before, which is known as Alperin's fusion theorem for fusion systems.

Theorem 2.1.13 ([5, Theorem I.3.5]). *Fix a finite p-group S and a saturated fusion system $\mathcal{F}$ over S. Then*

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(P) \mid P = S \text{ or } P \text{ is } \mathcal{F}\text{-essential} \rangle_S$$

At this moment, we have replaced the situation of having a finite group G and a Sylow subgroup $S \in \mathrm{Syl}_p(G)$ by having a saturated fusion system $\mathcal{F}$ over S. Then, we want to translate the notions of local theory of groups to notions about fusion systems. The most important one is that of being a normal subgroup, which translate to several notions in fusion systems. If we fix our attention to subgroups, there are several properties related to conjugation that a subgroup in a fusion system can verify.

Definition 2.1.14 ([5, Definition I.4.1])**.** Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.

- A subgroup $P \leq S$ is *normal* in $\mathcal{F}$ if $P \trianglelefteq S$ and, for all $Q, R \leq S$ and all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, φ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(QP, RP)$ such that $\overline{\varphi}(P) = P$. The maximal normal p-subgroup of $\mathcal{F}$ is denoted by $O_p(\mathcal{F})$.
- A subgroup $P \leq S$ is *strongly closed* in $\mathcal{F}$ if no element of P is $\mathcal{F}$-conjugate to an element of $S \setminus P$.

Remark 2.1.15. Note that if $P \trianglelefteq S$ is normal in $\mathcal{F}$ it is also strongly closed. Also, since $\mathcal{F}$ contains all conjugation morphisms, if P is strongly closed in $\mathcal{F}$ then P must be a normal subgroup of S.

Now, if we fix our attention to fusion systems and subsystems, we can also consider the notion of normality. There are several properties related with normality and several equivalent definitions. The following one is due to Aschbacher.

Definition 2.1.16 ([16, Definition 5.30 and Definition 8.2])**.** Let S be a finite p-group, $\mathcal{F}$ a saturated fusion system over S, and $\mathcal{F}' \leq \mathcal{F}$ a subsystem over a subgroup $S' \leq S$. Then, $\mathcal{F}'$ is *normal* in $\mathcal{F}$ if the following conditions are satisfied.

(N1) S' is strongly closed in $\mathcal{F}$.

(N2) For each $P \leq Q \leq S$ and each $\gamma \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, the map that sends each morphism $f \in \mathrm{Hom}_{\mathcal{F}'}(P, Q)$ to $\gamma \circ f \circ \gamma^{-1}$ defines a bijection between the sets $\mathrm{Hom}_{\mathcal{F}'}(P, Q)$ and $\mathrm{Hom}_{\mathcal{F}'}(\gamma(P), \gamma(Q))$.

(N3) $\mathcal{F}'$ is a saturated fusion system over S'.

(N4) Each $f \in \mathrm{Aut}_{\mathcal{F}'}(S')$ extends to some $\widetilde{f} \in \mathrm{Aut}_{\mathcal{F}}(S'C_S(S'))$ such that

$$[\widetilde{f}, C_S(S')] = \{\widetilde{f}(g) \cdot g^{-1} \mid g \in C_S(S')\} \leq Z(S')$$

Once we have the notion of normal subsystem, the notion of simplicity translates perfectly from groups to fusion systems.

Definition 2.1.17 ([5, Definition I.6.1])**.** A saturated fusion system $\mathcal{F}$ is *simple* if it contains no proper nontrivial normal fusion subsystems.

The focal and hyperfocal subgroups of a finite group are very important in local group theory and the understanding of the fusion in a group, so it seems natural to generalize the definition to fusion systems.

Definition 2.1.18 ([5, Definition I.7.1])**.** For any saturated fusion system $\mathcal{F}$ over a finite p-group S, the *focal subgroup* $\mathfrak{foc}(\mathcal{F})$ and the *hyperfocal subgroup* $\mathfrak{hyp}(\mathcal{F})$ are defined by setting

$$\mathfrak{foc}(\mathcal{F}) = \langle g^{-1}\alpha(g) \mid g \in P \leq S, \alpha \in \mathrm{Aut}_{\mathcal{F}}(P)\rangle$$
$$\mathfrak{hyp}(\mathcal{F}) = \langle g^{-1}\alpha(g) \mid g \in P \leq S, \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P))\rangle$$

Now we turn our attention to extensions of fusion systems. If we have a fusion system $\mathcal{F}$ and a subsystem $\mathcal{E}$, we would like to have the notion of index of $\mathcal{E}$ in $\mathcal{F}$. This is not possible in general, but we can control the index of a subsystem in some sense.

Definition 2.1.19 ([9, Definition 3.1])**.** Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S, and a fusion subsystem $\mathcal{E}$ in $\mathcal{F}$ over $T \leq S$.

- The subsystem $\mathcal{E}$ has *p-power index* in $\mathcal{F}$ if $T \geq \mathfrak{hyp}(\mathcal{F})$ and, for each $P \leq T$, we have $\mathrm{Aut}_{\mathcal{E}}(P) \geq O^p(\mathrm{Aut}_{\mathcal{F}}(P))$. Equivalently, a saturated fusion subsystem over $T \geq \mathfrak{hyp}(\mathcal{F})$ has p-power index if it contains all $\mathcal{F}$-automorphisms or order prime to p of subgroups of T.

- The subsystem $\mathcal{E}$ has *index prime to p* in $\mathcal{F}$ if $T = S$ and, for each $P \leq S$, we have $\mathrm{Aut}_{\mathcal{E}}(P) \geq O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$. Equivalently, a saturated fusion subsystem has index prime to p if it contains all $\mathcal{F}$-automorphisms of p-power order.

The next result shows one restriction about the normal subsystems of a saturated fusion system that will be useful in Chapter 3.

Lemma 2.1.20 ([16, Lemma 5.72])**.** *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S and $\mathcal{E}$ a normal subsystem of $\mathcal{F}$. Then $\mathcal{E}$ has index prime to p in $\mathcal{F}$.*

Later in the text we will be interested in simple fusion systems, so it will be useful to have some criterion to prove this property. One of these criteria has to do with minimal subsystems of p-power index and index prime to p, so we need to be sure these subsystems exist and try to have control over them.

The next two statements will allow us to prove when there are no proper normal subsystems of p-power index.

Theorem 2.1.21 ([9, Theorem 4.3]). *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then, there is a unique minimal saturated fusion subsystem of p-power index, denoted by $O^p(\mathcal{F})$, over the subgroup $\mathfrak{hyp}(\mathcal{F})$.*

Proposition 2.1.22 ([5, Corollary I.7.5]). *For any saturated fusion system $\mathcal{F}$ over a finite p-group S, $O^p(\mathcal{F}) = \mathcal{F}$ if and only if $\mathfrak{foc}(\mathcal{F}) = S$.*

We can have much more control over the subsystems of index prime to p, as the next theorem shows.

Theorem 2.1.23 ([9, Theorem 5.4]). *Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S. Set $\mathcal{E}_0 = \langle O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \mid P \leq S \rangle$, as a fusion system over S, and write*

$$\mathrm{Out}^0_{\mathcal{F}}(S) = \langle \alpha \in \mathrm{Out}_{\mathcal{F}}(S) \mid \alpha|_P \in \mathrm{Hom}_{\mathcal{E}_0}(P, S),\ \textit{for some } \mathcal{F}\textit{-centric } P \leq S \rangle$$

Then, there is a bijective correspondence between subgroups of $\mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S)$ and saturated fusion subsystems of $\mathcal{F}$ of index prime to p. In particular, there is a unique minimal saturated fusion subsystem of index prime to p, denoted by $O^{p'}(\mathcal{F})$.

We only need to define one more property about saturated fusion systems.

Definition 2.1.24 ([33, Definition 1.2]). A saturated fusion system $\mathcal{F}$ is *reduced* if $O_p(\mathcal{F}) = 1$ and $O^p(\mathcal{F}) = \mathcal{F} = O^{p'}(\mathcal{F})$.

Remark 2.1.25. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. As pointed out in [33], by Lemma 2.1.20 we have that a reduced fusion system is simple if it has no proper nontrivial strongly closed subgroups.

2.2. p-local finite groups

Now we want to introduce some topological spaces associated to saturated fusion systems. For this, we need to introduce the geometric realization of a category, the classifying space of a topological group and the p-completion functor. With these tools we will be able to define a topological space to any fusion system of a finite group with very interesting properties. We will define then the concept of centric linking system associated to a saturated fusion system to construct a topological space from it. At the end we will see that it is always possible to associate a topological space to any saturated fusion system in this way.

The construction of the geometric realization of a category will be done in two steps. First, we will define simplicial spaces and its the geometric realization. Then, we will define a simplicial set associated to any small category $\mathcal{C}$ and use its geometric realization to associate a topological space to $\mathcal{C}$.

First we need to define the category of ordered sets and ordered morphisms.

Definition 2.2.1 ([36, Section 1]). The *simplicial category*, Δ, is the category whose objects are the sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and whose morphisms are the order preserving maps between objects. For each n, there are $n + 1$ *face morphisms* $d_n^i \in \mathrm{Mor}_\Delta([n-1], [n])$ $(0 \leq i \leq n)$, where d_n^i is the unique injective morphism whose image does not contain i. Also, there are n *degeneracy morphisms* $s_n^i \in \mathrm{Mor}_\Delta([n], [n-1])$ $(0 \leq i \leq n-1)$, where s_n^i is the unique surjective morphism such that $s_n^i(i) = s_n^i(i+1) = i$.

The concept of simplicial set it is quite abstract, but very simple to define using the simplicial category.

Definition 2.2.2 ([36, Section 1]). A *simplicial set* is a functor $K\colon \Delta^{\mathrm{op}} \to \mathrm{Sets}$. If K is a simplicial set, it is usual to write $K_n = K([n])$.

In order to associate a topological space to any simplicial set we need first to define the basic building blocks of topological spaces, the standard simplices.

Definition 2.2.3 ([29, Section 1]). Let Δ^n denote the *standard n-simplex*, defined as

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum_{i=0}^{n} t_i = 1 \right\}$$

For any $\varphi \in \mathrm{Mor}_\Delta([n], [m])$, write $\varphi_*\colon \Delta^n \to \Delta^m$ for the map which sends a vertex $e_i \in \Delta^n$ to $e_{\varphi(i)} \in \Delta^m$, where $\{e_0, \ldots, e_n\}$ denotes the canonical basis for $\mathbb{R}^{n+1}$.

Now we can take one standard simplex for each simplex in the simplicial set and glue them together according to the combinatorics of the simplicial set.

Definition 2.2.4 ([29, Section 1]). The *geometric realization* $|K|$ of a simplicial set K is defined by setting

$$|K| = \left(\coprod_{n=0}^{\infty} K_n \times \Delta^n \right) \Big/ \sim$$

with the quotient topology, where $(\sigma, \varphi_*(\tau)) \sim (\varphi^*(\sigma), \tau)$ for all $\sigma \in K_m$, $\tau \in \Delta^n$ and $\varphi \in \mathrm{Mor}_\Delta([n], [m])$.

Once we have associated a topological space to any simplicial set, we can construct a simplicial set to any small category, obtaining this way a geometric realization for the category.

Definition 2.2.5 ([36, Section 2])**.** Let $\mathcal{C}$ be a small category. The *nerve* of $\mathcal{C}$ is the simplicial set $\mathcal{N}(\mathcal{C})$, defined by

$$\mathcal{N}(\mathcal{C})_n = \{c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n \mid c_i \in \mathrm{Ob}(\mathcal{C}), \alpha_i \in \mathrm{Mor}(\mathcal{C})\}$$

and where $\varphi \in \mathrm{Mor}_\Delta([n], [m])$ sends $\mathcal{N}(\mathcal{C})_n$ to $\mathcal{N}(\mathcal{C})_m$ by composing morphisms or inserting identity morphisms as appropriate. The geometric realization $|\mathcal{C}|$ of the category $\mathcal{C}$ is defined to be the geometric realization $|\mathcal{N}(\mathcal{C})|$ of its nerve.

One application of the geometric realization for small categories is an explicit construction of the classifying space for topological groups satisfying the condition of being locally contractible. In particular, any finite group is locally contractible, so it will fit in the following definition.

Definition 2.2.6 ([36, Section 3], [37, Proposition A.1])**.** For any locally contractible topological group G, define the category $\mathcal{B}(G)$ with only one object $*$ and morphisms $\mathrm{Mor}_{\mathcal{B}(G)}(*, *) = G$. Then, $BG = |\mathcal{B}(G)|$ is called the *classifying space* of G.

The classifying space of a group G is a topological space containing much of the information of the group, but if we are interested in a particular prime p, we would like to isolate the p-information of the group G. This is done with the p-completion functor of Bousfield-Kan.

Definition 2.2.7 ([8, Chapter I, Section 4])**.** The *Bousfield-Kan p-completion functor* is a functor from spaces to spaces, denoted $X \to X^\wedge_p$, together with a natural transformation $\phi\colon \mathrm{Id} \to (-)^\wedge_p$.

The interesting point about the Bousfield-Kan p-completion functor is that it isolates the information at the prime p, in the sense made precise in the next lemma.

Lemma 2.2.8 ([8, Lemma I.5.5])**.** *A continuous map $f\colon X \to Y$ induces an isomorphism in cohomology with coefficients in $\mathbb{F}_p$ if and only if $f^\wedge_p\colon X^\wedge_p \to Y^\wedge_p$ is a homotopy equivalence.*

When working with p-completion, it is essential to know if the spaces we are interested in behave well with respect to the p-completion functor.

Definition 2.2.9 ([8, Definition I.5.1])**.** A space X is

- *p-complete* if $\phi_X\colon X \to X^\wedge_p$ is a homotopy equivalence.
- *p-good* if ϕ_X induces an isomorphism in cohomology with coefficients in $\mathbb{F}_p$.

Since we don't want to lose any information on the prime p, we are interested only in p-good spaces. Next proposition shows that in these cases we obtain p-complete spaces when we apply the p-completion functor.

Proposition 2.2.10 ([8, Proposition I.5.2]). *Let X be a space. Then, the following are equivalent:*

(*a*) *X is p-good.*

(*b*) *$X^\wedge_p$ is p-complete.*

(*c*) *$X^\wedge_p$ is p-good.*

Lastly, in order to isolate the p-information of a finite group by applying the p-completion functor to its classifying space, we need to be sure that the classifying spaces of finite groups are p-good, which is shown in the next proposition.

Proposition 2.2.11 ([8, Proposition VII.5.1]). *A connected space X is p-good if $\pi_1(X)$ is finite.*

In particular, if G is a finite group, its classifying space BG is p-good, since $\pi_1(BG) \cong G$, so $BG^\wedge_p$ is a p-complete space.

We also have the next natural result about classifying spaces of p-groups.

Proposition 2.2.12 ([5, Proposition III.1.10]). *The classifying space of any p-group is p-complete.*

Now we have all general constructions we need to define a topological space associated to saturated fusion systems. For this, we will first construct the classifying space for fusion systems of groups and then we will construct it in general through the definition of centric linking system.

Let G be a finite group. First, we define the notion of p-centric subgroup, which is the motivation for the notion of $\mathcal{F}$-centric subgroup in Definition 2.1.9.

Definition 2.2.13 ([10, Section 1]). If G is a finite group and p is a prime, then a p-subgroup $P \leq G$ is *p-centric* if $C_G(P) = Z(P) \times C'_G(P)$, where $C'_G(P)$ has order prime to p.

Now we can define the centric linking category associated to a group. It is a category similar to the fusion system of a group, but with a special set of subgroups as objects and more information on morphisms than the fusion system. This extra information allows us to do some homotopy theory and prove interesting results relating the centric linking category with the p-completed classifying space of the group.

Definition 2.2.14 ([10, Section 1])**.** If G is a finite group and $S \in \mathrm{Syl}_p(G)$, the *centric linking category* of G over S is the category $\mathcal{L}_S^c(G)$ whose objects are the subgroups of S which are p-centric in G, and whose morphism sets are given by

$$\mathrm{Mor}_{\mathcal{L}_S^c(G)}(P,Q) = N_G(P,Q)/C'_G(P)$$

where $N_G(P,Q) = \{g \in G \mid gPg^{-1} \leq Q\}$.

The key property of the centric linking category of a finite group G is that in contains all the homotopy information of the classifying space of G at the prime p.

Proposition 2.2.15 ([10, Proposition 1.1])**.** *For any finite group G and any $S \in \mathrm{Syl}_p(G)$,*

$$BG_p^\wedge \simeq |\mathcal{L}_S^c(G)|_p^\wedge$$

Furthermore, the centric linking categories themselves allow us to distinguish between p-completed classifying spaces of finite groups.

Theorem 2.2.16 ([10, Theorem 2.9])**.** *For any prime p and any pair G_1, G_2 of finite groups, ${BG_1}_p^\wedge \simeq {BG_2}_p^\wedge$ if and only if for some $S_i \in \mathrm{Syl}_p(G_i)$, $\mathcal{L}_{S_1}^c(G_1)$ and $\mathcal{L}_{S_2}^c(G_2)$ are equivalent as categories.*

In view of the centric linking category for a finite group, we want to define a category associated to an abstract saturated fusion system with similar properties that allow us to prove interesting results in homotopy.

Definition 2.2.17 ([11, Definition 1.7])**.** Let $\mathcal{F}$ be a fusion system over a finite p-group S. A *centric linking system* associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor

$$\pi \colon \mathcal{L} \to \mathcal{F}^c$$

and distinguished monomorphisms $P \xrightarrow{\delta_p} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions.

(A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P,Q)$$

(B) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow{\scriptstyle \delta_P(g)} & & \downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{f} & Q \end{array}$$

We are now ready to define a p-local finite group.

Definition 2.2.18 ([11, Definition 1.8])**.** A *p-local finite group* is a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a finite p-group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The *classifying space* of the p-local finite group is the space $|\mathcal{L}|^\wedge_p$.

Remark 2.2.19. It is easy to check that for every finite group G and $S \in \mathrm{Syl}_p(G)$, $\mathcal{L}^c_S(G)$ is a centric linking system associated to $\mathcal{F}_S(G)$. So, for any finite group G and $S \in \mathrm{Syl}_p(G)$ we have an associated p-local finite group $(S, \mathcal{F}_S(G), \mathcal{L}^c_S(G))$.

Similarly to the case of saturated fusion systems, the converse to the previous remark is not true, that is, not all p-local finite groups can be constructed from finite groups and Sylow subgroups, so we can give an analogous definition for exotic p-local finite groups.

Definition 2.2.20 ([11, Section 9])**.** A p-local finite group $(S, \mathcal{F}, \mathcal{L})$ will be called *realizable* if $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}^c_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$, and will be called *exotic* otherwise.

All the examples mentioned in Remark 2.1.8 about exotic saturated fusion systems give rise to exotic p-local finite groups. This was not completely trivial until Chermak proved Theorem 2.2.22 in 2013, since one should prove the existence of a centric linking system associated to an exotic saturated fusion system in order to obtain an exotic p-local finite group.

Similarly to the case of centric linking categories of groups, for a general p-local finite group $(S, \mathcal{F}, \mathcal{L})$, the homotopy type of $|\mathcal{L}|^\wedge_p$ completely determines the p-local finite group.

Theorem 2.2.21 ([11, Theorem 7.4])**.** *If $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are two p-local finite groups such that $|\mathcal{L}|^\wedge_p \simeq |\mathcal{L}'|^\wedge_p$, then $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are isomorphic as p-local finite groups.*

In order to work with a p-local finite group we should deal with the saturated fusion system and the centric linking system associated to it, but the next theorem shows that we can focus our attention only into the saturated fusion systems. This is why, through the rest of this part, we will work mainly with saturated fusion systems over p-groups, without making explicit the correspondence between saturated fusion systems and p-local finite groups.

Theorem 2.2.22 ([15])**.** *Each saturated fusion system $\mathcal{F}$ over a finite p-group S has an associated centric linking system $\mathcal{L}$, which is unique up to isomorphism.*

The original proof of Theorem 2.2.22 uses the classification of finite simple groups in a very specific point, but Glauberman-Lynd provided a patch to avoid the use of the classification in [22].

Remark 2.2.23. Note that from Theorem 2.2.22 we can deduce that a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is exotic if and only if $\mathcal{F}$ is an exotic saturated fusion system over S.

CHAPTER 3

New examples of exotic p-local finite groups

In 2004, Albert Ruiz and Antonio Viruel classified in [35] all the p-local finite groups over the extraspecial group of order p^3 and exponent p. As a result of this classification, they found three new examples of exotic 7-local finite groups.

Later, in 2007, the same authors joint with Antonio Díaz generalized the classification to all p-local finite groups of rank two for odd prime p in [17]. In addition to the exotic examples over the extraspecial group, they found several infinite families of exotic 3-local finite groups.

In the first section of this chapter we will recall the construction of one of the families of exotic 3-local finite groups found in [17]. In the second section we will generalize the construction of the p-local finite groups for $p \geq 5$ over p-groups of maximal class similar to those for $p = 3$ and we will prove that these new examples are also exotic.

3.1. A family of examples for $p = 3$

We will work with saturated fusion systems over a particular class of 3-groups which, by [6], are of maximal class and can be presented as

$$\begin{aligned} S_{3,k} = \langle s, s_1, s_2, \ldots, s_{2k} \,|\, & [s, s_{i-1}] = s_i \text{ for } i = 2, \ldots, 2k, \\ & [s_1, s_i] = 1 \text{ for } i = 2, \ldots, 2k, \\ & s^3 = 1, \\ & s_i^3 s_{i+1}^3 s_{i+2} = 1 \text{ for } i = 1, \ldots, 2k \rangle \end{aligned}$$

where, in the last relation, we assume $s_j = 1$ if $j > 2k$.

In [17, Proposition A.9], it is proven that the groups $S_{3,k}$ fit in a split extension

$$1 \to \mathbb{Z}/3^k \times \mathbb{Z}/3^k \to S_{3,k} \to \mathbb{Z}/3 \to 1$$

where s is a generator of $\mathbb{Z}/3$, the subgroup $\mathbb{Z}/3^k \times \mathbb{Z}/3^k$ is generated by s_1 and s_2, and the action of $\mathbb{Z}/3$ on $\mathbb{Z}/3^k \times \mathbb{Z}/3^k$, which is conjugation by s in $S_{3,k}$, is given by the matrix

$$c_s = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}$$

To simplify the notation, we will write $T_{3,k} = \mathbb{Z}/3^k \times \mathbb{Z}/3^k \leq S_{3,k}$ for the maximal torus. It is also proven in [17] that the center of $S_{3,k}$ is isomorphic to $\mathbb{Z}/3$. We will write ζ for a generator of the center and we will write $V = \langle s, \zeta \rangle \cong \mathbb{Z}/3 \times \mathbb{Z}/3$.

We will need, at the end of this section, to know which elements of $S_{3,k} \setminus T_{3,k}$ are conjugated to the element s.

Lemma 3.1.1. *Let $s_1^i s_2^j s \in S_{3,k}$ be an element not in the maximal torus. Then, $s_1^i s_2^j s$ is conjugated to s if and only if $i \equiv 0 \mod 3$.*

Proof. Note that if we conjugate s by any element $s_1^\alpha s_2^\beta s$ we obtain $s_1^{3\beta} s_2^{-\alpha+3\beta} s$, and $3\beta \equiv 0 \mod 3$.

Conversely, if $i = 3l$, conjugating s by $s_1^{i-j} s_2^l s$ we obtain $s_1^i s_2^j s$. □

We will define a fusion system over $S_{3,k}$ by adding automorphisms to itself and the subgroups $T_{3,k}$ and V. For this, it will be useful to have certain control over the possibilities for the groups of outer automorphisms.

Lemma 3.1.2 ([17, Lemma 5.5])**.** *Let $\mathcal{F}$ be a saturated fusion system over $S_{3,k}$ such that $T_{3,k}$ and V are $\mathcal{F}$-essential subgroups. Then,*

(*a*) $\mathrm{Out}_{\mathcal{F}}(S_{3,k}) \leq C_2 \times C_2$.

(*b*) $\mathrm{Out}_{\mathcal{F}}(T_{3,k}) = \mathrm{SL}_2(\mathbb{F}_3)$ *or* $\mathrm{Out}_{\mathcal{F}}(T_{3,k}) = \mathrm{GL}_2(\mathbb{F}_3)$.

(*c*) $\mathrm{Out}_{\mathcal{F}}(V) = \mathrm{SL}_2(\mathbb{F}_3)$ *or* $\mathrm{Out}_{\mathcal{F}}(V) = \mathrm{GL}_2(\mathbb{F}_3)$.

In the proof of this lemma it is shown that there exist two automorphisms $\eta, \omega \in \mathrm{Aut}(S_{3,k})$ such that their projection onto $\mathrm{Out}(S_{3,k})$ generate a $C_2 \times C_2$ subgroup. By an abuse of notation we will write also η and ω for their projection on $\mathrm{Out}(S_{3,k})$.

It is also shown in [17] that there are automorphisms of $T_{3,k}$ and V such that $\mathrm{Out}_{\mathcal{F}}(T_{3,k}) = \mathrm{GL}_2(\mathbb{F}_3)$ and $\mathrm{Out}_{\mathcal{F}}(V) = \mathrm{GL}_2(\mathbb{F}_3)$.

With this information about the outer automorphisms groups we can state the main theorem of this section.

Theorem 3.1.3. *There are saturated fusion systems $\mathcal{F}_{3,k}$ over $S_{3,k}$, for all $k \geq 2$, characterized by the morphisms*

- $\mathrm{Out}_{\mathcal{F}_{3,k}}(S_{3,k}) = \langle \eta, \omega \rangle \cong C_2 \times C_2$
- $\mathrm{Out}_{\mathcal{F}_{3,k}}(T_{3,k}) = \mathrm{GL}_2(\mathbb{F}_3)$
- $\mathrm{Out}_{\mathcal{F}_{3,k}}(V) = \mathrm{GL}_2(\mathbb{F}_3)$

Moreover, the fusion systems $\mathcal{F}_{3,k}$ are simple and exotic.

Proof. The saturation and exoticness properties are proven in [17, Theorem 5.10]. For the simplicity property, note that $\mathcal{F}_{3,k}$ has no proper nontrivial strongly closed subgroups. Indeed, let $P \trianglelefteq S_{3,k}$ a nontrivial strongly closed subgroup. By [2, Theorem 8.1], P must intersect the center in a nontrivial subgroup. Since the center of $S_{3,k}$ has order 3, we must have $Z(S_{3,k}) \leq P$. Moreover, since ζ is $\mathcal{F}_{3,k}$-conjugated to s, we must have also $s \in P$, since P is strongly closed. Then, by [6, Lemma 2.2], P must be of index at most 3 in $S_{3,k}$.

In fact, by Lemma 3.1.1, P must contain the subgroup generated by s and all elements $s_1^{3l} s_2^{j} \in T_{3,k}$, which is an index 3 subgroup of $S_{3,k}$. Then, using that the automorphism group of $T_{3,k}$ is all $\mathrm{GL}_2(\mathbb{F}_3)$, we can conjugate, for example, the element $s_1^{-3} s_2^{-2}$ to $s_1 s_2$, obtaining that P contains also elements not conjugated to s. Since P had index at most 3, we obtain that P must be equal to $S_{3,k}$.

So, if there is a proper nontrivial normal subsystem of $\mathcal{F}_{3,k}$, it has to be over the same group $S_{3,k}$, by condition N1 of Definition 2.1.16. Then, by Lemma 2.1.20, we have that the normal subsystem has to be of index prime to p in $\mathcal{F}_{3,k}$, but by the classification in [17, Theorem 5.10], there is no subsystem of index prime to p in $\mathcal{F}_{3,k}$. □

3.2. Two families of examples for $p \geq 5$

Now we want to generalize the saturated fusion systems $\mathcal{F}_{3,k}$ to $p \geq 5$. First we have to define the p-groups $S_{p,k}$ to work with, and then we will deal with the fusion over $S_{p,k}$.

In [6], Blackburn studies the group $S_{3,k}$ from the point of view of being of maximal class, rather than of being of p-rank two, as in [17]. In this way, there is a natural generalization of $S_{3,k}$ to groups of maximal class $S_{p,k}$ which have rank $p-1$. The presentation is

$$\begin{aligned} S_{p,k} = \langle s, s_1, s_2, \ldots, s_{(p-1)k} \mid & [s, s_{i-1}] = s_i \text{ for } i = 2, \ldots, (p-1)k, && (3.1)\\ & [s_1, s_i] = 1 \text{ for } i = 2, \ldots, (p-1)k, && (3.2)\\ & s^p = 1, && (3.3)\\ & s_i^{\binom{p}{1}} s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1}^{\binom{p}{p}} = 1 \text{ for } i = 1, \ldots, (p-1)k \rangle && (3.4) \end{aligned}$$

where, in the last relation, we assume $s_j = 1$ if $j > (p-1)k$. We want to prove now that the groups $S_{p,k}$ are a semidirect product.

Proposition 3.2.1. *Consider the maximal torus subgroup $T_{p,k} = \langle s_1, \ldots, s_{(p-1)k} \rangle \leq S_{p,k}$. Then, the following holds:*

(*a*) *The subgroup $T_{p,k}$ is isomorphic to $(\mathbb{Z}/p^k)^{p-1}$ with generators $s_1, \ldots, s_{p-1}$.*

(*b*) *The group $S_{p,k}$ fits in a split extension*

$$1 \to (\mathbb{Z}/p^k)^{p-1} \to S_{p,k} \to \mathbb{Z}/p \to 1$$

where s is a generator of $\mathbb{Z}/p$ and the subgroup $(\mathbb{Z}/p^k)^{p-1}$ is generated by $s_1, \ldots, s_{p-1}$.

Proof. To prove (a) we will see first that $T_{p,k}$ is abelian by proving that $Z(T_{p,k}) = T_{p,k}$. From Equation (3.2) we obtain $s_1 \in Z(T_{p,k})$. Using Equation (3.4) with $i = (p-1)(k-1)$ we have

$$s_{(p-1)k} = s_{(p-1)k-1}^{-\binom{p}{p-1}} \cdots s_{(p-1)k-(p-1)}^{-\binom{p}{1}}$$

and joining it with Equation (3.1) we see that conjugation by s is an automorphism of $T_{p,k}$, hence $c_s(Z(T_{p,k})) = Z(T_{p,k})$. Using again Equation (3.1) we obtain $c_s(s_1) = s_1 s_2 \in Z(T_{p,k})$, therefore, $s_2 \in Z(T_{p,k})$. Since $c_s(s_i) = s_i s_{i+1}$ we can iterate this argument to obtain $Z(T_{p,k}) = T_{p,k}$.

Now, by Equation (3.4), we obtain $s_{(p-1)k}^p = 1$. Taking then $i = (p-1)k - j$ for $1 \leq j < p-1$, and using again Equation (3.4), we get $s_{(p-1)k-j}^p = 1$ for $1 \leq j < p-1$. Thus,

$$A_1 = \langle s_m \rangle_{(p-1)(k-1)+1 \leq m \leq (p-1)k} \cong (\mathbb{Z}/p)^{p-1}$$

By taking $i = (p-1)(k-1)$ we obtain $s_{(p-1)(k-1)}^p = s_{(p-1)k}^{-1}$, so $s_{(p-1)(k-1)}^{p^2} = 1$ and, repeating the argument, we get $s_{(p-1)k-j}^p = s_{(p-1)k-(j-(p-1))}^{-1}$ for $p-1 \leq j < 2(p-1)-1$.

By an inductive procedure we get

$$A_l = \langle s_m \rangle_{(p-1)(k-l)+1 \leq m \leq (p-1)(k-l+1)} \cong (\mathbb{Z}/p^l)^{p-1}$$

with $A_l \leq A_{l+1}$ for all $l = 1, \ldots, k-1$. Hence, $T_{p,k} = A_k \cong (\mathbb{Z}/p^k)^{p-1}$ with generators $s_1, \ldots, s_{p-1}$.

To prove (b) note that the fact that c_s is an automorphism of $T_{p,k}$ implies $T_{p,k} \trianglelefteq S_{p,k}$. Since $\langle s \rangle \cap T_{p,k} = \{e\}$, we have that $S_{p,k}$ is the semidirect product of $T_{p,k}$ by $\langle s \rangle$, by definition of semidirect product. □

From the presentation of the group $S_{p,k}$ we obtain that the action of s over $T_{p,k}$ with generators $\{s_1, \ldots, s_{p-1}\}$ it is given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -\binom{p}{1} \\ 1 & 1 & 0 & \cdots & 0 & -\binom{p}{2} \\ 0 & 1 & 1 & \cdots & 0 & -\binom{p}{3} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & -\binom{p}{p-2} \\ 0 & 0 & 0 & \cdots & 1 & 1-\binom{p}{p-1} \end{pmatrix}$$

Now we want to define the fusion over $S_{p,k}$. To do this, we first want to see $S_{p,k}$ as a Sylow subgroup of a bigger group, and then add morphism to some subgroups of $S_{p,k}$.

Consider the group $(\mathbb{Z}/p^k)^p$ generated by $e_1, \ldots, e_p$ with the action of Σ_p given by permutation of the generators. This action leaves invariant a subgroup T, isomorphic to $(\mathbb{Z}/p^k)^{p-1}$, generated by the elements $v_1, \ldots, v_{p-1}$, where $v_k = e_k - e_{k+1}$.

Fix now the permutation $\sigma = (1, 2, \ldots, p) \in \Sigma_p$ and consider the extension

$$1 \to (\mathbb{Z}/p^k)^{p-1} \to S_{p,k} \to \mathbb{Z}/p \to 1$$

where σ is a generator of $\mathbb{Z}/p$ and the subgroup $(\mathbb{Z}/p^k)^{p-1}$ is generated by $v_1, \ldots, v_{p-1}$. With this basis, the action is given by the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

Lemma 3.2.2. *The split extensions of the type*

$$1 \to (\mathbb{Z}/p^k)^{p-1} \to S_{p,k} \to \mathbb{Z}/p \to 1$$

where the action is given by the matrices A and B are equivalent.

Proof. Let M be the matrix of change of base given by the formula $v_j = \sum_{m=0}^{j-1} \binom{j-1}{m} s_{m+1}$, for $j = 1, \ldots, p-1$. Then, $A = MBM^{-1}$, so the induced semidirect products are isomorphic. □

Remark 3.2.3. Note that the change of basis it is also valid for $p=3$ and we also obtain an isomorphism which fixes s and leaves invariant the maximal torus.

Remark 3.2.4. The difference between the case $p=3$ and $p\geq 5$ comes from the fact that $\mathbb{Z}/3$ is a normal subgroup of Σ_3, hence the action of Σ_3 on $\mathbb{Z}/3^k\times\mathbb{Z}/3^k$ will not produce a saturated fusion system with $T_{3,k}$ an $\mathcal{F}$-essential subgroup.

The next lemma proves some properties about the group $S_{p,k}$ that will become useful later when dealing with the fusion systems.

Lemma 3.2.5. *Let $v_1^{i_1}\cdots v_{p-1}^{i_{p-1}}\in T_{p,k}\leq S_{p,k}$. Then, the following holds:*

(*a*) $s\cdot v_1^{i_1}\cdots v_{p-1}^{i_{p-1}}\cdot s^{-1}=v_1^{-i_{p-1}}v_2^{i_1-i_{p-1}}v_3^{i_2-i_{p-1}}\cdots v_{p-1}^{i_{p-2}-i_{p-1}}$.

(*b*) $v_1^{i_1}\cdots v_{p-1}^{i_{p-1}}\cdot s\cdot(v_1^{i_1}\cdots v_{p-1}^{i_{p-1}})^{-1}=v_1^{i_{p-1}+i_1}v_2^{i_{p-1}+i_2-i_1}\cdots v_{p-1}^{i_{p-1}+i_{p-1}-i_{p-2}}s$.

(*c*) *The elements $v_1^{i_1}\cdots v_{p-1}^{i_{p-1}}s$ and $v_1^{j_1}\cdots v_{p-1}^{j_{p-1}}s$ are $S_{p,k}$-conjugate if and only if the sums of exponents verify*

$$\sum_{l=1}^{p-1}i_l\equiv\sum_{l=1}^{p-1}j_l\pmod p$$

(*d*) *The center of $S_{p,k}$ is generated by $\zeta=(v_1^1v_2^2\cdots v_{p-1}^{p-1})^{p^{k-1}}$, so $Z(S_{p,k})\cong\mathbb{Z}/p$.*

Proof. Points (a) and (b) are direct computations using the action of s on $T_{p,k}$. To see (c), note first that

$$s\cdot v_1^{i_1}\cdots v_{p-1}^{i_{p-1}}s\cdot s^{-1}=v_1^{-i_{p-1}}v_2^{i_1-i_{p-1}}v_3^{i_2-i_{p-1}}\cdots v_{p-1}^{i_{p-2}-i_{p-1}}s$$

and

$$(v_1^{k_1}\cdots v_{p-1}^{k_{p-1}})\cdot v_1^{i_1}\cdots v_{p-1}^{i_{p-1}}s\cdot(v_1^{k_1}\cdots v_{p-1}^{k_{p-1}})^{-1}=\\ v_1^{i_1+k_{p-1}+k_1}v_2^{i_2+k_{p-1}+k_2-k_1}\cdots v_{p-1}^{i_{p-1}+k_{p-1}+k_{p-1}-k_{p-2}}s$$

So we see that in both cases the sum of the exponents are congruent modulo p. Assume now that the sums of the exponents are congruent modulo p. If we write $v_1^{k_1}\cdots v_{p-1}^{k_{p-1}}$ for an unknown element and we set the system of equations to conjugate the two elements, we get

$$\begin{pmatrix}1&0&\cdots&0&1\\-1&1&\ddots&\vdots&1\\0&-1&\ddots&0&\vdots\\\vdots&\ddots&\ddots&1&1\\0&\cdots&0&-1&2\end{pmatrix}\begin{pmatrix}k_1\\\vdots\\\vdots\\\vdots\\k_{p-1}\end{pmatrix}=\begin{pmatrix}i_1-j_1\\\vdots\\\vdots\\\vdots\\i_{p-1}-j_{p-1}\end{pmatrix}$$

Then, we can reduce the system by adding each row to the next, obtaining

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & \ddots & \ddots & \vdots & 2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 1 & p-2 \\ 0 & \cdots & \cdots & 0 & p \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ \vdots \\ \vdots \\ k_{p-1} \end{pmatrix} = \begin{pmatrix} i_1 - j_1 \\ \vdots \\ \vdots \\ \vdots \\ \sum_{l=1}^{p-1} i_l - \sum_{l=1}^{p-1} j_l \end{pmatrix}$$

And the system is compatible because we are assuming that $\sum_{l=1}^{p-1} i_l - \sum_{l=1}^{p-1} j_l$ is divisible by p. Note that we have proved that, if the two elements are $S_{p,k}$-conjugate, then they are conjugate by an element of the maximal torus.

Finally, to compute a generator of the center consider the action of Σ_p on the generators $\{e_1, \ldots, e_p\}$. Then, the elements of the type $\lambda(e_1 + \cdots + e_p)$ are invariant under the action of Σ_p (in particular, by the action of $\mathbb{Z}/p$). Now, if λ is a multiple of p^{k-1}, this element belongs to $T_{p,k}$, and in the basis $\{v_1, \ldots, v_{p-1}\}$ and with multiplicative notation can be written as in the statement. □

Now we want to define fusion systems over the p-groups $S_{p,k}$ for each $k \geq 2$. As in the case for $p = 3$, it is helpful to have some restriction for the outer automorphism group of $S_{p,k}$.

Proposition 3.2.6. *Let $\mathcal{F}$ be a saturated fusion system over $S_{p,k}$. Then, the outer automorphism group satisfies $\mathrm{Out}_{\mathcal{F}}(S_{p,k}) \leq C_{p-1} \times C_{p-1}$.*

Proof. Since $\mathcal{F}$ is saturated, the group $\mathrm{Out}_{\mathcal{F}}(S_{p,k})$ must be a subgroup of $\mathrm{Out}(S_{p,k})$ of order prime to p.

Consider now the Frattini subgroup $\Phi(S_{p,k})$. The kernel of the map

$$\rho\colon \mathrm{Out}(S_{p,k}) \to \mathrm{Out}(S_{p,k}/\Phi(S_{p,k}))$$

is a p-group by [24, Section 1.3]. Therefore, $\rho(\mathrm{Out}_{\mathcal{F}}(S_{p,k}))$ must be isomorphic to a subgroup of $\mathrm{Out}(S_{p,k}/\Phi(S_{p,k}))$. Then, since the minimum set for generators of $S_{p,k}$ is $\{s, v_1\}$, we have that $S_{p,k}/\Phi(S_{p,k})$ is a rank two elementary abelian group, and we can consider it as an $\mathbb{F}_p$ vector space with basis $\{\overline{s}, \overline{v_1}\}$.

In [6] it is shown that $T_{p,k}$ is a characteristic subgroup of $S_{p,k}$, so $\rho(\mathrm{Out}_{\mathcal{F}}(S_{p,k}))$ must be included in the lower triangular matrices of $\mathrm{GL}_2(\mathbb{F}_p)$. Now, since p cannot divide the order of $\mathrm{Out}_{\mathcal{F}}(S_{p,k})$, we obtain the result. □

As we want to work with concrete examples, we will consider $\phi, \psi \in \mathrm{Aut}(S_{p,k})$ of order $p-1$ defined as follows:

- The normalizer of $\langle s \rangle$ in Σ_p is isomorphic to $C_p \rtimes C_{p-1}$. Consider $\phi \in \Sigma_p$ an element of order $p-1$ normalizing $\langle s \rangle$ and consider the action over $S_{p,k} = T_{p,k} \rtimes \langle s \rangle$ as the induced by conjugation as a subgroup of $T_{p,k} \rtimes \Sigma_p$. This action sends $s \mapsto s^\lambda$, with λ a generator of $\mathbb{F}_p^\times$, while it fixes the center, that is $\phi(\zeta) = \zeta$.

- Consider $\mu \in \mathrm{Aut}(\mathbb{Z}/p^k) \cong (\mathbb{Z}/p^k)^\times$ an element of order $p-1$, and define ψ as the element in $\mathrm{Aut}(S_{p,k})$ which restricts to $\mu \times \cdots \times \mu$ in the maximal torus and to the identity on $\langle s \rangle$. To get easier computations, we can fix μ such that it reduces to λ modulo p. With this definition, $\psi(\zeta) = \zeta^\lambda$.

By definition, it is clear that $\langle \phi, \psi \rangle \cong C_{p-1} \times C_{p-1}$. Moreover, if we write also ϕ and ψ for their projections to the outer automorphisms group, we have that the subgroup $\langle \phi, \psi \rangle \leq \mathrm{Out}(S_{p,k})$ is isomorphic also to $C_{p-1} \times C_{p-1}$.

Now we can state the main result of this section, which provides two exotic saturated fusion systems over $S_{p,k}$ for each $k \geq 2$ and $p \geq 5$. Moreover, we show the relation of inclusion between the two fusion systems and prove that one of them is simple.

Theorem 3.2.7. *Consider the finite p-group $S_{p,k}$. Then,*

(a) For each $k \geq 2$ and $p \geq 5$ there is an exotic saturated fusion system $\widetilde{\mathcal{F}}_{p,k}$ over $S_{p,k}$ characterized by the morphisms

- $\mathrm{Out}_{\widetilde{\mathcal{F}}_{p,k}}(S_{p,k}) = \langle \phi, \psi \rangle \cong C_{p-1} \times C_{p-1}$
- $\mathrm{Out}_{\widetilde{\mathcal{F}}_{p,k}}(T_{p,k}) = \Sigma_p \times \langle \psi \rangle \cong \Sigma_p \times C_{p-1}$
- $\mathrm{Out}_{\widetilde{\mathcal{F}}_{p,k}}(V) = \mathrm{GL}_2(\mathbb{F}_p) \cong \mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{p-1}$

All other morphisms are given by composition and restriction of these morphisms and inner automorphisms of $S_{p,k}$.

(b) Each fusion system $\widetilde{\mathcal{F}}_{p,k}$ contains an exotic simple saturated fusion subsystem $\mathcal{F}_{p,k}$ of index 2, generated by the morphisms

- $\mathrm{Out}_{\mathcal{F}_{p,k}}(S_{p,k}) = \langle \phi^2, \psi\phi^{-1} \rangle \cong C_{\frac{p-1}{2}} \times C_{p-1}$
- $\mathrm{Out}_{\mathcal{F}_{p,k}}(T_{p,k}) = A_p \rtimes \langle \psi\phi^{-1} \rangle \cong A_p \rtimes C_{p-1}$
- $\mathrm{Out}_{\mathcal{F}_{p,k}}(V) = \mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}} < \mathrm{GL}_2(\mathbb{F}_p)$

and the conjugation by the elements in S.

Proof. To prove (a), the existence of the saturated fusion systems $\widetilde{\mathcal{F}}_{p,k}$ over $S_{p,k}$ can be found in [11, Example 9.3]. In the same result the authors also prove that these examples are exotic.

To get (b), we can proceed classifying all the saturated fusions subsystems of $\widetilde{\mathcal{F}}_{p,k}$ of index prime to p as in Theorem 2.1.23. To do this, we need to compute $\mathcal{E}_0$, the fusion system generated by $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k}}(P))$ for all $\widetilde{\mathcal{F}}_{p,k}$-centric subgroups P:

- For $P = S_{p,k}$, $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k}}(S_{p,k})) = \mathrm{Inn}(S_{p,k})$, since all inner automorphisms of $S_{p,k}$ have order a power of p.
- For $P = T_{p,k}$, $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k}}(T_{p,k})) \cong A_p$, since the elements of order p in Σ_p for odd prime p generate the alternating group A_p.
- For $P = V$, $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k}}(V)) \cong \mathrm{SL}_2(\mathbb{F}_p)$, since for odd prime p the elements of order p in $\mathrm{GL}_2(\mathbb{F}_p)$ generate $\mathrm{SL}_2(\mathbb{F}_p)$.

Now we have to detect the elements in $\mathrm{Out}_{\widetilde{\mathcal{F}}_{p,k}}(S_{p,k})$ which restrict to morphisms in $\mathcal{E}_0$ of some $\widetilde{\mathcal{F}}_{p,k}$-centric subgroups P:

- ϕ is an odd permutation, so the restriction to $T_{p,k}$ does not give an element of $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k}}(T_{p,k}))$, and it does not restrict to an automorphism of determinant one in V, so $\phi \notin \mathrm{Out}^0_{\widetilde{\mathcal{F}}_{p,k}}(S_{p,k})$.
- ϕ^2 is an even permutation, so it restricts to an element in $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k}}(T_{p,k}))$.
- ψ does not restrict to a permutation over $T_{p,k}$ since it does not fix the center, and neither it restricts to an automorphism of determinant one in V, so $\psi \notin \mathrm{Out}^0_{\widetilde{\mathcal{F}}_{p,k}}(S_{p,k})$.
- $\psi^i\phi^{-i}$ restricts to an automorphism of determinant one in V.
- Any other morphism $\psi^i\phi^j$ with $j \neq i^{-1}$ does not restrict to a permutation over $T_{p,k}$ neither to an automorphism of determinant one in V, so $\psi^i\phi^j \notin \mathrm{Out}^0_{\widetilde{\mathcal{F}}_{p,k}}(S_{p,k})$ for $j \neq i^{-1}$.

These computations show that

$$\mathrm{Out}_{\widetilde{\mathcal{F}}_{p,k}}(S_{p,k}) / \mathrm{Out}^0_{\widetilde{\mathcal{F}}_{p,k}}(S_{p,k}) \cong \mathbb{Z}/2$$

so there is just one proper nontrivial saturated fusion subsystem of index prime to p, and it is of index 2. Now, to get this saturated fusion system we need to add the necessary morphisms to $\mathcal{E}_0$ in order to verify saturation axioms. This consist in restricting the morphism $\psi^i\phi^{-i}$ to $T_{p,k}$ and restricting the morphism ϕ^2 to V, obtaining the stated description of the fusion system $\mathcal{F}_{p,k}$.

Let us see now that $\mathcal{F}_{p,k}$ is simple. By Remark 2.1.25, it is enough to check that $O_p(\mathcal{F}_{p,k}) = 1$ and $O^p(\mathcal{F}_{p,k}) = \mathcal{F}_{p,k} = O^{p'}(\mathcal{F}_{p,k})$:

- $O_p(\mathcal{F}_{p,k}) = 1$: there is not any proper nontrivial strongly closed subgroup in $\mathcal{F}_{p,k}$. Indeed, let $P \trianglelefteq S_{p,k}$ a nontrivial strongly closed subgroup. By [2, Theorem 8.1], P must intersect the center in a nontrivial subgroup. Since the center of $S_{p,k}$ has order p, we must have $Z(S_{p,k}) \leq P$. Moreover, since ζ is $\mathcal{F}_{p,k}$-conjugated to s by a morphism in $\mathrm{Aut}_{\mathcal{F}_{p,k}}(V)$, we must have also $s \in P$, since P is strongly closed. Then, by [6, Lemma 2.2], P must be of index at most p in $S_{p,k}$.

  In fact, by Lemma 3.2.5 (c), the subgroup P must contain the subgroup generated by s and all elements $v_1^{i_i} \cdots v_{p-1}^{i_{p-1}} \in T_{p,k}$ whose sum of exponents is congruent to 0 modulo p, which is an index p subgroup of $S_{p,k}$.

  Then, let φ be the automorphism of $T_{p,k}$ induced by the cycle $(123) \in A_p \cong \mathrm{Aut}_{\mathcal{F}_{p,k}}(T_{p,k})$. We have that

$$
\begin{aligned}
\varphi(v_1) &= v_2 \\
\varphi(v_2) &= v_1^{-1}v_2^{-1} \\
\varphi(v_3) &= v_1v_2v_3 \\
\varphi(v_i) &= v_i, \text{ for } 4 \leq i \leq p-1
\end{aligned}
$$

  So, taking for example the element $v_2^{-1}v_4$, which is in P since the sum of exponents is 0, we have that $\varphi(v_2^{-1}v_4) = v_1v_2v_4$. Thus, $v_1v_2v_4$ must lie also in P, but since the sum of the exponents is not 0 modulo p for $p \geq 5$, and using that the index of P is at most p, we get that $P = S_{p,k}$.

  So, by Remark 2.1.15, there is not any proper nontrivial normal subgroup in $\mathcal{F}_{p,k}$.

- $O^p(\mathcal{F}_{p,k}) = \mathcal{F}_{p,k}$: by Proposition 2.1.22, we need to show that $\mathfrak{foc}(\mathcal{F}_{p,k}) = S$. First, note that there are elements $\varphi, \varphi' \in \mathrm{Aut}_{\mathcal{F}_{p,k}}(V)$ such that $\varphi(s) = s\zeta$ and also $\varphi'(\zeta) = s\zeta$, hence $V \subseteq \mathfrak{foc}(\mathcal{F}_{p,k})$.

  The action of C_{p-1} on the maximal torus $T_{p,k}$ includes a morphism φ such that $\varphi(v) = v^{-1}$ for all $v \in T_{p,k}$, so we have also all elements $\langle v_1^2, \ldots, v_{p-1}^2 \rangle \subseteq \mathfrak{foc}(\mathcal{F}_{p,k})$.

  Taking now the expression $\zeta = (v_1^1v_2^2 \cdots v_{p-1}^{p-1})^{p^{k-1}}$, we get $v_1v_3 \cdots v_{p-2} \in \mathfrak{foc}(\mathcal{F}_{p,k})$. Then, conjugating this element by s we get $v_2v_4 \cdots v_{p-1} \in \mathfrak{foc}(\mathcal{F}_{p,k})$.

Now, let φ be the conjugation by s and apply it to the element v_{p-1}^{-1}. This tells us that $v_{p-1}\varphi(v_{p-1}^{-1}) = v_1 v_2 \dots v_{p-2} v_{p-1}^2$ is also in $\mathfrak{foc}(\mathcal{F}_{p,k})$. Hence, $v_1 v_2 \dots v_{p-2} \in \mathfrak{foc}(\mathcal{F}_{p,k})$ and we get that $v_{p-1} \in \mathfrak{foc}(\mathcal{F}_{p,k})$, since $v_1 v_2 \dots v_{p-1}$ is also in $\mathfrak{foc}(\mathcal{F}_{p,k})$.

Finally, conjugating $v_1 v_2 \dots v_{p-2}$ by s and using that $v_{p-1} \in \mathfrak{foc}(\mathcal{F}_{p,k})$ we get $v_1 \in \mathfrak{foc}(\mathcal{F}_{p,k})$, concluding that $S = \langle s, v_1 \rangle \subseteq \mathfrak{foc}(\mathcal{F}_{p,k})$.

- $O^{p'}(\mathcal{F}_{p,k}) = \mathcal{F}_{p,k}$: since the centric subgroups of $\widetilde{\mathcal{F}}_{p,k}$ coincide with the centric subgroups of $\mathcal{F}_{p,k}$, the computations made to classify the subsystems of index prime to p of $\widetilde{\mathcal{F}}_{p,k}$ prove that $O^{p'}(\mathcal{F}_{p,k}) = O^{p'}(\widetilde{\mathcal{F}}_{p,k}) = \mathcal{F}_{p,k}$.

Finally, by [33, Lemma 1.5], since the saturated fusion systems $\mathcal{F}_{p,k}$ are reduced, if they were realizable, they would be realized by a finite simple group, but these examples do not appear in the tables of [11, Proposition 9.5], where there are all finite simple groups which have $S_{p,k}$ as a Sylow p-subgroup. □

Part II

The compact case

CHAPTER 4

Fusion on infinite groups

In this chapter we will make a quick review of fusion on a certain class of infinite groups, the class of discrete p-toral groups. All finite p-groups are discrete p-toral groups, so this chapter is in fact a generalization of Chapter 2. This means that all definitions and results of this chapter also apply to fusion over finite groups, although some of these statements are presented only in this chapter since we use them only in the compact case. Also, those concepts and theorems needed in this part that were stated in Chapter 2 are restated in this chapter for the sake of completion.

In the first section we will define the concept of discrete p-toral group and generalize the notion of saturated fusion systems to this class of p-groups. In the second section we will recall the definition of centric linking system and generalize the definition of p-local finite groups to p-local compact groups. The third and fourth sections are devoted to present the two most important class of groups which give rise to p-local compact groups, the compact Lie groups and the p-compact groups.

As for Chapter 2, all results in this chapter are well known, but we provide a proof to those which are difficult to find in the literature.

4.1. Saturated fusion systems

To work with fusion on infinite groups we need to restrict to some class of p-groups to be able to manage all information in the fusion systems. One motivation to the generalization of saturated fusion systems to infinite groups is the study of compact Lie groups and p-compact groups, so it seems natural to consider the p-groups which act as Sylow subgroups of this kind of groups.

Let $\mathbb{Z}/p^\infty$ denote the union of the cyclic p-groups $\mathbb{Z}/p^n$ under the obvious inclusions, known as the Prüfer p-group.

Definition 4.1.1 ([12, Definition 1.1])**.** A *discrete p-toral group* is a group S, with normal subgroup $S_0 \trianglelefteq S$, such that S_0 is isomorphic to a finite product of copies of $\mathbb{Z}/p^\infty$, and S/S_0 is a finite p-group. The subgroup S_0 will be called the *identity component* of S, and S will be called *connected* if $S = S_0$. Set $\pi_0(S) = S/S_0$, the *group of components* of S.

Define $\mathrm{rk}(S) = k$ if $S_0 \cong (\mathbb{Z}/p^\infty)^k$, and set

$$|S| = (\mathrm{rk}(S), |\pi_0(S)|) = (\mathrm{rk}(S), |S/S_0|)$$

with the lexicographical order. Thus, $|S| \leq |S'|$ if and only if $\mathrm{rk}(S) < \mathrm{rk}(S')$, or $\mathrm{rk}(S) = \mathrm{rk}(S')$ and $|\pi_0(S)| \leq |\pi_0(S')|$.

Remark 4.1.2. Note that with this definition it is clear that a finite p-group is a discrete p-toral group of rank 0.

The main two properties of discrete p-toral groups, which in fact characterize it, are that they are artinian and locally finite p-groups.

Definition 4.1.3 ([26, Chapter 1, Section E])**.** A group G is *artinian* if every nonempty set of subgroups of G, partially ordered by inclusion, has a minimal element.

Definition 4.1.4 ([26, Chapter 1, Section A])**.** A group G is *locally finite* if every finitely generated subgroup of G is finite, and is a *locally finite p-group* if every finitely generated subgroup of G is a finite p-group.

Proposition 4.1.5 ([12, Proposition 1.2])**.** *A group is a discrete p-toral group if and only if it is artinian and a locally finite p-group.*

The class of discrete p-toral groups inherits the property of being closed under taking subgroups or quotients from the locally finite and artinian properties.

Lemma 4.1.6 ([12, Lemma 1.3])**.** *Any subgroup or quotient group of a discrete p-toral group is a discrete p-toral group.*

Now we can define the notion of fusion system over a discrete p-toral group, which is exactly the same as the Definition 2.1.1 in the finite case.

Definition 4.1.7 ([12, Definition 2.1])**.** A *fusion system* $\mathcal{F}$ over a discrete p-toral group S is a category whose objects are the subgroups of S, and whose morphisms sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ satisfy the following conditions:

(a) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ for all $P, Q \leq S$.

(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

As in the finite case, two subgroups $P, P' \leq S$ are called $\mathcal{F}$-conjugated if $\mathrm{Iso}_{\mathcal{F}}(P, P') \neq \emptyset$, and we denote by $P^{\mathcal{F}}$ the set of subgroups of S which are $\mathcal{F}$-conjugated to P.

The definition of saturated fusion system over discrete p-toral groups will be very similar to the one over finite groups, so we need first to define some similar properties about normalizers and centralizers.

Definition 4.1.8 ([12, Definition 2.2], [13, Definition 1.6])**.** Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S.

- A subgroup $P \leq S$ is *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.
- A subgroup $P \leq S$ is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.
- A subgroup $P \leq S$ is *fully automized* in $\mathcal{F}$ if the index of $\mathrm{Aut}_S(P)$ in $\mathrm{Aut}_{\mathcal{F}}(P)$ is finite and prime to p.
- A subgroup $P \leq S$ is *receptive* in $\mathcal{F}$ if for each $Q \leq S$ and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$, if we set
$$N_\varphi = \{g \in N_S(Q) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(P)\}$$
then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_Q = \varphi$.

Since we are dealing with infinite groups, it is natural to ask some coherence condition to morphisms which have an increasing chain of subgroups as a source. As we can see, this is the only extra condition added to the definition of saturation.

Definition 4.1.9 ([12, Definition 2.2])**.** Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S. Then $\mathcal{F}$ is a *saturated fusion system* if the following three conditions hold:

(I) Each subgroup $P \leq S$ which is fully normalized in $\mathcal{F}$ is also fully centralized and fully automized in $\mathcal{F}$.

(II) Each subgroup $P \leq S$ which is fully centralized in $\mathcal{F}$ is also receptive in $\mathcal{F}$.

(III) If $P_1 \leq P_2 \leq P_3 \leq \cdots$ is an increasing sequence of subgroups of S, with $P_\infty = \bigcup_{n=1}^{\infty} P_n$, and if $\varphi \in \mathrm{Hom}(P_\infty, S)$ is any homomorphism such that $\varphi|_{P_n} \in \mathrm{Hom}_{\mathcal{F}}(P_n, S)$ for all n, then $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P_\infty, S)$.

One way to see the need for the saturation axioms also in the infinite case, apart from being a pure generalization of the finite case, is the fact that, even with the slight modifications in the definitions, Proposition 2.1.6 is still valid also for fusion systems over discrete p-toral groups.

Lemma 4.1.10 ([13, Lemma 1.7]). *The following hold for any fusion system $\mathcal{F}$ over a discrete p-toral group S.*

(*a*) *If $P \leq S$ is fully automized and receptive in $\mathcal{F}$, then it is fully normalized.*

(*b*) *Every receptive subgroup of S is fully centralized.*

As in the finite case, we use the notation $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ and $\mathrm{Out}_S(P) = \mathrm{Aut}_S(P)/\mathrm{Inn}(P)$ for each $P \leq S$, thus $\mathrm{Out}_S(P) \leq \mathrm{Out}_{\mathcal{F}}(P) \leq \mathrm{Out}(P)$.

Remark 4.1.11. If $\mathcal{F}$ is a saturated fusion system over a discrete p-toral group S, then $\mathrm{Out}_{\mathcal{F}}(P)$ is finite for all $P \leq S$.

The definitions of $\mathcal{F}$-centric and $\mathcal{F}$-radical apply to subgroups of discrete p-toral groups without changes if $\mathcal{F}$ is a saturated fusion system.

Definition 4.1.12 ([12, Definition 2.6]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S.

- A subgroup P of S is $\mathcal{F}$-*centric* if $C_S(Q) = Z(Q)$ for all $Q \in P^{\mathcal{F}}$. Equivalently, P is $\mathcal{F}$-centric if P is fully centralized in $\mathcal{F}$ and $C_S(P) = Z(P)$.
- A subgroup P of S is $\mathcal{F}$-*radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ is p-reduced, i.e., if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$.

The next theorem, similar to Theorem 2.1.13 and also known as Alperin fusion theorem, proves that the fully normalized, centric, radical subgroups are enough to generate a saturated fusion system.

Theorem 4.1.13 ([12, Theorem 3.6]). *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then for each $\phi \in \mathrm{Iso}_{\mathcal{F}}(P, P')$, there exist sequences of subgroups of S*

$$P = P_0, P_1, \ldots, P_k = P' \qquad Q_1, Q_2, \ldots, Q_k,$$

and elements $\varphi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$, such that

(*a*) *Q_i is fully normalized in $\mathcal{F}$, $\mathcal{F}$-radical, and $\mathcal{F}$-centric for each i.*

(*b*) *$P_{i-1}, P_i \leq Q_i$ and $\varphi_i(P_{i-1}) = P_i$ for each i.*

(*c*) *$\varphi = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1$.*

A key point in Chapter 7 will be to study normal subsystems of saturated fusion systems over discrete p-toral groups, so we recall here the definitions about normality.

The concept of strongly closed subgroup can be applied to infinite fusion systems without changes.

Definition 4.1.14 ([5, Definition I.4.1])**.** Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. A subgroup $P \leq S$ is *strongly closed* in $\mathcal{F}$ if no element of P is $\mathcal{F}$-conjugate to an element of $S \setminus P$.

Remark 4.1.15. Note that, since $\mathcal{F}$ contains all conjugation morphisms, if P is strongly closed in $\mathcal{F}$ then P must be a normal subgroup of S.

Definition 2.1.16, dealing with normal fusion subsystems over finite p-groups, is due to Aschbacher, but it can be generalized without change to fusion systems over discrete p-toral groups.

Definition 4.1.16. Let S be a discrete p-toral group, $\mathcal{F}$ a saturated fusion system over S, and $\mathcal{F}' \leq \mathcal{F}$ a subsystem over a subgroup $S' \leq S$. Then, $\mathcal{F}'$ is *normal* in $\mathcal{F}$ if the following conditions are satisfied.

(N1) S' is strongly closed in $\mathcal{F}$.

(N2) For each $P \leq Q \leq S$ and each $\gamma \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, the map that sends each morphism $f \in \mathrm{Hom}_{\mathcal{F}'}(P, Q)$ to $\gamma \circ f \circ \gamma^{-1}$ defines a bijection between the sets $\mathrm{Hom}_{\mathcal{F}'}(P, Q)$ and $\mathrm{Hom}_{\mathcal{F}'}(\gamma(P), \gamma(Q))$.

(N3) $\mathcal{F}'$ is a saturated fusion system over S'.

(N4) Each $f \in \mathrm{Aut}_{\mathcal{F}'}(S')$ extends to some $\widetilde{f} \in \mathrm{Aut}_{\mathcal{F}}(S' C_S(S'))$ such that

$$[\widetilde{f}, C_S(S')] = \{\widetilde{f}(g) \cdot g^{-1} \,|\, g \in C_S(S')\} \leq Z(S')$$

The notion of simple saturated fusion system over discrete p-toral groups is slightly different to the definition of simple saturated fusion systems over finite groups. The situation is analogous to the different definitions for simple finite group and simple compact Lie group, where a similar phenomenon occurs.

Definition 4.1.17. Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then, $\mathcal{F}$ is *simple* if it satisfies one of the following conditions:

(a) $\mathrm{rk}(\mathcal{F}) = 0$ and $\mathcal{F}$ contains no proper nontrivial normal fusion subsystems.

(b) $\mathrm{rk}(\mathcal{F}) \geq 1$ and every proper normal fusion subsystem of $\mathcal{F}$ is finite.

Now we will show how to construct a fusion subsystem for a given saturated fusion system, playing the role of centralizers in group theory. This will be a very important construction to prove our results in Chapter 7.

Definition 4.1.18 ([13, Definition 2.1])**.** Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S and fix a subgroup $P \leq S$. Then, the *centralizer fusion system* $C_{\mathcal{F}}(P)$ is defined as the fusion system over $C_S(P)$ where, for $Q, R \leq C_S(P)$

$$\operatorname{Hom}_{C_{\mathcal{F}}(P)}(Q,R) = \{\varphi \in \operatorname{Hom}_{\mathcal{F}}(Q,R) \mid \text{ there exists } \overline{\varphi} \in \operatorname{Hom}_{\mathcal{F}}(QP,RP) \text{ with } \overline{\varphi}|_Q = \varphi \text{ and } \overline{\varphi}|_P = \operatorname{Id}\}$$

Finally, the next theorem suggests that the definition of centralizer fusion system is the right one. We will see another evidence of the coherence of the definition of centralizer fusion system in Theorem 7.1.1.

Theorem 4.1.19 ([13, Theorem 2.3])**.** *Fix a saturated fusion system $\mathcal{F}$ over a discrete p-toral group S. Assume $P \leq S$ is fully centralized in $\mathcal{F}$. Then, $C_{\mathcal{F}}(P)$ is a saturated fusion system over $C_S(P)$.*

4.2. p-local compact groups

Now we want to generalize the notion of p-local finite group to the case of saturated fusion systems over discrete p-toral groups. Since the choice of discrete p-toral groups was made because of their role as Sylow subgroups of compact Lie groups and p-compact groups, we will refer to the new construction as p-local compact groups.

In order to associate a topological space to a saturated fusion system over a discrete p-toral group, we can use exactly the same definition for centric linking system as in the finite case.

Definition 4.2.1 ([12, Definition 4.1])**.** Let $\mathcal{F}$ be a fusion system over a discrete p-toral group S. A *centric linking system* associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor

$$\pi \colon \mathcal{L} \to \mathcal{F}^c$$

and distinguished monomorphisms $P \xrightarrow{\delta_p} \operatorname{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions.

(A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\operatorname{Mor}_{\mathcal{L}}(P,Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \operatorname{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\operatorname{Mor}_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{F}}(P,Q)$$

(B) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle \delta_P(g)}\downarrow & & \downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{f} & Q \end{array}$$

Now the definition of p-local finite group generalizes to the one of p-local compact group in a natural way.

Definition 4.2.2 ([12, Definition 4.2]). A *p-local compact group* is a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a discrete p-toral group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The *classifying space* of the p-local compact group is the space $|\mathcal{L}|^\wedge_p$.

The next theorem shows that the homotopy type of the classifying space of a p-local compact group also completely determines it.

Theorem 4.2.3 ([12, Theorem 7.4]). *If $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are two p-local compact groups such that $|\mathcal{L}|^\wedge_p \simeq |\mathcal{L}'|^\wedge_p$, then $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are isomorphic as p-local compact groups.*

Similarly to the case of fusion systems over finite p-groups, we don't have to concern about the existence and uniqueness of the centric linking systems associated to the saturated fusion systems. This allows us to work mostly with saturated fusion systems over discrete p-toral groups when we want to study p-local compact groups. In this part, though, we will make explicit when we have the centric linking system for two reasons: because in Chapter 5 and Chapter 6 we don't assume the saturation axioms, so in theses cases the existence of the centric linking system, and so the p-local compact group, is not guaranteed; and because we will explicitly use some properties of the centric linking system in Chapter 7.

Theorem 4.2.4 ([27]). *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group. Then there exists a centric linking system associated to $\mathcal{F}$ which is unique up to isomorphism.*

Remark 4.2.5. Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S and $P \leq S$ a fully centralized subgroup in $\mathcal{F}$. Then, by Theorem 4.1.19, the fusion system $C_{\mathcal{F}}(P)$ is a saturated fusion system over $C_S(P)$, so we have that there exists a unique centric linking system associated to it. We will denote this centric linking system by $C_{\mathcal{L}}(P)$ and, consequently, we have a centralizer p-local compact group denoted by $(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P))$.

While the concept of exotic p-local finite group is clearly defined, we will see in the following sections that in the compact case there are several families of groups giving rise to p-local compact groups, blurring this way the condition of being exotic. Moreover, there cannot be a general notion of exoticness since, as the next proposition shows, the p-local compact groups are always realizable by some group.

Proposition 4.2.6 ([23, Proposition 2.6]). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Then, there exists a group G such that $S \in \mathrm{Syl}_p(G)$, $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}_S(G)$.*

However, this group G is constructed as an amalgam of groups and presented as a colimit of a tree of groups, resulting in a huge group which has nothing to do with the original motivation to construct p-local compact groups as a generalization for compact Lie groups and p-compact groups. This is why we will focus our attention on proving that some p-local compact groups cannot be realized by compact Lie groups nor by p-compact groups, but we will not deal with other classes of groups.

4.3. Compact Lie groups

In this section we will consider a compact Lie group G and study its discrete p-toral groups that play the role of Sylow subgroups. Then, we will study how normal subgroups of G inherit the Sylow subgroups, a property which will be very useful in Chapter 7. Finally, we state the theorem which ensures that we can construct a p-local compact group from every compact Lie group.

For any compact Lie group G, we will denote by G_0 the connected component of G which contains the neutral element. Recall that a compact Lie group $\mathbf{P}$ is called p-toral if $\mathbf{P}_0$ is a torus and if its group of components is a p-group.

Remark 4.3.1. Note that if G is a compact Lie group and $P \leq G$ is a discrete p-toral subgroup, then the topological closure $\overline{P}$ is a p-toral group.

To better differentiate between discrete and continuous subgroups, we will denote p-toral groups as $\mathbf{P}$, in contrast with discrete p-toral subgroups, denoted as P.

We will consider discrete Sylow subgroups of G, but we will define them by using p-toral subgroups whose identity component is a maximal torus.

Definition 4.3.2 ([12, Definition 9.1])**.** Let G be a compact Lie group.

(a) For any p-toral group $\mathbf{P}$, $\mathrm{Syl}_p(\mathbf{P})$ denotes the set of discrete p-toral subgroups $P \leq \mathbf{P}$ such that $P \cdot \mathbf{P}_0 = \mathbf{P}$ and P contains all p-power torsion in $\mathbf{P}_0$.

(b) A discrete p-toral subgroup $P \leq G$ is *snugly embedded* if $P \in \mathrm{Syl}_p(\overline{P})$.

(c) $\overline{\mathrm{Syl}}_p(G)$ denotes the set of all p-toral subgroups $\mathbf{S} \leq G$ such that the identity component $\mathbf{S}_0$ is a maximal torus of G and $\mathbf{S}/\mathbf{S}_0 \in \mathrm{Syl}_p(N(\mathbf{S}_0)/\mathbf{S}_0)$.

(d) $\mathrm{Syl}_p(G)$ denotes the set of all discrete p-toral subgroups $P \leq G$ such that $\overline{P} \in \overline{\mathrm{Syl}}_p(G)$ and $P \in \mathrm{Syl}_p(\overline{P})$.

The previous definition refers to conditions about maximality subgroups. In finite group theory, being a maximal p-subgroup is equivalent to contain all p-subgroups up to conjugacy, but with infinite groups this is not true in general. The next proposition shows that using the notions of Definition 4.3.2, Sylow p-subgroups of compact Lie groups contain all discrete p-toral subgroups up to conjugacy. In particular, it states that there is only one conjugacy class of Sylow subgroups for any compact Lie group G.

Proposition 4.3.3 ([12, Proposition 9.3])**.** *Let G be a compact Lie group, then any two subgroups in $\mathrm{Syl}_p(G)$ are G-conjugate, and each discrete p-toral subgroup $P \leq G$ is contained in some subgroup $S \in \mathrm{Syl}_p(G)$.*

Now we want to study how are the Sylows subgroups of normal subgroups of compact Lie groups. For this, we need to prove the converse of Proposition 4.3.3, that is, that besides all Sylow subgroups being conjugated, any conjugate of a Sylow subgroup is again a Sylow subgroup. This result will be a direct check of Definition 4.3.2 (d), but we need first to prove two topological lemmas relating conjugation morphisms with closures and components in compact Lie groups.

Lemma 4.3.4. *Let G be a compact Lie group and $P \leq G$ a subgroup. Then, for any $g \in G$, $\overline{gPg^{-1}} = g\overline{P}g^{-1}$.*

Proof. Since conjugation by g is an homeomorphism and $\overline{P}$ is a closed subgroup of G, we have that $g\overline{P}g^{-1}$ is a closed subgroup. Since $gPg^{-1} \leq g\overline{P}g^{-1}$, we conclude that $\overline{gPg^{-1}} \leq g\overline{P}g^{-1}$.

Now let $gxg^{-1} \in g\overline{P}g^{-1}$ and V an open neighborhood. Again, since conjugation by g is homeomorphism, we have that $g^{-1}Vg$ is an open neighborhood of x. Then, there exists $y \in (g^{-1}Vg) \cap P$, because $x \in \overline{P}$. That is, $gyg^{-1} \in V \cap (gPg^{-1})$, so $gxg^{-1} \in \overline{gPg^{-1}}$. □

Lemma 4.3.5. *Let G be a compact Lie group and $P \leq G$ a closed subgroup. Then, for any $g \in G$, $(gPg^{-1})_0 = gP_0g^{-1}$.*

Proof. Since P_0 is a connected subspace of G and conjugation by g is an homeomorphism, we have that gP_0g^{-1} is connected. Since gP_0g^{-1} contains the neutral element, and $gP_0g^{-1} \subseteq gPg^{-1}$, we have that $gP_0g^{-1} \subseteq (gPg^{-1})_0$.

Now let $gxg^{-1} \in (gPg^{-1})_0$, then, there exists a continuous map $\mu\colon I \to gPg^{-1}$ such that $\mu(0) = e$ and $\mu(1) = gxg^{-1}$. Composing with conjugation by g^{-1} we get a continuous map

$$\begin{array}{ccccc} I & \to & gPg^{-1} & \to & P \\ 0 & \mapsto & e & \mapsto & e \\ 1 & \mapsto & gxg^{-1} & \mapsto & x \end{array}$$

So we have that $x \in P_0$, hence $gxg^{-1} \in gP_0g^{-1}$. □

Now conditions of Definition 4.3.2 (d) follow easily for any conjugate of a Sylow subgroup.

Lemma 4.3.6. *Let G be a compact Lie group and $P \in \mathrm{Syl}_p(G)$ a p-Sylow subgroup of G. Then, $gPg^{-1} \in \mathrm{Syl}_p(G)$ for any $g \in G$.*

Proof. We must show that $\overline{gPg^{-1}} \in \overline{\mathrm{Syl}}_p(G)$ and $gPg^{-1} \in \mathrm{Syl}_p(\overline{gPg^{-1}})$.

First, by Lemmas 4.3.4 and 4.3.5, we have that $(\overline{gPg^{-1}})_0 = (g\overline{P}g^{-1})_0 = g\overline{P}_0g^{-1}$. Since $\overline{P}_0$ its a maximal torus of G by hypothesis, so it's $g\overline{P}_0g^{-1}$.

Now, we have the isomorphisms

$$\overline{P}/\overline{P}_0 \cong g\overline{P}g^{-1}/g\overline{P}_0g^{-1} \qquad \text{and} \qquad N_G(\overline{P})/\overline{P}_0 \cong N_G(g\overline{P}g^{-1})/g\overline{P}_0g^{-1}$$

So, if $\overline{P}/\overline{P}_0$ is a Sylow subgroup of $N_G(\overline{P})/\overline{P}_0$, we have that $g\overline{P}g^{-1}/g\overline{P}_0g^{-1}$ is a Sylow subgroup of $N_G(g\overline{P}g^{-1})/g\overline{P}_0g^{-1}$.

For the second condition, we have the chain of equalities

$$gPg^{-1}(\overline{gPg^{-1}})_0 = gPg^{-1}g\overline{P}_0g^{-1} = gP\overline{P}_0g^{-1} = g\overline{P}g^{-1}$$

where in the last one we use that $P \in \mathrm{Syl}_p(\overline{P})$. Finally, let $gxg^{-1} \in g\overline{P}_0g^{-1}$ an element of p-power order. Then, $x \in \overline{P}_0$ has p-power order, so we have, by hypothesis, that $x \in P$. Therefore, $gxg^{-1} \in gPg^{-1}$. □

By using that any conjugate of a Sylow subgroup is still a Sylow subgroup we can extend a result about Sylow subgroups of normal subgroups in finite groups to compact Lie groups.

We will use the next proposition in the second section of Chapter 7, when we prove that the identity component of a compact Lie group produces normal fusion subsystems according to Definition 4.1.16.

Proposition 4.3.7. *Let G be a compact Lie group and $S \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup. Let $H \trianglelefteq G$ a closed normal subgroup, then $R = S \cap H \in \mathrm{Syl}_p(H)$.*

Proof. Since H is closed it is also a Lie group, so it has Sylow p-subgroups and we can take $P \in \mathrm{Syl}_p(H)$. The group P is a discrete p-toral subgroup of G, so we know, by Proposition 4.3.3, that there exists $g \in G$ such that $gPg^{-1} \leq S$, since $S \in \mathrm{Syl}_p(G)$. Then, we have $gPg^{-1} \leq R$, because H is a normal subgroup of G.

Now, since R is a discrete p-toral subgroup and $P \in \mathrm{Syl}_p(H)$, there exists $h \in H$ such that $hRh^{-1} \leq P$. Hence, we have the chain of inclusions

$$hRh^{-1} \leq P \leq g^{-1}Rg$$

This implies $gh \in N_G(R)$, and then,

$$h^{-1}Ph \leq h^{-1}g^{-1}Rgh \leq R$$

We conclude that $P \leq hRh^{-1}$, so in fact $R = h^{-1}Ph$. Then, by Lemma 4.3.6, $R \in \mathrm{Syl}_p(H)$. □

By Proposition 2.2.15 and Remark 2.2.19, we know that any finite group G with Sylow subgroup $S \in \mathrm{Syl}_p(G)$ gives rise to a p-local finite group $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ such that $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$. The next theorem shows that we can extend this result to compact Lie groups and p-local compact groups.

Theorem 4.3.8 ([12, Lemma 9.5 and Theorem 9.10]). *Fix a compact Lie group G and a maximal discrete p-toral subgroup $S \in \mathrm{Syl}_p(G)$. Then, $\mathcal{F}_S(G)$ is a saturated fusion system over S and there exists a centric linking system $\mathcal{L}_S^c(G)$ associated to $\mathcal{F}_S(G)$ such that $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ is a p-local compact group with classifying space $|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$.*

As we have seen in the previous section, there is no notion of exoticness for p-local compact groups as clear as it is for p-local finite groups. Then, since we cannot talk about exotic p-local compact groups in general, we have to keep trace of which class of groups we are considering when studying the exoticness of the p-local compact groups.

Definition 4.3.9. We will say that a p-local compact group $(S, \mathcal{F}, \mathcal{L})$ is *realizable by a compact Lie group* if $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}_S^c(G)$ for some compact Lie group G with $S \in \mathrm{Syl}_p(G)$.

4.4. p-compact groups

A p-compact group is a homotopy theoretic version of a compact Lie group, but with all the information at the prime p, in the sense of p-completion defined in Chapter 2. They were defined by Dwyer-Wilkerson in [20], and later they were classified in [3] for p odd and independently in [4] and [31, 32] for $p = 2$.

The aim of this section is to present some properties about p-compact groups and the fact that any p-compact group give rise to a p-local compact group with equivalent classifying space.

Definition 4.4.1 ([20, Definition 2.3])**.** A *p-compact group* is a triple (X, BX, e) where X is a space such that $H^*(X; \mathbb{F}_p)$ is finite, BX is a pointed p-complete space, and $e\colon X \to \Omega(BX)$ is a homotopy equivalence.

Remark 4.4.2. If G is a compact Lie group such that the group of components $\pi_0(G)$ is a finite p-group, then the space BG is p-good by Proposition 2.2.11, therefore the space $BG^\wedge_p$ is p-complete and, upon setting $B\widehat{G} = BG^\wedge_p$ and $\widehat{G} = \Omega(B\widehat{G})$, the triple $(\widehat{G}, B\widehat{G}, \mathrm{Id})$ is a p-compact group.

Remark 4.4.3. Not all p-compact groups come from p-completions of compact Lie groups with the group of components a p-group. For $p = 2$, the first (and essentially the only) example of 2-compact group not coming from the 2-completion of a compact Lie group was shown by Dwyer-Wilkerson in [19]. For p odd, there are several examples of these type of p-compact groups, as shown for example in [1] and [38].

Usually we will denote the p-compact group (X, BX, e) only by X, and refer to BX as the classifying space of the p-compact group.

The homomorphisms are one of the most important concepts in group theory, and they can be extended to p-compact groups by using maps between the classifying spaces.

Definition 4.4.4 ([20, Definition 3.1])**.** A *homomorphism* $f\colon X \to Y$ of p-compact groups is a pointed map $Bf\colon BX \to BY$. Two homomorphisms $f, g\colon X \to Y$ are *conjugate* if Bf and Bg are freely homotopic.

The properties of injectivity and surjectivity of homomorphisms can also be extended to p-compact groups, this time by using well known constructions of homotopy theory.

Definition 4.4.5 ([20, Definition 3.2])**.** Suppose that $f\colon X \to Y$ is a homomorphism of p-compact groups. The homogeneous space $Y/f(X)$ (denoted Y/X if f is understood) is defined to be the homotopy fiber of Bf over the basepoint of BY. The space Y/X is pointed by the basepoint of BX. The homomorphism f is said to be a *monomorphism* if $H^*(Y/X; \mathbb{F}_p)$ is finite, and an *epimorphism* if $\Omega(Y/X)$ is a p-compact group.

Similarly to the notion of torus in compact Lie groups and its generalization to toral subgroups, we have the notion of p-compact torus and p-compact toral groups.

Definition 4.4.6 ([20, Definition 6.3])**.** If $T \cong (S^1)^r$ is a torus of rank r, then the p-completion $\widehat{T} = \Omega(BT_p^\wedge)$ of T is called a *p-compact torus* of rank r.

A *p-compact toral group* is a p-compact group $(\widehat{P}, B\widehat{P}, e)$ such that $\pi_1(B\widehat{P})$ is a p-group, and the identity component of $\widehat{P}$ is a p-compact torus with classifying space the universal cover of $B\widehat{P}$.

In order to consider fusion systems defined by p-compact groups we need to define Sylows of p-compact groups. Following the idea of Definition 4.3.2, we will first relate the discrete p-toral subgroups with the p-compact toral groups.

Definition 4.4.7 ([20, Definition 6.7])**.** If $\widehat{P}$ is an arbitrary p-compact toral group, a *discrete approximation* to $\widehat{P}$ is a pair (P, f), where P is a discrete p-toral group and $Bf\colon BP \to B\widehat{P}$ induces an isomorphism in mod p cohomology.

Remark 4.4.8. By [20, Proposition 6.9], every p-compact toral group has a discrete approximation. Moreover, each discrete p-toral group is a discrete approximation of $(\widehat{P}, B\widehat{P}, \mathrm{Id})$, where $B\widehat{P} = BP_p^\wedge$ and $\widehat{P} = \Omega(B\widehat{P})$.

The fundamental concept in the first section of Chapter 7 will be the centralizer of a discrete p-toral subgroup of a p-compact group.

Definition 4.4.9 ([20, Definition 3.5])**.** If $f\colon X \to Y$ is a homomorphism of p-compact groups, the *centralizer* of f in Y is defined to be the triple $(C_Y(X,f), BC_Y(X,f), \mathrm{Id})$, where

$$BC_Y(X,f) = \mathrm{Map}(BX, BY)_{Bf} \qquad \text{and} \qquad C_Y(X,f) = \Omega(BC_Y(X,f))$$

Whenever f is understood, we simply write $C_Y(X)$ for $C_Y(X,f)$. The homomorphism f is said to be *central* if the homomorphism $C_Y(X,f) \to Y$ induced by evaluation at the basepoint of BX is an equivalence.

Definition 4.4.10 ([12, Section 10])**.** A *discrete p-toral subgroup* of a p-compact group X is a pair (P, f), where P is a discrete p-toral group and $f\colon \widehat{P} \to X$ is a monomorphism. We write $BC_X(P,f) = BC_X(\widehat{P}, f) = \mathrm{Map}(BP, BX)_{Bf}$ and $C_X(P,f) = C_X(\widehat{P}, f)$ for short. A subgroup (P, f) is called *central* if f is a central homomorphism.

It is important to know that the centralizer of any discrete p-toral subgroup is a p-compact group.

Proposition 4.4.11 ([20, Proposition 5.1]). *The centralizer $C_X(P, f)$ of any discrete p-toral subgroup $f\colon P \to X$ is again a p-compact group.*

In order to be able to consider Sylow subgroups in p-compact groups as the maximal discrete p-toral subgroups, we need first to show that in fact there are always maximal discrete p-toral subgroups for any p-compact group and that they are all conjugated.

Proposition 4.4.12 ([12, Proposition 10.1]). *Let X be any p-compact group. Then, X has a maximal discrete p-toral subgroup $f\colon S \to X$. If $u\colon P \to X$ is any other discrete p-toral subgroup of X, then $Bu \simeq Bf \circ B\psi$ for some $\psi \in \mathrm{Hom}(P, S)$.*

We are now ready to define the fusion system of a p-compact group X. Since in a p-compact group there are no actual elements to operate with, we need to define the fusion system of X in terms of maps between classifying spaces of subgroups.

Definition 4.4.13 ([12, Definition 10.2]). For any p-compact group X with Sylow p-subgroup $f\colon S \to X$, let $\mathcal{F}_{S,f}(X)$ be the category whose objects are the subgroups of S, and where for $P, Q \leq S$,

$$\mathrm{Mor}_{\mathcal{F}_{S,f}(X)}(P, Q) = \{\varphi \in \mathrm{Hom}(P, Q) \mid Bf|_{BQ} \circ B\varphi \simeq Bf|_{BP}\}$$

Next proposition will be a key point to prove Theorem 7.1.1, the most important tool in the first section of Chapter 7.

Proposition 4.4.14 ([12, Proposition 10.4]). *Let X be a p-compact group, $S \xrightarrow{f} X$ be a Sylow p-subgroup and fix $P \leq S$. Write $f|_P\colon P \to X$ for the morphism corresponding to the map $Bf|_{BP}$ given by the composition of the inclusion $BP \hookrightarrow BS$ with Bf. Then, $C_S(P)$ is a discrete p-toral subgroup of $C_X(P, f|_P)$ and P is fully centralized in $\mathcal{F}_{S,f}(X)$ if and only if $C_S(P)$ is a Sylow p-subgroup of $C_X(P, f|_P)$.*

Finally, similarly to the case of finite groups and compact Lie groups, every p-compact group also gives rise to a p-local compact group with the same classifying space.

Theorem 4.4.15 ([12, Proposition 10.5 and Theorem 10.7]). *Let X be a p-compact group, and let $S \xrightarrow{f} X$ be a Sylow p-subgroup. Then, $\mathcal{F}_{S,f}(X)$ is a saturated fusion system over S and there exists a centric linking system $\mathcal{L}^c_{S,f}(X)$ associated to $\mathcal{F}_{S,f}(X)$ such that $(S, \mathcal{F}_{S,f}(X), \mathcal{L}^c_{S,f}(X))$ is a p-local compact group with classifying space $|\mathcal{L}^c_{S,f}(X)|^\wedge_p \simeq BX$.*

Similarly to the case of compact Lie groups, we can talk about p-local compact groups realized by p-compact groups.

Definition 4.4.16. We will say that a p-local compact group $(S, \mathcal{F}, \mathcal{L})$ is *realizable by a* p*-compact group* if $\mathcal{F} = \mathcal{F}_{S,f}(X)$ and $\mathcal{L} = \mathcal{L}^c_{S,f}(X)$ for some p-compact group X with $S \xrightarrow{f} X$ a Sylow p-subgroup.

CHAPTER 5

Limits of fusion systems

In this chapter we will make precise the notion of infinite union of fusion systems. Actually, we will define a more general notion of limit of fusion systems and we will consider the special case when all the morphisms are injective as the infinite union.

First, to fix the notation, we will recall some definitions and results about direct limits of groups. Then we will define the notion of morphisms of fusion systems following [5] and use it to define the limit of fusion systems. In the third section, we will prove that the limit of fusion systems it is again a fusion system if the direct limit of the underlying p-groups is a discrete p-toral group. Before ending the section we will discuss the relation between the definition of limit we make here and the concept of direct limit in category theory.

Finally, in the last section we will define new examples of fusion systems over discrete p-toral groups by considering the infinite union of the exotic fusion systems discussed in Chapter 3.

5.1. Direct limits of groups

We begin with the definition of directed set, which is the base to define direct systems and direct limits. All definitions in this section can be found for sets in [7], and it is no difficult to see that all definitions there extend to groups.

Definition 5.1.1. A preordered set I is said to be a *directed set* if for every two elements $i, j \in I$, there exists an element $k \in I$ such that $i \leq k$ and $j \leq k$.

Now we can define direct systems of groups as families of groups and morphisms indexed by directed sets such that the morphisms satisfy certain compatibility conditions.

Definition 5.1.2. Let I be a directed set and let $(E_i)_{i\in I}$ be a family of groups indexed by I. For each pair (i,j) of elements of I such that $i \leq j$, let $f_{ij}\colon E_i \to E_j$ be a morphism of groups. The pair (E_i, f_{ij}) is called a *direct system* over I if:

(a) $f_{ii}\colon E_i \to E_i$ is the identity morphism.

(b) $f_{ik} = f_{jk} \circ f_{ij}$ for every $i \leq j \leq k$.

Finally, for any direct system of groups we can construct a new group which satisfies the universal property of being, in some sense, the smallest group containing all groups in the direct system in a way compatible with the morphisms of the system.

Definition 5.1.3. Let (E_i, f_{ij}) be a direct system of groups. The *direct limit* of the system (E_i, f_{ij}) is the group $\varinjlim E_i$, defined as

$$\varinjlim E_i = \bigcup_{i\in I} (E_i \times \{i\})/\sim$$

where $(x,i) \sim (y,j)$ if there exists $k \in I$ such that $k \geq i$, $k \geq j$ and $f_{ik}(x) = f_{jk}(y)$.

The product operation on $\varinjlim E_i$ is defined as

$$[x,i][y,j] = [f_{ik}(x)f_{jk}(y), k]$$

for any $k \geq i, j$. For each $i \in I$, write f_i for the morphism

$$\begin{array}{rccc} f_i\colon & E_i & \to & \varinjlim E_i \\ & x & \mapsto & [x,i] \end{array}$$

Remark 5.1.4. If all the morphisms f_{ij} are injective, then f_i is injective for all i by definition, so we can identify E_i with $f(E_i)$ and therefore we can consider the limit $\varinjlim E_i$ as the union $\bigcup_{i\in I} E_i$.

Remark 5.1.5. By the definition of the direct limit and the maps f_i, we have $f_j(f_{ij}) = f_i$ for all $i \leq j$.

Remark 5.1.6. Let (E_i, f_{ij}) be a direct system of groups over I and $E = \varinjlim E_i$ its direct limit. If we have $F_i \leq E_i$ for all $i \in I$ such that $f_{ij}(F_i) \leq F_j$ for all $i \leq j$, then the family (F_i, g_{ij}) is a direct system of subgroups of E_i, where g_{ij} is the restriction of f_{ij} to F_i. Then there exists a unique morphism $j\colon \varinjlim F_i \to E$, which is indeed a monomorphism, so we can identify $\varinjlim F_i$ with a subgroup of E.

The following lemma will be useful in order to construct a direct system of groups whose limit is a given fixed group.

Lemma 5.1.7. *Let (E_i, f_{ij}) be a direct system of groups over I and $E = \varinjlim E_i$ its direct limit. Let $F \leq E$ be any subgroup. Then, there exist $F_i \leq E_i$ for all $i \in I$ such that $\varinjlim F_i = F$.*

Proof. For any $i \in I$ let $F_i = f_i^{-1}(F \cap f_i(E_i))$. Then, $F_i \leq E_i$ for all $i \in I$ and for any $x \in F_i$ we have

$$f_j(f_{ij}(x)) = f_i(x) \in F \cap f_i(E_i) = F \cap f_j(f_{ij}(E_i)) \leq F \cap f_j(E_j)$$

so $f_{ij}(F_i) \leq F_j$ for all $i \leq j$. Finally, we have $\varinjlim F_i = F$ by construction. □

5.2. Morphisms of fusion systems

In the next section we will define direct systems of fusion systems and a notion of limit fusion system. For this, we need first the concept of morphism of fusion systems. The following definition can be seen in [5, Definition II.2.2] for fusion systems over finite p-groups, but can be used without change for fusion systems over discrete p-toral groups.

If $\mathcal{F}$ is a fusion system over a discrete p-toral group S, we will write the fusion system in a compact form as $(S, \mathcal{F})$.

Definition 5.2.1. A *morphism* $\alpha\colon (S,\mathcal{F}) \to (\tilde{S}, \widetilde{\mathcal{F}})$ *of fusion systems* from $(S, \mathcal{F})$ to a system $(\tilde{S}, \widetilde{\mathcal{F}})$ is a family $(\alpha, \alpha_{P,Q})_{P,Q \in \mathcal{F}}$, such that $\alpha\colon S \to \tilde{S}$ is a group homomorphism, and $\alpha_{P,Q}\colon \operatorname{Hom}_{\mathcal{F}}(P,Q) \to \operatorname{Hom}_{\widetilde{\mathcal{F}}}(\alpha(P), \alpha(Q))$ are maps making the diagram

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & \alpha(P) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\alpha_{P,Q}(\varphi)} \\ Q & \xrightarrow{\alpha} & \alpha(Q) \end{array}$$

commutative for all $P, Q \leq S$.

Remark 5.2.2. Note that if $\alpha\colon (S,\mathcal{F}) \to (\tilde{S}, \widetilde{\mathcal{F}})$ is a morphism of fusion systems defined by a family $(\alpha, \alpha_{P,Q})_{P,Q \in \mathcal{F}}$, then the maps $\alpha_{P,Q}$ are uniquely determined by the group homomorphism α and the commutativity property.

Remark 5.2.3. If $\alpha\colon S \to \tilde{S}$ is an injective group homomorphism we have $P \cong \alpha(P)$ for all $P \leq S$. Then, for each $\varphi \in \operatorname{Hom}_{\mathcal{F}}(P,Q)$, the map $\alpha_{P,Q}$ must be defined as $\alpha_{P,Q}(\varphi) = \alpha\varphi\alpha^{-1}$, so the condition for $(\alpha, \alpha_{P,Q})$ being a morphism of fusion systems is $\alpha\varphi\alpha^{-1} \in \operatorname{Hom}_{\widetilde{\mathcal{F}}}(\alpha(P), \alpha(Q))$. Note also that in this case, the maps $\alpha_{P,Q}$ are all injective maps.

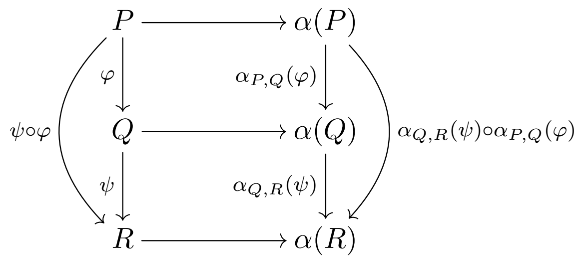
The next proposition, joint with Theorem 4.1.13 will allow us to prove that a certain group homomorphism $\alpha\colon S \to \tilde{S}$ is a morphism of fusion systems from $(S, \mathcal{F})$ to $(\tilde{S}, \widetilde{\mathcal{F}})$ by restricting our attention only to morphisms between certain subgroups of S.

Proposition 5.2.4. *Let $\alpha\colon (S, \mathcal{F}) \to (\tilde{S}, \widetilde{\mathcal{F}})$ be a morphism of fusion systems. Then, the following holds*

(*a*) *If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, then $\alpha_{P,R}(\psi \circ \varphi) = \alpha_{Q,R}(\psi) \circ \alpha_{P,Q}(\varphi)$.*

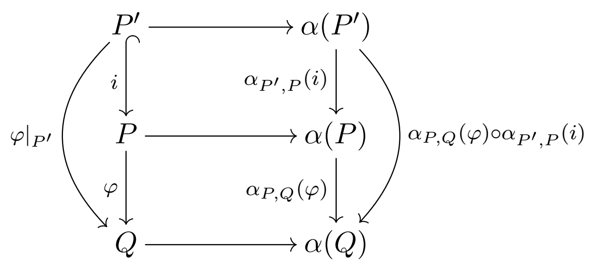
(*b*) *If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and $P' \leq P$, then $\alpha_{P,Q}(\varphi|_{P'}) = \alpha_{P,Q}(\varphi)|_{\alpha(P')}$.*

Proof. To prove (a), let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$. Then, $\alpha_{P,Q}(\varphi) \in \mathrm{Hom}_{\widetilde{\mathcal{F}}}(\alpha(P), \alpha(Q))$ and $\alpha_{Q,R}(\psi) \in \mathrm{Hom}_{\widetilde{\mathcal{F}}}(\alpha(Q), \alpha(R))$. Moreover, we have the commutative diagram



Since α is a morphism of fusion systems, the two little squares are commutative. Therefore, the big rectangle is also commutative. Then, since $\alpha_{P,R}(\psi \circ \varphi)$ is uniquely determined by the morphism α, we must have $\alpha_{P,R}(\psi \circ \varphi) = \alpha_{Q,R}(\psi) \circ \alpha_{P,Q}(\varphi)$.

The argument to prove (b) is similar. Let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and $P' \leq P$. In this case, we have the commutative diagram



By part (a) we have $\alpha_{P',Q}(\varphi|_{P'}) = \alpha_{P,Q}(\varphi) \circ \alpha_{P',P}(i)$ and again, since $\alpha_{P',P}(i)$ is uniquely determined by α, we have $\alpha_{P',P}(i) = i$. Hence, $\alpha_{P,Q}(\varphi|_{P'}) = \alpha_{P,Q}(\varphi)|_{\alpha(P')}$. $\square$

At the end of next section we will compare the notion of limit of fusion systems to that of direct limit in category theory. To do this, we need to consider the category of fusion systems and morphisms of fusion systems. We will see that better results follow if we focus only on saturated fusion systems.

Definition 5.2.5. Since the identity is a morphism of fusion systems and the associativity property of morphism of groups is inherited by morphisms of fusion systems we can define the category of fusion systems $\mathcal{FS}$ where the objects are pairs $(S, \mathcal{F})$ with $\mathcal{F}$ a fusion system over S and the morphisms are morphisms of fusion systems. If we restrict our attention to saturated fusion systems, we can consider the full subcategory $\mathcal{SFS}$ where the objects are the pairs $(S, \mathcal{F})$ such that $\mathcal{F}$ is a saturated fusion system over S.

5.3. Limits of fusion systems

We can now define a direct system of fusion systems in exactly the same way as we did for groups in Definition 5.1.2.

Definition 5.3.1. Let I be a directed set and let $(S_i, \mathcal{F}_i)_{i\in I}$ be a family of fusion systems indexed by I. For each pair (i, j) of elements of I such that $i \leq j$, let $\alpha_{ij} \colon (S_i, \mathcal{F}_i) \to (S_j, \mathcal{F}_j)$ be a morphism of fusion systems. The pair $((S_i, \mathcal{F}_i), \alpha_{ij})$ is called a *direct system* over I if:

(a) $\alpha_{ii} \colon (S_i, \mathcal{F}_i) \to (S_i, \mathcal{F}_i)$ is the identity morphism.

(b) $\alpha_{ik} = \alpha_{jk} \circ \alpha_{ij}$ for every $i \leq j \leq k$.

Remark 5.3.2. Note that if $((S_i, \mathcal{F}_i), \alpha_{ij})$ is a direct system of fusion systems, then, in particular, (S_i, α_{ij}) is a direct system of groups and we can consider its direct limit.

One important notion to define morphism in the limit fusion system will be the notion of compatible system of morphisms in a direct system.

Definition 5.3.3. Let $((S_i, \mathcal{F}_i), \alpha_{ij})$ be a direct system of fusion systems over a directed set I and $P_i, Q_i \leq S_i$ two families of subgroups such that $\alpha_{ij}(P_i) \leq P_j$ and $\alpha_{ij}(Q_i) \leq Q_j$ for all $i \leq j$. We say that a family of morphisms $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, Q_i)$ is a *compatible system of morphisms* if the diagram

$$\begin{array}{ccc} \alpha_{ij}(P_i) & \hookrightarrow & P_j \\ {\scriptstyle {\alpha_{ij}}_{P_i,Q_i}(\varphi_i)}\downarrow & & \downarrow{\scriptstyle \varphi_j} \\ \alpha_{ij}(Q_i) & \hookrightarrow & Q_j \end{array}$$

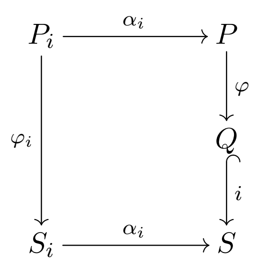
is commutative for all $i \leq j$.

We can proceed then with the definition of the limit of a direct system of fusion systems. Since we have defined fusion systems only over discrete p-toral groups, we should ask that the direct limit of the underlying groups in the system is a discrete p-toral group.

Recall that if $((S_i, \mathcal{F}_i), \alpha_{ij})$ is a direct system of fusion systems, we write by α_i the morphism from each S_i to the direct limit $\varinjlim S_i$.

Definition 5.3.4. Let $((S_i, \mathcal{F}_i), \alpha_{ij})$ be a direct system of fusion systems over a directed set I such that $S = \varinjlim S_i$ is a discrete p-toral group. Then, the *limit* of the direct system is defined to be $(S, \mathcal{F})$, where $\mathcal{F}$ is defined as

- Obj: All subgroups $P \leq S$.
- Mor: $\mathrm{Hom}_{\mathcal{F}}(P, Q) = \{\varphi \in \mathrm{Hom}(P, Q) \mid \exists P_i \leq S_i$ for all $i \in I$, with $P = \varinjlim P_i$ and $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, S_i)$ a compatible system of morphisms such that the diagram



  is commutative for all $i \in I\}$.

Now we must prove that this definition indeed verifies the properties to be a fusion system according to Definition 4.1.7.

Proposition 5.3.5. *Let $((S_i, \mathcal{F}_i), \alpha_{ij})$ be a direct system of fusion systems over a directed set I such that $S = \varinjlim S_i$ is a discrete p-toral group. Then, the limit $(S, \mathcal{F})$ of the direct system $((S_i, \mathcal{F}_i), \alpha_{ij})$ is a fusion system.*

Proof. The first thing we must prove is that $\mathcal{F}$ is indeed a category, then we will prove that it satisfies the axioms to be a fusion system.

To prove that $\mathcal{F}$ is a category we must show that the identity morphism is in $\mathrm{Hom}_{\mathcal{F}}(P, P)$ for all $P \leq S$, that we have a composition of morphisms, and that this composition is associative.

Let $P \leq S$ be a subgroup, and $\varphi = \mathrm{Id} \in \mathrm{Hom}(P, P)$ the identity morphism. By Lemma 5.1.7, we have that there exist subgroups $P_i \leq S_i$ for every $i \in I$ such that $P = \varinjlim P_i$. By taking φ_i to be the inclusion of the subgroup P_i into S_i, we have $\mathrm{Id} \in \mathrm{Hom}_{\mathcal{F}}(P, P)$.

Now let $P, Q, R \leq S$ be three subgroups of S and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ two morphisms in $\mathcal{F}$. We must prove that $\phi \circ \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R)$. We know, by the definition of limit fusion system, that there exist subgroups $P_i, Q_i \in S_i$ and morphisms

$\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, S_i)$ and $\phi_i \in \mathrm{Hom}_{\mathcal{F}_i}(Q_i, S_i)$ such that the diagrams

$$\begin{array}{ccc} P_i & \xrightarrow{\alpha_i} & P \\ & & \downarrow{\scriptstyle \varphi} \\ \downarrow{\scriptstyle \varphi_i} & & Q \\ & & \downarrow{\scriptstyle i} \\ S_i & \xrightarrow{\alpha_i} & S \end{array} \qquad \begin{array}{ccc} Q_i & \xrightarrow{\alpha_i} & Q \\ & & \downarrow{\scriptstyle \phi} \\ \downarrow{\scriptstyle \phi_i} & & R \\ & & \downarrow{\scriptstyle i} \\ S_i & \xrightarrow{\alpha_i} & S \end{array}$$

are commutative for all $i \in I$. Since φ_i is injective for all $i \in I$, we can define the subgroups $A_i = \varphi_i^{-1}(\varphi_i(P_i) \cap Q_i) \le P_i$. Note that we have $\alpha_{ij}(A_i) \le A_j$. Indeed, let $x \in A_i$, we must prove that $\varphi_j(\alpha_{ij}(x)) \in \varphi_j(P_j) \cap Q_j$. Since $x \in A_i \le P_i$ and $\alpha_{ij}(P_i) \le P_j$ it is obvious that $\varphi_j(\alpha_{ij}(x)) \in \varphi_j(P_j)$.

Using now that $\varphi_i(x) \in Q_i$, we have that $\alpha_{ij}(\varphi_i(x)) \in Q_j$ and, by the compatibility of the family $\{\varphi_i\}$, we know that $\alpha_{ij}(\varphi_i(x)) = \varphi_j(\alpha_{ij}(x))$. This proves that the subgroups A_i form a direct subsystem of the P_i, so we can consider the limit group $A = \varinjlim A_i$. Also, we have by construction that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\alpha_i} & A \\ & & \downarrow{\scriptstyle \phi\circ\varphi} \\ \downarrow{\scriptstyle \phi_i\circ\varphi_i} & & R \\ & & \downarrow{\scriptstyle i} \\ S_i & \xrightarrow{\alpha_i} & S \end{array}$$

is commutative for all $i \in I$. To finish the proof of $\phi \circ \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R)$ we must show that, in fact, $A = P$. Since $A_i \le P_i$ for all $i \in I$, we have that there exists a monomorphism $u \colon A \to P$.

We must show now that u is also an epimorphism. To do this, it is enough to prove that $P = \bigcup_{i \in I} \alpha_i(A_i)$. Let $[x, j] \in P$, then $\varphi([x, j]) \in Q$, so there exists $k \in I$ and $y \in Q_k$ such that $\varphi([x, j]) = [y, k]$. By the commutativity property of the family $\{\varphi_i\}$ with φ, we have that $\varphi([x, j]) = [\varphi_j(x), j]$. Now, since $[\varphi_j(x), j] = [y, k]$, there exists $l \in I$ such that $l \ge j, k$ and $\alpha_{kl}(y) = \alpha_{jl}(\varphi_j(x))$.

Since $y \in Q_k$, we have that $\alpha_{kl}(y) \in Q_l$, so we will be finished if we see that $\alpha_{kl}(y)$ is also in $\varphi_l(P_l)$. Indeed, this follows immediately from the compatibility condition on $\{\varphi_i\}$, since $\alpha_{kl}(y) = \alpha_{jl}(\varphi_j(x)) = \varphi_l(\alpha_{jl}(x))$ and $\alpha_{jl}(x) \in P_l$.

Note that is trivial to see then that $\alpha_{jl}(x) \in A_l$ and $\alpha_l(\alpha_{jl}(x)) = \alpha_j(x) = [x, j]$, so u is an epimorphism.

This proves that the composition of two morphisms in $\mathcal{F}$ it is again in $\mathcal{F}$. To finish the proof of $\mathcal{F}$ being a category, note that the associativity condition on the morphisms is satisfied since each α_{ij} is a group homomorphisms.

Now we have to check that $\mathcal{F}$ satisfies the axioms to be a fusion system. That is, we have to show that, for every $P, Q \leq S$, $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ satisfies

(a) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$.

(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

If φ is a morphism in $\mathrm{Hom}_{\mathcal{F}}(P, Q)$, it is clear that $\varphi \in \mathrm{Inj}(P, Q)$ because of the commutativity conditions with the family φ_i, provided that all $\mathcal{F}_i$ are fusion systems so all φ_i are injective morphisms.

We must then show that $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ contains all the conjugations from P to Q. Recall that $N_S(P, Q)$ denotes the set of elements $[x, j] \in S$ such that $[x, j]P[x, j]^{-1} \leq Q$. Let then $[x, j] \in N_S(P, Q)$. We have that $x \in S_j$ and, by Lemma 5.1.7, there exists $P_i \leq S_i$ for every $i \in I$ such that $P = \varinjlim P_i$. Consider the subset of I defined as $K = \{k \in I \mid j \leq k\}$, and define

$$P'_i = \begin{cases} \{e\} & \text{, if } i \notin K \\ P_i & \text{, if } i \in K \end{cases}$$

Since K is a cofinal subset of I, we have $P = \varinjlim P_i = \varinjlim P'_i$. Now, we can consider the morphisms

$$\varphi_i = \begin{cases} i & \text{, if } i \notin K \\ c_{\alpha_{ji}(x)} & \text{, if } i \in K \end{cases}$$

which proves that $c_{[x,j]} \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$.

To prove that every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion observe first that all inclusions are morphisms in $\mathcal{F}$, since they are the same as conjugation by the neutral element. So we must prove that if $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, then also $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, \varphi(P))$ and $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(P), P)$.

By hypothesis, we have that there exist $P_i \leq S_i$ such that $P = \varinjlim P_i$ and $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, S_i)$ such that the diagram

$$\begin{array}{ccc} P_i & \xrightarrow{\alpha_i} & P \\ & & \downarrow \varphi \\ \varphi_i \downarrow & & Q \\ & & \downarrow i \\ S_i & \xrightarrow{\alpha_i} & S \end{array}$$

is commutative for all $i \in I$. It is clear that the same P_i and φ_i will serve to prove that $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, \varphi(P))$. To see that $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(P), P)$, consider $\varphi_i(P_i)$, together with the morphisms $i \circ \varphi_i^{-1} \in \mathrm{Hom}_{\mathcal{F}_i}(\varphi_i(P_i), S_i)$. Then we have, by the compatibility condition of the family $\{\varphi_i\}$, that $\varphi_i(P_i)$ is a direct system and, by the commutativity of the φ_i with φ, that $\varinjlim \varphi_i(P_i) = \varphi(P)$ and that the diagram

$$\begin{array}{ccc} \varphi(P_i) & \xrightarrow{\alpha_i} & \varphi(P) \\ & & \downarrow{\scriptstyle \varphi^{-1}} \\ \downarrow{\scriptstyle i \circ \varphi_i^{-1}} & & P \\ & & \downarrow{\scriptstyle i} \\ S_i & \xrightarrow{\alpha_i} & S \end{array}$$

is commutative for all $i \in I$. □

We want to compare now the definition of limit of fusion systems given here and the notion of direct limit in category theory. For this we want to study if there are morphisms $\alpha_i \colon (S_i, \mathcal{F}_i) \to (S, \mathcal{F})$ commuting with α_{ij} and if $(S, \mathcal{F})$ is universal in the sense that for any other fusion system $(T, \mathcal{E})$ with morphisms $\psi_i \colon (S_i, \mathcal{F}_i) \to (T, \mathcal{E})$ commuting with α_{ij} there is a unique morphism $u \colon (S, \mathcal{F}) \to (T, \mathcal{E})$.

The next lemma proves a technical result that will be useful in the next proposition, which will answer the first question above in the affirmative.

Lemma 5.3.6. *Let $((S_i, \mathcal{F}_i), \alpha_{ij})$ be a direct system of fusion systems over a directed set I such that $S = \varinjlim S_i$ is a discrete p-toral group. Write $(S, \mathcal{F})$ for the limit fusion system and let $\varphi \in \mathrm{Hom}(P, Q)$ be a morphism for some $P, Q \leq S$. Finally, let $J \subset I$ be a totally-ordered subset and assume we have a compatible system of morphisms $\varphi_j \in \mathrm{Hom}_{\mathcal{F}_j}(P_j, S_j)$ with $P = \varinjlim P_j$ such that the diagram*

$$\begin{array}{ccc} P_j & \xrightarrow{\alpha_j} & P \\ & & \downarrow{\scriptstyle \varphi} \\ \downarrow{\scriptstyle \varphi_j} & & Q \\ & & \downarrow{\scriptstyle i} \\ S_j & \xrightarrow{\alpha_j} & S \end{array}$$

is commutative for all $j \in J$. Then, we can extend the family to a compatible system of morphisms for all $i \in I$ such that $P = \varinjlim P_i$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$.

Proof. For each $i \in I$, set $A_i = \{j \in J \mid j \leq i\}$. Then, consider the system

$$P_i = \begin{cases} \{e\} & \text{, if } i \notin J \text{ and } A_i = \emptyset \\ P_i & \text{, if } i \in J \\ \alpha_{ji}(P_j) & \text{, } \max j \in A_i \text{ if } i \notin J \end{cases}$$

along with the morphisms

$$\varphi_i = \begin{cases} i & \text{, if } i \notin J \text{ and } A_i = \emptyset \\ \varphi_i & \text{, if } i \in J \\ i \circ \alpha_{ji_{P_j,S_j}}(\varphi_j) & \text{, } \max j \in A_i \text{ if } i \notin J \end{cases}$$

Since J is totally ordered, the value $\max j \in A_i$ is well defined if $A_i \neq \emptyset$. Note also that $A_i \subseteq A_j$ if $i \leq j$. By construction, we have $P = \varinjlim P_j = \varinjlim P_i$.

It is clear that the new system extends the old one, now we have to prove that the new family is also a compatible system of morphisms. That is, we must prove that, for all $i \leq j$, the diagram

$$\begin{array}{ccc} \alpha_{ij}(P_i) & \hookrightarrow & P_j \\ {\scriptstyle \alpha_{ij_{P_i,S_i}}(\varphi_i)} \downarrow & & \downarrow {\scriptstyle \varphi_j} \\ \alpha_{ij}(S_i) & \hookrightarrow & S_j \end{array}$$

is commutative.

Let $i \leq j$, and assume $i \notin J$ and $j \notin J$. If $A_j = \emptyset$, then clearly A_i is also empty and the associated diagram is commutative in a trivial way. If $A_j \neq \emptyset$ but $A_i = \emptyset$, then the associated diagram also commutes trivially.

If both $A_j \neq \emptyset$ and $A_i \neq \emptyset$ we must consider different options. Set k_i and k_j for the maximum index in A_i and A_j, respectively. Then, we must prove the commutativity of the diagram

$$\begin{array}{ccc} \alpha_{ij}(\alpha_{k_i i}(P_{k_i})) & \hookrightarrow & \alpha_{k_j j}(P_{k_j}) \\ {\scriptstyle \alpha_{ij_{\alpha_{k_i i}(P_{k_i}),S_i}}(i \circ \alpha_{k_i i}(\varphi_{k_i}))} \downarrow & & \downarrow {\scriptstyle i \circ \alpha_{k_j j_{P_{k_j},S_{k_j}}}(\varphi_{k_j})} \\ \alpha_{ij}(S_i) & \hookrightarrow & S_j \end{array}$$

Since the set J is totally ordered and $A_i \subseteq A_j$, we must have $k_i \leq k_j$. If $k_i = k_j$, we have that $\alpha_{ij}(\alpha_{k_i i}(P_{k_i})) = \alpha_{k_i j}(P_{k_i}) = \alpha_{k_j j}(P_{k_j})$, so the diagram trivially commutes.

If $k_i < k_j$, we have that $\alpha_{k_i k_j}(P_{k_i}) \leq P_{k_j}$ and $\alpha_{k_j j} \circ \alpha_{k_i k_j} = \alpha_{k_i j} = \alpha_{ij} \circ \alpha_{k_i i}$, so $\alpha_{ij}(\alpha_{k_i i}(P_{k_i})) = \alpha_{k_j j}(\alpha_{k_i k_j})(P_{k_i})) \leq \alpha_{k_j j}(P_{k_j})$, and the diagram is commutative by the hypothesis of φ_j being a compatible system of morphisms for J.

The case where $i \in J$ or $j \in J$ is included in the previous diagram, just considering the corresponding morphisms as the identity.

The last thing we must prove is that the new morphisms still commute with φ. Let $i \in I$ and consider the diagram

$$\begin{array}{ccc} P_i & \xrightarrow{\alpha_i} & P \\ & & \downarrow{\scriptstyle \varphi} \\ \downarrow{\scriptstyle \varphi_i} & & Q \\ & & \hookdownarrow{\scriptstyle i} \\ S_i & \xrightarrow{\alpha_i} & S \end{array}$$

If $i \in J$, then the diagram commutes by hypothesis. Otherwise, $P_i = \alpha_{k_i i}(P_{k_i})$ with $k_i \in J$, and we can extend the diagram to

$$\begin{array}{ccccc} P_{k_i} & \xrightarrow{\alpha_{k_i i}} & \alpha_{k_i i}(P_{k_i}) & \xrightarrow{\alpha_i} & P \\ & & & & \downarrow{\scriptstyle \varphi} \\ \downarrow{\scriptstyle \varphi_{k_i}} & & \downarrow{\scriptstyle i \circ \alpha_{k_i i}(\varphi_{k_i})} & & Q \\ & & & & \hookdownarrow{\scriptstyle i} \\ S_{k_i} & \xrightarrow{\alpha_{k_i i}} & S_i & \xrightarrow{\alpha_i} & S \end{array}$$

By hypothesis, the big rectangle is commutative, since $\alpha_{k_i} = \alpha_i \circ \alpha_{k_i i}$ and $k_i \in J$. Also, we have just proved before that the left square is commutative, because the extended family is still a compatible system of morphisms. Since the map $P_{k_i} \mapsto \alpha_{k_i i}(P_{k_i})$ is clearly surjective, we have that the right square is also commutative. $\square$

Proposition 5.3.7. *Let $((S_i, \mathcal{F}_i), \alpha_{ij})$ be a direct system of fusion systems over a directed set I. Assume that $S = \varinjlim S_i$ is a discrete p-toral group and let $(S, \mathcal{F})$ be the limit of the system. Then, there exist morphisms $\alpha_i \colon (S_i, \mathcal{F}_i) \to (S, \mathcal{F})$ such that the diagrams*

$$\begin{array}{ccc} (S_i, \mathcal{F}_i) & \xrightarrow{\alpha_{ij}} & (S_j, \mathcal{F}_j) \\ & {\scriptstyle \alpha_i} \searrow \quad \swarrow {\scriptstyle \alpha_j} & \\ & (S, \mathcal{F}) & \end{array}$$

are commutative for all $i \leq j$.

Proof. Since S is defined as the direct limit of the S_i, there exist morphisms $\alpha_i \colon S_i \to S$ for all $i \in I$ by Definition 5.1.3. By Remark 5.1.5, the morphisms α_i are commutative with α_{ij}. Then, we need to prove that each α_i induces maps $\alpha_{i_{P,Q}}\colon \mathrm{Hom}_{\mathcal{F}_i}(P,Q) \to \mathrm{Hom}_{\mathcal{F}}(\alpha_i(P), \alpha_i(Q))$ for all $P, Q \leq S$ satisfying the commutativity condition of Definition 5.2.1.

For any $\varphi \in \mathrm{Hom}_{\mathcal{F}_i}(P,Q)$, we can define $\alpha_{i_{P,Q}}(\varphi)$ as follows. For $x \in \alpha_i(P)$, choose $y \in P$ such that $\alpha_i(y) = x$ and define $\alpha_{i_{P,Q}}(\varphi)(x) = \alpha_i(\varphi(y))$. Since α_i could be non injective, we must prove that $\alpha_{i_{P,Q}}(\varphi)(x)$ does not depend on the choice of y in order to have $\alpha_{i_{P,Q}}(\varphi)$ well defined.

Let $y_1, y_2 \in P$ such that $\alpha_i(y_1) = \alpha_i(y_2)$, then, there exists $j \geq i$ such that $\alpha_{ij}(y_1) = \alpha_{ij}(y_2)$. Since α_{ij} is a morphism of fusion systems, we have the following commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\alpha_{ij}} & \alpha_{ij}(P) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\alpha_{ij_{P,Q}}(\varphi)} \\ Q & \xrightarrow{\alpha_{ij}} & \alpha_{ij}(Q) \end{array}$$

So for any $y \in P$ we have $\alpha_{ij}(\varphi(y)) = \alpha_{ij_{P,Q}}(\varphi)(\alpha_{ij}(y))$. Hence,

$$\begin{aligned} \alpha_i(\varphi(y_1)) &= \alpha_j(\alpha_{ij}(\varphi(y_1))) \\ &= \alpha_j(\alpha_{ij_{P,Q}}(\varphi)(\alpha_{ij}(y_1))) \\ &= \alpha_j(\alpha_{ij_{P,Q}}(\varphi)(\alpha_{ij}(y_2))) \\ &= \alpha_j(\alpha_{ij}(\varphi(y_2))) \\ &= \alpha_i(\varphi(y_2)) \end{aligned}$$

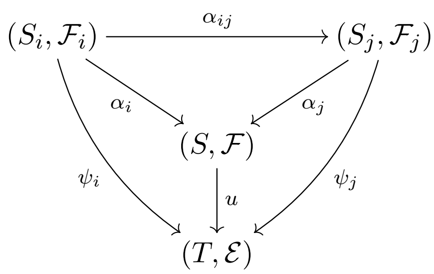
Therefore, $\alpha_{i_{P,Q}}(\varphi)$ is well defined. Then, by taking $J = \{i\}$ in Lemma 5.3.6, we have that $\alpha_{i_{P,Q}}(\varphi) \in \mathrm{Hom}_{\mathcal{F}}(\alpha_i(P), \alpha_i(Q))$, so we can conclude that $\alpha_i\colon (S_i, \mathcal{F}_i) \to (S, \mathcal{F})$ is a morphism of fusion systems. Moreover, the α_i satisfy the commutative diagram in the statement since they do at the level of groups and $\alpha_{i_{P,Q}}$ is determined by α_i as noted in Remark 5.2.2. □

For the second question of the comparison between the limit of fusion systems and the direct limit of a category, we must study when there is a unique morphism $u\colon (S, \mathcal{F}) \to (T, \mathcal{E})$ between the two fusion systems playing the role of the limit.

The morphism u will always exist at the level of groups and it will be unique because S is a direct limit of groups. But maybe u is not a morphism of fusion systems because there are no maps $u_{P,Q}\colon \mathrm{Hom}_{\mathcal{F}}(P,Q) \to \mathrm{Hom}_{\mathcal{E}}(u(P), u(Q))$ satisfying the required properties. We prove in the next proposition that if we work with the category $\mathcal{SFS}$, then u it is indeed a morphism of fusion systems.

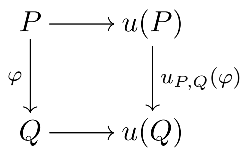
Proposition 5.3.8. *Let $((S_i, \mathcal{F}_i), \alpha_{ij})$ be a direct system of fusion systems over a directed set I such that $S = \varinjlim S_i$ is a discrete p-toral group. Assume that the limit fusion system $(S, \mathcal{F})$ is saturated. Then, $(S, \mathcal{F})$ is the categorical direct limit in the category $\mathcal{SFS}$.*

Proof. Assume that we have a diagram of the form



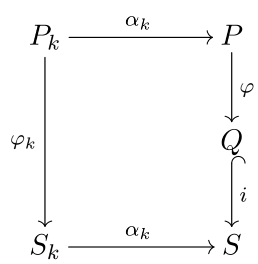
for every $i \leq j$, satisfying $\psi_i = \psi_j \circ \alpha_{ij}$. Then, u is given by the property of S being the direct limit of the S_i and it is unique. We must prove then that u it is also a morphism of fusion systems. Note that we will also have $\psi_i = u \circ \alpha_i$, since S is a direct limit of groups.

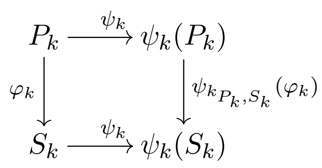
Let $P, Q \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. Then, we can define a morphism $u_{P,Q}(\varphi)\colon u(P) \to u(Q)$ such that the diagram



is commutative as follows. Let $x \in u(P)$ and choose $y = [y, i] \in P$ such that $u(y) = x$. Then, define $u_{P,Q}(\varphi)(x) = u(\varphi(y))$. Next we show that this morphism is well defined. Let $y_1 = [y_1, i_1] \in P$ and $y_2 = [y_2, i_2] \in P$ such that $u(y_1) = x = u(y_2)$, we must prove that $u(\varphi(y_1)) = u(\varphi(y_2))$. Since $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, there exist $P_i \leq S_i$ and $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(P_i, S_i)$ such that $P = \varinjlim P_i$ and the diagrams of Definition 5.3.4 are commutative. Then, there exists $k \in I$ big enough such that there exist $\widetilde{y_1}, \widetilde{y_2} \in P_k$ with $y_1 = \alpha_k(\widetilde{y_1})$ and $y_2 = \alpha_k(\widetilde{y_2})$. Then, since $u(\alpha_k(\widetilde{y_1})) = u(\alpha_k(\widetilde{y_2}))$, we have $\psi_k(\widetilde{y_1}) = \psi_k(\widetilde{y_2})$.

Now, since $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and ψ_k is a morphism of fusion systems, we have these two commutative diagrams



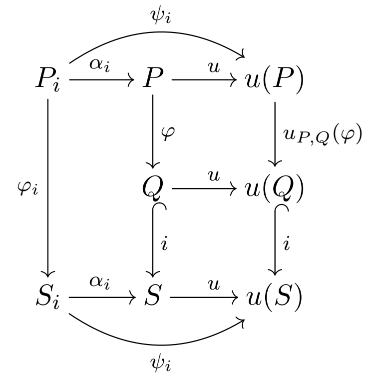


Therefore, we can consider the following chain of equalities

$$\begin{aligned}
u(\varphi(y_1)) &= u(\varphi(\alpha_k(\widetilde{y_1}))) \\
&= u(\alpha_k(\varphi_k(\widetilde{y_1}))) \\
&= \psi_k(\varphi_k(\widetilde{y_1})) \\
&= \psi_{k_{P_k,S_k}}(\varphi_k)(\psi_k(\widetilde{y_1})) \\
&= \psi_{k_{P_k,S_k}}(\varphi_k)(\psi_k(\widetilde{y_2})) \\
&= \psi_k(\varphi_k(\widetilde{y_2})) \\
&= u(\alpha_k(\varphi_k(\widetilde{y_2}))) \\
&= u(\varphi(\alpha_k(\widetilde{y_2}))) \\
&= u(\varphi(y_2))
\end{aligned}$$

Finally, we must prove that $u_{P,Q}(\varphi) \in \mathrm{Hom}_{\mathcal{E}}(u(P), u(Q))$. For this, we will use axiom (III) of saturated fusion systems from Definition 4.1.9.

Note that, by the properties of φ and u, we have the diagram



where all the squares, rectangles and curved triangles are commutative. By the uniqueness of the definition of $\psi_{i_{P_i,S_i}}(\varphi_i)$, we must have

$$u_{P,Q}(\varphi)|_{\psi_i(P_i)} = \psi_{i_{P_i,S_i}}(\varphi_i) \in \mathrm{Hom}_{\mathcal{E}}(\psi_i(P_i), \psi_i(S_i))$$

for all $i \in I$.

Now, since $u(P)$ is a countable group, we can numerate its elements $u(P) = \{u_1, u_2, \dots\}$. Note also that $u(P) = \bigcup_{i \in I} \psi_i(P_i)$ and the $\psi_i(P_i)$ form a direct system with the inclusion, so we can take, by the axiom of choice, a set of subgroups $\{\psi_{i_1}(P_{i_1}), \psi_{i_2}(P_{i_2}), \dots\}$ such that $\{u_1, \dots, u_k\} \in \psi_{i_k}(P_{i_k})$ for all $k \in \mathbb{N}$.

Finally, by discarding some subgroups if necessary, we obtain an increasing sequence of subgroups $\psi_{i_j}(P_{i_j}) \leq T$ whose union is $u(P)$ and such that the restriction of $u_{P,Q}(\varphi)$ to these subgroups is in $\mathcal{E}$. Therefore, by axiom (III) of saturation, we must have $u_{P,Q}(\varphi) \in \mathrm{Hom}_{\mathcal{E}}(u(P), T)$. Since the image of $u_{P,Q}(\varphi)$ is in clearly in $u(Q)$, we obtain that $u_{P,Q}(\varphi) \in \mathrm{Hom}_{\mathcal{E}}(u(P), u(Q))$. □

5.4. Definition of $(S_3, \mathcal{F}_3)$, $(S_p, \mathcal{F}_p)$ and $(S_p, \widetilde{\mathcal{F}}_p)$

In this section we will use the theory we have developed in the previous section to construct new examples of fusion systems over discrete p-toral groups by taking the limit of the fusion systems defined in Chapter 3.

To construct these new examples we will first prove that the fusion systems over the finite groups of Chapter 3 form direct systems of fusion systems. Then we will prove a general result about limits of fusion systems which fits exactly for the structure of our new examples.

Recall from the definition of the fusion systems $\mathcal{F}_{3,k}$ over $S_{3,k}$ and $\mathcal{F}_{p,k}$, $\widetilde{\mathcal{F}}_{p,k}$ over $S_{p,k}$ that we have, for $p = 3$, two automorphisms $\eta, \omega \in \mathrm{Aut}_{\mathcal{F}_{3,k}}(S_{3,k})$ and, for $p \geq 5$, two automorphisms $\phi, \psi \in \mathrm{Aut}_{\mathcal{F}_{p,k}}(S_{p,k})$. Then, by considering these automorphisms as generators for the corresponding outer automorphisms groups, we can see in Table 5.1 the isomorphism type of these groups for the fusion systems $\mathcal{F}_{3,k}$, $\mathcal{F}_{p,k}$ and $\widetilde{\mathcal{F}}_{p,k}$.

$\mathcal{F}$	$\mathrm{Out}_{\mathcal{F}}(S_{p,k})$	$\mathrm{Out}_{\mathcal{F}}(T_{p,k})$	$\mathrm{Out}_{\mathcal{F}}(V)$	prime
$\mathcal{F}_{3,k}$	$C_2 \times C_2$	$\mathrm{GL}_2(\mathbb{F}_3)$	$\mathrm{GL}_2(\mathbb{F}_3)$	$p = 3$
$\mathcal{F}_{p,k}$	$C_{\frac{p-1}{2}} \times C_{p-1}$	$A_p \rtimes C_{p-1}$	$\mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}}$	$p \geq 5$
$\widetilde{\mathcal{F}}_{p,k}$	$C_{p-1} \times C_{p-1}$	$\Sigma_p \times C_{p-1}$	$\mathrm{GL}_2(\mathbb{F}_p)$	

Table 5.1.: Isomorphism type for the outer automorphisms groups of the centric and radical subgroups of $\mathcal{F}_{3,k}$, $\mathcal{F}_{p,k}$ and $\widetilde{\mathcal{F}}_{p,k}$.

The next proposition proves that the Frobenius map from $T_{p,k}$ to $T_{p,k+1}$ induces morphisms of fusion systems.

Proposition 5.4.1. *The morphism $\rho\colon S_{p,k} \to S_{p,k+1}$ defined by $\rho(v) = v^p$ for $v \in T_{p,k}$ and $\rho(s) = s$ induces morphisms of fusion systems $(S_{3,k}, \mathcal{F}_{3,k}) \to (S_{3,k+1}, \mathcal{F}_{3.k+1})$, $(S_{p,k}, \mathcal{F}_{p,k}) \to (S_{p,k+1}, \mathcal{F}_{p.k+1})$ and $(S_{p,k}, \widetilde{\mathcal{F}}_{p.k}) \to (S_{p,k+1}, \widetilde{\mathcal{F}}_{p,k+1})$, for any $k \geq 2$.*

Proof. The proof is exactly the same for the cases of $\mathcal{F}_{p,k}$ and $\widetilde{\mathcal{F}}_{p,k}$, and completely analogous for $\mathcal{F}_{3,k}$. Therefore we will write the argument only for $\widetilde{\mathcal{F}}_{p,k}$.

Since $\rho = p\,\mathrm{Id}$, we have $\rho \in Z(\mathrm{GL}_{p-1}(\mathbb{Z}/p^k))$, so ρ is compatible with the action of s over $T_{p,k}$ and therefore, we can extend it to a morphism denoted also by $\rho\colon S_{p,k} \to S_{p,k+1}$. Note that ρ is a monomorphism so, by Remark 5.2.3, we only need to check that $\rho \circ \varphi \circ \rho^{-1} \in \mathrm{Hom}_{\widetilde{\mathcal{F}}_{p,k+1}}(\rho(P), \rho(Q))$ for any $\varphi \in \mathrm{Hom}_{\widetilde{\mathcal{F}}_{p,k}}(P, Q)$.

By Proposition 5.2.4, we only need to consider $\varphi \in \mathrm{Aut}_{\widetilde{\mathcal{F}}_{p.k}}(P)$ for $P = S_{p,k}, T_{p,k}$ and V. Moreover, if $\varphi = c_x \in \mathrm{Hom}_{\widetilde{\mathcal{F}}_{p,k}}(P, Q)$ with $x \in S_k$, then, $\rho \circ \varphi \circ \rho^{-1} = c_{\rho(x)} \in \mathrm{Hom}_{\widetilde{\mathcal{F}}_{p,k+1}}(\rho(P), \rho(Q))$ so, in the case of the full group $S_{p,k}$, we only need to deal with the outer automorphisms group.

By abuse of notation, we will write with the same symbol the morphisms for all k.

($T_{p,k}$) Let $\varphi \in \mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k}}(T_{p,k})$, then φ can be seen as a matrix in $\mathrm{GL}_{p-1}(\mathbb{Z}/p^k)$. Since $\rho = p\,\mathrm{Id}$, it commutes with φ, so $\rho \circ \varphi \circ \rho^{-1} = \varphi|_{\rho(T_{p,k})} \in \mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k+1}}(\rho(T_{p,k}))$, because φ is also in $\mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k+1}}(T_{p,k+1})$.

(V) Let $\varphi \in \mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k}}(V)$. Note that, in the base $\langle \zeta, s\rangle$, the morphism ρ is the identity. Then, $\rho \circ \varphi \circ \rho^{-1} = \varphi \in \mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k+1}}(V)$.

($S_{p,k}$) In this case we only need to check the morphisms ϕ and ψ.

- ϕ) Since $\phi(T_{p,k}) = T_{p,k}$, we have that $\phi(vs^i) = \phi(v)\phi(s^i) = \phi(v)s^{i\lambda}$. Then, $\rho(\phi(vs^i)) = \phi(v)^p s^{i\lambda}$. On the other hand, $\phi(\rho(vs^i)) = \phi(v^p s^i) = \phi(v)^p s^{i\lambda}$. So ϕ and ρ commute, hence $\rho \circ \phi \circ \rho^{-1} = \phi|_{\rho(S_{p,k})} \in \mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k+1}}(\rho(S_{p,k}))$.
- ψ) The same argument of ϕ works for ψ, since $\psi(T_{p,k}) = T_{p,k}$ and then, $\rho(\psi(vs^i)) = \rho(\psi(v)\psi(s)^i) = \rho(v^\lambda s^i) = v^{p\lambda} s^i = \psi(v^p s^i) = \psi(\rho(vs^i))$. Then, $\rho \circ \psi \circ \rho^{-1} = \psi|_{\rho(S_{p,k})} \in \mathrm{Aut}_{\widetilde{\mathcal{F}}_{p,k+1}}(\rho(S_{p,k}))$. □

With this proposition we can consider the morphisms of fusion systems given by ρ and its compositions and define some direct systems of fusion systems. Note that, since ρ is a monomorphism, in all the cases the corresponding limit fusion system can be considered as the infinite union of the fusion systems in the direct system. More precisely, let $I = \mathbb{N} \setminus \{1\}$ and consider the direct systems of fusion systems given by:

- $((S_{3,k}, \mathcal{F}_{3,k}), \alpha_{ij})$ where $\alpha_{ij} = \rho^{j-i}$ for $i \leq j$.
- $((S_{p,k}, \mathcal{F}_{p,k}), \alpha_{ij})$ where $\alpha_{ij} = \rho^{j-i}$ for $i \leq j$.
- $((S_{p,k}, \widetilde{\mathcal{F}}_{p,k}), \alpha_{ij})$ where $\alpha_{ij} = \rho^{j-i}$ for $i \leq j$.

Then, the direct limit S_p of $S_{p,k}$ fits in a split extension

$$1 \to (\mathbb{Z}/p^\infty)^{p-1} \to S_p \to \mathbb{Z}/p \to 1$$

hence it is a discrete p-toral group. As usual, we will write $T_p = (\mathbb{Z}/p^\infty)^{p-1} \leq S_p$, the maximal torus. Then, by Proposition 5.3.5, we can take the limit of the direct systems to obtain fusion systems $\mathcal{F}_3$ over S_3 and $\mathcal{F}_p$ and $\widetilde{\mathcal{F}}_p$ over S_p for $p \geq 5$.

The next proposition allows us to characterize the limit fusion system in the cases where the morphisms of the fusion systems in the direct systems are very well understood. The hypothesis in the proposition are motivated by properties satisfied by the three direct systems of fusion systems giving rise to $\mathcal{F}_3$, $\mathcal{F}_p$ and $\widetilde{\mathcal{F}}_p$.

Proposition 5.4.2. *Let $((S_i, \mathcal{F}_i), \alpha_{ij})$ be a direct system of fusion systems over a directed set I such that $S = \varinjlim S_i$ is a discrete p-toral group. Assume that, for each $i \in I$ there are subgroups of S_i, denoted as $P_1^i, \ldots, P_n^i$, and finite groups $G_1, \ldots, G_n$ such that the following conditions are verified:*

(i) $\mathcal{F}_i = \langle \mathrm{Aut}_{\mathcal{F}_i}(P_1^i), \ldots, \mathrm{Aut}_{\mathcal{F}_i}(P_n^i) \rangle$ *for all $i \in I$ and every isomorphism in $\mathcal{F}_i$ is the restriction of some of these automorphisms.*

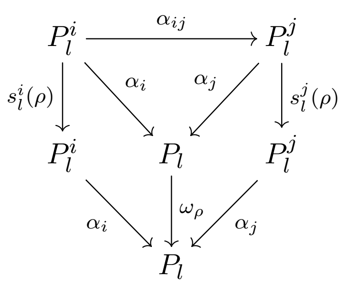
(ii) $\mathrm{Out}_{\mathcal{F}_i}(P_l^i) \cong G_l$ *for all $i \in I$. Moreover, there exist families of set-theoretic sections* $s_l^i \colon \mathrm{Out}_{\mathcal{F}_i}(P_l^i) \to \mathrm{Aut}_{\mathcal{F}_i}(P_l^i)$ *such that $\{s_l^i(\rho)\}$ is a compatible system of morphisms for all $\rho \in G_l$.*

(iii) If $P_l^i \leq P_k^i$, then $\mathrm{Out}_{\mathcal{F}_i}(P_k^i) \leq \mathrm{Out}_{\mathcal{F}_i}(P_l^i)$. *More precisely, for any* $\rho \in \mathrm{Out}_{\mathcal{F}_i}(P_k^i)$ *we have that* $s_k^i(\rho)|_{P_l^i} \in \mathrm{Aut}_{\mathcal{F}_i}(P_l^i)$ *and* $s_l^i(\left[s_k^i(\rho)|_{P_l^i}\right]) = s_k^i(\rho)|_{P_l^i}$.

(iv) $\alpha_{ij}(P_l^i) \leq P_l^j$ *for any $i \leq j$ and $P \not\leq P_l^i$ implies $\alpha_{ij}(P) \not\leq P_l^j$.*

Then, the limit fusion system $(S, \mathcal{F})$ is generated by $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(P_1), \ldots, \mathrm{Aut}_{\mathcal{F}}(P_n) \rangle$ *and* $\mathrm{Out}_{\mathcal{F}}(P_l) \cong G_l$, *where $P_l = \varinjlim P_l^i$.*

Proof. First note that there are morphisms $\omega_\rho \in \mathrm{Aut}_{\mathcal{F}}(P_l)$ for all $\rho \in G_l$. Indeed, by condition (ii), there is a compatible system of morphisms $\{s_l^i(\rho)\}$ which fit in the diagram



The morphism ω_ρ is given by the property of direct limits, and it is an automorphism because all of the $s_l^i(\rho)$ are automorphisms and we can make the symmetric argument.

Moreover, ω_ρ belongs to $\mathcal{F}$ by definition of limit fusion system. Note that, if $\rho, \rho' \in G_l$ are different, then ω_ρ and $\omega_{\rho'}$ don't differ in a conjugation by an element of S, since this would imply $\rho = \rho'$.

Now we prove that, if $K \subseteq I$ is a cofinal subset, the limit of conjugation morphism for $k \in K$ is a conjugation morphism in the limit. Let $P_i \leq S_i$ be subgroups such that $\alpha_{ij}(P_i) \leq P_j$ and write $P = \varinjlim P_i$. Consider $x_k \in P_k$ for all $k \in K$ such that $\{c_{x_k}\}$ is a compatible system of morphisms. For any $i \in I$, let $k \in K$ such that $k \geq i$ (which exists since K is cofinal) and define $\omega_i = \alpha_k \circ c_{x_k} \circ \alpha_{ik} \colon P_i \to P$. These morphisms are well defined because different choices of k will lead to the same morphism ω_i.

Then, there exists a morphism $\omega \colon P \to P$ commuting with the ω_i. Note that ω is an automorphism since we can make the symmetric argument with the conjugation by the inverse elements to obtain ω^{-1}. Then, consider all the subgroups $\alpha_k(P_k) \leq P$, which satisfy that $\alpha_k(P_k) \leq \alpha_l(P_l)$ if $k \leq l$, and therefore $C_P(\alpha_k(P_k)) \geq C_P(\alpha_l(P_l))$.

By the compatibility of the morphisms c_{x_k}, we have that $c_{\alpha_l(x_l)}|_{\alpha_k(P_k)} = c_{\alpha_k(x_k)}$ if $k \leq l$. Hence, $\alpha_l(x_l)^{-1}\alpha_k(x_k) \in C_P(\alpha_k(P_k))$. Since P is artinian, there exists m such that $C_P(\alpha_k(P_k)) = C_P(\alpha_m(P_m))$ if $k \geq m$. Then, for $[g,k] \in P$, we have $\omega([g,k]) = c_{\alpha_k(x_k)}([g,k]) = c_{\alpha_m(x_m)}([g,k])$, so ω is a conjugation by an element of P. Since all conjugations belong to any fusion system, we have $\omega \in \mathrm{Hom}_{\mathcal{F}}(P,P)$.

Now we want to prove $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(P_1), \ldots, \mathrm{Aut}_{\mathcal{F}}(P_n) \rangle$. For this, let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R,Q)$ and recall that, by definition of limit fusion system, there exist $R_i \leq S_i$ and $\varphi_i \in \mathrm{Hom}_{\mathcal{F}_i}(R_i, S_i)$ a compatible system of morphisms such that $R = \varinjlim R_i$ and the diagrams

$$\begin{array}{ccc} R_i & \longrightarrow & R \\ & & \downarrow{\scriptstyle \varphi} \\ \downarrow{\scriptstyle \varphi_i} & & Q \\ & & \downarrow{\scriptstyle i} \\ S_i & \longrightarrow & S \end{array}$$

are commutative for all $i \in I$.

By condition (i), there exists an automorphism $\psi_i \in \mathrm{Aut}_{\mathcal{F}_i}(P_l^i)$ such that $R_i, \varphi_i(R_i) \leq P_l^i$ and $\varphi_i = i \circ \psi_i|_{R_i}$. Let $\rho_i = [\psi_i] \in \mathrm{Out}_{\mathcal{F}_i}(P_l^i)$ and write $\omega_{\rho_i} = s_l^i(\rho_i)$ for short. Then, we have $\psi_i = \omega_{\rho_i} \circ c_{x_i}$ for some $x_i \in P_l^i$.

Since n is finite, there exists some $\lambda \in \{1, \ldots, n\}$ such that for all $i \in I$, there exists $j \geq i$ with $\psi_j \in \mathrm{Aut}_{\mathcal{F}_j}(P_\lambda^j)$. Therefore, the set

$$J = \{ j \in I \mid \psi_j \in \mathrm{Aut}_{\mathcal{F}_j}(P_\lambda^j) \}$$

is a cofinal subset of I. Then, fixed λ, since G_λ is a finite group, there exists $\rho \in G_\lambda$ such that for all $j \in J$ there exists $k \geq j$ with $\rho = [\psi_k]$. Therefore, the set

$$K = \{k \in J \mid [\psi_k] = \rho \in G_\lambda\}$$

is a cofinal subset of J, hence a cofinal subset of I.

Now consider the morphisms $\omega_{\rho_k}^{-1} \circ \varphi_k = c_{x_k}$ with the appropriate restrictions. Since this forms a compatible system of morphisms, we have that there exists some element $[z, j] \in P_\lambda$ such that the diagrams

$$\begin{array}{ccc} R_k & \longrightarrow & R \\ & & \downarrow{\scriptstyle c_{[z,j]}} \\ {\scriptstyle \omega_{\rho_k}^{-1}\circ\varphi_k}\downarrow & & Q \\ & & \downarrow{\scriptstyle i} \\ S_k & \longrightarrow & S \end{array}$$

are commutative for all $k \in K$. Then, we have $\varphi = (\omega_\rho \circ c_{[z,j]})|_R$ and therefore φ is a restriction of an automorphism of P_λ.

Finally, we will prove that $\mathrm{Out}_\mathcal{F}(P_l) \cong G_l$ for $l = 1, \ldots, n$. For this, we need to prove that, for every morphism $\varphi \in \mathrm{Aut}_\mathcal{F}(P_l)$, there exist $\rho \in G_l$ and $[z, j] \in P_l$ such that $\varphi = \omega_\rho \circ c_{[z,j]}$.

By the previous argument, if $\varphi \in \mathrm{Aut}_\mathcal{F}(P_l)$, there exists $\psi \in \mathrm{Aut}_\mathcal{F}(P_k)$ such that $P_l \leq P_k$, $\varphi = \psi|_{P_k}$ and $\psi = \omega_\rho \circ c_{[z,j]}$, for $\rho \in \mathrm{Out}_\mathcal{F}(P_k)$ and $[z, j] \in P_k$. If there is no P_k such that $P_l \leq P_k$, then $\psi \in \mathrm{Aut}_\mathcal{F}(P_l)$ and $[z, j] \in P_l$, so in this case we have, by construction, $\mathrm{Out}_\mathcal{F}(P_l) \cong G_l$.

If $P_l \leq P_k$ for some l, we can assume, by an induction process, that $\mathrm{Out}_\mathcal{F}(P_k) \cong G_k$. Let then $\rho \in G_k$, we have, by conditions (ii) and (iii), that $\mathrm{Out}_{\mathcal{F}_i}(P_k^i) \cong G_k$ and $\mathrm{Out}_{\mathcal{F}_i}(P_k^i) \leq \mathrm{Out}_{\mathcal{F}_i}(P_l^i)$, so $\rho \in \mathrm{Out}_{\mathcal{F}_i}(P_l^i)$. Taking the limit of $s_l^i(\rho)$ we have a morphism in $\mathrm{Aut}_\mathcal{F}(P_l)$ such that projects to ρ in the outer automorphism group, hence $\mathrm{Out}_\mathcal{F}(P_k) \leq \mathrm{Out}_\mathcal{F}(P_l)$.

Then, since $\rho \in \mathrm{Out}_\mathcal{F}(P_k)$, we have also $\rho \in \mathrm{Out}_\mathcal{F}(P_l)$ and this implies $c_{[z,j]} \in \mathrm{Aut}_\mathcal{F}(P_l)$. Therefore, to prove $\mathrm{Out}_\mathcal{F}(P_l) \cong G_l$, we are left to prove that the class $\left[c_{[z,j]}\right] \in \mathrm{Out}_\mathcal{F}(P_l)$ is a limit of elements in G_l, because then $c_{[z,j]} = \omega_{\rho'} \circ c_{[z',j']}$ for $\rho' \in G_l$ and $[z', j'] \in P_l$, so we will obtain the desired decomposition

$$\varphi = (\omega_\rho \circ \omega_{\rho'}) \circ c_{[z',j']} = (\omega_{\rho\rho'} \circ c_{[z'',j'']}) \circ c_{[z',j']} = \omega_{\rho\rho'} \circ c_{[z'',j''][z',j']}$$

If $[z, j] \in P_l$ it is trivial, so assume $[z, j] \notin P_l$. Then, we have $z \notin P_l^i$ for all $i \in I$, by condition (iv). Now, by the same hypothesis, and using that $c_{[z,j]}(P_l) = P_l$, we have $c_{\alpha_{ji}(z)} \in \mathrm{Aut}_{\mathcal{F}_i}(P_l^i)$ for $i \geq j$. Then, the morphism $c_{[z,j]}$ is the limit of $c_{\alpha_{ji}(z)}$ and the class $\left[c_{[z,j]}\right] \in \mathrm{Out}_{\mathcal{F}}(P_l)$ is the limit of the outer automorphisms $\left[c_{\alpha_{ji}(z)}\right] \in G_l$. Note that, if not all of them are compatible, we can take a cofinal subset in which they are compatible, since the outer automorphism groups are finite. □

The previous proposition proves, in particular, that there are automorphisms $\eta, \omega \in \mathrm{Aut}_{\mathcal{F}_3}(S_3)$ and $\phi, \psi \in \mathrm{Aut}_{\mathcal{F}_p}(S_p)$ for $p \geq 5$. Moreover, by considering this automorphisms as generators for the corresponding outer automorphism groups, we can see in Table 5.2 that the isomorphism type of these groups are completely similar to those in Table 5.1.

$\mathcal{F}$	$\mathrm{Out}_{\mathcal{F}}(S_p)$	$\mathrm{Out}_{\mathcal{F}}(T_p)$	$\mathrm{Out}_{\mathcal{F}}(V)$	prime
$\mathcal{F}_3$	$C_2 \times C_2$	$\mathrm{GL}_2(\mathbb{F}_3)$	$\mathrm{GL}_2(\mathbb{F}_3)$	$p = 3$
$\mathcal{F}_p$	$C_{\frac{p-1}{2}} \times C_{p-1}$	$A_p \rtimes C_{p-1}$	$\mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}}$	$p \geq 5$
$\widetilde{\mathcal{F}}_p$	$C_{p-1} \times C_{p-1}$	$\Sigma_p \times C_{p-1}$	$\mathrm{GL}_2(\mathbb{F}_p)$	

Table 5.2.: Isomorphism type for the outer automorphisms groups of the centric and radical subgroups of $\mathcal{F}_3$, $\mathcal{F}_p$ and $\widetilde{\mathcal{F}}_p$.

In the next chapter we will prove that these three fusion systems are saturated, so they are indeed the direct limit of the direct systems from a categorical point of view, by Proposition 5.3.8.

CHAPTER 6

Saturation of fusion systems over discrete p-toral groups

In the previous chapter we have defined three new fusion systems over discrete p-toral groups denoted by $(S_3, \mathcal{F}_3)$, $(S_p, \mathcal{F}_p)$ and $(S_p, \widetilde{\mathcal{F}}_p)$. The aim of this chapter is to prove that all of this new fusion systems are saturated, giving rise then to p-local compact groups by Theorem 4.2.4.

To prove the saturation of the given fusion systems, in the first section we will prove a saturation criterion by generalizing a known result for fusion systems over finite p-groups proved by Levi-Oliver. Then, in the second section we will study some properties about the p-groups S_3 and S_p and about the fusion systems, and we will prove the saturation using the main theorem of Section 6.1.

6.1. A saturation criterion

In this section we will show a criterion to prove that a given fusion system is saturated. It is a generalization of [28, Proposition 1.1] to the case of fusion systems over discrete p-toral groups. This result shows that it is enough to prove the saturation for some centralizer fusion subsystems if the elements in the fusion system are conjugated in a way that behaves well with the centralizers of the elements.

To prove the criterion we need first two lemmas dealing with the existence of fixed points of some action over a discrete p-toral group. These results are clear in the case of finite groups, but we need also to apply them to discrete p-toral group with an infinite number of elements.

Lemma 6.1.1. *Let π be a finite p-group, G a locally finite p-group and $\rho\colon \pi \to \mathrm{Aut}(G)$ an action of π on G. Then, G has a fixed point of order p.*

Proof. Since π is finite, we can write $\pi = \{g_1, \ldots, g_n\}$. Then, choose an element $x \in G$ and consider the subgroup $H = \langle \rho(g_1)(x), \ldots, \rho(g_n)(x) \rangle$. Since H is finitely generated and G is locally finite, we have that H is finite. Observe that, by construction, the action of π restricts to the subgroup H:

$$\begin{aligned}\rho(g_j)[\rho(g_{k_1})(x)\cdots\rho(g_{k_l})(x)] &= \rho(g_j)[\rho(g_{k_1})(x)]\cdots\rho(g_j)[\rho(g_{k_l})(x)]\\ &= \rho(g_jg_{k_1})(x)\cdots\rho(g_jg_{k_l})(x)\\ &= \rho(g_{m_1})(x)\cdots\rho(g_{m_l})(x) \in H\end{aligned}$$

Then, the action of π on H has fixed points, since we know that a finite p-group acting on any other finite p-group always have fixed points. Finally, we have that the fixed points form a subgroup of H, so, by Cauchy's theorem, there exists an element of order p in the subgroup of fixed points. □

Lemma 6.1.2. *Let S be a discrete p-toral group and $P \leq S$ any subgroup. Then, there exists an element $x \in Z(P)$ of order p which is fixed by all $\varphi \in \mathrm{Aut}_S(P)$.*

Proof. Set $\pi = \mathrm{Out}_S(P)$ and $\rho\colon \pi \to \mathrm{Aut}(Z(P))$ the action given by choosing a representative and restricting to $Z(P)$, which is well defined because $Z(P)$ is characteristic in P and the inner automorphisms of P restrict to the identity in $Z(P)$.

Since $Z(P) \leq P \leq S$ is a discrete p-toral group, $Z(P)$ is locally finite. By Lemma 6.1.1, there exists an element $x \in Z(P)$ of order p fixed by all morphisms in $\mathrm{Out}_S(P)$, because $\mathrm{Out}_S(P)$ is finite. Finally, since every $\psi \in \mathrm{Inn}(P)$ restricts to the identity in $Z(P)$, we have that the element x is fixed by all $\varphi \in \mathrm{Aut}_S(P)$. □

We are now ready to prove the criterion for saturation of fusion systems over discrete p-toral groups.

Theorem 6.1.3. *Let $(S, \mathcal{F})$ be a fusion system over a discrete p-toral group. Then, $\mathcal{F}$ is saturated if and only if it satisfies axiom (III) of saturated fusion systems and there exists a set $\mathfrak{X}$ of elements of order p in S such that the following conditions hold:*

(*i*) *Each $x \in S$ of order p is $\mathcal{F}$-conjugate to some $y \in \mathfrak{X}$.*

(*ii*) *If x, y are $\mathcal{F}$-conjugate and $y \in \mathfrak{X}$, then there is some morphism*

$$\rho \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$$

such that $\rho(x) = y$.

(*iii*) *For each $x \in \mathfrak{X}$, $C_{\mathcal{F}}(x)$ is a saturated fusion system over $C_S(x)$.*

Proof. First, if $\mathcal{F}$ is saturated, the set $\mathfrak{X}$ of all elements $x \in S$ of order p such that $\langle x \rangle$ is fully centralized in $\mathcal{F}$ satisfies the conditions in the statement. Suppose then that $\mathcal{F}$ satisfies axiom (III) and that such a set $\mathfrak{X}$ exists. In order to prove saturation we need to prove that $\mathcal{F}$ also satisfies axioms (I) and (II) of Definition 4.1.9.

We will prove first axiom (II) of saturation, since we will need to use it when proving axiom (I).

Axiom (II): Every subgroup of S which is fully centralized in $\mathcal{F}$ it is also receptive. That is, let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ be such that $P' = \varphi(P)$ is fully centralized in $\mathcal{F}$. Then, there exists $\widetilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\varphi = \widetilde{\varphi}|_P$, and where

$$N_\varphi = \{g \in N_S(P) \mid \varphi \circ c_g \circ \varphi^{-1} \in \mathrm{Aut}_S(P')\}$$

Choose $x' \in Z(P')$ of order p and which is fixed under the action of $\mathrm{Aut}_S(P')$, which exists by Lemma 6.1.2. Write $x = \varphi^{-1}(x') \in Z(P)$ and note that, for all $g \in N_\varphi$, the morphism $\varphi \circ c_g \circ \varphi^{-1} \in \mathrm{Aut}_S(P')$ fixes x', thus $c_g(x) = x$. Hence,

(A) $x \in Z(N_f)$, which implies $N_f \leq C_S(x)$.

Let $y \in \mathfrak{X}$ be $\mathcal{F}$-conjugate to x and x', whose existence is guaranteed by property (i) of the set $\mathfrak{X}$. Also, by property (ii) of $\mathfrak{X}$, there exist $\rho \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$ and $\rho' \in \mathrm{Hom}_{\mathcal{F}}(C_S(x'), C_S(y))$ such that $\rho(x) = y = \rho'(x')$. Set also $Q = \rho(P)$ and $Q' = \rho'(P')$. Since P is fully $\mathcal{F}$-centralized and $C_S(P) \leq C_S(x)$, it follows that

(B) $\rho'(C_{C_S(x')}(P')) = \rho'(C_S(P')) = C_S(Q') = C_{C_S(y)}(Q')$.

Set $\omega = \rho' \circ f \circ \rho^{-1} \in \mathrm{Iso}_{\mathcal{F}}(Q, Q')$. By construction, $\omega(y) = y$, and thus $\omega \in \mathrm{Iso}_{C_{\mathcal{F}}(y)}(Q, Q')$. Since P' is fully centralized in $\mathcal{F}$, (B) implies that Q' is fully centralized in $C_{\mathcal{F}}(y)$. Then, we can apply axiom (II) of saturated fusion systems on ω as a morphism in $C_{\mathcal{F}}(y)$, which is a saturated fusion system by property (iii) of $\mathfrak{X}$. We obtain that ω extends to some $\widetilde{\omega} \in \mathrm{Hom}_{C_{\mathcal{F}}(y)}(N_\omega, C_S(y))$, where

$$N_\omega = \{g \in N_{C_S(y)}(Q) \mid \omega \circ c_g \circ \omega^{-1} \in \mathrm{Aut}_{C_S(y)}(Q')\}$$

Note that, by (A), for all $g \in N_f \leq C_S(x)$ we have $\rho(g) \in N_\omega$, so we can consider the morphism $c_{\widetilde{\omega}(\rho(g))}$, which satisfies

$$\begin{aligned} c_{\widetilde{\omega}(\rho(g))} &= \omega \circ c_{\rho(g)} \circ \omega^{-1} \\ &= (\omega \circ \rho) \circ c_g \circ (\omega \circ \rho)^{-1} \\ &= (\rho' \circ \varphi) \circ c_g \circ (\rho' \circ \varphi)^{-1} \\ &= c_{\rho'(h)} \in \mathrm{Aut}_{C_S(y)}(Q') \end{aligned}$$

for some $h \in N_S(P')$ such that $\varphi \circ c_g \circ \varphi^{-1} = c_h$. In particular, $\rho(N_\varphi) \leq N_\omega$, and, by (B), we obtain $\widetilde{\omega}(\rho(N_\varphi)) \leq \rho'(N_{C_S(x)}(P'))$.

We can then define

$$\widetilde{\varphi} = (\rho')^{-1} \circ (\widetilde{\omega} \circ \rho)|_{N_\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$$

which clearly satisfies axiom (II) above.

Axiom (I): For all $P \leq S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

First, we need to define two sets and to prove two auxiliary results. Consider the sets U and U_0 defined as

$$\begin{aligned} U = \{(P, x) \mid\ & P \leq S \text{ is finite, } x \in Z(P)^\Gamma \text{ has order } p, \text{ and } \Gamma \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P)) \\ & \text{is such that } \mathrm{Aut}_S(P) \leq \Gamma\} \\ U_0 = \{(P, x) \in\ & U \mid x \in \mathfrak{X}\} \end{aligned}$$

Note that for each nontrivial finite subgroup $P \leq S$, there is some $x \in P$ such that $(P, x) \in U$, since every action of a finite p-group on $Z(P)$ has nontrivial fixed set. Then, we have the following:

(C) If $(P, x) \in U_0$ and P is fully centralized in $C_{\mathcal{F}}(x)$, then P is fully centralized in $\mathcal{F}$.

Assume otherwise and let $P' \in P^{\mathcal{F}}$ be fully centralized in $\mathcal{F}$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, P')$. Write also $x' = \varphi(x) \in Z(P')$. By property (ii) of the set $\mathfrak{X}$, there is $\rho \in \mathrm{Hom}_{\mathcal{F}}(C_S(x'), C_S(x))$ such that $\rho(x') = x$, since we are assuming $x \in \mathfrak{X}$. Note that $P' \leq C_S(x')$ and set then $P'' = \rho(P')$. In particular, $\rho \circ \varphi \in \mathrm{Iso}_{C_{\mathcal{F}}(x)}(P, P'')$ and therefore P'' is $C_{\mathcal{F}}(x)$-conjugate to P. Also, since $\langle x' \rangle \leq P'$, we have $C_S(P') \leq C_S(x')$ and then ρ sends $C_S(P')$ injectively into $C_S(P'')$, obtaining that

$$|C_S(P)| < |C_S(P')| \leq |C_S(P'')|$$

But we have the equalities $C_S(P) = C_{C_S(x)}(P)$ and $C_S(P'') = C_{C_S(x)}(P'')$, which contradict the assumption that P is fully centralized in $C_{\mathcal{F}}(x)$. This proves (C).

Note that, by definition, $N_S(P) \leq C_S(x)$ for all $(P, x) \in U$, and hence

$$\mathrm{Aut}_{C_S(x)}(P) = \mathrm{Aut}_S(P)$$

Also, if $(P, x) \in U$ and $\Gamma \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ is as in the definition of U, then $\Gamma \leq \mathrm{Aut}_{C_{\mathcal{F}}(x)}(P)$. In particular, we have

(D) For all $(P, x) \in U$,

$$\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P)) \Longleftrightarrow \mathrm{Aut}_{C_S(x)}(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{C_{\mathcal{F}}(x)}(P))$$

We are ready to check that $\mathcal{F}$ satisfies axiom (I) of saturated fusion systems.

Fix $P \leq S$ a finite subgroup fully normalized in $\mathcal{F}$. By definition, $|N_S(P)| \geq |N_S(P')|$ for all $P' \in P^{\mathcal{F}}$. Choose $x \in Z(P)$ such that $(P, x) \in U$ and let $\Gamma \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ be such that $\mathrm{Aut}_S(P) \leq \Gamma$ and such that $x \in Z(P)^{\Gamma}$. Then, by properties (i) and (ii) of the set $\mathfrak{X}$, there is some $y \in \mathfrak{X}$ and $\rho \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$ such that $\rho(x) = y$. Set $P' = \rho(P)$ and $\Gamma' = \rho \circ \Gamma \circ \rho^{-1} \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P'))$.

Note that, since P is assumed to be fully normalized in $\mathcal{F}$, it follows that $\rho(N_S(P)) = N_S(P')$. Therefore, $\mathrm{Aut}_S(P') \leq \Gamma'$ and $y \in Z(P')^{\Gamma'}$, hence $(P', y) \in U_0$. Since $N_S(P') \leq C_S(y)$, the maximality of $|N_S(P')| = |N_{C_S(y)}(P')|$ implies that P' is fully normalized in $C_{\mathcal{F}}(y)$.

Now, by property (iii) of $\mathfrak{X}$, the fusion system $C_{\mathcal{F}}(y)$ is saturated. Then, since P' is fully normalized in $C_{\mathcal{F}}(y)$, we have that P' is fully centralized in $C_{\mathcal{F}}(y)$ and $\mathrm{Aut}_{C_S(y)}(P') \in \mathrm{Syl}_p(\mathrm{Aut}_{C_{\mathcal{F}}(y)}(P'))$. Therefore, by (C) and (D), P' is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P') \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P'))$.

Since P' is fully centralized in $\mathcal{F}$, it is also receptive by axiom (II). Then, by [13, Lemma 2.2], since P is $\mathcal{F}$-conjugated to P' and P is fully normalized in $\mathcal{F}$, we have that also P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Finally, it is shown in [12] that if axiom (I) holds for all finite fully normalized subgroups, then axiom (I) holds for all fully normalized subgroups. □

6.2. Proof of saturation for $(S_3, \mathcal{F}_3)$, $(S_p, \mathcal{F}_p)$ and $(S_p, \widetilde{\mathcal{F}}_p)$

As an application of Theorem 6.1.3, we can prove that the fusion systems $(S_3, \mathcal{F}_3)$ and $(S_p, \mathcal{F}_p)$, $(S_p, \widetilde{\mathcal{F}}_p)$ for $p \geq 5$ are saturated. First, we need to prove some properties satisfied by the elements of S_3 and S_p.

By Remark 3.2.3, and since S_3 and S_p are direct limits of $S_{3,k}$ and $S_{p,k}$ respectively, the next lemma can be applied both to S_3 and to S_p at once.

Lemma 6.2.1. *Let $vs^i \in S_p$ an element not in the maximal torus, that is, such that $v \in T_p$ and $s^i \neq 1$. Then, the following holds:*

(a) *The element vs^i has order p.*

(b) *There exists an element $w \in T_p$ such that $\langle vs^i \rangle = \langle ws \rangle$.*

Proof. For any $v \in T_p$ we know that there is k large enough such that we can write $v = v_1^{i_1} \cdots v_{p-1}^{i_{p-1}}$. Then,

$$(vs^i)^p = vc_{s^i}(v)c_{s^{2i}}(v) \cdots c_{s^{(p-1)i}}(v)s^p$$

Note that $s^p = 1$ and, since T_p is commutative, we can write

$$\prod_{j=0}^{p-1} c_{s^j}(v) = \prod_{j=0}^{p-1}\prod_{l=1}^{p-1} c_{s^j}(v_l^{i_l}) = \prod_{l=1}^{p-1}\prod_{j=0}^{p-1}(c_{s^j}(v_l))^{i_l}$$

Finally, note that $c_{s^j}(v_l)$ verifies

$$c_{s^j}(v_l) = \begin{cases} v_{(l+j) \mod p} & , \text{ if } l \neq p-j \\ v_1^{-1} \cdots v_{p-1}^{-1} & , \text{ if } l = p-j \end{cases}$$

Then, as the index j ranges from 0 to $p-1$, we obtain

$$\prod_{j=0}^{p-1} c_{s^j}(v_l) = v_1 v_2 \cdots v_{p-1} v_1^{-1} v_2^{-1} \cdots v_{p-1}^{-1} = 1$$

Hence, $(vs^i)^p = 1$ and, since $vs^i \neq 1$ and p is prime, we obtain that vs^i has order p.

Now point (b) follows directly from point (a). Indeed, if $j \equiv i^{-1} \mod p$, we can take $ws = (vs^i)^j$ and, since both ws and vs^i have order p, we obtain $\langle vs^i \rangle = \langle ws \rangle$. □

Now we prove a result about the conjugacy classes of elements not in the maximal torus that we will need in the proof of saturation for $(S_3, \mathcal{F}_3)$, $(S_p, \mathcal{F}_p)$ and $(S_p, \widetilde{\mathcal{F}}_p)$.

Lemma 6.2.2. *The elements not contained in T_p belong to a single $\mathcal{F}$-conjugacy class for $\mathcal{F}$ the fusion system $\mathcal{F}_3$, $\mathcal{F}_p$ or $\widetilde{\mathcal{F}}_p$. Moreover, the elements of this class are $\mathcal{F}$-conjugated to the maximal torus.*

Proof. Let $v \in T_p$ and vs^i, with $i \neq 0$, an element not contained in T_p. By Lemma 6.2.1 (b) we have that $\langle vs^i \rangle = \langle ws \rangle$ for some $w \in T_p$. Since $T_p = (\mathbb{Z}/p^\infty)^{p-1}$, we can write $w = w_1^{pi_1} \cdots w_{p-1}^{pi_{p-1}}$, for some $w_1, \ldots, w_{p-1}$ in $\mathbb{Z}/p^k \subset \mathbb{Z}/p^\infty$, for k big enough. Then, by Lemma 3.2.5 (c), the elements $w_1^{pi_1} \cdots w_{p-1}^{pi_{p-1}} s$ and s are conjugated in $S_{p,k}$, so they are also conjugated in S_p. Therefore, we have vs^i conjugated to s^j for some $j \neq 0$. Finally, as $\mathrm{Out}_{\mathcal{F}_3}(V) = \mathrm{GL}_2(\mathbb{F}_3)$, $\mathrm{Out}_{\mathcal{F}_p}(V) = \mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}}$ for $p \geq 5$ and $\mathrm{Out}_{\widetilde{\mathcal{F}}_p}(V) = \mathrm{GL}_2(\mathbb{F}_p)$ for $p \geq 5$, we have that s is conjugated to s^j for all $j \neq 0$.

Then, every element not in the maximal torus is conjugated to s, and using again the description of the outer automorphisms groups, we obtain that s is conjugated $\zeta \in T_p$. □

Finally, we are ready to prove the saturation of the new fusion systems using the main theorem of the previous section.

Theorem 6.2.3. *The fusion systems $(S_3, \mathcal{F}_3)$ and $(S_p, \mathcal{F}_p)$, $(S_p, \widetilde{\mathcal{F}}_p)$ for $p \geq 5$ are saturated.*

Proof. Fix and odd prime p and let $\mathcal{F}$ be $\mathcal{F}_3$, $\mathcal{F}_p$ or $\widetilde{\mathcal{F}}_p$ for $p \geq 5$ as in the statement. We want to apply Theorem 6.1.3 to $\mathcal{F}$.

Note that, by construction, $\mathcal{F}$ satisfies axiom (III) of saturated fusion systems, so we have to prove the existence of a set $\mathfrak{X}$ verifying the hypothesis of Theorem 6.1.3. Let

$$\mathfrak{X} = \{v \in T_p \leq S_p \mid v \text{ has order } p\}$$

Then, $\mathfrak{X}$ verifies condition (i) since $\zeta \in \mathfrak{X}$ and, by Lemma 6.2.2, every element of order p not in the maximal torus is $\mathcal{F}$-conjugated to $\zeta \in T_p$.

We proceed now checking condition (ii). We will consider three different cases: elements in the center, elements in the maximal torus but not in the center, and elements not in the maximal torus. Observe that, for $v \in T_p$, the centralizer verifies $C_{S_p}(v) = T_p$ if $v \notin \langle\zeta\rangle$ and $C_{S_p}(v) = S_p$ if $v \in \langle\zeta\rangle$.

Let first $v \in T_p$ be and element of order p not in $\langle\zeta\rangle$. If v is $\mathcal{F}$-conjugated to other element $v' \in T_p$, then, by construction, there is an automorphism $\rho \in \mathrm{Aut}_{\mathcal{F}}(T_p)$ such that $\rho(v) = v'$.

If $v \in \langle\zeta\rangle$, then $v = \zeta^\lambda$, and ζ^λ is conjugated to ζ^μ for all $\lambda, \mu \neq 0$ by an $\mathcal{F}$-automorphism of the whole group S_p.

Finally, let vs^i be an element of order p not in the maximal torus. Then, by Lemma 6.2.2 and Lemma 3.2.5 (c), we know that vs^i is S_p-conjugated to s^j for some $j \neq 0$. Therefore, it is enough to prove that there is some $\rho \in \mathrm{Hom}_{\mathcal{F}}(C_{S_p}(s^j), C_{S_p}(\zeta^\lambda))$ such that $\rho(s^j) = \zeta^\lambda$ for some $\lambda \neq 0$. Recall that $C_{S_p}(s^j) = V$ and $C_{S_p}(\zeta^\lambda) = S_p$. By construction, there is an automorphism $\rho \in \mathrm{Aut}_{\mathcal{F}}(V)$ sending s^j to ζ^λ. Then, composing ρ with the inclusion in S_p, we obtain

$$i \circ \rho \in \mathrm{Hom}_{\mathcal{F}}(C_{S_p}(s^j), C_{S_p}(\zeta^\lambda))$$

and thus $\mathfrak{X}$ satisfies condition (ii).

We are left to check condition (iii), that is, for each $v \in \mathfrak{X}$, the fusion system $C_{\mathcal{F}}(v)$ is saturated. By definition, $C_{\mathcal{F}}(v) \subseteq \mathcal{F}$ is the fusion subsystem over $C_{S_p}(v)$ whose morphisms are those morphisms in $\mathcal{F}$ which fix the element v. A careful inspection of the generating morphisms of $\mathcal{F}$ for $\mathcal{F}$ being $\mathcal{F}_3$, $\mathcal{F}_p$ or $\widetilde{\mathcal{F}}_p$ shows that $C_{\mathcal{F}}(v) = \mathcal{F}_{C_{S_p}(v)}(T_p \rtimes L)$, where $L \leq W$ is the subgroup fixing the element v in W, with W being $\mathrm{GL}_2(\mathbb{F}_3)$, A_p or Σ_p, respectively. In either case, $C_{\mathcal{F}}(v)$ is saturated by [12, Theorem 8.7]. $\square$

Once we have proved the saturation for the fusion systems $(S_3, \mathcal{F}_3)$, $(S_p, \mathcal{F}_p)$, $(S_p, \widetilde{\mathcal{F}}_p)$ for $p \geq 5$, we have, by Theorem 4.2.4, three centric linking systems $\mathcal{L}_3$, $\mathcal{L}_p$ and $\widetilde{\mathcal{L}}_p$ associated to each fusion system respectively. Moreover, these three centric linking systems are unique up to isomorphism, so we have obtained three new examples of p-local compact groups $(S_3, \mathcal{F}_3, \mathcal{L}_3)$, $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ and $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$.

CHAPTER 7

On the exoticness of $(S_3, \mathcal{F}_3, \mathcal{L}_3)$, $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ and $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$

In the finite case, the concept of exotic p-local finite group is clearly defined as those not realized by any finite group. In the compact case, however, we have seen in Proposition 4.2.6 that there always exists a group realizing any p-local compact group. For this reason, we have to restrict ourselves to certain classes of groups in order to study whether certain p-local compact groups can be realized by groups in theses classes or not. From the point of view of homotopy theory, the most developed frameworks are the compact Lie groups and the p-compact groups, so in this chapter we will prove that the p-local compact groups $(S_3, \mathcal{F}_3, \mathcal{L}_3)$, $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ or $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$ cannot be realized by any compact Lie group nor by any p-compact group.

This result shows that the class of p-local compact groups is strictly larger than the class formed by compact Lie groups and p-compact groups.

We will denote the new p-local compact groups obtained at the end of the previous chapter by $\mathcal{G}_3 = (S_3, \mathcal{F}_3, \mathcal{L}_3)$, $\mathcal{G}_p = (S_p, \mathcal{F}_p, \mathcal{L}_p)$ and $\widetilde{\mathcal{G}}_p = (S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$, understanding that $p \geq 5$ without explicit mention.

In the first section of this chapter we will prove that there is no p-compact group realizing any of the p-local compact groups $\mathcal{G}_3$, $\mathcal{G}_p$ or $\widetilde{\mathcal{G}}_p$. Note that this also eliminates the chance for these p-local compact groups to be realized by compact Lie groups whose group of components is a p-group, since this kind of compact Lie groups give rise to p-compact groups, by Remark 4.4.2.

In the second section we will eliminate also the possibility to realize the p-local compact groups $\mathcal{G}_3$, $\mathcal{G}_p$ or $\widetilde{\mathcal{G}}_p$ by any compact Lie group. The proof will be based in proving that, if some of them is realized by a compact Lie group, then it is also realized by a compact connected Lie group, which in turn gives rise to a p-compact group, and therefore we can reduce to the results of the first section.

7.1. Exoticness as p-compact groups

To prove that there is no p-compact group realizing the p-local compact groups $\mathcal{G}_3$, $\mathcal{G}_p$ or $\widetilde{\mathcal{G}}_p$ we need to prove first a general result comparing the centralizers of p-compact groups and the ones of the p-local compact groups they generate. The next theorem states that if a p-local compact group $(S, \mathcal{F}, \mathcal{L})$ is realized by a p-compact group X, then the centralizer of a fully centralized subgroup $P \leq S$ as a p-local compact group coincides with the centralizer of the same subgroup P as a p-compact group.

Theorem 7.1.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group realized by a p-compact group (X, BX, e). That is, we have $f\colon S \to X$ a Sylow subgroup of X and $(S, \mathcal{F}, \mathcal{L}) \cong (S, \mathcal{F}_{S,f}(X), \mathcal{L}^c_{S,f}(X))$. Then, for any subgroup $P \leq S$ fully centralized in $\mathcal{F}$, the group $C_S(P)$ is a Sylow subgroup of $C_X(P)$ and*

$$(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P)) \cong (C_S(P), \mathcal{F}_{C_S(P),g}(C_X(P)), \mathcal{L}^c_{C_S(P),g}(C_X(P)))$$

Proof. Let $(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P))$ be the p-local compact group defined as the centralizer of P in $(S, \mathcal{F}, \mathcal{L})$. This p-local compact group exists by Remark 4.2.5, since P is fully centralized in $\mathcal{F}$.

By Proposition 4.4.11, the centralizer $C_X(P)$ is again a p-compact group and, by Proposition 4.4.14, since P is fully centralized in $\mathcal{F} \cong \mathcal{F}_{S,f}(X)$, we have that $g\colon C_S(P) \to C_X(P)$ is a Sylow p-subgroup, where Bg is, by construction, the adjoint map of $Bf|_{B(PC_S(P))} \circ B\mu$, with $\mu\colon P \times C_S(P) \to PC_S(P)$ being the multiplication morphism.

Hence, by Theorem 4.4.15, the p-compact group $C_X(P)$ gives rise to the p-local compact group $(C_S(P), \mathcal{F}_{C_S(P),g}(C_X(P)), \mathcal{L}^c_{C_S(P),g}(C_X(P)))$.

Write $\mathcal{F}' = \mathcal{F}_{C_S(P),g}(C_X(P))$ for short. We will prove that $\mathcal{F}' \cong C_{\mathcal{F}}(P)$ and therefore, by Theorem 4.2.4, we will obtain the isomorphism of the statement.

Let $Q, Q' \leq C_S(P)$ be two subgroups and $\varphi\colon Q \to Q'$ a group homomorphism. Then, we must prove that φ belongs to $\mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q, Q')$ if and only if it belongs to $\mathrm{Hom}_{\mathcal{F}'}(Q, Q')$.

From Definition 4.1.18, we know that a group morphism $\varphi\colon Q \to Q'$ belongs to $\mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q, Q')$ if and only if there exists $\widetilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, PQ')$ such that $\widetilde{\varphi}|_Q = \varphi$ and $\widetilde{\varphi}|_P = \mathrm{Id}_P$. Since $\mathcal{F}$ is the saturated fusion system generated by a p-compact group X over a Sylow $f\colon S \to X$, we have, by Definition 4.4.13, that such a morphism $\widetilde{\varphi}$ is in $\mathrm{Hom}_{\mathcal{F}}(PQ, PQ')$ if and only if the following diagram is homotopy commutative

$$\begin{array}{ccc} BPQ & \xrightarrow{\;B\widetilde{\varphi}\;} & BPQ' \\ & {\scriptstyle Bf|_{BPQ}} \searrow \quad \swarrow {\scriptstyle Bf|_{BPQ'}} & \\ & BX & \end{array} \qquad (7.1)$$

On the other side, a morphism $\varphi\colon Q \to Q'$ belongs to $\mathrm{Hom}_{\mathcal{F}'}(Q,Q')$ if and only if $Bg|_{BQ} \simeq Bg|_{BQ'} \circ B\varphi$. And, by construction of Bg, if we consider adjoint maps this is equivalent to verify that the compositions $Bf|_{BPQ} \circ B\mu$ and $Bf_{BPQ'} \circ B\mu' \circ \mathrm{Id} \times B\varphi$ are homotopy equivalent, where $\mu\colon P \times Q \to PQ$ and $\mu'\colon P \times Q' \to PQ'$ are the corresponding multiplications. That is, $\varphi \in \mathrm{Hom}_{\mathcal{F}'}(Q,Q')$ if and only if the following diagram is homotopy commutative

$$\begin{array}{ccc} BP \times BQ & \xrightarrow{\mathrm{Id} \times B\varphi} & BP \times BQ' \\ \downarrow{\scriptstyle B\mu} & & \downarrow{\scriptstyle B\mu'} \\ BPQ & & BPQ' \\ & \searrow^{Bf|_{BPQ}} \quad {}^{Bf|_{BPQ'}}\swarrow & \\ & BX & \end{array} \tag{7.2}$$

Note that, if a group homomorphism $\varphi\colon Q \to Q'$ satisfies $\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q,Q')$, there exists, by construction of $C_{\mathcal{F}}(P)$, a morphism $\widetilde{\varphi}$ such that the following diagram is homotopy commutative

$$\begin{array}{ccc} BP \times BQ & \xrightarrow{\mathrm{Id} \times B\varphi} & BP \times BQ' \\ \downarrow{\scriptstyle B\mu} & & \downarrow{\scriptstyle B\mu'} \\ BPQ & \xrightarrow[B\widetilde{\varphi}]{} & BPQ' \end{array} \tag{7.3}$$

Then, for any $\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q,Q')$ we have that Diagrams 7.1 and 7.3 are homotopy commutative, so Diagram 7.2 is also homotopy commutative, and therefore $\varphi \in \mathrm{Hom}_{\mathcal{F}'}(Q,Q')$.

Conversely, note that if $\varphi \in \mathrm{Hom}_{\mathcal{F}'}(Q,Q')$ there exists also a morphism $\widetilde{\varphi}$ making Diagram 7.3 homotopy commutative. Indeed, we have that $P \cap Q \leq Z(P)$ and the homomorphism $Bg|_{B(P\cap Q)}$ is central so, by [21, Lemma 6.5] and [12, Lemma 1.10], we obtain $\varphi|_{P\cap Q} = \mathrm{Id}|_{P\cap Q}$, since $P \cap Q$ is abelian. Therefore, we can define the extension $\widetilde{\varphi}\colon P \times Q \to PQ$ as $\widetilde{\varphi}(pq) = p\varphi(q)$.

Now, to finish the proof, we must prove that if Diagrams 7.2 and 7.3 are homotopy commutative, then Diagram 7.1 is also homotopy commutative.

For this, consider K to be the kernel of μ. Then, the map from BK to BX, which is the composition $Bf|_{BPQ} \circ B\mu|_{BK}$, is constant, so it is a central homomorphism in BX by [30, Theorem 5.4], hence

$$\mathrm{Map}(BK, BX)_{Bf|_{BPQ} \circ B\mu|_{BK}} \simeq BX$$

Now, by [18, Proposition 3.5], known as Zabrodsky's Lemma, we get that $B\mu$ induces an bijection

$$\pi_0(\mathrm{Map}(BPQ, BX)) \cong \pi_0(\mathrm{Map}(BP \times BQ, BX)_{[BK]})$$

where $\mathrm{Map}(BP \times BQ, BX)_{[BK]}$ is the space of maps $BP \times BQ \to BX$ which are homotopic to constant maps when restricted to BK.

The class $[Bf|_{BPQ}] \in \pi_0(\mathrm{Map}(BPQ, BX))$ corresponds to the class $[Bf|_{BPQ} \circ B\mu] \in \pi_0(\mathrm{Map}(BP \times BQ, BX))$, and the class $[Bf|_{BPQ'} \circ B\widetilde{\varphi}]$ corresponds to $[Bf|_{BPQ'} \circ B\widetilde{\varphi} \circ B\mu]$. Then, using Diagrams 7.2 and 7.3, we have

$$\begin{aligned} [Bf|_{BPQ}] &= [Bf|_{BPQ} \circ B\mu] && \text{(by Zabrodsky's Lemma)} \\ &= \big[Bf|_{BPQ'} \circ B\mu' \circ (\mathrm{Id} \times B\varphi)\big] && \text{(by Diagram 7.2)} \\ &= \big[Bf|_{BPQ'} \circ B\widetilde{\varphi} \circ B\mu\big] && \text{(by Diagram 7.3)} \\ &= \big[Bf|_{BPQ'} \circ B\widetilde{\varphi}\big] && \text{(by Zabrodsky's Lemma)} \end{aligned}$$

This implies that Diagram 7.1 is also homotopy commutative. Therefore, we have $\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q, Q')$ if and only if $\varphi \in \mathrm{Hom}_{\mathcal{F}'}(Q, Q')$. □

The next proposition was proved by Ishiguro in 2001 and we need it to obtain a corollary used to prove later the exoticness as p-compact groups for the p-local compact groups $\mathcal{G}_3$, $\mathcal{G}_p$ and $\widetilde{\mathcal{G}}_p$.

Proposition 7.1.2 ([25, Proposition 3.1]). *Suppose G is a compact Lie group. If the loop space of $(BG)^\wedge_p$ is homotopy equivalent to a p-compact group, then $\pi_0(G)$ is p-nilpotent.*

Corollary 7.1.3. *Let p be a prime number and H a finite non p-nilpotent group acting on a torus T. Then, there does not exist any p-compact group realizing the fusion system of $T \rtimes H$ over the prime p.*

Proof. Consider the compact Lie group $G = T \rtimes H$. By Theorem 4.3.8 there is a p-local compact group $(S, \mathcal{F}, \mathcal{L})$ with $\mathcal{F}$ the fusion system of G over a Sylow p-subgroup S and $|\mathcal{L}|^\wedge_p \simeq BG^\wedge_p$. Assume there is a p-compact group X realizing also the p-local compact group $(S, \mathcal{F}, \mathcal{L})$. Then, by Theorem 4.4.15, $|\mathcal{L}|^\wedge_p \simeq BX$, and therefore $BG^\wedge_p \simeq BX$. In this case, by Proposition 7.1.2, the group of components of G must be a p-nilpotent group, in contradiction with the hypothesis in H. □

Now we are ready to prove the exoticness result.

Theorem 7.1.4. *There does not exist any p-compact group realizing the p-local compact groups $\mathcal{G}_3$, $\mathcal{G}_p$ or $\widetilde{\mathcal{G}}_p$.*

Proof. Let $\mathcal{G} = (S_p, \mathcal{F}, \mathcal{L})$ be any of the p-local compact groups $\mathcal{G}_3$, $\mathcal{G}_p$ or $\widetilde{\mathcal{G}}_p$ and assume that there exists some p-compact group X realizing $\mathcal{G}$.

Let Z be the center of S_p, which is isomorphic to $\mathbb{Z}/p$. By definition, Z it is a subgroup fully centralized in $\mathcal{F}$, so we can construct the centralizer p-local compact group of Z in $\mathcal{G}$, which we denote by $(S_p, C_{\mathcal{F}}(Z), C_{\mathcal{L}}(Z))$. By Theorem 7.1.1, this centralizer p-local compact group is realized by $C_X(Z)$. But, if we compute $C_{\mathcal{F}}(Z)$, we get that it corresponds to the saturated fusion system of the groups $T_3 \rtimes \Sigma_3$ for $\mathcal{F} = \mathcal{F}_3$, $T_p \rtimes A_p$ for $\mathcal{F} = \mathcal{F}_p$, and $T_p \rtimes \Sigma_p$ for $\mathcal{F} = \widetilde{\mathcal{F}}_p$. As neither Σ_p for $p \geq 3$, nor A_p for $p \geq 5$ are p-nilpotent, it follows by Corollary 7.1.3 that none of these fusion systems can be realized by a p-compact group, getting a contradiction. □

7.2. Exoticness as compact Lie groups

We have seen in Corollary 7.1.3 that it is easy to construct examples of p-local compact groups impossible to realize by p-compact groups, but this type of construction always give rise to p-local compact groups realizable by compact Lie groups.

In particular, we saw in the previous section that the p-local compact groups $\mathcal{G}_3$, $\mathcal{G}_p$ or $\widetilde{\mathcal{G}}_p$ cannot be realized by p-compact groups. In this section we will prove that these p-local compact groups cannot be realized neither by compact Lie groups.

In order to do this, we will study first the possible normal subsystems of the corresponding saturated fusion systems $\mathcal{F}_3$, $\mathcal{F}_p$ and $\widetilde{\mathcal{F}}_p$. Then, we will be able to reduce the proof of the exoticness as compact Lie groups to the case of compact connected Lie groups. Since every compact connected Lie group gives rise to a p-compact group, we will use then Theorem 7.1.4 to complete the proof.

First, we prove a general result about saturated fusion systems realized by compact Lie groups. This result will be a key point in the proof of Theorem 7.2.4.

Proposition 7.2.1. *Let G be a compact Lie group and $S \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup. Let $H \trianglelefteq G$ a closed normal subgroup and write $R = S \cap H \in \mathrm{Syl}_p(H)$. Then, the saturated fusion system $\mathcal{F}_R(H)$ is normal in $\mathcal{F}_S(G)$.*

Proof. We know, by Proposition 4.3.7 that $R \in \mathrm{Syl}_p(H)$, so we are left to prove that $(R, \mathcal{F}_R(H)) \subseteq (S, \mathcal{F}_S(G))$ satisfies the properties of Definition 4.1.16.

(N1) R is strongly closed in $\mathcal{F}_S(G)$.

Let $a \in R = H \cap S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}_S(G)}(\langle a \rangle, S)$. Then, we have that $\varphi = c_g$ for some $g \in G$ and, since H is normal in G, we have that $c_g(a) \in H$. Moreover, $c_g \in \mathrm{Hom}_{\mathcal{F}_S(G)}(\langle a \rangle, S)$, so we also have $c_g(a) \in S$. Hence $c_g(a) \in H \cap S = R$.

(N2) For each $P \leq Q \leq S$ and each $\gamma \in \mathrm{Hom}_{\mathcal{F}_S(G)}(Q, S)$, the map that sends each morphism $f \in \mathrm{Hom}_{\mathcal{F}_R(H)}(P, Q)$ to $\gamma \circ f \circ \gamma^{-1}$ defines a bijection between the sets $\mathrm{Hom}_{\mathcal{F}_R(H)}(P, Q)$ and $\mathrm{Hom}_{\mathcal{F}_R(H)}(\gamma(P), \gamma(Q))$.

Fix $\gamma \in \mathrm{Hom}_{\mathcal{F}_S(G)}(Q, S)$, so we have $\gamma = c_g$ for some $g \in G$. For every morphism $f \in \mathrm{Hom}_{\mathcal{F}_R(H)}(P, Q)$ we have that $f = c_h$ for some $h \in H$, so $\gamma \circ f \circ \gamma^{-1} = c_{ghg^{-1}}$ for $ghg^{-1} \in H$, since H is normal in G. Moreover, $(ghg^{-1})\gamma(P)(ghg^{-1})^{-1} \leq \gamma(Q)$, so we can define a map

$$\begin{array}{rccc} \gamma^*\colon & \mathrm{Hom}_{\mathcal{F}_R(H)}(P, Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{F}_R(H)}(\gamma(P), \gamma(Q)) \\ & f & \mapsto & \gamma \circ f \circ \gamma^{-1} \end{array}$$

To prove the injectivity of γ^* let $f, f' \in \mathrm{Hom}_{\mathcal{F}_R(H)}(P, Q)$ such that $\gamma^*(f) = \gamma^*(f')$. Note that $f = c_h$ and $f' = c_{h'}$ for some $h, h' \in H$ and $\gamma^*(f) = c_{ghg^{-1}}$ while $\gamma^*(f') = c_{gh'g^{-1}}$. Then, it is clear that $c_h = c_{h'}$, and hence $f = f'$.

For the surjectivity of γ^*, let $f \in \mathrm{Hom}_{\mathcal{F}_R(H)}(\gamma(P), \gamma(Q))$. We know that $f = c_h$ for some $h \in H$, and since $H \trianglelefteq G$, we can take $c_{g^{-1}hg} \in \mathrm{Hom}_{\mathcal{F}_S(G)}(Q, S)$ so we have that $\gamma^*(c_{g^{-1}hg}) = f$.

(N3) $\mathcal{F}_R(H)$ is a saturated fusion system over R.

Since H is a closed subgroup of G, it is itself a compact Lie group. Moreover, $R \in \mathrm{Syl}_p(H)$, so $\mathcal{F}_R(H)$ is a saturated fusion system over R by Theorem 4.3.8.

(N4) Each $f \in \mathrm{Aut}_{\mathcal{F}_R(H)}(R)$ extends to some $\widetilde{f} \in \mathrm{Aut}_{\mathcal{F}_S(G)}(RC_S(R))$ such that

$$[\widetilde{f}, C_S(R)] = \{\widetilde{f}(g) \cdot g^{-1} \,|\, g \in C_S(R)\} \leq Z(R)$$

Let $\overline{S}$ be the topological closure of S. We will prove first that $[N_H(R), C_{\overline{S}}(R)] \leq C_H(R)$. Indeed, let $g \in N_H(R)$ and $x \in C_{\overline{S}}(R)$, then $gxg^{-1}x^{-1} \in H$ since H normal in G implies $xg^{-1}x^{-1} \in H$. Moreover, $gxg^{-1}x^{-1} \in C_G(R)$, hence $gxg^{-1}x^{-1} \in H \cap C_G(R) = C_H(R)$.

Therefore, we have that $C_H(R)C_{\overline{S}}(R)$ is a normal subgroup of $N_H(R)C_{\overline{S}}(R)$. Since R is strongly closed in $\mathcal{F}_S(G)$, it is also fully centralized in $\mathcal{F}_S(G)$, so we have $C_S(R) \in \mathrm{Syl}_p(C_G(R))$ by [12, Lemma 9.5]. Then, we have that $C_S(R) \in \mathrm{Syl}_p(C_H(R)C_{\overline{S}}(R))$ and we can apply the Frattini argument to obtain

$$N_H(R)C_{\overline{S}}(R) = C_H(R)C_{\overline{S}}(R)N_{N_H(R)C_{\overline{S}}(R)}(C_S(R))$$

Finally, let $f \in \mathrm{Aut}_{\mathcal{F}_R(H)}(R)$, then $f = c_g$ with $g \in N_H(R)$. By the previous decomposition, we can write $g = xh$ for some $x \in C_H(R)C_{\overline{S}}(R)$ and $h \in N_{N_H(R)C_{\overline{S}}(R)}(C_S(R))$. Then, take $\widetilde{f} = c_h$. It is clear that $\widetilde{f} \in \mathrm{Aut}_{\mathcal{F}_S(G)}(RC_S(R))$ and it is an extension of f. Moreover, if $g \in C_S(R)$, $\widetilde{f}(g) \cdot g^{-1} = hgh^{-1}g^{-1}$ is in H because $H \trianglelefteq G$ and it is in $C_S(R)$ because h normalizes $C_S(R)$. Then, we obtain

$$\widetilde{f}(g) \cdot g^{-1} \in H \cap C_S(R) = C_R(R) = Z(R)$$ □

The next step is to prove that, in our examples, condition N4 of Definition 4.1.16 is vacuous. This will be very useful to study the normal subsystems.

Lemma 7.2.2. *The saturated fusion systems $\mathcal{F}_3$, $\mathcal{F}_p$ and $\widetilde{\mathcal{F}}_p$ have no proper nontrivial strongly closed subgroups.*

Proof. Let $P \leq S_p$ be a strongly closed nontrivial subgroup. In particular, P is normal in S_p and, if we write $P_k = S_{p,k} \cap P$, we have that P_k is normal in $S_{p,k}$. Since P is nontrivial, there exists k such that P_k is also nontrivial and, by [2, Theorem 8.1], the center $Z(S_{p,k})$ intersects P_k in a non trivially way. Since the center has order p, we must have $Z(S_{p,k}) \leq P_k$. This implies that $Z(S_p) \leq P$.

In all three cases, ζ, the generator of the center, is $\mathcal{F}$-conjugated to s, so $s \in P$, since P is strongly closed by assumption. Moreover, we saw in the proof of Lemma 6.2.2 that all elements not in the maximal torus are conjugated to s, so all elements not in the maximal torus must belong also to P.

Finally, if P contains s and all the elements not in the maximal torus it also contains all elements of the maximal torus, so the only possibility is $P = S_p$, and then P is not proper. □

Now we prove that the structure of the normal subsystems of $\mathcal{F}_3$, $\mathcal{F}_p$ and $\widetilde{\mathcal{F}}_p$ is very simple. This result is the other key point in the proof of Theorem 7.2.4.

Proposition 7.2.3. *Consider the saturated fusion systems $\mathcal{F}_3$, $\mathcal{F}_p$ and $\widetilde{\mathcal{F}}_p$. Then, the following holds:*

(*a*) *$\mathcal{F}_3$ and $\mathcal{F}_p$ are simple.*

(*b*) *$\mathcal{F}_p$ is the only proper nontrivial normal subsystem of $\widetilde{\mathcal{F}}_p$.*

Proof. We first prove that $\mathcal{F}_p$ is simple. For this, we will assume that there is a normal subsystem of $\mathcal{F}_p$ over a subgroup of S_p different to the trivial subgroup and we will prove that the subsystem must be equal to all $\mathcal{F}_p$.

Let then $(R, \mathcal{E}) \trianglelefteq (S_p, \mathcal{F}_p)$. Since $(S_p, \mathcal{F}_p)$ has no proper nontrivial strongly closed subgroups by Lemma 7.2.2, R must be equal to S_p by condition N1 of Definition 4.1.16.

For any $\gamma \in \mathrm{Hom}_{\mathcal{F}_p}(T_p, T_p)$, we have, by condition N2 of Definition 4.1.16, that the map

$$\begin{array}{rccc} \gamma^*: & \mathrm{Hom}_{\mathcal{E}}(T_p, T_p) & \to & \mathrm{Hom}_{\mathcal{E}}(T_p, T_p) \\ & \varphi & \mapsto & \gamma \circ \varphi \circ \gamma^{-1} \end{array}$$

must be a bijection. Let $s \in S_p$, we know that $c_s \in \mathrm{Hom}_{\mathcal{E}}(T_p, T_p)$, by definition of fusion system. Then, consider the subgroup $H \leq A_p$ defined as

$$H = \langle \sigma \circ c_s \circ \sigma^{-1} \mid \sigma \in A_p \rangle$$

Note that, by taking γ to be any of the permutations in A_p, we have $H \leq \mathrm{Hom}_{\mathcal{E}}(T_p, T_p)$. Now, it is clear that $H \trianglelefteq A_p$ by construction, and, since A_p is simple for $p \geq 5$, we must have $H = A_p$. Therefore, we have $A_p \leq \mathrm{Aut}_{\mathcal{E}}(T)$.

Take now $\gamma \in \mathrm{Hom}_{\mathcal{F}_p}(V, V)$, we have then that the map

$$\begin{array}{rccc} \gamma^*: & \mathrm{Hom}_{\mathcal{E}}(V, V) & \to & \mathrm{Hom}_{\mathcal{E}}(V, V) \\ & \varphi & \mapsto & \gamma \circ \varphi \circ \gamma^{-1} \end{array}$$

must be a bijection, again by condition N2 of Definition 4.1.16. Fix $\lambda \in \{1, \ldots, p-1\}$ and set $i_j = (\frac{j}{2}(j - p + 2))\lambda p^{k-1}$. Then, it is easy to see that the element $w_\lambda = v_1^{i_1} \cdots v_{p-1}^{i_{p-1}}$ belongs to the normalizer of V and, in the basis $\langle \zeta, s \rangle$, we have

$$c_{w_\lambda} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

Therefore, we must have $c_{w_\lambda} \in \mathrm{Hom}_{\mathcal{E}}(V, V)$. Consider now the morphism $\gamma \in \mathrm{Hom}_{\mathcal{F}_p}(V, V)$ defined by the matrix

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_p)$$

If we set $\lambda = 1$ and conjugate the automorphism c_{w_1} by γ we obtain a new automorphism of V, given by the matrix

$$\gamma \circ c_{w_1} \circ \gamma^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Note that $\gamma \circ c_{w_1} \circ \gamma^{-1} \notin \mathrm{Aut}_{S_p}(V)$, since conjugation by elements of S_p must fix the center, but we must have $\gamma \circ c_{w_1} \circ \gamma^{-1} \in \mathrm{Aut}_{\mathcal{E}}(V)$, by condition N2 of Definition 4.1.16.

Also, note that

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{p-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathrm{SL}_2(\mathbb{F}_p) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

Therefore, we obtain that $\mathrm{Aut}_{\mathcal{E}}(V)$ must contain, among others, the two automorphisms c_{w_1} and $\gamma^*(c_{w_1})$, which generate themselves all the special linear group. Hence, we have $\mathrm{SL}_2(\mathbb{F}_p) \leq \mathrm{Aut}_{\mathcal{E}}(V)$.

Now we use condition N3 of Definition 4.1.16, requiring $\mathcal{E}$ to be saturated. Let

$$\varphi = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_p) \leq \mathrm{Aut}_{\mathcal{E}}(V)$$

If $w_\lambda \in N_{S_p}(V)$, we have $\varphi \circ c_{w_\lambda} \circ \varphi^{-1} = c_{w_\lambda^{\lambda^2}} \in \mathrm{Aut}_{S_p}(V)$, so φ must extend to an strict overgroup of V, by axiom (II) of saturation. Since there is no fully normalized, $\mathcal{F}_p$-centric and $\mathcal{F}_p$-radical subgroup of S_p containing V, we have that φ must extend to all S_p, by Theorem 4.1.13.

Let $\widetilde{\varphi} \in \mathrm{Aut}_{\mathcal{E}}(S_p)$ be the morphism extending φ. Since $\mathcal{E} \leq \mathcal{F}_p$, we have that the map $\widetilde{\varphi}$ must be $\psi\phi^{-1}$, and by restricting $\widetilde{\varphi}$ to T_p, we obtain $\mathrm{Aut}_{\mathcal{E}}(T_p) = A_p \rtimes C_{p-1}$.

Finally, let $\tau \in N_{\Sigma_p}((12\ldots p))$ be a morphism of order $p-1$ which normalizes s. Then, $\tau^2 \in A_p$ and, if we compute the group N_{τ^2} from the axiom (II) of saturation, we obtain $N_{\tau^2} = S_p$, by construction. In this case, using that $\mathcal{E} \leq \mathcal{F}_p$, we get that the extension of τ^2 is ϕ^2. Then, by restricting τ^2 to V we obtain $\mathrm{Aut}_{\mathcal{E}}(V) = \mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}}$.

This implies $\mathcal{E} = \mathcal{F}_p$ and so $\mathcal{F}_p$ is simple.

The proof of the simplicity for $\mathcal{F}_3$ is completely analogous. By condition N2 of Definition 4.1.16 we obtain that $\mathrm{SL}_2(\mathbb{F}_3) \leq \mathrm{Aut}_{\mathcal{E}}(T)$ and $\mathrm{SL}_2(\mathbb{F}_3) \leq \mathrm{Aut}_{\mathcal{E}}(V)$. Then, by extending $-\mathrm{Id}$ from both groups we obtain a morphism of determinant -1 in both groups, showing that, in fact, $\mathrm{Aut}_{\mathcal{E}}(T) = \mathrm{GL}_2(\mathbb{F}_3)$ and $\mathrm{Aut}_{\mathcal{E}}(V) = \mathrm{GL}_2(\mathbb{F}_3)$.

To prove (b) note that the computations just done prove that a normal subsystem of $\widetilde{\mathcal{F}}_p$ must contain $\mathcal{F}_p$. In fact, it is possible to have $\mathcal{F}_p$ as a normal subsystem, since in the definition of H we still have $H \leq A_p$, even taking $\sigma \in \Sigma_p$, and also $\gamma \circ c_{w_\lambda} \circ \gamma^{-1} \in \mathrm{SL}_2(\mathbb{F}_p)$ for all $\gamma \in \mathrm{GL}_2(\mathbb{F}_p)$.

Let then $\mathcal{E}$ be a normal subsystem of $\widetilde{\mathcal{F}}_p$ different from $\mathcal{F}_p$. Since the group $A_p \rtimes C_{p-1}$ has index 2 in $\Sigma_p \times C_{p-1}$, adding any extra automorphism to T_p we obtain $\mathrm{Aut}_{\mathcal{E}}(T_p) = \Sigma_p \times C_{p-1}$. Then, consider as before the morphism τ, which normalizes s and has order $p-1$. We have, again by axiom (II) of saturation, that τ extends to all S_p. Therefore, we must have $\phi \in \mathrm{Aut}_{\mathcal{E}}(S_p)$ and, by restricting ϕ to V, we obtain automorphisms of V represented by matrices not in $\mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}}$. This implies $\mathrm{Aut}_{\mathcal{E}}(V) = \mathrm{GL}_2(\mathbb{F}_p)$.

On the other hand, if we add an extra morphism to $\mathrm{Aut}_{\mathcal{E}}(V)$, we obtain $\mathrm{Aut}_{\mathcal{E}}(V) = \mathrm{GL}_2(\mathbb{F}_p)$, because $\mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}}$ has also index 2 in $\mathrm{GL}_2(\mathbb{F}_p)$. We can then consider the matrix in $\mathrm{Aut}_{\mathcal{E}}(V) = \mathrm{GL}_2(\mathbb{F}_p)$ corresponding to the restriction of ϕ, which must extend also to all S_p. This means that we obtain again $\phi \in \mathrm{Aut}_{\mathcal{E}}(S_p)$ and, by restricting ϕ to T_p we obtain a permutation of odd order, hence $\mathrm{Aut}_{\mathcal{E}}(T_p) = \Sigma_p \times C_{p-1}$.

This implies $\mathcal{E} = \widetilde{\mathcal{F}}_p$ and so the only proper nontrivial subsystem of $\widetilde{\mathcal{F}}_p$ is $\mathcal{F}_p$. □

Finally, we are ready to prove the result about the exoticness.

Theorem 7.2.4. *There does not exist any compact Lie group realizing the p-local compact groups $\mathcal{G}_3$, $\mathcal{G}_p$ or $\widetilde{\mathcal{G}}_p$.*

Proof. Assume that there is a compact Lie group G realizing $\mathcal{G}_p$, that is, such that $(S_p, \mathcal{F}_p, \mathcal{L}_p) \cong (S_p, \mathcal{F}_{S_p}(G), \mathcal{L}^c_{S_p}(G))$ for $S_p \in \mathrm{Syl}_p(G)$. Let $G_0 \trianglelefteq G$ be the connected component of the identity in G. By Proposition 7.2.1, we have that $S_p \cap G_0$ is strongly closed in $\mathcal{F}_{S_p}(G)$, but, by Lemma 7.2.2, $\mathcal{F}_p$ has no proper nontrivial strongly closed subgroups, hence $S_p \leq G_0$. Then, again by Proposition 7.2.1, $\mathcal{F}_{S_p}(G_0) \trianglelefteq \mathcal{F}_{S_p}(G)$, but, since $\mathcal{F}_p$ is a simple saturated fusion system by Proposition 7.2.3 (a), we must have $\mathcal{F}_{S_p}(G_0) \cong \mathcal{F}_{S_p}(G)$. This is impossible since a connected compact Lie group gives rise to a p-compact group, and the saturated fusion system $\mathcal{F}_p$ is not realized by any p-compact group by Theorem 7.1.4.

The exact same argument is valid for $\mathcal{G}_3$, since $\mathcal{F}_3$ is also a simple saturated fusion system.

Assume then that there is a compact Lie group $\widetilde{G}$ with $S_p \in \mathrm{Syl}_p(\widetilde{G})$ and such that $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p) \cong (S_p, \mathcal{F}_{S_p}(\widetilde{G}), \mathcal{L}^c_{S_p}(\widetilde{G}))$. Let again $\widetilde{G}_0 \trianglelefteq \widetilde{G}$ be the connected component of the identity in $\widetilde{G}$. As before, by Proposition 7.2.1, we have $S_p \leq \widetilde{G}_0$ and $\mathcal{F}_{S_p}(\widetilde{G}_0) \trianglelefteq \mathcal{F}_{S_p}(\widetilde{G})$. Then, we know, by Proposition 7.2.3 (b), that $\mathcal{F}_p$ is the only proper nontrivial normal subsystem of $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$. Therefore, in this case we must have $\mathcal{F}_{S_p}(\widetilde{G}_0) \cong \mathcal{F}_p$ or $\mathcal{F}_{S_p}(\widetilde{G}_0) \cong \widetilde{\mathcal{F}}_p$, but we have proved in Theorem 7.1.4 that there is no p-compact group realizing any of these two fusion systems, hence $\widetilde{G}$ cannot exist. □

7.3. Exoticness of limit fusion systems

In this section we make a summary of examples of limit fusion systems and their possibility to be realized by compact Lie groups or by p-compact groups.

In this thesis we have constructed two direct systems of exotic saturated fusion systems which, after taking the limit fusion system, have produced two examples of saturated fusion systems over the discrete p-toral group S_p that cannot be realized by any compact

Lie group nor by any p-compact group. There are other examples of limit fusion systems that we can construct having other exoticness properties, such as the following:

- If we consider the trivial fusion systems $\mathcal{F}_{S_{p,k}}(S_{p,k})$ for $k \geq 2$, it is obvious that they fit in a direct system with the same morphism ρ of Proposition 5.4.1. Then, if we consider the limit fusion system, we obtain the trivial saturated fusion system $\mathcal{F}_{S_p}(S_p)$. It is clear that this fusion system is realized by the compact Lie group given as the topological closure of S_p. Moreover, the group of components of this compact Lie group is a p-group, so the saturated fusion system $\mathcal{F}_{S_p}(S_p)$ is also realized by a p-compact group.

- As we said in Chapter 3, Díaz-Ruiz-Viruel classified in [17, Theorem 5.10] all saturated fusion systems over the family of 3-groups $S_{3,k}$. We have presented one of them in Section 3.1, obtaining in Section 5.4 the saturated fusion system $\mathcal{F}_3$. However, we can see that there are other exotic saturated fusion systems of the classification which fit in a direct system using the same morphism ρ. For example, the ones denoted in [17, Table 6] by $\mathcal{F}(3^{2k+1}, 2)$ and $3.\mathcal{F}(3^{2k}, 2).2$. Although these examples don't verify the hypothesis of Proposition 5.4.2, it is easy to see the following:
  - The limit fusion system of the family $\mathcal{F}(3^{2k+1}, 2)$, which we will denote by $\mathcal{F}'_3$, is realized by the p-compact group DI_2, by [14, Proposition 10.2]. Therefore, since DI_2 cannot be constructed as a 3-completion of any compact Lie group, we have that $\mathcal{F}'_3$ cannot be realized by any compact Lie group.
  - We can see also in the classification theorem of [17] that $\mathcal{F}'_3$, the limit fusion system of the family $\mathcal{F}(3^{2k+1}, 2)$, coincides with the direct limit of the family of fusion systems realized by the finite simple group ${}^2F_4(q)$, for a certain prime power q. This is because the extra outer automorphisms of the extraspecial subgroups in $S_{3,k}$ become conjugation by elements of $S_{3,k+1}$ when we apply the morphism ρ. Therefore, the saturated fusion system realized by the p-compact group DI_2 can also be obtained as the limit of realizable fusion systems.
  - The limit fusion system of the family $3.\mathcal{F}(3^{2k}, 2).2$, which we will denote by $\mathcal{F}''_3$, is the fusion system of $S_3 \rtimes (C_2 \times C_2)$, again because the extra outer automorphisms of the extraspecial subgroups are lost when we take the limit.

- Finally, we can also construct the realizable fusion systems $T_{p,k} \rtimes \Sigma_p$ for $k \geq 2$, which fit in a direct system with limit denoted $\mathcal{F}'_p$ and realized by $T_p \rtimes \Sigma_p$. Note that the saturated fusion system $\mathcal{F}'_p$ is realized by a compact Lie group, but, by Corollary 7.1.3, it cannot be realized by a p-compact group.

We can see in Table 7.1 a grid with a review of these examples depending on the exoticness of the fusion systems in the direct system and the exoticness of the limit fusion system.

Limit / System	Realizable by		Exotic as		
	compact Lie	p-compact	compact Lie	p-compact	both
Realizable	$\mathcal{F}_{S_p}(S_p), \mathcal{F}'_p$	$\mathcal{F}_{S_p}(S_p), \mathcal{F}'_3$	$\mathcal{F}'_3$	$\mathcal{F}'_p$	?
Exotic	$\mathcal{F}''_3$	$\mathcal{F}'_3, \mathcal{F}''_3$	$\mathcal{F}'_3, \mathcal{F}_3, \mathcal{F}_p, \widetilde{\mathcal{F}}_p$	$\mathcal{F}_3, \mathcal{F}_p, \widetilde{\mathcal{F}}_p$	$\mathcal{F}_3, \mathcal{F}_p, \widetilde{\mathcal{F}}_p$

Table 7.1.: Summary of examples of p-local compact groups associated to limit fusion systems and their exoticness.

Remark 7.3.1. If we could prove that it is impossible to find examples for the unknown cell in Table 7.1, that is, to prove that a limit of realizable fusion systems is always realizable by a compact Lie group or by a p-compact group, we would have, using Theorem 7.1.4 and Theorem 7.2.4, a proof for the exoticness of the p-local finite groups $\mathcal{F}_{3,k}$, $\mathcal{F}_{p,k}$ and $\widetilde{\mathcal{F}}_{p,k}$ without using the classification of finite simple groups.

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Abstract

In 2003, Broto-Levi-Oliver introduced the concept of p-local finite group, which is a generalization for p-completed classifying spaces of finite groups. Later, the same authors introduced also the notion of p-local compact group, which is a generalization for p-completed classifying spaces of compact Lie groups and p-compact groups. While the concept of exotic p-local finite group is clearly defined, in the compact case there are several families of groups which give rise to p-local compact groups, blurring this way the notion of exoticness.

In this thesis we construct new examples of exotic p-local finite groups for every $p \geq 5$. Moreover, we prove that these new examples are simple in the sense that they contain no proper nontrivial normal subsystems.

Then, we develop the theory of limits of fusion systems. We prove that, for any family of fusion systems satisfying certain compatibility properties, we can construct a related fusion system over a discrete p-toral group. Moreover, we prove that this limit construction coincides with the direct limit from a categorical point of view under saturation hypothesis.

Using the new examples of p-local finite groups for $p \geq 5$, as well as other families of examples discovered by Broto-Levi-Oliver and Díaz-Ruiz-Viruel, we apply the limit construction to produce two new examples of fusion systems over discrete p-toral groups for each $p \geq 5$ and one new example for $p = 3$.

Once we have the new fusion systems, we generalize a saturation criterion known for p-local finite groups to the compact case. Then, we use this criterion to prove the saturation of the new examples we have created, giving rise in this way to new examples of p-local compact groups.

Finally, we prove that neither the new example of 3-local compact group nor the new two examples of p-local compact groups for $p \geq 5$ can be realized by compact Lie groups or by p-compact groups.