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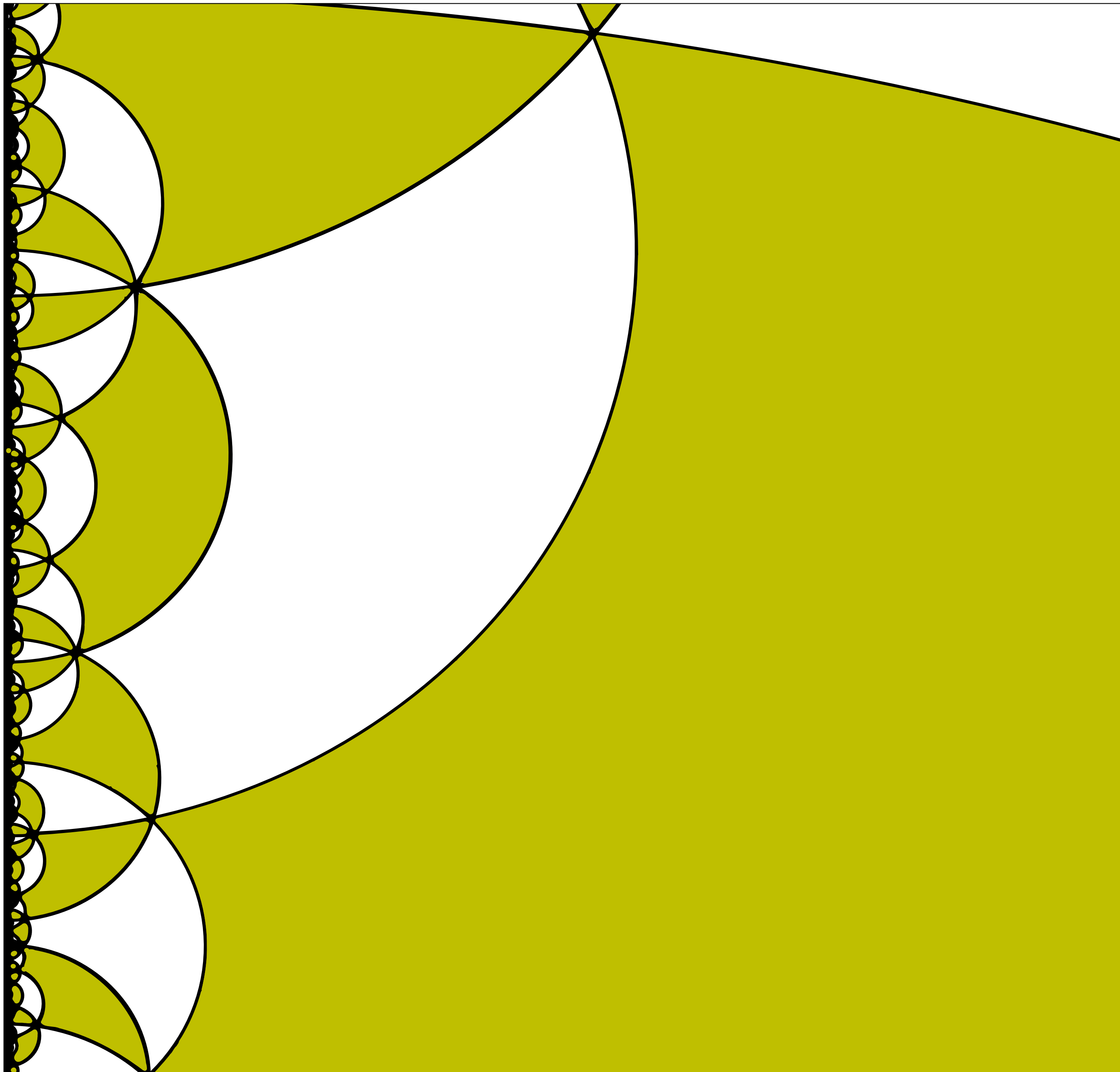
# On the Hyperbolic Uniformization of Shimura Curves with an Atkin-Lehner Quotient of Genus 0

Joan Nualart Riera



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per a aspirar al grau de
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FA CONSTAR
que aquesta memòria ha estat realitzada sota la seva direcció
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Barcelona, octubre de 2015

Signat: Artur Travesa Grau

*Als meus pares*

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# Introduction

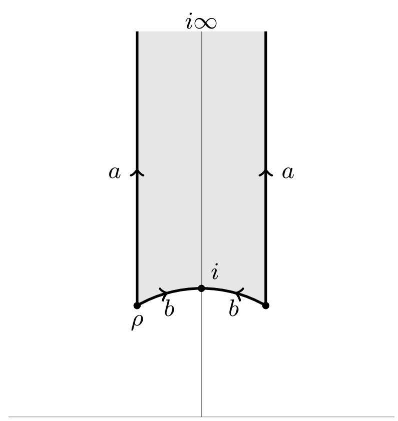
The main goal of this thesis is to contribute to the explicit hyperbolic uniformization of Shimura curves. We will restrict to the case of curves attached to Eichler orders in rational quaternion algebras whose maximal Atkin-Lehner quotient has genus 0, which despite multiple differences bears some resemblance to the classical modular case. We will provide an approach to obtain an explicit uniformization of these curves and some of their covers, together with several applications. We will illustrate all the applications with plenty of examples.

In order to get a better understanding of the problem, we will begin by introducing some historical background. Then, we will explain in detail our contributions and the content of the monograph.

The fuchsian groups and the corresponding automorphic functions are an important topic of study in the mathematics of the 19th century, which reaches a culminating point with the works of Poincaré [Poi95], Klein [Kle23, Kle79], Fricke [Fri92, Fri93, Fri94, Fri99] and Fricke and Klein [KF90, KF92, FK97, FK12]. However, some particular cases of fuchsian groups and automorphic functions, for example those attached to congruence subgroups $\Gamma_0(N)$ of the modular group $\Gamma_0(1) = \mathbf{SL}(2, \mathbb{Z})$, can be traced back to the works of Gauss, Abel or Dedekind. It is in this setting that the most well-known automorphic function makes its appearance, namely the $j$ function (or $j$ invariant), commonly called Klein $j$ function, but which was already introduced, up to scaling, by Dedekind [Ded77] as the function *Valenz*, and that, according to Klein [Kle79], stems from the works of Gauss. The $j$ function is an automorphic function for the modular group $\mathbf{SL}(2, \mathbb{Z})$, and therefore, as for all subgroups $\Gamma_0(N)$, is traditionally called a modular elliptic function.

If $\Gamma$ is a fuchsian group such that $\Gamma\backslash\mathcal{H}$ has finite hyperbolic volume (or more generally a fuchsian group of the first kind) and we consider its action on the upper half-plane, then the automorphic functions with respect to $\Gamma$ are the meromorphic functions on $\mathcal{H}$ which are invariant under the action of $\Gamma$ and which satisfy certain growth conditions. On the other hand, the action of $\Gamma$ on $\mathcal{H}$ induces a structure of Riemann surface on the quotient space $\Gamma\backslash\mathcal{H}$ and it allows to identify $\Gamma\backslash\mathcal{H}$ with a Zariski open subset of the complex points of an algebraic curve. In particular, for the case of congruence subgroups of the modular group, this gives rise to the classically called modular curves $X_0(N)$.

The field of rational functions of the modular curve $X_0(1)$ can be naturally identified with $\mathbb{C}(j)$. An important characteristic of the Riemann surface $\Gamma_0(N)\backslash\mathcal{H}$ is that it is not compact. For example, if $N = 1$, a fundamental domain for the action of $\mathbf{SL}(2,\mathbb{Z})$ on $\mathcal{H}$ is the well-known Gauss' fundamental domain, which originally appeared in the context of the reduction of binary quadratic forms. In particular, this allows to consider Fourier expansions for the modular functions.




For example, for the case of the $j$ function, the Fourier expansion yields the well-known expression

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots$$

where $q = e^{2\pi i\tau}$. In particular, the rationality (and integrality) conditions of the coefficients of these expansions translate into properties of the curves $X_0(N)$. For example, the fact that

$$\Gamma_0(N) = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathbf{SL}(2,\mathbb{Z}) \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \cap \mathbf{SL}(2,\mathbb{Z})$$

allows us to write $\mathbb{C}(X_0(N)) = \mathbb{C}(j, j(N\cdot))$. Moreover, there exists a monic symmetric polynomial of degree $\psi(N)$ in each variable, $\Phi_N(X,Y) \in \mathbb{Z}[X,Y]$, such that $\Phi_N(j, j(N\cdot)) = 0$, which provides an equation for $X_0(N)$ and is classically called the *modular equation of level* $N$. This polynomial has roots $(j, j \circ \alpha_N)$, where $\alpha_N$ runs over the transformations corresponding to a set of representatives of primitive elements of norm $N$ in $\mathbf{M}(2,\mathbb{Z})$ modulo the action of $\mathbf{SL}(2,\mathbb{Z})$. This allows to explicitly compute $\Phi_N(X,Y)$ using the Fourier expansion of $j$, since it is possible to choose a complete set of representatives $\alpha_N$ fixing $\infty$. In addition, the polynomial $\Phi_N(X,X)$ can be factored into smaller pieces known as class equations, each of those corresponding to the values of the $j$ function at certain points. In this way, the values of the $j$ function at a set of points in the upper half-plane known as complex multiplication (CM) points, which consists of the quadratic points in $\mathcal{H}$, can be determined. The value of $j$ at such a point $\tau$ is not only an algebraic number but it generates a certain abelian extension of the field $\mathbb{Q}(\tau)$. In particular, this provides a way to explicitly obtain polynomials defining these extensions.

On the other hand, it is well-known that the points of $X_0(1)$ correspond to isomorphism classes of generalized elliptic curves. A point $\tau \in \mathbf{SL}(2,\mathbb{Z})\backslash\mathcal{H}$ corresponds to the isomorphism class of the elliptic curve represented by $\mathbb{C}/\langle 1,\tau\rangle$ and the coefficients of a Weierstrass model for it are given by means of modular elliptic forms. Then, the function $j$ is a rational function of the coefficients of the Weierstrass model. Hence, from an algebraic point of view, the $j$ function can be identified with an absolute invariant of binary cubic (or quartic) forms which classifies isomorphism classes of elliptic curves.

Many more properties of the modular functions have been studied; for example: the reduction of the polynomial $\Phi_p(X,Y)$ modulo $p$, which is given by the Kronecker congruence formula; some alternative functions generating the same abelian extensions generated by $j$ at the CM points, which give to rise simpler defining equations, like Weber modular functions, etc. Modular forms, as we have seen, also appear in this context, either in the form of theta functions (Jacobi) or intervening in the expansions of modular functions (Eisenstein), even though they were only later formalized by Hecke [Hec24]. It was also Hecke who introduced the $L$-series attached to an eigenform and proved some of its important properties, like the existence of an Euler product expansion. Introducing in this way a notion which has played an important role in some of the most important results in number theory since the second half of the 20th century.

However, not only automorphic functions and forms for congruence subgroups of $\mathbf{SL}(2,\mathbb{Z})$ have been systematically considered. For example, one of the first generalizations which was introduced is that one corresponding to the Hilbert modular group $\mathbf{SL}(2,\mathcal{O}_K)$, where $\mathbb{Z}$ is replaced by the ring of integers of a totally real number field, giving rise in this way to Hilbert modular functions and forms.

Similarly, if we replace the upper half-plane by the set of all square matrices of dimension $g$ over $\mathbb{C}$ with positive definite imaginary part and consider the group $\mathbf{Sp}(2g,\mathbb{Z})$ acting on it, we obtain Siegel modular functions and forms of genus $g$, introduced by Siegel [Sie35, Sie39] after the work of Riemann [Rie57] on Riemann surfaces of genus $g$. For $g=1$, this notion coincides with the classical modular case. For $g=2$, a detailed study of the field of modular functions and the graded ring of modular forms was carried out by Igusa [Igu60, Igu62, Igu64, Igu67], including their interpretation as invariants of sextic forms. Generators in this case are known as Igusa invariants.

It is in this context, and after the work of Eichler [Eic37, Eic38, Eic55a, Eic55b] on quaternion algebras, that Shimura [Shi67] considers the fuchsian groups obtained through an embedding from the elements of norm 1 in certain orders of a rational quaternion algebra $\mathbb{H}$ such that $\mathbb{H}\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbf{M}(2,\mathbb{R})$, giving rise to the Shimura curves which we are going to study in this thesis. Shimura considers as well the case of quaternion algebras $\mathbb{H}$ over totally real number fields of degree $d$ satisfying that $\mathbb{H}\otimes\mathbb{R}\cong\mathbf{M}(2,\mathbb{R})^r\times\mathbb{H}_{\mathbb{R}}^{d-r}$ for some integer $0<r\leq d$, giving rise to Shimura varieties of dimension $r$. And even more generally, in [Shi75a] the construction is extended to the case of reductive groups acting on bounded symmetric domains. This construction was later reinterpreted by Deligne [Del71, Del79], whose approach is commonly used nowadays.

We will restrict to the rational case and we will assume that $\mathbb{H}\not\cong\mathbf{M}(2,\mathbb{Q})$. In this case we will consider an indefinite rational quaternion algebra of discriminant $D$, an embedding $\Phi:\mathbb{H}\hookrightarrow\mathbf{M}(2,\mathbb{R})$ and the group $\Gamma\subset\mathbf{SL}(2,\mathbb{R})$ (or its image in $\mathbf{PSL}(2,\mathbb{R})$) corresponding to the image in $\mathbf{M}(2,\mathbb{R})$ of the elements of norm 1 of a suitable order $\mathcal{O}\subset\mathbb{H}$. An important difference between this case and the modular case is that in this situation the corresponding Riemann surface is always compact. In particular, there is no way to obtain Fourier expansions for the corresponding automorphic functions. Thanks to Shimura [Shi67], we know that there exists a *canonical model* for the Riemann surface attached to $\Gamma$, which is a projective curve $X(\Gamma)$ defined over $\mathbb{Q}$ together with an analytic map $j_\Gamma:\Gamma\backslash\mathcal{H}\to X(\Gamma)(\mathbb{C})$, giving an isomorphism with its image, with suitable conditions for its values at certain points (CM points). In order to do so, Shimura proves that $X(\Gamma)$ is a coarse moduli space. More precisely, the points in $X(\Gamma)(\mathbb{C})$ correspond to isomorphism classes of principally polarized abelian surfaces with quaternionic multiplication by a maximal order containing $\mathcal{O}$ (which are usually called fake elliptic curves) together with a certain level structure. It is worth observing that the construction of Shimura is a generalization which comprises the classical modular case and the function $j_\Gamma$ plays the role of the Klein $j$ function. But, if $\mathbb{H} \not\cong \mathbf{M}(2, \mathbb{Q})$, it is proved by Shimura himself [Shi75a] that any corresponding Shimura curve does not have real points. Therefore, the function $j_\Gamma$ will not just be an automorphic function in a suitable affine chart, but a tuple of several such functions.

The arithmetical and diophantine properties of Shimura curves have been widely studied since its introduction in the 60s, for example: a $p$-adic uniformization was given by Čerednik [Čer76] and Drinfeld [Dri76]; the study of integral models, by Morita [Mor81], Boutot and Carayol [BC91], etc.; the study of local points, by Jordan and Livné [JL85], Ogg [Ogg85], etc.; the existence of rational points on certain (Atkin-Lehner) quotients, by Jordan [Jor81], Ogg [Ogg83], etc. In addition they have played an important role, together with modular curves, in the proof of some of the most important conjectures in number theory in the last times, essentially through the proof of the modularity of all elliptic curves over $\mathbb{Q}$, thanks to the work of Eichler, Shimura and Wiles, among others. In this way, both modular and Shimura curves are coarse moduli spaces of (fake) elliptic curves and, at the same time, provide modular parametrizations for all the elliptic curves over $\mathbb{Q}$.

Note however that the function $j_\Gamma$ is not explicit in Shimura's construction. Even forgetting about the function $j_\Gamma$, computing equations for the curve $X(\Gamma)$ does not have an obvious approach and is in general a difficult problem. The equation of a (non-modular) Shimura curve was first computed by Ihara [Iha79] and since then several other equations have been computed using different techniques, starting by Kurihara [Kur79], Jordan [Jor81] and Roberts [Rob89] and continuing with Elkies [Elk98], Gonzàlez and Rotger [GR04, GR06], Molina [Mol12], etc. Most of these authors make use of the Čerednik-Drinfeld uniformization to compute the equations. However, Elkies uses the complex uniformization provided by the Shimura canonical model. In this way, he recovers as well the CM points corresponding to certain orders. A mixture of both approaches has been considered as well by Yang [Yan13] and Tu [Tu14], extending the cases computed by Elkies to the curves such that suitable covers of genus 0 exist, adding the knowledge of some equations for them, essentially those computed by Gonzàlez and Rotger mentioned before.

In order to compute with automorphic forms and functions corresponding to Shimura curves attached to non-split quaternion algebras, some other type of power expansions, which replace the Fourier expansions usually used in the modular case, have to be considered. The fundamental ideas of the expansions considered can be traced back to Shimura [Shi75a], and have been studied under different viewpoints by Mori [Mor94, Mor95, Mor11] and by Bayer [Bay02], Bayer and Travesa [BT07a, BT07b, BT08]. In particular, expansions for some modular and automorphic functions corresponding to the modular curve and also to the Shimura curve attached to a maximal order in the rational quaternion algebra of discriminant $D = 6$ have been computed in [BT07a, BT07b], and the case of automorphic forms, also in the case $D = 6$, has been studied by Baba and Granath [BG12]. These have also been used to obtain expansions for automorphic forms using a numerical approach by Voight and Willis [VW14]. The computation of CM points in the case of triangle groups through numeric approximations is also considered by Voight [Voi06] and the determination of automorphic forms attached to Shimura curves through Jacquet-Langlands correspondence is considered by Voight in [Voi10].

Finally, the problem of giving equations for the genus 2 curves giving rise to the abelian surfaces corresponding to points in $X(\Gamma)$ has been studied by Rotger in [Rot04] and a numerical approach has also been given by Bayer and Guàrdia [BG05]. In [BG08], Baba and Granath obtain a generating function (Hauptmodul) for the rational function field of the maximal Atkin-Lehner quotient of the Shimura curves attached to quaternion algebras of discriminants $D = 6, 10$ by means of Igusa invariants using the Hashimoto-Murabayashi families [HM95], yielding therefore a formula which is similar in essence to the one relating $j$ to the coefficients of a Weierstrass model of the corresponding elliptic curve. In [BT08] this Hauptmodul is related to the one computed in [BT07b].

Therefore, it is natural to consider whether we can make use of the expansions of the automorphic functions around CM points studied in [BT07b] to help us in the determination of an explicit uniformization of a genus 0 Atkin-Lehner quotient of a Shimura curve and contributing therefore to a better understanding of the open problems posed by Elkies [Elk98, §5.5], especially those concerning the computation of covers and CM points. This is the main problem we consider; in parallel, we will also deal with several other problems we will find in the way, as detailed next.

In Chapter 1 we collect the basic concepts and results which will be required later. It is broken down into three sections, which deal with quaternion algebras and orders, the Shimura curve $X(D, N)$ attached to an Eichler order of level $N$ in a quaternion algebra of discriminant $D$ and automorphic forms, respectively.

In Chapter 2, we introduce the kroneckerian polynomials, which are the natural generalization to the case of Shimura curves of the classical modular equations. In order to do so, a complete study of the set of primitive elements in an Eichler order of a given norm coprime to the level of the order is carried out. Then, given a certain Atkin-Lehner quotient $X(D, M)^W$ of genus 0 of a Shimura curve $X(D,M)$, a Hauptmodul $t$ for the curve $X(D,M)^W$ over a certain number field $K$ and a level $N$ coprime to $DM$, the kroneckerian polynomial $\Psi(X,Y) = \Psi_{\Gamma(D,M)\langle W\rangle,N,t}(X,Y) \in \mathbb{C}[X,Y]$ is defined. In Theorem 2.1, we prove that it is a symmetric (if $N > 1$), irreducible polynomial with coefficients in $K$ of degree $\psi(N)$ in each variable. Moreover, we see that it provides an equation for $X(D,MN)^W$ over $K$ and that an equation for $X(D,MN)^{\langle W,\omega_N\rangle}$ over $K$ can be deduced from it. In Proposition 2.5, we study the factorization of $\Psi(X,X)$ and we relate its zeros to the values of the Hauptmodul $t$ at certain CM points, similarly to the classical factoritzation of the modular equation $\Phi(X,X)$ in terms of class equations. Next, we study the singularities of the kroneckerian polynomial of a certain level. We see that the set of singular points consists entirely of CM points and we provide a description of the number of branches through a point, which we simply call multiplicity, in Theorem 2.2. In Theorem 2.3, we focus on the case of prime power level, and we give a complete description of the set of singular points together with a formula for the multiplicity.

In the second part of Chapter 2, we focus on those Shimura curves $X(D,N)$ which have genus 0, which are the curves $X(D,1)$ with $D = 6, 10, 22$. In this case, in Theorem 2.4 we give a unified approach to the construction of Hauptmoduln for these three curves. All these Hauptmoduln are defined over quartic fields. However, we see that the corresponding kroneckerian polynomials are defined over $\mathbb{Q}$ and satisfy certain additional symmetry conditions corresponding to the action of the Atkin-Lehner involutions. Finally, in Theorem 2.5, for a kroneckerian polynomial of prime level attached to any of these Hauptmoduln, we study its reduction modulo $p$ and deduce a formula, similar to that given by Kronecker congruence formula in the modular case, from Shimura reciprocity law.

In Chapter 3, we begin by giving an algorithmic approach to the computation of fundamental domains for the groups $\Gamma(D,N)$ based on Ford's method. Then, we use it to compute fundamental domains for $\Gamma(D,1)\langle W_D\rangle$ for $D = 6, 10, 22$. Next, we introduce the $q$-expansions and the differential equations satisfied by the Hauptmoduln together with their relation to linear differential equations, and use them together with the kroneckerian polynomials, to obtain an explicit uniformization of the curve $X(22,1)^{W_{22}}$, which to our knowledge, was not known before. We recover as well uniformizations for the curves $X(D,1)^{W_D}$ for $D = 6, 10$ which had already been obtained by [Elk98] and [BT07b]. All these uniformizations are given in Theorem 3.1. Finally, we express our Hauptmodul for the curve $X(22,1)^{W_{22}}$ in terms of Igusa invariants. In order to do so, we embed our curve in Igusa's threefold and identify its image as a certain factor of an intersection of Humbert surfaces. This allows us to obtain an expression for our Hauptmodul in terms of Igusa invariants in Theorem 3.10 and to write an explicit family of genus 2 curves parametrized by automorphic functions in $\mathbb{Q}(X(22,1))$ giving rise to the corresponding abelian surfaces at each point (except for a finite set of exceptional points which is also computed). Finally the potential good reduction of the curves in the family is also studied.

In Chapter 4 we begin by computing the whole set of couples $(D, N)$ for which the curve $X(D,N)^{W_{DN}}$ has genus 0, which is the set of curves for which we could follow the approach introduced in Chapter 3 in order to obtain an explicit uniformization. In Proposition 4.1, we give a list of all the 271 curves which have an Atkin-Lehner quotient of genus 0 together with all the information necessary to compute the genus and the number of elliptic cycles (and the corresponding quadratic orders) of any quotient of the curve $X(D, N)$. Those with squarefree $N$ had already been computed in [LMR06]. Moreover, we prove that all such curves $X(D,N)^{W_{DN}}$ are isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$. We can also readily see that the number of elliptic cycles of the curve $X(D,N)^{W_{DN}}$ is bigger or equal than that of the curve $X(D,1)^{W_D}$, which also has genus 0. Since the computational complexity of the approach given in Chapter 3 depends basically on the number of elliptic cycles, it is therefore better to obtain an explicit uniformization for the curve $X(D,1)^{W_D}$ and then use this one to obtain an uniformization for $X(D,N)^{W_{DN}}$. In this chapter, we will see how this can be done using again the kroneckerian polynomials introduced in Chapter 2 and the ideas used in [Elk98]. To illustrate the approach, we will consider the curves studied in Chapter 3, and we will uniformize the curves $X(D, N)$, $D = 6, 10, 22$, together with all their quotients, for the 3 smallest possible values of $N > 1$ in each case. We will provide Hauptmoduln for all the rational quotients, and equations for all the remaining curves of genus $0, 1, 2$ or hyperelliptic of higher genus. To our knowledge, explicit uniformizations for the cases $(6, 11)$, $(10, 9)$ and those of discriminant 22 had not been obtained before. In this case, the complexity of the computations depends on the degree of the cover $X(D,N)^{W_D} \to X(D,1)^{W_D}$, which is $\psi(N)$ but, if $N$ is composite (or a prime power), it can be broken down into smaller pieces. Moreover, we can work with numeric (or $p$-adic) approximations and we could even resort to some computer algebra software to deal with the easier cases.

In Chapter 5 we study the algebraic properties of the coefficients of the $q$-expansions of automorphic forms attached to the groups $\Gamma(D, N)$ around CM points. In order to do so, we introduce the type of $q$-expansions we are going to consider and relate the coefficients to the values of certain operators, closely related to Shimura-Maaß differential operators. Then, following the ideas in [Shi75a], in Theorem 5.1 we obtain an algebraicity result for the coefficients of the $q$-expansions, when suitably normalized, formulated over precise number fields. In Corollary 5.2, we consider the algebraicity of the coefficients of the $q$-expansions of automorphic functions attached to $\Gamma(D, N)$, generalizing to an arbitrary $D$ and over precise number fields the algebraicity result given in [BT08]. In Theorem 5.2 we prove the other implication, characterizing the field of definition of a form in terms of the coefficients of the $q$-expansion at a CM point. In Corollary 5.3, we obtain a different proof for a result by Ihara [Iha74], namely that the automorphic derivative of an automorphic function defined over a field $K$ is as well an automorphic function defined over the same field. Finally, in Theorem 5.3, we deal with the action of the Galois group on the coefficients of the $q$-expansions, extending in this way the Shimura reciprocity law, which deals with the action of the Galois group on the values of the automorphic functions, to the whole set of coefficients.

In Chapter 6 we deal with some of the computational applications of the tools introduced in the previous chapters. It is broken down into four different sections. In Section 6.1 we study the explicit computation of kroneckerian polynomials of a certain level when $q$-expansions around certain CM points are known and we give some examples in the case $D = 6$. More polynomials have been computed in order to deal with the later applications, more precisely for $D = 6, 10, 22$ we have computed the kroneckerian polynomials of prime level for all primes not dividing $D$ up to 41. In Section 6.2 we deal with the problem of computing the values of the Hauptmodul at a given CM point in a deterministic way without making use of approximations by using Proposition 2.5. In particular, we apply this to the computation of the rational and quadratic CM points of the curves $X(D, 1)^{W_D}$, obtaining in particular a new proof that the rational values given by Elkies in [Elk98] for the cases $D = 6, 10$ are correct (we give as well an omitted value in the case $D = 10$). A different proof following a completely different method has been given by Errthum [Err11]. Our approach can also be used to check the correctness of the tables given in [Elk98] for the cases $D = 14, 15$. We do not restrict ourselves to the case of the values of the Hauptmoduln and we provide a method to obtain as well the first few terms of a $q$-expansion at a given CM point. After that, more terms can be computed using the differential equation satisfied by the Haupmodul. We provide as well tables for the first few terms of the $q$-expansions around all the rational CM points. However, to make these expansions really functional, the local parameter with respect to which we consider the expansion has to be computed and this is what we do in Section 6.3. Note that the results in this section, with the exception of three values in the case $D = 6$, are obtained by approximation and recognition and, even though we are confident that they are correct, we know no way to prove it and therefore we can only be certain that they are good approximations (at least up to 150 digital digits, but more digits can be checked) of the real values. In any case, this suffices in general to obtain numeric approximations for the value of the function or its derivative in a neighborhood of the point. Finally, in Section 6.4, we deal with the problem of obtaining $q$-expansions of automorphic forms. In order to do so, we assume that $X(D, N)$ is such that $X(D, N)^{W_{DN}}$ has genus 0 and we compute a basis for the space of weight $k$ automorphic forms corresponding to a curve $X(D, N)$ in terms of a Hauptmodul for $X(D, N)^{W_{DN}}$. We study as well how to compute the action of a certain Hecke operator in the space of automorphic forms of a certain weight. Then, we provide a set of generators and relations for the graded algebra of automorphic forms for $D = 6, 10, 22$ and the levels for which we computed uniformizations in the previous chapters. Moreover, some examples of $q$-expansions are also provided.

Finally, we must add that almost all the computations in this thesis have been carried out using Magma [BCP97] or Mathematica [Wol08], but at some point we have also used Sage [Dev12], PARI [The14] and Macaulay2 [GS].

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# Chapter 1

# Quaternion algebras, Shimura curves and automorphic forms

In this chapter we are going to introduce the basic notions and some well-known results which will be required as a background. We will break this chapter into three sections. The first one deals with quaternion algebras, quaternion orders and some other related notions. Based on this, we introduce Shimura curves and some of their properties in Section 1.2 and automorphic forms attached to groups of quaternion units in Section 1.3. Some references for the results in this chapter are [Vig80], [Shi71], [Ogg83], [AB04].

## 1.1 Quaternion algebras

Let $K$ be a field of characteristic different from 2.

**Definition.** A quaternion $K$-algebra $\mathbb{H}$ is an unitary and associative $K$-algebra such that:

(a) $\dim_K \mathbb{H} = 4$;

(b) it is central, i.e. its center is $K$, $Z(\mathbb{H}) = K$;

(c) it is simple, i.e. $\mathbb{H}$ has no non-trivial bilateral ideals.

We denote by $\mathbb{H}^*$ its group of units, i.e.

$$\mathbb{H}^* = \{h \in \mathbb{H} : \exists h' \in \mathbb{H}, h'h = hh' = 1\}.$$

As in the case of general $K$-algebras, a morphism of quaternion $K$-algebras is a $K$-linear morphism of unitary rings and we can naturally consider $K$ embedded into $\mathbb{H}$.

Given an element $k \in \mathbb{H}^*$, we can consider an automorphism of $\mathbb{H}$ given by $h \mapsto khk^{-1}$ for every $h \in \mathbb{H}$. These automorphisms of $\mathbb{H}$ are called inner automorphisms. The well-known Theorem of Skolem-Noether states that any automorphism of $\mathbb{H}$ is inner.

In our case, since the characteristic of $K$ is not 2, every quaternion $K$-algebra $\mathbb{H}$ admits the following description: there exists a $K$-basis $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{ij}\}$ of $\mathbb{H}$ such that $\boldsymbol{i}^2 = a$, $\boldsymbol{j}^2 = b$ and $\boldsymbol{ij} = -\boldsymbol{ji}$ for some $a, b \in K^*$. Conversely, a $K$-basis with the previous properties, and 1 as unity, together with the associativity, defines a quaternion $K$-algebra. This algebra will be denoted, as usual, by $\left(\frac{a,b}{K}\right)$.

**Remark.** Note that different choices of $a, b \in K^*$ can yield isomorphic quaternion $K$-algebras. For example, $\left(\frac{a,b}{K}\right) \cong \left(\frac{b,a}{K}\right)$.

A quaternion $K$-algebra $\mathbb{H} = \left(\frac{a,b}{K}\right)$ is endowed with an involutive anti-automorphism called conjugation which is denoted by $w \mapsto \overline{w}$. If we consider the natural $K$-basis on $\mathbb{H}$ and we write $w = x + y\boldsymbol{i} + z\boldsymbol{j} + t\boldsymbol{ij}$ with $x, y, z, t \in K$, then $\overline{w} = x - y\boldsymbol{i} - z\boldsymbol{j} - t\boldsymbol{ij}$. Now that the conjugation has been introduced, we can immediately define the (reduced) trace $\mathrm{tr}(w) = w + \overline{w}$ and the (reduced) norm $\mathrm{nr}(w) = w\overline{w}$. Using the previous expression of $w$ in the basis $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{ij}\}$, one has that $\mathrm{tr}(w) = 2x$ and $\mathrm{nr}(w) = x^2 - ay^2 - bz^2 + abt^2$. The group of units $\mathbb{H}^*$ is exactly the set of elements with non-zero norm.

**Remark.** Since we will not use the general notions of trace and norm for $K$-algebras (cf. [Bou58]), whenever we refer to traces and norms in quaternion algebras, it will be implied that they are the reduced traces and norms above.

Given an integral domain $R$, we will denote by $\mathbf{M}(2, R)$ the ring of $2 \times 2$ matrices over $R$, by $\mathbf{GL}(2, R)$ the corresponding linear group of invertible matrices in $\mathbf{M}(2, R)$, by $\mathbf{SL}(2, R)$ the corresponding special linear group of determinant 1 matrices and by $\mathbf{PGL}(2, R) = \mathbf{GL}(2, R)/R^*$, $\mathbf{PSL}(2, R) = \mathbf{SL}(2, R)/\{\pm 1\}$ the corresponding projective groups.

**Example.** If $\mathbb{H} = \mathbf{M}(2, K)$ and $w = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$, then $\overline{w} = \left(\begin{smallmatrix} t & -y \\ -z & x \end{smallmatrix}\right)$ is the transposed adjoint matrix of $w$. In particular, the reduced trace and norm coincide with the trace and determinant of the matrix.

If a quaternion $K$-algebra is isomorphic to $\mathbf{M}(2, K)$, it will simply be called a matrix $K$-algebra. If that is not the case, then the quaternion $K$-algebra is a skew field and it will be called a division $K$-algebra.

From now on we will assume that $K$ is a number field and we will denote its completion at a place $\nu$ by $K_\nu$. We will write $\mathcal{O}_K$ and $\mathcal{O}_{K,\nu}$ their respective rings of integers.

**Definition.** Let $\mathbb{H}$ be a quaternion $K$-algebra. For each place $\nu$ of $K$, let $\mathbb{H}_\nu = K_\nu \otimes_K \mathbb{H}$ be the corresponding quaternion $K_\nu$-algebra. If $\mathbb{H}_\nu$ is a matrix algebra, we will say that $\mathbb{H}$ is unramified (or does not ramify) at $\nu$. Otherwise, if $\mathbb{H}_\nu$ is a division algebra, we will say that $\mathbb{H}$ is ramified (or ramifies) at $\nu$.

Given $\mathbb{H} = \left(\frac{a,b}{K}\right)$ and a place $\nu$ of $K$, the Hasse invariant at $\nu$ is defined as

$$\epsilon\left(\frac{a, b}{K}\right)_\nu = \begin{cases} 1, & \text{if } \nu \text{ does not ramify in } \mathbb{H}, \\ -1, & \text{if } \nu \text{ ramifies in } \mathbb{H}. \end{cases}$$

In the particular case $K = \mathbb{Q}$, the Hasse invariant at $p$, $\epsilon\left(\frac{a,b}{\mathbb{Q}}\right)_p$, coincides with the Hilbert symbol $(a, b)_p$, and therefore it admits an explicit description in terms of Legendre symbols (cf. [Ser73]).

Now we can state the well-known classification theorem.

**Theorem 1.1.** *(1) A quaternion $K$-algebra $\mathbb{H}$ ramifies at a finite and even number of places.*

*(2) Two quaternion $K$-algebras are isomorphic if and only if they ramify at the same places.*

*(3) Given an even number of non-complex places of $K$, there exists a quaternion $K$-algebra which ramifies at exactly these places.* □

**Definition.** The (reduced) discriminant $D_{\mathbb{H}}$ of a quaternion $K$-algebra $\mathbb{H}$ is the integral ideal of $\mathcal{O}_K$ which is equal to the product of the prime ideals that ramify in $\mathbb{H}$.

Therefore, two quaternion $K$-algebras are isomorphic if and only if they have the same discriminant. In particular, a quaternion $K$-algebra is a matrix algebra if and only if it has discriminant $\mathcal{O}_K$.

Another notion playing an important role is that of splitting fields.

**Definition.** Let $\mathbb{H}$ be a quaternion $K$-algebra and $F$ a field extension of $K$. The field $F$ splits $\mathbb{H}$ if and only if $F \otimes_K \mathbb{H}$ is a matrix $F$-algebra.

As we will see later on, among the fields splitting a quaternion $K$-algebra, those which are quadratic extensions of $K$ will be of special interest, and therefore the following criterion will be particularly useful.

**Theorem 1.2.** *Let $\mathbb{H}$ be a quaternion $K$-algebra and $F|K$ a quadratic extension. Then, the following statements are equivalent:*

*(i) $F$ splits $\mathbb{H}$.*

*(ii) $F$ is isomorphic as a $K$-algebra to a maximal subfield of $\mathbb{H}$ containing $K$.*

*(iii) There exists an embedding of $K$-algebras $F \hookrightarrow \mathbb{H}$.*

*(iv) Every place $\nu$ in $K$ which ramifies in $\mathbb{H}$ is not totally split in $F$.* □

The case we will be interested in is the case $K = \mathbb{Q}$, that is, the case of rational quaternion algebras.

**Definition.** Let $\mathbb{H}$ be a rational quaternion algebra. If $\mathbb{R}$ splits $\mathbb{H}$, we say that $\mathbb{H}$ is indefinite. Otherwise, we say that $\mathbb{H}$ is definite.

**Remark.** Given a rational quaternion algebra $\mathbb{H} = \left(\frac{a,b}{\mathbb{Q}}\right)$, $\mathbb{H}$ is indefinite if and only if at least one of $a$ or $b$ is positive.

In the case of rational quaternion algebras, since the discriminant is an ideal of $\mathbb{Z}$, we can identify it with a positive integer generating this ideal. The definite or indefinite character of a rational quaternion algebra $\mathbb{H}$ can then be read from the number of prime factors of $D_{\mathbb{H}}$, namely an odd number of prime factors corresponds to the definite case and an even number to the indefinite one. On the other hand, the indefinite character of $\mathbb{H}$ is equivalent to the existence of an embedding $\mathbb{H} \hookrightarrow \mathbf{M}(2, \mathbb{R})$.

Let $\mathbb{H} = \left(\frac{a,b}{\mathbb{Q}}\right)$ be an indefinite rational quaternion algebra. Permuting $a$ and $b$, if necessary, we can assume that $a > 0$. Then $\Phi : \mathbb{H} \hookrightarrow \mathbf{M}(2, \mathbb{R})$ defined by

$$\Phi(x + y\boldsymbol{i} + z\boldsymbol{j} + t\boldsymbol{ij}) = \begin{pmatrix} x + y\sqrt{a} & z + t\sqrt{a} \\ b(z - t\sqrt{a}) & x - y\sqrt{a} \end{pmatrix}$$

is an embedding.

### 1.1.1 Quaternion orders

Let $R$ be a Dedekind domain and $K$ its fraction field. Let $\mathbb{H}$ be a quaternion $K$-algebra.

**Definition.** An element $h \in \mathbb{H}$ is said to be integral over $K$ if both $\mathrm{tr}(h)$ and $\mathrm{nr}(h)$ lie in $R$.

In contrast to the abelian case, the set of integral elements in a quaternion algebra is not a ring except in some particular cases. For this reason the notion of order is introduced. First of all we consider the notion of ideal.

**Definition.** An $R$-lattice $\Lambda$ of $\mathbb{H}$ is a finitely generated $R$-module contained in $\mathbb{H}$. An $R$-ideal $I$ of $\mathbb{H}$ is an $R$-lattice such that $K \cdot I = \mathbb{H}$. The inverse of an $R$-ideal $I$ is the $R$-ideal

$$I^{-1} = \{h \in \mathbb{H} : IhI \subset I\}.$$

An $R$-ideal $I$ is integral if all its elements are integral.

**Definition.** A subset $\mathcal{O}$ of $\mathbb{H}$ is an $R$-order if it satisfies any of the three following equivalent conditions:

(a) $\mathcal{O}$ is a subring of $\mathbb{H}$ which is a free $R$-module of rank 4.

(b) $\mathcal{O}$ is an $R$-ideal which is a ring.

(c) $\mathcal{O}$ is a ring containing $R$, whose elements are integral and such that $K\mathcal{O} = \mathbb{H}$.

Given an $R$-ideal $I$ in $\mathbb{H}$, its associated right and left orders are defined as follows:

$$\mathcal{O}_r(I) = \{h \in \mathbb{H} : Ih \subseteq I\} \text{ and } \mathcal{O}_l(I) = \{h \in \mathbb{H} : hI \subseteq I\}.$$

The $R$-ideal $I$ is integral if and only if it is contained in the associated orders. Then $I$ is a right (resp. left) ideal of $\mathcal{O}_r(I)$ (resp. $\mathcal{O}_l(I)$). We say that $I$ is bilateral if $\mathcal{O}_r(I) = \mathcal{O}_l(I)$. The $R$-ideal $I$ is principal if there exists $h \in \mathbb{H}$ such that $I = h\mathcal{O}_r(I) = \mathcal{O}_l(I)h$.

**Definition.** Given an $R$-order $\mathcal{O}$, its different $\mathcal{D}_{\mathcal{O}}$ is the bilateral $R$-ideal of $\mathbb{H}$ defined as the inverse of the codifferent,

$$\mathcal{D}_{\mathcal{O}}{}^{-1} = \{\alpha \in \mathbb{H} : \operatorname{tr}(\alpha\mathcal{O}) \subseteq R\}.$$

The (reduced) discriminant $D_{\mathcal{O}}$ of $\mathcal{O}$ is the reduced norm of its different $\mathcal{D}_{\mathcal{O}}$, namely, the fractional ideal of $R$ generated by the norms of the elements in $\mathcal{D}_{\mathcal{O}}$.

The discriminant of a given $R$-order can be computed easily using the following result.

**Proposition 1.1.** *Let $\mathcal{O}$ be an $R$-order in $\mathbb{H}$.*

*(1) If $\{v_1, v_2, v_3, v_4\}$ is an $R$-basis of $\mathcal{O}$, then $D_{\mathcal{O}}^2 = R\det(\operatorname{tr}(v_i v_j))$.*

*(2) Let $\mathcal{O} \subseteq \mathcal{O}'$ be $R$-orders in $\mathbb{H}$. Then $D_{\mathcal{O}'}$ divides $D_{\mathcal{O}}$, and the two are equal if and only if the orders are equal.*

*(3) Given $\psi : \mathbb{H} \to \mathbb{H}$ an automorphism of $\mathbb{H}$, then $D_{\psi(\mathcal{O})} = D_{\mathcal{O}}$.* □

In any quaternion $K$-algebra, every $R$-order is contained in a maximal one. Maximal orders will play an important role as soon as we introduce Shimura curves. However, we will consider them in the bigger family of Eichler orders.

**Definition.** An Eichler $R$-order in a quaternion $K$-algebra $\mathbb{H}$ is the $R$-order obtained as the intersection of two maximal $R$-orders in $\mathbb{H}$.

Now, we are going to study in more detail the properties and structure of these orders. First of all, it is obvious that the class of Eichler orders is invariant under isomorphisms. In particular, the set of Eichler orders of a quaternion algebra is closed under conjugation. What is not so obvious is when two such orders are indeed conjugated.

We will split the study of Eichler orders in two cases: we will begin by the local Eichler orders and then we will move on to the global ones.

### Local Eichler orders

Let $\nu$ be a finite place of a number field $K$ and let $K_\nu$ the corresponding completion. Let $\mathbb{H}_\nu = K_\nu \otimes_K \mathbb{H}$. Let $R_\nu$ be the ring of integers of $K_\nu$ and let $\varpi$ be a uniformizer of $R_\nu$.

Now two possible cases appear depending on whether $\mathbb{H}_\nu$ is a division algebra or a matrix algebra. For this reason, we will summarize the classification of Eichler orders in two different theorems.

**Theorem 1.3.** *Let $\mathbb{H}_\nu$ be a local division $K_\nu$-algebra. Then,*

$$\mathcal{O}_\nu = \{h \in \mathbb{H}_\nu : \mathrm{nr}(h) \in R_\nu\}$$

*is the unique maximal order. Hence, $\mathcal{O}_\nu$ is the unique Eichler $R_\nu$-order in $\mathbb{H}_\nu$. Moreover, the ideal $\varpi R_\nu$ ramifies in $\mathcal{O}_\nu$; more precisely, $\varpi \mathcal{O}_\nu = \mathfrak{m}^2$, where $\mathfrak{m}$ is the unique maximal ideal of $\mathcal{O}_\nu$.* □

**Theorem 1.4.** *Let $\mathcal{O}_\nu \subset \mathbf{M}(2, K_\nu)$ be an $R_\nu$-order. Then, the following are equivalent:*

*(i) $\mathcal{O}_\nu$ is an Eichler order.*

*(ii) There exists a unique pair of maximal orders $\{\mathcal{O}_1, \mathcal{O}_2\}$ in $\mathbf{M}(2, K_\nu)$ such that $\mathcal{O}_\nu = \mathcal{O}_1 \cap \mathcal{O}_2$.*

*(iii) There exists an integer $n \in \mathbb{N}$ such that $\mathcal{O}_\nu$ is conjugated to the order*

$$\begin{pmatrix} R_\nu & R_\nu \\ \varpi^n R_\nu & R_\nu \end{pmatrix}.$$

*Moreover, $n$ is uniquely determined.*

*(iv) The order $\mathcal{O}_\nu$ contains a subring conjugated to*

$$\begin{pmatrix} R_\nu & 0 \\ 0 & R_\nu \end{pmatrix}.$$

*The ideal $N_{\mathcal{O}_\nu} = \varpi^n R_\nu$ is called the level of the local Eichler order $\mathcal{O}_\nu$.* □

Therefore, we can now define the general notion of level of a local Eichler order.

**Definition.** Let $\mathcal{O}_\nu$ be a local Eichler order in a quaternion $K_\nu$-algebra $\mathbb{H}_\nu$. The level of $\mathcal{O}_\nu$ is the ideal

$$N(\mathcal{O}_\nu) = \begin{cases} R_\nu, & \text{if } \mathbb{H}_\nu \text{ is a division algebra,} \\ N_{\mathcal{O}_\nu}, & \text{if } \mathbb{H}_\nu \text{ is a matrix algebra.} \end{cases}$$

**Remark.** For any local Eichler order $\mathcal{O}_\nu$, $D_{\mathcal{O}_\nu} = N(\mathcal{O}_\nu)$.

### Global Eichler orders

Let $\mathbb{H}$ be a quaternion algebra over a number field $K$ with ring of integers $R$ and let $\mathcal{O}$ be an $R$-order in $\mathbb{H}$. For any place $\nu$ of $K$, let $\mathcal{O}_\nu = R_\nu \otimes_R \mathcal{O}$.

For a finite place $\nu$, $\mathcal{O}_\nu$ is a local $R_\nu$-order and for an infinite place $\nu$, we consider $R_\nu = K_\nu$ and then $\mathcal{O}_\nu = \mathbb{H}_\nu$. In particular, considering the natural inclusions $\mathbb{H} \hookrightarrow \mathbb{H}_\nu$, we obtain that $\mathcal{O} = \bigcap_\nu (\mathbb{H} \cap \mathcal{O}_\nu)$. Moreover, it can be proved that $(D_\mathcal{O})_\nu = D_{\mathcal{O}_\nu}$.

**Proposition 1.2.** *$\mathcal{O}$ is a maximal order if and only if $\mathcal{O}_\nu$ is an $R_\nu$-maximal order for every finite place $\nu$. Moreover, the order $\mathcal{O}$ is maximal if and only if $D_\mathcal{O} = D_\mathbb{H}$.* □

Observe that the previous result gives a simple procedure to check whether a given order is maximal or not. The local character of the maximality is also true for Eichler orders.

**Proposition 1.3.** *$\mathcal{O}$ is an Eichler $R$-order if and only if $\mathcal{O}_\nu$ is a local Eichler $R_\nu$-order for every finite place $\nu$.* □

Thus, this proposition provides a very useful way to recognize Eichler orders using the local characterization we have seen before.

Now, since we have a notion of level for local Eichler orders, it can be shown that we can pack them together in order to define the level of a global Eichler order.

**Definition.** The level $N_\mathcal{O}$ of the global Eichler order $\mathcal{O}$ is the unique ideal $N$ in $R$ such that $N_\nu = R_\nu \otimes_R N$ is the level of the local Eichler $R_\nu$-order $\mathcal{O}_\nu$.

Eichler orders are not characterized in general in terms of their discriminant, as it was the case for maximal orders. However, some conditions and particular cases can still be obtained.

**Proposition 1.4.** *Let $\mathcal{O}$ be an order in a rational quaternion algebra $\mathbb{H}$.*

*(1) The level of an Eichler order in $\mathbb{H}$ is coprime to $D_\mathbb{H}$.*

*(2) If $D_\mathcal{O} = D_\mathbb{H} N$ with $N$ a squarefree integer, then $\mathcal{O}$ is an Eichler order of level $N$.* □

Also in the case $K = \mathbb{Q}$, we will denote by $\mathcal{O}(D, N)$ an Eichler order of level $N$ in the algebra of discriminant $D$. In this situation, we can give an equivalent interpretation of the notion of level.

**Proposition 1.5.** *Let $\mathcal{O}(D,N) \subseteq \mathcal{O}(D,1)$. Then, the index of $\mathcal{O}(D,N)$ in $\mathcal{O}(D,1)$ as $\mathbb{Z}$-modules is $N$.* □

Finally, we recall some results regarding the existence and the unicity of Eichler orders of a fixed level modulo conjugation.

**Proposition 1.6.** *Let $\Lambda$ be a lattice in a quaternion $K$-algebra $\mathbb{H}$. For each finite place $\nu$ of $K$, let $L_\nu$ be a local lattice in the local quaternion $K_\nu$-algebra $\mathbb{H}_\nu$. Assume that $L_\nu = \Lambda_\nu$ for all but a finite number of $\nu$. Then, there exists a lattice $\Lambda'$ in $\mathbb{H}$ such that $\Lambda'_\nu = L_\nu$ for all $\nu$.* □

**Corollary 1.1.** *Let $\mathbb{H}$ be a quaternion $\mathbb{Q}$-algebra of discriminant $D$. Then, for every $N$ coprime to $D$ there exist Eichler orders of level $N$.* □

**Theorem 1.5.** *Let $\mathbb{H}$ be an indefinite quaternion $\mathbb{Q}$-algebra. Then, there is only one conjugacy class of Eichler orders having the same level and all (left or right) ideals of an Eichler order in $\mathbb{H}$ are principal.* □

### Bilateral ideals and normalizer of Eichler orders

Given an order $\mathcal{O}$ in $\mathbb{H}$, its normalizer is

$$\mathrm{Nor}(\mathcal{O}) = \{\sigma \in \mathbb{H}^* : \sigma\mathcal{O}\sigma^{-1} = \mathcal{O}\}.$$

Let $\mathcal{O} = \mathcal{O}(D,N)$ be an Eichler order of level $N$ in a rational quaternion algebra of discriminant $D$. For any $m \| DN$, i.e. $m|DN$ and $\gcd(m, DN/m) = 1$, there exists an ideal $I = I(m)$ such that $I^2 = m\mathcal{O}$, which can be described locally as follows:

- If $p \nmid m$, $I_p = \mathcal{O}_p$.
- If $p|m$ and $p|D$, $I_p = \mathcal{O}_p \setminus \mathcal{O}_p^*$.
- If $p^k \| m$ and $p^k \| N$ and we identify $\mathcal{O}_p$ with

$$\left\{ \begin{pmatrix} a & b \\ cp^k & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}_p \right\},$$

  then $I_p$ is identified with $\begin{pmatrix} 0 & 1 \\ p^k & 0 \end{pmatrix} \mathcal{O}_p = \mathcal{O}_p \begin{pmatrix} 0 & 1 \\ p^k & 0 \end{pmatrix}$.

Since all ideals in $\mathcal{O}$ are principal, we can write $I(m) = w_m\mathcal{O} = \mathcal{O}w_m$, for a certain $w_m \in \mathcal{O}$ of norm $m$. In particular,

$$\mathcal{O} = w_m\mathcal{O}w_m^{-1}.$$

**Theorem 1.6.** *Let $\mathcal{O} = \mathcal{O}(D,N)$ be an Eichler order of level $N$ in a rational quaternion algebra of discriminant $D$. Then,*

$$\mathcal{O}^*\backslash\mathrm{Nor}(\mathcal{O}) \cong (\mathbb{Z}/2\mathbb{Z})^r,$$

*where $r$ is the number of distinct primes dividing $DN$. The elements $w_m$ above provide representatives for the non-trivial classes.* □

**Embeddings of quadratic orders**

**Definition.** Consider $\mathbb{H}$ an indefinite quaternion $\mathbb{Q}$-algebra and $F$ a quadratic field. Consider as well a quaternion order $\mathcal{O} \subset \mathbb{H}$ and a quadratic order $\Lambda \subset F$. An embedding of $\Lambda$ into $\mathcal{O}$ is an embedding $\phi : F \hookrightarrow \mathbb{H}$ such that $\phi(\Lambda) \subset \mathcal{O}$. It is called optimal if $\phi(F) \cap \mathcal{O} = \phi(\Lambda)$. We will denote $\mathcal{E}(\mathcal{O},\Lambda)$ and $\mathcal{E}^*(\mathcal{O},\Lambda)$ the set of embeddings and optimal embeddings respectively.

Given a subgroup $G \subset \mathrm{Nor}(\mathcal{O})$, we can consider the action

$$\begin{array}{rcl} \mathcal{E}(\mathcal{O},\Lambda) \times G & \to & \mathcal{E}(\mathcal{O},\Lambda) \\ (\phi,\sigma) & \mapsto & \phi^\sigma, \end{array}$$

where $\phi^\sigma(\alpha) = \sigma^{-1}\phi(\alpha)\sigma$ for all $\alpha \in F$. This action restricts to $\mathcal{E}^*(\mathcal{O},\Lambda)$ and therefore induces equivalence relations on both sets. We denote by $\nu(\mathcal{O},\Lambda;G)$ the number of equivalence classes of optimal embeddings modulo $G$. The following theorem gives an explicit formula for this number when $G = \mathcal{O}^*$.

**Theorem 1.7.** *Let $\mathcal{O} = \mathcal{O}(D,N)$ be an Eichler order of level $N$ in an indefinite quaternion $\mathbb{Q}$-algebra of discriminant $D$. Let $\Lambda(d,m)$ be the quadratic order of conductor $m$ in $\mathbb{Q}(\sqrt{d})$. Then,*

$$\nu(\mathcal{O},\Lambda(d,m);\mathcal{O}^*) = h(d,m)\prod_{p|DN}\nu_p(\mathcal{O},\Lambda(d,m);\mathcal{O}^*),$$

*where $h(d,m)$ denotes the ideal class number of the order $\Lambda(d,m)$, $\psi$ denotes the Euler $\psi$ function and*

- *If* $p|D$, $\nu_p(\mathcal{O},\Lambda(d,m);\mathcal{O}^*) = \begin{cases} 1-\left(\frac{D_F}{p}\right), & \text{if } p\nmid m, \\ 0, & \text{otherwise.} \end{cases}$

- *If* $p\|N$, $\nu_p(\mathcal{O},\Lambda(d,m);\mathcal{O}^*) = \begin{cases} 1+\left(\frac{D_F}{p}\right), & \text{if } p\nmid m, \\ 2, & \text{otherwise.} \end{cases}$

- *Assume* $N = p^r u_1$, $p\nmid u_1$, $r\geq 2$, *and* $m = p^k u_2$, $p\nmid u_2$*:*
  - *If* $r \geq 2k+2$, $\nu_p(\mathcal{O},\Lambda(d,m);\mathcal{O}^*) = \begin{cases} 2\psi(p^k), & \text{if } \left(\frac{D_F}{p}\right)=1, \\ 0, & \text{otherwise .} \end{cases}$
  - *If* $r = 2k+1$, $\nu_p(\mathcal{O},\Lambda(d,m);\mathcal{O}^*) = \begin{cases} 2\psi(p^k), & \text{if } \left(\frac{D_F}{p}\right)=1, \\ p^k, & \text{if } \left(\frac{D_F}{p}\right)=0, \\ 0, & \text{otherwise.} \end{cases}$
  - *If* $r = 2k$, $\nu_p(\mathcal{O},\Lambda(d,m);\mathcal{O}^*) = p^{k-1}\left(p+1+\left(\frac{D_F}{p}\right)\right)$.
  - *If* $r \leq 2k-1$, $\nu_p(\mathcal{O},\Lambda(d,m);\mathcal{O}^*) = \begin{cases} p^{k/2-1}(p+1), & \text{if } k \text{ is even,} \\ 2p^{(k-1)/2}, & \text{if } k \text{ is odd.} \end{cases}$

□

## 1.2 Shimura curves

### 1.2.1 The upper half-plane

Let $\mathcal{H} = \{z = x+iy \in \mathbb{C} : y > 0\}$ be the Poincaré upper half-plane endowed with the structure provided by the hyperbolic metric,

$$\frac{1}{y^2}((dx)^2+(dy)^2).$$

The hyperbolic distance between a couple of points $z, w \in \mathcal{H}$ is given by

$$d_{\mathcal{H}}(z,w) = \operatorname{arccosh}\left(1+\frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}\right) = 2\operatorname{arctanh}\left(\left|\frac{z-w}{z-\overline{w}}\right|\right).$$

Then, the set of hyperbolic lines are the semilines which are parallel to the imaginary axis and the semicircles whose center lies on the real axis. Moreover, the hyperbolic circle of center $z = x + iy$ and radius $r$ is the euclidean circle of center $x + i\cosh(r)y$ and radius $\sinh(r)y$.

A hyperbolic polygon is a closed subset of $\mathcal{H}$ bounded by a simple curve formed by a finite number of segments of hyperbolic lines. The hyperbolic volume, with respect to the corresponding measure

$$dw = \frac{dxdy}{y^2},$$

of a hyperbolic polygon can be computed easily by means of the following result.

**Theorem 1.8** (Lambert). *Let $\mathcal{P}$ be a hyperbolic polygon with vertices $\{v_i\}_{i=1}^n$ and internal angle $\theta_i$ at the vertex $v_i$, $i = 1, \ldots, n$. Then, the hyperbolic volume of $\mathcal{P}$ is $V_{\mathcal{H}}(\mathcal{P}) = (n-2)\pi - \sum_{i=1}^n \theta_i$.* □

### 1.2.2 Homographic transformations

An homographic transformation $\gamma : \mathcal{H} \to \mathcal{H}$ is a transformation given by

$$\gamma(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R},\ ad - bc > 0.$$

This set of transformations with the composition is a group isomorphic to $\mathbf{PSL}(2, \mathbb{R})$ by means of $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \dfrac{az+b}{cz+d}$. Therefore, we have a faithful action of $\mathbf{PSL}(2, \mathbb{R})$ on $\mathcal{H}$, which gives rise to an action on $\mathcal{H}$ of $\mathbf{GL}^+(2, \mathbb{R}) \subset \mathbf{GL}(2, \mathbb{R})$, the subgroup of matrices of positive determinant. When there is no risk of confusion, we will write $\gamma$ for both the transformation and a representative matrix.

More generally, given $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbf{PGL}(2, \mathbb{R})$, we define the corresponding transformation as $\gamma(z) = \dfrac{az+b}{cz+d}$, if $ad - bc > 0$, and $\gamma(z) = \dfrac{a\overline{z}+b}{c\overline{z}+d}$, if $ad - bc < 0$. Observe that this definition does not depend on the representative.

**Definition.** Let $\gamma$ be an homographic transformation different from $id$.

(1) $\gamma$ is elliptic if it has a fixed point $z \in \mathcal{H}$.

(2) $\gamma$ is hyperbolic if it has two fixed points in $\mathbb{R} \cup \{\infty\}$.

(3) $\gamma$ is parabolic if it has a unique fixed point in $\mathbb{R} \cup \{\infty\}$.

Let $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$ be a discrete subgroup (it is usually called a fuchsian group) and consider the natural action of $\Gamma$ on the upper half-plane, which is properly discontinuous. This group gives rise therefore to an equivalence relation on the upper half-plane, namely, $z, z' \in \mathcal{H}$ are equivalent if and only if there exists $\gamma \in \Gamma$ such that $\gamma(z) = z'$.

A point $z \in \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ is called:

(1) elliptic by $\Gamma$ if there exists $\gamma \in \Gamma$ elliptic such that $\gamma(z) = z$; in particular, $z \in \mathcal{H}$. In this case we denote by $\Gamma_z = \{\gamma \in \Gamma : \gamma(z) = z\}$ its isotropy group and we define the order of the elliptic point as the order of its isotropy group;

(2) hyperbolic by $\Gamma$ if there exists $\gamma \in \Gamma$ hyperbolic such that $\gamma(z) = z$; in particular, $z \in \mathbb{R} \cup \{\infty\}$;

(3) parabolic (or a cusp) by $\Gamma$ if there exists $\gamma \in \Gamma$ parabolic such that $\gamma(z) = z$; in particular, $z \in \mathbb{R} \cup \{\infty\}$.

### 1.2.3 Fundamental domains

Let $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$ be a discrete subgroup and let $\mathcal{H}^*$ be the union of $\mathcal{H}$ and the cusps of $\Gamma$. Assume further that $\Gamma$ is cocompact, i.e. $\Gamma \backslash \mathcal{H}^*$ is compact.

**Definition.** A convex and closed hyperbolic polygon $\mathcal{D} \subset \mathcal{H}^*$ will be called a fundamental domain for $\Gamma$ if the following two conditions are satisfied:

(a) For all $z \in \mathcal{H}$, there exists $\gamma \in \Gamma$ such that $\gamma z \in \mathcal{D}$.

(b) If there exist $z \in \mathcal{D}$ and $\gamma \in \Gamma \smallsetminus \{1\}$ such that $\gamma z \in \mathcal{D}$, then $z \in \mathcal{D} \smallsetminus \mathrm{int}(\mathcal{D})$.

We will see the existence of a fundamental domain for $\Gamma$, together with an explicit description of the corresponding polygon, in Theorem 1.9.

However, we will not only consider the previous set, but we will consider as well the action of the group on it. Given a fundamental domain $\mathcal{D}$ for a group $\Gamma$, the action of $\Gamma$ induces a pairing between the sides of $\mathcal{D}$ (it may be necessary to split some sides by adding additional vertices, for example if a side is identified with itself). Whenever we refer to a fundamental domain we will associate to it a pairing between its sides and the corresponding identification between its vertices.

A cycle of a fundamental domain is an orbit of vertices under the action of $\Gamma$. A cycle is called elliptic of order $k$ if it consists of elliptic vertices of order $k$. A cycle is called parabolic if it consists of parabolic vertices (we say that its order is $\infty$). A cycle is called accidental if it is neither elliptic nor parabolic. The length of a cycle is the number of vertices in it.

**Remark.** Since the property of being elliptic of a given order (or parabolic) is invariant under the action of $\Gamma$, every cycle is in one of the cases above. The number of elliptic and parabolic cycles is an intrinsic property of the group, but the number of accidental cycles is not.

Moreover, if we consider the transformations identifying sides together with the relations provided by the cycles, we obtain a presentation of the group.

There are various methods to compute fundamental domains for the group $\Gamma$, cf. [For29], [Leh64], [AB04], [Kat92]. We will follow Ford's method and in order to make it work in all situations we will need to move to the slightly more general setting of groups conjugated to fuchsian groups inside $\mathbf{PSL}(2,\mathbb{C})$.

**Definition.** For $a,b,c,d\in\mathbb{C}$, $ad-bc=1$, consider the homographic transformation

$$\begin{array}{rcl} \gamma:\mathbb{P}^1_{\mathbb{C}}=\mathbb{C}\cup\{\infty\} & \to & \mathbb{P}^1_{\mathbb{C}}=\mathbb{C}\cup\{\infty\}, \\ z & \mapsto & \gamma(z)=\frac{az+b}{cz+d}. \end{array}$$

Assume further that $c\neq 0$, or equivalently $\gamma(\infty)\neq\infty$. Then we define the isometric circle attached to $\gamma$ as

$$C_\gamma=\{z\in\mathbb{C}:|cz+d|\leq 1\}.$$

We denote as well by $E_\gamma=\mathbb{P}^1_{\mathbb{C}}\smallsetminus C_\gamma$ the exterior of the isometric circle.

**Theorem 1.9** ([For29], [Leh64]). *Let $z_0\in\mathcal{H}$ be a non-elliptic point for $\Gamma$ and let $\gamma_0\in\mathbf{PSL}(2,\mathbb{C})$ be the complex homographic involution which permutes $z_0$ and $\infty$. Then,*

$$\overline{\gamma_0\Big(\bigcap_{\gamma\in\Gamma}E_{\gamma_0\gamma\gamma_0}\Big)}\cap\mathcal{H},$$

*where $\overline{\cdot}$ denotes the complex topological closure, is a fundamental domain for $\Gamma$.* $\square$

Some examples and more details on the computation will be given in Chapter 3.

### 1.2.4 Groups of quaternion units and the corresponding Riemann surfaces

Now we are going to use the quaternion orders previously introduced to obtain the groups $\Gamma$ we will be working with.

Let $\mathbb{H} = \left(\frac{a,b}{\mathbb{Q}}\right)$ be a rational indefinite quaternion algebra of discriminant $D$ and fix $\Phi : \mathbb{H} \hookrightarrow \mathbf{M}(2,\mathbb{R})$. Let also $N$ be a natural number such that $\gcd(D,N) = 1$. Let $\mathcal{O}(D,N)$ be an Eichler order in $\mathbb{H}$ of level $N$. Consider now $\mathcal{O}(D,N)_1 = \{\alpha \in \mathcal{O}(D,N) : \mathrm{nr}(\alpha) = 1\}$. This is a group of index 2 in $\mathcal{O}(D,N)^*$. Observe, moreover, that it is defined by $D$ and $N$ up to isomorphism.

Now, we define $\Gamma(D,N) = \Phi(\mathcal{O}(D,N)_1)/\{\pm 1\} \subset \mathbf{PSL}(2,\mathbb{R})$. $\Gamma(D,N)$ is then a cocompact fuchsian group and its action on the upper half-plane provides a quotient set $\Gamma(D,N)\backslash\mathcal{H}^*$, which naturally inherits an analytical structure from that of $\mathcal{H}$, after suitably modifying it at the elliptic and parabolic points (cf. [Shi71]). Therefore, $\Gamma(D,N)\backslash\mathcal{H}^*$ acquires a natural structure of compact Riemann surface and the fundamental domain together with the identifications provides a plane model of it.

Fix $\Gamma(D,N)$ and a fundamental domain $\mathcal{D}(D,N)$ for it. Let $e_i(D,N)$ denote the number of elliptic cycles of order $i$, $e_\infty$ the number of parabolic cycles, $V(D,N) = V_{\mathcal{H}}(\mathcal{D}(D,N))/(2\pi)$ the normalized volume and $g(D,N)$ the genus of $\Gamma(D,N)\backslash\mathcal{H}^*$.

**Theorem 1.10.** *Consider the group $\Gamma(D,N)$. Then, only elliptic points of orders 2 and 3 can occur and cusps exist if and only if $D = 1$. Moreover, we have the following formulas for all these invariants:*

*(1)* $e_2(D,N) = \begin{cases} \prod_{p|D}\left(1-\left(\frac{-4}{p}\right)\right)\prod_{p|N}\left(1+\left(\frac{-4}{p}\right)\right), & \textit{if } 4 \nmid N,\\ 0, & \textit{otherwise.}\end{cases}$

*(2)* $e_3(D,N) = \begin{cases} \prod_{p|D}\left(1-\left(\frac{-3}{p}\right)\right)\prod_{p|N}\left(1+\left(\frac{-3}{p}\right)\right), & \textit{if } 9 \nmid N,\\ 0, & \textit{otherwise.}\end{cases}$

*(3)* $e_\infty(1,N) = \sum_{\substack{d|N\\ d>0}} \phi(\gcd(d,N/d))$, *where $\phi$ is the Euler $\phi$-function.*

*(4)* $V(D,N) = \frac{N}{6}\prod_{p|D}(p-1)\prod_{p|N}(1+\frac{1}{p})$.

*(5)* $2-2g(D,N) = -V(D,N) + \frac{1}{2}e_2(D,N) + \frac{2}{3}e_3(D,N) + e_\infty(D,N)$. □

**Remark.** The formula relating the genus, the volume and the number of elliptic cycles is a particular case of a formula which is valid for more general fuchsian groups, $\Gamma$, such that $V_{\mathcal{H}}(\Gamma\backslash\mathcal{H}^*) < \infty$. More precisely, if we denote by $e_\infty(\Gamma)$ the number of parabolic cycles by $\Gamma$ and $e_i(\Gamma)$ the number of elliptic cycles of order $i$ by $\Gamma$, then

$$\frac{1}{2\pi}V_{\mathcal{H}}(\Gamma\backslash\mathcal{H}^*) = 2g(\Gamma\backslash\mathcal{H}^*) - 2 + e_\infty(\Gamma) + \sum_i \tfrac{i-1}{i}e_i(\Gamma).$$

### 1.2.5 Complex Multiplication (CM) points

Consider $\mathcal{O}(D,N)$ an Eichler order of level $N$ in an indefinite quaternion algebra $\mathbb{H}$ of discriminant $D$ together with an embedding $\Phi : \mathbb{H} \hookrightarrow \mathbf{M}(2,\mathbb{R})$.

**Definition.** Let $F$ be an imaginary quadratic field and $\Lambda \subset F$ be an order such that $\mathcal{E}^*(\mathcal{O}(D,N),\Lambda) \neq \varnothing$. Let $\phi \in \mathcal{E}^*(\mathcal{O}(D,N),\Lambda)$ and consider $z_\phi \in \mathcal{H}$ the common fixed point by all the transformations in $\Phi(\phi(F^*))$. We say that $z_\phi$ is a complex multiplication (CM) point by $\Lambda$.

It is worth noting that, if $z \in \mathcal{H}$ is fixed by the transformation attached to $\gamma = \Phi(h)$, for $h \in \mathcal{O}(D,N) \smallsetminus \mathbb{Z}$, then $z$ is a CM point by a certain order in the field $\mathbb{Q}(\sqrt{d})$, with $d = \mathrm{tr}(\gamma)^2 - 4\mathrm{nr}(\gamma) < 0$.

A very particular case of CM points are the elliptic ones. Namely, $z \in \mathcal{H}$ is elliptic of order 2 (resp. 3) if and only if it is a CM point by the ring of integers of $\mathbb{Q}(i)$ (resp. $\mathbb{Q}(\sqrt{-3})$).

Moreover, the set of CM points by a certain order $\Lambda$ is invariant under the action of $\Gamma(D,N)$, therefore we can consider $\mathrm{CM}(D,N,d,m)$ the set of CM points in $\Gamma(D,N)\backslash\mathcal{H}$ by the order $\Lambda$ of conductor $m$ in the quadratic field of discriminant $d<0$. If $D,N$ are clear from the context, then we will simply write $\mathrm{CM}(\Lambda) = \mathrm{CM}(D,N,d,m)$. We will denote $\mathrm{cm}(D,N,d,m) = \#\mathrm{CM}(D,N,d,m)$.

**Theorem 1.11.** *Given an Eichler order* $\mathcal{O}(D,N)$,

$$\mathrm{cm}(D,N,d,m) = \nu(\mathcal{O}(D,N),\Lambda(d,m);\mathcal{O}(D,N)^*),$$

*as introduced in Theorem 1.7.* □

We refer to [AB04] for the details on the proof of the statements above and on the computation of CM points.

### 1.2.6 Ring Class Fields

Let $K$ be an imaginary quadratic field, $\mathcal{O}_K$ its ring of integers and $\mathfrak{m} \subset \mathcal{O}_K$ an ideal of $K$.

**Definition.** Let $I(\mathcal{O}_K, \mathfrak{m})$ be the group of all fractional ideals in $K$ coprime to $\mathfrak{m}$ and let $P_1(\mathcal{O}_K, \mathfrak{m})$ be the subgroup of $I(\mathcal{O}_K, \mathfrak{m})$ generated by the principal ideals $\alpha\mathcal{O}_K$, with $\alpha \equiv 1 \pmod{\mathfrak{m}}$. Then, we define the ideal class group of modulus $\mathfrak{m}$ as

$$C(\mathcal{O}_K, \mathfrak{m}) = I(\mathcal{O}_K, \mathfrak{m})/P_1(\mathcal{O}_K, \mathfrak{m}).$$

Note that for $\mathfrak{m} = (1)$, it is just the classical ideal class group. As in this case, for the general one we have as well the following result.

**Proposition 1.7.** *The group $C(\mathcal{O}_K, \mathfrak{m})$ is finite.* $\square$

A similar construction to the one above is given by the ideal class groups corresponding to quadratic orders.

**Definition.** Let $\Lambda \subset \mathcal{O}_K$ be the order of conductor $f$. The ideal class group of $\Lambda$, $C(\Lambda)$, is defined as follows

$$C(\Lambda) = I(\mathcal{O}_K, f)/P_{\mathbb{Z}}(\mathcal{O}_K, f),$$

where $I(\mathcal{O}_K, f)$ denotes the subgroup of the group of fractional ideals in $K$ generated by the integral ideals of norm prime to $f$ and $P_{\mathbb{Z}}(\mathcal{O}_K, f)$ is the subgroup generated by the principal ideals of the form $\alpha\mathcal{O}_K$, with $\alpha \equiv a \pmod{f\mathcal{O}_K}$ for some integer $a$ coprime to $f$. We denote by $h(\Lambda)$ its order.

**Remark.** This group can be rewritten in terms of proper ideals in $\Lambda$, but the presentation given above will be enough for our purposes.

**Theorem 1.12.** *Let $\Lambda$ be the order of conductor $f$ in $K$. Then,*

$$h(\Lambda) = \frac{h(\mathcal{O}_K) f}{[\mathcal{O}_K^* : \Lambda^*]} \prod_{p \mid f} \left(1 - \left(\frac{D_K}{p}\right) \frac{1}{p}\right),$$

*where $D_K$ stands for the discriminant of $K$ and $\left(\frac{D_K}{\cdot}\right)$ denotes the Kronecker symbol.* $\square$

**Proposition 1.8.** *Let $L|K$ be a finite abelian field extension, $\mathfrak{p}$ a prime of $K$ unramified in $L$ and $\mathfrak{P}$ a prime of $L$ over $\mathfrak{p}$. Then, there exists a unique element $\left(\frac{L|K}{\mathfrak{p}}\right) \in \mathrm{Gal}(L|K)$ such that for any $a \in \mathcal{O}_L$,*

$$\left(\frac{L|K}{\mathfrak{p}}\right)(a) \equiv a^{\mathrm{Norm}(\mathfrak{p})} \pmod{\mathfrak{P}}.$$

*Moreover, this element is independent of the choice of $\mathfrak{P}$.* □

**Definition.** Let $L|K$ be a finite abelian extension and $\mathfrak{m}$ an ideal of $K$ which is divisible by all ramified primes in $L$. Given

$$\mathfrak{a} = \prod_{i=1}^{r} \mathfrak{p}_i^{r_i}, \quad r_i \in \mathbb{Z},$$

for certain unramified primes $\mathfrak{p}_i \nmid \mathfrak{m}$. We define the Artin symbol as

$$\left(\frac{L|K}{\mathfrak{a}}\right) = \prod_{i=1}^{r} \left(\frac{L|K}{\mathfrak{p}_i}\right)^{r_i}.$$

**Theorem 1.13.** *Let $L|K$ be a finite abelian extension and $\mathfrak{m}$ an ideal of $K$ which is divisible by all ramified primes in $L$.*

*(1) The map*

$$\begin{array}{rcl} \Phi_{\mathfrak{m}} : I(\mathcal{O}_K, \mathfrak{m}) & \to & \mathrm{Gal}(L|K), \\ \mathfrak{a} & \mapsto & \left(\frac{L|K}{\mathfrak{a}}\right) \end{array}$$

*is a surjective morphism.*

*(2) If the exponents of the primes in $\mathfrak{m}$ are large enough,*

$$P_1(\mathcal{O}_K, \mathfrak{m}) \subset \ker(\Phi_{\mathfrak{m}}).$$

□

The ideal $\mathfrak{m}$ for which the second part of the previous theorem holds is not unique, but can be chosen in a minimal way according to the following theorem.

**Theorem 1.14.** *Given $L|K$ a finite abelian extension, there exists $\mathfrak{f} = \mathfrak{f}(L|K)$ such that:*

*(1) A prime of $K$ ramifies in $L$ if and only if it divides $\mathfrak{f}$.*

*(2) If $\mathfrak{m}$ is an ideal which is divisible by all the primes of $K$ ramified in $L$, then $P_1(\mathcal{O}_K, \mathfrak{m}) \subset \ker(\Phi_{\mathfrak{m}})$ if and only if $\mathfrak{f}|\mathfrak{m}$.*

*$\mathfrak{f}$ is called the conductor of $L|K$.* □

The last central result of the theory is that we can go the other way around.

**Theorem 1.15.** *Let $\mathfrak{m}$ be an ideal of $\mathcal{O}_K$ and let $H$ be a subgroup of $I(\mathcal{O}_K, \mathfrak{m})$ such that*

$$P_1(\mathcal{O}_K, \mathfrak{m}) \subset H \subset I(\mathcal{O}_K, \mathfrak{m}).$$

*Then, there exists a unique abelian extension $L|K$ such that all ramified primes divide $\mathfrak{m}$ and $H = \ker(\Phi_{\mathfrak{m}})$.* □

Therefore, we have a way to attach finite abelian extensions of $K$ to subgroups $H$ of the form

$$P_1(\mathcal{O}_K, \mathfrak{m}) \subset H \subset I(\mathcal{O}_K, \mathfrak{m}).$$

**Definition.** Given an ideal $\mathfrak{m}$, we define the ray class field attached to $\mathfrak{m}$ as the finite abelian extension of $K$, $K^{\mathfrak{m}}|K$, corresponding to $H = P_1(\mathcal{O}_K, \mathfrak{m})$. Its Galois group is isomorphic to $C(\mathcal{O}_K, \mathfrak{m})$. If $\mathfrak{m} = (1)$, it is called the Hilbert Class Field and its Galois group is isomorphic to the ideal class group.

**Definition.** Given an order $\Lambda \subset \mathcal{O}_K$ of conductor $f$, we have

$$P_1(\mathcal{O}_K, f) \subset P_{\mathbb{Z}}(\mathcal{O}_K, f) \subset I(\mathcal{O}_K, f),$$

and therefore there exists an abelian extension of $K$, $K^{\Lambda}|K$, contained in $K^f$ whose Galois group is isomorphic to $C(\Lambda)$. $K^{\Lambda}|K$ is called the Ring Class Field attached to $\Lambda$.

Note that Theorem 1.12 provides a formula for the degree of the Ring Class Field.

This theory, which has been presented up to now in terms of ideals, can be restated in terms of ideles. At some point it will be useful to have this interpretation and therefore we are going to introduce it now. We will keep talking about imaginary quadratic fields for coherence.

**Definition.** An adele $\alpha$ of $K$ is a family $\alpha = (\alpha_{\mathfrak{p}})_{\mathfrak{p}}$, with $\alpha_{\mathfrak{p}} \in K_{\mathfrak{p}}$, indexed by the primes $\mathfrak{p}$ of $K$, such that $\alpha_{\mathfrak{p}}$ is integral except for a finite number of $\mathfrak{p}$. We denote by $\mathbb{A}_K$ the set of adeles of $K$, which has structure of ring by the operations induced componentwise. We denote by $\mathbb{I}_K = \mathbb{A}_K^*$ the ideles of $K$. The group $C_K = \mathbb{I}_K/K^*$ is called the idele class group.

Given an ideal $\mathfrak{m} = \prod_{i=1}^n \mathfrak{p}_i^{n_i}$, we define

$$\mathbb{I}_K^{\mathfrak{m}} = \{\alpha = (\alpha_{\mathfrak{p}}) \in \mathbb{I}_K : \alpha_{\mathfrak{p}} \in \mathcal{O}_{K_{\mathfrak{p}}}, \alpha_{\mathfrak{p}_i} \in 1 + \mathfrak{p}_i^{n_i}\}.$$

We define as well $C_K^{\mathfrak{m}} = \mathbb{I}_K^{\mathfrak{m}} K^*/K^*$.

**Theorem 1.16.** *The morphism*

$$\begin{array}{rcl} \kappa : \mathbb{I}_K & \to & I(\mathcal{O}_K, 1), \\ \alpha & \mapsto & \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_{\mathfrak{p}}(\alpha_{\mathfrak{p}})}, \end{array}$$

*induces an isomorphism*

$$\kappa_m : C_K/C_K^{\mathfrak{m}} \to C(\mathcal{O}_K, \mathfrak{m}).$$

□

Therefore, given $a \in C_K$ we can attach to it Artin symbols. In the adelic notation, they are denoted $(a, L|K)$.

### 1.2.7 The Shimura curve $X(D, N)$

Let $D, N$ be positive integers. Assume that $D$ is the product of an even number of primes and that $N$ is coprime to $D$.

Now consider an indefinite quaternion algebra $\mathbb{H}$ over $\mathbb{Q}$ of discriminant $D$, an embedding $\Phi : \mathbb{H} \hookrightarrow \mathbf{M}(2, \mathbb{R})$ and $\mathcal{O}(D, N) \subset \mathbb{H}$ an Eichler order of level $N$. Denote by $\Gamma(D, N)$ the image in $\mathbf{PSL}(2, \mathbb{R})$ of $\Phi(\mathcal{O}(D, N)_1) \subset \mathbf{SL}(2, \mathbb{R})$. As we have seen in Section 1.2.4, $\Gamma(D, N)$ is a fuchsian group acting on $\mathcal{H}^*$ and the quotient set $\Gamma(D, N)\backslash\mathcal{H}^*$ has structure of compact Riemann surface. Therefore, it can be seen as the set of points of a projective curve over $\mathbb{C}$. However, this Riemann surface can be naturally identified with the set of complex points of a projective curve over $\mathbb{Q}$, according to the following theorem, which is a particular case of a more general construction due to Shimura (cf. [Shi67], [Shi70]).

**Theorem 1.17.** *There exists a projective curve $X(D,N)$ defined over $\mathbb{Q}$ and a map $j_{D,N} : \mathcal{H}^* \to X(D,N)(\mathbb{C})$ with the following properties:*

*(1) The map $j_{D,N}$ gives an analytic isomorphism between $\Gamma(D,N)\backslash\mathcal{H}^*$ and $X(D,N)(\mathbb{C})$.*

*(2) Let $K$ be a quadratic imaginary field and $\Lambda \subset \mathcal{O}_K$ an order such that $\mathcal{E}^*(\mathcal{O}(D,N),\Lambda) \neq \varnothing$. Let $\phi \in \mathcal{E}^*(\mathcal{O}(D,N),\Lambda)$ and denote by $z_\phi \in \mathcal{H}$ the corresponding CM point. Then*

$$K^\Lambda = K(j_{D,N}(z_\phi)),$$

*i.e. the coordinates of the point $j_{D,N}(z_\phi)$ generate the Ring Class Field attached to $\Lambda$ over $K$.*

*The curve $X(D,N)$ is uniquely determined up to isomorphism by these conditions. We call the couple $(X(D,N), j_{D,N})$ a canonical model for the Shimura curve.* □

**Remark.** This correspondence between the set of complex points of the curve and the Riemann surface allows us to translate the notions we had on $\mathcal{H}$ to $X(D,N)$. For example we will talk about CM points indistinctly on $\mathcal{H}$ or $X(D,N)(\mathbb{C})$.

**Example.** The most simple example of canonical model is the one provided by $X_0(1) = X(1,1)$, the classical modular curve, together with the Klein $j$-function.

The key point in giving $\Gamma(D,N)\backslash\mathcal{H}^*$ a structure of curve over $\mathbb{Q}$ is that its points can be identified with isomorphism classes of principally polarized abelian surfaces with quaternionic multiplication by $\mathcal{O}(D,1)$ and a level $N$ structure. More details can be found in Section 3.5.1, where we are going to explicitly discuss how to recover these polarized abelian surfaces in the case $N = 1$.

The conditions in the theorem are more than enough to characterize the canonical model; if we wished, we could even restrict to the set of CM points by maximal orders. This model has several additional properties which will be useful later on.

**Proposition 1.9.** *If $D > 1$, the curve $X(D,N)$ has no real points.* □

Moreover, we can give a description of the field of rational functions of the curves $X(D,N)$ in terms of functions in $\mathcal{H}$ as follows.

**Proposition 1.10.** *Let $F \subset \mathbb{C}$ be a field. The field $\{g \circ j_{D,N} : g \in F(X(D,N))\}$ coincides with the field of meromorphic functions $f : \mathcal{H}^* \to \mathbb{C} \cup \{\infty\}$ such that $f \circ \gamma = f$ for all $\gamma \in \Gamma(D,N)$ (i.e. automorphic functions by $\Gamma(D,N)$ as defined in page 24) and such that $f(z_\phi) \in F \cdot K^\Lambda \cup \{\infty\}$, for any $\phi \in \mathcal{E}^*(\mathcal{O}(D,N),\Lambda)$.* □

From now on, we will identify the field $\mathbb{Q}(X(D,N))$ with the field of (meromorphic) automorphic functions with the above condition on the values at CM points.

Another property which we will use later on is the Shimura reciprocity law. We will now give an explicit version of it.

Given a quadratic imaginary field $K$ and an order $\Lambda \subset \mathcal{O}_K$, denote by $G = \mathrm{Gal}(K^\Lambda | K) = C(\Lambda)$. Then, there exists an action of $G$ on the set of equivalence classes of optimal embeddings $\mathcal{E}^*(\mathcal{O}(D,N),\Lambda)/\mathcal{O}(D,N)^*$, which we can describe as follows.

Fix $\phi \in \mathcal{E}^*(\mathcal{O}(D,N),\Lambda)$ and denote by $z_\phi \in \mathcal{H}$ the corresponding fixed point. Let $\sigma \in G$ and consider a fractional ideal $\mathfrak{a}$ such that $\left(\frac{K^\Lambda|K}{\mathfrak{a}}\right) = \sigma$. Then, $I = \phi(\mathfrak{a})\mathcal{O}(D,N)$ is an $\mathcal{O}(D,N)$-right ideal and therefore principal, by Theorem 1.5. Let $\alpha_\sigma \in \mathbb{H}^*$ (in fact in $\mathcal{O}(D,N)$) be an element of positive norm such that $I = \alpha_\sigma \mathcal{O}(D,N)$.

**Theorem 1.18** (Shimura reciprocity law). *With the notations above, the element $\sigma \in G$ maps the optimal embedding $i$ with corresponding fixed point $z_\phi \in \mathcal{H}$ to the optimal embedding $i^\sigma = \alpha_\sigma^{-1} i \alpha_\sigma$ with fixed point $\alpha_\sigma^{-1} z$. Equivalently, if $f \in \mathbb{Q}(X(D,N))$, then $f(z_\phi)^\sigma = f(\alpha_\sigma^{-1} z_\phi)$.* □

In Theorem 1.6 we introduced elements $w_m \in \mathrm{Nor}(\mathcal{O}(D,N))$, for every $m \| DN$. Consequently, the corresponding elements $\omega_m \in \mathbf{PSL}(2,\mathbb{R})$ normalize $\Gamma(D,N)$ and therefore induce analytic involutions on the Riemann surface $\Gamma(D,N)\backslash\mathcal{H}^*$.

**Proposition 1.11.** *The automorphisms induced on the curve $X(D,N)$ by the analytic involutions $\omega_m$ are defined over $\mathbb{Q}$. The group generated by these involutions is called the Atkin-Lehner group, and will be denoted by $W_{DN}$. Moreover $W_{DN} \cong (\mathbb{Z}/2\mathbb{Z})^r$, where $r$ is the number of different primes dividing $DN$.* □

We will both refer to the involution on $\Gamma(D,N)\backslash\mathcal{H}^*$ and on $X(D,N)$ by $\omega_m$. A lot of properties regarding the action of these involutions have been widely studied: their fixed points (cf. [Ogg83]), which we will describe and use in Section 4.1; their relation with the action of the Galois group of the Ring

Class Field when acting on a set of CM points by a given order (cf. [Jor81], [GR06]), which we will describe and use in Section 6.2.1, etc.

As we observed above, the Klein $j$-function provides a canonical model for the classical modular curve $X_0(1)$. For this function, a classical congruence involving the Atkin-Lehner involution of prime level $p$ holds, namely

$$(j - j^p \circ \omega_p) \cdot (j \circ \omega_p - j^p) \equiv 0 \pmod{p}.$$

This congruence generalizes to the well-known Eichler-Shimura congruence formula.

**Theorem 1.19** (Eichler-Shimura congruence formula, [Shi67]). *Let $j_D$ be a canonical model for the Shimura curve $X(D,1)$. There exists a finite set of primes $S$ containing all those primes dividing $D$ such that, if $p \notin S$ and $j_D^p$ is the function we obtain from $j_D$ when we rise every component to the power $p$, then*

$$(j_D - j_D^p \circ \omega_p) \cdot (j_D \circ \omega_p - j_D^p) \equiv 0 \pmod{p}.$$

□

**Remark.** There are several equivalent ways to interpret the congruence mod $p$ in the previous theorem. The easiest way is to understand it as the reduction of the correspondence given by the image of $(j_D, j_D \circ \omega_p)$ in $X(D,1) \times X(D,1)$ (cf. [Shi67], [Shi71]). But it can also be understood as a congruence between the values of the functions at certain CM points (cf. [Shi67], [Mor81]).

## 1.3 Automorphic forms

Let $\Gamma$ be a cocompact fuchsian group and $\Gamma\backslash\mathcal{H}^*$ the corresponding Riemann surface as introduced in the previous section.

Given $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}(2,\mathbb{R})$ and $z \in \mathbb{C}$, we define

$$j(\sigma, z) = cz + d.$$

For every integer $k$, every $\sigma \in \mathbf{GL}^+(2,\mathbb{R})$, and every function $f : \mathcal{H} \to \mathbb{C}$, we define

$$(f|_k\sigma)(z) = \det(\sigma)^{k/2} j(\sigma, z)^{-k} f(\sigma(z)).$$

Then, for another $\tau \in \mathbf{GL}^+(2,\mathbb{R})$, we have that

$$f|_k\sigma\tau = (f|_k\sigma)|_k\tau.$$

**Definition.** Let $k$ be an even integer. A $\mathbb{C}$-valued function $f$ on $\mathcal{H}$ is called an (meromorphic) automorphic form of weight $k$ with respect to $\Gamma$ if it satisfies the following three conditions:

(a) $f$ is meromorphic on $\mathcal{H}$;

(b) $f|_k\gamma = f$ for all $\gamma \in \Gamma$;

(c) $f$ is meromorphic at every cusp of $\Gamma$.

**Remark.** Note that since $k$ is even, $f|_k\gamma$ does not depend on the representative in $\mathbf{SL}(2,\mathbb{R})$ we choose.

The last condition only applies when $\Gamma$ has cusps. For example, when we deal with the groups $\Gamma(D,N)$ introduced above, this translates into $D=1$.

The condition of being meromorphic at every cusp can be translated as follows. For a given cusp $s$, we choose a transformation $\rho$ sending $s$ to $\infty$. Then, the function $f|_k\rho^{-1}$ is invariant by a translation of length $h$. Therefore, there exists a suitable neighborhood of 0 such that $f|_k\rho^{-1}(z) = \Phi(e^{2\pi iz/h})$. We say that $f$ is meromorphic at $s$ if $\Phi$ is meromorphic at 0. In particular, if we consider a power expansion of $\Phi$ around 0, we obtain a Fourier expansion for $f|_k\rho^{-1}$,

$$f|_k\rho^{-1}(z) = \sum_{n\geq n_0} c_n e^{2\pi inz/h}.$$

If $k=0$, we recover the notion of automorphic function with respect to $\Gamma$.

**Definition.** Let us define $A_k(\Gamma)$ as the set of all (meromorphic) automorphic forms of weight $k$ with respect to $\Gamma$, in particular $A_0(\Gamma)$ is the set of automorphic functions. We denote by $G_k(\Gamma)$ the set of all holomorphic (including the cusps) automorphic forms of weight $k$ with respect to $\Gamma$ and by $S_k(\Gamma)$ the subset of $G_k(\Gamma)$ consisting of those forms vanishing at the cusps. The elements of $S_k(\Gamma)$ are usually called cusp forms.

Note that if $\Gamma$ has no cusps, which is the case we are more interested in, $G_k(\Gamma) = S_k(\Gamma)$. In the case of the groups $\Gamma(1,N)$, instead of referring to these functions as automorphic functions and automorphic forms, we call them modular functions and modular forms.

Observe that $A_0(\Gamma)$ is the field of automorphic functions and can be identified with the field of meromorphic functions on the Riemann surface $\Gamma\backslash\mathcal{H}^*$.

Therefore it is a field of transcendence degree 1 over $\mathbb{C}$. Moreover all the spaces $A_k(\Gamma)$ are vector spaces of dimension one over $A_0(\Gamma)$.

On the other hand, $G_k(\Gamma), S_k(\Gamma)$ are $\mathbb{C}$-vector spaces of finite dimension. A formula for the dimension of these spaces is given in the following theorem.

**Theorem 1.20.** *Let $g$ be the genus of $\Gamma\backslash\mathcal{H}^*$, $e_\infty$ the number of inequivalent cusps and $e_1,\dots,e_r$ the orders of the elliptic cycles of $\Gamma$. Then, for an even integer $k$,*

$$\dim_{\mathbb{C}} G_k(\Gamma) = \begin{cases} (k-1)(g-1) + \frac{k}{2}e_\infty + \sum_{i=1}^{r}\lfloor\frac{k(e_i-1)}{2e_i}\rfloor, & \text{if } k > 2, \\ g + e_\infty - 1, & \text{if } k = 2, e_\infty > 0, \\ g, & \text{if } k = 2, e_\infty = 0, \\ 1, & \text{if } k = 0, \\ 0, & \text{if } k < 0, \end{cases}$$

*and*

$$\dim_{\mathbb{C}} S_k(\Gamma) = \begin{cases} (k-1)(g-1) + (\frac{k}{2}-1)e_\infty + \sum_{i=1}^{r}\lfloor\frac{k(e_i-1)}{2e_i}\rfloor, & \text{if } k > 2, \\ g, & \text{if } k = 2, \\ 1, & \text{if } k = 0, e_\infty = 0, \\ 0, & \text{if } k = 0, e_\infty > 0, \\ 0, & \text{if } k < 0. \end{cases}$$

□

From now on we will restrict ourselves to the case of the groups $\Gamma(D,N)$. Hence, we fix an indefinite quaternion algebra of discriminant $D$ and an Eichler order $\mathcal{O} = \mathcal{O}(D,N)$ of level $N$, together with an embedding $\Phi : \mathbb{H} \to \mathbf{M}(2,\mathbb{R})$ and the corresponding group $\Gamma(D,N) \subset \mathbf{PSL}(2,\mathbb{R})$.

For the field of automorphic functions, $A_0(\Gamma(D,N))$, we have already seen that there exists a canonical subfield $A_0(\Gamma(D,N))_{\mathbb{Q}} = \mathbb{Q}(X(D,N))$ such that $A_0(\Gamma(D,N)) = A_0(\Gamma(D,N))_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. Now we are going to see how the same is done with $A_k(\Gamma(D,N))$. In the classical modular case, this is done using the coefficients of the Fourier expansions introduced above. However, in the general case, Fourier expansions are no longer available.

Notice that $A_k(\Gamma(D,N))$ is a vector space of dimension 1 over $A_0(\Gamma(D,N))$ and the same has to happen for the corresponding subspaces over $\mathbb{Q}$. Therefore, we only need to find a suitable element for each of these spaces. Moreover, if $f \in A_2(\Gamma(D,N))_{\mathbb{Q}}$, then $f^k \in A_{2k}(\Gamma(D,N))_{\mathbb{Q}}$, therefore it will be sufficient to choose an element in $A_2(\Gamma(D,N))_{\mathbb{Q}}$. In order to do so, we only need to note that, if $f \in A_0(\Gamma(D,N))$, then $\frac{df}{dz} \in A_2(\Gamma(D,N))$. Therefore we could take $\frac{df}{dz}A_0(\Gamma(D,N))_{\mathbb{Q}}$ for any non-constant function $f \in A_0(\Gamma(D,N))_{\mathbb{Q}}$. The fact that $A_0(\Gamma(D,N))_{\mathbb{Q}}$ is an extension of $\mathbb{Q}$ of transcendence degree 1 and such that $\mathbb{Q}^{ab} \cap A_0(\Gamma(D,N))_{\mathbb{Q}} = \mathbb{Q}$ ensures the independence of the chosen $f$. However, in order for the definition to agree with the modular classical structure we will take $A_2(\Gamma(D,N))_{\mathbb{Q}} = \frac{1}{2\pi i}\frac{df}{dz}A_0(\Gamma(D,N))_{\mathbb{Q}}$ for any non-constant function $f \in A_0(\Gamma(D,N))_{\mathbb{Q}}$, for which all previous assertions also hold. From this, we obtain sets $G_k(\Gamma(D,N))_{\mathbb{Q}}$ and $S_k(\Gamma(D,N))_{\mathbb{Q}}$ intersecting with the set of holomorphic forms or holomorphic forms vanishing at the cusps. Replacing $\mathbb{Q}$ with a number field $K$ we obtain the definition of automorphic forms over $K$.

From the definition it is clear that $A_k(\Gamma(D,N)) = A_k(\Gamma(D,N))_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. However, although it is not obvious in any way, even in the case of modular forms, we can choose a basis of any of the spaces $G_k(\Gamma(D,N))$ and $S_k(\Gamma(D,N))$ consisting of elements of the corresponding spaces over $\mathbb{Q}$.

Finally we are going to introduce a family of Hecke operators acting on the corresponding spaces of automorphic forms. Given a positive integer $m$ such that $\gcd(m, DN) = 1$, we consider $\mathcal{O}(D,N)'_m$ the set of primitive elements of norm $m$ and we choose a set $\alpha_i \in \mathbf{GL}^+(2,\mathbb{R})$, $i = 1,\dots,r$, corresponding by the embedding $\Phi$ to a system of representatives of the quotient set $\mathcal{O}(D,N)_1\backslash\mathcal{O}(D,N)'_m$. Then, we can define an operator

$$T_m : A_k(\Gamma(D,N)) \to A_k(\Gamma(D,N))$$

such that

$$T_m f = \det(\alpha)^{k/2-1}\sum_{i=1}^{r} f|_k\alpha_i.$$

This operator acts as well on the corresponding sets of holomorphic automorphic forms and of cusp forms. It can be seen that there exists a basis for $S_k(\Gamma(D,N))_{\mathbb{Q}}$ consisting of eigenvectors for all the $T_m$, usually called eigenforms. We will not go into detail in the properties of the Hecke operators since just the definition will suffice for our purposes. More details on how to explicitly compute their action on the spaces of cusps forms will be given in Chapter 6.

# Chapter 2

# Kroneckerian polynomials

Let $\mathcal{H}$ be the upper half-plane and $j : \mathcal{H} \to \mathbb{C}$ the classical modular $j$-function, which is a generator of the field of rational functions of the modular curve $X_0(1)$ over $\mathbb{Q}$. For any integer $N \geq 1$, there exists an irreducible polynomial $\Phi_N(X,Y) \in \mathbb{Z}[X,Y]$ such that $\Phi_N(j(\tau), j(N\tau)) = 0$ for all $\tau \in \mathcal{H}$. For $N = 1$, $\Phi_1(X,Y) = X - Y$, and for $N > 1$, $\Phi_N(X,Y)$ is a symmetric polynomial which provides a singular model for the modular curve $X_0(N)$ in $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$. The equation $\Phi_N(X,Y) = 0$ was already studied by Kronecker and is classically called the modular equation of level $N$ (cf. [Lan87]).

Let $\mathbb{H}_D$ be a rational indefinite quaternion algebra of discriminant $D > 1$, and fix an embedding $\Phi : \mathbb{H}_D \hookrightarrow \mathbf{M}(2, \mathbb{R})$. For a positive integer $M$ coprime to $D$, let $X(D,M)$ be the Shimura curve attached to an Eichler order of level $M$ in $\mathbb{H}_D$, $\mathcal{O} = \mathcal{O}(D,M)$, and assume that the quotient $X$ of $X(D,M)$ by a certain subgroup $W$ of the Atkin-Lehner group of level $DM$ has genus 0. $X$ is a curve defined over $\mathbb{Q}$ attached to a certain group $\Gamma$, but in contrast to the classical modular case we introduced above, this curve may not have rational or even real points. Therefore, there exists in general no generator of the field of rational functions defined over $\mathbb{Q}$. Moreover, the classical description of modular functions by means of Fourier expansions cannot be extended to this case due to the absence of cusps. Let us take $X$ as our base curve. For a positive integer $N$ coprime to $DM$ and a complex generator $f$ of the field of rational functions, which we will call a Hauptmodul, we construct a polynomial $\Psi_{\Gamma,N,f}(X,Y) \in \mathbb{C}[X,Y]$ playing the role of the modular polynomial of level $N$, which we call the kroneckerian polynomial of level $N$. It provides a complex model for the corresponding quotient of the Shimura curve of discriminant $D$ and level $MN$, $X(D, MN)^W$. Note that this curve has in general genus greater than 0. In Theorem 2.1 we prove that this polynomial is irreducible, symmetric (if $N > 1$) and that, if the function $f$ is defined over a number field $K$, then its coefficients can also be taken in $K$. In Proposition 2.5 we study the roots of the polynomial $\Psi_{\Gamma,N,f}(X, X)$ together with their multiplicities and we obtain a formula for its decomposition similar to the factorization in class equations which holds in the modular case, providing therefore a way to compute the exact values of the function $f$ at certain CM points.

Then, for the particular curves $X(D, 1)$, $D = 6, 10, 22$, we introduce some specific Hauptmoduln $t_D$ defined over some explicit quartic number fields. In the case $D = 6$, this Hauptmodul is the one introduced in [BT07b]. For these choices, we prove that the model provided by $\Psi_{\Gamma,N,t_D}$ descends to $\mathbb{Q}$, even though these Hauptmoduln are not defined over $\mathbb{Q}$, and that a congruence similar to the Kronecker congruence formula holds.

Our approach is similar to the usual construction for the modular case, cf. [Cox89, Lan87], except for the proof of the congruence relation, which in our case relies on the Shimura reciprocity law. In particular, the explicit description of the polynomials $\Psi_{\Gamma,N,f}$ will allow us to prove additional arithmetic properties and to effectively compute them without making any assumption on the genus of $X(D, MN)^W$, using only a differential equation satisfied by the Hauptmodul, which provides expansions for it around suitable CM points. The explicit computation of these polynomials, together with its application to the determination of the values of $f$ at certain CM points, will be considered in Chapter 6.

## 2.1 The polynomials $\Psi_{\Gamma,N,f}$

Let $\mathbb{H}_D$ be the rational indefinite quaternion algebra of discriminant $D > 1$ and $\Phi : \mathbb{H}_D \hookrightarrow \mathbf{M}(2, \mathbb{R})$ an embedding of $\mathbb{H}_D$ into the real algebra of $2 \times 2$ matrices. Moreover, we also fix a positive integer $M$ coprime to $D$, a maximal order $\mathcal{O}(D, 1)$ inside $\mathbb{H}_D$ and an Eichler order of level $M$, $\mathcal{O} = \mathcal{O}(D, M) \subseteq \mathcal{O}(D, 1)$, with $\mathbb{Z}$-basis $\{1, I, J, K\}$. We will be mainly interested in the case $M = 1$; however, at some points it will be useful to have considered this more general setting, which only amounts to slightly more involved notations and statements.

For every positive integer $N$ coprime to $DM$, we consider Eichler orders of level $MN$ contained in $\mathcal{O}(D, M)$. Denote $\mathcal{O} = \mathcal{O}(1)$ and $\mathcal{O}(N) = \mathcal{O}(D, MN)$;

then, for every integer $k$, we write $\mathcal{O}(N)_k$ for the set of quaternions of norm $k$ in $\mathcal{O}(N)$. In particular, $\mathcal{O}(N)_1$ coincides with the group of units of norm 1 of $\mathcal{O}(N)$. This group will be identified with its image in $\mathbf{SL}(2,\mathbb{R})$ under $\Phi$. We denote $\Gamma(N) = \Gamma(D, MN)$, the image of $\Phi(\mathcal{O}(N)_1)$ by the standard projection $\mathbf{SL}(2,\mathbb{R}) \to \mathbf{PSL}(2,\mathbb{R})$. If $k = l$ is a prime, the notation $\mathcal{O}_l$ can both denote the elements of norm $l$ or $\mathcal{O} \otimes \mathbb{Z}_l$. When confusion may arise, we will denote $\mathcal{O}_{(l)} = \mathcal{O} \otimes \mathbb{Z}_l$.

Consider in $\mathcal{H}$ the usual action of the group $\mathbf{PSL}(2,\mathbb{R})$ via homographic transformations. Then, as we have seen in Section 1.2.7, the corresponding action of the group $\Gamma(N)$ gives rise to a compact Riemann surface, $\Gamma(N)\backslash\mathcal{H}$, which we can identify canonically with the set of complex points of a complete non-singular curve defined over $\mathbb{Q}$, which we denote by $X(N) = X(D, MN)$, the Shimura curve of discriminant $D$ and level $MN$.

Let $m \geq 1$ be an integer such that $m$ divides exactly $DMN$, namely, $m$ divides $DMN$ and it is coprime to $DMN/m$. In this situation, we choose an element $w_m \in \mathcal{O}(N)_m$ which normalizes $\mathcal{O}(N)$. This element is not unique but, for a fixed $m$, all these $w_m$ define the same automorphism of $\Gamma(N)\backslash\mathcal{H}$. This automorphism is involutive (the identity when $m = 1$). We will denote by $\omega_m$ the corresponding element of $\mathbf{PSL}(2,\mathbb{R})$ as well as the corresponding automorphism of $\Gamma(N)\backslash\mathcal{H}$. It follows from the description given before Theorem 1.6, that the element $w_N$ allows us to write the order $\mathcal{O}(N)$ as $\mathcal{O}(N) = \mathcal{O} \cap w_N^{-1}\mathcal{O}w_N$ and, therefore, its group of units of norm 1 as $\mathcal{O}(N)_1 = \mathcal{O}_1 \cap w_N^{-1}\mathcal{O}_1 w_N$.

Recall that, by Proposition 1.11, the involutions $\omega_m$ such that $m$ divides exactly $DMN$ form the Atkin-Lehner group $W_{DMN}$. The structure of this group is given by $W_{DMN} \cong C_2^r$, where $C_2$ is the cyclic group of order 2 and $r$ is the number of distinct primes dividing $DMN$. These involutions are, in fact, rational automorphisms. For any subgroup $W \subseteq W_{DMN}$, we denote by $X(N)^W$ the corresponding quotient curve by $W$. In the situation above, the particular involution $\omega_N$ will play a central role.

To define the kroneckerian polynomials of level $N$ we work with the groups $\Gamma(D,M)\langle W\rangle$, the groups generated inside $\mathbf{PSL}(2,\mathbb{R})$ by $\Gamma(D,M)$ and a subgroup $W$ of the full Atkin-Lehner group $W_{DM}$ of level $DM$. For a subgroup $W \subseteq W_{DM}$ we will abbreviate $\Gamma = \Gamma(D,M)\langle W\rangle$ when there is no risk of confusion.

For every positive integer $N$ coprime to $DM$, let $\mathcal{O}'_N \subseteq \mathcal{O}_N \subseteq \mathcal{O}$ be the subset of primitive elements of norm $N$,

$$\mathcal{O}'_N = \{a + bI + cJ + dK \in \mathcal{O}_N : \gcd(a,b,c,d) = 1\} \subseteq \mathcal{O}_N.$$

Note that this definition does not depend on the $\mathbb{Z}$-basis $\{1, I, J, K\}$ which we consider in $\mathcal{O}$ and that, if $N$ is squarefree, then $\mathcal{O}'_N = \mathcal{O}_N$.

Now, as before, let $w_N \in \mathcal{O}_N$ be an element that represents the involution of level $N$ on $X(N)$ and consider the sets of orbits by the action of the groups $\mathcal{O}(N)_1$ and $\mathcal{O}_1$

$$R(N) = \mathcal{O}(N)_1 \backslash \mathcal{O}_1 \qquad \text{and} \qquad \tilde{R}(N) = \mathcal{O}_1 \backslash \mathcal{O}_1 w_N \mathcal{O}_1.$$

By abuse of notation, we also write $R(N)$ and $\tilde{R}(N)$ for a set of representatives of each of these sets. The first result we prove is the following.

**Proposition 2.1.** *Let $N$ be a positive integer coprime to $DM$. Then,*

*(1) There exists a bijection between $R(N)$ and $\tilde{R}(N)$.*

*(2) The cardinality of the sets $R(N)$ and $\tilde{R}(N)$ is given by the classical $\psi$ function:*

$$\#R(N) = \#\tilde{R}(N) = \psi(N) = N \prod_{\substack{p|N \\ p \text{ prime}}} (1 + p^{-1}).$$

*(3) $\mathcal{O}_1 w_N \mathcal{O}_1 = \mathcal{O}'_N$.*

*Proof.* (1) Consider the map $r : \mathcal{O}_1 \longrightarrow \mathcal{O}_1 \backslash \mathcal{O}_N$ defined by $r(\gamma) = \mathcal{O}_1 w_N \gamma$, for any $\gamma \in \mathcal{O}_1$. Then, $r$ factors through the quotient set $R(N)$ and the corresponding map $\overline{r} : R(N) \longrightarrow \mathcal{O}_1 \backslash \mathcal{O}_N$ is injective: for $\gamma$, $\gamma' \in \mathcal{O}_1$, the equality $\mathcal{O}_1 w_N \gamma = \mathcal{O}_1 w_N \gamma'$ holds if and only if $\gamma' \gamma^{-1} \in \mathcal{O}_1 \cap w_N^{-1} \mathcal{O}_1 w_N = \mathcal{O}(N)_1$, and this is equivalent to $\mathcal{O}(N)_1 \gamma' = \mathcal{O}(N)_1 \gamma$. Moreover, the image of $r$ is, clearly, $\tilde{R}(N)$, which yields the first statement.

(2) The cardinality of $R(N)$, and therefore, that of $\tilde{R}(N)$, is computed in [Vig80] and [AB04].

(3) From the local description of the ideal $w_N \mathcal{O}(N)$ given in page 9, it follows that $w_N \in \mathcal{O}'_N$; then, we have that $\mathcal{O}_1 w_N \mathcal{O}_1 \subseteq \mathcal{O}'_N$ and this fact gives us an inclusion

$$\tilde{R}(N) \hookrightarrow \mathcal{O}_1 \backslash \mathcal{O}'_N. \qquad (*)$$

Now we will see that $\tilde{R}(N) = \mathcal{O}_1 \backslash \mathcal{O}'_N$ by showing that the cardinality of the two sets coincides. To compute the cardinality of the right hand side of $(*)$, we will use again the local description of the orders and the involutions $w_N$. If we denote by $(\mathcal{O}\otimes\mathbb{Z}_l)_N$ the set of elements in $\mathcal{O}\otimes\mathbb{Z}_l$ which have norm $N$ up to units of $\mathbb{Z}_l$, then the natural map

$$\mathcal{O}_N \longrightarrow \prod_{l \text{ prime}} (\mathcal{O}\otimes\mathbb{Z}_l)^*\backslash(\mathcal{O}\otimes\mathbb{Z}_l)_N$$

factors through $\mathcal{O}_1\backslash\mathcal{O}_N$ giving rise to an inclusion. Denote by $(\mathcal{O}\otimes\mathbb{Z}_l)'_N$ the set of primitive elements of $(\mathcal{O}\otimes\mathbb{Z}_l)_N$; then, the restriction to $\mathcal{O}'_N$ gives an injection

$$\mathcal{O}_1\backslash\mathcal{O}'_N \longrightarrow \prod_{l \text{ prime}} (\mathcal{O}\otimes\mathbb{Z}_l)^*\backslash(\mathcal{O}\otimes\mathbb{Z}_l)'_N.$$

To finish the proof, it suffices to prove that the cardinality of the right hand side is $\psi(N)$.

If $l \nmid N$, we have that $(\mathcal{O}\otimes\mathbb{Z}_l)'_N = (\mathcal{O}\otimes\mathbb{Z}_l)^*$ and hence the quotient is trivial. Otherwise, if $k \geq 1$ is the exact power of $l$ that divides $N$, then the set $(\mathcal{O}\otimes\mathbb{Z}_l)'_N = (\mathcal{O}\otimes\mathbb{Z}_l)'_{l^k}$ can be identified with the set of elements $\{X \in \mathbf{M}(2,\mathbb{Z}_l)' : \det X \in l^k\mathbb{Z}_l^*\}$ when $(\mathcal{O}\otimes\mathbb{Z}_l)^*$ is identified with $\mathbf{M}(2,\mathbb{Z}_l)^*$. Proceeding in the same way as it is computed in the modular case (cf. [Lan87]), a set of representatives for this quotient is

$$\left\{\begin{pmatrix} l^r & b \\ 0 & l^{k-r}\end{pmatrix} \in \mathbf{M}(2,\mathbb{Z}) :\, 0 \leq r \leq k,\, 0 \leq b < l^{k-r},\, \gcd(l^k, l^{k-r}, b) = 1\right\},$$

which has cardinality $\psi(l^k)$. Therefore, the set

$$\prod_{l \text{ prime}} (\mathcal{O}\otimes\mathbb{Z}_l)^*\backslash(\mathcal{O}\otimes\mathbb{Z}_l)'_N$$

has cardinality $\psi(N)$, as we wanted to see. $\square$

Now, let $G \subset \mathbf{PSL}(2,\mathbb{R})$ be a subgroup and consider the set of orbits

$$R(G,N) = (G \cap \omega_N^{-1} G \omega_N)\backslash G.$$

As before, it will denote both the set itself and a set of representatives. For a subgroup $W \subseteq W_{DM}$, write $\Gamma = \Gamma(D,M)\langle W\rangle$.

**Corollary 2.1.**

*(1) The projection $\mathcal{O}_1 \longrightarrow \Gamma(D,M)$ induces a bijection*

$$R(N) \longrightarrow R(\Gamma(D,M),N)$$

*which is compatible with the right action of $\mathcal{O}_1$ and $\Gamma(D,M)$, respectively.*

*(2) The inclusion* $\Gamma(D,M) \hookrightarrow \Gamma$ *induces a bijection*

$$R(\Gamma(D,M),N) \longrightarrow R(\Gamma,N)$$

*which is compatible with the right action of* $\Gamma(D,M)$ *on both sets.*

*Proof.* (1) The proof of this statement is obtained by applying the projection $\mathbf{SL}(2,\mathbb{R}) \longrightarrow \mathbf{PSL}(2,\mathbb{R})$.

(2) Consider the composition $i : \Gamma(D,M) \hookrightarrow \Gamma \longrightarrow R(\Gamma,N)$. It follows from Theorem 1.6 that any element $\alpha \in \Gamma$ can be written as $\alpha = \gamma\omega$ for $\gamma \in \Gamma(D,M)$, $\omega \in W$ in a unique manner. Moreover, if we choose $W$ as a subgroup inside $W_{DMN}$, then $\omega_N$ commutes with the elements of $W$ up to an element in $\Gamma(D,M) \cap \omega_N^{-1}\Gamma(D,M)\omega_N$. Therefore, we have that

$$\Gamma \cap \omega_N^{-1}\Gamma\omega_N = (\Gamma(D,M) \cap \omega_N^{-1}\Gamma(D,M)\omega_N)\langle W\rangle$$

and $i$ is easily seen to be surjective. Moreover, $i$ factors through the quotient set $R(\Gamma(D,M),N)$ giving rise to a bijection which preserves the right action of $\Gamma(D,M)$, as desired. □

We will need as well some results that show us how to write elements of a given prime power norm as products of elements of smaller norms in a somehow unique way. First, we will prove the following lemma.

**Lemma 2.1.** *Let* $k \leq r$ *be positive integers,* $p$ *a prime not dividing* $DM$ *and* $\mathcal{O}(p^k) \supset \mathcal{O}(p^r)$ *Eichler orders of levels* $Mp^k$ *and* $Mp^r$*, respectively, contained in a fixed Eichler order* $\mathcal{O}$ *of level* $M$*. Then, for any choice of representatives of the corresponding Atkin-Lehner involutions,* $w_{p^k}$ *and* $w_{p^r}$*, we have that* $w_{p^k}w_{p^r}, w_{p^r}w_{p^k} \in p^k\mathcal{O}'_{p^{r-k}}$*. In particular,* $w_{p^r} \in \mathcal{O}'_{p^{r-k}}w_{p^k} \cap w_{p^k}\mathcal{O}'_{p^{r-k}}$*.*

*Proof.* By Proposition 1.3 and Theorem 1.4, we can consider an isomorphism $\varphi_p : \mathcal{O} \otimes \mathbb{Z}_p \cong \mathbf{M}(2,\mathbb{Z}_p)$ such that $\varphi_p(\mathcal{O}(p^k) \otimes \mathbb{Z}_p) = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^k\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ and $\varphi_p(\mathcal{O}(p^r) \otimes \mathbb{Z}_p) = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^r\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$. Then,

$$\varphi_p(w_{p^k}) \in \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^k\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}\begin{pmatrix} 0 & 1 \\ p^k & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p^k & 0 \end{pmatrix}\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^k\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$$

and

$$\varphi_p(w_{p^r}) \in \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^r\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}\begin{pmatrix} 0 & 1 \\ p^r & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p^r & 0 \end{pmatrix}\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^r\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

Therefore, $\varphi_p(w_{p^k}w_{p^r}) \in p^k\varphi_p(\mathcal{O}\otimes\mathbb{Z}_p)'$ and we obtain the first claim.

As a consequence, if we replace the representative $w_{p^k}$ by $\overline{w_{p^k}}$, we obtain $w_{p^r}\overline{w_{p^k}} \in p^k\mathcal{O}'_{p^{r-k}}$, and therefore

$$w_{p^r} \in \mathcal{O}'_{p^{r-k}} w_{p^k}.$$

□

**Proposition 2.2.** *For an Eichler order $\mathcal{O}$ of level $M$, a positive integer $k$ and a prime $p$ not dividing $DM$, consider Eichler orders of levels $Mp^k$ and $Mp$, $\mathcal{O}(p^k) \subset \mathcal{O}(p) \subset \mathcal{O}$, and an element $w_{p^k} \in \mathcal{O}(p^k)'_{p^k}$ giving rise to the Atkin-Lehner involution of level $p^k$. Denote by $\tilde\Gamma(p^k) \subset \tilde\Gamma(p) \subset \tilde\Gamma$ the corresponding groups of units of norm 1 and $\tilde\Gamma' = \tilde\Gamma \setminus \tilde\Gamma(p)$. Fix as well a set of representatives $R(\tilde\Gamma, p^k)$ of $\tilde\Gamma(p^k)\backslash\tilde\Gamma$ and $R'(\tilde\Gamma, p^k)$ a set of representatives of $\tilde\Gamma(p^k)\backslash\tilde\Gamma'$. Now, for a positive integer $n \geq 2$, consider the map*

$$\begin{array}{rcl} \varpi : \tilde\Gamma \times \overset{n+1}{\cdots} \times \tilde\Gamma & \to & \mathcal{O}_{p^{nk}} \\ (\gamma_0, \ldots, \gamma_n) & \mapsto & \gamma_0 w_{p^k}\gamma_1 w_{p^k}\cdots w_{p^k}\gamma_{n-1}w_{p^k}\gamma_n. \end{array}$$

*Then,*

*(1) $\varpi$ is a surjective map.*

*(2) $\varpi$ induces a bijection between $\tilde\Gamma\backslash\mathcal{O}'_{p^{kn}}$ and $\{1\} \times R'(\tilde\Gamma, p^k)^{n-1} \times R(\tilde\Gamma, p^k)$.*

*(3) $\mathcal{O}'_{p^{kn}} = \varpi(\tilde\Gamma \times (\tilde\Gamma')^{n-1} \times \tilde\Gamma) = \varpi(\tilde\Gamma \times R'(\tilde\Gamma, p^k)^{n-1} \times R(\tilde\Gamma, p^k))$.*

*(4) For $i \leq k$, an element $\alpha = \varpi(\gamma_0, \ldots, \gamma_n) \in \mathcal{O}'_{p^{kn}}$ can be written as $\alpha = p^i\alpha_0$ for some integral element $\alpha_0$ if and only if $\gamma_n\gamma_0 \in \tilde\Gamma(p^i)$.*

*Proof.* (1) By Proposition 2.1, we have that $\mathcal{O}'_{p^k} = \tilde\Gamma w_{p^k}\tilde\Gamma$, or, equivalently, that $\mathcal{O}'_{p^k} = \tilde\Gamma\alpha\tilde\Gamma$, for any $\alpha \in \mathcal{O}'_{p^k}$. Consequently, it is only necessary to prove that for any $0 \leq i \leq nk$, there exists a primitive element of norm $p^i$ in the image of $\varpi$, which follows from the previous lemma.

(2) The inclusion $\mathcal{O}'_{p^{kn}} \subset \varpi(\tilde\Gamma \times (\tilde\Gamma')^{n-1} \times \tilde\Gamma)$ follows from (1) together with the lemma above. Recall as well that the quotient set $\tilde\Gamma\backslash\mathcal{O}'_{p^{kn}}$ has $\psi(p^{kn})$ elements and this coincides with the cardinality of $\{1\}\times R'(\tilde\Gamma, p^k)^{n-1}\times R(\tilde\Gamma, p^k)$.

Now, we claim that $\varpi$ induces a surjection from $\{1\}\times R'(\tilde\Gamma, p^k)^{n-1}\times R(\tilde\Gamma, p^k)$ to $\tilde\Gamma\backslash\mathcal{O}'_{p^{kn}}$ as follows. We have seen that any element in $\mathcal{O}'_{p^{kn}}$ can be written as

$\gamma_0 w_{p^k}\gamma_1 w_{p^k}\cdots\gamma_{n-1}w_{p^k}\gamma_n$ with $\gamma_1,\ldots,\gamma_{n-1}\in\tilde\Gamma'$. Then, since $w_{p^k}$ normalizes $\tilde\Gamma(p^k)$, we can replace $\gamma_n$ by an element in $R(\tilde\Gamma,p^k)$, which changes $\gamma_{n-1}\in\tilde\Gamma'$ by $\gamma'_{n-1}\in\tilde\Gamma'$. Now, in the same manner, we can replace $\gamma'_{n-1}$ by an element in $R'(\tilde\Gamma,p^k)$. Iterating, we obtain the claim.

Finally, since both sets are finite with the same cardinal we can conclude that the map is indeed a bijection.

(3) It follows immediately from (2), where we have already proved that $\mathcal{O}'_{p^{kn}}\subset\varpi(\tilde\Gamma\times(\tilde\Gamma')^{n-1}\times\tilde\Gamma)=\varpi(\tilde\Gamma\times R'(\tilde\Gamma,p^k)^{n-1}\times R(\tilde\Gamma,p^k))$ and also that $\varpi(\tilde\Gamma\times R'(\tilde\Gamma,p^k)^{n-1}\times R(\tilde\Gamma,p^k))\subset\mathcal{O}'_{p^{kn}}$.

(4) First of all, observe that, if $\alpha=\varpi(\gamma_0,\ldots,\gamma_n)$ with $\gamma_n\gamma_0\in\tilde\Gamma(p^i)$, then $\alpha=(w_{p^i}\gamma_n)^{-1}w_{p^i}\gamma_n\alpha=(w_{p^i}\gamma_n)^{-1}w_{p^i}\gamma_n\gamma_0 w_{p^k}\gamma_1\cdots\gamma_{n-1}w_{p^k}w_{p^i}^{-1}(w_{p^i}\gamma_n)$ and since $\gamma_n\gamma_0\in\tilde\Gamma(p^i)$, $w_{p^i}\gamma_n\gamma_0 w_{p^k}\in p^i\mathcal{O}_{p^{k-i}}$ and $w_{p^k}w_{p^i}^{-1}\in\mathcal{O}_{p^{k-i}}$. Thus we have that $\alpha\in p^i(w_{p^i}\gamma_n)^{-1}\mathcal{O}_{p^{n-2}}(w_{p^i}\gamma_n)$ and the result follows.

To obtain the other implication, fix $\gamma_0,\ldots,\gamma_n$ such that $\alpha=\varpi(\gamma_0,\ldots,\gamma_n)$. Now, we are going to prove that the set

$$\{\gamma\in\tilde\Gamma:\varpi(\gamma,\gamma_1,\ldots,\gamma_{n-1},\gamma_n)=p^i\beta,\text{ for a certain integer }\beta\}$$

constitutes a unique class in $\tilde\Gamma/\tilde\Gamma(p^i)$. Since both $\gamma_0$ and $\gamma_n^{-1}$ satisfy this property, we will be done. By Proposition 1.3 and Theorem 1.4, completing at $p$, we can identify $\mathcal{O}\otimes\mathbb{Z}_p$ with $\mathbf{M}(2,\mathbb{Z}_p)$ and $(\mathcal{O}(p^i)\otimes\mathbb{Z}_p)_1$ with $\Gamma_0(p^i)_p$. This gives a bijection between $(\mathcal{O}\otimes\mathbb{Z}_p)_1\backslash(\mathcal{O}\otimes\mathbb{Z}_p)'_{p^{kn}}$ and $\mathbf{SL}(2,\mathbb{Z}_p)\backslash\mathbf{M}(2,\mathbb{Z}_p)'_{p^{kn}}$ and we can identify $w_{p^{kn}}$ with $\begin{pmatrix}0&1\\p^{kn}&0\end{pmatrix}$. An element of the form $\varpi(\gamma,\gamma_1,\ldots,\gamma_{n-1},\gamma_n)$ is $p^i$ times an integer, if and only if the same holds in the completion, and therefore we can work in this new setting. The usual set of representatives of $\mathbf{SL}(2,\mathbb{Z}_p)\backslash\mathbf{M}(2,\mathbb{Z}_p)'_{p^{kn}}$ is given by the matrices

$$\left\{\begin{pmatrix}p^s&x\\0&p^{kn-s}\end{pmatrix}:\ \begin{matrix}0\le s\le kn,\ 0\le x<p^{kn-s},\\ \gcd(p^s,p^{kn-s},x)=1\end{matrix}\right\},$$

but it will be better for our purposes to consider

$$\begin{aligned}&\left\{A_{s,x}=\begin{pmatrix}p^s&x\\0&p^{kn-s}\end{pmatrix}:\ \begin{matrix}0\le s\le(n-1)k,\ 0\le x<p^{kn-s},\\ \gcd(p^s,x)=1\end{matrix}\right\}\\ \cup&\left\{A_{s,x}=\begin{pmatrix}yp^s&-1\\p^{kn}&0\end{pmatrix}:\ \begin{matrix}(n-1)k<s\le kn,\ 0\le x,y<p^{kn-s},\\ \gcd(p^{kn-s},x)=1,\ xy+1\equiv 0\,(\mathrm{mod}\ p^{kn-s})\end{matrix}\right\}.\end{aligned}$$

This last system of representatives has the additional property that it corresponds to elements in the completion of $\varpi(\{1\} \times (\tilde{\Gamma}')^{n-1} \times \tilde{\Gamma})$. Therefore, all we have to prove is that for each one of these representatives, $A_{s,x}$, the set of $\gamma \in \mathbf{SL}(2, \mathbb{Z}_p)$ such that $p^i | \operatorname{tr}(\gamma \cdot A_{s,x})$ is a class in $\mathbf{SL}(2, \mathbb{Z}_p)/\Gamma_0(p^i)_p$. Write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2, \mathbb{Z}_p)$ and observe that $p^i | \operatorname{tr}(\gamma \cdot A_{s,x})$ if and only if

$$\begin{cases} \gamma \in \Gamma_0(p^i)_p, \text{ if } i \le s, \\ c = p^s c_0, \text{ i.e. } \gamma \in \Gamma_0(p^s)_p, \text{ and } p^{i-s} | a + c_0 x, \text{ if } i > s. \end{cases}$$

Since for $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathbf{SL}(2, \mathbb{Z}_p)$ such that $c' = p^s c_0'$ and $p^{i-s} | a' + c_0' x$, we have $\gamma'^{-1}\gamma \in \Gamma_0(p^i)_p$, we obtain the result. □

**Proposition 2.3.** *Let $D$ and $M$ be integers as before and fix $\mathcal{O}$ an Eichler order of level $M$. Let $\Lambda$ be an order in a quadratic imaginary field and $\phi : \Lambda \to \mathcal{O}$ an optimal embedding. Let $s$ be an integer dividing exactly $DM$, that is $s|DM$ and $\gcd(s, DM/s) = 1$. Then, if $\lambda \in \Lambda$ has norm $sN$ and $w_s \cdot \phi(\lambda) \in s\mathcal{O}$, we have that $\lambda^2 \in s\Lambda$. Reciprocally, if $s$ is squarefree and $\lambda$ is such that $\lambda^2 \in s\Lambda$, then $w_s \cdot \phi(\lambda) \in s\mathcal{O}$.*

*Proof.* The first assertion is clear: if $w_s \cdot \phi(\lambda) \in s\mathcal{O}$, it follows that $\phi(\lambda) \in w_s\mathcal{O}$ and therefore that $\phi(\lambda)^2 = \phi(\lambda^2) \in s\mathcal{O}$, since $w_s$ normalizes $\mathcal{O}$. Thus, since $\phi$ is optimal, $\lambda^2 \in s\Lambda$.

On the other hand, if we assume that $s$ is squarefree, we need to show that we can go the other way around. We can work locally for every $l|s$. If $l|D$, then it is clear that $\phi(\lambda)_l \in \mathcal{O}_{(l)} \setminus \mathcal{O}_{(l)}^* = \omega_l \mathcal{O}_{(l)}$. Otherwise, if $l|M$, then the element $\phi(\lambda)_l = \gamma_0 w_l \gamma_1$ for certain $\gamma_0, \gamma_1 \in \mathcal{O}(D,1)_{(l)}^*$. In particular, $\phi(\lambda)_l \in \mathcal{O}(D,l)_{(l)} = \mathcal{O}(D,M)_{(l)}$ if and only if $w_l \phi(\lambda)_l w_l \in l\mathcal{O}(D,M)_{(l)}$, which by the proposition above holds if and only if $\gamma_0$ or $\gamma_1 \in \mathcal{O}(D,M)_{(l)}^*$. However, since $\lambda^2 \in s\Lambda$, we obtain that $\gamma_0\gamma_1 \in \mathcal{O}(D,M)_{(l)}^*$. Therefore $\phi(\lambda)_l \in \omega_l \mathcal{O}_{(l)}$ and the result follows. □

**Remark.** In the non-squarefree case, if we fix $M'$ such that $M/M'^2$ is squarefree, the elements $\lambda$ such that $\lambda^2 \in s\Lambda$ are of the form $\overline{\gamma}\omega_s\mathcal{O}\gamma$ with $\gamma \in \Gamma(M/M')$. Since we consider the embeddings up to conjugation by $\Gamma(M)$, the condition $\omega_s \cdot \phi(\lambda) \in s\mathcal{O}$, fixes a conjugation class among $\Gamma(M)\backslash\Gamma(M/M')$.

From now on, fix $W \subseteq W_{DM}$, consider the group $\Gamma = \Gamma(D,M)\langle W\rangle$ and assume that the curve $X(D,M)^W$ has genus 0. Thus, there exists a Hauptmodul, i.e. a $\Gamma$-automorphic function $f : \mathcal{H} \to \mathbb{C} \cup \{\infty\}$ such that the field of rational functions of the complex curve $X(D,M)^W_{\mathbb{C}}$ is $\mathbb{C}(f)$.

**Definition.** Let $f$ be a Hauptmodul for $X(D,M)^W$. For each positive integer $N$ coprime to $DM$, we define

$$\tilde{\Psi}_{\Gamma,N,f}(X,f) = \prod_{\gamma \in R(\Gamma,N)} (X - f \circ \omega_N \circ \gamma),$$

where $\omega_N$ is the transformation attached to an element in $\mathcal{O}_N$ representing the Atkin-Lehner involution of level $N$ on $X(D,MN)$.

The kroneckerian polynomial of level $N$ for $f$ will be a modification of it, but first we need to study some of the properties of this one, which we collect in the following proposition.

**Proposition 2.4.** *The polynomial* $\tilde{\Psi}_{\Gamma,N,f}(X,f)$ *belongs to* $\mathbb{C}(f)[X]$ *and it is irreducible, monic and has degree* $\psi(N)$.

*Proof.* The left action of $\Gamma$ on $\mathcal{H}$ gives rise to a right action on the field of meromorphic functions $\mathcal{H} \longrightarrow \mathbb{P}^1(\mathbb{C})$. Being $f$ a $\Gamma$-automorphic function, this action is trivial on the field $\mathbb{C}(f)$. On the other hand, by using Corollary 2.1 and Proposition 2.1, we see that the action can be restricted to the set

$$\Omega = \{f \circ \omega_N \circ \gamma : \gamma \in R(\Gamma,N)\}$$

and that it is transitive on it. Therefore, $\Gamma$ acts on the right as a group of automorphisms on the field $\mathbb{C}(f,\Omega)$.

Now, the coefficients of $\tilde{\Psi}_{\Gamma,N,f}(X,f)$, as a polynomial in $X$, are the elementary symmetric polynomials of $\Omega$; thus, we obtain that these symmetric polynomials are invariant under transformations $z \mapsto \delta z$, for $\delta \in \Gamma$, and therefore automorphic with respect to this group. Since any $\Gamma$-automorphic function is a rational function of $f$, we get that $\tilde{\Psi}_{\Gamma,N,f}(X,f) \in \mathbb{C}(f)[X]$. Moreover, by the transitivity of the action on the set $\Omega$, one has that $\tilde{\Psi}_{\Gamma,N,f}(X,f)$ is the minimal polynomial of $f \circ \omega_N$ over the fixed field by $\Gamma$, $\mathbb{C}(f,\Omega)^\Gamma$, and, since $\mathbb{C}(f) \subseteq \mathbb{C}(f,\Omega)^\Gamma$, it follows that

$$\tilde{\Psi}_{\Gamma,N,f}(X,f) = \mathrm{Irr}(f \circ \omega_N, \mathbb{C}(f)).$$

Thus, it is irreducible. $\square$

Now, $\tilde{\Psi}_{\Gamma,N,f}(X,f)$ is a monic polynomial in $\mathbb{C}(f)[X]$ and therefore we can multiply it by a certain polynomial in $\mathbb{C}[f]$ to remove the denominators of the coefficients without adding any common factors; in this way, we obtain an irreducible polynomial $\Psi_{\Gamma,N,f}(X,Y) \in \mathbb{C}[X,Y]$.

**Definition.** For a subgroup $W \subseteq W_{DM}$, we take the group $\Gamma = \Gamma(D,M)\langle W\rangle$ and a Hauptmodul $f$ for $\Gamma$. Then, for a positive integer $N$ coprime to $DM$, we call a polynomial $\Psi_{\Gamma,N,f}(X,Y) \in \mathbb{C}[X,Y]$ as above a kroneckerian polynomial of level $N$ attached to the Hauptmodul $f$ for the group $\Gamma$.

Note that, for every positive integer $N$ coprime to $DM$, the polynomial $\Psi_{\Gamma,N,f}(X,Y)$ is unique up to multiplication by a non-zero complex constant. Therefore, all statements involving kroneckerian polynomials should be understood up to multiplication by such a constant, even if it is not made completely explicit.

We state now the basic properties of the kroneckerian polynomials.

**Theorem 2.1.** *Let $f$ be a Hauptmodul for $X(D,M)^W_{\mathbb{C}}$. For every positive integer $N$ coprime to $DM$, the kroneckerian polynomial attached to $f$, $\Psi_{\Gamma,N,f}(X,Y)$, is an irreducible polynomial of degree $\psi(N)$ in each indeterminate. If $N=1$, we can take $\Psi_{\Gamma,1,f}(X,Y) = X-Y$; if $N>1$, then $\Psi_{\Gamma,N,f}(X,Y)$ is symmetric. Moreover, if $f$ is a Hauptmodul defined over a number field $K$, then, after a suitable choice of the constant of its definition, $\Psi_{\Gamma,N,f}(X,Y) \in K[X,Y]$.*

*Proof.* The irreducibility has already been established. Moreover, it is clear that, if $N=1$, then we can take $\Psi_{\Gamma,1,f}(X,Y) = X-Y$. So, we assume from now on that $N>1$. To prove that $\Psi_{\Gamma,N,f}(X,Y)$ has degree $\psi(N)$ in each indeterminate, it suffices to see that it is symmetric, because, by construction, it has degree $\psi(N)$ as a polynomial in $X$. Now, to prove the symmetry, we will use that $\omega_N$ is an involution. $\Psi_{\Gamma,N,f}(X,f)$ is an irreducible polynomial over $\mathbb{C}(f)$ which has $f\circ\omega_N$ as a root. Thus, we have the identity $\Psi_{\Gamma,N,f}(f(\omega_N z), f(z)) = 0$, for each $z \in \mathcal{H}$. Applying this to $\omega_N z \in \mathcal{H}$, we have $\Psi_{\Gamma,N,f}(f(\omega_N^2 z), f(\omega_N z)) = 0$; therefore, $\Psi_{\Gamma,N,f}(f(z), f(\omega_N z)) = 0$, for all $z \in \mathcal{H}$. That is, $f\circ\omega_N$ is a root of the polynomial $\Psi_{\Gamma,N,f}(f,X)$, which is the polynomial we started with, but with the variables exchanged. By the irreducibility of $\Psi_{\Gamma,N,f}(X,f)$, we obtain that it divides $\Psi_{\Gamma,N,f}(f,X)$ in $\mathbb{C}(f)[X]$. Thus, there exists $g(X,Y) \in \mathbb{C}(Y)[X]$ such that

$$\Psi_{\Gamma,N,f}(f,X) = g(X,f)\Psi_{\Gamma,N,f}(X,f);$$

using Gauss' Lemma, we see that $g(X,Y) \in \mathbb{C}[Y,X]$. Then, we have that

$$\Psi_{\Gamma,N,f}(f,X) = g(X,f)g(f,X)\Psi_{\Gamma,N,f}(f,X),$$

which implies that $g(X,Y)=\pm 1$. If it were $-1$, then

$$\Psi_{\Gamma,N,f}(f,X)=-\Psi_{\Gamma,N,f}(X,f),$$

which would yield that $\Psi_{\Gamma,N,f}(f,f)=0$ and, by the irreducibility, the degree in $X$ would be 1 and this is impossible since it has degree $\psi(N)>1$. Therefore, $g(X,f)=1$ and we get the symmetry.

Finally, assume that $f$ is a Hauptmodul defined over a number field $K\subset\mathbb{C}$, in the sense that the field of $K$-rational functions of the curve $X(D,M)^W$ is $K(f)$. Since the involutions $\omega_N$ are automorphisms of $X(D,MN)^W$ defined over $\mathbb{Q}$, both $f$ and $f\circ\omega_N$ are defined over $K$; in other words, we have an extension of fields $K(f,f\circ\omega_N)\mid K(f)$. Since the minimal polynomial of $f\circ\omega_N$ for this extension is also $\tilde{\Psi}_{\Gamma,N,f}(X,f)$, it is now obvious that $\Psi_{\Gamma,N,f}(X,f)$ can be taken such that $\Psi_{\Gamma,N,f}(X,f)\in K[X,f]$. □

**Corollary 2.2.** *Let $K$ be a number field and let $f$ be a Hauptmodul for the curve $X(D,M)^W$ defined over $K$. Then, for every positive integer $N$ coprime to $DM$, the fields of $K$-rational functions of $X(N)^W$ and $X(N)^{\langle W,\omega_N\rangle}$ are the fields $K(X(N)^W)=K(f,f\circ\omega_N)$ and*

$$K(X(N)^{\langle W,\omega_N\rangle})=K(f\cdot f\circ\omega_N, f+f\circ\omega_N).$$

*Proof.* Since $K(f)=K(X(D,M)^W)$ and $K(f,f\circ\omega_N)\subseteq K(X(N)^W)$, the first result follows from the equality of degrees

$$[K(f,f\circ\omega_N):K(f)]=[K(X(N)^W):K(f)]=\psi(N).$$

The other equality of fields follows from the fact that we can identify the function field of $K(X(N)^{\langle W,\omega_N\rangle})$ as the subfield of $K(X(N)^W)$ fixed by $\omega_N$, which acts interchanging $f$ and $f\circ\omega_N$. □

**Corollary 2.3.** *Let $K$ be a number field and let $f$ be a Hauptmodul defined over $K$ for $X(D,M)^W$. For every integer $N>1$ coprime to $DM$, the polynomial*

$$\Psi_{\Gamma,N,f}(X+\sqrt{Y},X-\sqrt{Y})$$

*belongs to $K[X,Y]$, is irreducible over $\mathbb{C}$ and provides a model over $K$ for the curve $X(N)^{\langle W,\omega_N\rangle}$.*

*Proof.* The expression $p(X,\sqrt{Y})=\Psi_{\Gamma,N,f}(X+\sqrt{Y},X-\sqrt{Y})$ clearly belongs to $K[X,\sqrt{Y}]$. Moreover, because of the symmetry of $\Psi_{\Gamma,N,f}(X,Y)$, we obtain that $p(X,\sqrt{Y}) \in K(X,\sqrt{Y})$ is fixed by the Galois group of $K(X,\sqrt{Y})|K(X,Y)$. Hence, it lies in $K(X,Y) \cap K[X,\sqrt{Y}] = K[X,Y]$. Now we will prove the irreducibility in $\mathbb{C}[X,\sqrt{Y}]$, which implies the same result in $\mathbb{C}[X,Y]$. Consider the isomorphism of algebras $\phi : \mathbb{C}[X,\sqrt{Y}] \to \mathbb{C}[T,U]$ such that

$$\phi(X) = (T+U)/2, \quad \phi(\sqrt{Y}) = (T-U)/2$$

and then $\phi(p(X,\sqrt{Y})) = p((T+U)/2,(T-U)/2) = \Psi_{\Gamma,N,f}(T,U)$, for which we already have proved the irreducibility. Finally, in the previous corollary we have seen that

$$\begin{aligned} K(X(D,MN)^{\langle W,\omega_N\rangle}) &= K(f \cdot f \circ \omega_N, f + f \circ \omega_N) \\ &= K((f + f\circ\omega_N)/2, ((f - f\circ\omega_N)/2)^2) \end{aligned}$$

and $p(\frac{f+f\circ\omega_N}{2}, \sqrt{((f-f\circ\omega_N)/2)^2}) = 0$, from which we obtain the last assertion. □

### 2.1.1 Relation to class equations

As in the modular case, the roots of the polynomial $\Psi_{\Gamma,N,f}(X,X)$ are related to the values of $f$ at certain CM points. In particular, this provides a method to compute the values of $f$ at an explicit set of CM points, which we will exploit in Chapter 6. Some of the most basic properties of the $q$-expansions introduced in Section 3.3 will be used in the proof.

**Proposition 2.5.** *Let $W$ be a subgroup of $W_{DM}$, $\Gamma = \Gamma(D,M)\langle W\rangle$ and $f$ a Hauptmodul for the curve $X(D,M)^W_{\mathbb{C}}$. Consider the set $S = \{s : \omega_s \in W\}$.*

*Given $\tau_0$ a CM point by an order $\Lambda$ in a quadratic field $K$, the multiplicity of $f(\tau_0)$ as a root of $\Psi_{\Gamma,N,f}(X,X)$ is given by*

$$\begin{aligned} r(\Lambda,N,S) &= \#\{x \in \Lambda' : \omega_s\phi(x) \in s\mathcal{O}(D,M)_N \text{ for a certain } s \in S\}/\sim \\ &= \#\{x \in \Lambda' : \omega_s\phi(x) \in s\mathcal{O}(D,M)_N \text{ for a certain } s \in S\}/2e(\tau_0), \end{aligned}$$

*where $x \sim y$ if and only if $x\overline{y} \in N\Lambda$, $e(\tau_0)$ denotes the elliptic order of $\tau_0$ and $\overline{\cdot}$ denotes the conjugation.*

*In particular, $z_0 \in \mathbb{C}$ is a root of the polynomial $\Psi_{\Gamma,N,f}(X,X) \in \mathbb{C}[X]$ if and only if $z_0 = f(\tau_0)$ for a complex multiplication point $\tau_0 \in \mathcal{H}$ attached to an optimal embedding $\phi : \Lambda \hookrightarrow \mathcal{O}(D,M)$ such that there exists a primitive element $x \in \Lambda$ of norm $sN$ for a certain $s \in S$ for which $\omega_s\phi(x) \in s\mathcal{O}(D,M)$ (cf. Proposition 2.3).*

*Proof.* First of all, assume that $f \circ \omega_N \circ \gamma(\tau_0) \in \mathbb{C}$ for all $\gamma \in R(\Gamma, N)$. Then, since $\Psi_{\Gamma,N,f}(X, f) = P(f) \prod_\gamma (X - f \circ \omega_N \circ \gamma)$ for some $P \in \mathbb{C}[X]$ and we have that $f(\tau_0), f \circ \omega_N \circ \gamma(\tau_0) \in \mathbb{C}$, the order of $f(\tau_0)$ as a zero of $\Psi_{\Gamma,N,f}(X, f(\tau_0))$ is equal to that of $\prod_\gamma (X - f \circ \omega_N \circ \gamma(\tau_0))$. And now, this multiplicity coincides with the formula given in the statement, after identifying the elements in $\mathcal{O}(D, M)$ which fix $\tau_0$ with the corresponding order $\Lambda \subset K$ by means of the embedding $\phi$. Therefore, we only need to prove that the multiplicity of $f(\tau_0)$ as a root of $\prod_\gamma (X - f \circ \omega_N \circ \gamma(\tau_0))$ is equal to that of $\Psi_{\Gamma,N,f}(X, X)$. For this purpose, we study

$$\begin{aligned} \lim_{\tau \to \tau_0} \frac{\prod_\gamma (f(\tau) - f \circ \omega_N \circ \gamma(\tau_0))}{\Psi_{\Gamma,N,f}(f(\tau), f(\tau))} &= \lim_{\tau \to \tau_0} \frac{\prod_\gamma (f(\tau) - f \circ \omega_N \circ \gamma(\tau_0))}{\prod_\gamma (f(\tau) - f \circ \omega_N \circ \gamma(\tau))} \\ &= \lim_{\tau \to \tau_0} \prod_\gamma \frac{f(\tau) - f \circ \omega_N \circ \gamma(\tau_0)}{f(\tau) - f \circ \omega_N \circ \gamma(\tau)}. \end{aligned}$$

Now, if $f \circ \omega_N \circ \gamma(\tau_0) \neq f(\tau_0)$, the corresponding quotient is clearly 1. To study the case when $f \circ \omega_N \circ \gamma(\tau_0) = f(\tau_0)$, we consider a local parameter around $\tau_0$, $q(z) = \frac{z - \tau_0}{z - \overline{\tau_0}}$, and the corresponding expansion $f(q) = f(\tau_0) + a_e q^e + \dots$ where $e$ is the order of the point $\tau_0$ and $a_e \in \mathbb{C}^*$. Then, $q \circ \omega_N \circ \gamma = aq$ for a certain $a \in \mathbb{C}^*$, and therefore the value of the corresponding quotient is $\frac{a_e}{a_e - a_e a^e} \in \mathbb{C}^*$, since $a^e = 1$ would imply that $f \circ \omega_N \circ \gamma = f$.

Finally, if any of the values $f \circ \omega_N \circ \gamma(\tau_0)$ is not finite, we can always reduce to the previous case applying a suitable homographic transformation to $f$ so that the values of the new Hauptmodul satisfy the required condition. □

### 2.1.2 Singularities of the kroneckerian polynomials of prime power level

We are also interested in describing, as precisely as we can, the set of singularities of the kroneckerian polynomial of a given prime power level $N = p^r$ attached to a certain Hauptmodul of a genus 0 Shimura curve (including the case of Atkin-Lehner quotients of genus 0). This result can also be used to describe the set of non-cuspidal singularities of the classical modular polynomials. First of all we are going to characterize the set of singularities and, in particular, we will obtain that it is contained in the set of complex multiplication points attached to rings containing non-trivial elements of suitable norms. The rest of the proof will deal with giving an explicit description of this set of points. If we were only interested in the prime level case, then the other inclusion would hold and everything could be proved in a rather straightforward way (cf. [Gro87] for the modular case). Moreover, we are going to provide formulas for the multiplicity, in a certain sense, of each of these singularities.

**Lemma 2.2.** *Let $X(D,M)^W$ be the Atkin-Lehner quotient by $W \subset W_{DM}$ of the Shimura curve of discriminant $D$ and level $M$ and assume that it has genus $0$. Let us denote as well the corresponding Eichler order of level $M$ by $\mathcal{O}$ and fix an arbitrary integer $N > 1$ coprime to $DM$. Fix an Atkin-Lehner involution $\omega_N$ and consider the kroneckerian polynomial attached to a certain Hauptmodul $f$ of $X(D,M)^W_{\mathbb{C}}$, $\Psi(X,Y) = \Psi_{\Gamma,N,f}(X,Y) \in \mathbb{C}[X,Y]$. A point of the curve $\Psi(X,Y) = 0$ is singular if and only if it is of the form $(f(P), f(w_N P))$ for $P \in \mathcal{H}$ such that $\overline{\gamma_0}\,\overline{\omega_N}\gamma_1\omega_N\omega_s P = P$, where $\gamma_0, \gamma_1 \in \Gamma(D,M) \smallsetminus \Gamma(D,MN)$ and $\omega_s \in W$. In particular, $P$ gives a CM point in $X(D,MN)$ by an order $\Lambda$ containing an element of norm $sN^2$ not in $N\Lambda$, for some $s \in S$.*

*Proof.* Every point of the curve $\Psi(X,Y) = 0$ is of the form $(f(P), f(w_N P))$ for a certain $P \in \mathcal{H}$. Moreover, since $X(D,MN)^W_{\mathbb{C}}$ is non-singular and its set of complex points can be identified with $\langle\Gamma(D,MN),W\rangle\backslash\mathcal{H}$, $(f(P), f(w_N P))$ is singular if and only if there exists $P' \in \mathcal{H}$ such that $P \neq P'$ in $\langle\Gamma(D,MN),W\rangle\backslash\mathcal{H}$ and $(f(P), f(w_N P)) = (f(P'), f(w_N P'))$, which translates into $P = \gamma_0 P'$ and $\omega_N P' = \gamma_1\omega_N\omega_s P$ for certain elements $\gamma_0, \gamma_1 \in \Gamma(D,M) \smallsetminus \Gamma(D,MN)$ and $w_s \in W$. Therefore we obtain the condition in the statement of the proposition. $\square$

**Definition.** With the notations of the previous lemma, we define the multiplicity of $\Psi$ at $(P, \omega_N P)$, $\mathrm{mult}_{(P,\omega_N P)}\Psi$, as

$$\#\{P' \in X(D,MN)^W(\mathbb{C}) : f(P') = f(P), f(\omega_N P') = f(\omega_N(P))\},$$

i.e. the number of branches of the curve $\Psi(X,Y) = 0$ through $(f(P), f(\omega_N P))$.

The first result we will prove relates this notion of multiplicity to a certain order of vanishing.

**Theorem 2.2.** *Let $X(D,M)^W$ be a genus $0$ curve with a Hauptmodul $f$. Denote as well by $\mathcal{O}$ the corresponding order and $\Gamma = \langle\Gamma(D,M),W\rangle$ the corresponding group. For a positive integer $N$ coprime to $DM$ fix an Eichler order of level $MN$ inside $\mathcal{O}$, $\mathcal{O}(N)$, the corresponding group $\Gamma(N) = \langle\Gamma(D,MN),W\rangle \subset \Gamma$, and $\omega_N$ the corresponding involution of level $N$. Now, for a point $P \in \mathcal{H}$, denote by*

$$\Gamma_P = \{\gamma \in \Gamma : \gamma(P) = P\}$$

*the isotropy of $P$ and*

$$(\Gamma\backslash\mathcal{O}'_N)_P = \{\Gamma\alpha \in \Gamma\backslash\mathcal{O}'_N : \alpha(P) = \omega_N(P) \text{ in } \Gamma\backslash\mathcal{H}\}.$$

*Then,*

*(1)* $(\Gamma_P \cap \Gamma(N))\backslash\Gamma_P$ *acts on the right on* $(\Gamma\backslash\mathcal{O}'_N)_P$*, as follows*

$$\begin{array}{rcl} \cdot[\epsilon] : (\Gamma\backslash\mathcal{O}'_N)_P & \to & (\Gamma\backslash\mathcal{O}'_N)_P \\ \Gamma\alpha & \mapsto & \Gamma\alpha\epsilon. \end{array}$$

*Moreover, this action is free, i.e., if* $[\epsilon] \neq [1]$*, then* $\cdot[\epsilon]$ *fixes no element.*

*(2) The branches of the polynomial* $\Psi(X,Y) = \Psi_{\Gamma,N,f}(X,Y)$ *through* $(P, \omega_N P)$ *are in bijective correspondence with the orbits of this action. The elements in a fixed orbit,* $\alpha_i$*, give the elements* $(f, f \circ \alpha_i)$ *which define this branch. In particular,*

$$\mathrm{ord}_{f(\omega_N P)}\Psi(f(P), X) = \frac{e_1(P)}{e_N(P)}\mathrm{mult}_{(P,\omega_N(P))}\Psi(X,Y),$$

*where* $e_a(P)$ *denotes the elliptic order of the point* $P$ *in the* $X(D, Ma)^W$*.*

*Proof.* First of all, consider the right action by multiplication of $\Gamma_P$ on $(\Gamma\backslash\mathcal{O}'_N)_P$, which is clearly well defined. Let us now prove that this action factors through $(\Gamma_P \cap \Gamma(N))\backslash\Gamma_P$. Observe that any class in $(\Gamma\backslash\mathcal{O}'_N)_P$ can be written as $\Gamma\alpha$ for a certain $\alpha \in \mathcal{O}'_N$ such that $\alpha(P) = \omega_N(P)$ in $\mathcal{H}$. Hence, consider $\Gamma\alpha \in (\Gamma\backslash\mathcal{O}'_N)_P$ with $\alpha(P) = \omega_N(P)$. Then, if $\gamma \in \Gamma_P$, $\gamma$ and $\omega_N^{-1}\alpha$ are two elements fixing $P$ and therefore they commute. Consider $\omega_N^{-1}\alpha$ and $\omega_N^{-1}\alpha\gamma = \omega_N^{-1}\alpha'$. We will see that $\alpha' \in \Gamma\alpha$ if and only if $\gamma \in \Gamma(N)$, proving in this way that it factors through a free action. Observe that

$$\gamma = (\omega_N^{-1}\alpha)^{-1}\omega_N^{-1}\alpha' = \omega_N^{-1}\alpha'(\omega_N^{-1}\alpha)^{-1} = \omega_N^{-1}\alpha'\alpha^{-1}\omega_N.$$

Hence, we obtain that

$$\omega_N^{-1}\alpha\gamma\alpha^{-1}\omega_N = \gamma \in \Gamma.$$

Thus, if $\gamma \in \Gamma(N)$, we obtain that $\alpha\gamma\alpha^{-1} \in \Gamma(N)$ and therefore, $\alpha\gamma \in \Gamma(N)\alpha$ as desired. Otherwise, if $\gamma \in \Gamma \setminus \Gamma(N)$, then it follows from the above equality that $\alpha\gamma\alpha^{-1} \notin \Gamma$ and therefore $\alpha\gamma \notin \Gamma\alpha$ which proves the equivalence.

In order to relate this expression with the branches through $(P, \omega_N P)$, note that the branches through $P$ can all be parametrized as $(f, f \circ \omega_N \circ \gamma)$ for those $\gamma$

such that $\omega_N\gamma(P) = \omega_N(P)$. Moreover, for any elliptic transformation $\rho$ fixing $P$,

$$(f(\rho z), f(\omega_N\gamma\rho z)) = (f(z), f(\omega_N\gamma\rho z))$$

and therefore the branch attached to $\omega_N\gamma\rho$ coincides with the branch attached to $\omega_N\gamma$. Reciprocally, if the branches $(f, f\circ\omega_N)$ and $(f, f\circ\omega_N\gamma)$ coincide, this means that there exists a non-constant analytic map $h$ around $P$ and fixing $P$ such that $(f, f\circ\omega_N) = (f\circ h, f\circ\omega_N\gamma h)$. In particular, $h = \rho$ is an elliptic transformation fixing $P$. Thus, the bijection between branches and orbits is now clear. Moreover, the elements in the orbit correspond exactly to the transformations $\gamma_i \in \Gamma(N)\backslash\Gamma$ such that $(f, f\circ\omega_N\gamma_i)$ parametrizes the corresponding branch. Therefore,

$$\begin{aligned}\#\,(\Gamma\backslash\mathcal{O}'_N)_P &= \#\,((\Gamma_P\cap\Gamma(N))\backslash\Gamma_P)\,\mathrm{mult}_{(P,\omega_N(P))}\Psi(X,Y)\\ &= \frac{e_1(P)}{e_N(P)}\mathrm{mult}_{(P,\omega_N(P))}\Psi(X,Y).\end{aligned}$$

On the other hand, $\#\,(\Gamma\backslash\mathcal{O}'_N)_P = \mathrm{ord}_{f(\omega_N P)}\Psi(f(P),X)$ is automatic from the definition of the set $(\Gamma\backslash\mathcal{O}'_N)_P$. □

**Remark.** Observe that

$$\mathrm{mult}_{(P,\omega_N(P))}\Psi(X,Y) \le \min(\mathrm{ord}_{f(\omega_N P)}\Psi(f(P),X), \mathrm{ord}_{f(P)}\Psi(f(\omega_N P),X)),$$

and the equality holds, for example, if $N$ is the power of an odd prime (since in this case $e_N(P) = \min(e_1(P), e_1(\omega_N P))$).

**Theorem 2.3.** *Assume that $M$ is either squarefree or a prime power. Let $Y(1)$ be the curve $X(D,M)^W$ attached to the Eichler order $\mathcal{O}$ of level $M$ and the Atkin-Lehner subgroup $W$. Let $N = p^r$, for an odd prime $p$ coprime to $DM$ and some integer $r \ge 1$. Fix Eichler orders $\mathcal{O}\supset\mathcal{O}(p)\supset\cdots\supset\mathcal{O}(p^{r-1})\supset\mathcal{O}(p^r)$ and for any $0\le k\le r$ denote by $Y(p^k)$ the corresponding curve $X(D,Mp^k)^W$. Write $S = \{s : \omega_s\in W\}$. Given $1\le t\le r$, consider $Z_t = \bigcup_{k=0}^{t}\omega_{p^t}Z_{t,k}$, where $Z_{t,k}$ denotes the set of CM points in $Y(p^t)$ corresponding to conjugation classes by $\Gamma(D,Mp^t)$ of optimal embeddings $\phi$ of $\Lambda$ into $\mathcal{O}(p^k)$ such that:*

- *$p\nmid\mathrm{cond}(\Lambda)$, or, equivalently, $\mathrm{cond}_p(\Lambda) = 1$, where $\mathrm{cond}_p$ denotes the $p$-part of the conductor.*
- *If $t > k \ge 0$, then $\phi(\Lambda)\not\subset\mathcal{O}(p^{k+1})$, or, equivalently, $\phi(\Lambda)\cap\mathcal{O}(p^{k+1}) = \mathbb{Z}+p\phi(\Lambda)$.*

- *At least one of the following two conditions holds (cf. Proposition 2.3):*
  - *The point in $Y(p^k)$ corresponding to $\phi$ is elliptic.*
  - *There exists an element $\lambda \in \Lambda'$ of norm $sp^{2k}$, for a certain $s \in S$, such that $\lambda^2 \in s\Lambda$. Moreover, if $s$ is not squarefree, $w_s\phi(\lambda) \in s\mathcal{O}$.*

*We will denote by $Z^1_{t,k}$ the subset of elements for which an element of norm $sp^{2k}$ as above exists.*

*Consider as well the following natural projections: $\pi_{p^\nu,p^\mu} : Y(p^\nu) \to Y(p^\mu)$ for integers $\nu \geq \mu$.*

*Then, the set of points of $Y(p^r)$ giving rise to singularities of the kroneckerian polynomial $\Psi_{p^r}(X,Y)$ is a subset of*

$$\bigcup_{t=1}^{r} \pi^{-1}_{p^r,p^t} Z_t.$$

*Given $P \in \pi^{-1}_{p^r,p^t}\omega_{p^t}P_0$, with $P_0 \in Z_{t,k}$ a CM point in the previous set, denote by $\Lambda, \Lambda_0$ and $\Lambda_1$ the orders corresponding to the CM points $P_0, P$ and $\omega_{p^r}P$ in $Y(1)$, respectively. Then, we have*

$$a \cdot \operatorname{mult}_{(P,\omega_{p^r}P)}\Psi_{p^r}(X,Y) = A,$$

*where*

$$A = \begin{cases} 2e_1(\omega_{p^t}P), & \text{if } k > 0, P_0 \in Z^1_{t,k} \text{ and } \operatorname{cond}_p(\Lambda_1) = p^{r-t}, \\ 1, & \text{if } k = 0 \text{ and } \operatorname{cond}_p(\Lambda_1) < p^{r-t}, \\ e_1(\omega_{p^t}P), & \text{otherwise,} \end{cases}$$

*$a = \max(e_1(P), e_1(\omega_{p^r}P))$, $e_1(Q)$ denotes the elliptic order of the point $Q$ in the curve $Y(1)$ and $\operatorname{mult}_{(P,\omega_{p^r}P)}\Psi_{p^r}(X,Y)$ denotes the number of branches of the curve through the point $(P,\omega_{p^r}P)$.*

*Moreover, all the branches at a singular point have different tangents if and only if $e_1(P) = e_1(\omega_{p^r}P)$. Otherwise, all branches are tangent at $P$.*

**Remark.** In Proposition 4.1 we will see that the restriction on $M$ is not an actual restriction since all curves $X(D,M)^W$ of genus 0 satisfy this condition.

*Proof.* Given a positive integer $k$, we will denote $\Gamma(p^k) = \Gamma(D,Mp^k)$ and $\Gamma(p^k)' = \Gamma(D,Mp^k) \smallsetminus \Gamma(D,Mp^{k+1})$.

To begin with, let us prove that any singular point can be obtained as predicted by the statement.

Consider $P \in \mathcal{H}$ giving a CM point in $Y(p^r)$ and assume it is fixed by an element of the form $\overline{\gamma_0}\,\overline{\omega_N}\gamma_1\omega_N\omega_s$ with $\gamma_1 \in \Gamma(p^{s_1})'$, $\gamma_0 \in \Gamma(p^{s_0})'$, $s \in S$ and $0 \leq s_0, s_1 \leq r$. Then, using Proposition 2.2, we obtain that the image of $P$ in $Y(1)$ is a CM point by an order $\Lambda_0$ such that the $p$-part of its conductor is $p^{s_0}$, if $s_0 + s_1 \leq r$, and $p^{r-s_1}$, if $s_0 + s_1 \geq r$. Similarly, the point $\omega_N P$ gives a CM point in $Y(p^r)$ and it is fixed by the transformation $\omega_N\overline{\gamma_0}\,\overline{\omega_N}\gamma_1\omega_s$ corresponding to an element in $\mathcal{O}(p^r)$. Therefore, the image of $\omega_N P$ in $Y(1)$ is a CM point by an order $\Lambda_1$ such that the $p$-part of its conductor is $p^{s_1}$, if $s_0 + s_1 \leq r$, and $p^{r-s_0}$, if $s_0 + s_1 \geq r$. Observe that the case $s_0 + s_1 \geq r$ is equivalent to requiring the algebraic element corresponding to the transformation to be divisible by $p^r$ in the ring of integers. Moreover, if we denote by $p^{f_0}, p^{f_1}$ the $p$-part of the conductor of the orders corresponding to $P$, $\Lambda_0$, and $\omega_N P$, $\Lambda_1$, in $Y(1)$ as above, we have that $f_0 + f_1 \leq r$.

Fix $P \in \mathcal{H}$ representing a point in $Y(N)$ such that $(P, \omega_N P)$ yields a point in $Y(1) \times Y(1)$ which is a singularity of $\Psi_N$ or, equivalently, such that $\overline{\gamma_0}\,\overline{\omega_N}\gamma_1\omega_N\omega_s P = P$ for certain $\gamma_0, \gamma_1 \in \Gamma(1) \smallsetminus \Gamma(N)$ and $s \in S$. In particular, $P$ is a CM point and denote by $p^{f_0}$, resp. $p^{f_1}$, the $p$-part of the conductor of the order $\Lambda_0$, resp. $\Lambda_1$, corresponding to the projection of the point $P$, resp. $\omega_N P$, to $Y(1)$. Then, $\overline{\gamma_0}\,\overline{\omega_N}\gamma_1\omega_N\omega_s$ corresponds to an element $\lambda \in \Lambda_0 \smallsetminus N\Lambda_0$ of norm $sN^2$ for some $s \in S$ and it follows from the previous paragraph that either

$$\gamma_0 \in \Gamma(p^{f_0}) \text{ and } \gamma_1 \in \Gamma(p^{f_1}),$$

or

$$\gamma_0 \in \Gamma(p^{r-f_1}) \text{ and } \gamma_1 \in \Gamma(p^{r-f_0}).$$

Then, we define

$$t = r - f_1, \quad k = r - f_0 - f_1.$$

In the first case, we can write

$$\overline{\gamma_0}\,\overline{\omega_N}\gamma_1\omega_N\omega_s = p^{f_1}\overline{\gamma_0}\,\overline{\omega_{p^t}}\gamma_1'\omega_{p^t}\omega_s.$$

Now, since $\gamma_0 \in \Gamma(p^{f_0})$, we obtain that $\omega_{p^t}P$ is fixed by

$$\omega_{p^t}\overline{\gamma_0}\,\overline{\omega_{p^t}}\gamma_1'\omega_s = p^{f_0}\omega_{p^k}\overline{\gamma_0'}\,\overline{\omega_{p^k}}\gamma_1'\omega_s.$$

If we prove that the element in $\mathcal{O}$ corresponding to $\omega_{p^k}\overline{\gamma_0'}\,\overline{\omega_{p^k}}\gamma_1'\omega_s$ does not belong to $\mathcal{O}(p^{k+1})$ (we will simply write $\omega_{p^k}\overline{\gamma_0'}\,\overline{\omega_{p^k}}\gamma_1'\omega_s \notin \mathcal{O}(p^{k+1})$) whenever

$t > k$, we can deduce that

$$P \in \pi^{-1}_{p^r,p^t}\omega_{p^t}Z_{t,k}.$$

In order to prove this, observe that $\omega_{p^k}\,\overline{\gamma_0'}\,\overline{\omega_{p^k}}\gamma_1'\omega_s \notin \mathcal{O}(p^{k+1})$ if and only if $\overline{\omega_{p^{k+1}}}\omega_{p^k}\,\overline{\gamma_0'}\,\overline{\omega_{p^k}}\gamma_1'\omega_s\omega_{p^{k+1}} \notin p^{k+1}\mathcal{O}$, and since $\gamma_1' \in \Gamma(1)'$, this is equivalent to $\omega_{p^k}\,\overline{\gamma_0'}\,\overline{\omega_{p^k}}\gamma_1' \notin \omega_{p^{k+1}}\mathcal{O}'_{p^{k-1}}$. In order to prove this assertion, observe that $\omega_{p^t}\,\overline{\gamma_0}\,\overline{\omega_{p^t}}\gamma_1' = p^{t-k}\omega_{p^k}\,\overline{\gamma_0'}\,\overline{\omega_{p^k}}\gamma_1'$ and multiplying on the left by $\overline{\omega_{p^{k+1}}}$, if $t > k$, we obtain that

$$\overline{\omega_{p^{k+1}}}\omega_{p^t}\,\overline{\gamma_0}\,\overline{\omega_{p^t}}\gamma_1' = p^{k+1}\gamma_2\omega_{p^{t-k-1}}\,\overline{\gamma_0}\omega_{p^t}\gamma_1' \in p^t\Gamma(1)'\omega_{p^{k+1}}\Gamma(1)'.$$

From this, we deduce that $\overline{\omega_{p^{k+1}}}\omega_{p^k}\,\overline{\gamma_0'}\,\overline{\omega_{p^k}}\gamma_1' \in p^k\mathcal{O}'_{p^{k+1}}$ and the result follows.

Similarly, in the second case, using that $f_1 \le r - f_0$, we can write

$$\overline{\gamma_0}\,\overline{\omega_N}\gamma_1\omega_N\omega_s = p^{f_1}\overline{\gamma_0}\,\overline{\omega_{p^{r-f_1}}}\gamma_1'\omega_{p^{r-f_1}}\omega_s,$$

with $\gamma_1' \in \Gamma(p^{r-f_0-f_1})$ and since $\gamma_0 \in \Gamma(p^{r-f_1})$, we obtain that

$$P \in \pi^{-1}_{p^r,p^{r-f_1}}\omega_{p^{r-f_1}}Z_{r-f_1,r-f_1-f_0} = \pi^{-1}_{p^r,p^t}\omega_{p^t}Z_{t,k}.$$

Now, let us prove the formula providing the multiplicity of the singularities.

If we assume that a point $P \in \mathcal{H}$ corresponding to a CM point in $Y(p^r)$ is fixed by a transformation which can be written as $\overline{\gamma_0}\,\overline{\omega_N}\gamma_1\omega_N\omega_s$, then for any other transformation $\overline{\gamma_0}\,\overline{\omega_N}\rho_1\omega_N\omega_{s'}$, multiplying one of them by the conjugate of the other, we obtain a third transformation $\overline{\omega_{s'}\omega_N}\,\overline{\rho_1}\gamma_1\omega_N\omega_s$. Therefore we can conclude that either both elements are equal, or $\omega_N P$ is elliptic. In any case, there are $e_1(\omega_{p^r}P)$ such transformations. Moreover, we have seen in the previous theorem that the branch attached to $\omega_N\gamma'$ coincides with the branch attached to $\omega_N\gamma$ if and only if they are related by an elliptic transformation fixing $P$. Therefore, we obtain $e_1(P)/e_{p^r}(P)$ additional transformations for any of the transformations above, where $e_{p^r}(P)$ denotes the order of the point $P$ in the curve $Y(p^r)$. As a result, we obtain the number of different transformations giving the same branch, namely, $e_1(P)e_1(\omega_{p^r}P)/e_{p^r}(P) = \max(e_1(P), e_1(\omega_{p^r}P))$.

In the first part of the proof we have seen that any transformation of the form $\overline{\gamma_0}\,\overline{\omega_{p^r}}\gamma_1\omega_{p^r}\omega_s$ fixing $P$ can be obtained from a transformation $\overline{\rho_0}\,\overline{\omega_{p^k}}\rho_1\omega_{p^k}\omega_s$ fixing $P_0$. Now, we are going to study when we can go the other way around. First of all, assume that we start with an element of the form $\overline{\rho_0}\,\overline{\omega_{p^k}}\rho_1\omega_{p^k}\omega_s$, $k \ge 0$, which is primitive in the ring of integers. This gives an element of the form $\overline{\gamma_0'}\overline{\omega_{p^t}}\gamma_1'\omega_{p^t}\omega_s$ fixing $P$. Now, if we can write $p^{r-t}\overline{\gamma_0'}\overline{\omega_{p^t}}\gamma_1'\omega_{p^t}\omega_s = \overline{\gamma_0\omega_{p^r}}\gamma_1\omega_{p^r}\omega_s$

it will follow from the first part of the proof that $\Lambda_1$ has $p$-conductor $p^{r-t}$. Let us now prove that, if the $p$-conductor of $\Lambda_1$ is $p^{r-t}$, then a primitive element like the one we started from lifts to an element as desired. In order to do that, we will first of all study when $p^{r-t}\overline{\gamma_0'}\,\overline{\omega_{p^t}}\gamma_1'\omega_{p^t}\omega_s$ can be written as $\overline{\gamma_0}\,\overline{\omega_{p^r}}\gamma_1\omega_{p^r}\omega_s$. This is always true if $t=r$. Otherwise, if $t<r$,

$$p^{r-t}\overline{\gamma_0'}\,\overline{\omega_{p^t}}\gamma_1'\omega_{p^t}\omega_s = \overline{\gamma_0}\,\overline{\omega_{p^r}}\gamma_1\omega_{p^r}\omega_s$$

holds for certain $\gamma_0, \gamma_1$ if and only if, multiplying both sides on the right by $\overline{\omega_{p^r}}$,

$$p^{-t}\overline{\gamma_0'}\,\overline{\omega_{p^t}}\gamma_1'\omega_{p^t}\overline{\omega_{p^r}}$$

is primitive in $\mathcal{O}$, which holds if and only if $\overline{\gamma_0'}\,\overline{\omega_{p^t}}\gamma_1'\omega_{p^t}$ cannot be written as

$$\overline{\gamma_0'}\,\overline{\omega_{p^{t-1}}}\gamma_1''\omega_{p^{t+1}}.$$

Thus, we only need to prove that, if $P\in Y(p^r)$ is fixed by $\overline{\gamma_0'}\,\overline{\omega_{p^{t-1}}}\gamma_1''\omega_{p^{t+1}}\omega_s$, then the $p$-part of the conductor of $\Lambda_1$ is strictly smaller than $p^{r-t}$. Observe that since this element is not divisible by $p$ in $\mathcal{O}$, $\gamma_1''\notin\Gamma(p)$. If we consider

$$\omega_{p^{t+1}}\overline{\gamma_0'}\,\overline{\omega_{p^{t-1}}}\gamma_1''\omega_{p^{t+1}}\omega_s\overline{\omega_{p^{t+1}}} = p^{t+1}\omega_{p^{t+1}}\overline{\gamma_0'}\,\overline{\omega_{p^{t-1}}}\gamma_1''\omega_s,$$

since $\gamma_1''\notin\Gamma(p)$ we obtain that $\omega_{p^{t+1}}P$ is a CM point in $Y(1)$ by a $p$-maximal order. Therefore, since $\omega_{p^{t+1}}P$ and $\omega_{p^r}P$ are related by a transformation attached to an element of norm $p^{r-t-1}$, the $p$-part of the conductor of $\Lambda_1$ is at most $p^{r-t-1}$.

Secondly, assume now that we start from an element of the form $\overline{\omega_{p^t}}\gamma_1'\omega_{p^t}\omega_s$ fixing $P$ with $\gamma_1'\in\Gamma(p^k)$, $k>0$. We are going to prove that all these elements can be written as $p^{r-t}\overline{\omega_{p^t}}\gamma_1'\omega_{p^t}\omega_s = \overline{\gamma_0}\,\overline{\omega_{p^r}}\gamma_1\omega_{p^r}\omega_s$. As in the previous case, the only elements for whom this does not happen are those which can be written as

$$\overline{\omega_{p^t}}\gamma_1'\omega_{p^t}\omega_s = \overline{\omega_{p^{t-1}}}\gamma_1''\omega_{p^{t+1}}\omega_s.$$

In particular, if $t=k=1$, the condition translates into

$$\overline{\omega_p}\gamma_1'\omega_p\omega_s = \gamma_1''\omega_{p^2}\omega_s.$$

Since $\gamma_1'\in\Gamma(p)$, the lefthand side is not primitive in $\mathcal{O}$, while the righthand side is. Therefore we reach a contradiction. Otherwise, if $t>1$, we can write $\overline{\omega_{p^t}}\gamma_1'\omega_{p^t}\omega_s = p\overline{\omega_{p^{t-1}}}\gamma_1''\omega_{p^{t-1}}\omega_s$. Therefore this element does not lift in the desired fashion if and only if $\overline{\omega_{p^{t-1}}}\gamma_1''\omega_{p^{t-1}}\omega_s = \overline{\omega_{p^{t-2}}}\gamma_1'''\omega_{p^t}\omega_s$. But putting together both equalities,

$$\overline{\omega_{p^t}}\gamma_1' = p\overline{\omega_{p^{t-2}}}\gamma_1'''$$

we have an equality between an element which is primitive in the quadratic order and another one which is not.

Summing up, we have already seen that

$$a(\mathrm{mult}_{(P,\omega_{p^r}P)}\Psi_{p^r}(X,Y) - 1)$$

coincides with the number of transformations of the form

$$\overline{\gamma_0}\,\overline{\omega_{p^r}}\gamma_1\omega_{p^r}\omega_s$$

fixing $P$. Now, if $P \in \pi^{-1}_{p^r,p^t}\omega_{p^t}Z_{t,k}$, all such transformations give elements in the order corresponding to $\omega_{p^t}P$ of norm $s$ or $sp^{2k}$, for some $s \in S$. It is worth noting that, since all these elements must lie in a quadratic $p$-maximal order, given two elements $\lambda, \lambda'$ of norm in $Sp^{2k}$, then $p^{2k}$ either divides $\lambda\lambda'$ or $\lambda\overline{\lambda'}$. Hence, it follows that the transformations attached to these elements are related through an elliptic transformation. As a result, if $P \in \pi^{-1}_{p^r,p^t}\omega_{p^t}Z^1_{t,k}$ is such that $\omega_{p^r}P$ has $p$-conductor $p^{r-t}$, then the number of transformations coming from elements of norm in $Sp^{2k}$ of the form $\overline{\gamma_0}\,\overline{\omega_{p^r}}\gamma_1\omega_{p^r}\omega_s$ is $e_1(\omega_{p^t}P)$. On the other hand, the number of transformations of the same form coming from elliptic elements is as well $e_1(\omega_{p^t}P)$. Therefore, if $k > 0$ we obtain $2e_1(\omega_{p^t}P)$ transformations and if $k = 0$, $e_1(\omega_{p^t}P)$. On the other hand, if $\omega_{p^r}P$ has $p$-conductor strictly smaller than $p^{r-t}$, none of the transformations coming from elements of norm in $Sp^{2k}$ rise to elements of the expected form, and thus what we obtain are $e_1(\omega_{p^t}P)$ transformations if $k > 0$ and 1 if $k = 0$ (only the one corresponding to the identity). Now, we only have to find out how many of these transformations do not correspond to branches, and that is those which correspond to elliptic transformations fixing $P$ or $\omega_{p^r}P$, that is $e_1(P) + e_1(\omega_{p^r}P) - e_{p^r}(P)$, which coincides with $a$. Thus we obtain that the number of branches through a singularity coincides with the formula given in the statement.

Now, it only remains to study the tangents of the different branches through a singularity. In order to study this we will make use of expansions which will be introduced later on, when expansions are studied on their own, in Section 3.3. Denote $e = e_1(P)$ and $e' = e_1(\omega_{p^r}P)$. Then, the branches of the curve $\Psi_{p^r}(X,Y) = 0$ attached to the Hauptmodul $f$ through $P$ can be obtained as $(f, f \circ \omega_{p^r} \circ \overline{\lambda})$ where $\lambda$ runs over transformations $\overline{\gamma_0}\,\overline{\omega_{p^r}}\gamma_1\omega_{p^r}\omega_s$ giving the different branches which we have computed before. Since $f$ is a Hauptmodul, we can choose a local parameter of the form $q = \kappa\frac{z-P}{z-\overline{P}}$ for a certain $\kappa \in \mathbb{C}^*$ such that $f = a_0 + q^e + o(q^e)$. Then, with this same parameter, we can consider an expansion of $f \circ \omega_{p^r}$ around $P$. Since it is a Hauptmodul for the curve corresponding to the quaternion order $w_{p^r}^{-1}\mathcal{O}w_{p^r}$ and for which $P$ is an elliptic point of order $e'$, then

$$f \circ \omega_{p^r} = b_0 + b_1 q^{e'} + o(q^{e'})$$

with $b_1 \in \mathbb{C}^*$. Now, if $\alpha$ is the element in the quadratic order corresponding to the transformation $\lambda$ fixing $P$,

$$q \circ \overline{\lambda} = \frac{\mathrm{Norm}\alpha}{\alpha^2} q = \frac{\overline{\alpha}}{\alpha} q$$

(cf. Proposition 3.6). Observe that, since $\alpha$ lies in a quadratic field but not in $\mathbb{Q}$, this constant factor is at most quadratic and different from 1. Therefore, if it is a root of unity, it must be primitive of order $2, 3, 4$ or 6. The only primitive elements for which this holds are $\alpha \in \{1 \pm i, \frac{\pm 3 \pm \sqrt{-3}}{2}\} \cup \{\sqrt{-D}\}$ for positive squarefree integers $D > 1$, which have norm $2, 3, D$, respectively, and none of these has the form $sp^{2k}$ for a prime $p$, $s \in S$ and some $k > 0$.

Now, we will deal with the different cases one by one. If $e_1(\omega_{p^r}P) > 1$, whenever we have a singularity it comes from a primitive element $\overline{\lambda}$ of norm $sp^{2k}$, $k > 0$, and we have two branches through $P$, which can be parametrized as follows:

$$(f, f \circ \omega_{p^r}) = (a_0 + q^e + \dots, b_0 + b_1 q^{e'} + \dots),$$
$$(f, f \circ \omega_{p^r}\gamma) = (a_0 + q^e + \dots, b_0 + b_1 \tfrac{\overline{\lambda}^{e'}}{\lambda^{e'}} q^{e'} + \dots).$$

Therefore, if $e = 1$, both have common tangent $Y = 0$ and, if $e = e'$, the tangents are different, $b_1 X - Y = 0$ and $b_1 \frac{\overline{\lambda}^e}{\lambda^e} X - Y = 0$, respectively. The case $e > 1, e' = 1$ is totally analogous to the case $e = 1, e' > 1$ reversing the role of $X$ and $Y$. Finally, in the case $e = e' = 1$, $f = a_0 + q + \cdots$ and $f \circ \omega_{p^r} = b_0 + b_1 q + \cdots$ and every non-trivial transformation fixing $P$ changes $q$ non-trivially. □

**Remark.** It is worth noting that, if either $P$ or $\omega_{p^r}P$ are elliptic, then there are at most two branches through $(P, \omega_{p^r}P)$. Moreover, if both are elliptic, then these two branches have different tangents. Even though this is not in general the only type of points related to elliptic points having these properties, we will exploit this fact later on.

**Corollary 2.4.** *The condition* $\mathrm{cond}_p(\Lambda_1) < p^{r-t}$ *in the previous theorem can be rewritten as:*

- *$e_1(\omega_{p^t}P) = e_1(\omega_{p^{t+1}}P) > 1$, or*
- *$e_1(\omega_{p^t}P) = 1$ and*

$$\mathrm{mult}_{(P,\omega_{p^{t+1}}P)}\Psi_{p^{t-1}} > \mathrm{mult}_{(P,\omega_{p^t}P)}\Psi_{p^{t-2}} - \mathrm{mult}_{(P,\omega_{p^{t+1}}P)}\Psi_{p^{t-3}}.$$

*Recall that $\Psi_1(X,Y) = X - Y$, and we simply put $\Psi_u(X,Y) = 1$ if $u$ is not a positive integer.*

*Proof.* In order to prove the equivalence, recall that, as we have seen in Theorem 2.3, the condition $\mathrm{cond}_p(\Lambda_1) < p^{r-t}$ is equivalent to the existence of an element of the form $\overline{\gamma_0}\,\overline{\omega_{p^{t-1}}}\gamma_1\omega_{p^{t+1}}\omega_s$ fixing $P$ with $\gamma_1 \notin \Gamma(p)$.

First, let us assume that $\mathrm{cond}_p(\Lambda_1) < p^{r-t}$. Then, there exists a transformation of the form $\overline{\gamma_0}\,\overline{\omega_{p^{t-1}}}\gamma_1\omega_{p^{t+1}}\omega_s$ fixing $P$ with $\gamma_1 \notin \Gamma(p)$. Therefore, $\omega_{p^{t+1}}\overline{\gamma_0}\,\overline{\omega_{p^{t-1}}}\gamma_1\omega_s$ is an element fixing $\omega_{p^{t+1}}P$. Since $\gamma_1 \notin \Gamma(p)$, we obtain that the quadratic order corresponding to $\omega_{p^{t+1}}P$ has conductor coprime to $p$, and therefore it coincides with the order attached to $\omega_{p^t}P$. Since both of these points are related by a transformation coming from an element in $\mathcal{O}$, we can conclude that, if $\omega_{p^t}P$ is elliptic, so is $\omega_{p^{t+1}}P$. Reciprocally, if $e_1(\omega_{p^t}P) = e_1(\omega_{p^{t+1}}P) > 1$, we obtain that the conductor of $\omega_{p^r}P$ is divisible at most by $p^{r-t-1}$.

Otherwise, in the case $e_1(\omega_{p^t}P) = 1$, it will suffice to prove that the number of elements of the form $\overline{\gamma_0}\,\overline{\omega_{p^{t-1}}}\gamma_1\omega_{p^{t+1}}\omega_s$ fixing $P$ with $\gamma_1 \notin \Gamma(p)$ is

$$\mathrm{mult}_{(P,\omega_{p^{t+1}}P)}\Psi_{p^{t-1}} - (\mathrm{mult}_{(P,\omega_{p^t}P)}\Psi_{p^{t-2}} - \mathrm{mult}_{(P,\omega_{p^{t+1}}P)}\Psi_{p^{t-3}}).$$

Consider the set

$$\{\overline{\gamma_0}\,\overline{\omega_{p^{t-1}}}\gamma_1\omega_{p^{t+1}}\omega_s : \text{it fixes } P \text{ and } \gamma_1 \notin \Gamma(p)\}.$$

Even though we will not make use of it, it is easy to see that this set has cardinality either 0 or 1. Now, if we consider the bigger set

$$\{\overline{\gamma_0}\,\overline{\omega_{p^{t-1}}}\gamma_1\omega_{p^{t+1}}\omega_s : \text{it fixes } P\},$$

since we are in the non-elliptic case, it has cardinality $\mathrm{mult}_{(P,\omega_{p^{t+1}}P)}\Psi_{p^{t-1}}$. Moreover, the two sets differ by

$$\{\overline{\gamma_0}\,\overline{\omega_{p^{t-1}}}\gamma_1\omega_{p^{t+1}}\omega_s : \text{it fixes } P \text{ and } \gamma_1 \in \Gamma(p)\},$$

and every such element can be written as

$$\overline{\gamma_0}\,\overline{\omega_{p^{t-2}}}\gamma_1\omega_{p^t}\omega_s.$$

The number of such elements is $\mathrm{mult}_{(P,\omega_{p^t}P)}\Psi_{p^{t-2}}$. However, not all these elements come from one of the form $\overline{\gamma_0}\,\overline{\omega_{p^{t-1}}}\gamma_1\omega_{p^{t+1}}\omega_s$. The set of those for whom it is not possible, as we have seen in the proof of Theorem 2.3, is $\{\overline{\gamma_0}\,\overline{\omega_{p^{t-3}}}\gamma_1\omega_{p^{t+1}}\omega_s$ : it fixes $P\}$, which has cardinality $\mathrm{mult}_{(P,\omega_{p^{t+1}}P)}\Psi_{p^{t-3}}$. As a result, the expression above follows. □

## 2.2 Kroneckerian polynomials for the Hauptmoduln $t_D$ of $X(D,1)$

In the previous section, we dealt with the case of curves $X(D,M)^W$ of genus 0. Now we are going to restrict our study to the case $M = 1, W = \{id\}$, which amounts to $D = 6, 10, 22$. In this situation, these are genus 0 curves with no real points and therefore, there exist no Hauptmoduln defined over $\mathbb{Q}$. In the previous section we studied the properties satisfied by a kroneckerian polynomial attached to a generic Hauptmodul defined over a generic number field. Now we are going to introduce some particular Hauptmoduln for all these curves and prove that the corresponding kroneckerian polynomials are defined over $\mathbb{Q}$ even though the Hauptmoduln are not, together with some additional arithmetic properties concerning their symmetries and their reduction. First of all, we are going to introduce the Hauptmodul $t_D$, which can be seen as an extension to the cases $D = 10, 22$ of the construction given in [BT07b] for the case $D = 6$ (cf. [Kur79], [Jor81]). Similarly, we could reproduce the construction given in this section to Atkin-Lehner quotients $X(D,M)^W$ for a certain subgroup $W \subset W_{DM}$ of index at least 4 such that $X(D,M)^W$ has genus 0 and no rational points, which is the case, for example for $D = 6, N = 5$ and $W = \{\omega_2\}$ or $W = \{\omega_{10}\}$.

**Theorem 2.4.** *Fix $D = 6, 10, 22$ and for each of these values of $D$ consider the prime integer*

$$c(D) = \begin{cases} 3, & D = 6, \\ 2, & D = 10, \\ 11, & D = 22. \end{cases}$$

*Then, there exists a Hauptmodul $t_D$ for the curve $X(D,1)$ satisfying the following properties:*

*(1) Attached to an element of norm $-D$ in $\mathcal{O}$, there is a hyperbolic symmetry*

*s on the upper half-plane such that, for every $z \in \mathcal{H}$,*

$$t_D(z) = \overline{t_D(s(z))}.$$

*(2) Let $\alpha_1, \alpha_2, \alpha_3$ be the three non-trivial involutions of $X(D,1)$ in $W_D$, then $\{t_D \circ \alpha_j\}_{j=1,2,3} = \{-t_D, t_D^{-1}, -t_D^{-1}\}$. More precisely,*

| $D$ | $\alpha_1$ | $t_D \circ \alpha_1$ | $\alpha_2$ | $t_D \circ \alpha_2$ | $\alpha_3$ | $t_D \circ \alpha_3$ |
|---|---|---|---|---|---|---|
| 6 | $\omega_2$ | $-t_6$ | $\omega_3$ | $t_6^{-1}$ | $\omega_6$ | $-t_6^{-1}$ |
| 10 | $\omega_2$ | $t_{10}^{-1}$ | $\omega_5$ | $-t_{10}$ | $\omega_{10}$ | $-t_{10}^{-1}$ |
| 22 | $\omega_2$ | $-t_{22}$ | $\omega_{11}$ | $t_{22}^{-1}$ | $\omega_{22}$ | $-t_{22}^{-1}$ |

*(3) The function $t_D$ is a Hauptmodul over the field $\mathbb{Q}(i, \sqrt{c(D)})$ for the curve $X(D,1)$.*

*(4) The canonical model $X(D,1)$ is given by a function $j_D : \mathcal{H} \to \mathbb{P}^2(\mathbb{C})$ which, in projective coordinates, can be written as $j_D = (u_{D,1} : u_{D,2} : 1)$, where its coordinate functions relate to $t_D$ as follows:*

$$u_{D,1} = \sqrt{-c(D)}\,\frac{t_D^2+1}{2t_D}, \quad u_{D,2} = \sqrt{c(D)}\,\frac{t_D^2-1}{2t_D}.$$

*Moreover, they satisfy the equation $u_{D,1}^2 + u_{D,2}^2 + c(D) = 0$.*

*(5) The functions $u_{D,1}$, $u_{D,2}$ and $u_{D,3} = u_{D,1}/u_{D,2}$ are Hauptmoduln defined over $\mathbb{Q}$ for the three quotients of $X(D,1)$ by a non-trivial Atkin-Lehner involution:*

| $D$ | $u_{D,1}$ | $u_{D,2}$ | $u_{D,3}$ |
|---|---|---|---|
| 6 | $\omega_3$ | $\omega_6$ | $\omega_2$ |
| 10 | $\omega_2$ | $\omega_{10}$ | $\omega_5$ |
| 22 | $\omega_{11}$ | $\omega_{22}$ | $\omega_2$ |

*(6) A Hauptmodul defined over $\mathbb{Q}$ for the quotient $X(D,1)^{W_D} = X(D,1)^+$ is*

*given by $t_D^+$ such that*

| $D$ | $u_{D,1}^2$ | $u_{D,2}^2$ | $u_{D,3}^2$ |
|---|---|---|---|
| 6 | $\frac{3t_6^+}{1-t_6^+}$ | $\frac{3}{t_6^+-1}$ | $-t_6^+$ |
| 10 | $-2t_{10}^+$ | $2(t_{10}^+-1)$ | $\frac{t_{10}^+}{1-t_{10}^+}$ |
| 22 | $\frac{11t_{22}^+}{1-t_{22}^+}$ | $\frac{11}{t_{22}^+-1}$ | $-t_{22}^+$ |

*Proof.* For the moment, let us denote $X = X(D,1)$, $X_1, X_2, X_3$ the quotient curves corresponding to $\alpha_1, \alpha_2, \alpha_3$ respectively and $X^+$ the double quotient curve. The Riemann-Hurwitz formula applied to the Galois cover $X \to X^+$ yields that this morphism is ramified exactly at 3 points $x_1, x_2, x_3$ of $X^+$ with ramification index 2, each of them corresponding respectively to the projection to $X^+$ of the ramification divisors of $X \to X_i$. Since the morphisms $\alpha_i$ are defined over $\mathbb{Q}$, we obtain therefore that $x_1, x_2, x_3 \in X^+(\mathbb{Q})$. Now we can consider a Hauptmodul $t^+ : X^+(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $t^+(x_1) = 1$, $t^+(x_2) = \infty$, $t^+(x_3) = 0$, which is therefore defined over $\mathbb{Q}$.

Similarly, $x_j \in X_l(\mathbb{Q})$ if $1 \leq j, l \leq 3$, $j \neq l$, and therefore $X_l \cong P^1_{\mathbb{Q}}$, $1 \leq l \leq 3$. Therefore, there exist Hauptmoduln $\tilde{u}_l$ for $X_l$ defined over $\mathbb{Q}$. Now we are going to give expressions for a certain choice of these $\tilde{u}_l$ in terms of $t^+$. The morphism $X_1 \to X^+$ has degree 2 and ramifies exactly at $x_2, x_3$, which are rational points of $X^+$ where $t^+$ takes the values $\infty$ and 0, respectively. Thus, we can choose $\tilde{u}_1 \in \mathbb{Q}(X_1)$ such that $\tilde{u}_1^2 = k_1 t^+$ for a suitable squarefree integer $k_1$ which we are going to determine next. Then, since $\mathbb{Q}(t^+) \subset \mathbb{Q}(\tilde{u}_1)$, we will automatically have that $\mathbb{Q}(X_1) = \mathbb{Q}(\tilde{u}_1)$. Similarly, $\tilde{u}_2, \tilde{u}_3$ can be taken such that $\tilde{u}_2^2 = k_2 \dfrac{t^+ - 1}{t^+}$ and $\tilde{u}_3^2 = k_3(t^+ - 1)$.

Note that we can take $\tilde{u}_3 = \tilde{u}_1 \tilde{u}_2$, that is $k_3 = k_1 k_2$. It is worth mentioning as well that $k_3 \tilde{u}_1^2 - k_1 \tilde{u}_3^2 - k_1 k_3 = 0$ and, therefore, this pair of functions provides and affine chart for a canonical model of the Shimura curve $X$. In particular, since $X(\mathbb{R}) = \varnothing$, we obtain that $k_1, k_2 < 0$ and $k_3 > 0$. By definition, $\tilde{u}_l \circ \alpha_l = \tilde{u}_l$ and $\tilde{u}_l \circ \alpha_j = -\tilde{u}_l$ for $j \neq l$. Moreover, we can compute the value of these functions at the points $x_j$:

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $\tilde{u}_1$ | $\pm\sqrt{k_1}$ | $\infty$ | $0$ |
| $\tilde{u}_2$ | $0$ | $\pm\sqrt{k_2}$ | $\infty$ |
| $\tilde{u}_3$ | $0$ | $\infty$ | $\pm\sqrt{-k_3}$ |

Let us now discuss the three cases separately:

- $D = 6$: Choose $\alpha_1 = \omega_2$, $\alpha_2 = \omega_3$ and $\alpha_3 = \omega_6$. It follows from [Ogg83], cf. Theorem 4.1, that

$$x_1 \in \mathrm{CM}(\mathbb{Z}[i]),\ x_2 \in \mathrm{CM}(\mathbb{Z}[\tfrac{1+\sqrt{-3}}{2}]),\ x_3 \in \mathrm{CM}(\mathbb{Z}[\sqrt{-6}]).$$

  Therefore,

$$\tilde{u}_l(x_1) \in HCF(\mathbb{Q}(i)) = \mathbb{Q}(i),\ \tilde{u}_l(x_1) \in HCF(\mathbb{Q}(\sqrt{-3})) = \mathbb{Q}(\sqrt{-3}),$$

  which taking into account the restriction on the signs of the $k_i$'s we have observed before, yields that $k_1 = -1, k_2 = -3, k_3 = 3$.

  Define $u_{6,1} = 3/\tilde{u}_2$ and $u_{6,2} = 3/\tilde{u}_3$.

- $D = 10$: Choose $\alpha_1 = \omega_2$, $\alpha_2 = \omega_5$ and $\alpha_3 = \omega_{10}$. As in the previous case, we obtain that

$$x_1 \in \mathrm{CM}(\mathbb{Z}[\sqrt{-2}]),\ x_2 \in \mathrm{CM}(\mathbb{Z}[\sqrt{-5}]),\ x_3 \in \mathrm{CM}(\mathbb{Z}[\sqrt{-10}]).$$

  Therefore,

$$\begin{aligned}\tilde{u}_l(x_1) &\in HCF(\mathbb{Q}(\sqrt{-2})) = \mathbb{Q}(\sqrt{-2}),\\ \tilde{u}_l(x_2) &\in HCF(\mathbb{Q}(\sqrt{-5})) = \mathbb{Q}(i,\sqrt{-5}),\end{aligned}$$

  which having the signs of the $k_i$'s in mind yields that

$$k_1 = -2, k_2 = -1, -5, k_3 = 2, 10.$$

  Observe, however, that if we had $k_1 = -2, k_2 = -5, k_3 = 10$, then the values $\tilde{u}_1(x_3) = 0, u_3(x_3) = \pm\sqrt{-10}$ would not generate the Hilbert Class Field $HCF(\mathbb{Q}(\sqrt{-10})) = \mathbb{Q}(\sqrt{-10}, \sqrt{-2})$ over $\mathbb{Q}(\sqrt{-10})$ in contradiction with the properties of the canonical model. Therefore $k_1 = -2$, $k_2 = -1$, $k_3 = 2$.

  Define $u_{10,1} = \tilde{u}_1$ and $u_{10,2} = \tilde{u}_3$.

- $D = 22$: This case is totally analogous to the case $D = 6$. In this case $k_1 = -1, k_2 = -11, k_3 = 11$ and $u_{22,1} = 11/\tilde{u}_2$ and $u_{22,2} = 11/\tilde{u}_3$.

We can now define the Hauptmodul

$$t_D = \frac{\sqrt{c(D)}}{iu_{D,1} + u_{D,2}} = \frac{iu_{D,1} - u_{D,2}}{\sqrt{c(D)}},$$

which is clearly defined over $\mathbb{Q}(i, \sqrt{c(D)})$. With these definitions, the statements (3), (4) and (5) are now immediate and it only remains to check that the first two hold too.

It follows from [Shi75b] that for any $f \in \mathbb{Q}(X(D,1))$, if we consider the symmetry attached to an element of norm $-1$ in $\mathcal{O}(D,1)$, then $f \circ s = \overline{f}$. Therefore, if we consider the symmetry $s \circ \omega_D$ corresponding to an element of norm $-D$, from the values of the corresponding functions at the points $x_1$, $x_2$, $x_3$, we obtain that $u_{D,1} \circ (s \circ \omega_D) = -\overline{u_{D,1}}$ and $u_{D,2} \circ (s \circ \omega_D) = \overline{u_{D,2}}$. Now, (1) follows trivially.

Finally, in order to prove (2), we observe that $u_{D,j} \circ \omega_l = \pm u_{D,j}$. Then, using the explicit description of $t_D$ in terms of $u_{D,j}$ and that $u_{D,1}^2 + u_{D,2}^2 + c(D) = 0$, the result follows. □

These properties of $t_D$ will allow us to prove more properties of the corresponding kroneckerian polynomials $\Psi_{\Gamma,N,t_D}(X,Y) \in \mathbb{Q}(i, \sqrt{c(D)})[X,Y]$.

**Proposition 2.6.** *For every integer $N > 1$ coprime to $D$, the polynomial $\Psi_{\Gamma,N,t_D}(X,Y)$ can be taken as a primitive polynomial in $\mathbb{Z}[X,Y]$. Also, the global degrees of all the non-zero monomials of $\Psi_{\Gamma,N,t_D}(X,Y)$ are even. Moreover, if we write*

$$\Psi_{\Gamma,N,t_D}(X,Y) = \sum_{j,k=0}^{\psi(N)} a_{jk} X^j Y^k, \qquad a_{j,k} \in \mathbb{Z},$$

*then the coefficients satisfy the following symmetry conditions:*

$$a_{jk} = a_{kj} = a_{\psi(N)-k,\psi(N)-j} = a_{\psi(N)-j,\psi(N)-k}.$$

*Proof.* The symmetry property of $t_D$ given in Theorem 2.4 is equivalent to the following symmetry properties of $u_{D,1}$ and $u_{D,2}$:

$$\overline{u_{D,1}(s(z))} = -u_{D,1}(z), \quad \overline{u_{D,2}(s(z))} = u_{D,2}(z).$$

If we define $u = iu_{D,1} + u_{D,2} = \dfrac{\sqrt{c(D)}}{t_D}$, then $u$ is a Hauptmodul for $X(D,1)$ defined over $\mathbb{Q}(i)$. Therefore, for every positive integer $N$ coprime to $D$, we can consider $\Psi_{\Gamma,N,u}(X,Y) \in \mathbb{Z}[i][X,Y]$ a kroneckerian polynomial attached to the Hauptmodul $u$. Recall that $\Psi_{\Gamma,N,u}(X,u)$ is irreducible over $\mathbb{Q}(i)(u)$, and has $u \circ \omega_N$ as a root. Then, for all $z \in \mathcal{H}$,

$$0 = \overline{\Psi_{\Gamma,N,u}(u(z), u(\omega_N z))} = \overline{\Psi_{\Gamma,N,u}}(\overline{u(z)}, \overline{u(\omega_N z)});$$

and, taking $sz$ for $z$,

$$0 = \overline{\Psi_{\Gamma,N,u}}(\overline{u(sz)}, \overline{u(\omega_N sz)}).$$

We note that $\overline{u(s(z))} = u(z)$ and

$$\overline{u(\omega_N s(z))} = \overline{u(s(s^{-1}\omega_N s)(z))} = u((s^{-1}\omega_N s)z).$$

Since the element $s^{-1}w_N s$ has norm $N$ and is primitive, it belongs to $\mathcal{O}_1 w_N \gamma$ for some $\gamma \in \mathcal{O}_1$. Being $u$ a $\Gamma$-automorphic function, for every $z \in \mathcal{H}$, we have $u((s^{-1}\omega_N s)z) = u(\omega_N \gamma z)$ and

$$0 = \overline{\Psi_{\Gamma,N,u}}(u(z), u(\omega_N \gamma z)),$$

which applying the map $z \mapsto \gamma^{-1}z$ gives

$$0 = \overline{\Psi_{\Gamma,N,u}}(u(z), u(\omega_N z)).$$

Since the conjugated polynomial $\overline{\Psi_{\Gamma,N,u}}(X,Y)$ is irreducible, it follows that $\overline{\Psi_{\Gamma,N,u}}(X,u)$ is irreducible over $\mathbb{Q}(i)(u)$ and has $u \circ \omega_N$ as a root, which implies that we can take $\Psi_{\Gamma,N,u}(X,Y) \in \mathbb{Z}[X,Y]$.

Similarly, if we choose $\alpha$ an Atkin-Lehner involution according to Theorem 2.4 so that $t_D \circ \alpha = -t_D$ for every $z \in \mathcal{H}$, we have, in addition, the identity

$$0 = \Psi_{\Gamma,N,u}(u(\alpha z), u(\omega_N \alpha z)) = \Psi_{\Gamma,N,u}(-u(z), -u(\omega_N z)).$$

Therefore, we obtain that $\Psi_{\Gamma,N,u}(-X,-Y) = \pm\Psi_{\Gamma,N,u}(X,Y)$. Observe that if $\Psi_{\Gamma,N,u}(-X,-Y) = -\Psi_{\Gamma,N,u}(X,Y)$, then $\Psi_{\Gamma,N,u}(X,-X) = -\Psi_{\Gamma,N,u}(-X,X) = -\Psi_{\Gamma,N,u}(X,-X)$, which yields $\Psi_{\Gamma,N,u}(X,-X) = 0$ and therefore that $X+Y$ is a factor of $\Psi_{\Gamma,N,u}(X,Y)$ which is a contradiction because of the irreducibility, since $N > 1$. As a result, the global degrees of all the non-zero monomials of the polynomial $\Psi_{\Gamma,N,u}(X,Y)$ are even.

Now, since $t_D = \sqrt{c(D)}u^{-1}$ and the monomials of the kroneckerian polynomial $\Psi_{\Gamma,N,t_D}(X,Y)$ are even, we can take $\Psi_{\Gamma,N,t_D}(X,Y) \in \mathbb{Z}[X,Y]$.

It only remains to prove that $a_{jk} = a_{\psi(N)-j,\psi(N)-k}$ for $0 \le j,k \le \psi(N)$.

We consider now an involution $\alpha$ such that $t_D \circ \alpha = 1/t_D$. Replacing $z$ by $\alpha(z)$ in the identity $0 = \Psi_{\Gamma,N,t_D}(t_D(z), t_D(\omega_N z))$, it follows that

$$0 = t_D(z)^{\psi(N)} t_D(\omega_N z)^{\psi(N)} \Psi_{\Gamma,N,t_D}(1/t_D(z), 1/t_D(\omega_N z)),$$

for every $z \in \mathcal{H}$. Therefore, up to multiplication by $\pm 1$, the polynomial $X^{\Psi(N)} Y^{\Psi(N)} \Psi_{\Gamma,N,t_D}(X^{-1}, Y^{-1})$ must be the polynomial $\Psi_{\Gamma,N,t_D}(X,Y)$. However, if $X^{\Psi(N)} Y^{\Psi(N)} \Psi_{\Gamma,N,t_D}(X^{-1}, Y^{-1}) = -\Psi_{\Gamma,N,t_D}(X,Y)$, we would have $\Psi_{\Gamma,N,t_D}(X^{-1}, X) = -\Psi_{\Gamma,N,t_D}(X, X^{-1})$ which yields that $\Psi_{\Gamma,N,t_D}(X, X^{-1}) = 0$ and therefore that $XY - 1$ is a factor of $\Psi_{\Gamma,N,t_D}(X,Y)$ which, as before, is a contradiciton. This gives the claimed relation between the coefficients of $\Psi_{\Gamma,N,t_D}(X,Y)$. $\square$

From now on, we will fix a canonical model for $X(D,1)$, $j_D$, and we will restrict ourselves to the case $N = p \nmid D$. In the analogous situation when $D = 1$, there is a well-known formula for the reduction modulo $p$ of the kroneckerian polynomial of level $p$, namely the Kronecker congruence formula. Its analog in the case of Shimura curves is the Eichler-Shimura congruence formula, introduced in Theorem 1.19.

We will deduce an analog to the Kronecker congruence formula for the reduction of the polynomial $\Psi_{\Gamma,p,t_D}$, from the Shimura reciprocity law. We will first prove some auxiliary results.

**Lemma 2.3.** *Let $\mathbb{H}_D$ be an indefinite rational quaternion algebra of discriminant $D$ and $p \nmid D$ be a prime number. Then, for any sufficiently large prime $r$, there exists a quadratic imaginary field $K$ satisfying:*

*(a) $p$ decomposes in $K$, $(p) = \mathfrak{p}\mathfrak{p}'$;*

*(b) $K$ splits $\mathbb{H}_D$;*

*(c) The residual degree of $\mathfrak{p}$ in the extension $K^{(1)}|K$ is $r$, where $K^{(1)}$ denotes the Hilbert Class Field of $K$.*

*Proof.* Let $S_0$ be a finite set of primes containing all positive primes smaller than $4p$, not dividing $D$ and different from $p$. If $p \equiv 1 \pmod 4$, assume further that $S_0$ contains a prime $q \nmid D$ such that $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = -1$. Define $\alpha_0 = \prod_{p \in S_0} p$ and let $r \ge 3$ be a prime such that $4D\alpha_0 < p^{r/2}$.

For any prime $q \mid D$, define an integer $0 \leq A_q < q$ if $q \neq 2$ and $A_2 \leq 4$ if $q = 2 \mid D$ such that:

- In the case $q = 2$, when $2 \mid D$:
  - if $p \equiv \pm 3 \pmod 8$, then $A_2 = 4$;
  - otherwise, $A_2 = 2$;
- In the case $q \neq 2$, $q \mid D$, we define $A_q$ such that $A_q^2 - p^r$ is not a square modulo $q$. This can be done, for example, in the following way:
  - if $-p$ is not a square modulo $q$, then $A_q = 0$;
  - if $-p$ is a square modulo $q$, $-p \equiv b^2 \pmod q$, let $a_q$ be an integer such that $a_q^2 + 1$ is not a square modulo $q$. Such an integer exists, otherwise the set of squares would be invariant by adding 1, which would mean that it is the whole $\mathbb{F}_q$. Now define $A_q \equiv b^r a_q \pmod q$ which satisfies the desired condition.

In this situation, the Chinese Remainder Theorem grants the existence of an integer $0 < A \leq 4D$ such that:

$$\begin{cases} \alpha_0 A \equiv A_q \pmod q, & \text{for } q \mid D,\ q \neq 2, \text{ and} \\ \alpha_0 A \equiv A_2 \pmod 8, & \text{for } 2 \mid D. \end{cases}$$

Now define $\alpha = A\alpha_0$ and consider integers $\beta, d$ with $d$ squarefree such that $\beta^2 d = \alpha^2 - p^r$. Since $\alpha < p^{r/2}$, we obtain that $d < 0$. Since, by construction, there exists a prime in $S$ such that $-d$ is not a square modulo this prime (or modulo 4 if this prime is 2), then $d \neq -1$; moreover, it is not divisible by any prime in $S_0 \cup \{q : q \mid D\} \cup \{p\}$. Therefore $-d > 4p$. Consider the quadratic imaginary field $K = \mathbb{Q}(\sqrt{d})$, whose discriminant $D_K$ is either $d$ or $4d$ depending on $d \pmod 4$.

We are going to prove that this quadratic field $K$ together with the prime $r$ satisfy the properties we were looking for.

(a) Consider the ideal $(p, \alpha + \beta\sqrt{d})$ with $\alpha, \beta$ those given above. Then, since $\gcd(p, 2\alpha) = 1$, $(p) = (p, \alpha + \beta\sqrt{d})(p, \alpha - \beta\sqrt{d})$. Therefore, $p$ is decomposed in $K$. Let $\mathfrak{p} = (p, \alpha + \beta\sqrt{d})$.

(b) $K$ splits $\mathbb{H}_D$ if and only if the Kronecker symbols $\left(\frac{D_K}{q}\right)$ are not 1, for any $q \mid D$. This follows immediately by the choice of $\alpha$.

(c) The Hilbert Class Field $K^{(1)}|K$ is the class field corresponding to the ideal class group of $K$, $C(\mathcal{O}_K)$. In $C(\mathcal{O}_K)$ we consider the class of the ideal $\mathfrak{p}$. First of all, observe that $\mathfrak{p}^r \subseteq (p^r, \alpha+\beta\sqrt{d}) = (\alpha+\beta\sqrt{d})$ and because both ideals have the same norm we conclude that both sides are equal, and therefore $\mathfrak{p}^r$ is principal. Since $r$ is prime, we conclude that the order of $\mathfrak{p}$ in the class group is either 1 or $r$. But there are no elements of norm $p$ in $\mathcal{O}_K$, because $-d > 4p$. Thus, $\mathfrak{p}$ has order $r$ in $\mathrm{Cl}_K$, and by the well-known decomposition theorem of class field theory (cf. [Neu99]), this order is equal to the residual degree of $\mathfrak{p}$ as we wanted. □

**Lemma 2.4.** *Let $p, R$ be odd primes such that $R > (p+1)^2$ and consider $\Psi(X,Y) \in \mathbb{Z}_p[X,Y]$ a primitive symmetric polynomial of degree $p+1$ in each variable. Assume further that there exists $\alpha \in \mathbb{F}_{p^R} \setminus \mathbb{F}_p$ such that $\Psi(\alpha, \alpha^p) = 0$. Then, there exists $a \in \mathbb{Z} \setminus p\mathbb{Z}$ such that*

$$\Psi(X,Y) \equiv a(X - Y^p)(Y - X^p) \pmod{p}.$$

*Proof.* Let $\Psi(X,Y) = \sum_{0 \le i,j \le p+1} a_{i,j} X^i Y^j$, with $a_{i,j} = a_{j,i}$. Then,

$$0 = \Psi(\alpha, \alpha^p) = \sum_{0 \le i,j \le p+1} a_{i,j} \alpha^{i+jp} = \sum_{k=0}^{(p+1)^2} \Big( \sum_{\substack{0 \le i,j \le p+1 \\ i+jp=k}} a_{i,j} \Big) \alpha^k.$$

Define, therefore,

$$c_{i,j} = \sum_{\substack{0 \le i,j \le p+1 \\ i+jp=k}} a_{i,j} \in \mathbb{Z}_p.$$

Since $\alpha \in \mathbb{F}_{p^R} \setminus \mathbb{F}_p$ and $R > (p+1)^2$ is prime, we have that $\{1, \alpha, \alpha^2, \ldots, \alpha^{(p+1)^2}\}$ are linearly independent over $\mathbb{F}_p$ and therefore all these coefficients must be 0 modulo $p$. Observe, that if $k \not\equiv 0, 1 \pmod{p}$, there exists only one coefficient $a_{i,j}$ with $i + jp = k$, namely, that with $(i,j)$ corresponding to the representation of the integer $k$ in base $p$. Similarly, if $k = 0, 1, (p+1)^2 - 1, (p+1)^2$, the same is true. Therefore, $a_{i,j} \equiv 0 \pmod{p}$ if $i \not\equiv 0, 1 \pmod{p}$ and $a_{0,0}, a_{1,0}, a_{p,p+1}, a_{p+1,p+1}$ are 0 modulo $p$ as well.

Now, if $k \equiv 0 \pmod{p}$, we have that $c_{pk'} = a_{p,k'-1} + a_{0,k'}$. Note that $a_{0,k'} = a_{k',0} \equiv 0 \pmod{p}$ if $k' \neq p, p+1$, but if $k' = p$, $c_{p^2} = a_{p,p-1} + a_{0,p}$ and $a_{p,p-1} = a_{p-1,p} \equiv 0 \pmod{p}$ which implies that $a_{0,p}$ is a multiple of $p$ too.

Therefore, only $a_{0,p+1}$ can be different from 0 modulo $p$ and it satisfies that $a_{p,p} + a_{0,p+1} \equiv 0 \pmod p$.

Similarly, if $k \equiv 1 \pmod p$, $c_{pk'+1} = a_{p+1,k'-1} + a_{1,k'}$, studying the coefficients $a_{1,k'}$ we reach the conclusion that only $a_{1,1}$ can be non-zero modulo $p$ and it satisfies that $a_{1,1} + a_{p+1,0} \equiv 0 \pmod p$.

Therefore, all coefficients $a_{i,j}$ are multiples of $p$ except for $a_{0,p+1}, a_{1,1}, a_{p,p}$, $a_{p+1,0}$ and these satisfy $-a_{1,1} \equiv a_{p+1,0} \equiv a_{0,p+1} \equiv -a_{p,p} \pmod p$ as desired. $\square$

**Lemma 2.5.** *Let $p \geq 3$ be a prime, $a \in \mathbb{Z}_p^*$ and $r$ a positive integer. Let $\Psi_1(X,Y), \Psi_2(X,Y) \in \mathbb{Z}_p[X,Y]$ be primitive symmetric polynomials of degree $p+1$ in each variable such that $p^s\Psi_1(X+2a/p^r, Y+2a/p^r) = \Psi_2(X,Y)$, for a suitable positive integer $s$. If $\Psi_1(X,Y) \equiv (X-Y^p)(Y-X^p) \pmod p$, then $\Psi_2(X,Y) \not\equiv b(X-Y^p)(Y-X^p) \pmod p$ for any $b \in \mathbb{Z} \setminus p\mathbb{Z}$.*

*Proof.* Assume that $\Psi_2(X,Y) \equiv b(X-Y^p)(Y-X^p) \pmod p$ and we will reach a contradiction. We will begin by computing some of the coefficients $d_{i,j}$ of the polynomial $p^{2rp}\Psi_1(X+2a/p^r, Y+2a/p^r)$. Let us write

$$\Psi_1(X,Y) = (X-Y^p)(Y-X^p) + p \sum_{0\leq i,j\leq p+1} c_{i,j}X^iY^j,$$

where $c_{i,j} \in \mathbb{Z}_p$, and then

$$\begin{aligned}
d_{0,0} \in& c_{p+1,p+1}(2a)^{2p+2}p^{1-2r} + 2c_{p+1,p}(2a)^{2p+1}p^{1-r} + (2a)^{2p} + p^{1+r}\mathbb{Z}_p,\\
d_{1,0} \in& c_{p+1,p+1}(2a)^{2p+1}(p+1)p^{1-r} + c_{p+1,p}(2a)^{2p}(2p+1)p + (2a)^{2p-1}p^{1+r}\\
&+ p^{2+r}\mathbb{Z}_p,\\
d_{1,1} \in& c_{p+1,p+1}(2a)^{2p}(p+1)^2p + 2c_{p+1,p}(2a)^{2p-1}(p+1)p^{2+r}\\
&+ 2c_{p+1,p-1}(2a)^{2p-2}(p^2-1)p^{1+2r} + p^{2+2r}\mathbb{Z}_p,\\
d_{p,p} =& c_{p+1,p+1}(2a)^2(p+1)^2p^{2rp+1-2r} + 2c_{p+1,p}(2a)(p+1)p^{2rp-r+1}\\
&+ p^{2rp} + c_{p,p}p^{2rp+1},\\
d_{p+1,0} \in& c_{p+1,p+1}(2a)^{p+1}p^{rp-r+1} + c_{p+1,p}(2a)^pp^{rp+1} + p^{rp+r+1}\mathbb{Z}_p.
\end{aligned}$$

The assumption we have made implies that

$$\nu_p(d_{1,1}) = \nu_p(d_{p,p}) = \nu_p(d_{p+1,0}) < \nu_p(d_{0,0}), \nu_p(d_{1,0})$$

and the description of $d_{p,p}$ provides that $\nu_p(d_{p,p}) \geq 2r(p-1)+1 \geq 1+2r$. Therefore, $p^{2r+1}$ must divide all the coefficients above. In particular, $\nu_p(c_{p+1,p+1}) \geq r$ and we define $c'_{p+1,p+1} = c_{p+1,p+1}p^{-r}$.

Now, each of the $d_{i,j}$ above yields a condition on the $p$-adic valuation of the coefficients, namely:

*(i)* $d_{0,0}$: $\nu_p(c'_{p+1,p+1}(2a)^2 + 2c_{p+1,p}(2a) + p^{r-1}) \geq r+1$;

*(ii)* $d_{1,0}$: $\nu_p(c'_{p+1,p+1}(2a) + c_{p+1,p}(2p+1)) \geq r$;

*(iii)* $d_{p+1,0}$: $\nu_p(c'_{p+1,p+1}(2a) + c_{p+1,p}) \geq r$.

Now, from *(i)* and *(iii)* we obtain that $\nu_p(c_{p+1,p}) = r-1$ which using *(ii)* gives that $\nu_p(c'_{p+1,p+1}) = r-1$. Using this information on $d_{1,1}$, it gives that $\nu_p(d_{1,1}) = 2r < 1+2r$ and therefore we reach a contradiction. □

**Proposition 2.7.** *Let $X(D,1)$ be the Shimura curve of discriminant $D$ for $D = 6, 10, 22$ with canonical model $j_D : \mathcal{H} \to \mathbb{P}^2$, given by the affine chart $j_D = (u_{D,1} : u_{D,2} : 1)$ such that $u_{D,1}^2 + u_{D,2}^2 + c(D) = 0$. Then, if $p, R$ are odd primes, with $p \nmid c(D)$ and $R \gg p$, there exists a CM point $P$ such that $u_{D,1}(P), u_{D,2}(P)$ are algebraic $p$-integers and the reduction modulo $p$ of any of them generates a finite field of degree $R$.*

*Proof.* Let $u_{D,3} = \frac{u_{D,1}}{u_{D,2}}$. For $u = u_{D,j}$, $j = 1,2,3$, there exists $\omega = \omega_q$, $q|D$, an Atkin-Lehner involution of level $q$ such that $u \circ \omega = -u$. Let also $\Psi(X,Y)$ be the kroneckerian polynomial of level $p$, for some prime $p$ coprime to $D$. Following Lemma 2.3, consider a CM point $P$ attached to the ring of integers of a quadratic field $K$, where $p$ decomposes, and such that the residual degree at $p$ of $K^{(1)}|K$ is $R$. Fix $\mathfrak{p}$ a prime of $K$ above $p$, and from now on we will work in the $\mathfrak{p}$-adic completion of $K$, which is identified throughout with $\mathbb{Q}_p$.

If $u(P)$ is a $p$-adic algebraic integer but its reduction modulo $p$ lies in $\mathbb{F}_p^*$, then, we can write $u(P) = a + bp^r$, for an integer $0 < a < p^r$, $p \nmid a$, and $b$ whose reduction lies in $\mathbb{F}_{p^R} \setminus \mathbb{F}_p$. Now, choose $\omega$ such that $u \circ \omega = -u$ and, in particular, $u(\omega P) = -a - bp^r$. In this way, we obtain functions

$$u' = \frac{u-a}{p^r}, \quad u'' = \frac{u+a}{p^r},$$

whose kroneckerian polynomials of level $p$, $\Psi_{u'}, \Psi_{u''}$ lie in $\mathbb{Z}_p[X,Y]$. Now, Shimura reciprocity law ensures that there exists a transformation $\omega_p$ of norm

$p$ such that $u'(\omega_p P) = \mathrm{Frob}_p(u'(P))$ and $u''(\omega_p P) = \mathrm{Frob}_p(u''(P))$, giving rise to points of the form $(\alpha, \alpha^p)$ with $\alpha \in \mathbb{F}_{p^R}$ of the corresponding kroneckerian polynomials. Therefore, we can apply Lemma 2.4 to obtain the congruence $\Psi_{u'}(X,Y) \equiv \Psi_{u''}(X,Y) \equiv (X - Y^p)(Y - X^p) \pmod p$. Since $u'' = u' + \frac{2a}{p^r}$, it follows that $c\Psi_{u'}(X,Y) = \Psi_{u''}(X + \frac{2a}{p^r}, Y + \frac{2a}{p^r})$ and therefore, by Lemma 2.5, we reach a contradiction.

Assume now, that $u(P)$ is a $p$-adic algebraic integer whose reduction modulo $p$ is 0. Then, one of the functions $u_{D,1}, u_{D,2}, u_{D,3}$ is in the previous case, and therefore a contradiction is also reached.

Finally, if $u_{D,1}(P), u_{D,2}(P)$ are not $p$-adic integers, then $u_{D,1}(P) = \frac{u'_{D,1}(P)}{p^r}$ and $u_{D,2}(P) = \frac{u'_{D,2}(P)}{p^s}$ with $u'_{D,1}(P), u'_{D,2}(P) \not\equiv 0 \pmod p$. It follows easily from $u_{D,1}^2 + u_{D,2}^2 + c(D) = 0$ that $r = s$ and $u'_{D,1}(P)^2 + u'_{D,2}(P)^2 \equiv 0 \pmod p$ and therefore $u_{D,3}(P) = \frac{u'_{D,1}(P)}{u'_{D,2}(P)}$ must be a $p$-adic algebraic integer whose reduction modulo $p$ lies in $\mathbb{F}_p^*$, which is not possible by the discussion above.

Therefore, the point $P$ gives us the result. □

**Corollary 2.5.** *Let $D = 6, 10, 22$ and $p$ a prime not dividing $D$. Then,*

$$\Psi_{\Gamma,p,u_{D,j}} \equiv a(X - Y^p)(Y - X^p) \pmod p,$$

*for some $a \in \mathbb{Z} \smallsetminus p\mathbb{Z}$.*

*Proof.* It follows immediately after using the output of the previous proposition in Lemma 2.4. □

**Theorem 2.5.** *Let $D = 6, 10, 22$ and $p$ a prime not dividing $D$. Then, the general form of the reduction modulo $p$ of the kroneckerian polynomial $\Psi_{\Gamma,p,t_D}(X,Y)$ depends only on $p \pmod{4c(D)}$. More precisely, up to a non-zero multiplicative constant, one has that*

$$\Psi_{\Gamma,p,t_D}(X,Y) \equiv \begin{cases} (X - Y^p)(Y - X^p) \pmod p, & \text{if } p \equiv 1 \pmod 4 \text{ and } \left(\frac{c(D)}{p}\right) = 1, \\ (X + Y^p)(Y + X^p) \pmod p, & \text{if } p \equiv 1 \pmod 4 \text{ and } \left(\frac{c(D)}{p}\right) = -1, \\ (1 - XY^p)(1 - YX^p) \pmod p, & \text{if } p \equiv 3 \pmod 4 \text{ and } \left(\frac{c(D)}{p}\right) = -1, \\ (1 + XY^p)(1 + YX^p) \pmod p, & \text{if } p \equiv 3 \pmod 4 \text{ and } \left(\frac{c(D)}{p}\right) = 1. \end{cases}$$

*Proof.* According to Theorem 2.4, $u_{D,1}$, $u_{D,2}$ are Hauptmoduln for certain quotient curves of $X(D,1)$; therefore, we can consider the kroneckerian polynomials

$\Psi_{\Gamma_1,p,u_{D,1}}(X,Y)$, $\Psi_{\Gamma_2,p,u_{D,2}}(X,Y) \in \mathbb{Z}[X,Y]$, such that

$$\Psi_{\Gamma_1,p,u_{D,1}}(u_{D,1}, u_{D,1}\circ w_p) = 0, \quad \Psi_{\Gamma_2,p,u_{D,2}}(u_{D,2}, u_{D,2}\circ w_p) = 0.$$

Now, by the previous corollary, for these kroneckerian polynomials we have congruences

$$\Psi_{\Gamma_k,p,u_{D,j}}(X,Y) \equiv (X - Y^p)(Y - X^p) \pmod{p}, \quad j = 1,2,$$

up to a non-zero multiplicative constant.

At this point, we recall the expression $u_{D,1} = \sqrt{-c(D)}\,\dfrac{t_D^2+1}{2t_D}$, and consider the ring homomorphism

$$\tilde{\varphi}_1 : \mathbb{Z}[X,Y] \longrightarrow \mathbb{Z}[\sqrt{-c(D)}, \tfrac{1}{2}][X, X^{-1}, Y, Y^{-1}]$$

defined by $\tilde{\varphi}_1(X) := \sqrt{-c(D)}\,\dfrac{X^2+1}{2X}$, $\tilde{\varphi}_1(Y) := \sqrt{-c(D)}\,\dfrac{Y^2+1}{2Y}$. It induces a homomorphism between the quotient rings

$$\overline{\varphi}_1 : \frac{\mathbb{Z}[X,Y]}{(\Psi_{\Gamma_1,p,u_{D,1}}(X,Y))} \longrightarrow \frac{\mathbb{Z}[\sqrt{-c(D)}, \frac{1}{2}][X, X^{-1}, Y, Y^{-1}]}{(\Psi_{\Gamma,p,t_D}(X,Y))}.$$

If we fix a prime of $\mathbb{Q}(\sqrt{-c(D)})$ lying above $p$ and reduce, we obtain a ring homomorphism

$$\varphi_1 : \frac{\mathbb{Z}/(p)[X,Y]}{((X-Y^p)(Y-X^p))} \longrightarrow \frac{\mathbb{F}[X,X^{-1},Y,Y^{-1}]}{(\overline{\Psi}_{\Gamma,p,t_D}(X,Y))},$$

for the finite field $\mathbb{F}$ of characteristic $p$ generated by a square root of $-c(D)$. Moreover, the trivial equality

$$\varphi_1((X-Y^p)(Y-X^p)) = 0 \in \frac{\mathbb{F}[X,X^{-1},Y,Y^{-1}]}{(\overline{\Psi}_{\Gamma,p,t_D}(X,Y))}$$

can be read as

$$\begin{aligned}\frac{-c(D)}{4X^{p+1}Y^{p+1}} &\cdot (Y - (-c(D))^{(p-1)/2}X^p)\cdot(X - (-c(D))^{(p-1)/2}Y^p)\\ &\cdot ((-c(D))^{(p-1)/2} - YX^p)\cdot((-c(D))^{(p-1)/2} - XY^p) = 0,\end{aligned}$$

and this one translates into a divisibility condition on $\mathbb{F}[X,Y]$, and then on $\mathbb{Z}/(p)[X,Y]$: the polynomial $\overline{\Psi}_{\Gamma,p,t_D}(X,Y)$ divides the four-term product

$$(Y-(-c(D))^{(p-1)/2}X^p)\cdot(X-(-c(D))^{(p-1)/2}Y^p)$$
$$\cdot((-c(D))^{(p-1)/2}-YX^p)\cdot((-c(D))^{(p-1)/2}-XY^p).$$

Now, remember that the polynomial $\overline{\Psi}_{\Gamma,p,t_D}(X,Y)$ must be symmetric and of degree at most $p+1$ in each variable. Since all four factors of the product are irreducible and $\mathbb{Z}/(p)[X,Y]$ is a unique factorization domain, this leaves us with only the two possibilities

$$\overline{\Psi}_{\Gamma,p,t_D}(X,Y)=(Y-(-c(D))^{(p-1)/2}X^p)\cdot(X-(-c(D))^{(p-1)/2}Y^p)$$

or

$$\overline{\Psi}_{\Gamma,p,t_D}(X,Y)=((-c(D))^{(p-1)/2}-YX^p)\cdot((-c(D))^{(p-1)/2}-XY^p),$$

up to a non-zero multiplicative constant.

Next, we take into account the expression $u_{D,2}=\sqrt{c(D)}\,\dfrac{t_D^2-1}{2t_D}$ and, working as above with the ring homomorphism

$$\tilde{\varphi}_2:\mathbb{Z}[X,Y]\longrightarrow\mathbb{Z}[\sqrt{c(D)},\tfrac{1}{2}][X,X^{-1},Y,Y^{-1}]$$

defined by $\tilde{\varphi}_2(X):=\sqrt{c(D)}\,\dfrac{X^2-1}{2X}$, $\tilde{\varphi}_2(Y):=\sqrt{c(D)}\,\dfrac{Y^2-1}{2Y}$, we reach a similar condition; namely, that the same polynomial $\overline{\Psi}_{\Gamma,p,t_D}(X,Y)$ divides the four-term product

$$(Y-c(D)^{(p-1)/2}X^p)\cdot(X-c(D)^{(p-1)/2}Y^p)$$
$$\cdot(c(D)^{(p-1)/2}+YX^p)\cdot(c(D)^{(p-1)/2}+XY^p)$$

in $\mathbb{Z}/(p)[X,Y]$, and this leaves us only with the two possibilities

$$\overline{\Psi}_{\Gamma,p,t_D}(X,Y)=(Y-c(D)^{(p-1)/2}X^p)\cdot(X-c(D)^{(p-1)/2}Y^p)$$

or

$$\overline{\Psi}_{\Gamma,p,t_D}(X,Y)=(c(D)^{(p-1)/2}+YX^p)\cdot(c(D)^{(p-1)/2}+XY^p),$$

up to a non-zero multiplicative constant.

Finally, dealing with the possible values of $\left(\frac{-1}{p}\right)$ and $\left(\frac{c(D)}{p}\right)$ and taking into account that $\left(\frac{-1}{p}\right)=(-1)^{(p-1)/2}$ and $\left(\frac{c(D)}{p}\right)\equiv c(D)^{(p-1)/2} \pmod p$, we obtain the desired result. $\square$

# Chapter 3

# The curves $X(D,1)$ of genus 0

In this chapter we study in detail the curves $X(D,1)$ of genus 0, which are those with $D = 6, 10, 22$, including their Atkin-Lehner quotients. First of all, we focus on the problem of computing fundamental domains for the groups $\langle \Gamma(D,N), W \rangle \subset \mathbf{PSL}(2,\mathbb{R})$, for $W \subset W_{D,N}$. There are different approaches to effectively compute fundamental domains for them, for example, [Als99], [Joh00], [Voi09]. We are going to give an algorithm to compute them based on Ford's method (cf. [For29]). We will use it to explicitly compute fundamental domains for the groups $\langle \Gamma(D,1), W_D \rangle$, $D = 6, 10, 22$.

Next, we introduce $q$-expansions of automorphic functions for $\Gamma(D,N)$ with respect to some suitable parameters $q$ around CM points in $\mathcal{H}$ following the ideas in [Elk98], [BT07b]. In particular, this approach has been partially used to explicitly uniformize the curves $X(D,N)^{W_{DN}}$ of genus 0 such that there exists $N' > 1$ with $X(D,NN')^{W_{DN}}$ of genus 0, cf. [Yan13], [Tu14]. In the following section we study how it can be done in the cases $D = 6, 10$, $N = 1$ and use the ideas in [SD77] together with the properties of the kroneckerian polynomials introduced in the previous section to uniformize $X(22,1)^{W_{22}}$, for which there is no $N > 1$ with $g(X(22,N)^{W_{22}}) = 0$. Even though the complexity of the computations increases dramatically whenever the number of elliptic cycles of the curve $X(D,N)^{W_{DN}}$ increases, this approach does not rely on the existence of a suitable cover of genus 0, and therefore it can be used, at least, to uniformize Shimura curves $X(D,N)^{W_{DN}}$ of genus 0 with a small number of elliptic cycles. For more details see Chapter 4.

Finally we review how the Hauptmoduln constructed relate to Igusa invariants in the cases $D = 6, 10$, cf. [BT08], [BG08], and extend the results to the case $D = 22$. In order to do so, we have to extend some results in [HM95] to this case using the Humbert surfaces computed in [Gru10].

## 3.1 Fundamental domains for $\Gamma(D,N)$

Let $\mathbb{H} = \left(\frac{\alpha,\beta}{\mathbb{Q}}\right)$, $\alpha > 0$, be a rational indefinite quaternion algebra of discriminant $D$ and fix the embedding $\Phi : \mathbb{H} \hookrightarrow \mathbf{M}(2,\mathbb{R})$ given by

$$\Phi(x + y\boldsymbol{i} + z\boldsymbol{j} + t\boldsymbol{ij}) = \begin{pmatrix} x + y\sqrt{\alpha} & z + t\sqrt{\alpha} \\ \beta(z - t\sqrt{\alpha}) & x - y\sqrt{\alpha} \end{pmatrix}.$$

Let also $N$ be a natural number such that $\gcd(D,N) = 1$. Let $\mathcal{O}(D,N)$ be an Eichler order in $\mathbb{H}$ of level $N$. Let $\Gamma = \Gamma(D,N) \subset \mathbf{PSL}(2,\mathbb{R})$ be the corresponding fuchsian group.

We will briefly deal with the computational part of the determination of fundamental domains for these groups as introduced in Section 1.2.3. Even though fundamental domains could be mostly avoided in the computations we will need later on, they provide a useful way to work with the set of complex points of the Shimura curves from an analytic viewpoint, for example, when dealing with numerical approximations.

Since the hyperbolic volume of a fundamental domain is a known invariant of the group (Theorem 1.10) and the hyperbolic volume of a hyperbolic polygon can be easily computed by means of the Lambert formula (Theorem 1.8), Theorem 1.9 allows us to effectively compute fundamental domains for these fuchsian groups.

However, since Theorem 1.9 involves the intersection of an infinite number of sets from which only a finite number are relevant, it is convenient to study how to order the sets involved.

Fix $t > 0$ such that $ti$ is not elliptic by $\Gamma$ and consider $\gamma_0 = \begin{pmatrix} 1 & 0 \\ \frac{1}{ti} & -1 \end{pmatrix}$. Then, we are interested in computing

$$\overline{\gamma_0(\bigcap_{\gamma\in\Gamma} E_{\gamma_0\gamma\gamma_0})} \cap \mathcal{H} = \mathcal{H} \bigcap_{\gamma\in\Gamma} \gamma_0(\overline{E_{\gamma_0\gamma\gamma_0}}),$$

where

$$E_\gamma = (\mathbb{C} \smallsetminus C_\gamma) \cup \{\infty\}$$

and

$$C_\gamma = \{z \in \mathbb{C} : |cz + d| \leq 1\},$$

where $\begin{pmatrix} a & b \\ c & b \end{pmatrix}$, with $ad - bc = 1$, is a representative for $\gamma$.

**Lemma 3.1.** *Assume that $\gamma$ can be represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc = 1$. Define $O, r \in \mathbb{R}$, $r > 0$, by*

$$O = -\frac{ab + cdt^2}{a^2 + c^2t^2 - 1},$$
$$r^2 = -\frac{b^2 + (d^2 - 1)t^2}{a^2 + c^2t^2 - 1} + O^2 = \frac{c^2t^4 + (a^2 + d^2 - 2)t^2 + b^2}{(a^2 + c^2t^2 - 1)^2}.$$

*Then,*

$$\mathcal{H} \cap \gamma_0(\overline{E_{\gamma_0\gamma\gamma_0}}) = \begin{cases} \{z \in \mathcal{H} : |z - O| \geq r\}, & \textit{if } a^2 + ct^2 - 1 > 0, \\ \{z \in \mathcal{H} : |z - O| \leq r\}, & \textit{if } a^2 + ct^2 - 1 < 0. \end{cases}$$

*Proof.* If we write $\gamma_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \gamma_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

$$\gamma_0(\overline{E_{\gamma_0\gamma\gamma_0}}) = \{z \in \mathbb{C} : |C\gamma_0(z) + D| \leq 1\}$$

and the result follows from a straightforward computation. The second expression for $r^2$ also follows from a direct computation if we take into account that $ad - bc = 1$. $\square$

Once we have found some reasonable expressions for the sets appearing in the intersection, we are interested in choosing a suitable order to intersect the sets. Since the point $ti$ is contained by construction in the interior of the fundamental domain, we will make the intersection according to the hyperbolic distance $d_{\mathcal{H}}$ from the point $ti$ to the boundary of the circles.

**Lemma 3.2.** *Let $O, r \in \mathbb{R}$, $r > 0$, and denote by $\mathrm{Circ}_O(r)$ the circumference of center $O$ and radius $r$. Then,*

$$d_{\mathcal{H}}(ti, \mathrm{Circ}_O(r)) = \operatorname{arcsinh}\left(\frac{|O^2 + t^2 - r^2|}{2rt}\right).$$

*Proof.* As we recalled in Section 1.2.1, the hyperbolic circumference of center $ti$ and radius $R$ coincides with the euclidean circumference of center $\cosh(R)ti$ and radius $\sinh(R)t$. Thus, the hyperbolic distance of $ti$ to $\mathrm{Circ}_O(r)$ is the only value of $R$ for which these two circumferences are tangent. The two circumferences are tangent if and only if

$$d(O, \cosh(R)ti) = r \pm \sinh(R)t,$$

which squaring both sides yields

$$O^2 + \cosh^2(R)t^2 = r^2 + \sinh^2(R)t^2 \pm 2r\sinh(R)t$$

and the desired formula is obtained. $\square$

Finally, given a certain distance $R > 0$, we need to compute all the transformations yielding isometric circles whose boundary is within distance $R$ from $ti$, which according to the previous lemma is equivalent to finding all transformations such that

$$|O^2 + t^2 - r^2| \leq 2rt\sinh(R).$$

Using the expressions above for $O$ and $r$ in terms of $a, b, c, d$, we obtain that

$$|O^2 + t^2 - r^2| = \left|\frac{b^2 + (d^2-1)t^2}{a^2 + c^2t^2 - 1} + t^2\right| = |a^2 + c^2t^2 - 1|r^2.$$

Therefore $|O^2 + t^2 - r^2| \leq 2rt\sinh(R)$ if and only if

$$|a^2 + c^2t^2 - 1|r^2 \leq 2rt\sinh(R),$$

which is equivalent to

$$(|a^2 + c^2t^2 - 1|r)^2 = c^2t^4 + (a^2 + d^2 - 2)t^2 + b^2 \leq 4t^2\sinh^2(R).$$

In particular, we obtain that $a^2 + d^2 - 2 \leq 4\sinh^2(R)$. Since $a$ and $d$ are conjugated quadratic with bounded denominators, this yields a finite number of possible choices for $a, d$. For any of these values the inequality above provides a bound for $c^2t^4 + b^2 = (\frac{c}{\beta})^2(\beta t^2)^2 + b^2 \geq \min(1, \beta^2t^4)((\frac{c}{\beta})^2 + b^2)$. Since $\frac{c}{\beta}$ and $b$ are conjugated quadratic as before, we obtain as well a finite number of choices for $b, c$. Consequently, we obtain the following result.

**Proposition 3.1.** *Fix $\Gamma = \Gamma(D, N)$, $t$ and $\gamma_0$ as before. Then, for a given $R > 0$, the transformations $\gamma$ such that the hyperbolic distance from $ti$ to the boundary of $\gamma_0(\overline{E_{\gamma_0\gamma\gamma_0}})$ is less or equal than $R$ can be explicitly computed.* $\square$

Thus, after fixing $R$, all this procedure can be carried out and, if the volume of the region enclosing $ti$ is equal to the volume of the fundamental domain, we have obtained indeed a fundamental domain for the group. Otherwise, $R$ has to be increased, the new circles have to be added and the new volume has to be computed. Since the fundamental domain is limited by a finite number of circumferences for a big enough $R$ the process ends.

It is also worth noting that when we deal with the cases $D > 1$, the fundamental domain will be a compact polygon in $\mathcal{H}$. Thus, as soon as we have a compact polygon in $\mathcal{H}$, we could prove that it is indeed a fundamental domain, without using the volume, choosing $R$ such that this polygon is contained in the hyperbolic disc of center $ti$ and radius $R$. Then, if we check that for all transformations giving isometric circles within distance $R$ from $ti$ they do not further reduce the polygon, we can conclude as well that it is indeed a fundamental domain.

This procedure can be easily modified to make it work for groups obtained attaching Atkin-Lehner involutions to the groups $\Gamma(D,N)$. We will not write it down explicitly because we will not make use of it. Essentially, for every involution of level $m$ in the group, we would have to additionally consider matrices as above with all entries divided by $\sqrt{m}$. Note that even though the entries of the matrices would be no longer quadratic all the assertions still hold.

Next, we will give explicitly a fundamental domain for $\Gamma(22,1)$ which we have computed using this procedure. The details of the presentation of the quaternion algebra and the Eichler order chosen are given in the next section, together with some other cases. Here we will only illustrate the procedure above and show the resulting region.

First of all, we choose

$$t = \tfrac{1}{2}, \quad \sinh(R) = 6$$

and then:

- the number of circles within distance $R$ is 36;

- the number of circles bordering the fundamental domain is 12.

With these values, the figure we obtain when we print all 36 circles is the following:

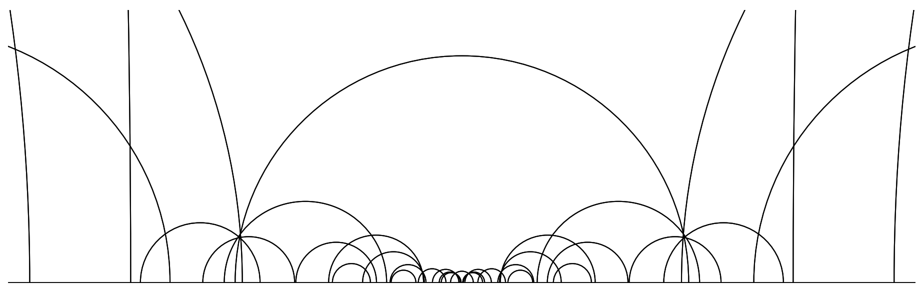


Figure 3.1: All circles within distance $R$

and when we only consider the relevant circles we obtain a fundamental domain for $\Gamma(22,1)$:

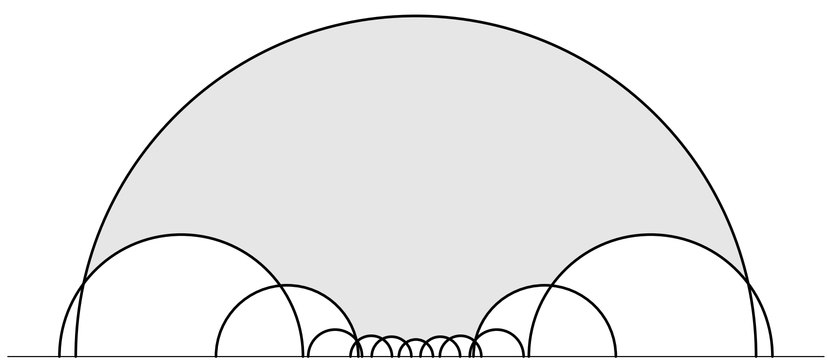


Figure 3.2: The circles in the border of the fundamental domain

It can be checked that the volume of this polygon coincides with the volume of a fundamental domain. Notice that there exist two tiny sides, in an euclidean sense, on the lower central part of the polygon.

## 3.2 Some explicit fundamental domains in the cases $D = 6, 10, 22$

We will now give fundamental domains for some of the groups $\langle\Gamma(D,1), W\rangle$, where $W \subset W_D$, which will be used later on. Fundamental domains for $\Gamma(6,1)$ and $\Gamma(10,1)$ can be found in [AB04]. We will not reproduce them below and we will give instead fundamental domains for $\langle\Gamma(6,1), W_6\rangle$ and $\langle\Gamma(10,1), W_{10}\rangle$. These can easily be deduced from the former ones studying the action of $W_D$ on them. We will begin by giving a fundamental domain for $\Gamma(22,1)$ and deducing from it a fundamental domain for $\langle\Gamma(22,1), W_{22}\rangle$. We could directly use the method above to compute a fundamental domain for the group $\langle\Gamma(D,1), W\rangle$, however it is faster and more convenient, in order to study the morphisms between the corresponding quotient curves, to compute them this way. Next we will state the results for the cases $D = 6$ and $D = 10$.

As we have seen before, in order to compute fundamental domains we need to have an explicit description of the group $\Gamma$, and this means an explicit choice of the quaternion algebra $\mathbb{H}$ together with an order $\mathcal{O}\subset\mathbb{H}$ and an explicit embedding $\Phi : \mathcal{O} \hookrightarrow \mathbf{M}(2,\mathbb{R})$. Recall that if we have $\mathbb{H} = \left(\frac{a,b}{\mathbb{Q}}\right)$, and we write $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{ij}\}$ the corresponding $\mathbb{Q}$-basis with $\boldsymbol{i}^2 = a, \boldsymbol{j}^2 = b, \boldsymbol{ij} = -\boldsymbol{ji}$, then

$$\begin{array}{rcl} \Phi : \mathbb{H} & \to & \mathbf{M}(2,\mathbb{R}) \\ x + y\boldsymbol{i} + z\boldsymbol{j} + t\boldsymbol{ij} & \mapsto & \begin{pmatrix} x + y\sqrt{a} & z + t\sqrt{a} \\ b(z - t\sqrt{a}) & x - y\sqrt{a} \end{pmatrix} \end{array}$$

provides an embedding. In the cases we are interested in, we can choose the following particular representative of the isomorphism class of quaternion algebras of discriminant $D$

| $D$ | $\mathbb{H}_D$ |
|---|---|
| 6 | $\left(\frac{3,-1}{\mathbb{Q}}\right)$ |
| 10 | $\left(\frac{2,5}{\mathbb{Q}}\right)$ |
| 22 | $\left(\frac{11,-1}{\mathbb{Q}}\right)$ |

and in these quaternion algebras we can choose a maximal order $\mathcal{O}(D,1)$ with

$\mathbb{Z}$-basis $\{1, I, J, K\}$ as follows:

| $D$ | $I$ | $J$ | $K$ |
|---|---|---|---|
| 6 | $\boldsymbol{i}$ | $\boldsymbol{j}$ | $(1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{ij})/2$ |
| 10 | $\boldsymbol{i}$ | $(1+\boldsymbol{j})/2$ | $(\boldsymbol{i}+\boldsymbol{ij})/2$ |
| 22 | $\boldsymbol{i}$ | $\boldsymbol{j}$ | $(1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{ij})/2$ |

Then, the group $\Gamma = \Gamma(D,1)$ is the image in $\mathbf{PSL}(2,\mathbb{R})$ of $\Phi(\mathcal{O}(D,1)_1)$. We will denote by $(a,b,c,d)_D$ the element $a+bI+cJ+dK$ and $[a,b,c,d]_D$ the image of $\Phi(a+bI+cJ+dK)$ in $\mathbf{PSL}(2,\mathbb{R})$. We will drop the subscripts when no confusion arises.

Finally, we need to give an explicit description of the elements in the group $W_D = \{\omega_k\}_{k|D}$, where $\omega_1 = id$ and $\omega_k$ is the transformation attached to a certain normalizing element of norm $k$ in $\mathcal{O}(D,1)$. The action induced on $\Gamma(D,1)\backslash\mathcal{H}$ is involutive and independent of the chosen element. We choose the following representatives:

| $D$ | $\omega_2$ | $\omega_{D/2}$ | $\omega_D = \omega_{D/2}\omega_2$ |
|---|---|---|---|
| 6 | $[1,0,1,0]_6$ | $[2,1,2,-1]_6$ | $[0,1,3,0]_6$ |
| 10 | $[0,0,0,1]_{10}$ | $[-1,1,2,-2]_{10}$ | $[4,-2,2,-1]_{10}$ |
| 22 | $[1,0,1,0]_{22}$ | $[4,-3,-7,3]_{22}$ | $[11,-3,0,0]_{22}$ |

**Proposition 3.2.** *Let $\mathcal{D}(22,1)$ be the hyperbolic decagon with vertices $(Q_{22,i})_{i=1}^{10}$ which, if no confusion is possible, we will simply denote by $(Q_i)_i$:*

$$Q_{22,1} = i, \qquad Q_{22,2} = \tfrac{1}{10}\left(i - 3\sqrt{11}\right), \qquad Q_{22,3} = \tfrac{i\sqrt{3}-5\sqrt{11}}{17+\sqrt{11}},$$
$$Q_{22,4} = \tfrac{i\sqrt{3}-\sqrt{11}}{5+\sqrt{11}}, \qquad Q_{22,5} = \tfrac{i\sqrt{3}-\sqrt{11}}{17+5\sqrt{11}}, \qquad Q_{22,6} = i\left(10-3\sqrt{11}\right),$$
$$Q_{22,6+j} = -\overline{Q_{22,6-j}}, \quad j = 1,\dots,4.$$

*(1) The internal angles at its vertices are, respectively,*

$$\left(\pi, \tfrac{\pi}{4}, \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{\pi}{3}, \tfrac{\pi}{2}, \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{\pi}{3}, \tfrac{\pi}{4}\right).$$

*(2) All its vertices are elliptic points, $Q_1, Q_2, Q_6, Q_{10}$ of order 2 and $Q_3, Q_4, Q_5, Q_7, Q_8, Q_9$ of order 3. The corresponding transformations of $\Gamma(22,1)$ fixing*

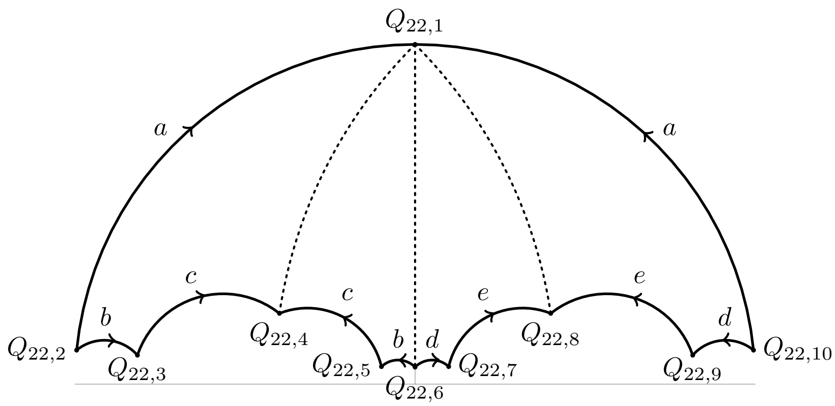



Figure 3.3: $\Gamma(22,1)\backslash\mathcal{H}$

*them are:*

$$\gamma_{22,1} = [0,0,1,0]_{22}, \qquad \gamma_{22,2} = [0,-3,-10,0]_{22},$$

$$\gamma_{22,3} = [0,3,9,-1]_{22}, \qquad \gamma_{22,4} = [0,-1,-3,1]_{22},$$

$$\gamma_{22,5} = [2,3,11,-5]_{22}, \qquad \gamma_{22,6} = [3,3,13,-6]_{22},$$

$$\gamma_{22,7} = [2,2,11,-5]_{22}, \qquad \gamma_{22,8} = [0,0,-3,1]_{22},$$

$$\gamma_{22,9} = [0,-2,9,-1]_{22}, \qquad \gamma_{22,10} = [0,3,-10,0]_{22}.$$

*(3) There are* 6 *elliptic cycles, two of order* 2*:* $\{Q_1\}$, $\{Q_2, Q_6, Q_{10}\}$*; and four of order* 3*:* $\{Q_3, Q_5\}$, $\{Q_4\}$, $\{Q_7, Q_9\}$, $\{Q_8\}$.

*(4) The identifications between edges are given by:*

$$\begin{array}{rll}
(Q_1Q_2, Q_1Q_{10}) & \textit{by means of} & \gamma_{22,1}, \\
(Q_2Q_3, Q_6Q_5) & \textit{by means of} & \gamma_{22,5}\cdot\gamma_{22,4} = [3,-3,-7,3]_{22}, \\
(Q_3Q_4, Q_5Q_4) & \textit{by means of} & \gamma_{22,4}, \\
(Q_6Q_7, Q_{10}Q_9) & \textit{by means of} & \gamma_{22,8}\cdot\gamma_{22,7} = [3,0,-7,3]_{22}, \\
(Q_7Q_8, Q_9Q_8) & \textit{by means of} & \gamma_{22,8}.
\end{array}$$

*(5) The transformations* $\{\gamma_{22,1}, \gamma_{22,5}, \gamma_{22,4}, \gamma_{22,7}, \gamma_{22,8}\}$ *together with the relations*

$$1 = \gamma_{22,1}^2 = \gamma_{22,4}^3 = \gamma_{22,8}^3 = (\gamma_{22,8} \cdot \gamma_{22,7} \cdot \gamma_{22,5} \cdot \gamma_{22,4} \cdot \gamma_{22,1})^2 = \gamma_{22,5}^3 = \gamma_{22,7}^3$$

*give a presentation for the group* $\Gamma(22,1)$. □

**Remark.** This fundamental domain was computed after the polygon found in the previous section. Some parts of the polygon were moved around in order to eliminate accidental cycles and to obtain a more workable polygon, without those tiny sides (in an euclidean sense). However, once the above set is given, all the listed properties, which are easily checked, already ensure that it is indeed a fundamental domain for $\Gamma(22,1)$.

In order to deduce a fundamental domain for $\langle\Gamma(22,1), W_{22}\rangle$, we study the action of the group $W_{22}$ on the domain $\mathcal{D}(22,1)$. Using the explicit choices of $\omega_k$ we have made above, we obtain:

$$\begin{aligned}
&\omega_2(Q_1, Q_2, Q_3, Q_4, Q_5, Q_6) = (Q_1, Q_6, Q_7, Q_8, Q_9, Q_{10}),\\
&\omega_{11}(Q_1, Q_2, Q_3, Q_4) = (Q_6, Q_1, Q_4, Q_5),\\
&\omega_{11}\gamma_{22,1}(Q_1, Q_8, Q_9, Q_{10}) = (Q_6, Q_7, Q_8, Q_1).
\end{aligned}$$

It is now clear that $(Q_1, Q_4, Q_5, Q_6)$ constitutes a fundamental domain for $\langle\Gamma(22,1), W_{22}\rangle$.

**Proposition 3.3.** *Let* $\mathcal{D}(22,1)^+$ *be the hyperbolic hexagon with vertices* $(Q_{22,i}^+)_{i=1}^6$ *which, if no confusion is possible, we will simply denote by* $(Q_i^+)_i$*:*

$$\begin{aligned}
&Q_{22,1}^+ = Q_{22,1}, \quad Q_{22,2}^+ = Q_{22,4}, \quad Q_{22,3}^+ = \tfrac{-1+i}{3+\sqrt{11}},\\
&Q_{22,4}^+ = Q_{22,5}, \quad Q_{22,5}^+ = Q_{22,6}, \quad Q_{22,6}^+ = \tfrac{i\sqrt{2}}{3+\sqrt{11}}.
\end{aligned}$$

*(1) The internal angles at its vertices are, respectively,*

$$\left(\tfrac{\pi}{4}, \tfrac{\pi}{3}, \pi, \tfrac{\pi}{3}, \tfrac{\pi}{4}, \pi\right).$$

*(2) All its vertices are elliptic points:* $Q_3^+, Q_6^+$ *of order* 2*,* $Q_2^+, Q_4^+$ *of order* 3 *and* $Q_1^+, Q_5^+$ *of order* 4*. The corresponding transformations of* $\langle\Gamma(22,1), W_{22}\rangle$ *fixing them are:*

$$\begin{aligned}
&\gamma_{22,1}^+ = \omega_2, && \gamma_{22,2}^+ = \gamma_{22,4},\\
&\gamma_{22,3}^+ = \omega_{11} \cdot \gamma_{22,4}^2, && \gamma_{22,4}^+ = \gamma_{22,5},\\
&\gamma_{22,5}^+ = \omega_2 \cdot [-7,-3,-6,3]_{22}, && \gamma_{22,6}^+ = \omega_{22} \cdot \gamma_{22,1}.
\end{aligned}$$

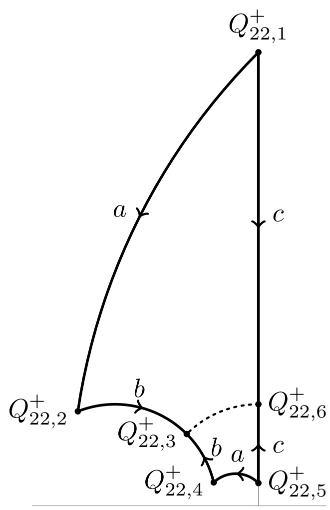



Figure 3.4: $\langle \Gamma(22,1), W_{22}\rangle\backslash\mathcal{H}$

*(3) There are 4 elliptic cycles, two of order 2: $\{Q_3^+\}$, $\{Q_6^+\}$; one of order 3: $\{Q_2^+, Q_4^+\}$; and one of order 4: $\{Q_1^+, Q_5^+\}$. $Q_3^+, Q_6^+$ are CM points by the orders $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ and $\mathbb{Z}[\sqrt{-22}]$ respectively.*

*(4) The identifications between edges are given by:*

$$\begin{array}{lll} (Q_1^+Q_2^+, Q_5^+Q_4^+) & \textit{by means of} & \omega_{11}, \\ (Q_2^+Q_3^+, Q_4^+Q_3^+) & \textit{by means of} & \gamma^+_{22,3}, \\ (Q_5^+Q_6^+, Q_1^+Q_6^+) & \textit{by means of} & \gamma^+_{22,6}. \end{array}$$

*(5) The transformations in (4) together with the relations*

$$1 = (\gamma^+_{22,3})^2 = (\gamma^+_{22,6})^2 = (\gamma^+_{22,3}\cdot\omega_{11})^3 = (\gamma^+_{22,6}\cdot\omega_{11})^4$$

*give a presentation for the group $\langle\Gamma(22,1), W_{22}\rangle$.*

*(6) Any element of norm $-1$ in $\mathcal{O}(22,1)$ induces a symmetry of the fundamental domain, which is the inversion with respect to the circumference with center at the origin and which passes through $Q_3^+, Q_6^+$.* $\square$

We are going now to reproduce the result for the cases $D = 6, 10$. The case $D = 6$ can be found in full detail in [BT07b], but it will be useful for future reference and to fix the notations to have all these cases together.

**Proposition 3.4.** *Let $\mathcal{D}(6,1)^+$ be the hyperbolic quadrilateral with vertices $(Q_{6,i}^+)_{i=1}^4$ which, if no confusion is possible, we will simply denote by $(Q_i^+)_i$:*

$$Q_{6,1}^+ = i, \qquad Q_{6,2}^+ = \tfrac{-1+i}{1+\sqrt{3}},$$
$$Q_{6,3}^+ = (2-\sqrt{3})i, \quad Q_{6,4}^+ = \tfrac{\sqrt{2}i}{1+\sqrt{3}}.$$

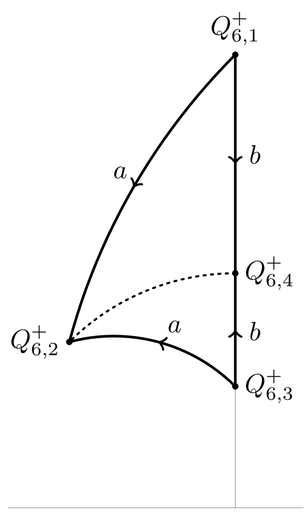



Figure 3.5: $\langle \Gamma(6,1), W_6 \rangle \backslash \mathcal{H}$

*(1) The internal angles at its vertices are, respectively,*

$$\left(\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{4}, \pi\right).$$

*(2) All its vertices are elliptic points: $Q_4^+$ of order 2, $Q_1^+, Q_3^+$ of order 4 and $Q_2^+$ of order 3. The corresponding transformations of $\langle \Gamma(6,1), W_6 \rangle$ fixing*

*them are:*

$$\gamma^+_{6,1} = \omega_2, \qquad \gamma^+_{6,2} = \omega_3,$$
$$\gamma^+_{6,3} = \omega_2 \cdot [2,1,1,-1]_6, \qquad \gamma^+_{6,4} = \omega_2 \cdot \omega_3 = \omega_6 \cdot [2,1,1,-1]_6.$$

(3) *There are* 3 *elliptic cycles, one of order* 2, $\{Q^+_4\}$, *one of order* 4, $\{Q^+_1, Q^+_3\}$, *and one of order* 6, $\{Q^+_2\}$. $Q^+_4$ *is a CM point by* $\mathbb{Z}[\sqrt{-6}]$.

(4) *The identifications between edges are given by:*

$$(Q^+_1 Q^+_2, Q^+_3 Q^+_2) \quad \text{by means of} \quad \gamma^+_{6,2},$$
$$(Q^+_3 Q^+_4, Q^+_1 Q^+_4) \quad \text{by means of} \quad \gamma^+_{6,4}.$$

(5) *The transformations in (4) together with the relations*

$$1 = (\gamma^+_{6,4})^2 = (\gamma^+_{6,4} \cdot \gamma^+_{6,2})^4 = (\gamma^+_{6,2})^6$$

*give a presentation for the group* $\langle \Gamma(6,1), W_6 \rangle$.

(6) *Any element of norm* $-1$ *in* $\mathcal{O}(6,1)$ *induces a symmetry of the fundamental domain, which is the inversion with respect to the circumference with center at the origin and which passes through* $Q^+_2, Q^+_4$. □

**Proposition 3.5.** *Let* $\mathcal{D}(10,1)^+$ *be the hyperbolic hexagon with vertices* $(Q^+_{10,i})_{i=1}^6$ *which, if no confusion is possible, we will simply denote by* $(Q^+_i)_i$*:*

$$Q^+_{10,1} = \frac{i\left(1+\sqrt{2}\right)}{\sqrt{5}}, \qquad Q^+_{10,2} = \frac{2-i\sqrt{6}}{5\left(-2+\sqrt{2}\right)},$$
$$Q^+_{10,3} = -\frac{1}{5} + \frac{2i}{5}, \qquad Q^+_{10,4} = \frac{-2+i\sqrt{6}}{5\left(2+\sqrt{2}\right)},$$
$$Q^+_{10,5} = \frac{i\left(-1+\sqrt{2}\right)}{\sqrt{5}}, \qquad Q^+_{10,6} = \frac{i}{\sqrt{5}}.$$

(1) *The internal angles at its vertices are, respectively,*

$$\left(\frac{\pi}{2}, \frac{\pi}{3}, \pi, \frac{\pi}{3}, \frac{\pi}{2}, \pi\right).$$

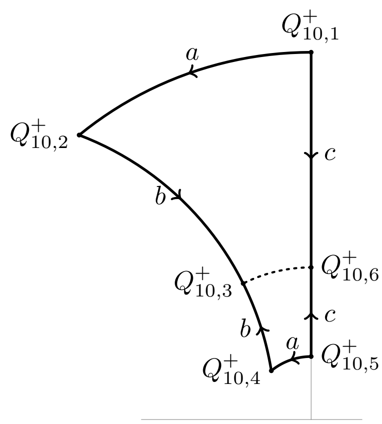



Figure 3.6: $\langle\Gamma(10,1), W_{10}\rangle\backslash\mathcal{H}$

*(2) All its vertices are elliptic points:* $Q_1^+, Q_3^+, Q_5^+, Q_6^+$ *of order* 2 *and* $Q_2^+, Q_4^+$ *of order* 3*. The corresponding transformations of* $\langle\Gamma(10,1), W_{10}\rangle$ *fixing them are:*

$$\gamma_{10,1}^+ = \omega_5 \cdot [3,-2,0,0]_{10}, \qquad \gamma_{10,2}^+ = [0,0,1,1]_{10},$$
$$\gamma_{10,3}^+ = \omega_2, \qquad \gamma_{10,4}^+ = [1,0,-1,1]_{10},$$
$$\gamma_{10,5}^+ = \omega_5, \qquad \gamma_{10,6}^+ = \omega_{10} \cdot [1,0,-1,-1]_{10}.$$

*(3) There are* 4 *elliptic cycles, three of order* 2*:* $\{Q_1^+, Q_5^+\}$*,* $\{Q_3^+\}$*,* $\{Q_6^+\}$*; and one of order* 3*,* $\{Q_2^+, Q_4^+\}$*.* $Q_1^+, Q_5^+, Q_3^+$ *and* $Q_6^+$ *are CM points by* $\mathbb{Z}[\sqrt{-5}]$*,* $\mathbb{Z}[\sqrt{-5}]$*,* $\mathbb{Z}[\sqrt{-2}]$ *and* $\mathbb{Z}[\sqrt{-10}]$*, respectively.*

*(4) The identifications between edges are given by:*

$$(Q_1^+Q_2^+, Q_5^+Q_4^+) \quad \textit{by means of} \quad \gamma_{10,4}^2 \cdot \omega_2,$$
$$(Q_2^+Q_3^+, Q_4^+Q_3^+) \quad \textit{by means of} \quad \gamma_{10,3}^+,$$
$$(Q_5^+Q_6^+, Q_1^+Q_6^+) \quad \textit{by means of} \quad \gamma_{10,6}^+.$$

*(5) The transformations in (4) together with the relations*

$$1 = (\gamma_{10,6}^+ \cdot \gamma_{10,4}^2 \cdot \omega_2)^2 = (\gamma_{10,3}^+)^2 = (\gamma_{10,6}^+)^2 = (\gamma_{10,3}^+ \cdot \gamma_{10,4}^2 \cdot \omega_2)^3$$

*give a presentation for the group* $\langle \Gamma(10,1), W_{10}\rangle$.

*(6) Any element of norm* $-1$ *in* $\mathcal{O}(10,1)$ *induces a symmetry of the fundamental domain, which is the inversion with respect to the circumference with center at the origin and which passes through* $Q_3^+, Q_6^+$. □

## 3.3 $q$-expansions of automorphic functions

Now, we are going to introduce the $q$-expansions of automorphic functions. First of all, we define the local parameters around a given point $P \in \mathcal{H}$, which we will use to obtain expansions for automorphic functions.

**Definition** ([BT07b])**.** For $P \in \mathcal{H}$ and $\kappa \in \mathbb{C}^*$, we define the local parameter around $P$ with local constant $\kappa$, $q = q_{P,\kappa} : \mathcal{H} \to \mathbb{C}$, as

$$q(z) = \kappa \frac{z - P}{z - \overline{P}}.$$

We will consider power series expansions of automorphic functions and forms with respect to these parameters, which we will suitably normalize later on. First of all, let us deal with the case of elliptic points.

**Proposition 3.6.** *Let* $P \in \mathcal{H}$, $q = q_{P,\kappa}$ *and* $\gamma \in \mathrm{GL}^+(2,\mathbb{R})$ *such that* $\gamma(P) = P$. *Then, there exists* $\epsilon \in \mathbb{C}^*$ *such that* $q \circ \gamma = \epsilon q$. *Moreover,* $\epsilon = \frac{\det \gamma}{\alpha^2}$, *where* $\alpha \in \mathbb{C}$ *is a root of the characteristic polynomial of* $\gamma$. *In particular, if* $P$ *is a CM point by a quadratic imaginary field* $K$, *and we consider an embedding* $\phi : K \hookrightarrow \{\delta \in \mathrm{SL}(2,\mathbb{R}) : \delta(P) = P\}$, *then* $\overline{\alpha} = \phi^{-1}(\gamma)$.

*Proof.* The existence of $\epsilon$ together with an expression for it are a simple computation and the rest of the properties can be checked immediately from it. □

Observe, in particular, that if $\Gamma$ is a fuchsian group and $P$ is an elliptic point of order $e$ by $\Gamma$, then an automorphic function with respect to $\Gamma$ has an expansion as a power series in $q^e$.

We will compute expansions of automorphic functions as solutions of a certain type of differential equations. Let us now introduce the automorphic derivative, as introduced in [BT07a], which is closely related to the classical schwarzian derivative.

**Definition.** The automorphic derivative of a meromorphic function $f$ is

$$Da(f,z) = \frac{2f'f''' - 3(f'')^2}{(f')^4},$$

where $f' = \frac{df}{dz}$ is the usual derivative.

Some basic properties of this differential operator are the following.

**Proposition 3.7.**

*(1) If $g$ is automorphic with respect to $\Gamma$, the same is true for $Da(g,z)$.*

*(2) If $g$ and $h$ are meromorphic functions and its composition is defined,*

$$Da(h \circ g, z) = Da(h, g(z)) + \frac{Da(g,z)}{(g')^2}.$$

*(3) If $\gamma$ is an homographic transformation,*

$$Da(\gamma(z), z) = 0.$$

*(4) Let $X$, $Y$ be meromorphic functions such that there exists a polynomial $P$ in two indeterminates which satisfies $P(X,Y) = 0$. Then, it holds that*

$$Da(X,z) = \frac{P_X^2}{P_Y^2} Da(Y,z) + \frac{\Theta(X,Y)}{P_X^2 P_Y^4},$$

*where the subscripts of $P$ denote partial derivatives and*

$$\begin{aligned}\Theta(X,Y) =& 6P_X^2 P_Y^3 P_{XXY} + 3P_{XX}^2 P_Y^4 - 2P_X P_Y^3(3P_{XX}P_{XY} + P_{XXX}P_Y) \\ & - 6P_X^3 P_Y(P_{XYY}P_Y - P_{XY}P_{YY}) - P_X^4(3P_{YY}^2 - 2P_Y P_{YYY}).\end{aligned}$$

□

Observe, in particular, that given a function $f$ and a local parameter $q_P$ around a point $P$, $Da(f,z) = Da(f, q_P(z))$.

The following result is a reformulation of the classical result concerning the conformal mapping of hyperbolic polygons which can be found, for example, in [Neh75]. When these polygons are obtained from suitable cocompact fuchsian groups, see [BT07b] for a more clear approach along this lines. A discussion in the lines of our formulation can be found in [Elk98].

**Proposition 3.8.** *Let $\Gamma$ be a fuchsian group with elliptic cycles $c_1, \ldots, c_r$ of orders $e_1, \ldots, e_r$. Assume that $\Gamma\backslash\mathcal{H}$ is compact and has genus 0 and that $f$ is a generator of the function field $A_0(\Gamma)/\mathbb{C}$. Then,*

$$Da(f, z) + R(f) = 0$$

*for $R(f) \in \mathbb{C}(f)$ of the form*

$$R(f) = \sum_i \frac{1 - 1/e_i^2}{(f - a_i)^2} + \sum_i \frac{B_i}{f - a_i},$$

*where the sums run over the elliptic cycles where $f$ takes finite values $a_i$. Moreover, if the values at all the elliptic cycles are finite, the following conditions on $B_i$ hold:*

*(1)* $\sum_i B_i = 0$,

*(2)* $\sum_i a_i B_i + \sum_i (1 - 1/e_i^2) = 0$,

*(3)* $\sum_i a_i^2 B_i + \sum_i a_i (1 - 1/e_i^2) = 0$.

*On the other hand, if $f$ is infinite at an elliptic cycle of order $e$, then these conditions must be replaced by:*

*(1)* $\sum_i B_i = 0$,

*(2)* $\sum_i a_i B_i + \sum_i (1 - 1/e_i^2) - (1 - 1/e^2) = 0$.

*Proof.* Let $P_i \in \mathcal{H}$, $i = 1, \ldots, r$, be a point representing the elliptic cycle $c_i$ and let $q_i = \kappa_i \frac{z - P_i}{z - \overline{P}_i}$, $\kappa_i \in \mathbb{C}$, be a local parameter around the point $P_i$. The value of $\kappa_i$ will be fixed later.

Since $Da(f(z), z)$ is a meromorphic automorphic function with respect to $\Gamma$ and $f$ is a generator of the field of automorphic functions with respect to $\Gamma$, $Da(f(z), z) \in \mathbb{C}(f(z))$. It is immediate to check that the poles of $f$ are zeros of $Da(f(z), z)$, therefore the set of singularities of the function

$$Da(f(z), z) = \frac{2f'(z)f'''(z) - 3(f''(z))^2}{(f'(z))^4}$$

is contained in the set of double points of $f$ which are not poles, and this is nothing else than the set of elliptic points of $\Gamma$ where $f$ takes finite values.

Fix $P_i$ and assume that $f(P_i) = a_i \in \mathbb{C}$. There exists $\kappa_i \in \mathbb{C}$ such that $f = a_i + q_i^{e_i} + a_{i,2} q_i^{2e_i} + \dots$ and then an expansion for $Da(f(z), z) = Da(f(q_i), q_i)$ around $P_i$ has leading term $-(1-1/e_i^2) q_i^{-2e_i}$ equal to that of $-(1-1/e_i^2)\frac{1}{(f-a_i)^2}$. Therefore, there exists $B_i$ such that

$$Da(f(z), z) + \frac{1 - 1/e_i^2}{(f - a_i)^2} + \frac{B_i}{f - a_i}$$

has the same singularities as $Da(f(z), z)$ except for $P_i$. Thus, repeating this process for all the points $P_i$ where $f$ takes a finite value $a_i$, we obtain that

$$Da(f(z), z) + \sum_i \frac{1 - 1/e_i^2}{(f - a_i)^2} + \sum_i \frac{B_i}{f - a_i}$$

is holomorphic and automorphic, and therefore constant. That is, there exists $\gamma \in \mathbb{C}$ such that

$$Da(f(z), z) + \sum_i \frac{1 - 1/e_i^2}{(f - a_i)^2} + \sum_i \frac{B_i}{f - a_i} + \gamma = 0. \tag{$*$}$$

Let us now study this equality around a point $P \in \mathcal{H}$ where $f$ has a pole. We will make a distinction on whether $P$ is an elliptic point for $\Gamma$ or not. We will give the proof in the case $P$ non-elliptic. The case $P$ elliptic is totally analogous.

Consider an expansion $f = \frac{1}{q} + \sum_k c_k q^k$. In this situation, $Da(f(z), z)$ has a zero of order at least 4 at $P$. If we compute the corresponding expansion of $(*)$, we obtain

$$\begin{aligned}
0 =& Da(f(z), z) + \sum_i \frac{1 - 1/e_i^2}{(f - a_i)^2} + \sum_i \frac{B_i}{f - a_i} + \gamma \\
=& \gamma + \big(\sum_i B_i\big) q \\
&+ \big(\sum_i (1 - 1/e_i^2) + \sum_i a_i B_i - c_0 \sum_i B_i\big) q^2 \\
&+ \big(2\sum_i (1 - 1/e_i^2) a_i + \sum_i a_i^2 B_i - 2c_0 (\sum_i (1 - 1/e_i^2) + \sum_i a_i B_i)\big) q^3 + \dots
\end{aligned}$$

from which $\gamma = 0$ and the equations in the statement of the proposition follow. $\square$

**Remark.** Observe that, if there are $r$ elliptic cycles, then $R$ depends on $2r$ parameters (or $2r-1$ if $f$ is not finite at an elliptic cycle). The Hauptmodul $f$ is determined by its value at 3 different points of $\Gamma\backslash\mathcal{H}$, which we can choose to be 3 of the elliptic cycles, and then the conditions on the $B_i$ leaves us with $2(r-3)$ unknown parameters. In conclusion, there are no unknown parameters after choosing $f$ properly, if and only if $\Gamma$ has only 3 elliptic cycles.

This situation is exploited in [Elk98] and [BT07b] to deal with the case $\Gamma = \langle\Gamma(6,1), W_6\rangle$. However, neither the case $D = 10$ nor $D = 22$ fall into this particular situation, since in both cases we end up with 4 elliptic cycles. The exact computation of the parameters in this differential equation, classically called accessory parameters, is a problem to which no general solution is known (cf. [FK12, SD77]). We will see later on how we can exploit the algebraic structures present in the curves we are interested in to explicitly compute these unknown values.

It is worth observing that for a Hauptmodul $f$ defined over $\mathbb{Q}$ of any of the curves $X(D,1)^W$, $D = 6, 10, 22$, $\{id\} \neq W \subset W_D$, and $\Gamma$ the corresponding fuchsian group, the set of points of $\Gamma\backslash\mathcal{H}$ where $f$ takes real values, which corresponds to the set of real points of the curve, can be identified with the set of fixed points in $\Gamma\backslash\mathcal{H}$ of the transformation attached to any element in $\mathcal{O}(D,1)_{-1}$. This set of fixed points, when the fundamental domain is properly chosen, yields the sides of an hyperbolic polygon, hence providing the link between the statement given above, and the classical result on conformal mapping.

Another crucial observation which we will use extensively later on is the following. Given $f$ an automorphic function for $\Gamma$ and $P \in \mathcal{H}$ an elliptic point of order $e \geq 2$, then there exists a local parameter $q = q_{P,\kappa}$ for a certain $\kappa \in \mathbb{C}^*$ such that $f(q) = a_0 + q^e + \sum_{k\geq 2} a_k q^{ke}$. In particular, if we know the corresponding value $a_0$ of the function $f$ at $P$, and we know the automorphic derivative of $f$, we can explicitly compute the expansion of $f$ up to a desired order with respect to this parameter $q$. We will see later on that the determination of the value $a_0$ can be done exactly, however we can only compute in general the value of $\kappa$ through approximations. In the case $D = 6$ the values of $\kappa$ for all the elliptic points can be found in [BT08]: it is a trascendental value, whose trascendency class can be obtained by means of the Chowla-Selberg formula (cf. [SC67]), and its exact value is a special value of a suitable hypergeometric series. However, this approach using hypergeometric series can not be extended to the cases $D = 10, 22$. In Chapter 6 we will see as well how to compute expansions around arbitrary CM points, for suitable local parameters $q$, starting from the information around the elliptic points and making use of the kroneckerian polynomials. The exact value of the constants of the local parameters, as in the case of the elliptic points, can be approximated up to a desired precision, but no approach to obtain the exact value is known.

### 3.3.1 Solutions of automorphic differential equations by means of order 2 linear differential equations

Series expansions around a given CM point for the Hauptmoduln we introduced before can be computed directly from the automorphic differential equation once the initial conditions are known. However, the recursion formula is rather involved and for some computations it will be useful to obtain these series expansions as solutions of simpler differential equations. These are classical results which go back to Schwarz and whose relation to the theory of automorphic functions was already well-known at the end of the 19th century.

**Proposition 3.9** ([FK12, For02]). *Let $R(t) \in \mathbb{C}(t)$ be a rational function and $f : \mathcal{H} \to \mathbb{C} \cup \{\infty\}$ a meromorphic function. Consider a branch of the functional inverse of $f$, $f^{-1} : \mathbb{C} \cup \{\infty\} \to \mathcal{H}$, and denote $w = f(z)$. Then, $f$ is a solution of the automorphic differential equation $Da(f(z), z) + R(f(z)) = 0$ if and only if $f^{-1}(w) = \frac{u_1(w)}{u_2(w)}$, where $u_1, u_2$ are solutions of the second order linear differential equation $u'' + \frac{R(w)}{4}u = 0$.* □

Let $f$ be a Hauptmodul for a certain fuchsian group and assume that it is a solution of the automorphic differential equation $Da(f(z), z) + R(f(z)) = 0$, for a certain rational function $R(t)$. Let $P \in \mathcal{H}$ be a CM point and denote by $e$ its order ($e = 1$ if the point is non-elliptic). For simplicity, assume $f(P) = a \in \mathbb{C}$ and consider a local parameter $q = q_P$ such that $f = a + q^e + bq^{2e} + \dots$ By the invariance of $Da$ with respect to homographic transformations, we have that $Da(f(q), q) + R(f(q)) = 0$. Now we need to distinguish between the case $e = 1$ and $e > 1$:

- $\boldsymbol{e = 1}$: In this case, we can choose a couple of solutions $u_1$, $u_2$ such that $u_1(a) = u_2'(a) = 0$, $u_1'(a) = u_2(a) = 1$, and then $f^{-1} = u_1/(bu_1 + u_2)$. Therefore, if we compute expansions of $u_1, u_2$ in $\mathbb{C}[[w - a]]$, we obtain an expansion for $f$ with respect to $q$ by finding the inverse of the series corresponding to $u_1/(bu_1 + u_2)$.
- $\boldsymbol{e > 1}$: In this case, the linear differential equation has a singularity at the point $w = a$. However, since the leading term of the rational function $R(w)/4$

around $w = a$ is $\frac{1-1/e^2}{4(w-a)^2}$, there exist solutions of this differential equation in $(w-a)^{(e-1)/(2e)}\mathbb{C}[[(w-a)^{1/e}]]$ such that

$$u_1 = (w-a)^{(e-1)/(2e)}((w-a)^{1/e} + \sum_{k\geq 2} a_{1,k}(w-a)^{k/e})$$

and

$$u_2 = (w-a)^{(e-1)/(2e)}(1 + \sum_{k\geq 2} a_{2,k}(w-a)^{k/e}).$$

Then, $f^{-1} = u_1/u_2$ and computing the inverse series, we obtain an expansion for $f$ with respect to $q^e$.

The case $f(P) = \infty$ can be reduced to the above discussion by considering the function $1/f$ which satisfies a differential equation of the same type with a different $R$, which according to Proposition 3.7 is $R(1/t)/t^4$.

## 3.4 Hauptmoduln: $t_D^+$

### 3.4.1 An overview of the case $D = 6$

The case $D = 6$ can be found in detail in [BT07b]. We present here the outline of the process in this case, which we will reproduce next for the cases $D = 10, 22$, where an additional step will be needed. Let $X(6,1)^+$ be the quotient curve of $X(6,1)$ by $W_6$ and $\Gamma(6,1)^+$ the corresponding group $\langle\Gamma(6,1), W_6\rangle$. This group has three elliptic cycles corresponding to the CM points by $\mathbb{Z}[i]$ (order 4), $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ (order 6) and $\mathbb{Z}[\sqrt{-6}]$ (order 2). Recall, in this situation, that in the proof of Theorem 2.4, we already made a choice for a Hauptmodul $t_6^+$ which took values 1 at the elliptic cycle corresponding to $\mathbb{Z}[i]$, $\infty$ at that corresponding to $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ and 0 at that corresponding to $\mathbb{Z}[\sqrt{-6}]$. We fix this Hauptmodul from now on. Proposition 3.8 tells us that there exists $R_6^+(f) \in \mathbb{C}(f)$ such that $Da(t_6^+, z) + R_6^+(t_6^+) = 0$ and

$$R_6^+(f) = \frac{1-\frac{1}{16}}{(f-1)^2} + \frac{1-\frac{1}{4}}{f^2} + \frac{B_1}{f-1} + \frac{B_3}{f}$$

where $B_1 + B_3 = 0$, $B_1 + (1-\frac{1}{16}) + (1-\frac{1}{4}) - (1-\frac{1}{36}) = 0$ which means

$$R_6^+(f) = \frac{108 - 113f + 140f^2}{144(-1+f)^2 f^2}.$$

Then, following the construction given in the proof of Theorem 2.4, we obtain the canonical model of $X(6,1)$ and the remaining quotients. That is, we define functions $u_{6,1}, u_{6,2}, u_{6,3} = u_{6,1}/u_{6,2}$ such that $u_{6,1}^2 = -3\frac{t_6^+}{t_6^+-1}$ and $u_{6,2}^2 = \frac{3}{t_6^+-1}$, which provide canonical models for the curves $X(6,1)^{\langle\omega_3\rangle}$, $X(6,1)^{\langle\omega_6\rangle}$ and $X(6,1)^{\langle\omega_2\rangle}$, respectively. Then, an affine chart of the canonical model of $X(6,1)$ is given by $(u_{6,1} : u_{6,2} : 1)$. Moreover, since we have computed the automorphic derivative of $t_6^+$, the automorphic derivatives of $u_{6,k}$ follow immediately from Proposition 3.7. For instance, in order to compute the automorphic derivative of $u_{6,1}$, we consider the polynomial

$$P(X,Y) = X^2(Y-1) + 3Y$$

which satisfies that $P(u_{6,1}, t_6^+) = 0$ and we obtain that

$$Da(u_{6,1}, z) = \frac{(2u_{6,1}(t_6^+ - 1))^2}{(u_{6,1}^2 + 3)^2}(-R_6^+(t_6^+(z))) + \frac{3}{u_{6,1}^2}.$$

Now using that $t_6^+ = \frac{u_{6,1}^2}{3+u_{6,1}^2}$,

$$Da(u_{6,1}, z) + \frac{-113 + 9u_{6,1}^2}{12\left(3 + u_{6,1}^2\right)^2} = 0.$$

Similarly,

$$Da(u_{6,2}, z) + \frac{96 - 49u_{6,2}^2 + 9u_{6,2}^4}{12u_{6,2}^2\left(3 + u_{6,2}^2\right)^2} = 0,$$

$$Da(u_{6,3}, z) + \frac{-103 + 32u_{6,3}^2}{36\left(1 + u_{6,3}^2\right)^2} = 0,$$

$$Da(t_6, z) + \frac{27 + 74t_6^2 + 27t_6^4}{36(-1+t_6)^2 t_6^2 (1+t_6)^2} = 0.$$

Therefore, $q$-expansions for all these functions can be computed around any point in $\mathcal{H}$ which corresponds to an elliptic one in $X(6,1)^+(\mathbb{C})$. In Chapter 6, we will recall some of these expansions and explain how to compute expansions around arbitrary CM points.

**Remark.** It is worth noting that the function $t_6^+$ we have introduced above is not exactly the one given under the same name in [BT07b], which in terms of our $t_6^+$ is $t_6^+/(t_6^+-1)$. On the other hand, our functions $u_{6,1}, u_{6,2}, u_{6,3} = \frac{u_{6,1}}{u_{6,2}}$ and $t_6$ do correspond exactly to the functions $u_3, u_6, u_2$ and $t_6$.

### 3.4.2 The cases $D = 10$ and $D = 22$

In the previous section we have briefly introduced the case $D = 6$, the only one of the three we are dealing with which corresponds to a triangle group, and we have computed the automorphic derivatives for a set of functions providing the canonical models of $X(6,1)$ and all its quotients. The key point has been the knowledge of the automorphic derivative of a Hauptmodul of the genus 0 curve $X(6,1)^+$, from which the automorphic derivatives of all the other functions involved are easily computed. The situation is exactly the same for the cases we are now going to deal with, with the important difference that none of the curves $X(D,1)$, $D = 10, 22$, nor their quotients is triangular; the corresponding groups are still generated by elliptic transformations, but the number of elliptic cycles is always greater than 3.

Firstly, we are going to treat the case $D = 22$ to show how we resolve the computation of the unknown parameters, and next we are going to sketch the case $D = 10$, which can be computed following exactly the same procedure, but has already been computed by other means in [Elk98] using that the curve $X(10,3)^{W_{10}}$ is a genus 0 curve which gives a covering of $X(10,1)^+$ of degree 4 and studying its ramification. In the case $D = 22$ this approach cannot be used straightforwardly since the curve $X(22,N)^{W_{22}}$ does not have genus 0 for any $N$.

**Proposition 3.10.** *Let $l \neq 11$ be an odd prime and $r$ a positive integer. Then, the genus of the curve $X(22,l^r)^{W_{22}}$ is equal to*

$$\begin{cases} \frac{1}{24}\left(5l^{r-1}(l+1)+3-6\left(\frac{-11}{l}\right)-6\left(\frac{-22}{l}\right)-9\left(\frac{-1}{l}\right)\right), & \textit{if } l=3, r\geq 2,\\ \frac{1}{24}\left(5l^{r-1}(l+1)-5-6\left(\frac{-11}{l}\right)-6\left(\frac{-22}{l}\right)-8\left(\frac{-3}{l}\right)-9\left(\frac{-1}{l}\right)\right), & \textit{otherwise.}\end{cases}$$

*In particular, $g(X(22,N)) \geq 1$ for any integer $N > 1$ coprime to 22.*

*Proof.* The genus of the curve $X(22,l^r)$ can be computed using the formula in Theorem 1.10 and we obtain that

$$2-2g(X(22,l^r)) = \begin{cases} -\frac{5}{3}l^{r-1}(l+1)+1+\left(\frac{-1}{l}\right), & \text{if } l=3, r\geq 2,\\ -\frac{5}{3}l^{r-1}(l+1)+\frac{11}{3}+\left(\frac{-1}{l}\right)+\frac{8}{3}\left(\frac{-3}{l}\right), & \text{otherwise.}\end{cases}$$

Now, we want to apply the Riemann-Hurwitz formula and therefore we need to study the ramification points of the morphism $X(22,l^r) \to X(22,l^r)^{W_{22}}$, which has degree 4, or equivalently the set of fixed points by the transformations in $W_{22}$. According to Theorem 4.1, the set of fixed points by the transformations in $W_{22}$ are the CM points corresponding to the following rings of integers:

$$\begin{aligned}&\text{Fixed points by } \omega_2 : \mathrm{CM}(\mathbb{Z}[i]),\\ &\text{Fixed points by } \omega_{11} : \mathrm{CM}(\mathbb{Z}[\tfrac{1+\sqrt{-11}}{2}]),\\ &\text{Fixed points by } \omega_{22} : \mathrm{CM}(\mathbb{Z}[\sqrt{-22}]).\end{aligned}$$

Finally, we have to compute how many CM points by these rings there are. For this purpose, we use the formula given in Theorem 1.7 and obtain that

$$\begin{aligned}&\#\mathrm{CM}(\mathbb{Z}[i]) = 2\left(1+\left(\tfrac{-1}{l}\right)\right),\\ &\#\mathrm{CM}(\mathbb{Z}[\tfrac{1+\sqrt{-11}}{2}]) = 2\left(1+\left(\tfrac{-11}{l}\right)\right),\\ &\#\mathrm{CM}(\mathbb{Z}[\sqrt{-22}]) = 2\left(1+\left(\tfrac{-22}{l}\right)\right).\end{aligned}$$

Since the ramification index of any of these points is 2, Riemann-Hurwitz formula gives the desired expression for the genus of $X(22,l^r)^{W_{22}}$. □

Let us, for the moment, try to follow the same procedure we used in the case $D=6$. Let $X(22,1)^+$ be the quotient curve of $X(22,1)$ by $W_{22}$ and $\Gamma(22,1)^+$ be the corresponding group $\langle \Gamma(22,1), W_{22}\rangle$. This group has four elliptic cycles corresponding to the CM points by $\mathbb{Z}[i]$ (order 4), $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ (order 3), $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ (order 2) and $\mathbb{Z}[\sqrt{-22}]$ (order 2). Recall, in this situation, that in the proof of Theorem 2.4, we already made a choice for a Hauptmodul $t_{22}^+$ which took values 1 at the elliptic cycle corresponding to $\mathbb{Z}[i]$, $\infty$ at that corresponding to $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ and 0 at that corresponding to $\mathbb{Z}[\sqrt{-22}]$. Proposition 3.7 tells us that there exists $R_{22}^+(f) \in \mathbb{C}(f)$ such that $Da(t_{22}^+(z),z) + R_{22}^+(t_{22}^+(z)) = 0$ and

$$R_{22}^+(f) = \frac{1-\frac{1}{16}}{(f-1)^2} + \frac{1-\frac{1}{4}}{f^2} + \frac{1-\frac{1}{9}}{(f-a)^2} + \frac{B_1}{f-1} + \frac{B_3}{f} + \frac{B}{f-a}$$

where $B_1 + B_3 + B = 0, B_1 + aB + (1-\frac{1}{16}) + (1-\frac{1}{4}) + (1-\frac{1}{9}) - (1-\frac{1}{4}) = 0$. Therefore, $B_1 = \frac{1}{144}(-263 - 144aB)$ and $B_3 = -B_1 - B$. Then, if we write $b = -216 + 47a - 144aB + 144a^2B$, we obtain that

$$R_{22}^+(f) = \frac{108a^2 + abf + \left(20 - 263a + 27a^2 - b - ab\right)f^2 + (7 - 7a + b)f^3 + 108f^4}{144(f-1)^2f^2(f-a)^2}.$$

Now we have to compute the values of $a, b$ and, in order to do so, we will use the kroneckerian polynomials introduced in Chapter 2. Since the function $t_{22}^+$ takes rational values at 3 of the four elliptic cycles, it takes a rational value at the fourth one as well and therefore $a$ must be rational. Looking at the fundamental domain, we can see moreover that $a \in (0,1)$. Let us fix $p = 3 \nmid D$ and consider the kroneckerian polynomial of level 3 attached to the Hauptmodul $t_{22}^+$ of $X(22,1)^+$, $\Psi(X,Y) = \Psi_{\Gamma(22,1)^+, t_{22}^+, 3}(X,Y) \in \mathbb{Q}[X,Y]$, which is symmetric and has degree 4 in each of the variables, that is $\Psi(X,Y) = \sum_{i,j=0}^4 c_{i,j} X^i Y^j$ with $c_{i,j} = c_{j,i} \in \mathbb{Q}$. In this case, among the four elliptic cycles, only those by $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ and $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ are fixed by a transformation in $\mathcal{O}_3 \cdot W_{22}$. Thus, we could now use Proposition 2.5 to impose some conditions on the $c_{i,j}$, that is $(X-a)X^2 \mid \Psi(X,X)$. We need to consider expansions around these two elliptic cycles,

$$
\begin{aligned}
t_{22}^+(q_3) &= a + q_3^3 + \sum_{k \geq 2} d_{3,k} q_3^{3k}, \\
t_{22}^+(q_{11}) &= q_{11}^2 + \sum_{k \geq 2} d_{11,k} q_{11}^{2k},
\end{aligned}
$$

where $q_3(z) = \kappa_3 \frac{z - P_3}{z - \overline{P_3}}$, $q_{11}(z) = \kappa_{11} \frac{z - P_{11}}{z - \overline{P_{11}}}$ are suitable local parameters around the points $P_3, P_{11} \in \mathcal{H}$ representing the corresponding elliptic cycles. Using the automorphic differential equation we have computed above, we can find the values $d_{3,k}, d_{11,k} \in \mathbb{Q}(a,b)$ up to a desired value of $k$.

Moreover, expansions for a function $t_{22}^+ \circ \omega_3 \circ \gamma$, $\gamma \in \Gamma(22,1)$, around these two points can be obtained easily from the ones for $t_{22}^+$. Let $P$ be one of these two points and $q$ the corresponding local parameter. Since there exists an element $\omega \in \mathcal{O}_3 \cdot W_{22}$ fixing $P$, then $t_{22}^+ \circ \omega_3 \circ \gamma = t_{22}^+ \circ \omega$ and $q \circ \omega / q \in \mathbb{C}$. After Proposition 3.6, no computation is necessary to obtain the value of this quotient. In this particular case, the transformations $\omega$ can be chosen such these quotients are $-1$ in the case of $q_3$ and $\frac{-5-\sqrt{-11}}{6}$ in the case of $q_{11}$.

Finally, we consider the equations

$$
\begin{aligned}
0 = &\Psi(t_{22}^+(q_3), t_{22}^+(-q_3)) \\
= &\Psi(t_{22}^+(q_{11}), t_{22}^+(\frac{1}{6}\left(-5 - \sqrt{-11}\right) q_{11}))
\end{aligned}
$$

which provide a system of linear homogeneous equations in the unknown coefficients of $\Psi$ with coefficients in $\mathbb{Q}(a,b)$ which has to be indeterminate. The values of $a, b$ for which this occurs can be obtained in a straightforward (but lengthy) way, computing some minors and the corresponding resultants. Because of their size, none of the polynomials and resultants involved in the process can be reproduced here. The only possible rational value of $a \in (1,\infty)$ is $a = 27/16$, giving only a possible value $b = -5899/16$ and this yields

$$R_{22}^{+}(f) = \frac{8748 - 17697f + 18316f^2 - 10624f^3 + 3072f^4}{16(-1+f)^2 f^2(-27+16f)^2}.$$

We obtain as well the kroneckerian polynomial of level 3 attached to $t_{22}^{+}$, namely

$$\begin{aligned}\Psi(X,Y) =& 55130625 - 260884800X + 494913024X^2 - 440745984X^3 \\ &+ 157351936X^4 - 260884800Y + 848782592XY \\ &- 1089896448X^2Y + 802750464X^3Y - 308281344X^4Y \\ &+ 494913024Y^2 - 1089896448XY^2 + 693043200X^2Y^2 \\ &- 245366784X^3Y^2 + 150994944X^4Y^2 - 440745984Y^3 \\ &+ 802750464XY^3 - 245366784X^2Y^3 - 117440512X^3Y^3 \\ &+ 157351936Y^4 - 308281344XY^4 + 150994944X^2Y^4.\end{aligned}$$

Now we can, as in the case $D = 6$, recover the automorphic derivatives for the rest of the functions $u_{22,1}, u_{22,2}, u_{22,3}, t_{22}$ introduced in Theorem 2.4 obtaining a whole set of generators for the curve $X(22,1)$ and its quotients, cf. Theorem 3.1.

The outline of the case $D = 10$ is completely analogous, except for the computations being fairly easier. Let $X(10,1)^+$ be the quotient curve of $X(10,1)$ by $W_{10}$. There are also four elliptic cycles, three of them of order 2 corresponding to $\mathbb{Z}[\sqrt{-2}], \mathbb{Z}[\sqrt{-5}]$ and $\mathbb{Z}[\sqrt{-10}]$ and one of order 3 and therefore attached to $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. In Theorem 2.4 we chose a Hauptmodul $t_{10}^{+}$ such that it takes the value 1 at the elliptic cycle corresponding to $\mathbb{Z}[\sqrt{-2}]$, $\infty$ at that corresponding to $\mathbb{Z}[\sqrt{-5}]$ and 0 at that corresponding to $\mathbb{Z}[\sqrt{-10}]$. Then,

$$Da(t_{10}^{+}(z), z) + R_{10}^{+}(t_{10}^{+}(z))$$

for a certain $R_{10}^{+}(f) \in \mathbb{C}(f)$ which can be written as

$$R_{10}^{+}(f) = \frac{27a^2 + abf + (5 - 59a - b - ab)f^2 + (-5 + 5a + b)f^3 + 27f^4}{36(-1+f)^2 f^2(-a+f)^2}.$$

As in the previous case, $a$ must be a rational number, and in this case, observing the fundamental domain, we obtain that $a > 1$. We fix $p = 3 \nmid 10$ and we consider the kroneckerian polynomial of level 3 attached to $t_{10}^+$,

$$\Psi(X,Y) = \Psi_{\Gamma,3,t_{10}^+}(X,Y) = \sum_{0\le i,j\le 4} c_{i,j} X^i Y^j,$$

with $c_{i,j} = c_{j,i}$. In this case, three of the elliptic cycles are fixed by transformations in $\mathcal{O}_3 \cdot W_{10}$, namely those attached to $\mathbb{Z}[\sqrt{-2}]$, $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ and $\mathbb{Z}[\sqrt{-5}]$. Let $P_2, P_3, P_5$ respectively be points representing these elliptic cycles and consider local parameters as above $q_2, q_3, q_5$ such that

$$\begin{aligned}
t_{10}^+(q_2) &= 1 + q_2^2 + \sum_{k\ge 2} d_{2,k} q_2^{2k},\\
t_{10}^+(q_3) &= a + q_3^3 + \sum_{k\ge 2} d_{3,k} q_3^{3k},\\
t_{10}^+(q_5) &= q_5^{-2} + \sum_{k\ge 0} d_{5,k} q_5^{2k}.
\end{aligned}$$

Hence, we obtain, as above,

$$\begin{aligned}
0 =& \Psi(t_{10}^+(q_2), t_{10}^+(\tfrac{-1-2\sqrt{-2}}{3} q_2)),\\
0 =& \Psi(t_{10}^+(q_3), t_{10}^+(-q_3)),\\
0 =& \Psi(t_{10}^+(q_5), t_{10}^+(\tfrac{-2-\sqrt{-5}}{3} q_5)).
\end{aligned}$$

Finally, when we impose that this system of linear equations is indeterminate, we obtain the values $a = 27/2$, $b = -1061/2$, from which

$$R_{10}^+(f) = \frac{2187 - 3183f + 3067f^2 - 208f^3 + 12f^4}{4(-1+f)^2 f^2 (-27+2f)^2}$$

and

$$\begin{aligned}
\Psi(X,Y) =& 29160000 - 19926000X + 6525225X^2 - 1066410X^3 + 83521X^4\\
&- 19926000Y - 19848850XY + 9014010X^2Y - 21744X^3Y\\
&- 69360X^4Y + 6525225Y^2 + 9014010XY^2 + 2121246X^2Y^2\\
&- 271440X^3Y^2 + 14400X^4Y^2 - 1066410Y^3 - 21744XY^3\\
&- 271440X^2Y^3 + 3200X^3Y^3 + 83521Y^4 - 69360XY^4\\
&+ 14400X^2Y^4.
\end{aligned}$$

Let us sum up all these results in the following theorem, together with some additional properties we have skipped in the discussion above.

**Theorem 3.1.** *Let $D = 6, 10, 22$.*

*(1) Let $f$ be any of the functions previously introduced in Theorem 2.4 or any of the functions $t_D^+$ considered above. Then, there exists a rational function $R(X) \in \mathbb{Q}[X]$ such that $Da(f(z), z) + R(f(z)) = 0$ and the precise value of $R(X)$ is given in the following table:*

| $f$ \ $D$ | 6 | 10 |
|---|---|---|
| $t_D^+$ | $\frac{108-113f+140f^2}{144(-1+f)^2f^2}$ | $\frac{2187-3183f+3067f^2-208f^3+12f^4}{4(-1+f)^2f^2(-27+2f)^2}$ |
| $u_{D,1}$ | $\frac{-113+9f^2}{12(3+f^2)^2}$ | $-\frac{10\left(303-22f^2+7f^4\right)}{(2+f^2)^2(27+f^2)^2}$ |
| $u_{D,2}$ | $\frac{96-49f^2+9f^4}{12f^2(3+f^2)^2}$ | $\frac{2\left(-1075+614f^2+29f^4\right)}{(-5+f)^2(5+f)^2(2+f^2)^2}$ |
| $u_{D,3}$ | $\frac{-103+32f^2}{36(1+f^2)^2}$ | $-\frac{2859+5522f^2+2675f^4}{(1+f^2)^2(27+25f^2)^2}$ |
| $t_D$ | $\frac{27+74f^2+27f^4}{36(-1+f)^2f^2(1+f)^2}$ | $\frac{128\left(1+23f^2+f^4\right)}{(1-52f^2+f^4)^2}$ |

| $f$ \ $D$ | 22 |
|---|---|
| $t_D^+$ | $\frac{8748-17697f+18316f^2-10624f^3+3072f^4}{16(-1+f)^2f^2(-27+16f)^2}$ |
| $u_{D,1}$ | $\frac{-80619-6743f^2-193f^4+3f^6}{4(11+f^2)^2(27+f^2)^2}$ |
| $u_{D,2}$ | $\frac{-2816+8800f^2+61f^4+3f^6}{4(-4+f)^2(4+f)^2(11+f^2)^2}$ |
| $u_{D,3}$ | $-\frac{10167+14240f^2+5888f^4}{4(1+f^2)^2(27+16f^2)^2}$ |
| $t_D$ | $\frac{11\left(33+508f^2+3014f^4+508f^6+33f^8\right)}{4f^2(11-86f^2+11f^4)^2}$ |

*(2) The values of the functions $t_D^+$ at the CM points by the rings of integers of some quadratic imaginary fields of small discriminants are given in the following table. The symbol $*$ indicates that no CM points by the given field*

*exist in the curve.*

| | $-3$ | $-4$ | $-8$ | $-11$ | $-19$ | $-20$ | $-24$ | $-40$ |
|---|---|---|---|---|---|---|---|---|
| $t_6^+$ | $\infty$ | 1 | $*$ | $*$ | $\frac{3211}{1024}$ | $*$ | 0 | $\frac{2312}{125}$ |
| $t_{10}^+$ | $\frac{27}{2}$ | $*$ | 1 | $*$ | $*$ | $\infty$ | $*$ | 0 |
| $t_{22}^+$ | $\frac{27}{16}$ | 1 | $*$ | $\infty$ | $*$ | $\frac{5}{16}$ | $*$ | $*$ |

□

**Remark.** The values of the functions above at a certain CM point can be exactly computed using Proposition 2.5 once a suitable kroneckerian polynomial is computed. In Chapter 6 we will discuss in detail the computation of kroneckerian polynomials and the determination of the values of a Hautpmodul at a given CM point.

## 3.5 Expression of $t_D^+$ in terms of absolute Igusa invariants

Let $D = 6, 10, 22$ and let $t_D^+$ be the Hauptmoduln for $\langle \Gamma(D,1), W_D \rangle$ introduced in Section 2.4. In [BG08], the uniformization of the curves $X(D,1)^+$, $D = 6, 10$, is carried out embedding these curves into Igusa's threefold (cf. [Rot04]) and obtaining a Hauptmodul for them in terms of absolute Igusa invariants. In order to do so, they make use of the parametric families of genus 2 curves whose jacobian has quaternionic multiplication by $\mathcal{O}(D,1)$, $D = 6, 10$, computed by Hashimoto and Murabayashi in [HM95]. Then, all this information is used in [BT08] to write the Hauptmodul, which according to our notations would be $t_6^+/(t_6^+ - 1)$, in terms of Igusa invariants and Thetanullwerte.

Firstly we are going to introduce some classical results on the Siegel upper half-plane and Igusa's threefold and the natural embeddings of certain Atkin-Lehner quotients of Shimura curves into Igusa's threefold.

Next we are going to reproduce briefly the case $D = 6$ and then we will apply the same approach to the case $D = 10$, where all the tools used when $D = 6$ are also available. Finally we will deal with this problem for $D = 22$. We will begin by computing a parametric family of genus 2 curves playing the role of the Hashimoto-Murabayashi family in our case, using some of the explicit equations of Humbert surfaces computed in [Gru10], and then we will be able to follow the outline used in [BG08] to obtain a Hauptmodul for $X(22,1)^+$ in terms of Igusa invariants. Finally, a comparison between the values of this Hauptmodul and the one we have already computed at certain CM points will allow us to write $t_{22}^+$ in terms of Igusa invariants.

In this way, given a point $P \in \Gamma(D,1)\backslash\mathcal{H} \cong X(D,1)(\mathbb{C})$, we will obtain an equation for a genus 2 curve in terms of $t_D^+(P)$ whose jacobian corresponds to the abelian surface with quaternionic multiplication represented by $P$. Moreover, some additional properties of these curves will be discussed, for example, which of them have additional automorphisms or have simultaneously quaternionic multiplication by two of the orders $\mathcal{O}(D,1)$, $D = 6, 10, 22$, we are dealing with.

### 3.5.1 From a point on $X(D,1)$ to the corresponding abelian surface

Given an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $D > 1$, $\mathbb{H}_D$, an embedding $\Phi : \mathbb{H}_D \to \mathrm{M}(2,\mathbb{R})$ and a maximal order $\mathcal{O} \subset \mathbb{H}_D$, we have introduced the group $\Gamma(D,1)$ and considered the quotient space $\Gamma(D,1)\backslash\mathcal{H}$, which according to Shimura, cf. [Shi67], can be identified with the set of complex points of an algebraic curve defined over $\mathbb{Q}$, $X(D,1)$, and interpreted as a (coarse) moduli space of principally polarized abelian surfaces with quaternionic multiplication by $\mathcal{O}$. In order to make this identification, we need an auxiliary element $\rho \in \mathcal{O}$ such that $\rho^2 + D = 0$, which necessarily normalizes $\mathcal{O}$. In this way, the application

$$\begin{array}{rcl} \mathbb{H}_D & \to & \mathbb{H}_D \\ \alpha & \mapsto & \alpha^* = \rho^{-1}\overline{\alpha}\rho \end{array}$$

is a positive involution acting on $\mathcal{O}$. Then, the points of $X(D,1)(\mathbb{C})$ are in one to one correspondence with the set of isomorphism classes of $(A,\iota,\mathcal{L})$, where $(A,\mathcal{L})$ is a principally polarized abelian surface and $\iota : \mathcal{O} \to \mathrm{End}(A)$ is an injective ring homomorphism such that the Rosati involution with respect to $\mathcal{L}$ coincides with $*$ on $\iota(\mathcal{O})$. Note that the isomorphism class of the algebraic curve $X(D,1)$ does not depend on $\rho$, but the moduli interpretation, in general, does.

On the other hand, we can consider the moduli space of principally polarized abelian surfaces, $\mathcal{A}_2$, which is an affine scheme of dimension 3 and contains, via the Torelli embedding, the moduli space of genus 2 curves, $\mathcal{M}_2$, as a Zariski open and dense subset. Similarly to the case of Shimura curves and therefore to the classical modular case, the set of complex points of $\mathcal{A}_2$ can be identified with $\mathbf{Sp}(4,\mathbb{Z})\backslash\mathcal{H}_2$. Generators for the field of Siegel modular functions and the graded ring of modular forms where given by Igusa, cf. [Igu60, Igu62, Igu64], both in terms of Thetanullwerte and in terms of invariants of binary sextic forms in the subspace $\mathcal{M}_2$. In this case, the construction of the complex tori attached to the abelian surfaces is carried out in the same way it is done in the modular case.

Therefore, we can naturally construct a map $X(D,1) \to \mathcal{A}_2$ such that $[(A,\iota,\mathcal{L})] \mapsto [(A,\mathcal{L})]$. This map, obtained when we drop the quaternionic structure, has been studied by Rotger in [Rot04] where it is proved that it factors through an Atkin-Lehner quotient of $X(D,1)$ giving rise to a birrational map. In all the cases $D = 6, 10, 22$, this quotient is the full Atkin-Lehner quotient, since all these cases are *twisting* cases: for $D = 6, 10, 22$, $\mathbb{H}_D \cong \left(\frac{-D,2}{\mathbb{Q}}\right)$. Moreover, if we consider the subset $\mathcal{Q}_D$ of $\mathcal{A}_2$ consisting of those principally polarized abelian surfaces $[(A,\mathcal{L})]$ such that $\mathcal{O} \hookrightarrow \mathrm{End}(A)$, it is obvious that this set contains the image of the forgetful map. The forgetful map depends on the choice of $\rho$ and $\mathcal{Q}_D$ can have multiple components coming from these different choices. Rotger gives a formula for the number of irreducible components of $\mathcal{Q}_D$, which in all the cases $D = 6, 10, 22$ turns out to be 1, that is, the set $\mathcal{Q}_D$ is irreducible, and therefore the image of the above map is the whole quaternionic locus.

Finally, for a given $D$, we will briefly introduce the embedding $\epsilon_D : \mathcal{H} \to \mathcal{H}_2$ compatible with the actions of $\Gamma(D,1)$ and $\mathbf{Sp}(4,\mathbb{Z})$ through an embedding $\mathcal{O}(D,1)_1 \hookrightarrow \mathbf{Sp}(4,\mathbb{Z})$. The procedure can be found, for example, in [Has95, BG05]. Consider the non-degenerate alternate symmetric form on $\mathbb{H}_D$ given by

$$T(\alpha,\beta) = \frac{1}{D}\,\mathrm{tr}(\rho\alpha\overline{\beta}),$$

which takes integral values on $\mathcal{O}(D,1)$. Now, let us consider a symplectic basis of $\mathcal{O}(D,1)$ with respect to $T$ and denote it by $\{\eta_1,\ldots,\eta_4\}$. Given an element $\gamma \in \mathcal{O}(D,1)_1$, note that $\{\eta_1\gamma,\eta_2\gamma,\eta_3\gamma,\eta_4\gamma\}$ is also a symplectic basis of $\mathcal{O}(D,1)$ with respect to $T$. Therefore, there exists a matrix $M_\gamma \in \mathbf{Sp}(4,\mathbb{Z})$ such that $\{\eta_1,\eta_2,\eta_3,\eta_4\}\gamma = \{\eta_1,\eta_2,\eta_3,\eta_4\}^t M_\gamma$.

Given $\tau \in \mathcal{H}$, consider the matrices $\Omega_1(\tau) = \left(\Phi(\eta_1)\begin{pmatrix}\tau\\1\end{pmatrix}, \Phi(\eta_2)\begin{pmatrix}\tau\\1\end{pmatrix}\right)$, $\Omega_2(\tau) = \left(\Phi(\eta_3)\begin{pmatrix}\tau\\1\end{pmatrix}, \Phi(\eta_4)\begin{pmatrix}\tau\\1\end{pmatrix}\right)$ and $\Omega(\tau) = \Omega_2(\tau)^{-1}\cdot\Omega_1(\tau)$.

**Proposition 3.11.** *The map $\epsilon_D : \mathcal{H} \to \mathcal{H}_2, \tau \mapsto \Omega(\tau)$, is an holomorphic* *embedding compatible with the actions of* $\Gamma(D,1)$ *and* $\mathbf{Sp}(4,\mathbb{Z})$ *through the embedding* $\mathcal{O}(D,1)_1 \to \mathbf{Sp}(4,\mathbb{Z})$, $\gamma \mapsto M_\gamma$. $\square$

Since this embedding does depend on several choices of elements of $\mathcal{O}(D,1)$, let us fix all these necessary elements and the resulting embeddings in the cases $D = 6, 10, 22$ for the orders which in these cases we have already fixed in Section 3.1. We choose the values of $\rho$ and a corresponding symplectic basis as follows:

| $D$ | $\rho$ | $\eta_1$ | $\eta_2$ | $\eta_3$ | $\eta_4$ |
|---|---|---|---|---|---|
| 6 | $(0,1,3,0)_6$ | $(1,0,0,0)_6$ | $(-1,0,-1,1)_6$ | $(0,0,-1,0)_6$ | $(-1,1,0,0)_6$ |
| 10 | $(0,1,0,-2)_{10}$ | $(1,0,0,0)_{10}$ | $(0,1,0,0)_{10}$ | $(0,0,0,-1)_{10}$ | $(0,0,-1,0)_{10}$ |
| 22 | $(0,3,11,0)_{22}$ | $(1,0,0,0)_{22}$ | $(0,1,3,0)_{22}$ | $(0,0,1,0)_{22}$ | $(-2,0,1,1)_{22}$ |

With these choices, the explicit formulas for the embeddings $\epsilon_D : \mathcal{H} \to \mathcal{H}_2$, where $\epsilon_D(\tau) = \begin{pmatrix} A(\tau) & B(\tau) \\ B(\tau) & C(\tau) \end{pmatrix}$, are the following:

| $D$ | $A(\tau)$ | $B(\tau)$ |
|---|---|---|
| 6 | $\frac{1+\sqrt{3}-2\sqrt{3}\tau+(1-\sqrt{3})\tau^2}{1-\sqrt{3}-2\sqrt{3}\tau+(1+\sqrt{3})\tau^2}$ | $\frac{2(1+\tau^2)}{1-\sqrt{3}-2\sqrt{3}\tau+(1+\sqrt{3})\tau^2}$ |
| 10 | $-\frac{(1-5\tau^2)\sqrt{2}}{2(1+6\tau+5\tau^2)}$ | $\frac{1+2\tau+5\tau^2}{1+6\tau+5\tau^2}$ |
| 22 | $-\frac{3+\sqrt{11}+2\sqrt{11}\tau-\left(-3+\sqrt{11}\right)\tau^2}{-3-\sqrt{11}+2\sqrt{11}\tau+\left(-3+\sqrt{11}\right)\tau^2}$ | $\frac{2\left(1+\tau^2\right)}{-3-\sqrt{11}+2\sqrt{11}\tau+\left(-3+\sqrt{11}\right)\tau^2}$ |

| $D$ | $C(\tau)$ |
|---|---|
| 6 | $\frac{-4\sqrt{3}\tau}{1-\sqrt{3}-2\sqrt{3}\tau+(1+\sqrt{3})\tau^2}$ |
| 10 | $-\frac{\sqrt{2}\left(1-5\tau^2\right)}{1+6\tau+5\tau^2}$ |
| 22 | $\frac{2\sqrt{11}\left(-1+\tau^2\right)}{-3-\sqrt{11}+2\sqrt{11}\tau+\left(-3+\sqrt{11}\right)\tau^2}$ |

Finally, we will define a set of coordinates in $\mathbf{Sp}(4,\mathbb{Z})\backslash\mathcal{H}_2$ and some additional functions we will need when dealing with the case $D = 22$.

**Definition.** For $m^t = \begin{pmatrix} m' \\ m'' \end{pmatrix} \in \{0,1\}^4$, we define the function $\vartheta_m : \mathcal{H}_2 \to \mathbb{C}$ such that for every $\tau \in \mathcal{H}_2$,

$$\vartheta_m(\tau) = \sum_{n \in \mathbb{Z}^2} e^{\pi i \left((n+m'/2)^t \cdot \tau \cdot (n+m'/2) + m'' \cdot n^t\right)}$$

which is classically called the Thetanullwerte (or thetaconstant) of characteristic $m$. The characteristic is said to be even (resp. odd) if $m'^t \cdot m''$ is even (resp. odd).

We introduce as well the Thetanullwerte transformation formula, which we will use next to write a set of generators of the ring of Siegel modular forms of genus 2.

**Theorem 3.2** (Thetanullwerte transformation formula)**.** *Consider* $\tau \in \mathcal{H}_2$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbf{Sp}(4, \mathbb{Z})$ *and* $m = \begin{pmatrix} m' \\ m'' \end{pmatrix}^t$. *Then,*

$$\vartheta_{Mm}(M\tau) = \nu(M) e^{2\pi i \phi_m(M)} |C\tau + D|^{1/2} \vartheta_m(\tau),$$

*where*

- $(Mm)^t = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} m^t + \begin{pmatrix} (CD^t)_\Delta \\ (AB^t)_\Delta \end{pmatrix}$,
- $\nu(M)^4 = e^{\pi i \, \mathrm{tr}(BC^t)}$,
- $\phi_m(M) = -\frac{1}{8}(m'^t B^t D m' + m''^t A^t C m'' - 2m'^t B^t C m'' - 2(AB^t)_\Delta (Dm' - Cm''))$,
- $(\cdot)_\Delta$ *denotes the diagonal written as a column vector.* □

**Theorem 3.3** ([Igu62, Igu67])**.** *The graded ring of Siegel modular forms of even weight and genus* 2 *is*

$$\mathbb{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}],$$

*where*

$$2^2 \psi_4 = \sum_{m \ even} \vartheta_m^8,$$
$$2^2 \psi_6 = \sum_{M \in \Gamma_2/\Gamma_2(2)} M(\vartheta_{m_1} \vartheta_{m_2} \vartheta_{m_3})^4,$$

$$
\begin{aligned}
-2^{14}\chi_{10} &= \prod_{m\ even} \vartheta_m^2, \\
2^{17}3\chi_{12} &= 2^{-15}3^{-4}11^{-1}(2^3\psi_6^2 - 2^2 11\psi_4^3 + 3^2 \sum_{m\ even} \vartheta_m^{24}).
\end{aligned}
$$

*The whole graded ring of Siegel modular forms is obtained adding a form of weight* 35*, which can be written as well in terms of Thetanullwerte.* □

In this way, we obtain the (Baily-Borel-)Satake compactification of the moduli space of principally polarized abelian surfaces,

$$\mathrm{Proj}\mathbb{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}] \simeq \mathbb{P}_{\mathbb{C}}(2, 3, 5, 6),$$

and the moduli space of genus 2 curves can be identified with the subspace $\mathrm{Proj}\mathbb{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}] \smallsetminus \{\chi_{10} = 0\}$. Moreover, similarly to what happens in the case of elliptic curves and classical modular forms, the values of the above Siegel modular forms can be related to values of projective invariants of sextic forms.

Given a sextic form $u(X) = u_0 X^6 + u_1 X^5 + \cdots + u_6$, denote by $\xi_1, \ldots, \xi_6$ his roots and write $(ij) = \xi_i - \xi_j$. Then we define the Igusa's projective invariants $A, B, C, D$ through their irrational expressions as:

$$
\begin{aligned}
A(u) &= u_0^2 \sum_{\text{fifteen}} (12)^2(34)^2(56)^2, \\
B(u) &= u_0^4 \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2, \\
C(u) &= u_0^6 \sum_{\text{sixty}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2, \\
D(u) &= u_0^{10} \prod_{j<k} (jk)^2,
\end{aligned}
$$

where the sums run over the set of permutations of the roots giving different summands and the subscripts in the sums denote the number of summands. Since these expressions are symmetric in the roots of $u(X)$, they admit a polynomial expression in the coefficients $u_i$. Moreover, they are generators of the graded ring of projective invariants of sextic forms of even degree.

Given a sextic form $u(X)$ without multiple roots, i.e. $D(u) \neq 0$, we can consider the corresponding genus 2 curve $Y^2 = u(X)$, which gives a point in the moduli space of genus 2 curves, and vice versa. This identification can be used to construct a map from the space of Siegel modular forms to the space of projective invariants of sextic forms (cf. [Igu67]) which in our case yields:

$$2^2\psi_4 = B, \quad 2^2\psi_6 = (AB - 3C)/2, \quad -2^{14}\chi_{10} = D, \quad 2^{17}3\chi_{12} = AD.$$

Since the invariants $A, B, C, D$ do not work well modulo $2, 3, 5$ another set of invariants, and therefore of Siegel modular forms, is usually introduced:

$$\begin{aligned}
J_2 &= 2^{-3}A,\\
J_4 &= 2^{-5}3^{-1}(4J_2^2 - B),\\
J_6 &= 2^{-6}3^{-2}(8J_2^3 - 160J_2J_4 - C),\\
J_8 &= 2^{-2}(J_2J_6 - J_4^2),\\
J_{10} &= 2^{-12}D.
\end{aligned}$$

These provide a set of generators of projective invariants of sextic forms of even weight working in all characteristics. However, note that, if working in odd characteristic, $J_8$ is redundant.

Finally, we define another set of coordinates used in [Gru10]:

$$\begin{aligned}
s_2 &= 12\psi_4,\\
s_3 &= 12\psi_6,\\
s_5 &= 60\psi_4\psi_6 - 2^{14}3^5 5\chi_{10},\\
s_6 &= 108\psi_4^3 + 24\psi_6^2 + 2^{15}3^7\chi_{12}.
\end{aligned}$$

### 3.5.2 The case of $t_6^+$

We will briefly review the results for the case $D = 6$.

**Theorem 3.4** ([BG08]). *Let $\epsilon_6 : \mathcal{H} \to \mathcal{H}_2$ be the natural embedding introduced on page 96. Then, $g_6 : \mathcal{H} \to \mathbb{C} \cup \{\infty\}$ defined as*

$$g_6(z) = \frac{(12^5 J_{10}(\epsilon(z)))^2}{(J_2(\epsilon(z)) - 24J_4(\epsilon(z)))^5}$$

*gives a Hauptmodul for $X(6,1)^+$.* □

In particular, we can relate this Hautpmodul $g_6$ to $t_6^+$ by comparing their values at three different points.

**Theorem 3.5.** *It holds that* $t_6^+ = \dfrac{27}{16} g_6 + 1$.

*Proof.* Since both $g_6$ and $t_6^+$ are generators of the function field $\mathbb{Q}(X(6,1)^+)$, we have that $t_6^+ = \frac{ag_6+b}{cg_6+d}$ for $a,b,c,d \in \mathbb{Q}$, $ad - bc \neq 0$. Then, we take into account the values of these two functions at the elliptic points. The values of $t_6^+$ at the elliptic points, which are the CM points by the ring of integers of certain quadratic fields $K$, were fixed when we defined the function and the values of $g_6$ at the same points have been computed in [BG08]:

| $K$ | $t_6^+$ | $g_6$ |
|---|---|---|
| $\mathbb{Q}(i)$ | 1 | 0 |
| $\mathbb{Q}(\sqrt{-3})$ | $\infty$ | $\infty$ |
| $\mathbb{Q}(\sqrt{-6})$ | 0 | $-\frac{16}{27}$ |

The claim now follows. □

**Remark.** Recall that our function $t_6^+$ does not agree with the function of the same name defined in [BT07b]. The function used there is equal to the function $t_6^+/(t_6^+ - 1)$ in our setting, which recovers the expression given in [BT08].

**Theorem 3.6** ([BG08])**.** *The curve* $C : y^2 = f(x)$, *where*

$$f(x) = (s-1)x^6 - 6tx^5 - 3(3s+7)tx^4 - 4t^2x^3 - 3(3s-7)t^2x^2 - 6t^3x + (s+1)t^3,$$

$t = 2j_0$ *and* $s = \sqrt{-2j_0+2}$, *is a genus 2 curve defined over* $\mathbb{Q}(s)$ *such that* $t_6^+(C) = j_0$. □

### 3.5.3 The case of $t_{10}^+$

**Theorem 3.7** ([BG08])**.** *Let* $\epsilon_{10} : \mathcal{H} \to \mathcal{H}_2$ *be the natural embedding introduced on page 96. Then,* $g_{10} : \mathcal{H} \to \mathbb{C} \cup \{\infty\}$ *defined as*

$$g_{10}(z) = \frac{23751C^2 - 501060ABC + 2641541A^2B^2 - 37046420B^3}{2169C^2 - 34404ABC - 16709A^2B^2 + 37046420B^3},$$

*where*

$$\begin{aligned} A &= 5J_2(\epsilon(z)), \\ B &= J_2(\epsilon(z))^2 - 24J_4(\epsilon(z)), \\ C &= 5(33J_3(\epsilon(z))^2 - 992J_2(\epsilon(z))J_4(\epsilon(z)) + 3600J_6(\epsilon(z))), \end{aligned}$$

*gives a Hauptmodul for* $X(10,1)^+$. □

**Theorem 3.8.** *It holds that* $t_{10}^+ = \frac{1}{2}\dfrac{8g_{10}-27}{4g_{10}-1}$.

*Proof.* The outline of the proof is totally analogous to that given in the case $D = 6$. In this case, the values of $t_{10}^+$ at three of the elliptic points were fixed when we defined the function (the value at the other elliptic point has already been computed but it is not necessary for the result) and the values of $g_{10}$ at the same points have been computed in [BG08]:

| $K$ | $t_{10}^+$ | $g_{10}$ |
|---|---|---|
| $\mathbb{Q}(\sqrt{-2})$ | 1 | $\infty$ |
| $\mathbb{Q}(\sqrt{-5})$ | $\infty$ | $\frac{1}{4}$ |
| $\mathbb{Q}(\sqrt{-10})$ | 0 | $\frac{27}{8}$ |
| $\mathbb{Q}(\sqrt{-3})$ | $\frac{27}{2}$ | 0 |

□

**Theorem 3.9** ([BG08])**.** *The curve* $C : y^2 = f(x)$*, where*

$$\begin{aligned} f(x) =&(2s^2+2s-5)t^3x^6 - 4(4s-15)t^3x^5 - 3(2s^2-10s+75+7)t^2x^4 \\ &- 200t^2x^3 + 3(2s^2+10s+75)tx^2 - 4(4s-15)tx - (2s^2-2s-5), \end{aligned}$$

*with* $t = j_0$ *and* $s = \sqrt{-5(j_0-1)}$*, is a curve of genus* 2 *defined over* $\mathbb{Q}(s)$ *such that* $t_{10}^+(C) = j_0$. □

### 3.5.4 The case of $t_{22}^+$

As noted in the beginning of this section, the results in [HM95] and [BG08] do not cover this case. As a consequence we will start by constructing a Hauptmodul for the curve given by the image of $X(22,1)$ inside Igusa's threefold. In order to do so, we will follow the outline used in [HM95] to study the cases $D = 6, 10$ and this means writing this curve as a component in a suitable intersection of Humbert surfaces.

Let us first recall the definition of Humbert surfaces and their relation to Shimura curves. More details can be found in [Has95, HM95]. Equations for those Humbert surfaces needed in [HM95] date back to Humbert, cf. [Hum99], but more cases have been computed recently by Gruenewald in his Phd thesis, cf. [Gru08, Gru10].

**Definition.** Let $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathcal{H}_2$. We say that $\tau$ has a singular relation of discriminant $\Delta$, if there exist relatively prime integers $a, b, c, d, e$ such that $\Delta = b^2 - 4ac - 4de$ satisfying

$$a\tau_1 + b\tau_2 + c\tau_3 + d(\tau_2^2 - \tau_1\tau_3) + e = 0.$$

**Proposition 3.12** ([Hum99])**.** *Let $A$ be a principally polarized abelian surface. The following are equivalent:*

*(1) $A \cong A_\tau$ for $\tau \in \mathcal{H}$ satisfying a singular relation of discriminant $\Delta$.*

*(2) There exists an endomorphism of $A$ fixed by the Rosati involution whose characteristic polynomial has discriminant $\Delta$.* □

Consider $N_\Delta = \{\tau \in \mathcal{H}_2 : \tau \text{ has a singular relation with discriminant } \Delta\}$ and $H_\Delta$ its image under the projection $\mathcal{H}_2 \to \mathrm{Sp}(4,\mathbb{Z})\backslash\mathcal{H}_2$. It can be seen that for $\Delta < 0$, $H_\Delta$ is empty and for $\Delta = 0$ it is the whole $\mathcal{H}_2$.

**Definition.** Let $\Delta \equiv 0, 1 \pmod 4$ be a positive integer. $H_\Delta$ is called the Humbert surface of discriminant $\Delta$.

**Proposition 3.13.** *The image of $X(22,1)$ in $\mathbf{Sp}(4,\mathbb{Z})\backslash\mathcal{H}_2$ through the embedding $\epsilon_{22}$ lies in the Humbert surface of discriminant $\Delta$, $H_\Delta$, if and only if $\Delta$ is represented by the form $\Delta_{22}(x,y) = 8x^2 + 8xy + 13y^2$.*

*Proof.* A generic element in $\epsilon_{22}(\mathcal{H})$ only satisfies the singular relations

$$(2x+y)\tau_1 + 3y\tau_2 + (-x-y)\tau_3 + x(\tau_2^2 - \tau_1\tau_3) + y = 0$$

parametrized by $x, y$, whose determinant is $8x^2 + 8xy + 13y^2$. Then, the claim follows. □

**Remark.** In [Has95], the above problem is studied for a family of basis for Eichler orders in a quaternion algebra of fixed discriminant.

**Corollary 3.1.** *We have an inclusion $\epsilon_{22}(X(22,1)) \subset H_8 \cap H_{13}$.*

*Proof.* The quadratic form $\Delta_{22}$ trivially represents 8 and 13 and the result follows from the previous proposition. Moreover, it is the smallest pair of integers represented by $\Delta_{22}$. $\square$

**Remark.** In the cases $D = 6, 10$, the image of these two curves into the Igusa's threefold lies in $H_5 \cap H_8$ and equations for these two surfaces where already computed in [Hum99], where Humbert surfaces where introduced. The knowledge of these equations is of key importance in the computations of the Hashimoto-Murabayashi families.

At this point we will make use of the equations for $H_{13}$ computed in [Gru10], where more considerations on the computational aspects of Humbert surfaces can be found.

Hashimoto and Murabayashi, when dealing with the cases $D = 6, 10$, work essentially with Rosenhain models (level 2 structure) for their curves in order to obtain nice expressions for the coefficients of the sextic forms they are interested in. However, in our case, the equation for $H_{13}$ makes the whole picture considerably more involved. Therefore, since we are only interested in the computation of a Hauptmodul, we will work in level 1. Equations in level 1 are not any easier than the ones in level 2 but we have the important extra property that the curve we want to study has genus 0 in this level. We will at the end use Mestre's algorithm to compute a parametric family of curves whose jacobians are abelian surfaces with complex multiplication by $\mathcal{O}(22, 1)$.

We will extend the notations used in [BG08] to the case $D = 22$ and denote $\tilde{E}_{22}$ the image of $X(22, 1)$ in $\mathcal{A}_2$ and $E_{22}$ the intersection of $\tilde{E}_{22}$ with $\mathcal{M}_2$.

**Proposition 3.14.** *An equation for the image $\tilde{E}_{22}$ of $X(22, 1)$ in the Satake compactification of $\mathbf{Sp}(4, \mathbb{Z})\backslash\mathcal{H}_2$ identified with $\mathbb{P}(2, 3, 5, 6)$ by means of $s_2, s_3, s_5, s_6$ is determined by*

$$\begin{aligned}
&-674398835230312 5s_2^{11}s_3 + 16611508151988750s_2^{10}s_5 + 197357460342931850000s_2^8s_3^3\\
&\quad - 14448990617808589725 00s_2^7s_3^2s_5 + 3526536364630885314000s_2^6s_3s_5^2\\
&\quad - 801745426117687324710 9375s_2^5s_3^5 - 2869378793038923598800s_2^5s_5^3\\
&\quad + 987253760115657692911 68750s_2^4s_3^4s_5 - 4862453032078707557912 31000s_2^3s_3^3s_5^2\\
&\quad - 8537760000000000000s_2^2s_3^7 + 119736425255533943143 1497200s_2^2s_3^2s_5^3\\
&\quad + 427375872000000000000s_2s_3^6s_5 - 1474148784379420073160316080s_2s_3s_5^4\\
&\quad - 534830376960000000000s_3^5s_5^2 + 7259227623551639991728 34144s_5^5.
\end{aligned}$$

*A parametrization of $\tilde{E}_{22}$ is given by $[s_2 : s_3 : s_5 : s_6]$, where*

$$
\begin{aligned}
s_2 &= -2^2 \cdot 3 \cdot 11 f(11+73f)^2,\\
s_3 &= 2^2 \cdot 3 \cdot 11 f^2(11+73f)^2 \left(429 + 5774f + 19513f^2\right),\\
s_5 &= 2^3 \cdot 3 \cdot 5 \cdot 11^2 f^3(11+73f)^4 \left(-209 - 2800f - 9299f^2 + 800f^3\right),\\
s_6 &= 2^2 \cdot 3 f^3(11+73f)^4 \left(-1771561 + 16368638f + 1034587631f^2\right.\\
&\quad \left.+11546213976f^3 + 52870650096f^4 + 90192389796f^5\right),
\end{aligned}
$$

*and its inverse by*

$$
\begin{aligned}
f = &(11(239282477543678678 5s_2^4 s_3 - 306475931766864060 00s_2 s_3^3\\
&- 3582508007923968630 s_2^3 s_5 + 79794269444084577696 s_3^2 s_5\\
&- 5916450736450366200 0s_2 s_3 s_6 + 1154732324129323917 12s_5 s_6))/\\
&(2528 s_2(211603945426835 s_2^3 s_3 - 114771506545713600 s_3^3\\
&- 106795942633963098 s_2^2 s_5 + 6136436684643408 00s_3 s_6)).
\end{aligned}
$$

*Proof.* Consider the set of components of $H_8 \cap H_{13}$ computed in [Gru10]. Using Thethanullwerte we can compute $s_i$ up to a desired precision and therefore we can choose a suitable point $P \in \mathcal{H}$, and then check in which component lies $\epsilon_{22}(P)$. For example, we have chosen $P = \frac{-11+3\sqrt{-3}}{5+\sqrt{11}}$ and computed the values $s_i(\epsilon_{22}(P))$ with 250 significant digits, which is more than enough to identify the component in the statement. Then, we consider the affine chart with coordinates $x = s_2 s_3/s_5$ and $y = s_2^5/s_5^2$ and the rest of the claims in the statement can be obtained after a computation in Magma and a slight simplification. It is worth noting, however, that this computation needs not be done again since it is enough to check that the above expressions for $s_i$ in terms of $f$ satisfy indeed the equation given and that the expression for $f$ in terms of the $s_i$ also holds. $\square$

**Corollary 3.2.** *Let $\epsilon_{22} : \mathcal{H} \to \mathcal{H}_2$ the natural embedding introduced on page 96, then $g_{22} : \mathcal{H} \to \mathbb{C} \cup \{\infty\}$ defined as*

$$g_{22}(z) = \frac{g_{22,1}}{g_{22,2}},$$

*where*

$$
\begin{aligned}
g_{22,1} =\,& 11(2392824775436786785 s_2(\epsilon_{22}(z))^4 s_3(\epsilon_{22}(z)) \\
& - 30647593176686406000 s_2(\epsilon_{22}(z)) s_3(\epsilon_{22}(z))^3 \\
& - 3582508007923968630 s_2(\epsilon_{22}(z))^3 s_5(\epsilon_{22}(z)) \\
& + 79794269444084577696 s_3(\epsilon_{22}(z))^2 s_5(\epsilon_{22}(z)) \\
& - 59164507364503662000 s_2(\epsilon_{22}(z)) s_3(\epsilon_{22}(z)) s_6(\epsilon_{22}(z)) \\
& + 115473232412932391712 s_5(\epsilon_{22}(z)) s_6(\epsilon_{22}(z))), \\
g_{22,2} =\,& 2528 s_2(\epsilon_{22}(z))(211603945426835 s_2(\epsilon_{22}(z))^3 s_3(\epsilon_{22}(z)) \\
& - 114771506545713600 s_3(\epsilon_{22}(z))^3 \\
& - 106795942633963098 s_2(\epsilon_{22}(z))^2 s_5(\epsilon_{22}(z)) \\
& + 613643668464340800 s_3(\epsilon_{22}(z)) s_6(\epsilon_{22}(z))),
\end{aligned}
$$

*gives a Hauptmodul for* $X(22,1)^+$. *Moreover,*

$$
\begin{aligned}
s_2 =\,& 12(J_2^2 - 24J_4), \\
s_3 =\,& 12(J_2^3 - 36J_2J_4 + 216J_6), \\
s_5 =\,& 60(82944J_{10} + J_2^5 - 60J_2^3J_4 + 864J_2J_4^2 + 216J_2^2J_6 - 5184J_4J_6), \\
s_6 =\,& 12(497664J_{10}J_2 + 11J_2^6 - 792J_2^4J_4 + 18144J_2^2J_4^2 - 124416J_4^3 + 864J_2^3J_6 \\
& - 31104J_2J_4J_6 + 93312J_6^2).
\end{aligned}
$$

*Proof.* This follows immediately from the above proposition together with the explicit expressions we have given for all the functions involved. □

In order to compare this Hauptmodul $g_{22}$ to the one we have computed before, $t_{22}^+$, we need to compute the values of the function $g_{22}$ at some points, at least three, where we know the values of the function $t_{22}^+$. In order to do so we will use roughly the same tools used in [BG08]. We will begin by computing the values of $g_{22}$ at some points corresponding to the elliptic cycles of the curve $X(22,1)^+$, which are the CM points attached to the integer rings of $\mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-11}), \mathbb{Q}(\sqrt{-22})$. The tools we will be using will depend on which of the points we are dealing with.

**Proposition 3.15.** *The curve of genus* 2 *corresponding to the CM point by* $\mathbb{Z}[\sqrt{-22}]$ *is*

$$Y^2 = 1 + \left(-6189 + 5184\sqrt{2}\right) X^2 + \left(-6189 - 5184\sqrt{2}\right) X^4 + X^6.$$

*The abelian surfaces corresponding to the integer rings of* $\mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$ *are not jacobians of genus 2 curves. The corresponding values of* $g_{22}$ *are* $-\frac{176}{125}$, $0$ *and* $-\frac{11}{73}$, *respectively.*

*Proof.* Let us first deal with the case of $\mathrm{CM}(\mathbb{Z}[\sqrt{-22}])$. The class number of $\mathbb{Z}[\sqrt{-22}]$ is 2 and a representative of the non-trivial class in the class group is given by $(2, \sqrt{-22})$. Thus, there are two isomorphism classes of elliptic curves with CM by $\mathbb{Z}[\sqrt{-22}]$, namely $\mathcal{E}_1 = \mathbb{C}/\mathbb{Z}[\sqrt{-22}]$ and $\mathcal{E}_2 = \mathbb{C}/(2, \sqrt{-22})$. The corresponding $j$-invariants are, respectively, $216000(14571395 + 10303524\sqrt{2})$ and $216000(14571395 - 10303524\sqrt{2})$.

Consider now the cubic polynomial

$$f(X) = 1 + \left(-6189 + 5184\sqrt{2}\right) X + \left(-6189 - 5184\sqrt{2}\right) X^2 + X^3$$

and the corresponding genus 2 curve $C : Y^2 = f(X^2)$. The Igusa invariants of the sextic form $f(X^2)$ can be computed easily and we obtain that the curve $C$ corresponds to a point in $E_{22}$ such that the corresponding value of $g_{22}$ is $-\frac{176}{925}$. This curve has two obvious non-hyperelliptic involutions, namely $s_1(X,Y) = (-X,Y)$ and $s_2(X,Y) = (-X,-Y)$. The quotient curve $C^{\langle s_1 \rangle}$ has equation $Z^2 = f(T)$, where $Z = Y$, $T = X^2$, which is an elliptic curve with $j$-invariant $216000(14571395{+}10303524\sqrt{2})$, which means that $C^{\langle s_1 \rangle} \cong \mathcal{E}_1$. Similarly, $C^{\langle s_2 \rangle}$ has equation $Z^2 = T^3 f(1/T)$, where $Z = Y/X^3$, $T = 1/X^2$, and has $j$-invariant equal to that of $j(\mathcal{E}_2)$, i.e. $C^{\langle s_2 \rangle} \cong \mathcal{E}_2$. Now, if we consider the natural projections $C \to \mathcal{E}_i$, for $i = 1, 2$, we obtain a natural isomorphism of abelian surfaces $\pi : Jac(C) \to \mathcal{E}_1 \times \mathcal{E}_2$. In particular,

$$\mathrm{End}(Jac(C)) = \begin{pmatrix} \mathbb{Z}[\sqrt{22}] & (2, \sqrt{-22}) \\ (2, \sqrt{-22})^{-1} & \mathbb{Z}[\sqrt{22}] \end{pmatrix}$$

has complex multiplication by $\mathbb{Z}[\sqrt{-22}]$.

On the other hand, to deal with the cases $\mathrm{CM}(\mathbb{Z}[i])$ and $\mathrm{CM}(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$, it follows from [HN65] that in these cases the corresponding principally polarized abelian surfaces are not jacobians from genus 2 curves and therefore the corresponding points in $\mathcal{A}_2$ satisfy that $\chi_{10} = 0$. Therefore, if we use the expressions we have for the explicit parametrization of the image of the Shimura curve inside $\mathcal{A}_2$, we can compute the values of $g_{22}$ giving points in $\mathcal{A}_2 \smallsetminus \mathcal{M}_2$, and the values are $\{-\frac{1}{5}, 0, -\frac{11}{64}, -\frac{11}{73}\}$. Therefore a numerical computation using the expressions in terms of Thetanullwerte is enough to decide which values correspond to these two points and we find that the value of $g_{22}$ corresponding to $\mathbb{Z}[i]$ is 0 and that corresponding to $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ is $-\frac{11}{73}$. $\square$

**Theorem 3.10.** *It holds that* $t_{22}^+ = \dfrac{925g_{22}+176}{1024g_{22}+176}$.

*Proof.* As we have done in the cases $D = 6, 10$, we only need to compare the values of these two functions at three different points. The values of $t_{22}^+$ at three of the elliptic points were fixed when we defined the function and the values of $g_{22}$ we computed in the previous proposition:

| $K$ | $t_{22}^+$ | $g_{22}$ |
|---|---|---|
| $\mathbb{Q}(i)$ | 1 | 0 |
| $\mathbb{Q}(\sqrt{-3})$ | $\frac{27}{16}$ | $-\frac{11}{73}$ |
| $\mathbb{Q}(\sqrt{-22})$ | 0 | $-\frac{176}{925}$ |

The claim now follows. $\square$

**Corollary 3.3.** *The points in $\tilde{E}_{22} \smallsetminus E_{22}$ are the points in $\tilde{E}_{22}$ corresponding to CM points by $\mathbb{Z}[i]$, $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$, $\mathbb{Z}[\sqrt{-5}]$ and $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$.*

*Proof.* In Theorem 3.1 we obtained the values of $t_{22}^+$ at the CM points by $\mathbb{Z}[\sqrt{-5}]$ and $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$, namely $\frac{5}{16}$ and $\infty$. The corresponding values of $g_{22}$ are $-\frac{1}{5}$ and $-\frac{11}{64}$ and these are the two values which together with those by $\mathbb{Z}[i]$ and $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ we have proved in Proposition 3.15 that give points in $\tilde{E}_{22} \smallsetminus E_{22}$. $\square$

Now, for any point in $X(22,1)^+(\mathbb{C})$ different from the ones in the previous corollary, we are going to give an hyperelliptic curve of genus 2 giving the abelian surface corresponding to this point in terms of $t_{22}^+$.

In order to do this, we are going to follow Mestre's construction, cf. [Mes91]. In first place, we are going to introduce a new set of generators of the graded ring of projective invariants of even degree of a sextic form introduced by Clebsch, [Cle72]. Since both [Igu60] and [Cle72] define invariants $A, B, C, D$ which are actually not the same, we are going to follow Igusa's notation and call $A', B', C', D'$ the invariants introduced by Clebsch in [Cle72], contrary to the choice made in [Mes91]. Both sets of invariants are related as follows:

$$A' = -\frac{A}{120}, \quad B' = \frac{A^2 + 20B}{135000}, \quad C' = \frac{-A^3 - 80AB + 600C}{121500000},$$
$$D' = \frac{-9A^5 - 700A^3B + 12400AB^2 + 3600A^2C - 48000BC - 10800000D}{49207500000000}.$$

Given a set of invariants $A', B', C', D'$ such that the corresponding Igusa invariant $D$ does not vanish, Mestre constructs a conic $L$ and a cubic $M$ in $\mathbb{P}^2$ depending on the invariants as follows:

$$L : A_{11}x_1^2 + A_{22}x_2^2 + A_{33}x_3^2 + 2A_{13}x_1x_3 + 2A_{12}x_1x_2 + 2A_{23}x_2x_3 = 0,$$
$$M : \sum_{1 \le i,j,k \le 3} a_{ijk}x_ix_jx_k = 0,$$

where

$$\begin{aligned} A_{11} &= 2C' + A'B'/3, & A_{23} &= B'(B'^2 + A'C')/3 + C'(6C' + A'B')/9, \\ A_{22} &= D', & A_{31} &= D', \\ A_{33} &= B'D'/2 + 2C'(B'^2 + A'C')/9, & A_{12} &= 2(B'^2 + A'C')/3, \end{aligned}$$

and

$$\begin{aligned}
36a_{111} &= 8(A'^2C' - 6B'C' + 9D'), \\
36a_{112} &= 4(2B'^3 + 4A'B'C' + 12C'^2 + 3A'D'), \\
36a_{113} &= a_{122} = 4(A'B'^3 + 4A'^2B'C'/3 + 4B'^2C' + 6A'C'^2 + 3B'D'), \\
36a_{123} &= 2(2B'^4 + 4A'B'^2C' + 4A'^2C'^2/3 + 4B'C'^2 + 3A'B'D' + 12C'D'), \\
36a_{133} &= 2(A'B'^4 + 4A'^2B'^2C'/3 + 16B'^3C'/3 + 26A'B'C'^2/3 + 8C'^3 \\
&\quad + 3B'^2D' + 2A'C'D'), \\
36a_{222} &= 4(3B'^4 + 6A'B'^2C' + 8A'^2C'^2/3 + 2B'C'^2 - 3C'D'), \\
36a_{223} &= 2(-2B'^3C'/3 - 4A'B'C'^2/3 - 4C'^3 + 9B'^2D' + 8A'C'D'), \\
36a_{233} &= 2(B'^5 + 2A'B'^3C' + 8A'^2B'C'^2/9 + 2B'^2C'^2/3 - B'C'D' + 9D'^2), \\
36a_{333} &= -2B'^4C' - 4A'B'^2C'^2 - 16A'^2C'^3/9 - 4B'C'^3/3 + 9B'^3D' \\
&\quad + 12A'B'C'D' + 20C'^2D'.
\end{aligned}$$

In [Mes91] further normalizations are carried out in order to obtain a sextic form with exactly the invariants $A', B', C', D'$. Observe, in particular, that the pair of curves obtained above change if we choose an equivalent set of invariants $r^2A', r^4B', r^6C', r^{10}D'$ for some $r \neq 0$, but there exists an automorphism of $\mathbb{P}^2$ mapping one pair to the other. Since we will only need a form with the above invariants up to equivalence, the above construction will suffice.

Given a sextic form over a field $k$ with non-trivial automorphisms, then it has either a non-trivial involution, which is equivalent to $L$ being degenerate, or it has an automorphism of order 5, which is equivalent to $A = B = C = 0$, $D \neq 0$. In the first case, the sextic is isomorphic over $\overline{k}$ to a sextic of the form $X^6 + uX^4 + vX^2 + 1 \in k[X]$. In the second case, the sextic is $\overline{k}$-isomorphic to $X^6 + X$.

In general, for a given set of invariants $A', B', C', D' \in k$ such that $L$ is non-degenerate, one can find a sextic form corresponding to them over a field $K|k$ if and only if $L(K) \neq \emptyset$. In order to do that, we consider a parametrization $\varphi : \mathbb{P}^1_K \to L_K$, $[x : y] \mapsto [X_1 : X_2 : X_3]$, then when we substitute this parametrization into the equation defining $M$, we obtain a sextic form defined over $K$ with a set of invariants equivalent to the prescribed $A', B', C', D'$.

In our situation, let $f$ be the Hauptmodul for $\tilde{E}_{22}$ we have computed in Theorem 3.14. Using the formulas relating $A, B, C, D$ with $s_2, s_3, s_5, s_6$ we obtain the following expression for the Igusa invariants restricted to $\tilde{E}_{22}$ in terms of $f$:

$$\begin{aligned}
A =& -8(161051 + 4465505f + 50233271f^2 + 286721842f^3 + 830345956f^4 \\
&+ 975422551f^5),\\
B =& -5797836f(1+5f)^4(11+64f)^2(11+73f)^2,\\
C =& -3865224f(1+5f)^4(11+64f)^2(11+73f)^2(-644204 - 16149023f\\
&- 150781004f^2 - 561784366f^3 + 78064712f^4 + 5932994381f^5\\
&+ 11333150400f^6),\\
D =& \, 37661140520652f^3(1+5f)^{12}(11+64f)^6(11+73f)^4.
\end{aligned}$$

It is worth recalling that $r^2A, r^4B, r^6C, r^{10}D$, $r \in \mathbb{C}(f)$, yield the same sextic form (up to isomorphism).

Now we are going to apply Mestre's algorithm, to obtain an expression for a sextic form with this set of Igusa invariants (up to equivalence). We consider the conic given above with matrix $L = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \in \mathbf{M}(2, \mathbb{Q}[t])$ and we obtain a matrix $N \in \mathbf{M}(2, \mathbb{Z}[f])$ with determinant

$$\begin{aligned}\det N = &-179481912335677617187500 f^2 \left(1+5f\right)^9 \left(11+64f\right)^6 \left(11+67f\right)\\ &\cdot \left(11+73f\right)^4 \left(11+100f\right)^2 \left(121+1210f+2377f^2\right)^2\\ &\cdot \left(121+1331f+3664f^2\right)^2 \left(1331+37389f+296340f^2+732304f^3\right)^2\end{aligned}$$

such that

$$\frac{1}{\lambda(f)} N \cdot L \cdot N^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -22f(11+64f) & 0 \\ 0 & 0 & -11f(176+925f) \end{pmatrix}$$

where

$$\begin{aligned}\lambda(f) = &-369116838242910252 f^3 \left(1+5f\right)^{14} \left(11+64f\right)^8 \left(11+67f\right)^2 \left(11+73f\right)^6\\ &\cdot \left(11+100f\right)^2 \left(121+1210f+2377f^2\right)^2 \left(121+1331f+3664f^2\right)^2\\ &\cdot \left(1331+37389f+296340f^2+732304f^3\right)^2.\end{aligned}$$

As a consequence, we obtain the following two propositions:

**Proposition 3.16.** *The curves in $E_{22}$ with non-trivial automorphisms are those corresponding to CM points by: $\mathbb{Z}[3i]$, $\mathbb{Z}[\frac{1+3\sqrt{-3}}{2}]$, $\mathbb{Z}[\sqrt{-22}]$, $\mathbb{Z}[\sqrt{-14}]$, $\mathbb{Z}[\frac{1+5\sqrt{-3}}{2}]$ and $\mathbb{Z}[\frac{1+\sqrt{-59}}{2}]$.*

*Proof.* The determinant of the coefficient matrix of the conic $L$ in terms of $f$ is:

$$\begin{aligned}&-2^3 3^{-7} 5^{-20} 11^{24} f^7 (1+5f)^{24} (11+64f)^{13} (11+73f)^{10}\\ &\quad\cdot (11+67f)^4 (11+100f)^2 (176+925f)\\ &\quad\cdot \left(121+1210f+2377f^2\right)^2\\ &\quad\cdot \left(121+1331f+3664f^2\right)^2\\ &\quad\cdot \left(1331+37389f+296340f^2+732304f^3\right)^2\end{aligned}$$

As we have seen above, the first four roots correspond to points lying outside $\mathcal{M}_2$. For the rest of them, if we study the corresponding values of $t_{22}^{+}$ we can find suitable candidates for fields giving rise to them, and a computation of class equations yields that each of the irreducible factors from the fifth one onewards correspond, respectively, to $\mathbb{Z}[3i]$, $\mathbb{Z}[\frac{1+3\sqrt{-3}}{2}]$, $\mathbb{Z}[\sqrt{-22}]$, $\mathbb{Z}[\sqrt{-14}]$, $\mathbb{Z}[\frac{1+5\sqrt{-3}}{2}]$, $\mathbb{Z}[\frac{1+\sqrt{-59}}{2}]$. □

**Proposition 3.17** ([Mes91]). *Given $P \in E_{22}$ such that the corresponding genus 2 curve does not have non-trivial automorphisms, then this curve can be defined over a field $K \supset \mathbb{Q}(f(P))$ if and only if $K$ splits the quaternion algebra*

$$(22f(P)(11+64f(P)), 11f(P)(176+925f(P)))_{\mathbb{Q}(f(P))}.$$

*This quaternion algebra is usually called the Mestre obstruction.* □

**Remark.** If the curve corresponding to $P \in E_{22}$ has non-trivial automorphisms, the curves can be defined over the field of moduli, cf. [CQ05].

Now, if we choose $s = \sqrt{22f(11+64f)}$, a parametrization of $L$ is given by $\phi(x,y) = (s(x^2+1936fy^2+10175f^2y^2), -x^2+1936fy^2+10175f^2y^2, 2xys)\cdot N$. Then, substituting this expression into the cubic $M$ we obtain the following result.

**Theorem 3.11.** *Consider $S^+$ the set of points of $X(22,1)^+(\mathbb{C})$ giving points in $\tilde{E}_{22} \smallsetminus E_{22}$, namely*

$$S^+ = \bigcup_K \mathrm{CM}(\mathcal{O}_K), \quad \textit{for } K \in \{\mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\sqrt{-11})\}.$$

*The curve $C : y^2 = f(x)$ corresponding to the point $P \in X(22,1)^+(\mathbb{C}) \smallsetminus S^+$ with $t_{22}^+(P) = t_0$ is given by*

$$\begin{aligned} f(X) =& 8\left(-7+5s_0-208t_0-96s_0t_0+512t_0^2+256s_0t_0^2\right)X^6 \\ &+16t_0\left(759+160s_0-1824t_0-512s_0t_0+768t_0^2\right)X^5 \\ &-12t_0\left(1393+335s_0-1872t_0-1312s_0t_0+512t_0^2+768s_0t_0^2\right)X^4 \\ &+16t_0^2\left(2141-2144t_0+256t_0^2\right)X^3 \\ &-6t_0^2\left(-1393+335s_0+1872t_0-1312s_0t_0-512t_0^2+768s_0t_0^2\right)X^2 \\ &+4t_0^3\left(759-160s_0-1824t_0+512s_0t_0+768t_0^2\right)X \\ &+t_0^3\left(7+5s_0+208t_0-96s_0t_0-512t_0^2+256s_0t_0^2\right) \end{aligned}$$

*where $s_0 = \sqrt{2(1-t_0)}$.*

*Proof.* The result follows after applying Mestre's algorithm to obtain an equation in terms of $t$ and then using the relation between $f \circ \epsilon_{22} = g_{22}$ and $t_{22}^+$, together with some slight simplifications. □

**Corollary 3.4.** *Consider $S$ the set of points of $X(22,1)(\mathbb{C})$ giving points in $\tilde{E}_{22} \smallsetminus E_{22}$, i.e. $S = \bigcup_K \mathrm{CM}(\mathcal{O}_K)$, for $K \in \{\mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\sqrt{-11})\}$. The functions $v_1 = u_{22,3}$ and $v_2 = \frac{1}{u_{22,2}}$ satisfy $v_1^2 + 11v_2^2 + 1 = 0$ and give another affine chart of the canonical model for $X(22,1)$ given in Theorem 2.4. Then, a genus 2 curve giving the abelian surface corresponding to a point in $X(22,1)(\mathbb{C}) \smallsetminus S$ is given by*

$$
\begin{aligned}
Y^2 = & -512\sqrt{2}X\left(3-2X^2+3X^4\right)v_1^5 \\
& + 256\sqrt{-22}(-1+X)(1+X)\left(1+10X^2+X^4\right)v_1^4v_2 \\
& + 512(1+X^2)^3v_1^4 \\
& - 2048\sqrt{-11}(1+X^2)(-1+X)X(1+X)v_1^3v_2 \\
& - 64\sqrt{2}X\left(57-134X^2+57X^4\right)v_1^3 \\
& + 96\sqrt{-22}(-1+X)(1+X)\left(1+42X^2+X^4\right)v_1^2v_2 \\
& + 208\left(1+X^2\right)\left(1+26X^2+X^4\right)v_1^2 \\
& - 640\sqrt{-11}(1+X^2)(-1+X)X(1+X)v_1v_2 \\
& - 2\sqrt{2}X\left(759-4282X^2+759X^4\right)v_1 \\
& + 5\sqrt{-22}(-1+X)(1+X)\left(1+202X^2+X^4\right)v_2 \\
& - 7\left(1+X^2\right)\left(1-598X^2+X^4\right).
\end{aligned}
$$

□

**Proposition 3.18.** *The pairwise intersections of $\tilde{E}_6$, $\tilde{E}_{10}$ and $\tilde{E}_{22}$ are:*

*(1) ([BG08]) $\tilde{E}_6 \cap \tilde{E}_{10} = \{P_3, P_{43}\}$ where $P_3$ is the point corresponding to $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ (not in $\mathcal{M}_2$) and $P_{43}$ the CM point corresponding to $\mathbb{Z}[\frac{1+\sqrt{-43}}{2}]$.*

*(2) $\tilde{E}_6 \cap \tilde{E}_{22}$ consists only of the CM points attached to $\mathbb{Z}[i]$ and $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$, both of them in $\mathcal{A}_2 \smallsetminus \mathcal{M}_2$.*

*(3) $\tilde{E}_{10} \cap \tilde{E}_{22}$ contains only the points corresponding to CM points by $\mathbb{Z}[\sqrt{-3}]$ (not in $\mathcal{M}_2$), $\mathbb{Z}[\frac{1+3\sqrt{-3}}{2}]$(corresponding to $g_{22} = -\frac{11}{100}$) and $\mathbb{Z}[\frac{1+\sqrt{-67}}{2}]$ (corresponding to $g_{22} = -\frac{99}{625}$).*

*Proof.* We consider the equations for $\tilde{E}_6$ and $\tilde{E}_{10}$ computed for example in [BG08] or [Gru10]. When we substitute the parametrization of $\tilde{E}_{22}$ we have computed above we obtain the values of $f$ giving points in the intersection. Then, we only need to find the corresponding values of $t_{22}^+$ and check to which CM points they correspond. □

**Proposition 3.19.** *Given a point in $X(22,1)^+(\overline{\mathbb{Q}})$ giving rise to a point in $\mathcal{M}_2$, the corresponding curve has potential good reduction at $p \neq 2, 11$ if and only if $\nu_p((t_{22}^+ - 1)(16t_{22}^+ - 27)(16t_{22}^+ - 5)) = 0$.*

*Proof.* If the valuation is 0, the statement follows from the model given in Theorem 3.11 taking $u = 4\sqrt{-t}$, and introducing into $X$ all common factors. Then $\mathrm{disc}(C) = 2^{49}11^6(t_{22}^+ - 1)^3(16t_{22}^+ - 27)^4(16t_{22}^+ - 5)^{12}$ and all these factors are coprime to $p$, hence the claim follows.

Reciprocally, assume that $C$ has potential good reduction, that is, we can consider an integral model over a number field $K$ where the curve has good reduction, which translates into the fact that $J_2, J_4, J_6, J_8, J_{10}$ are integral and $\nu_p(J_{10}) = 0$. Now, using the expressions we have computed above, we can write the point $[J_2 : J_4 : J_6 : J_8 : J_{10}] \in \mathbb{P}_{\overline{\mathbb{Q}}}(2,4,6,8,10)$ in terms of the value $t_0$ of the function $t_{22}^+$ at the corresponding point:

$$
\begin{aligned}
J_2 =& -1930497 - 4560208t_0 + 21187328t_0^2 - 24879104t_0^3 + 12845056t_0^4 - 3145728t_0^5,\\
J_4 =& \frac{1}{8}(970295139003 + 10245702245584t_0 - 49583541376256t_0^2\\
&+ 72597620043776t_0^3 - 10488611930112t_0^4 - 115879149830144t_0^5\\
&+ 204872221720576t_0^6 - 185104232087552t_0^7 + 96087008346112t_0^8\\
&- 26938034880512t_0^9 + 3298534883328t_0^{10}),\\
J_6 =& -\frac{1}{16}(-980299 + 1287984t_0 - 1083136t_0^2 + 1925120t_0^3 - 2359296t_0^4 + 1048576t_0^5)\\
&\cdot(-989801 - 6587112497008t_0 + 29334461323520t_0^2 - 56750053199872t_0^3\\
&+ 74537865510912t_0^4 - 93423622160384t_0^5 + 107641342787584t_0^6\\
&- 88124944285696t_0^7 + 43241730736128t_0^8 - 10995116277760t_0^9\\
&+ 1099511627776t_0^{10}),
\end{aligned}
$$

$$\begin{aligned}
J_8 =& \frac{1}{256}(-941465164120709389721997 + 299807980845201038251551 68t_0 \\
&- 1785376874326436039071321 60t_0^2 + 4250377966638331835158364 16t_0^3 \\
&+ 2350874102977980229759139 84t_0^4 - 5917929433784854286346223 616t_0^5 \\
&+ 2587973606305498997359588 1472t_0^6 - 6969705328745145762858192 0768t_0^7 \\
&+ 1316022500370900464507174 25664t_0^8 - 1814523235933769923739024 09728t_0^9 \\
&+ 1862210972218930063471332 55680t_0^{10} - 1443258722923570918567863 91040t_0^{11} \\
&+ 8654980760744907925976881 5616t_0^{12} - 4273086013319475466922570 5472t_0^{13} \\
&+ 1971677026355195828987808 9728t_0^{14} - 9456126262869180795961999 360t_0^{15} \\
&+ 4355398614609521978283917 312t_0^{16} - 1604332516704356361977200 640t_0^{17} \\
&+ 4044234655929564161009254 40t_0^{18} - 5923736516111682956060262 4t_0^{19} \\
&+ 3626777458843887524118528t_0^{20}), \\
J_{10} =& -2^2 11^6(-1+t_0)^3(-27+16t_0)^4(-5+16t_0)^{12}.
\end{aligned}$$

First of all, assume that $t_0$ is an algebraic $p$-integer. Then,

$$\begin{aligned}
&\nu_p(-2^2 11^6(-1+t_0)^3(-27+16t_0)^4(-5+16t_0)^{12}) \\
&\qquad = 3\nu_p(-1+t_0) + 4\nu_p(-27+16t_0) + 12\nu_p(-5+16t_0) \geq 0
\end{aligned}$$

and the three terms are non-negative. If they are all zero, we obtain the claim in the statement of the proposition. Assume that this valuation is strictly positive, which means that one of the terms has strictly positive valuation. Note now, that the expression for the coordinate of $J_6$ in terms of $t_0$ can be written as

$$\begin{aligned}
J_6 &= -11^{15}2^{-4} + (-1+t_0)P_1(t_0) \\
&= -2^8 11^{15} + (-27+16t_0)P_2(t_0) \\
&= -2^2 11^{15} + (-5+16t_0)P_3(t_0)
\end{aligned}$$

where $P_i(t_0) \in \mathcal{O}_K[\frac{1}{2}, t_0]$, which implies that the corresponding value has valuation 0. Therefore, since all the expressions of the $J_{2i}$ in terms of $t_0$ are $p$-integers we obtain that the two expressions cannot match. Thus, we reach a contradiction.

Similarly, if the value $t_0$ is not an algebraic $p$-integer, we can obtain an equivalent point multiplying the components by $r^2, r^4, r^6, r^8, r^{10}$, respectively, for a suitable $r \in \overline{\mathbb{Q}}$ to make them $p$-integers, and a simple calculation yields in this case as well that the equality is impossible.

Hence, we obtain the desired result. $\square$

# Chapter 4

# Uniformization of some Shimura curves with full Atkin-Lehner quotient of genus $0$

In the previous chapter we have reviewed the uniformization of the curves $X(D,1)^+$ for $D = 6, 10$ carried out in [BT07b] and [Elk98], and we have combined the techniques used in these papers with those introduced in [SD77] to compute a uniformization of $X(22,1)^+$. In this chapter, we are going to begin by determining the set of $(D,N)$ such that $X(D,N)^+ = X(D,N)^{W_{DN}}$ has genus 0, which for squarefree values of $N$ was obtained in [LMR06]. This gives the set of curves for which the technique used in the previous chapter may work from a theoretical point of view. However, the complexity of the computations involved increases fastly with the number of elliptic cycles of the curve. In particular, for a given $D$ in the list, the easiest case is always $(D,1)$. Therefore, in the second part of the chapter we will see how to obtain a Hauptmodul for $X(D,N)^+$ from that of $X(D,1)^+$ using again the kroneckerian polynomials and we will apply it to obtain explicit uniformizations of $X(D,N)$, for $D = 6, 10, 22$ and the three smallest possible values of $N > 1$.

## 4.1 All the curves $X(D,N)$ with an Atkin-Lehner quotient of genus $0$

The method we used to uniformize the Shimura curves in the previous chapter was not really dependent on the genus of the curves $X(D,1)$ but on the existence of a genus 0 quotient. As we have noted previously, the list of Shimura curves $X(D,N)$ of genus 0 only consists of those three curves we have already studied, but the list of Shimura curves having an Atkin-Lehner quotient of genus 0 is much longer. In order to compute a list of such curves, it is worth recalling that, since we know the genus of $X(D,N)$, Riemann-Hurwitz formula allows us to compute the genus of the full Atkin-Lehner group once we know the set of ramification points of the morphism $X(D,N) \to X(D,N)^+$ and the corresponding ramification indices. However, a point $P \in X(D,N)$ is a ramification point if and only if it is fixed by $\omega \in W_{DN} \smallsetminus \{id\}$ and the ramification index corresponding to this point is exactly the number of transformations in $W_{DN}$ fixing it. Therefore, we obtain all this information from the following result by Ogg (cf. [Ogg83]).

**Theorem 4.1.** *The fixed points by the Atkin-Lehner involution of level $m \| DN$ acting on $X(D,N)$ are CM points by*

$$\begin{cases} \mathbb{Z}[i], \mathbb{Z}[\sqrt{-2}], & \text{if } m = 2, \\ \mathbb{Z}[\frac{1+\sqrt{-m}}{2}], \mathbb{Z}[\sqrt{-m}], & \text{if } m \equiv 3 \pmod 4, \\ \mathbb{Z}[\sqrt{-m}], & \text{otherwise.} \end{cases}$$

*The number of fixed points corresponding to any of the previous orders, R, is given by*

$$h(R) \prod_{p \mid \frac{DN}{m}} \nu_p(\mathcal{O}(D,N), R; \mathcal{O}(D,N)^*),$$

*where the values of $\nu_p(\mathcal{O}(D,N), R; \mathcal{O}(D,N)^*)$ can be found in Theorem 1.7.* □

Note that the sets of fixed points by different non-trivial Atkin-Lehner involutions are disjoint. Thus, the ramification index of any ramified point is 2.

We list on the following table the Shimura curves $X(D,N)$ having its full Atkin-Lehner quotient of genus 0. The values of $D,N$ for squarefree $N$, which amount to 260 out of the 271 total cases were computed in [LMR06]. The rest, to our knowledge, had not been computed before. Let us briefly explain the contents of the table:

- 1st column: value of $D$.
- 2nd column: value of $N$.
- 3rd column: genus of $X(D,N)$.
- 4th column: number of elliptic cycles on the quotient $\langle\Gamma(D,N), W_{DN}\rangle\backslash\mathcal{H}$, $n_W$. As we have seen, this determines the complexity of the procedure we have carried out in the previous chapter.
- 5th and 6th columns: number of elliptic cycles in $\Gamma(D,N)\backslash\mathcal{H}$ of order 2 and 3, $e_2$, $e_3$ respectively.
- 7th column: This column contains a list of expressions of the form $\{m, L_m\}$ where $m$ is a divisor of $DN$ such that $\gcd(m, DN/m) = 1$ and $L_m$ is a list containing the number of fixed points of the Atkin-Lehner involution. If $\omega_m$ has no fixed points, we omit the term altogether. As we have seen before, if $m > 2$ the fixed points by $\omega_m$ are CM points by certain orders inside $\mathbb{Q}(\sqrt{-m})$. We list the number of fixed points corresponding to each order separately and ordered in function of the absolute value of the corresponding discriminants. In the case $m = 2$ we list first the number of fixed points corresponding to $\mathrm{CM}(\mathbb{Z}[i])$ and then those corresponding to $\mathrm{CM}(\mathbb{Z}[\sqrt{-2}])$.

**Proposition 4.1.** *The following table contains all the pairs of integers $D > 1$, $N \geq 1$ with $(D,N) = 1$ giving rise to Shimura curves $X(D,N)^+$ of genus 0:*

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 6 | 1 | 0 | 3 | 2 | 2 | $\{2,\{2,0\}\}$, $\{3,\{2,0\}\}$, $\{6,\{2\}\}$ |
| | 5 | 1 | 4 | 4 | 0 | $\{2,\{4,0\}\}$, $\{6,\{4\}\}$, $\{10,\{4\}\}$, $\{30,\{4\}\}$ |
| | 7 | 1 | 4 | 0 | 4 | $\{3,\{4,0\}\}$, $\{6,\{4\}\}$, $\{21,\{4\}\}$, $\{42,\{4\}\}$ |
| | 11 | 3 | 5 | 0 | 0 | $\{6,\{4\}\}$, $\{22,\{4\}\}$, $\{33,\{4\}\}$, $\{66,\{8\}\}$ |
| | 13 | 1 | 4 | 4 | 4 | $\{2,\{4,0\}\}$, $\{3,\{4,0\}\}$, $\{13,\{4\}\}$, $\{78,\{4\}\}$ |
| | 17 | 3 | 5 | 4 | 0 | $\{2,\{4,0\}\}$, $\{34,\{8\}\}$, $\{51,\{4,0\}\}$, $\{102,\{4\}\}$ |
| | 19 | 3 | 5 | 0 | 4 | $\{3,\{4,0\}\}$, $\{19,\{4,0\}\}$, $\{57,\{4\}\}$, $\{114,\{8\}\}$ |
| | 23 | 5 | 6 | 0 | 0 | $\{46,\{8\}\}$, $\{69,\{8\}\}$, $\{138,\{8\}\}$ |
| | 25 | 5 | 6 | 4 | 0 | $\{2,\{4,0\}\}$, $\{6,\{4\}\}$, $\{25,\{4\}\}$, $\{75,\{4,0\}\}$, $\{150,\{8\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 6 | 29 | 5 | 6 | 4 | 0 | $\{2,\{4,0\}\}$, $\{6,\{4\}\}$, $\{58,\{4\}\}$, $\{174,\{12\}\}$ |
| | 31 | 5 | 6 | 0 | 4 | $\{3,\{4,0\}\}$, $\{6,\{4\}\}$, $\{93,\{4\}\}$, $\{186,\{12\}\}$ |
| | 35 | 9 | 6 | 0 | 0 | $\{6,\{8\}\}$, $\{10,\{8\}\}$, $\{21,\{8\}\}$, $\{70,\{8\}\}$, $\{105,\{8\}\}$, $\{210,\{8\}\}$ |
| | 37 | 5 | 6 | 4 | 4 | $\{2,\{4,0\}\}$, $\{3,\{4,0\}\}$, $\{37,\{4\}\}$, $\{222,\{12\}\}$ |
| | 41 | 7 | 7 | 4 | 0 | $\{2,\{4,0\}\}$, $\{82,\{8\}\}$, $\{123,\{4,0\}\}$, $\{246,\{12\}\}$ |
| | 43 | 7 | 7 | 0 | 4 | $\{3,\{4,0\}\}$, $\{43,\{4,0\}\}$, $\{129,\{12\}\}$, $\{258,\{8\}\}$ |
| | 47 | 9 | 8 | 0 | 0 | $\{94,\{16\}\}$, $\{141,\{8\}\}$, $\{282,\{8\}\}$ |
| | 49 | 9 | 8 | 0 | 4 | $\{3,\{4,0\}\}$, $\{6,\{4\}\}$, $\{49,\{8\}\}$, $\{147,\{4,0\}\}$, $\{294,\{12\}\}$ |
| | 53 | 9 | 8 | 4 | 0 | $\{2,\{4,0\}\}$, $\{6,\{4\}\}$, $\{106,\{12\}\}$, $\{318,\{12\}\}$ |
| | 55 | 13 | 7 | 0 | 0 | $\{6,\{8\}\}$, $\{10,\{8\}\}$, $\{30,\{8\}\}$, $\{66,\{16\}\}$, $\{165,\{8\}\}$, $\{330,\{8\}\}$ |
| | 59 | 11 | 9 | 0 | 0 | $\{6,\{4\}\}$, $\{118,\{12\}\}$, $\{177,\{4\}\}$, $\{354,\{16\}\}$ |
| | 61 | 9 | 8 | 4 | 4 | $\{2,\{4,0\}\}$, $\{3,\{4,0\}\}$, $\{61,\{12\}\}$, $\{366,\{12\}\}$ |
| | 65 | 13 | 7 | 8 | 0 | $\{2,\{8,0\}\}$, $\{10,\{8\}\}$, $\{30,\{8\}\}$, $\{130,\{8\}\}$, $\{195,\{8,0\}\}$, $\{390,\{16\}\}$ |
| | 67 | 11 | 9 | 0 | 4 | $\{3,\{4,0\}\}$, $\{67,\{4,0\}\}$, $\{201,\{12\}\}$, $\{402,\{16\}\}$ |
| | 71 | 13 | 10 | 0 | 0 | $\{142,\{8\}\}$, $\{213,\{8\}\}$, $\{426,\{24\}\}$ |
| | 73 | 11 | 9 | 4 | 4 | $\{2,\{4,0\}\}$, $\{3,\{4,0\}\}$, $\{6,\{4\}\}$, $\{73,\{8\}\}$, $\{219,\{8,0\}\}$, $\{438,\{8\}\}$ |
| | 77 | 17 | 8 | 0 | 0 | $\{6,\{8\}\}$, $\{21,\{8\}\}$, $\{33,\{8\}\}$, $\{66,\{16\}\}$, $\{154,\{16\}\}$, $\{462,\{8\}\}$ |
| | 79 | 13 | 10 | 0 | 4 | $\{3,\{4,0\}\}$, $\{6,\{4\}\}$, $\{237,\{12\}\}$, $\{474,\{20\}\}$ |
| | 83 | 15 | 11 | 0 | 0 | $\{6,\{4\}\}$, $\{166,\{20\}\}$, $\{249,\{12\}\}$, $\{498,\{8\}\}$ |
| | 85 | 17 | 8 | 8 | 0 | $\{2,\{8,0\}\}$, $\{30,\{8\}\}$, $\{34,\{16\}\}$, $\{51,\{8,0\}\}$, $\{85,\{8\}\}$, $\{510,\{16\}\}$ |
| | 89 | 15 | 11 | 4 | 0 | $\{2,\{4,0\}\}$, $\{178,\{16\}\}$, $\{267,\{4,0\}\}$, $\{534,\{20\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 6 | 91 | 17 | 8 | 0 | 8 | $\{3,\{8,0\}\}$, $\{13,\{8\}\}$, $\{42,\{8\}\}$, $\{91,\{8,0\}\}$, $\{273,\{8\}\}$, $\{546,\{24\}\}$ |
| | 95 | 21 | 9 | 0 | 0 | $\{10,\{8\}\}$, $\{19,\{8,0\}\}$, $\{114,\{16\}\}$, $\{190,\{8\}\}$, $\{285,\{16\}\}$, $\{570,\{16\}\}$ |
| | 97 | 15 | 11 | 4 | 4 | $\{2,\{4,0\}\}$, $\{3,\{4,0\}\}$, $\{6,\{4\}\}$, $\{97,\{8\}\}$, $\{291,\{8,0\}\}$, $\{582,\{16\}\}$ |
| | 107 | 19 | 13 | 0 | 0 | $\{6,\{4\}\}$, $\{214,\{12\}\}$, $\{321,\{20\}\}$, $\{642,\{16\}\}$ |
| | 109 | 17 | 12 | 4 | 4 | $\{2,\{4,0\}\}$, $\{3,\{4,0\}\}$, $\{109,\{12\}\}$, $\{654,\{28\}\}$ |
| | 113 | 19 | 13 | 4 | 0 | $\{2,\{4,0\}\}$, $\{226,\{16\}\}$, $\{339,\{12,0\}\}$, $\{678,\{20\}\}$ |
| | 115 | 25 | 10 | 0 | 0 | $\{10,\{8\}\}$, $\{30,\{8\}\}$, $\{46,\{16\}\}$, $\{69,\{16\}\}$, $\{115,\{8,0\}\}$, $\{345,\{8\}\}$, $\{690,\{16\}\}$ |
| | 119 | 25 | 10 | 0 | 0 | $\{21,\{8\}\}$, $\{34,\{16\}\}$, $\{42,\{8\}\}$, $\{238,\{16\}\}$, $\{357,\{8\}\}$, $\{714,\{24\}\}$ |
| | 137 | 23 | 15 | 4 | 0 | $\{2,\{4,0\}\}$, $\{274,\{24\}\}$, $\{411,\{12,0\}\}$, $\{822,\{20\}\}$ |
| | 143 | 29 | 11 | 0 | 0 | $\{13,\{8\}\}$, $\{22,\{8\}\}$, $\{66,\{16\}\}$, $\{286,\{24\}\}$, $\{429,\{16\}\}$, $\{858,\{16\}\}$ |
| | 145 | 29 | 11 | 8 | 0 | $\{2,\{8,0\}\}$, $\{6,\{8\}\}$, $\{30,\{8\}\}$, $\{145,\{16\}\}$, $\{174,\{24\}\}$, $\{435,\{8,0\}\}$, $\{870,\{16\}\}$ |
| | 155 | 33 | 12 | 0 | 0 | $\{6,\{8\}\}$, $\{30,\{8\}\}$, $\{186,\{24\}\}$, $\{310,\{16\}\}$, $\{465,\{16\}\}$, $\{930,\{24\}\}$ |
| | 161 | 33 | 12 | 0 | 0 | $\{21,\{8\}\}$, $\{42,\{8\}\}$, $\{69,\{16\}\}$, $\{138,\{16\}\}$, $\{322,\{16\}\}$, $\{483,\{8,0\}\}$, $\{966,\{24\}\}$ |
| | 203 | 41 | 14 | 0 | 0 | $\{6,\{8\}\}$, $\{42,\{8\}\}$, $\{174,\{24\}\}$, $\{406,\{32\}\}$, $\{609,\{16\}\}$, $\{1218,\{24\}\}$ |
| | 235 | 49 | 16 | 0 | 0 | $\{10,\{8\}\}$, $\{30,\{8\}\}$, $\{94,\{32\}\}$, $\{141,\{16\}\}$, $\{235,\{8,0\}\}$, $\{705,\{24\}\}$, $\{1410,\{32\}\}$ |
| 10 | 1 | 0 | 4 | 0 | 4 | $\{2,\{0,2\}\}$, $\{5,\{2\}\}$, $\{10,\{2\}\}$ |
| | 3 | 1 | 4 | 0 | 4 | $\{2,\{0,4\}\}$, $\{3,\{4,0\}\}$, $\{5,\{4\}\}$, $\{30,\{4\}\}$ |
| | 7 | 1 | 5 | 0 | 8 | $\{5,\{4\}\}$, $\{10,\{4\}\}$, $\{35,\{4,0\}\}$, $\{70,\{4\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 10 | 9 | 5 | 6 | 0 | 0 | $\{2,\{0,4\}\}$, $\{5,\{4\}\}$, $\{18,\{4\}\}$, $\{45,\{4\}\}$, $\{90,\{8\}\}$ |
| | 11 | 5 | 6 | 0 | 0 | $\{2,\{0,4\}\}$, $\{10,\{4\}\}$, $\{22,\{4\}\}$, $\{110,\{12\}\}$ |
| | 13 | 3 | 6 | 0 | 8 | $\{10,\{4\}\}$, $\{13,\{4\}\}$, $\{65,\{8\}\}$, $\{130,\{4\}\}$ |
| | 17 | 7 | 7 | 0 | 0 | $\{2,\{0,4\}\}$, $\{17,\{8\}\}$, $\{85,\{4\}\}$, $\{170,\{12\}\}$ |
| | 19 | 5 | 7 | 0 | 8 | $\{2,\{0,4\}\}$, $\{10,\{4\}\}$, $\{38,\{12\}\}$, $\{190,\{4\}\}$ |
| | 21 | 9 | 6 | 0 | 8 | $\{3,\{8,0\}\}$, $\{5,\{8\}\}$, $\{35,\{8,0\}\}$, $\{42,\{8\}\}$, $\{105,\{8\}\}$, $\{210,\{8\}\}$ |
| | 23 | 9 | 8 | 0 | 0 | $\{5,\{4\}\}$, $\{10,\{4\}\}$, $\{115,\{4,0\}\}$, $\{230,\{20\}\}$ |
| | 29 | 11 | 9 | 0 | 0 | $\{5,\{4\}\}$, $\{58,\{4\}\}$, $\{145,\{8\}\}$, $\{290,\{20\}\}$ |
| | 31 | 9 | 9 | 0 | 8 | $\{62,\{16\}\}$, $\{155,\{8,0\}\}$, $\{310,\{8\}\}$ |
| | 33 | 17 | 8 | 0 | 0 | $\{2,\{0,8\}\}$, $\{30,\{8\}\}$, $\{33,\{8\}\}$, $\{110,\{24\}\}$, $\{165,\{8\}\}$, $\{330,\{8\}\}$ |
| | 37 | 11 | 10 | 0 | 8 | $\{10,\{4\}\}$, $\{37,\{4\}\}$, $\{185,\{16\}\}$, $\{370,\{12\}\}$ |
| | 39 | 17 | 8 | 0 | 8 | $\{3,\{8,0\}\}$, $\{30,\{8\}\}$, $\{65,\{16\}\}$, $\{78,\{8\}\}$, $\{195,\{8,0\}\}$, $\{390,\{16\}\}$ |
| | 41 | 15 | 11 | 0 | 0 | $\{2,\{0,4\}\}$, $\{5,\{4\}\}$, $\{10,\{4\}\}$, $\{82,\{8\}\}$, $\{205,\{8\}\}$, $\{410,\{16\}\}$ |
| | 49 | 17 | 13 | 0 | 8 | $\{5,\{4\}\}$, $\{10,\{4\}\}$, $\{98,\{16\}\}$, $\{245,\{12\}\}$, $\{490,\{12\}\}$ |
| | 51 | 25 | 10 | 0 | 0 | $\{2,\{0,8\}\}$, $\{17,\{16\}\}$, $\{30,\{8\}\}$, $\{102,\{8\}\}$, $\{170,\{24\}\}$, $\{510,\{16\}\}$ |
| | 53 | 19 | 13 | 0 | 0 | $\{10,\{4\}\}$, $\{53,\{12\}\}$, $\{265,\{8\}\}$, $\{530,\{28\}\}$ |
| | 57 | 25 | 10 | 0 | 8 | $\{2,\{0,8\}\}$, $\{3,\{8,0\}\}$, $\{38,\{24\}\}$, $\{57,\{8\}\}$, $\{285,\{16\}\}$, $\{570,\{16\}\}$ |
| | 61 | 19 | 14 | 0 | 8 | $\{5,\{4\}\}$, $\{122,\{20\}\}$, $\{305,\{16\}\}$, $\{610,\{12\}\}$ |
| | 69 | 33 | 12 | 0 | 0 | $\{5,\{8\}\}$, $\{30,\{8\}\}$, $\{138,\{16\}\}$, $\{230,\{40\}\}$, $\{345,\{8\}\}$, $\{690,\{16\}\}$ |
| | 77 | 33 | 12 | 0 | 0 | $\{10,\{8\}\}$, $\{35,\{8,0\}\}$, $\{77,\{16\}\}$, $\{110,\{24\}\}$, $\{385,\{8\}\}$, $\{770,\{32\}\}$ |
| | 93 | 41 | 14 | 0 | 8 | $\{3,\{8,0\}\}$, $\{30,\{8\}\}$, $\{62,\{32\}\}$, $\{93,\{8\}\}$, $\{155,\{16,0\}\}$, $\{465,\{16\}\}$, $\{930,\{24\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 14 | 1 | 1 | 4 | 2 | 0 | $\{2,\{2,2\}\}$, $\{14,\{4\}\}$ |
| | 3 | 3 | 5 | 0 | 0 | $\{2,\{0,4\}\}$, $\{14,\{8\}\}$, $\{21,\{4\}\}$, $\{42,\{4\}\}$ |
| | 5 | 3 | 5 | 4 | 0 | $\{2,\{4,0\}\}$, $\{14,\{8\}\}$, $\{35,\{4,0\}\}$, $\{70,\{4\}\}$ |
| | 9 | 7 | 7 | 0 | 0 | $\{2,\{0,4\}\}$, $\{9,\{4\}\}$, $\{14,\{8\}\}$, $\{18,\{4\}\}$, $\{126,\{8\}\}$ |
| | 11 | 7 | 7 | 0 | 0 | $\{2,\{0,4\}\}$, $\{11,\{4,0\}\}$, $\{22,\{4\}\}$, $\{77,\{8\}\}$, $\{154,\{8\}\}$ |
| | 13 | 7 | 7 | 4 | 0 | $\{2,\{4,0\}\}$, $\{14,\{8\}\}$, $\{91,\{4,0\}\}$, $\{182,\{12\}\}$ |
| | 15 | 13 | 7 | 0 | 0 | $\{14,\{16\}\}$, $\{21,\{8\}\}$, $\{30,\{8\}\}$, $\{35,\{8,0\}\}$, $\{105,\{8\}\}$, $\{210,\{8\}\}$ |
| | 19 | 11 | 9 | 0 | 0 | $\{2,\{0,4\}\}$, $\{14,\{8\}\}$, $\{133,\{4\}\}$, $\{266,\{20\}\}$ |
| | 23 | 13 | 10 | 0 | 0 | $\{14,\{8\}\}$, $\{46,\{8\}\}$, $\{161,\{16\}\}$, $\{322,\{8\}\}$ |
| | 25 | 15 | 11 | 4 | 0 | $\{2,\{4,0\}\}$, $\{14,\{8\}\}$, $\{25,\{4\}\}$, $\{50,\{12\}\}$, $\{350,\{16\}\}$ |
| | 29 | 15 | 11 | 4 | 0 | $\{2,\{4,0\}\}$, $\{29,\{12\}\}$, $\{58,\{4\}\}$, $\{203,\{8,0\}\}$, $\{406,\{16\}\}$ |
| | 37 | 19 | 13 | 4 | 0 | $\{2,\{4,0\}\}$, $\{37,\{4\}\}$, $\{74,\{20\}\}$, $\{259,\{8,0\}\}$, $\{518,\{16\}\}$ |
| | 39 | 29 | 11 | 0 | 0 | $\{14,\{16\}\}$, $\{42,\{8\}\}$, $\{78,\{8\}\}$, $\{182,\{24\}\}$, $\{273,\{8\}\}$, $\{546,\{24\}\}$ |
| | 43 | 23 | 15 | 0 | 0 | $\{2,\{0,4\}\}$, $\{43,\{4,0\}\}$, $\{86,\{20\}\}$, $\{301,\{8\}\}$, $\{602,\{24\}\}$ |
| | 57 | 41 | 14 | 0 | 0 | $\{2,\{0,8\}\}$, $\{14,\{16\}\}$, $\{21,\{8\}\}$, $\{57,\{8\}\}$, $\{114,\{16\}\}$, $\{266,\{40\}\}$, $\{798,\{16\}\}$ |
| 15 | 1 | 1 | 4 | 0 | 2 | $\{3,\{2,2\}\}$, $\{15,\{2,2\}\}$ |
| | 2 | 3 | 5 | 0 | 0 | $\{3,\{0,4\}\}$, $\{10,\{4\}\}$, $\{15,\{4,4\}\}$, $\{30,\{4\}\}$ |
| | 4 | 5 | 6 | 0 | 0 | $\{3,\{0,4\}\}$, $\{12,\{4\}\}$, $\{15,\{4,8\}\}$, $\{60,\{4\}\}$ |
| | 7 | 5 | 6 | 0 | 4 | $\{3,\{4,4\}\}$, $\{7,\{4,4\}\}$, $\{105,\{8\}\}$ |
| | 8 | 9 | 8 | 0 | 0 | $\{15,\{4,12\}\}$, $\{40,\{8\}\}$, $\{120,\{8\}\}$ |
| | 11 | 9 | 8 | 0 | 0 | $\{33,\{8\}\}$, $\{55,\{8,8\}\}$, $\{165,\{8\}\}$ |
| | 13 | 9 | 8 | 0 | 4 | $\{3,\{4,4\}\}$, $\{13,\{8\}\}$, $\{195,\{4,12\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 15 | 14 | 17 | 8 | 0 | 0 | $\{3,\{0,8\}\}$, $\{7,\{8,8\}\}$, $\{10,\{8\}\}$, $\{42,\{8\}\}$, $\{70,\{8\}\}$, $\{105,\{8\}\}$, $\{210,\{8\}\}$ |
| | 17 | 13 | 10 | 0 | 0 | $\{15,\{4,4\}\}$, $\{85,\{8\}\}$, $\{255,\{12,12\}\}$ |
| | 19 | 13 | 10 | 0 | 4 | $\{3,\{4,4\}\}$, $\{15,\{4,4\}\}$, $\{57,\{8\}\}$, $\{285,\{16\}\}$ |
| | 22 | 25 | 10 | 0 | 0 | $\{10,\{8\}\}$, $\{22,\{8\}\}$, $\{30,\{8\}\}$, $\{33,\{8\}\}$, $\{55,\{16,16\}\}$, $\{165,\{8\}\}$, $\{330,\{8\}\}$ |
| | 26 | 29 | 11 | 0 | 0 | $\{3,\{0,8\}\}$, $\{10,\{8\}\}$, $\{13,\{8\}\}$, $\{30,\{8\}\}$, $\{78,\{8\}\}$, $\{130,\{8\}\}$, $\{195,\{0,24\}\}$, $\{390,\{16\}\}$ |
| | 29 | 21 | 14 | 0 | 0 | $\{87,\{12,12\}\}$, $\{145,\{16\}\}$, $\{435,\{4,12\}\}$ |
| | 34 | 37 | 13 | 0 | 0 | $\{15,\{8,8\}\}$, $\{30,\{8\}\}$, $\{85,\{8\}\}$, $\{102,\{8\}\}$, $\{255,\{24,24\}\}$, $\{510,\{16\}\}$ |
| | 41 | 29 | 18 | 0 | 0 | $\{123,\{4,12\}\}$, $\{205,\{16\}\}$, $\{615,\{20,20\}\}$ |
| 21 | 1 | 1 | 5 | 4 | 0 | $\{7,\{2,2\}\}$, $\{21,\{4\}\}$ |
| | 2 | 3 | 5 | 4 | 0 | $\{2,\{4,0\}\}$, $\{7,\{4,4\}\}$, $\{21,\{4\}\}$, $\{42,\{4\}\}$ |
| | 4 | 7 | 7 | 0 | 0 | $\{4,\{4\}\}$, $\{7,\{4,8\}\}$, $\{28,\{4\}\}$, $\{84,\{8\}\}$ |
| | 5 | 5 | 7 | 8 | 0 | $\{15,\{4,4\}\}$, $\{21,\{8\}\}$, $\{105,\{8\}\}$ |
| | 10 | 17 | 8 | 8 | 0 | $\{2,\{8,0\}\}$, $\{15,\{8,8\}\}$, $\{21,\{8\}\}$, $\{30,\{8\}\}$, $\{70,\{8\}\}$, $\{105,\{8\}\}$, $\{210,\{8\}\}$ |
| | 11 | 13 | 10 | 0 | 0 | $\{7,\{4,4\}\}$, $\{21,\{8\}\}$, $\{231,\{12,12\}\}$ |
| | 13 | 13 | 11 | 8 | 0 | $\{39,\{8,8\}\}$, $\{91,\{4,12\}\}$, $\{273,\{8\}\}$ |
| | 19 | 21 | 14 | 0 | 0 | $\{21,\{8\}\}$, $\{57,\{8\}\}$, $\{133,\{8\}\}$, $\{399,\{16,16\}\}$ |
| | 22 | 37 | 13 | 0 | 0 | $\{7,\{8,8\}\}$, $\{21,\{8\}\}$, $\{22,\{8\}\}$, $\{154,\{16\}\}$, $\{231,\{24,24\}\}$, $\{462,\{8\}\}$ |
| | 26 | 41 | 14 | 8 | 0 | $\{2,\{8,0\}\}$, $\{39,\{16,16\}\}$, $\{42,\{8\}\}$, $\{78,\{8\}\}$, $\{91,\{0,24\}\}$, $\{273,\{8\}\}$, $\{546,\{24\}\}$ |
| 22 | 1 | 0 | 4 | 2 | 4 | $\{2,\{2,0\}\}$, $\{11,\{2,0\}\}$, $\{22,\{2\}\}$ |
| | 3 | 3 | 5 | 0 | 4 | $\{3,\{4,0\}\}$, $\{11,\{4,0\}\}$, $\{33,\{4\}\}$, $\{66,\{8\}\}$ |
| | 5 | 5 | 6 | 4 | 0 | $\{2,\{4,0\}\}$, $\{5,\{4\}\}$, $\{11,\{4,0\}\}$, $\{110,\{12\}\}$ |
| | 7 | 5 | 7 | 0 | 8 | $\{14,\{8\}\}$, $\{77,\{8\}\}$, $\{154,\{8\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 22 | 13 | 9 | 9 | 4 | 8 | $\{2,\{4,0\}\}$, $\{22,\{4\}\}$, $\{26,\{12\}\}$, $\{286,\{12\}\}$ |
| | 15 | 21 | 9 | 0 | 0 | $\{5,\{8\}\}$, $\{11,\{8,0\}\}$, $\{66,\{16\}\}$, $\{110,\{24\}\}$, $\{165,\{8\}\}$, $\{330,\{8\}\}$ |
| | 17 | 15 | 11 | 4 | 0 | $\{2,\{4,0\}\}$, $\{34,\{8\}\}$, $\{187,\{4,0\}\}$, $\{374,\{28\}\}$ |
| | 19 | 15 | 12 | 0 | 8 | $\{22,\{4\}\}$, $\{38,\{12\}\}$, $\{209,\{20\}\}$, $\{418,\{8\}\}$ |
| | 21 | 25 | 10 | 0 | 8 | $\{3,\{8,0\}\}$, $\{14,\{16\}\}$, $\{33,\{8\}\}$, $\{42,\{8\}\}$, $\{66,\{16\}\}$, $\{77,\{16\}\}$, $\{462,\{8\}\}$ |
| | 35 | 41 | 14 | 0 | 0 | $\{5,\{8\}\}$, $\{14,\{16\}\}$, $\{70,\{8\}\}$, $\{110,\{24\}\}$, $\{154,\{16\}\}$, $\{385,\{8\}\}$, $\{770,\{32\}\}$ |
| 26 | 1 | 2 | 5 | 0 | 0 | $\{2,\{0,2\}\}$, $\{13,\{2\}\}$, $\{26,\{6\}\}$ |
| | 3 | 5 | 6 | 0 | 0 | $\{2,\{0,4\}\}$, $\{6,\{4\}\}$, $\{26,\{12\}\}$, $\{78,\{4\}\}$ |
| | 5 | 7 | 7 | 0 | 0 | $\{5,\{4\}\}$, $\{26,\{12\}\}$, $\{65,\{8\}\}$, $\{130,\{4\}\}$ |
| | 7 | 9 | 8 | 0 | 0 | $\{13,\{4\}\}$, $\{26,\{12\}\}$, $\{91,\{4,0\}\}$, $\{182,\{12\}\}$ |
| | 9 | 13 | 10 | 0 | 0 | $\{2,\{0,4\}\}$, $\{18,\{4\}\}$, $\{26,\{12\}\}$, $\{117,\{8\}\}$, $\{234,\{12\}\}$ |
| | 15 | 25 | 10 | 0 | 0 | $\{5,\{8\}\}$, $\{6,\{8\}\}$, $\{26,\{24\}\}$, $\{65,\{16\}\}$, $\{195,\{8,0\}\}$, $\{390,\{16\}\}$ |
| | 17 | 19 | 13 | 0 | 0 | $\{2,\{0,4\}\}$, $\{13,\{4\}\}$, $\{26,\{12\}\}$, $\{34,\{8\}\}$, $\{221,\{16\}\}$, $\{442,\{8\}\}$ |
| | 21 | 33 | 12 | 0 | 0 | $\{6,\{8\}\}$, $\{21,\{8\}\}$, $\{26,\{24\}\}$, $\{182,\{24\}\}$, $\{273,\{8\}\}$, $\{546,\{24\}\}$ |
| 33 | 1 | 1 | 5 | 4 | 2 | $\{3,\{2,2\}\}$, $\{33,\{4\}\}$ |
| | 2 | 5 | 6 | 4 | 0 | $\{2,\{4,0\}\}$, $\{3,\{0,4\}\}$, $\{22,\{4\}\}$, $\{33,\{4\}\}$, $\{66,\{8\}\}$ |
| | 5 | 9 | 9 | 8 | 0 | $\{15,\{4,4\}\}$, $\{55,\{8,8\}\}$, $\{165,\{8\}\}$ |
| | 7 | 13 | 10 | 0 | 4 | $\{3,\{4,4\}\}$, $\{33,\{8\}\}$, $\{231,\{12,12\}\}$ |
| | 10 | 29 | 11 | 8 | 0 | $\{2,\{8,0\}\}$, $\{15,\{8,8\}\}$, $\{55,\{16,16\}\}$, $\{66,\{16\}\}$, $\{165,\{8\}\}$, $\{330,\{8\}\}$ |
| | 14 | 41 | 14 | 0 | 0 | $\{3,\{0,8\}\}$, $\{33,\{8\}\}$, $\{42,\{8\}\}$, $\{66,\{16\}\}$, $\{154,\{16\}\}$, $\{231,\{24,24\}\}$, $\{462,\{8\}\}$ |
| 34 | 1 | 1 | 5 | 0 | 4 | $\{17,\{4\}\}$, $\{34,\{4\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 34 | 3 | 5 | 6 | 0 | 4 | $\{3,\{4,0\}\}$, $\{6,\{4\}\}$, $\{17,\{8\}\}$, $\{51,\{4,0\}\}$, $\{102,\{4\}\}$ |
| | 5 | 9 | 8 | 0 | 0 | $\{5,\{4\}\}$, $\{10,\{4\}\}$, $\{34,\{8\}\}$, $\{85,\{4\}\}$, $\{170,\{12\}\}$ |
| | 7 | 9 | 9 | 0 | 8 | $\{14,\{8\}\}$, $\{17,\{8\}\}$, $\{34,\{8\}\}$, $\{238,\{8\}\}$ |
| | 11 | 17 | 12 | 0 | 0 | $\{11,\{4,0\}\}$, $\{17,\{8\}\}$, $\{22,\{4\}\}$, $\{187,\{4,0\}\}$, $\{374,\{28\}\}$ |
| 35 | 1 | 3 | 6 | 0 | 0 | $\{7,\{2,2\}\}$, $\{35,\{2,6\}\}$ |
| | 2 | 7 | 7 | 0 | 0 | $\{2,\{0,4\}\}$, $\{7,\{4,4\}\}$, $\{35,\{0,12\}\}$, $\{70,\{4\}\}$ |
| | 3 | 9 | 8 | 0 | 0 | $\{15,\{4,4\}\}$, $\{35,\{4,12\}\}$, $\{105,\{8\}\}$ |
| | 4 | 13 | 10 | 0 | 0 | $\{7,\{4,8\}\}$, $\{28,\{4\}\}$, $\{35,\{0,12\}\}$, $\{140,\{12\}\}$ |
| | 6 | 25 | 10 | 0 | 0 | $\{2,\{0,8\}\}$, $\{15,\{8,8\}\}$, $\{30,\{8\}\}$, $\{35,\{0,24\}\}$, $\{42,\{8\}\}$, $\{105,\{8\}\}$, $\{210,\{8\}\}$ |
| | 13 | 29 | 18 | 0 | 0 | $\{35,\{4,12\}\}$, $\{65,\{16\}\}$, $\{455,\{20,20\}\}$ |
| 38 | 1 | 2 | 5 | 2 | 0 | $\{2,\{2,0\}\}$, $\{19,\{2,0\}\}$, $\{38,\{6\}\}$ |
| | 3 | 7 | 7 | 0 | 0 | $\{6,\{4\}\}$, $\{38,\{12\}\}$, $\{57,\{4\}\}$, $\{114,\{8\}\}$ |
| | 7 | 13 | 10 | 0 | 0 | $\{19,\{4,0\}\}$, $\{38,\{12\}\}$, $\{133,\{4\}\}$, $\{266,\{20\}\}$ |
| | 13 | 21 | 14 | 4 | 0 | $\{2,\{4,0\}\}$, $\{26,\{12\}\}$, $\{38,\{12\}\}$, $\{494,\{28\}\}$ |
| 39 | 1 | 3 | 6 | 0 | 0 | $\{13,\{4\}\}$, $\{39,\{4,4\}\}$ |
| | 2 | 7 | 7 | 0 | 0 | $\{6,\{4\}\}$, $\{13,\{4\}\}$, $\{39,\{8,8\}\}$, $\{78,\{4\}\}$ |
| | 4 | 13 | 10 | 0 | 0 | $\{39,\{8,16\}\}$, $\{52,\{8\}\}$, $\{156,\{8\}\}$ |
| | 5 | 13 | 10 | 0 | 0 | $\{15,\{4,4\}\}$, $\{39,\{8,8\}\}$, $\{195,\{4,12\}\}$ |
| | 7 | 17 | 12 | 0 | 0 | $\{7,\{4,4\}\}$, $\{13,\{8\}\}$, $\{21,\{8\}\}$, $\{91,\{4,12\}\}$, $\{273,\{8\}\}$ |
| | 10 | 37 | 13 | 0 | 0 | $\{6,\{8\}\}$, $\{15,\{8,8\}\}$, $\{39,\{16,16\}\}$, $\{130,\{8\}\}$, $\{195,\{0,24\}\}$, $\{390,\{16\}\}$ |
| 46 | 1 | 1 | 5 | 2 | 4 | $\{2,\{2,2\}\}$, $\{46,\{4\}\}$ |
| | 3 | 7 | 7 | 0 | 4 | $\{2,\{0,4\}\}$, $\{3,\{4,0\}\}$, $\{6,\{4\}\}$, $\{69,\{8\}\}$, $\{138,\{8\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m,\text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 46 | 5 | 11 | 9 | 4 | 0 | $\{2,\{4,0\}\}$, $\{46,\{8\}\}$, $\{115,\{4,0\}\}$, $\{230,\{20\}\}$ |
| 51 | 1 | 3 | 6 | 0 | 2 | $\{3,\{2,2\}\}$, $\{51,\{2,6\}\}$ |
| | 2 | 9 | 8 | 0 | 0 | $\{3,\{0,4\}\}$, $\{6,\{4\}\}$, $\{34,\{8\}\}$, $\{51,\{0,12\}\}$, $\{102,\{4\}\}$ |
| | 5 | 17 | 12 | 0 | 0 | $\{51,\{4,12\}\}$, $\{85,\{8\}\}$, $\{255,\{12,12\}\}$ |
| | 10 | 49 | 16 | 0 | 0 | $\{6,\{8\}\}$, $\{10,\{8\}\}$, $\{34,\{16\}\}$, $\{51,\{0,24\}\}$, $\{85,\{8\}\}$, $\{255,\{24,24\}\}$, $\{510,\{16\}\}$ |
| 55 | 1 | 3 | 7 | 0 | 4 | $\{5,\{4\}\}$, $\{55,\{4,4\}\}$ |
| | 2 | 11 | 9 | 0 | 0 | $\{5,\{4\}\}$, $\{22,\{4\}\}$, $\{55,\{8,8\}\}$, $\{110,\{12\}\}$ |
| | 3 | 13 | 10 | 0 | 4 | $\{3,\{4,4\}\}$, $\{5,\{8\}\}$, $\{15,\{4,4\}\}$, $\{33,\{8\}\}$, $\{165,\{8\}\}$ |
| 57 | 1 | 3 | 7 | 4 | 0 | $\{19,\{2,6\}\}$, $\{57,\{4\}\}$ |
| | 2 | 9 | 8 | 4 | 0 | $\{2,\{4,0\}\}$, $\{6,\{4\}\}$, $\{19,\{0,12\}\}$, $\{57,\{4\}\}$, $\{114,\{8\}\}$ |
| | 7 | 25 | 16 | 0 | 0 | $\{7,\{4,4\}\}$, $\{19,\{4,12\}\}$, $\{133,\{8\}\}$, $\{399,\{16,16\}\}$ |
| 58 | 1 | 2 | 6 | 0 | 4 | $\{2,\{0,2\}\}$, $\{29,\{6\}\}$, $\{58,\{2\}\}$ |
| | 3 | 9 | 8 | 0 | 4 | $\{2,\{0,4\}\}$, $\{3,\{4,0\}\}$, $\{29,\{12\}\}$, $\{174,\{12\}\}$ |
| | 5 | 15 | 11 | 0 | 0 | $\{10,\{4\}\}$, $\{29,\{12\}\}$, $\{145,\{8\}\}$, $\{290,\{20\}\}$ |
| 62 | 1 | 3 | 6 | 2 | 0 | $\{2,\{2,2\}\}$, $\{62,\{8\}\}$ |
| | 3 | 11 | 9 | 0 | 0 | $\{2,\{0,4\}\}$, $\{62,\{16\}\}$, $\{93,\{4\}\}$, $\{186,\{12\}\}$ |
| | 7 | 21 | 14 | 0 | 0 | $\{14,\{8\}\}$, $\{62,\{16\}\}$, $\{217,\{8\}\}$, $\{434,\{24\}\}$ |
| 65 | 1 | 5 | 8 | 0 | 0 | $\{5,\{4\}\}$, $\{13,\{4\}\}$, $\{65,\{8\}\}$ |
| | 3 | 17 | 12 | 0 | 0 | $\{5,\{8\}\}$, $\{15,\{4,4\}\}$, $\{65,\{16\}\}$, $\{195,\{4,12\}\}$ |
| 69 | 1 | 3 | 7 | 4 | 2 | $\{3,\{2,2\}\}$, $\{69,\{8\}\}$ |
| | 2 | 11 | 9 | 4 | 0 | $\{2,\{4,0\}\}$, $\{3,\{0,4\}\}$, $\{6,\{4\}\}$, $\{46,\{8\}\}$, $\{69,\{8\}\}$, $\{138,\{8\}\}$ |
| 74 | 1 | 4 | 7 | 0 | 0 | $\{2,\{0,2\}\}$, $\{37,\{2\}\}$, $\{74,\{10\}\}$ |
| | 3 | 13 | 10 | 0 | 0 | $\{2,\{0,4\}\}$, $\{6,\{4\}\}$, $\{74,\{20\}\}$, $\{222,\{12\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 74 | 5 | 19 | 13 | 0 | 0 | $\{5,\{4\}\}$, $\{74,\{20\}\}$, $\{185,\{16\}\}$, $\{370,\{12\}\}$ |
| 77 | 1 | 5 | 9 | 4 | 0 | $\{11,\{2,6\}\}$, $\{77,\{8\}\}$ |
| | 2 | 15 | 11 | 4 | 0 | $\{2,\{4,0\}\}$, $\{11,\{0,12\}\}$, $\{14,\{8\}\}$, $\{22,\{4\}\}$, $\{77,\{8\}\}$, $\{154,\{8\}\}$ |
| | 3 | 21 | 14 | 0 | 0 | $\{11,\{4,12\}\}$, $\{77,\{16\}\}$, $\{231,\{12,12\}\}$ |
| 82 | 1 | 3 | 7 | 0 | 4 | $\{41,\{8\}\}$, $\{82,\{4\}\}$ |
| | 3 | 13 | 10 | 0 | 4 | $\{3,\{4,0\}\}$, $\{6,\{4\}\}$, $\{41,\{16\}\}$, $\{123,\{4,0\}\}$, $\{246,\{12\}\}$ |
| 85 | 1 | 5 | 9 | 0 | 4 | $\{5,\{4\}\}$, $\{17,\{8\}\}$, $\{85,\{4\}\}$ |
| | 3 | 21 | 14 | 0 | 4 | $\{3,\{4,4\}\}$, $\{5,\{8\}\}$, $\{17,\{16\}\}$, $\{255,\{12,12\}\}$ |
| 86 | 1 | 4 | 7 | 2 | 0 | $\{2,\{2,0\}\}$, $\{43,\{2,0\}\}$, $\{86,\{10\}\}$ |
| | 3 | 15 | 11 | 0 | 0 | $\{6,\{4\}\}$, $\{86,\{20\}\}$, $\{129,\{12\}\}$, $\{258,\{8\}\}$ |
| 87 | 1 | 5 | 8 | 0 | 2 | $\{3,\{2,2\}\}$, $\{87,\{6,6\}\}$ |
| | 2 | 15 | 11 | 0 | 0 | $\{3,\{0,4\}\}$, $\{58,\{4\}\}$, $\{87,\{12,12\}\}$, $\{174,\{12\}\}$ |
| 93 | 1 | 5 | 9 | 4 | 0 | $\{31,\{6,6\}\}$, $\{93,\{4\}\}$ |
| | 2 | 15 | 11 | 4 | 0 | $\{2,\{4,0\}\}$, $\{31,\{12,12\}\}$, $\{93,\{4\}\}$, $\{186,\{12\}\}$ |
| 94 | 1 | 3 | 7 | 2 | 4 | $\{2,\{2,2\}\}$, $\{94,\{8\}\}$ |
| 95 | 1 | 7 | 10 | 0 | 0 | $\{5,\{4\}\}$, $\{95,\{8,8\}\}$ |
| | 2 | 19 | 13 | 0 | 0 | $\{5,\{4\}\}$, $\{38,\{12\}\}$, $\{95,\{16,16\}\}$, $\{190,\{4\}\}$ |
| | 3 | 25 | 16 | 0 | 0 | $\{5,\{8\}\}$, $\{57,\{8\}\}$, $\{95,\{16,16\}\}$, $\{285,\{16\}\}$ |
| 106 | 1 | 4 | 8 | 0 | 4 | $\{2,\{0,2\}\}$, $\{53,\{6\}\}$, $\{106,\{6\}\}$ |
| 111 | 1 | 7 | 10 | 0 | 0 | $\{37,\{4\}\}$, $\{111,\{8,8\}\}$ |
| | 2 | 19 | 13 | 0 | 0 | $\{6,\{4\}\}$, $\{37,\{4\}\}$, $\{111,\{16,16\}\}$, $\{222,\{12\}\}$ |
| 115 | 1 | 7 | 11 | 0 | 4 | $\{23,\{6,6\}\}$, $\{115,\{2,6\}\}$ |
| | 2 | 23 | 15 | 0 | 0 | $\{2,\{0,4\}\}$, $\{23,\{12,12\}\}$, $\{115,\{0,12\}\}$, $\{230,\{20\}\}$ |
| 118 | 1 | 4 | 8 | 2 | 4 | $\{2,\{2,0\}\}$, $\{59,\{6,0\}\}$, $\{118,\{6\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 119 | 1 | 9 | 12 | 0 | 0 | $\{7,\{2,2\}\}$, $\{119,\{10,10\}\}$ |
| | 2 | 25 | 16 | 0 | 0 | $\{7,\{4,4\}\}$, $\{14,\{8\}\}$, $\{119,\{20,20\}\}$, $\{238,\{8\}\}$ |
| 122 | 1 | 6 | 9 | 0 | 0 | $\{2,\{0,2\}\}$, $\{61,\{6\}\}$, $\{122,\{10\}\}$ |
| 129 | 1 | 7 | 11 | 4 | 0 | $\{43,\{2,6\}\}$, $\{129,\{12\}\}$ |
| 134 | 1 | 6 | 9 | 2 | 0 | $\{2,\{2,0\}\}$, $\{67,\{2,0\}\}$, $\{134,\{14\}\}$ |
| | 3 | 23 | 15 | 0 | 0 | $\{6,\{4\}\}$, $\{134,\{28\}\}$, $\{201,\{12\}\}$, $\{402,\{16\}\}$ |
| 143 | 1 | 11 | 14 | 0 | 0 | $\{11,\{2,6\}\}$, $\{143,\{10,10\}\}$ |
| | 2 | 31 | 19 | 0 | 0 | $\{11,\{0,12\}\}$, $\{26,\{12\}\}$, $\{143,\{20,20\}\}$, $\{286,\{12\}\}$ |
| 146 | 1 | 7 | 10 | 0 | 0 | $\{73,\{4\}\}$, $\{146,\{16\}\}$ |
| 159 | 1 | 9 | 12 | 0 | 2 | $\{3,\{2,2\}\}$, $\{159,\{10,10\}\}$ |
| | 2 | 27 | 17 | 0 | 0 | $\{3,\{0,4\}\}$, $\{106,\{12\}\}$, $\{159,\{20,20\}\}$, $\{318,\{12\}\}$ |
| 161 | 1 | 11 | 15 | 4 | 0 | $\{23,\{6,6\}\}$, $\{161,\{16\}\}$ |
| 166 | 1 | 6 | 10 | 2 | 4 | $\{2,\{2,0\}\}$, $\{83,\{6,0\}\}$, $\{166,\{10\}\}$ |
| 178 | 1 | 7 | 11 | 0 | 4 | $\{89,\{12\}\}$, $\{178,\{8\}\}$ |
| 183 | 1 | 11 | 14 | 0 | 0 | $\{61,\{12\}\}$, $\{183,\{8,8\}\}$ |
| 194 | 1 | 9 | 12 | 0 | 0 | $\{97,\{4\}\}$, $\{194,\{20\}\}$ |
| 202 | 1 | 8 | 12 | 0 | 4 | $\{2,\{0,2\}\}$, $\{101,\{14\}\}$, $\{202,\{6\}\}$ |
| 206 | 1 | 9 | 12 | 2 | 0 | $\{2,\{2,2\}\}$, $\{206,\{20\}\}$ |
| 210 | 1 | 5 | 5 | 0 | 0 | $\{30,\{8\}\}$, $\{42,\{8\}\}$, $\{70,\{8\}\}$, $\{105,\{8\}\}$, $\{210,\{8\}\}$ |
| | 11 | 49 | 10 | 0 | 0 | $\{22,\{16\}\}$, $\{30,\{16\}\}$, $\{105,\{16\}\}$, $\{165,\{16\}\}$, $\{330,\{16\}\}$, $\{385,\{16\}\}$, $\{462,\{16\}\}$, $\{1155,\{16,0\}\}$, $\{2310,\{32\}\}$ |
| | 13 | 57 | 11 | 0 | 0 | $\{30,\{16\}\}$, $\{42,\{16\}\}$, $\{78,\{16\}\}$, $\{105,\{16\}\}$, $\{130,\{16\}\}$, $\{273,\{16\}\}$, $\{910,\{32\}\}$, $\{1365,\{16\}\}$, $\{2730,\{32\}\}$ |
| | 17 | 73 | 13 | 0 | 0 | $\{30,\{16\}\}$, $\{42,\{16\}\}$, $\{70,\{16\}\}$, $\{85,\{16\}\}$, $\{102,\{16\}\}$, $\{238,\{32\}\}$, $\{357,\{16\}\}$, $\{595,\{16,0\}\}$, $\{1785,\{32\}\}$, $\{3570,\{32\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 210 | 19 | 81 | 14 | 0 | 0 | $\{57,\{16\}\}$, $\{70,\{16\}\}$, $\{105,\{16\}\}$, $\{133,\{16\}\}$, $\{190,\{16\}\}$, $\{798,\{32\}\}$, $\{1330,\{48\}\}$, $\{1995,\{16,0\}\}$, $\{3990,\{48\}\}$ |
| 215 | 1 | 15 | 18 | 0 | 0 | $\{43,\{2,6\}\}$, $\{215,\{14,14\}\}$ |
| 237 | 1 | 13 | 17 | 4 | 0 | $\{79,\{10,10\}\}$, $\{237,\{12\}\}$ |
| 314 | 1 | 14 | 17 | 0 | 0 | $\{2,\{0,2\}\}$, $\{157,\{6\}\}$, $\{314,\{26\}\}$ |
| 330 | 1 | 5 | 5 | 0 | 8 | $\{3,\{8,0\}\}$, $\{22,\{8\}\}$, $\{33,\{8\}\}$, $\{165,\{8\}\}$, $\{330,\{8\}\}$ |
| | 7 | 49 | 10 | 0 | 16 | $\{3,\{16,0\}\}$, $\{33,\{16\}\}$, $\{42,\{16\}\}$, $\{70,\{16\}\}$, $\{210,\{16\}\}$, $\{385,\{16\}\}$, $\{462,\{16\}\}$, $\{1155,\{16,0\}\}$, $\{2310,\{32\}\}$ |
| | 13 | 89 | 15 | 0 | 16 | $\{3,\{16,0\}\}$, $\{22,\{16\}\}$, $\{78,\{16\}\}$, $\{130,\{16\}\}$, $\{165,\{16\}\}$, $\{390,\{32\}\}$, $\{715,\{16,0\}\}$, $\{858,\{32\}\}$, $\{2145,\{32\}\}$, $\{4290,\{48\}\}$ |
| 390 | 1 | 9 | 6 | 0 | 0 | $\{13,\{8\}\}$, $\{78,\{8\}\}$, $\{130,\{8\}\}$, $\{195,\{8,0\}\}$, $\{390,\{16\}\}$ |
| | 7 | 65 | 12 | 0 | 0 | $\{13,\{16\}\}$, $\{70,\{16\}\}$, $\{195,\{16,0\}\}$, $\{210,\{16\}\}$, $\{273,\{16\}\}$, $\{390,\{32\}\}$, $\{910,\{32\}\}$, $\{1365,\{16\}\}$, $\{2730,\{32\}\}$ |
| 462 | 1 | 9 | 6 | 8 | 0 | $\{2,\{8,0\}\}$, $\{22,\{8\}\}$, $\{42,\{8\}\}$, $\{154,\{16\}\}$, $\{462,\{8\}\}$ |
| | 5 | 57 | 11 | 16 | 0 | $\{2,\{16,0\}\}$, $\{70,\{16\}\}$, $\{154,\{32\}\}$, $\{165,\{16\}\}$, $\{210,\{16\}\}$, $\{330,\{16\}\}$, $\{385,\{16\}\}$, $\{1155,\{16,0\}\}$, $\{2310,\{32\}\}$ |
| 510 | 1 | 9 | 6 | 0 | 8 | $\{3,\{8,0\}\}$, $\{10,\{8\}\}$, $\{85,\{8\}\}$, $\{102,\{8\}\}$, $\{510,\{16\}\}$ |
| | 7 | 81 | 14 | 0 | 16 | $\{3,\{16,0\}\}$, $\{10,\{16\}\}$, $\{105,\{16\}\}$, $\{210,\{16\}\}$, $\{238,\{32\}\}$, $\{357,\{16\}\}$, $\{510,\{32\}\}$, $\{595,\{16,0\}\}$, $\{1785,\{32\}\}$, $\{3570,\{32\}\}$ |
| 546 | 1 | 13 | 7 | 0 | 0 | $\{21,\{8\}\}$, $\{78,\{8\}\}$, $\{91,\{8,0\}\}$, $\{273,\{8\}\}$, $\{546,\{24\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 546 | 5 | 73 | 13 | 0 | 0 | $\{21,\{16\}\}$, $\{70,\{16\}\}$, $\{91,\{16,0\}\}$, $\{130,\{16\}\}$, $\{210,\{16\}\}$, $\{546,\{48\}\}$, $\{910,\{32\}\}$, $\{1365,\{16\}\}$, $\{2730,\{32\}\}$ |
| 570 | 1 | 13 | 7 | 0 | 0 | $\{30,\{8\}\}$, $\{57,\{8\}\}$, $\{190,\{8\}\}$, $\{285,\{16\}\}$, $\{570,\{16\}\}$ |
| | 7 | 97 | 16 | 0 | 0 | $\{42,\{16\}\}$, $\{133,\{16\}\}$, $\{210,\{16\}\}$, $\{285,\{32\}\}$, $\{570,\{32\}\}$, $\{798,\{32\}\}$, $\{1330,\{48\}\}$, $\{1995,\{16,0\}\}$, $\{3990,\{48\}\}$ |
| 690 | 1 | 13 | 7 | 0 | 8 | $\{3,\{8,0\}\}$, $\{115,\{8,0\}\}$, $\{138,\{16\}\}$, $\{345,\{8\}\}$, $\{690,\{16\}\}$ |
| 714 | 1 | 17 | 8 | 0 | 0 | $\{51,\{8,0\}\}$, $\{102,\{8\}\}$, $\{238,\{16\}\}$, $\{357,\{8\}\}$, $\{714,\{24\}\}$ |
| 770 | 1 | 21 | 9 | 0 | 0 | $\{22,\{8\}\}$, $\{70,\{8\}\}$, $\{77,\{16\}\}$, $\{385,\{8\}\}$, $\{770,\{32\}\}$ |
| | 3 | 81 | 14 | 0 | 0 | $\{42,\{16\}\}$, $\{77,\{32\}\}$, $\{165,\{16\}\}$, $\{210,\{16\}\}$, $\{330,\{16\}\}$, $\{462,\{16\}\}$, $\{770,\{64\}\}$, $\{1155,\{16,0\}\}$, $\{2310,\{32\}\}$ |
| 798 | 1 | 17 | 8 | 8 | 0 | $\{2,\{8,0\}\}$, $\{42,\{8\}\}$, $\{57,\{8\}\}$, $\{114,\{16\}\}$, $\{133,\{8\}\}$, $\{798,\{16\}\}$ |
| 858 | 1 | 21 | 9 | 0 | 0 | $\{33,\{8\}\}$, $\{78,\{8\}\}$, $\{286,\{24\}\}$, $\{429,\{16\}\}$, $\{858,\{16\}\}$ |
| 870 | 1 | 17 | 8 | 0 | 8 | $\{3,\{8,0\}\}$, $\{10,\{8\}\}$, $\{58,\{8\}\}$, $\{145,\{16\}\}$, $\{435,\{8,0\}\}$, $\{870,\{16\}\}$ |
| 910 | 1 | 25 | 10 | 0 | 0 | $\{2,\{0,8\}\}$, $\{65,\{16\}\}$, $\{70,\{8\}\}$, $\{130,\{8\}\}$, $\{182,\{24\}\}$, $\{910,\{16\}\}$ |
| 930 | 1 | 21 | 9 | 0 | 0 | $\{10,\{8\}\}$, $\{93,\{8\}\}$, $\{310,\{16\}\}$, $\{465,\{16\}\}$, $\{930,\{24\}\}$ |
| 966 | 1 | 21 | 9 | 8 | 0 | $\{2,\{8,0\}\}$, $\{46,\{16\}\}$, $\{322,\{16\}\}$, $\{483,\{8,0\}\}$, $\{966,\{24\}\}$ |
| 1110 | 1 | 25 | 10 | 0 | 0 | $\{37,\{8\}\}$, $\{222,\{24\}\}$, $\{370,\{24\}\}$, $\{555,\{8,0\}\}$, $\{1110,\{16\}\}$ |
| 1122 | 1 | 25 | 10 | 0 | 8 | $\{3,\{8,0\}\}$, $\{22,\{8\}\}$, $\{34,\{16\}\}$, $\{102,\{8\}\}$, $\{187,\{8,0\}\}$, $\{561,\{16\}\}$, $\{1122,\{16\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 1190 | 1 | 33 | 12 | 0 | 0 | $\{85,\{8\}\}$, $\{170,\{24\}\}$, $\{238,\{16\}\}$, $\{595,\{8,0\}\}$, $\{1190,\{40\}\}$ |
| 1218 | 1 | 29 | 11 | 0 | 0 | $\{21,\{8\}\}$, $\{58,\{8\}\}$, $\{406,\{32\}\}$, $\{609,\{16\}\}$, $\{1218,\{24\}\}$ |
| 1230 | 1 | 25 | 10 | 0 | 8 | $\{3,\{8,0\}\}$, $\{30,\{8\}\}$, $\{82,\{16\}\}$, $\{123,\{8,0\}\}$, $\{205,\{16\}\}$, $\{1230,\{24\}\}$ |
| 1254 | 1 | 29 | 11 | 8 | 0 | $\{2,\{8,0\}\}$, $\{66,\{16\}\}$, $\{114,\{16\}\}$, $\{418,\{16\}\}$, $\{627,\{8,0\}\}$, $\{1254,\{24\}\}$ |
| 1290 | 1 | 29 | 11 | 0 | 0 | $\{10,\{8\}\}$, $\{43,\{8,0\}\}$, $\{258,\{16\}\}$, $\{430,\{24\}\}$, $\{645,\{16\}\}$, $\{1290,\{16\}\}$ |
| 1302 | 1 | 29 | 11 | 8 | 0 | $\{2,\{8,0\}\}$, $\{93,\{8\}\}$, $\{186,\{24\}\}$, $\{217,\{16\}\}$, $\{651,\{16,0\}\}$, $\{1302,\{16\}\}$ |
| 1326 | 1 | 33 | 12 | 0 | 0 | $\{6,\{8\}\}$, $\{34,\{16\}\}$, $\{78,\{8\}\}$, $\{102,\{8\}\}$, $\{442,\{16\}\}$, $\{1326,\{40\}\}$ |
| 1410 | 1 | 29 | 11 | 0 | 8 | $\{3,\{8,0\}\}$, $\{235,\{8,0\}\}$, $\{282,\{16\}\}$, $\{705,\{24\}\}$, $\{1410,\{32\}\}$ |
| 1590 | 1 | 33 | 12 | 0 | 8 | $\{3,\{8,0\}\}$, $\{30,\{8\}\}$, $\{265,\{16\}\}$, $\{318,\{24\}\}$, $\{795,\{8,0\}\}$, $\{1590,\{32\}\}$ |
| 1722 | 1 | 41 | 14 | 0 | 0 | $\{123,\{8,0\}\}$, $\{246,\{24\}\}$, $\{574,\{32\}\}$, $\{861,\{24\}\}$, $\{1722,\{24\}\}$ |
| 1770 | 1 | 37 | 13 | 0 | 8 | $\{3,\{8,0\}\}$, $\{118,\{24\}\}$, $\{177,\{8\}\}$, $\{885,\{24\}\}$, $\{1770,\{40\}\}$ |
| 1794 | 1 | 45 | 15 | 0 | 0 | $\{6,\{8\}\}$, $\{13,\{8\}\}$, $\{46,\{16\}\}$, $\{78,\{8\}\}$, $\{138,\{16\}\}$, $\{598,\{16\}\}$, $\{897,\{16\}\}$, $\{1794,\{32\}\}$ |
| 1914 | 1 | 45 | 15 | 0 | 8 | $\{3,\{8,0\}\}$, $\{58,\{8\}\}$, $\{66,\{16\}\}$, $\{174,\{24\}\}$, $\{957,\{16\}\}$, $\{1914,\{48\}\}$ |
| 1938 | 1 | 49 | 16 | 0 | 0 | $\{6,\{8\}\}$, $\{57,\{8\}\}$, $\{102,\{8\}\}$, $\{114,\{16\}\}$, $\{646,\{32\}\}$, $\{969,\{24\}\}$, $\{1938,\{32\}\}$ |
| 1974 | 1 | 45 | 15 | 8 | 0 | $\{2,\{8,0\}\}$, $\{21,\{8\}\}$, $\{42,\{8\}\}$, $\{141,\{16\}\}$, $\{282,\{16\}\}$, $\{658,\{16\}\}$, $\{987,\{16,0\}\}$, $\{1974,\{32\}\}$ |
| 2010 | 1 | 45 | 15 | 0 | 0 | $\{10,\{8\}\}$, $\{67,\{8,0\}\}$, $\{402,\{32\}\}$, $\{670,\{24\}\}$, $\{1005,\{16\}\}$, $\{2010,\{32\}\}$ |

| $D$ | $N$ | $g$ | $n_W$ | $e_2$ | $e_3$ | $\{m, \text{fixed points by } \omega_m\}$ |
|---|---|---|---|---|---|---|
| 2130 | 1 | 45 | 15 | 0 | 8 | $\{3,\{8,0\}\}$, $\{10,\{8\}\}$, $\{30,\{8\}\}$, $\{142,\{16\}\}$, $\{213,\{16\}\}$, $\{355,\{16,0\}\}$, $\{1065,\{16\}\}$, $\{2130,\{32\}\}$ |

In order to minimize the required computations to prove that the previous table contains all the curves with maximal Atkin-Lehner quotient of genus 0, we will reduce to the squarefree level case found in [LMR06] thanks to the following lemma.

**Lemma 4.1.** *Consider the Shimura curve $X(D,N)^+$ and write $N = N'l^{2r}$ such that $l^2 \nmid N'$ for a certain prime $l$ and an integer $r > 0$. If $g(X(D,N)^+) = 0$, then $g(X(D,N')^+) = 0$.*

*Proof.* The Euler characteristic of $X(D,N')^+$ can be computed by the general formula which relates the genus, volume and order of the elliptic cycles (cf. Theorem 1.10),

$$\chi(X(D,N')^+) = -V(D,N')^+ + \frac{1}{2}(d_2' + e_2' + \frac{3}{2}e_4') + \frac{2}{3}e_3' + \frac{5}{6}e_6',$$

where $V(D,N')^+$ is the normalized volume of the quotient $\Gamma(D,N')^+\backslash\mathcal{H}$ and $e_i'$ denotes the number of elliptic cycles of order $i$ coming from elliptic cycles in $X(D,N')$ and $d_2'$ the number of elliptic cycles of order 2 which come from ordinary cycles in $X(D,N')$. Note that in such a quotient the only possible orders for elliptic points are $2,3,4,6$. Moreover, using the formulas for the number of elliptic cycles of orders 2 and 3 in the curve $X(D,N)$ and the formulas for the number of fixed points by the Atkin-Lehner involutions, we obtain that

$$e_6' = \begin{cases} 1, & \text{if there exist elliptic cycles of order 3 in } X(D,N') \text{ and } 3\|DN', \\ 0, & \text{otherwise,} \end{cases}$$

$$e_4' = \begin{cases} 1, & \text{if there exist elliptic cycles of order 2 in } X(D,N') \text{ and } 2\|DN', \\ 0, & \text{otherwise,} \end{cases}$$

$$e_3' = \begin{cases} 1, & \text{if there exist elliptic cycles of order 3 in } X(D,N') \text{ and } 3 \nmid DN', \\ 0, & \text{otherwise,} \end{cases}$$

$$e_2' = \begin{cases} 1, & \text{if there exist elliptic cycles of order 2 in } X(D,N') \text{ and } 2 \nmid DN', \\ 0, & \text{otherwise,} \end{cases}$$

and

$$d_2' = 2^{-k+1} \sum_{\substack{m \| DN' \\ m \neq 1}} \sum_R h(R) \prod_{p | \frac{DN'}{m}} \nu_p(R, \mathcal{O}(D, N')),$$

where $k$ denotes the number of different primes dividing $DN'$ and for each $m$ the sum runs over the corresponding orders $R$ described in Theorem 4.1, except for $\mathbb{Z}[i]$ if $m = 2$ and $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ if $m = 3$. Observe, in particular, that $e_2' + e_4' \leq 1$ and $e_3' + e_6' \leq 1$ and therefore, $g(X(D,N)^+) = 0$ if and only if $2V(D,N)^+ < d_2' + e_2' + \frac{3}{2}e_4'$ or equivalently $2V(D,N)^+ < d_2'$.

We will separate the proof in two cases depending on the parity of $\nu_l N$.

In first place we will deal with the case of even valuation. This means that $N = N' l^{2r}$ and $l \nmid N'$. Then,

$$\chi(X(D,N)^+) = -\frac{V(D,N')^+ \psi(l^{2r})}{2} + \frac{1}{2}(d_2 + e_2 + \frac{3}{2}e_4) + \frac{2}{3}e_3 + \frac{5}{6}e_6,$$

where $e_i, d_2$ are defined as above and can be computed by the above formulas replacing $N'$ by $N$. In particular $e_i \leq e_i'$ for $i = 2, 3, 4, 6$ and

$$\begin{aligned} d_2 = 2^{-k} & \sum_{\substack{m | DN' \\ m > 1}} \sum_R h(R) \prod_{p | \frac{DN' l^{2r}}{m}} \nu_p(R, \mathcal{O}(D, N)) \\ & + 2^{-k} \sum_{l^{2r} | m | DN' l^{2r}} \sum_R h(R) \prod_{p | \frac{DN' l^{2r}}{m}} \nu_p(R, \mathcal{O}(D, N)) \\ \leq & d_2' + \frac{1}{2}\psi(l^r)(d_2' + \frac{1}{2}(e_2' + e_4')) \\ \leq & d_2' + \frac{1}{2}\psi(l^r)(d_2' + e_2' + \frac{3}{2}e_4'). \end{aligned}$$

Thus, if we assume that $g(X(D,N)^+) = 0$, we obtain that

$$0 < -V(D,N')^+ \frac{\psi(l^{2r})}{2} + \frac{1}{2}(d_2 + e_2 + \frac{3}{2}e_4)$$

which replacing $d_2, e_2$ and $e_4$ by the upper bounds we have just computed, becomes

$$2V(D,N')^+ \frac{\psi(l^{2r})}{2 + \psi(l^r)} < d_2' + e_2' + \frac{3}{2}e_4'.$$

Now observing that $\frac{\psi(l^{2r})}{2+\psi(l^r)} = \frac{l^r \psi(l^r)}{2+\psi(l^r)} \geq \frac{l^r \psi(l^r)}{2\psi(l^r)} \geq 1$, we obtain that

$$2V(D,N')^+ < d_2' + e_2' + \frac{3}{2}e_4',$$

and therefore the result follows.

On the other hand, if $N$ has odd $l$-adic valuation, we have $N = N'l^{2r}$ with $l \| N'$. The discussion of this case follows the same lines above, but in this case the number of distinct primes dividing both $DN$ and $DN'$ is the same. Thus,

$$\chi(X(D,N)^+) = -\frac{V(D,N')^+\psi(l^{2r})}{2} + \frac{1}{2}(d_2+e_2) + \frac{2}{3}e_3 + \frac{3}{4}e_4 + \frac{5}{6}e_6,$$

and $e_i \le e_i'$ for $i = 2,3,4,6$. Similarly,

$$\begin{aligned}
d_2 &= 2^{-k+1} \sum_{\substack{m|DN \\ m>1 \\ (m,l)=1}} \sum_R h(R) \prod_{p|\frac{DN'l^{2r}}{m}} \nu_p(R,\mathcal{O}(D,N)) \\
&\quad + 2^{-k+1} \sum_{l^{2r+1}|m|DN'l^{2r}} \sum_R h(R) \prod_{p|\frac{DN'l^{2r}}{m}} \nu_p(R,\mathcal{O}(D,N)) \\
&= 2^{-k+1} \sum_{\substack{m|DN' \\ m>1 \\ (m,l)=1}} \sum_R h(R) \prod_{p|\frac{DN'}{m}} \nu_p(R,\mathcal{O}(D,N')) \\
&\quad + 2^{-k+1}\psi(l^r) \sum_{l|m|DN'} \sum_R h(R) \prod_{p|\frac{DN'}{m}} \nu_p(R,\mathcal{O}(D,N')) \\
&\le \frac{1}{2}\psi(l^r)d_2'.
\end{aligned}$$

Therefore, $g(X(D,N)^+) = 0$ if and only if $\frac{V(D,N')^+\psi(l^{2r})}{2} < \frac{1}{2}d_2 \le \frac{1}{4}\psi(l^r)d_2'$, which immediately yields that $2V(D,N)^+ < d_2'$ as desired. □

*Proof of the Proposition 4.1.* The values of $D, N$ with squarefree $N$ are given in [LMR06]. Therefore, we only need to find the lacking non-squarefree $N$ and to do so we can follow two different approaches, the first one would be to follow the same method used in the aforementioned article, namely that the normalized volume of such a quotient of the upper half-plane must be smaller than $64/3$, which provides a fairly short list from which we can select the genus 0 ones computing it in every case. However, the previous lemma allows us to restrict to pairs of the form $D, N'(N'')^2$ with squarefree $N'$, and $D, N'$, now with $N'$ squarefree in the list. Next we are going to use the bound on the volume to reduce the possible values of $N''$ to a short list. Let us fix a couple $D, N'$ with $N'$ squarefree such that $X(D,N')^+$ has genus zero.

We want to compute the integers $N''$ such that the curve $X(D,N)^+$ with $N = N'(N'')^2$ has genus 0 too. If we denote by $V(D,N')^+$ the normalized volume of the corresponding quotient of the upper half-plane and $k$ the number of different primes dividing $N''$, the volume of the quotient attached to $X(D,N)^+$ is $V(D,N)^+ = \frac{\psi(N)}{2^k\psi(N')}V(D,N')^+ \geq \frac{N''^2}{2^k}V(D,N')^+$. The Selberg-Zograf bound guarantees that $64/3 \geq V(D,N)^+ \geq \frac{N''^2}{2^k}V(D,N')^+$, which gives $\frac{N''}{2^{k/2}} \leq \sqrt{\frac{64}{3V(D,N')}}$, and therefore we obtain a small number of cases to check for each of the curves in [LMR06]. After carrying out these computations, we find the 11 curves of non-squarefree level which can be found in the list above. □

**Remark.** Using the Riemann-Hurwitz formula, we can compute the genus of any Atkin-Lehner quotient from the information on the following table. Moreover, we can compute as well the number of elliptic cycles, $n_W$, corresponding to the Atkin-Lehner quotient by $W \neq \{id\}$ and their orders as follows. Let us denote $m(W) = \{m : \omega_m \in W\}$ and $r(W)$ such that $2^{r(W)} = \#W$.

$$2^{r(W)-1}n_W = (e'_2 + e'_3)/2 + \sum_{m \in m(W)} \sum_{n \in L_m} n,$$

where, for $k = 2, 3$, $e'_k = e_k$ if $k \notin m(W)$ and 0 otherwise. Moreover, all these elliptic cycles are of order two except for the following ones:

$$\begin{cases} \frac{e_2}{2^{r(W)-1}} \text{ of order } 4, & \text{if } 2 \in m(W), \\ \frac{e_3}{2^{r(W)}} \text{ of order } 3, & \text{if } 3 \notin m(W), \\ \frac{e_3}{2^{r(W)-1}} \text{ of order } 6, & \text{if } 3 \in m(W). \end{cases}$$

**Proposition 4.2.** *The maximal Atkin-Lehner quotient of all the curves in the list of Proposition 4.1 has a rational CM point. In particular, they are all isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$.*

*Proof.* Since the maximal Atkin-Lehner quotients of all the curves in the list are genus 0 curves, we only need to prove that all of them do have a rational CM point. All the curves in the list but 15 have a rational elliptic cycle. For these 15 values of $(D,N)$ there exists a rational CM point, for example the CM point by the order $\Lambda$ of discriminant $D(\Lambda)$ given in the following table:

| $(D,N)$ | $D(\Lambda)$ | $(D,N)$ | $D(\Lambda)$ | $(D,N)$ | $D(\Lambda)$ |
|---|---|---|---|---|---|
| $(6,47)$ | $-52$ | $(6,71)$ | $-52$ | $(14,15)$ | $-11$ |
| $(14,23)$ | $-11$ | $(15,11)$ | $-7$ | $(21,19)$ | $-15$ |
| $(39,1)$ | $-7$ | $(39,4)$ | $-7$ | $(62,7)$ | $-20$ |
| $(65,1)$ | $-7$ | $(95,1)$ | $-7$ | $(95,3)$ | $-120$ |
| $(111,1)$ | $-19$ | $(143,2)$ | $-20$ | $(194,1)$ | $-20$ |

$\square$

## 4.2 A detailed case: the curve $X(22,3)$

In this section we are going to deal with the explicit uniformization of the Shimura curve $X(22,3)$. It follows from [Ogg83] that it is an hyperelliptic curve of genus 3, but it can be seen as well from the table in Proposition 4.1, since the quotient by $\omega_{66}$ has genus 0. At the same time we will study as well all its Atkin-Lehner quotients and in order to do so we will start by its maximal quotient, $X(22,3)^+$.

We are going to use that both $X(22,1)^+$ and $X(22,3)^+$ are curves isomorphic to $\mathbb{P}^1_{\mathbb{Q}}$, even though $X(22,3)^{W_{22}}$ has genus 1. In the previous chapter we computed a Hauptmodul defined over $\mathbb{Q}$ for $X(22,1)^+$, $t_{22}^+$. Together with this function we computed the corresponding kroneckerian polynomial of level 3, $\Psi_3(X,Y)=\Psi_{\Gamma(22,1)^+,3,t_{22}^+}(X,Y)$, which provided a singular model for the curve $X(22,3)^{W_{22}}$, and allowed us to obtain as well a singular model for $X(22,3)^+$, cf. Corollary 2.3. Now, we are going to make use of both this kroneckerian polynomial and the explicit equation for $X(22,3)^+$. Recall that the Hauptmodul $t_{22}^+$ was determined by the values at (three of) the elliptic cycles:

| $P$ | $t_{22}^+(P)$ |
|---|---|
| $\mathrm{CM}(\mathbb{Z}[i])$ | $1$ |
| $\mathrm{CM}(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}])$ | $\frac{27}{16}$ |
| $\mathrm{CM}(\mathbb{Z}[\frac{1+\sqrt{-11}}{2}])$ | $\infty$ |
| $\mathrm{CM}(\mathbb{Z}[\sqrt{-22}])$ | $0$ |

First of all, we need to properly choose a Hauptmodul for $X(22,3)^+$, $t^+_{22,3}$, and to do so we will take three rational CM points in $X(22,3)^+$. Since we are interested in relating this function to $t^+_{22}$, it will be convenient to choose, whenever possible, three CM points which are preimages of the elliptic cycles of $X(22,1)^+$. In order to decide on the rationality of these points in $X(22,3)^+$, and since the polynomial $\Psi_3(X,Y)$ is known, the following lemma will be useful.

**Lemma 4.2.** *Consider the curve $X(D,M)^W$. Assume that it is isomorphic to the projective line over $\mathbb{Q}$ and let $f$ be a Hauptmodul over $\mathbb{Q}$. Let $N=p^r$ for an odd prime $p$ not dividing $DM$. Let $P\in X(D,MN)^W(\mathbb{C})$ be such that its image in $X(D,M)^W$ is elliptic and consider the natural birrational map*

$$\begin{array}{rcl} X(D,MN)^W & \to & X(D,M)^W\times X(D,M)^W \\ \tau & \mapsto & (\tau,\omega_N\tau). \end{array}$$

*Then, the image of the point $P$ is rational in $X(D,MN)^{\langle W,\omega_N\rangle}$ if any of the following conditions holds:*

*(1) $[\mathbb{Q}(f(P)):\mathbb{Q}]\le 2$, $\mathrm{Gal}(\mathbb{Q}(f(P))|\mathbb{Q})=\langle\sigma\rangle$, $f(P)=\sigma(f(\omega_N P))$ and $(f(P),f(\omega_N P))$ is not a singularity of $\Psi_{\Gamma,N,f}(X,Y)$.*

*(2) $f(P)=f(\omega_N P)\in\mathbb{Q}\cup\{\infty\}$.*

**Remark.** In the hypotheses of the proposition above, Theorems 2.2 and 2.3 yield that the number of branches through $(f(P),f(\omega_N P))$ is at most 2, and provides a description of the singularities. Since in this case we explicitly know the polynomial $\Psi_{\Gamma,N,f}(X,Y)$, we can compute the number of branches through $(f(P),f(\omega_N P))$ as

$$\frac{e_N(P)}{e_1(P)}\mathrm{ord}_{f(\omega_N P)}\Psi_{\Gamma,N,f}(f(P),X)=\frac{e_1(\omega_N P)}{e_1(P)}\mathrm{ord}_{f(\omega_N P)}\Psi_{\Gamma,N,f}(f(P),X).$$

*Proof.* In the first case, the result is clear. The only case which needs to be considered is the second one when $(P,\omega_N P)$ is a singularity of $\Psi_{\Gamma,N,f}(X,Y)$. In this situation, Theorem 2.3 gives us that there exist two non-tangent branches through $(P,\omega_N P)$ and therefore there exist a couple of points

$$P',P''\in X(D,MN)^W(\mathbb{C})$$

corresponding to this singularity. Moreover $P'+P''$ is a rational divisor and therefore it is enough to prove that these points are not fixed by $\omega_N$.

The elliptic points in $X(D,M)^W$ can be of two different types, the ones coming from elliptic points in $X(D,M)$ and the fixed points by elements in $W$. Let us begin by the latter ones, that is, assume that the image of $P$ in $X(D,M)^W$ is fixed by an element $\omega_m \in W$, $m > 1$, $m \| DM$. Then, Theorem 4.1 yields that it must come from a CM point by an order of discriminant $-m$ or $-4m$ in $X(D,M)$. However, if $P$ is fixed by $\omega_N$ it must come from a CM point by an order of discriminant $-N$ or $-4N$ in $X(D,MN)$ and therefore its image in $X(D,M)$ will be obtained from a CM point by an order containing this one and therefore of discriminant $-N'$ or $-4N'$ for some $N'$ such that $N/N'$ is a square. Since $\gcd(m,N) = 1$, this is clearly a contradiction.

Notice that it was not necessary in the previous case, but the condition $f(P) = f(\omega_N P)$ yields that the order corresponding to $P$ is the same both in $X(D,M)^W$ and $X(D,MN)^W$. Keeping this in mind, we will now focus on the case that the image of $P$ in $X(D,M)^W$ (and in $X(D,MN)^W$) is a CM point by $\mathbb{Z}[i]$ or $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. In order for it to be fixed by $\omega_N$, we must have $N = 2$ or $N = 3$, respectively. The first case cannot happen since we are assuming that $N$ is odd. Therefore, we only have to deal with the case $N = 3$. Assume that $(P, \omega_N P)$ corresponds to a singular point. In this case we reach as well a contradiction, since $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ is a principal ideal domain and the singularity provides a primitive element of norm 9 in this ring, which does not exist. $\square$

Let us call $\pi^+_{22,3} : X(22,3)^{W_{22}} \to X(22,1)^+$ the natural projection, which is a morphism of curves ramified, at most, at the preimages of the elliptic cycles of $X(22,1)^+$. Since the kroneckerian polynomial gives a model for $X(22,3)^{W_{22}}$ inside $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ and the corresponding morphism to $X(22,1)^+$ with the model provided by $t^+_{22}$ is given by the projection on the first component, we can easily study the ramification of $\pi^+_{22,3}$. Denote by $P_i$ the CM points in $X(22,1)^+$ corresponding to orders of discriminant $-i$, for $i = 3, 4, 11, 88$, which correspond to the elliptic cycles in this curve. Recall that the values of the Hauptmodul $t^+_{22}$ at these points are $\frac{27}{16}, 1, \infty, 0$ respectively. For any of these values, $t^+_{22}(P_i)$, the factorization of $\Psi_3(f(P_i), X)$ is as follows:

| $i$ | $\Psi_3(t^+_{22}(P_i), X)$ |
|---|---|
| 3 | $16(-27+16X)(-75+64X)^3$ |
| 4 | $(-49+16X)^4$ |
| 11 | $65536(-49+48X)^2$ |
| 88 | $\left(7425 - 17568X + 12544X^2\right)^2$ |

Thus, we obtain a set of rational points on the curve $\Psi_3(X,Y)=0$. Recall that the elliptic order of the point $P_i$ in the curve $X(22,1)^+$ is given by

$$e_1(P_i)=\begin{cases}2, & i=11,22,\\ 3, & i=3,\\ 4, & i=4.\end{cases}$$

Then using the above remark we get that each of these points corresponds to a certain number of points on the curve $X(22,3)^{W_{22}}$ as stated in the following table:

| $(P,\omega_3 P)$ | $\mathrm{mult}_{(P,\omega_3P)}\Psi_3$ |
|---|---|
| $(\frac{27}{16},\frac{27}{16})$ | 1 |
| $(\frac{27}{16},\frac{75}{64})$ | 1 |
| $(1,\frac{49}{16})$ | 1 |
| $(\infty,\infty)$ | 2 |
| $(\infty,\frac{49}{48})$ | 1 |

In particular, by Lemma 4.2, all of them define rational points on the curve $X(22,3)^+$. Now, since this is a curve of genus 0 with rational points we can define a Hauptmodul $t^+_{22,3}$ over $\mathbb{Q}$ by assigning three different rational values at 3 of the above points. Let us fix, once and for all, $t^+_{22,3}$ such that

| $P$ | $t^+_{22,3}(P)$ |
|---|---|
| $(\frac{27}{16},\frac{75}{64})$ | 1 |
| $(1,\frac{49}{16})$ | 0 |
| $(\infty,\infty)$ | $\infty$ |

**Lemma 4.3.** *Assume that $X(D,M)^W$ and $X(D,MN)^{\langle W,\omega_N\rangle}$ have both genus 0 and let $f,g$ be Hauptmoduln for these curves over a field $K$. Then for $a,b\in K$, $a\neq b$, there exist $R_1(g),R_2(g)\in K(g)$ such that $(f-a)(f\circ\omega_N-a)=R_1(g)$, $(f-b)(f\circ\omega_N-b)=R_2(g)$. Moreover,*

$$R_1(g)(f-b)-R_2(g)(f-a)+(a-b)(f-a)(f-b)=0.$$

*Proof.* It is only necessary to observe that $(f - a)(f \circ \omega_N - a)$ is invariant by $\langle \Gamma(D, MN), W \rangle$. $\square$

**Remark.** It is worth noting that the functions $R_i(g)$ can be written as a quotient of two coprime polynomials of degree at most $\psi(N)$ with both functions sharing the same denominator. The last equation in the lemma allows us to recover the values of $f$ from those of $g$.

In the case of the pole of $g$, if it corresponds to a point such that it is both a pole of $f$ and $f \circ \omega_N$, then since $\frac{R_1(g)}{R_2(g)} = \frac{f-a}{f-b}\frac{f\circ\omega_N - a}{f\circ\omega_N - b}$,

$$\lim_{g\to\infty} \frac{R_1(g)}{R_2(g)} = 1.$$

Otherwise, if $f$ takes a finite value $f_0$ at a point corresponding to a pole of $g$ but $f \circ \omega_N$ has a pole or the other way around,

$$\lim_{g\to\infty} \frac{R_1(g)}{R_2(g)} = \frac{f_0 - a}{f_0 - b}.$$

Finally, if both $f$ and $f \circ \omega_N$ take finite values $f_0$ and $f_0'$ at a point corresponding to a pole of $g$, we have that

$$\lim_{g\to\infty} \frac{R_1(g)}{R_2(g)} = \frac{f_0 - a}{f_0 - b}\frac{f_0' - a}{f_0' - b}.$$

Back to our case, we consider the function $(t_{22}^{+} - 1)(t_{22}^{+} \circ \omega_3 - 1) \in \mathbb{Q}(t_{22,3}^{+})$ and comparing its zeros and poles with those of $t_{22,3}^{+}$, we get

$$R_1(t_{22,3}^{+}) = (t_{22}^{+} - 1)(t_{22}^{+} \circ \omega_3 - 1) = a_0 \frac{(t_{22,3}^{+})^4}{(t_{22,3}^{+} - b_0)^2}$$

for certain $a_0, b_0 \in \mathbb{Q}^*$. Similarly,

$$R_2(t_{22,3}^{+}) = (t_{22}^{+} - \tfrac{27}{16})(t_{22}^{+} \circ \omega_3 - \tfrac{27}{16}) = a_1 \frac{(t_{22,3}^{+} - 1)^3 (t_{22,3}^{+} - a_2)}{(t_{22,3}^{+} - b_0)^2}$$

for certain $a_1, a_2 \in \mathbb{Q}^*$. Moreover, by the remark above, $a_0 = a_1$.

**Lemma 4.4.** *Let $K$ be a number field and $f$ a Hauptmodul for $X(D, M)^W$ defined over $K$. Then, for any $N > 1$ coprime to $DM$ and any $a, b \in K$, $a \neq b$, the polynomial*

$$\Psi\left(\frac{X - Y}{2(b-a)} + \frac{b+a}{2}, \frac{bX - aY}{b-a} + ab\right)$$

*where*

$$\Psi(X,Y) = \Psi_{\Gamma,N,f}(X+\sqrt{X^2-Y}, X-\sqrt{X^2-Y})$$

*has*

$$((f-a)(f\circ\omega_N - a), (f-b)(f\circ\omega_N - b))$$

*as a root and provides a model for $X(D,MN)^{\langle W,\omega_N\rangle}$ over $K$.*

*Proof.* It follows from Corollary 2.3, after writing

$$\frac{f+f\circ\omega_N}{2} = \frac{(f-a)(f\circ\omega_N-a)-(f-b)(f\circ\omega_N-b)}{2(b-a)} + \frac{b+a}{2}$$

$$\left(\frac{f-f\circ\omega_N}{2}\right)^2 = \left(\frac{f+f\circ\omega_N}{2}\right)^2 - f\cdot f\circ\omega_N$$

$$f\cdot f\circ\omega_N = \frac{b(f-a)(f\circ\omega_N-a)-a(f-b)(f\circ\omega_N-b)}{b-a} + ab.$$

□

Now, using this lemma, we can explicitly compute a polynomial $P\in\mathbb{Q}[X,Y]$ such that

$$P((t_{22}^{+}-1)(t_{22}^{+}\circ\omega_3-1), (t_{22}^{+}-\tfrac{27}{16})(t_{22}^{+}\circ\omega_3-\tfrac{27}{16})) = 0.$$

Imposing that

$$P(R_1(t_{22,3}^{+}), R_2(t_{22,3}^{+})) = 0$$

we obtain a unique solution for $a_0, a_2, b_0 \in \mathbb{Q}$, namely,

$$(R_1(X), R_2(X)) = \left(\frac{121X^4}{64(3X+1)^2}, \frac{121(X-1)^3(2X+1)}{128(3X+1)^2}\right).$$

The unicity of the solution could be predicted thanks to the following well-known result on the unicity up to linear rational transformations of proper parametrizations of rational irreducible algebraic curves, cf. [SWPD08]. Recall that, as usual, the degree of a rational function is the maximum of the degrees of the numerator and the denominator, after common factors are removed.

**Theorem 4.2.** *Consider an algebraic irreducible affine plane curve $C$ of genus 0 over $\mathbb{C}$ with equation $\Psi(X,Y)=0$, with $\Psi(X,Y)\in\mathbb{C}[X,Y]$ irreducible. Then, there exist $Q_1(t), Q_2(t)\in\mathbb{C}(t)$ with $\deg Q_1 = \deg_Y \Psi$, $\deg Q_2 = \deg_X\Psi$ such that $\Psi(Q_1(t),Q_2(t))=0$. Moreover, $Q_1$ and $Q_2$ with these conditions are unique up to homographic transformation of $t$.* □

According to this theorem, $R_1(X)$ and $R_2(X)$ are unique up to homographic transformations of the variable $X$, but no such transformation maps $R_1$ and $R_2$ to rational fractions of the same shape, in particular, we obtain that the solution had to be unique.

Now, using the equation given in Lemma 4.3 together with Proposition 3.7, we can easily compute the automorphic derivative of $t^+_{22,3}$ from that of $t^+_{22}$.

**Proposition 4.3.** *$Da(t^+_{22,3}) + R(t^+_{22,3}) = 0$, where*

$$R(t) = \frac{1452t^6 - 2684t^5 + 4327t^4 + 1300t^3 + 5046t^2 - 2880t - 81}{(2t-3)^2(2t+1)^2(11t^2 - 2t + 3)^2}.$$

$\square$

Observe that the zeros of the denominator of $R$ are the values of $t^+_{22,3}$ at the elliptic points in $X(22,3)^+$, and thus we obtain the values at the CM points by $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$, $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$, $\mathbb{Z}[\sqrt{-33}]$, $\mathbb{Z}[\sqrt{-66}]$, namely $-\frac{1}{2}, \infty, \frac{3}{2}$ and the zeros of $11X^2 - 2X + 3$ respectively. However as many values as desired can be computed as long as we know the value of $t^+_{22}$ at the corresponding point, for example we know that $t^+_{22}$ takes the value $\frac{5}{16}$ at the CM point by $\mathbb{Z}[\sqrt{-5}]$. Then, since

$$\Psi_3(\tfrac{5}{16}, X) = 64(-5 + 16X)^2\left(6241 - 10381X + 4624X^2\right),$$

we can choose the point corresponding to $(\frac{5}{16}, \frac{5}{16})$, which is a CM point by $\mathbb{Z}[\sqrt{-5}]$ in $X(22,3)^+$, the value of $t^+_{22,3}$ can be obtained solving the system

$$((\tfrac{5}{16} - 1)^2, (\tfrac{5}{16} - \tfrac{27}{16})^2) = \left(\frac{121X^4}{64(3X+1)^2}, \frac{121(X-1)^3(2X+1)}{128(3X+1)^2}\right)$$

which gives the value $-1$.

Now, we are interested in computing uniformizing functions giving the canonical model and the corresponding equations for all the Atkin-Lehner quotients of $X(22,3)$. We will illustrate the procedure with a couple of cases and we will sum up the information for the whole set of 16 curves. In order to do so, we begin by studying the genus of the different curves and the ramification of the natural projections between them. Since the ramification points of the projection coincide with the fixed points of the involutions we are adding, which are controlled by Theorem 4.1, and its number can be deduced from the number of such CM points in $X(22,3)$, which is given by Theorem 1.7, we can easily obtain the genus from the genus of $X(22,3)$, which is 3, using Riemann-Hurwitz formula.

In particular, among the 16 curves we are considering, 6 have genus 0, including $X(22,3)^+$, which we have already studied. We will begin by the 5 genus 0 curves remaining and will compute a Hauptmodul over $\mathbb{Q}$ for those having one, and an equation as a conic for those who do not.

**Proposition 4.4.** *The curves $X(22,3)^W$, $W \subsetneq W_{66}$, having genus 0 are those with $W \in \{\langle\omega_{66}\rangle, \langle\omega_2,\omega_{33}\rangle, \langle\omega_3,\omega_{22}\rangle, \langle\omega_6,\omega_{11}\rangle, \langle\omega_3,\omega_{11}\rangle\}$.*

*(1) $X(22,3)^{\langle\omega_3,\omega_{11}\rangle}$ does not have rational (or even real) points. Its field of rational functions is*

$$\mathbb{Q}\left(t^+_{22,3}, \sqrt{-11(t^+_{22,3})^2 + 2t^+_{22,3} - 3}\right)$$

*and an equation for this curve over $\mathbb{Q}$ is*

$$Y^2 + 11X^2 - 2X + 3 = 0.$$

*(2) The rest of these curves have rational points. There exist functions $U_i$ such that $\mathbb{Q}(X(22,3)^{\langle\omega_i,\omega_{66}\rangle}) = \mathbb{Q}(U_i)$, $i = 1,2,3,6$, defined as follows:*

| $i$ | $U_i^2$ | $-Da(U_i)$ |
|---|---|---|
| 6 | $\frac{2t^+_{22,3}+1}{2t^+_{22,3}-3}$ | $\frac{648-285U_6^2-12044U_6^4+69794U_6^6+26796U_6^8+8019U_6^{10}}{(-1+U_6)^2U_6^2(1+U_6)^2\left(27+50U_6^2+99U_6^4\right)^2}$ |
| 3 | $-2t^+_{22,3}+3$ | $\frac{4\left(50787-64284U_3^2+29266U_3^4-5852U_3^6+451U_3^8\right)}{(-2+U_3)^2(2+U_3)^2\left(99-62U_3^2+11U_3^4\right)^2}$ |
| 2 | $-2t^+_{22,3}-1$ | $-\frac{4\left(-2592+771U_2^2+9908U_2^4+6690U_2^6+1716U_2^8+275U_2^{10}\right)}{U_2^2\left(4+U_2^2\right)^2\left(27+26U_2^2+11U_2^4\right)^2}$ |
| 1 | $\frac{1-U_6}{1+U_6}$ | $\frac{32\left(11-132U_1^2+388U_1^4-516U_1^6+579U_1^8-516U_1^{10}+388U_1^{12}-132U_1^{14}+11U_1^{16}\right)}{(-1+U_1)^2(1+U_1)^2\left(11-18U_1^2+41U_1^4-18U_1^6+11U_1^8\right)^2}$ |

*Moreover, we can rewrite*

$$U_1 = \frac{U_3 - U_2}{2}.$$

*The action of the Atkin-Lehner involution $\omega_d$ on such a Hauptmodul $U$ is given by a rational linear function $R_d(X) \in \mathbb{Q}(X)$ such that $U \circ \omega_d = R_d(U)$*

*as described in the following table:*

| | $R_2$ | $R_3$ | $R_6$ | $R_{11}$ | $R_{22}$ | $R_{33}$ | $R_{66}$ |
|---|---|---|---|---|---|---|---|
| $U_6$ | $-X$ | $-X$ | $X$ | $X$ | $-X$ | $-X$ | $X$ |
| $U_3$ | $-X$ | $X$ | $-X$ | $-X$ | $X$ | $-X$ | $X$ |
| $U_2$ | $X$ | $-X$ | $-X$ | $-X$ | $-X$ | $X$ | $X$ |
| $U_1$ | $-\frac{1}{X}$ | $\frac{1}{X}$ | $-X$ | $-X$ | $\frac{1}{X}$ | $-\frac{1}{X}$ | $X$ |

*Proof.* This proof runs along the same lines we followed when uniformizing the curves of level $N=1$ in the previous chapter.

In the case of $X(22,3)^{\langle\omega_3,\omega_{11}\rangle}$, the map to $X(22,3)^+$ ramifies at the CM points by $\mathbb{Z}[\sqrt{-66}]$. Since we know the values of the function $t^+_{22,3}$ at these points, namely the zeros of $11X^2-2X+3$, we can take a function

$$\sqrt{k(11(t^+_{22,3})^2-2t^+_{22,3}+3)}$$

for a certain integral and squarefree value of $k$ for which it is defined over $\mathbb{Q}$. In order to find the value of $k$ we will evaluate this function at certain CM points to obtain conditions on $k$. For example, for the CM points by $\mathbb{Z}[\sqrt{-5}]$, $t^+_{22,3}$ takes the value $-1$, so the function above takes the value $4\sqrt{k}$ which must lie in the Hilbert Class Field of $\mathbb{Q}(\sqrt{-5})$ which is $\mathbb{Q}(\sqrt{5},i)$. Similarly, if we evaluate the function $\frac{\sqrt{k(11(t^+_{22,3})^2-2t^+_{22,3}+3)}}{t^+_{22,3}}$ at the CM points by $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ it takes the value $\sqrt{11k}$ which must lie in $\mathbb{Q}(\sqrt{-11})$, since this field has class number 1. Therefore we conclude that $k=-1$ and the result follows.

For the curve $X(22,3)^{\langle\omega_6,\omega_{66}\rangle}$, its projection to $X(22,3)^+$ ramifies at the CM points by $\mathbb{Z}[\sqrt{-33}]$, where $t^+_{22,3}$ is $\frac{3}{2}$, and by $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$, where $t^+_{22,3}$ is $-\frac{1}{2}$. Thefore, we can take $U_6=\sqrt{k\frac{2t^+_{22,3}+1}{2t^+_{22,3}-3}}$. Now, if we evaluate this function at the CM point by the ring of integers of $\mathbb{Q}(\sqrt{-11})$, $t^+_{22,3}$ is $\infty$, hence we obtain that $\sqrt{k}\in\mathbb{Q}(\sqrt{-11})$. Similarly, for the CM points by $\mathbb{Z}[\sqrt{-5}]$, we obtain $\sqrt{\frac{k}{5}}\in\mathbb{Q}(\sqrt{5},i)$. Therefore $k=1$.

For the curve $X(22,3)^{\langle\omega_3,\omega_{66}\rangle}$, the projection to $X(22,3)^+$ ramifies at the CM points by $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ and $\mathbb{Z}[\sqrt{-33}]$, so we can consider $U_3=\sqrt{k(2t^+_{22,3}-3)}$. Evaluating this function at the CM points by the ring of integers of $\mathbb{Q}(\sqrt{-5})$

and $\mathbb{Q}(\sqrt{-3})$, we obtain $k=-1$. Now, for the curve $X(22,3)^{\langle\omega_2,\omega_{66}\rangle}$, we can take $U_2=U_3U_6$.

For the curve $X(22,3)^{\langle\omega_{66}\rangle}$, we consider the projection to $X(22,3)^{\langle\omega_6,\omega_{66}\rangle}$, which ramifies at the CM points by the ring of integers of $\mathbb{Q}(\sqrt{-11})$, where $U_6$ takes the values $\pm1$. Therefore, we can take $U_1=\sqrt{k\frac{U_6-1}{U_6+1}}$. Now evaluating at the CM points by the rings of integers of $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-5})$, where $U_6$ takes the values $-\frac{1}{2},\pm\frac{\sqrt{5}}{5}$ respectively, we obtain that $k=-1$.

The automorphic derivatives are obtained using Proposition 3.7. □

The equations for all the remaining curves can be computed using these functions. Let us show how to obtain an hyperelliptic model for $X(22,3)$. Consider the projection to $X(22,3)^{\langle\omega_{66}\rangle}$ which ramifies at the 8 CM points by $\mathbb{Z}[\sqrt{-66}]$. The values of $U_1$ at these points are the roots of the polynomial $11X^8-18X^6+41X^4-18X^2+11$. Therefore,

$$\mathbb{Q}(X(22,3))=\mathbb{Q}\left(U_1,\sqrt{k(11U_1^8+18U_1^6+41U_1^4-18U_1^2+11)}\right).$$

To find out the value of $k$, we evaluate the function at the CM points by the rings of integers of $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-11})$, where $U_1$ takes the values $\pm1$ and $0,\infty$, respectively. In this way we find that $k=-1$. Similarly, we can work out all the remaining curves. For some of them it is useful to use the CM points by the order $\mathbb{Z}[3i]$, where $t^+_{22,3}$ takes the value 0 and the corresponding Ring Class Field is $\mathbb{Q}(i,\sqrt{-3})$.

We collect in the following proposition a set of generators providing an elliptic or hyperelliptic model for all the curves $X(22,3)^W$ which are not of genus 0.

**Proposition 4.5.** *(1) For $W\in\{\langle\omega_2,\omega_{11}\rangle,\langle\omega_2,\omega_3\rangle,\langle\omega_6,\omega_{22}\rangle\}$, $X(22,3)^W$ are elliptic curves. For each of them, the polynomial $P_W(X)$ in the following table provides a system of generators of the field of rational functions*

$$\mathbb{Q}(X(22,3)^W)=\mathbb{Q}\left(t^+_{22,3},\sqrt{P_W(t^+_{22,3})}\right)$$

*and the corresponding equation for the curve*

$$Y^2=P_W(X).$$

| $W$ | $P_W(X)$ |
|---|---|
| $\langle\omega_2,\omega_{11}\rangle$ | $-(2X+1)(2X-3)(11X^2-2X+3)$ |
| $\langle\omega_2,\omega_3\rangle$ | $(2X-3)(11X^2-2X+3)$ |
| $\langle\omega_6,\omega_{22}\rangle$ | $(2X+1)(11X^2-2X+3)$ |

*(2) The curves $X(22,3)^{\langle\omega_i\rangle}$, for $i=3,11,33$, are genus 1 curves without real points. For each of them, the polynomial $P_i(X)$ and the function $f_i$ in the following table provide a system of generators of the field of rational functions*

$$\mathbb{Q}(X(22,3)^{\langle\omega_i\rangle})=\mathbb{Q}\left(f_i,\sqrt{P_i(f_i)}\right)$$

*and the corresponding equation for the curve,*

$$Y^2=P_i(X).$$

| $i$ | $f_i$ | $P_i(X)$ |
|---|---|---|
| 3 | $U_3$ | $-(11X^4-62X^2+99)$ |
| 11 | $U_6$ | $-(99X^4+50X^2+27)$ |
| 33 | $U_2$ | $-(11X^4+26X^2+27)$ |

*(3) The curves $X(22,3)^{\langle\omega_i\rangle}$, for $i=2,6,22$, are genus 2 curves. For each of them, the polynomial $P_i(X)$ and the function $f_i$ in the following table provide a system of generators of the field of rational functions*

$$\mathbb{Q}(X(22,3)^{\langle\omega_i\rangle})=\mathbb{Q}\left(f_i,\sqrt{P_i(f_i)}\right)$$

*and the corresponding equation for the curve*

$$Y^2=P_i(X).$$

| $i$ | $f_i$ | $P_i(X)$ |
|---|---|---|
| 2 | $U_2$ | $-(X^2+4)(11X^4+26X^2+27)$ |
| 6 | $U_6$ | $(X^2-1)(99X^4+50X^2+27)$ |
| 22 | $U_3$ | $-(X^2-4)(11X^4-62X^2+99)$ |

*(4) The curve $X(22,3)$ is an hyperelliptic curve of genus 3. Its field of rational functions is*

$$\mathbb{Q}(X(22,3)) = \mathbb{Q}\left(U_1, \sqrt{P(U_1)}\right)$$

*and the corresponding equation for the curve is*

$$Y^2 = -(11X^8 - 18X^6 + 41X^4 - 18X^2 + 11).$$

□

## 4.3 Other cases

Once we have studied the uniformization of the curve $X(22,3)$ in detail, other curves $X(D,N)$ where a Hauptmodul for $X(D,1)^+$ is known and such that $X(D,N)^+$ has genus 0 can be studied in the same way. We will compilate in this section some more cases. Since we have focused on discriminants $D = 6, 10, 22$ we will restrict to these three discriminants. Moreover, we will study the three smallest values of $N$ in each of these three cases, including the case $(22,3)$ above. Some of these cases for $D = 6, 10$ have already been studied before, namely $(6,5), (6,7), (10,3), (10,7)$. The three first ones can be found in [Elk98] and the last one in [Yan13], [Tu14]. As far as we know, Hauptmoduln for the cases $(6,11)$ and $(10,9)$ have not been obtained before. We will use the tools we have already seen to recover or obtain Hauptmoduln for all those quotients of $X(D,N)$ of genus 0 over $\mathbb{Q}$. In some cases the approach above can be simplified by relating the Hauptmodul to the Hauptmodul of some genus 0 intermediate curves, which happens for all the curves studied in [Elk98] and [Tu14]. We will illustrate how to deal with these cases with the curve $X(6,5)$. Another situation where we can simplify the computations is that of composite levels, we will deal with this situation for the curve $X(10,9)$.

Sometimes, we will obtain genus 0 curves which are given as a conic. In this case, if such a curve has a rational point, then it is indeed the projective line over $\mathbb{Q}$, by means, for example, of the natural parametrization by lines through one rational point. We resume in the following lemma the equations involved, which allow us to build the Hauptmodul from the component functions and to recover the component functions from the Hauptmodul.

**Lemma 4.5.** *Given a conic in the projective plane of the form*

$$C : X^2 + \alpha Y^2 - \beta Z^2 = 0$$

*with a point* $[a : b : 1]$ *over a field* $K$*, then it admits the following proper parametrization over* $K$*,* $f : \mathbb{P}^1_K \to C$*, with*

$$f(t) = [-at^2 - 2\alpha bt + \alpha a : bt^2 - 2at - \alpha b : t^2 + \alpha]$$

*which has inverse*

$$f^{-1}([x : y : 1]) = \frac{x-a}{y-b}.$$

□

However, in some cases it will be more convenient to use other even more obvious parametrizations, for example, for the conics of the form $XY = \alpha$ or $(X - Y)(X + Y) = \alpha$.

### 4.3.1 The curve $X(22, 5)$

This case is very similar to the model case we studied above. On the one hand, all the curves appearing of genus bigger than 2 are hyperelliptic. On the other hand, the computation of the Hauptmodul $t^+_{22,5}$ for $X(22, 5)^+$ involves only slightly harder computations.

The uniformization of $X(22, 5)^+$ is given in the following proposition.

**Proposition 4.6.** *There exists a Hauptmodul for the curve* $X(22, 5)^+$*,* $t^+_{22,5}$*, defined by the following values*

| $(t^+_{22}, t^+_{22} \circ \omega_5)$ | $t^+_{22,5}$ |
|---|---|
| $(1, 1)$ | $0$ |
| $(1, \frac{729}{784})$ | $1$ |
| $(\infty, \infty)$ | $\infty$ |

*and it satisfies*

$$(t^+_{22} - 1)(t^+_{22} \circ \omega_5 - 1) = \frac{11^2 25^3 (t^+_{22,5} - 1)^4 (t^+_{22,5})^2}{256(125(t^+_{22,5})^2 - 175t^+_{22,5} + 64)^2},$$

$$(t^+_{22} - \tfrac{27}{16})(t^+_{22} \circ \omega_5 - \tfrac{27}{16}) = \frac{121(25(t^+_{22,5})^2 - 35t^+_{22,5} + 16)^3}{256(125(t^+_{22,5})^2 - 175t^+_{22,5} + 64)^2}.$$

*Moreover,* $Da(t^+_{22,5}) + R(t^+_{22,5}) = 0$, *where*

$$R(X) = \frac{P(X)}{4X^2(5X-4)^2(1375X^3 - 3650X^2 + 3295X - 1024)^2}$$

*and*

$$\begin{aligned} P(X) =& 5(28359375X^8 - 162250000X^7 + 421556250X^6 \\ & - 669567500X^5 + 731069375X^4 - 564580300X^3 \\ & + 294187120X^2 - 91344896X + 12582912). \end{aligned}$$

*In the following table we give the values of* $t^+_{22,5}$ *at* $\mathrm{CM}(\Lambda)$ *by certain orders* $\Lambda$*:*

| $\Lambda$ | $t^+_{22,5}$ |
|---|---|
| $\mathbb{Z}[i]$ | $0$ |
| $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ | $\infty$ |
| $\mathbb{Z}[\sqrt{-5}]$ | $\frac{4}{5}$ |
| $\mathbb{Z}[\sqrt{-110}]$ | $\alpha : 1375\alpha^3 - 3650\alpha^2 + 3295\alpha - 1024 = 0$ |
| $\mathbb{Z}[5i]$ | $1$ |
| $\mathbb{Z}[\frac{1+5\sqrt{-3}}{2}]$ | $\frac{1\pm\sqrt{-15}}{10}$ |

□

Next we will give Hauptmoduln for the four remaining genus 0 curves, since in this case all of them have rational points.

**Proposition 4.7.** *There exist functions* $U_i = U_{i,110}$ *such that*

$$\mathbb{Q}(X(22,5)^{\langle \omega_i, \omega_{110} \rangle}) = \mathbb{Q}(U_i), \quad i = 1, 2, 5, 10,$$

*defined as follows:*

| $i$ | $U_i^2$ |
|---|---|
| 10 | $\frac{5t^+_{22,5}}{5t^+_{22,5}-4}$ |
| 5 | $-5t^+_{22,5}$ |
| 2 | $-5t^+_{22,5}+4$ |

*and*

$$U_1 = \frac{U_5}{U_2 + 2}.$$

*The action of the Atkin-Lehner involution $\omega_d$ on such a Hauptmodul $U$ is given by a rational linear function $R_d(X) \in \mathbb{Q}(X)$ such that $U \circ \omega_d = R_d(U)$ as described in the following table:*

| | $R_2$ | $R_5$ | $R_{10}$ | $R_{11}$ | $R_{22}$ | $R_{55}$ | $R_{110}$ |
|---|---|---|---|---|---|---|---|
| $U_{10}$ | $-X$ | $-X$ | $X$ | $X$ | $-X$ | $-X$ | $X$ |
| $U_5$ | $-X$ | $X$ | $-X$ | $-X$ | $X$ | $-X$ | $X$ |
| $U_2$ | $X$ | $-X$ | $-X$ | $-X$ | $-X$ | $X$ | $X$ |
| $U_1$ | $-X$ | $-\frac{1}{X}$ | $\frac{1}{X}$ | $\frac{1}{X}$ | $-\frac{1}{X}$ | $-X$ | $X$ |

□

**Remark.** The expressions for the automorphic derivatives can be computed from that of $t_{22,5}^{+}$ as in the model case using Proposition 3.7. We do not include them now because they are considerably larger.

Finally, models for the rest of the curves in this case are given in the following proposition.

**Proposition 4.8.** *(1) For $W \in \{\langle\omega_5, \omega_{11}\rangle, \langle\omega_2, \omega_5\rangle, \langle\omega_2, \omega_{11}\rangle\}$, $X(22,5)^W$ are elliptic curves. For each of them, the polynomial $P_W(X)$ in the following table provides a system of generators of the field of rational functions*

$$\mathbb{Q}(X(22,5)^W) = \mathbb{Q}\left(t_{22,5}^{+}, \sqrt{P_W(t_{22,5}^{+})}\right)$$

*and the corresponding equation for the curve*

$$Y^2 = P_W(X).$$

| $W$ | $P_W(X)$ |
|---|---|
| $\langle\omega_5, \omega_{11}\rangle$ | $-5X(1375X^3 - 3650X^2 + 3295X - 1024)$ |
| $\langle\omega_2, \omega_5\rangle$ | $1375X^3 - 3650X^2 + 3295X - 1024$ |
| $\langle\omega_2, \omega_{11}\rangle$ | $(-5X + 4)(1375X^3 - 3650X^2 + 3295X - 1024)$ |

*(2) The curves $X(22,5)^W$, for $W \in \{\langle\omega_{10},\omega_{22}\rangle, \langle\omega_i\rangle\}_{i=2,5,11}$, are genus 2 curves. For each of them, the polynomials $P_W(X)$ and the function $f_W$ in the following table provide a system of generators of the field of rational functions*

$$\mathbb{Q}(X(22,5)^W) = \mathbb{Q}\left(f_W, \sqrt{P_W(f_W)}\right)$$

*and the corresponding equation for the curve*

$$Y^2 = P_W(X).$$

| $W$ | $f_W$ | $P_W(X)$ |
|---|---|---|
| $\langle\omega_{10},\omega_{22}\rangle$ | $t^+_{22,5}$ | $5X(5X-4)(1375X^3-3650X^2+3295X-1024)$ |
| $\langle\omega_2\rangle$ | $U_2$ | $-(11X^6+14X^4+19X^2+20)$ |
| $\langle\omega_5\rangle$ | $U_5$ | $-(11X^6+146X^4+659X^2+1024)$ |
| $\langle\omega_{11}\rangle$ | $U_{10}$ | $5X^6-34X^4+109X^2+256)$ |

*(3) The curves $X(22,5)^{\langle\omega_i\rangle}$, for $i=10,22,55$, are genus 3 hyperelliptic curves. For each of them, the polynomial $P_i(X)$ and the function $f_i$ in the following table provide a system of generators of the field of rational functions*

$$\mathbb{Q}(X(22,5)^{\langle\omega_i\rangle}) = \mathbb{Q}\left(f_i, \sqrt{P_i(f_i)}\right)$$

*and the corresponding equation for the curve*

$$Y^2 = P_i(X).$$

| $i$ | $f_i$ | $P_i(X)$ |
|---|---|---|
| 10 | $U_{10}$ | $-(X^2-1)(5X^6-34X^4+109X^2-256)$ |
| 22 | $U_5$ | $-(X^2+4)(11X^6+146X^4+659X^2+1024)$ |
| 55 | $U_2$ | $-(X^2-4)(11X^6+14X^4+19X^2+20)$ |

*(4) The curve $X(22,5)$ is an hyperelliptic curve of genus 5. Its field of rational functions is*

$$\mathbb{Q}(X(22,5)) = \mathbb{Q}\left(U_1, \sqrt{P(U_1)}\right)$$

*and the corresponding equation for the curve is*

$$Y^2 = -(64X^{12}+275X^{10}+660X^8+818X^6+660X^4+275X^2+64).$$

$\square$

### 4.3.2 The curve $X(22,7)$

The case of level 7, even though quite similar from a theoretical point of view to the cases we have already seen, presents already some more computational difficulties, in part due to the higher degree of the projection but also to the fact that most of the ramification points are no longer rational. However it is still possible to follow a similar approach to obtain a Hauptmodul for $X(22,7)^+$.

**Proposition 4.9.** *There exists a Hauptmodul for the curve $X(22,7)^+$, $t^+_{22,7}$, defined by the following values*

| $(t^+_{22}, t^+_{22}\circ\omega_7)$ | $t^+_{22,7}$ |
|---|---|
| $(\frac{27}{16}, \frac{27}{16})$ | 0 |
| $(\frac{75}{64}, \frac{75}{64})$ | 1 |
| $(\frac{5}{16}, \frac{5}{16})$ | $\infty$ |

*and it satisfies*

$$(t^+_{22}-1)(t^+_{22}\circ\omega_7-1)=\frac{121(18(t^+_{22,7})^2-18t^+_{22,7}+1)^4}{256(324(t^+_{22,7})^4-486(t^+_{22,7})^3+171(t^+_{22,7})^2-6t^+_{22,7}+1)^2},$$

$$(t^+_{22}-\tfrac{27}{16})(t^+_{22}\circ\omega_7-\tfrac{27}{16})=\frac{1089(t^+_{22,7})^2(36(t^+_{22,7})^2-42t^+_{22,7}+7)^3}{256(324(t^+_{22,7})^4-486(t^+_{22,7})^3+171(t^+_{22,7})^2-6t^+_{22,7}+1)^2}.$$

*Moreover, $Da(t^+_{22,7})+R(t^+_{22,7})=0$, where*

$$R(X)=\frac{P(X)}{X^2(36X^2-36X+1)^2(45X^2-48X+4)^2(48X^2-56X+9)^2}$$

*and*

$$\begin{aligned}P(X)=&4(1310940288X^{10}-5781798144X^9+10240292160X^8\\&-9287887680X^7+4594990032X^6-1260832608X^5\\&+201956472X^4-18590520X^3+953761X^2-23680X+288).\end{aligned}$$

*The values of $t^+_{22,7}$ at the CM points by certain orders $\Lambda$ are given in the following* *table:*

| $\Lambda$ | $t^+_{22,7}$ |
|---|---|
| $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $0$ |
| $\mathbb{Z}[\sqrt{-5}]$ | $\infty$ |
| $\mathbb{Z}[\frac{1+3\sqrt{-3}}{2}]$ | $1$ |
| $\mathbb{Z}[\sqrt{-14}]$ | $\alpha : 36\alpha^2 - 36\alpha + 1 = 0$ |
| $\mathbb{Z}[\sqrt{-77}]$ | $\alpha : 45\alpha^2 - 48\alpha + 4 = 0$ |
| $\mathbb{Z}[\sqrt{-154}]$ | $\alpha : 48\alpha^2 - 56\alpha + 9 = 0$ |

□

**Proposition 4.10.** *(1) For $W \in \{\langle\omega_2, \omega_{77}\rangle, \langle\omega_{11}, \omega_{14}\rangle, \langle\omega_{14}, \omega_{22}\rangle\}$, the curves $X(22,7)^W$ are genus 0 curves. The first two of them have rational points and the third one does not.*

*There exist functions $U_i = U_{i,154}$ such that $\mathbb{Q}(X(22,7)^{\langle\omega_i,\omega_{154}\rangle}) = \mathbb{Q}(U_i)$, $i = 2, 11$, defined as follows:*

| $i$ | $U_i$ |
|---|---|
| 2 | $\frac{1+\sqrt{36(t^+_{22,7})^2-36t^+_{22,7}+1}}{3t^+_{22,7}}$ |
| 11 | $\frac{2+\sqrt{45(t^+_{22,7})^2-48t^+_{22,7}+4}}{3t^+_{22,7}}$ |

*The action of the Atkin-Lehner involution $\omega_d$ on such a Hauptmodul $U$ is given by a rational linear function $R_d(X) \in \mathbb{Q}(X)$ such that $U \circ \omega_d = R_d(U)$ as described in the following table:*

| | $R_2$ | $R_7$ | $R_{11}$ | $R_{14}$ | $R_{22}$ | $R_{77}$ | $R_{154}$ |
|---|---|---|---|---|---|---|---|
| $U_2$ | $X$ | $\frac{4-6X}{6-X}$ | $\frac{4-6X}{6-X}$ | $\frac{4-6X}{6-X}$ | $\frac{4-6X}{6-X}$ | $X$ | $X$ |
| $U_{11}$ | $\frac{-5+4X}{-4+X}$ | $\frac{-5+4X}{-4+X}$ | $X$ | $X$ | $\frac{-5+4X}{-4+X}$ | $\frac{-5+4X}{-4+X}$ | $X$ |

*On the other hand,*

$$\mathbb{Q}(X(22,7)^{\langle\omega_{14},\omega_{22}\rangle}) = \mathbb{Q}\left(12t^+_{22,7} - 7, \sqrt{-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9)}\right),$$

*and the corresponding equation is*

$$X^2 + Y^2 - 22 = 0.$$

*(2) The curves $X(22,7)^W$ for $W$ in the following table are hyperelliptic curves. For each of them, the polynomial $P_W(X)$ provides a system of generators of the field of rational functions*

$$\mathbb{Q}(X(22,7)^W) = \mathbb{Q}\left(U_W, \sqrt{P_W(U_W)}\right)$$

*and a corresponding equation for the curve*

$$Y^2 = P_W(X).$$

| $W$ | $U_W$ | $P_W(X)$ |
|---|---|---|
| $\langle\omega_2,\omega_7\rangle$ | $t^+_{22,7}$ | $-3(45X^2-48X+4)(48X^2-56X+9)$ |
| $\langle\omega_7,\omega_{11}\rangle$ | $t^+_{22,7}$ | $-3(36X^2-36X+1)(48X^2-56X+9)$ |
| $\langle\omega_7,\omega_{22}\rangle$ | $t^+_{22,7}$ | $(45X^2-48X+4)(36X^2-36X+1)$ |
| $\langle\omega_{14}\rangle$ | $U_{11}$ | $-(27X^4-224X^3+882X^2-928X+291)$ |
| $\langle\omega_{77}\rangle$ | $U_2$ | $-(27X^4-112X^3+520X-320X+48)$ |
| $\langle\omega_{154}\rangle$ | $U_2$ | $X^4-8X^3+45X^2-28X+4$ |
| $\langle\omega_2,\omega_{11}\rangle$ | $t^+_{22,7}$ | $-3(36X^2-36X+1)(45X^2-48X+4)(48X^2-56X+9)$ |
| $\langle\omega_2\rangle$ | $U_2$ | $-(X^4-8X^3+45X^2-28X+4)(27X^4-112X^3+520X^2-320X+48)$ |
| $\langle\omega_{11}\rangle$ | $U_{11}$ | $-(X^4-48X^3+246X^2-272X+89)(27X^4-224X^3+882X^2-928X+291)$ |

*(3) The remaining curves $X(22,7)^W$ are non-hyperelliptic curves.*

*(a) The natural map of $X(22,7)^{\langle\omega_{22}\rangle}$ to*

$$X(22,7)^{\langle\omega_7,\omega_{22}\rangle} \underset{X(22,7)^+}{\times} X(22,7)^{\langle\omega_{14},\omega_{22}\rangle}$$

*provides a model for this curve in $\mathbb{P}^3$ with affine equations*

$$Y^2 = (45X^2-48X+4)(36X^2-36X+1), \quad Z^2 = 22 - X^2.$$

*(b) The natural map of $X(22,7)$ to*

$$X(22,7)^{\langle\omega_2,\omega_{77}\rangle} \underset{X(22,7)^+}{\times} X(22,7)^{\langle\omega_{11},\omega_{14}\rangle} \underset{X(22,7)^+}{\times} X(22,7)^{\langle\omega_{14},\omega_{22}\rangle}$$

*provides a model for this curve in $\mathbb{P}^4$ as the intersection of the 3 hyperquadrics*

$$\begin{aligned} Y^2 &= (36X^2 - 36X + 1), \\ Z^2 &= (45X^2 - 48X + 4), \\ T^2 &= -3(48X^2 - 56X + 9). \end{aligned}$$

*(c) The natural map of $X(22,7)^{\langle\omega_7\rangle}$ to*

$$X(22,7)^{\langle\omega_2,\omega_7\rangle} \underset{X(22,7)^+}{\times} X(22,7)^{\langle\omega_7,\omega_{22}\rangle}$$

*provides a model for this curve in $\mathbb{P}^3$ with affine equations*

$$\begin{aligned} Y^2 &= -3(45X^2 - 48X + 4)(48X^2 - 56X + 9), \\ Z^2 &= (45X^2 - 48X + 4)(36X^2 - 36X + 1). \end{aligned}$$

*The birrational map to $\mathbb{P}^2$ such that*

$$[X : Y : Z : 1] \mapsto [Y : Z : 2(48X^2 - 56X + 9)] = [U : V : 1]$$

*has image the non-singular planar quartic curve*

$$7 + 18Y^2 + 7Y^4 - 6Z^2 - 38Y^2Z^2 - 49Z^4 = 0.$$

□

**Remark.** Some of the genus 1 curves above have some obvious rational points which would allow us to compute a Weierstrass model for them, for example, we have seen that $X(22,7)^{\langle\omega_7,\omega_{22}\rangle}$ has equation

$$Y^2 = (45X^2 - 48X + 4)(36X^2 - 36X + 1)$$

and $(0,2)$ is an obvious rational point. A Weierstrass model for it is

$$Y^2 = X(X^2 + 150X - 7)$$

and the expression for the function corresponding to the variable $X$ in this model, instead of $t^+_{22,7}$, is

$$\frac{8 - 192t + 189t^2 + 4\sqrt{4 - 192t + 1917t^2 - 3348t^3 + 1620t^4}}{9t^2},$$

where $t = t^+_{22,7}$. Note that even though the equation is somewhat simpler, the corresponding functions are fairly more involved. Therefore, since we are both interested in the functions and the corresponding equations, we will restrict to the models given above.

### 4.3.3 The curve $X(6,5)$

This case is easier than the ones above, since $X(6,5)^{\langle\omega_2,\omega_3\rangle}$ has genus 0, and therefore there exists a morphism $X(6,5)^{\langle\omega_2,\omega_3\rangle} \to X(6,1)^{\langle\omega_2,\omega_3\rangle} = X(6,1)^+$ between curves of genus 0. For discriminant $D = 6$ this happens for $N = 5, 7, 13$.

In this situation, we can break the computation of a Hauptmodul for $X(6,5)^+$ into two steps. First we follow the approach found in [Elk98] to compute directly a Hauptmodul for the curve $X(6,5)^{\langle\omega_2,\omega_3\rangle}$. In order to do so, we can make use of the knowledge of the polynomial $\Psi_5(X,Y) = \Psi_{\Gamma(6,1)^+,5,t_6^+}(X,Y)$ to simplify the discussion of cases. Secondly, we can obtain a Hauptmodul for $X(6,5)^+$ together with some additional properties from this Hauptmodul. Even though the case $N = 5$ is quite straightforward, the same approach can be used to solve as well the cases $D = 6, N = 7$ and $D = 10, N = 3, 7$.

We will begin by choosing a Hauptmodul defined over $\mathbb{Q}$ for the curve $X(6,5)^{\langle\omega_2,\omega_3\rangle}$, we will simply call it $f$, since we will later normalize it and rename it, as follows:

| $(t_6^+, t_6^+ \circ \omega_5)$ | $f$ |
|---|---|
| $(1, \frac{421850521}{1771561})$ | $0$ |
| $(\frac{421850521}{1771561}, 1)$ | $1$ |
| $(\infty, \frac{152881}{138240})$ | $\infty$ |

It can be checked as we did before, that all the points we chose to define the Hauptmodul are non-singular on the curve. Therefore, $f$ is a Hauptmodul defined over $\mathbb{Q}$. Consequently,

$$t_6^+ - 1 = af^4(f^2 + bf + c), \quad a, b, c \in \mathbb{Q},$$

and $f^2 + bf + c$ vanishes at the couple of points where $t_6^+(P) = t_6^+ \circ \omega_5(P) = 1$, or equivalently at the elliptic points by $\mathbb{Z}[i]$. Moreover, since $f \circ \omega_5$ is a Hauptmodul and $\omega_5$ is involutive, there exist $x, y, z \in \mathbb{Q}$ such that

$$f \circ \omega_5 = \frac{xf + y}{zf - x}$$

and by the choice of the values above, we obtain that $f \circ \omega_5 = \frac{f-1}{df-1}$ for some $d \in \mathbb{Q}$. Therefore,

$$t_6^+ \circ \omega_5 - 1 = \frac{a'(f-1)^4(f^2 + bf + c)}{(df-1)^6}.$$

Observe that the quadratic polynomial in the numerator is the same in both expressions since its corresponding divisor is fixed by $\omega_5$. In particular, if we evaluate both equations at suitable points where we know the values of $f$, one of the expressions will become an identity, and the other will yield an equation relating these parameters:

- evaluating at $f = 0$: $\frac{421850521}{1771561} - 1 = a'c$,
- evaluating at $f = \infty$: $\frac{152881}{138240} - 1 = \frac{a'}{d^6}$,
- evaluating at $f = 1$: $\frac{421850521}{1771561} - 1 = a(1 + b + c)$,
- evaluating at $f = 1/d$: $\frac{152881}{138240} - 1 = \frac{a(cd^2+db+1)}{d^6}$.

If we solve these equations, we obtain that we can write all the parameters in terms of $d$, namely,

| | |
|---|---|
| $a$ | $-\frac{14641d^6}{138240(-1+d)}$ |
| $b$ | $-\frac{58071715430400+25937424601d^5}{25937424601d^5}$ |
| $c$ | $\frac{58071715430400}{25937424601d^6}$ |
| $a'$ | $\frac{14641d^6}{138240}$ |

Then, we can either substitute the resulting expressions for $t_6^+$ and $t_6^+ \circ \omega_5$ into $\Psi_5(t_6^+, t_6^+ \circ \omega_5) = 0$ or use that the above expression of $t_6^+$ in terms of $f$ must contain a quadratic square factor. Either way, we obtain a unique possible value for $d$, namely, $d = -\frac{504}{121}$. In particular,

$$t_6^+ = \frac{\left(605 - 2772t + 5292t^2\right)\left(3025 + 6930t + 10584t^2\right)^2}{5536128125}$$

and

$$f \circ \omega_5 = \frac{121(1-f)}{121 + 504f}.$$

This allows us to compute the value of $f$ at any point where we know the values of $t_6^+$ and $t_6^+ \circ \omega_5$. In particular, we obtain the following values:

| $D(\Lambda)$ | $(t_6^+, t_6^+ \circ \omega_5)$ | $f$ |
|---|---|---|
| $-4$ | $(1,1)$ | $\alpha : 7056\alpha^2 + 5544\alpha + 3025 = 0$ |
| $-24$ | $(0,0)$ | $\alpha : 5292\alpha^2 - 2772\alpha + 605 = 0$ |
| $-40$ | $(\frac{2312}{125}, \frac{2312}{125})$ | $-\frac{11}{14}$ |
| $-120$ | $(\frac{5776}{3375}, \frac{5776}{3375})$ | $\frac{11}{36}$ |

In addition, we obtain a Hauptmodul for $X(6,5)^+$, for example, we can take $F = f + f \circ \omega_5 = \frac{121+505f^2}{121+504f}$. We could also have taken $f \cdot f \circ \omega_5 = -\frac{121}{504}(F-1)$.

Now, consider $t_{6,5}^+$ the Hauptmodul for $X(6,5)^+$ defined by the following three values:

| $D(\Lambda)$ | $t_{6,5}^+$ |
|---|---|
| $-4$ | $\infty$ |
| $-24$ | $1$ |
| $-40$ | $-1$ |

Comparing the values of $F$ and $t_{6,5}^+$ at these three points we obtain that

$$t_{6,5}^+ = \frac{7F+33}{2(14F+11)} = \frac{4\left(605 + 2079f + 441f^2\right)}{3025 + 5544f + 7056f^2},$$

and in particular we obtain that the value of $t_{6,5}^+$, for example, at the CM point by $\mathbb{Z}[\sqrt{-30}]$ is $\frac{61}{64}$. Now, since the morphism $X(6,5)^{\langle \omega_2, \omega_3 \rangle} \to X(6,5)^+$ ramifies at the CM points by $\mathbb{Z}[\sqrt{-10}]$ and $\mathbb{Z}[\sqrt{-30}]$, we can take a Hauptmodul of $X(6,5)^{\langle \omega_2, \omega_3 \rangle}$ such that

$$U_{2,3}^2 = k \frac{t_{6,5}^+ - \frac{61}{64}}{t_{6,5}^+ + 1}$$

for a certain squarefree integer $k$. Then, substituting $t_{6,5}^+$ in this expression in terms of $f$, yields that $k = -1$ and we can take

$$U_{2,3} = \sqrt{-\frac{t_{6,5}^+ - \frac{61}{64}}{t_{6,5}^+ + 1}} = \frac{7}{24}\frac{36f-1}{14f+11}.$$

Moreover, for this Hauptmodul the action of $\omega_5$ is $U_{2,3} \circ \omega_5 = -U_{2,3}$ and the expression of $t_6^+$ in terms of $U_{2,3}$ is

$$t_6^+ = \frac{\left(3 + 128U_{2,3}^2\right)\left(228 + 117U_{2,3} + 272U_{2,3}^2\right)^2}{125(-3 + 4U_{2,3})^6}.$$

Then, using Proposition 3.7 we can obtain the automorphic derivative of $U_{2,3}$ in terms of that of $t_6^+$ and from that of $U_{2,3}$, using the same result, we can obtain the automorphic derivative of $t_{6,5}^+$. In the following propositions we give these results together with Hauptmoduln for all the genus 0 curves and generating functions and equations for the others.

**Proposition 4.11.** *There exists a Hauptmodul for the curve $X(6,5)^+$, $t_{6,5}^+$, defined by the following values*

| $D(\Lambda)$ | $t_{6,5}^+$ |
|---|---|
| $-4$ | $\infty$ |
| $-24$ | $1$ |
| $-40$ | $-1$ |

*and it satisfies*

$$t_6^+ \cdot t_6^+ \circ \omega_5 = \frac{(-1 + t_{6,5}^+)^2 \left(919 - 173t_{6,5}^+ + 64(t_{6,5}^+)^2\right)^2}{(-1 + 4t_{6,5}^+)^6},$$
$$(t_6^+ - 1)(t_6^+ \circ \omega_5 - 1) = \frac{729(4 - 5t_{6,5}^+)^4}{(1 - 4t_{6,5}^+)^6}.$$

*Moreover, $Da(t_{6,5}^+) + R(t_{6,5}^+) = 0$, where*

$$R(X) = \frac{3\left(5120X^4 - 6928X^3 + 12841X^2 - 24304X + 13307\right)}{4(X-1)^2(X+1)^2(64X-61)^2}.$$

*The values of $t_{6,5}^+$ at the CM points by certain orders $\Lambda$ are given in the following table:*

| $\Lambda$ | $t^+_{6,5}$ |
|---|---|
| $\mathbb{Z}[\sqrt{-30}]$ | $\frac{61}{64}$ |
| $\mathbb{Z}[5i]$ | $\frac{4}{5}$ |
| $\mathbb{Z}[\frac{1+5\sqrt{-3}}{2}]$ | $\frac{1}{4}$ |

$\square$

Next we will give Hauptmoduln for all the remaining genus 0 curves with rational points.

**Proposition 4.12.** *There exist functions $U_K = U_K^{(6,5)}$ such that*

$$\mathbb{Q}(X(6,5)^{\langle \omega_k : k \in K \rangle}) = \mathbb{Q}(U_K)$$

*for $K \in \{\{2,3\},\{2,5\},\{6,10\},\{2,15\},\{3,10\},\{5,6\},\{6\},\{30\}\}$. They can be chosen as follows*

| $K$ | $U_K^2$ |
|---|---|
| $\{2,3\}$ | $-\frac{t^+_{6,5}-\frac{61}{64}}{t^+_{6,5}+1}$ |
| $\{2,5\}$ | $-\frac{t^+_{6,5}-\frac{61}{64}}{t^+_{6,5}-1}$ |
| $\{6,10\}$ | $-(t^+_{6,5}-\frac{61}{64})$ |
| $\{2,15\}$ | $\frac{t^+_{6,5}+1}{t^+_{6,5}-1}$ |
| $\{3,10\}$ | $t^+_{6,5}-1$ |
| $\{5,6\}$ | $t^+_{6,5}+1$ |

*and*

$$U_6 = \frac{U_{6,10}-\frac{11}{8}}{U_{5,6}-\frac{1}{4}}, \quad U_{30} = U_{5,6} + U_{3,10}.$$

*The action of the Atkin-Lehner involution $\omega_d$ on such a Hauptmodul $U$ is given by a rational linear function $R_d(X) \in \mathbb{Q}(X)$ such that $U \circ \omega_d = R_d(U)$ as* *described in the following table:*

| | $R_2$ | $R_3$ | $R_5$ | $R_6$ | $R_{10}$ | $R_{15}$ | $R_{30}$ |
|---|---|---|---|---|---|---|---|
| $U_{2,3}$ | $X$ | $X$ | $-X$ | $X$ | $-X$ | $-X$ | $-X$ |
| $U_{2,5}$ | $X$ | $-X$ | $X$ | $-X$ | $-X$ | $X$ | $-X$ |
| $U_{6,10}$ | $-X$ | $-X$ | $-X$ | $X$ | $X$ | $X$ | $-X$ |
| $U_{2,15}$ | $X$ | $-X$ | $-X$ | $-X$ | $-X$ | $X$ | $X$ |
| $U_{3,10}$ | $-X$ | $X$ | $-X$ | $-X$ | $X$ | $-X$ | $X$ |
| $U_{5,6}$ | $-X$ | $-X$ | $X$ | $X$ | $-X$ | $-X$ | $X$ |
| $U_6$ | $-\frac{1}{X}$ | $\frac{2+11X}{-11+2X}$ | $\frac{11-2X}{2+11X}$ | $\frac{11-2X}{2+11X}$ | $\frac{2+11X}{-11+2X}$ | $-\frac{1}{X}$ | $X$ |
| $U_{30}$ | $-X$ | $-\frac{2}{X}$ | $\frac{2}{X}$ | $\frac{2}{X}$ | $-\frac{2}{X}$ | $-X$ | $X$ |

*Moreover, for a suitable determination of the root we have that*

$$t_6^+ = 1 - \frac{27(7+24U_{2,3})^4\left(1+U_{2,3}^2\right)}{125(-3+4U_{2,3})^6}.$$

□

**Remark.** The expressions for the automorphic derivatives can be computed from that of $t_{6,5}^+$ as in the previous cases using Proposition 3.7.

Finally, models for the rest of the curves in this case are given in the following proposition.

**Proposition 4.13.** *(1) For $i = 2, 10$, $X(6,5)^{\langle\omega_i\rangle}$ are conics without real points. We have*

$$\mathbb{Q}(X(6,5)^{\langle\omega_2\rangle}) = \mathbb{Q}(U_{2,15}, U_{2,5})$$
$$\mathbb{Q}(X(6,5)^{\langle\omega_{10}\rangle}) = \mathbb{Q}(U_{6,10}, U_{3,10})$$

*and the corresponding equations are*

$$3X^2 + 128Y^2 + 125 = 0, \quad X^2 + Y^2 + \frac{3}{64} = 0,$$

*respectively.*

*(2) The curves $X(6,5)^W$, for $W \subset \langle \omega_3, \omega_5 \rangle$, are genus 1 curves. For each of them, functions $f_W, g_W$ providing a system of generators of the function field $\mathbb{Q}(X(6,5)^W)$ and a polynomial $P_W$ giving the corresponding equation for the curve*

$$f_W^2 = P_W(g_W)$$

*are displayed in the following table:*

| $W$ | $f_W, g_W$ | $P_W(X)$ |
|---|---|---|
| $\langle \omega_3, \omega_5 \rangle$ | $U_{2,3}U_{3,10}(t_{6,5}^+ + 1), t_{6,5}^+$ | $-(X-1)(X+1)(X-\frac{61}{64})$ |
| $\langle \omega_3 \rangle$ | $U_{2,3}(U_{3,10}^2 + 2), U_{3,10}$ | $-(X^2+2)(X^2+\frac{3}{64})$ |
| $\langle \omega_5 \rangle$ | $U_{2,5}(U_{5,6}^2 - 2), U_{5,6}$ | $-(X^2-\frac{125}{64})(X^2-2)$ |
| $\langle \omega_{15} \rangle$ | $U_{2,15}(U_{6,10}^2 + \frac{3}{64}), U_{6,10}$ | $(X^2-\frac{125}{64})(X^2+\frac{3}{64})$ |
| $\langle 1 \rangle$ | $8U_{6,10}U_{30}, U_{30}$ | $-16X^4+61X^2-64$ |

□

### 4.3.4 The curve $X(6,7)$

This case is, apart from some increase in the computational complexity, analogous to the previous one.

**Proposition 4.14.** *There exists a Hauptmodul for the curve $X(6,7)^+$, $t_{6,7}^+$, defined by the following values*

| $D(\Lambda)$ | $t_{6,7}^+$ |
|---|---|
| $-3$ | $\infty$ |
| $-24$ | $0$ |
| $-84$ | $1$ |

*and it satisfies*

$$t_6^+ \cdot t_6^+ \circ \omega_7 = \frac{(t_{6,7}^+)^2 \left(-2016 + 3129t_{6,7}^+ - 518(t_{6,7}^+)^2 + 81(t_{6,7}^+)^3\right)^2}{16(-4+7t_{6,7}^+)^6},$$

$$(t_6^+ - 1)(t_6^+ \circ \omega_7 - 1) = \frac{\left(16 + 3t_{6,7}^+ + 9(t_{6,7}^+)^2\right)^4}{16(4-7t_{6,7}^+)^6}.$$

*Moreover,* $Da(t^+_{6,7}) + R(t^+_{6,7}) = 0$, *where*

$$R(X) = \frac{3\left(1024 - 5152X + 13777X^2 - 15753X^3 + 8505X^4\right)}{4(-1+X)^2X^2(-32+81X)^2}.$$

*The values of* $t^+_{6,7}$ *at the CM points by certain orders* $\Lambda$ *are given in the following table:*

| $\Lambda$ | $t^+_{6,7}$ |
|---|---|
| $\mathbb{Z}[\sqrt{-42}]$ | $\frac{32}{81}$ |
| $\mathbb{Z}[\frac{1+7\sqrt{-3}}{2}]$ | $\frac{4}{7}$ |
| $\mathbb{Z}[\sqrt{-10}]$ | $\frac{1}{3}$ |
| $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ | $\frac{4}{3}$ |

□

Next we will give Hauptmoduln for all the remaining genus 0 curves with rational points.

**Proposition 4.15.** *There exist functions* $U_K = U_K^{(6,7)}$ *such that*

$$\mathbb{Q}(X(6,7)^{\langle \omega_k : k \in K\rangle}) = \mathbb{Q}(U_K)$$

*for* $K \in \{\{2,3\},\{2,21\},\{3,7\},\{3,14\},\{6,7\},\{6,14\},\{42\}\}$. *They can be chosen as follows*

| $K$ | $U_K^2$ |
|---|---|
| $\{2,3\}$ | $-3\frac{t^+_{6,7}-1}{t^+_{6,7}-\frac{32}{81}}$ |
| $\{2,21\}$ | $3t^+_{6,7}$ |
| $\{3,7\}$ | $-3\frac{t^+_{6,7}}{t^+_{6,7}-\frac{32}{81}}$ |
| $\{3,14\}$ | $\frac{t^+_{6,7}}{t^+_{6,7}-1}$ |
| $\{6,7\}$ | $3(t^+_{6,7}-1)$ |
| $\{6,14\}$ | $-(t^+_{6,7}-\frac{32}{81})$ |

*and*

$$U_{42} = U_{2,21} + U_{6,7}.$$

*The action of the Atkin-Lehner involution $\omega_d$ on such a Hauptmodul $U$ is given by a rational linear function $R_d(X) \in \mathbb{Q}(X)$ such that $U \circ \omega_d = R_d(U)$ as described in the following table:*

| | $R_2$ | $R_3$ | $R_6$ | $R_7$ | $R_{14}$ | $R_{21}$ | $R_{42}$ |
|---|---|---|---|---|---|---|---|
| $U_{2,3}$ | $X$ | $X$ | $X$ | $-X$ | $-X$ | $-X$ | $-X$ |
| $U_{2,21}$ | $X$ | $-X$ | $-X$ | $-X$ | $-X$ | $X$ | $X$ |
| $U_{3,7}$ | $-X$ | $X$ | $-X$ | $X$ | $-X$ | $X$ | $-X$ |
| $U_{3,14}$ | $-X$ | $X$ | $-X$ | $-X$ | $X$ | $-X$ | $X$ |
| $U_{6,7}$ | $-X$ | $-X$ | $X$ | $X$ | $-X$ | $-X$ | $X$ |
| $U_{6,14}$ | $-X$ | $-X$ | $X$ | $-X$ | $X$ | $X$ | $-X$ |
| $U_{42}$ | $\frac{3}{X}$ | $-X$ | $-\frac{3}{X}$ | $-\frac{3}{X}$ | $-X$ | $\frac{3}{X}$ | $X$ |

*Moreover, for a suitable determination of the root we have that*

$$t_6^+ = 1 - \frac{49\left(162 + 27U_{2,3} + 44U_{2,3}^2\right)^4}{4(-27 + 10U_{2,3})^6\left(3 + U_{2,3}^2\right)}.$$

□

**Remark.** The expressions for the automorphic derivatives can be computed from that of $t_{6,7}^+$ as in the previous cases using Proposition 3.7.

Finally, models for the rest of the curves in this case are given in the following proposition.

**Proposition 4.16.** *(1) For $i = 3, 6, 21$, $X(6,7)^{\langle \omega_i \rangle}$ are conics without rational points. We have*

$$\mathbb{Q}(X(6,7)^{\langle \omega_3 \rangle}) = \mathbb{Q}(U_{2,3}, U_{3,7})$$
$$\mathbb{Q}(X(6,7)^{\langle \omega_6 \rangle}) = \mathbb{Q}(U_{6,7}, U_{6,14})$$
$$\mathbb{Q}(X(6,7)^{\langle \omega_{21} \rangle}) = \mathbb{Q}(U_{2,21}, U_{6,14})$$

*and the corresponding equations are, respectively,*

$$32X^2 + 49Y^2 + 3\cdot 81 = 0, \quad X^2 + 3Y^2 + \frac{49}{27} = 0, \quad X^2 + 3Y^2 - \frac{32}{27} = 0.$$

(2) *The curves $X(6,7)^W$, for $W \subset \langle\omega_2, \omega_7\rangle$, are genus 1 curves. For each of them, functions $f_W, g_W$ providing a system of generators of the function field $\mathbb{Q}(X(6,7)^W)$ and a polynomial $P_W$ giving the corresponding equation for the curve*

$$f_W^2 = P_W(g_W)$$

*are displayed in the following table:*

| $W$ | $f_W, g_W$ | $P_W(X)$ |
|---|---|---|
| $\langle\omega_2,\omega_7\rangle$ | $U_{2,3}U_{2,21}(t^+_{6,7} - \frac{32}{81})/3, t^+_{6,7}$ | $-X(X-1)(X-\frac{32}{81})$ |
| $\langle\omega_2\rangle$ | $U_{2,3}(U^2_{2,21} - \frac{32}{27}), U_{2,21}$ | $-3(X^2-3)(X^2-\frac{32}{27})$ |
| $\langle\omega_7\rangle$ | $U_{3,7}(U^2_{6,7} + \frac{49}{27}), U_{6,7}$ | $-3(X^2+3)(X^2+\frac{49}{27})$ |
| $\langle\omega_{14}\rangle$ | $U_{3,14}(U^2_{6,14} + \frac{49}{81}), U_{6,14}$ | $(X^2-\frac{32}{81})(X^2+\frac{49}{81})$ |
| $\{id\}$ | $18U_{6,14}U_{42}, U_{42}$ | $-27X^4 - 34X^2 - 243$ |

□

### 4.3.5 The curve $X(6,11)$

Now, we are going to deal with the curve $X(6,11)$. From a theoretical point of view it is similar to the previous cases, however the determination of a Hauptmodul is considerably harder from a computational point of view.

**Proposition 4.17.** *There exists a Hauptmodul for the curve $X(6,11)^+$, $t^+_{6,11}$, defined by the following values*

| $D(\Lambda)$ | $t^+_{6,11}$ |
|---|---|
| $-24$ | $0$ |
| $-88$ | $\infty$ |
| $-132$ | $1$ |

*and it satisfies*

$$t_6^+ \cdot t_6^+ \circ \omega_{11} = \frac{A(t_{6,11}^+)}{\left(3 + 22t_{6,11}^+ + 275(t_{6,11}^+)^2\right)^6}$$

$$(t_6^+ - 1)(t_6^+ \circ \omega_{11} - 1) = \frac{729\left(-1 - 41t_{6,11}^+ - 751(t_{6,11}^+)^2 + 441(t_{6,11}^+)^3\right)^4}{\left(3 + 22t_{6,11}^+ + 275(t_{6,11}^+)^2\right)^6},$$

*where*

$$A(X) = 16X^2\left(-891 - 36036X - 367202X^2 \; -231132X^3 - 9164859X^4 + 3886472X^5\right)^2.$$

*Moreover,* $Da(t_{6,11}^+) + R(t_{6,11}^+) = 0$*, where*

$$R(X) = \frac{3\left(9 + 507X + 13531X^2 + 285618X^3 + 2261179X^4 - 2224101X^5 + 1771561X^6\right)}{4(-1+X)^2X^2\left(3 + 118X + 1331X^2\right)^2}.$$

*In the following table we give the values of* $t_{6,11}^+$ *at* $\mathrm{CM}(\Lambda)$ *by certain orders* $\Lambda$*:*

| $\Lambda$ | $t_{6,11}^+$ | $\Lambda$ | $t_{6,11}^+$ |
|---|---|---|---|
| $\mathbb{Z}[\frac{1+11\sqrt{-3}}{2}]$ | $\alpha : 3 + 22\alpha + 275\alpha^2 = 0$ | $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ | $-1$ |
| $\mathbb{Z}[\sqrt{-66}]$ | $\alpha : 3 + 118\alpha + 1331\alpha^2 = 0$ | $\mathbb{Z}[\sqrt{-30}]$ | $\frac{3}{5}$ |
| $\mathbb{Z}[\sqrt{-10}]$ | $-\frac{1}{9}$ | $\mathbb{Z}[\frac{1+\sqrt{-43}}{2}]$ | $-\frac{1}{49}$ |

□

In this case the computations are harder and therefore, since the solution has rational coefficients, it is useful to find $p$-adic approximations. Then we can check that the approximated value up to a certain precision is indeed a solution. Finally we can conclude that it is the solution we are looking for thanks to Theorem 4.2. In this case, we worked with 101-adic approximations.

Next we will give Hauptmoduln for all the remaining genus 0 curves with rational points.

**Proposition 4.18.** *There exist functions* $U_K = U_K^{(6,11)}$ *such that*

$$\mathbb{Q}(X(6,11)^{\langle \omega_k : k \in K\rangle}) = \mathbb{Q}(U_K)$$

*for $K \in \{\{2,33\},\{3,22\},\{6,11\},\{66\}\}$. They can be chosen as follows*

| $K$ | $U_K^2$ |
|---|---|
| $\{2,33\}$ | $-t^+_{6,11}$ |
| $\{3,22\}$ | $2\frac{t^+_{6,11}}{t^+_{6,11}-1}$ |
| $\{6,11\}$ | $-2(t^+_{6,11}-1)$ |

*and*

$$U_{66} = \frac{U_{2,33}-1}{U_{6,11}-2}.$$

*The action of the Atkin-Lehner involution $\omega_d$ on such a Hauptmodul $U$ is given by a rational linear function $R_d(X) \in \mathbb{Q}(X)$ such that $U \circ \omega_d = R_d(U)$ as described in the following table:*

| | $R_2$ | $R_3$ | $R_6$ | $R_{11}$ | $R_{22}$ | $R_{33}$ | $R_{66}$ |
|---|---|---|---|---|---|---|---|
| $U_{2,33}$ | $X$ | $-X$ | $-X$ | $-X$ | $-X$ | $X$ | $X$ |
| $U_{3,22}$ | $-X$ | $X$ | $-X$ | $-X$ | $X$ | $-X$ | $X$ |
| $U_{6,11}$ | $-X$ | $-X$ | $X$ | $X$ | $-X$ | $-X$ | $X$ |
| $U_{66}$ | $\frac{-1+X}{-1+2X}$ | $\frac{1}{2X}$ | $\frac{2X-1}{2(X-1)}$ | $\frac{2X-1}{2(X-1)}$ | $\frac{1}{2X}$ | $\frac{-1+X}{-1+2X}$ | $X$ |

□

**Remark.** The expressions for the automorphic derivatives can be computed from that of $t^+_{6,11}$ as in the previous cases using Proposition 3.7.

Finally, models for the rest of the curves in this case are given in the following proposition.

**Proposition 4.19.** *(1) $X(6,11)^{\langle \omega_6,\omega_{22}\rangle}$ is a conic without real points. We have*

$$\mathbb{Q}(X(6,11)^{\langle \omega_6,\omega_{22}\rangle}) = \mathbb{Q}\left(t^+_{6,11}, \sqrt{-(1331(t^+_{6,11})^2+118t^+_{6,11}+3)}\right)$$

*and the corresponding equation is*

$$1331X^2+118X+Y^2+3=0.$$

*(2) For* $W \in \{\langle\omega_2,\omega_3\rangle, \langle\omega_2,\omega_{11}\rangle, \langle\omega_3,\omega_{11}\rangle, \langle\omega_6\rangle, \langle\omega_{22}\rangle, \langle\omega_{33}\rangle\}$*, the corresponding curves* $X(6,11)^W$ *are genus* 1 *curves. For each of them, functions* $f_W, g_W$ *providing a system of generators of the function field* $\mathbb{Q}(X(6,11)^W)$ *and a polynomial* $P_W$ *giving the corresponding equation for the curve*

$$f_W^2 = P_W(g_W)$$

*are displayed in the following table:*

| $W$ | $f_W, g_W$ | $P_W(X)$ |
|---|---|---|
| $\langle\omega_2,\omega_3\rangle$ | $U_{6,11}\sqrt{-(1331(t^+_{6,11})^2+118t^+_{6,11}+3)}, t^+_{6,11}$ | $2(X-1)(3+118X+1331X^2)$ |
| $\langle\omega_2,\omega_{11}\rangle$ | $U_{2,33}U_{6,11}\sqrt{-(1331(t^+_{6,11})^2+118t^+_{6,11}+3)}, t^+_{6,11}$ | $-2X(X-1)(3+118X+1331X^2)$ |
| $\langle\omega_3,\omega_{11}\rangle$ | $U_{2,33}\sqrt{-(1331(t^+_{6,11})^2+118t^+_{6,11}+3)}, t^+_{6,11}$ | $X(3+118X+1331X^2)$ |
| $\langle\omega_6\rangle$ | $2\sqrt{-(1331(t^+_{6,11})^2+118t^+_{6,11}+3)}, U_{6,11}$ | $-5808+5560X^2-1331X^4$ |
| $\langle\omega_{22}\rangle$ | $(U_{3,22}^2-2)\sqrt{-(1331(t^+_{6,11})^2+118t^+_{6,11}+3)}, U_{3,22}$ | $-4\left(3-62X^2+363X^4\right)$ |
| $\langle\omega_{33}\rangle$ | $\sqrt{-(1331(t^+_{6,11})^2+118t^+_{6,11}+3)}, U_{2,33}$ | $-3+118X^2-1331X^4$ |

*(3) The curves* $X(6,11)^W$*, for* $W \in \{\langle\omega_2\rangle, \langle\omega_3\rangle, \langle\omega_{11}\rangle\}$*, are genus* 2 *curves. For each of them, functions* $f_W, g_W$ *providing a system of generators of the function field* $\mathbb{Q}(X(6,11)^W)$ *and a polynomial* $P_W$ *giving the corresponding equation for the curve*

$$f_W^2 = P_W(g_W)$$

*are displayed in the following table:*

| $W$ | $f_W, g_W$ | $P_W(X)$ |
|---|---|---|
| $\langle\omega_2\rangle$ | $U_{6,11}\sqrt{-(1331(t^+_{6,11})^2+118t^+_{6,11}+3)}, U_{2,33}$ | $-2\left(1+X^2\right)\left(3-118X^2+1131X^4\right)$ |
| $\langle\omega_3\rangle$ | $4\frac{\sqrt{-(1331(t^+_{6,11})^2+118t^+_{6,11}+3)}}{U_{6,11}^3}, U_{3,22}$ | $\left(-2+X^2\right)\left(3-62X^2+313X^4\right)$ |
| $\langle\omega_{11}\rangle$ | $4U_{2,33}\sqrt{-(1331(t^+_{6,11})^2+118t^+_{6,11}+3)}, U_{6,11}$ | $-2\left(-2+X^2\right)\left(5008-4760X^2+1131X^4\right)$ |

*(4) The curve* $X(6,11)$ *is an hyperelliptic curve of genus* 3*. Its function field is generated by the functions*

$$\mathbb{Q}(X(6,11)) = \mathbb{Q}\left(\frac{(-1+2U_6^2)^2}{8}\sqrt{-(1331(t^+_{6,11})^2+118t^+_{6,11}+3)}, U_6\right)$$

*and the corresponding equation is*

$$Y^2 = -(19 - 318X + 2133X^2 - 7350X^3 + 13943X^4 - 14700X^5 + 8532X^6 - 2544X^7 + 304X^8).$$

□

### 4.3.6 The curve $X(10,3)$

Finally, we will move on to the discriminant 10 case. Therefore, we will use the Hauptmodul $t_{10}^{+}$ for the curve $X(10,1)^{+}$ introduced in Theorem 2.4 and further studied in Theorem 3.1. Since the map $X(10,3)^{W_{10}} \to X(10,1)^{+}$ has degree 4, the computations can be carried out without any special difficulties.

**Proposition 4.20.** *There exists a Hauptmodul for the curve $X(10,3)^{+}$, $t_{10,3}^{+}$, defined by the following values*

| $D(\Lambda)$ | $t_{10,3}^{+}$ |
|---|---|
| $-3$ | $0$ |
| $-120$ | $\infty$ |
| $-20$ | $1$ |

*and it satisfies*

$$t_{10}^{+} \cdot t_{10}^{+} \circ \omega_3 = \frac{729\left(1 - 11t_{10,3}^{+} + 64(t_{10,3}^{+})^2\right)^2}{4(-1 + t_{10,3}^{+})^2(1 + 80t_{10,3}^{+})^2}$$

$$(t_{10}^{+} - 1)(t_{10}^{+} \circ \omega_3 - 1) = \frac{(5 - 32t_{10,3}^{+})^2(5 + 49t_{10,3}^{+})^2}{4(-1 + t_{10,3}^{+})^2(1 + 80t_{10,3}^{+})^2}.$$

*Moreover, $Da(t_{10,3}^{+}) + R(t_{10,3}^{+}) = 0$, where*

$$R(X) = \frac{875 - 8475X + 44403X^2 - 44768X^3 + 27648X^4}{36(-1 + X)^2X^2(-5 + 32X)^2}.$$

*The values of $t_{10,3}^{+}$ at the CM points by certain orders $\Lambda$ are given in the following* *table:*

| $\Lambda$ | $t^+_{10,3}$ |
|---|---|
| $\mathbb{Z}[\sqrt{-2}]$ | $\frac{5}{32}$ |
| $\mathbb{Z}[\frac{1+3\sqrt{-3}}{2}]$ | $-\frac{5}{4}$ |
| $\mathbb{Z}[3\sqrt{-2}]$ | $-\frac{5}{49}$ |
| $\mathbb{Z}[3\sqrt{-5}]$ | $-\frac{1}{80}$ |
| $\mathbb{Z}[\frac{1+\sqrt{-35}}{2}]$ | $\frac{1}{28}$ |

□

Next we will give Hauptmoduln for all the remaining genus 0 curves with rational points.

**Proposition 4.21.** *There exist functions $U_K = U_K^{(10,3)}$ such that*

$$\mathbb{Q}(X(10,3)^{\langle \omega_k : k \in K \rangle}) = \mathbb{Q}(U_K)$$

*for $K = \{\{2,3\},\{2,5\},\{2,15\},\{3,5\},\{3,10\},\{5,6\},\{30\}\}$. They can be chosen as follows:*

| $K$ | $U_K^2$ |
|---|---|
| $\{2,3\}$ | $3(t^+_{10,3}-1)$ |
| $\{2,5\}$ | $-5t^+_{10,3}$ |
| $\{2,15\}$ | $30(t^+_{10,3}-\frac{5}{32})$ |
| $\{3,5\}$ | $-15\frac{t^+_{10,3}-1}{t^+_{10,3}}$ |
| $\{3,10\}$ | $10\frac{t^+_{10,3}-\frac{5}{32}}{t^+_{10,3}-1}$ |
| $\{5,6\}$ | $-6\frac{t^+_{10,3}-\frac{5}{32}}{t^+_{10,3}}$ |

*and*

$$U_{30} = U_{2,15} + 4U_{5,6}.$$

*The action of the Atkin-Lehner involution $\omega_d$ on such a Hauptmodul $U$ is given by a rational linear function $R_d(X) \in \mathbb{Q}(X)$ such that $U \circ \omega_d = R_d(U)$ as* *described in the following table:*

| | $R_2$ | $R_3$ | $R_5$ | $R_6$ | $R_{10}$ | $R_{15}$ | $R_{30}$ |
|---|---|---|---|---|---|---|---|
| $U_{2,3}$ | $X$ | $X$ | $-X$ | $X$ | $-X$ | $-X$ | $-X$ |
| $U_{2,5}$ | $X$ | $-X$ | $X$ | $-X$ | $X$ | $-X$ | $-X$ |
| $U_{2,15}$ | $X$ | $-X$ | $-X$ | $-X$ | $-X$ | $X$ | $X$ |
| $U_{3,5}$ | $-X$ | $X$ | $X$ | $-X$ | $-X$ | $X$ | $-X$ |
| $U_{3,10}$ | $-X$ | $X$ | $-X$ | $-X$ | $X$ | $-X$ | $X$ |
| $U_{5,6}$ | $-X$ | $-X$ | $X$ | $X$ | $-X$ | $-X$ | $X$ |
| $U_{30}$ | $\frac{81}{X}$ | $-X$ | $-\frac{81}{X}$ | $-\frac{81}{X}$ | $-X$ | $\frac{81}{X}$ | $X$ |

*Moreover, for a suitable determination of the root we have that*

$$t_{10}^{+} = \frac{27}{2} - \frac{27U_{2,5}(-5+2U_{2,5})^3}{5(-1+4U_{2,5})^2\left(5+U_{2,5}^2\right)}.$$

□

**Remark.** The expressions for the automorphic derivatives can be computed from that of $t_{10,3}^{+}$ as in the previous cases using Proposition 3.7.

Finally, models for the rest of the curves in this case are given in the following proposition.

**Proposition 4.22.** *(1) For $i = 2, 3, 5$, $X(10,3)^{\langle\omega_i\rangle}$ are conics without rational points. We have*

$$\mathbb{Q}(X(10,3)^{\langle\omega_2\rangle}) = \mathbb{Q}(U_{2,3}, U_{2,5})$$
$$\mathbb{Q}(X(10,3)^{\langle\omega_3\rangle}) = \mathbb{Q}(U_{2,3}, U_{3,5})$$
$$\mathbb{Q}(X(10,3)^{\langle\omega_5\rangle}) = \mathbb{Q}(U_{2,5}, U_{3,5})$$

*and the corresponding equations are, respectively,*

$$5X^2 + 3Y^2 + 15 = 0, \quad 10X^2 - Y^2 + \frac{5 \cdot 81}{16} = 0, \quad 6X^2 + Y^2 + \frac{75}{16} = 0.$$

*(2) The curves $X(10,3)^W$, for $W \subset \langle\omega_6,\omega_{10}\rangle$, are genus 1 curves. For each of them, functions $f_W, g_W$ providing a system of generators of the function field $\mathbb{Q}(X(10,3)^W)$ and a polynomial $P_W$ giving the corresponding equation for the curve*

$$f_W^2 = P_W(g_W)$$

*are displayed in the following table:*

| $W$ | $f_W, g_W$ | $P_W(X)$ |
|---|---|---|
| $\langle\omega_6,\omega_{10}\rangle$ | $\frac{U_{2,3}U_{2,5}U_{3,5}}{15}, t_{10,3}^+$ | $-2X(X-1)(X-\frac{5}{32})$ |
| $\langle\omega_6\rangle$ | $U_{5,6}(U_{2,3}^2+3), U_{2,3}$ | $-6(X^2+3)(X^2+\frac{81}{32})$ |
| $\langle\omega_{10}\rangle$ | $U_{3,10}(U_{2,5}^2+5), U_{2,5}$ | $10(X^2+5)(X^2+\frac{25}{32})$ |
| $\langle\omega_{15}\rangle$ | $U_{2,15}(U_{3,5}^2+\frac{75}{16}), U_{3,5}$ | $-15(X^2-\frac{5\cdot 81}{16})(X^2+\frac{75}{16})$ |
| $\{id\}$ | $30\frac{U_{30}}{U_{2,5}}, U_{30}$ | $-3(X^4+222X^2+6561)$ |

□

### 4.3.7 The curve $X(10,7)$

Next, we deal with the curve $X(10,7)^+$. Like in the cases $D=6,22$, the computations become more involved as the degree of the map $X(10,7)^{W_{10}} \to X(10,1)^+$ increases, even though, they can still be carried out. Moreover, Theorem 4.2 provides a way to check the result without having to go back to the computations.

**Proposition 4.23.** *There exists a Hauptmodul for the curve $X(10,7)^+$, $t_{10,7}^+$, defined by the following values*

| $D(\Lambda)$ | $t_{10,7}^+$ |
|---|---|
| $-35$ | $0$ |
| $-280$ | $\infty$ |
| $-20$ | $1$ |

*and it satisfies*

$$t_{10}^+ \cdot t_{10}^+ \circ \omega_7 = \frac{(-25+32t_{10,7}^+)^2\left(2000-1016t_{10,7}^+-6859(t_{10,7}^+)^2+6561(t_{10,7}^+)^3\right)^2}{4(-1+t_{10,7}^+)^2\left(10000-20200t_{10,7}^+-919(t_{10,7}^+)^2+13520(t_{10,7}^+)^3\right)^2},$$

$$(t_{10}^{+}-\frac{27}{2})(t_{10}^{+}\circ\omega_7-\frac{27}{2})=\frac{(-20+27t_{10,7}^{+})^2\left(400-835t_{10,7}^{+}+484(t_{10,7}^{+})^2\right)^3}{(-1+t_{10,7}^{+})^2\left(10000-20200t_{10,7}^{+}-919(t_{10,7}^{+})^2+13520(t_{10,7}^{+})^3\right)^2}.$$

*Moreover,* $Da(t_{10,7}^{+})+R(t_{10,7}^{+})=0$*, where* $R(X)\in\mathbb{Q}(X)$ *is*

$$\frac{750000-4135000X+10178675X^2-15135155X^3+14698603X^4-8589408X^5+2239488X^6}{4(-1+X)^2X^2(-20+27X)^2(-25+32X)^2}.$$

*In the following table we give the values of* $t_{10,7}^{+}$ *at* $\mathrm{CM}(\Lambda)$ *by certain orders* $\Lambda$*:*

| $\Lambda$ | $t_{10,7}^{+}$ | $\Lambda$ | $t_{10,7}^{+}$ |
|---|---|---|---|
| $\mathbb{Z}[\sqrt{-10}]$ | $\frac{25}{32}$ | $\mathbb{Z}[3\sqrt{-5}]$ | $\frac{100}{121}$ |
| $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ | $\frac{20}{27}$ | $\mathbb{Z}[\frac{1+3\sqrt{-3}}{2}]$ | $\frac{5}{12}$ |
| $\mathbb{Z}[\sqrt{-52}]$ | $\frac{20}{13}$ | $\mathbb{Z}[\frac{1+\sqrt{-115}}{2}]$ | $\frac{16}{23}$ |

□

Now, we are going to provide Hauptmoduln for all the quotients of genus 0 with rational points.

**Proposition 4.24.** *There exist functions* $U_K=U_K^{(10,7)}$ *such that*

$$\mathbb{Q}(X(10,7)^{\langle\omega_k:k\in K\rangle})=\mathbb{Q}(U_K)$$

*for* $K=\{\{2,5\},\{2,35\},\{5,7\},\{5,14\},\{7,10\},\{10,14\},\{70\}\}$*. They can be chosen as follows:*

| $K$ | $U_K^2$ |
|---|---|
| $\{2,5\}$ | $-5t_{10,7}^{+}$ |
| $\{2,35\}$ | $10\frac{t_{10,7}^{+}-\frac{25}{32}}{t_{10,7}^{+}-1}$ |
| $\{5,7\}$ | $70(t_{10,7}^{+}-\frac{25}{32})$ |
| $\{5,14\}$ | $-14\frac{t_{10,7}^{+}-\frac{25}{32}}{t_{10,7}^{+}}$ |
| $\{7,10\}$ | $-35\frac{t_{10,7}^{+}-1}{t_{10,7}^{+}}$ |
| $\{10,14\}$ | $7(t_{10,7}^{+}-1)$ |

*and*

$$U_{70} = \frac{U_{5,14} - \frac{7}{2}}{U_{7,10} - 7}.$$

*The action of the Atkin-Lehner involution $\omega_d$ on such a Hauptmodul $U$ is given by a rational linear function $R_d(X) \in \mathbb{Q}(X)$ such that $U \circ \omega_d = R_d(U)$ as described in the following table:*

| | $R_2$ | $R_5$ | $R_7$ | $R_{10}$ | $R_{14}$ | $R_{35}$ | $R_{70}$ |
|---|---|---|---|---|---|---|---|
| $U_{2,5}$ | $X$ | $X$ | $-X$ | $X$ | $-X$ | $-X$ | $-X$ |
| $U_{2,35}$ | $X$ | $-X$ | $-X$ | $-X$ | $-X$ | $X$ | $X$ |
| $U_{5,7}$ | $-X$ | $X$ | $X$ | $-X$ | $-X$ | $X$ | $-X$ |
| $U_{5,14}$ | $-X$ | $X$ | $-X$ | $X$ | $-X$ | $-X$ | $X$ |
| $U_{7,10}$ | $-X$ | $-X$ | $X$ | $X$ | $-X$ | $-X$ | $X$ |
| $U_{10,14}$ | $-X$ | $-X$ | $-X$ | $X$ | $X$ | $X$ | $-X$ |
| $U_{70}$ | $\frac{5}{16X}$ | $\frac{-5+10X}{-10+16X}$ | $\frac{-5+8X}{8(-1+2X)}$ | $\frac{-5+8X}{8(-1+2X)}$ | $\frac{-5+10X}{-10+16X}$ | $\frac{5}{16X}$ | $X$ |

*Moreover, for a suitable determination of the root we have that*

$$t_{10}^{+} = \frac{27}{2} - \frac{\left(100 + 15U_{2,5} + 22U_{2,5}^2\right)^3 \left(100 + 27U_{2,5}^2\right)}{5\left(5 + U_{2,5}^2\right)\left(500 + 200U_{2,5} + 141U_{2,5}^2 + 52U_{2,5}^3\right)^2}.$$

□

**Remark.** The expressions for the automorphic derivatives can be computed from that of $t_{10,7}^{+}$ as in the previous cases using Proposition 3.7.

Finally, models for the rest of the curves in this case are given in the following proposition.

**Proposition 4.25.** *1. For $i = 5, 10, 35$, $X(10,7)^{\langle \omega_i \rangle}$ are conics without rational points. We have*

$$\mathbb{Q}(X(10,7)^{\langle \omega_5 \rangle}) = \mathbb{Q}(U_{2,5}, U_{5,7})$$
$$\mathbb{Q}(X(10,7)^{\langle \omega_{10} \rangle}) = \mathbb{Q}(U_{2,5}, U_{10,14})$$
$$\mathbb{Q}(X(10,7)^{\langle \omega_{35} \rangle}) = \mathbb{Q}(U_{5,7}, U_{10,14})$$

*and the corresponding equations are*

$$14X^2+Y^2+\frac{7\cdot 125}{16}=0, \quad 7X^2+5Y^2+35=0, \quad X^2-10Y^2-\frac{5\cdot 49}{16}=0,$$

*respectively.*

2. *The curves $X(10,7)^W$, for $W\subset\langle\omega_2,\omega_7\rangle$, are genus 1 curves. For each of them, functions $f_W, g_W$ providing a system of generators of the function field $\mathbb{Q}(X(10,7)^W)$ and a polynomial $P_W$ giving the corresponding equation for the curve*

$$f_W^2=P_W(g_W)$$

*are displayed in the following table:*

| $W$ | $f_W, g_W$ | $P_W(X)$ |
|---|---|---|
| $\langle\omega_2,\omega_7\rangle$ | $\frac{U_{2,5}U_{5,7}U_{10,14}}{35}, t^+_{10,7}$ | $-2X(X-1)(X-\frac{25}{32})$ |
| $\langle\omega_2\rangle$ | $U_{2,35}(U_{2,5}^2+5), U_{2,5}$ | $10(X^2+5)(X^2+\frac{125}{32})$ |
| $\langle\omega_7\rangle$ | $U_{7,10}(U_{5,7}^2+\frac{7\cdot125}{16}), U_{5,7}$ | $-35(X^2+\frac{7\cdot125}{16})(X^2-\frac{5\cdot49}{16})$ |
| $\langle\omega_{14}\rangle$ | $U_{5,14}(U_{10,14}^2+7), U_{10,14}$ | $-14(X^2+7)(X^2+\frac{49}{32})$ |
| $\{id\}$ | $\frac{5(16U_{70}^2-5)}{2U_{2,5}}, U_{70}$ | $-(75-280X+528X^2-896X^3+768X^4)$ |

□

### 4.3.8 The curve $X(10,9)$

In this case, since we have a composite level it is easier to consider the map $X(10,9)^{W_{10}}\to X(10,3)^+$, or even better in this case $X(10,9)^{W_{10}}\to X(10,3)^{W_{10}}$, since we have already seen that $X(10,3)^{W_{10}}$ also has genus 0.

We have already computed a Hauptmodul for $X(10,3)^{W_{10}}$, namely $U_{2,5}^{(10,3)}$. For simplicity, we will denote it by $u$. In order to carry out the computations explicitly, even though it will not matter in the final result, it is necessary to choose one of the two determinations of the root for $u$. We will choose the one such that

| $D(\Lambda)$ | $u$ |
|---|---|
| $-3$ | $0$ |
| $-3\cdot 3^2$ | $-\frac{5}{2}$ |
| $-20\cdot 3^2$ | $-\frac{1}{4}$ |

Then, we take the kroneckerian polynomial of level 9, $\Psi_{\Gamma(10,1)^+,9,t_{10}^+}(X,Y)$, and recall that $\Psi_{\Gamma(10,1)^+,9,t_{10}^+}(t_{10}^+,X)\in\mathbb{Q}(X(10,1)^+)[X]$ is the irreducible polynomial of $t_{10,3}\circ\omega_9$ over $\mathbb{Q}(X(10,1)^+)$. In particular, when considering this polynomial over the field $\mathbb{Q}(X(10,3)^{W_{10}})$, it is divisible by the irreducible polynomial of $t_{10,3}^+\circ\omega_9$ over $\mathbb{Q}(X(10,3)^{W_{10}})$, which is a factor of degree 3. However, it is more convenient to replace $t_{10}^+$ by $u$ and then we obtain a certain polynomial which could be defined as the kroneckerian polynomial of level 9 corresponding to the Hauptmodul $u$ of $X(10,3)^{W_{10}}$, that is, we find an irreducible polynomial

$$\begin{aligned}\Psi(X,Y)=&-625-750X-300X^2-40X^3\\&-750Y+3825XY-1440X^2Y+1680X^3Y\\&-300Y^2-1440XY^2-1980X^2Y^2+1752X^3Y^2\\&-40Y^3+1680XY^3+1752X^2Y^3+896X^3Y^3\end{aligned}$$

such that $\Psi(u,u\circ\omega_9)=0$. We could indeed define a kroneckerian polynomial in this setting which would match this one, but since this situation will not appear again, having this explicit polynomial together with the properties of $\Psi_{\Gamma(10,1)^+,3,t_{10}^+}(X,Y)$, $\Psi_{\Gamma(10,1)^+,9,t_{10}^+}(X,Y)$ will suffice.

Then, we consider the Hautpmodul $t_{10,9}^+$ of $X(10,9)^+$ defined by the values

| $D(\Lambda)$ | $(u(P),u(\omega_9P))$ | $t_{10,9}^+(P)$ |
|---|---|---|
| $-3\cdot 3^2$ | $(0,-\frac{5}{2})$ | $0$ |
| $-3\cdot 3^2$ | $(-\frac{5}{2},-\frac{5}{2})$ | $\infty$ |
| $-20\cdot 3^2$ | $(-\frac{1}{4},-\frac{1}{4})$ | $1$ |

and then we can relate $t^+_{10,9}$ and $u$ as follows:

$$\begin{aligned} u\cdot u\circ\omega_9 =& a_0\frac{(t^+_{10,9})^3}{(t^+_{10,9})^3+b(t^+_{10,9})^2+ct^+_{10,9}+d},\\ (u+5/2)(u\circ\omega_9+5/2) =& a_1\frac{t^+_{10,9}}{(t^+_{10,9})^3+b(t^+_{10,9})^2+ct^+_{10,9}+d}. \end{aligned}$$

Now, evaluating at the points such that $t^+_{10,9}(P)=\infty,1$, we obtain that

$$a_0=\frac{25}{4},\quad a_1=\frac{25\cdot 81}{4}.$$

Moreover,

$$(t^+_{10,9})^2=81\frac{u\cdot u\circ\omega_9}{(u+5/2)(u\circ\omega_9+5/2)}.$$

In particular, the values of $t^+_{10,9}$ at the poles of $u$ and $u\circ\omega_9$ are obtained evaluating the above expression at $u=\infty$ and the corresponding value of $u\circ\omega_9$, which can be computed from $\Psi$ as a root of $112X^3+219X^2+210X-5$. Thus, the corresponding value of $t^+_{10,9}$ is a root of either

$$X^3+9X^2+54X-54,\quad X^3-9X^2+54X+54.$$

Evaluating any of the two equalities relating the functions at the point where $t^+_{10,9}$ takes the value 1, we obtain that the value of this polynomial at 1 must be 100 and therefore,

$$\begin{aligned} u\cdot u\circ\omega_9 =& \frac{25}{4}\frac{(t^+_{10,9})^3}{(t^+_{10,9})^3-9(t^+_{10,9})^2+54t^+_{10,9}+54},\\ (u+5/2)(u\circ\omega_9+5/2) =& \frac{25\cdot 81}{4}\frac{t^+_{10,9}}{(t^+_{10,9})^3-9(t^+_{10,9})^2+54t^+_{10,9}+54}. \end{aligned}$$

Then, using that

$$t^+_{10}=\frac{27\left(5+5u+8u^2\right)^2}{10(-1+4u)^2\left(5+u^2\right)},$$

we obtain that

$$\begin{aligned} t^+_{10}\cdot t^+_{10}\circ\omega_9 =& \frac{A(t^+_{10,9})}{C(t^+_{10,9})},\\ (t^+_{10}-1)\cdot(t^+_{10}\circ\omega_9-1) =& \frac{B(t^+_{10,9})}{C(t^+_{10,9})}, \end{aligned}$$

where $A(X), B(X), C(X) \in \mathbb{Q}[X]$ are the following:

$$A(X) = 9\left(99144 - 107892X + 22356X^2 + 17064X^3 - 2214X^4 + 153X^5 + 289X^6\right)^2,$$
$$B(X) = 625(3+X)^2\left(6 - 4X + X^2\right)^2\left(-540 + 594X + 90X^2 + 25X^3\right)^2,$$
$$C(X) = 4(-6+X)^2\left(6 + 2X + X^2\right)^2\left(594 - 216X - 99X^2 + 121X^3\right)^2.$$

Finally we can recover the expression for the automorphic derivative of $t_{10,9}^{+}$ from that of $u$ or from that of $t_{10,9}^{+}$, using Proposition 3.7.

**Proposition 4.26.** *There exists a Hauptmodul for the curve $X(10,9)^{+}$, $t_{10,9}^{+}$, defined by the following values*

| $D(\Lambda)$ | $(t_{10}^{+}(P), t_{10}^{+}(\omega_9 P))$ | $t_{10,9}^{+}(P)$ |
|---|---|---|
| $-3 \cdot 3^2$ | $(\frac{27}{2}, \frac{867}{242})$ | $0$ |
| $-3 \cdot 3^2$ | $(\frac{867}{242}, \frac{867}{242})$ | $\infty$ |
| $-20 \cdot 3^2$ | $(\frac{289}{120}, \frac{289}{120})$ | $1$ |

*and it satisfies*

$$t_{10}^{+} \cdot t_{10}^{+} \circ \omega_9 = \frac{A(t_{10,9}^{+})}{C(t_{10,9}^{+})},$$
$$(t_{10}^{+} - 1) \cdot (t_{10}^{+} \circ \omega_9 - 1) = \frac{B(t_{10,9}^{+})}{C(t_{10,9}^{+})},$$

*where $A(X), B(X), C(X) \in \mathbb{Q}[X]$ are the following:*

$$A(X) = 9\left(99144 - 107892X + 22356X^2 + 17064X^3 - 2214X^4 + 153X^5 + 289X^6\right)^2,$$
$$B(X) = 625(3+X)^2\left(6 - 4X + X^2\right)^2\left(-540 + 594X + 90X^2 + 25X^3\right)^2,$$
$$C(X) = 4(-6+X)^2\left(6 + 2X + X^2\right)^2\left(594 - 216X - 99X^2 + 121X^3\right)^2.$$

*Moreover, $Da(t_{10,9}^{+}) + R(t_{10,9}^{+}) = 0$, where $R(X) \in \mathbb{Q}(X)$ is*

$$\frac{4\left(11664 - 8208X + 9108X^2 + 888X^3 - 3024X^4 + 148X^5 + 253X^6 - 38X^7 + 9X^8\right)}{(-6+X)^2(-1+X)^2(2+X)^2(3+X)^2\left(6 - 3X + X^2\right)^2}.$$

*The values of $t^{+}_{10,9}$ at the CM points by certain orders $\Lambda$ are given in the following table:*

| $\Lambda$ | $t^{+}_{10,9}$ |
|---|---|
| $\mathbb{Z}[\sqrt{-2}]$ | $-3$ |
| $\mathbb{Z}[\frac{1+\sqrt{-35}}{2}]$ | $\frac{3}{2}$ |
| $\mathbb{Z}[3\sqrt{-2}]$ | $-2$ |
| $\mathbb{Z}[\sqrt{-5}]$ | $6$ |
| $\mathbb{Z}[3\sqrt{-10}]$ | $\alpha : \alpha^2 - 3\alpha + 6 = 0$ |

$\square$

Next, we move on to the genus 0 quotients. In this case, only two more quotients have genus 0 and both of them have rational points.

**Proposition 4.27.** *There exist functions $U_K = U_K^{(10,9)}$ such that*

$$\mathbb{Q}(X(10,9)^{\langle \omega_k : k \in K\rangle}) = \mathbb{Q}(U_K)$$

*for $K = \{\{2,45\},\{5,18\}\}$. They can be chosen as follows:*

| $K$ | $U_K^2$ |
|---|---|
| $\{2,45\}$ | $\frac{t^{+}_{10,9}-6}{t^{+}_{10,9}+2}$ |
| $\{5,18\}$ | $\frac{t^{+}_{10,9}-1}{t^{+}_{10,9}+3}$ |

*The action of the Atkin-Lehner involution $\omega_d$ on such a Hauptmodul $U$ is given by a rational linear function $R_d(X) \in \mathbb{Q}(X)$ such that $U \circ \omega_d = R_d(U)$ as described in the following table:*

| | $R_2$ | $R_5$ | $R_9$ | $R_{10}$ | $R_{18}$ | $R_{45}$ | $R_{90}$ |
|---|---|---|---|---|---|---|---|
| $U_{2,45}$ | $X$ | $-X$ | $-X$ | $-X$ | $-X$ | $X$ | $X$ |
| $U_{5,18}$ | $-X$ | $X$ | $-X$ | $-X$ | $X$ | $-X$ | $X$ |

$\square$

**Remark.** The expressions for the automorphic derivatives can be computed from that of $t_{10,9}^+$ as in the previous cases using Proposition 3.7.

Finally, models for the elliptic and hyperelliptic quotients are given in the following proposition.

**Proposition 4.28.** *(1) The curves $X(10,9)^W$, for $W \in \{\langle\omega_2,\omega_9\rangle,\ \langle\omega_2,\omega_5\rangle,\ \langle\omega_5,\omega_9\rangle,\ \langle\omega_9,\omega_{10}\rangle,\ \langle\omega_{10},\omega_{18}\rangle,\ \langle\omega_{90}\rangle\}$, are genus 1 curves. For each of them, functions $f_W, g_W$ providing a system of generators of the function field $\mathbb{Q}(X(10,9)^W)$ and a polynomial $P_W$ giving the corresponding equation for the curve*

$$f_W^2 = P_W(g_W)$$

*are displayed in the following table:*

| $W$ | $f_W, g_W$ | $P_W(X)$ |
|---|---|---|
| $\langle\omega_2,\omega_9\rangle$ | $\sqrt{-3(t_{10,9}^+-1)(t_{10,9}^+-6)((t_{10,9}^+)^2-3t_{10,9}^++6)}, t_{10,9}^+$ | $-3(X-6)(X-1)(X^2-3X+6)$ |
| $\langle\omega_2,\omega_5\rangle$ | $\sqrt{-3(t_{10,9}^+-1)(t_{10,9}^++2)((t_{10,9}^+)^2-3t_{10,9}^++6)}, t_{10,9}^+$ | $-3(X-6)(X+2)(X^2-3X+6)$ |
| $\langle\omega_5,\omega_9\rangle$ | $\sqrt{-3(t_{10,9}^++2)(t_{10,9}^++3)((t_{10,9}^+)^2-3t_{10,9}^++6)}, t_{10,9}^+$ | $-3(X+2)(X+3)(X^2-3X+6)$ |
| $\langle\omega_9,\omega_{10}\rangle$ | $\sqrt{(t_{10,9}^+-6)(t_{10,9}^+-1)(t_{10,9}^++2)(t_{10,9}^++3)}, t_{10,9}^+$ | $(X-6)(X-1)(X+2)(X+3)$ |
| $\langle\omega_{10},\omega_{18}\rangle$ | $\sqrt{-3(t_{10,9}^+-6)(t_{10,9}^++3)((t_{10,9}^+)^2-3t_{10,9}^++6)}, t_{10,9}^+$ | $-3(X-6)(X+3)(X^2-3X+6)$ |
| $\langle\omega_{90}\rangle$ | $U_{5,18}(U_{2,45}^2-9), U_{2,45}$ | $-(X+3)(X-3)(3X^2+5)$ |

*(2) The curves $X(10,9)^W$, for $W \in \{\langle\omega_2\rangle, \langle\omega_5\rangle, \langle\omega_{18}\rangle, \langle\omega_{45}\rangle\}$, are genus 2 curves. For each of them, functions $f_W, g_W$ providing a system of generators of the function field $\mathbb{Q}(X(10,9)^W)$ and a polynomial $P_W$ giving the corresponding equation for the curve*

$$f_W^2 = P_W(g_W)$$

*are displayed in the following table:*

| $W$ | $f_W, g_W$ |
|---|---|
| $\langle\omega_2\rangle$ | $\frac{(U_{2,45}^2-1)^2}{8}\sqrt{-3(t_{10,9}^+-1)(t_{10,9}^++2)((t_{10,9}^+)^2-3t_{10,9}^++6)}, U_{2,45}$ |
| $\langle\omega_5\rangle$ | $\frac{(U_{5,18}^2-1)^2}{4}\sqrt{-3(t_{10,9}^++2)(t_{10,9}^++3)((t_{10,9}^+)^2-3t_{10,9}^++6)}, U_{5,18}$ |
| $\langle\omega_{18}\rangle$ | $\frac{(U_{5,18}^2-1)^2}{4}\sqrt{-3(t_{10,9}^+-6)(t_{10,9}^++3)((t_{10,9}^+)^2-3t_{10,9}^++6)}, U_{5,18}$ |
| $\langle\omega_{45}\rangle$ | $\frac{(U_{2,45}^2-1)^2}{8}\sqrt{-3(t_{10,9}^++2)(t_{10,9}^++3)((t_{10,9}^+)^2-3t_{10,9}^++6)}, U_{2,45}$ |

| $W$ | $P_W(X)$ |
|---|---|
| $\langle\omega_2\rangle$ | $-3\left(5+3X^2\right)\left(3+3X^2+2X^4\right)$ |
| $\langle\omega_5\rangle$ | $-3\left(3+X^2\right)\left(1-3X^2+6X^4\right)$ |
| $\langle\omega_{18}\rangle$ | $-3\left(-5+9X^2\right)\left(1-3X^2+6X^4\right)$ |
| $\langle\omega_{45}\rangle$ | $3(-3+X)(3+X)\left(3+3X^2+2X^4\right)$ |

$\square$

# Chapter 5

# Automorphic form expansions: arithmetic properties

In the classical theory of modular forms, the existence of cusps, and therefore of Fourier expansions, plays a central role when carrying out explicit computations. In the case of Shimura curves attached to rational non-split indefinite quaternion algebras, the absence of cusps prevents the existence of Fourier expansions. Alternative expansions around CM points, which provide in some sense a suitable substitute in the general case, have been introduced in [Bay02] and further studied in [BT07a] for the modular case and in [BT07b, BT08] for the case of the Shimura curve of discriminant 6. Explicit expansions for some functions around certain CM points have already been computed in these papers. Some properties of these expansions, including the algebraicity of the coefficients in the case $D = 6$, have been studied in [BT08] as well. Expansions around CM points from a more geometric point of view have also been considered in [Mor94, Mor95, Mor11].

In this chapter, we generalize those expansions considered in [BT07b] to provide expansions for quaternionic automorphic forms. Then we relate the coefficients of these expansions to the values of certain differential operators, closely related to those studied in [Shi75a, Shi79] for certain types of modular forms, and then we follow a similar approach to the one in these papers to obtain some arithmetic properties of the coefficients. More precisely, we prove, for any discriminant and level, that the field generated by the coefficients of the expansion corresponding to an automorphic form is closely related to its field of definition and that, when suitably normalized, Shimura reciprocity law extends to all its coefficients. In this way, we recover some of the results in [Shi79] and [BT08], formulated over precise number fields instead of over $\overline{\mathbb{Q}}$. Using the algebraicity results of the coefficients of these expansions we can deduce the rationality of the automorphic derivative of an automorphic function, which is a classical result proved by Ihara [Iha74] using a completely different approach.

In the next chapter, we will study how to compute explicit $q$-expansions of cusp forms for the cases we have studied in the previous chapter, and we will provide some explicit examples.

## 5.1 Shimura-Maaß operators and coefficients

In this section, we will recall the definition and some properties of the Shimura-Maaß differential operators introduced in [Shi75a], a slight modification of a differential operator already considered by Maaß in [Maa52]. These differential operators, when applied to an automorphic form, yield (generically) non-holomorphic functions which behave as automorphic forms of higher weight. We will prove that the values obtained when evaluating these functions at a CM point $P$ is related to the coefficients of the $q$-expansion of the original form around $P$. However, in order to preserve the holomorphy, we will replace every Shimura-Maaß differential operator with a family of holomorphic differential operators, one for every CM point, and we will show that, for those corresponding to a certain CM point $P$, their values when applied to an automorphic form and evaluated at $P$ coincide with those of the Shimura-Maaß. Moreover, we will prove some properties of these operators, including a transformation law, playing the same role as the automorphic behaviour of the original Shimura-Maaß operators.

Recall that in Section 3.3, we considered $q$-expansions of automorphic functions with respect to the following local parameter $q_P$:

**Definition.** For $P \in \mathcal{H}$ and $\kappa \in \mathbb{C}^*$, we define the local parameter around $P$ with local constant $\kappa$, $q = q_{P,\kappa} : \mathcal{H} \to \mathbb{C}$, as

$$q(z) = \kappa \frac{z - P}{z - \overline{P}}.$$

**Lemma 5.1.** *Let $q = q_{P,\kappa}$ be a local parameter with constant $\kappa$ at $P \in \mathcal{H}$ and $t$ a non-negative integer. Let also $\delta_{2t} = \delta_{2t}^{\kappa,P} = \dfrac{q-\kappa}{\kappa(P-\overline{P})}\left((q-\kappa)\dfrac{d}{dq} - 2t\right)$.*

*(1) For every non-negative integer $t$ and $F = F(q)$ analytic in a neighborhood of 0,*

$$\delta_{2t}\left(\left(\frac{(q-\kappa)^2}{\kappa(P-\overline{P})}\right)^t F\right) = \left(\frac{(q-\kappa)^2}{\kappa(P-\overline{P})}\right)^{t+1}\frac{dF}{dq}.$$

*(2) Let $F = \sum_{n\geq 0} a_n q^n$ be an analytic function in a neighborhood of $q = 0$. Then,*

$$\delta_{2t+2s}\circ\cdots\circ\delta_{2t+2}\circ\delta_{2t}\left(\left(\frac{(q-\kappa)^2}{\kappa(P-\overline{P})}\right)^t F\right) = \left(\frac{(q-\kappa)^2}{\kappa(P-\overline{P})}\right)^{t+s+1}\frac{d^{s+1}F}{dq^{s+1}}.$$

*In particular,*

$$\left(\delta_{2t+2s}\circ\cdots\circ\delta_{2t+2}\circ\delta_{2t}\left(\left(\frac{(q-\kappa)^2}{\kappa(P-\overline{P})}\right)^t F\right)\right)(P) = \frac{\kappa^{t+s+1}(s+1)!}{(P-\overline{P})^{t+s+1}}a_{s+1}.$$

*Proof.* The first equality is a direct computation and the second one follows immediately from the first one. □

Given $\Gamma$ the group of units of norm 1 of an Eichler order in a rational indefinite quaternion algebra, we recall some of the notions introduced in Section 1.3 concerning automorphic functions and forms. $A_k(\Gamma)_K$ denotes the set of weight $k$ (meromorphic) automorphic forms defined over a field $K \subset \mathbb{C}$, which is in fact an $A_0(\Gamma)_K$-vector space of dimension 1. We will denote by $S_k(\Gamma)_K$ the $K$-vector space of weight $k$ cusp forms defined over $K$ and $S_*(\Gamma)_K$ the graded $K$-algebra of cusp forms. If $K = \mathbb{C}$, we will omit the subscript. In the case of a non-split indefinite quaternion algebra, since there are no cusps, all holomorphic automorphic forms are cusp forms.

Now, consider the Shimura-Maaß operator $\tilde{\delta}_{2t} = \dfrac{1}{2\pi i}\left(\dfrac{d}{dz} + \dfrac{2t}{z-\overline{z}}\right)$, where $\dfrac{d}{dz} = \dfrac{1}{2}\left(\dfrac{d}{dx} - i\dfrac{d}{dy}\right)$, $z = x + iy$, cf. [Shi75a]. Let us also consider

$$f = \frac{1}{(2\pi i)^t}\left(\frac{(q-\kappa)^2}{\kappa(P-\overline{P})}\right)^t \sum_{n\geq 0} a_n q^n \in A_{2t}(\Gamma)_K.$$

**Proposition 5.1.** $(\tilde{\delta}_{2t+2s} \circ \cdots \circ \tilde{\delta}_{2t})(f)(P) = \left( \dfrac{\kappa}{2\pi i(P - \overline{P})} \right)^{t+s+1} (s+1)! a_{s+1}.$

*Proof.* Since $\dfrac{d\overline{z}}{dz} = 0$, given a smooth function $h = h(z, \overline{z})$ and a point $P \in \mathcal{H}$ we have the equality $\tilde{\delta}_t h(P) = \dfrac{1}{2\pi i} \left( \dfrac{d}{dz} + \dfrac{t}{z - \overline{P}} \right) h(z, \overline{P})$. And the results follows from the previous lemma after observing that $\delta_t$ is obtained from $\dfrac{d}{dz} + \dfrac{t}{z - \overline{P}}$ after applying the change of variables $q(z) = \kappa \dfrac{z - P}{z - \overline{P}}$. □

Now let

$$D^e_{2t,\kappa,P} = \delta^{\kappa,P}_{2t+2(e-1)} \circ \cdots \circ \delta^{\kappa,P}_{2t}$$

and

$$\tilde{D}^e_{2t} = \tilde{\delta}_{2t+2(e-1)} \circ \cdots \circ \tilde{\delta}_{2t}.$$

Finally, for $e = 1$ we will not usually write the superindex and if, in addition, $t = 0$, we will abbreviate

$$D = D_0 = \tilde{D}_0 = \frac{1}{2\pi i} \frac{d}{dz}.$$

**Proposition 5.2.** *(1) Let* $f = \dfrac{1}{(2\pi i)^t} \left( \dfrac{(q-\kappa)^2}{\kappa(P - \overline{P})} \right)^t \sum_{n \geq 0} a_n \dfrac{q^n}{n!}$ *be analytic in a neighborhood of $z = P$. Then,*

$$(D^e_{2t,\kappa,P} f)(P) = a_e \left( \frac{\kappa}{2\pi i(P - \overline{P})} \right)^{t+e}.$$

*(2) $D_{t+s,\kappa,P}(fg) = D_{t,\kappa,P}(f)g + f D_{s,\kappa,P}(g)$, for any pair of functions $f, g$ analytic at $z = P$.*

*(3) $D_{t+s,\kappa,P}((z - \overline{P})^{-t} f) = (z - \overline{P})^{-t} D_{s,\kappa,P}(f)$, for any function $f$ analytic at $z = P$.*

*(4) $D_{t,\kappa,P'}(f|_t \alpha) = (D_{t,\kappa,P}(f))|_{t+2}\alpha$ for any function $f$ analytic at $z = P$ and any $\alpha \in \mathbf{GL}^+_2(\mathbb{R})$ such that $\alpha(P') = P$.*

*(5) For any number field $K$, $D_0 : A_0(\Gamma)_K \to A_2(\Gamma)_K$.*

*Proof.* The first assertion has been proved in the previous lemma and the second one is easily checked using the definition.

To prove (3), we apply (2) to obtain the equality

$$D_{t+s,\kappa,P}((z-\overline{P})^{-t}f) = D_{t,\kappa,P}((z-\overline{P})^{-t})f + (z-\overline{P})^{-t}D_{s,\kappa,P}(f)$$

and observe that

$$(z-\overline{P})^{-t} = \left(\frac{q-\kappa}{\kappa(P-\overline{P})}\right)^{t}.$$

Now, by definition of $D_{t,\kappa,P}$, we obtain that $D_{t,\kappa,P}\left(\left(\frac{q-\kappa}{\kappa(P-\overline{P})}\right)^{t}\right) = 0$ and therefore the result.

To prove (4), let $f$ be as in (1), this is $f(z) = \left(\frac{dq_{P,\kappa}(z)}{dz}\right)^{t} F(q_{P,\kappa}(z))$. Then, $f|_{2t}\alpha = a^{t}\left(\frac{dq_{P',\kappa}(z)}{dz}\right)^{t} F(aq_{P',\kappa}(z))$ for $a \in \mathbb{C}^{*}$ such that $q_{P,\kappa}(\alpha z) = aq_{P',\kappa}(z)$. According to the previous lemma, we have

$$\begin{aligned}
D_{2t,\kappa,P'}(f|_{2t}\alpha) &= a^{t}\left(\frac{dq_{P',\kappa}(z)}{dz}\right)^{t+1}\frac{F(aq_{P',\kappa}(z))}{dq_{P',\kappa}(z)}\\
&= \left(\frac{dq_{P,\kappa}(\alpha z)}{dz}\right)^{t+1}\frac{F(q_{P,\kappa}(\alpha z))}{dq_{P,\kappa}(\alpha z)}\\
&= (D_{2t,\kappa,P}f)|_{2t+2}\alpha.
\end{aligned}$$

Finally, (5) follows immediately, since, by definition, $Df \in A_2(\Gamma)_K$ for any $f \in A_0(\Gamma)_K$. □

## 5.2 Arithmeticity principle

In this section we will study the field of definition of the coefficients of suitably normalized expansions of automorphic forms around CM points and we will show that Shimura reciprocity law relates expansions of an automorphic form at CM points by the same quadratic order, in the same way it relates the values of functions. The outline of the proof is essentialy the one in [Shi75a], which deals with a similar problem in a different case.

Let $\mathbb{H}$ be a rational indefinite quaternion algebra of discriminant $D$ and let $\mathcal{O} = \mathcal{O}(D,N) \subset \mathbb{H}$ be an Eichler order of level $N$, $\gcd(D,N) = 1$. Fix an embedding $\Phi : \mathbb{H} \hookrightarrow \mathbf{M}(2,\mathbb{R})$. Let also $K|\mathbb{Q}$ be a quadratic imaginary field and $\Lambda \subset \mathcal{O}_K$ be an order such that there exists an optimal embedding $\phi : \Lambda \hookrightarrow \mathcal{O}$. We denote by $K^\Lambda$ the Ring Class Field of $K$ corresponding to the order $\Lambda$. Let $\varepsilon \in \Lambda$ be such that $\Lambda = \mathbb{Z}[\varepsilon]$ and let $P \in \mathcal{H}$ be the common fixed point by $\Phi(\phi(\Lambda))$. Consider $\rho = \Phi(\phi(\varepsilon))$. Finally, given $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{M}(2,\mathbb{R})$ and $z \in \mathcal{H}$, we define $\lambda(A,z) = cz + d$.

**Lemma 5.2.** *Given $\alpha \in \mathcal{O}$ a primitive element of norm $N'$, $\gcd(N', ND) = 1$, then $\mathcal{O} \cap \alpha^{-1}\mathcal{O}\alpha$ is an Eichler order of level $NN'$.*

*Proof.* Recall that $\mathcal{O}$ is an Eichler order of level $N'$ if and only if for every prime $p \in \mathbb{Z}$, $\mathcal{O}_p = \mathcal{O} \otimes \mathbb{Z}_p$ is conjugated to $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N'\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ if $p|N'$ and after $\mathbb{H} \otimes \mathbb{Q}_p$ is identified with $\mathbf{M}(2,\mathbb{Q}_p)$, and is a maximal $\mathbb{Z}_p$-order otherwise.

On the one hand, if $p \nmid N'$, then $N' \in \mathbb{Z}_p^*$. Therefore, $\alpha \in \mathcal{O}_p^*$, which gives that $\alpha^{-1}\mathcal{O}_p\alpha = \mathcal{O}_p$. Hence, $\alpha^{-1}\mathcal{O}_p\alpha \cap \mathcal{O}_p$ is a maximal $\mathbb{Z}_p$-order if $p \nmid N$ and is conjugated to $\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ NN'\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$ otherwise.

On the other hand, if $p \mid N'$, then $\mathcal{O}_p$ is maximal, since $p \nmid N$. Hence, $\alpha^{-1}\mathcal{O}_p\alpha \cap \mathcal{O}_p$ is the intersection of two maximal orders and, therefore, an Eichler order. To compute its level, since $\alpha$ is primitive and $\mathcal{O}_p$ is maximal, $\mathcal{O}_{p,1}\alpha\mathcal{O}_{p,1}$ is the set of all primitive elements of norm $N'$. In particular, $\gamma_1 \begin{pmatrix} 0 & -1 \\ N' & 0 \end{pmatrix} \gamma_2 = \alpha$ for $\gamma_1, \gamma_2 \in \mathcal{O}_{p,1}$. Now, a direct computation shows that

$$\alpha^{-1}\mathcal{O}_p\alpha \cap \mathcal{O}_p = \gamma_2^{-1} \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N'\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \gamma_2$$

and, consequently,

$$\alpha^{-1}\mathcal{O}_p\alpha \cap \mathcal{O}_p \cong \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N'\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ NN'\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

□

**Remark.** An analogous result also holds if we let the norm of $\alpha$ be divisible by primes $p \mid D$.

**Lemma 5.3.** *Fix* $b \in \mathbb{Z} \setminus \{0\}$. *There exist an integer* $n$ *and a finite set* $S \subset \mathbb{Z}$ *such that for every* $a \in \mathbb{Z} \setminus S$ *satisfying* $\gcd(\det(a\,\mathrm{Id}+b\rho), n) = 1$*, then*

$$(\det(a\,\mathrm{Id}+b\rho))^{-1/2}\,\lambda(a\,\mathrm{Id}+b\rho, P)$$

*is not a root of unity.*

*Proof.* Firstly, we observe that $\lambda(a\,\mathrm{Id}+b\rho, P) \in K$. More precisely, we have that $\lambda(a\,\mathrm{Id}+b\rho, P)$ is a root of the characteristic polynomial of $a\,\mathrm{Id}+b\rho$, a polynomial defining $K$. Thus, $(\det(a\,\mathrm{Id}+b\rho))^{-1/2}\,\lambda(a\,\mathrm{Id}+b\rho, P) \in K((\det(a\,\mathrm{Id}+b\rho))^{1/2})$ and it is an extension of $\mathbb{Q}$ of degree at most 4.

Consequently, if $(\det(a\,\mathrm{Id}+b\rho))^{-1/2}\,\lambda(a\,\mathrm{Id}+b\rho, P)$ is a root of unity, it is a root of unity lying in an extension of degree 4 and there are only a finite number of these. In particular, there exist $m \in \mathbb{Z}$ and an $m$th-root of unity $\xi_m$ such that $(\det(a\,\mathrm{Id}+b\rho))^{-1/2}\,\lambda(a\,\mathrm{Id}+b\rho, P) \in \mathbb{Q}(\xi_m)$. Hence it holds that $(\det(a\,\mathrm{Id}+b\rho))^{1/2} \in K(\xi_m)$.

Now, we observe that $\det(a\,\mathrm{Id}+b\rho)$ is not a square except for a finite number of $a$, as $\det(a\,\mathrm{Id}+b\rho) = \mathrm{Norm}_{K|\mathbb{Q}}(a+b\varepsilon) = (a+b\frac{\mathrm{tr}(\varepsilon)}{2})^2 + b^2(\mathrm{Norm}_{K|\mathbb{Q}}\varepsilon - \frac{\mathrm{tr}(\varepsilon)^2}{4})$ and $b^2(\mathrm{Norm}_{K|\mathbb{Q}}\varepsilon - \frac{\mathrm{tr}(\varepsilon)^2}{4})$ is a non-zero rational number (it is the discriminant of $K$ up to a non-zero square factor) independent of $a$. Let $S$ be this set of integers.

Let now $a$ be an integer not in $S$. Then, $(\det(a\,\mathrm{Id}+b\rho))^{1/2}$ defines a quadratic subextension of $K(\xi_m)$. Therefore, there exists $n$ (e.g. the product of the discriminants of the quadratic subextensions of $K(\xi_m)$), such that if we have $\gcd(\det(a\,\mathrm{Id}+b\rho), n) = 1$, then $(\det(a\,\mathrm{Id}+b\rho))^{1/2} \notin K(\xi_m)$ and, in particular,

$$(\det(a\,\mathrm{Id}+b\rho))^{-1/2}\,\lambda(a\,\mathrm{Id}+b\rho, P)$$

is not a root of unity. □

**Lemma 5.4.** *For every positive integer* $N$*, there exists* $b \in \mathbb{Z} \setminus \{0\}$ *such that* $\gcd(\det(a\,\mathrm{Id}+b\rho), N) = 1$ *for infinitely many integers* $a$.

*Proof.* Choosing $b$ properly, we can rewrite the set $\{a+b\varepsilon\}_a$ as $\{a+b'\sqrt{D_K}\}_a$ for some integer $b'$. Then, we need prove that there exist infinitely many $a$ such that $\gcd(\det(\Phi(\phi(a+b'\sqrt{D_K}))), N) = 1$. Since $\det(\Phi(\phi(a+b'\sqrt{D_K}))) = a^2 - b'^2 D_K$, it is enough to prove that for every integer $t$, there exist infinitely many $a$ such that $\gcd(a^2+t, N) = 1$. And this follows easily, assuming first that $N$ is prime and then using the Chinese Remainer Theorem to obtain the general case. □

**Proposition 5.3.** *Following the notations above, there exists $\alpha \in \mathbb{H}^*$ such that $\alpha(P) = P$, $\alpha^{-1}\mathcal{O}\alpha \cap \mathcal{O}$ is an Eichler order admitting $\phi$ as an optimal embedding for $\Lambda$ and $(\det(\alpha))^{-1/2}\,\lambda(\alpha, P)$ is not a root of unity.*

*Proof.* Follows immediately from the previous lemmas, taking $\alpha \in \phi(\Lambda)$. □

Let $L$ be a number field. Let also $\Gamma$ be the image in $\mathbf{PSL}_2(\mathbb{R})$ of $\Phi(\mathcal{O}_1)$. Given $a, w \in \mathbb{Z}$, $w > 0$, we fix the representative $[a]_w \in \{0, \ldots, w-1\}$ of the residue class $a \pmod{w}$.

**Theorem 5.1.** *Let $f \in A_{2r}(\Gamma)_L$ and $g \in A_{2r+2e}(\Gamma)_L$ with $r, e$ non-negative integers. Let $P \in \mathcal{H}$ be a complex multiplication point by the order $\Lambda \subset K$ and let $w$ be its elliptic order (by $\Gamma$). Then, if $f, g$ are holomorphic at $P$ and $\operatorname{ord}_P g = [-r-e]_w$,*

$$(g^{-1}D^e f)(P) \in LK(j(P)) = L \cdot K^{\Lambda}.$$

**Remark.**

- $(D^e f)(P) = 0$ if $e + r \not\equiv 0 \pmod{w}$.
- If $e + r \not\equiv 0 \pmod{w}$, then there exists $g \in A_{2r+2e}(\Gamma)_{L\cdot K^{\Lambda}}$ such that $g(P) \neq 0$.
- More generally, if $f \in A_{2r}(\Gamma)_{L\cdot K^{\Lambda}}$ is holomorphic at $P$ and $e \geq 0$, then there exists $g \in A_{2r+2e}(\Gamma)_{L\cdot K^{\Lambda}}$ such that $\operatorname{ord}_P D^e f = \operatorname{ord} g$.

*Proof.* Let $n_0 = \operatorname{ord}_P f \geq 0$. We will proceed by induction on $e$ in two steps. We will begin by proving the cases $0 \leq e \leq n_0$, reducing them to the case $e = 0$, and then we will prove the cases $e > n_0$ by reduction to those ones.

- $\boldsymbol{e = 0}$: We need to prove that $(f/g)(P) \in L \cdot K^{\Lambda}$, but this is obvious, since $f/g \in A_0(\Gamma)_{L\cdot K^{\Lambda}}$ and it is holomorphic at $P$.

  When we rewrite this in terms of the coefficients of the $q$-expansions of $f$ and $g$, we obtain that: if we consider $f = Q^r \sum_{n \geq n_0} a_n \frac{q^n}{n!}$, $g = Q^r \sum_{n \geq n_0} b_n \frac{q^n}{n!}$ with $a_{n_0}, b_{n_0} \neq 0$ and $Q = \frac{(q-\kappa)^2}{2\pi i \kappa (P - \overline{P})}$, then $a_{n_0}/b_{n_0} \in L \cdot K^{\Lambda}$.

- $\boldsymbol{0 < e \leq n_0}$: We will reduce this situation to the case $e = 0$. With the notations for the expansions which we gave in the previous case, we have

$$D^e f = Q^{r+e} \sum_{n \geq n_0} a_n \frac{q^{n-e}}{(n-e)!}.$$

Let $g_1 \in A_{2r+2e}(\Gamma)_{L\cdot K^\Lambda}$ be such that $\mathrm{ord}_P\, g_1 = \mathrm{ord}_P\, D^e f = n_0 - e$. Let $t \in A_0(\Gamma)_{L\cdot K^\Lambda}$ be such that $t(P) = 0$. Then, $Dt/t \in A_2(\Gamma)_{L\cdot K^\Lambda}$ has a simple pole at $P$ and this allows us to write $g_1 = g\left(\frac{\mathrm{ord}_P\, tDt}{t}\right)^e g_2$, where $g_2 \in A_0(\Gamma)_{L\cdot K^\Lambda}$ is holomorphic and non-vanishing at $P$.

Therefore, evaluating $((D^e f)/g_1)(P) = \frac{a_{n_0} n_0!}{g_2(P)(n_0-e)! b_{n_0}}$ which lies in $K \cdot L^\Lambda$, since $a_{n_0}/b_{n_0}$ and $g_2(P)$ do, as we wanted to prove.

- $\boldsymbol{e > n_0}$: Let $\alpha \in \mathbb{H}^*$ be the element given by Proposition 5.3. Then, if we consider $h = \frac{f|_{2r}\alpha}{f}$, its value at $P$ is not a root of unity, more precisely, $h(P) = \lambda^{-2r-2n_0}$. Now, applying $D_{2r}^e$ to both sides of the equality $f|_{2r}\alpha = fh$ yields
$$(D^e f)|_{2r+2e}\alpha = (D^e f)h + \sum_p C_p (D^{e-p} f)(D^p h),$$
which we can rewrite as:
$$(D^e f)|_{2r+2e}\alpha - (D^e f)h = \sum_{p=1}^{e-1} C_p (D^{e-p} f)(D^p h) + f D^e h.$$
Note that $\mathrm{ord}_P((D^{e-p} f) D^p h) \geq [-r-e+p]_w + [-p]_w \geq [-r-e]_w$ and for every $p \in \{1, \ldots, e\}$ we choose $g_p \in A_{2p}(\Gamma)_{L\cdot K^\Lambda}$ with order at $P$ equal to $[-p]_w$, which is less or equal than $\mathrm{ord}_P\, D^p h$. Therefore, we obtain that $\mathrm{ord}_P(D^{e-p} f) \geq [-r-e+p]_w \geq [-r-e]_w - [-p]_w = \mathrm{ord}_P(g g_p^{-1})$.

Putting everything together, it follows that
$$\frac{(D^e f)|_{2r+2e}\alpha - (D^e f)h}{g} = \sum_{p=1}^{e-1} C_p \frac{(D^{e-p} f)}{g g_p^{-1}} \frac{(D^p h)}{g_p} + \frac{f}{g g_e^{-1}} \frac{D^e h}{g_e}.$$
Now, by induction hypothesis, the value of the right-hand side at $P$ lies in $L \cdot K^\Lambda$. On the other hand, the value of the left-hand side at $P$ can be computed as
$$\begin{aligned}\frac{(D^e f)|_{2r+2e}\alpha - (D^e f)h}{g}(P) &= \left(\frac{D^e f}{g}\right)(P) \left(\frac{(D^e f)|_{2r+2e}\alpha}{D^e f}(P) - h(P)\right) \\ &= \left(\frac{D^e f}{g}\right)(P) \lambda^{-2r-n_0}(\lambda^{-2(e+\mathrm{ord}_P\, D^e f - n_0)} - 1).\end{aligned}$$
Since $e > n_0$ and $D^e f$ is analytic at $P$, $e + \mathrm{ord}_P\, D^e f - n_0 > 0$ and therefore $\lambda^{-2r-n_0}(\lambda^{-2(e+\mathrm{ord}_P\, D^e f - n_0)} - 1) \in L \cdot K^\Lambda \setminus \{0\}$, which yields the result. $\square$

**Corollary 5.1.** *Let $f \in A_0(\Gamma)_{L\cdot K^\Lambda}$ and fix $P$ a CM point by $\Lambda \subset \mathcal{O}_K$. If $g \in A_0(\Gamma)_L$ is analytic at $P$ and $w \in \mathbb{Z}_{\geq 0}$ is such that $(D^w g)(P) \neq 0$, then for every $e \in \mathbb{Z}_{\geq 0}$,*

$$((D^w g)^{-e}(D^{we} f))(P) \in L \cdot K^\Lambda.$$

*Proof.* Let $g_1 \in A_{2w}(\Gamma)_{L\cdot K^\Lambda}$ such that $g_1(P) \neq 0$. Note that it exists because $(D^w g)(P) \neq 0$. Then, the theorem yields

$$\frac{D^w g}{g_1}(P) \in (L \cdot K^\Lambda)^*,$$

and also

$$\frac{D^{we} g}{g_1^e}(P) \in L \cdot K^\Lambda.$$

Therefore,

$$\frac{D^{we} f}{(D^w g)^e}(P) = \left(\left(\frac{D^w g}{g_1}\right)^{-e} \frac{D^{we} f}{g_1^e}\right)(P) \in L \cdot K^\Lambda.$$

$\square$

**Corollary 5.2.** *Let $P$ be a CM point by an order $\Lambda \subset \mathcal{O}_K$. Then, there exists $\kappa = \kappa_P \in \mathbb{C}^*$ such that for all $f \in A_{2r}(\Gamma)_{L\cdot K^\Lambda}$,*

$$f = (q')^r \sum_n a_n q^n, \quad q(z) = \kappa \frac{z - P}{z - \overline{P}}, \quad a_n \in L \cdot K^\Lambda.$$

*Moreover, $\kappa \pi_{D_K}^{-1} \in \overline{\mathbb{Q}}$, for*

$$\pi_{D_K} = \left( \prod_{m=1}^{|D_K|} \Gamma\left(\frac{m}{|D_K|}\right)^{\left(\frac{D_K}{m}\right)} \right)^s,$$

*where*

$$s = \begin{cases} 2, & \textit{if } K = \mathbb{Q}(i), \\ 3, & \textit{if } K = \mathbb{Q}(\sqrt{-3}), \\ 1/h(K), & \textit{otherwise.} \end{cases}$$

*Proof.* It is enough to prove the result in the case $r = 0$, because any form of weight $2r$ can be written as $(Dt_1)^r t_2$ for some functions $t_1, t_2$. We can further assume that $f$ is holomorphic at $P$.

Let $q_1 = \frac{z-P}{z-\overline{P}}$. Then, every function in $A_0(\Gamma)_{L\cdot K^\Lambda}$ holomorphic at $P$ admits an expansion in $\mathbb{C}[\![q_1^w]\!]$, where $w$ is the elliptic order of $P$. We can choose $t \in A_0(\Gamma)_{L\cdot K^\Lambda}$ such that $t = a_1 q_1^w + \dots, a_1 \neq 0$. Then, define $\kappa = \sqrt[w]{a_1}$ and $q = \kappa q_1$. Thus, $t = q^w + \dots$ and, if $t_1 = \sum_n b_n q^{wn}$, then it follows from Corollary 5.1 that

$$\left(\frac{D^{we} t_1}{(D^w t)^e}\right)(P) = b_n \in L \cdot K^\Lambda.$$

Therefore, $t_1$ admits an expansion in $L \cdot K^\Lambda[\![q^w]\!]$.

The fact that $\kappa \pi_{D_K}^{-1} \in \overline{\mathbb{Q}}$ follows from [BT08] and [Shi79]. □

**Theorem 5.2.** *Let $P$ be a CM point by $\Lambda \subset \mathcal{O}_K$, let $\kappa \in \mathbb{C}^*$ be as in the previous corollary and let $q$ be the corresponding local parameter. Let also $f \in A_{2r}(\Gamma)$. Then, $f \in A_{2r}(\Gamma)_{L\cdot K^\Lambda}$ if and only if $f = (q')^r \sum_n a_n q^n$ with $a_n \in L \cdot K^\Lambda$ for all $n$.*

*Proof.* The rightwards implication follows from the previous corollary. To prove the other, let $f \in A_{2r}(\Gamma)$ be such that $f = (q')^r \sum_n a_n q^n$ with $a_n \in L \cdot K^\Lambda$. Moreover, $f$ can be written as $f = (Dt_1)^r t_2$ for some $t_1 \in A_0(\Gamma)_{L\cdot K^\Lambda}$ and $t_2 \in A_0(\Gamma)$. Thus, we can assume that $r = 0$. In this situation, let $\{f_i\}_i$ be a $\mathbb{C}$-base of $A_0(\Gamma)$ with elements lying in $A_0(\Gamma)_{L\cdot K^\Lambda}$, which exists since $A_0(\Gamma) = (A_0(\Gamma)_{L\cdot K^\Lambda}) \otimes \mathbb{C}$. Therefore, $t_2 = \sum_i' a_i f_i$, $a_i \in \mathbb{C}$ uniquely.

Reordering the functions, we can assume $t_2 = a_1 f_1 + \dots + a_n f_n$ uniquely with $f_1, \dots, f_n \in \{f_i\}_i$. Now, since the $q$-expansions of all these functions have coefficients in $L \cdot K^\Lambda$, the constants $a_i \in L \cdot K^\Lambda$. □

**Corollary 5.3.** *Let $f \in A_0(\Gamma)_L$. Then $Da(f, z) \in A_0(\Gamma)_L$, where $Da(f, z)$ denotes the automorphic derivative of $f$ introduced in Section 3.3, namely,*

$$Da(f, z) = \frac{2f'f''' - 3(f'')^2}{(f')^4},$$

*where $f' = \frac{df}{dz}$.*

*Proof.* Let $P$ be a CM point and $\Lambda \subset \mathcal{O}_K$ its corresponding order. Let also $\kappa_P$ be a local constant corresponding to this point, and $q_P$ the corresponding local parameter. Then, $f$ has an expansion lying in $L \cdot K^\Lambda[\![q_P]\!]$ and therefore also

$Da(f,z)$. Consequently, the previous theorem yields that $Da(f,z) \in A(\Gamma)_{L\cdot K^\Lambda}$. Therefore, we only need to prove that $\bigcap_P L\cdot K^\Lambda = L$. We are going to prove that this intersection, when running over a much more restrictive family of points is already $L$. Let $K = \mathbb{Q}(\sqrt{-p})$, $p > 0$, $p \equiv 3 \pmod 4$. Eichler Embedding Theorem ensures that if $\left(\frac{-p}{q}\right) = -1$ for every $q|D_{\mathbb{H}}$, then $K$ embeds into $\mathbb{H}$. Moreover, if $\left(\frac{-p}{q}\right) = 1$ for every $q|N$, there exists an optimal embedding of the ring of integers into the Eichler order of level $N$.

Then, the Theorem of Primes in Arithmetic Progression grants us the existence of infinitely many primes satisfiying the above hypotheses. Note that the condition $p \equiv 3 \pmod 4$ excludes no cases, since $\left(\frac{\cdot}{2}\right)$ has order 8. We choose two such primes, and let $K_1, K_2$ be its corresponding quadratic imaginary fields, $K_1^{(1)}, K_2^{(1)}$ its respective Hilbert Class Fields, and $\tilde{K}_1^{(1)}, \tilde{K}_2^{(1)}$ its Galois closures over $\mathbb{Q}$. Let $\tilde{L}$ be the Galois closure of $L|\mathbb{Q}$. Now, since the ramification sets of $\tilde{K}_1^{(1)}, \tilde{K}_2^{(1)}$ and $\tilde{L}$ are disjoints and these extensions are Galois, they are linearly disjoint. And the result now follows immediately. $\square$

### 5.2.1 Shimura reciprocity

In Theorem 1.18 we gave an explicit version of Shimura reciprocity law. This theorem gives an explicit description of the action of $\mathrm{Gal}(K^\Lambda|K)$ in the set of CM points by the order $\Lambda$. Equivalently, this can be reinterpreted as a description of the action of $\mathrm{Gal}(K^\Lambda|K)$ on the values of the functions in $A_0(\Gamma)_{\mathbb{Q}}$ at the CM points by the order $\Lambda$.

Our goal in this section is to prove a similar result not only for the values of functions but for the whole set of coefficients of expansions of quaternionic automorphic forms. The proof is similar to the one we have given to find the field where those coefficients lie, we only need to realize that this reciprocity is compatible with the construction in the proof given before.

**Lemma 5.5.** *Let $P$ be a CM point by $\Lambda \subset \mathcal{O}_K$ and $\phi : \Lambda \hookrightarrow \mathcal{O}(D,N)$ an optimal embedding corresponding to $P$. Let $\sigma \in Gal(K^\Lambda|K)$ and $\rho \in \mathcal{O}(D,N)$ the element corresponding to $\sigma$ by Shimura reciprocity law. Then, if $\alpha$ is the one given by Proposition 5.3, $\mathcal{O}(D,N) \cap (\rho\alpha\rho^{-1})^{-1}\mathcal{O}(D,N)(\rho\alpha\rho^{-1})$ is an Eichler order of level equal to the one of $\mathcal{O}(D,N) \cap \alpha^{-1}\mathcal{O}(D,N)\alpha$ and $\rho^{-1}P$ is CM by $\Lambda \subset \mathcal{O}_K$ with corresponding optimal embedding*

$$\phi^\rho : \Lambda \hookrightarrow \mathcal{O}(D,N) \cap (\rho\alpha\rho^{-1})^{-1}\mathcal{O}(D,N)(\rho\alpha\rho^{-1}),$$

*such that* $\phi^\rho = \rho^{-1}\phi\rho$.

*Proof.* Shimura reciprocity law yields that $\phi^\rho : \Lambda \hookrightarrow \mathcal{O}(D,N)$ is optimal and its fixed point is $\rho^{-1}P$. However, by the properties of $\alpha$, this embedding factors through $\mathcal{O}(D,N)\cap(\rho\alpha\rho^{-1})^{-1}\mathcal{O}(D,N)(\rho\alpha\rho^{-1})$. Moreover, since $\rho\alpha\rho^{-1}$ satisfies all the properties we required to $\alpha$, it is an Eichler order of the desired level. □

**Theorem 5.3.** *Let $P$ be a non-elliptic CM point by $\Lambda \subset \mathcal{O}_K$. Consider also $f \in A_{2k}(\Gamma)_{L\cdot K}$ holomorphic at $P$ and $g \in A_{2r+2e}(\Gamma)_{L\cdot K}$ holomorphic and non-vanishing at $P$. Then, for every $\sigma \in Gal(K^\Lambda|K)$,*

$$\left[\frac{D^e f}{g}(P)\right]^\sigma = \frac{D^e f}{g}(P^\sigma).$$

*Proof.* We will proceed by induction on $e$. The case $e = 0$ is just standard Shimura reciprocity law, Theorem 1.18.

We will begin by showing in detail how to prove the case $e = 1$ and then we will sketch the general case.

Let $h = \frac{f|_{2k}}{f} \in A_0(\Gamma\cap\alpha^{-1}\Gamma\alpha)$. Hence, $(Df)h + f(Dh) = (Df)|_{2k+2}\alpha$, which evaluated at $P$ yields

$$\left(\frac{Df}{g}\right)(P)C(\lambda) = \frac{f(Dh)}{g}(P),$$

with $C(\lambda) \in (L\cdot K)^*$. Therefore, since the right-hand side is an element of $A_0(\Gamma \cap \alpha^{-1}\Gamma\alpha)$ evaluated at a CM point $P$, the case $e = 0$ applies and we obtain

$$\left[\frac{Df}{g}(P)\right]^\sigma C(\lambda) = \left[\frac{Df}{g}(P)C(\lambda)\right]^\sigma = \left[\frac{fDh}{g}(P)\right]^\sigma = \frac{fDh}{g}(P^\sigma).$$

Now, define $F = f|_{2k}\rho, G = g|_{2k}\rho$ and $H = \frac{F|_{2k}\alpha}{F}$, and as above

$$\frac{(DF)H}{G}(P) + \frac{F(DH)}{G}(P) = \frac{(DF)|_{2k+2}\alpha}{G}(P).$$

Moreover, $H(P) = \frac{f|_{2k}(\rho\alpha\rho^{-1})}{f}(\rho P) \in (L\cdot K)^*$ and indeed is equal to $h(P)$. Therefore,

$$\frac{DF}{G}(P)C(\lambda) = \frac{F(DH)}{G}(P).$$

Putting together the equalities we have found, we obtain the result.

In the case $e > 1$, we keep the notations of the previous case. Then it holds that

$$\frac{D^e f|_{2r+2e}\alpha - (D^e f)h}{g} = \sum_{p=1}^{e-1} C_p \frac{D^{e-p} f}{g g_p^{-1}} \frac{D^p h}{g_p} + \frac{f}{g g_e^{-1}} \frac{D^e h}{g_e},$$

for $g_p \in A_{2p}(\Gamma)_{L\cdot K}$ not vanishing at $P$ (which is possible since $P$ is not elliptic). Now, evaluation at $P$, using induction hipothesis on the right-hand side and putting this together with the above formula for $F, G, H$, yields the result. $\square$

# Chapter 6

# Computational applications: kroneckerian polynomials and expansions of automorphic functions and forms

In this last chapter we study how to compute some of the objects introduced in the previous chapters and we provide several examples. Moreover, additional applications are given as well.

In the first section we devote ourselves to the explicit computation of kroneckerian polynomials and we give some examples for the case $D = 6$. In the second section, we focus on the computation of the values of a Hauptmodul at a given CM point and we use this to compute all the rational and quadratic CM points of the curves $X(D,1)^+$, $D = 6, 10, 22$. In particular, we are able to give another proof that the estimated values for the rational CM points of $X(D,1)^+$, $D = 6, 10$, given in [Elk98] are correct (cf. [Err11] for a different approach). In the next section we compute as well the first terms of a $q$-expansion (those necessary to obtain the rest using the differential equation given by the automorphic derivatives in Theorem 3.1) around all the previously computed rational CM points for $X(D,1)^+$, $D = 6, 10, 22$, together with estimates for the corresponding local constants. Finally, in the last section, we focus on the study of automorphic forms, we provide generators of the $\mathbb{Q}$-algebras of cusp forms corresponding to the curves considered in Chapters 3 and 4. Moreover, we study how to compute the action of the Hecke operators and give examples of $q$-expansions for some of the eigenforms.

## 6.1 Computation of kroneckerian polynomials

Let $X(D,N)$ be a Shimura curve such that its quotient by the Atkin-Lehner subgroup $W \subset W_{DN}$ has genus 0 (cf. Proposition 4.1). Assume that we know a Hauptmodul $t$ for $X(D,N)^W$, and by known we mean that both the values of the Hauptmodul at the elliptic cycles and the corresponding automorphic derivative are known. In particular, given an elliptic point $P$ of elliptic order $e$, we can compute $q$-expansions for the function $t = \sum_i a_i q_P^{ei}$ with $q_P = \kappa_P \frac{z-P}{z-\overline{P}}$ for a certain $\kappa_P \in \mathbb{C}^*$ such that $a_1 = 1$.

In this section we are going to see how this information can be used to explicitly compute kroneckerian polynomials of a given level, under some assumptions on the levels considered, and we will provide some examples. In the next section we are going to find a possible way to overcome this limitation.

Fix $P \in \mathcal{H}$ an elliptic point by $\Gamma = \langle \Gamma(D,N), W \rangle$. In particular, $P$ is a CM point by a quadratic order $\Lambda$, where $\Lambda$ is either $\mathbb{Z}[i]$, $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ or is one of the orders appearing in the list of CM points fixed by $\omega_m \in W$, according to Theorem 4.1. Denote by $\phi \in \mathcal{E}^*(\mathcal{O}(D,N),\Lambda)$ an optimal embedding corresponding to $P$. Then, assume that there exists an element $\lambda \in \Lambda'$ of norm $mk$ such that $m|\lambda^2$ in $\Lambda$, for some $m$ such that $\omega_m \in W$ and $k$ such that $\gcd(k, DN) = 1$. If $m$ is not squarefree, assume further that $w_m\phi(\lambda)$ is divisible by $m$ in $\mathcal{O}(D,N)$. Now, it follows from Proposition 2.1 and Proposition 2.3, that $\phi(\lambda) = w_m\gamma_0 w_k \gamma_1$, for certain $\gamma_0, \gamma_1 \in \mathcal{O}(D,N)_1$ and representatives $w_m, w_k$ of the Atkin-Lehner involutions of levels $m, k$.

In particular, $t \circ \phi(\lambda) = t \circ (w_k\gamma_1)$ is one of the zeros of the polynomial $\Psi_{\Gamma,k,t}(t,X)$. On the other hand, if we consider a $q$-expansion of $t$, $t = \sum_i a_i q^i$, since $\phi(\lambda)$ fixes $P$ we have that $q \circ \phi(\lambda) = \epsilon q$, where the expression for $\epsilon$ can be found in Proposition 3.6. Therefore, we automatically obtain a $q$-expansion for the function $t \circ (w_k\gamma_1)$. In particular, we can consider a symmetric polynomial in two variables, of degree $p+1$ in each variable and indeterminate coefficients, and then use the $q$-expansions of $t$ and $t \circ (w_k\gamma_1)$ in order to obtain a linear system for the unknown coefficients of this polynomial, when we impose that it vanishes at $(t, t \circ (w_k\gamma_1))$.

There are formulas to obtain the kroneckerian polynomial of a certain level $N$ from the ones having levels the prime factors of $N$. For example, the classical formulas for the modular polynomial of level $p^r$ (cf. [Cox89],[Web08]) also hold for the kroneckerian polynomials. Namely, if we denote by $\Psi_m(X,Y)$ the kroneckerian polynomials for a certain group of quaternionic units and a certain Hauptmodul, then

$$\Psi_{p^r}(X,Y) = \begin{cases} \dfrac{\mathrm{Res}_Z(\Psi_p(X,Z), \Psi_{p^{r-1}}(Y,Z))}{\Psi_{p^{r-2}}(X,Y)^p}, & \text{if } r > 2, \\ \dfrac{\mathrm{Res}_Z(\Psi_p(X,Z), \Psi_p(Y,Z))}{(X-Y)^{p+1}}, & \text{if } r = 2. \end{cases}$$

Similarly, the same is true for the formula of a kroneckerian polynomial of level $nm$, $\gcd(n,m)=1$, in terms of the kroneckerian polynomials of levels $n, m$:

$$\Psi_{nm}(X,Y) = \mathrm{Res}_Z(\Psi_n(X,Z), \Psi_m(Y,Z)).$$

All these expressions involve essentially resultants, which makes them difficult to apply in a setting where the polynomials have high degree and large coefficients. However we will see in the next section how we can use them to compute explicitly the values of a Hauptmodul at certain points. In order to do so, the following results will be useful.

**Proposition 6.1.** *Assume that $n, m$ are positive integers with* $\gcd(n,m)=1$ *or that $n=m=p$ is a prime number. Then,*

$$\Psi_{nm}(X,X) = \begin{cases} \mathrm{Res}_Z(\Psi_n(X,Z), \Psi_m(X,Z)), & \text{if } \gcd(n,m)=1, \\ c\mathrm{Res}_Z(\Psi_p(X,Z), \frac{\partial}{\partial X}(\Psi_p(X,Z))), & \text{if } n=m=p, \end{cases}$$

*for a certain $c \in \mathbb{Z}$.*

*Proof.* The first statement follows immediately from the expression for the kroneckerian polynomial above. In order to prove the second one, observe that we can write

$$\Psi_p(X,Z) = \Psi_p(Y,Z) + (X-Y)Q(X,Y,Z),$$

for a certain polynomial $Q(X,Y,Z)$ such that $Q(X,X,Z) = \frac{\partial}{\partial X}(\Psi_p(X,Z))$. Then,

$$\mathrm{Res}_Z(\Psi_p(X,Z), \Psi_p(Y,Z)) = c(X)^\alpha \mathrm{Res}_Z(\Psi_p(X,Z) - \Psi_p(Y,Z), \Psi_p(Y,Z))$$

where $c(X)$ denotes the leading coefficient of $\Psi_p(X,Z)$ as a polynomial in $Z$ and

$$\alpha = \deg_Z P(X,Z) - \deg_Z(P(X,Z) - P(Y,Z)).$$

Note that, if $c(X)$ is non-constant, then $\alpha = 0$. Anyway, $c(X)^\alpha = c \in \mathbb{Z}$. On the other hand,

$$\begin{aligned}\mathrm{Res}_Z(\Psi_p(X,Z) - \Psi_p(Y,Z), \Psi_p(Y,Z)) &= \mathrm{Res}_Z((X-Y)Q(X,Y,Z), \Phi_p(Y,Z)) \\ &= \mathrm{Res}_Z((X-Y), \Psi_p(Y,Z))\mathrm{Res}_Z(Q(X,Y,Z), \Psi_p(Y,Z)) \\ &= (X-Y)^{p+1}\mathrm{Res}_Z(Q(X,Y,Z), \Psi_p(Y,Z)).\end{aligned}$$

Therefore,

$$\Psi_{p^2}(X,Y) = c\mathrm{Res}_Z(Q(X,Y,Z), \Psi_p(Y,Z))$$

and the expression on the statement follows after taking $X = Y$ and reordering the factors. □

**Proposition 6.2.** *Let $X(D,M)^W$ be a Shimura curve of genus $0$ with a Hauptmodul $t$ and fix a prime $p$ coprime to $DM$. Let $\Lambda_0$ be an order inside a quadratic imaginary field $K$ of conductor coprime to $p$ and consider the order $\Lambda = \mathbb{Z} + p^k\Lambda_0$. Assume that $P \in \mathcal{H}$ is a CM point by $\Lambda$ and denote by $\phi$ a corresponding embedding. Assume that there exists $\lambda \in \Lambda' \cap p^k\Lambda_0$ of norm $mNp^{2k}$ for an integer $N$ coprime to $DMp$ and $m \| DM$ such that $\omega_m \in W$ and, if $m$ is not squarefree, $w_m\phi(\lambda) \in m\mathcal{O}(D,M)$. Then, there exists $Q$ a CM point by $\Lambda_0$ such that $\Psi_{\Gamma,p^k,t}(t(P), t(Q)) = 0$. In particular, $t(Q)$ is a root of $\Psi_{\Gamma,N,t}(X,X)$.*

*Proof.* Thanks to Propositions 2.1 and 2.3, we can write $\phi(\lambda) = \overline{\gamma}\,\overline{w_{p^k}}\alpha w_{p^k}\gamma$ for a certain $\alpha \in w_m\mathcal{O}(D,M)'_N$, and $\gamma \in \mathcal{O}(D,M)_1$. In particular, $\alpha$ fixes $\omega_{p^k}\gamma P$, which is a CM point by $\Lambda_0$, since it is conjugated to $P$ by an element of norm $p^k$ and the corresponding order contains the element $\lambda/p^k$. Now, by construction of the kroneckerian polynomial, the result follows. □

Finally, in order to find the value of the Hauptmodul $t$ of the curve $X(D,M)^W$ at a CM point by $\Lambda$, $P$, which is fixed by a transformation attached an element $\lambda \in \Lambda'$ of norm $mN$, with $m\|DM$ but $\omega_m \notin W$, in the conditions of the previous proposition, we could consider the quotient $X(D,M)^{\langle W,\omega_m\rangle}$. Then, we could compute the value of a Hauptmodul $t'$ at $P$ using Proposition 6.2 and deduce the value of $t$ at $P$ from that of $t'$. However, the same proof of Proposition 2.5 can be used to obtain the following result, which will be useful in our computations.

**Proposition 6.3.** *Let $W$ be a subgroup of $W_{DM}$, $\Gamma = \Gamma(D,M)\langle W\rangle$ and $t$ a Hauptmodul for the curve $X(D,M)_{\mathbb{C}}^{W}$. Denote $S = \{s : \omega_s \in W\}$ and let $m\|DM$. Consider $Q(X) = \frac{Q_0(X)}{Q_1(X)} \in \mathbb{C}(X)$, with $Q_i(X)$ polynomials of degree 1, such that $t\circ\omega_m = Q(t)$. Then, given $P$ a CM point by an order $\Lambda$ in a quadratic field $K$, the multiplicity of $t(P)$ as a root of $Q_1(X)^{\psi(N)}\Psi_{\Gamma,N,f}(X,Q(X))$ is given by*

$$\begin{aligned} r(\Lambda,N,S) &= \#\{\lambda\in\Lambda' : \omega_s\omega_m\phi(\lambda)\in sm\mathcal{O}(D,M)_N \text{ for a certain } s\in S\}/\sim \\ &= \#\{\lambda\in\Lambda' : \omega_s\omega_m\phi(\lambda)\in sm\mathcal{O}(D,M)_N \text{ for a certain } s\in S\}/e(P), \end{aligned}$$

*where $\lambda_0\sim\lambda_1$ if and only if $\lambda_0\overline{\lambda_1}\in N\Lambda$, and $e(P)$ denotes the elliptic order of $P$.*

*In particular, the point $z_0\in\mathbb{C}$ is a root of the polynomial*

$$Q_1^{\psi(N)}\Psi_{\Gamma,N,t}(X,Q(X))\in\mathbb{C}[X]$$

*if and only if $z_0 = t(P)$ for a CM point $P\in\mathcal{H}$ which corresponds to an optimal embedding $\phi:\Lambda\hookrightarrow\mathcal{O}(D,M)$ such that there exists a primitive element $\lambda\in\Lambda$ of norm $smN$ for a certain $s\in S$ with $\omega_s\omega_m\phi(\lambda)\in sm\mathcal{O}(D,M)$ (cf. Proposition 2.3).* □

Now we are going to present some concrete examples of kroneckerian polynomials computed using the method above. We will choose some of the simpler cases, which are $D=6$ and $N=5,7,11,13$.

### 6.1.1 Examples in the case $D=6$

Let us consider the Hauptmodul $t=t_6$ for $\Gamma=\Gamma(6,1)$ given in Theorem 2.4, further studied in Theorem 3.1.

**The case $p=5$**

The point $P=i\in\mathcal{H}$ is a CM point by $\mathbb{Z}[i]$ and therefore it is an elliptic point of order 2 for $\Gamma(6,1)$. Denote by $\phi$ the corresponding optimal embedding. Moreover, there exists an element $\lambda=2+i\in\mathbb{Z}[i]$ of norm 5. Thus, for

$$q_i(z)=\kappa_i\frac{z-i}{z+i},\qquad \kappa_i\in\mathbb{C},$$

we have

$$q_i(\phi(\lambda)) = \frac{5}{\lambda^2} q_i = \frac{\lambda}{\bar{\lambda}} q_i.$$

Then, we have an expansion

$$t_6(z) = \sum_{n \geq -2} a_n q_i(z)^n,$$

where the first few coefficients $a_n \in \mathbb{Q}$ are computed explicitly. This gives us

$$t_6(\phi(\lambda)(z)) = \sum_{n \geq -2} a_n q_i(\phi(\lambda)(z))^n = \sum_{n \geq -2} a'_n q_i(z)^n,$$

where $a'_n = a_n \left(\frac{2+i}{2-i}\right)^n$. Therefore, we can substitute both series for $X$, $Y$ in a polynomial

$$\Psi_{\Gamma,5,t_6}(X,Y) = \sum_{j,k=0}^{6} a_{jk} X^j Y^k,$$

with indeterminate coefficients $a_{jk}$, and solve the resulting system. The non-zero coefficients of this polynomial are listed in Table 6.1. It is a trivial verification that the reduction modulo 5 of this polynomial is $(X + Y^5)(Y + X^5)$, as stated in Theorem 2.5.

| | |
|---|---|
| $a_{02} = a_{20} = a_{46} = a_{64}$ | 119574225 |
| $a_{04} = a_{40} = a_{26} = a_{62}$ | −210039480 |
| $a_{06} = a_{60}$ | 92236816 |
| $a_{11} = a_{55}$ | 66961566 |
| $a_{13} = a_{31} = a_{35} = a_{53}$ | −582922980 |
| $a_{15} = a_{51}$ | 550309200 |
| $a_{22} = a_{44}$ | −785133000 |
| $a_{24} = a_{42}$ | 1102896150 |
| $a_{33}$ | 1712753300 |

Table 6.1: Coefficients of $\Psi_{\Gamma,5,t_6}$

### The case $p = 13$

This case is similar to the previous one since $\lambda = 3 + 2i \in \mathbb{Z}[i]$ has norm 13. Therefore we can take the point $P = i \in \mathcal{H}$ as above. Then,

$$q_i \circ \phi(\lambda) = \frac{3+2i}{3-2i} q_i.$$

Moreover, from the above expansion for $t_6$ around $P = i$ we obtain

$$t_6(\phi(\lambda)(z)) = \sum_{n \geq -2} a_n q_i(\phi(\lambda)(z))^n = \sum_{n \geq -2} a'_n q_i(z)^n,$$

where $a'_n = a_n \left(\frac{3+2i}{3-2i}\right)^n$. Therefore, we can substitute both series for $X$, $Y$ in a polynomial

$$\Psi_{\Gamma,13,t_6}(X,Y) = \sum_{j,k=0}^{14} a_{jk} X^j Y^k,$$

with indeterminate coefficients $a_{jk}$, and solve the resulting system. The non-zero coefficients of this polynomial are listed in Table 6.2. It is a trivial verification that the reduction modulo 13 of this polynomial is $-(X - Y^{13})(Y - X^{13})$, as stated in Theorem 2.5.

| | |
|---|---|
| $a_{0,2} = a_{2,0} = a_{12,14} = a_{14,12}$ | 157321474567113614046314215229221867776 |
| $a_{0,4} = a_{4,0} = a_{10,14} = a_{14,10}$ | $-$223329048165612085653790111234085115168 |
| $a_{0,6} = a_{6,0} = a_{8,14} = a_{14,8}$ | 639558329083494880075337606636408903937 |
| $a_{0,8} = a_{8,0} = a_{6,14} = a_{14,6}$ | $-$339885081390651885727867107320969704176 |
| $a_{0,10} = a_{10,0} = a_{4,14} = a_{14,4}$ | 457846149604433581004932521042517595232 |
| $a_{0,12} = a_{12,0} = a_{2,14} = a_{14,2}$ | 102941842560971611003135897276198473984 |
| $a_{0,14} = a_{14,0}$ | 5310433027391928548082272255815434496 |
| $a_{1,1} = a_{13,13}$ | 221553319212858225698359664050620145152 |
| $a_{1,3} = a_{3,1} = a_{11,13} = a_{13,11}$ | $-$3907822344001165260505767172808382272640 |
| $a_{1,5} = a_{5,1} = a_{9,13} = a_{13,9}$ | 5358810648261307600800023159686628064186 |
| $a_{1,7} = a_{7,1} = a_{7,13} = a_{13,7}$ | 14188639516967683311243184784903680944504 |
| $a_{1,9} = a_{9,1} = a_{5,13} = a_{13,5}$ | $-$8509532670217823471919483578243457096096 |
| $a_{1,11} = a_{11,1} = a_{3,13} = a_{13,3}$ | 1763597638222966767233053435039160321664 |
| $a_{1,13} = a_{13,1}$ | $-$709719245999343588656896320744946706432 |
| $a_{2,2} = a_{12,12}$ | $-$5582267572500765247322729910827058588864 |
| $a_{2,4} = a_{4,2} = a_{10,12} = a_{12,10}$ | 15566671962647972437246849875069036512271 |

| | |
|---|---|
| $a_{2,6} = a_{6,2} = a_{8,12} = a_{12,8}$ | 17788682738348192950267058609087486547792 |
| $a_{2,8} = a_{8,2} = a_{6,12} = a_{12,6}$ | 44512203475303030585138782936649232745468 |
| $a_{2,10} = a_{10,2} = a_{4,12} = a_{12,4}$ | $-45481786563481670680883365110716938016016$ |
| $a_{2,12} = a_{12,2}$ | 10772996380747037651874095877574912677184 |
| $a_{3,3} = a_{11,11}$ | 22667327663068646397348695143993699978860 |
| $a_{3,5} = a_{5,3} = a_{9,11} = a_{11,9}$ | 103292709255177545224740743281321684248456 |
| $a_{3,7} = a_{7,3} = a_{7,11} = a_{11,7}$ | $-141385872491557640165418198331988931171624$ |
| $a_{3,9} = a_{9,3} = a_{5,11} = a_{11,5}$ | 156229489001319601347721493609328430457544 |
| $a_{3,11} = a_{11,3}$ | $-46932210288112374010605471441783240964288$ |
| $a_{4,4} = a_{10,10}$ | 123613067875122986869852989122780505527616 |
| $a_{4,6} = a_{6,4} = a_{8,10} = a_{10,8}$ | $-234603178607061580963128257263524955945884$ |
| $a_{4,8} = a_{8,4} = a_{6,10} = a_{10,6}$ | 151963375795261173964103278270125202052688 |
| $a_{4,10} = a_{10,4}$ | 104364071921642313047444412533434295993990 |
| $a_{5,5} = a_{9,9}$ | $-460109713337629190182764691769863407901680$ |
| $a_{5,7} = a_{7,5} = a_{7,9} = a_{9,7}$ | 545160957003294505782278920619623236703800 |
| $a_{5,9} = a_{9,5}$ | $-316049176219180561461317385354087527471140$ |
| $a_{6,6} = a_{8,8}$ | 326166687776446607181743146779777359568000 |
| $a_{6,8} = a_{8,6}$ | $-460420624875331284131007926726632542448230$ |
| $a_{7,7}$ | $-1086940199760004041936334880320578594960120$ |

Table 6.2: Coefficients of $\Psi_{\Gamma,13,t_6}$

**The case $p = 7$**

Now, the point $P = \frac{1+i}{1+\sqrt{3}} \in \mathcal{H}$ is a CM point by the ring of integers of $\mathbb{Q}(\sqrt{-3})$ and thus an elliptic point of order 3 for $\Gamma(6,1)$. Moreover, there exists an element $\lambda = 2 + \sqrt{-3} \in \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ of norm 7. Then, if we denote by $\phi$ the optimal embedding corresponding to $P$, we have that

$$q_P \circ \phi(\lambda) = \frac{2+\sqrt{-3}}{2-\sqrt{-3}} q_P$$

and therefore, given the expansion

$$t_6(z) = \sum_{n \geq 0} a_n q_P(z)^n,$$

where the first few coefficients $a_n \in \mathbb{Q}$ are computed explicitly, we obtain an expansion for

$$t_6(\phi(\lambda)(z)) = \sum_{n\geq 0} a_n q_P(\phi(\lambda)(z))^n = \sum_{n\geq 0} a'_n q_P(z)^n,$$

where $a'_n = a_n \left(\frac{2+\sqrt{-3}}{2-\sqrt{-3}}\right)^n$. Therefore, we can substitute both series for $X$, $Y$ in a polynomial

$$\Psi_{\Gamma,7,t_6}(X,Y) = \sum_{j,k=0}^{8} a_{jk} X^j Y^k,$$

with indeterminate coefficients $a_{jk}$, and solve the resulting system. The non-zero coefficients of this polynomial are listed in Table 6.3. It is a trivial verification that the reduction modulo 7 of this polynomial is $(1-XY^7)(1-YX^7)$, as stated in Theorem 2.5.

| | |
|---|---|
| $a_{00} = a_{88}$ | 366028679279376 |
| $a_{02} = a_{20} = a_{68} = a_{86}$ | $-175148154031032$ |
| $a_{04} = a_{40} = a_{48} = a_{84}$ | 83697102786465 |
| $a_{06} = a_{60} = a_{28} = a_{82}$ | $-15011938285344$ |
| $a_{08} = a_{80}$ | 2688917803264 |
| $a_{11} = a_{77}$ | $-2484877824079056$ |
| $a_{13} = a_{31} = a_{57} = a_{75}$ | 2326773665288268 |
| $a_{15} = a_{51} = a_{37} = a_{73}$ | 632022013171836 |
| $a_{17} = a_{71}$ | $-193115439357632$ |
| $a_{22} = a_{66}$ | 669210475564134 |
| $a_{24} = a_{42} = a_{46} = a_{64}$ | $-3793573444895640$ |
| $a_{26} = a_{62}$ | 3577214930233798 |
| $a_{33} = a_{55}$ | 9188300872159416 |
| $a_{35} = a_{53}$ | $-12672039590642936$ |
| $a_{44}$ | 6858140981941060 |

Table 6.3: Coefficients of $\Psi_{\Gamma,7,t_6}$

**The case $p = 11$**

This case is more complicated since neither $\mathbb{Z}[i]$ nor $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ contain elements of norm 11. However, if we move to the quotient $X(6,1)^+$, the CM points by

$\mathbb{Z}[\sqrt{-6}]$ have become elliptic and therefore we can compute an expansion of $t_6^+$ around the CM point by $\mathbb{Z}[\sqrt{-6}]$ given by $P = \frac{(\sqrt{6}-\sqrt{2})i}{2}$, where $t_6^+(P) = 0$. From this expansion we obtain an expansion for $t_6$ around the point $P$, where $t_6(P) = i$, using that

$$t_6^+ = \left(\frac{t_6^2+1}{t_6^2-1}\right)^2.$$

The ring $\mathbb{Z}[\sqrt{-6}]$ does not contain any element of norm 11, but it does contain an element of norm 22, namely $\lambda = 4+\sqrt{-6}$, such that $2|\lambda^2$ in $\mathbb{Z}[\sqrt{-6}]$. Therefore we could use the procedure above to compute the kroneckerian polynomial of level 11 attached to $t_6^+$ or even for the Hauptmodul corresponding to $X(6,1)^{\langle\omega_2\rangle}$, $u_{6,3}$. However, since we know explicitly the action of the Atkin-Lehner involutions on $t_6$, in our case $t_6 \circ w_2 = -t_6$, we can directly compute the kroneckerian polynomial of level 11 corresponding to $t_6$, using Proposition 6.3.

Note that $t_6 \circ \phi(\lambda) = t_6(\omega_2\gamma_0 w_{11}\gamma_1) = -t_6(w_{11}\gamma_1)$ and therefore we have that $t_6(w_{11}\gamma_1) = -t_6 \circ \phi(\lambda)$. Then, for

$$q_P(z) = \kappa_P \frac{z-P}{z-\overline{P}}, \qquad \kappa_P \in \mathbb{C}^*$$

we have

$$q_P(\phi(\lambda)) = \frac{\lambda}{\overline{\lambda}} q_P.$$

Moreover, we have an expansion

$$t_6(z) = \sum_{n\geq 0} a_n q_P(z)^n,$$

where the first few coefficients $a_n \in \mathbb{Q}$ are computed explicitly. This gives us a relation

$$t_6(w_{11}\gamma_1(z)) = \sum_{n\geq 0} a'_n q_P(z)^n,$$

where $a'_n = -a_n \left(\frac{4+\sqrt{-6}}{4-\sqrt{-6}}\right)^n$. Therefore, we can substitute these series

$$\sum_{n\geq 0} a_n q_P(z)^n, \qquad \sum_{n\geq 0} a'_n q_P(z)^n,$$

for $X$, $Y$ in a polynomial

$$\Psi_{\Gamma,11,t_6}(X,Y) = \sum_{j,k=0}^{12} a_{jk} X^j Y^k,$$

with indeterminate coefficients $a_{jk}$, and solve the resulting system. The non-zero coefficients of this polynomial are listed in Table 6.4. It is a trivial verification that the reduction modulo 11 of this polynomial is $(1 + XY^{11})(1 + YX^{11})$, as stated in Theorem 2.5.

| | |
|---|---|
| $a_{0,0} = a_{12,12}$ | 161479346678868884449383016704 |
| $a_{0,2} = a_{2,0} = a_{10,12} = a_{12,10}$ | $-700259488320028961583165392640$ |
| $a_{0,4} = a_{4,0} = a_{8,12} = a_{12,8}$ | 919912311217388018171537308512 |
| $a_{0,6} = a_{6,0} = a_{6,12} = a_{12,6}$ | 76545318714518090586092693904 |
| $a_{0,8} = a_{8,0} = a_{4,12} = a_{12,4}$ | $-881662021257852546155119386711$ |
| $a_{0,10} = a_{10,0} = a_{2,12} = a_{12,2}$ | 211560066253894917848033517696 |
| $a_{0,12} = a_{12,0}$ | 279732396586128790436714745856 |
| $a_{1,1} = a_{11,11}$ | $-5152213998401374154602359608832$ |
| $a_{1,3} = a_{3,1} = a_{9,11} = a_{11,9}$ | 18511567572675167012497768926336 |
| $a_{1,5} = a_{5,1} = a_{7,11} = a_{11,7}$ | $-25899527501613233338738322103264$ |
| $a_{1,7} = a_{7,1} = a_{5,11} = a_{11,5}$ | 28376635514179567967013595298904 |
| $a_{1,9} = a_{9,1} = a_{3,11} = a_{11,3}$ | $-23743359785030634825102696625956$ |
| $a_{1,11} = a_{11,1}$ | 7438491727516340412718850925312 |
| $a_{2,2} = a_{10,10}$ | 59328230855330299186274825945664 |
| $a_{2,4} = a_{4,2} = a_{8,10} = a_{10,8}$ | $-164185592320146228724455084990864$ |
| $a_{2,6} = a_{6,2} = a_{6,10} = a_{10,6}$ | 201983865932424685349472770205756 |
| $a_{2,8} = a_{8,2} = a_{4,10} = a_{10,4}$ | $-125853332231546454366142736028912$ |
| $a_{2,10} = a_{10,2}$ | 31648366312546597884262004848374 |
| $a_{3,3} = a_{9,9}$ | $-316120114238429618184282492087360$ |
| $a_{3,5} = a_{5,3} = a_{7,9} = a_{9,7}$ | 646552456939854174209188752276840 |
| $a_{3,7} = a_{7,3} = a_{5,9} = a_{9,5}$ | $-544407496253705110435610320239120$ |
| $a_{3,9} = a_{9,3}$ | 211652433379326599389385977311760 |
| $a_{4,4} = a_{8,8}$ | 905214213307076899036139519181750 |
| $a_{4,6} = a_{6,4} = a_{6,8} = a_{8,6}$ | $-1330514179243565505915761972442000$ |
| $a_{4,8} = a_{8,4}$ | 733895948770622934801403475408040 |
| $a_{5,5} = a_{7,7}$ | $-1772204569903587914287480135652760$ |
| $a_{5,7} = a_{7,5}$ | 1637057782264734851739512604044400 |
| $a_{6,6}$ | 2296209946830565387198646731768260 |

Table 6.4: Coefficients of $\Psi_{\Gamma,11,t_6}$

## 6.2 Computation of the values of a Hauptmodul and its expansions at a given CM point

First of all, given a Hauptmodul $t$ for a genus 0 curve $X(D,N)^W$ corresponding to the group $\Gamma = \langle \Gamma(D,N), W \rangle$, we are going to describe a method to compute the value of $t$ around an arbitrary CM point using only the expansions around elliptic points and the corresponding kroneckerian polynomials we can compute from them according to the previous section. Then we are going to see how to extend this method to compute as well expansions around these points. Note that in this way we can go back to the previous section and use these new expansions to compute new kroneckerian polynomials which we could not compute directly using only the expansions at elliptic points.

Assume we are given a non-elliptic CM point $P$ by an order $\Lambda_K$ of conductor $f$ in a quadratic imaginary field $K$ and denote by $\phi_K$ a corresponding optimal embedding. Then we need to compute $a_0, a_2 \in \overline{\mathbb{Q}}$ such that $t = a_0 + q + a_2 q + \dots$ for a certain local parameter $q = \kappa \frac{z-P}{z-\overline{P}}$. If these two values are known, then we can use the expression of the automorphic derivative to compute as many coefficients of the $q$-expansions around $P$ as desired.

In order to compute this couple of algebraic numbers we make the following choices:

- An elliptic point $Q$ by an order $\Lambda_F$ in a quadratic imaginary field $F$. Denote by $\phi_F$ an optimal embedding corresponding to $Q$.
- An odd prime $p$ such that:
  - $p$ decomposes in both $F$ and $K$;
  - $\gcd(p, f) = 1$.

Then, there exists $\lambda_F \in \Lambda_F'$ of norm $s_0 p^{r_0}$ for some positive integers $s_0, r_0$ such that $\omega_{s_0} \in W$, $\lambda_F^2 \in s_0 \Lambda_F$ and, if $s_0$ is not squarefree, we require in addition that $\omega_{s_0} \phi_F(\lambda_F) \in s_0 \mathcal{O}(D,N)$ (cf. Proposition 2.3).

Similarly, there exists $\lambda_K \in \Lambda_K'$ of norm $s_1 p^{r_1}$ for some positive integers $s_1, r_1$ such that $\omega_{s_1} \in W$, $\lambda_K^2 \in s_1 \Lambda_K$ and, if $s_1$ is not squarefree, we have as well that $\omega_{s_1} \phi_K(\lambda_K) \in s_1 \mathcal{O}(D,N)$.

Then, we define $r = \operatorname{lcm}(r_0, r_1)$. Since we will be looking for an $r$ which is as small as possible, it will be convenient in most cases to take $s_0 = r_0 = 1$ (for the existence of such primes in this case, cf. [Cox89]). Note that some choices of $F$ and $p$ can lead to more computationally complex situations than others.

Let $\Psi_{\Gamma,p^r,t}(X,Y) \in \mathbb{C}[X,Y]$ be the kroneckerian polynomial of level $p^r$ attached to a Hauptmodul $t$. By construction, this polynomial can be computed using expansions around the elliptic point $Q$. Since $P$ is not elliptic, we can fix a local constant $\kappa = \kappa_P \in \mathbb{C}$ and a local parameter $q = q_P(z) = \kappa \frac{z-P}{z-\overline{P}}$ such that the expansion of $t$ around $P$ with respect to this parameter is

$$t(q) = a_0 + q + a_2 q^2 + \dots$$

Now, since $\alpha = \phi_F(\lambda_F)$ fixes $P$, $\frac{q \circ \alpha}{q} = \epsilon \in \mathbb{C}^*$ and therefore

$$(t \circ \alpha)(q) = a_0 + \epsilon q + \epsilon^2 a_2 q^2 + \dots$$

By definition, $\Psi_{\Gamma,p^r,t}(t, t\circ\alpha) = 0$ and thus $a_0$ is a zero of $\Psi_{\Gamma,p^r,t}(X,X)$. More details on the zeros of $\Psi_{\Gamma,p^r,t}(X,X)$ and their multiplicity are given in Proposition 2.5. This fact, together with an approximation of the value at $P$, if necessary, allows us to compute the exact value of $a_0$.

In order to compute the value of $a_2$, we start by observing that due to the choices above, Theorem 2.3 gives us that $\mathrm{mult}_{(P,\omega_{p^r}\gamma P)}\Psi_{\Gamma,p^r,t} = 2$, where $\gamma$ is such that $\phi_K(\lambda_K^{r/r_0}) = \omega_{s_0}^{r/r_0}\gamma' w_{p^r}\gamma$. In particular, since $\phi_K(\lambda_K^{r/r_0})$ fixes $P$, $P$ defines the same point as $\omega_{p^r}\gamma P$ in $\langle\Gamma(D,N),W\rangle\backslash\mathcal{H}$. Therefore, we obtain that $\mathrm{mult}_{(P,P)}\Psi_{\Gamma,p^r,t} = 2$ and, by Theorem 2.2, that $\mathrm{ord}_{f(P)}\Psi(f(P),X) = 2$. In particular, all first order derivatives of $\Psi_{\Gamma,p^r,t}(X,Y)$ vanish at $(a_0,a_0)$ and there exists a second order derivative of $\Psi_{\Gamma,p^r,t}(X,Y)$ not vanishing at $(a_0,a_0)$.

Using that $\Psi_{\Gamma,p^r,t}(t(q), t\circ\alpha(q)) = 0$ and that $\Psi_{\Gamma,p^r,t}$ is symmetric we obtain, differentiating twice with respect to $q$ and evaluating at $q=0$,

$$(1+\epsilon^2)(\partial_{XX}\Psi_{\Gamma,p^r,t})(a_0,a_0) + 2\epsilon(\partial_{XY}\Psi_{\Gamma,p^r,t})(a_0,a_0) = 0.$$

Similarly, differentiating three times and evaluating at $q=0$, we obtain an expression containing $a_2$, powers of $\epsilon$ and derivatives up to order three of $\Psi_{\Gamma,p^r,t}$ evaluated at $(a_0,a_0)$. The coefficient of $a_2$ in this expression is

$$(1+\epsilon^3)(\partial_{XX}\Psi_{\Gamma,p^r,t})(a_0,a_0) + (\epsilon+\epsilon^2)(\partial_{XY}\Psi_{\Gamma,p^r,t})(a_0,a_0).$$

If it is not zero, then we will be able to compute $a_2$ from $\epsilon$, $a_0$ and the values of the kroneckerian polynomial. To prove that this expression is really non-zero we need to check that $\epsilon \neq 0, 1, -1$. By Proposition 3.6, $\epsilon = \frac{\lambda_K}{\overline{\lambda_K}}$, and therefore it is obvious that $\epsilon \neq 0, 1$; if $\epsilon$ were $-1$, then $\lambda_K = \sqrt{-s_1 p^{r_1}}$ and since $p$ does not ramify in $K$, $r_1 \in 2\mathbb{Z}$ and moreover it is primitive in $\Lambda_K$, therefore $\Lambda_K = \mathbb{Z}[p^{r_1'}\sqrt{-s_1}]$ and it has conductor coprime to $p$ from what we obtain a contradiction, namely $0 < r_1 = 2r_1' = 0$. Hence, we get the value of $a_2$ as desired.

**Remark.** The above choice of $p$ is convenient to have a general approach to the problem, but it is not always the most suitable approach to our particular case. To compute the value of $t$ at a certain CM point $P$ by $\Lambda_K$ as above, it is enough to know $\Psi_{\Gamma,N,t}(X,X)$ for a certain $N$ such that there exists $\lambda_K \in \Lambda_K'$ of norm $s_1 N$ with $\omega_{s_1} \in W$ such that $\lambda_K^2 \in s_1\Lambda_K$ and, if $s_1$ is not squarefree, $\omega_{s_1}\phi(\lambda_K) \in s_1\mathcal{O}(D,N)$. Then, Proposition 2.5 yields that $t(P)$ is a root of this polynomial.

Similarly, to compute the value of $a_2$ it suffices to have $\Psi_{\Gamma,p^r,t}(X,Y)$ for a certain $p^r$ with $\gcd(p,f) = 1$ such that there exists $\lambda_K \in \Lambda_K'$ of norm $s_1 p^r$ with $\omega_{s_1} \in W$, $\lambda_K^2 \in s_1\Lambda_K$ and, if $s_1$ is not squarefree, $\omega_{s_1}\phi(\lambda_K) \in s_1\mathcal{O}(D,N)$. Then, Theorem 2.3, ensures that $\mathrm{mult}_{(P,P)}\Psi_{\Gamma,p^r,t} \leq 2$ and therefore, if the multiplicity is 1 the value of $a_2$ is obtained immediately and if it is 2 we can proceed as described in the method above.

### 6.2.1 All rational and quadratic CM points on the curves $X(D,1)^+$ for $D = 6, 10, 22$

As an application of the previous algorithm we are going to compute the set of rational and (non-rational) quadratic CM points on the curves $X(D,1)^+$, $D = 6, 10, 22$. We denote the corresponding group by $\Gamma^+$. In particular we obtain that the tables for $D = 6, 10$ of rational CM points given by Elkies in [Elk98], where some of the values were only numerically correct, are indeed correct. There is only one missing value in [Elk98] in the case $D = 10$. A completely different approach to check that these values in the cases $D = 6, 10$ are correct can be found in [Err11]. The case $D = 22$ was not studied in Elkies' paper, and the other two cases studied there, $D = 14, 15$, can also be checked using our approach.

To do so, in all these cases, we compute the kroneckerian polynomials $\Psi_{\Gamma^+,p,t_D^+}(X,Y)$ for all primes $3 \leq p \leq 41$ coprime to $D$, which can be obtained using the method we have seen. From these, the polynomials $\Psi_{\Gamma(D,1),p,t_D}(X,Y)$ can also be deduced. We do not reproduce all these polynomials, because they are too large to be printed here.

Then, we consider the list of orders which may give rational or quadratic CM points on $X(D,1)^+$. In particular, all these CM points must be attached to fields having class number $1,2,4$ or $8$. The list of quadratic imaginary fields with class numbers 4 are given in [Arn92] and those of class number 8 can be computed thanks to the bounds given in [Wat04].

The list of such fields is given in the following table:

| $h$ | $\lvert D(K)\rvert$ for imaginary quadratic $K$ with $h_K = h$ |
|---|---|
| 1 | 3, 4, 7, 8, 11, 19, 43, 67, 163 |
| 2 | 15, 20, 24, 35, 40, 51, 52, 88, 91, 115, 123, 148, 187, 232, 235, 267, 403, 427 |
| 4 | 39, 55, 56, 68, 84, 120, 132, 136, 155, 168, 184, 195, 203, 219, 228, 259, 280, 291, 292, 312, 323, 328, 340, 355, 372, 388, 408, 435, 483, 520, 532, 555, 568, 595, 627, 667, 708, 715, 723, 760, 763, 772, 795, 955, 1003, 1012, 1027, 1227, 1243, 1387, 1411, 1435, 1507, 1555 |
| 8 | 95, 111, 164, 183, 248, 260, 264, 276, 295, 299, 308, 371, 376, 395, 420, 452, 456, 548, 552, 564, 579, 580, 583, 616, 632, 651, 660, 712, 820, 840, 852, 868, 904, 915, 939, 952, 979, 987, 995, 1032, 1043, 1060, 1092, 1128, 1131, 1155, 1195, 1204, 1240, 1252, 1288, 1299, 1320, 1339, 1348, 1380, 1428, 1443, 1528, 1540, 1635, 1651, 1659, 1672, 1731, 1752, 1768, 1771, 1780, 1795, 1803, 1828, 1848, 1864, 1912, 1939, 1947, 1992, 1995, 2020, 2035, 2059, 2067, 2139, 2163, 2212, 2248, 2307, 2308, 2323, 2392, 2395, 2419, 2451, 2587, 2611, 2632, 2667, 2715, 2755, 2788, 2827, 2947, 2968, 2995, 3003, 3172, 3243, 3315, 3355, 3403, 3448, 3507, 3595, 3787, 3883, 3963, 4123, 4195, 4267, 4323, 4387, 4747, 4843, 4867, 5083, 5467, 5587, 5707, 5947, 6307 |

For any of these fields, the set of orders we have to consider can be computed using Theorem 1.12.

Then, among this orders we can select those giving rational or quadratic CM points thanks to the following result, cf. [Jor81, GR06]. In [GR06], a version of this result with squarefree level is stated. Since we are interested in the level 1 case, we will restrict to this case.

**Proposition 6.4.** *Let $X(D,1)$ be the Shimura curve of discriminant $D$. For a quadratic order $\Lambda$ of conductor $f$ in a quadratic imaginary field $K$, consider the subgroup of Atkin-Lehner involutions given by*

$$W(\Lambda) = \langle \omega_p : p \nmid f, \left(\tfrac{K}{p}\right) = -1 \rangle.$$

*Then, $W(\Lambda) \times \mathrm{Gal}(K^{\Lambda}|K)$ acts freely and transitively in the set of CM points by $\Lambda$.* □

Note, in particular, that the CM points by $\Lambda$ are rational in $X(D,1)^+$ if and only if there exists only one representative of such points on $\langle \Gamma(D,1), W_D \rangle \backslash \mathcal{H}$.

### The case $D = 6$

First of all, we have to consider among those fields above, the ones which are embeddable into $\mathbb{H}_6$. Then we check which of those yield rational and quadratic CM points. We give them in the following table:

| $f$ | 1 | 5 | 7 | 11 | 13 |
|---|---|---|---|---|---|
| $\lvert D(K)\rvert$ s.t. the order of conductor $f$ gives rational CM points | 3, 4, 19, 24, 40, 43, 51, 52, 67, 84, 88, 120, 123, 132, 148, 163, 168, 228, 232, 267, 312, 372, 408, 708 | 3, 4 | 3 | | |
| $\lvert D(K)\rvert$ s.t. the order of conductor $f$ gives quadratic CM points | 91, 115, 136, 184, 187, 195, 219, 235, 264, 276, 280, 291, 292, 328, 340, 388, 403, 420, 427, 435, 456, 483, 520, 532, 552, 555, 564, 568, 627, 660, 723, 760, 772, 795, 840, 852, 1012, 1032, 1092, 1128, 1227, 1320, 1380, 1428, 1752, 1848, 1992 | 24 | 4 | 3 | 3 |

Now that we have the list of CM points for which the values of $t_6^+$ need to be computed, we have to use Proposition 2.5 to identify the value of $t_6^+$ at each of them as a zero of $\Psi_{\Gamma^+,N,t_6^+}(X,X)$, for some $N$ such that we can explicitly compute $\Psi_{\Gamma^+,N,t_6^+}(X,X)$. We can use Proposition 6.1 to deal with the composite levels. In this case, Proposition 6.2 will not be needed.

In Table 6.5, for each order $\Lambda$ with discriminant $D(\Lambda)$ such that the points in $\mathrm{CM}(\Lambda)$ are rational in $X(6,1)^+$, according to the table above, we provide a list of some $N$ such that $\Psi_{\Gamma^+,N,t_6^+}(X,X)$ has $t(P)$ as a root, for all points $P \in \mathrm{CM}(\Lambda)$. In this list, we do not provide values of $N$ for the rings of integers of $\mathbb{Q}(\sqrt{-d})$, for $d = 1,3,6$, since the values at these points are already known by construction of the Hauptmodul $t_6^+$.

| $D(\Lambda)$ | $N$ |
|---|---|
| $-40$ | $5, 7, 11, 13$ |
| $-84$ | $5, 7, 11, 17, 19, 23, 27$ |
| $-19$ | $5, 7, 11, 17, 19, 23$ |
| $-120$ | $5, 37$ |
| $-51$ | $5(11)$ |
| $-52$ | $7, 11, 13$ |
| $-132$ | $7, 11$ |
| $-168$ | $7, 23$ |
| $-3 \cdot 5^2$ | $7(13)$ |
| $-43$ | $11, 13, 17$ |
| $-88$ | $11, 13$ |
| $-123$ | $11, 17$ |

| $D(\Lambda)$ | $N$ |
|---|---|
| $-228$ | $11(19)$ |
| $-312$ | $13, 19, 29$ |
| $-3 \cdot 7^2$ | $13, 19$ |
| $-4 \cdot 5^2$ | $13(29)$ |
| $-67$ | $17, 19$ |
| $-372$ | $17, 29$ |
| $-408$ | $17(37)$ |
| $-148$ | $19(23)$ |
| $-267$ | $23(41)$ |
| $-232$ | $29(31)$ |
| $-708$ | $31$ |
| $-163$ | $41$ |

Table 6.5: Some $N$ for which $\Psi_{\Gamma^+,N,t_6^+}(X,X)$ vanishes at $t_6^+(\mathrm{CM}(\Lambda))$

Moreover, the order in which $\Lambda$ appears in the list can be used to determine the values of $t_6^+$ at all the corresponding CM points. The factorization of all these polynomials can be computed in a reasonable time and therefore the corresponding values are obtained. For some orders where only one $N$ is needed to determine the value, another one is given into parentheses whenever possible.

Since the Hauptmodul $t_6^+$ we have considered coincides with the one used in [Elk98], the values we have computed are the same which appear there. They can be found in Table 6.6.

| $D(\Lambda)$ | $t_6^+(P)$, $P \in \mathrm{CM}(\Lambda)$ |
|---|---|
| $-3$ | $\infty$ |
| $-4$ | $1$ |
| $-24$ | $0$ |
| $-40$ | $\frac{2312}{125}$ |
| $-84$ | $-\frac{169}{27}$ |
| $-19$ | $\frac{3211}{1024}$ |
| $-120$ | $\frac{5776}{3375}$ |
| $-51$ | $-\frac{1377}{1024}$ |
| $-52$ | $\frac{6877}{15625}$ |
| $-132$ | $\frac{13689}{15625}$ |
| $-168$ | $-\frac{701784}{15625}$ |
| $-3 \cdot 5^2$ | $\frac{152881}{138240}$ |
| $-43$ | $\frac{21250987}{16000000}$ |
| $-88$ | $\frac{15545888}{20796875}$ |

| $D(\Lambda)$ | $t_6^+(P)$, $P \in \mathrm{CM}(\Lambda)$ |
|---|---|
| $-123$ | $-\frac{296900721}{16000000}$ |
| $-228$ | $\frac{66863329}{11390625}$ |
| $-312$ | $\frac{27008742384}{27680640625}$ |
| $-3 \cdot 7^2$ | $-\frac{1073152081}{3024000000}$ |
| $-4 \cdot 5^2$ | $\frac{421850521}{1771561}$ |
| $-67$ | $\frac{77903700667}{1024000000}$ |
| $-372$ | $-\frac{455413074649}{747377296875}$ |
| $-408$ | $-\frac{32408609436736}{55962140625}$ |
| $-148$ | $\frac{69630712957}{377149515625}$ |
| $-267$ | $-\frac{5766681714488721}{1814078464000000}$ |
| $-232$ | $\frac{66432278483452232}{56413239012828125}$ |
| $-708$ | $\frac{71475755554842930369}{224337327397603890625}$ |
| $-163$ | $\frac{69969023945136070506 7}{68417881400334400000 0}$ |

Table 6.6: Rational values of $t_6^+$ at the CM points by the order of discriminant $D(\Lambda)$

Similarly, we can reproduce the procedure above to compute the value of $t_6^+$ at the quadratic CM points. First of all, we begin by finding values of $N$ such that $\Psi_{\Gamma^+,N,t_6^+}(X,X)$ vanishes at $t_6^+(\mathrm{CM}(\Lambda))$; a suitable selection of such values is given in Table 6.7.

| $D(\Lambda)$ | $N$ |
|---|---|
| $-264$ | $11(17)$ |
| $-276$ | $13(23)$ |
| $-136$ | $17, 19$ |
| $-195$ | $17(23)$ |
| $-219$ | $19, 37$ |
| $-420$ | $19, 31$ |
| $-456$ | $19(41)$ |
| $-91$ | $23, 5\cdot 7$ |
| $-552$ | $23, 29$ |
| $-184$ | $23(31)$ |
| $-4\cdot 7^2$ | $29, 37$ |
| $-115$ | $29, 31$ |
| $-660$ | $29(41)$ |
| $-3\cdot 11^2$ | $31, 37$ |
| $-24\cdot 5^2$ | $31, 7\cdot 11$ |
| $-291$ | $31(5\cdot 17)$ |
| $-280$ | $37, 5\cdot 7$ |
| $-292$ | $37, 41$ |
| $-435$ | $37, 5\cdot 11$ |
| $-564$ | $37, 5\cdot 19$ |
| $-852$ | $37(7^2)$ |
| $-483$ | $41, 7\cdot 11$ |
| $-840$ | $41, 5\cdot 7$ |
| $-328$ | $41(7\cdot 13)$ |
| $-340$ | $5\cdot 11, 5\cdot 17$ |
| $-1320$ | $5\cdot 11$ |

| $D(\Lambda)$ | $N$ |
|---|---|
| $-235$ | $5\cdot 13, 5\cdot 23, 13^2$ |
| $-555$ | $5\cdot 13, 13^2$ |
| $-520$ | $5\cdot 13(5\cdot 23)$ |
| $-795$ | $5\cdot 17$ |
| $-1380$ | $5\cdot 19, 5\cdot 23$ |
| $-760$ | $5\cdot 19(5\cdot 29)$ |
| $-187$ | $7\cdot 11, 7^2$ |
| $-1848$ | $7\cdot 11$ |
| $-532$ | $7\cdot 13, 7\cdot 19$ |
| $-1092$ | $7\cdot 13(7\cdot 23)$ |
| $-1428$ | $7\cdot 17, 7\cdot 19$ |
| $-427$ | $7\cdot 17$ |
| $-3\cdot 13^2$ | $7\cdot 19, 13^2$ |
| $-388$ | $7\cdot 19(7^2)$ |
| $-1032$ | $7\cdot 23(7^2)$ |
| $-627$ | $11\cdot 13, 13\cdot 17$ |
| $-1128$ | $11\cdot 13(13^2)$ $*$ |
| $-568$ | $11\cdot 13(13^2)$ $*$ |
| $-403$ | $11\cdot 13(11^2)$ |
| $-723$ | $11\cdot 17, 11^2$ |
| $-1012$ | $11\cdot 17$ |
| $-772$ | $11\cdot 19(11^2)$ |
| $-1227$ | $11\cdot 29(11^2)$ |
| $-1752$ | $13\cdot 17(13^2)$ |
| $-1992$ | $13^2$ |

Table 6.7: Some $N$ for which $\Psi_{\Gamma^+,N,t_6^+}(X,X)$ vanishes at $t_6^+(\mathrm{CM}(\Lambda))$

Moreover, studying the factors of the corresponding polynomials in the order given in the table also allows us to find out which factor corresponds to which order, except for a couple of orders marked in the list. Note as well that for all the orders for which only one polynomial is necessary to find the corresponding quadratic factor, another one is given when available to help in the computation.

To determine which quadratic factor corresponds to each of the polynomials marked with $*$, we can study the factor of $\Psi_{\Gamma^+,11\cdot 13,t_6^+}(X,X)$ coming from $\Psi_{\Gamma,11\cdot 13,t_6}(X,X)$, since it vanishes at $t_6^+(P)$ for $P$ a CM point by the order of discriminant $-568$ but not for those attached to the order of discriminant $-1128$. Otherwise, we can consider the extension defined by the possible polynomials over the $\mathbb{Q}(\Lambda)$. Since it must be contained in the Hilbert Class Field it has to be unramified and it only happens for one of the choices.

Finally, studying the factors of these polynomials we obtain the values of $t_6^+(P)$, for all those quadratic CM points in $X(6,1)^+$. We give the irreducible polynomials of $t_6^+(P)$ for all these $P$ in Table 6.8.

| $D(\Lambda)$ | $\mathrm{Irr}(t_6^+(P),X)$ for $P\in \mathrm{CM}(\Lambda)$ |
|---|---|
| $-264$ | $42719296 - 53148528X + 26198073X^2$ |
| $-276$ | $-206586207 + 328736070X + 1771561X^2$ |
| $-136$ | $52236012608 + 9087757328X + 8703679193X^2$ |
| $-195$ | $884218628241 - 1812440448000X + 1048576000000X^2$ |
| $-219$ | $-128030514913249 + 125096383466496X + 1320903770112X^2$ |
| $-420$ | $3732357388761 - 1357411709250X + 377149515625X^2$ |
| $-456$ | $31050765799488 - 43342656704496X + 42761175875209X^2$ |
| $-91$ | $86743200841609 - 37494741699584X + 118887446216704X^2$ |
| $-552$ | $-5477054994808384 + 3075898907250000X + 2165464599609375X^2$ |
| $-184$ | $384963067803392 - 412829614079584X + 293681802023X^2$ |
| $-4\cdot 7^2$ | $642574792185481 - 402872644381538X + 262254607552729X^2$ |
| $-115$ | $2241128724615668409 - 261027932841728000X + 29025255424000000X^2$ |
| $-660$ | $4314790752945768481 - 927163113067208250X + 1229255326344515625X^2$ |
| $-3\cdot 11^2$ | $22527082795349027281 - 61407732535776000000X + 55427328000000000000X^2$ |
| $-24\cdot 5^2$ | $200192797421660160 - 25808161838902560X + 3573226485213841X^2$ |
| $-291$ | $-65127370229218609 - 383293130981529600X + 36563462560677888X^2$ |
| $-280$ | $14831567427889598464 + 15754661879632468000X + 16465090583970015625X^2$ |
| $-292$ | $19486892954181586609 - 19814023185757653125 0X + 3614157446289 0625X^2$ |
| $-435$ | $5694812018915900250026 41 - 58615713496201735411 2000X$ $+184510450772213760000 00X^2$ |
| $-564$ | $-389128161774795183 + 391849027410973158X + 262254607552729X^2$ |
| $-852$ | $-1425186124386322245327 9 - 457708406701301766562 50X$ $+640270037970529785156 25X^2$ |

| | |
|---|---|
| $-483$ | $5744038397002835046920001 + 58622007476214561600000000X$ $+61792176640000000000000X^2$ |
| $-840$ | $3722258877373681280671136 - 7248828831090893333382000X$ $+355078782317139783890625X^2$ |
| $-328$ | $113894136694228860l6064 - 97279142799405175000 0X$ $+406148777599853515625X^2$ |
| $-340$ | $5905190877157885941 2809 - 9319996166918436099425 0X$ $+441480121832469431406 25X^2$ |
| $-1320$ | $59399623661988395071714457480256-12712639971271419094681148221 8000X$ $+55262357411041748169986282889 0625X^2$ |
| $-235$ | $98428582062197494896026775289 - 498595493614954928624852480 00X$ $+172648988241769431040000 00X^2$ |
| $-555$ | $59118881838118852885218 21681 - 33860736130198556936666880 00X$ $+37467995349595885404160000 00X^2$ |
| $-520$ | $2798410605022250285773843920 3904 - 1896138255071810162542518810 2000X$ $+55461466273082794940961225156 25X^2$ |
| $-795$ | $670604658282845649851776925381 4186961$ $-160002920168153943633891060703 3728000X$ $+935315369583235360022436249600 0000X^2$ |
| $-1380$ | $226206117696572800390670174828 27761$ $-891771271600987348989896203598 9250X$ $+107957398635427511337789151406 25X^2$ |
| $-760$ | $452886502050801398701510350192484964948224$ $-445437509849522458916545214029502351588000X$ $+47251249624981144624431585390862656 25X^2$ |
| $-187$ | $128180352766314405305128 9 + 146223250516439200000000X$ $+378971875840000000000000X^2$ |
| $-1848$ | $809411280400369595393196069633732611484 16$ $+756673051185329976750536206449333750000 0X$ $+143525908830815777957639943419165039062 5X^2$ |
| $-532$ | $373928904209401066844409019456 09 + 28610007024431602155530809375 0X$ $+3218720433250754250244140625X^2$ |
| $-1092$ | $881180666024379209413236481 - 20721089994540456985607812 50X$ $+14977069034820694650878906 25X^2$ |
| $-1428$ | $508072398562305356625657318510 5841$ $+259513863267103817430821811384 3750X$ $+111004163771598482150634765625X^2$ |
| $-427$ | $236345145177884150466328274020723680186889$ $-249208777711953197450743796334028592000000X$ $+95235372311230160330277535744000000000000 0X^2$ |
| $-3 \cdot 13^2$ | $1036729644580793755626091 21 - 2181052417963780483200000 0X$ $+42980054008320000000000000X^2$ |

| | |
|---|---|
| $-388$ | $9912915372060849777348889 - 14660860130078731447531250X$ <br> $+4662118527211564697265625X^2$ |
| $-1032$ | $20833341886935507108358707 84 - 56573541798919546839750000X$ <br> $+20544676316626467285156 25X^2$ |
| $-627$ | $104817080583432592888674903921 - 1598794807192779432139200000000X$ <br> $+6713717953349862400000000000000X^2$ |
| $-1128$ | $-43494050993600195677408743 04 - 50200592664335641050667500 00X$ <br> $+1099130201839780444335937 5X^2$ |
| $-568$ | $5958733561785019472052856 4631296$ <br> $-1170945197372990495475641 00500000X$ <br> $+2273504269256093456667749 0234375X^2$ |
| $-403$ | $4402921782424728674818063256213417644489$ <br> $-175120941766664486593304207549360000000X$ <br> $+5368960405883858220639846400000000000000X^2$ |
| $-723$ | $-9038300822385717397069614840001$ <br> $-13097519021225312562204763 2000000X$ <br> $+13983849700491504844800000000000000X^2$ |
| $-1012$ | $2570187045545914742463370102372216333917137449$ <br> $-898508775411634816629467939210605913963419 06250X$ <br> $+874834691102984776775279738455357398986816 40625X^2$ |
| $-772$ | $4708627161358049627738157747286403827249$ <br> $-5224321697045595217012469095718953531250X$ <br> $+4984110844092335739042410447463378906 25X^2$ |
| $-1227$ | $-448593784649197685633914462760558347855141 28161$ <br> $+21434298178618274731026952272896386716192000000X$ <br> $+85525366214177410333043640607488000000000000X^2$ |
| $-1752$ | $-808584070739564828143582108025597706 24$ <br> $+589490967995486694183850006334265000 00X$ <br> $+219917758720767766489798343088378906 25X^2$ |
| $-1992$ | $2570187045545914742463370102372216333917137449$ <br> $-898508775411634816629467939210605913963419 06250X$ <br> $+874834691102984776775279738455357398986816 40625X^2$ |

Table 6.8: Irreducible polynomials of the values of $t_6^+$ at the CM points by the order of discriminant $D(\Lambda)$

**The case $D = 10$**

We follow the same approach which we used in the case $D = 6$. We begin by finding the orders giving rational and quadratic CM points on the curve $X(10,1)^+$:

| $f$ | 1 | 3 | 7 |
|---|---|---|---|
| $\|D(K)\|$ s.t. the order of conductor $f$ gives rational CM points | 3, 8, 20, 35, 40, 43, 52, 67, 88, 115, 120, 148, 163, 232, 235, 280, 340, 520, 760 | 3, 8, 20 | |
| $\|D(K)\|$ s.t. the order of conductor $f$ gives quadratic CM points | 91, 115, 136, 184, 187, 195, 219, 235, 264, 276, 280, 291, 292, 328, 340, 388, 403, 420, 427, 435, 456, 483, 520, 532, 552, 555, 564, 568, 627, 660, 723, 760, 772, 795, 840, 852, 1012, 1032, 1092, 1128, 1227, 1320, 1380, 1428, 1752, 1848, 1992 | 35, 40 | 3 |

Now that we have the list of CM points we have to work with, we find suitable values of $N$ such that the value of $t_{10}^+$ at each of them is a zero of $\Psi_{\Gamma^+,N,t_{10}^+}(X,X)$, where this polynomial can be explicitly computed. In Table 6.9, for each order $\Lambda$ with discriminant $D(\Lambda)$ such that the points in $\mathrm{CM}(\Lambda)$ are rational in $X(10,1)^+$, according to the table above, we provide a list of some $N$ such that $\Psi_{\Gamma^+,N,t_{10}^+}(X,X)$ has $t_{10}^+(P)$ as a root, for all CM points by $\Lambda$, $P$. We provide as well the corresponding values of the Hauptmodul $t_{10}^+$. Note that in this case our Hauptmodul does not match the one used in [Elk98]. If we denote by $f$ the one there, we have the following relation:

$$f = \frac{2t_{10}^+ - 27}{t_{10}^+ - 1}.$$

Using this we recover all the values given in [Elk98], together with the missing value for $t_{10}^+(\mathrm{CM}(\mathbb{Z}[\frac{1+\sqrt{-135}}{2}]))$.

Next, we move to the case of CM points were $t_{10}^+$ takes quadratic values and provide the results in Table 6.10.

| $D(\Lambda)$ | $N$ | $t_{10}^{+}(\mathrm{CM}(\Lambda))$ | $D(\Lambda)$ | $N$ | $t_{10}^{+}(\mathrm{CM}(\Lambda))$ |
|---|---|---|---|---|---|
| $-3$ | $(3)$ | $\frac{27}{2}$ | $-43$ | $11, 13, 17$ | $\frac{31347}{722}$ |
| $-8$ | $(3)$ | $1$ | $-88$ | $11, 13$ | $\frac{729}{3179}$ |
| $-20$ | $(3)$ | $\infty$ | $-8\cdot 3^2$ | $11, 19$ | $\frac{3844}{169}$ |
| $-40$ | $(7)$ | $0$ | $-340$ | $11(17)$ | $-\frac{18225}{22984}$ |
| $-35$ | $3, 7$ | $-\frac{5}{2}$ | $-520$ | $13, 31$ | $\frac{12131289}{874225}$ |
| $-120$ | $3(31)$ | $\frac{54}{5}$ | $-235$ | $13(23)$ | $-\frac{3065445}{87362}$ |
| $-52$ | $7, 11$ | $\frac{729}{104}$ | $-67$ | $17(19)$ | $\frac{439587}{232562}$ |
| $-280$ | $7, 17, 19$ | $-\frac{6561}{845}$ | $-148$ | $19, 23$ | $\frac{5517801}{342176}$ |
| $-3\cdot 3^2$ | $7, 13$ | $\frac{867}{242}$ | $-760$ | $19(29)$ | $\frac{294523290}{531898939}$ |
| $-115$ | $7, 17$ | $-\frac{3645}{242}$ | $-232$ | $29(31)$ | $\frac{106571781}{171374281}$ |
| $-20\cdot 3^2$ | $7(23)$ | $\frac{289}{120}$ | $-163$ | $41$ | $\frac{183003204987}{15685392962}$ |

Table 6.9: Some $N$ for which $\Psi_{\Gamma^+,N,t_{10}^+}(X,X)$ vanishes at $t_{10}^{+}(\mathrm{CM}(\Lambda))$ and value of $t_{10}^{+}(\mathrm{CM}(\Lambda))$

| $D(\Lambda)$ | $N$ | $\mathrm{Irr}(t_{10}^{+}(P), X)$ for $P \in \mathrm{CM}(\Lambda)$ |
|---|---|---|
| $-195$ | $11(3\cdot 7)$ | $700569 - 594540X + 36100X^2$ |
| $-260$ | $13, 29$ | $36125 - 5120X + 208X^2$ |
| $-68$ | $13, 17$ | $14297 + 1056X + 272X^2$ |
| $-420$ | $13(3\cdot 7)$ | $210681 - 259200X + 115600X^2$ |
| $-132$ | $17, 3\cdot 7$ | $2712609 - 399168X + 327184X^2$ |
| $-580$ | $17, 37$ | $15411789 - 1982880X + 338000X^2$ |
| $-35\cdot 3^2$ | $17(7\cdot 13)$ | $89397025 - 17972620X + 1387684X^2$ |
| $-40\cdot 3^2$ | $19, 23$ | $5400 - 1845X + 289X^2$ |
| $-660$ | $19, 3\cdot 11$ | $5061611025 - 344413080X + 8952064X^2$ |
| $-355$ | $19, 29$ | $-37732311 - 227783340X + 11628100X^2$ |
| $-155$ | $19(3\cdot 13)$ | $-26071 - 60460X + 12100X^2$ |
| $-168$ | $23, 29$ | $72795024 + 3601827X + 165649X^2$ |
| $-435$ | $23, 3\cdot 11$ | $32871503025 - 68044642380X + 39633642724X^2$ |
| $-820$ | $23(41)$ | $-2832049089 - 370536120X + 43808000X^2$ |
| $-228$ | $29, 41$ | $2503100961 + 2500698528X + 1336341136X^2$ |
| $-555$ | $29, 3\cdot 13$ | $201138115488489 - 33082706982060X + 971190540100X^2$ |
| $-1060$ | $29(7\cdot 19)$ | $625041046125 - 341752050720X + 56700102992X^2$ |
| $-123$ | $31, 37$ | $64126952289 - 44678870028X + 397683364X^2$ |
| $-840$ | $31, 3\cdot 7$ | $145138140900 - 6918248205X + 1527480889X^2$ |
| $-1240$ | $31, 11\cdot 17$ | $-884370968100 + 942728817780X + 20146684519X^2$ |
| $-595$ | $31(41)$ | $1566735366249 + 1473678646740X + 15205356100X^2$ |
| $-292$ | $37, 41$ | $395749763793 - 21388556736X + 948183568X^2$ |
| $-1380$ | $37, 3\cdot 23$ | $187881086227445361 - 21968541318636000X + 803465102611600X^2$ |
| $-715$ | $37, 11\cdot 13$ | $15069341705625 - 11139015359340X + 8161866182404X^2$ |
| $-3\cdot 7^2$ | $37, 3\cdot 13$ | $156652098849 - 186645543948X + 203778210724X^2$ |
| $-795$ | $41, 3\cdot 17$ | $706183354362148569 - 683747285443080300X + 19465258372202500X^2$ |
| $-312$ | $41, 3\cdot 13$ | $4616391719241 - 1024325855982X + 13185699241X^2$ |
| $-1540$ | $41, 7\cdot 11$ | $17208167475809025 - 2412703826676000X + 85157386661776X^2$ |
| $-328$ | $41(7\cdot 13)$ | $177545935044 - 260353891413X + 19261831369X^2$ |
| $-1320$ | $3\cdot 11(3\cdot 37)$ | $4253917632420096 + 222068375568360X + 152919316263025X^2$ |
| $-372$ | $3\cdot 17, 3\cdot 29$ | $612210160180161 - 422027880219072X + 26809032929536X^2$ |
| $-408$ | $3\cdot 17, 3\cdot 23$ | $692582067838116 + 294235182711468X + 42878648216041X^2$ |
| $-267$ | $3\cdot 23(3\cdot 29)$ | $8209389990443736249 - 1639217860896435948X + 78471426340840324X^2$ |
| $-708$ | $3\cdot 31$ | $300082949530934595113169 + 392412040297979729948512X$ $+319612844688157889 44X^2$ |
| $-187$ | $7\cdot 11(7\cdot 17)$ | $13005854087769 - 2684967194988X + 553443747844X^2$ |
| $-532$ | $7\cdot 13(7\cdot 19)$ | $69021381726708849 - 18231570739375848X + 1604357522842624X^2$ |
| $-427$ | $7\cdot 17(17^2)$ | $186406496917342222 8249 - 602881837984322727 48X + 511922609014091685124X^2$ |
| $-388$ | $7\cdot 19, 7\cdot 23$ | $92700984533337 - 17805086855424X + 1109805837712X^2$ |
| $-1435$ | $7\cdot 19$ | $345927391230371738 0625 + 881880798911158166 74740X$ $+370557220051807928164X^2$ |
| $-568$ | $11\cdot 13, 13^2$ | $8285001402009564 - 12863267765246628X + 3482283956596439X^2$ |
| $-403$ | $11\cdot 13, (11^2)$ | $2165643130620871566729 - 2190647179775859697 08X + 612006435839793604X^2$ |
| $-1012$ | $11\cdot 17(11\cdot 23)$ | $208523918999894083 3847889 + 627189273993351994880472X$ $+174238926938195499787264X^2$ |
| $-772$ | $11\cdot 19(11^2)$ | $2302252639397514677 7 - 11207080634279747904X + 78906790753194116752X^2$ |
| $-2020$ | $11\cdot 23(11^2)$ | $485942503741395741 + 164603068156444320X + 212641567145618000X^2$ |
| $-1780$ | $13\cdot 19(13^2)$ | $-7284723384049281 + 1837303480723680X + 4187652728072000X^2$ |
| $-955$ | $7^2$ | $-577517467114068471 - 897234444403808940X + 76097089966140100X^2$ |
| $-1555$ | $17^2(17\cdot 23)$ | $-1693960065777294863416 71 + 2823037333492147559 8740X$ $+82799772443611404841 00X^2$ |

Table 6.10: Some $N$ for which $\Psi_{\Gamma^+,N,t_{10}^+}(X, X)$ vanishes at $t_{10}^+(\mathrm{CM}(\Lambda))$ and the irreducible polynomial of $t_{10}^+(\mathrm{CM}(\Lambda))$

**The case $D = 22$**

In this case, the orders whose corresponding points on the curve $X(22,1)^+$ are rational or quadratic are given in the following table:

| $f$ | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $\lvert D(K)\rvert$ s.t. the order of conductor $f$ gives rational CM points | 3, 4, 11, 20, 67, 88, 132, 148, 163, 187, 232, 1012 | 3, 4, 11 | 4 | |
| $\lvert D(K)\rvert$ s.t. the order of conductor $f$ gives quadratic CM points | 56, 91, 115, 136, 168, 235, 264, 267, 280, 308, 312, 328, 372, 388, 408, 427, 520, 532, 616, 627, 660, 708, 715, 760, 1243, 1320, 1507, 1540, 1672, 1848 | 20, 88 | 3, 11 | 3, 4 |

Now, following the same approach as in the two previous cases, we find suitable values of $N$ for each of the different possible $\Lambda$ giving rational points in $X(22,1)^+$ such that $\Psi_{\Gamma^+,N,t_{22}^+}(X,X)$ vanishes at $t_{22}^+(P)$ for all $P \in \mathrm{CM}(\Lambda)$. These values of $N$ together with the corresponding value of the Hauptmodul $t_{22}^+$ at $\mathrm{CM}(\Lambda)$ are given in Table 6.11. Note that the values at the elliptic points of $X(22,1)^+$ are already known, three of them by definition of $t_{22}^+$, and the other, $t_{22}^+(\mathrm{CM}(\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]))$, was obtained during the computation of the automorphic derivative in Section 3.4.2, cf. Theorem 3.1.

| $D(\Lambda)$ | $N$ | $t_{22}^+(\mathrm{CM}(\Lambda))$ |
|---|---|---|
| $-3$ | $(3)$ | $\frac{27}{16}$ |
| $-4$ | $(5)$ | $1$ |
| $-11$ | $(3)$ | $\infty$ |
| $-88$ | $(13)$ | $0$ |
| $-20$ | $3,5$ | $\frac{5}{16}$ |
| $-132$ | $3(7)$ | $\frac{25}{16}$ |
| $-4\cdot 3^2$ | $5,13$ | $\frac{49}{16}$ |
| $-11\cdot 3^2$ | $5(23)$ | $\frac{49}{48}$ |

| $D(\Lambda)$ | $N$ | $t_{22}^+(\mathrm{CM}(\Lambda))$ |
|---|---|---|
| $-3\cdot 3^2$ | $7,13$ | $\frac{75}{64}$ |
| $-187$ | $7(17)$ | $-\frac{17}{64}$ |
| $-4\cdot 5^2$ | $13,17$ | $\frac{729}{784}$ |
| $-67$ | $17,19$ | $\frac{1675}{784}$ |
| $-1012$ | $17(23)$ | $-\frac{34225}{8624}$ |
| $-148$ | $19(23)$ | $\frac{1813}{2704}$ |
| $-232$ | $29(31)$ | $\frac{2312}{1421}$ |
| $-163$ | $41$ | $\frac{118827}{92416}$ |

Table 6.11: Some $N$ for which $\Psi_{\Gamma^+,N,t_{22}^+}(X,X)$ vanishes at $t_{22}^+(\mathrm{CM}(\Lambda))$ and value of $t_{22}^+(\mathrm{CM}(\Lambda))$

Similarly, the values at the quadratic CM points of the Hauptmodul $t_{22}^{+}$ together with values of $N$ providing some $\Psi_{\Gamma^+,N,t_{22}^+}(X,X)$ having the corresponding value as a root are given in Table 6.12.

| $D(\Lambda)$ | $N$ | $\mathrm{Irr}(t_{22}^{+}(P),X)$ for $P\in\mathrm{CM}(\Lambda)$ |
|---|---|---|
| $-264$ | $3(17)$ | $297-544X+256X^2$ |
| $-616$ | $7,29$ | $-6561+800X+1792X^2$ |
| $-308$ | $7,37$ | $-175-32X+256X^2$ |
| $-56$ | $7(3\cdot5)$ | $2023-3936X+1792X^2$ |
| $-660$ | $13(3\cdot5)$ | $1225-1685X+784X^2$ |
| $-136$ | $17,19$ | $12393-6944X+4352X^2$ |
| $-627$ | $17(3\cdot13)$ | $893025-813888X+200704X^2$ |
| $-3\cdot5^2$ | $19,31$ | $2809+2896X+4096X^2$ |
| $-715$ | $19,5\cdot13$ | $893025-1689280X+802816X^2$ |
| $-1672$ | $19(7^2)$ | $29452329-42546464X+15462656X^2$ |
| $-168$ | $23,3\cdot7$ | $133225-135968X+12544X^2$ |
| $-91$ | $23,29$ | $35721-30016X+4096X^2$ |
| $-20\cdot3^2$ | $23,5\cdot29$ | $6241-10381X+4624X^2$ |
| $-1540$ | $23(5\cdot7)$ | $5267025-17575840X+12503296X^2$ |
| $-115$ | $29,41$ | $164025-327280X+173056X^2$ |
| $-4\cdot7^2$ | $29,37$ | $6561-167584X+92416X^2$ |
| $-88\cdot3^2$ | $29(31)$ | $7425-17568X+12544X^2$ |
| $-11\cdot5^2$ | $31,3\cdot23$ | $14161-16880X+5120X^2$ |
| $-1243$ | $31(17\cdot19)$ | $-51480625+28113200X+7614464X^2$ |
| $-280$ | $37,5\cdot7$ | $1225449-3335584X+2119936X^2$ |
| $-1320$ | $37,3\cdot5$ | $20638849-12202784X+2119936X^2$ |
| $-3\cdot7^2$ | $37,3\cdot13$ | $2175625-3349568X+1183744X^2$ |
| $-1507$ | $37(13^2)$ | $-332150625-140259600X+159509504X^2$ |
| $-312$ | $41,3\cdot13$ | $22325625-34538400X+12503296X^2$ |
| $-328$ | $41(7\cdot13)$ | $7405625-10400800X+3789056X^2$ |
| $-1848$ | $3\cdot7$ | $1586030625-2350404000X+765296896X^2$ |
| $-372$ | $3\cdot17,3\cdot29$ | $893025+242163X+26896X^2$ |
| $-408$ | $3\cdot17(3\cdot37)$ | $45630025-84318368X+62220544X^2$ |
| $-267$ | $3\cdot23(3\cdot29)$ | $583947225-621513648X+39337984X^2$ |
| $-708$ | $3\cdot31$ | $9365400625-11007447425X+3148107664X^2$ |
| $-520$ | $5\cdot13,5\cdot31$ | $393824025-330322720X+79995136X^2$ |
| $-235$ | $5\cdot13(5\cdot23)$ | $24950025-5043040X+3625216X^2$ |
| $-760$ | $5\cdot19(5\cdot29)$ | $3318336025-25543986080X+24131758336X^2$ |
| $-532$ | $7\cdot13(7\cdot19)$ | $34164025-308664608X+275294464X^2$ |
| $-427$ | $7\cdot17(7\cdot31)$ | $46831124025-71808216928X+26254369024X^2$ |
| $-388$ | $7\cdot19(7\cdot23)$ | $455625-642025X+29584X^2$ |

Table 6.12: Some $N$ for which $\Psi_{\Gamma^+,N,t_{22}^+}(X,X)$ vanishes at $t_{22}^{+}(\mathrm{CM}(\Lambda))$ and the irreducible polynomial of $t_{22}^{+}(\mathrm{CM}(\Lambda))$

## 6.3 Computation of the local parameters

Given a CM point $P \in \mathcal{H}$, we consider expansions around $P$ with respect to the local parameter $q_P(z) = \kappa_P \dfrac{z-P}{z-\overline{P}}$, for a suitable local constant $\kappa_P \in \mathbb{C}^*$, of the form

$$f(q_P) = \sum_{i \geq n_0} a_i q_P^i.$$

Moreover, given a Hauptmodul $t$ of the corresponding curve we will choose $\kappa_P$ such that, if $P$ has order $e$, then $a_e = 1$, if $P$ is not a pole of $t$, and $a_{-e} = 1$, if $P$ is a pole of $t$.

We will restrict to the cases $D = 6, 10, 22$ and will provide $q$-expansions around all the points where $t_D^+$ admits a rational $q$-expansion, which coincides, as we have seen, with the set of points where $t_D^+$ takes rational values and which has just been computed. The $q$-expansions for the case $D = 6$ can be found in Table 6.13, those for $D = 10$ in Table 6.14 and those for $D = 22$ in Table 6.15.

Next we are going to deal with the problem of finding the local parameters which correspond to all the expansions we have given in this section. In order to do so, it will be useful to recall Corollary 5.2 which gives more information about the transcendental nature of the local constant $\kappa_P$ corresponding to a CM point $P$. Apart from the values at a very limited set of points in the case $D = 6$ all of them have been computed numerically at high precision and then, after taking care of the transcendental part, the algebraic part is recognized. We do not know any way to certify that the given values are indeed exactly correct, but all of them match the real ones up to 150 decimal digits. Given a Hauptmodul $t$ for the curve $X(D,1)^W$ and a non-elliptic CM point $P \in \mathcal{H}$ for an order $\Lambda$ in a quadratic imaginary field $K$ of discriminant $D_K$ such that $t(P) \neq \infty$, we consider $\kappa_P \in \mathbb{C}^*$ such that $t(z) = \sum_n a_n q_P(z)^n$ with $a_1 = 1$ as introduced above. In this situation,

$$\kappa_P = (P - \overline{P}) \frac{\partial f}{\partial z}(P).$$

In particular, if we know an expansion around a point $Q$, $f = \sum_n b_n q_Q(z)^n$, which converges at $P$, we can obtain an approximation of $k_P$ as follows:

$$\kappa_P = \kappa_Q \frac{(P - \overline{P})(Q - \overline{Q})}{(P - \overline{Q})^2} \frac{\partial f}{\partial q_Q}(q_Q(P)).$$

Note that it does not matter if $Q$ is elliptic or not.

| $D(\Lambda)$ | $t_6^+(q), P \in \mathrm{CM}(\Lambda)$ |
|---|---|
| $-3$ | $\frac{1}{q^6} + \frac{167}{280} + \frac{85453}{1281280}q^6 + \cdots$ |
| $-4$ | $1 + q^4 + \frac{103}{270}q^8 + \cdots$ |
| $-24$ | $q^2 - \frac{103}{216}q^4 + \frac{141191}{933120}q^6 + \cdots$ |
| $-40$ | $\frac{2312}{125} + q + \frac{222875}{6741792}q^2 + \ldots$ |
| $-84$ | $-\frac{169}{27} + q - \frac{23103}{264992}q^2 + \ldots$ |
| $-19$ | $\frac{3211}{1024} + q + \frac{572672}{2340819}q^2 + \ldots$ |
| $-120$ | $\frac{5776}{3375} + q + \frac{36147375}{55472704}q^2 + \ldots$ |
| $-51$ | $-\frac{1377}{1024} + q - \frac{1103104}{3306177}q^2 + \cdots$ |
| $-52$ | $\frac{6877}{15625} + q - \frac{14171875}{160426656}q^2 + \cdots$ |
| $-132$ | $\frac{13689}{15625} + q - \frac{573859375}{212015232}q^2 + \cdots$ |
| $-168$ | $-\frac{701784}{15625} + q - \frac{26070484375}{2013864630624}q^2 + \cdots$ |
| $-3 \cdot 5^2$ | $\frac{152881}{138240} + q + \frac{8307475200}{2238330721}q^2 + \cdots$ |
| $-43$ | $\frac{21250987}{16000000} + q + \frac{48292724000000}{37196218824723}q^2 + \cdots$ |
| $-88$ | $\frac{15545888}{20796875} + q - \frac{122816340453125}{108841674388608}q^2 + \cdots$ |
| $-123$ | $-\frac{296900721}{16000000} + q - \frac{2869087636000000}{92900449666319841}q^2 + \cdots$ |
| $-228$ | $\frac{66863329}{11390625} + q + \frac{3379935300046875}{29672717264572928}q^2 + \cdots$ |
| $-312$ | $\frac{27008742384}{27680640625} + q - \frac{1096497273887224859375}{72588505997726986176}q^2 + \cdots$ |
| $-3 \cdot 7^2$ | $-\frac{1073152081}{3024000000} + q - \frac{4227725246724000000}{4396867281898630561}q^2 + \cdots$ |
| $-4 \cdot 5^2$ | $\frac{421850521}{1771561} + q + \frac{1167053473908421}{472561408365701760}q^2 + \cdots$ |
| $-67$ | $\frac{77903700667}{1024000000} + q + \frac{15557107471616000000}{19964043960435509414963}q^2 + \cdots$ |
| $-372$ | $-\frac{455413074649}{747377296875} + q - \frac{274460303965213535349359375}{438213169003166284716060 8}q^2 + \cdots$ |
| $-408$ | $-\frac{32408609436736}{55962140625} + q - \frac{4252350276837884702390625}{4208526483126612133037334784}q^2 + \cdots$ |
| $-148$ | $\frac{69630712957}{377149515625} + q + \frac{5165633692245434557812 5}{5710067593988222338473 6}q^2 + \cdots$ |
| $-267$ | $-\frac{5766681714488721}{1814078464000000} + q - \frac{70424994788924137771594240000 00}{437158311032151602411389802158 41}q^2 + \cdots$ |
| $-232$ | $\frac{66432278483452232}{56413239012828125} + q + \frac{2023469428190655366812375638296875}{887450160332267339060091949542432}q^2 + \cdots$ |
| $-708$ | $\frac{71475755554842930369}{22433732739760389062 5} + q + \frac{21512411779178971197301432724041060578125}{8740717074209794704697203014425107531571 2}q^2 + \cdots$ |
| $-163$ | $\frac{69969023945136070506 7}{68417881400334400000 0} + q + \frac{6032162610260815870963908610646939600000 0}{3617730995318246123710922456278437158163}q^2 + \cdots$ |

Table 6.13: First terms of the $q$-expansion of $t_6^+$ at the CM points by the order of discriminant $D(\Lambda)$

| $D(\Lambda)$ | $t_{10}^{+}(q)$ for $P \in \mathrm{CM}(\Lambda)$ |
|---|---|
| $-3$ | $\frac{27}{2} + q^3 + \frac{34}{675}q^6 + \cdots$ |
| $-8$ | $1 + q^2 + \frac{43}{150}q^4 + \cdots$ |
| $-20$ | $\frac{1}{q^2} + \frac{35}{6} + \frac{611}{48}q^2 + \cdots$ |
| $-40$ | $q^2 - \frac{505}{1458}q^4 + \cdots$ |
| $-35$ | $-\frac{5}{2} + q - \frac{101}{560}q^2 + \cdots$ |
| $-120$ | $\frac{54}{5} + q - \frac{815}{10584}q^2 + \cdots$ |
| $-52$ | $\frac{729}{104} + q + \frac{9854}{455625}q^2 + \cdots$ |
| $-280$ | $-\frac{6561}{845} + q - \frac{18136085435}{258697238184}q^2 + \cdots$ |
| $-3 \cdot 3^2$ | $\frac{867}{242} + q + \frac{361427}{2890000}q^2 + \cdots$ |
| $-115$ | $-\frac{3645}{242} + q - \frac{72319643}{1813518720}q^2 + \cdots$ |
| $-20 \cdot 3^2$ | $\frac{289}{120} + q + \frac{15629310}{65007371}q^2 + \cdots$ |
| $-43$ | $\frac{31347}{722} + q + \frac{319034111}{15360030000}q^2 + \cdots$ |
| $-88$ | $\frac{729}{3179} + q + \frac{664789301}{893025000}q^2 + \cdots$ |
| $-8 \cdot 3^2$ | $\frac{3844}{169} + q + \frac{384157787}{7063350000}q^2 + \cdots$ |
| $-340$ | $-\frac{18225}{22984} + q - \frac{4201228426538}{9137830982175}q^2 + \cdots$ |
| $-520$ | $\frac{12131289}{874225} + q + \frac{121521147690975249 25}{13322510436874687776}q^2 + \cdots$ |
| $-235$ | $-\frac{3065445}{87362} + q - \frac{14365338484281983}{7597271753450719 20}q^2 + \cdots$ |
| $-67$ | $\frac{439587}{232562} + q + \frac{67591577001553}{182010997350000}q^2 + \cdots$ |
| $-148$ | $\frac{5517801}{342176} + q + \frac{5790060120973864}{38010789573631875}q^2 + \cdots$ |
| $-760$ | $\frac{294523290}{531898939} + q - \frac{186335570578283025417256 31}{142644863435237646735755160}q^2 + \cdots$ |
| $-232$ | $\frac{106571781}{171374281} + q - \frac{5079255084064624453628 3}{1806420259450170510300 00}q^2 + \cdots$ |
| $-163$ | $\frac{183003204987}{15685392962} + q - \frac{904016711289308170317618459 47}{652077042863923852878502800000}q^2 + \cdots$ |

Table 6.14: First terms of the $q$-expansion of $t_{10}^{+}$ at the CM points by the order of discriminant $D(\Lambda)$

| $D(\Lambda)$ | $t_{22}^{+}(q)$ for $P \in \mathrm{CM}(\Lambda)$ |
|---|---|
| $-3$ | $\frac{27}{16} + q^3 + \frac{232}{297}q^6 + \cdots$ |
| $-4$ | $1 + q^4 - \frac{91}{330}q^8 + \cdots$ |
| $-11$ | $\frac{1}{q^2} + \frac{23}{24} + \frac{559}{3840}q^2 + \cdots$ |
| $-88$ | $q^2 - \frac{3389}{5832}q^4 + \cdots$ |
| $-20$ | $\frac{5}{16} + q + \frac{8}{55}q^2 + \cdots$ |
| $-132$ | $\frac{25}{16} + q - \frac{424}{225}q^2 + \cdots$ |
| $-4 \cdot 3^2$ | $\frac{49}{16} + q + \frac{664}{1617}q^2 + \cdots$ |
| $-11 \cdot 3^2$ | $\frac{49}{48} + q + \frac{1683}{98}q^2 + \cdots$ |
| $-3 \cdot 3^2$ | $\frac{75}{64} + q + \frac{432}{275}q^2 + \cdots$ |
| $-187$ | $-\frac{17}{64} + q - \frac{24304}{19125}q^2 + \cdots$ |
| $-4 \cdot 5^2$ | $\frac{729}{784} + q - \frac{205016}{40095}q^2 + \cdots$ |
| $-67$ | $\frac{1675}{784} + q + \frac{336973}{331650}q^2 + \cdots$ |
| $-1012$ | $-\frac{34225}{8624} + q - \frac{1880397403192}{11922213277575}q^2 + \cdots$ |
| $-148$ | $\frac{1813}{2704} + q - \frac{60159944}{67307625}q^2 + \cdots$ |
| $-232$ | $\frac{2312}{1421} + q - \frac{1636930897}{343332000}q^2 + \cdots$ |
| $-163$ | $\frac{118827}{92416} + q + \frac{216379056448}{392292487125}q^2 + \cdots$ |

Table 6.15: First terms of the $q$-expansion of $t_{22}^{+}$ at the CM points by the order of discriminant $D(\Lambda)$

Therefore, we will obtain the values of $\kappa_P$ for all the points $P$ above, from the values at a more restricted set of CM points, which we will take as the set of elliptic points. In order to obtain approximated values for the local constants at the set of elliptic points, we will use Section 3.3.1. In particular, if we assume that $t$ is a Hauptmodul satisfying the differential equation

$$Da(t,z) + R(f(z)) = 0$$

and that $P$ is an elliptic point such that $t(P) = a \neq \infty$, in Section 3.3.1, we have seen that

$$\kappa_P \frac{Q-P}{Q-\overline{P}} q_P(Q) = \frac{u_1(t(Q))}{u_2(t(Q))},$$

where $u_1$, $u_2$ are solutions of the linear differential equation $u''(w)+R(w)/4 = 0$ with initial conditions $u_1(a) = 0, u_1'(a) = 1$, $u_2(a) = 1, u_2'(a) = 0$ and $Q$ lies in a suitable neighborhood of $P$.

Note that assuming that we know the values of $t$ at the elliptic points is no restriction after assuming that we know $R$, since these values are the zeros of the denominator of $R$. Moreover, we have a method to compute the value of $t$ at a given CM point, in particular $t(Q)$ for a suitable $Q$.

Finally, for the case $t(P) = \infty$, we consider the Hauptmodul $1/t$, which reduces to the situation above.

Let $P \in \mathcal{H}$ be a non-elliptic CM point by an order $\Lambda$ in a quadratic field $K$ of discriminant $D_K$ where $t_D^+$ takes a real value. Assume that we have $t_D^+(q_P(z)) = \sum_n a_n q_P^n$, with $a_1 = 1$. Then, since $t_D^+$ is defined over $\mathbb{Q}$, it follows from [Shi75b] that $t_D^+(s(z)) = \overline{t_D^+(z)}$, where $s$ is the symmetry attached to an element of norm $-1$ in $\mathcal{O}(D,1)$. The set of fixed points of this symmetry coincides with the pictured lines in the fundamental domains of $\langle\Gamma(D,1), W_D\rangle$ given in Figures 3.5, 3.6 and 3.4 and corresponds to the set of points where $t_D^+$ takes real values. In particular, the point $P$ must lie in one of these hyperbolic lines (and only one if the point is not elliptic), assume that it is an euclidian circle of center $a$. Note as well that $a \in \overline{\mathbb{Q}} \cup \{\infty\}$ since this is the set of fixed points of a symmetry given by a matrix with algebraic coefficients. Then, the directional derivative along this line must be real and this allows us to break down $\kappa_P$ a little more. More precisely, we fix a tangent vector to the hyperbolic line of center $a$ through $P$, which for convenience we are going to choose as follows: $iv_{a,P}$, where $v_{a,P} = P/a - 1$ if $a \neq 0$ and $v_{a,P} = \overline{P}/\mathrm{Re}(P)$ if $a = 0$.

Then, the directional derivative along $iv_{a,P}$ at $P$ is $\left(\frac{\partial f}{\partial z}\right)(P) \cdot iv_{a,P} \in \mathbb{R}$, which can be computed as

$$\left(\frac{\partial f}{\partial z}\right)(P)\cdot iv_{a,P} = \left(\frac{\partial f}{\partial q_P}\right)(0)q_P'(P)\cdot iv_{a,P} = \kappa_P\frac{v_{a,P}}{2\mathrm{Im}(P)}.$$

In particular $\kappa_P \in \overline{v_{a,P}}\mathbb{R}$. Therefore, since $a$ is algebraic, we have

$$\frac{\kappa_P}{\overline{v_{a,P}}\pi_{D_K}} \in \mathbb{R}\cap\overline{\mathbb{Q}},$$

where $\pi_{D_K}$ is the constant introduced in Corollary 5.2.

Note that if we change $P\in\mathcal{H}$ by $Q=\gamma(P)$ for a certain $\gamma\in\Gamma^+$, the expansion at $Q$, $\sum_n a_n q_Q^n$, has the same coefficients as the expansion at $P$ but the local constant at $Q$ is not exactly the same: if we write $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, then we have that $\kappa_Q = \frac{a-cQ}{a-c\overline{Q}}\kappa_P$. For this reason we are going to choose representatives of all the points above in the fundamental domain $\mathcal{D}(D,1)^+$ and for each of these points we will give the corresponding local constant. More precisely, in Tables 6.16, 6.17, 6.18, for the non-elliptic rational CM points, we provide a representative point $P\in\mathcal{D}(D,1)^+$, the corresponding values of $a$ and $\frac{\kappa_P}{\overline{v_{a,P}}\pi_{D_K}}$, which determine the values of $\kappa_P$ corresponding to the expansions for $t_D^+$ given above. Similarly, for an elliptic CM point of order $e$ we provide as well a representative point $P\in\mathcal{D}(D,1)^+$ and the value of $\left(\frac{\kappa_P}{\pi_{D_K}}\right)^e$. It is worth noting that from one of these expansions at a point $P$ we can obtain expansions for any of the functions $u_{D,i}$ and $t_D$ at those points mapped to $P$, since these functions are algebraically related to $t_D^+$ by means of the polynomials given in Theorem 2.4.

Finally, let us remark that the only three values which are unconditionally correct are the ones corresponding to the elliptic points in the case $D=6$. In this case there are only three elliptic cycles and this allows to rewrite the function $u_1/u_2$ in terms of hypergeometric series. More details can be found in [BT07b], where these three local constants at the elliptic points together with the corresponding expansions are computed using this fact.

| $D_K$ | $f(\Lambda)$ | $P$ | $e(P)$ | $(\kappa_P/\pi_{D_K})^{e(P)}$ |
|---|---|---|---|---|
| $-3$ | 1 | $Q^+_{6,2}=\frac{-1+i}{1+\sqrt{3}}$ | 6 | $\frac{2i}{729}$ |
| $-4$ | 1 | $Q^+_{6,1}=i$ | 4 | $-\frac{3}{64}$ |
| $-24$ | 1 | $Q^+_{6,4}=\frac{i\sqrt{2}}{1+\sqrt{3}}$ | 2 | $\frac{1}{4}$ |

| $D_K$ | $f(\Lambda)$ | $P$ | $a$ | $\frac{\kappa_P}{\overline{v_{a,P}}\pi_{D_K}}$ |
|---|---|---|---|---|
| $-40$ | 1 | $\frac{-\sqrt{3}i\sqrt{10}}{4+\sqrt{3}}$ | 1 | $\frac{8262\sqrt{3}}{625}$ |
| $-84$ | 1 | $-\frac{-1+i\sqrt{7}}{2(1+\sqrt{3})}$ | 0 | $\frac{91\sqrt{6}}{81}$ |
| $-19$ | 1 | $\frac{-\sqrt{3}+i\sqrt{19}}{5+\sqrt{3}}$ | 1 | $\frac{3159\sqrt{3}}{2048}$ |
| $-120$ | 1 | $\frac{-1+i\sqrt{10}}{1+2\sqrt{3}}$ | 1 | $\frac{13034\sqrt{6}}{50625}$ |
| $-51$ | 1 | $\frac{-1+i\sqrt{17}}{3(1+\sqrt{3})}$ | 0 | $\frac{1029}{2048}$ |
| $-52$ | 1 | $\frac{i\sqrt{13}}{4+\sqrt{3}}$ | $\infty$ | $-\frac{22356\sqrt{3}}{78125}$ |
| $-132$ | 1 | $\frac{i\sqrt{33}}{6+\sqrt{3}}$ | $\infty$ | $-\frac{10296\sqrt{2}}{78125}$ |
| $-168$ | 1 | $\frac{-2+i\sqrt{14}}{3(1+\sqrt{3})}$ | 0 | $\frac{2124276\sqrt{3}}{78125}$ |
| $-3$ | 5 | $\frac{-1+5i}{1+3\sqrt{3}}$ | 1 | $\frac{520421}{276480\sqrt{3}5^{5/6}}$ |
| $-43$ | 1 | $\frac{-\sqrt{3}+i\sqrt{43}}{7+\sqrt{3}}$ | 1 | $\frac{58594347\sqrt{3}}{160000000}$ |
| $-88$ | 1 | $\frac{i\sqrt{22}}{5+\sqrt{3}}$ | $\infty$ | $-\frac{232377012\sqrt{6}}{1143828125}$ |
| $-123$ | 1 | $\frac{-3+i\sqrt{41}}{5(1+\sqrt{3})}$ | 0 | $\frac{18992838501}{800000000}$ |
| $-228$ | 1 | $\frac{-2+i\sqrt{19}}{2+3\sqrt{3}}$ | 1 | $\frac{852632144\sqrt{2}}{170859375}$ |
| $-312$ | 1 | $\frac{i\sqrt{78}}{9+\sqrt{3}}$ | $\infty$ | $-\frac{109824062796\sqrt{3}}{1522435234375}$ |
| $-3$ | 7 | $\frac{-1+7i}{5(1+\sqrt{3})}$ | 0 | $\frac{530508320243}{1058400000000\sqrt{3}7^{1/6}}$ |
| $-4$ | 5 | $\frac{-2\sqrt{3}+5i}{7+2\sqrt{3}}$ | 1 | $\frac{13695240888\sqrt{3}}{194871715^{1/4}}$ |
| $-67$ | 1 | $\frac{-3\sqrt{3}+i\sqrt{67}}{11+3\sqrt{3}}$ | 1 | $\frac{3782858460381\sqrt{3}}{20480000000}$ |
| $-372$ | 1 | $\frac{-1+i\sqrt{31}}{4(1+\sqrt{3})}$ | 0 | $\frac{49217478940721}{41105751328125\sqrt{6}}$ |
| $-408$ | 1 | $\frac{-4+i\sqrt{34}}{5(1+\sqrt{3})}$ | 0 | $\frac{15485222546427233 6\sqrt{2}}{71351729296875}$ |
| $-148$ | 1 | $\frac{i\sqrt{37}}{7+2\sqrt{3}}$ | $\infty$ | $-\frac{19250322046956\sqrt{3}}{32057708828125}$ |
| $-267$ | 1 | $\frac{-3+i\sqrt{89}}{7(1+\sqrt{3})}$ | 0 | $\frac{2802694559631731037}{399097262080000000}$ |
| $-232$ | 1 | $\frac{-\sqrt{3}+i\sqrt{58}}{8+\sqrt{3}}$ | 1 | $\frac{1386799984631153432 34\sqrt{3}}{1881381521077817968 75}$ |
| $-708$ | 1 | $\frac{i\sqrt{59}}{4+5\sqrt{3}}$ | $\infty$ | $-\frac{151313303299065300013 4208\sqrt{2}}{552991512035093590390625}$ |
| $-163$ | 1 | $\frac{-\sqrt{3}+i\sqrt{163}}{13+\sqrt{3}}$ | 1 | $\frac{38910568342784299532 0343\sqrt{3}}{12794143821862532800 00000}$ |

Table 6.16: Local constants around the rational CM points in $X(6,1)^+$ normalized for the Hauptmodul $t_6^+$

| $D_K$ | $f(\Lambda)$ | $P$ | $e(P)$ | $(\kappa_P/\pi_{D_K})^{e(P)}$ |
|---|---|---|---|---|
| $-3$ | 1 | $Q_{10,2}^+ = \frac{2-i\sqrt{6}}{5(-2+\sqrt{2})}$ | 3 | $\frac{1}{10}(-14+3i\sqrt{6})$ |
| $-8$ | 1 | $Q_{10,3}^+ = \frac{-1+2i}{5}$ | 2 | $\frac{3}{16} - \frac{i}{4}$ |
| $-20$ | 1 | $Q_{10,1}^+ = \frac{i(1+\sqrt{2})}{\sqrt{5}}$ | 2 | $-\frac{1}{100}$ |
| $-40$ | 1 | $Q_{10,6}^+ = \frac{i}{\sqrt{5}}$ | 2 | $-\frac{81}{400}$ |

| $D_K$ | $f(\Lambda)$ | $P$ | $a$ | $\frac{\kappa_P}{\overline{v_{a,P}}\pi_{D_K}}$ |
|---|---|---|---|---|
| $-35$ | 1 | $\frac{i(1+2\sqrt{2})}{\sqrt{35}}$ | $\infty$ | $\frac{4\sqrt{10}}{5}$ |
| $-120$ | 1 | $\frac{2i\sqrt{3}-\sqrt{5}}{5\sqrt{5}-2\sqrt{10}}$ | $-1$ | $-\frac{189}{50\sqrt{2}}$ |
| $-52$ | 1 | $\frac{3-i\sqrt{26}}{5(-3+\sqrt{2})}$ | $-1$ | $-\frac{30375}{5408\sqrt{2}}$ |
| $-280$ | 1 | $\frac{i(3+\sqrt{2})}{\sqrt{35}}$ | $\infty$ | $\frac{2028807}{54925\sqrt{10}}$ |
| $-3$ | 3 | $\frac{2\sqrt{2}-3i\sqrt{3}}{5(1-2\sqrt{2})}$ | $-1$ | $-\frac{4250\sqrt{2}}{1331}$ |
| $-115$ | 1 | $\frac{i(3+4\sqrt{2})}{\sqrt{115}}$ | $\infty$ | $\frac{50544\sqrt{10}}{6655}$ |
| $-20$ | 3 | $\frac{5-3i\sqrt{10}}{5(-5+\sqrt{2})}$ | $-1$ | $-\frac{26741}{7200\sqrt{6}}$ |
| $-43$ | 1 | $\frac{\sqrt{2}-i\sqrt{43}}{15(1-\sqrt{2})}$ | 0 | $\frac{113400}{6859}$ |
| $-88$ | 1 | $\frac{1}{15}(-1+2i\sqrt{11})$ | 0 | $-\frac{70875}{594473\sqrt{2}}$ |
| $-8$ | 3 | $\frac{-1+3i}{5(2-\sqrt{2})}$ | 0 | $\frac{27125}{2197\sqrt{6}}$ |
| $-340$ | 1 | $\frac{i(1+3\sqrt{2})}{\sqrt{85}}$ | $\infty$ | $\frac{26095041}{10158928\sqrt{2}}$ |
| $-520$ | 1 | $\frac{i(5i+\sqrt{65})}{15(2-\sqrt{2})}$ | 0 | $\frac{2891123361}{817400375\sqrt{5}}$ |
| $-235$ | 1 | $\frac{i(5+6\sqrt{2})}{\sqrt{235}}$ | $\infty$ | $\frac{2109899988\sqrt{10}}{45646645}$ |
| $-67$ | 1 | $\frac{3\sqrt{2}-i\sqrt{67}}{5(1-3\sqrt{2})}$ | $-1$ | $-\frac{82923750\sqrt{2}}{39651821}$ |
| $-148$ | 1 | $\frac{2\sqrt{2}-i\sqrt{37}}{15(1-\sqrt{2})}$ | 0 | $\frac{1939862925}{430457408}$ |
| $-760$ | 1 | $\frac{1}{45}(-5+2i\sqrt{95})$ | 0 | $-\frac{53379126511389}{53471268438731\sqrt{2}}$ |
| $-232$ | 1 | $\frac{1}{25}(-3+2i\sqrt{29})$ | 0 | $-\frac{1457018179155}{2243460712571}$ |
| $-163$ | 1 | $\frac{i(12i+\sqrt{326})}{5(12-5\sqrt{2})}$ | $-1$ | $-\frac{14925693317535000\sqrt{2}}{694541357660879}$ |

Table 6.17: Local constants around the rational CM points in $X(10,1)^+$ normalized for the Hauptmodul $t_{10}^+$

| $D_K$ | $f(\Lambda)$ | $P$ | $e(P)$ | $(\kappa_P/\pi_{D_K})^{e(P)}$ |
|---|---|---|---|---|
| $-3$ | 1 | $Q^+_{22,2} = \frac{i\sqrt{3}-\sqrt{11}}{5+\sqrt{11}}$ | 3 | $\frac{(P-1)^3}{32\sqrt{11}}$ |
| $-4$ | 1 | $Q^+_{22,1} = i$ | 4 | $-\frac{1}{704}$ |
| $-11$ | 1 | $Q^+_{22,3} = \frac{-1+i}{3+\sqrt{11}}$ | 2 | $-\frac{8i}{121}$ |
| $-88$ | 1 | $Q^+_{22,6} = \frac{i\sqrt{2}}{3+\sqrt{11}}$ | 2 | $\frac{81}{484}$ |

| $D_K$ | $f(\Lambda)$ | $P$ | $a$ | $\frac{\kappa_P}{v_{a,P}\pi_{D_K}}$ |
|---|---|---|---|---|
| $-20$ | 1 | $\frac{i\sqrt{5}}{4+\sqrt{11}}$ | $\infty$ | $-\frac{\sqrt{11}}{16}$ |
| $-132$ | 1 | $\frac{-2+i\sqrt{3}}{2+\sqrt{11}}$ | 1 | $\frac{15}{176\sqrt{2}}$ |
| $-4$ | 3 | $\frac{3i-\sqrt{11}}{8+2\sqrt{11}}$ | $3-\sqrt{11}$ | $\frac{73^{1/4}\sqrt{11}}{8\sqrt{2}}$ |
| $-11$ | 3 | $\frac{-1+3i}{1+\sqrt{11}}$ | 1 | $\frac{7}{198\sqrt{3}}$ |
| $-3$ | 3 | $\frac{3i\sqrt{3}-\sqrt{11}}{7+\sqrt{11}}$ | 1 | $\frac{5\sqrt{11}}{128}$ |
| $-187$ | 1 | $\frac{-1+i\sqrt{17}}{9+3\sqrt{11}}$ | 0 | $\frac{225}{1408}$ |
| $-4$ | 5 | $\frac{5i}{6+\sqrt{11}}$ | $\infty$ | $-\frac{243\sqrt{11}}{54885^{1/4}}$ |
| $-67$ | 1 | $\frac{-3\sqrt{11}+i\sqrt{67}}{21+5\sqrt{11}}$ | $3-\sqrt{11}$ | $\frac{135\sqrt{11}}{343}$ |
| $-1012$ | 1 | $\frac{-3+i\sqrt{23}}{12+4\sqrt{11}}$ | 0 | $\frac{289854855}{2656192\sqrt{22}}$ |
| $-148$ | 1 | $\frac{i\sqrt{37}}{9+2\sqrt{11}}$ | $\infty$ | $-\frac{4725\sqrt{11}}{35152}$ |
| $-232$ | 1 | $\frac{-3\sqrt{11}+i\sqrt{58}}{16+3\sqrt{11}}$ | 1 | $\frac{22950\sqrt{11}}{288463}$ |
| $-163$ | 1 | $\frac{-3\sqrt{11}+i\sqrt{163}}{19+3\sqrt{11}}$ | 1 | $\frac{2083725\sqrt{11}}{7023616}$ |

Table 6.18: Local constants around the rational CM points in $X(22,1)^+$ normalized for the Hauptmodul $t^+_{22}$

## 6.4 Automorphic forms, Hecke operators and $q$-expansions

In this section we will consider the graded $\mathbb{Q}$-algebra of cusp forms $S_*(\Gamma(D,N))$, which we introduced in Section 1.3, for $D = 6$, $N = 1, 5, 7, 11$, $D = 10$, $N = 1, 3, 7, 9$ and $D = 22$, $N = 1, 3, 5, 7$. In all these cases we will give a set of generating forms for this $\mathbb{Q}$-algebra. In order to do so we will use the Hauptmoduln $t_D^+$, which we introduced in Theorem 2.4 together with the properties given in Theorem 3.1 and in Section 6.2.1, and $t_{D,N}^+$, $N > 1$, which we computed and studied in Chapter 4. The study of the Hecke operators will be carried out, when necessary, through a construction similar to that of the kroneckerian polynomials. The computation of $q$-expansions for automorphic forms around a CM point will then follow from the knowledge of the $q$-expansions for $t_D^+$ and $t_{D,N}^+$ around this point. The ring of automorphic forms corresponding to $D = 6, N = 1$ has been studied in [BG08] and $q$-expansions around the elliptic points in this case can be found in [BG12], where they are computed following the approach given in [Zag08] and a recursion formula for the coefficients is provided.

Given an automorphic form of weight $2t$ (or a meromorphic form which is analytic at $P$), we will consider expansions around a CM point $P$ of the form

$$f = \frac{1}{(2\pi i)^t}\left(\frac{(q-\kappa)^2}{\kappa(P-\overline{P})}\right)^t \sum_{n\geq 0} ca_n \frac{q^n}{n!},$$

where $q = \kappa\frac{z-P}{z-\overline{P}}, \kappa \in \mathbb{C}^*$, is a suitable local parameter and $c \in \mathbb{C}^*$ is a normalizing constant. Note that these are just a normalized version of those considered in Chapter 5 and coincide as well, up to normalization, with those considered in [Zag08] and [BG12].

**Example.** If we consider the modular curve of level 1, $X_0(1)$, its function field is generated by the classical $j$-function, which is usually identified with its Fourier expansion

$$j(q_\infty) = \frac{1}{q_\infty} + 744 + 196884q_\infty + 21493760q_\infty^2 + \dots$$

where $q_\infty(z) = e^{2\pi iz}$ for $z \in \mathcal{H}$. Moreover, the algebra of modular forms is generated by $E_4, E_6, \Delta$, where $E_4$ and $E_6$ denote the normalized Eisenstein series of weights 4 and 6, respectively, and $\Delta$, the modular discriminant.

Then,

$$E_4 = \left(\frac{1}{2\pi i}\frac{dj}{dz}\right)^2 \frac{1}{j^2 - 1728j},$$
$$E_6 = -\left(\frac{1}{2\pi i}\frac{dj}{dz}\right)^3 \frac{1}{j^2(j - 1728)},$$
$$\Delta = \left(\frac{1}{2\pi i}\frac{dj}{dz}\right)^6 \frac{1}{j^4(j - 1728)^3}.$$

This can be easily seen comparing the first terms of the Fourier expansions and checking that the divisors of both sides agree. To do that, we only need to remember that the only multiple points of $j$ in $\mathcal{H}$ are: $\rho = e^{2\pi i/6}$, where $j(\rho) = 0$ with multiplicity 3, and $i$, where $j(i) = 1728$ with multiplicity 2. Note also that $\frac{1}{2\pi i}\frac{d}{dz}$ is nothing else than the differential operator more usually written as $q_\infty \frac{d}{dq_\infty}$.

Therefore, from the $q$-expansion of $j$ around a CM point (or the Fourier expansion at $\infty$) we can easily compute $q$-expansions (or Fourier expansions) for these forms. Expansions for $j$ around certain CM points are given in [BT07a] and expansions around other CM points can be computed using the procedure described in Section 6.2.

Expansions around the elliptic point $i$, of order 2, and $\rho$, of order 3, and also around $\sqrt{-6}$, a CM point by the ring of integers of $\mathbb{Q}(\sqrt{-6})$ can be found in Tables 6.19, 6.20, 6.21. The values of the corresponding local constants for the CM points by the rings of integers of $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-6})$ are deduced from the ones computed in [BT07a].

| | $c$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $j$ | 576 | 3 | 0 | 36 | 0 | 1104 | 0 | 49816 | 0 | 2973824 | 0 | 227654496 | 0 |
| $E_4$ | 8 | 3 | 0 | 10 | 0 | 30 | 0 | 360 | 0 | 2280 | 0 | 66960 | 0 |
| $E_6$ | $-96$ | 0 | 3 | 0 | 14 | 0 | 168 | 0 | 1764 | 0 | 43848 | 0 | 864864 |
| $\Delta$ | 8 | 1 | 0 | $-2$ | 0 | 6 | 0 | 48 | 0 | $-1584$ | 0 | 6048 | 0 |

Table 6.19: Coefficients of the expansions around $i$ with $\kappa_i = \frac{1}{2\sqrt{2}}\pi_{-4}$

| | $c$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $j$ | 1728 | 0 | 0 | 0 | 8 | 0 | 0 | −460 | 0 | 0 | 53334 | 0 | 0 | −10612635 | 0 |
| $E_4$ | −6 | 0 | 2 | 0 | 0 | −5 | 0 | 0 | 30 | 0 | 0 | −600 | 0 | 0 | 15360 |
| $E_6$ | 18 | 2 | 0 | 0 | −7 | 0 | 0 | 63 | 0 | 0 | −1428 | 0 | 0 | 52248 | 0 |
| $\Delta$ | $-3/4$ | 1 | 0 | 0 | 1 | 0 | 0 | 8 | 0 | 0 | 120 | 0 | 0 | 372 | 0 |

Table 6.20: Coefficients of the expansions around $\rho$ with $\kappa_\rho = -\frac{\sqrt[3]{6}}{3}\pi_{-3}$

| | $c$ | 0 |
|---|---|---|
| $j$ | 1728 | $1399 + 988\sqrt{2}$ |
| $E_4$ | $-18\left(-1954045 + 1381718\sqrt{2}\right)$ | 68 |
| $E_6$ | $216\left(-5463983723 + 3863619942\sqrt{2}\right)$ | 92 |
| $\Delta$ | $-108\left(-83452309924610425346859 + 59009694253373452368026\sqrt{2}\right)$ | 2 |

| | 1 | 2 |
|---|---|---|
| $j$ | 79764 | $-204\left(-15774652 + 11145963\sqrt{2}\right)$ |
| $E_4$ | $-40\left(-8971 + 6339\sqrt{2}\right)$ | $-20\left(-879357913 + 621799902\sqrt{2}\right)$ |
| $E_6$ | $-84\left(-25090 + 17737\sqrt{2}\right)$ | $-42\left(-663024313 + 468828932\sqrt{2}\right)$ |
| $\Delta$ | $12\left(-1147 + 809\sqrt{2}\right)$ | $6\left(-32318107 + 22852414\sqrt{2}\right)$ |

| | 3 | 4 |
|---|---|---|
| $j$ | $-6\left(-41259885695815 + 29175135775248\sqrt{2}\right)$ | $-36\left(-648890835855607688 + 458835110231329411\sqrt{2}\right)$ |
| $E_4$ | $-60\left(-1693258199391 + 1197314355013\sqrt{2}\right)$ | $-30\left(-389630223646822311 + 275510173295898148\sqrt{2}\right)$ |
| $E_6$ | $-168\left(-11737526010111 + 8299684235999\sqrt{2}\right)$ | $-756\left(-65827732575595575 + 46547236094338036\sqrt{2}\right)$ |
| $\Delta$ | $-6\left(-1643944362399 + 1162444207295\sqrt{2}\right)$ | $-153\left(-328527266679847 + 232303858073740\sqrt{2}\right)$ |

Table 6.21: Coefficients of the expansions around $P = \sqrt{-6}$ with $\kappa_P = \frac{\left(2765+1956\sqrt{2}\right)}{79764}\pi_{-24}$

Let us now move on to the non-modular case. Apart from the case $D = 6$, $N = 1$ where the ring of automorphic forms is studied in [BG08], we have a priori no known generators of the $\mathbb{Q}$-algebra of automorphic forms. However, in all the cases we are going to work with, we have that $X(D,N)^+$ has genus 0 and we already know a rational Hauptmodul for this curve, $t^+_{D,N}$. Therefore, the following result will be useful to explicitly determine the cusp forms of a certain weight.

**Proposition 6.5.** *Let $X(D,N)$ be a Shimura curve attached to an Eichler order of level $N$ in a non-split indefinite quaternion algebra of discriminant $D$ such that $g(X(D,N)^+) = 0$ (cf. Proposition 4.1). Then, given a positive integer $t$,*

$$S_{2t}(\Gamma(D,N))_{\mathbb{Q}} = S_{2t}(\Gamma(D,N)^+)_{\mathbb{Q}} \oplus \bigoplus_{W} S_{2t}(\langle \Gamma(D,N), W \rangle)^-_{\mathbb{Q}},$$

*where $W$ runs over the set of subgroups of $W_{DN}$ of index 2 and*

$$S_{2t}(\langle\Gamma(D,N),W\rangle)_{\mathbb{Q}}^{-}\subset S_{2t}(\langle\Gamma(D,N),W\rangle)_{\mathbb{Q}}$$

*is the eigenspace of eigenvalue $-1$ by the non-trivial element in $W_{DN}/W$. Let $t^{+}_{D,N}$ be a Hauptmodul for $X(D,N)^{+}$ defined over $\mathbb{Q}$ and fix $W\subset W_{DN}$ a subgroup of index at most 2. Denote by*

$$\{P_i\}_i\subset\Gamma(D,N)^{+}\backslash\mathcal{H}$$

*the set of points ramified by the quotient map*

$$X(D,N)^{W}(\mathbb{C})\to X(D,N)^{+}(\mathbb{C})$$

*and by $\{Q_i\}$ the remaining elliptic points in $X(D,N)^{+}(\mathbb{C})$. Then, if $W$ has index 2, we consider $u_W\in\mathbb{Q}(X(D,N)^{W})$ such that*

$$u_W^2=u_{W,0}\prod_i(t^{+}_{D,N}-t^{+}_{D,N}(P_i)),$$

*where the product runs over the set of $P_i$ where $t^{+}_{D,N}$ is finite and $u_{W,0}\in\mathbb{Z}$ is a squarefree integer. In the case $W=W_{DN}$, we set $u_{W_{DN}}=1$. Then, either $S_{2t}(\langle\Gamma(D,N),W\rangle)_{\mathbb{Q}}^{-}$, if $W\subset W_{DN}$ has index 2, or $S_{2t}(\Gamma(D,N)^{+})_{\mathbb{Q}}$, if $W=W_{DN}$, can be written as*

$$\frac{D(t^{+}_{D,N})^{t}}{u_W}\frac{\Phi(t^{+}_{D,N})}{\prod\limits_{i=1}^{r}(t^{+}_{D,N}-t^{+}_{D,N}(P_i))^{\alpha_i}\prod\limits_{i=1}^{s}(t^{+}_{D,N}-t^{+}_{D,N}(Q_i))^{\beta_i}},$$

*where the products are restricted to the points where $t^{+}_{D,N}$ is finite, and*

$$\alpha_i=\left\lfloor\frac{t(e(P_i)-1)}{e(P_i)}-\frac{1}{2}\right\rfloor,\quad \beta_i=\left\lfloor\frac{t(e(Q_i)-1)}{e(Q_i)}\right\rfloor,$$

*where $e(P)$ denotes the elliptic order of $P\in X(D,N)^{+}(\mathbb{C})$ and $\Phi(X)\in\mathbb{Q}[X]$ is a non-zero polynomial of degree at most*

$$g(X(D,N)^{W})+1-2t+\sum_i\alpha_i+\sum_i\beta_i.$$

**Remark.** We could use the above proposition to recover the formulas for the dimension of $S_k(\Gamma(D,N))$ given in Theorem 1.20 in this case.

The functions $u_W$ for all the cases we will deal with can be found in Theorem 2.4, if $N = 1$, and in Chapter 4, if $N > 1$.

The expression we obtain in the case $W = W_{D,N}$ is the classical description of the ring of automorphic forms for genus 0 curves in terms of a Hauptmodul, which is classically found in the setting of modular forms (cf. [Sch74]), but can also be found in the case of Shimura curves, cf. [Yan13].

*Proof.* The space of meromorphic automorphic forms of weight $2t$ with respect to $\Gamma(D,N)$ which are defined over $\mathbb{Q}$, $A_{2t}(\Gamma(D,N))_{\mathbb{Q}}$, can be written as $D(t^+_{D,N})^t\mathbb{Q}(X(D,N))$. But not only this, since $S_{2t}(\Gamma(D,N))_{\mathbb{Q}}$ has a basis of eigenforms, and the action of the Hecke operators is compatible with the action of the Atkin-Lehner involutions, we can decompose a cusp form into a sum of forms in $S_{2t}(\Gamma(D,N)^+)_{\mathbb{Q}}$ and $S_{2t}(\langle\Gamma(D,N),W\rangle)^-_{\mathbb{Q}}$ for all the $W \subset W_{DN}$ such that $[W_{DN} : W] = 2$. Therefore, to write down the elements in $S_{2t}(\Gamma(D,N))_{\mathbb{Q}}$ it suffices to determine the holomorphic elements in the various spaces $D(t^+_{D,N})^t\mathbb{Q}(X(D,N)^W)$. Observe now that $\mathbb{Q}(X(D,N)) = \mathbb{Q}(t^+_{D,N})$ and, since $\mathbb{Q}(X(D,N)^W)$ is a quadratic extension of $\mathbb{Q}(t^+_{D,N})$,

$$\mathbb{Q}(X(D,N)^W) = \mathbb{Q}(t^+_{D,N}, u_W),$$

for a certain $u_W \in \mathbb{Q}(X(D,N)^W)$ such $u_W^2 = u_{W,0}\prod_i(t^+_{D,N} - t^+_{D,N}(P_i))$, where $u_{W,0} \in \mathbb{Z}$ is a suitable squarefree integer and $\{P_i\}_i \subset X(D,N)^+(\mathbb{C})$ denotes the set of points where $t^+_{D,N}$ is finite and which ramify in the quotient map $X(D,N)^W(\mathbb{C}) \to X(D,N)^+(\mathbb{C})$. The number of points of $X(D,N)^+(\mathbb{C})$ which are ramified by this quotient map can be computed using the Riemann-Hurwitz formula and it is equal to $r = 2g(X(D,N)^W) + 2$. In particular, the subspace $S_{2t}(\langle\Gamma(D,N),W\rangle)^-_{\mathbb{Q}}$ coincides with the set of holomorphic elements in $\frac{D(t^+_{D,N})^t}{u_W}\mathbb{Q}(t^+_{D,N})$. Now, an element of this last set can be written as

$$\frac{D(t^+_{D,N})^t}{u_W}\frac{\Phi_0(t^+_{D,N})}{\Phi_1(t^+_{D,N})},$$

for some coprime polynomials $\Phi_0(X), \Phi_1(X) \in \mathbb{Q}[X]$, and therefore the set of zeros of $\Phi_1(t^+_{D,N})$ must be contained in the set of zeros of $\frac{D(t^+_{D,N})^t}{u_W}$ and since this is a subset of $\{P_i, Q_j\}$, we obtain that it can be written as

$$F(\{\alpha_i\},\{\beta_i\},\Phi) = \frac{D(t^+_{D,N})^t}{u_W}\frac{\Phi(t^+_{D,N})}{\prod_i(t^+_{D,N} - t^+_{D,N}(P_i))^{\alpha_i}\prod_i(t^+_{D,N} - t^+_{D,N}(Q_i))^{\beta_i}},$$

for some positive integers $\alpha_i, \beta_i$. In particular, we already have that this expression is holomorphic away from $\{P_i, Q_j\}$ and the pole of $t^+_{D,N}$. Thus, we only have to find out which conditions have to be imposed on $\alpha_i, \beta_i$ and $\deg\Phi$ so that it is holomorphic at these points as well. If we compute $\operatorname{ord}_{P_i} F(\{\alpha_i\},\{\beta_i\},\Phi)$, we obtain

$$\begin{cases} e(P_i)\left(t\frac{e(P_i)-1}{e(P_i)} - \frac{1}{2} - \alpha_i\right), & \text{if } t^+_{D,N}(P_i) \neq \infty, \\ e(P_i)\left(-t\frac{e(P_i)+1}{e(P_i)} - \deg\Phi + \frac{r-1}{2} + \sum_{j\neq i}\alpha_j + \sum_j \beta_j\right), & \text{if } t^+_{D,N}(P_i) = \infty. \end{cases}$$

Similarly, the order of the function $F(\{\alpha_i\},\{\beta_i\},\Phi)$ at any of the two points in $X(D,N)^W(\mathbb{C})$ which map to $Q_i$ is equal to

$$\begin{cases} e(Q_i)\left(t\frac{e(Q_i)-1}{e(Q_i)} - \beta_i\right), & \text{if } t^+_{D,N}(Q_i) \neq \infty, \\ e(Q_i)\left(-t\frac{e(Q_i)+1}{e(Q_i)} - \deg\Phi + \frac{r}{2} + \sum_j \alpha_j + \sum_{j\neq i}\beta_j\right), & \text{if } t^+_{D,N}(Q_i) = \infty. \end{cases}$$

Otherwise, if the pole $P$ of $t^+_{D,N}$ is not in $\{P_i, Q_i\}$,

$$\operatorname{ord}_P F(\{\alpha_i\},\{\beta_i\},\Phi) = -2t - \deg\Phi + \frac{r}{2} + \sum_j \alpha_j + \sum_j \beta_j.$$

Therefore, we obtain that

$$\alpha_i \leq \left\lfloor \frac{t(e(P_i)-1)}{e(P_i)} - \frac{1}{2} \right\rfloor, \quad \beta_i \leq \left\lfloor \frac{t(e(Q_i)-1)}{e(Q_i)} \right\rfloor,$$

and

$$\deg\Phi \leq g + 1 - 2t + \sum_j \alpha_j + \sum_j \beta_j.$$

Finally, if we no longer assume that $\Phi(t^+_{D,N})$ is coprime to the denominator, multiplying the numerator and the denominator by the necessary factors $\prod_i(t^+_{D,N} - t^+_{D,N}(P_i))$, $\prod_i(t^+_{D,N} - t^+_{D,N}(Q_i))$, we can always assume that

$$\alpha_i = \left\lfloor \frac{t(e(P_i)-1)}{e(P_i)} - \frac{1}{2} \right\rfloor, \quad \beta_i = \left\lfloor \frac{t(e(Q_i)-1)}{e(Q_i)} \right\rfloor,$$

which is the expression given in the statement of the proposition. □

Using this result we can explicitly compute the space $S_{2t}(\Gamma(D,N))_{\mathbb{Q}}$ for a fixed positive integer $t$. However, we are interested in computing a set of generators of $S_*(\Gamma(D,N))_{\mathbb{Q}}$, the algebra of cusp forms. In order to do so, we will compute the elements of this algebra up to a certain weight, and then we need to check whether they generate the full algebra or not. We could use the explicit descriptions given in the previous proposition to prove that the elements up to a certain weight suffice to generate the algebra. However, it will be more convenient to use the following criterion.

**Proposition 6.6.** *Let $f_1,\dots,f_r \in S_*(\Gamma(D,N))_{\mathbb{Q}}$, $r \geq 2$, be forms of weights $k_1,\dots,k_r \geq 2$. Consider the graded polynomial ring $\mathbb{Q}[X_1,\dots,X_r]$ with weights $(k_1,\dots,k_r)$ and the natural surjective morphism*

$$\phi : \mathbb{Q}[X_1,\dots,X_r] \to \mathbb{Q}[f_1,\dots,f_r] \subset S_*(\Gamma(D,N))$$

*such that $\phi(X_i) = f_i$. Let $I \subset \mathbb{Q}[X_1,\dots,X_n]$ be an ideal and assume that*

*(a)* $f_1^{k_2} f_2^{-k_1} \notin \mathbb{Q}$,

*(b)* $I \subset \ker\phi$,

*(c)* $\dim(\mathbb{Q}[X_1,\dots,X_r]/I) = 2$,

*(d) $I$ is prime.*

*Then, $I = \ker\phi$. In particular, $\mathbb{Q}[f_1,\dots,f_r] = S_*(\Gamma(D,N))_{\mathbb{Q}}$ if and only if both rings have the same Hilbert series, in which case $\mathbb{Q}[X_1,\dots,X_r]/I$ gives a presentation of $S_*(\Gamma(D,N))_{\mathbb{Q}}$.*

*Proof.* $(a)$ is equivalent to $f_1^{k_2}$ and $f_2^{k_1}$ linearly independent over $\mathbb{Q}$. Let us now see that two linearly independent forms $g_1$, $g_2$ of the same weight, are algebraically independent. Given a polynomial $P(x,y) \in \mathbb{Q}[x,y]$ such that $P(g_1,g_2) = 0$, if we write $P(X,Y) = P_1(X,Y) + \cdots + P_s(X,Y)$ as a sum of homogenous polynomials of distinct degrees, the transformation law for automorphic forms, yields that $P_i(g_1,g_2) = 0$ for all $i$. In particular, we can assume that $P$ is homogeneous. Now, it follows that $P(g_1g_2^{-1},1) = 0$ which implies that $g_1g_2^{-1}$ is constant and therefore it lies in $\mathbb{Q}$. We have proved indeed that $\mathbb{Q}(f_1,\dots,f_r)$ is an extension of transcendence degree 1 over a non-constant subextension of $\mathbb{Q}(X(D,N))$ which also has transcendence degree 1. In particular, the Krull dimension $\dim\mathbb{Q}[f_1,\dots,f_r]$ is 2.

Now, by $(b)$, there exists an exhaustive morphism of $\mathbb{Q}[X_1,\dots,X_r]/I$ into $\mathbb{Q}[f_1,\dots,f_r]$. Since $\mathbb{Q}[f_1,\dots,f_r]$ is a domain, because it is a subalgebra of $S_*(\Gamma(D,N))_{\mathbb{Q}}$, we obtain that $\ker\phi$ is a prime ideal containing $I$. In particular, it follows from $(c)$ and $(d)$ that, if $I \subsetneq \ker\phi$, then $\dim(\mathbb{Q}[X_1,\dots,X_r]/\ker\phi) < 2$ and we reach a contradiction, since it surjects into $\mathbb{Q}[f_1,\dots,f_r]$ which has Krull dimension 2.

The final claim follows from the fact that, given an inclusion of graded rational algebras $B \subset A$, then $B = A$ if and only if $\dim_{\mathbb{Q}} B_k = \dim_{\mathbb{Q}} A_k$ for all $k$, which is equivalent to the equality between their Hilbert series, i.e. the generating functions of the sequences $(\dim_{\mathbb{Q}} B_k)_k$, $(\dim_{\mathbb{Q}} A_k)_k$. □

More information on the notions of dimension theory of $K$-algebras and Hilbert series involved in the proof can be found, for example, in [ZS60] or [Mat87].

Finally, once we have a set of generators of the algebra, we are interested in studying how the Hecke operators act on these elements. For this purpose, given a cusp form of weight $2t$, $f \in S_{2t}(\Gamma(D,N))_{\mathbb{Q}}$, and a positive integer $m$ coprime to $DN$, we consider the polynomial

$$\Phi_{m,f}(X) = \prod_{\omega\in\mathcal{O}(D,N)_1\backslash\mathcal{O}(D,N)'_m} \left(X - m^t f|_{2t}\omega\right),$$

which is analogous to the construction of kroneckerian polynomials we considered for functions in this setting of forms.

**Proposition 6.7.**

*(1) For $f \in S_{2t}(\Gamma(D,N))_{\mathbb{Q}}$, then $\Phi_{m,f}(X) \in S_*(\Gamma(D,N))_{\mathbb{Q}}[X]$. More precisely, it is a polynomial in $X$ of degree $\psi(m)$,*

$$\Phi_{m,f}(X) = \sum_{i=0}^{\psi(m)} c_i X^{\psi(m)-i},$$

*such that $c_i \in S_{2ti}(\Gamma(D,N))_{\mathbb{Q}}$.*

*(2) The coefficients of the polynomial $\Phi_{m,f}(X)$ can be expressed in terms of the Hecke operators acting on the powers of $f$ as follows:*

$$c_i = \sum_{\substack{j_1+2j_2+\cdots+\psi(m)j_{\psi(m)}=i\\ j_1,\dots,j_{\psi(m)}\geq 0}} \left(\frac{-1}{m}\right)^{j_1+\cdots+j_{\psi(m)}} \frac{(T_m f)^{j_1}\cdots(T_m f^{\psi(m)})^{j_{\psi(m)}}}{j_1!j_2!\cdots j_{\psi(m)}!1^{j_1}2^{j_2}\cdots\psi(m)^{j_{\psi(m)}}}.$$

*In particular,*

$$c_1 = -\frac{1}{m}T_m f.$$

*Proof.* The degree of $\Phi_{m,f}(X)$ follows from the computation of the cardinal of $\mathcal{O}(D,N)_1\backslash\mathcal{O}(D,N)'_m$ in Proposition 2.1. The rest of the assertions in the proposition are a consequence of the formula given in (2), which follows from the expression of the elementary symmetric polynomials in terms of power sums, usually known as Newton polynomials. □

If we assume that $g(X(D,N)^+)=0$, we can say something more about this polynomial.

**Proposition 6.8.** *Under the assumptions of Proposition 6.5 and using the same notations, let $f \in S_{2t}(\Gamma(D,N)^W)^-_{\mathbb{Q}}$ or $f \in S_{2t}(\Gamma(D,N)^+_{\mathbb{Q}})$. Then if $m=p$ is prime, $\Phi_{m,f}(X)$ is either irreducible over the fraction field of $S_{2t}(\Gamma(D,N))_{\mathbb{Q}}$ or is the power of a factor of degree one.*

*Proof.* By Proposition 2.1, we have that the group $\Gamma(D,N)$ acts transitively on the set $\{f|_{2t}\omega\}_{\omega\in\mathcal{O}(D,N)_1\backslash\mathcal{O}(D,N)'_m}$. Therefore, if all the elements in this set are different, as occurrs in the case of the kroneckerian polynomials, this polynomial is irreducible over the fraction field of $S_{2t}(\Gamma(D,N))_{\mathbb{Q}}$. Otherwise, we are going to see that this set, since $m$ is prime, consists of a single element. In order to do so, let us write $f = D(t^+_{D,N})^t g$ for some $g \in \mathbb{Q}(X(D,N)^W)$. Then, if we had $f|_{2t}\omega = f|_{2t}\omega'$, for some $\omega,\omega' \in \mathcal{O}(D,N)_1\backslash\mathcal{O}(D,N)'_m$ corresponding to different classes, dividing by $D(t^+_{D,N})^t$, we would have that

$$\frac{D(t^+_{D,N}\circ\omega)^t}{D(t^+_{D,N})^t}g\circ\omega = \frac{D(t^+_{D,N}\circ\omega')^t}{D(t^+_{D,N})^t}g\circ\omega'.$$

Now we observe that

$$\frac{D(t^+_{D,N}\circ\omega)^t}{D(t^+_{D,N})^t} \in \mathbb{Q}(t^+_{D,N}, t^+_{D,N}\circ\omega) = A_0(\omega^{-1}\Gamma(D,N)\omega\cap\Gamma(D,N))_{\mathbb{Q}},$$

which can be explicitly written down by differentiating the corresponding kroneckerian polynomial. Since the same holds replacing $\omega$ by $\omega'$, we obtain

$$\frac{D(t^+_{D,N}\circ\omega)^t}{D(t^+_{D,N})^t}(g\circ\omega) \in A_0(\omega^{-1}\Gamma(D,N)\omega\cap\Gamma(D,N))_{\mathbb{Q}}\cap A_0(\omega'^{-1}\Gamma(D,N)\omega'\cap\Gamma(D,N))_{\mathbb{Q}},$$

and since $\omega$ and $\omega'$ correspond to different classes, this yields

$$\frac{D(t^+_{D,N}\circ\omega)^t}{D(t^+_{D,N})^t}(g\circ\omega) \in A_0(\Gamma(D,N))_{\mathbb{Q}}.$$

Therefore, $f|_{2t}\omega \in S_{2t}(\Gamma(D,N))_{\mathbb{Q}}$ and, by the transitivity of the action, all the elements are equal. □

### 6.4.1 Cusp forms attached to Shimura curves of discriminant 6

We will begin by providing a set of generators of the $\mathbb{Q}$-algebra of cusp forms with respect to $\Gamma(6,1)$. In order to do so, recall the automorphic functions for $\Gamma(6,1)$ introduced in Theorem 2.4, which coincide with those given in [BT07b], and further studied in Theorem 3.1. In particular, the function $t_6$ is a Hauptmodul over the field $\mathbb{Q}(\sqrt{-3},i)$, i.e. $\mathbb{Q}(\sqrt{-3},i)(X(6,1))=\mathbb{Q}(\sqrt{-3},i)(t_6)$, and it satisfies the differential equation

$$Da(t_6,z)+\frac{27t_6^4+74t_6^2+27}{36t_6^2(t_6^2-1)^2}=0,$$

together with the functions

$$u_{6,1}=\sqrt{-3}\frac{t_6^2+1}{2t_6},\qquad u_{6,2}=\sqrt{3}\frac{t_6^2-1}{2t_6},\qquad u_{6,3}=i\frac{t_6^2+1}{t_6^2-1},$$

which lie in $\mathbb{Q}(X(6,1))$ and provide Hauptmoduln over $\mathbb{Q}$ for the Atkin-Lehner quotients. We also have a Hauptmodul for $X(6,1)^+$ given by

$$t_6^+=\left(\frac{t_6^2+1}{t_6^2-1}\right)^2,$$

which satisfies the differential equation

$$Da(t_6^+,z)+R(t_6^+)=0,$$

where

$$R(t)=\frac{108-113t+140t^2}{144(-1+t)^2t^2}.$$

**Proposition 6.9.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to $\Gamma(6,1)$*

*is generated by the eigenforms*

$$h_4 = \frac{(Dt_6^+)^2}{\sqrt{3(t_6^+ - 1)t_6^+(t_6^+ - 1)}} = \frac{8}{\sqrt{3}} \frac{(Dt_6)^2}{t_6(t_6^2 - 1)},$$

$$h_6 = \frac{1}{3} \frac{(Dt_6^+)^3}{\sqrt{-t_6^+ t_6^+ (t_6^+ - 1)^2}} = \frac{32i}{3} \frac{(Dt_6)^3}{t_6(t_6^2 - 1)^2},$$

$$h_{12} = \frac{1}{3} \frac{(Dt_6^+)^6}{\sqrt{-3t_6^+(t_6^+ - 1)(t_6^+)^2(t_6^+ - 1)^4}} = \frac{512\sqrt{-3}}{9} \frac{(Dt_6)^6(t_6^2 + 1)}{t_6^3(t_6^2 - 1)^4},$$

*which satisfy the relation* $0 = h_4^6 + 3h_6^4 + h_{12}^2$.

*Proof.* The expressions for these forms are obtained thanks to Proposition 6.5, which ensures as well that they are defined over $\mathbb{Q}$. Moreover the relation between these forms follows immediately from the given expressions.

In order to prove that they are indeed generators of the whole algebra, we will use Proposition 6.6. In this case, it is clear that $h_4^6, h_6^4$ are not proportional, that the ideal $I = (x_1^6 + 3x_2^4 + x_3^{12})$ is prime in $\mathbb{Q}[x_1, x_2, x_3]$, since the generating polynomial is irreducible, and that the Krull dimension of the quotient is 2. Therefore, it is clear that $\mathbb{Q}[x_1, x_2, x_3]/(x_1^6 + 3x_2^4 + x_3^{12}) \cong \mathbb{Q}[h_4, h_6, h_{12}]$ and it only remains to see that the dimension of the subspace of weigth $k$ has the same dimension as $S_k(\Gamma(6,1))_{\mathbb{Q}}$. This can be done using a computational algebra software, but in this case, since we only have one relation it is easily seen that they agree up to degree 12 and they increase in 2 when the weight increases in 12 (cf. Theorem 1.20 for the dimension of the space $S_k(\Gamma(6,1))_{\mathbb{Q}}$).

Now, $h_4$ and $h_6$ are obviously eigenforms, because the space of automorphic forms of weights 4 and 6 are both 1-dimensional. On the other hand, $h_{12}$ is also an eigenform, since it lies in the 1-dimensional space $S_{12}(\langle \Gamma(6,1), \omega_3 \rangle)_{\mathbb{Q}}^-$. □

Finally, in order to compute the expansions of these forms around a CM point $P$ all we need is the expansion of $t_6^+$ around this point. In particular, if $P$ is rational, it can be obtained using the information on Table 6.13. To show how the coefficients of these expansions look like we will give expansions around the following points in $\mathcal{H}$: $Q_{6,1}^+ = i$, an elliptic point of order 2, cf. Table 6.22; $Q_{6,2}^+ = \frac{-1+i}{1+\sqrt{3}}$, an elliptic point of order 3, cf. Table 6.23, and $Q_{6,4}^+ = \frac{1}{2} i \left(-\sqrt{2} + \sqrt{6}\right)$, a CM point by the ring of integers of $\mathbb{Q}(\sqrt{-6})$, cf. Table 6.24.

| | $c$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $h_4$ | $48$ | $0$ | $0$ | $1$ | $0$ | $0$ | $0$ | $-30$ | $0$ | $0$ | $0$ | $-6120$ | $0$ |
| $h_6$ | $576i$ | $0$ | $0$ | $0$ | $1$ | $0$ | $0$ | $0$ | $84$ | $0$ | $0$ | $0$ | $-40824$ |
| $h_{12}$ | $110592i$ | $0$ | $0$ | $1$ | $0$ | $0$ | $0$ | $234$ | $0$ | $0$ | $0$ | $61776$ | $0$ |

Table 6.22: Expansions around $Q^+_{6,1} = i$
with $\kappa_{Q^+_{6,1}} = \frac{1-i}{4}\pi_{-4}$

| | $c$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $h_4$ | $-4i\sqrt{3}$ | $0$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-5$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-4980$ | $0$ | $0$ |
| $h_6$ | $24$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $7$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-84$ | $0$ | $0$ | $0$ |
| $h_{12}$ | $576i\sqrt{3}$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-26$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-18252$ | $0$ | $0$ | $0$ |

Table 6.23: Expansions around $Q^+_{6,2}$
with $\kappa_{Q^+_{6,2}} = -\frac{1+i}{\sqrt[3]{18}}\pi_{-3}$

| | $c$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $h_4$ | $96i\sqrt{3}$ | $1$ | $0$ | $-10$ | $0$ | $-270$ | $0$ | $-13320$ | $0$ | $1371960$ |
| $h_6$ | $-1152\sqrt{2}$ | $1$ | $0$ | $21$ | $0$ | $-252$ | $0$ | $40572$ | $0$ | $352296$ |
| $h_{12}$ | $-15925248i\sqrt{6}$ | $0$ | $1$ | $0$ | $13$ | $0$ | $3432$ | $0$ | $-8424$ | $0$ |

Table 6.24: Expansions around $Q^+_{6,4} = \frac{1}{2}i\left(-\sqrt{2}+\sqrt{6}\right)$
with $\kappa_{Q^+_{6,4}} = \dfrac{1}{12\sqrt{2}}i\pi_{-24}$

Similarly, we can deal with the case of level $N = 5$. The Shimura curve of discriminant $D = 6$ and level $N = 5$ is a genus 1 curve defined over $\mathbb{Q}$ which has been studied in detail in Section 4.3.3, where a Hauptmodul for the curve $X(6,5)^+$ together with its automorphic derivative have been computed. Moreover, generating functions for all the fields $\mathbb{Q}(X(6,5)^W)$ have also been computed. In particular, those corresponding to subgroups $W \subset W_{6,5}$ of index 2 will be used to prove the following result.

**Proposition 6.10.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to $\Gamma(6,5)$ is generated by the eigenforms*

$$f_2 = \frac{Dt_{6,5}^+}{2\sqrt{61-64t_{6,5}^+}\sqrt{t_{6,5}^+-1}\sqrt{t_{6,5}^+ +1}},$$

$$f_{4,1} = \frac{(Dt_{6,5}^+)^2}{4(t_{6,5}^+-1)\sqrt{t_{6,5}^+ +1}(-61+64t_{6,5}^+)},$$

$$f_{4,2} = \frac{(Dt_{6,5}^+)^2}{4(1+t_{6,5}^+)(-61+64t_{6,5}^+)\sqrt{t_{6,5}^+ -1}},$$

$$f_{4,3} = \frac{(Dt_{6,5}^+)^2}{4\sqrt{61-64t_{6,5}^+}\left(t_{6,5}^+-1\right)\left(t_{6,5}^+ +1\right)},$$

*which satisfy the relations*

$$0 = f_{4,1}^2 - f_{4,2}^2 - 2f_2^4, \qquad 0 = 64f_{4,1}^2 + f_{4,2}^2 - 125f_2^4.$$

*The subalgebra of cusp forms for $\Gamma(6,1)$, $S_*(\Gamma(6,1))_{\mathbb{Q}} \subset S_*(\Gamma(6,5))_{\mathbb{Q}}$, is explicitly determined by the equalities*

$$h_4 = 12(6f_{4,1} + f_{4,3}),\ h_6 = -72(7f_{4,1} - 3f_{4,3})f_2,$$

$$h_{12} = 1728(4250f_2^4 - 352f_{4,1}^2 - 117f_{4,1}f_{4,3})f_{4,2}.$$

*Proof.* The functions above can be obtained as in the case of level 1 by means of Proposition 6.5 and the relations can be easily checked using the expression of these functions.

In order to prove that they are generators, we use again Proposition 6.6. In this case, the verifications are not so easy as in the previous case, but they can be carried out by hand. However, they can also be checked immediately using a computational algebra software such as Magma or Macaulay2.

Finally, to prove that all these forms are eigenforms we can use, as in the previous case, the Atkin-Lehner involutions. In the following table, for a given form, we will write $+$, resp. $-$, if the form is an eigenvector of eigenvalue 1, resp. $-1$:

| | $f_2^2$ | $f_{4,1}$ | $f_{4,2}$ | $f_{4,3}$ |
|---|---|---|---|---|
| $\omega_2$ | $+$ | $-$ | $-$ | $-$ |
| $\omega_6$ | $+$ | $+$ | $-$ | $+$ |
| $\omega_5$ | $+$ | $+$ | $-$ | $-$ |

Therefore, there exists a unique (up to scalar factors) common basis of eigenvectors for these involutions, which implies that they are indeed eigenforms.

The identification of the generators of $S_*(\Gamma(6,1))_{\mathbb{Q}}$ as elements in $S_*(\Gamma(6,5))_{\mathbb{Q}}$ can be carried out easily comparing their expansions around a given CM point, for example, any of the ones which we are going to compute next. $\square$

In order to complete our exposition we will give the first few terms of the expansions (following the same notations as in the previous cases) of these functions and forms around some special points, namely $P_4 = (2+\sqrt{3})i$, an elliptic point of order 2, $P_{4\cdot 5^2} = i$, a CM point by $\mathbb{Q}(i)$ of level 5, $P_{120} = \frac{-1+\sqrt{10}i}{1+2\sqrt{3}}$, a CM point by the ring of integers of $\mathbb{Q}(\sqrt{-30})$, $P_{40} = \frac{3+\sqrt{30}i}{4\sqrt{3}-3}$, a CM point by the ring of integers of $\mathbb{Q}(\sqrt{-10})$, $P_{24} = \frac{\sqrt{2}i}{1+\sqrt{3}}$, a CM point by the ring of integers of $\mathbb{Q}(\sqrt{-6})$, and $P_{19}\frac{\sqrt{3}-i\sqrt{19}}{-5+\sqrt{3}}$, a CM point by the ring of integers of $\mathbb{Q}(\sqrt{-19})$. Note that in this case we can use Table 6.13 together with the relation between the functions $t_{6,5}^+$ and $t_6^+$ given in Proposition 4.11 to obtain the first coefficients of the expansion of $t_{6,5}^+$ around any point of $X(6,5)^+$ related to a point in the table and then use the automorphic derivative to compute as many coefficients as desired. This also allows us to obtain the values of the local constants.

| | $c$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| $f_2$ | 2 | 0 | 1 | 0 | 0 | 0 | $-474$ | 0 | 0 | 0 |
| $f_{4,1}$ | $i/2$ | 1 | 0 | 0 | 0 | $-1170$ | 0 | 0 | 0 | 483480 |
| $f_{4,2}$ | $-i/2$ | 1 | 0 | 0 | 0 | 366 | 0 | 0 | 0 | 2934936 |
| $f_{4,3}$ | $-4$ | 1 | 0 | 0 | 0 | 330 | 0 | 0 | 0 | 4767480 |

Table 6.25: Expansions around $P_4 = (2+\sqrt{3})i$ with $\kappa_{P_4} = \frac{2-i}{10\sqrt{6}}\pi_{-4}$

| | $c$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| $f_2$ | $3i\sqrt{5}$ | 1 | $-9$ | 45 | 29160 | 1212570 |
| $f_{4,1}$ | $27\sqrt{5}$ | 1 | $-25$ | $-450$ | $-8775$ | $-4872150$ |
| $f_{4,2}$ | $9i\sqrt{5}$ | 1 | 45 | 2160 | $-93555$ | $-1630530$ |
| $f_{4,3}$ | $9\sqrt{5}$ | 7 | 450 | 8100 | 157950 | $-102145050$ |

Table 6.26: Expansions around $P_{4\cdot 5^2} = i$ with $\kappa_{P_{4\cdot 5^2}} = \frac{i}{30\sqrt[4]{-180}}\pi_{-4}$

| | $c$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| $f_2$ | $-2\sqrt{2}$ | 1 | 0 | 43 | 0 | 29094 | 0 |
| $f_{4,1}$ | $-5\sqrt{5}$ | 1 | 0 | 110 | 0 | $-9630$ | 0 |
| $f_{4,2}$ | $-i\sqrt{3}$ | 1 | 0 | $-914$ | 0 | 285282 | 0 |
| $f_{4,3}$ | $-80i\sqrt{30}$ | 0 | 1 | 0 | $-675$ | 0 | $-181980$ |

Table 6.27: Expansions around $P_{120} = \frac{-1+\sqrt{10}i}{1+2\sqrt{3}}$ with $\kappa_{P_{120}} = \frac{-\sqrt{10}+2\sqrt{30}-10i-2\sqrt{-3}}{1320}\pi_{-120}$

| | $c$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| $f_2$ | $-i\sqrt{2}$ | 1 | 0 | 83 | 0 | $-6666$ |
| $f_{4,1}$ | $20\sqrt{5}$ | 0 | 1 | 0 | 75 | 0 |
| $f_{4,2}$ | $2i\sqrt{2}$ | 1 | 0 | $-84$ | 0 | 14502 |
| $f_{4,3}$ | $10\sqrt{5}$ | 1 | 0 | $-90$ | 0 | 9570 |

Table 6.28: Expansions around $P_{40} = \frac{3+\sqrt{30}i}{4\sqrt{3}-3}$ with $\kappa_{P_{40}} = \frac{12-10\sqrt{3}-3i\sqrt{10}-4i\sqrt{30}}{1560}\pi_{-40}$

| | $c$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| $f_2$ | $-\sqrt{2}$ | 1 | 0 | 13 | 0 | $-138$ |
| $f_{4,1}$ | $-2\sqrt{2}$ | 1 | 0 | 20 | 0 | 5670 |
| $f_{4,2}$ | $-4i\sqrt{3}$ | 0 | 1 | 0 | $-171$ | 0 |
| $f_{4,3}$ | $2i\sqrt{3}$ | 1 | 0 | $-230$ | 0 | 19170 |

Table 6.29: Expansions around $P_{24} = \frac{\sqrt{2}i}{1+\sqrt{3}}$ with $\kappa_{P_{24}} = \frac{3\sqrt{2}-2\sqrt{-2}+2\sqrt{3}-2\sqrt{-3}}{120}\pi_{-24}$

| | $c$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| $f_2$ | $-3i\sqrt{19}$ | 1 | 27 | $-1044$ | 63666 | $-37804968$ |
| $f_{4,1}$ | $513/2$ | 1 | 35 | 2790 | $-599940$ | 4665600 |
| $f_{4,2}$ | $-171/2$ | 1 | $-117$ | 4158 | $-152604$ | $-121803264$ |
| $f_{4,3}$ | $171i\sqrt{19}$ | 1 | $-90$ | 7290 | 588060 | 43545600 |

Table 6.30: Expansions around $P_{19} = \frac{\sqrt{3}-i\sqrt{19}}{-5+\sqrt{3}}$ with $\kappa_{P_{19}} = \frac{-3\left(7+19\sqrt{3}\right)+i\left(9\sqrt{19}-7\sqrt{57}\right)}{37620}\pi_{-19}$

Now we are going to deal with the cases $N = 7, 11$. Both of them are analogous to the cases above.

**Proposition 6.11.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to $\Gamma(6,7)$ is generated by the forms*

$$
\begin{aligned}
f_2 &= \frac{Dt^+_{6,7}}{\sqrt{3t^+_{6,7}}\sqrt{3t^+_{6,7}-3}\sqrt{-t^+_{6,7}+\frac{32}{81}}},\\
f_{4,1} &= \frac{(Dt^+_{6,7})^2}{\sqrt{3t^+_{6,7}}(3t^+_{6,7}-3)(-t^+_{6,7}+\frac{32}{81})},\\
f_{4,2} &= \frac{(Dt^+_{6,7})^2}{3t^+_{6,7}\sqrt{3t^+_{6,7}-3}(-t^+_{6,7}+\frac{32}{81})},\\
f_{4,3} &= \frac{(Dt^+_{6,7})^2}{3t^+_{6,7}(3t^+_{6,7}-3)\sqrt{-t^+_{6,7}+\frac{32}{81}}},\\
f_{6,1} &= \frac{(Dt^+_{6,7})^3}{3t^+_{6,7}(3t^+_{6,7}-3)(-t^+_{6,7}+\frac{32}{81})\sqrt{-t^+_{6,7}+\frac{32}{81}}},\\
f_{6,2} &= \frac{(Dt^+_{6,7})^3}{3t^+_{6,7}(3t^+_{6,7}-3)(-t^+_{6,7}+\frac{32}{81})\sqrt{3t^+_{6,7}-3}},\\
f_{6,3} &= \frac{(Dt^+_{6,7})^3}{3t^+_{6,7}(3t^+_{6,7}-3)(-t^+_{6,7}+\frac{32}{81})\sqrt{3t^+_{6,7}}},\\
f_{6,4} &= \frac{(Dt^+_{6,7})^3}{(3t^+_{6,7}-3)(-t^+_{6,7}+\frac{32}{81})\sqrt{3t^+_{6,7}}\sqrt{3t^+_{6,7}-3}\sqrt{-t^+_{6,7}+\frac{32}{81}}},
\end{aligned}
$$

*which satisfy the relations*

$$
\begin{aligned}
0 &= -f_{4,1}f_{4,2} + f_2 f_{6,1}, & 0 &= f_{4,1}f_{6,2} - f_{4,3}f_{6,4},\\
0 &= -f_{4,1}f_{4,3} + f_2 f_{6,2}, & 0 &= f_{4,3}f_{6,1} - f_{4,2}f_{6,2},\\
0 &= -f_{4,2}f_{4,3} + f_2 f_{6,3}, & 0 &= f_{4,3}f_{6,1} - f_{4,1}f_{6,3},\\
0 &= -f_{4,1}^2 + f_2 f_{6,4}, & 0 &= -f_{4,1}f_{6,1} + f_{4,2}f_{6,4},\\
0 &= 3f_2^4 - f_{4,1}^2 + f_{4,2}^2, & 0 &= -f_{6,1}f_{6,2} + f_{6,3}f_{6,4},\\
0 &= -32f_2^4 + 27f_{4,1}^2 + 81f_{4,3}^2, & &
\end{aligned}
$$

$$0 = -3f_2^3 f_{4,3} + f_{4,1} f_{6,2} - f_{4,2} f_{6,3},$$

$$0 = -(32/27) f_2^3 f_{4,2} + 3 f_{4,3} f_{6,3} + f_{4,2} f_{6,4},$$

$$0 = 3f_2^3 f_{4,1} + f_{4,2} f_{6,1} - f_{4,1} f_{6,4},$$

$$0 = -(32/27) f_2^3 f_{4,1} + 3 f_{4,3} f_{6,2} + f_{4,1} f_{6,4},$$

$$0 = 3f_2^2 f_{4,1} f_{4,3} + f_{6,1} f_{6,3} - f_{6,2} f_{6,4},$$

$$0 = -(32/27) f_2^2 f_{4,1} f_{4,2} + 3 f_{6,2} f_{6,3} + f_{6,1} f_{6,4},$$

$$0 = 3f_2^2 f_{4,1}^2 + f_{6,1}^2 - f_{6,4}^2,$$

$$0 = (32/27) f_2^6 - (32/81) f_2^2 f_{4,1}^2 + (1/3) f_{6,1}^2 + f_{6,3}^2,$$

$$0 = (49/81) f_2^2 f_{4,1}^2 + (1/3) f_{6,1}^2 + f_{6,2}^2.$$

*A basis of eigenforms for* $S_4(\Gamma(6,7))_{\mathbb{Q}}$ *is*

$$(f_2^2,\ f_{4,1},\ f_{4,2},\ f_{4,3})$$

*and one for* $S_6(\Gamma(6,7))_{\mathbb{Q}}$ *is given by*

$$(f_2 f_{4,1},\ f_2 f_{4,2},\ f_2 f_{4,3},\ f_{6,1},\ f_{6,2},\ f_{6,3},\ -3f_{6,4} + 8f_2^3,\ 5f_{6,4} + 34 f_2^3).$$

*The subalgebra of cusp forms with respect to* $\Gamma(6,1)$ *inside* $S_*(\Gamma(6,7))_{\mathbb{Q}}$ *is explicitly determined by the equalities*

$$\begin{aligned} h_4 &= -20/3 f_{4,2} + 18 f_{4,3}, \\ h_6 &= -136/3 f_2^3 - 20/3 f_{6,4} + 18 f_{6,3}, \\ h_{12} &= 1508 f_{6,1} f_{6,3} - 2793 f_{6,1} f_{6,4} - 9099 f_{6,2} f_{6,3} - 5096/3 f_{6,2} f_{6,3}. \end{aligned}$$

*Proof.* In this case, everything can be done as in the two previous cases, except for finding eigenforms in the space $S_6(\langle\Gamma(6,7),\omega_2,\omega_7\rangle)^-_{\mathbb{Q}} = \langle f_2^3, f_{6,4}\rangle_{\mathbb{Q}}$.

In order to do so, we can use the expansions for $t_6^+$ and $t_6^+ \circ \omega$ for $\omega$ corresponding to a set of representatives of $\mathcal{O}(D,1)_1 \backslash \mathcal{O}(D,1)_5$ and then use the relation between $t_6^+$ and $t_{6,7}^+$ to obtain expansions for $t_{6,7}^+$ and $t_{6,7}^+ \circ \omega$. From these we can deduce expansions for $f_2$ and $f_{6,4}$ and $T_5 f_2^3$, $T_5 f_{6,4}$. Comparing those, we find that $-3f_{6,4} + 8f_2^3, 5f_{6,4} + 34 f_2^3$ are eigenforms of eigenvalues $-76$ and 66, respectively.

Similarly, from expansions for $t_{6,7}^+$ and $t_{6,7}^+\circ\omega_5$ we could compute the polynomial given in Proposition 6.7, starting from a polynomial with indeterminate coefficients in the corresponding spaces of cusp forms of weights given in the proposition. Then, the action of $T_5$ can be read from the coefficients. $\square$

**Proposition 6.12.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to $\Gamma(6,11)$ is generated by the eigenforms*

$$f_{2,1}=\frac{Dt_{6,11}^+}{\sqrt{-2t_{6,11}^+ +2\sqrt{-1331(t_{6,11}^+)^2-118t_{6,11}^+-3}}},$$

$$f_{2,2}=\frac{Dt_{6,11}^+}{\sqrt{-t_{6,11}^+}\sqrt{-1331(t_{6,11}^+)^2-118t_{6,11}^+-3}},$$

$$f_{2,3}=\frac{Dt_{6,11}^+}{\sqrt{-t_{6,11}^+}\sqrt{-2t_{6,11}^+ +2\sqrt{-1331(t_{6,11}^+)^2-118t_{6,11}^+-3}}},$$

$$f_4=\frac{(Dt_{6,11}^+)^2}{-t_{6,11}^+(-2t_{6,11}^+ +2)\sqrt{-1331(t_{6,11}^+)^2-118t_{6,11}^+-3}},$$

*which satisfy the relations*

$$0=-2f_{2,1}^2+f_{2,2}^2-2f_{2,3}^2,\qquad 0=f_4^2+3f_{2,3}^4-118f_{2,3}^2f_{2,1}^2+1331f_{2,1}^4.$$

*The subalgebra of cusp forms with respect to $\Gamma(6,1)$ inside $S_*(\Gamma(6,11))_{\mathbb{Q}}$ is explicitly determined by the equalities*

$$\begin{aligned}
h_4 &= 288f_{2,2}f_{2,3}-120f_4,\\
h_6 &= -\,69408f_{2,2}^2f_{2,3}-3024f_{2,2}f_4+139392f_{2,3}^3,\\
h_{12} &= -\,7514726400f_{2,1}f_{2,2}^3f_{2,3}^2-968827392f_{2,1}f_{2,2}^2f_{2,3}f_4\\
&\quad +15736799232f_{2,1}f_{2,2}f_{2,3}^4+38541312f_{2,1}f_{2,2}f_4^2+2348476416f_{2,1}f_{2,3}^3f_4.
\end{aligned}$$

$\square$

### 6.4.2 Cusp forms attached to Shimura curves of discriminant 10

In this section we are going to provide generators for the graded algebras of cusp forms corresponding to the Shimura curves of discriminant 10 and levels

$N = 1, 3, 7, 9$. All the results we give are checked in the same way we dealt with the cases of discriminant 6. We start with the case of level 1.

**Proposition 6.13.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to* $\Gamma(10, 1)$ *is generated by the eigenforms*

$$f_4 = \frac{(Dt_{10}^+)^2}{(-2t_{10}^+)(2t_{10}^+ - 2)(t_{10}^+ - 27/2)},$$
$$f_{6,1} = \frac{(Dt_{10}^+)^3}{-2t_{10}^+(2t_{10}^+ - 2)\sqrt{-2t_{10}^+}\sqrt{2t_{10}^+ - 2}(t_{10}^+ - 27/2)^2},$$
$$f_{6,2} = \frac{(Dt_{10}^+)^3}{-2t_{10}^+(2t_{10}^+ - 2)\sqrt{2t_{10}^+ - 2}(t_{10}^+ - 27/2)^2},$$
$$f_{6,3} = \frac{(Dt_{10}^+)^3}{-2t_{10}^+(2t_{10}^+ - 2)\sqrt{-2t_{10}^+}(t_{10}^+ - 27/2)^2},$$

*which satisfy the relations*

$$0 = f_{6,3}^2 + f_{6,2}^2 + 2f_{6,1}^2, \qquad 0 = f_4^3 + 27/2 f_{6,1}^2 + 1/2 f_{6,2}^2.$$

□

In the case $N = 3$, we have the following result.

**Proposition 6.14.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to* $\Gamma(10, 3)$ *is generated by*

$$f_2 = \frac{(Dt_{10,3}^+)}{(\sqrt{-5t_{10,3}^+}\sqrt{3t_{10,3}^+ - 3}\sqrt{30t_{10,3}^+ - 75/16)}},$$
$$f_{4,1} = \frac{(Dt_{10,3}^+)^2}{-5t_{10,3}^+\sqrt{-5t_{10,3}^+}\sqrt{3t_{10,3}^+ - 3}(30t_{10,3}^+ - 75/16)},$$
$$f_{4,2} = \frac{(Dt_{10,3}^+)^2}{-5t_{10,3}^+\sqrt{-5t_{10,3}^+}(3t_{10,3}^+ - 3)\sqrt{30t_{10,3}^+ - 75/16}},$$
$$f_{4,3} = \frac{(Dt_{10,3}^+)^2}{-5t_{10,3}^+\sqrt{-5t_{10,3}^+}(3t_{10,3}^+ - 3)(30t_{10,3}^+ - 75/16)},$$

$$f_{6,1} = \frac{(Dt^+_{10,3})^3}{(-5t^+_{10,3})^2\sqrt{-5t^+_{10,3}}(3t^+_{10,3}-3)\sqrt{3t^+_{10,3}-3}\sqrt{30t^+_{10,3}-75}},$$
$$f_{6,2} = \frac{(Dt^+_{10,3})^3}{(-5t^+_{10,3})^2\sqrt{-5t^+_{10,3}}(3t^+_{10,3}-3)\sqrt{3t^+_{10,3}-3}(30t^+_{10,3}-75)},$$
$$f_{6,3} = \frac{(Dt^+_{10,3})^3}{(-5t^+_{10,3})^2\sqrt{-5t^+_{10,3}}(3t^+_{10,3}-3)(30t^+_{10,3}-75)},$$
$$f_{6,4} = \frac{(Dt^+_{10,3})^3}{(-5t^+_{10,3})^2\sqrt{-5t^+_{10,3}}(3t^+_{10,3}-3)(30t^+_{10,3}-75)\sqrt{30t^+_{10,3}-75}},$$

*which satisfy the relations*

$$\begin{aligned}
0 &= f_{4,2}^2 - f_2 f_{6,1}, \\
0 &= f_{4,1}f_{4,2} - f_2 f_{6,3}, \\
0 &= f_{4,1}f_{4,3} - f_2 f_{6,4}, \\
0 &= f_{4,2}f_{4,3} - f_2 f_{6,2}, \\
0 &= 3f_2^4 + 5f_{4,1}^2 + 15f_{4,3}^2, \\
0 &= 96f_2^4 + 16f_{4,2}^2 + 75f_{4,3}^2,
\end{aligned}$$

$$\begin{aligned}
0 &= f_{4,2}f_{6,3} - f_{4,1}f_{6,1}, \\
0 &= f_{4,2}f_{6,4} - f_{4,1}f_{6,2}, \\
0 &= -f_{4,3}f_{6,3} + f_{4,2}f_{6,4}, \\
0 &= f_{4,3}f_{6,1} - f_{4,2}f_{6,2}, \\
0 &= f_{6,4}f_{6,1} - f_{6,3}f_{6,2},
\end{aligned}$$

$$\begin{aligned}
0 &= 96f_2^3 f_{4,2} + 16f_{4,2}f_{6,1} + 75f_{4,3}f_{6,2}, \\
0 &= 3f_2^3 f_{4,2} + 5f_{4,1}f_{6,3} + 15f_{4,3}f_{6,2}, \\
0 &= 81f_2^3 f_{4,3} - 25f_{4,1}f_{6,4} + 16f_{4,2}f_{6,2}, \\
0 &= 96f_2^3 f_{6,3} + 16f_{6,3}f_{6,1} + 75f_{6,4}f_{6,2}, \\
0 &= 160f_{6,3}^2 - 16f_{6,1}^2 + 405f_{6,2}^2, \\
0 &= -25f_{6,3}^2 + 81f_2^3 f_{6,1} + 16f_{6,1}^2, \\
0 &= -25f_{6,3}f_{6,4} + 81f_2^3 f_{6,2} + 16f_{6,1}f_{6,2}, \\
0 &= 96f_2^3 f_{4,1} + 75f_{4,3}f_{6,4} + 16f_{4,1}f_{6,1}, \\
0 &= 209952f_2^6 + 23600f_{6,3}^2 + 50625f_{6,4}^2 - 8192f_{6,1}^2.
\end{aligned}$$

□

Next, we move to the case of level 7, for which 16 generators and 104 relations are needed.

**Proposition 6.15.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to $\Gamma(10,7)$ is generated by*

$$f_2 = \frac{Dt^+_{10,7}}{\sqrt{-5t^+_{10,7}}\sqrt{70(t^+_{10,7}-25/32)}\sqrt{7(t^+_{10,7}-1)}},$$

$$f_{4,1} = \frac{(Dt^+_{10,7})^2}{-5t^+_{10,7}(70(t^+_{10,7}-25/32))\sqrt{7(t^+_{10,7}-1)}(t^+_{10,7}-20/27)},$$

$$f_{4,2} = \frac{(Dt^+_{10,7})^2}{-5t^+_{10,7}(70(t^+_{10,7}-25/32))(7(t^+_{10,7}-1))(t^+_{10,7}-20/27)},$$

$$f_{4,3} = \frac{(Dt^+_{10,7})^2}{-5t^+_{10,7}\sqrt{70(t^+_{10,7}-25/32)}\sqrt{7(t^+_{10,7}-1)}(t^+_{10,7}-20/27)},$$

$$f_{4,4} = \frac{(Dt^+_{10,7})^2}{-5t^+_{10,7}\sqrt{70(t^+_{10,7}-25/32)}(7(t^+_{10,7}-1))(t^+_{10,7}-20/27)},$$

$$f_{4,5} = \frac{(Dt^+_{10,7})^2}{\sqrt{-5t^+_{10,7}}(70(t^+_{10,7}-25/32))\sqrt{7(t^+_{10,7}-1)}(t^+_{10,7}-20/27)},$$

$$f_{4,6} = \frac{(Dt^+_{10,7})^2}{\sqrt{-5t^+_{10,7}}(70(t^+_{10,7}-25/32))(7(t^+_{10,7}-1))(t^+_{10,7}-20/27)},$$

$$f_{4,7} = \frac{(Dt^+_{10,7})^2}{\sqrt{-5t^+_{10,7}}\sqrt{70(t^+_{10,7}-25/32)}(7(t^+_{10,7}-1))(t^+_{10,7}-20/27)},$$

$$f_{6,1} = \frac{(Dt^+_{10,7})^3}{-5t^+_{10,7}(70(t^+_{10,7}-25/32))(7(t^+_{10,7}-1))(t^+_{10,7}-20/27)^2},$$

$$f_{6,2} = \frac{(Dt^+_{10,7})^3}{-5t^+_{10,7}(70(t^+_{10,7}-25/32))(7(t^+_{10,7}-1))\sqrt{7(t^+_{10,7}-1)}(t^+_{10,7}-20/27)^2},$$

$$f_{6,3} = \frac{(Dt^+_{10,7})^3}{-5t^+_{10,7}(70(t^+_{10,7}-25/32))\sqrt{70(t^+_{10,7}-25/32)}(7(t^+_{10,7}-1))(t^+_{10,7}-20/27)^2},$$

$$f_{6,4} = \frac{(Dt^+_{10,7})^3}{-5t^+_{10,7}(70(t^+_{10,7}-25/32))\sqrt{70(t^+_{10,7}-25/32)}(7(t^+_{10,7}-1))\sqrt{7(t^+_{10,7}-1)}(t^+_{10,7}-20/27)^2},$$

$$f_{6,5} = \frac{(Dt^+_{10,7})^3}{-5t^+_{10,7}\sqrt{-5t^+_{10,7}}(70(t^+_{10,7}-25/32))(7(t^+_{10,7}-1))\sqrt{7(t^+_{10,7}-1)}(t^+_{10,7}-20/27)^2},$$

$$f_{6,6} = \frac{(Dt^+_{10,7})^3}{-5t^+_{10,7}\sqrt{-5t^+_{10,7}}(70(t^+_{10,7}-25/32))(7(t^+_{10,7}-1))(t^+_{10,7}-20/27)^2},$$

$$f_{6,7} = \frac{(Dt^+_{10,7})^3}{-5t^+_{10,7}\sqrt{-5t^+_{10,7}}(70(t^+_{10,7}-25/32))\sqrt{70(t^+_{10,7}-25/32)}(7(t^+_{10,7}-1))\sqrt{7(t^+_{10,7}-1)}} \cdot \frac{1}{(t^+_{10,7}-20/27)^2},$$

$$f_{6,8} = \frac{(Dt^+_{10,7})^3}{-5t^+_{10,7}\sqrt{-5t^+_{10,7}}\sqrt{70(t^+_{10,7}-25/32)}(7(t^+_{10,7}-1))(t^+_{10,7}-20/27)^2},$$

*which satisfy the relations*

$$\begin{array}{lll}
0 = f_{4,2}f_{4,7} - f_2 f_{6,2}, & 0 = f_{4,3}f_{6,6} - f_{4,1}f_{6,8}, & 0 = f_{4,3}f_{6,2} - f_{4,6}f_{6,8}, \\
0 = f_{4,1}f_{4,7} - f_2 f_{6,1}, & 0 = f_{4,2}f_{6,6} - f_{4,3}f_{6,7}, & 0 = f_{4,2}f_{6,2} - f_{4,7}f_{6,7}, \\
0 = f_{4,4}f_{4,6} - f_2 f_{6,2}, & 0 = f_{4,6}f_{6,5} - f_{4,7}f_{6,7}, & 0 = f_{4,1}f_{6,2} - f_{4,6}f_{6,6}, \\
0 = f_{4,3}f_{4,6} - f_2 f_{6,1}, & 0 = f_{4,5}f_{6,5} - f_{4,6}f_{6,6}, & 0 = f_{4,4}f_{6,1} - f_{4,6}f_{6,8}, \\
0 = f_{4,2}f_{4,6} - f_2 f_{6,4}, & 0 = f_{4,3}f_{6,5} - f_{4,2}f_{6,8}, & 0 = f_{4,3}f_{6,1} - f_{4,5}f_{6,8}, \\
0 = f_{4,1}f_{4,6} - f_2 f_{6,3}, & 0 = f_{4,2}f_{6,5} - f_{4,4}f_{6,7}, & 0 = f_{4,2}f_{6,1} - f_{4,6}f_{6,6}, \\
0 = f_{4,4}f_{4,5} - f_2 f_{6,1}, & 0 = f_{4,1}f_{6,5} - f_{4,3}f_{6,7}, & 0 = f_{4,1}f_{6,1} - f_{4,5}f_{6,6}, \\
0 = f_{4,2}f_{4,5} - f_2 f_{6,3}, & 0 = f_{4,4}f_{6,4} - f_{4,7}f_{6,7}, & 0 = f_{6,5}f_{6,6} - f_{6,7}f_{6,8}, \\
0 = f_{4,3}f_{4,4} - f_2 f_{6,8}, & 0 = f_{4,3}f_{6,4} - f_{4,6}f_{6,6}, & 0 = f_{6,4}f_{6,6} - f_{6,1}f_{6,7}, \\
0 = f_{4,2}f_{4,4} - f_2 f_{6,5}, & 0 = f_{4,2}f_{6,4} - f_{4,6}f_{6,7}, & 0 = f_{6,2}f_{6,6} - f_{6,4}f_{6,8}, \\
0 = f_{4,1}f_{4,4} - f_2 f_{6,6}, & 0 = f_{4,1}f_{6,4} - f_{4,5}f_{6,7}, & 0 = f_{6,1}f_{6,6} - f_{6,3}f_{6,8}, \\
0 = f_{4,2}f_{4,3} - f_2 f_{6,6}, & 0 = f_{4,4}f_{6,3} - f_{4,6}f_{6,6}, & 0 = f_{6,4}f_{6,5} - f_{6,2}f_{6,7}, \\
0 = f_{4,2}^2 - f_2 f_{6,7}, & 0 = f_{4,3}f_{6,3} - f_{4,5}f_{6,6}, & 0 = f_{6,3}f_{6,5} - f_{6,1}f_{6,7}, \\
0 = f_{4,7}f_{6,6} - f_{4,6}f_{6,8}, & 0 = f_{4,2}f_{6,3} - f_{4,5}f_{6,7}, & 0 = f_{6,1}f_{6,5} - f_{6,4}f_{6,8}, \\
0 = f_{4,4}f_{6,6} - f_{4,2}f_{6,8}, & 0 = f_{4,4}f_{6,2} - f_{4,7}f_{6,5}, &
\end{array}$$

$$\begin{array}{ll}
0 = 30240 f_2^2 f_{4,6} - 432 f_{4,4}f_{4,7} - 1225 f_2 f_{6,4}, & 0 = 80 f_{4,3}^2 - 7840 f_{4,6}^2 + 112 f_{4,7}^2 - 30625 f_2 f_{6,7}, \\
0 = 160 f_{4,5}^2 + 245 f_{4,6}^2 - 16 f_{4,7}^2, & 0 = 5 f_{4,1}f_{4,3} + 7 f_{4,6}f_{4,7} + 35 f_2 f_{6,5}, \\
0 = 27 f_{4,3}f_{4,5} - 189 f_2^2 f_{4,7} + 49 f_2 f_{6,2}, & 0 = 135 f_2^2 f_{4,3} + 27 f_{4,5}f_{4,7} + 100 f_2 f_{6,6}, \\
0 = 160 f_{4,1}f_{4,5} - 16 f_{4,4}f_{4,7} + 245 f_2 f_{6,4}, & 0 = 875 f_{4,1}f_{4,2} + 224 f_{4,5}f_{4,6} + 16 f_2 f_{6,8}, \\
0 = 30240 f_2^2 f_{4,5} - 432 f_{4,3}f_{4,7} - 1225 f_2 f_{6,3}, & 0 = 135 f_2^2 f_{4,2} + 27 f_{4,6}^2 + 100 f_2 f_{6,7}, \\
0 = 16 f_{4,4}^2 + 224 f_{4,6}^2 + 875 f_2 f_{6,7}, & 0 = 5 f_{4,1}^2 + 7 f_{4,6}^2 + 35 f_2 f_{6,7}, \\
0 = 135 f_2^2 f_{4,4} + 27 f_{4,6}f_{4,7} + 100 f_2 f_{6,5}, & 0 = 4725 f_2^2 f_{4,1} + 49 f_{4,5}f_{4,6} - 64 f_2 f_{6,8}, \\
 & 0 = 30240 f_2^2 f_{6,6} - 1225 f_{4,3}f_{6,7} - 432 f_{4,4}f_{6,8},
\end{array}$$

$$0 = 432f_{4,4}f_{6,5} - 30240f_2^2f_{6,7} + 1225f_{4,2}f_{6,7},$$
$$0 = 189f_2^2f_{6,5} - 27f_{4,1}f_{6,6} - 49f_{4,4}f_{6,7},$$
$$0 = 7f_{4,7}f_{6,4} + 5f_{4,1}f_{6,6} + 35f_{4,4}f_{6,7},$$
$$0 = 27f_{4,6}f_{6,4} + 135f_2^2f_{6,7} + 100f_{4,2}f_{6,7},$$
$$0 = 224f_{4,5}f_{6,4} + 875f_{4,1}f_{6,7} + 16f_{4,2}f_{6,8},$$
$$0 = 30240f_2^2f_{6,4} - 432f_{4,7}f_{6,5} - 1225f_{4,6}f_{6,7},$$
$$0 = 224f_{4,7}f_{6,3} + 875f_{4,3}f_{6,7} + 16f_{4,4}f_{6,8},$$
$$0 = 224f_{4,6}f_{6,3} + 875f_{4,1}f_{6,7} + 16f_{4,2}f_{6,8},$$
$$0 = 160f_{4,1}f_{6,3} - 16f_{4,7}f_{6,5} + 245f_{4,6}f_{6,7},$$
$$0 = 30240f_2^2f_{6,3} - 1225f_{4,5}f_{6,7} - 432f_{4,6}f_{6,8},$$
$$0 = 7f_{4,6}f_{6,2} + 5f_{4,1}f_{6,6} + 35f_{4,4}f_{6,7},$$
$$0 = 224f_{4,5}f_{6,2} + 875f_{4,3}f_{6,7} + 16f_{4,4}f_{6,8},$$
$$0 = 189f_2^2f_{6,2} - 27f_{4,5}f_{6,6} - 49f_{4,7}f_{6,7},$$
$$0 = 27f_{4,7}f_{6,1} + 135f_2^2f_{6,8} + 100f_{4,2}f_{6,8},$$
$$0 = 224f_{4,6}f_{6,1} + 875f_{4,3}f_{6,7} + 16f_{4,4}f_{6,8},$$
$$0 = 224f_{4,5}f_{6,1} + 875f_{4,1}f_{6,6} + 16f_{4,3}f_{6,8},$$
$$0 = 30240f_2^2f_{6,1} - 1225f_{4,6}f_{6,6} - 432f_{4,7}f_{6,8},$$
$$0 = 189f_2f_{4,4}f_{4,7} - 49f_{4,7}f_{6,5} - 27f_{4,5}f_{6,8},$$
$$0 = 1225f_{6,6}^2 - 30240f_2f_{4,1}f_{6,8} + 432f_{6,8}^2,$$
$$0 = 27f_{6,3}f_{6,6} - 189f_2f_{4,7}f_{6,7} + 49f_{6,2}f_{6,7},$$
$$0 = 432f_{6,5}^2 - 30240f_2f_{4,2}f_{6,7} + 1225f_{6,7}^2,$$
$$0 = 27f_{6,4}^2 + 135f_2f_{4,2}f_{6,7} + 100f_{6,7}^2,$$
$$0 = 49f_{6,3}f_{6,4} + 4725f_2f_{4,1}f_{6,7} - 64f_{6,7}f_{6,8},$$
$$0 = 27f_{6,2}f_{6,4} + 135f_2f_{4,4}f_{6,7} + 100f_{6,5}f_{6,7},$$
$$0 = 224f_{6,1}f_{6,4} + 875f_{6,6}f_{6,7} + 16f_{6,5}f_{6,8},$$
$$0 = 224f_{6,2}f_{6,3} + 875f_{6,6}f_{6,7} + 16f_{6,5}f_{6,8},$$
$$0 = 27f_{6,1}f_{6,2} + 135f_2f_{4,2}f_{6,8} + 100f_{6,7}f_{6,8},$$
$$0 = 49f_{6,1}^2 + 4725f_2f_{4,1}f_{6,8} - 64f_{6,8}^2,$$

$$0 = 30240f_2f_{4,5}f_{6,7} - 1225f_{6,3}f_{6,7} - 432f_{6,4}f_{6,8},$$
$$0 = 30240f_2f_{4,3}f_{6,7} - 1225f_{6,6}f_{6,7} - 432f_{6,5}f_{6,8},$$
$$0 = 30240f_2f_{4,6}f_{6,6} - 1225f_{6,1}f_{6,7} - 432f_{6,2}f_{6,8},$$
$$0 = 432f_{6,2}f_{6,5} - 30240f_2f_{4,6}f_{6,7} + 1225f_{6,4}f_{6,7},$$
$$0 = 4082400f_2^4 - 571725f_{4,6}^2 + 11664f_{4,7}^2 - 2240000f_2f_{6,7},$$
$$0 = 6048f_{4,5}f_{6,3} + 165375f_2^2f_{6,7} - 42875f_{4,2}f_{6,7} + 432f_{4,1}f_{6,8},$$
$$0 = 3024f_{4,7}f_{6,2} + 1058400f_2^2f_{6,7} - 42875f_{4,2}f_{6,7} + 2160f_{4,1}f_{6,8},$$
$$0 = 42336f_2f_{4,6}f_{4,7} + 31465f_{4,1}f_{6,6} + 54880f_{4,4}f_{6,7} + 432f_{4,3}f_{6,8},$$
$$0 = 1143072f_2f_{4,6}^2 + 5946885f_2^2f_{6,7} - 60025f_{4,2}f_{6,7} + 11664f_{4,1}f_{6,8},$$
$$0 = 193536f_2f_{4,5}f_{4,6} + 30625f_{4,1}f_{6,7} + 13824f_2^2f_{6,8} + 10800f_{4,2}f_{6,8},$$
$$0 = 7001316000f_2^3f_{6,7} - 2098774125f_2f_{4,2}f_{6,7} + 73530625f_{6,7}^2 - 352719360f_2f_{4,1}f_{6,8} + 5038848f_{6,8}^2,$$
$$0 = 816480f_2f_{4,5}f_{6,6} - 231525f_2f_{4,7}f_{6,7} + 60025f_{6,2}f_{6,7} - 11664f_{6,1}f_{6,8},$$
$$0 = 816480f_2f_{4,1}f_{6,6} - 231525f_2f_{4,4}f_{6,7} + 60025f_{6,5}f_{6,7} - 11664f_{6,6}f_{6,8},$$
$$0 = 81648f_2f_{4,7}f_{6,5} - 1481760f_2f_{4,6}f_{6,7} + 60025f_{6,4}f_{6,7} - 11664f_{6,3}f_{6,8},$$

$$0 = 7408800f_{6,3}^2 + 202584375f_2f_{4,2}f_{6,7} - 52521875f_{6,7}^2 + 13063680f_2f_{4,1}f_{6,8} - 186624f_{6,8}^2,$$

$$0 = 6048f_{6,1}f_{6,3} + 165375f_2f_{4,4}f_{6,7} - 42875f_{6,5}f_{6,7} + 432f_{6,6}f_{6,8},$$

$$0 = 740880f_{6,2}^2 + 259308000f_2f_{4,2}f_{6,7} - 10504375f_{6,7}^2 + 13063680f_2f_{4,1}f_{6,8} - 186624f_{6,8}^2. \qquad \square$$

Finally, we deal with the case $N = 9$. Note that a canonical model for the curve $X(10,9)^+$, which we did not give in Section 4.3.8, can be deduced from the equations given in the proposition.

**Proposition 6.16.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to $\Gamma(10,9)$ is generated by the eigenforms*

$$f_{2,1} = \frac{Dt_{10,9}^+}{\sqrt{t_{10,9}^+ + 3}\sqrt{t_{10,9}^+ - 6}\sqrt{t_{10,9}^+ + 2}\sqrt{t_{10,9}^+ - 1}},$$

$$f_{2,2} = \frac{Dt_{10,9}^+}{\sqrt{t_{10,9}^+ - 6}\sqrt{t_{10,9}^+ - 1}\sqrt{-3((t_{10,9}^+)^2 - 3t_{10,9}^+ + 6}},$$

$$f_{2,3} = \frac{Dt_{10,9}^+}{\sqrt{t_{10,9}^+ + 2}\sqrt{t_{10,9}^+ - 1}\sqrt{-3((t_{10,9}^+)^2 - 3t_{10,9}^+ + 6}},$$

$$f_{2,4} = \frac{Dt_{10,9}^+}{\sqrt{t_{10,9}^+ + 3}\sqrt{t_{10,9}^+ + 2}\sqrt{-3((t_{10,9}^+)^2 - 3t_{10,9}^+ + 6}},$$

$$f_{2,5} = \frac{Dt_{10,9}^+}{\sqrt{t_{10,9}^+ + 3}\sqrt{t_{10,9}^+ - 6}\sqrt{-3((t_{10,9}^+)^2 - 3t_{10,9}^+ + 6}},$$

*which satisfy the relations*

$$0 = -3f_{2,3}^2 - 5f_{2,2}^2 - f_{2,4}^2 + 9f_{2,5}^2, \quad 0 = f_{2,2}^2 + 2f_{2,4}^2 + f_{2,1}^2, \quad 0 = f_{2,2}f_{2,4} - f_{2,3}f_{2,5}.$$

$\square$

### 6.4.3 Cusp forms attached to Shimura curves of discriminant 22

Now we move on to the case of discriminant 22 and levels $N = 1, 3, 5, 7$. Everything will follow the same pattern as for the other discriminants we have considered. We start with the case of level 1.

**Proposition 6.17.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to $\Gamma(22,1)$ is generated by the eigenforms*

$$f_{4,1} = \frac{(Dt_{22}^+)^2}{-t_{22}^+(11t_{22}^+-11)(t_{22}^+-27/16)},$$
$$f_{4,2} = \frac{(Dt_{22}^+)^2}{-t_{22}^+(11t_{22}^+-11)\sqrt{11t_{22}^+-11}(t_{22}^+-27/16)},$$
$$f_{4,3} = \frac{(Dt_{22}^+)^2}{\sqrt{-t_{22}^+}(11t_{22}^+-11)\sqrt{11t_{22}^+-11}(t_{22}^+-27/16)},$$
$$f_{6,1} = \frac{(Dt_{22}^+)^3}{-t_{22}^+\sqrt{-t_{22}^+}(11t_{22}^+-11)^2(t_{22}^+-27/16)^2},$$
$$f_{6,2} = \frac{(Dt_{22}^+)^3}{-t_{22}^+\sqrt{-t_{22}^+}(11t_{22}^+-11)(t_{22}^+-27/16)^2},$$
$$f_{6,3} = \frac{(Dt_{22}^+)^3}{-t_{22}^+(11t_{22}^+-11)\sqrt{11t_{22}^+-11}(t_{22}^+-27/16)^2},$$
$$f_{6,4} = \frac{(Dt_{22}^+)^3}{-t_{22}^+(11t_{22}^+-11)^2(t_{22}^+-27/16)^2},$$
$$f_{6,5} = \frac{(Dt_{22}^+)^3}{-t_{22}^+\sqrt{-t_{22}^+}(11t_{22}^+-11)\sqrt{11t_{22}^+-11}(t_{22}^+-27/16)^2},$$

*which satisfy the relations*

$$0 = 11f_{4,3}^2 + f_{4,1}^2 + 11f_{4,2}^2,$$
$$0 = -f_{4,2}f_{6,2} + f_{4,1}f_{6,5},$$
$$0 = -f_{4,2}f_{6,5} + f_{4,1}f_{6,1},$$
$$0 = f_{4,2}f_{6,4} - f_{4,3}f_{6,1},$$
$$0 = f_{4,1}f_{6,4} - f_{4,3}f_{6,5},$$
$$0 = -f_{4,2}f_{6,3} + f_{4,1}f_{6,4},$$
$$0 = f_{4,1}f_{6,3} - f_{4,3}f_{6,2},$$
$$0 = 11f_{4,3}f_{6,4} + f_{4,1}f_{6,5} + 11f_{4,2}f_{6,1},$$
$$0 = 11f_{4,3}f_{6,3} + f_{4,1}f_{6,2} + 11f_{4,2}f_{6,5},$$
$$0 = -f_{6,5}^2 + f_{6,2}f_{6,1},$$
$$0 = -f_{6,4}f_{6,5} + f_{6,3}f_{6,1},$$
$$0 = f_{6,4}f_{6,2} - f_{6,3}f_{6,5},$$
$$0 = -11f_{6,3}^2 + 121f_{6,4}^2 - f_{6,2}^2 + 121f_{6,1}^2,$$
$$0 = 11f_{6,3}f_{6,4} + f_{6,2}f_{6,5} + 11f_{6,5}f_{6,1},$$
$$0 = 11f_{6,3}^2 + f_{6,2}^2 + 11f_{6,5}^2,$$
$$0 = 176f_{4,1}^3 - 121f_{6,3}^2 - 27f_{6,2}^2,$$

$$0 = 176f_{4,1}^2f_{4,2} - 121f_{6,3}f_{6,4} - 27f_{6,2}f_{6,5},$$
$$0 = 1936f_{4,1}f_{4,2}^2 + 297f_{6,3}^2 - 1331f_{6,4}^2 + 27f_{6,2}^2,$$

$$
\begin{aligned}
0 &= 176 f_{4,3} f_{4,1} f_{4,2} - 16 f_{6,4} f_{6,2} + 121 f_{6,4} f_{6,1},\\
0 &= 176 f_{4,3} f_{4,1}^2 - 16 f_{6,3} f_{6,2} + 121 f_{6,3} f_{6,1}. \qquad \square
\end{aligned}
$$

Next, we give the result for level $N = 3$, where we already need 12 generators and 67 relations.

**Proposition 6.18.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to $\Gamma(22,3)$ is generated by*

$$
\begin{aligned}
f_{2,1} &= \frac{Dt_{22,3}^{+}}{\sqrt{-2t_{22,3}^{+}-1}\sqrt{-11(t_{22,3}^{+})^2+2t_{22}^{+}-3}},\\
f_{2,2} &= \frac{Dt_{22,3}^{+}}{\sqrt{-2t_{22,3}^{+}-1}\sqrt{-2t_{22,3}^{+}+3}\sqrt{-11(t_{22,3}^{+})^2+2t_{22}^{+}-3}},\\
f_{2,3} &= \frac{Dt_{22,3}^{+}}{\sqrt{-2t_{22,3}^{+}+3}\sqrt{-11(t_{22,3}^{+})^2+2t_{22}^{+}-3}},\\
f_{4,1} &= \frac{(Dt_{22,3}^{+})^2}{(-2t_{22,3}^{+}-1)\sqrt{-2t_{22,3}^{+}-1}\sqrt{-2t_{22,3}^{+}+3}\sqrt{-11(t_{22,3}^{+})^2+2t_{22}^{+}-3}},\\
f_{4,2} &= \frac{(Dt_{22,3}^{+})^2}{(-2t_{22,3}^{+}-1)\sqrt{-2t_{22,3}^{+}-1}(-2t_{22,3}^{+}+3)\sqrt{-11(t_{22,3}^{+})^2+2t_{22}^{+}-3}},\\
f_{4,3} &= \frac{(Dt_{22,3}^{+})^2}{(-2t_{22,3}^{+}-1)(-2t_{22,3}^{+}+3)\sqrt{-11(t_{22,3}^{+})^2+2t_{22}^{+}-3}},\\
f_{4,4} &= \frac{(Dt_{22,3}^{+})^2}{(-2t_{22,3}^{+}-1)\sqrt{-2t_{22,3}^{+}-1}\sqrt{-2t_{22,3}^{+}+3}(-11(t_{22,3}^{+})^2+2t_{22}^{+}-3)},\\
f_{4,5} &= \frac{(Dt_{22,3}^{+})^2}{(-2t_{22,3}^{+}-1)\sqrt{-2t_{22,3}^{+}-1}(-2t_{22,3}^{+}+3)(-11(t_{22,3}^{+})^2+2t_{22}^{+}-3)},\\
f_{6,1} &= \frac{(Dt_{22,3}^{+})^3}{(-2t_{22,3}^{+}-1)^2\sqrt{-2t_{22,3}^{+}-1}\sqrt{-2t_{22,3}^{+}+3}(-11(t_{22,3}^{+})^2+2t_{22}^{+}-3)\sqrt{-11(t_{22,3}^{+})^2+2t_{22}^{+}-3}},\\
f_{6,2} &= \frac{(Dt_{22,3}^{+})^3}{(-2t_{22,3}^{+}-1)^2\sqrt{-2t_{22,3}^{+}-1}(-2t_{22,3}^{+}+3)(-11(t_{22,3}^{+})^2+2t_{22}^{+}-3)\sqrt{-11(t_{22,3}^{+})^2+2t_{22}^{+}-3}},\\
f_{6,3} &= \frac{(Dt_{22,3}^{+})^3}{(-2t_{22,3}^{+}-1)^2\sqrt{-2t_{22,3}^{+}-1}(-2t_{22,3}^{+}+3)(-11(t_{22,3}^{+})^2+2t_{22}^{+}-3)},\\
f_{6,4} &= \frac{(Dt_{22,3}^{+})^3}{(-2t_{22,3}^{+}-1)^2\sqrt{-2t_{22,3}^{+}-1}(-2t_{22,3}^{+}+3)\sqrt{-2t_{22,3}^{+}+3}(-11(t_{22,3}^{+})^2+2t_{22}^{+}-3)},
\end{aligned}
$$

*which satisfy the relations*

$$0 = -f_{2,3}^2 + f_{2,1}^2 - 4f_{2,2}^2,$$
$$0 = f_{2,2}^3 - f_{2,3}f_{4,5},$$
$$0 = f_{2,1}f_{2,2}^2 - f_{2,3}f_{4,4},$$
$$0 = f_{2,2}f_{4,3} - f_{2,3}f_{4,2},$$
$$0 = f_{2,1}f_{4,3} - f_{2,3}f_{4,1},$$
$$0 = -f_{2,2}f_{4,4} + f_{2,1}f_{4,5},$$
$$0 = -f_{2,2}f_{4,1} + f_{2,1}f_{4,2},$$
$$0 = f_{2,3}f_{2,2}^2 - f_{2,1}f_{4,4} + 4f_{2,2}f_{4,5},$$
$$0 = f_{2,3}f_{4,3} - f_{2,1}f_{4,1} + 4f_{2,2}f_{4,2},$$
$$0 = -f_{2,2}f_{6,3} + f_{2,1}f_{6,4},$$
$$0 = -f_{4,4}f_{4,2} + f_{2,1}f_{6,4},$$
$$0 = -f_{4,1}f_{4,5} + f_{2,1}f_{6,4},$$
$$0 = -f_{2,2}f_{6,1} + f_{2,1}f_{6,2},$$
$$0 = -f_{4,4}^2 + f_{2,1}f_{6,2},$$
$$0 = -f_{4,4}f_{4,5} + f_{2,2}f_{6,2},$$
$$0 = -f_{4,2}f_{4,5} + f_{2,2}f_{6,4},$$
$$0 = -f_{4,3}f_{4,5} + f_{2,3}f_{6,4},$$
$$0 = f_{4,3}f_{4,4} - f_{2,3}f_{6,3},$$
$$0 = -f_{4,1}f_{4,4} + f_{2,1}f_{6,3},$$
$$0 = -f_{2,1}f_{2,2}f_{4,2} + f_{2,3}f_{6,3},$$
$$0 = -f_{2,1}f_{2,2}f_{4,5} + f_{2,3}f_{6,2},$$
$$0 = -f_{2,2}^2f_{4,2} + f_{2,3}f_{6,4},$$
$$0 = -f_{2,1}^2f_{4,5} + f_{2,3}f_{6,1},$$
$$0 = f_{2,2}^2f_{4,3} - f_{2,1}f_{6,3} + 4f_{2,2}f_{6,4},$$
$$0 = f_{4,3}^2 - f_{4,1}^2 + 4f_{4,2}^2,$$
$$0 = f_{2,1}f_{2,2}^3 - f_{2,1}f_{6,1} + 4f_{2,2}f_{6,2},$$
$$0 = 11f_{2,3}^4 + 26f_{2,3}^2f_{2,2}^2 + 27f_{2,2}^4 + 4f_{4,3}^2,$$
$$0 = f_{2,1}f_{2,2}^4 - f_{2,3}^2f_{6,2},$$
$$0 = f_{2,2}^3f_{4,3} - f_{2,3}^2f_{6,4},$$
$$0 = f_{4,2}f_{6,1} - f_{4,1}f_{6,2},$$
$$0 = f_{4,2}f_{6,1} - f_{4,4}f_{6,3},$$
$$0 = f_{4,5}f_{6,1} - f_{4,4}f_{6,2},$$
$$0 = -f_{4,5}f_{6,3} + f_{4,2}f_{6,2},$$
$$0 = f_{4,2}f_{6,2} - f_{4,4}f_{6,4},$$
$$0 = f_{4,2}f_{6,3} - f_{4,1}f_{6,4},$$
$$0 = -f_{4,3}f_{6,2} + f_{2,1}f_{2,2}f_{6,4},$$
$$0 = -f_{4,1}f_{6,1} + 4f_{4,2}f_{6,2} + f_{2,3}f_{2,1}f_{6,4},$$
$$0 = f_{2,3}^2f_{2,2}^3 + 4f_{2,2}^5 - f_{2,3}^2f_{6,1},$$
$$0 = -f_{4,4}f_{6,3} + f_{2,3}f_{2,2}f_{6,4} + 4f_{4,5}f_{6,4},$$
$$0 = -f_{4,3}f_{6,1} + f_{2,3}^2f_{6,4} + 4f_{2,2}^2f_{6,4},$$
$$0 = -f_{6,3}f_{6,2} + f_{6,1}f_{6,4},$$
$$0 = -f_{6,1}f_{6,3} + f_{2,3}f_{4,4}f_{6,4} + 4f_{6,2}f_{6,4},$$
$$0 = -f_{6,1}^2 + f_{2,3}f_{4,4}f_{6,2} + 4f_{6,2}^2,$$
$$0 = -286f_{2,3}^4 - 379f_{2,3}^2f_{2,2}^2 - 131f_{4,3}^2 + 27f_{4,1}^2 + 729f_{4,5}^2,$$
$$0 = 44f_{2,3}^2f_{2,1}f_{2,2} + 77f_{2,1}f_{2,2}^3 + 16f_{4,1}f_{4,2} + 27f_{2,1}f_{6,1},$$
$$0 = -847f_{2,3}^4 - 814f_{2,3}^2f_{2,2}^2 - 416f_{4,3}^2 + 108f_{4,1}^2 + 729f_{2,1}f_{6,2},$$
$$0 = 44f_{2,3}^3f_{2,2} + 77f_{2,3}f_{2,2}^3 + 16f_{4,3}f_{4,2} + 27f_{2,3}f_{6,1},$$
$$0 = 11f_{2,3}^3f_{2,1} + 26f_{2,3}f_{2,1}f_{2,2}^2 + 4f_{4,3}f_{4,1} + 27f_{2,3}f_{6,2},$$
$$0 = -f_{2,3}f_{2,1}f_{2,2}^3 + f_{2,3}f_{2,1}f_{6,1} - 4f_{4,4}f_{6,1} + 16f_{4,5}f_{6,2},$$

$$0 = 99f_{2,3}f_{2,1}f_{2,2}^3 + 77f_{2,3}f_{2,1}f_{6,1} + 108f_{4,4}f_{6,1} + 64f_{4,2}f_{6,3},$$

$$0 = -11f_{2,3}^4f_{2,1} - 26f_{2,3}^2f_{2,1}f_{2,2}^2 - 27f_{2,1}f_{2,2}^4 - 4f_{2,1}f_{4,1}^2 + 16f_{4,3}f_{6,3},$$

$$0 = -847f_{2,3}^3f_{2,2}^2 - 814f_{2,3}f_{2,2}^4 + 729f_{4,5}f_{6,1} - 308f_{4,1}f_{6,3} + 1664f_{4,2}f_{6,4},$$

$$0 = -99f_{2,3}f_{4,5}^2 + 44f_{2,3}f_{2,1}f_{6,2} + 27f_{4,4}f_{6,2} + 16f_{4,2}f_{6,4},$$

$$0 = 99f_{2,2}f_{4,5}^2 + 11f_{2,3}^2f_{6,1} - 18f_{2,1}f_{2,2}f_{6,2} + 4f_{4,3}f_{6,4},$$

$$0 = 27f_{6,1}^2 + 44f_{2,3}^2f_{2,1}f_{6,2} + 77f_{2,3}f_{4,4}f_{6,2} + 16f_{2,2}f_{4,3}f_{6,4} + 64f_{6,4}^2,$$

$$0 = 44f_{2,3}^2f_{2,2}f_{6,2} + 77f_{2,3}f_{4,5}f_{6,2} + 27f_{6,1}f_{6,2} + 16f_{6,3}f_{6,4},$$

$$0 = 27f_{6,1}^2 + 16f_{6,3}^2 + 44f_{2,3}^2f_{2,1}f_{6,2} + 77f_{2,3}f_{4,4}f_{6,2}. \qquad \square$$

The result for $N = 5$ is very similar to the previous one, as can be seen in the following proposition.

**Proposition 6.19.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to $\Gamma(22, 5)$ is generated by*

$$f_{2,1} = \frac{Dt^+_{22,5}}{\sqrt{1375(t^+_{22,5})^3 - 3650(t^+_{22,5})^2 + 3295t^+_{22,5} - 1024}},$$

$$f_{2,2} = \frac{Dt^+_{22,5}}{\sqrt{-5t^+_{22,5}}\sqrt{-5t^+_{22,5}4}\sqrt{1375(t^+_{22,5})^3 - 3650(t^+_{22,5})^2 + 3295t^+_{22,5} - 1024}},$$

$$f_{2,3} = \frac{Dt^+_{22,5}}{\sqrt{-5t^+_{22,5}}\sqrt{1375(t^+_{22,5})^3 - 3650(t^+_{22,5})^2 + 3295t^+_{22,5} - 1024}},$$

$$f_{2,4} = \frac{Dt^+_{22,5}}{\sqrt{-5t^+_{22,5}4}\sqrt{1375(t^+_{22,5})^3 - 3650(t^+_{22,5})^2 + 3295t^+_{22,5} - 1024}},$$

$$f_{2,5} = \frac{\sqrt{-5t^+_{22,5}}(Dt^+_{22,5})}{\sqrt{-5t^+_{22,5}4}\sqrt{1375(t^+_{22,5})^3 - 3650(t^+_{22,5})^2 + 3295t^+_{22,5} - 1024}},$$

$$f_{4,1} = \frac{(Dt^+_{22,5})^2}{-5t^+_{22,5}\sqrt{-5t^+_{22,5}}\sqrt{-5t^+_{22,5}4}(1375(t^+_{22,5})^3 - 3650(t^+_{22,5})^2 + 3295t^+_{22,5} - 1024)},$$

$$f_{4,2} = \frac{(Dt^{+}_{22,5})^2}{-5t^{+}_{22,5}\sqrt{-5t^{+}_{22,5}}(1375(t^{+}_{22,5})^3 - 3650(t^{+}_{22,5})^2 + 3295t^{+}_{22,5} - 1024)},$$

$$f_{4,3} = \frac{(Dt^{+}_{22,5})^2}{\sqrt{-5t^{+}_{22,5}}(-5t^{+}_{22,5}4)\sqrt{1375(t^{+}_{22,5})^3 - 3650(t^{+}_{22,5})^2 + 3295t^{+}_{22,5} - 1024}},$$

$$f_{4,4} = \frac{(Dt^{+}_{22,5})^2}{-5t^{+}_{22,5}\sqrt{-5t^{+}_{22,5}}\sqrt{1375(t^{+}_{22,5})^3 - 3650(t^{+}_{22,5})^2 + 3295t^{+}_{22,5} - 1024}},$$

$$f_{4,5} = \frac{(Dt^{+}_{22,5})^2}{-5t^{+}_{22,5}\sqrt{-5t^{+}_{22,5}4}\sqrt{1375(t^{+}_{22,5})^3 - 3650(t^{+}_{22,5})^2 + 3295t^{+}_{22,5} - 1024}},$$

$$f_{4,6} = \frac{(Dt^{+}_{22,5})^2}{-5t^{+}_{22,5}(-5t^{+}_{22,5}4)\sqrt{1375(t^{+}_{22,5})^3 - 3650(t^{+}_{22,5})^2 + 3295t^{+}_{22,5} - 1024}},$$

$$f_{4,7} = \frac{(Dt^{+}_{22,5})^2}{-5t^{+}_{22,5}\sqrt{-5t^{+}_{22,5}}\sqrt{-5t^{+}_{22,5}4}\sqrt{1375(t^{+}_{22,5})^3 - 3650(t^{+}_{22,5})^2 + 3295t^{+}_{22,5} - 1024}},$$

*which satisfy the relations*

$$0 = -f_{2,4}f_{2,3} + f_{2,1}f_{2,2},$$
$$0 = f_{2,1}f_{2,4} - f_{2,3}f_{2,5},$$
$$0 = -f_{2,4}^2 + f_{2,2}f_{2,5},$$
$$0 = f_{2,1}^2 - 4f_{2,3}^2 + 16f_{2,2}^2 - f_{2,5}^2,$$
$$0 = -f_{2,1}f_{2,3} + 4f_{2,4}f_{2,2} + f_{2,4}f_{2,5},$$
$$0 = -f_{2,1}^2 + 4f_{2,4}^2 + f_{2,5}^2,$$
$$0 = -f_{2,1}f_{4,3} + f_{4,5}f_{2,5},$$
$$0 = -f_{2,4}f_{4,3} + f_{4,6}f_{2,5},$$
$$0 = -f_{2,1}f_{4,5} + f_{4,4}f_{2,5},$$
$$0 = -f_{2,1}f_{2,3}f_{2,2} + f_{4,2}f_{2,5},$$
$$0 = -f_{2,1}f_{4,6} + f_{4,7}f_{2,5},$$
$$0 = -f_{2,4}f_{4,5} + f_{4,7}f_{2,5},$$
$$0 = -f_{2,3}f_{4,3} + f_{4,7}f_{2,5},$$
$$0 = -f_{2,1}f_{2,2}^2 + f_{4,1}f_{2,5},$$
$$0 = -f_{2,2}f_{4,3} + f_{2,4}f_{4,6},$$
$$0 = -f_{2,4}f_{4,4} + f_{2,1}f_{4,7},$$
$$0 = -f_{2,3}f_{4,5} + f_{2,1}f_{4,7},$$
$$0 = -f_{2,4}f_{4,2} + f_{2,1}f_{4,1},$$
$$0 = -f_{2,3}f_{4,6} + f_{2,4}f_{4,7},$$
$$0 = -f_{2,2}f_{4,5} + f_{2,4}f_{4,7},$$
$$0 = -f_{2,3}f_{2,2}^2 + f_{2,4}f_{4,1},$$
$$0 = -f_{2,2}f_{4,4} + f_{2,3}f_{4,7},$$
$$0 = -f_{2,2}f_{4,2} + f_{2,3}f_{4,1},$$
$$0 = -f_{2,3}f_{4,2} + 4f_{2,2}f_{4,1} + f_{4,1}f_{2,5},$$
$$0 = -f_{2,3}f_{4,4} + 4f_{2,2}f_{4,7} + f_{4,7}f_{2,5},$$
$$0 = 4f_{2,2}^3 - f_{2,1}f_{4,1} + f_{2,2}^2f_{2,5},$$
$$0 = 4f_{2,2}f_{4,6} - f_{2,1}f_{4,7} + f_{4,6}f_{2,5},$$
$$0 = -f_{2,1}f_{4,2} + 4f_{2,4}f_{4,1} + f_{2,3}f_{2,2}f_{2,5},$$
$$0 = -f_{2,1}f_{4,4} + 4f_{2,4}f_{4,7} + f_{4,5}f_{2,5},$$
$$0 = 4f_{2,4}f_{4,6} + f_{4,3}f_{2,5} - f_{4,4}f_{2,5},$$
$$0 = -f_{4,5}^2 + f_{4,3}f_{4,4},$$
$$0 = -f_{4,3}f_{4,2} + f_{2,1}f_{2,2}f_{4,7},$$
$$0 = -f_{4,5}f_{4,6} + f_{4,3}f_{4,7},$$

$$0 = f_{2,4}f_{2,2}f_{4,7} - f_{4,3}f_{4,1},$$
$$0 = f_{4,6}f_{4,4} - f_{4,5}f_{4,7},$$
$$0 = f_{2,3}f_{2,2}f_{4,7} - f_{4,5}f_{4,1},$$
$$0 = -f_{4,6}f_{4,2} + f_{2,3}f_{2,2}f_{4,7},$$
$$0 = f_{2,2}^2 f_{4,7} - f_{4,6}f_{4,1},$$
$$0 = -f_{4,2}f_{4,7} + f_{4,4}f_{4,1},$$
$$0 = -f_{4,4}f_{4,2} + f_{2,1}f_{2,2}f_{4,7} + 4f_{4,7}f_{4,1},$$
$$0 = f_{4,3}f_{4,4} - f_{4,4}^2 + 4f_{4,7}^2,$$
$$0 = -f_{4,5}f_{4,2} + 4f_{2,2}^2 f_{4,7} + f_{2,2}f_{4,7}f_{2,5},$$
$$0 = f_{4,3}f_{4,5} - f_{4,5}f_{4,4} + 4f_{4,6}f_{4,7},$$
$$0 = f_{4,3}^2 + 4f_{4,6}^2 - f_{4,3}f_{4,4},$$
$$0 = 825f_{2,1}^4 - 328f_{4,3}f_{4,4} + 403f_{4,4}^2 + 284777f_{2,1}f_{2,3}f_{4,2} + 1650688f_{4,1}^2 + 25382f_{2,1}^2 f_{2,2}f_{2,5},$$
$$0 = 659f_{2,1}f_{2,2}f_{4,2} + f_{4,4}f_{4,7} + 1024f_{4,2}f_{4,1} + 11f_{2,1}^2 f_{2,3}f_{2,5} + 146f_{2,1}f_{4,2}f_{2,5},$$
$$0 = -1991f_{2,1}^4 - 584f_{4,3}f_{4,4} + 403f_{4,4}^2 + 116073f_{2,1}f_{2,3}f_{4,2} + 412672f_{4,2}^2 - 730f_{2,1}^2 f_{2,2}f_{2,5},$$
$$0 = -25382f_{2,1}^3 f_{2,2} - 256f_{4,3}f_{4,5} + 5f_{4,5}f_{4,4} + 20480f_{2,2}^2 f_{4,1} - 2761f_{2,1}^3 f_{2,5} - 60961f_{2,3}f_{4,2}f_{2,5},$$
$$0 = 11f_{2,1}^3 f_{2,3} + f_{4,6}f_{4,4} + 102f_{2,1}^2 f_{4,2} + 1024f_{2,2}^2 f_{4,2} + 251f_{2,2}f_{4,2}f_{2,5},$$
$$0 = 146f_{2,1}^3 f_{2,2} + f_{4,5}f_{4,4} + 1024f_{2,3}f_{2,2}f_{4,2} + 11f_{2,1}^3 f_{2,5} + 659f_{2,3}f_{4,2}f_{2,5},$$
$$0 = -1727f_{2,1}^4 + 32240f_{2,2}^4 - 403f_{4,3}^2 + 246f_{4,3}f_{4,4} + 60961f_{2,1}f_{2,3}f_{4,2} + 1718f_{2,1}^2 f_{2,2}f_{2,5},$$
$$0 = 251f_{2,1}f_{2,2}f_{4,2} + f_{4,3}f_{4,7} + 20f_{2,4}f_{2,2}f_{4,1} + 11f_{2,1}^2 f_{2,3}f_{2,5} + 102f_{2,1}f_{4,2}f_{2,5},$$
$$0 = -11f_{2,1}^4 - f_{4,3}f_{4,4} - 659f_{2,1}f_{2,3}f_{4,2} + 1612f_{2,4}f_{2,2}f_{4,2} - 102f_{2,1}^2 f_{2,2}f_{2,5},$$
$$0 = 11f_{2,1}^3 f_{2,3} + 20f_{2,4}f_{2,2}^3 + f_{4,3}f_{4,6} + 58f_{2,1}^2 f_{4,2} + 19f_{2,2}f_{4,2}f_{2,5}. \qquad \square$$

Finally, we deal with the case of level 7, where 21 generators and 287 relations are needed.

**Proposition 6.20.** *The graded $\mathbb{Q}$-algebra of cusp forms with respect to $\Gamma(22,7)$ is generated by*

$$f_{2,1} = \frac{Dt_{22,7}^+}{\sqrt{45(t_{22,7}^+)^2 - 48t_{22,7}^+ + 4}\sqrt{-3(48(t_{22,7}^+)^2 - 56t_{22,7}^+ + 9)}},$$
$$f_{2,2} = \frac{Dt_{22,7}^+}{\sqrt{36(t_{22,7}^+)^2 - 36t_{22,7}^+ + 1}\sqrt{-3(48(t_{22,7}^+)^2 - 56t_{22,7}^+ + 9)}},$$
$$f_{2,3} = \frac{Dt_{22,7}^+}{\sqrt{36(t_{22,7}^+)^2 - 36t_{22,7}^+ + 1}\sqrt{45(t_{22,7}^+)^2 - 48t_{22,7}^+ + 4}},$$
$$f_{2,4} = \frac{t_{22,7}^+(Dt_{22,7}^+)}{\sqrt{36(t_{22,7}^+)^2 - 36t_{22,7}^+ + 1}\sqrt{45(t_{22,7}^+)^2 - 48t_{22,7}^+ + 4}\sqrt{-3(48(t_{22,7}^+)^2 - 56t_{22,7}^+ + 9)}},$$

$$f_{2,5} = \frac{Dt^+_{22,7}}{\sqrt{36(t^+_{22,7})^2 - 36t^+_{22,7} + 1}\sqrt{45(t^+_{22,7})^2 - 48t^+_{22,7} + 4}\sqrt{-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9)}},$$

$$f_{4,1} = \frac{(Dt^+_{22,7})^2}{t^+_{22,7}\sqrt{36(t^+_{22,7})^2 - 36t^+_{22,7} + 1}\sqrt{45(t^+_{22,7})^2 - 48t^+_{22,7} + 4}\sqrt{-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9)}},$$

$$f_{4,2} = \frac{(Dt^+_{22,7})^2}{t^+_{22,7}\sqrt{36(t^+_{22,7})^2 - 36t^+_{22,7} + 1}\sqrt{45(t^+_{22,7})^2 - 48t^+_{22,7} + 4}(-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9))},$$

$$f_{4,3} = \frac{(Dt^+_{22,7})^2}{t^+_{22,7}\sqrt{36(t^+_{22,7})^2 - 36t^+_{22,7} + 1}(45(t^+_{22,7})^2 - 48t^+_{22,7} + 4)\sqrt{-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9)}},$$

$$f_{4,4} = \frac{(Dt^+_{22,7})^2}{t^+_{22,7}(36(t^+_{22,7})^2 - 36t^+_{22,7} + 1)\sqrt{45(t^+_{22,7})^2 - 48t^+_{22,7} + 4}\sqrt{-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9)}},$$

$$f_{4,5} = \frac{(Dt^+_{22,7})^2}{t^+_{22,7}(45(t^+_{22,7})^2 - 48t^+_{22,7} + 4)\sqrt{-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9)}},$$

$$f_{4,6} = \frac{(Dt^+_{22,7})^2}{t^+_{22,7}\sqrt{36(t^+_{22,7})^2 - 36t^+_{22,7} + 1}(-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9))},$$

$$f_{4,7} = \frac{(Dt^+_{22,7})^2}{t^+_{22,7}\sqrt{45(t^+_{22,7})^2 - 48t^+_{22,7} + 4}(-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9))},$$

$$f_{4,8} = \frac{(Dt^+_{22,7})^2}{t^+_{22,7}(45(t^+_{22,7})^2 - 48t^+_{22,7} + 4)(-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9))},$$

$$f_{6,1} = \frac{(Dt^+_{22,7})^3}{(t^+_{22,7})^2(36(t^+_{22,7})^2 - 36t^+_{22,7} + 1)\left(\sqrt{45(t^+_{22,7})^2 - 48t^+_{22,7} + 4}\right)^3} \cdot \frac{1}{\left(\sqrt{-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9)}\right)^3},$$

$$f_{6,2} = \frac{(Dt^+_{22,7})^3}{(t^+_{22,7})^2\left(\sqrt{36(t^+_{22,7})^2 - 36t^+_{22,7} + 1}\right)^3(45(t^+_{22,7})^2 - 48t^+_{22,7} + 4)} \cdot \frac{1}{\left(\sqrt{-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9)}\right)^3},$$

$$f_{6,3} = \frac{(Dt^+_{22,7})^3}{(t^+_{22,7})^2\left(\sqrt{36(t^+_{22,7})^2 - 36t^+_{22,7} + 1}\right)^3\left(\sqrt{45(t^+_{22,7})^2 - 48t^+_{22,7} + 4}\right)^3} \cdot \frac{1}{(-3(48(t^+_{22,7})^2 - 56t^+_{22,7} + 9)},$$

$$f_{6,4} = \frac{(Dt_{22,7}^{+})^3}{(t_{22,7}^{+})^2 \left(\sqrt{36(t_{22,7}^{+})^2 - 36t_{22,7}^{+} + 1}\right)^3 \left(\sqrt{45(t_{22,7}^{+})^2 - 48t_{22,7}^{+} + 4}\right)^3} \cdot \frac{1}{\left(\sqrt{-3(48(t_{22,7}^{+})^2 - 56t_{22,7}^{+} + 9)}\right)^3},$$

$$f_{6,5} = \frac{(Dt_{22,7}^{+})^3}{(t_{22,7}^{+})^2(36(t_{22,7}^{+})^2 - 36t_{22,7}^{+} + 1)(45(t_{22,7}^{+})^2 - 48t_{22,7}^{+} + 4)(-3(48(t_{22,7}^{+})^2 - 56t_{22,7}^{+} + 9)},$$

$$f_{6,6} = \frac{(Dt_{22,7}^{+})^3}{(t_{22,7}^{+})^2(36(t_{22,7}^{+})^2 - 36t_{22,7}^{+} + 1)(45(t_{22,7}^{+})^2 - 48t_{22,7}^{+} + 4) \left(\sqrt{-3(48(t_{22,7}^{+})^2 - 56t_{22,7}^{+} + 9)}\right)^3},$$

$$f_{6,7} = \frac{(Dt_{22,7}^{+})^3}{(t_{22,7}^{+})^2(36(t_{22,7}^{+})^2 - 36t_{22,7}^{+} + 1) \left(\sqrt{45(t_{22,7}^{+})^2 - 48t_{22,7}^{+} + 4}\right)^3 (-3(48(t_{22,7}^{+})^2 - 56t_{22,7}^{+} + 9)},$$

$$f_{6,8} = \frac{(Dt_{22,7}^{+})^3}{(t_{22,7}^{+})^2 \left(\sqrt{36(t_{22,7}^{+})^2 - 36t_{22,7}^{+} + 1}\right)^3 (45(t_{22,7}^{+})^2 - 48t_{22,7}^{+} + 4)(-3(48(t_{22,7}^{+})^2 - 56t_{22,7}^{+} + 9)},$$

*which satisfy the relations*

$$\begin{array}{lll}
0 = f_{2,1}f_{4,6} - f_{4,7}f_{2,2}, & 0 = f_{4,1}f_{4,2} - f_{4,7}f_{4,4}, & 0 = f_{2,2}f_{6,4} - f_{2,5}f_{6,2}, \\
0 = f_{2,1}f_{4,1} - f_{4,5}f_{2,2}, & 0 = f_{4,1}f_{4,3} - f_{4,5}f_{4,4}, & 0 = f_{2,3}f_{6,4} - f_{2,5}f_{6,3}, \\
0 = f_{2,1}f_{4,1} - f_{4,7}f_{2,3}, & 0 = f_{2,1}f_{6,5} - f_{4,2}f_{4,3}, & 0 = f_{4,5}f_{6,6} - f_{4,8}f_{6,5}, \\
0 = f_{2,1}f_{4,2} - f_{4,8}f_{2,2}, & 0 = f_{2,1}f_{6,5} - f_{4,8}f_{4,4}, & 0 = f_{4,5}f_{6,6} - f_{4,7}f_{6,7}, \\
0 = f_{2,1}f_{4,2} - f_{4,7}f_{2,5}, & 0 = f_{2,2}f_{6,5} - f_{4,2}f_{4,4}, & 0 = f_{4,8}f_{6,6} - f_{4,7}f_{6,1}, \\
0 = f_{2,1}f_{4,3} - f_{4,8}f_{2,3}, & 0 = f_{2,3}f_{6,5} - f_{4,3}f_{4,4}, & 0 = f_{4,5}f_{6,1} - f_{4,8}f_{6,7}, \\
0 = f_{2,1}f_{4,3} - f_{4,5}f_{2,5}, & 0 = f_{2,1}f_{6,8} - f_{2,2}f_{6,7}, & 0 = f_{4,1}f_{6,6} - f_{4,2}f_{6,5}, \\
0 = f_{2,2}f_{4,1} - f_{4,6}f_{2,3}, & 0 = f_{2,1}f_{6,8} - f_{2,3}f_{6,6}, & 0 = f_{4,1}f_{6,6} - f_{4,6}f_{6,7}, \\
0 = f_{2,2}f_{4,2} - f_{4,6}f_{2,5}, & 0 = f_{2,1}f_{6,8} - f_{2,5}f_{6,5}, & 0 = f_{4,1}f_{6,6} - f_{4,7}f_{6,8}, \\
0 = f_{2,1}f_{4,4} - f_{2,2}f_{4,3}, & 0 = f_{2,1}f_{6,2} - f_{2,2}f_{6,1}, & 0 = f_{4,2}f_{6,6} - f_{4,6}f_{6,1}, \\
0 = f_{2,1}f_{4,4} - f_{2,3}f_{4,2}, & 0 = f_{2,1}f_{6,2} - f_{2,5}f_{6,6}, & 0 = f_{4,2}f_{6,6} - f_{4,7}f_{6,2}, \\
0 = f_{2,1}f_{4,4} - f_{2,5}f_{4,1}, & 0 = f_{2,1}f_{6,3} - f_{2,3}f_{6,1}, & 0 = f_{4,1}f_{6,7} - f_{4,3}f_{6,5}, \\
0 = f_{4,7}f_{4,1} - f_{4,5}f_{4,6}, & 0 = f_{2,1}f_{6,3} - f_{2,5}f_{6,7}, & 0 = f_{4,1}f_{6,7} - f_{4,5}f_{6,8}, \\
0 = f_{4,7}f_{4,2} - f_{4,8}f_{4,6}, & 0 = f_{2,1}f_{6,4} - f_{2,5}f_{6,1}, & 0 = f_{4,1}f_{6,1} - f_{4,2}f_{6,7}, \\
0 = f_{4,8}f_{4,1} - f_{4,5}f_{4,2}, & 0 = f_{2,4}f_{6,8} - f_{2,5}^2f_{4,4}, & 0 = f_{4,1}f_{6,1} - f_{4,3}f_{6,6}, \\
0 = f_{4,8}f_{4,1} - f_{4,7}f_{4,3}, & 0 = f_{2,2}f_{6,3} - f_{2,3}f_{6,2}, & 0 = f_{4,1}f_{6,1} - f_{4,5}f_{6,2}, \\
0 = f_{4,1}f_{4,2} - f_{4,6}f_{4,3}, & 0 = f_{2,2}f_{6,3} - f_{2,5}f_{6,8}, & 0 = f_{4,1}f_{6,1} - f_{4,8}f_{6,8}, \\
 & & 0 = f_{4,1}f_{6,1} - f_{4,7}f_{6,3}, \\
 & & 0 = f_{4,2}f_{6,1} - f_{4,8}f_{6,2},
\end{array}$$

$$0 = f_{4,2}f_{6,1} - f_{4,7}f_{6,4},$$
$$0 = f_{4,3}f_{6,7} - f_{4,5}f_{6,3},$$
$$0 = f_{4,3}f_{6,1} - f_{4,5}f_{6,4},$$
$$0 = f_{4,3}f_{6,1} - f_{4,8}f_{6,3},$$
$$0 = f_{4,1}f_{6,8} - f_{4,4}f_{6,5},$$
$$0 = f_{4,1}f_{6,2} - f_{4,2}f_{6,8},$$
$$0 = f_{4,1}f_{6,2} - f_{4,4}f_{6,6},$$
$$0 = f_{4,1}f_{6,2} - f_{4,6}f_{6,3},$$
$$0 = f_{4,2}f_{6,2} - f_{4,6}f_{6,4},$$
$$0 = f_{4,1}f_{6,3} - f_{4,3}f_{6,8},$$
$$0 = f_{4,1}f_{6,3} - f_{4,4}f_{6,7},$$
$$0 = f_{4,1}f_{6,4} - f_{4,2}f_{6,3},$$
$$0 = f_{4,1}f_{6,4} - f_{4,3}f_{6,2},$$
$$0 = f_{4,1}f_{6,4} - f_{4,4}f_{6,1},$$
$$0 = f_{6,5}f_{6,1} - f_{6,6}f_{6,7},$$
$$0 = f_{6,5}f_{6,2} - f_{6,6}f_{6,8},$$
$$0 = f_{6,5}f_{6,3} - f_{6,7}f_{6,8},$$
$$0 = f_{6,1}f_{6,8} - f_{6,7}f_{6,2},$$
$$0 = f_{6,1}f_{6,8} - f_{6,6}f_{6,3},$$
$$0 = f_{6,1}f_{6,8} - f_{6,5}f_{6,4},$$
$$0 = f_{6,1}f_{6,2} - f_{6,6}f_{6,4},$$
$$0 = f_{6,1}f_{6,3} - f_{6,7}f_{6,4},$$
$$0 = f_{6,2}f_{6,3} - f_{6,8}f_{6,4},$$
$$0 = f_{2,1}f_{2,2}f_{2,3} - f_{4,1}f_{2,4},$$
$$0 = f_{2,1}f_{2,2}f_{2,5} - f_{2,4}f_{4,2},$$
$$0 = f_{2,1}f_{2,3}f_{2,5} - f_{2,4}f_{4,3},$$
$$0 = f_{2,2}f_{2,3}f_{2,5} - f_{2,4}f_{4,4},$$
$$0 = f_{2,1}f_{2,5}f_{4,4} - f_{2,4}f_{6,5},$$

$$0 = -6f_{2,1}^2 + 8f_{2,2}^2 + f_{2,3}^2 + f_{2,5}^2,$$
$$0 = -71f_{2,1}^2 + 92f_{2,2}^2 + 11f_{2,3}^2 + 12f_{2,5}f_{2,4},$$
$$0 = -52f_{2,1}^2 + 67f_{2,2}^2 + 8f_{2,3}^2 + 9f_{2,4}^2,$$
$$0 = -6f_{4,7}^2 + 8f_{4,6}^2 + f_{4,1}^2 + f_{4,2}^2,$$
$$0 = -71f_{4,7}^2 + 12f_{2,2}f_{2,5}f_{4,7} + 92f_{4,6}^2 + 11f_{4,1}^2,$$
$$0 = -52f_{4,7}^2 + 9f_{2,2}f_{2,4}f_{4,7} + 67f_{4,6}^2 + 8f_{4,1}^2,$$
$$0 = -6f_{2,1}f_{4,7} + 8f_{2,2}f_{4,6} + f_{2,3}f_{4,1} + f_{2,5}f_{4,2},$$

$$0 = 71f_{2,5}^3 - 89460f_{2,4}^2f_{2,5} + 92016f_{2,4}^3 - 108f_{2,1}f_{4,6} - 16f_{4,5}f_{2,3} - 2567f_{4,8}f_{2,4},$$
$$0 = 23004f_{2,4}^3 - 23004f_{2,5}f_{2,4}^2 + 639f_{2,5}^2f_{2,4} - 624f_{4,8}f_{2,4} - 27f_{2,1}f_{4,6} - 4f_{4,5}f_{2,3},$$
$$0 = 373f_{2,3}f_{2,5}^2 - 396f_{2,3}f_{2,4}^2 + 11f_{4,5}f_{2,4} + 16f_{2,1}f_{4,3} - 4f_{2,2}f_{4,4},$$
$$0 = -3456f_{2,3}f_{2,4}^2 + 96f_{4,5}f_{2,4} + 3357f_{2,3}f_{2,5}f_{2,4} + 4f_{2,1}f_{4,3} - f_{2,2}f_{4,4},$$
$$0 = 781f_{2,2}f_{2,5}^2 - 828f_{2,2}f_{2,4}^2 + 23f_{4,7}f_{2,4} + 27f_{2,1}f_{4,2} + f_{2,3}f_{4,4},$$
$$0 = -28944f_{2,2}f_{2,4}^2 + 804f_{4,7}f_{2,4} + 28116f_{2,2}f_{2,5}f_{2,4} + 27f_{2,1}f_{4,2} + f_{2,3}f_{4,4},$$
$$0 = -36f_{2,2}f_{2,3}f_{2,5} + f_{4,4}f_{2,5} - f_{2,1}f_{4,1} + 36f_{2,2}f_{2,3}f_{2,4},$$
$$0 = 12f_{2,1}f_{2,5}^2 - 71f_{2,1}f_{4,8} + 92f_{2,2}f_{4,2} + 11f_{2,3}f_{4,3},$$
$$0 = -52f_{2,1}f_{4,8} + 9f_{2,1}f_{2,5}f_{2,4} + 67f_{2,2}f_{4,2} + 8f_{2,3}f_{4,3},$$
$$0 = 135f_{2,1}f_{2,4}^2 - 3f_{4,6}f_{2,4} - 761f_{2,1}f_{4,8} + 980f_{2,2}f_{4,2} + 117f_{2,3}f_{4,3},$$
$$0 = 4f_{2,1}f_{4,5} - 3f_{2,2}f_{4,1} - 9f_{2,1}f_{2,3}f_{2,4} + 8f_{2,5}f_{4,3},$$
$$0 = 4f_{2,1}f_{4,5} + 96f_{2,1}f_{2,3}f_{2,5} - f_{2,2}f_{4,1} - 99f_{2,1}f_{2,3}f_{2,4},$$
$$0 = -71f_{2,1}f_{4,7} + 92f_{2,2}f_{4,6} + 12f_{2,1}f_{2,2}f_{2,5} + 11f_{2,3}f_{4,1},$$
$$0 = -52f_{2,1}f_{4,7} + 67f_{2,2}f_{4,6} + 8f_{2,3}f_{4,1} + 9f_{2,1}f_{2,2}f_{2,4},$$

$$0 = 16f_{2,1}f_{4,6} + 5f_{4,5}f_{2,3} + 71f_{4,8}f_{2,5} - 72f_{4,8}f_{2,4},$$
$$0 = -8f_{4,5}f_{4,8} - 207f_{2,3}f_{2,4}f_{4,8} - 5f_{4,1}f_{4,2} - 384f_{2,2}f_{2,5}f_{4,4} + 576f_{2,2}f_{2,4}f_{4,4} + 28f_{2,3}f_{6,4},$$
$$0 = -12f_{4,5}f_{4,8} - 1623f_{2,3}f_{2,4}f_{4,8} + 3f_{4,1}f_{4,2} - 128f_{2,2}f_{2,5}f_{4,4} + 1536f_{2,2}f_{2,4}f_{4,4} + 224f_{2,4}f_{6,3},$$
$$0 = 94016f_{4,7}^2 - 575f_{4,5}^2 - 7448f_{4,8}^2 - 123648f_{4,6}^2 - 14676f_{4,1}^2 - 84f_{4,4}^2 + 3024f_{2,2}f_{2,3}f_{4,4} + 147f_{2,5}f_{6,4},$$
$$0 = 441f_{2,5}^4 + 267024f_{4,7}^2 - 1650f_{4,5}^2 - 21266f_{4,8}^2 - 351232f_{4,6}^2 - 41684f_{4,1}^2 - 245f_{4,4}^2 + 8568f_{2,2}f_{2,3}f_{4,4},$$
$$0 = 12328f_{4,7}^2 + 50f_{4,5}^2 + 1127f_{4,8}^2 - 15456f_{4,6}^2 - 1788f_{4,1}^2 - 84f_{4,4}^2 + 1008f_{2,2}f_{2,3}f_{4,4} + 147f_{2,3}f_{6,3},$$
$$0 = -448f_{4,7}f_{4,8} + 576f_{4,6}f_{4,2} + 69f_{4,1}f_{4,3} + 12f_{2,3}f_{2,5}f_{4,4} - 36f_{2,3}f_{2,4}f_{4,4} + 7f_{2,2}f_{6,4},$$
$$0 = -1309f_{4,7}f_{4,8} + 1684f_{4,6}f_{4,2} + 201f_{4,1}f_{4,3} + 3f_{2,3}f_{2,5}f_{4,4} - 72f_{2,3}f_{2,4}f_{4,4} + 21f_{2,4}f_{6,2},$$
$$0 = 4f_{4,7}f_{4,5} - 3f_{4,6}f_{4,1} - 9f_{4,7}f_{2,3}f_{2,4} + 288f_{2,5}f_{2,4}f_{4,4} - 288f_{2,4}f_{6,8} + 8f_{2,2}f_{6,3},$$
$$0 = -f_{4,1}f_{4,3} - 36f_{2,3}f_{2,5}f_{4,4} + 36f_{2,3}f_{2,4}f_{4,4} + f_{2,3}f_{6,8},$$
$$0 = -448f_{4,7}^2 + 576f_{4,6}^2 + 68f_{4,1}^2 + f_{4,4}^2 - 24f_{2,2}f_{2,3}f_{4,4} + 7f_{2,2}f_{6,2},$$
$$0 = -f_{4,1}f_{4,2} - 36f_{2,2}f_{2,5}f_{4,4} + 36f_{2,2}f_{2,4}f_{4,4} + f_{2,2}f_{6,8},$$
$$0 = 192f_{2,1}f_{4,8}f_{2,3} - 384f_{4,6}f_{2,5}f_{2,3} + 576f_{4,6}f_{2,4}f_{2,3} - 7f_{4,7}f_{4,1} - 405f_{2,1}f_{4,5}f_{2,4} + 28f_{2,1}f_{6,3},$$
$$0 = 9f_{2,1}f_{4,8}f_{2,3} - 4f_{4,6}f_{2,5}f_{2,3} + 48f_{4,6}f_{2,4}f_{2,3} - 60f_{2,1}f_{4,5}f_{2,4} + 7f_{2,4}f_{6,7},$$
$$0 = 29547f_{2,1}f_{4,8}f_{2,4} + 52416f_{4,6}f_{2,5}f_{2,4} + 729f_{4,7}f_{4,2} + 12328f_{4,5}f_{4,3} - 9152f_{4,1}f_{4,4} + 24388f_{2,3}f_{6,7},$$
$$0 = -199f_{4,7}f_{4,6} - 4992f_{2,2}f_{2,5}f_{4,6} + 7488f_{2,2}f_{2,4}f_{4,6} - 16f_{4,5}f_{4,1} - 2709f_{2,1}f_{4,7}f_{2,4} + 364f_{2,1}f_{6,2},$$
$$0 = -81f_{4,7}f_{4,6} - 832f_{2,2}f_{2,5}f_{4,6} + 9984f_{2,2}f_{2,4}f_{4,6} - 12f_{4,5}f_{4,1} - 10563f_{2,1}f_{4,7}f_{2,4} + 1456f_{2,4}f_{6,6},$$
$$0 = -60f_{2,1}f_{4,7}f_{2,3} + 48f_{2,2}f_{4,6}f_{2,3} + 9f_{4,8}f_{4,1} - 4f_{4,6}f_{4,4} + 7f_{2,1}f_{6,8},$$
$$0 = -165f_{2,1}f_{4,7}f_{2,3} + 132f_{2,2}f_{4,6}f_{2,3} + 16f_{4,8}f_{4,1} - 4f_{4,6}f_{4,4} + 21f_{2,1}f_{2,5}f_{4,4},$$
$$0 = -72f_{4,6}f_{2,5}f_{2,4} + 14f_{4,7}f_{4,2} + 3f_{4,1}f_{4,4} + 67f_{2,2}f_{6,6},$$
$$0 = 804f_{4,6}f_{2,5}^2 - 828f_{4,6}f_{2,4}f_{2,5} + 27f_{4,7}f_{4,2} + f_{4,1}f_{4,4},$$
$$0 = -36f_{4,6}f_{2,3}f_{2,5} - f_{4,7}f_{4,1} + 36f_{4,6}f_{2,3}f_{2,4} + f_{2,2}f_{6,5},$$
$$0 = 1480f_{4,7}^2 - 25f_{4,5}^2 - 301f_{4,8}^2 - 2016f_{4,6}^2 - 240f_{4,1}^2 + 42f_{2,1}f_{6,1},$$
$$0 = 17288f_{4,7}^2 - 275f_{4,5}^2 - 3521f_{4,8}^2 - 23520f_{4,6}^2 + 504f_{4,8}f_{2,5}^2 - 2808f_{4,1}^2,$$
$$0 = 6298f_{4,7}^2 - 100f_{4,5}^2 - 1288f_{4,8}^2 - 8568f_{4,6}^2 - 1023f_{4,1}^2 + 189f_{4,8}f_{2,5}f_{2,4},$$
$$0 = 4f_{4,5}f_{4,8} - 9f_{2,3}f_{2,4}f_{4,8} - 3f_{4,1}f_{4,2} + 8f_{2,1}f_{6,7},$$
$$0 = 4f_{4,5}f_{4,8} + 96f_{2,3}f_{2,5}f_{4,8} - 99f_{2,3}f_{2,4}f_{4,8} - f_{4,1}f_{4,2},$$
$$0 = 536f_{4,7}^2 + 25f_{4,5}^2 + 49f_{4,8}^2 - 672f_{4,6}^2 - 96f_{4,1}^2 + 42f_{4,3}^2,$$
$$0 = -6f_{4,7}f_{4,8} + 8f_{4,6}f_{4,2} + f_{4,1}f_{4,3} + f_{2,1}f_{6,6},$$

$$0 = 12f_{4,7}f_{2,5}^2 - 71f_{4,7}f_{4,8} + 92f_{4,6}f_{4,2} + 11f_{4,1}f_{4,3},$$

$$0 = -52f_{4,7}f_{4,8} + 9f_{4,7}f_{2,5}f_{2,4} + 67f_{4,6}f_{4,2} + 8f_{4,1}f_{4,3},$$

$$0 = 4f_{4,7}f_{4,5} - 3f_{4,6}f_{4,1} - 9f_{4,7}f_{2,3}f_{2,4} + 8f_{2,1}f_{6,5},$$

$$0 = 4f_{4,7}f_{4,5} + 96f_{4,7}f_{2,3}f_{2,5} - f_{4,6}f_{4,1} - 99f_{4,7}f_{2,3}f_{2,4},$$

$$0 = 624f_{2,1}f_{4,8}f_{2,5} - 639f_{2,1}f_{4,8}f_{2,4} + 27f_{4,7}f_{4,2} + 4f_{4,5}f_{4,3},$$

$$0 = -48f_{2,1}f_{4,8}f_{2,3} - f_{4,7}f_{4,1} + 45f_{2,1}f_{4,5}f_{2,4} + 4f_{4,8}f_{4,3},$$

$$0 = 7f_{4,7}f_{4,6} + 2f_{4,5}f_{4,1} - 27f_{2,1}f_{4,7}f_{2,4} + 26f_{4,8}f_{4,2},$$

$$0 = 27f_{4,7}f_{4,6} + 624f_{2,1}f_{4,7}f_{2,5} + 4f_{4,5}f_{4,1} - 639f_{2,1}f_{4,7}f_{2,4},$$

$$0 = 5184f_{6,8}f_{2,4}^2 - 135f_{2,1}f_{6,5}f_{2,4} - 5040f_{2,5}f_{6,8}f_{2,4} + 3f_{4,6}f_{6,5} - 16f_{4,1}f_{6,1} + 4f_{4,4}f_{6,4},$$

$$0 = 48f_{6,8}f_{2,5}^2 - 1728f_{2,4}f_{6,8}f_{2,5} + f_{4,6}f_{6,5} - 45f_{2,1}f_{2,4}f_{6,5} - 4f_{4,1}f_{6,1} + 1728f_{2,4}^2f_{6,8},$$

$$0 = -f_{4,1}f_{6,7} - 36f_{2,3}f_{2,5}f_{6,8} + 36f_{2,3}f_{2,4}f_{6,8} + f_{4,4}f_{6,3},$$

$$0 = -f_{4,1}f_{6,6} - 36f_{2,2}f_{2,5}f_{6,8} + 36f_{2,2}f_{2,4}f_{6,8} + f_{4,4}f_{6,2},$$

$$0 = -3f_{4,7}f_{6,5} + 108f_{2,2}f_{2,4}f_{6,5} - 9f_{2,5}f_{2,4}f_{6,7} + 4f_{4,5}f_{6,1} - 108f_{2,1}f_{2,2}f_{6,8} + 8f_{4,3}f_{6,4},$$

$$0 = 96f_{6,7}f_{2,5}^2 - 99f_{2,4}f_{6,7}f_{2,5} - f_{4,7}f_{6,5} + 36f_{2,2}f_{2,4}f_{6,5} + 4f_{4,5}f_{6,1} - 36f_{2,1}f_{2,2}f_{6,8},$$

$$0 = 968f_{4,7}f_{6,6} + 6048f_{2,2}f_{2,4}f_{6,6} + 1675f_{4,5}f_{6,7} + 3283f_{4,8}f_{6,1} - 1056f_{4,1}f_{6,8} + 2814f_{4,3}f_{6,3},$$

$$0 = 10648f_{4,7}f_{6,6}+66528f_{2,2}f_{2,4}f_{6,6}+4355f_{4,5}f_{6,7}+33768f_{2,3}f_{2,5}f_{6,7}+36113f_{4,8}f_{6,1}-360f_{4,1}f_{6,8},$$

$$0 = 3872f_{4,7}f_{6,6}+24192f_{2,2}f_{2,4}f_{6,6}+1072f_{4,5}f_{6,7}+12663f_{2,3}f_{2,4}f_{6,7}+13132f_{4,8}f_{6,1}-3f_{4,1}f_{6,8},$$

$$0 = f_{4,5}f_{6,5} - 36f_{2,3}f_{2,4}f_{6,5} - 6f_{4,8}f_{6,6} + 36f_{2,1}f_{2,3}f_{6,8} + 8f_{4,6}f_{6,2} + f_{4,2}f_{6,4},$$

$$0 = 12f_{6,6}f_{2,5}^2 + 11f_{4,5}f_{6,5} - 396f_{2,3}f_{2,4}f_{6,5} - 71f_{4,8}f_{6,6} + 396f_{2,1}f_{2,3}f_{6,8} + 92f_{4,6}f_{6,2},$$

$$0 = 8f_{4,5}f_{6,5} - 288f_{2,3}f_{2,4}f_{6,5} - 52f_{4,8}f_{6,6} + 9f_{2,5}f_{2,4}f_{6,6} + 288f_{2,1}f_{2,3}f_{6,8} + 67f_{4,6}f_{6,2},$$

$$0 = 4f_{4,5}f_{6,6} - 3f_{4,6}f_{6,8} - 9f_{2,1}f_{2,4}f_{6,8} + 8f_{4,1}f_{6,4},$$

$$0 = 4f_{4,5}f_{6,6} - f_{4,6}f_{6,8} + 96f_{2,1}f_{2,5}f_{6,8} - 99f_{2,1}f_{2,4}f_{6,8},$$

$$0 = -f_{4,5}f_{6,5} + 36f_{2,3}f_{2,4}f_{6,5} - 36f_{2,1}f_{2,3}f_{6,8} + f_{4,1}f_{6,3},$$

$$0 = 14f_{4,7}f_{6,6} - 72f_{2,2}f_{2,4}f_{6,6} + 3f_{4,1}f_{6,8} + 67f_{4,2}f_{6,2},$$

$$0 = 27f_{4,7}f_{6,6} + 804f_{2,2}f_{2,5}f_{6,6} - 828f_{2,2}f_{2,4}f_{6,6} + f_{4,1}f_{6,8},$$

$$0 = -f_{4,7}f_{6,5} + 36f_{2,2}f_{2,4}f_{6,5} - 36f_{2,1}f_{2,2}f_{6,8} + f_{4,1}f_{6,2},$$

$$0 = -f_{4,1}f_{6,6} - 48f_{2,1}f_{2,5}f_{6,7} + 45f_{2,1}f_{2,4}f_{6,7} + 4f_{4,3}f_{6,1},$$

$$0 = 7f_{4,6}f_{6,6} - 27f_{2,1}f_{2,4}f_{6,6} + 2f_{4,1}f_{6,7} + 26f_{4,2}f_{6,1},$$

$$0 = 27f_{4,6}f_{6,6} + 624f_{2,1}f_{2,5}f_{6,6} - 639f_{2,1}f_{2,4}f_{6,6} + 4f_{4,1}f_{6,7},$$

$$0 = 24f_{2,5}f_{4,7}^2 + 6f_{2,3}f_{4,4}f_{4,7} + f_{4,1}f_{6,5} - 30f_{2,1}f_{2,2}f_{6,6} + 27f_{4,2}f_{6,6},$$

$$0 = -1260f_{6,8}f_{2,5}^3 + 1296f_{2,4}f_{6,8}f_{2,5}^2 - 36f_{4,8}f_{6,8}f_{2,5} - f_{6,5}f_{6,1} + f_{6,2}f_{6,3},$$

$$0 = -36f_{2,3}f_{6,8}f_{2,5}^2 - 11f_{6,5}f_{6,7} + 180f_{4,8}f_{2,3}f_{6,8} - 144f_{2,2}f_{4,4}f_{6,8} + 11f_{6,8}f_{6,3},$$

$$0 = -36f_{2,2}f_{6,8}f_{2,5}^2 + 144f_{4,7}f_{6,8}f_{2,5} - 23f_{6,5}f_{6,6} + 36f_{2,3}f_{4,4}f_{6,8} + 23f_{6,8}f_{6,2},$$

$$0 = -f_{6,5}^2 + f_{6,8}^2 - 36f_{2,5}f_{4,4}f_{6,8} + 36f_{2,4}f_{4,4}f_{6,8},$$

$$0 = 192f_{4,8}f_{2,3}f_{6,7} - 405f_{4,5}f_{2,4}f_{6,7} + 28f_{6,3}f_{6,7} + 324f_{4,1}f_{2,4}f_{6,8} - 132f_{2,1}f_{4,4}f_{6,8} - 7f_{6,5}f_{6,8},$$

$$0 = 768f_{4,7}f_{2,5}f_{6,6} - 2880f_{4,7}f_{2,4}f_{6,6} + 189f_{6,2}f_{6,6} - 720f_{4,1}f_{2,4}f_{6,8} + 72f_{2,1}f_{4,4}f_{6,8} + 7f_{6,5}f_{6,8},$$

$$0 = -4f_{6,5}f_{6,6} - 87f_{2,1}f_{4,8}f_{6,7} + 27f_{2,3}f_{4,3}f_{6,7} + 16f_{6,7}f_{6,1} + 156f_{4,7}f_{2,5}f_{6,8},$$

$$0 = -4f_{6,5}^2 + 16f_{6,7}^2 + 75f_{2,1}f_{4,5}f_{6,7} - 27f_{2,5}f_{4,3}f_{6,7} - 60f_{4,7}f_{2,3}f_{6,8},$$

$$0 = -156f_{2,1}f_{4,8}f_{6,6} + 240f_{4,6}f_{2,5}f_{6,6} + 27f_{6,1}f_{6,6} + f_{6,5}f_{6,7} + 36f_{4,8}f_{2,3}f_{6,8},$$

$$0 = f_{6,5}^2 + 27f_{6,6}^2 - 156f_{2,1}f_{4,7}f_{6,6} + 240f_{2,2}f_{4,6}f_{6,6} + 36f_{4,7}f_{2,3}f_{6,8},$$

$$\begin{aligned}0 = {}&2195416f_{4,7}^2 - 13525f_{4,5}^2 - 174391f_{4,8}^2 - 2887584f_{4,6}^2 - 342696f_{4,1}^2 - 2016f_{4,4}^2 + 70560f_{2,2}f_{2,3}f_{4,4}\\&+ 3528f_{2,4}f_{6,4},\end{aligned}$$

$$\begin{aligned}0 = {}&-181503f_{2,1}f_{4,8}f_{2,4} + 157248f_{4,6}f_{2,5}f_{2,4} - 2101f_{4,7}f_{4,2} - 1072f_{4,5}f_{4,3} + 416f_{4,1}f_{4,4}\\&+ 24388f_{2,1}f_{6,4},\end{aligned}$$

$$\begin{aligned}0 = {}&-2123163f_{2,1}f_{4,8}f_{2,4} + 1834560f_{4,6}f_{2,5}f_{2,4} + 292656f_{6,1}f_{2,4} - 10665f_{4,7}f_{4,2} - 2412f_{4,5}f_{4,3}\\&+ 208f_{4,1}f_{4,4},\end{aligned}$$

$$\begin{aligned}0 = {}&3699f_{4,7}f_{4,6} + 109824f_{2,2}f_{2,5}f_{4,6} - 112320f_{2,2}f_{2,4}f_{4,6} + 184f_{4,5}f_{4,1} + 441f_{2,1}f_{4,7}f_{2,4}\\&+ 364f_{2,3}f_{6,5},\end{aligned}$$

$$\begin{aligned}0 = {}&6144f_{2,5}f_{4,7}^2 + 1536f_{2,3}f_{4,4}f_{4,7} + 256f_{4,1}f_{6,5} + 28224f_{2,1}f_{2,2}f_{6,6} + 7545f_{2,1}f_{2,3}f_{6,7} - 688f_{4,3}f_{6,7}\\&+ 2940f_{2,2}f_{2,3}f_{6,8} - 84f_{4,4}f_{6,8} + 147f_{2,5}^2f_{6,4},\end{aligned}$$

$$\begin{aligned}0 = {}&17952f_{2,5}f_{4,7}^2 + 4488f_{2,3}f_{4,4}f_{4,7} + 748f_{4,1}f_{6,5} + 82488f_{2,1}f_{2,2}f_{6,6} + 22080f_{2,1}f_{2,3}f_{6,7}\\&- 2012f_{4,3}f_{6,7} + 8568f_{2,2}f_{2,3}f_{6,8} - 245f_{4,4}f_{6,8} + 441f_{2,5}f_{2,4}f_{6,4},\end{aligned}$$

$$\begin{aligned}0 = {}&4992f_{2,5}f_{4,7}^2 + 1248f_{2,3}f_{4,4}f_{4,7} + 208f_{4,1}f_{6,5} + 24864f_{2,1}f_{2,2}f_{6,6} + 9720f_{2,1}f_{2,3}f_{6,7} - 783f_{4,3}f_{6,7}\\&+ 1323f_{4,8}f_{6,4},\end{aligned}$$

$$\begin{aligned}0 = {}&196f_{6,5}^2 + 21168f_{6,4}^2 + 25939968f_{2,1}f_{4,7}f_{6,6} - 33374208f_{2,2}f_{4,6}f_{6,6} + 479925f_{2,1}f_{4,5}f_{6,7}\\&+ 941787f_{2,5}f_{4,3}f_{6,7} - 4328388f_{4,7}f_{2,3}f_{6,8} + 435456f_{2,4}f_{4,4}f_{6,8},\end{aligned}$$

$$\begin{aligned}0 = {}&16128f_{2,2}f_{6,8}f_{2,5}^2 + 150492f_{4,7}f_{6,8}f_{2,5} - 644f_{6,5}f_{6,6} - 114471f_{2,1}f_{4,8}f_{6,7} + 16491f_{2,3}f_{4,3}f_{6,7}\\&+ 1536f_{2,3}f_{4,4}f_{6,8} + 2576f_{6,3}f_{6,4},\end{aligned}$$

$$\begin{aligned}0 = {}&-28f_{6,5}^2 + 112f_{6,3}^2 - 675f_{2,1}f_{4,5}f_{6,7} - 717f_{2,5}f_{4,3}f_{6,7} + 540f_{4,7}f_{2,3}f_{6,8} - 1536f_{2,5}f_{4,4}f_{6,8}\\&+ 2304f_{2,4}f_{4,4}f_{6,8},\end{aligned}$$

$$\begin{aligned}0 = {}&-6804f_{2,3}f_{6,8}f_{2,5}^2 + 175296f_{4,6}f_{6,6}f_{2,5} - 136752f_{2,1}f_{4,8}f_{6,6} + 77f_{6,5}f_{6,7} + 18972f_{4,8}f_{2,3}f_{6,8}\\&+ 1296f_{2,2}f_{4,4}f_{6,8} + 2079f_{6,2}f_{6,4},\end{aligned}$$

$$
\begin{aligned}
0 &= 7f_{6,5}^2 + 189f_{6,2}^2 - 12432f_{2,1}f_{4,7}f_{6,6} + 15936f_{2,2}f_{4,6}f_{6,6} + 1872f_{4,7}f_{2,3}f_{6,8} + 324f_{2,5}f_{4,4}f_{6,8} \\
&\quad - 972f_{2,4}f_{4,4}f_{6,8}, \\
0 &= -264192f_{4,7}f_{2,5}f_{6,6} + 1363968f_{4,7}f_{2,4}f_{6,6} - 28512f_{4,8}f_{2,3}f_{6,7} + 287955f_{4,5}f_{2,4}f_{6,7} \\
&\quad + 110628f_{4,1}f_{2,4}f_{6,8} - 1764f_{2,1}f_{4,4}f_{6,8} + 49f_{6,5}f_{6,8} + 5292f_{6,1}f_{6,4}, \\
0 &= 192f_{4,8}f_{2,5}f_{6,7} - 405f_{4,8}f_{2,4}f_{6,7} + 135f_{2,1}f_{4,7}f_{6,8} - 108f_{2,2}f_{4,6}f_{6,8} + 165f_{2,5}f_{4,2}f_{6,8} \\
&\quad - 7f_{6,5}f_{6,2} + 28f_{6,1}f_{6,3}, \\
0 &= -640f_{4,7}f_{2,2}f_{6,6} - 2072f_{4,8}f_{2,5}f_{6,6} - 500f_{2,1}f_{4,5}f_{6,8} + 240f_{4,6}f_{2,3}f_{6,8} - 588f_{2,5}f_{4,3}f_{6,8} \\
&\quad + 189f_{6,1}f_{6,2} + 7f_{6,5}f_{6,3}, \\
0 &= 196f_{6,5}^2 + 21168f_{6,1}^2 + 2466816f_{2,1}f_{4,7}f_{6,6} - 3182592f_{2,2}f_{4,6}f_{6,6} + 58725f_{2,1}f_{4,5}f_{6,7} \\
&\quad + 134379f_{2,5}f_{4,3}f_{6,7} - 424836f_{4,7}f_{2,3}f_{6,8}. \qquad \square
\end{aligned}
$$

# Resum en català

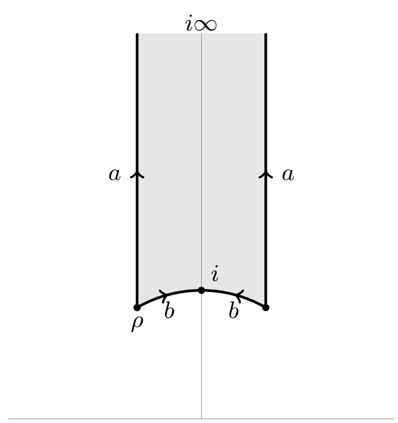
L'objectiu principal d'aquesta tesi és contribuir a la uniformització hiperbòlica explícita de les corbes de Shimura. Ens restringim a les corbes associades a ordres d'Eichler dins d'àlgebres de quaternions racionals tals que el seu quocient pel grup d'involucions d'Atkin-Lehner és de gènere 0. Aquest cas, tot i que presenta nombroses diferències amb el cas modular clàssic, també hi té certes similituds. Utilitzem aquest fet per a discutir una aproximació al problema de l'obtenció d'uniformitzacions hiperbòliques explícites d'aquestes corbes i d'alguns recobriments, així com també algunes aplicacions, que il·lustrem amb abundants exemples.

Per a entendre millor el problema, començarem introduint breument el seu rerefons històric. Després explicarem en detall les nostres contribucions i el contingut de la memòria.

Els grups fuchsians i les funcions automorfes que hi estan associades són un tema d'estudi molt important en les matemàtiques del segle XIX i assoleixen un punt àlgid amb les obres de Poincaré [Poi95], de Klein [Kle23, Kle79], de Fricke [Fri92, Fri93, Fri94, Fri99] i de Fricke i Klein [KF90, KF92, FK97, FK12]. Tot i això, alguns casos particulars de grups fuchsians i funcions automorfes, per exemple les que estan associades a subgrups de congruència $\Gamma_0(N)$ del grup modular $\Gamma_0(1) = \mathbf{SL}(2,\mathbb{Z})$, ja es poden veure en les obres de Gauss, Abel o Dedekind. És en aquest context que apareix la funció automorfa més coneguda, la funció $j$ (o invariant $j$), que també es coneix habitualment com la funció $j$ de Klein, tot i que ja havia estat estudiada abans. Per exemple, Dedekind [Ded77] considera la funció *Valenz*, que coincideix, llevat d'una constant multiplicativa, amb la funció $j$, i fins i tot Klein [Kle79] afirma que els orígens d'aquesta funció es remunten a Gauss. La funció $j$ és una funció automorfa per al grup modular $\mathbf{SL}(2,\mathbb{Z})$, i per tant, de la mateixa manera que per a tots els subgrups $\Gamma_0(N)$, s'acostuma a anomenar una funció modular (el·líptica).

Si $\Gamma$ és un grup fuchsià tal que $\Gamma\backslash\mathcal{H}$ té volum hiperbòlic finit (o més generalment, si és un grup fuchsià de primer tipus) i considerem la seva acció en el semiplà superior, aleshores les funcions automorfes respecte $\Gamma$ són les funcions meromorfes en $\mathcal{H}$ que són invariants per l'acció de $\Gamma$ i que satisfan certes condicions de creixement. Per altra banda, l'acció de $\Gamma$ en $\mathcal{H}$ indueix una estructura de superfície de Riemann en l'espai quocient $\Gamma\backslash\mathcal{H}$ que permet identificar $\Gamma\backslash\mathcal{H}$ amb un obert de Zariski del conjunt de punts complexos d'una corba algebraica. En particular, per a alguns dels subgrups de congruència del grup modular, s'obtenen les corbes modulars $X_0(N)$.

El cos de funcions racionals de la corba modular $X_0(1)$ s'identifica naturalment amb $\mathbb{C}(j)$. Una característica important de la superfície de Riemann $\Gamma_0(N)\backslash\mathcal{H}$ és que no és compacta. Per exemple, per a $N = 1$, un domini fonamental per a l'acció de $\mathbf{SL}(2,\mathbb{Z})$ en $\mathcal{H}$ és el conegut domini fonamental de Gauss, que apareix originalment en el context de la reducció de formes quadràtiques binàries. En particular, això permet considerar desenvolupaments de Fourier per a les funcions modulars.




Per exemple, per a la funció $j$, els primers termes del desenvolupament de Fourier corresponent són

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots,$$

on $q = e^{2\pi i\tau}$. En particular, la racionalitat (i la integralitat) dels coeficients d'aquests desenvolupaments es tradueix en propietats de la corba $X_0(N)$. Per exemple, el fet que

$$\Gamma_0(N) = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathbf{SL}(2,\mathbb{Z}) \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \cap \mathbf{SL}(2,\mathbb{Z})$$

ens permet escriure $\mathbb{C}(X_0(N)) = \mathbb{C}(j, j(N\cdot))$. A més a més, existeix un polinomi mònic irreductible i simètric (si $N > 1$) de grau $\psi(N)$ en cada indeterminada,

$\Phi_N(X,Y) \in \mathbb{Z}[X,Y]$, tal que $\Phi_N(j, j(N\cdot)) = 0$ i que proporciona una equació per a $X_0(N)$. Aquest polinomi es coneix habitualment com a *equació modular de nivell* $N$. Les arrels del polinomi $\Phi_N(j, X)$ es poden escriure com $j \circ \alpha_N$, on $\alpha_N$ recorre les transformacions corresponents a un sistema de representants dels elements primitius de norma $N$ en $\mathbf{M}(2,\mathbb{Z})$ mòdul $\mathbf{SL}(2,\mathbb{Z})$. En aquest cas, a més, és possible escollir els representants $\alpha_N$ de manera que fixin $\infty$, fet que permet obtenir també desenvolupaments de Fourier de les funcions $j \circ \alpha_N$ a partir dels desenvolupaments de $j$ i, en particular, calcular explícitament el polinomi $\Phi_N(X,Y)$. A més a més, el polinomi $\Phi_N(X,X)$ factoritza com a producte d'uns polinomis coneguts com a polinomis de les classes i que tenen com arrels els valors de $j$ en uns punts determinats. D'aquesta manera es poden obtenir explícitament els valors de la funció $j$ en un conjunt de punts del semiplà superior coneguts com a punts de multiplicació complexa (punts CM) i que en aquest cas no són res més que els punts quadràtics de $\mathcal{H}$. A més, el valor de $j$ en un punt CM, $\tau$, no és només un valor algebraic sinó que genera una extensió abeliana del cos $\mathbb{Q}(\tau)$. En particular, a partir del polinomi $\Phi_N(X,Y)$ per a un $N$ adequat, podem obtenir una equació definidora d'aquesta extensió.

Per altra banda, també és un resultat clàssic que els punts de la corba $X_0(1)$ corresponen a classes d'isomorfisme de corbes el·líptiques generalitzades. Si excloem la punta $i\infty$, un punt $\tau \in \mathbf{SL}(2,\mathbb{Z})\backslash\mathcal{H}$ correspon a la classe d'isomorfisme de la corba el·líptica que està representada per $\mathbb{C}/\langle 1,\tau\rangle$ i els coeficients d'un model de Weierstrass estan donats per mitjà de formes modulars el·líptiques en $\tau$. Aleshores, la funció $j$ es pot escriure com una funció racional dels coeficients del model de Weierstrass. En particular, des d'un punt de vista algebraic, la funció $j$ s'identifica amb un invariant absolut de formes cúbiques (o quàrtiques) binàries que classifica la classe d'isomorfisme de corbes el·líptiques.

S'han estudiat moltes més propietats de les funcions modulars; per exemple, la reducció del polinomi de nivell $p$ primer, $\Phi_p(X,Y)$, mòdul $p$ està donada per la fórmula de congruència de Kronecker; algunes funcions que generen les mateixes extensions abelianes que la funció $j$ en els punts CM però que proporcionen equacions definidores més senzilles, com ara les funcions modulars de Weber, etc. Les formes modulars, com hem vist, també apareixen en aquest context, ja sigui com a funcions theta de Jacobi o intervenint en el desenvolupament de funcions modulars (Eisenstein). Va ser Hecke [Hec24], però, qui va formalitzar-les i també qui va introduir les sèries $L$ associades a formes pròpies i va demostrar algunes de les seves propietats més importants, com ara l'existència de desenvolupaments en forma de producte d'Euler. D'aquesta manera, apareix un concepte clau que ha jugat un paper central en alguns dels resultats més importants de la teoria de nombres des de la segona meitat del segle XX.

Tot i això, no només s'han estudiat sistemàticament les funcions i formes automorfes associades a subgrups de $\mathbf{SL}(2,\mathbb{Z})$. Una de les primeres generalitzacions que es va considerar és la que correspon al grup modular de Hilbert $\mathbf{SL}(2,\mathcal{O}_K)$, en què $\mathbb{Z}$ es substitueix per l'anell d'enters d'un cos de nombres totalment real. D'aquesta manera s'obtenen les funcions i les formes modulars de Hilbert.

De la mateixa manera, si substituïm el semiplà superior pel conjunt de les matrius quadrades de dimensió $g$ sobre $\mathbb{C}$ amb part imaginària definida positiva i considerem el grup $\mathbf{Sp}(2g,\mathbb{Z})$ que hi actua, obtenim les funcions i formes modulars de Siegel, introduïdes per Siegel [Sie35, Sie39] a partir dels treballs de Riemann [Rie57] sobre superfícies de Riemann de gènere $g$. Per a $g=1$, aquesta noció coincideix amb el cas modular clàssic. Per a $g=2$, Igusa [Igu60, Igu62, Igu64, Igu67] realitzà un estudi detallat del cos de funcions i de l'anell graduat de formes modulars de Siegel, incloent la seva interpretació com a invariants de formes sèxtiques. Els generadors, en aquest cas, es coneixen com a invariants d'Igusa.

És en aquest context, i després dels treballs d'Eichler [Eic37, Eic38, Eic55a, Eic55b] sobre àlgebres de quaternions, que Shimura [Shi67] considera els grups fuchsians obtinguts a través d'una immersió a $\mathbf{SL}(2,\mathbb{R})$ dels elements de norma 1 en certs ordres d'una $\mathbb{Q}$-àlgebra de quaternions, $\mathbb{H}$, tal que $\mathbb{H}\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbf{M}(2,\mathbb{R})$, introduint així les corbes de Shimura que considerem en aquest treball. Shimura també considera el cas d'àlgebres de quaternions $\mathbb{H}$ sobre cossos totalment reals de grau $d$ sobre $\mathbb{Q}$ tals que $\mathbb{H}\otimes\mathbb{R}\cong\mathbf{M}(2,\mathbb{R})^r\times\mathbb{H}_{\mathbb{R}}^{d-r}$ per a un enter $0<r\leq d$, que dóna lloc a varietats de Shimura de dimensió $r$. I encara més generalment, en [Shi75a] la construcció s'estén al cas de grups reductius que actuen en certs dominis. Aquesta construcció fou reformulada per Deligne [Del71, Del79], donant lloc a una reinterpretació que és habitualment utilitzada avui en dia.

Nosaltres ens restringirem al cas racional, és a dir, considerarem una àlgebra de quaternions racional, indefinida i no escindida de discriminant $D>1$, $\mathbb{H}_D$, i una immersió $\Phi:\mathbb{H}_D\to\mathbf{M}(2,\mathbb{R})$. Aleshores, com és habitual, fixarem un ordre d'Eichler de nivell $N$, $\mathcal{O}(D,N)\subset\mathbb{H}_D$, i considerarem el grup $\Gamma(D,N)\subset\mathbf{PSL}(2,\mathbb{R})$, projecció en $\mathbf{PSL}(2,\mathbb{R})$ del subgrup de $\mathbf{SL}(2,\mathbb{R})$ obtingut a partir dels elements de norma 1 de $\mathcal{O}(D,N)$ a través de $\Phi$. Com en el cas modular, un cop fixada la immersió, obtenim uns punts destacats en $\mathcal{H}$, que corresponen a elements fixos per transformacions de $\mathbb{H}_D\smallsetminus\mathbb{Q}$, que també anomenarem punts CM. És convenient observar que si considerem un punt CM, els elements de $\mathbb{H}_D$ que el fixen són, de fet, un cos quadràtic $K$ dins $\mathbb{H}_D$ i, per tant, direm que el punt és CM pel cos $K$. De fet es pot veure que fixat $\mathcal{O}(D,N)$, els punts CM estan naturalment associats a ordres dins de cossos quadràtics mitjançant la noció d'immersió optimal, però, per a introduir les idees generals i explicar de manera resumida els resultats d'aquesta memòria, aquesta identificació menys precisa ja ens serà suficient. Una diferència destacable entre aquest cas i el cas modular és que en aquesta situació la superfície de Riemann $\Gamma(D,N)\backslash\mathcal{H}$ corresponent és sempre compacte. En particular, no existeixen desenvolupaments de Fourier per a les funcions automorfes corresponents. Gràcies a Shimura [Shi67], sabem que existeix un *model canònic* per a la superfície de Riemann corresponent a $\Gamma(D,N)$, és a dir, existeix una corba projectiva $X(D,N)$ definida sobre $\mathbb{Q}$ juntament amb un isomorfisme analític $j_{D,N} : \Gamma(D,N)\backslash\mathcal{H} \to X(D,N)(\mathbb{C})$, tal que els seus valors en un punt CM per un cos quadràtic $K$ generen determinats cossos de classes d'anell sobre $K$, en particular, s'obtenen extensions abelianes de $K$ que contenen el cos de classes de Hilbert de $K$. Per tal de demostrar això, Shimura identifica $X(D,N)$ com un espai groller de moduli. Més precisament, els punts en $X(D,N)(\mathbb{C})$ corresponen a classes d'isomorfisme de superfícies abelianes principalment polaritzades amb multiplicació quaterniònica per $\mathcal{O}(D,1)$ (habitualment anomenades corbes el·líptiques falses) i amb una certa estructura de nivell corresponent al nivell $N$ de la corba. Val la pena esmentar que la construcció de Shimura també engloba el cas modular clàssic i la funció $j_{D,N}$ juga el paper de la funció $j$ de Klein. Tot i així, si $\mathbb{H} \ncong \mathbf{M}(2,\mathbb{Q})$, Shimura mateix [Shi75a] demostra que la corba de Shimura corresponent no té punts reals. Per tant, la funció $j_{D,N}$, en una carta afí, no serà una funció automorfa sinó una tupla de funcions automorfes.

També cal destacar l'existència d'un grup d'involucions racionals de la corba $X(D,N)$ que denotem $W_{DN}$ i anomenem grup d'involucions d'Atkin-Lehner. Com a grup abstracte, aquest grup és un producte de grups cíclics d'ordre 2, $C_2^r$, on $r$ és el nombre de primers diferents que divideixen $DN$. A més està generat per involucions $\omega_{p^k}$ on $p^k \| DN$ i la seva acció en els punts complexos de la corba correspon a l'acció d'un determinat element de norma $p^k$ en $\mathcal{O}(D,N)$. En particular, per a cada subgrup $W \subset W_{DN}$, obtenim un quocient $X(D,N)^W$ de la corba $X(D,N)$ que anomenem quocient d'Atkin-Lehner.

Les propietats aritmètiques i diofàntiques de les corbes de Shimura han estat àmpliament estudiades des de la seva introducció als anys 60, per exemple: una uniformització $p$-àdica va ser obtinguda per Čerednik [Čer76] i Drinfeld [Dri76]; l'estudi de models enters, per Morita [Mor81], Boutot i Carayol [BC91], etc.; l'estudi dels punts locals, per Jordan i Livné [JL85], Ogg [Ogg85], etc.; l'existència de punts racionals en certs quocients d'Atkin-Lehner, per Jordan [Jor81], Ogg [Ogg83], etc. A més a més, han tingut un paper important, juntament amb les corbes modulars, en la demostració d'algunes de les conjectures més importants de la teoria de nombres dels últims temps, essencialment a través de la demostració de la modularitat de totes les corbes el·líptiques sobre $\mathbb{Q}$, gràcies als treballs d'Eichler, Shimura i Wiles, entre d'altres. D'aquesta manera, tant les corbes modulars com les de Shimura són espais (grollers) de moduli de corbes el·líptiques (falses) i, al mateix temps, donen parametritzacions modulars per a totes les corbes el·líptiques sobre $\mathbb{Q}$.

És convenient fer palès que la funció $j_{D,N}$ no és explícita en la construcció de Shimura, però fins i tot obviant la funció $j_{D,N}$, determinar equacions per a la corba $X(D,N)$ és en general un problema complicat. La primera equació d'una corba de Shimura (no modular) va ser calculada per Ihara [Iha79] i des de llavors s'han obtingut moltes més equacions seguint diferents tècniques, començant per Kurihara [Kur79], Jordan [Jor81] i Roberts [Rob89] i seguint amb Elkies [Elk98], Gonzàlez i Rotger [GR04, GR06], Molina [Mol12], etc. La majoria d'aquests autors fan ús de la uniformització de Čerednik-Drinfeld per a determinar algunes equacions. Elkies, però, utilitza la uniformització complexa donada pel model canònic de Shimura. D'aquesta manera, recupera també els punts CM corresponents a certs ordres quadràtics. Una combinació de les dues tècniques també ha estat utilitzada per Yang [Yan13] i Tu [Tu14], estenent d'aquesta manera els casos considerats per Elkies a les corbes que tenen certs recobriments per corbes de gènere 0, afegint el coneixement de certes equacions per a aquests recobriments, essencialment les obtingudes per Gonzàlez i Rotger que acabem de mencionar.

Per tal de treballar directament amb formes i funcions automorfes corresponents a corbes de Shimura associades a àlgebres de quaternions indefinides i no escindides sobre $\mathbb{Q}$ és convenient considerar algun tipus de desenvolupament en sèrie de potències que substitueixi els desenvolupaments de Fourier que s'utilitzen habitualment en el cas modular. Les idees subjacents d'aquests tipus de desenvolupaments es poden remuntar a Shimura [Shi75a] i han estat estudiats des de diferents punts de vista per Mori [Mor94, Mor95, Mor11] i per Bayer [Bay02] i Bayer i Travesa [BT07a, BT07b, BT08]. En particular, en [BT07a, BT07b] es calculen desenvolupaments explícits per a algunes funcions modulars i automorfes al voltant de punts CM per a la corba modular i també per a la corba de Shimura associada a un ordre maximal en l'àlgebra de quaternions racional de discriminant 6. El cas de formes automorfes per al cas de discriminant 6 també ha estat estudiat per Baba i Granath [BG12]. Per altra banda, Voight i Willis [VW14] han utilitzat tècniques numèriques per a obtenir desenvolupaments de formes automorfes al voltant de punts CM. De la mateixa manera, Voight [Voi06] ja havia considerat l'ús d'aproximacions per al càlcul de punts CM en el cas triangular. Per altra banda, Voight [Voi10] també considera la determinació de formes automorfes associades a corbes de Shimura mitjançant la correspondència de Jacquet-Langlands.

Finalment, el problema de donar equacions per a les corbes de gènere 2 que donen lloc a les superfícies abelianes corresponents als punts de $X(D,N)(\mathbb{C})$ ha estat estudiat per Rotger [Rot04] i Bayer i Guàrdia [BG05] han considerat el seu càlcul explícit utilitzant aproximacions numèriques. Baba i Granath [BG08] obtenen funcions generadores (Hauptmoduln) per al cos de funcions racionals del quocient maximal d'Atkin-Lehner de les corbes de discriminants 6 i 10 mitjançant invariants d'Igusa a partir de les famílies de Hashimoto-Murabayashi [HM95]. D'aquesta manera, obtenen una fórmula que és similar en essència a la que relaciona $j$ amb els coeficients d'un model de Weierstrass de la corresponent corba el·líptica. En [BT08] aquest Hauptmodul, per a $D=6$, es relaciona amb l'obtingut a [BT07b].

Per tant, resulta natural plantejar-se si podem utilitzar els desenvolupaments de funcions automorfes al voltant de punts CM considerats en [BT07b] per ajudar en la determinació d'una uniformització explícita d'un quocient d'Atkin-Lehner de gènere 0 d'una corba de Shimura, contribuint d'aquesta manera a una millor comprensió dels problemes oberts plantejats per Elkies [Elk98, §5.5], en especial aquells referits al càlcul de recobriments i de punts CM. Aquest és el principal tema que hem considerat en aquesta tesi, però, en paral·lel, també tractem altres problemes que ens apareixeran i que comentarem a continuació.

El Capítol 1 d'aquesta memòria està dedicat a introduir alguns conceptes i resultats bàsics que resulten necessaris al llarg de la tesi.

En el Capítol 2 introduïm la generalització natural de les equacions modulars al nostre context de corbes de Shimura, donant lloc als que denominem polinomis kroneckerians. Per tal de fer això, analitzem algunes de les propietats dels elements de norma $N$ en un ordre d'Eichler $\mathcal{O}(D,M)$, amb $\mathrm{mcd}(DM,N)=1$.

Per a construir els polinomis kroneckerians, fixem un subgrup del grup d'involucions d'Atkin-Lehner $W \subseteq W_{DM}$, considerem el grup $\Gamma = \Gamma(D,M)\langle W\rangle$ i suposem que la corba $X(D,M)^W$ és de gènere 0. En particular, existeix un Hauptmodul $f:\Gamma\backslash\mathcal{H}\to\mathbb{C}\cup\{\infty\}$.

Aleshores, veiem que el polinomi

$$\tilde{\Psi}_{\Gamma,N,f}(X,f) = \prod_{\gamma\in(\Gamma\cap\omega_N^{-1}\Gamma\omega_N)\backslash\Gamma} (X - f\circ\omega_N\circ\gamma),$$

on $\omega_N$ és la transformació associada a un element primitiu de norma $N$ en $\mathcal{O}(D,M)$ que representa la involució d'Atkin-Lehner de nivell $N$ en $X(D,MN)$, pertany a $\mathbb{C}(f)[X]$.

**Definició.** Anomenem polinomi kroneckerià de nivell $N$ associat al Hauptmodul $f$ pel grup $\Gamma$ a un polinomi $\Psi_{\Gamma,N,f}(X,Y)\in\mathbb{C}[X,Y]$ primitiu com a polinomi en la variable $Y$ i tal que $\Psi_{\Gamma,N,f}(X,f)=P(f)\tilde{\Psi}_{\Gamma,N,f}(X,f)$, on $P(f)\in\mathbb{C}(f)$.

En particular, convé notar que aquests polinomis estan determinats únicament mòdul una constant multiplicativa de $\mathbb{C}^*$. Les seves propietats més bàsiques estan recollides en el Teorema 2.1.

**Teorema 2.1.** *Sigui $f$ un Hauptmodul per a la corba $X(D,M)^W_{\mathbb{C}}$. Per a tot enter positiu $N$ coprimer amb $DM$, el polinomi $\Psi_{\Gamma,N,f}(X,Y)$ és un polinomi irreductible de grau $\psi(N)$ en cada indeterminada. Si $N=1$, podem prendre $\Psi_{\Gamma,1,f}(X,Y)=X-Y$; si $N>1$, aleshores $\Psi_{\Gamma,N,f}(X,Y)$ és simètric. A més a més, si $f$ és un Hauptmodul definit sobre un cos de nombres $K$, aleshores, per a una elecció adequada de la constant multiplicativa de la definició, $\Psi_{\Gamma,N,f}(X,Y)\in K[X,Y]$.* □

En particular, el polinomi $\Psi_{\Gamma,N,f}(X,Y)$ proporciona un model pla (i singular) per a la corba $X(D,MN)^W$ sobre $K$, si el Hauptmodul està definit sobre $K$, però no només això ja que també podem obtenir com a corol·lari un model per a la corba $X(D,MN)^{\langle W,\omega_N\rangle}$ sobre $K$.

**Corol·lari 2.3.** *Siguin $K$ un cos de nombres i $f$ un Hauptmodul definit sobre $K$ de $X(D,M)^W$. Per a tot enter $N>1$ coprimer amb $DM$, el polinomi*

$$\Psi_{\Gamma,N,f}(X+\sqrt{Y},X-\sqrt{Y})$$

*pertany a $K[X,Y]$, és irreductible sobre $\mathbb{C}$ i proporciona un model sobre $K$ per a la corba $X(D,MN)^{\langle W,\omega_N\rangle}$.* □

Com en el cas clàssic, en la Proposició 2.5 demostrem que, a partir dels polinomis kroneckerians, també podem obtenir els valors del Hauptmodul corresponents a uns punts CM determinats.

**Proposició 2.5.** *Siguin $W$ un subgrup de $W_{DM}$, $\Gamma=\Gamma(D,M)\langle W\rangle$ i $f$ un Hauptmodul per la corba $X(D,M)^W_{\mathbb{C}}$. Considerem el conjunt $S=\{s:\omega_s\in W\}$.*

*Donat $\tau_0 \in \mathcal{H}$ un punt CM per l'ordre $\Lambda$ en el cos quadràtic $K$, la multiplicitat $f(\tau_0)$ com a arrel de $\Psi_{\Gamma,N,f}(X,X)$ és donada per*

$$\begin{aligned} r(\Lambda, N, S) &= \#\{\lambda \in \Lambda' : \omega_s\phi(\lambda) \in s\mathcal{O}(D,M)_N \text{ per a un cert } s \in S\}/\sim \\ &= \#\{\lambda \in \Lambda' : \omega_s\phi(\lambda) \in s\mathcal{O}(D,M)_N \text{ per a un cert } s \in S\}/2e(\tau_0), \end{aligned}$$

*on $x \sim y$ si i només si $x\overline{y} \in N\Lambda$, $e(\tau_0)$ denota l'ordre el·líptic del punt $\tau_0$ i $\overline{\cdot}$ denota la conjugació.*

*En particular, $z_0 \in \mathbb{C}$ és una arrel del polinomi $\Psi_{\Gamma,N,f}(X,X) \in \mathbb{C}[X]$ si i només si $z_0 = f(\tau_0)$ per a un punt CM, $\tau_0 \in \mathcal{H}$, associat a una immersió optimal $\phi : \Lambda \hookrightarrow \mathcal{O}(D,M)$ tal que existeix un element primitiu $\lambda \in \Lambda$ de norma $sN$ per a un cert $s \in S$ per al qual $\omega_s\phi(\lambda) \in s\mathcal{O}(D,M)$ (cf. Proposició 2.3).* $\square$

Finalment, pel que fa a les propietats generals dels polinomis kroneckerians, ens centrem a estudiar les seves singularitats. Comencem observant que totes les seves singularitats provenen de punts CM i aleshores ens concentrem en l'estudi del nombre de branques per un punt CM determinat. Més precisament, anomenem multiplicitat de $\Psi$ en $(P, \omega_N P)$, i la denotem $\mathrm{mult}_{(P,\omega_N P)}\Psi$, al nombre de branques de $\Psi(X,Y) = 0$ a través de $(f(P), f(\omega_N P))$.

En el Teorema 2.2 relacionem aquesta multiplicitat amb un determinat ordre d'anul·lació.

**Teorema 2.2.** *Sigui $X(D,M)^W$ una corba de gènere 0 amb un Hauptmodul $f$. Denotem per $\mathcal{O}$ i $\Gamma = \langle\Gamma(D,M), W\rangle$ l'ordre i el grup corresponents. Per a un enter positiu $N$ coprimer amb $DM$, fixem un ordre d'Eichler de nivell $MN$ dins $\mathcal{O}$, $\mathcal{O}(N)$, i considerem $\Gamma(N) = \langle\Gamma(D,MN), W\rangle \subset \Gamma$ i $\omega_N$ la involució de nivell $N$ corresponent. Per a un punt $P \in \mathcal{H}$, denotem per*

$$\Gamma_P = \{\gamma \in \Gamma : \gamma(P) = P\}$$

*la isotropia de $P$ i*

$$(\Gamma\backslash\mathcal{O}'_N)_P = \{\Gamma\alpha \in \Gamma\backslash\mathcal{O}'_N : \alpha(P) = \omega_N(P) \text{ en } \Gamma\backslash\mathcal{H}\}.$$

*Aleshores,*

*(1) $(\Gamma_P \cap \Gamma(N))\backslash\Gamma_P$ actua per la dreta en $(\Gamma\backslash\mathcal{O}'_N)_P$, de la manera següent*

$$\begin{array}{rcl} \cdot[\epsilon] : (\Gamma\backslash\mathcal{O}'_N)_P & \to & (\Gamma\backslash\mathcal{O}'_N)_P \\ \Gamma\alpha & \mapsto & \Gamma\alpha\epsilon. \end{array}$$

*A més a més, aquesta acció és lliure, és a dir, si $[\epsilon] \neq [1]$, llavors $\cdot[\epsilon]$ no fixa cap element.*

*(2) Les branques del polinomi $\Psi(X,Y) = \Psi_{\Gamma,N,f}(X,Y)$ a través de $(P, \omega_N P)$ estan en correspondència bijectiva amb les òrbites d'aquesta acció. Els elements en una òrbita fixada, $\alpha_i$, donen els elements $(f, f \circ \alpha_i)$ que defineixen una branca. En particular,*

$$\mathrm{ord}_{f(\omega_N P)} \Psi(f(P), X) = \frac{e_1(P)}{e_N(P)} \mathrm{mult}_{(P, \omega_N(P))} \Psi(X,Y),$$

*on $e_a(P)$ denota l'ordre el·líptic del punt $P$ en $X(D, Ma)^W$.* □

En el Teorema 2.3, ens centrem en el cas de polinomis kroneckerians de nivell potència de primer i donem una descripció explícita dels punts singulars i de la seva multiplicitat.

En la segona part del Capítol 2 ens centrem en les corbes $X(D,N)$ de gènere 0, que són exactament $X(D,1)$, $D = 6, 10, 22$. En el Teorema 2.4 donem una construcció unificada dels Hauptmoduln $t_D$ per a aquestes tres corbes i $u_{D,1}, u_{D,2}, u_{D,3}$ i $t_D^+$ per als seus quocients d'Atkin-Lehner. Els Hauptmoduln $t_D$ estan definits sobre cossos quàrtics i la resta sobre $\mathbb{Q}$. Tot i que $t_D$ no estigui definit sobre $\mathbb{Q}$, en la Proposició 2.6 provem que els polinomis kroneckerians corresponents sí que estan definits sobre $\mathbb{Q}$.

**Proposició 2.6.** *Per a tot $N > 1$ coprimer amb $D$, el polinomi $\Psi_{\Gamma,N,t_D}(X,Y)$ es pot prendre com un polinomi primitiu en $\mathbb{Z}[X,Y]$. A més, el grau global de tots els monomis no nuls de $\Psi_{\Gamma,N,t_D}(X,Y)$ és parell i, si escrivim*

$$\Psi_{\Gamma,N,t_D}(X,Y) = \sum_{j,k=0}^{\psi(N)} a_{jk} X^j Y^k, \qquad a_{j,k} \in \mathbb{Z},$$

*aleshores se satisfan les relacions següents:*

$$a_{jk} = a_{kj} = a_{\psi(N)-k, \psi(N)-j} = a_{\psi(N)-j, \psi(N)-k}.$$

□

Finalment, en el Teorema 2.5 estudiem la reducció mòdul $p$ dels polinomis $\Psi_{\Gamma,p,t_D}(X,Y)$ a partir de la llei de reciprocitat de Shimura i de les relacions concretes entre les funcions $u_{D,1}$, $u_{D,2}$, $u_{D,3}$. Obtenim així un anàleg en aquest cas de la fórmula de congruència de Kronecker.

**Teorema 2.5.** *Siguin $D = 6, 10, 22$ i $p$ un primer que no divideixi $D$. Aleshores, la forma general de la reducció mòdul $p$ del polinomi kroneckerià $\Psi_{\Gamma,p,t_D}(X,Y)$*

*només depèn de p* (mod $4c(D)$)*. Més precisament, llevat d'una constant multiplicativa no nul·la, es té que*

$$\Psi_{\Gamma,p,t_D}(X,Y) \equiv \begin{cases} (X-Y^p)(Y-X^p)\,(\mathrm{mod}\,p), & si\ p\equiv 1\,(\mathrm{mod}4)\ i\ \left(\frac{c(D)}{p}\right)=1,\\ (X+Y^p)(Y+X^p)\,(\mathrm{mod}\,p), & si\ p\equiv 1\,(\mathrm{mod}4)\ i\ \left(\frac{c(D)}{p}\right)=-1,\\ (1-XY^p)(1-YX^p)\,(\mathrm{mod}\,p), & si\ p\equiv 3\,(\mathrm{mod}4)\ i\ \left(\frac{c(D)}{p}\right)=-1,\\ (1+XY^p)(1+YX^p)\,(\mathrm{mod}\,p), & si\ p\equiv 3\,(\mathrm{mod}4)\ i\ \left(\frac{c(D)}{p}\right)=1, \end{cases}$$

*on*

$$c(D)=\begin{cases} 3, & si\ D=6,\\ 2, & si\ D=10,\\ 11, & si\ D=22. \end{cases}$$

□

En el Capítol 3 comencem introduint l'algoritme que hem utilitzat per a calcular dominis fonamentals per als grups $\Gamma(D,N)$ basant-nos en el mètode de Ford i l'utilitzem per al càlcul de dominis fonamentals per als grups $\Gamma(22,1)$ i $\Gamma(D,1)\langle W_D\rangle$ per a $D=6,10,22$. A continuació, introduïm els paràmetres respecte els quals farem els desenvolupaments al voltant d'un punt de multiplicació complexa $P$, $q(z)=\kappa\frac{z-P}{z-\overline{P}}$, $\kappa\in\mathbb{C}^*$, i les equacions diferencials satisfetes pels Hauptmoduln: donat un Hauptmodul $f:\mathcal{H}\to\mathbb{C}\cup\{\infty\}$ corresponent a una corba de gènere 0, aleshores existeix una funció racional $R(X)\in\mathbb{C}(X)$ tal que

$$Da(f(z),z)+R(f(z))=0,$$

on

$$Da(f(z),z)=\frac{2f'(z)f'''(z)-3(f''(z))^2}{(f'(z))^4}$$

i $f'(z)=\frac{df(z)}{dz}$. La Proposició 3.8 dóna una descripció més explícita de la funció $R$, però llevat d'algun cas molt concret, no se'n té una expressió concreta en general. També estudiem la relació d'aquesta equació amb equacions diferencials lineals. Llavors, utilitzem aquests desenvolupaments, l'equació diferencial satisfeta per un Hauptmodul fixat i un polinomi kroneckerià corresponent de nivell petit per a determinar una uniformització explícita de la corba $X(22,1)^{W_{22}}$, que pel que sabem, no havia estat calculada abans. També recuperem les uniformitzacions de les corbes $X(D,1)^{W_D}$, per a $D=6,10$, obtingudes amb anterioritat per Elkies [Elk98] i Bayer i Travesa [BT07b]. Resumim totes aquestes uniformitzacions en el Teorema 3.1.

**Teorema 3.1.** *Sigui $D = 6, 10, 22$.*

*(1) Sigui $f$ qualsevol de les funcions introduïdes en el Teorema 2.4. Aleshores, existeix una funció racional $R(X) \in \mathbb{Q}[X]$ tal que $Da(f(z), z) + R(f(z)) = 0$ i el valor exacte de $R(X)$ es pot trobar en la taula següent:*

| $f$ \ $D$ | 6 | 10 |
|---|---|---|
| $t_D^+$ | $\frac{108-113f+140f^2}{144(-1+f)^2f^2}$ | $\frac{2187-3183f+3067f^2-208f^3+12f^4}{4(-1+f)^2f^2(-27+2f)^2}$ |
| $u_{D,1}$ | $\frac{-113+9f^2}{12(3+f^2)^2}$ | $-\frac{10\left(303-22f^2+7f^4\right)}{(2+f^2)^2(27+f^2)^2}$ |
| $u_{D,2}$ | $\frac{96-49f^2+9f^4}{12f^2(3+f^2)^2}$ | $\frac{2\left(-1075+614f^2+29f^4\right)}{(-5+f)^2(5+f)^2(2+f^2)^2}$ |
| $u_{D,3}$ | $\frac{-103+32f^2}{36(1+f^2)^2}$ | $-\frac{2859+5522f^2+2675f^4}{(1+f^2)^2(27+25f^2)^2}$ |
| $t_D$ | $\frac{27+74f^2+27f^4}{36(-1+f)^2f^2(1+f)^2}$ | $\frac{128\left(1+23f^2+f^4\right)}{(1-52f^2+f^4)^2}$ |

| $f$ \ $D$ | 22 |
|---|---|
| $t_D^+$ | $\frac{8748-17697f+18316f^2-10624f^3+3072f^4}{16(-1+f)^2f^2(-27+16f)^2}$ |
| $u_{D,1}$ | $\frac{-80619-6743f^2-193f^4+3f^6}{4(11+f^2)^2(27+f^2)^2}$ |
| $u_{D,2}$ | $\frac{-2816+8800f^2+61f^4+3f^6}{4(-4+f)^2(4+f)^2(11+f^2)^2}$ |
| $u_{D,3}$ | $-\frac{10167+14240f^2+5888f^4}{4(1+f^2)^2(27+16f^2)^2}$ |
| $t_D$ | $\frac{11\left(33+508f^2+3014f^4+508f^6+33f^8\right)}{4f^2(11-86f^2+11f^4)^2}$ |

*(2) Els valors de les funcions $t_D^+$ en els punts CM per l'anell d'enters d'alguns cossos quadràtics imaginaris de discriminants petits es poden trobar a la taula següent. El símbol $*$ indica que no hi ha punts CM per aquest cos en*

*la corba corresponent.*

| | $-3$ | $-4$ | $-8$ | $-11$ | $-19$ | $-20$ | $-24$ | $-40$ |
|---|---|---|---|---|---|---|---|---|
| $t_6^+$ | $\infty$ | $1$ | $*$ | $*$ | $\frac{3211}{1024}$ | $*$ | $0$ | $\frac{2312}{125}$ |
| $t_{10}^+$ | $\frac{27}{2}$ | $*$ | $1$ | $*$ | $*$ | $\infty$ | $*$ | $0$ |
| $t_{22}^+$ | $\frac{27}{16}$ | $1$ | $*$ | $\infty$ | $*$ | $\frac{5}{16}$ | $*$ | $*$ |

$\square$

Aleshores, busquem una expressió del Hauptmodul $t_{22}^+$ en funció dels invariants d'Igusa. Per tal de fer això considerem una immersió de la corba $X(22,1)^{W_{22}}$ en la varietat d'Igusa i identifiquem la imatge com un cert factor d'una intersecció de superfícies de Humbert. Això ens permet obtenir una expressió per al Hauptmodul $t_{22}^+$ mitjançant invariants d'Igusa tal com recollim en el Teorema 3.10.

**Teorema 3.10.** *Siguin $\epsilon_{22} : \mathcal{H} \to \mathcal{H}_2$ la immersió de $\mathcal{H}$ en el semiplà superior de Siegel de gènere 2 donada a la pàgina 96, i $g_{22} : \mathcal{H} \to \mathbb{C} \cup \{\infty\}$ definida com*

$$g_{22}(z) = \frac{g_{22,1}}{g_{22,2}},$$

*on*

$$\begin{aligned}
g_{22,1} =\, & 11(2392824775436786785 s_2(\epsilon_{22}(z))^4 s_3(\epsilon_{22}(z)) \\
& - 3064759317668640600 0 s_2(\epsilon_{22}(z)) s_3(\epsilon_{22}(z))^3 \\
& - 3582508007923968630 s_2(\epsilon_{22}(z))^3 s_5(\epsilon_{22}(z)) \\
& + 7979426944408457769 6 s_3(\epsilon_{22}(z))^2 s_5(\epsilon_{22}(z)) \\
& - 5916450736450366200 0 s_2(\epsilon_{22}(z)) s_3(\epsilon_{22}(z)) s_6(\epsilon_{22}(z)) \\
& + 1154732324129323917 12 s_5(\epsilon_{22}(z)) s_6(\epsilon_{22}(z))), \\
g_{22,2} =\, & 2528 s_2(\epsilon_{22}(z))(211603945426835 s_2(\epsilon_{22}(z))^3 s_3(\epsilon_{22}(z)) \\
& - 114771506545713600 s_3(\epsilon_{22}(z))^3 \\
& - 1067959426339630 98 s_2(\epsilon_{22}(z))^2 s_5(\epsilon_{22}(z)) \\
& + 613643668464340800 s_3(\epsilon_{22}(z)) s_6(\epsilon_{22}(z))),
\end{aligned}$$

*i*

$$\begin{aligned}
s_2 &= 12(J_2^2 - 24J_4),\\
s_3 &= 12(J_2^3 - 36J_2J_4 + 216J_6),\\
s_5 &= 60(82944J_{10} + J_2^5 - 60J_2^3J_4 + 864J_2J_4^2 + 216J_2^2J_6 - 5184J_4J_6),\\
s_6 &= 12(497664J_{10}J_2 + 11J_2^6 - 792J_2^4J_4 + 18144J_2^2J_4^2 - 124416J_4^3 + 864J_2^3J_6\\
&\quad - 31104J_2J_4J_6 + 93312J_6^2),
\end{aligned}$$

*on $J_2, J_4, J_6, J_{10}$ són els invariants d'Igusa projectius. Aleshores, el Hauptmodul $t_{22}^+$ admet l'expressió següent: $t_{22}^+ = \dfrac{925g_{22}+176}{1024g_{22}+176}$.* □

Llavors, utilitzant el mètode de Mestre, podem recuperar equacions per a una corba que dóna lloc a la superfície abeliana en funció de $t_{22}^+$ o, equivalentment, en funció de $u_{22,2}$, $u_{22,3}$. Per a poder fer això, hem d'eliminar un conjunt finit de punts que també determinem explícitament, tal com apareix en el Corol·lari 3.4.

**Corol·lari 3.4.** *Sigui $S$ el conjunt de punts CM associats als anells d'enters dels cossos quadràtics $K \in \{\mathbb{Q}(i), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-5}), \mathbb{Q}(\sqrt{-11})\}$. Les funcions $v_1 = u_{22,3}$ i $v_2 = \frac{1}{u_{22,2}}$ satisfan $v_1^2 + 11v_2^2 + 1 = 0$ i donen una altra carta afí per al model canònic de $X(22,1)$ donat al Teorema 2.4. Aleshores, una corba de gènere 2 que dóna lloc a la superfície abeliana corresponent a un punt de $X(22,1)(\mathbb{C}) \smallsetminus S$ és donada per*

$$\begin{aligned}
Y^2 = &- 512\sqrt{2}X\left(3 - 2X^2 + 3X^4\right)v_1^5\\
&+ 256\sqrt{-22}(-1+X)(1+X)\left(1 + 10X^2 + X^4\right)v_1^4v_2\\
&+ 512(1+X^2)^3v_1^4\\
&- 2048\sqrt{-11}(1+X^2)(-1+X)X(1+X)v_1^3v_2\\
&- 64\sqrt{2}X\left(57 - 134X^2 + 57X^4\right)v_1^3\\
&+ 96\sqrt{-22}(-1+X)(1+X)\left(1 + 42X^2 + X^4\right)v_1^2v_2\\
&+ 208\left(1+X^2\right)\left(1 + 26X^2 + X^4\right)v_1^2\\
&- 640\sqrt{-11}(1+X^2)(-1+X)X(1+X)v_1v_2\\
&- 2\sqrt{2}X\left(759 - 4282X^2 + 759X^4\right)v_1\\
&+ 5\sqrt{-22}(-1+X)(1+X)\left(1 + 202X^2 + X^4\right)v_2\\
&- 7\left(1+X^2\right)\left(1 - 598X^2 + X^4\right).
\end{aligned}$$

□

Finalment, també estudiem la bona reducció potencial de les corbes obtingudes.

**Proposició 3.19.** *Donat un punt de $X(22,1)^+(\overline{\mathbb{Q}})$, la corba corresponent té bona reducció potencial en $p \neq 2, 11$ si i només si*

$$\nu_p((t_{22}^+ - 1)(16t_{22}^+ - 27)(16t_{22}^+ - 5)) = 0.$$

□

En el Capítol 4 comencem calculant totes les parelles $(D, N)$ tals que la corba $X(D,N)^{W_{DN}}$ té gènere 0, que és el conjunt de corbes per a les quals podríem seguir el mateix mètode que hem utilitzat en el Capítol 3 per a obtenir-ne una uniformització explícita, almenys des d'un punt de vista teòric. En la Proposició 4.1, donem una llista de les 271 corbes $X(D, N)$ que tenen un quocient d'Atkin-Lehner de gènere 0 juntament amb tota la informació necessària per a calcular el gènere i el nombre de cicles el·líptics (i els corresponents ordres quadràtics) de qualsevol dels seus quocients d'Atkin-Lehner. Aquelles corbes amb $N$ lliure de quadrats ja han estat determinades a [LMR06]. A més a més, demostrem que en tots aquests casos, la corba $X(D,N)^{W_{DN}}$ és isomorfa a $\mathbb{P}^1_{\mathbb{Q}}$. També podem observar directament que el nombre de cicles el·líptics de la corba $X(D,N)^{W_{DN}}$ és sempre més gran o igual que el de la corba $X(D,1)^{W_D}$, que també és de gènere 0. Com que la complexitat computacional del procediment descrit en el Capítol 3 depèn en gran part del nombre de cicles el·líptics, és millor determinar una uniformització explícita de la corba $X(D,1)^{W_D}$ i utilitzar-la després per a obtenir una uniformització de $X(D,N)^{W_{DN}}$. En aquest capítol, veiem com podem combinar els polinomis kroneckerians amb les idees utilitzades en [Elk98] per a fer-ho. Per a il·lustrar aquest procediment, considerem les corbes estudiades en el Capítol 3 i uniformitzem les corbes $X(D, N)$, $D = 6, 10, 22$, juntament amb els seus quocients, per als tres valors més petits possibles de $N > 1$ en cada cas. Donem Hauptmoduln per als quocients racionals i equacions per a la resta de corbes de gèneres $0, 1, 2$ o hiperel·líptiques de gènere superior. En la Secció 4.2 estudiem el cas $(22, 3)$ en detall i en les seccions posteriors estudiem la resta de casos, on ja posem l'èmfasi en les particularitats de cada cas. Pel que sabem, no es tenien uniformitzacions explícites de les corbes corresponents a $(6, 11)$, $(10, 9)$ ni per a cap de les de discriminant 22. En aquest cas la complexitat del càlcul depèn en gran part del grau del recobriment $X(D,N)^{W_D} \to X(D,1)^{W_D}$, que és $\psi(N)$. Ara bé, si $N$ és compost (o potència de primer), podem trencar aquest càlcul en parts més petites (per exemple per a $X(10, 9)$). A més a més, també podem utilitzar aproximacions numèriques o

$p$-àdiques, i fins i tot podem utilitzar programari d'àlgebra computacional per a resoldre els casos més senzills.

En el Capítol 5 estudiem les propietats aritmètiques dels coeficients dels desenvolupaments, respecte paràmetres locals adequats, de les formes (i funcions) automorfes associades als grups $\Gamma(D,N)$ al voltant dels punts CM. Per tal de fer això, donats un punt CM, $P$, i una forma automorfa de pes $2r$, $f$, considerem un paràmetre local en $P$ de la forma $q(z) = \kappa\frac{z-P}{z-\overline{P}}$ amb $\kappa \in \mathbb{C}^*$ i aleshores el corresponent desenvolupament

$$f = \frac{1}{(2\pi i)^r}\left(\frac{(q-\kappa)^2}{\kappa(P-\overline{P})}\right)^r \sum_{n\geq 0} a_n q^n \in A_{2r}(\Gamma).$$

En primer lloc, relacionem els coeficients $a_n$ amb els valors de certs operadors diferencials, íntimament relacionats amb els operadors de Shimura-Maaß. Aleshores, procedint de manera similar a [Shi75a], obtenim un resultat d'algebraicitat per als coeficients $a_n$, després de normalitzar-los adequadament, formulat sobre cossos de nombres explícits, que enunciem en el Teorema 5.1. En particular, demostrem l'algebraicitat dels coeficients d'aquests desenvolupaments de funcions automorfes per $\Gamma(D,N)$, generalitzant així a $D$ arbitrari i sobre cossos de nombres explícits els corresponents resultats donats en [BT08].

**Corol·lari 5.2.** *Sigui $P$ un punt CM per un ordre $\Lambda \subset \mathcal{O}_K$. Denotem $K^\Lambda$ el corresponent cos de classes d'anell. Aleshores, existeix $\kappa = \kappa_P \in \mathbb{C}^*$ tal que per a tot $f \in A_{2r}(\Gamma)_{L\cdot K^\Lambda}$, $a_n \in L\cdot K^\Lambda$, per a tot $n$. A més a més, $\kappa\pi_{D_K}^{-1} \in \overline{\mathbb{Q}}$, per a*

$$\pi_{D_K} = \left(\prod_{m=1}^{|D_K|}\Gamma\left(\frac{m}{|D_K|}\right)^{\left(\frac{D_K}{m}\right)}\right)^s,$$

*on*

$$s = \begin{cases} 2, & si\ K = \mathbb{Q}(i), \\ 3, & si\ K = \mathbb{Q}(\sqrt{-3}), \\ 1/h(K), & altrament. \end{cases}$$

□

En el Teorema 5.2 demostrem en certa manera l'altra implicació, donant una caracterització del cos de definició d'una forma o funció a partir dels coeficients del desenvolupament al voltant d'un punt CM.

**Teorema 5.2.** *Siguin $P$ un punt CM per $\Lambda \subset \mathcal{O}_K$, $\kappa \in \mathbb{C}^*$ com en el corol·lari anterior i $q$ el corresponent paràmetre local. Sigui també $f \in A_{2r}(\Gamma)$. Aleshores, $f \in A_{2r}(\Gamma)_{L \cdot K^\Lambda}$ si i només si $a_n \in L \cdot K^\Lambda$ per a tot $n$.* □

A partir d'aquest resultat, en el Corol·lari 5.3 obtenim una nova prova d'un resultat d'Ihara [Iha74] que ens assegura que els coeficients de l'equació diferencial corresponent a un Hauptmodul definit sobre un cos també té coeficients en el mateix cos.

**Corol·lari 5.3.** *Sigui $f \in A_0(\Gamma)_L$. Aleshores, $Da(f, z) \in A_0(\Gamma)_L$.* □

Finalment, en el Teorema 5.3, estudiem l'acció del grup de Galois en els coeficients d'aquests desenvolupaments i obtenim d'aquesta manera una extensió de la llei de reciprocitat de Shimura a tots els coeficients del desenvolupaments d'una funció i no només per als valors de la funció, que podem interpretar com la llei de reciprocitat de Shimura clàssica.

En el Capítol 6 tractem algunes aplicacions més computacionals de les eines i els objectes introduïts en els capítols previs. En la Secció 6.1 estudiem el càlcul explícit de polinomis kroneckerians d'un determinat nivell quan es tenen desenvolupaments al voltant de certs punts CM i a més a més donem alguns exemples per a $D = 6$. Tot i això, per a algunes de les aplicacions posteriors, hem calculat alguns polinomis més que per motius d'espai no hem reproduït, més precisament per a $D = 6, 10, 22$ hem calculat els polinomis kroneckerians de nivell primer per a tots els primers coprimers amb $D$ i menors o iguals que 41.

En la Secció 6.2 considerem el problema de calcular els valors d'un Hauptmodul donat en un punt CM concret de manera determinista, sense fer ús d'aproximacions, gràcies a la Proposició 2.5. En particular, ho utilitzem per a calcular tots els punts CM racionals i quadràtics de les corbes $X(D, 1)^{W_D}$, $D = 6, 10, 22$, en la Secció 6.2.1. Així obtenim una nova demostració que els valors racionals calculats per Elkies a [Elk98] per als casos $D = 6, 10$ són correctes (també donem un valor omès en [Elk98] per al cas $D = 10$). Una altra demostració utilitzant eines completament diferents va ser obtinguda per Errthum [Err11]. El nostre procediment també es pot utilitzar per a comprovar si els valors de les taules en [Elk98] per als casos $D = 14, 15$ són correctes. A més a més, no ens restringim al cas dels valors dels Hauptmoduln i també proporcionem un mètode per al càlcul dels primers termes d'un desenvolupament en un punt CM arbitrari. Un cop es disposa dels primers termes, es pot utilitzar l'equació diferencial per a calcular tants termes com calguin.

En la Secció 6.3 comencem donant els primers termes dels desenvolupaments al voltant de tots els punts CM racionals per als Hauptmoduln de cadascuna de les corbes $X(D,1)^{W_D}$ que hem determinat anteriorment. Tot i això, per a poder treballar realment amb aquestes funcions no només calen els desenvolupaments, sinó també el paràmetre $q$ respecte al qual hem calculat el desenvolupament; també donem aquests valors en la Secció 6.3. Cal destacar, però, que aquestes constants, amb l'excepció de tres valors per al cas $D = 6$, s'han obtingut mitjançant aproximació i identificació i per tant, tot i que creiem fermament que els valors que donem són correctes, com que no coneixem cap mètode per a demostrar que ho són, només podem garantir que són aproximacions dels valors reals (com a mínim fins a 150 xifres decimals, però en cada cas es poden comprovar més dígits). En qualsevol cas, això és suficient, en general, per a obtenir aproximacions dels valors de la funció o de la seva derivada en un entorn del punt.

Finalment, en la Secció 6.4 tractem el problema de calcular desenvolupaments per a formes automorfes. Amb aquest propòsit, suposem que $X(D,N)$ és tal que $X(D,N)^{W_{DN}}$ és de gènere 0 i en la Proposició 6.5 calculem una base de l'espai de formes de pes $k$ per a la corba $X(D,N)$ a partir d'un Hauptmodul per a $X(D,N)^{W_{DN}}$. També estudiem com determinar l'acció d'un operador de Hecke determinat en l'espai de formes d'un pes determinat. Aleshores, en les seccions 6.4.1, 6.4.2, 6.4.3 donem un sistema de generadors i les relacions corresponents per a l'àlgebra graduada de formes automorfes per a $D = 6, 10, 22$ i els nivells per als quals hem calculat uniformitzacions en els capítols anteriors. També donem alguns exemples de desenvolupaments d'aquestes formes, juntament amb (aproximacions de) els pàrametres locals corresponents.

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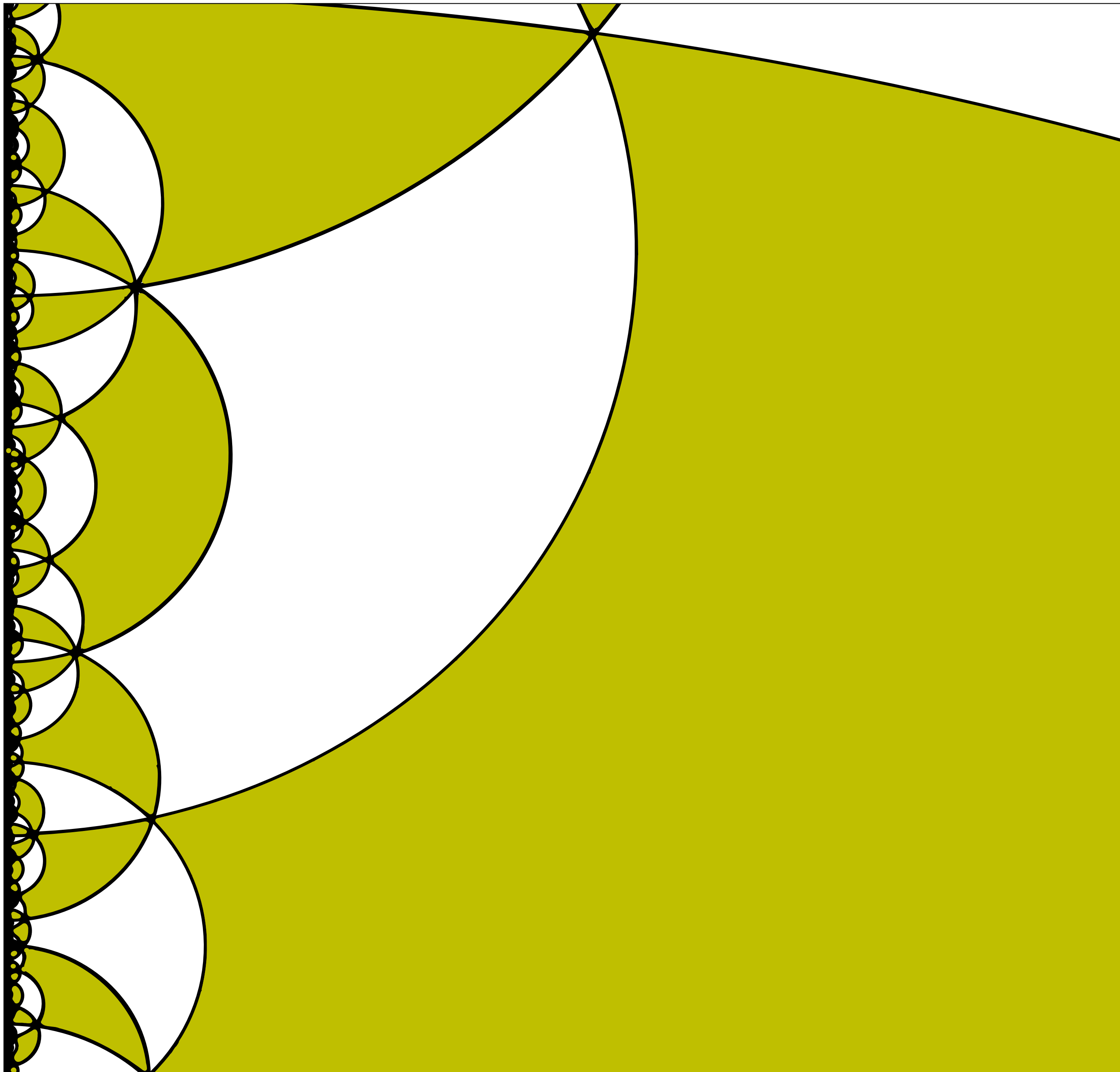

On the Hyperbolic Uniformization of Shimura Curves
with an Atkin-Lehner Quotient of Genus 0

Joan Nualart Riera