Numerical Computation of Invariant Objects with Wavelets

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Abstract

This dissertation is divided into two parts. The first one (Chapters 1 – 4) is devoted to recall and introduce all the theory and the methodology which is used into the second part (Chapters 5 – 7). In Chapter 1 a self-contained and general introduction to wavelets it is done. However, since we will use Daubechies wavelets with $p \geq 1$ vanishing moments, we will focus in such wavelets (in $\mathbb{R}$ or $\mathbb{S}^1$). Actually, we have performed the translation from the $\mathbb{R}$-Daubechies wavelets language to $\mathbb{S}^1$. As usual we have call it ψ^{PER}. Also, in Chapter 1 a crash course on the notion of regularity, in terms of the functional spaces $\mathscr{B}^s_{2,2}(\mathbb{S}^1)$ and $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$, and the relationship between them and the wavelets coefficients, $\langle f, \psi^{\text{PER}}_{-j,n}\rangle$, is done.

The main topic of Chapter 2 is the calculation of the wavelets coefficients using three different techniques; namely, Fast Wavelet Transform, quadrature rules and the solution of (non)-linear systems of equations. In a more concrete terms, we have performed an algorithm to calculate, in an efficient and fast way, ψ^{PER} on a mesh of points of $\mathbb{S}^1$. The method is based on the Daubechies–Lagarias algorithm (see [DL91, DL92]). Such computation, which is a key point of this disquisition, will be the main part of the Chapter 2. Despite of this, we do a Gauss–Lobatto quadrature in $\mathbb{R}$. Indeed, following methodologies similar as those ones in [BBDK01, BBD$^+$02], we set $\psi(x) + c$ to be the weight of the quadrature.

The last two chapters of this first part are devoted to give a compilation of the theoretical framework where this dissertation is dealt. In Chapter 4 it is shown the machinery and we try to characterize the mechanisms involved in the geometric properties of a particular family of quasi periodically forced skew products on the cylinder. As a matter of fact, combining them with ideas from [AM08, Har12] we extend the results of [Sta97, Sta99] to a more weird class of functions. Also we derive "*theoretically*" the regularity, in terms of $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$, for φ of the Keller-GOPY model. Such effort is to justify the use of the software in other cases of SNA's among the study of them in terms of regularity.

Moving to the second part of this Thesis, two different exercises are done. The first one, following [dlLP02], is the development of an algorithm to estimate regularities, in terms of $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$, for $s \in \mathbb{R}$. Such algorithm is the main goal of Chapter 5, and it is used in many situations, but not only with the Fast Wavelet Transform. Indeed, in Chapter 6 we perform the same kind of regularity assessment with a different methodology to obtain $\langle \varphi, \psi^{\text{PER}}_{-j,n}\rangle$.

Certainly, in the last Chapter(s), we have focused to solve the Invariance Equation for several situations. The iterative solvers are the main topic of such Chapter(s). Due to the good properties of the Daubechies wavelet family (see e.g [Dau92, HW96, Tri06]) we have derived two strategies to find $\langle \varphi, \psi^{\text{PER}}_{-j,n}\rangle$. Both of them are similar but, due to the simplicity of the Haar wavelet ($p = 1$), the first strategy give us a close to explicit method calculate the Haar wavelet coefficients. As mentioned, beyond the "*numerical approximation*" of φ, we have estimated the regularity, in terms of $\mathscr{B}^s_{2,2}(\mathbb{S}^1)$ and $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$, of a certain models of skew products using the Haar wavelet. Both goals have been repeated with other Daubechies wavelet. Nevertheless, when $p > 1$ the core of the iterative method is the aforementioned massive evaluation of ψ^{PER} and, also, the splitting of the left conditioned discretization of the Transfer Operator. As an extra point, a numerical exploration of the Lyapunov exponent of a particular instance of the kind of systems in [AM08] it is done.

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