UNIVERSITAT DE BARCELONA

New insights into holography from supersymmetric localization

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NEW INSIGHTS INTO HOLOGRAPHY FROM SUPERSYMMETRIC LOCALIZATION

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Universitat de Barcelona

April 2016

Universitat de Barcelona

Institut de Ciències del Cosmos

Programa de Doctorat en Física

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A dissertation submitted by

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to the University of Barcelona
for the degree of Doctor of Philosophy

I certify that this thesis was prepared under

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Barcelona, April 2016

“ Als meus pares, a qui dec un suport incondicional.
Al meu germà, a voltes un Sòcrates incansable.
Als qui seguiu al meu costat, amb agraïment i estima.
A tothom qui cerca incansablement la comprensió. ”

Acknowledgements

Along the path towards the completion of this thesis many people have given me good advice, ideas and courage, and, in some cases, also logistical support. Although it is not feasible to list them all here I would like to acknowledge in these lines some contributions that made the journey more placid, and without which these thesis would have never reached port.

First and foremost, I want to emphasise the value of the guidance task carried out by my supervisor, Tomeu Fiol. Not only he has been a supportive, affable and wise advisor; I also feel greatly indebted to him for his endless patience, his clairvoyant physical insights and his commitment to the projects we shared. I also would like to thank Domènec Espriu for agreeing to be the tutor of my thesis.

I want to express my gratitude to the entire string theoretic research group in Barcelona as well, and specially to Roberto Emparan, David Mateos and Tomeu. Besides all the physics they have taught me, they have done a great job in creating a rich academical environment, with a dynamical Journal Club, plenty of colloquia and seminars, and visits from world-leading researchers. In this fruitful atmosphere, I have enjoyed enriching discussions with Jorge Russo, Jorge Casalderrey, Pablo Cámara, Javier Tarrio, Miguel Zilhão, Markus B. Fröb, Alejandro Barranco, Blai Garolera, Miquel Triana, Marina Martínez.

I would like to thank Veronika Hubeny for her warm hospitality during my visit to Durham University, and to the COST network for its economical support, which made possible this visit and the subsequent interesting discussions with Veronika, Mukund Rangamani, Henry Maxfield, Jyotirmoy Bhattacharya, Felix Haehl, Massimiliano Rota and Vaios Ziogas.

None of this would have been possible without the financial support of the FI-DGR-2013 grant and the support of the secretaries of the department of *Física Fonamental*, Olga, Cristina and Bea.

My conception of physics has been greatly influenced by many interesting discussions with close friends that I have found in the academic world. In what follows I would like to mention those I am most indebted with:

Blai Garolera, my academical brother, has always been a reference to me. I have always envied his abilities of coming up with lucid physical qualitative pictures and of anticipating connections between different phenomena. I am greatly indebted to him, and I must say it has been a pleasure to work with him shoulder to shoulder.

Dani and Markus were finishing their phd when I entered the group, and enlightened my first steps through an initiative they called the *Awesome Physics Journal Club*. I must say with shame and sorrow that us, your fellow phd students, let that initiative die not long after your departure. I want to apologise for my piece of guilt in this tragedy, and thank you for all what you taught me.

Matteo Baggioli lies, in the landscape of opinions, at an antipodal point from my position. That's probably why our friendship and countless discussions have been so interesting and helpful. I wish you the best in Crete!

This list would go on with other friends and colleagues, including Adriana, Albert, Alejandro, Blai P., Carmen, Carla, David P., David R., Eli, Guille, Isa, Ivan, Joan C., Jesús, Lluís H., Marina, Markus, Martin G., Matteo B., Miquel, Miguel, Núria, Ricard, Sam and Xin, and it is probably too extense to present it here. Thank you all for all what we shared.

Vull donar-te gràcies, Marta, pel coratge que ens hem donat mútuament en passar alhora pel feixuc tràmit d'haver d'escriure una tesi a contrarellotge, i per la passió que m'has encomanat per construïr aquesta tesi com un objecte preciós i delicat del qual estar orgullós.

Estimats companys de pis que m'heu aguantat durant l'escriptura: Dèlia, Joan, Maria i Pau, s'agraeix moltíssim la paciència. És un plaer sentir-vos la meva família adoptiva, juntament amb als cosins Gené, en Roger i l'Artemi.

Gràcies per la paciència, cantaires de la coral de física, Cor Cantus Firmus, colla Els Llitons i grup de divulgació científica C3.

No vull deixar d'esmentar tota la gent per qui m'he sentit recolzat fora el món acadèmic: l'Aina, l'Alba, l'Albert, els germans Alsina, l'Anna V., l'Aran, en Bareda, la Berta, les Blanques, l'Eduard, l'Elena, en Ferran, en Gabi, en Gerard, l'Ingrid, la

Irene, l'Isaac, en Jime, en Marc, la Magalí, la Maria, en Martí, en Miquel, la Mireia, la Mònica, l'Oriol, la Paula, en Pere, la Pili, en Raul, en Sergi, en Víctor i la Xènia.

Dedico aquesta tesi a la meva família, a qui dec el caràcter que m'ha dut a escriure-la: amb la passió per la ciència que m'ha transmès el meu pare, el perfeccionisme i la perseverança que he copiat a la meva mare i la necessitat de posar-ho tot en dubte que he après del meu germà.

Contents

Chapter 1

Introduction

Physics has reached an unprecedented degree of precision and descriptive power through perturbative field theory analysis. The accomplishments of this versatile framework are extended through a wide spectrum of systems, scales and constituents, more often than not providing accurate predictions that surpass or challenge the best available experimental setups, both in particle physics and condensed matter models. Nevertheless, this success is somehow delusive: perturbation theory is applicable only when certain conditions are satisfied, and that is generally not the case.

To begin with, even in weakly interacting field theories, perturbative methods totally fail to account for a certain class of phenomena that have consequently been denominated *nonperturbative effects*. This is reflected in the fact that perturbative series are in general asymptotic, and therefore their precision is in general bounded. It is nonetheless often possible to add complementary terms to perturbative expansions to encompass the missing physics, and indeed the ongoing research on resurgent methods might soon offer a way to make this analysis systematic. Most of the known nonperturbative effects can be dealt with by studying the instantons in the theory, and therefore, they are often under control.

That is not the end of the story, though. Whenever the theory under consideration does not admit a weakly coupled quasiparticle description at the regime of interest no perturbative expansion is applicable and, on general grounds, we lack the mathematical technology to analyse it. Admittedly, this question might have been considered of little interest by most physicists if such *nonperturbative field theories* were not realized in nature, but in fact numerous physical systems are known or expected to fall into this class. They include a case of central interest in nuclear physics, for instance: the theory of quantum chromodynamics in the regime where it becomes strongly coupled, which is precisely the parametric region where the infamous confinement crossover, the so-called *hadronization*, takes place. Another theory thought to require a nonperturbative field description is the effective description of superconductors with high critical temperatures. To my knowledge, there is no known theory that successfully accounts for their transport properties, but a minutious analysis of the experimental results clearly disfavours any description in terms of (weakly-coupled) quasiparticles. Let me point out that, in contrast to what happens with strongly coupled chromodynamics, there is in general no reason to think that a nonperturbative field theory ought to be the extension of a weakly coupled field theory beyond the validity of the power-series in the coupling, or in regimes where perturbative effects become dominant. In fact, in large families of known nonperturbative field theories, a weakly coupled regime where one can define particles is not even expected.

To make things worse, there is an important sector of physics where even the paradigm of quantum field theory is at stake: quantum gravity. There is no evidence that the quantum theory for a dynamical geometry can be build upon a full-fledged field theoretical description, at least not in terms of its own dynamical fields. It is known that, if any quantum ultraviolet completion exists for Einstein's equations, this fixed point will not be Gaussian. While the search for this hypothetical nonperturbative quantum completion of gravity is an open line of research, in this case there are reasons to suspect not only the *perturbative* adjective but also the *field theory* label. Indeed, why should we expect quantum field theory, a framework that relies on placid, semiclassical geometrical backgrounds, to be able to withstand such a dramatic deformation as the quantization of the spacetime it lives on? In this regard, string theory arises as a good alternative: in an oversimplifying, heuristic, way, this framework delegates the quantization of the fields that describe spacetime to a field theory defined in another spacetime with nondynamical metric. This framework comes with (at least) a drawback: To the best of my knowledge, we are not yet able to formulate a full-fledged nonperturbative string theory, and all our string theoretical constructions can be tracked back to asymptotic perturbative series, possibly with nonperturbative corrections. This fact can be somehow discouraging from a philosophical point of view, but it does not invalidate the theory as long as the accessible regimes cover our descriptive needs. In fact, many nonperturbative string regimes can be studied using a type of relation between string constructions known as *duality*.

The main character in this thesis precisely ties together the problems of nonperturbative field theories and quantum gravity, and offers resources to address them. It goes under many names: *gravitational holography*, *Maldacena's conjecture*, *AdS/CFT*, *gauge-strings correspondence*, and sometimes consistent reorganizations of these words. For the purposes of this introduction it is sufficient to state that this conjectured equivalence relates gauge field theories to certain realizations of string theory. The way this relationship is established makes the duality extremely interesting, and indeed it has remained one of the central subjects of study of theoretical physics for almost two full decades, principally for two reasons: On the one hand, this correspondence maps specific nonperturbative regimes of certain gauge theories to the semiclassical dynamics of hyperbolic gravitational spacetimes, which equips us with analytical and numerical well-understood geometrical tools to explore the former. On the other, the possibility to rephrase the regime of perturbative string theory in a field theoretical language gives us a way of meaningfully completing the string realization beyond its perturbative regime, and encodes the mysterious domain of string-theoretical quantum gravity into a well-defined quantum field theory on a rigid spacetime. The AdS/CFT construction, therefore, opens two windows towards quantum regimes that would otherwise be considered almost unattainable. Two of the widest gaps in our understanding of quantum theories can in principle be filled by better deciphering this conceptual Rosetta Stone.

Unfortunately, in practice, this duality is hard to explore. Like other well-known correspondences involving nonperturbative regimes, it eludes a formal proof or formulation, because its two sides are seldom simultaneously accessible, and sometimes not even simultaneously well-defined. Needless to say, that's precisely why holography is expected to become so useful once we understand it, and therefore waiting for a hypothetical nonperturbative string completion or mathematical control on nonperturbative field theories to derive it from first principles would defy its main purposes. Instead, research in AdS/CFT aims at gaining intuition for the duality by looking at specific cases where symmetries, and more specifically supersymmetry and conformal symmetry, enhance our visibility. Through these landmarks we progressively learn how to approach less symmetric setups. Let me emphasise, though, that the directory of regimes and observables that can be used for this program is, as of today, very short.

To sum up, holography is a bridge in the making. Some provisional structure makes it practicable for some purposes, yet we are far from the solid reliable building we envision. Theoretical physicists have made slow but steady progress in its construction for almost two decades now, even in nontrivial regimes. The construction level is reached by stepping on auxiliary scaffolds; stable narrow platforms attached to solid grounds on their sides, but sustained over the void out with the help of high amounts of symmetry for almost all their length.

This thesis is ultimately a laborer's contribution to this colossal project. It focuses on a very precise target: the exploration of how the duality can benefit from the exact predictions of a field theoretic technique known as supersymmetric localization. More precisely, its content addresses a specific type of observables amenable to this technique: supersymmetric loop operators of the *Wilsonian* type. The results I will present in this dissertation are based on the research I conducted in collaboration with Blai Garolera and Bartomeu Fiol, and which resulted in the publications [1, 2, 3, 4], where we provided:

- A benchmark for the study of strongly coupled field theories through holographic supergravity.
- Precision tests for the holographic conjecture.

- A refinement on the bubbling geometry description of the gravitational duals to theories with $\mathfrak{so}$ and $\mathfrak{sp}$ algebras.
- String theory predictions for the nonorientable sectors of the string topological expansion at all orders.
- A list of potential diagnostic methods for semiclassical holographical duals in field theories.
- Nonperturbative predictions for the dissipation and the radiative properties of highly energetic particles moving through strongly coupled backgrounds.

These results are contextualized and detailed in the central part of this thesis. More specifically, chapter 3 elaborates on the interplay between localization and holographic techniques for one of the simplest configurations at hand, namely, $\mathcal{N} = 4$ super Yang-Mills theory at large number of $SU(N)$ colors, while chapters 4 and 5 study the implications on holography of supersymmetric localization in more general settings: $\mathcal{N} = 4$ theories of generic classical Lie algebras of finite rank and $\mathcal{N} = 2$ superconformal field theories, respectively.

These chapters are preceded by an overview of several concepts and methods of relevance in this dissertation (chapter 2). In particular, I considered most important to present:

- A short digression on the roles played by objects of reduced dimensionality in field and string theories.
- A brief introduction to the framework of the holographic conjecture.
- A detailed characterization of the field theoretical observables of interest: Wilson loop operators that preserve some amount of symmetry in 4-dimensional $\mathcal{N} = 4$ nonabelian gauge theories and their closer relatives.
- A general overview of the supersymmetric localization method, its predictions for $\mathcal{N} = 2$ Lagrangian supersymmetric field theories on flat 4-dimensional spacetimes, and the mathematical resources one can use to analyse the resulting matrix models.

The reader should nonetheless bear in mind that this text is not fully self-contained. Modern physics refuses to stick to the “tree of knowledge” conventional (and frequently inappropriate) picture for science, forming instead a less story-friendly “net of knowledge”. Bethe-like descriptions of this net are specially inefficient and misleading in the edges of our understanding, where the interplay between many related frameworks becomes an important driving force for research. In my efforts to keep this dissertation compact, the links of minor importance for my purposes have been either omitted or referred to the literature. I have also assumed in the writing that the reader is familiar with Lie algebras and their representations, quantum field theory, linear response analysis, conformal symmetry, supersymmetry, and differential geometry.

I would like to make a few disclaimers before going into details:

- The taming of nonperturbative effects can somehow help us extend a weakly coupled field theory to stronger couplings, reaching regimes where the theory becomes nonperturbative. Consequently, I have allowed myself to include both methods used to describe nonperturbative effects and generic approaches to nonperturbative theories under the same label: "nonperturbative methods".

- From the preceding text the reader might get the impression that the characterization of nonperturbative physics is a recent line of research, motivated by the saturation of the applicability of perturbative methods. That vision would not only be wrong, it would be totally unfair as well. Some nonperturbative methods to characterize both perturbative and nonperturbative effects and theories are as old as perturbative theory itself and very useful and necessary complements to it. Their list includes the field theoretical identities obtained from symmetries and functional field theory methods relying on analyticity. Other nonperturbative methods were not born with quantum field theories, but they are still older than holography, including among others lattice discretizations of quantum field theories, the bootstrap equations on conformal field theory, the so-called Seiberg-Witten formalism, and instanton calculus. Even when it is not designed as a non-perturbative tool itself, an immediate predecessor of holography called "large N" expansionthat will be mentioned later in this document can also be a useful framework to obtain some nonperturbative results.

- In the same manner, I should warn the reader that holography is not the only recently opened window towards nonperturbative physics. Together with the other main character in this dissertation, namely supersymmetric localization, and other strategies including integrability, resurgent analysis, and several dualities derived from geometrical tools in higher dimensions, they set the scenario for a new type of approach towards nonperturbative physics. They no longer aim only for qualitative targets, such as analytical structure or anomaly characterization, but address specific computations of nontrivial nonperturbative observables with quantitative precision. As of today, only supersymmetric configurations are at reach for most purposes, but hopefully the regions where the applicability of different methods overlap will permit symbiosis amongst them and help us perfect tools that we can later use in more realistic settings.

Chapter 2

General concepts and analytical tools

2.1 Defects and probes

This section is devoted to the study of defects in relativistic quantum theories. The word *defect* denotes here any object implemented through an embedded submanifold. It will help us to avoid, when necessary, the subtle linguistic distinctions between state, operator, theory, and physical object, which often have little or no support in the quantum mechanical framework.

Two applications of defects are cornerstones of this thesis:

- Wilson loop operators in gauge theories constitute its main testing ground. These objects will be discussed in general in 2.1.3. Later, in 2.3 we will stipulate which specific Wilson operators will be of interest in this thesis.
- String theory sets a conceptual background from which the implications of Wilson loop results are inferred. The key objects in our setting are not strings themselves, but through another sort of constituents of these theories: D-branes. These objects will be introduced in 2.1.4 from string theoretical dualities.

Before going into this specific details, some general contextualization on defects will be provided.

2.1.1 Fauna and flora of the defect operator landscape

As their name indicates, quantum field theories (QFTs) are defined in general in terms of fields. However, this does not exclude the possibility that they develop effective objects of reduced dimensionality. It neither diminishes the importance of geometrical quantities defined on submanifolds, such as fluxes across virtual surfaces or charges in a volume, through which we characterize some field configurations. For these and many other reasons, defects are useful objects in in quantum theories, and they are in general implemented through operators with support on the locus where the submanifold resides.

Some of these operators can be defined in terms of integrals over the defect of local fields, or, more properly, of their restrictions and pull-backs. When physical defects are given by a worldspace Lagrangian are treated perturbatively, for instance, they end up as a series of n-points of such operators. These insertions to the theory are harmless in the sense that they are algebraically defined from local operators, and, thus, we don't have to introduce any new rule to work with them in quantum field theory. I should warn the reader that one should not be careless with these operators, however: for noncompact manifolds the integral over the worldspace is usually IR-divergent, and their reduced dimensionality requires a specific treatment of the UV divergences of their fields[1]. Both types of divergences can be properly addressed using conventional methods.

Another word of caution is needed: the class of operators presented in the previous paragraph does not cover the full set of defect operators one can build. As Kapustin argued in [5], in the UV fixed point where the theory flows from, conformal symmetry can be used to map the problem to a theory where the defect is located at a boundary. In this language, one can freely specify which symmetry-preserving boundary conditions for the fields are imposed by the defect. Although in general this does not exempt us from including the defect contribution in the action or inserting the defect operator, the amount of possible choices for these boundary conditions enlarges the set of defect operators we are able to define. This is a central topic in the study of defects, and the list of operators that need this formalism for their construction includes important examples such as the 't Hooft loops. However, this direction will not be further explored in this dissertation.

Not every defect operator $\mathscr{D}$ can represent a sensible physical insertion or deformation of the theory. A necessary consistency condition for this to happen is that the v.e.v.s of observables in presence of the defect, which would read

$$\langle \mathcal{O} \rangle_{\mathscr{D}} = \frac{\langle \mathcal{O}\mathscr{D} \rangle}{\langle \mathscr{D} \rangle} \,, \tag{2.1}$$

define a unitary theory. This is guaranteed if the defects are built from combinations of physical objects in a unitary theory. All the defects we will consider in this thesis are of this type.

For some defects, quantities of the form (2.1) can be computed in perturbation theory, but this is not the general rule. The behaviour of correlators with defects depends largely on the properties of the theory and the defect, and for this reason very few things can be stated about it in general. In fact, Despite their definition in terms of a submanifold, it is not guaranteed that their repercussion on the theory will be confined to a small neighbourhood of their location. For specific defects their insertion can even change the moduli structure of vacua.

For semiclassical theories, an interesting question to ask regarding this issue is whether the backreaction of the defect, i.e., the change it causes to the solution of the equations of motion, is local. If this is the case, there will be scales where backreaction can be

[1] This issue is specially confusing when, as it happens for Wilson loops, the defect operator is built purely in terms of the charges, fields and couplings of the field theory: The condition for the absence of divergences is different if the field insertion is integrated over d dimensions instead of D dimensions, and there is an apparent contradiction between the renormalization equations in the worldspace and that of the target space. The way out of this paradox is to notice that the apparent change of nature of the field is caused by the fact that these operators can only be meaningful if either they are later integrated over the path integral of the embedding fields or the path they are defined on is a saddle point of this second integral. Heuristically, in both cases the renormalization of bulk charges and fields is divided in two steps, *longitudinal* and *transverse*. The "renormalization of the dimensional integral" is just half the whole story.

neglected, and we reach a regime where the insertion can be used as a measuring tool for the defect-less theory. This approximation is concordantly called the *probe approximation*, and the defects where it is appropriate are referred to as *probes*.

2.1.2 Defects as building blocks?

So far, we have discussed how effective defects can be inserted into QFTs and which type of operators are they related to. A totally different matter is whether one can build a relativistic quantum theory out of fundamental defects. We know the answer for particle theories: only partially.

In perturbative field theory we can replace propagators in terms of free particles using Schwinger's proper time formalism;

$$G(x,y) = \langle x|\frac{i}{p^2-m^2}|y\rangle = \int_0^\infty dT \int_{\substack{r(0)=x \\ r(T)=y}} [Dr]\, e^{-im\int_0^T dt\sqrt{\dot{r}^2}} \tag{2.2}$$

for the case of free propagation, and more involved 2-point effective actions

$$\Gamma(x,y) = \int_0^\infty dT \int_{\substack{r(0)=x \\ r(T)=y}} [Dr]\, e^{-i\int_0^T dt\left(m(r)\sqrt{\dot{r}^2(t)}+\lambda_i\phi^i(r(t))\right)} \tag{2.3}$$

for generic backgrounds[2]. The propagation of a gauge-dependent degree of freedom, for instance, can be expressed in this formalism through a "minimally coupled" expression that implements the gauge transport, matching the equivalent degrees of freedom at both ends:

$$\Gamma(x,y) = \int_0^\infty dT \int_{\substack{r(0)=x \\ r(T)=y}} [Dr]\, e^{-i\int_0^T dt\left(m\sqrt{\dot{r}^2}+qA_\mu\dot{r}^\mu\right)}\,. \tag{2.4}$$

In 2.1.3 we will examine the phase in this integrand from a deeper point of view.

Some questions are in practice impossible to address in the language of particles, though: nontrivial vacua and renormalization, nonperturbative solutions and systems among them. At best, we can give some intuition for them as "particle condensates". In addition, the field theoretical framework deals automatically with a set of issues in

[2] Seen as the effective action obtained by integrating out some degrees of freedom in the path integral, expression (2.3) is not at odds with the quantization of its ϕ^i fields. This expression appears as a factor inside the path integral for a specific set of Feynman diagrams: those involving the particle propagation in question.

the construction of the theory that would require the introduction of ad hoc rules from the particle description. Causality and the structure of Feynman diagrams are the two clearest examples of this fact.

What happens for defects of higher dimensionality? In practice, only a specific type of 2-dimensional defect (the critical string) can be addressed with a particle-like construction, i.e., first identifying freely propagating objects, then introducing rules for their propagation, and afterwards extending its description to nontrivial backgrounds sourced by condensation of their "quanta". No nonperturbatively complete theory is presently at reach for this structure, and yet its study is highly appealing: critical string theories include descriptions of gravity that are free from the UV pathologies inherent to QFT.

Let us now address the construction of generic theories with higher dimensional defects as the building blocks and observe where the special properties of critical string theories arise from. The full-detailed discussion of this issue is full of important subtleties, which are extensively addressed in string theory books and are not essential for the remaining of this dissertation. I will therefore content myself with presenting a story that, although incomplete, captures the most colourful parts of the puzzle.

To keep this discussion bounded we restrict ourselves to the construction of a free nontopological defect: the action will include the worldvolume form coming from the pulled-back metric of the target space:

$$V = \sqrt{\det |P\,[g]|} d^n\sigma ; \;\; P\,[g]_{ij} = \partial_i x^M \partial_j x^N g_{MN} \tag{2.5}$$

For later reference, I should mention that for the specific case of a string ($d = 2$) this action term is known as the Nambu-Goto (NG) action.

Although the form (2.5) is often useful to study semiclassical effective actions, it is preferable to use a less involved equivalent expression instead if we wish to quantize the embedding fields. Through the introduction of an auxiliary nondynamical field γ_{ij} that we will call the auxiliary worldspace metric, we obtain

$$V_P = \sqrt{\det |\gamma|} \gamma^{ij} g_{MN} \partial_i x^M \partial_j x^N d^n\sigma \tag{2.6}$$

For particles reparametrization invariance allow us to gauge-fix $\gamma = \eta$. The only other type of defect for which this happens is the critical string, namely, a type of string where an additional symmetry $\gamma \to c\,(X)\,\gamma$ known as Weyl rescaling invariance is kept, not only classically but also at the quantum level. This fixes a specific dimension for the target space of the theory, $D = D_0$. It turns out that $D_0 \gg 4$. However, the amount of unobserved dimensions required by these objects is not an insurmountable problem: they can remain hidden in low energy physics via Kaluza-Klein compactification or a type of constructions known as "worldbrane universes".

There are some terms we can add to (2.6) without breaking the symmetry of reparametrization (and Weyl invariance):

- For $d = 1$, one can consider a "mass" term $-m^2\gamma$ and write the (bosonic) particle action as

$$S = \frac{1}{2}\int dt \left(\frac{\dot{x}^2}{\sqrt{\gamma}} - \sqrt{\gamma}m^2\right) \tag{2.7}$$

- For $d = 2$, one can add an Einstein-Hilbert term on γ without cosmological constant, any other option would violate Weyl rescaling invariance. This term turns out to be topological for $d = 2$. Therefore, with this insertion, γ has not been promoted to a dynamical field. This term in the action ultimately becomes to the Euler characteristic χ_Ω of the worldsheet Ω, with a specific prefactor. The critical string action in trivial backgrounds is therefore given by the sum of this term and the worldvolume term, which, in this context, is called the Polyakov action. The latter reads

$$S_P = \tfrac{1}{4\pi\alpha'}\int_\Omega d^2\sigma\sqrt{-\gamma}\gamma^{ij}\partial_i x^\mu \partial_j x^\nu \eta_{\mu\nu} \tag{2.8}$$

I should mention here that although the notation in the previous expressions suggests a bosonic euclidean target space, the same ideas can be applied to compact targets, and even to superspaces. In the latter case, the right and left-moving modes of a fermion can additionally have two types of boundary conditions:

- Ramond (R) modes, with periodic boundary conditions around the closed string or the same relative sign between right-moving and left-moving modes at both endpoints of the open string.
- Neveu-Schwarz (NS) modes, with antiperiodic conditions or different sigh at the two endpoints, respectively.

For closed strings, where the right and left moving sectors can be varied almost independently, we split the spectrum in four sectors: NSNS, NSR, RNS and RR. This distinction plays an important role in the study of the physical spectrum of the superstring.

Observe that, although the previous arguments restrict the Lagrangian of the free string to a very simple and rigid form, the action has still a nonlocal freedom: the global topology of the worldsheet Ω. In contrast to what happens in the $d = 1$ case, this topology can be highly nontrivial, independently of whether we considered compact manifolds (closed strings) or allow them to have spatial endpoints (open strings). Nontrivial topologies are seen from any time slicing as processes involving junctions and splitting of strings (see figure 2.1). We can therefore introduce string interactions in the theory without ad hoc rules. The theory will also be free from similar singular vertex-like loci: external states with different number of strings can be connected through a smooth evolution. The sum over amplitudes of processes with different topology arranges itself in a perturbative series because of a prefactor $g_s^{\chi_\Omega}$ coming from the Einstein-Hilbert term on the action. The coefficient g_s is known as the string coupling because it plays the role of a coupling constant in a "smoothed" Feynman diagram structure.

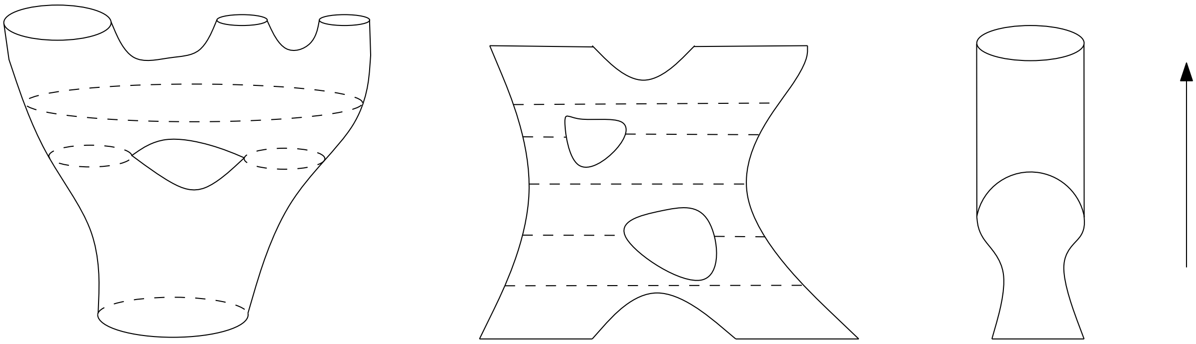

Figure 2.1: Examples of interacting diagrams in string theory. From left to right, the depicted diagram describes a closed string process, an open string process and an a diagram connecting an open string to a closed one. The arrow indicates the direction of time in some coordinates. The dotted lines are slicings of the worldsheet according to the same time, and help us observe that strings join and split, although the whole worldsheet is smooth

When we quantize either (2.7) or (2.8) we have to be specially careful with gauge reparametrization invariance and Weyl rescaling. For the bosonic particle case, it turns out it is sufficient to quantize the gauge-fixed action and impose the constraint $p^2 - m^2 = 0$ for the physical states. The quantization of the free critical string is comparatively more involved. Even after imposing the boundary conditions dictated by Ω (either restrictions on the endpoints of the open string or periodicity in the closed string) and fixing $\gamma = \eta$, we have some residual gauge symmetry. The quantization of the theory can be either be made using the BRST mechanism, or fixing the so-called *lightcone gauge*. In the former case we keep covariance at the cost of having to deal with ghosts. In the latter, we work with only physical states but ultimately need to justify the absence of anomalies to recover covariance. Both paths ultimately lead to the same results for D_0 and the physical spectrum. The gauge constraints are manifest in the result in the fact that the right-moving and left-moving states are maximal-weight representations of their Virasoro algebra. Additionally, physical states are restricted to have a nonvanishing component for the zero mode that lies on a mass-shell.For the bosonic closed string, for instance,

$$M^2 = p^\mu p_\mu = \frac{1}{\alpha'} \left(N + \bar{N} - 2 \right) \,, \tag{2.9}$$

where N and $\bar{N}$ are positive or zero labels related in a degenerated way to the left and right-moving configurations, respectively. These levels are generally not independent: for the open bosonic string, for instance, boundary conditions enforce the right and left-moving structure to be identical, while for closed bosonic strings we get a physical restriction known as level-matching that, in absence of compactification, reads $\bar{N} - N = 0$.

The spectrum obtained in all string theories as we have presented them has some generic characteristics:

Type	RR spectrum (lightcone)	Worldspace SUSY
IIA	A_{μ}, $A_{\mu,\nu,\rho}$	Non-Chiral, $\mathcal{N}=2$
IIB	A, $A_{\mu,\nu}$, $A_{\mu,\nu,\rho,\sigma}$ (self-dual)	Chiral $\mathcal{N}=2$
Het $SO(32)$	adj $(SO(32))$	Chiral $\mathcal{N}=1$
Het $E_8 \times E_8$	adj $(E_8 \times E_8)$	Chiral $\mathcal{N}=1$

Additionally, the bosonic spectrum always includes the NSNS sector, $g_{\mu,\nu}$, $B_{\mu,\nu}$, ϕ.

Table 2.1: Bosonic spectrum in the lightcone gauge and worldsheet supersymmetry for the four perturbative closed superstring theories.

- The massless part of the spectrum contains a tensor $T_{\mu\nu}$ that we generically decompose in its trace ϕ, its symmetric traceless part $g_{\mu\nu}$ and its antisymmetric part $B_{\mu\nu}$.
- For open strings, the spectrum contains additional massless states: a gauge field for the target space directions where the motion is free (Von Neumann boundary conditions) and a scalar field for the target space directions where the motion is fixed (Dirichlett boundary conditions). In 2.1.4 these fields will be reinterpreted in terms of D-brane objects, the string theory defects to which open strings endpoints can be restricted.
- The spectrum includes in general a pathological tachyonic state ($M^2 < 0$) signalling an instability.

A sensible theory should admit additional structure so that the part of the spectrum including the tachyon can be projected out. It turns out that this can be achieved only for supersymmetric strings, or superstrings for short. All the cases where this happens have $D_0 = 10$. For closed string theories we obtain four possibilities:

- Homotic string theories, with 8 transverse bosons and 8 Majorana fermions. In these theories, left-moving and right-moving sectors are treated symmetrically. Their perturbative string sector is known as “Type II” superstrings. The projection can be done in two manners, which are called type IIA and type IIB.
- Heterotic string theories, with 8 transverse bosons, 8 right-moving Majorana-Weyl fermions and 32 left-moving Majorana-Weyl fermions. Depending on the implementation of boundary conditions we can get two different algebraic structures for the gauge field in the spectrum: either an adjoint representation of $SO(32)$ or of $E_8 \times E_8$.

Additionally, a fifth type of perturbative superstring theory can be built if we include open strings with Von Neumann boundary conditions to type IIB strings and take the appropriate projections. This theory is known as type I strings.

We now address the question of string propagation in nontrivial backgrounds. In general it is not known how to consistently couple the string to arbitrary condensates of the bosonic modes in the superstring spectrum: the ghost dependence of fermions in the covariant formulation, the need of "picture changing operators" and the breaking of separate superconformal symmetry of the matter and ghost sectors are important obstacles that leave RR backgrounds out of reach. Conversely, if we consider only NSNS modes the following worldsheet effective action can be built:

$$S_{\mathrm{WS}} = \frac{1}{4\pi\alpha'} \int_{\Omega} d^2\sigma \sqrt{|\gamma|} \left(\left(\gamma^{ij} g_{MN}(x) + \epsilon^{ij} B_{MN}(x) \right) \partial_i X^M \partial_j X^N \right) + \alpha' \Phi(x) \, \mathsf{R}[\gamma] + \dots , \tag{2.10}$$

where the dots denote a fermionic part of the action. From the point of view of the worldspace theory, background classical fields act as an infinite set of couplings for the embedding function:

$$f(x) = \sum_{n=0}^{\infty} f_{\mu_1 \dots \mu_n} x^{\mu_1} \dots x^{\mu_n} . \tag{2.11}$$

When considered under this light, g, B and Φ would in general be expected to flow under RG. Yet, conformal invariance for the worldsheet is still needed to decouple pathological states from the spectrum. Consistency therefore demands that the β functions for these coupling fields become 0, and this imposes a set of equations on them. These equations coincide with the ones obtained in an Einstein-Hilbert gravity coupled to a gauge form B.

By carefully analysing the fermionic sector as well as the bosonic, we observe that type IIA and IIB reproduce the equations of motion for 10-dimensional type IIA and IIB $\mathcal{N}=2$ supergravity, respectively, while the heterotic closed theories and type I generate 10-dimensional $\mathcal{N}=1$ supergravities coupled to Yang-Mills. Since by compactification they can generate lower dimensional supergravities, almost all possible supergravities admit a string theoretical origin. We will comment in 2.1.4 that the obvious exception, 11-dimensional $\mathcal{N}=1$ supergravity, is related to certain nonperturbative regimes of string theory.

One of the questions one would like to ask the UV completion of quantum gravity is what happens at the singularities where the classical theory breaks down. If string theory truly provides an UV completion of supergravities, it should account for the black p-brane defects, namely, objects of p spacelike dimensions evolving in time and coupling to a background $p+1$-gauge form, that these theories can develop[3]. The simplest version

[3] The 10-dimensional supergravities have another type of brane also explained by string theory, the 5-dimensional solitonic brane, charged under the B NSNS field

of this object is the extremal black p-brane, described by the following configuration

$$\left\{ \begin{array}{c} ds^2 = H^{-\frac{1}{2}}(r)\, dx^\alpha dx_\alpha - H^{\frac{1}{2}}(r)\left(dr^2 + r^2 d\Omega_{8-p}^2\right) \\ H(r) \equiv 1 + \left(\frac{R}{r}\right)^{7-p} \end{array} \right\}, \tag{2.12}$$

and held together by a conserved radial F_{p+2} flux that fulfils

$$\left\{ \begin{array}{c} N \equiv \oint_{S^{8-p}} {}^\star F_{p+2} = \mathrm{ctt} \\ R^{7-p} = (4\pi\alpha')^{\frac{7-p}{2}} \frac{g_s N}{4\pi} \Gamma\left(\frac{7-p}{2}\right) \end{array} \right\}. \tag{2.13}$$

Their physically UV complete description, of course, should not be expected at the level of the effective action (2.10) where the theory essentially implements supergravity, but at the level of the nonperturbative completion of string theory. As we will review in 2.1.4 a specific nonsingular description of these objects is accessible through string dualities.

Let me make a final comment concerning string theory: naively one could expect that the theories obtained from superstrings will have very fine-tuned characteristics: very specific contents of matter and only two independent coupling parameters, g_s and α'. Indeed, they do have this structure in ten dimensions. However, the compactification to 4 dimensions adds more freedom than we could wish for. String theory offers a UV completion with gravity of such a vast amount of 4-dimensional low-energy theories that it is often accused of not being predictive for being able to predict virtually "anything" beyond the range of observed regimes. I wish to point out here that this situation has been the rule, not the exception, for the theoretical framework in physics through history. We come from a short interval in the XX-th century in which the geometrical possibilities for the universe were underestimated and UV pathologies were thought to be terminal, which left an unusually low (and fictitious) number of options for "fundamental" theories.

2.1.3 Wilson lines and loops

Local gauge transformations are not physical symmetries but artefacts of our description. Objects in nonsinglet gauge representations can only be combined directly in a gauge-invariant way if they lie at the same point: their "color" decomposition and/or "gauge phases" are meaningful only when we refer them to a local origin and basis choice. Any nonlocal measure should, therefore, include a parallel transport rule for the gauge field in order to "translate" the gauge "color" or "phase" along the manifold. And it is not possible in general to choose it to be parallelly transported everywhere: as we already know from classical electrodynamics, charged defects have prescribed nontrivial gauge holonomies [6]. In mathematical language, all these facts find a precise formulation in terms of fiber bundles.

From the previous paragraph we conclude that the gauge parallel transport rule that connects gauge-variant descriptions sitting at two different points needs to be defined along a path that does not cross neither wrap any defect. The gauge field is locally free and therefore it can be written as

$$A_\mu = \frac{i}{q}\Omega^{-1}\partial_\mu\Omega \tag{2.14}$$

From this equation in terms of the group variables, that we can solve for Ω by exponentiating the infinitesimal behaviour, we obtain:

$$\Omega = \mathsf{P}e^{-iq\int A_\mu dx^\mu}\Omega_0 \tag{2.15}$$

where P is a path-ordering operator. The quantum defect operator built with this recipe is known as a *Wilson line*.

Wilson line operators are ubiquitous in field theories. Their most straightforward application is the definition of gauge invariant operators, such as

- Meson operators, obtained by coupling the endpoints of the lines to charges in conjugate representations q and $\mathsf{q}^\dagger$ and letting A_μ act appropriately upon them,

  $$\mathsf{M}_C\left(x, y\right) = \mathsf{q}^\dagger\left(x\right)\mathsf{P}e^{-iq\int_C A_\mu dx^\mu}\mathsf{q}\left(x\right) \tag{2.16}$$

  When these operators are timelike, they are interpreted as the creation, propagation and annihilation of a charged particles.

- Wilson loops, obtained by closing a Wilson line upon itself and tracing it in any representation of the algebra to similarly close the gauge structure:

  $$\mathsf{W}_\mathcal{R}\left(C\right) = \mathrm{Tr}_\mathcal{R}\mathsf{P}e^{-iq\oint A_\mu dx^\mu} \tag{2.17}$$

In the absence of charges, operators of the form (2.17) are the only possible building blocks of gauge-invariant operators in the theory. Polyakov [7, 8] proved that free gauge theories could indeed be rephrased in terms of this observables, and explicitly discussed how the suppression of gauge redundancies in the original field theory lead to a dynamical theory defined on a "loop space" characterized by their paths C.

However, this was not the original purpose for which these operators were designed. Instead their motivation came from the study of the semiclassical limit of massive particles. The path integral of a charged particle propagator coupled to the gauge field with minimal coupling can be written in terms of propagators of the form (2.4). When the particle is massive enough, this integral can be approximated through a saddle point approximation and the factor concerning the gauge field evaluates to a factor of type (2.15) [9, 10]. I should point out that in general the particle will be coupled to other background fields and we will obtain generalizations of Wilson loops that include other fields in their effective action. This type of extensions is of special relevance in supersymmetry and for the purposes of this thesis.

$V\,(L \gg 1)$	Phase	$V_{\text{mag}}\,(L \gg 1)$
$\sim \frac{1}{L}$	Coulomb	$\sim \frac{1}{L}$
$\sim \frac{1}{L \log L\Lambda}$	Free electric	$\sim \frac{\log L\Lambda}{L}$
$\sim \frac{\log L\Lambda}{L}$	Free magnetic	$\sim \frac{1}{L \log L\Lambda}$
$\sim$ ctt	Higgs	$\sim \sigma L$
$\sim \sigma L$	Confining	$\sim$ ctt

Table 2.2: The behaviour of the charge-anticharge potential (and its magnetic dual when it exists) at large separation serves as an order parameter for some phases frequently found in 4-dimensional gauge theories. The free electric and free magnetic phase happen when screening effects source logarithmic corrections to the Coulomb potential. Notice the electric and magnetic charged are affected in a complementary way. Condensation of electric particles is known to join the magnetic field lines in flux tubes, thus confining the dual magnetic monopoles. This is known as the Meissner effect. Its electromagnetic dual has been proposed as the driving mechanism of quark confinement.

The dynamical picture of line operators described in the previous paragraph was precisely the one used by Wilson in his seminal paper [11], where he suggested rectangular Wilson loop operators as an order parameter to study confinement in strongly coupled field theories. In particular, he focused on the case where these rectangles have spatial sides of size L and temporal sides of size T, with $T \gg L$. Their v.e.v. can be interpreted as the amplitude for creating a quark-antiquark massive pair with separation L and annihilating it back after some time T. If T is sufficiently large, this amplitude is dominated by the interactions between the two "propagating" sides, and we can use it to evaluate the quark-antiquark effective potential:

$$\langle \mathsf{W}_{q\bar{q}} \rangle \sim e^{-TV(L)} \tag{2.18}$$

Whenever this potential decreases at large distances the process of pair creation will not be infinitely suppressed and deconfined quarks are observable. In contrast, if V increases with distance, one should only be able to detect the internal "partons" of a meson when probing them at very small resolution. In the simplest examples one can think of, deconfining regimes have no quark-antiquark interaction at large L and therefore the exponent scales with a perimeter law while for confining examples the pair sources interacting bosons across the loop and the exponent scales as the volume instead. In practice, more diverse and sophisticated qualitative behaviours can be found at large L. They are labelled in the literature as different "phases" for the underlying theories, although the "phase transitions" between them are sometimes continuous crossovers.

It is in this interpretation in terms of particle propagation where another of its interesting applications shows up. It was already observed in [8] that whenever we pick a path for a Wilson line that has a cusp at one point its exponent develops an additional logarithmic term that is highly sensitive to the IR and UV structure of the defect. This term, known as the *cusp anomalous dimension*, turns out to be independent of the general shape of the loop, and to characterize with its coefficient the radiative properties of the probe.

Let me illustrate with a toy example why should we expect to obtain a term with logarithmic dependence on the cutoffs and these properties: Consider a weakly coupled theory where we introduce a Wilson line straight segment of length L. If the coupling is sufficiently small, the contributions to the loop are dominated by the exponential of the propagator. The argument will still hold if this propagator is corrected by local physics as long as it has a $\sim \Delta x^{-2}$ behaviour. In the perturbative expansion, the UV behaviour clearly dominates the interactions on the loop itself: there is no solid angle to soften the Δx^{-2} divergence. Curvature in the shape would in general modify the distances between arbitrary loop points, but only curvatures of radius comparable to ϵ can modify the leading UV contributions. When we integrate over the two endpoints of the propagators we will get a ϵ^{-1} behaviour characterizing short-distance physics integrated along the loop, resulting in a contribution linear in $\frac{L}{\epsilon}$, which ultimately would endow the closed loop with the perimeter-law behaviour we expect in this deconfining regime. Schematically,

$$\int_0^L dx_1 \int_0^L dx_2 \frac{\Theta_{|x_1-x_2|-\epsilon}}{|x_1-x_2|^2} = 2\int_0^{L-\epsilon} dx_1 \int_\epsilon^{L-x_1} dx \frac{1}{x^2} = 2\frac{L}{\epsilon} - 2\ln\frac{L}{\epsilon} + \mathcal{O}(1) \qquad (2.19)$$

We could naively think that the log subdominant contribution is uninteresting because it arises from an artificial IR cutoff, but in fact it contains a lot of physical information. Observe for instance that (2.19) should still hold when we double the length L, and, by comparison with the collinear junction of two independent segments of length L, we learn that the leading contribution of the cross-interactions between these two segments can be determined from this precise logarithmic term. In this sense, the information it contains represents the equilibrium properties of the charge cloud polarized by the line defect. Whenever a junction like the previous is not done collinearly, the two $\sim \ln\frac{L}{\epsilon}$ terms that did cancel in the collinear case don't vanish identically anymore, and the discordant coefficients signal how the polarized cloud reequilibrates and emits radiation.

The detailed relations between Wilson loops, radiation and cusp anomalous dimensions will be clarified along this dissertation, but it is convenient to introduce at this point some specific terminology. In particular, we define the "Cusp anomalous dimension" [12] as the coefficient controlling the energy radiated by a particle in a homogeneous vacuum at small acceleration, which has the shape

$$\Delta E = -2\pi B \int dt a^\mu a_\mu \qquad (2.20)$$

Notice that for QED this equation becomes the Larmor-Liénard formula, with $B = \frac{e^2}{12\pi^2 c^3}$

Apart from the universal properties discussed here, observables involving Wilson loop operators convey, in specific cases, large amounts of interesting information. The list of relations that will not be exploited in this dissertation includes the possibility of computing gluon scattering amplitudes through light-like Wilson loop polygons [13] and the correspondence between the lightlike-to-lightlike cusp anomalous dimensions and the anomalous dimension of large spin operators of twist 2 [14, 15].

2.1.4 Dualities, their role in string theory and D-branes

It is often pointed out that a significant part of the success of modern physics comes from its ability to treat different physical systems with the same mathematical language, either exactly or as a controlled approximation. The harmonic oscillator and the plane wave are the most paradigmatic examples of classes of systems addressed with the same framework. In general, these analogies and equivalences are very useful to develop intuition and to set a language to study unfamiliar systems. The complementary situation is frequent (and fertile) as well: the description of the same physical system through seemingly independent mathematical frameworks provides a guideline to connect this alternative mathematical structures. When this happens we say the two setups are *dual*, or equivalently they are connected by a *duality*.

Dualities are of great help in formulating mathematical conjectures, developing new theorems and offering alternative manners of performing a computation, but they are not restricted to these uses. In fact, having more than one description of a physical system is seldom a luxury: more often than not the regimes of applicability of dual descriptions are not totally coincident and therefore dualities are frequently essential tools to explore all the accessible parametric configurations of the problem. Occasionally, they might even help us find how to improve one of the descriptions to enlarge its domain of applicability. Notice also that when a mathematical framework is part of both dualites and analogies the applications and implications of the former are extended to the analogue systems as well, beyond the physical system where they were identified.

One of the simplest examples of duality is the electric-magnetic duality of $4D$ electromagnetism, which essentially consists in rephrasing of the theory in terms of the Hodge-dual form of F, $*F$. The electric j^μ and magnetic k^μ currents of the theory are interchanged in the dual picture as $j^\mu \to k^\mu$, $k^\mu \to -j^\mu$. I should mention in the passing that at the quantum level the coexistence between electric and magnetic charges imposes severe restrictions on the theory. This can be seen from the fact that the quantum fundamental object in the field description is the gauge field A_μ, not F, and this form cannot describe continuously the sphere wrapping a particle with magnetic charge in any gauge: at best, a string-like singularity extends from the magnetic monopole to infinity, with a gauge-dependent position. The theory can only be consistent if this string defect is not physical, or, equivalently, if it does not affect the motion of electric charges. This condition can be imposed to a particle of purely electric q_e by studying its Wilson loop around the singular string and imposing that the holonomy vanishes:

$$1 = e^{i\frac{q_e}{\hbar}\oint A} = e^{i\frac{q_e}{\hbar}\int \vec{B}d\vec{S}} = e^{\frac{i}{\hbar}q_e q_m} \Rightarrow q_e q_m = 2\pi n\hbar \tag{2.21}$$

In other words, consistency demands that both electric and magnetic charges are quantized in terms of elementary charges, and that, in appropriate units, the electrical and magnetic coupling are inverse one of the other. Notice that because of the latter property strongly and weakly coupled descriptions get interchanged through the electric-magnetic duality. This weak-strong character makes this duality very useful to explore a theory beyond its perturbative regimes. Unfortunately, its extension to nonabelian gauge theories, that goes under the name S-duality is quite more involved. For $\mathcal{N} = 4$ super Yang-Mills theories this duality has successfully been built, and in its more general version [16, 17] it acts as a modular $SL(2, \mathbb{Z})$ transformation on a combination of the gauge coupling g and theta angle θ ($\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$) with a potential interchange of the charge weight lattice Λ_G by the corresponding lattice for the Langlands conjugated group, $\Lambda_{^L G}$. Again, a subtle issue is played by the charges admitted by the theory: not any population of the charge weight lattice is consistent, but consistency leaves in general more than one option. Different choices for the occupation of this lattice lead to different theories that often are interrelated by dualities [18]. Beyond $\mathcal{N} = 4$ super Yang-Mills, analogues of S-duality are thought to exist in many cases, but it is seldom known to explicitly implement them.

S-dualities appear frequently in superstring theory as well, by means of compact toric spaces [19]. In addition, superstrings have other types of dualities such as the so-called T-duality. In its simplest form, T-duality is just the observation that when we compactify circularly one of the target space dimensions of a closed string theory, we get two interchangeable towers of modes representing the wrapping and winding modes of the string around that circle. More specifically, the zero mode of a compact boson $x_{\mathrm{C}} \sim x_{\mathrm{C}} + 2\pi R n$ with a cylindrical worldspace, decomposes because of compactness in a sum of two towers of discrete modes that we interpret as wrapping and winding:

- *Wrapping* appears because the string can encircle the compact direction: $x_{\mathrm{C}}(\sigma) \sim x_{\mathrm{C}}(\sigma + 2\pi)$ decomposes under Fourier in modes of $e^{inR\sigma}$ type, $n \in \mathbb{Z}$.
- *Winding* appears through the quantization of the momentum caused by the periodic boundary conditions, $p = \alpha' \frac{m}{R}$, coming from modes of the $e^{i\alpha'\frac{m}{R}\tau}$ type, $m \in \mathbb{Z}$.

These two sectors ultimately have the same type of physical contributions. The mass spectrum for the string, for instance, is given by

$$M^2 = \frac{1}{\alpha'}\left(m^2\frac{\alpha'}{R^2} + n^2\frac{R^2}{\alpha'} + 2\left(N + \bar{N} - 2\right)\right); \left\{ \begin{array}{c} 0 \leq n, m, N, \bar{N} \in \mathbb{Z} \\ \bar{N} - N = nm \end{array} \right\} \tag{2.22}$$

The formal symmetry between these two towers of modes grants that the same physics can be described through a similar (T-dualized) construction, obtained by replacing the compact radius by $R' = \alpha'/R$ and interchanging the winding and wrapping modes

($n \leftrightarrow m$). Alternatively, the T-dualization can be viewed as a change of sign of the right-moving part of the compact boson.

Both S and T dualities play a crucial role in the modern viewpoint on string theory. During the mid-90s, compelling evidence was gathered showing that the five perturbative superstring theories presented above are not different theories but dual descriptions of the same underlying physics [20, 21, 22], which was dubbed "M-theory" and was seen to admit yet another semiclassical limit: eleven dimensional supergravity. The full discussion of dualities in string theory and M-theory surpasses by far the scope of this thesis, though, and will not be presented here. Instead, let us focus on another consequence of string theory dualities: the existence of a type of defect known as D-brane.

D-branes can precisely be introduced in string theory by building the T-dual description of open string theory. Since the open string does not topologically wrap the compact circle, we should make sense of the T-dualization as the sing interchange of the right-moving part of the compact boson. In practice, this is equivalent to changing Von Neumann boundary conditions into Dirichlett boundary conditions. The string is therefore forced to end in a very special locus known a Dirichlet-brane, or D-brane for short. In general, this object is described through an effective action that should include

- A gauge field accounting for the longitudinal string modes A_μ that generates a gauge theory on the brane, and appears only in the specific combination $\mathcal{F} = 2\pi\alpha' F - P\left[B\right]$ with $F = dA$, so that gauge transformations $2\pi\alpha' B \to 2\pi\alpha' B + d\Lambda$ can be absorbed by $A \to A + \Lambda$. The $2\pi\alpha'$ choice of normalization for the field A has been taken for convenience.

- A scalar field for any transverse modes of the brane (and the modes with Dirichlett boundary conditions for the strings) that describes the effective dynamics of its embedding. These scalars should be coupled through a volume form, and for consistency g should appear in combination with B as in (2.10).

- A tension that can be computed from the propagation of a closed string between two branes, or, equivalently, from a vacuum loop of open strings. One obtains

$$T_p = \frac{\sqrt{2\pi}}{\left(2\pi\sqrt{\alpha'}\right)^{p+1} g_s} \,. \tag{2.23}$$

- In the open string restricted to a hyperplane, the boundary conditions break half of the supersymmetry. D-branes will in general be curved, but locally they should still preserve half of the supersymmetries. Therefore the D-brane must be a BPS operator and this ultimately implies that it will couple to RR fields.

The effective action for this kind of object was built by Leigh et. al [23, 24] for specific backgrounds and later its general form with explicit Lorentz invariance (with the corresponding κ-symmetry projection to match the number of fermionic and bosonic degrees of freedom and allow for local supersymmetry) was developed in parallel by several groups [25, 26, 27]. The resulting action for a D-brane of p dimensions, or $D-p$-brane for short, is the sum of two parts, which are respectively called of "Dirac-Born-Infield" and "Wess-Zumino" in reference at their simpler analogues. They respectively read, in the notation of [28],

$$S_{DBI} = -T_p \int_\Omega d^{p+1}x e^{-\Phi} \sqrt{|\det\left(g_{MN}\partial_i \xi^M \partial_j \xi^N + \mathcal{F}_{ij}\right)|} \tag{2.24}$$

$$S_{WZ} = T_p \int_\Omega e^{\mathcal{F}} \wedge C \tag{2.25}$$

where the target space is in general taken to be a superspace and

$$\mathcal{F}_{ij} = 2\pi\alpha' F_{ij} - B_{MN}\partial_i \xi^M \partial_j \xi^N;\; C = \bigoplus_n C_n \tag{2.26}$$

Observe that in the weak curvature limit when the background is flat and for small $\mathcal{F}$ the action (2.24) admits a Taylor expansion and the resulting theory will have as the leading nontrivial action a Maxwell term coupled to scalars and their superpartners.

When we form a stack of N D-branes instead of considering them independent, endpoints of open strings inherit a label that tells which brane they are ending on. This indices, known as "Chan-Paton" indices, get translated through T-duality into the brane low effective action, where all the fields will be in the adjoint representation of a nonabelian $U(N)$ theory. In this case, the effective theory obtained from the low-deformation action around flat space becomes a Super-Yang-Mills (SYM) theory [29, 30].

By carefully analysing the degrees of freedom, dependence on the string coupling, symmetries and RR charge of D-brane stacks, Polchinski argued they provided dual description of the black brane solitons of supergravity [31]. This connexion is the seed where holography germinated from.

Let us close this section with a remark on D-branes that connects to the appreciations we made earlier about effective defects. I have already highlighted that the connection between D-branes and black branes explained in the previous paragraph constitutes a paradigmatic example of the effective defects we mentioned in 2.1.1. It is by no means the only example of this type of situation found in field theories: even field theories in flat spacetime are filled with solitons and instantons that, in the appropriate approximations, are better described as defects. However, this idea can be taken a step further, and D-branes is precisely one of the richest contexts to observe it. The issue is the following: from the worldspace point of view the embedding function is a field, and as such it can develop singular behaviour in a locus of reduced dimensionality [32, 33, 34, 35], which are sometimes called *BIons*. In the target space this type of solution will look like an infinite spike (with possible flat directions) developed by the brane in question, with the appropriate charges to be interpreted as another type of object of reduced dimensionality ending on the original brane. In a specific limit where the spike becomes

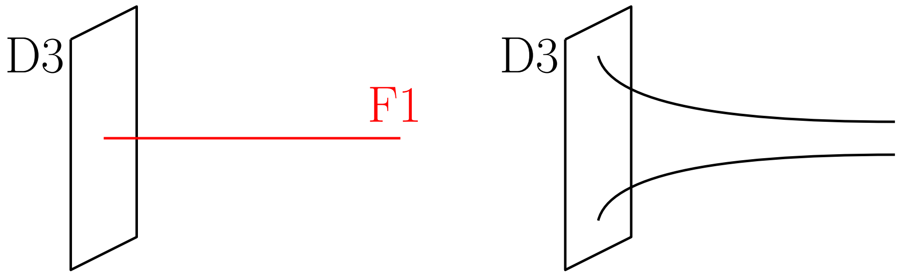

Figure 2.2: The D3 brane BIon is an efficient description of a large stack of fundamental strings being polarized by the Emparan-Myers effect near their endpoint in a D3 brane. For a single fundamental string, in contrast, we distrust the highly curved D3 singular embedding and resort to the conventional Nambu-Goto description for the string.

narrow, the defect-to-defect description will suffice to describe the system, while in the regimes where the embedding becomes smoother the spiked D-brane solution is more efficient: it captures the *polarization*, in the sense of the Emparan-Myers effect [36, 30] that transforms a lower-dimensional brane into a BIon.

For the purposes of this thesis, we will be specially interested in a specific implementation of the Myers effect, the polarization of D3 branes into fundamental strings, schematically depicted in figure 2.2. This structure, first considered in [33], is weakly curved and therefore trustful when it represents a stack of multiple strings (see [37, 38] for more precise details on this regard).

2.2 General holographic concepts

In this section we will focus on the central subject of study of this thesis: the conjectured correspondence between d-dimensional gauge field theories in locally flat space and a string theory compactified to a $d+1$ gravity. Let me begin this section by historically contextualizing its appearance.

Many symptoms of holography were already known before Juan Maldacena developed its first implementations [39] in 1997. Let me list here a sample of manifest consequences of this relation:

The algebraic structure of conformal algebras: Both the conformal group of a Minkowskian D-dimensional spacetime and the isometry group of a hyperbolic AdS_{D+1} spacetime are $SO\left(D,2\right)$. A straightforward way to see this is to depart from a $D+2$-dimensional flat spacetime of coordinates ξ^{α}, $\alpha\in\left(-1,D\right)$, with ξ^{-1} and ξ^{0} timelike. Then:

- One obtains AdS spacetime as the universal cover of the hyperboloid $\eta_{\alpha\beta}\xi^{\alpha}\xi^{\beta}=R^{2}$. Global coordinates for the resulting space, that has hypercylindrical topology, can be obtained through

$$\left\{\begin{array}{c}\xi^{-1}=R\cosh\rho\sin\tau\\ \xi^{0}=R\cosh\rho\cos\tau\\ \xi^{i}=R\sinh\rho\Omega_{D-1}^{(i)};\ \text{i}\in\{1,\ldots,\text{D}\}\end{array}\right\}\tag{2.27}$$

$$ds^{2}=R^{2}\left(d\rho^{2}-\cosh^{2}\rho d\tau^{2}+\sinh^{2}\rho d\Omega_{D-1}^{2}\right)\tag{2.28}$$

where $\Omega_{D-1}^{(i)}$ denotes the unitary vector in the i-th Cartesian coordinate in a D-dimensional space. The universal covering is obtained by continuing τ to $(-\infty,\infty)$. For our purposes, a specific region of this space will be of interest, that known as the "Poincaré patch". It can be obtained from the hyperboloid through the variable substitution:

$$\left\{\begin{array}{c}\xi^{-1}=\frac{R^{2}+z^{2}-\eta_{\mu\nu}x^{\mu}x^{\nu}}{2z}\\ \xi^{\mu}=\frac{R}{z}x^{\mu};\ \mu\in\{0,\ldots,D-1\}\\ \xi^{D}=\frac{R^{2}-z^{2}+\eta_{\mu\nu}x^{\mu}x^{\nu}}{2z}\end{array}\right\}\tag{2.29}$$

$$ds^{2}=R^{2}\left(\frac{\eta_{\mu\nu}dx^{\mu}dx^{\nu}-dz^{2}}{z^{2}}\right)\tag{2.30}$$

Regardless of the coordinate system used the isometries that leave the hyperboloid invariant are precisely the orthogonal transformations in the $(D,2)$ spacetime.

- At the projective hypercone $\eta_{\alpha\beta}\xi^\alpha\xi^\beta = 0$, we can study the effects of the $D+1$-dimensional rotations [40]

$$iJ^{\alpha\beta} = \xi^\alpha\partial^\beta - \xi^\beta\partial^\alpha \tag{2.31}$$

on

$$X^\mu = \frac{\xi^\mu}{\xi^{-1}+\xi^D} \tag{2.32}$$

They reproduce the effects of translations, special conformal transformations, dilatations and rotations. Respectively,

$$\left\{\begin{array}{c} P^\mu = J^{D\mu} + J^{-1\mu} \\ K^\mu = J^{D\mu} - J^{-1\mu} \\ J^{\mu\nu} \\ D = J^{D-1} \end{array}\right\} \to \left\{\begin{array}{c} iP^\mu X^\nu = \eta^{\mu\nu} \\ iK^\mu X^\nu = X^\mu X^\nu \\ iJ^{\mu\nu}X^\rho = X^\mu\eta^{\nu\rho} - X^\nu\eta^{\mu\rho} \\ iDX^\mu = 2X^\mu \end{array}\right\} \tag{2.33}$$

The conventional commutation relations for the conformal group can be obtained from those of the orthogonal group in $(D,2)$:

$$i\left[J^{\alpha\beta}, iJ^{\gamma\delta}\right] = \eta^{\beta\gamma}J^{\alpha\delta} + \eta^{\alpha\delta}J^{\beta\gamma} - \eta^{\beta\delta}J^{\alpha\gamma} - \eta^{\alpha\gamma}J^{\beta\delta} \tag{2.34}$$

An important observation here is that at the $z \to 0$ boundary of the Poincaré patch, at finite x^μ, we precisely encounter the projective space where the conformal algebra is realized.

Flux tubes: Quark-antiquark flux tubes have a stringy behaviour that endows light mesons with very particular characteristics, among which the celebrated “Regge-slope law” that relates mass and angular momentum for families of mesons:

$$J - J_0 = \alpha' M^2 \tag{2.35}$$

In fact, string theory was born as a justification of the “dual resonance models” that attempted to explain these properties before the advent of quantum chromodynamics [41]. The success of these archaic string theories was only partial and they were abandoned in favour of the gauge theory. Surprisingly, in its holographic implementation, string theory stands again as a good model to describe confining flux tubes, bypassing all the objections that caused its original demise by residing in additional dimensions, as we shall see in 2.3.3.

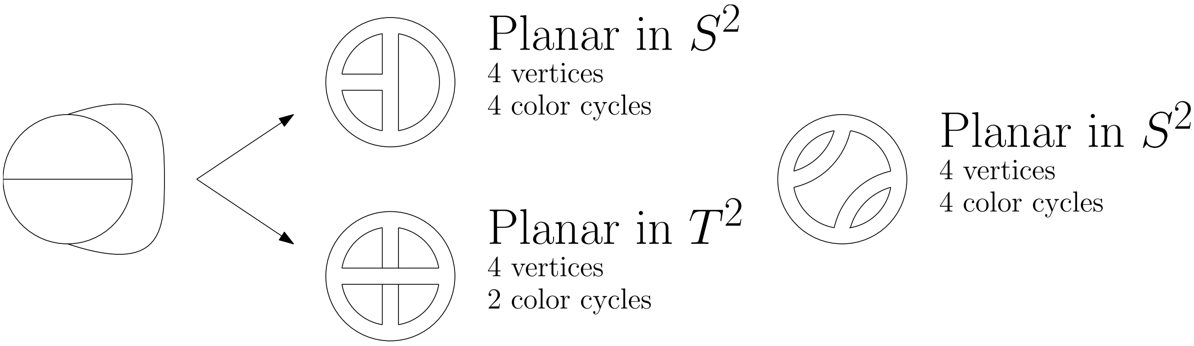

Figure 2.3: The double line notation specifies the color structure without the ambiguity inherent to the vertices-propagators graph. It also makes explicit the number of color traces in a diagram.

The gravitational entropy: The black hole entropy derived by Bekenstein [42] from black hole thermodynamics

$$S = \frac{c^3 \text{Area}}{4G\hbar} \, , \tag{2.36}$$

has to describe the maximal amount of entropy a region bounded by a specific area can have. Otherwise, black hole formation could in general violate the second principle of black hole thermodynamics. Elaborating on the suggestion of 't Hooft [43], Susskind [44] illustrated how this formula is consistent with a holographic construction of gravity, where all d-dimensional gravitational degrees of freedom are encoded in a screen or box of dimension $d-1$.

't Hooft's large N expansion for simple gauge theories: The rank of the gauge group N can be used for interacting gauge theories with simple classical gauge algebras as the inverse of an expansion parameter. This was first realized by 't Hooft [45], who observed that in the limit $N \to \infty$ (which has to be taken keeping $\lambda \equiv g^2 N$ fixed to keep the theory interacting), the theory becomes simpler: and all Feynman diagrams with nonplanar graphs become infinitely suppressed in comparison to the leading planar ones.

More precisely, 't Hooft studied a $U(N)$ gauge theory, for which the adjoint representation can be built as the product of two conjugate fundamental representations. This allows to rewrite the adjoint elements in a double-index notation, where we observe the following relation for the color generators in two vertices connected by a propagator.

$$\sum_i (T_i)^a_b \, (T_i)^c_d = \frac{C_2}{N} \delta^a_d \delta^b_c \tag{2.37}$$

If we represent graphically the propagation of the two indices of this notation independently, the number of independent color traces in a Feynman diagram becomes manifest as the number of independent fundamental index cycles in its double-line graph (see figure 2.3). A minutious analysis of the vertex and propagator factors show that

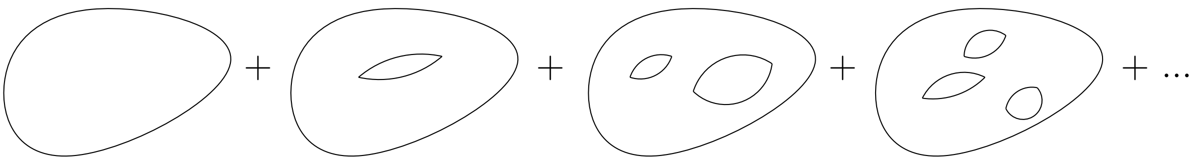

Figure 2.4: Both in perturbative string theory and gauge theories of large rank we can arrange the diagramatic expansion in terms of a topological sequence of surfaces of increasing complexity. For $U(N)$ theories containing only adjoint fields the only topological property that varies between the terms is the genus of the surface.

the order of a diagram in the N^{-1} expansion depends on the Euler characteristic of a compact 2-dimensional oriented surface related to the diagram. We obtain this surface as the topologically simplest surface where the diagram can be drawn planarly respecting the double line structure[4]. This argument shows that in this case the theory admits a topological expansion, and, at each order, an expansion in terms of the coupling. Schematically,

$$\langle \mathcal{O} \rangle = \sum_{\chi,n} \mathcal{O}_{\chi,n} N^{-\chi} \lambda^n \tag{2.38}$$

This double expansion resembles suspiciously the perturbative behaviour of string theories (see figure 2.4).

The generalization of this idea to other classical simple groups introduces additional terms to the expansion. For $SU(N)$ they are simply subleading N^{-1} contributions to each diagram. Their origin is the trace subtraction from the $U(N)$ structure generated by the product of conjugated fundamental representations. In turn, $SO(N)$ and $SP(N)$ include nonorientable terms emerging from matrices that raise and lower the double line indices. Consequently, the topological expansion in SO and SP includes an expansion on the number of crosscaps in addition to the expansion in the genus. A well-known topological result (Dyck's theorem) states that three crosscaps can always be traded for a handle and a single crosscap, and therefore 't Hooft's topological expansion can be arranged in terms of surfaces that contain two crosscaps at most. Table 2.3 summarizes the structure found for each of these simple groups.

By combining these ideas with that of gravitational entropy it became apparent that a gauge/gravity duality should exist [44]. Even in the solvable 2-dimensional gauge-matter theories, though, it was not clear how to formulate the string dual [46, 47].

Chern-Simons and Wess-Zumino-Witten:The observation that the 3-dimensional topological Chern-Simons field theory can be explicitly rewritten in terms of a conformal theory on its boundary, the Wess-Zumino-Witten model [48], sets an interesting precedent for AdS/CFT, though in this case the bulk description involves no metric (is topological).

[4]Equivalently, this 2-dimensional surface is defined by a triangulation provided by the diagram, where each double line defines a 1-simplex of the triangulation.

Group	Double line properties
$SU(N)$	$\sum_i (T_i)^a_b (T_i)^c_d \propto \delta^a_d \delta^b_c - \frac{1}{N}\delta^a_b \delta^d_c$
$SO(N)$	$\sum_i (T_i)^a_b (T_i)^c_d \propto \delta^a_d \delta^b_c - \delta_{bd}\delta^{ac}$
$SP(N)$	$\sum_i (T_i)^a_b (T_i)^c_d \propto \delta^a_d \delta^b_c - J_{bd}J^{ca}$

Table 2.3: Bosonic spectrum in the lightcone gauge and worldsheet supersymmetry for the four perturbative closed superstring theories. In the $SP(N)$ case J denotes the block matrix establishing the symplectic structure

2.2.1 The AdS/CFT correspondence

Let us now discuss how, by using the relation between D-branes and black branes that was discussed in section 2.1.4, Juan-Martín Maldacena was able to explicitly build examples of gauge-string duality [39].

The low-energy description of branes the degrees of freedom associated to the defect usually decouple from those of the background. In particular,

- In a supersymmetric D-brane configuration, the low-energy description of open strings ending on a specific D-brane stack is a supersymmetric gauge theory defined on the worldvolume, which decouples from the background fields at low energy.
- The low-energy (as seen from infinity) description of a black brane defect in supergravity includes generic modes on the gravitational "throat" whose energy is seen as "low" because of the redshift factor. This makes them invisible to the long-wavelenght modes of the asymptotically flat spacetime, and therefore the theory develops two decoupled sectors: a flat gravity and a near-horizon geometrical description.

Although the regimes where the D-brane description and black brane description are disjoint, they are in principle perturbative descriptions of the same type of object. In his seminal paper, [39] Maldacena made use of this fact to conjecture that there should be specific constructions where the very same object is described in both sides, and the two descriptions become dual low-energy descriptions. Since in both cases we obtain a decoupled flat gravitational sector, the conjecture ultimately relates the AdS-like spacetimes that emerge in the near-horizon limit of black brane solutions to the gauge CFT descriptions of the D-brane open string sector, that's why the conjecture is generically called "AdS/CFT" or "gauge-gravity".

To identify precise examples of such dualities we mostly rely on symmetry. Maldacena [39] already presented examples of the correspondence where symmetry totally fixes the form of the duality at both sides. Other maximally symmetric cases have been identified posteriorly. Less symmetrical examples of the theory can be constructed by deforming examples under control. Although most of the ideas exposed in what follows are in general valid for any holographic duality, I will on purpose exemplify them only for a specific example of the original list of [39].

The precise example that will be exposed in this thesis is the duality obtained by studying defects of $d = 3$ in type IIB strings. The supersymmetric case is described on the one hand by the superconformal phase of an $\mathcal{N} = 4$ SYM theory (in particular, with gauge group $SU(N)$), whose Lagrangian density is, in Euclidean signature,

$$\mathcal{L} = \frac{1}{g^2}\mathrm{Tr}\left\{\frac{1}{2}F_{\mu\nu}^2 + \frac{g^2\theta}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + (D_\mu\Phi_i)^2 - \frac{1}{2}\sum_{i,j}[\Phi_i,\Phi_j]^2 + i\bar{\Psi}\Gamma^\mu D_\mu\Psi + i\bar{\Psi}\Gamma^i[\Phi_i,\Psi]\right\}, \tag{2.39}$$

with $g = \sqrt{4\pi g_s}$. On the other hand, the near horizon description of a flat black 3-brane in supergravity becomes an $AdS_5 \times S^5$ spacetime, whose background is given by the near horizon limit $r \ll R$ of (2.12). In terms of $z = \frac{R^2}{r}$,

$$ds^2 = R^2\left(\frac{dx^\alpha dx_\alpha}{z^2} - \frac{dz^2}{z^2} - d\Omega_5^2\right) \tag{2.40}$$

To put this duality to work we need to understand better how the elements on one side can be translated to the other. The key ideas in this matching are:

The matching of parameters: Recall where the parameters in our construction come from: We depart from a string theory with string coupling g_s and tension $(2\pi\alpha')^{-1}$. Two descriptions of extremal D3-rane stacks are known:

- In the supergravity description of the string theory, at $g_s \ll 1$, the extremal brane construction (2.12) for $p = 3$ and at fixed F_5 form flux N, has a radius R that satisfies $R^4 = 4\pi g_s \alpha'^2 N$. At $R^4 \gg \sqrt{\alpha'}$, the semiclassical description is valid.
- The tension of a single effective $D3$-brane description is given by (2.23). Taking N branes in the low energy description, we identify the g^{-2} coupling for the Yang-Mills coefficient: $g^2 = 4\pi g_s N$.

Therefore, the relation between the AdS/CFT coefficients is given by

$$\lambda \equiv g^2 N = 4\pi g_s N = \frac{R^4}{\alpha'^2} \tag{2.41}$$

Notice that both approaches rely on $g_s \ll 1$. Also,

- The weak coupling expansion needs

$$\lambda = \frac{R^4}{\alpha'^2} \ll 1 \tag{2.42}$$

- Holography is semiclassical in the regime

$$\lambda = \frac{R^4}{\alpha'^2} \gg 1 \tag{2.43}$$

The matching of symmetries: We already discussed the coincidence of the isometry group of AdS_5 and the conformal group in $3+1$ dimensions. The isometries of the compact S^5 manifold coincide with the $SO(6)$ R-symmetry group of $\mathcal{N}=4$ SYM. Also, the superconformal $\mathcal{N}=4$ group matches the 32 Killing generators of a maximally supersymmetric type IIB supersymmetry.

The matching of observables: A more precise formulation of the conjecture [49, 50] for Euclidean signature follows from the idea that in the duality between the black brane and D-brane descriptions the physics should be the same and therefore the partition functions of the two systems should be equated[5]. In the supergravity regime, we can evaluate the gravitational partition function from its semiclassical action, and therefore

$$\mathcal{Z}_{\text{CFT}} \sim \exp\left(-S_{\text{SUGRA}}\right) \tag{2.44}$$

If we identify the sources introduced in a CFT with the boundary conditions of the corresponding field in $AdS_5 \times S^5$ the prescription to compute a local observables becomes

$$\left\langle \phi_J(x)\, e^{J\phi} \right\rangle_{J=0} = \left.\frac{\delta \mathcal{Z}_{\text{CFT}}}{\delta J}\right|_{J=0} \sim \left.\frac{\delta \exp\left(-S_{\text{SUGRA}}(J)\right)}{\delta J}\right|_{J=0} \tag{2.45}$$

where $S_{\text{SUGRA}}(J)$ denotes the action for the classical solution with a prescribed limit for the field associated to J at the $z \to 0$ limit. In particular, this field should satisfy

$$J(z) \to \epsilon^{d-\Delta} J \tag{2.46}$$

where

$$\Delta = \frac{d}{2} \pm \sqrt{\frac{d^2}{2} + R^2 m^2} \tag{2.47}$$

coincides with the scaling dimension of ϕ_J and d coincides with the number of dimensions of the AdS boundary. This procedure is known as the GKPW recipe.

The extension of this prescription to Lorentzian signature is not straightforward and was developed by Son and Starinets in [51]

[5] As Witten points out [50], for the partition function in the gravity side we have to restrict the possible metrics to those inducing the appropriate boundary behaviour.

2.2.2 Extensions of the conjecture

Known specific realizations of the AdS/CFT conjecture, as the one presented above, constitute landmarks in our exploration of the duality. We can obtain other holographic realizations from applying equivalent deformations on both sides. This can be done in several ways, including[6]:

Turning on a thermal background: In the holographic side this is studied by taking a nonextremal stack of branes instead of the extremal one and keeping the energy-to-nonextremality ratio fixed in the low energy limit. We obtain a Schwarzschild-AdS solution at finite temperature T. In this regime, interestingly, the theory develops many characteristics of generic field theories, including mass gap, phase transitions and confinement [50].

Moving along the moduli space: In the gravity side we know how to implement this idea directly on the Coulomb branch, where the spontaneous symmetry breaking can be realized by splitting the original stack of branes in different stacks at a finite distance. When this distance is large enough, the strings connecting different stacks of branes can be treated as semiclassical objects of infinite mass from the field theory side. We will discuss this limit in more detail in 2.3.3.

Adding a Lagrangian deformation: We can add generic deformations to any specific CFT and study how the theory behaves along the RG flow. The main guideline in this operation is the GKPW prescription, which imposes how the theory behaves near the $z \to 0$ boundary.

Observe that this type of deformation reproduces in z-slices qualitatively the properties of the RG flow caused by the UV deformation in the field theory side: It is not possible to make sense of irrelevant deformations in this framework, these deformations develop pathological divergences as $z \to 0$ that indicate we would need an UV completion of the theory including the new irrelevant operators. For relevant operators, in contrast, the effect at $z \sim 0$ can be studied as a perturbation. We obtain the deformed geometry evolving the original equations of motion with the appropriate boundary conditions at $z = 0$. In general, this evolution will eventually become inconsistent with supergravity at a certain depth, unless the deformation is fine-tuned to asymptote a specific well-behaved large z limit.

Folding the spacetime: We can perform further geometrical identifications in the string theory. A possibility to keep the dimensions along our field theory unaltered, is to consider in general foldings in the compact space in the gravity side.

An important family of duality constructions of this type was built by Kakhru and Silverstein [53], by replacing in the $D3$ construction of Maldacena the D-brane stack in a trivial background by the orbifold 3-plane resolution in terms of D3-branes [54]. The compact space in the AdS side becomes S^5/Γ, with Γ a finite group (a $\mathbb{Z}_n$ in the original

[6] Further details on the study of the holographic landscape by deforming known solutions can be found in [52]

G	$\theta_{NS} = 0$	$\theta_{NS} = 1/2$
$\theta_{RR} = 0$	$SO(2N)$	$SP(2N)$
$\theta_{RR} = 1/2$	$SO(2N+1)$	$SP(2N)$

Table 2.4: Correspondence between different choices for discrete torsion of 2-forms in the $\mathbb{Z}_2$ orientifolded geometry and the gauge algebra governing the field theory worldspace description

construction of [53]). In the low energy field theoretic limit these theories become quiver gauge theories. For $\Gamma = \mathbb{Z}_n$, for instance, the associated quiver is described by an A_{n-1} Dynkin diagram.

For the scope of this thesis, we will be interested in a specific subset of these constructions, namely, those with an antipodal $\Gamma = \mathbb{Z}_2$ action on the S^5, corresponding to the case where stack of branes sits at an orientifold 3-plane. Recall that when a $\mathbb{Z}_2$ identification acts on S^5 by identifying antipodal points the resulting manifold is a projective $\mathbb{RP}^5$ plane, and therefore in this case the geometry of the supergravitational construction will be $AdS_5 \times \mathbb{RP}^5$. For this particular case the gauge group of the field theory is still given by a simple gauge group, of $SO(N)$ or $SP(N)$ type. The additional ingredients that discriminate between $SO(2N)$, $SO(2N+1)$ and $SP(2N)$ are the possible choices of discrete torsion for the B and C_2 two-forms of the string theory, which become twisted two-forms. Since $H^3\left(\mathbb{RP}^5, \tilde{\mathbb{Z}}\right) = \mathbb{Z}_2$ their discrete torsion is classified respectively by θ_{NS} and θ_R, both with two possible values: 0 and $1/2$. The action of the $SL(2,\mathbb{Z})$ S-duality on these cases permits to identify which gauge algebra emerges in every case, see table 2.4.

2.3 Loops and superconformality

In this section we will discuss the characterization of specific classes of loop operators, and more specifically, Wilson loop operators, that preserve some of the symmetries of $\mathcal{N} = 4$.

2.3.1 Loops preserving conformal subgroups

A 2-dimensional defect on a D dimensional spacetime can preserve an $SL(2,\mathbb{R}) \times SO(D-1)$ subgroup of the conformal symmetry at most [5]. This subgroup is easy to identify for a defect lying on a straight line through the origin: denoting the longitudinal indices by i, j, ... and transverse indices by a, b, ..., we observe that the stability subgroup for the defect is generated by D, K_i, P_i, M_{ij} and M_{ab}. This structure of symmetries will remain unchanged under any inner automorphism, and therefore we can identify the same structure in any circular or hyperbolic loop that can be obtained through conformal transformations of a straight line defect [55, 56].

For these maximally conformal loops, conformal field theory allows to reduce all the dynamics of their "two-object" functions with local operators[7] into numerical coefficients. It is useful to define the coefficients f_{W} and h_{W} for the operator defect W through [5]

$$\frac{\langle \mathcal{L}(x)\,\mathsf{W}\rangle}{\langle \mathsf{W}\rangle} = \frac{f_W}{|x|^4} \tag{2.48}$$

$$\frac{\langle T_{\mu\nu}(x)\,\mathsf{W}\rangle}{\langle \mathsf{W}\rangle} = \frac{h_{\mathsf{W}}}{|x|^4} t_{\mu\nu}; \left\{ \begin{array}{c} t_{00} = 1 \\ t_{ij} = 2n_i n_j - \delta_{ij} \\ t_{0i} = 0 \end{array} \right\} \tag{2.49}$$

More generally, we can study the OPE structure of a theory in the presence of a defect[58]. Operators defined on the worldspace, such as the local operators for the transverse embedding functions, that are known as "displacement operators", have nontrivial relations amongst themselves and with the other operators in the theory, and therefore we have three sectors for the OPE that are called "bulk-to-bulk", "bulk-to-defect" and "defect-to-defect" in the literature.

In chapter 3 we will elaborate on another helpful aspect of conformal symmetry in the study of loop operators, namely, a relation between the properties of the cusp anomalous dimension at low angles, the Bremsstrahlung function for the radiatin and the coefficient controlling the local thermodynamic fluctuations of the loop path at the linearized level, originally observed in [12].

[7] More generic situations, including correlators with more than a local operator are discussed in [57]

2.3.2 Loops preserving supersymmetry

The 4-dimensional $\mathcal{N}=4$ SYM theory contains only adjoint matter, but fundamental W-bosons are generated in the coulomb branch, for instance, when the gauge group is broken from $SU(N+1)$ to $SU(N)\times U(1)$. In energies far lower than the W-boson mass, these objects can be efficiently described [8] by Wilson loops with coupling to the scalars, which we generically write as

$$\mathsf{W}_{\mathcal{R}} = \frac{1}{\dim \mathcal{R}} \mathrm{tr} \mathsf{P} \exp\left(i \int dx A_\mu \dot{x}^\mu + |\dot{x}|\, \theta^i \Phi_i\right) . \tag{2.50}$$

The normalization used in this expression has emerged as the canonical one in the literature, though its adequacy will depend on the physical purpose of the operator insertion in any case.

Using the construction of 4-dimensional $\mathcal{N}=4$ SYM as a dimensional reduction of 10-dimensional $\mathcal{N}=1$ SYM, Dymarsky and Pestun [60] classified the set of symmetry-preserving Wilson loops one can build in this theory. In this picture, the tangent vector to the loop and the coupling to the scalars are expressed in a single 10-dimensional vector v^μ, and supersymmetry with respect to the transformation generated by a killing spinor ϵ^a imposes

$$v^M(x)\,\Gamma_M \epsilon(x) = 0 \tag{2.51}$$

The set of 10-dimensional flat space Killing spinors are in turn fixed by

$$D_\mu \epsilon = \frac{1}{4}\Gamma_\mu \not{D}\epsilon \Rightarrow \epsilon = \epsilon_s + x^\mu \Gamma_\mu \epsilon_c \tag{2.52}$$

where ϵ_s and ϵ_c are constant spinors, respectively chiral and antichiral. For non-pure spinors, $u^M \equiv \epsilon\Gamma^M\epsilon \neq 0$, the equation (2.51) can only be solved by $v^M = \lambda u^\mu$. In the pure spinor case, $v^M = 0$, ϵ can be used to define an almost complex structure, and equation (2.51) is solved by its anti-holomorfic sector.

This systematic construction contains as particular cases of the pure spinor antiholomorphic case the two families of supersymmetric operators that were known at the time of its publication, namely,

The Wilson loops of Zarembo's type: For the loops proposed originally in [61], supersymmetry is obtained by imposing that the coupling to the scalars Φ_i is given in terms of the unitary 6-vector

$$\theta^i = M^i_\mu \frac{\dot{x}^\mu}{|\dot{x}|} \tag{2.53}$$

[8] see, for example, the appendix A of [59]

with M defines an $\mathbb{R}^4$hyperplane embedding in $\mathbb{R}^6$. This way, the equation for 4-dimensional supersymmetry is identical at every point of the loop, and, therefore, it is solved by the same spinor, that defines a global supersymmetry. The Wilson loop operators satisfying this condition can be expressed as

$$\mathsf{W} = \frac{1}{N}\mathrm{tr}\mathcal{P}e^{\oint dx^\mu \left(iA_\mu + M^i_\mu \Phi_i\right)} \tag{2.54}$$

If the shape of the Wilson loop is restricted to a $\mathbb{R}^n$ surface (with $n \leq 4$, the $n = 4$ case being unrestricted) the resulting Wilson loop will be at least of 2^{n-4}-BPS type. All the loops of this family, are conjectured to satisfy $\langle \mathsf{W} \rangle = 1$, even at the quantum level.

The Wilson loops of DGRT type: In this case [62, 63, 64], the coupling to the scalars is taken to be

$$\theta^i = \sigma^I_{\mu\nu} x^\nu M^{Ii} \tag{2.55}$$

In this expression indices I are Pauli matrix indices, M is a 3×6 matrix with $MM^T = \mathbb{I}_{3\times3}$, and $\sigma^I_{\mu\nu}$ relates the decomposition of anti-chiral Lorentz generators into Pauli matrices τ_I:

$$\frac{1}{2}\left(1 - \gamma^5\right)\gamma_{\mu\nu} = i\sigma^I_{\mu\nu}\tau_I \tag{2.56}$$

The loop becomes $\frac{1}{16}$-BPS when its path is restricted to lie on an S^3. Again, paths of enhanced symmetry provide more supersymmetric Wilson loops: on a maximal S^2 for instance, they become $\frac{1}{8}$-BPS, while for maximal circles we obtain $\frac{1}{2}$ BPS loops.

In contrast to what happens for the loops of the previous family, DGRT loops are in general nontrivial. Of special relevance are the loops that wind once around a circle on S^2, say, at a specific latitude θ_0. They are in general $\frac{1}{4}$-BPS[9], but it was argued in [65] and later discussed in [62, 63, 64, 66, 67, 68] that their v.e.v. can be related to the $\frac{1}{2}$-BPS one sitting at $\theta_0 = 0$ by a coupling redefinition:

$$\langle \mathsf{W}_{\theta_0} \rangle (\lambda) = \langle \mathsf{W}_\circ \rangle \left(\lambda \cos^2\theta_0\right) \ . \tag{2.57}$$

This fact was used in [12] to argue that the Bremsstrahlung function of $\mathcal{N} = 4$ SYM loops can be obtained from the $\frac{1}{2}$-BPS DGRT loop through

$$B = \frac{1}{2\pi^2}\lambda\partial_\lambda \ln \langle \mathsf{W}_0 \rangle \ . \tag{2.58}$$

In chapter 3 this quantity and its properties will be discussed with more generality.

[9] The limit $\theta_0 \to \frac{\pi}{2}$ in this construction connects to a $\frac{1}{2}$-BPS loop of the family defined by Zarembo.

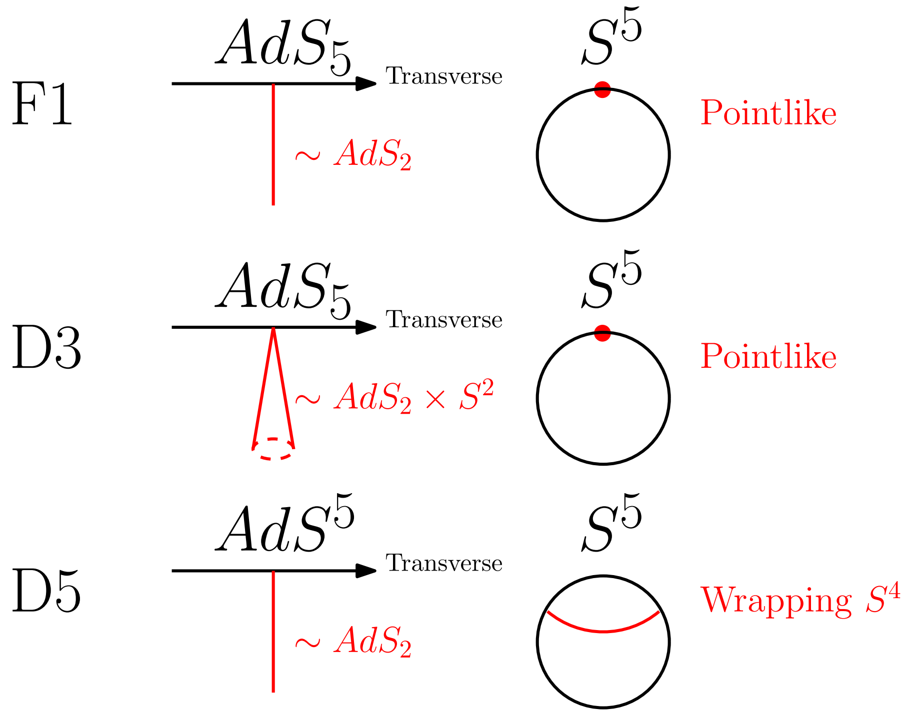

Figure 2.5: Structure of the probe embeddings for the three types of holographic probes we can describe as defects, i.e., fundamental strings, D3 brane spikes and D5 brane wrapping the compact sphere.

2.3.3 Holographic loops

Let us now briefly discuss how the Coulomb branch construction we mentioned in 2.3.2 can be implemented holographically and how can we use holography to compute Wilson loops.

An $SU(N+1) \to SU(N) \times U(1)$ spontaneous breaking corresponds in the $D3$-brane holographic construction of Maldacena to separating a single $D3$ from an N $D3$-brane stack. The role of heavy W-bosons is played in this case by strings hanging between the stack and the "Higgsed" brane.

A fundamental Wilson loop is therefore implemented holographically as a fundamental string with prescribed boundary conditions at the $z \to 0$ limit of AdS_5. Following the idea of the GKPW prescription, the v.e.v. for the Wilson loop should be computed as the effective string action in the gravitational side. In the supergravity limit the dominant contribution is given by the area of the string of extremal surface satisfying the appropriate boundary conditions [69, 70, 59], up to an infinite substraction needed to regularize this quantity. As noted in [59], though, only Wilson loops with local half-BPS symmetry, i.e., unitary coupling to the scalars in the proper time parametrization, can be consistently implemented this way.

What about charges of higher representation? In 2.1.4 we mentioned that string-like objects ending in D3 branes can be under certain circumstances described with a singularity in the D3 embedding function [33, 34, 35]. These type of solution, which was extended to AdS backgrounds in [70, 38], is reliable when the effective string it describes carries a large charge and provides plausible results for the leading order that connect to the fundamental string. Nonetheless, the total charge for a Wilson loop (and its corresponding dual string stack) does not distinguish different physical situations where the trace structure of the dual Wilson loop changes, and which should have different subleading features [38]. The same total charge occurs, for instance, in the superposition of k fundamental Wilson loops, when a fundamental Wilson loop is wrapped k times on itself, and in a k-symmetrical or k-antisymmetrical representation for the loop. Which solution is captured by the D3-brane polarized string?

This question was elucidated for generic representations in [71] after some examples [72, 73] illustrated that for specific cases the proper contribution was not the one predicted by the polarization of a D3 brane, but instead could be reproduced by a D5 brane wrapping an S^4 on the compact S^5 holographic sphere. Matching the symmetries of BPS configurations the paper [71] argued that the generic irreducible representation can always be described via a set of D3 brane stacks embedded in $AdS5$ or equivalently[10] a set of D5 brane stacks wrapping the aforementioned S^4 sphere in S^5. The cases we can address using the DBI+WZ actions, i.e., the single D3 and D5 brane stacks of charge k correspond, respectively, to the k-symmetric and k-antisymmetric representations, as summarized schematically in figure 2.5.

2.3.4 Early hints of localization

A surprising observation was made by Erickson, Semenoff and Zarembo (ESZ) about the $\frac{1}{2}$-BPS circular loop in $U(N)$ $\mathcal{N}=4$ SYM [74]. They found by perturbative analysis at low orders that all interacting diagrams cancelled in the Wilson loop v.e.v., which could therefore be evaluated at large N by summing only the contributions the set of diagrams of “rainbow” type, as the one depicted in 2.6. Furthermore, the supersymmetry and geometrical structure of the $\frac{1}{2}$-BPS Wilson loop makes all the propagators between two points of the loop independent of their endpoints once the scalar and vector contributions has been cancelled, and therefore the sum of rainbow contributions can be computed, up to an overall factor fixed by the single propagator case, by counting the number of diagrams. Conjecturing that the suppression of non-rainbow diagrams observed in their analysis was valid at all orders, the three authors obtained:

$$\langle \mathsf{W} \rangle = \frac{2}{\sqrt{\lambda}} I_1\left(\sqrt{\lambda}\right) + \mathcal{O}\left(\frac{\lambda}{N^2}\right) \tag{2.59}$$

matching the behaviour predicted from holography. They also observed that the coordinate dependence of the propagators mapped the problem to a matrix model.

[10] The two pictures are related by the framework of fermion bosonization

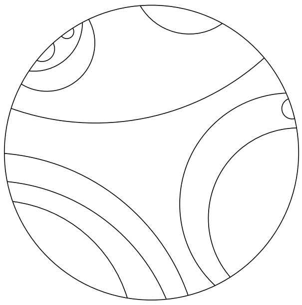

Figure 2.6: The term *rainbow diagram* for diagrams without circular vertices in circular Wilson loops is easy to justify.

Drukker and Gross [75] provided an explanation to the cancellations observed in [74] and the fact that the problem could be reduced to a 0-dimensional problem. The circular $\frac{1}{2}$-BPS Wilson loop can be obtained in these theory as a conformal transformation of the straight line Wilson loop, which has trivial v.e.v., $\langle \mathsf{W} \rangle = 1$. The transformation is anomalous, and an intuitive reason of why this should happen is that special conformal transformations interchange points of the theory with a point that is not part of it, the "point at infinity". The same authors computed the expression for the circular loop at all N, λ: From the matrix model of the ESZ conjecture

$$\langle \mathsf{W} \rangle = \frac{1}{N} \int [dM]\, e^{M} e^{-\frac{2N}{\lambda} \mathrm{Tr}\left(M^2\right)} \,, \tag{2.60}$$

where M runs over Hermitical matrices, they obtained

$$\langle \mathsf{W} \rangle = \frac{L^1_{N-1}\left(-\frac{\lambda}{4N}\right) e^{\frac{\lambda}{8N}}}{N} \,, \tag{2.61}$$

which reproduces (2.59) at large N.

2.4 Supersymmetric localization

In 2007 Pestun provided an elegant proof of the ESZ conjecture that relied on the symmetry of the configuration. His argument is valid $\forall\lambda$, N, even at the nonperturbative level, for generic $\mathcal{N}=2$ Lagrangian SYM theories. This thesis ultimately builds on this result, whose origin and applicability will be summarized in this section.

2.4.1 An overview of the localization method

Supersymmetric localization is an extension to path integrals of the localization technique of equivariant cohomology. Let us observe how it works, schematically: Suppose that for a specific type of integrals the integration measure and domain makes them invariant under the addition of Q-exact elements to the integrand, where Q is a generic linear field operator. Any combination of $\alpha+\mathsf{Q}\beta$ is equivalent under the integral to α, and therefore we can divide the possible integrands in classes of equivalent elements under Q-addition, in close analogy to cohomological classes. It is often computationally efficient to replace the original representative of this class by the representative that makes the integral the simplest. Localization takes this idea one step further. Let us define $\mathsf{Q}^2\equiv\mathcal{L}_B$ for convenience. For any Q-exact α and $\mathcal{L}_B$-exact V_F, $\alpha e^{-t\mathsf{Q}V_F}$ lies in the same cohomology class as α, for any $t\in\mathbb{R}$:

$$\frac{d}{dt}\left(\alpha e^{-t\mathsf{Q}V_F}\right)=\mathsf{Q}\left(\alpha V_F e^{-t\mathsf{Q}V_F}\right) \tag{2.62}$$

The key point in what follows is that if the bosonic part of $\mathsf{Q}V_F$ is positive definite, for the $t\to\infty$ cohomological representative the integral can be exactly evaluated through the saddle point technique, which becomes exact in this limit.

In its original formulation, namely, in equivariant cohomology, this idea was developed for spaces of polyforms, with $\mathsf{Q}\equiv\mathrm{d}+\xi\mathrm{i}_V$, and $\mathsf{Q}^2=\xi\left[\mathrm{d},\mathrm{i}_V\right]=\xi\mathcal{L}_V$. In this case $\mathcal{L}_V$ denotes a Lie derivative along V, and any integral on a closed domain of maximal dimensionality is invariant under the addition of Q-exact terms, defining the *equivariant cohomology classes*.

To extend this idea to the CFT, observe that if we take a nonanomalous supersymmetry of the theory Q,

$$\left\{\begin{array}{c}\mathsf{Q}S=0\\ \int\left[D\phi\right]\mathsf{Q}\left(f\left(\phi\right)\right)=0\end{array}\right\}, \tag{2.63}$$

the supersymmetric localization technique can be applied to any scalar observable under Q, provided we find a potential V_F such that $\mathsf{Q}V_F$ is positive definite and $\mathsf{Q}^2V_F=0$. The evaluation of this observable is restricted to an integral over the critical surface of $\mathsf{Q}V_F$, typically a locus of a reduced dimension, therefore the name *localization*. In summary,

$$\left\{\begin{array}{c} \mathsf{Q}S=0;\ \mathsf{Q}\mathcal{O}=0 \\ \mathrm{Re}\left(\mathsf{Q}V_F\right)\geq 0\forall\phi \\ \int\left[D\phi\right]\mathsf{Q}\left(f\left(\phi\right)\right)=0 \end{array}\right\}\Rightarrow\langle\mathcal{O}\rangle=\int_{\mathsf{Q}V_F=0}\left[D\phi_{\text{loc}}\right]\left(\mathcal{O}e^{-S}\right)_{\mathsf{Q}V_F=0}\mathcal{Z}_{1-\text{loop}} \qquad (2.64)$$

In this computation, $\mathcal{Z}_{1-\text{loop}}$ appears in the exact Gaussian evaluation and does not depend on $\mathcal{O}$, only on the theory and $\mathsf{Q}V_F$.

The success of the method relies on the possibility of having operators V_F with all the aforementioned properties for the appropriate Q. In supersymmetric field theories, we can use for this purpose a generator of supersymmetry Q in combination with $V_F=\left(\psi,\overline{\mathsf{Q}\psi}\right)$.

2.4.2 Localization on $\mathcal{N}$=2 4-dimensional theories:

The work of Pestun [76] implements supersymmetric localization on a set of $\mathcal{N}=2$ theories on a compact euclidean S^4 manifold. In particular, it considers $\mathcal{N}=2$ theories built from a gauge multiplet with gauge algebra G and a massive hypermultiplet whose fields are in a specific representation of this algebra. This set of theories includes the massive deformation of $4D$ $\mathcal{N}=4$ that flows to pure $\mathcal{N}=2$, namely, the $\mathcal{N}=2^\star$ 4-dimensional SYM family. The presence of spacetime curvature makes the supersymmetric structure more demanding, but introduces additional interplay between the localization potential $\mathsf{Q}V_F$ and the fields that leaves only a flat direction for a specific field in discrete points. More precisely, the zero modes that will not localize are the values of the scalar field at the poles of the sphere. Therefore, his result is a computational rule for supersymmetric observables through a Gaussian matrix model:

$$\langle\mathcal{O}\rangle=\frac{1}{\mathcal{Z}}\int d\Phi_0\mathcal{O}\left(\Phi_0\right)e^{-\frac{8\pi^2r^2\left(\Phi_0,\Phi_0\right)}{g^2}}\mathcal{Z}_{\text{1-loop}}\left(r\Phi_0\right)\mathcal{Z}_{\text{inst}}\left(\Phi_0,\frac{1}{r},\frac{1}{r},q\right)^2 , \qquad (2.65)$$

where (,) denotes the Killing form of the algebra, and r is the radius of S^4. The factor $\mathcal{Z}_{\text{inst}}\left(\Phi_0,\epsilon_1,\epsilon_2,q\right)$ denotes Nekrasov's instantonic partition function [77]. In turn, $\mathcal{Z}_{\text{1-loop}}$ is a factor obtained from determinants of the appropriate kinetic operators for the bosons and fermions integrated out by the localization process. Its general form is involved and in general it needs a regularization. For this dissertation, however, we will only need its expression for superconformal theories, which is finite and reads

$$\mathcal{Z}_{1-\text{loop}}\left(x\right)=\frac{\prod_{\alpha\in\Lambda_R(G)}H\left(\alpha\cdot x\right)}{\prod_{\omega\in\Lambda_\omega(G)}\left(H\left(\omega\cdot x\right)\right)^{n_{\mathcal{R}}(\omega)}} , \qquad (2.66)$$

where $\Lambda_R\left(G\right)$ and $\Lambda_\omega\left(G\right)$ designate respectively the set of roots and the weight lattice of the algebra G. The label $\mathcal{R}$ designates here the representation of the matter hypermultiplet of the $\mathcal{N}=2$ theory under consideration, and $n_{\mathcal{R}}(\omega)$ is the multiplicity of weight ω in this representation.

Without going into details, Pestun's analysis proceeds by:

- Identifying the Killing spinors on the sphere and deforming the flat space $\mathcal{N}=2$ action to keep the on-shell supersymmetry on this curved manifold.
- Introducing auxiliary fields through the Berkovitz construction in order to make the specific symmetry we will use for localization, δ, realized off shell.
- Covariantly gauge-fixing the theory through the BRST formalism. It is practical to build the Q generator from a specific combination of δ and δ_{BRST} that makes its action on the fields completely nontrivial.
- Computing the 1-loop factor $\mathcal{Z}_{1-\text{loop}}$, obtained from operator determinants, by using results of algebraic topology [78].
- Arguing that the instanton contributions $\mathcal{Z}_{inst}$, is given by Nekrasov's partition function [77], which was also derived using supersymmetric localization, locating the theory in flat space instead.

It should be noted that in the original publication [76], some points of the discussion are only analysed for the $\mathcal{N}=2^{\star}$ family. Most results can nevertheless be straightforwardly extended to generic Lagrangian $\mathcal{N}=2$ theories, in particular those of the superconformal field theories we will encounter in this thesis.

The results on the decompactification limit $R\to\infty$ are known to match the expected behaviour for many supersymmetric observables, among which, the $\frac{1}{2}$-BPS Wilson loops. In particular, the ESZ conjecture is verified by computing the v.e.v. of the equatorial half-BPS Wilson loop, which corresponds in the matrix model to the operator

$$\mathsf{W}_{\tilde{\mathcal{R}}}\left(\Phi_0\right)=\frac{1}{\dim\tilde{\mathcal{R}}}\mathrm{tr}_{\mathcal{R}}e^{2\pi r\Phi_0}\,. \tag{2.67}$$

In the large N limit, where the instantonic factor is supposed to become one, this observable becomes independent of r, as one can easily see with a change of variables that absorbs $2\pi r$ into Φ. The result coincides then with expression 2.61 for a loop in the fundamental representation of $SU(N)$ $\mathcal{N}=4$ SYM (where $\mathcal{Z}_{\text{1-loop}}=1$ and $(\Phi_0,\Phi_0)=\Phi_0^2$).

Nevertheless, one should take care when exploring the noncompact theory from a decompactification limit: In the compact sphere transitions between vacua of the theory happen with finite amplitude, and therefore the observables computed through localization will contain in the decompactification limit an average on their values on the vacua of the theory in flat space. This combination is likely to be dominated by a specific vacuum, as discussed in [79, 80, 81]. The problem might become milder for large N: it is known that such theories can develop phase transitions even in compact spaces [82].

Algebra	$\Delta_{FP}(x)$
$SU(N)$	$\prod_{i<j}(x_i - x_j)^2$
$SO(2N)$	$\prod_{i<j}\left(x_i^2 - x_j^2\right)^2$
$SO(2N+1)$	$\prod_{i<j}\left(x_i^2 - x_j^2\right)^2 \prod_i x_i^2$
$SP(2N)$	$\prod_{i<j}\left(x_i^2 - x_j^2\right)^2 \prod_i (2x_i)^2$

Table 2.5: Faddeev-Popov factors obtained in the gauge-fixing of matrix models to their Cartan subalgebra for nonexceptional gauge algebras. They follow straightforwardly from the root structure of these algebras.

2.4.3 Computational tools for matrix models

Matrix models such as (2.65) are ubiquitous in mathematics, statistics and quantum mechanics. In their simplest nondynamical version (which coincides with the form they take in localization), they are defined by the partition function

$$\mathcal{Z} \propto \int_M dm e^{-S(m)} , \tag{2.68}$$

where M is an $N \times N$ matrix algebra. Their action $S(m)$ is invariant under the action of the algebra itself, and the transformations it induces are considered to be gauge. For the specific case of localization they indeed have origin in a gauge symmetry: they represent the action of the gauge on the value at the north pole of the zero mode that survives localization. Since this theory is not dynamical, in practice we can gauge-fix the states independently. Using the conventional Faddeev-Popov technique to restrict the matrices to a chosen Cartan subalgebra, we obtain, up to an irrelevant overall normalization factor,

$$\mathcal{Z} = \int d^N x_i \Delta_{FP}(x) e^{-S(x_i)} . \tag{2.69}$$

The Faddeev-Popov determinant is given by

$$\Delta_{FP}(x) = \prod_{\alpha \in \Lambda_R(G)} (\alpha \cdot x) . \tag{2.70}$$

The particular form of this determinant for nonexceptional simple Lie algebras is listed in table 2.5.

Two simple techniques can be often used to study this type of model, both of them widely used throughout this dissertation. In what follows, I give an overview of both of them, for a more complete and detailed explanation I would refer the reader to [83].

Orthogonal polynomials: Observe that for any classical simple gauge algebra the Faddeev-Popov includes the square of the following factor, which is known as the Vandermonde determinant:

$$\Delta_V = \prod_{i>j} (x_i - x_j) = \det\left(x_i^{j-1}\right) = \det\left(p_{j-1}(x_i)\right) \ . \tag{2.71}$$

In the last step of 2.71, columns of the determinant have been recombined to form polynomials of arbitrary form, up to a fixed degree and normalization:

$$p_{j-1}(x) = x^{j-1} + \sum_{n=0}^{j-2} c_n x^n \ . \tag{2.72}$$

The c_n coefficients are in principle different and independent for every j. Needless to say, for SO and SP algebras, where the Vandermonde determinant appears in terms of the squared variables, this trick can still be applied but polynomial p_j will be of order $2j$ and contain only even powers in the original variables. Furthermore, for $SO(2N+1)$ and $SP(2N)$ we may let every p_j polynomial absorb an additional x_i factor from the Faddeev-Popov determinant, obtaining instead odd p_j polynomials of order $2j+1$. The main advantage in this transformation is that all the Faddeev-Popov determinant is rewritten as the determinant of polynomials p_j, without any additional factor.

It follows from the definition of a determinant that Δ_V can be evaluated as a sum of terms where the contributions of x_i are factorized, and the contributions for every x_i are identical. If the same is true for the remaining factors in $\mathcal{Z}$, the integrals of the matrix model can be evaluated by choosing conveniently the p_j family to be orthogonal under the appropriate integration weight.

Fortunately, the construction of orthogonal polynomials under the integration of a single variable with an arbitrary integral weight $d\mu(x)$ can be done recursively in j in a systematic way, as long as $\int d\mu(x) \neq 0$. It is sufficient to observe that in the normalization of 2.72, necessarily

$$p_j(x) = x p_{j-1}(x) - \sum_{n=0}^{j-1} \tilde{c}_{j;n} x^n \ , \tag{2.73}$$

and the coefficients $\tilde{c}_n$ can be fixed for p_j once all the orthogonal polynomials of smaller degree are known by the corresponding orthogonality conditions

$$\int d\mu(x)\, p_j(x)\, p_k(x) = h_j \delta_{jk} \ . \tag{2.74}$$

Substituting equation 2.73 into 2.74 one obtains the following recurrence rule:

$$\left\{ \begin{array}{c} p_0(x) = 1;\ p_1(x) = (x + s_0) \\ p_{j+1}(x) = (x + s_j)\, p_j(x) - \frac{h_j}{h_{j-1}} p_{j-1}(x) \\ s_j = \frac{1}{h_j} \int d\mu(x)\, x p_j^2(x) \end{array} \right\} . \tag{2.75}$$

In chapter 4 we will use the orthogonal polynomials obtained from this recurrence under a gaussian weight, known as Hermite polynomials.

Observe that this construction gives us an unexpected resource: The polynomials obtained from a specific $d\mu(x)$ can be used to compute $\int d\mu(x)\, x^k p_i(x)\, p_j(x)$ as well, by absorbing factors of x in one of the polynomials with an iterated use of 2.75.

Saddle point approximation: The integrals over x can be evaluated through a Gaussian approximation. The equations fixing the position of the saddle point intertwine all the eigenvalues of the model. In fact, this problem becomes analogous to the search for the stable equilibrium configurations of a system with N interacting particles moving on a line, and we will, in what follows, use the clarifying language of the latter to discuss the problem.

The action $S(x)$ will generate a background potential for the particles that keeps them at finite distance from the origin, but the interactions arising from the Faddeev-Popov determinant will in general include pair repulsion at short distance from the Vandermonde factors that avoid the condensation of eigenvalues. At intermediate distances the problem might develop other types of behaviour, depending on the shape of S and the Vandermonde factor.

If we restrict ourselves to the matrix model of a simple nonexceptional algebra, the Faddeev-Popov term will generate only two-body couplings and modifications to the background potential $S(x)$. In practice, we are able to solve the problem analytically in two cases: either very large or very small N. The large N case is addressed using the continuum approximation: we build the distribution

$$\rho(x) \equiv \frac{1}{N} \sum_{k=1}^{N} \delta(x - x_k) \tag{2.76}$$

and look for continuous $\tilde{\rho}(x)$ functions with some support Γ on the line that, in the distribution sense and for sufficiently smooth test functions, behave as $\rho(x)$. In what follows, to keep the notation simple, we will denote both $\tilde{\rho}$ and ρ with the symbol ρ. Although the continuum limit can be applied to any large N problem, its solvability for the case of simple classical algebras is due to the fact that the resulting equation is in this case of a Fredholm problem of second type:

$$\frac{d}{dx}\left(S(x) - K_1(x)\right) = N\mathcal{P} \int_\Gamma dy \rho(y)\, K_2(x, y) \ , \tag{2.77}$$

where K_1 and K_2 are to be determined from Δ_{F-P}. This equation should be supplemented with the appropriate normalization condition:

$$\int_\Gamma dy\rho\,(y) = 1\ . \tag{2.78}$$

The structure of Γ can in general be inferred from the physical problem. For a Gaussian action, for instance, with symmetrical $\vec{x} \to -\vec{x}$ interactions, we expect Γ to be a symmetric segment $[-\mu, \mu]$, where μ can be determined by normalization.

In order to illustrate the use of the large N saddle point method let us analyse one of the simplest cases at hand, known as the Wigner surmise. This matrix model, which was proposed by Wigner [84] in the context of quantum energy level statistics, is an $SU(N)$ Gaussian matrix model,

$$\mathcal{Z} = \int d^N x_i \prod_{i<j} (x_i - x_j)^2\, e^{-8\pi^2 N x^2/\lambda}\ . \tag{2.79}$$

The unusual way of writing the Gaussian variance in (2.79) will match the form in which this problem appears in the supersymmetric localization of $SU(N)$ $\mathcal{N} = 4$ SYM (among other theories, as we will clarify in chapter 5).

The large N saddle point equation for the Wigner surmise reads

$$\frac{8\pi^2 x}{\lambda} = ⨍_{-\mu}^{\mu} dy\rho\,(y)\,\frac{1}{x-y}\ . \tag{2.80}$$

In order to solve this problem it suffices to find an integral operator that is inverse to the principal part distribution $\mathcal{P}\,(x-y)^{-1}$. This can be rephrased as a Riemann-Hilbert problem in distributions using the resolvent method (see for instance [83]). One obtains through this path

$$\mathcal{P}^{-1}_{x\to z}\,[f\,(x)] = -\frac{1}{\pi^2} ⨍_{-\mu}^{\mu} \frac{dx}{z-x}\sqrt{\frac{\mu^2 - z^2}{\mu^2 - x^2}}\, f\,(x)\ , \tag{2.81}$$

which inverts the principal part integral operator in the following sense:

$$\mathcal{P}^{-1}_{x\to z}\left[⨍_{-\mu}^{\mu} dy\frac{\rho\,(y)}{x-y}\right] = \rho\,(z)\,;\ z \in [-\mu, \mu]\ . \tag{2.82}$$

The use of this functional takes us directly to the solution of Wigner's problem, which is known as the Wigner semicircle law:

$$\left\{ \begin{array}{c} \rho\,(y) = \frac{8\pi}{\lambda}\sqrt{\mu^2 - x^2} \\ \mu = \sqrt{\frac{\lambda}{4\pi^2}} \end{array} \right\}\ . \tag{2.83}$$

Chapter 3

Momentum fluctuations of strongly coupled particles

Predictions of localization for large N, large λ $\mathcal{N} = 4$ SYM

In this chapter, field theoretical results for supersymmetric half-BPS Wilson loops in the large N and strongly coupled limits of $\mathcal{N} = 4$ SYM are compared to the predictions of holography, focusing our attention on the transverse momentum fluctuations of heavy particles. Both holographic and localization results are available in this regime, and therefore the role of the correspondence is here almost testimonial. Nonetheless, beyond the sanity check that supposes the matching of the two approaches to the problem, the exact results of supersymmetric localization can be used to gauge the reliability of the probe and supergravitational approximations in the holographic side and provide nontrivial tests for holography. The content of this chapter follows closely the ideas of [1].

3.1 Linear response for line operators in generic conformal field theories

The statistical and quantum mechanical fluctuations of a heavy particle in a representation $\mathcal{R}$ of the gauge group can be studied using perturbative methods around its path C in the classical equilibrium solution. We already learned in 2.1.3 that the effective action for the particle can be described as a Wilson loop operator $\mathsf{W}_{\mathcal{R}}$. The linearized response describing the fluctuations in the particle's position around the classical saddle $\mathsf{W}_{\mathcal{R}}(C)$ can be described in terms of a certain set of adjoint operators that are known as displacement operators [85, 12] $\mathbb{D}_i$:

$$\delta \mathsf{W}_{\mathcal{R}}(C) = \mathrm{tr}_{\mathcal{R}} \mathsf{P}\left(\mathsf{W}(C) \int dt \delta x^j(t) \mathbb{D}_j(t)\right) . \tag{3.1}$$

These operators $\mathbb{D}_i$ are in general gauge-dependent on their own, but, evaluated over the world-line, they form gauge-invariant operators. For a generic and possibly nonlocal operator $\mathcal{O}$ with support on the defect, we denote with double brackets its normalized two-point function with the loop:

$$\langle\langle \mathcal{O} \rangle\rangle_{\mathsf{W}_{\mathcal{R}}(C)} \equiv \frac{\langle \mathrm{tr}_{\mathcal{R}} \mathsf{P} \mathcal{O} \mathsf{W} \rangle}{\langle \mathsf{W}_{\mathcal{R}}(C) \rangle} \tag{3.2}$$

known as the "insertion" of $\mathcal{O}$ into the loop. The subindex $\mathsf{W}_{\mathcal{R}}(C)$ will be omitted when it can be inferred from the context.

We can regard $\mathbb{D}_i$ as a force applied to the particle (or equivalently a source for δx fluctuations), although the quantum or statistical mechanical average on this fields makes this force random, and the problem becomes a generalization of a Brownian motion, which can be addressed, at the linear response level, by means of Langevin dynamics with white noise [86, 10, 87]. The interpretation as a force is specially transparent for the case of a particle coupled minimally to a $U(1)$ Maxwell field and moving with 4-velocity U^μ in the Lorentz gauge: the corresponding displacement operator becomes $\mathbb{D}_\mu = qF_{\mu\nu}U^\nu$, the Lorentz force.

Let us now focus in the transverse displacement operators in CFT theories. Notice that the dimensionality of δx^j and dt in (3.1) is the canonical one by definition, and therefore $\mathbb{D}_j$ is necessarily an operator of $\Delta = 2$, protected against quantum corrections. For the $SL(2,\mathbb{R}) \times SO(3)$-preserving defects discussed in [5, 55, 56], the three independent transverse displacement operators form an $SO(3)$ triplet $\mathbb{D}_i^T$ ($i \in \{1,2,3\}$), and, therefore, in the timelike case, where the conformally symmetric motion is hyperbolic, symmetry constraints the 2-insertions of displacement operators to

$$G_{ij}(\tau) = \left\langle\left\langle \mathbb{D}_i^T(\tau) \mathbb{D}_j^T(0) \right\rangle\right\rangle = \tilde{\gamma} \frac{\delta_{ij}}{16R^4 \sinh^4 \frac{\tau}{2R}} . \tag{3.3}$$

The result for the static particle (straight line) can be recovered when the limit $\tau \ll 2R$ is taken appropriately:

$$\left\langle\left\langle \mathbb{D}_i^T(t)\,\mathbb{D}_j^T(0)\right\rangle\right\rangle_{\text{straight}} = \tilde{\gamma}\frac{\delta_{ij}}{t^4} \tag{3.4}$$

In fact, the coefficient $\tilde{\gamma}$ will be found in this limit for any generic smooth timelike path. In this sense $\tilde{\gamma}$ is a "universal" characteristic of each type of particle in a CFT. In fact, it is not independent of other important coefficients with similar universality[1]. Let me digress on this idea by explaining how $\tilde{\gamma}$ is directly related with the Bremsstrahlung function, the leading angular dependence of the cusp anomalous dimension at low angles, the radiative properties of the particle and the coefficient h_{W}. The authors of [12] used there the conformal mapping between $\mathbb{R}^4$ and $S^3 \times \mathbb{R}^1$ to rewrite a cusp of angle ϕ between straight lines to a geometric configuration where two static particles sit across S^3 with a $\pi - \phi$ angular separation between them (a similar construction was considered in [85]). Then, expanding around the $\phi = 0$ (straight line) case, they provided a relation between the quadratic coefficient B on the Taylor expansion for the cusp with $\tilde{\gamma}$:

$$\Gamma \sim -\phi^2 B = -\phi^2\frac{\tilde{\gamma}}{12} \tag{3.5}$$

This relation will be verified holographically in section 3.3. In the same paper, it was argued that B coincides with the Bremsstrahlung function, by studying the probability that a Wilson loop absorbs or emits a quantum of energy. To this end, they computed the probability that a static quark absorbs in the interval $t \in (0, T)$ a quantum of energy associated to the displacement $\delta x = \eta\left(e^{i\omega t} + e^{-i\omega t}\right)$, which is given by the Born amplitude rule,

$$p_{\text{abs}} = \left\|\eta\int dte^{-i\omega t}\mathbb{D}\left|\mathsf{W}_{\text{static}}\right\rangle\right\|^2 = T\left\|\eta\right\|^2\int dte^{i\omega t}\left\langle\left\langle\mathbb{D}(t)\,\mathbb{D}(0)\right\rangle\right\rangle = \frac{\pi\omega^3}{3}T\left\|\eta\right\|^2 \;. \tag{3.6}$$

The argument is completed by the fact that the total acceleration caused by this mode is

$$\int_0^T dt\left\|\delta\ddot{x}\right\|^2 = 2\omega^4 T\left\|\eta\right\|^2 \;, \tag{3.7}$$

[1] For the coefficients h_{W} and h_{W} we have additional reasons to believe that the coefficient for an accelerated loop matches the coefficient for the straight line, besides the matching at $\tau \ll 2R$: For any Euclidean CFT, a conformal transformation maps the straight Wilson line to a circular one. It is well-known that there is a conformal anomaly associated with this mapping, and the vacuum expectation values of these two operators do not coincide [74, 75]. Nevertheless, the contribution of this anomaly is localized on the Wilson line, so it is reasonable to expect that it cancels in a normalized two-point function like the one above, and the same coefficient f also appears in a similarly normalized two-point function with the circular Wilson loop. This expectation is borne out by explicit computations [88, 89, 90, 57].

and by identifying ω with the amount of energy absorbed in this particular process:

$$\langle \Delta E \rangle = \omega p_{\text{abs}} = -\frac{\pi}{6}\tilde{\gamma}\int dt a^{\mu} a_{\mu} \; . \tag{3.8}$$

Intuition dictates that this radiative property should be intimately linked to the h_{W} coefficient controlling $\langle \mathsf{W} T_{\mu\nu} \rangle$ for conformal defects, and indeed, Lewkowycz and Maldacena observed in section 6 of [91] that one could evaluate $\langle \Delta E \rangle$ as a Larmor-like formula characterized by h_{W}. The only subtlety in the relation is that not all the energy-momentum tensor contribution to the energy flux corresponds to radiated energy, in general self-energy contributions have a contribution as well. In [91] the self energy subtraction was implemented for supersymmetric theories of generic dimensionality D, and it was argued that, for them

$$B_{\mathcal{R}} = \frac{4\pi^{\frac{D+1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)} \frac{D-1}{D-2} \frac{h_{\mathsf{W}_{\mathcal{R}}}}{4\pi^2} \tag{3.9}$$

For $D = 4$, this relation simply reads $B_{\mathcal{R}} = 3h_{\mathsf{W}_{\mathcal{R}}}$ and is totally consistent with the results found in the literature [92, 93].

In what follows, a more detailed study of the thermodynamic equilibrium between the background and the transverse displacement fluctuations will reveal yet another physical observable that is fully determined by the function $B_{\mathcal{R}}$ in generic CFTs: the momentum dissipation coefficient κ. It is in general defined from the Fourier-transformed retarded green function $G^{R}_{ij}(\omega)$ through a conventional Kubo formula [94]:

$$\kappa_{ij} = -\lim_{\omega\to 0} \frac{2T}{\omega} \mathrm{Im} G^{R}_{ij}(\omega) \; . \tag{3.10}$$

Nonetheless, in our transversely isotropic and thermodynamically equilibrated system [10, 87]

$$G(\omega) = -\coth\frac{\omega}{2T} \mathrm{Im} G^{R}(\omega) \; , \tag{3.11}$$

and therefore

$$\kappa_{ij} = \lim_{\omega\to 0} G_{ij}(\omega) = \lim_{\omega\to 0} \int_{-\infty}^{\infty} e^{-i\omega\tau} G_{ij}(\tau)\, d\tau = \kappa\delta_{ij} \tag{3.12}$$

A straightforward complex integral along the path shown in figure 3.1 gives us

$$G(\omega) = \tilde{\gamma}\delta_{ij} \frac{2\pi}{3!R^2} \frac{\omega + R^2\omega^3}{e^{2\pi\omega R} - 1} \tag{3.13}$$

and therefore,

$$\kappa = \frac{\tilde{\gamma}}{6R^3} = \frac{2B}{R^3} \tag{3.14}$$

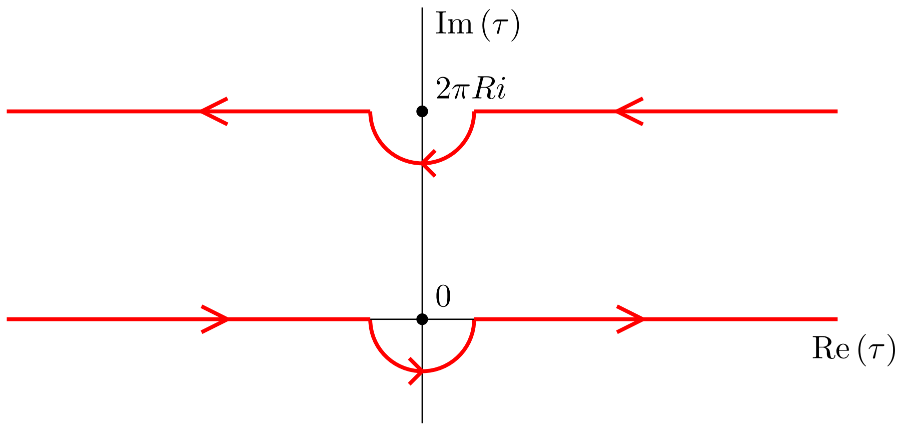

Figure 3.1: The Fourier transform of $\sinh^{-4}\left(\frac{\tau}{2R}\right)$ can be evaluated using the depicted circuit in $\mathbb{C}$.

for any generic CFT.

Observe that (3.13) displays a thermal behavior, where we identify the usual Unruh temperature [95, 96]

$$\beta = T^{-1} = 2\pi R \tag{3.15}$$

Through the Unruh effect, our accelerated particle perceives a thermal bath in its comoving frame, and its momentum fluctuations are controlled by this temperature:

$$\kappa = 16\pi^3 B T^3 \ . \tag{3.16}$$

Although the properties of this bath are known to be quantitatively different from those felt by static probes in a thermal bath (see for instance [96] and the review [97] for a discussion on this point), the two types of thermal fluctuations source cause qualitative behavior. In section 3.4 we will discuss up to which point the findings for the particle in hyperbolic motion can be extrapolated to the properties of a heavy particle moving through a thermal bath, and in particular to a charm or bottom quark moving through the quark-gluon plasma in LHC and RHIC colliders.

3.2 Field theoretical computation of the Bremsstrahlung coefficient

In the previous section, we presented a versatile radiative coefficient $B_{\mathcal{R}}$ and illustrated many of its applications, included its relation with the coefficients controlling the linearized response of a particle (γ), the energy and momentum sourced by a particle (h_W), and the transverse momentum dissipation caused by the background (κ).

We will now turn to the evaluation of these quantities using localization techniques in $\mathcal{N}=4$ SYM. In fact, the methods we will review here are specific of $\mathcal{N}=4$ SYM. For theories with less supersymmetry it is still possible to evaluate h_W, as we will discuss in chapter 5, but $\mathcal{N}=4$ SYM theories enjoy additional symmetries that simplify this computation. More specifically, in this maximally supersymmetric 4-dimensional theory the coefficients h_{W} and f_{W} are related because $T_{\mu\nu}$ and $\mathcal{L}$ lie at the same supermultiplet. This relation was used to extract f_{W} from B in [90] and indirectly checked in [12] when the conjecture of [65] was used to obtain the $\mathcal{N}=4$ Bremsstrahlung function. In practice, this identity implies that the Bremsstrahlung function can be evaluated using

$$B_{\mathcal{R}} = 4 f_{\mathsf{W}} = \frac{1}{2\pi^2}\lambda\partial_\lambda \ln \langle \mathsf{W}_{\mathcal{R}}\rangle \ . \tag{3.17}$$

The last identity can be justified following [4]: Since, schematically,

$$\langle \mathsf{W}\rangle = \frac{\int \mathcal{D}\phi \mathsf{W} e^{-\frac{1}{g^2}\int d^4x\mathcal{L}}}{\int \mathcal{D}\phi e^{-\frac{1}{g^2}\int d^4x\mathcal{L}}}\ , \tag{3.18}$$

$$g^2\partial_{g^2}\ln\langle \mathsf{W}\rangle = -\frac{1}{g^2}\int d^4x\frac{\langle \mathsf{W}\mathcal{L}\rangle}{\langle \mathsf{W}\rangle} \tag{3.19}$$

For conformal loops the integrand becomes $f_{\mathsf{W}} r^{-4}$, and after discarding a pole for the short distance regularization of the integral [2] in the conformal loop case we obtain the last identity of (3.17), which is valid for any Lagrangian 4d CFT, supersymmetric or not, as opposed to the first equality in the same expression.

But, what is the v.e.v. for a generic hyperbolic Wilson line? By analytic continuation of the equation for its trajectory, it can be related to the v.e.v. of a circular Wilson loop of the "maximal circle" half-BPS DGRT family. It is worth emphasizing that this argument would in principle require the starting Lorentzian description to include both branches of the hyperbolic Wilson line. In other words, an antiparticle should be added

[2] A convenient regularization was used in [91]. It consists of mapping the space to $S^1\times H_3$,

$$ds^2 = d\tau^2 + d\rho^2 + \sinh^2\rho\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.20}$$

and introducing a short distance cut-off ρ_c for the coordinate ρ. The divergence appears then as a pole $1/\rho_c$, which is discarded.

to the picture with a complementary motion to that of the particle. This issue will reappear in the holographic analysis in the next chapter. Nonetheless, in both cases this fact should not affect our conclusions: a straightforward analysis demonstrates that the particle-antiparticle pair under concern is causally disentangled, and therefore, the presence of the antiparticle should not affect in any way the local fluctuation properties of the particle and vice versa.

As we discussed in 2.4 the v.e.v. for half-BPS circular Wilson loops can be obtained using localization. From equation 2.61, we derive

$$B_{U(N)} = \frac{\lambda}{16\pi^2 N} \frac{L_{N-1}^2\left(-\frac{\lambda}{4N}\right) + L_{N-2}^2\left(-\frac{\lambda}{4N}\right)}{L_{N-1}^1\left(-\frac{\lambda}{4N}\right)} \tag{3.21}$$

where L_n^α are generalized Laguerre Polynomials. It is worth emphasizing that this formula is valid for any value of λ and N. In various limits, it can be checked using the AdS/CFT correspondence [92, 98] or Bethe ansatz techniques [99, 100, 101, 102]. To obtain the result for the $SU(N)$ theory, we have to subtract the $U(1)$ contribution [12]

$$B_{SU(N)} = B_{U(N)} - \frac{\lambda}{16\pi^2 N^2} \tag{3.22}$$

This results will be revisited and extended to other simple Lie groups and representations in chapter 4.

3.3 Holographic computation of momentum fluctuations

In the large N and large λ limits, the heavy Wilson loop of the previous section can be analysed using semiclassical holography with probe defects. The first step for this study is to identify the worldspaces corresponding to the classical solutions of the holographic objects that are dual to the hyperbolic BPS Wilson loop. Fortunately, for both strings and branes the relevant solution can be obtained from the simpler case of the static particle. In the boundary any hyperbolic Wilson line is related to the straight Wilson line through a special conformal transformation. Its extension to the bulk, given by the 6-dimensional isometry K^μ in the terminology of (2.33), defines an isometry on AdS_5 as well [55], and therefore when applied to the straight line solution produces an object that not only satisfies the boundary conditions set by the quark hyperbolic motion at $z \to 0$, but also stays an extremal area[3], and therefore a solution of the NG or DBI-WZ equations of motion.

The solution obtained in this manner describes a particle following the desired classical path, but this boundary condition alone does not make the problem well-posed in the bulk: other classical supergravitational solutions end up in the same trajectory. The full specification of the solution needs additional boundary conditions for the endpoints of the strings or branes in full AdS, or equivalently boundary conditions in the Poincaré horizon. How do we know that the solution obtained this way is the physical one? The answer in this case is that generic boundary conditions would break supersymmetry, and therefore the half-BPS preserving solution obtained by the special conformal transformation is the appropriate choice.

As an aside, let me point out that there is a specific type of boundary conditions at the Poincaré horizon for which the general solution of the problem is known for an arbitrary timelike Wilson line: purely infalling (or, with a change of sign, purely outgoing) propagation at this horizon. The problem was originally solved for strings [92], and has recently been extended to D3-brane stacks [103]. Nonetheless, as far as I am aware of, it is less clear that these solutions are always the right ones for a probe in thermal equilibrium in the generic case. The half-BPS case is invariant under timer reversal, and both infalling and outgoing boundary conditions lead to the right result. In fact, any mixed combination of infalling and outgoing conditions at the horizon would also reproduce the BPS solution. If the right prescription to take to describe a physical particle in thermal equilibrium was of this hybrid type, i.e., a combination of infalling and outgoing modes, we would not be able to obtain the defect shape for an arbitrary ending path using the known outgoing and infalling solutions: the problem is nonlinear. This objection does not apply to the linearized fluctuations around the half-BPS solution.

The problem ill-posedness is absent in the Euclidean version of this Wilson loops. In this case we obtain a string [58] or brane [38] of finite area solving a minimal area

[3]This argument is not generally true for DBI actions, but it will hold in the cases we are interested in because the nonzero flux for the Drukker-Fiol solution does not contribute to the spacetime dependence of the $g + \mathcal{F}$ determinant.

problem and ending at the prescribed circle at $z \to 0$ (except for the straight Wilson line, which can be thought as the infinite radius limit of this construction). However, the connection between these Euclidean solutions and the Lorentzian solutions that we will address below is established through the direct analytic continuation of the quadric equations of the Wilson line and the string/brane. This type of continuation in generic surfaces involves the complexification of embeddings, which makes it at first sight not suitable to solve the issue.

Another important question has been discussed in the literature about the purely infalling/outgoing constructions, and it concerns the hyperbolic half-BPS solution as a particular case. The issue is the following: for Wilson lines that asymptote a light-speed motion at $t \to \pm\infty$, the only way to avoid a singular behavior (or a "mid-air" ending) for the holographic dual object is to include a fully entangled antiparticle to the one described by the original Wilson line [104, 105, 106]. We have already discussed that this antiparticle is present in the half-BPS construction under study. From the holographic point of view, these two particles are connected through a worldspace whose induced metric reproduces a wormhole. The duality between a fully entangled (Einstein-Podolsky-Rosen) particle-antiparticle pair and a holographic wormhole (Einstein-Rosen bridge) has been thoroughly studied in the literature, under the name ER=EPR [107, 108, 109], after the seminal paper of Susskind and Maldacena [110]. The fact that in the hyperbolic construction under study the EPR pair in question is causally disconnected makes this interesting question irrelevant for the study of fluctuations, and it will not be further examined in this thesis.

Having clarified that the holographically extended special conformal transformation will lead us to the desired solution from the static case, let us proceed by constructing these static and hyperbolic solutions in the probe approximation explicitly.

- String solutions: In the coordinates of the Poincaré patch, whose metric we reproduce here for convenience,

$$ds^2_{AdS_5\times S^5} = R^2\left(\frac{dx^\mu dx_\mu}{z^2} - \frac{dz^2}{z^2} - d\Omega_5\right) \tag{3.23}$$

  the static string solution corresponds to a string extending only in t and z, i.e., given by

$$\text{static string: } \vec{x} = 0 \tag{3.24}$$

  and a fixed static location on the S^5 [70]. The result of the conformal transformation is better expressed in terms of Rindler coordinates, which in terms of the original AdS_5 variables are defined as $t = \rho \sinh\psi$, $x^1 = \rho\cosh\psi$, hence

$$ds^2_{AdS_5\times S^5} = \frac{R^2}{z^2}\left(\rho^2 d\psi^2 - d\rho^2 - d\left(x^2\right)^2 - d\left(x^3\right)^2 - dz^2\right) - R^2 d\Omega_5 \tag{3.25}$$

In the Rindler patch where these coordinates are valid, the hyperbolic solution reads simply

$$\text{hyperbolic string: } \rho = \sqrt{L^2 - z^2};\ x^2 = x^3 = 0 \tag{3.26}$$

The Euclidean version of the hyperbolic string was first considered in [58], see also [92, 111, 112, 113] for the Lorentzian construction. Recall that from these string solutions, it is straightforward to obtain a solution for a D5 brane stack wrapping an S^4 hyperplane in the compact S^5 space [114].

- Static D3 brane stack solution: Although the relevant static D3-brane solution was found in [70, 38] it is better expressed in the coordinates introduced in [115, 73, 116], which are obtained by expressing $\vec{x}$ in terms of a radius $\tilde{r}$ and a solid angle $\{\theta, \phi\}$ and redefining $z = r\text{sech}u$ and $\tilde{r} = r\tanh u$ to decouple the S_2 and AdS_2 manifest structures in the metric that we will use to build a worldspace with this structure:

$$\begin{gathered} ds^2_{AdS_5} = R^2\left(\cosh^2 u\, ds^2_H\,(t,r) - du^2 - \sinh^2 u d\Omega^2_2\,(\theta,\phi) - d\Omega^2_5\right) \\ ds^2_H = \tfrac{dt^2 - dr^2}{r^2};\ d\Omega^2_2 = d\theta^2 + \sin^2\theta d\phi^2 \end{gathered} \tag{3.27}$$

Using a static gauge with t, r, θ and ϕ as the worldspace coordinates, the solution for the $D3$-brane of flux N is given by

$$\text{static brane: } \sinh u = \frac{k\sqrt{\lambda}}{4N} \equiv K;\ F_{tr} = \frac{\sqrt{\lambda}}{2\pi}\frac{\sqrt{1+K^2}}{r^2} \tag{3.28}$$

For the hyperbolic solution in this case it is convenient to rewrite:

$$ds^2_H = \sinh^2\zeta d\psi^2 - d\zeta^2 \tag{3.29}$$

which makes the solution transformed under the extended special conformal symmetry

$$\text{hyperbolic brane: } \sinh u = K;\ F_{\zeta\psi} = \tfrac{\sqrt{\lambda}}{2\pi}\sqrt{1+K^2}\sinh\zeta \tag{3.30}$$

It is possible to analyze the string solutions in the coordinates adapted to the branes, but in what follows the notation will be kept referred to the Poincaré-Rindler sliced coordinates to make the connection to the field theoretical coordinates more explicit for them.

We now turn to the study of fluctuations of the string (NG) and brane (DBI+WZ) actions[4]. In our holographic construction, the S^2 part of the $D3$ worldspace (or equivalently

[4] A crucial identity in the expansion of the NG and DBI terms is

$$\sqrt{|M|} = \sqrt{|M_0|}\left(1 + \frac{\text{tr}\left(M_0^{-1}\delta M\right)}{2} + \frac{\text{tr}^2\left(M_0^{-1}\delta M\right)}{8} - \frac{\text{tr}\left(M_0^{-1}\delta M\right)^2}{4} + \mathcal{O}\left(\delta M\right)^3\right) \tag{3.31}$$

the S^4 part of the D5 worldsheet in the extension à la Hartnoll of the string solution) is independent of the holographic coordinate and does not appear parametrically in the Wilson line path. Consequently, the excitations in this subspace will be interpreted as modes for the point particle. The optimal way to analyze them is through their spherical harmonic expansion.

The quadratic expansion of string and brane actions is in general involved. Around arbitrary string/brane solutions in generic backgrounds, the quadratic expansion of the corresponding action includes tangled fluctuating modes ($\sim \partial\delta x \partial dy$) with massive ($\sim \delta x \delta y$) and drag ($\sim \delta x \partial \delta y$) terms in curved backgrounds. However, for the half-BPS objects presented above and in the adapted coordinate systems in which we wrote them, the the full spectrum of fluctuations for strings [117, 37, 118] and D-branes [119, 116] has been studied in the literature, S^5 and fermionic parts included.

Since we are expanding around solutions of the equations of motion, the linear terms will vanish and therefore the Lagrangian becomes quadratic. When we use the appropriate parametrization, additionally, the quadratic Lagrangian describe pure fluctuations, i.e., $\mathcal{L}_{\text{fluc}} \sim \frac{1}{2}\mathcal{G}^{ij}\partial_i x^a \partial_j x^b$, where a and b denote transverse directions and $\mathcal{G}^{ij}$ is a symmetric tensor that plays the role of the inverse of an "effective worldspace metric" and should not be confused with the induced metric. We obtain from the expansion of the Lagrangian:

$$\mathcal{L}_{\text{fluc}}^{\text{stat str}} = \frac{1}{2\pi\alpha'}\frac{R^2}{z^2}\left(\frac{(\partial_t \vec{x})^2 - (\partial_z \vec{x})^2}{2}\right) \tag{3.32}$$

$$\mathcal{L}_{\text{fluc}}^{\text{hyp str}} = \frac{1}{2\pi\alpha'}\frac{R^2}{z^2}\sum_{i=2,3}\left(\frac{1}{2}\frac{L}{L^2-z^2}\left(\partial_\psi \tilde{x}^i\right)^2 - \frac{1}{2L}\left(L^2-z^2\right)\left(\partial_z \tilde{x}^i\right)^2\right) \tag{3.33}$$

$$\mathcal{L}_{\text{fluc}}^{\text{branes}} = T_{D3}R^2\frac{\sqrt{|\mathscr{G}|}}{2}\left(\mathscr{G}^{ab}\partial_a u \partial_b u\right);\ \mathscr{G} = R^2 K\sqrt{1+K^2}\left(ds_H^2 - d\Omega_2^2\right) \tag{3.34}$$

where $\tilde{x}^{2,3} \equiv x^{2,3}$, but we have defined $\tilde{x}^1 \equiv \sqrt{1^2 - z^2L^{-2}}$ to avoid drag terms for the hyperbolic string fluctuations in the "longitudinal" direction[5]. Observe that the fluctuating Lagrangian for the branes is optimally written in terms of an auxiliary metric $\mathscr{G}$, which is neither the fluctuating metric $\mathcal{G}$ for the modes we want to study, nor the induced metric.

Let us check that the fluctuating modes of the strings match the $\Delta = 2$ displacement operators we introduced in the previous section: for the static string, the equations of

which follows from arithmetic manipulations of $[\exp \ln] \det(\mathbb{I} + A)$ in the basis where the nonsingular matrix $\mathbb{I} + A$ is diagonal.

[5]The term "longitudinal" is here potentially confusing. The coordinates $x^{1,2,3}$ or $\tilde{x}^{1,2,3}$are equally valid to describe the three types of physical deformations of the loop path, which are all "transverse". However, the motion in the 1 direction makes the x^1 and $\tilde{x}^1$ coordinates not directly normal to the worldline, that's why we call them "longitudinal" and need a specific treatment to hide the drag terms that appear for x^1.

Static:	Hyperbolic (define $\hat{z} \equiv \frac{z}{L}$)
$\frac{2}{z}\partial_z \vec{x} - \partial_z^2 \vec{x} = \omega^2 \vec{x}$	$\frac{2}{\hat{z}}\left(1-\hat{z}^2\right)\partial_{\hat{z}} x^{2,3} - \left(1-\hat{z}^2\right)^2 \partial_{\hat{z}}^2 x^{2,3} = \omega^2 x^{2,3}$
$\vec{x} = \vec{x}_0\left(\omega\right) e^{i\omega(z-t)}\left(1 - i\omega z\right)$	$\vec{x} = \vec{x}_0\left(\omega\right) e^{i\omega\left(\text{arctanh}\hat{z} - t\right)}\left(1 - i\omega \hat{z}\right)$
$G^R\left(\omega\right) = -\frac{R^2}{2\pi\alpha'}\left(\cancel{\frac{\omega^2}{z}} + i\omega^3\right)$ $G\left(\omega\right) = -\text{sign}\omega G^R\left(\omega\right) = \frac{\sqrt{\lambda}}{2\pi}\left\|\omega\right\|^3$ $G\left(t\right) = \frac{3\sqrt{\lambda}}{\pi^2 t^4} \to \gamma = \frac{3\sqrt{\lambda}}{\pi^2}$	$G^R = -\frac{R^2}{2\pi\alpha'}\left(\cancel{\frac{\omega^2}{z}} + i\omega\left(\frac{1}{L^2} + \omega^2\right)\right)$ $\kappa = \lim_{\omega\to 0}\frac{2T}{\omega}\text{Im}G^R = 4\pi\sqrt{\lambda}T^3$

Table 3.1: Summary of results for the physics of the quadratic fluctuations of a string dual to a particle static (left) and hyperbolic (right) motion. In both cases the first line displays the equations of motion for the explicitly transverse modes, the second line presents its solution with outgoing boundary conditions and the last line contains the retarded propagator of the displacement operators of the dual particle, extracted from the near-boundary behaviour of the outgoing solution. From this propagator some linear response coefficients are inferred.

motion read

$$\partial_t^2 \vec{x} - z^2 \partial_z \frac{\partial_z \vec{x}}{z^2} = 0 \Rightarrow \partial_t^2 \vec{\phi} - \partial_z^2 \vec{\phi} + \frac{2}{z^2}\vec{\phi} = 0;\ \vec{\phi} \equiv \frac{R}{z}\vec{x} \tag{3.35}$$

and thus we learn that these three transverse modes are massive $m^2 = 2L^{-2}$ scalars in the worldsheet. The usual relation ___, which can harmlessly be extended to operators on a defect (it is usually stated that "holography acts twice", see [120]), determines that, as we expected, these fluctuating modes correspond to $\Delta = 2$ CFT operators. In the $z \ll L$ the equations of motion for the hyperbolic motion become identical to (3.35), and therefore, the identification with the displacement operators works identically. For the D3 branes, we identify the corresponding $\Delta = 2$ modes in the $\ell = 1$ triplet in the spherical harmonic expansion.

We are now in position to solve for the modes describing the transverse fluctuations of the string or the brane. Following [37, 87] we solve the corresponding equations of motion using Fourier transformation methods in the temporal coordinate and keep the outgoing solution at the Poincaré horizon. From this solution we can obtain the displacement propagator [87] for the fluctuating variable $\tilde{x}$ through

$$\text{Im}G^R_{\vec{x}} = -\lim_{z\to 0}\mathcal{G}^{z\beta}\partial_\beta \ln\tilde{x}\ , \tag{3.36}$$

which follows from the GKPW prescription. Table 3.1 summarizes the results for the static and hyperbolic strings, first derived in [87] and [111], respectively.

It is not necessary to reproduce the full computation for the D3 brane analogs: it suffices to observe that the fluctuation Lagrangian for the kinetic triplet of interest is $k\sqrt{1+K^2}$ times the one computed with the corresponding string. Therefore, we obtain, for k-symmetric representations,

$$\begin{aligned} \tilde{\gamma}_{\mathcal{S}_k} &= \tfrac{3\sqrt{\lambda}}{\pi^2} k \sqrt{1+\tfrac{k^2\lambda}{16N^2}} \\ \kappa_{\mathcal{S}_k} &= 4\pi\sqrt{\lambda} k \sqrt{1+\tfrac{k^2\lambda}{16N^2}} T^3 \end{aligned} \tag{3.37}$$

These results for the string and brane match the holographic predictions obtained for the other coefficients. For completeness, let me briefly present these results and comment on how to obtain them:

- The half-BPS Wilson loop v.e.v. (or, more precisely, its logarithm), can be obtained in the supergravity and probe approximation limits simply from the evaluation of the defect action in the classical solution. In other words, we can make use of the Eulerian solutions of the minimal area problems for the string and the brane [58, 38] and compute

$$\begin{aligned} \text{string: } \ln\langle W\rangle &= \sqrt{\lambda} \\ \text{brane: } \ln\langle W\rangle &= \sqrt{\lambda}\sqrt{1+\tfrac{k^2\lambda}{16N^2}}\tfrac{k}{2} + 2N \text{arcsinh} \tfrac{k\sqrt{\lambda}}{4N} \end{aligned} \tag{3.38}$$

- The h_{W} and f_{W} coefficients of $\mathcal{N}=4$ SYM can be computed holographically in more than one way: One option is to take an Euclidean circular half-BPS Wilson loops and compute its correlator with the conformal primary $\mathcal{O}_2$, which is in the supermultiplet of both $T_{\mu\nu}$ and $\mathcal{L}$. This can be done in the probe semiclassical approximation by evaluating the propagation of supergravity states between the worldspace of the holographic representative of the loop and the $\mathcal{O}_2$ operator. In the limit where the distance between the loop and the primary is large in comparison to the circle of the loop one can extract from this quantity the second OPE coefficient, which is directly related to h_{W} and f_{W}. Alternatively, one can study the specific $\mathcal{O}_2$ channel in the correlation of two circular Wilson loops separated a large distance. This computation was done for strings in [58], and later extended to $D3$-branes in [88]. The direct computation of the Lorentzian f_W coefficient can be done instead studying the near-boundary decay of the dilaton sourced by the holographic representative of the loop. It was performed originally in [37] for strings and extended for $D3$-branes in [90].

- The Bremsstrahlung function can be evaluated in terms of the worldspace energy flowing through the causal horizon that the string or brane worldspace develops when for an accelerated Lorentzian Wilson line. This idea was used to compute the B function for strings in [92], and then extended to branes in [98].

There is another result matching that deserves a mention: Although at $k = 1$ the defect action of DBI develops large $\mathcal{F}$ gradients that should break down its validity, we still recover the string result at $k \to 1$, $N \gg \sqrt{\lambda}$ in the previous expressions [38, 98, 90]. Furthermore, at finite $\sqrt{\lambda}/4N$ the extrapolation of the brane results to $k = 1$ happen to reproduce the correct $N \gg 1$ limit of the localization results for these observables. Conjecturally, the high amount of symmetry in this situation protects the probe solution from corrections beyond the regime of applicability of supergravity. To my knowledge, there is still not a clear understanding of this effect.

3.4 Lessons for thermal plasmas?

In section 3.2 we have used the results of supersymmetric localization for the Bremsstrahlung function to obtain exact predictions for the momentum fluctuations of accelerated heavy quarks at zero temperature. As expected, the two-point function of displacement operators presents thermal behavior due to the Unruh effect. A question arises: Can we use our results to learn something about momentum fluctuations of heavy probes immersed in a thermal bath (at a finite temperature) of the same conformal field theory?

Besides its intrinsic interest, this question has broader relevance since it is expected that at finite temperature, conformal theories (even superconformal ones) share some properties with the high-temperature deconfined phase of confining gauge theories. More specifically, a particular CFT, $\mathcal{N}=4$ SYM at $T\neq 0$, has been used by means of the AdS/CFT correspondence to model the quark-gluon plasma experimentally observed at RHIC and at the LHC (see [121, 122, 123] for reviews). In particular, the momentum fluctuations of a heavy quark (either static or moving at constant velocity) in the quark-gluon plasma have been estimated by considering a dual trailing string in the background of a black Schwarzschild brane in an asymptotically AdS_5 background [124, 10, 125, 126]. The applicability of the holographic correspondence for the study of heavy quarks in strongly coupled plasmas is nonetheless currently limited to the large λ and large N regime of supergravity, and although some subleading corrections can be computed (see [127, 128] for a computation of the $\lambda^{-1/2}$ and $\lambda^{-3/2}$ corrections to the jet quenching parameter in $\mathcal{N}=4$ SYM, for example) it seems extremely hard to reach finite N and λ in such computations. For this reason, it would be very interesting if the study of an accelerated quarks in the vacuum of a conformal field theory, which as we have seen can be tackled at finite λ and N, can become an indirect route to the study of conformal $T\neq 0$ plasma.

However, as we anticipated in 3.1, while a probe accelerated in vacuum and a static probe in a thermal bath experience a non-zero temperature, the details of their response are not identical quantitatively, and under some circumstances even qualitatively [96, 97]. We can see this explicitly for the N = 4 SYM plasma, by comparing known expressions of the momentum diffusion coecients in various regimes:

- Weakly coupled field theory: the momentum diffusion coefficient of a heavy quark in a weakly coupled N = 4 SU(N) SYM plasma has been computed at leading and next-to-leading orders [129, 130]

$$\kappa_{\text{Thermal}} = \lambda T^3 \frac{N^2-1}{N^2}\frac{\lambda}{6\pi}\left(\log\frac{1}{\sqrt{\lambda}} + c_1 + c_2\sqrt{\lambda} + \mathcal{O}(\lambda)\right) \tag{3.39}$$

  where c_1 and c_2 are known coefficients [130]. Observe that the thermal series starts with an $\mathcal{O}(\lambda^2\log\lambda)$ term. This feature comes from the non-trivial coupling dependence of the Debye mass in the thermal bath [131]. Conversely, the weak coupling expansion of the result (3.16) with the Bremsstrahlung function presented in 3.2 for $SU(N)$ reads

$$\kappa_{\text{Unruh}} = \lambda T^3 \frac{N^2-1}{N^2} \pi \left(1 - \frac{\lambda}{24} + \mathcal{O}\left(\lambda^2\right)\right) \tag{3.40}$$

and presents a qualitatively different behavior, with a leading term of $\mathcal{O}(\lambda)$.

- Strongly coupled field theory: The holographic picture describing an accelerated probe in a thermal bath is rather different from the strings reaching the boundary at a hyperbola discussed in 3.3. The former consists of a string in the Schwarzschild-AdS background, and is characterized by qualitatively different retarded Green functions (see [132] for a discussion on this point. The result obtained for these correlators in [124, 10] is

$$\kappa^{SUGRA}_{\text{Thermal}} = \pi\sqrt{\lambda}T^3 \tag{3.41}$$

which is four times smaller than the supergravity result (or the localization result at $\lambda \to \infty$ with $N \sim \lambda$) for the similar transport coefficient for the Unruh background, which we reproduce here for convenience:

$$\kappa^{SUGRA}_{\text{Unruh}} = 4\pi\sqrt{\lambda}T^3 \tag{3.42}$$

This difference might be surprising at first, since it can be argued that transport coefficients can be read from the world-sheet horizon [133], and the two classical world-sheet metrics (i.e. accelerated string in AdS_5 versus hanging/trailing string in Schwarzschild-AdS_5) while clearly different, have the same near-horizon metric, 1+1 Rindler space. However, the different change of variables used to write these near-horizon metrics imply different normalizations of the corresponding wavefunctions, giving rise to this factor of four discrepancy between the respective transport coefficients.

Keeping these differences in mind, we nevertheless propose to use our exact results to make an educated guess of the impact of using SUGRA instead of the exact results for computing the momentum diffusion coefficient of a static heavy quark, κ_{Thermal}, in $\mathcal{N} = 4$ SYM at finite temperature. To that end, we start by evaluating the difference between the SUGRA (large λ, large N) and the exact (finite λ, $N = 3$) computations of the coefficient for the accelerated probe in vacuum. For the latter, we obtain simply by combining equations (3.21), (3.22) and (3.16),

$$\kappa_{SU(3)} = 4\pi \frac{\lambda}{18} \frac{\lambda^2 + 144\lambda + 3456}{\lambda^2 + 72\lambda + 864} T^3 \tag{3.43}$$

Observe that the linear growth in λ of the Bremsstrahlung function at finite N and large λ, is very different from the $\sim \sqrt{\lambda}$ dependence observed in (3.42), which describes the λ limit of the same function but when N scales with $N \sim \lambda$. In fact, for generic N we

"Obvious" scheme	$T_{\mathcal{N}=4} = T_{QCD}$	$\lambda = g^2_{\mathcal{N}=4} N = 12\pi\alpha_s \sim 6\pi$
"Alternative" scheme	$T_{\mathcal{N}=4} = 3^{-1/4} T_{QCD}$	$\lambda = g^2_{\mathcal{N}=4} N = 5.5$

Table 3.2: The different underlying physics of $\mathcal{N} = 4$ SYM and QCD makes the numerical comparison of their phenomena ambiguous. The two prescriptions listed here correspond to a direct matching of parameters ("obvious" scheme) and a correction proposed to account for the difference in the number of degrees of freedom ("alternative" scheme).

obtain a similar asymptotic λ linear growth,

$$\kappa^{\lambda \gg 1}_{SU(N)} = \frac{N-1}{N^2} \pi \lambda T^3 \tag{3.44}$$

which does not match at $N \sim \lambda$ the result (3.42). The limits of large N and large λ do not commute, and therefore, we should a priori mistrust the extrapolations to finite N of the $\lambda \to \infty$ holographic prediction.

That being said, observe that the quotient between (3.42) and (3.43),

$$\frac{\kappa^{EXACT}_{\text{Unruh}}}{\kappa^{SUGRA}_{\text{Unruh}}} = \frac{\sqrt{\lambda}}{18} \frac{\lambda^2 + 144\lambda + 3456}{\lambda^2 + 72\lambda + 864} \tag{3.45}$$

which reflects the $\lambda \to \infty$ discrepancy clearly, becomes finite and of $\mathcal{O}(1)$ for a large parametric range around $\lambda \sim 182.45$, i.e., the point where this monotonic function becomes exactly one. The parametric regime that has been considered when modeling the QCD quark-gluon plasma by $\mathcal{N} = 4$ SYM lies precisely in the regime where the holographic and exact predictions are of the same order.

Given the differences between $\mathcal{N} = 4$ SYM and QCD there are inherent ambiguities in choosing the parameters of the former that might best model the real world QCD plasma. A first choice [134] is to equate the temperature and coupling constant of both theories, but other prescriptions have been proposed in the literature. In particular, in order to ameliorate the impact of the obvious difference in the number of degrees of freedom, Gubser proposed to take an alternative scheme [135, 87], rescaling both the temperature and the coupling constant as indicated in the table 3.2, although this choice comes with its own limitations. Without entering the discussion about the adequacy of each scheme, let us take their two parametric setups as indicators of the values between which the best fit of $\mathcal{N} = 4$ to QCD should reside.

Within the interval $\lambda \in [5.5, 6\pi]$ the ratio (3.45) increases from 0.43 to 0.61, see figure 3.2. Roughly speaking, in this range of values for the 't Hooft coupling supergravity overestimates κ by a factor of about two.

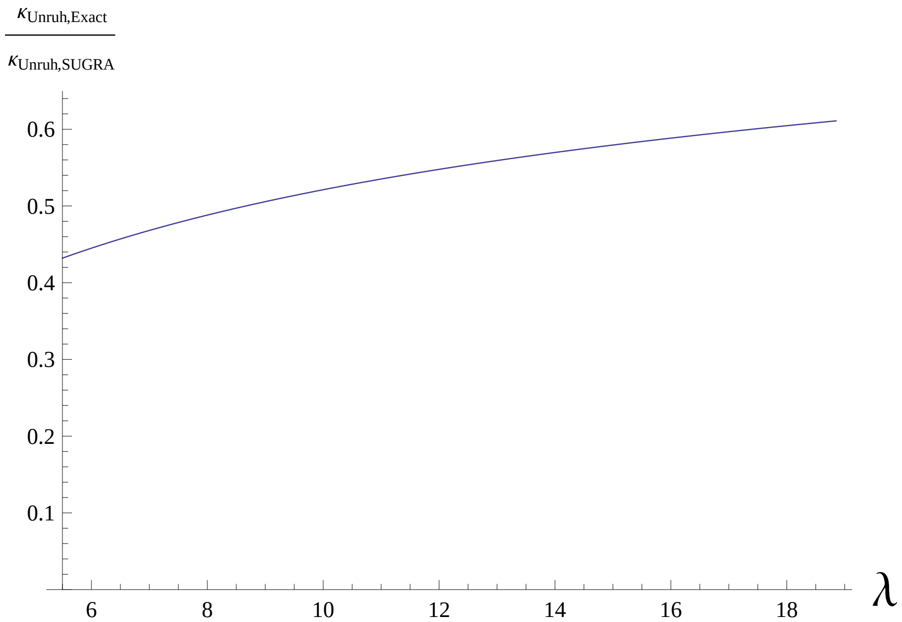

Figure 3.2: The relation between the exact momentum diffusion coefficient and the supergravity approximation for an accelerated quark in vacuum. The range of λ displayed corresponds to the one considered when modelling the quark-gluon plasma.

Clearly, we are not in a position to estimate κ_{Thermal} for $\mathcal{N} = 4$ SYM at arbitrary λ and N: the qualitative differences observed in the perturbative regime and the quantitative overall discrepancy in the supergravity regime are convincingly dissuasive in that regard. A more modest goal is to estimate it in the range of values singled out above, that appear when modeling the QCD quark-gluon plasma. Moving across the (λ, N) plane towards the region of validity of supergravity, the ratio (3.45) will uneventfully evolve from the values of ~ 0.5 found above to 1. In order to proceed, we are going to assume that roughly the same is true for κ_{Thermal}, so, along that path,

$$\frac{\kappa_{\text{Thermal}}^{EXACT}}{\kappa_{\text{Thermal}}^{SUGRA}} \approx \frac{\kappa_{\text{Unruh}}^{EXACT}}{\kappa_{\text{Unruh}}^{SUGRA}} \tag{3.46}$$

If this assumption is true it means that the supergravity computations [10, 124, 125] for κ_{Thermal} give an answer that is about twice the exact one. While we currently lack the solid arguments to substantiate this speculation, let's end by noting that if true, it would in turn imply that the diffusion constant $D = 2T^2/\kappa$ for the $\mathcal{N} = 4$ SYM plasma would be about twice the one obtained in supergravity, pushing it in the right direction to match the range of values suggested by RHIC [136].

Chapter 4

Generic line probes and orientifolded geometries

Predictions of localization for finite N and λ $\mathcal{N} = 4$ SYM

The previous chapter illustrates the relations between the v.e.v. of ciruclar half-BPS Wilson loops and the radiative and thermal properties of a particle in 4D $\mathcal{N} = 4$ SYM in semiclassical holography. The picture presented there is valid (up to small modifications for $SO(N)$ and $SP(N)$ groups) in the parametric configurations with a simple Lie gauge group of large rank and when the coupling is strong. Perturbative corrections in N^{-1} and $\lambda^{-1/2}$ are sometimes sufficient to extend the holographical description to finite N and λ, but, as perturbative series, their accuracy is limited, and they eventually break down.

In contrast, the results of supersymmetric localization keep their validity in the whole parametrical range of λ and N. In this chapter we will discuss the lessons we can extract from them about the holographic regimes that lie beyond supergravity, both perturbatively and nonperturbatively, following closely the discussions of [2] and [3].

4.1 Circular Wilson loops of arbitrary R and G

Let us in first place derive the exact results that will serve us as guiding principles in the rest of the chapter. In what follows, we systematically study the v.e.v.s of half-BPS circular Wilson loops for particles of generic representation $\mathcal{R}$ in $\mathcal{N}=4$ SYM theories of simple nonexceptional compact gauge algebra G. All the results presented below are exact at any N and λ. Before proceeding, I should display here a warning message: this section is rather technically involved and exempt of physical discussions. The reader might prefer to skip it and proceed to the other section of this chapter, which elaborates on the implications of the main results found here (equations (4.17), (4.29), (4.39) and table 4.2), for field theory and holography.

The fact that we can analyse the aforementioned family of Wilson loop observables in full generality is a direct consequence of the fact that the obtention of (2.65) in [76] is independent of G and $\mathcal{R}$. Additionally, this result becomes specially simple when the theory under consideration is $\mathcal{N}=4$ SYM: the instantonic contribution identically vanishes ($\mathcal{Z}_{\text{inst}}=1$), and the 1-loop determinant includes the very same factors in the numerator and denominator (and therefore $\mathcal{Z}_{\text{1-loop}}=1$). As a result, in this case, equivariant observables are computed in the matrix model using

$$\langle f(x)\rangle = \frac{1}{\mathcal{Z}}\int d^N x \Delta_{FP}(x)\, e^{-\frac{2}{g^2}(x,x)} f(x) \ . \tag{4.1}$$

Then, the v.e.v. of a generic half-BPS circular Wilson loop reads simply

$$\langle \mathsf{W}_{\mathcal{R}}\rangle = \frac{1}{\dim \mathcal{R}}\left\langle \sum_{\omega\in\Lambda_\omega(G)} n_{\mathcal{R}}(\omega)\, e^{\omega(x)}\right\rangle \ . \tag{4.2}$$

Notice that for convenience we have redefined in (4.1) and (4.2) the integration variable $x=2\pi r\Phi_0$ with respect to (2.65) and (2.67).

The trace over the representation $\mathcal{R}$ of the Wilson loop is implemented in (4.2) as a sum over the weights ω of the weight lattice $\Lambda_\omega(G)$ with their corresponding multiplicity in $\mathcal{R}$, $n_{\mathcal{R}}(\omega)$.

4.1.1 Perturbative expansion for arbitrary representations of $U(N)$

Let us first turn our attention to the case where the gauge group is $U(N)$. Recall that, for this case, $(x,x)=\sum_n x_n^2$, the Weyl group of the Cartan subalgebra is the permutation group S_N, and the weight lattice is $\mathbb{R}^N$ in the orthonormalized coordinates of the Cartan subalgebra. The traces over generic representations can be implemented with a useful combinatorial tool: Young tableaux.

Let me briefly elaborate on the connexion between representations of $U(N)$ and Young tableaux: Consider first the Young diagram Y_μ, where μ denotes a sequence

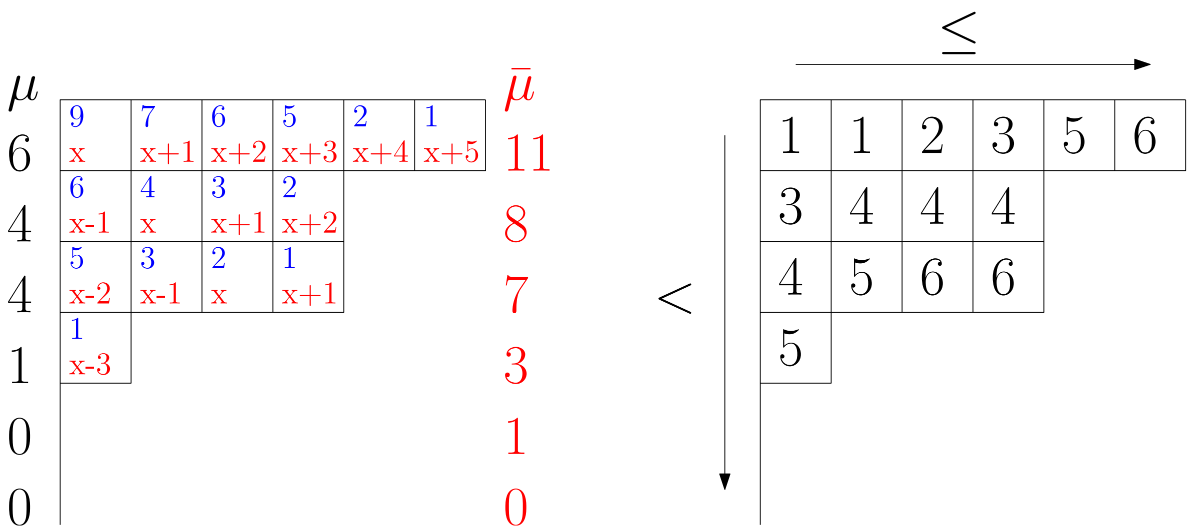

Figure 4.1: Left: Young diagrams (black) of $U(N)$ are labeled by a sequence of N weakly decreasing numbers μ that indicate the number of squares they contain in every row. Observe here the $(6,4,4,1,0,0)$ diagram of $U(6)$. For each square in the diagram it is useful to define the x-content (red) and the hook length (blue). The maximum N-content of each row defines the shifted sequence $\bar{\mu}$. The hook length is obtained by adding 1 to the sum of the squares either purely at the right of purely below of the considered square.
Right: A semistandard Young tableau is obtained by filling the squares of a Young diagram with integers in $[1, N]$ that are weakly and explicitly increasing along rows and columns, respectively. The tableau represented here represents the $(2,1,2,4,3,3)$ weight, or a monomial $x_1^2 x_2 x_3^2 x_4^4 x_5^3 x_6^3$ in a Schur polynomial.

of weakly decreasing row lengths $\mu = (\mu_1, \ldots, \mu_N)$. We will denote by $|\mu|$ the total number of squares in the diagram, or, equivalently, if this number of squares is k, we will write $\mu \vdash k$. Any tableau in this diagram $Y \hookrightarrow Y_\mu$ with entries in $\{1, \ldots, N\}$ defines a weight: in the orthonormal basis of the Cartan subalgebra, the k-th component of the 1-form ω is taken as the number of instances of k in the tableau, $C_k[Y]$ (see figure 4.1. Each representation of $U(N)$ can be univoquely implemented with the correct weight multiplicity by a diagram Y_μ, provided we consider all semistandard Young tableaux (i.e., weakly increasing along rows and strictly increasing down each column) we can write on Y_μ. In other words, the function implementing the trace over $\mathcal{R}_\mu$, is given by

$$\mathrm{tr}_{\mathcal{R}_\mu}(f(x)) = \sum_\omega n_{\mathcal{R}_\mu}(\omega) f\left(\omega^k x_k\right) = \sum_{Y \hookrightarrow Y_\mu} f\left(\sum_n C_n[Y] x_n\right) . \tag{4.3}$$

For our specific interests, where $f(x) = e^x$, the function implementing the trace becomes a Schur polynomial:

$$\mathcal{S}_\mu(e^{x_n}) \equiv \sum_{Y \hookrightarrow Y_\mu} \prod_{n=1}^{N} e^{C_n[Y] x_n} \tag{4.4}$$

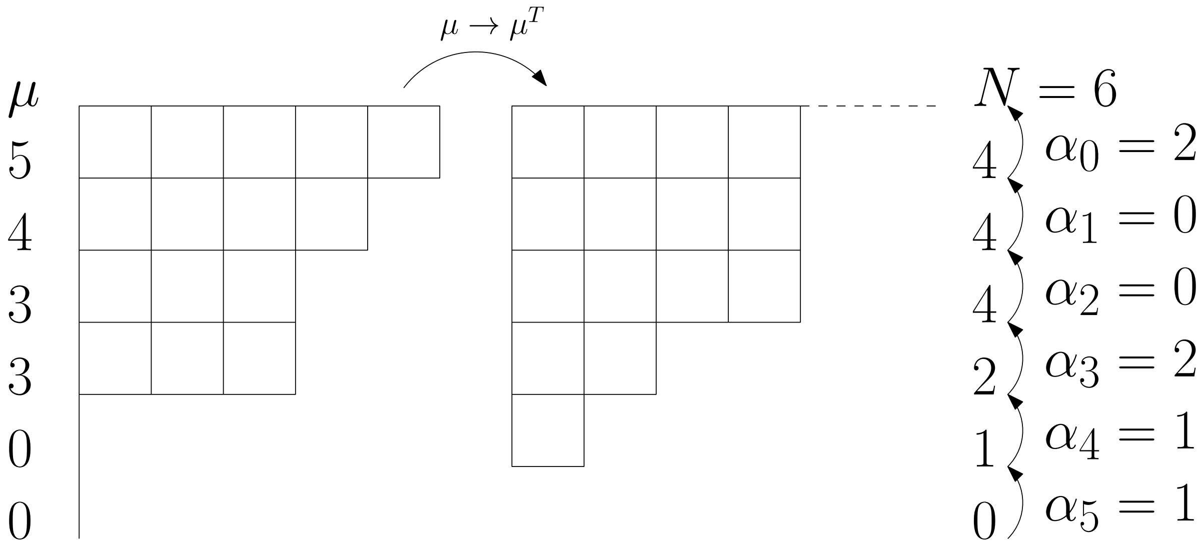

Figure 4.2: Observe how the transposition of a Young diagram encodes in its sequence the multiplicity of the entries in the original sequence of row lengths.

These polynomials are the characters of polynomial irreducible representations of $GL(N)$ groups. They form a basis of the space of symmetric functions of N variables.

Before addressing the explicit computation at hand, let us define two operations on sequences that will significantly simplify the notation on what follows:

- The shifting operation $\mu \to \bar{\mu}$ acts on the sequence μ of N elements as $\bar{\mu}_n \equiv \mu_n + N - n$. The resulting sequence $\bar{\mu}$ is strictly decreasing and it coincides with the maximal N-content[1] in every row of its diagram.

- The diagramatic transposition $\mu \to \mu^T$, which converts the sequence μ into the sequence describing the Young diagram $(Y_\mu)^T$, i.e., the Young diagram where the rows and columns have been interchanged. Notice that $\alpha_n(\mu) \equiv \mu_n^T - \mu_{n+1}^T$ (with $\mu_{N+1}^T = 0$) counts the number of instances of number n in the μ sequence. Additionally, we define $\alpha_0(\mu) \equiv N - \sum_{n=1}^{N} \alpha_n = N - \mu_1^T$, which counts the number of zeroes at the end of the partition. This operation is represented in figure 4.2.

Observe that one has to be careful with the length N of any sequence μ: the corresponding Schur polynomial and the sequence $\bar{\mu}$ will depend on this quantity, and we will seldom make it explicit.

Our first approach to the v.e.v. of $U(N)$ loops in the matrix model will be the scrutiny of its perturbative series in g. A helpful property for this purpose is the translation

[1] The x-shifted content, or x-content for short, of the n-th square in the m-th row of a Young diagram is given by $x + n - m$. This quantity defined for the squares of a diagram, which should not be confused with the content $C_n[Y]$ of a Young tableaux, is a very useful quantity in the study of combinatorics of Young diagrams and tableaux.

identity for Schur polynomials:

$$\left\{ \begin{array}{c} \mathcal{S}_\mu\left(1+y_1,\ldots,1+y_N\right) = \sum_\nu b_{\mu\nu}\mathcal{S}_\nu\left(y_1,\ldots,y_N\right) \\ b_{\mu\nu} \equiv \det \begin{pmatrix} \bar{\mu}_n \\ \bar{\nu}_m \end{pmatrix} \end{array} \right\} . \tag{4.5}$$

Here, and in the rest of this chapter, we will work with the understanding that $\frac{1}{a!} = 0$ for $-a \in \mathbb{N}$ to simplify the expressions. For this specific study it is also useful to rescale the integrating variables (the eigenvalues) of the matrix model as $x_i \to g\hat{x}_i/2$ in order to make the power series in g more transparent in the expansion and make the Gaussian exponent canonical. The translation identity becomes

$$\begin{aligned} \left\langle \mathcal{S}_\mu\left(e^{\hat{x}_i}\right)\right\rangle = b_{\mu\emptyset} + \tfrac{g}{2}\, b_{\mu\square} \begin{pmatrix} N \\ 1 \end{pmatrix} \langle \hat{x}_n \rangle + \\ + \left(\tfrac{g}{2}\right)^2 \left(\tfrac{b_{\mu\square}}{2} + b_{\mu\square\square}\right) \begin{pmatrix} N \\ 1 \end{pmatrix} \langle \hat{x}_n^2 \rangle + \left(\tfrac{g}{2}\right)^2 \left(b_{\mu\square\square} + b_{\mu\substack{\square\\\square}}\right) \begin{pmatrix} N \\ 2 \end{pmatrix} \langle \hat{x}_n \hat{x}_{m\neq n} \rangle + \ldots . \end{aligned} \tag{4.6}$$

The v.e.v.s in this series can be evaluated easily, using for instance orthogonal polynomials and their recurrence properties, as we discussed in 2.4.3. The terms of odd order in $\hat{x}_n$ will all vanish from the expansion because the simultaneous change of sign of all the $\hat{x}_n$ is a symmetry of the integral. Consequently, the expansion (4.6) is a perturbative series in g^2, not in g.

In order to express the power series in a representation-independent way, it is convenient to evaluate the $b_{\mu\nu}$ coefficients in terms of the Casimirs of the representation $\mathcal{R}_\mu$, $C\left(\mu\right)$, which for $U\left(N\right)$ can be read directly from their generating functions

$$G_\mu\left(t\right) = \sum_k C_k\left(\mu\right) t^k = \frac{1}{t}\left[1 - \prod_{n=1}^{N}\left(1 - \frac{t}{1-\mu_n t}\right)\right] . \tag{4.7}$$

At the linear order in g^2, the coefficients are easily evaluated (see table 4.1),

$$\left\langle \mathsf{W}_{\mathcal{R}_\mu}\right\rangle = \frac{\langle \mathcal{S}_\mu \rangle}{\dim \mathcal{R}_\mu} = 1 + \frac{C_2\left(\mu\right)}{8} g^2 + \ldots . \tag{4.8}$$

As one increases the computed order, however, these equations become more involved (and far from illuminating). Although there is no obvious way to rewrite generally the coefficients $b_{\mu\nu}$ in terms of Casimir invariants, we will now observe that the highest power of μ in every coefficient is a direct power of the second Casimir. Let us first notice that the Casimir C_p has degree p in μ and that the term g^{2k} contains at most terms of total degree $2k$. Furthermore, only those terms where ν is a partition of $2k$ can generate the

$b_{\mu\emptyset} = \dim \mathcal{R}_\mu$	
$b_{\mu\square} = \dim \mathcal{R}_\mu \frac{C_1(\mu)}{N}$	$\langle \hat{x}_i \rangle = 0$
$b_{\mu\square\square} = \frac{\dim \mathcal{R}_\mu}{2N(N+1)} \left(C_2(\mu) + C_1(\mu)^2 - (N+1) C_1(\mu) \right)$	$\langle \hat{x}_i^2 \rangle = N$
$b_{\mu\begin{smallmatrix}\square\\\square\end{smallmatrix}} = \frac{\dim \mathcal{R}_\mu}{2N(N-1)} \left(-C_2(\mu) + C_1(\mu)^2 + (N-1) C_1(\mu) \right)$	$\langle \hat{x}_i \hat{x}_{j\neq i} \rangle = -1$

Table 4.1: Binomial determinants and monomial correlators involved in the evaluation of the linear term in the Taylor series for the Wilson loop in representation $\mathcal{R}$.

maximal degree in μ at order g^{2k}. With this fact in mind we now make use of the relation between the $b_{\mu\nu}$ coefficients and shifted Schur polynomials [137] $\mathcal{S}^\star_\mu$,

$$\frac{b_{\mu\nu}}{b_{\mu\emptyset}} = \prod_{n=1}^{N} \frac{\bar{\emptyset}_n!}{\bar{\nu}_n!} \mathcal{S}^\star_\nu(\mu) = \prod_{n=1}^{N} \frac{(N-n)!}{\bar{\nu}_n!} \left(\mathcal{S}_\nu(\mu) + \ldots \right) . \tag{4.9}$$

In the last identity, we used the expansion of shifted Schur polynomials as ordinary Schur functions $\mathcal{S}_\mu$ plus lower degree polynomials [2] that don't contribute to the term we are considering [137]. To recapitulate, we have argued that the term with degree $2k$ in μ in the expansion of $\mathsf{W}_{\mathcal{R}_\mu}$ is, at order $(g/2)^{2k}$, given by

$$\sum_{\nu \vdash 2k} \prod_{n=1}^{N} \frac{(N-n)!}{\bar{\nu}_n!} \mathcal{S}_\mu(\nu) \langle \mathcal{S}_\nu(\hat{x}) \rangle . \tag{4.11}$$

We will now make use of the fact that Schur polynomials can be rephrased in terms of power sum polynomials, which makes their connection with permutation groups explicit. In short, power sum polynomials are an alternative to Schur polynomials as a basis of symmetric polynomials of N variables, defined on a partition λ by

$$p_\rho(x) = \prod_{i=1}^{N} \left(\sum_{j=1}^{N} x_j^{\rho_i} \right) . \tag{4.12}$$

The linear change of basis between Schur and power sum polynomials is given by

$$\mathcal{S}_\mu(\nu) = \frac{1}{(2k)!} \sum_{\rho \vdash 2k} \#[\rho] \, \chi_\mu[\rho] \, p_\rho(\nu) , \tag{4.13}$$

[2] This property is easier to see comparing (4.9) to the textbook definition of Schur polynomials:

$$\begin{gathered} S^\star_\nu \equiv \det\left[\bar{\mu}_n!/(\bar{\mu}_n - \bar{\nu}_m)!\right] / \det\left[\bar{\mu}_n!/\left(\bar{\mu}_n - \bar{\emptyset}_m\right)!\right] \\ S_\nu \equiv \det \mu_n^{\bar{\nu}_m} / \det \bar{\mu}_n^{\bar{\emptyset}_m} \end{gathered} \tag{4.10}$$

where $\#[\rho]$ denotes the number of elements in the conjugacy class of ρ in S_{2k}, which can be counted as

$$\#(\rho) = \frac{(2k)!}{\prod_n \rho_n! n^{\rho_n}} , \tag{4.14}$$

and χ_μ denotes the character. In this form, $\langle \mathcal{S}_\nu(\hat{x}) \rangle$ can be exactly evaluated in the Gaussian matrix model, where only the conjugacy class 2^k, with k disjoint 2-cycles, contributes to the expected value. One thus obtains [138]

$$\langle \mathcal{S}_\nu(\hat{x}) \rangle = \frac{1}{(2k)!} \# \left[2^k\right] \chi_\mu \left[2^k\right] \prod_{n=1}^{N} \frac{\bar{\nu}_n!}{(N-n)!} \tag{4.15}$$

The substitution of (4.13) and (4.15) into (4.11), and subsequent application of the orthogonality of characters, transforms it into

$$\frac{\dim \mathcal{R}_\mu}{(2k)!} \# \left[2^k\right] p_{\left[2^k\right]}(\mu) \tag{4.16}$$

Now, $p_{\left[2^k\right]}$ differs from C_2 only in lower degree terms, so for the purpose of computing the highest degree term, we can replace $p_{\left[2^k\right]} \to C_2$ and arrive at the first main result of this section:

$$\left\langle \mathsf{W}_{\mathcal{R}_\mu} \right\rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\left(\frac{C_2(\mathcal{R}_\mu)}{2} \right)^k + \dots \right] \left(\frac{g}{2} \right)^{2k} \tag{4.17}$$

To reiterate, the dots stand for terms that we are missing at every order in g^{2k}, that are of lower degree in Casimirs than the maximal $2k$. These terms we haven't computed come from different sources:

- Contributions from shifterd Schurs with $|\mu| < 2k$.
- Contributions from $S^\star_{\mu \vdash 2k}$ that we omitted when we replaced this functions with ordinary Schur polynomials.

4.1.2 Exact results for arbitrary representations of $U(N)$

Without departing the $U(N)$ algebra yet, let us now go back to (4.2) and consider the problem of the exact evaluation of $\left\langle \mathsf{W}_{\mathcal{R}_\mu} \right\rangle$ from a different perspective that will take us to a closed expression for the result. We will apply the method of orthogonal polynomials, and, as we anticipated in 2.4.3, for this task we will make use of the unnormalized

Hermite orthogonal polynomials

$$\left\{ \begin{array}{c} p_n(x) \equiv \left(\frac{g^2}{8}\right)^{\frac{n}{2}} H_n\left(\frac{\sqrt{2}x}{g}\right);\ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \\ \int dx p_m(x) p_n(x) e^{-\frac{2}{g^2}x^2} = \sqrt{2\pi} n! \left(\frac{g}{2}\right)^{2n+1} \delta_{mn} \equiv h_n \delta_{mn} \end{array} \right\} . \tag{4.18}$$

For this algebra, the Faddeev-Popov determinant is simply the square of the Vandermonde determinant, and therefore, in (4.2) we can replace

$$\Delta_{FP} = \Delta_V^2 = \left(\sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \prod_{n=1}^{N} p_{\sigma(n)-1}(x_n) \right)^2 \equiv \sum_{\sigma,\rho} (-1)^{\epsilon(\sigma+\rho)} \Delta_\sigma(x) \Delta_\rho(x) \tag{4.19}$$

With these expressions we can already evaluate the partition function:

$$\mathcal{Z} = N! \prod_{n=0}^{N-1} h_n , \tag{4.20}$$

since the orthogonality of the polynomials implies that only $\rho = \sigma$ will have a nonvanishing contribution, and all the terms that remain are identical up to relabelling of integration variables.

In order to compute Wilson loop v.e.v.s we need to work further. It is convenient to observe an appealing property of Hermite polynomials:

$$\frac{d}{dx} p_n(x) = n p_{n-1}(x) \tag{4.21}$$

With the help of this expression, it is no longer needed to expand the exponentials coming from the Wilson loop in power series to evaluate them, because we can use instead

$$\int dx p_n(x) p_m(x) e^{\tau x} e^{-\frac{2}{g^2}x^2} = \frac{2 h_n h_m \tau^{n+m}}{\sqrt{2\pi} g} e^{\frac{g^2}{8}\tau^2} \sum_{k=-\infty}^{\infty} \frac{(g\tau/2)^{-2k}}{k!(m-k)!(n-k)!} , \tag{4.22}$$

which is a finite polynomial despite the appearances: in practice the factorials in the denominator truncate the infinite sum. An equivalent but more familiar [139] way of writing this expression is

$$\int dx p_n(x) p_m(x) e^{\tau x} e^{-\frac{2}{g^2}x^2} = \sqrt{2\pi} \left(\frac{g}{2}\right)^{2m+1} n! \tau^{m-n} L_n^{m-n} \left(-\left(\frac{g\tau}{2}\right)^2 \right) e^{\frac{g^2}{8}\tau^2} , \tag{4.23}$$

where we identified the extended Laguerre polynomial

$$L_n^{m-n}(-x) = \sum_{k=0}^{n} \frac{m!}{(n-k)!(m-n+k)!k!} x^k . \tag{4.24}$$

The advantage of (4.22) over (4.23) is the fact that the symmetry between n and m remains explicit. Additionally, in (4.22) the dependences on n and m are explicitly factorized term by term. Notice, in any case, that for $\tau \to 0$ we recover the appropriate limit (4.18).

Before we can apply (4.22) (or (4.23)) onto (4.2) it is convenient to first organize the Schur polynomial appropriately. Not all of its summands are on equal footing in this computation: only those monomials related by a permutation of x labels are in principle reduced to the same value. It is therefore convenient to refer the Schur polynomials to yet another basis of the space of symmetric functions of N variables, the monomial basis, whose elements are labelled by a weakly decreasing partition μ, and which are defined by

$$m_\mu\left(x\right) \equiv \sum_{\sigma\in S_N}\prod_n x_n^{\mu_{\sigma(n)}} \ . \tag{4.25}$$

The coefficients $K_{\mu\nu}$ relating the $\mathcal{S}_\mu$ and m_μ functions,

$$\mathcal{S}_\mu\left(x\right) = \sum_{\nu\leq\mu} K_{\mu\nu} m_\nu\left(x\right) \ , \tag{4.26}$$

are positive integers known as the Kostka numbers. In this expression $\nu \leq \mu$ means that either $\mu = \nu$, or the first non-zero $\mu_i - \nu_i$ is positive[3].

We are now in position of facing (4.2) head-on, which after the translation to the monomial basis has become

$$\left\{\begin{array}{c} \left\langle \mathsf{W}_{\mathcal{R}_\mu}\right\rangle = \sum_{\nu\leq\mu}\frac{K_{\mu\nu}}{\dim\mathcal{R}_\mu}\left\langle m_\nu\left(e^x\right)\right\rangle \\ \left\langle m_\nu\left(e^x\right)\right\rangle = \frac{1}{\mathcal{Z}}\sum\limits_{\rho,\sigma,\beta\in S_N}\left(-1\right)^{\epsilon(\rho+\sigma)}\int d^N x\Delta_\rho\left(x\right)\Delta_\sigma\left(x\right)\exp\sum_n\left(\nu_\beta\left(n\right)x_n - \frac{2}{g^2}x_n^2\right) \\ \Delta_\rho\left(x\right)\equiv\prod_n p_{\rho(n)-1}\left(x_n\right) \end{array}\right\} \tag{4.27}$$

Using the results we presented for orthogonal polynomials the v.e.v. of the monomial element can be straightforwardly computed:

$$\left\{\begin{array}{c} \left\langle m_\nu\left(e^x\right)\right\rangle = \frac{(g/2)^{N(N-1)}}{N!\prod_{i=0}^{N-1}i!}\left(\sum_{\beta\in S_N}\sum_{\vec{k}\in\mathbb{N}^N}\left(\frac{g}{2}\right)^{-2\sum_n k_n}\prod_{n=1}^N k_n!\mathcal{D}_{\vec{k}\beta}^2\right)e^{\frac{g^2}{8}\sum_n\nu_n^2} \\ \mathcal{D}_{\vec{k}\beta} = \det\left(\left(\begin{array}{c} j-1 \\ k_i \end{array}\right)\nu_{\beta(i)}^{j-1-k_i}\right) \end{array}\right\} . \tag{4.28}$$

Notice that the expression for the matrix elements entering $\mathcal{D}_{\vec{k}\beta}$ is indeterminate for $\nu = 0$ and $j-1-k_i \leq 0$. A detailed analysis of the origin of this expression makes clear that they should be taken to be 0 except in the $j-1 = k_i$ case, where they evaluate to 1.

[3]This order relation is usually called the reverse lexicographic ordering

The structure of (4.28) can be summarized as

$$\left\langle \mathsf{W}_{\mathcal{R}_\mu} \right\rangle = \frac{1}{\dim \mathcal{R}_\mu} \sum_{\nu \leq \mu} K_{\mu\nu} P_\nu \left(\frac{g^2}{4} \right) e^{\frac{g^2}{8} \sum_n \nu_n^2} \ , \tag{4.29}$$

A relevant question concerning this observable, already answered in [140], is what is the largest exponent in (4.29), and it is immediate to see that it corresponds to $\sum_i \nu_i^2$ where ν is the highest weight of the representation $\mathcal{R}_\mu$.

The properties of the $P_\nu(x)$ polynomials can be inferred from the combinatorics in (4.28). From its explicit form, for instance, we conclude that the coefficients $d_{\nu;k}$ in its series expansion,

$$P_\nu (x) = \sum_{k \geq 0} d_{\nu;k} \frac{x^j}{j!} \ , \tag{4.30}$$

are always positive. Although these combinatorics are in general involved, there is an efficient and tidy way to present them: If one defines the infinite family of $N \times N$ matrices labelled by $k \in \mathbb{N}$

$$A_{nm}^{(k)} (x) = k^{m-n} L_n^{m-n} \left(-k^2 x \right) \ . \tag{4.31}$$

The v.e.v.s $\langle m_\nu (x) \rangle$ can be obtained from the generating function

$$\left\langle \prod_{n=1}^{N} \left(t + \sum_{k=1}^{\infty} y_k e^{k x_n} \right) \right\rangle = \det \left(t \delta_{mn} + \sum_{k=1}^{\infty} y_k A_{mn}^{(k)} \left(\frac{g^2}{4} \right) e^{k^2 \frac{g^2}{8}} \right) \tag{4.32}$$

Combining these v.e.v.s with the basis transformation matrix defined by the Kostka numbers, equation (4.26), we recover the form (4.29).

To gain further insight into the properties of $P_\nu(g)$ we will promptly explore the simplest nontrivial case, i.e., the case where ν contains a single nonzero entry. For this particular case, the monomial basis elements coincide with Schur polynomials corresponding to the same partition, so from this study we will obtain directly the results for antisymmetric Wilson loops.

4.1.3 Exact results for antisymmetric $U(N)$ representations

The study of Wilson loops in an antisymmetric representation is particularly simple and its results are most explicit, so we will focus our attention on it. For the k-antisymmetric representation, the Schur polynomial concides with a very specific element in the monomial basis, m_{1^k}, which is known as the k-th *elementary symmetric function*

$$\langle W_{\mathcal{A}_k}(g) \rangle = \frac{1}{\mathcal{Z}} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi} \prod_{i<j} |x_i - x_j|^2 e^{x_1 + \cdots + x_k} e^{-\frac{2}{g^2} \sum_{k=1}^{N} x_k^2} \ . \tag{4.33}$$

From this integral (4.33), it is straightforward to relate the vevs of the Wilson loops for the k-th and the (N-k)-th antisymmetric representations. To do so, complete the squares for the $x_1, \ldots, x_k$ eigenvalues in (4.33), and then change variables $\tilde{x}_i = x_i - g$. Except for the x_i-independent exponents generated by completing squares, the resulting integral is the one that yields the vev of the Wilson loop in the $(N-k)$-th representation, so we arrive at the relation

$$\langle \mathsf{W}_{\mathcal{A}_k} \rangle \, e^{-\frac{kg^2}{8}} = \left\langle \mathsf{W}_{\mathcal{A}_{N-k}} \right\rangle e^{-\frac{(N-k)g^2}{8}} \; . \tag{4.34}$$

From this expression a first property of the corresponding P_k polynomials that is not obvious at all[4] in (4.28) is their palindromy: $P_k(x) = P_{N-k}(x)$.

Let us now analyse these observables in more detail. We will first look at them from the generating functional language. Observe that if we want to restrict ourselves to the antisymmetric sector it suffices to consider the generating function (4.32) with a single nonzero y_k variable[5],

$$\left\{ \begin{array}{c} F_A(t) = \sum_{k=0}^{N} e_k t^{N-k} = \prod_{i=1}^{N} \left(t + e^{x_i}\right) \\ \langle F_A(t) \rangle = \sum_{k=0}^{N} t^{N-k} \binom{N}{k} \langle W_{\mathcal{A}_k} \rangle = \det \left(t\delta_{nm} + L_{n-1}^{m-n} \left(-\frac{g^2}{4}\right) e^{\frac{g^2}{8}} \right) \end{array} \right\} . \tag{4.35}$$

This generating functional can be interpreted from the point of view of a more formal framework: Its parametric continuation to the complex plane, i.e., the result of promoting t and x to complex variables, can be regarded as a spectral curve [141] for the set of of Wilson loop operators under consideration. The results for $\left\langle \mathsf{W}_{\mathcal{A}_{\|}} \right\rangle$ are referred to contour integrals of $t^{k-1-N} F_A$ around the origin in the complex plane of t. Thus, their study can be rephrased in terms of the analytic structure in t of the $\det\left(t + A\left(x\right)\right)$ determinant. Although the full analysis of this framework is well beyond the scope of the present dissertation, let us make some observations concerning this structure: The observed palindromy in in t for this determinant implies that roots come in pairs $t_i, 1/t_i$ (except $t_i = -1$, that appears unpaired for N odd)[6]. For x real, if t_i is a root, so is t_i^*. From the fact that all coefficients in $P_k(x)$ are real and positive, we learn that for $x > 0$, the roots of $|t + Le^{x/2}|$ can't be positive real numbers. Numerical experimentation suggests the following picture: for $x > 0$ all roots are real and negative; at $x = 0$ all eigenvalues are equal to -1, and as $x \to +\infty$, half of them tend to $-\infty$ as powers of x, while the other half are their pairs $1/t_i$ and tend to zero. This is consistent with the observation of [141] that at large N, the discrete zeros coalesce on a branch cut along the negative real axis.

It is nonetheless also possible in our construction to evade this complex calculus machinery and evaluate any specific antisimetric representation loop directly from (4.35),

[4] A proof of this property from the explicit combinatorics is provided in appendix C of [2]

[5] This generating function is closely related to the usual one, $E(t) = \sum_{k=0}^{N} e_k(y) t^k = \prod_{i=1}^{N} (1 + y_i t)$. Indeed, $F_A(t) = t^N E(1/t)$.

[6] If, in analogy with the analysis of the spectral curve of classical strings in $AdS_5 \times S^5$ (see [142] for a review), we define quasimomenta p_j by $t_j = e^{ip_j}$, this $\mathbb{Z}_2$ involution translates into the quasi-momenta coming in pairs $(p_j, -p_j)$.

by computing the appropriate minors of the determinant:

$$\langle \mathsf{W}_{\mathcal{A}_k} \rangle = \frac{e^{\frac{kg^2}{8}}}{N!} \sum_{\sigma \in S_N} \sum_{\mu \in S_k} (-1)^{\epsilon(\mu)} \prod_{m=1}^{k} L_{\sigma(m)-1}^{\mu(\sigma(m))-\sigma(m)}(-g^2/8) \ . \tag{4.36}$$

As a check, this result permits us to recover at $k=1$

$$\langle \mathsf{W}_{\mathcal{A}_k} \rangle = \frac{e^{\frac{kg^2}{8}}}{N!}(N-1)! \sum_{n=0}^{N-1} L_n^0(-g^2/4) = \frac{1}{N} L_{N-1}^1(-g^2/4) e^{\frac{g^2}{8}} \ ; \tag{4.37}$$

the result for the fundamental Wilson loop derived by [75], equation (2.61). Additional information can be extracted from 4.35: it allows us to determine that the coefficients $d_{k;i}$ in the series expansion of P_k is an integer, for instance. To prove this, observe that $L_{n-1}^{m-n}(x)$ is a linear combination of terms $x^i/i!$ and that in the computation of the determinant products, sums and subtractions of these terms keep the coefficients integer because

$$\frac{g^i}{i!}\frac{g^j}{j!} = \binom{i+j}{i} \frac{g^{i+j}}{(i+j)!} \ . \tag{4.38}$$

Another property easily derived from (4.35) is the particularization of (4.8) to antisymmetric loops: The dependence on k predicted by (4.8) is straightforward to reproduce by evaluating the appropriate minors of the determinant of the generating function.

In turn, the casuistics on (4.28) is also simplified considerably when we restrict ourselves to antisymmetric representations, i.e., when ν has a single nonzero row. The crucial points in the argument are the following:

- The determinants $\mathcal{D}_{\vec{k}\beta}$ vanish if two identical entries in $\nu_{\beta(i)}$ are assigned identical values of k_i. In other words, the k entries for k_i corresponding to $\nu_{\beta(i)} = 1$ form a strictly decreasing sequence of integers confined to the interval $[0, N-1]$.
- Similarly, the choices of k_i for the rows with $\nu_{\beta(i)} = 0$ are mutually exclusive. In addition, those rows will be made of zeroes, except for a single one at $j-1 = k_i$. In practice this implies that for this choice of k_i in any nonzero contribution to the determinant the column $j-1$ satisfying this condition has to be paired with row i and therefore antisymmetry forbids the pairing of this column to any other row in the determinant. Since we have $N-k$ exclusions of this type, we are left with a strictly decreasing sequence of $j-1$ values that can get paired with the $\nu = 1$ rows, which are of course also contained in $[0, N-1]$.
- The previous two strictly decreasing sequences of k elements can be viewed as shifted sequences of partitions of k entries, with the first value restricted to be at most $N-k$. We denote these two sequences σ and τ in order to distinguish them from the sequences of N entries μ, ν, λ, ... that we have encountered up to this point. Notice that the determinant $\mathcal{D}_{\vec{k}\beta}$ becomes the coefficient $b_{\sigma\tau}$ defined in (4.5), i.e., the shifted binomial determinant, of the partitions defined in this fashion.

From these considerations, we obtain

$$P_k(x) = \sum_{\tau,\sigma \in L(k,N-k)} x^{\sum_n(\sigma_n - \tau_n)} \prod_{n=1}^{k} \frac{\bar{\tau}_n!}{\bar{\sigma}_n!} b_{\sigma\tau}^2 \tag{4.39}$$

where $L(k, N-k)$ designates the set of Young diagrams drawin in a rectangle of $(N-k) \times (k)$ boxes. Collecting all the terms with g^n, we arrive then at a formula for the coefficients $d_{k,n}$:

$$d_{k;n} = n! \sum_{\tau,\sigma \in L(k,N-k)} b_{\sigma\tau}^2 \prod_{m=1}^{k} \frac{\bar{\tau}_m!}{\bar{\sigma}_m!} \delta_{\sum(\sigma_m - \tau_m),n} \,. \tag{4.40}$$

As a check, if we set $k = 1$ we arrive at

$$d_{1,n} = \sum_{\bar{\sigma}=n}^{N-1} \binom{\bar{\sigma}}{\bar{\sigma} - n} = \binom{N}{n+1} \tag{4.41}$$

reproducing the expansion of $P_1(x) = L_{N-1}^1(-x)$.

Let us close this analysis with two brief remarks on the result (4.39):

Combinatorial interpretation: Observe that the simple expression of (4.39) is written exclusively in terms of combinatorial quantities. It is tantalizing to look for a physical interpretation for them. Without venturing any guess, let us list here some well-known counting problems where $b_\sigma\tau$ and related quantities appear naturally.

Before, let us make an important observation: the sums above can be restricted to pairs of k-tuples such that $\sigma_i \geq \tau_i$ for $i = 1, \ldots, k$. The reason is that if for some j it happens that $\sigma_j < \tau_j$, the matrix with binomial coefficients in (4.40) has a zero block in the upper right corner. The full determinant is then the product of determinants of the diagonal blocks, but the determinant of the lower diagonal block is zero, since it has a zero row.

A first example of combinatorial problem resembling the counting in $d_{k;n}$ appears in the enumeration of nonintersecting paths on a two dimensional lattice with boundary condtions given by the k-tuples $\bar{\sigma}$ and $\bar{\tau}$ [143]: the number of paths satisfying all conditions is precisely given by a combinatorial determinant. A second possibility that connects to our problem is to map each Young diagram to a configuration of fermions in a 1-dimensional lattice(see e.g. [144]). Time-dependent processes for the evolution from a diagram τ to a diagram σ are described by standard (i.e., strictly decreasing in both directions) tableaus we can draw on the squares of $\sigma - \tau$, i.e., the complement of τ inside σ; the so-called skew Young diagram. The number of standard tableaux in a skew young diagram is precisely given by

$$f^{\sigma/\tau} = (|\sigma| - |\tau|)! \det \left(\frac{1}{(\bar{\sigma}_i - \bar{\tau}_j)!} \right) \,, \tag{4.42}$$

which is useful to recast in terms of a binomial determinant $b_{\sigma\tau}$ (a direct way to establish their relation is by pulling common factors out of the binomial determinant to keep only its $\bar{\sigma}-\bar{\tau}$ part). One can write, for instance,

$$d_{k;n} = \frac{1}{n!} \sum_{\substack{\tau\subseteq\sigma\in L(k,N-k),\\ |\sigma|-|\tau|=n}} \prod_{m=1}^{k} \frac{\bar{\sigma}!}{\bar{\tau}!} f^2_{\sigma/\tau} = \sum_{\substack{\tau\subseteq\sigma\in L(k,N-k),\\ |\sigma|-|\tau|=n}} f_{\sigma/\tau} b_{\sigma\tau} \tag{4.43}$$

Notice that each of the expressions in (4.43) and (4.40) suggestively points at a different combinatorial interpretation in terms of fermions and paths.

Evaluating the coefficients: The index n in $d_{k;n}$ runs from 0 to $k(N-k)$, and, when it takes arbitrary values, it seems doubtful that the sum can carried out explicitly. We will now evaluate these coefficients for a few values of j, close to the endpoints of its range.

- For $n=0$, both k-tuples have to be identical to contribute: $\bar{\sigma}_i = \bar{\tau}_i$. In this case the matrix with entries $\binom{\bar{\sigma}_i}{\bar{\tau}_j}$ is triangular, and its determinant is one. The prefactor cancels as well, so d_0 is given by the number of k-tuples,

$$d_{k;0} = \binom{N}{k} \tag{4.44}$$

  Alternatively, in the language of skew Young diagrams, $n=0$ corresponds to the case of $\bar{\sigma}=\bar{\tau}$ and $d_{k;0}$ is just counting the number of Young diagrams that fit into a rectangle with $(N-k)\times k$ boxes, which is precisely $\binom{N}{k}$ (proposition 6.3 in [145]).

- For $n=1$, given a k-tuple $\bar{\tau}_i$, the only k-tuples $\bar{\sigma}_i$ that contribute are those where all the $\bar{\sigma}_i = \bar{\tau}_i$, except for precisely one element $\bar{\sigma}_j = \bar{\tau}_j + 1$. For each of those cases the matrix with entries $\binom{\bar{\sigma}_i}{\bar{\tau}_j}$ is triangular, the determinant is $\bar{\sigma}_j$ and the contribution in each case is $\bar{\sigma}_j$. It remains to count how many such pairs of k-tuples there are, which is easily seen to be $\binom{N-2}{k-1}$. Adding all contributions we obtain

$$d_{k;1} = \binom{N}{2}\binom{N-2}{k-1} \tag{4.45}$$

  The two computations presented so far are sufficient to reproduce the result 4.8.

- For $n=2$, there are two types of contributions. There are contributions from pairs of k-tuples when all $\bar{\sigma}_i = \bar{\tau}_i$ except for a single $\bar{\sigma}_j = \bar{\tau}_j + 2$. There are also contributions from pairs of k-tuples when $\bar{\sigma}_m = \bar{\tau}_m$ except for two $\bar{\sigma}$s, $\bar{\sigma}_i = \tau_i + 1$ and $\bar{\sigma}_j = \tau_j + 1$, with $i<j$. It is convenient to treat separately the cases where $\bar{\tau}_j = \bar{\bar{\sigma}}_i$ (in which case the matrix fails to be lower triangular) and the case $\bar{\tau}_j > \bar{\bar{\sigma}}_i$. By arguments very similar to the ones in the previous cases we arrive at

$$d_{k;2} = \frac{N!}{12(k-1)!(N-k-1)!}(3k(N-k)-N-1)$$

This coefficient allows us to write the perturbative expansion of the antisymmetric Wilson loop to order g^4,

$$\langle W_{\mathcal{A}_k} \rangle = 1 + \frac{C_2 g^2(\mathcal{A}_k)}{8} + \left(\frac{1}{4} C_2(\mathcal{A}_k)^2 - \frac{N+1}{12} \left(C_2(\mathcal{A}_k) - C_1(\mathcal{A}_k) \right) \right) \frac{g^4}{32} + \dots \tag{4.46}$$

- Having computed the first three coefficients $d_{k,j}(k,N)$, we turn to the opposite end of the range, when j is close to $k(N-k)$. For $n = k(N-k)$, there is only one term that contributes: $\bar{\sigma}_i = N - k - 1 + i$, $\tau_j = j - 1$. The determinant is 1, as can be proven by induction on k, for N fixed. Therefore

$$d_{k,k(N-k)}(k,N) = (k(N-k))! \, \frac{0!1! \dots (k-1)!}{(N-1)!(N-2)! \dots (N-k)!} \tag{4.47}$$

- For $n = k(N-k) - 1$, there are two terms that contribute. The first one has $\bar{\sigma}_i = N - k - 1 + i$, $\tau_j = 0, 1, \dots, k-2, k$; the corresponding determinant is $N - k$. The second term has $\bar{\sigma}_i = N - k - 1, N - k + 1, \dots, N - 1$ and $\bar{\tau}_j = j - 1$; the corresponding determinant is k. Adding these two terms one obtains

$$d_{k,k(N-k)-1}(k,N) = (k(N-k))! \, \frac{0!1! \dots (k-1)!}{(N-1)!(N-2)! \dots (N-k)!} N \tag{4.48}$$

As a consistency check, all the explicit results we obtain satisfy $d_{k;n}(k,N) = d_{k;n}(N-k,N)$.

4.1.4 Exact results for nonexceptional simple Lie algebras

In this chapter we have so far considered Wilson loops in theories with $U(N)$ algebra. Although this is an interesting case of study and possibly the simplest nonabelian example at hand in many regards, this type of theory is actually not part of the list of theories we want to address: its $U(1)$ diagonal subsector is not reproduced by the holographic constructions we are considering. As we discussed in 2.2, the low energy description of a D-brane is an $SU(N)$ theory if this stack is in a trivial background, and an ADE quiver field theory with semisimple algebra if it is located at a singular spacetime folding. Therefore, in consonance with the goals of this thesis, our next step will be to extend the results obtained with the methods of localization so far to a sample of Wilson loops in theories with SU, SO and SP algebras, without considering quiver constructions. These algebraic choice corresponds to the specific cases where the large N limit is unique and well defined.

In many aspects, the analysis of these Wilson loops proceeds in the same fashion as in 4.1.2 and its results can be rewritten in the language we used so far:

- Since the Weyl group of the Cartan subalgebra of these cases contains an S_N subgroup (along with additional reflections for SO and SP groups), it is still convenient to decompose the weight sum of the trace on a generic representation in the monomial basis (4.25).

- The matrix model for $SU(N)$ can be built directly within the $U(N)$ framework: Let us rewrite the matrix model in variables where the decomposition of the latter in $SU(N)$ and $U(1)$ is explicit,

$$\begin{array}{c} \int d^N x_i \Delta_{FP}(x)\, e^{-k\sum_i x_i^2} \int dx_D \delta\left(\sum_i x_i - x_D\right) = \\ = \int d^N \tilde{x}_i \delta\left(\sum_i \tilde{x}_i\right) \Delta_{FP}(\tilde{x})\, e^{-k\sum \tilde{x}_i^2} \cdot \int dx_D e^{-\frac{k}{N^2} x_D^2} \end{array} \leftarrow \tilde{x}_i \equiv x_i - \frac{1}{N} x_D \,. \qquad (4.49)$$

The part for $U(1)$ can be integrated out, though for any observable of $SU(N)$ we express in the $U(N)$ terminology we will have to explicitly subtract the contribution of the trace to obtain its v.e.v. in the $SU(N)$ theory. For our Wilson loops, for instance, we have to modify the insertion to [75, 146]

$$\mathrm{tr}_{\mathcal{R}} e^X \to e^{-\frac{|\mathcal{R}|}{N}\mathrm{tr}X}\, \mathrm{tr}_{\mathcal{R}} e^X \qquad (4.50)$$

- As we discussed in 2.4.3, for SO and SP we can rewrite the full Faddeev-Popov determinant as $\det p_{j-1}(x_i)$. Despite the appearances this does not make these matrix models identical to the $U(N)$ case and among them: In $U(N)$ the polynomial p_j has to be of order j, while for $SO(2N)$ the argument in 2.4.3 restricts it to be an even polynomial of degree $2j$, while for $SO(2N+1)$ and $SP(2N)$ we argued that it must to odd and of degree $2j+1$. Fortunately, Hermite polynomials have defined parity, $p_n(-x) = (-1)^n p_n(x)$, and by virtue of this property we can still choose the appropriate polynomials in the Faddeev-Popov determinant from their list, and conveniently make use of (4.18) and (4.22). From this fact we conclude that the expressions for $\left\langle m_\nu\left(e^{x/2}\right)\right\rangle$ (where the additional quocient by two has been introduced here to account for generic representations of SO algebras, we will soon come back to this point) resembles (4.28), or, equivalently, follow from a generating function in the shape of (4.32). However, for SO and SP groups, the analogues of the matrices $\mathcal{D}$ and A^k_{mn} are built only from Hermite polynomials of defined parity, and therefore, they carry only the even or odd columns (for the case of A^k_{mn}, also only the even or odd rows) of the original matrices.

Despite all this facts, one should not be to hasty in extending the expressions for $U(N)$ Wilson loops, specially for the SO and SP examples. One should take care with the fact that the representation structure is very different for every algebra:

- This fact is of special notoriety in $SO(2N+1)$ and $SP(2N)$: for these two theories matrix model observables are computed in exactly the same fashion (the discrepancy in factors of two in their Faddeev-Popov determinants disappears when we normalize), and all the differences between these theories (in the localizable sector) comes from the different weight structures. The Wilson loop in fundamental representation for $SO\left(2N+1\right)$, for instance, includes a zero weight contribution that is absent in $SP\left(2N\right)$.

- The decomposition in terms of the monomial basis is no longer given by Kostka numbers; the weight structure and multiplicity of SO and SP algebras is no longer implemented by Schur polynomials and Young tableaux.

- For SO and SP groups , the weight lattice is symmetrical under the specular reflections $x \leftrightarrow -x$. The invariance of the matrix model under this transformation implies that this fact can simply be encoded in the multiplicity of each instance of a monomial term in a specific representation, i.e., the generalization of the Kostka numbers.

- SO groups contain spinor representations. We should consider in general $\left\langle m_\mu\left(e^{x/2}\right)\right\rangle$ with some restrictions on the parity of the μ partition to account for the refined Weight lattice that this fact endows the theory with. In addition, the matrix model action acquires an additional factor of 2 coming from the fact that the fundamental trace (x, x) contains weights of both signs. Therefore, the precise implementation aforementioned (4.28) and (4.32) correspondents for SO and SP should include a shift $g^2 \to g^2/2$ if we wish to respect the normalization of (2.65) for the coupling constant.

The difference in the representation structure will always be apparent in the results, but the normalization on the coupling constant can be absorbed in the definition of the coupling constant. For the sake of comparability it is often appropriate to define the

Algebra G	Fundamental weights	monomial v.e.v.
$SU\left(N\right)$	$\sum_{n=1}^{k} x_n$	$\exp\left(-\frac{g^2k^2}{8N^2}\right)\cdot\left\{SU\left(N\right)\text{ result}\right\}$
$SO\left(2N\right)$	$\sum_{n=1}^{k\leq N-2} x_n$	Generated by $\det\left(t\delta_{nm}+L_{2m-2}^{2n-2n}\left(-\frac{g^2}{8}\right)e^{g^2/16}\right)$
	$\frac{1}{2}\sum_{n=1}^{N}\left(-1\right)^{\delta_{nN}} x_n$	$\det\left(L_{2n-2}^{2m-2n}\left(-\frac{g^2}{32}\right)e^{g^2/64}\right)$
	$\frac{1}{2}\sum_{n=1}^{N} x_n$	$\det\left(L_{2n-2}^{2m-2n}\left(-\frac{g^2}{32}\right)e^{g^2/64}\right)$
$SP\left(2N\right)$	$\sum_{n=1}^{k} x_n$	Generated by $\det\left(t\delta_{nm}+L_{2m-1}^{2n-1}\left(-\frac{g^2}{8}\right)e^{g^2/16}\right)$
$SO\left(2N+1\right)$	$\sum_{n=1}^{k\leq N-1} x_n$	Generated by $\det\left(t\delta_{nm}+L_{2m-1}^{2n-1}\left(-\frac{g^2}{8}\right)e^{g^2/16}\right)$
	$\frac{1}{2}\sum_{n=1}^{N} x_n$	$\det\left(L_{2n-1}^{2m-2n}\left(-\frac{g^2}{32}\right)e^{g^2/64}\right)$

Table 4.2: Fundamental weights and the v.e.v. of the monomial element that contains them for generic simple nonexceptional algebras.

Algebra G	$\langle \mathsf{W}_\square \rangle$
$U(N)$	$\langle \mathsf{W}_\square \rangle_{U(N)} = \frac{1}{N} L^1_{N-1}(-\tilde{g})\, e^{\tilde{g}/2}$
$SU(N)$	$\langle \mathsf{W}_\square \rangle_{SU(N)} = \exp\left(-\frac{\tilde{g}}{2N^2}\right) \langle \mathsf{W}_\square(\tilde{g}) \rangle_{U(N)}$
$SO(2N)$	$\langle \mathsf{W}_\square \rangle_{SO(2N)} = \frac{1}{N} \sum_{k=0}^{N-1} L_{2k}(-\tilde{g})\, e^{\tilde{g}/2}$
$SP(2N)$	$\langle \mathsf{W}_\square \rangle_{SP(2N)} = \frac{1}{N} \sum_{k=0}^{N-1} L_{2k+1}(-\tilde{g})\, e^{\tilde{g}/2}$
$SO(2N+1)$	$\langle \mathsf{W}_\square \rangle_{SO(2N+1)} = \frac{1}{2N+1} \left(1 + 2 \sum_{k=0}^{N-1} L_{2k+1}(-\tilde{g})\, e^{\tilde{g}/2}\right)$

Table 4.3: V.e.v.s of fundamental Wilson loop operators for SU, SP and SO algebras

canonical Gaussian coupling $\tilde{g}$ as

$$\tilde{g} \equiv \begin{cases} g^2/4 & \text{if } G \in U, SU \\ g^2/8 & \text{if } G \in SO, SP \end{cases} \tag{4.51}$$

Table 4.2 presents the v.e.v.s of the monomial including every fundamental weight in the considered theories[7]. The results on this table are sufficient to compute the v.e.v. of Wilson loops in antisymmetric or spinorial representations, though in general they are no longer given by a single monomial contribution.

In order to exemplify this, observe in table 4.3 the v.e.v. of the Wilson loops in the fundamental representation of the considered groups. Since the fundamental representation of $SO(2N+1)$ contains an instance of the weight zero, its v.e.v. averages the contribution of two elements of the monomial basis.

[7] These monomials play here the role of the monomial elements with a single nonzero entry in $U(N)$, because the fundamental weights are the highest weights of that span the weight lattice via linear combinations of integer coefficients. The representations with a fundamental weight as a highest weight are often called *fundamental* in some contexts, though in Physics, the concept of *fundamental representation* often refers only to the representation with highest weight $\omega \cdot x = x_1$. For SU and SP the other representations with fundamental highest weight coincide with the antisymmetric representations, while for SO groups the antisymmetric representations of higher order are relegated from this role by the spinorial representations.

4.2 Learning from exact $\mathcal{N}$=4 localization results

With the exact results obtained in the previous section at our disposal, we will now proceed to examine them from more physical perspectives:

- We will start this analysis in 4.2.1 with the direct physical implications of the Wilson loop v.e.v.s for the particles they represent. In particular, we will compute their Bremsstrahlung function, the remarkable physical content of which has already been discussed in 3.2.
- In 4.2.2, still within the language of field theory, we will move closer to the purpose of this thesis by considering the 't Hooft $1/N$ rearrangement of the perturbative field theoretic series we reviewed in the beginning of 2.2, focusing our attention to the particular case of SO and SP algebras. Without computing any specific diagram, we will provide generic arguments relating different perturbative sectors of the theory and comment on their implications on the putative dual string theoretic worldsheet perturbation expansion, currently well out of reach.
- Finally, in 4.2.3 we will make an observation on the structure of the matrix model computations that will provide us with interesting insights on the AdS/CFT conjecture. In particular, we will observe that the dual theory to theories with SO and SP algebras can be interpreted in the language of Lin, Lunin, Maldacena (LLM)-like bubbling geometry constructions [147] for orientifold spacetimes [148], at least in the supergravity limit. This connection refines and clarifies the identification of these LLM-like geometries.

4.2.1 Lessons for the linear response of charged probes

Recall that as we argued in 3.1, for $\mathcal{N} = 4$ SYM theories the Bremsstrahlung coefficient can be computed through the formula (2.58), which we will transcribe here for convenience: For half-BPS circular Wilson loops W,

$$B = \frac{1}{2\pi^2} g^2 \partial_{g^2} \ln \langle \mathsf{W} \rangle \ . \tag{4.52}$$

The argument leading to this relation is independent of the representation, so we can put to use our results for $\langle \mathsf{W}_{\mathcal{R}} \rangle$. Since they are most explicit for the antisymmetric representation, let us start with this case. For the antisymmetric representations, the vev of the Wilson loop is a polynomial in g times an exponential, see eqs. (4.36). From this simple fact, it follows that the final answer is a rational function in the coupling,

$$B^{U(N)}_{\mathcal{A}_k} = \frac{\lambda}{16\pi^2 N} \frac{\sum_{j=0}^{k(N-k)} \frac{2d_{k;j+1} + k d_{k;j}}{j!} \left(\frac{\lambda}{4N}\right)^j}{\sum_{j=0}^{k(N-k)} \frac{d_{k;j}}{j!} \left(\frac{\lambda}{4N}\right)^j} \ , \tag{4.53}$$

with the understanding that $d_{k;k(N-k)+1} = 0$. For fixed N, both at weak and at strong 't Hooft coupling, the Bremsstrahlung function is linear in λ

$$B^{U(N)}_{\mathcal{A}_k} = \left\{ \begin{array}{ll} \frac{c_2(\mathcal{A}_k)}{16\pi^2 N}\lambda & \lambda \ll 1 \\ \frac{k}{16\pi^2 N}\lambda & \lambda \gg 1 \end{array} \right\} . \tag{4.54}$$

Let's now briefly discuss the case of general representations. Now $\langle \mathsf{W}_{\mathcal{R}} \rangle$ is given by a linear combination of $\langle m_\tau \rangle$, so it is a sum of polynomials times exponentials, (4.29). Since in general the exponents in these exponentials are different, the corresponding Bremsstrahlung functions are no longer rational in the coupling. On the other hand, it also follows from (2.58) that it is still true that for fixed N, both at weak and at strong 't Hooft coupling, the Bremsstrahlung function is linear in λ. The weak coupling result can be read off from (4.8). In the large coupling limit, the coefficient of λ is given by the largest exponent in the exponentials which as pointed out after (4.29) (see also [140]) for a representation $\mathcal{R}$ with partition ν this exponent is $(\sum_n \nu_n^2)g^2/8$, so

$$B^{U(N)}_{\mathcal{R}} = \left\{ \begin{array}{ll} \frac{c_2(\mathcal{R})}{16\pi^2 N}\lambda & \lambda \ll 1 \\ \frac{\sum_i \tau_i^2}{16\pi^2 N}\lambda & \lambda \gg 1 \end{array} \right\} . \tag{4.55}$$

4.2.2 Lessons for the topological string expansion

We already pointed out in 2.2 that gauge field theories with $U(N)$, $SO(N)$ and $SP(N)$ gauge groups could be analysed using 't Hooft large N expansions. We mentioned as well that the expansions for SO and SP groups contain, in addition to the terms one finds for the $U(N)$ case, terms coming from unoriented surfaces, which could always be reduced to a sector with one crosscap and another with two crosscaps. Additional arguments in the literature [149] determine that the quantities for $SO(2N)$ are related to those of $SP(2N)$ by the replacement $N \to -N$. And finally, we know that $SO(N)$ and $SP(2N)$ theories can be obtained from orientifolding $U(2N)$. All in all, these general arguments imply that v.e.v.s in the respective fundamental representations of various groups ought to be related by

$$\langle \mathsf{W}(\tilde{g}) \rangle_{\substack{SO(2N) \\ Sp(2N)}} = \langle \mathsf{W}(\tilde{g}) \rangle_{U(2N)} \pm \text{unoriented}_{\ c=1} + \text{unoriented}_{\ c=2} \tag{4.56}$$

where unoriented $_c$ refers to terms that in the large N limit arrange themselves into non-orientable surfaces with the number of crosscaps c fixed.

We are now going to show that indeed the exact results of table 4.3 follow the pattern expressed in (4.56). In the process, we will find a couple of additional features that do not follow from these general arguments.

To obtain the $1/N$ expansion of $\langle W(\tilde{g})\rangle_{SO(2N)}$ and $\langle \mathsf{W}(\tilde{g})\rangle_{SP(2N)}$, we can analyse them separately, following the steps of [75]. The details of this procedure can be found in the appendix C of [3]. However, it is much more efficient to consider their sum and their difference, and expand those:

- From the sum of $\langle \mathsf{W}(\tilde{\mathsf{g}})\rangle_{SO(2N)}$ and $\langle \mathsf{W}(\tilde{g})\rangle_{SP(2N)}$ it is immediate that

$$\langle \mathsf{W}(\tilde{g})\rangle_{SO(2N)} + \langle \mathsf{W}(\tilde{g})\rangle_{SP(2N)} = 2\,\langle \mathsf{W}(\tilde{g})\rangle_{U(2N)} \tag{4.57}$$

- As for the difference $\langle \mathsf{W}\rangle_{SP(2N)} - \langle \mathsf{W}(g)\rangle_{SO(2N)}$, using properties of the Laguerre polynomials, it is not difficult to prove from the explicit results in table 4.3 that the following exact relation holds

$$\frac{\partial}{\partial \tilde{g}}\Big(\langle \mathsf{W}(\tilde{g})\rangle_{SP(2N)} - \langle \mathsf{W}(\tilde{g})\rangle_{SO(2N)}\Big) = \langle \mathsf{W}(\tilde{g})\rangle_{U(2N)} \tag{4.58}$$

These last two relations, eqs. (4.57) and (4.58), can we rewritten in the following suggestive form

$$\langle \mathsf{W}(\tilde{g})\rangle_{\substack{SO(2N)\\ SP(2N)}} = \langle \mathsf{W}(\tilde{g})\rangle_{U(2N)} \mp \frac{1}{2}\int_0^{\tilde{g}} dg'\ \langle \mathsf{W}(g')\rangle_{U(2N)} \tag{4.59}$$

Recall that $\langle \mathsf{W}(\tilde{g})\rangle_{U(2N)}$ has a expansion in $1/N^2$. Furthermore, since $\tilde{g} \sim \frac{\lambda}{N}$, the integral brings an extra power of $1/N$. Therefore, equation (4.59) neatly splits the $1/N$ expansions of $\langle \mathsf{W}(\tilde{g})\rangle_{SO(2N)}$ and $\langle \mathsf{W}(\tilde{g})\rangle_{SP(2N)}$ into even and odd powers of $1/N$. The $1/N^{2k}$ terms coincide for both v.e.v.s and are given $\langle \mathsf{W}(\tilde{g})\rangle_{U(2N)}$; they correspond to orientable surfaces. Note in particular that since all even powers of $1/N$ come from orientable surfaces, there are no contributions from world-sheets with two crosscaps, as it can be already deduced from eqs. (4.56) and (4.57).

Turning now to the $1/N^{2k+1}$ terms in the expansion of $\langle \mathsf{W}(\tilde{g})\rangle_{SO(2N)}$ and $\langle \mathsf{W}(\tilde{g})\rangle_{SP(2N)}$, they come from the integral in eq. (4.59), so it is manifest that they differ just by a sign; this, together with the equality of the even terms in the expansions, proves that indeed $\langle \mathsf{W}(\tilde{g})\rangle_{SP(2N)}$ can be obtained from $\langle \mathsf{W}(\tilde{g})\rangle_{SO(2N)}$ by the substitution $N \to -N$, as it had to happen according to general arguments [149].

To recapitulate, the $1/N$ expansion of $\langle \mathsf{W}(\tilde{g})\rangle_{SO(2N)}$ and $\langle \mathsf{W}(\tilde{g})\rangle_{SP(2N)}$ could in principle involve contributions from three kinds of surfaces, with zero, one or two crosscaps. By a mix of generic arguments and exact field theory computations, we have found that for these quantities, and for any number of handles, contributions from surfaces with one crosscap are given by an integral of the contribution from surfaces without crosscaps, while there is no contribution from surfaces with two crosscaps, eq. (4.59).

The two features that we have just uncovered for the $1/N$ expansion of $\langle \mathsf{W}(\tilde{g})\rangle_{SO(2N)}$ and $\langle \mathsf{W}(\tilde{g})\rangle_{SP(2N)}$ bear certain resemblance with properties encountered in other instances of $1/N$ expansion of SO and Sp gauge theories. A first example is the computation of the effective glueball superpotential of $\mathcal{N}=1$ SYM theories with a scalar field in the adjoint, with an arbitrary tree-level polynomial superpotential, $\mathcal{W}(\Phi)$. Dijkgraaf and Vafa [150] pointed out that for $G=U(N)$, this computation reduces to an evaluation of the planar free energy of a one-matrix model with the matrix model potential given by the tree-level superpotential of the gauge theory. For $\mathcal{N}=1$ SYM with gauge groups $SO(N), SP(N)$ the corresponding matrix models are, like in the present work, valued on the Lie algebras [151]. It was found in [151] that the effective superpotential of the $\mathcal{N}=1$ SYM gauge theory is fully captured by the contributions from S^2 and $\mathbb{RP}^2$, so there is no contribution from the world-sheet with two crosscaps (Klein bottle); furthermore, the contribution to the free energy coming from $\mathbb{RP}^2$ is given by a derivative of the contribution from S^2,

$$\mathcal{F}_1 = \pm \frac{g_s}{4}\frac{\partial \mathcal{F}_0}{\partial S_0}$$

with S_0 (half) the 't Hooft coupling. Notice however that in this example the properties are only established for world-sheets without any handles or boundaries, while our arguments work for world-sheets with a single boundary and an arbitrary number of handles. A second example comes from the large N expansion of Chern-Simons theory on 3-manifolds. It was observed in [152] that the $1/N$ expansion of the free energy of Chern-Simons on S^3 with gauge groups $SO(N), SP(N)$ involves unoriented world-sheets with one cross-cap, but again world-sheets with two cross-caps are absent in this expansion. Moreover, the large N expansion of Chern-Simons with $G=SO(N), SP(N)$, via its connection with knot theory, displays non-trivial relations for the invariants of U(N) and SO/Sp links [153].

While it is interesting that the two features we have uncovered in the $1/N$ expansion of $\langle \mathsf{W}(\tilde{g})\rangle_{SO(2N)}$ and $\langle \mathsf{W}(\tilde{g})\rangle_{SP(2N)}$ have superficially similar incarnations in other gauge theories with gauge groups $SO(N), SP(N)$, we don't expect these two features to be generic for other observables of $\mathcal{N}=4$ SYM with $G=SO(N), SP(N)$. For instance, in the case we have studied, the absence of contributions coming from world-sheets with two crosscaps is a consequence of the exact relation (4.57), but this relation appears to be quite specific of vevs of Wilson loops in the respective fundamental representations, and we don't know of similar relations for vevs of Wilson loops in other representations. Not surprisingly, in Chern-Simons theory with $G=SO(N), SP(N)$, vevs of Wilson loops in higher representations do get contributions from world-sheets with two crosscaps [154].

Turning now to string theory, reproducing the actual $1/N$ expansion of $\langle \mathsf{W}(\tilde{g})\rangle_{SO(2N)}$ or $\langle \mathsf{W}(\tilde{g})\rangle_{SP(2N)}$ from world-sheet computations is as out of reach as for $\langle \mathsf{W}(\tilde{g})\rangle_{U(2N)}$. On the other hand, granting the AdS/CFT duality for any value of g_s and α'/L^2, our results are also exact results in string theory, even beyond the perturbative regime. It is tantalizing to suspect that the results we have found - *e.g.* the absence of contributions from world-sheets with two crosscaps and any number of handles - are in the string theory language consequences of some symmetry enjoyed by the particular quantities we are considering. Identifying this symmetry and the stringy argument beyond the relations we have found appears to be a more promising and illuminating task than attempting to reproduce them by carrying out the explicit world-sheet computations.

Everything we have said so far follows from the exact results we have computed, and the exact relations among them. We didn't even have to carry out the explicit $1/N$ expansion of the exact results to arrive at these conclusions. Nevertheless, it is still worth to obtain this $1/N$ expansion explicitly, and this task can be accomplished with very little effort, by combining the exact relation (4.58) with the results in [75]. Drukker and Gross [75] obtained the following $1/N$ expansion of $\langle \mathsf{W} \rangle_{U(N)}$, that we write for $U(2N)$,

$$\langle \mathsf{W} \rangle_{U(2N)} = \frac{2}{\sqrt{2\lambda}} I_1(\sqrt{2\lambda}) + \sum_{k=1}^{\infty} \frac{1}{N^{2k}} \sum_{i=0}^{k-1} X_k^i \left(\frac{\lambda}{2}\right)^{\frac{3k-i-1}{2}} I_{3k-i-1}(\sqrt{2\lambda})$$

where $I_\alpha(x)$ are modified Bessel functions of the first kind, and X_k^i are coefficients satisfying the recursion relation

$$4(3k-i)X_k^i = X_{k-1}^i + (3k-i-2)X_{k-1}^{i-1} \tag{4.60}$$

with initial values $X_1^0 = 1/12$ and $X_k^k = 0$. A trivial integration then yields

$$\langle \mathsf{W} \rangle_{\substack{SO(2N)\\ SP(2N)}} = \langle \mathsf{W} \rangle_{U(2N)} \mp \frac{1}{4N} \left[\left(I_0(\sqrt{2\lambda}) - 1 \right) + \sum_{k=1}^{\infty} \frac{1}{N^{2k}} \sum_{i=0}^{k-1} X_k^i \left(\frac{\lambda}{2}\right)^{\frac{3k-i}{2}} I_{3k-i}(\sqrt{2\lambda}) \right]$$

This result is valid for any λ. We can then use it to obtain a large λ expansion at every order in $1/N$

$$\langle \mathsf{W} \rangle_{SO(2N)} - \langle \mathsf{W} \rangle_{SP(2N)} = \sum_k \frac{1}{(2N)^{2k+1}} \frac{e^{\sqrt{2\lambda}} (2\lambda)^{\frac{6k-1}{4}}}{96^k k! \sqrt{2\pi}} \left(1 - \frac{36k^2 + 144k - 5}{40\sqrt{2\lambda}} + \dots \right)$$

Perhaps the most important feature of this result is that the exponent $(6k-3)/4$ obtained in [75] is now replaced by $(6k-1)/4$.

4.2.3 Lessons for holographic orientifold backgrounds

Consider the analogue of the LLM geometries [147] in $AdS_5 \times \mathbb{RP}^5$. Let's recall briefly that LLM [147] constructed an infinite family of ten dimensional IIB supergravity solutions, corresponding to the backreaction of $1/2$ BPS states associated to chiral primary operators built out of a single chiral scalar field. These ten dimensional solutions are completely determined by a single function $u(x_1, x_2)$ of two spacetime coordinates. For regular solutions, this function can take only the values $u(x_1, x_2) = 0, 1$ defining a "black-and-white" pattern on the x_1, x_2 plane[8]. On the field theory side, the dynamics of this sector of operators of $\mathcal{N} = 4$ $SU(N)$ SYM is controlled by the matrix quantum mechanics of N fermions on a harmonic potential [155, 156]. The one-fermion phase space (q, p) gets identified with the (x_1, x_2) plane displaying the "black-and-white" pattern. In particular, the ground state of the system is given by filling the first N states of the harmonic

[8]This function $u(x_1, x_2)$ is related to the function $z(x_1, x_2)$ of the original paper [147] by $u = 1/2 - z$.

oscillator; in the one-fermion phase space, this corresponds to a circular droplet, which in turn is the pattern giving rise to the $AdS_5 \times S^5$ solution in supergravity. The fermion picture can be inferred directly from the supergravity solutions [157, 158]. What is the similar sector for $\mathcal{N} = 4$ SYM with $G = SO(N), SP(N)$?

Before addressing this question let us make an observation on the matrix models for SO and SP we encountered in 4.1.4: for all of them, as we explained in 2.4.3, the Faddeev-Popov determinant can be substituted by the square of a determinant of orthogonal polynomials. In combination with the Gaussian exponent, the square root of this determinant is (up to a normalization factor) the Slater determinant that gives the wave-function of an N-fermion state,

$$|\Psi_N(x_1, \ldots, x_N)\rangle = C \left| H_i(x_j) e^{-\frac{1}{4g}x_j^2} \right| \,. \tag{4.61}$$

Consequently, in all cases the computations we perform can be thought of as normalized matrix elements for certain N-fermion states

$$\langle \mathcal{O} \rangle_{mm} = \frac{\langle \Psi_N \,|\, \mathcal{O} \,|\, \Psi_N \rangle}{\langle \Psi_N \,|\, \Psi_N \rangle} \,, \tag{4.62}$$

where the specific $|\Psi_N\rangle$ depends on the algebra G.

Observe that equation (4.62) implies that the groundstate of the LLM sector for $SU(N)$ is precisely the N-fermion state $|\Psi_N\rangle$ that appears in the matrix model obtained from localization. This led my collaborators and me to propose in [3] that for the other classical Lie algebras, it also holds that the corresponding $|\Psi_{\mathfrak{g}}\rangle$ in eq. (4.62) is the groundstate of the fermionic system dual to the LLM sector. We can imagine starting with the matrix model for $U(2N)$, so in the ground state the fermions fill up the first 2N energy levels, and then the orientifold projects out either the even or odd parity eigenstates, depending on the gauge group we consider. The LLM sectors are certainly richer than just the groundstate: they are given by a matrix quantum mechanics that allows for excitations. Our complete proposal is that the full LLM sectors are given by *any* N fermion state built from one-fermion eigenstates of fixed parity: even parity for $SO(2N)$ and odd parity for $SO(2N+1), SP(2N)$,

$$\psi(-x) = (-1)^s \psi(x) \tag{4.63}$$

where $s = 0, 1$ depending on the gauge group. This picture is especially easy to visualize for $SO(2N+1), SP(N)$ since in these cases we are keeping odd-parity eigenstates, which are the eigenstates of an elementary problem in 1d quantum mechanics: the "half harmonic oscillator" where we place an infinite wall at the origin of a harmonic oscillator potential. This identification between the orientifold in $AdS_5 \times \mathbb{RP}^5$ and the projection from the harmonic oscillator to the half harmonic oscillator was pointed out in [148], where it was suggested to hold for any $SO(N), SP(2N)$ group. According to our argument, this identification holds for $SO(2N+1), SP(2N)$, but it does not for $SO(2N)$, since in this case the states preserved by the orientifold action are the even parity ones.

We can formalize this identification as follows. In [148] it was argued that the orientifold projection acts in the (x_1, x_2) plane of LLM geometries as $(x_1, x_2) \sim (-x_1, -x_2)$. Since the (x_1, x_2) plane is identified with the one-fermion phase space, this identification amounts to implementing a parity projection in phase space. To do so, one can define [159] the following parity operator in phase space

$$\Pi_{q,p} = \int_{-\infty}^{\infty} ds \; e^{-2ips/\hbar} \; |q-s\rangle \langle q+s| \tag{4.64}$$

and the projectors

$$P^{\pm}_{q,p} = \frac{1}{2}\left(1 \pm \Pi_{q,p}\right) \tag{4.65}$$

In particular, $\Pi_{(0,0)}$ is the parity operator about the origin of phase space: it changes $\psi(q)$ into $\psi(-q)$ and $\hat{\psi}(p)$ into $\hat{\psi}(-p)$, so the similarity with the orientifold action is apparent. The projectors $P^{\pm}_{0,0}$ project on the space of wavefunctions symmetric or antisymmetric about the origin, and the orientifold projection amounts to keeping one of these subspaces.

Going forward with the argument, we note that $s = 0, 1$ in eq. (4.63), depending on the absence or presence of discrete torsion. We want to provide a new perspective on this discrete torsion, from the phase space point of view. We start by recalling that the function $u(x_1, x_2)$ is identified with the phase space density $u(p, q)$ of one of the fermions in the system of N fermions in a harmonic potential. To go beyond a purely classical description, one can consider a number of phase space quasi-distributions that replace the phase space density, as has been discussed in the LLM context in [160, 161]. One particular such distribution is the Wigner distribution, defined as the Wigner transform of the density matrix,

$$\mathcal{W}(p,q) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} dy \; e^{2ipy/\hbar} \, \langle q-y|\hat{\rho}|q+y\rangle \tag{4.66}$$

A salient feature of Wigner quasi-distributions is that they are not positive definite functions over phase space. For instance, if we consider a given eigenstate $|n\rangle$ of the harmonic oscillator, the corresponding Wigner distribution is given again by a Laguerre function [160, 161][9]

$$\mathcal{W}_n(p,q) = \frac{(-1)^n}{\pi\hbar} L_n\left(2\frac{q^2+p^2}{\hbar}\right) e^{-\frac{q^2+p^2}{\hbar}} \tag{4.67}$$

In particular, for the eigenstate $|n\rangle$, at the origin of phase space we have

$$\mathcal{W}_n(0,0) = (-1)^n \frac{1}{\pi\hbar} \tag{4.68}$$

[9]At this time, we regard the fact that Laguerre functions appear both in the vevs of circular Wilson loops and in Wigner distributions as merely fortuitous. In particular, note that the vevs of Wilson loops have negative argument, while for Wigner distributions the argument is positive.

so it can have either sign. More generally, the Wigner quasi-distribution is the expectation value of the parity operator defined in (4.64) [159]

$$\mathcal{W}(p,q) = \frac{1}{\pi\hbar}\langle \Pi_{p,q}\rangle \tag{4.69}$$

and in particular

$$\mathcal{W}(0,0) = \frac{1}{\pi\hbar}\langle \Pi_{0,0}\rangle \tag{4.70}$$

so it is clear that the sign of $\mathcal{W}(0,0)$ captures the parity properties of the wavefunction with respect to the origin of phase space[10]. For a generic N fermion state with eigenstates $\{j_1,\ldots,j_N\}$, the Wigner function is [160, 161],

$$\mathcal{W}(p,q) = \frac{1}{\pi\hbar}e^{-(q^2+p^2)/\hbar}\sum_{\{j_i\}}(-1)^{j_i}L_{j_i}\left(\frac{2}{\hbar}(q^2+p^2)\right) \tag{4.71}$$

For $G = SO(N), SP(N)$, the sign $(-1)^{j_i}$ is the same for all states, to it comes out of the sum. In particular, for any N fermion state, at the origin of phase space we get

$$(-1)^s = \text{sign}\ \mathcal{W}(0,0) \tag{4.72}$$

[10] Incidentally, negative values of the Wigner function at the origin of phase space have apparently been measured experimentally for single photon fields [162].

Chapter 5

Diagnosing holographic supergravity

Predictions of localization for large N, large λ $\mathcal{N}=2$ SYM

In this chapter, we take a step forward and extend our attention, which was up to this point centred in $\mathcal{N}=4$ SYM, to a more general set of $\mathcal{N}=2$ conformal field theories. We will restrict ourselves to a specific set of Lagrangian conformal field theories where the techniques of supersymmetric localization can still be used to reduce the v.e.v. of the appropriate observables to a matrix model. Following [4], we will discuss how the results for this v.e.v. can be used as a diagnostic tool to characterize the corresponding holographic dual theories.

Interestingly, the family of theories we will consider includes, besides $\mathcal{N}=4$ SYM, not only different theories with plausibly semiclassical regimes in their holographic dual, but also other theories where the existence of these regimes is precluded by strong arguments, at least in any GKPW-like implementation of holography. This diversity makes the set an ideal testing ground to identify candidate encodings of the geometrical structure of the holographic dual in the field theory side.

5.1 The Lagrangian 4-dimensional $\mathcal{N}$=2 family of superconformal field theories

The Lagrangian of $\mathcal{N} = 4$ SYM can be written explicitly in terms of $\mathcal{N} = 2$ supermultiplets, as the sum of a gauge multiplet and a massless hypermultiplet in the adjoint representation (see for instance [163] for a rewriting of $\mathcal{N} = 4$ where this becomes manifest). Any different choice for the hypermultiplet representation, or the addition of a mass term for this matter supermultiplet would break exactly half of the supersymmetry, leaving us with an $\mathcal{N} = 2$ Lagrangian theory. The family of Lagrangians we can obtain this way coincides with the flat space limit of the $\mathcal{N} = 2$ Lagrangians on an S^4 that Pestun considered in [76].

However, not all the theories obtained with the mentioned substitution on the $\mathcal{N} = 4$ Lagrangian are superconformal. The coupling will be truly marginal if and only if the one-loop contribution to their β function is zero [164]. In other words, they have to satisfy

$$\sum_{\alpha\in G} (\alpha \cdot x)^2 = \sum_{\omega\in\Lambda_G} n_{\mathcal{R}}(\omega)\,(\omega \cdot x)^2 \tag{5.1}$$

Since we are interested in superconformal field theories that admit a large N limit, we restrict to nonexceptional gauge groups. Also, the positivity of all summands in (5.1) leads to the conclusion that these families can only include matter content in representations with up to two indices: fundamental, 2-symmetric, 2-antisymmetric and adjoint. The complete list of such theories is well-known [165], and we present it in table 5.1.

The quantity

$$\nu \equiv \lim_{N\to\infty} \frac{n_f}{2N} \tag{5.2}$$

has a very interesting role to play in what follows. It counts what fraction of the matter in these theories belongs to the fundamental representation in the large N limit. Observe in table 5.1 that it can only take the values $\nu = 0, 1/2, 1$. In the next section we will observe that theories with a semiclassical dual are expected to reside in the $\nu = 0$ subset. In section 5.3 we will find that this quantity ν becomes the label that classifies the three qualitatively different behaviours that we will meet at large N and λ.

$SU(N)$				
$(n_{adj}, n_f, n_{S_2}, n_{A_2})$	c	a	$\delta \equiv (c-a)/c$	ν
$(1,0,0,0)$	$\frac{1}{4}N^2 - \frac{1}{4}$	$\frac{1}{4}N^2 - \frac{1}{4}$	0	0
$(0,0,1,1)$	$\frac{1}{4}N^2 - \frac{1}{6}$	$\frac{1}{4}N^2 - \frac{5}{24}$	$\frac{1}{6N^2} + \mathcal{O}(N^{-4})$	0
$(0,4,0,2)$	$\frac{1}{4}N(N+1) - \frac{1}{6}$	$\frac{1}{4}N(N+\frac{1}{2}) - \frac{5}{24}$	$\frac{1}{2N} + \mathcal{O}(N^{-2})$	0
$(0,2N,0,0)$	$\frac{1}{3}N^2 - \frac{1}{6}$	$\frac{7}{24}N^2 - \frac{5}{24}$	$\frac{1}{8} + \mathcal{O}(N^{-2})$	1
$(0,N+2,0,1)$	$\frac{7}{24}N^2 + \frac{1}{8}N - \frac{1}{6}$	$\frac{13}{48}N^2 + \frac{1}{16}N - \frac{5}{24}$	$\frac{1}{14} + \mathcal{O}(N^{-1})$	$\frac{1}{2}$
$(0,N-2,1,0)$	$\frac{7}{24}N^2 - \frac{1}{8}N - \frac{1}{6}$	$\frac{13}{48}N^2 - \frac{1}{16}N - \frac{5}{24}$	$\frac{1}{14} + \mathcal{O}(N^{-1})$	$\frac{1}{2}$

$SO(2N)$				
(n_{adj}, n_f, n_{S_2})	c	a	$\delta \equiv (c-a)/c$	ν
$(1,0,0)$	$\frac{1}{2}N^2 - \frac{1}{4}N$	$\frac{1}{2}N^2 - \frac{1}{4}N$	0	0
$(0,2N-2,0)$	$\frac{2}{3}N^2 - \frac{1}{2}N$	$\frac{7}{12}N^2 - \frac{3}{8}N$	$\frac{1}{8} - \frac{3}{32N} + \mathcal{O}(N^{-2})$	1

$SO(2N+1)$				
(n_{adj}, n_f, n_{S_2})	c	a	$\delta \equiv (c-a)/c$	ν
$(1,0,0)$	$\frac{1}{2}N^2 + \frac{1}{4}N$	$\frac{1}{2}N^2 + \frac{1}{4}N$	0	0
$(0,2N-1,0)$	$\frac{2}{3}N^2 + \frac{1}{6}N - \frac{1}{12}$	$\frac{7}{12}N^2 + \frac{5}{24}N - \frac{1}{24}$	$\frac{1}{8} - \frac{3}{32N} + \mathcal{O}(N^{-2})$	1

$Sp(2N)$				
(n_{adj}, n_f, n_{A_2})	c	a	$\delta \equiv (c-a)/c$	ν
$(1,0,0)$	$\frac{1}{2}N^2 + \frac{1}{4}N$	$\frac{1}{2}N^2 + \frac{1}{4}N$	0	0
$(0,4,1)$	$\frac{1}{2}N^2 + \frac{3}{4}N - \frac{1}{12}$	$\frac{1}{2}N^2 + \frac{1}{2}N - \frac{1}{24}$	$\frac{1}{2N} + \mathcal{O}(N^{-2})$	0
$(0,2N+2,0)$	$\frac{2}{3}N^2 + \frac{1}{2}N$	$\frac{7}{12}N^2 + \frac{3}{8}N$	$\frac{1}{8} + \frac{3}{32N} + \mathcal{O}(N^{-2})$	1

Table 5.1: List of 4d $\mathcal{N} = 2$ SCFT families admitting a large N limit for each nonexceptional simple Lie algebra. For later reference, the central charges a and c have been explicited for every construction, together with their combination δ.

5.2 Central charges of 4-dimensional superconformal field theories

Before analysing the Matrix models of the theories presented in the previous section, let me digress on why any qualitative difference observed in the exact results of localization and correlated with the classes labelled by ν may be of interest for us. An essential role in this discussion will be played by the 4-dimensional central charges a and c, so let me begin by revisiting their field theoretical origin and holographic implementation.

In even dimensional conformal field theories with curved backgrounds the simultaneous conservation of Weyl symmetry and diffeomorphism invariance is generically not consistent at the quantum level regardless of the chosen renormalization scheme. Notice that we already encountered this clash in 2.1.2 for $d = 2$.

Choosing to keep diffeomorphism invariance, the resulting Weyl anomaly is encoded [166] in the response of the effective action for a specific background $W\left[g_{\mu\nu}\right]$ (i.e., the effective action obtained integrating out all the physics in the chosen background) to the infinitesimal change $\delta g_{\mu\nu} = g_{\mu\nu}\phi\left(x\right)$, $\mathcal{A} \equiv \delta W/\delta\phi$. This quantity can be organized in the following three types of contribution to the anomaly:

- The parity-even topologic invariants of the background metric, which come from scale-independent terms in the effective action, and therefore satisfy $\int d^d x \mathcal{A} = 0$. In practice the only available candidate to this contribution is the Euler density E_d. The form and properties of this type of anomaly are completely analogous to those of the chiral anomaly.
- The contributions arising from geometric conformal invariants I_d whose effective action contains a scale, with $\int d^d x \mathcal{A} \neq 0$.
- Total derivative terms, which are not significant in practice, since they can be cancelled with the addition of local counterterms to the effective action.

Alternatively, the Weyl anomaly can be classified in terms of the cohomological characterization of the Weyl 1-cocycles [167].

In four dimensional CFTs, anomalies of the the second type are uniquely sourced by the square of the Weyl curvature. Therefore, for these theories (see for instance [168]),

$$\left\langle T^{\mu}_{\mu}\right\rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 \,, \tag{5.3}$$

where

$$\left\{ \begin{array}{l} E_4 = R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \\ I_4 = R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \end{array} \right\} . \tag{5.4}$$

The algebraic quantities a and c are known as the *central charges* of the 4-dimensional CFT. They are the 4-dimensional analogues of the central charge of the Virasoro algebra governing the 2-dimensional conformal anomaly.

Consistent quantum field theories cannot have arbitrary positive values for a and c. Physical constraints on the energy flux at infinity [169], or, equivalently, on the deep inelastic scattering amplitudes [170], or causality [171], impose that the central charges must lie at a wedge of the positive quadrant of the a, c plane. For generic conformal field theories,

$$\frac{31}{18} \geq \frac{a}{c} \geq 13 \,, \tag{5.5}$$

though for superconformal theories this locus is further reduced to the interval

$$\frac{3}{2} \geq \frac{a}{c} \geq 12 \,. \tag{5.6}$$

Observe that all the theories considered in table 5.1 explore only the $c > a$ region within this wedge.

How does conformal anomaly manifest itself in the holographic dual? This question was first addressed by Henningson and Skenderis in [172]. According the GKPW prescription the behaviour of the near-boundary metric is constrained by the appropriate conformal structure on the boundary. It is possible to expand the near-boundary behaviour in the gauge of Fefferman and Graham [173], which defines radial slices preserving this conformal structure. The UV regularization of the theory is implemented by restricting the path integral of the gravitational theory to exclude the contributions at depths lower than a certain $\rho = \epsilon$ in this slicing. The tight relation between the RG flow and the conformal anomaly allows then to relate the terms in the Fefferman-Graham expansion to the central charges of the theory. In [172] this analysis was restricted to a gravitational dual whose action has two derivatives (i.e. Einstein-Hilbert in the gravitational sector). The result one obtains in this analysis is that in the large N limit the central charges must satisfy

$$c, a \gg 1 \,, \qquad c - a = 0 + \mathcal{O}(1/N) \,. \tag{5.7}$$

If one relaxes the requirement that the gravitational action involves just two derivatives, and requires only a sensible higher derivative expansion, it is always possible to redefine the fields [174] so that the effective action at the order of the first correction reads

$$S = \frac{1}{2\ell_P^3} \int d^5x \sqrt{-g} \left[\frac{12}{L^2} + R + \alpha R_{abcd} R^{abcd} + \dots \right] \,. \tag{5.8}$$

The methods of Henninson and Skenderis determine that in this case the central charges satisfy

$$\begin{aligned} \alpha &= \tfrac{1}{8}\tfrac{c-a}{c} \\ \tfrac{L^3}{\ell_P^3} &= \tfrac{c}{\pi^2}\left(1-3\alpha\right) \end{aligned} \tag{5.9}$$

and therefore the constraint (5.7) on the large N value of the central charges is weakened to [174]

$$c, a \gg 1\,, \qquad \frac{|c-a|}{c} \ll 1\,. \tag{5.10}$$

Going through the list of theories considered in table 5.1, we observe that this condition is satisfied precisely by the $\nu = 0$ theories.

In other words, $\nu = 0$ is a direct symptom of a gravitational description with a sensible higher derivative expansion, and any particularity developped by the field theories with this property provides a hint on where to look for the encoding of the dual semiclassical geometry. In the remaining sections in this chapter we will show that having a Wigner eigenvalue density is one of these putative consequences of semiclassical gravity.

5.3 Saddle-point equation for the localized partition function

As in the previous chapters, half-BPS circular Wilson loops can be evaluated for the set of theories we are considering using the matrix model derived in [76]. Nonetheless, in this case we have to deal with an ingredient that was so far absent in our calculations: For generic $\mathcal{N}=2$ superconformal field theories the 1-loop determinant is no longer trivial. We will still assume, as it is costumary, that the contribution of instantons is negligible in the large N limit.

For convenience, let us collect here the expressions from 2.4 that summarize how to determine the circular Wilson loop v.e.v. using the localization technique in the large N limit of $\mathcal{N}=2$ superconformal Lagrangian field theories. We will write them in the gauge of the Cartan subalgebra

$$\left\{\begin{array}{c} \mathcal{Z}_{S^4} \propto \int d^N x \Delta_{FP}\left(x\right) \mathcal{Z}_{\text{1-loop}}\left(x\right) e^{-8\pi^2 (x,x) N/\lambda} \cancel{\mathcal{Z}_{\text{inst}}} \xrightarrow{} \text{Large } N \\ \Delta\left(x\right) = \prod_{\alpha\in\Lambda_R(G)} (\alpha\cdot x)^2 \\ \mathcal{Z}_{1-\text{loop}}\left(x\right) = \prod_{\alpha\in\Lambda_R(G)} H\left(\alpha\cdot x\right) / \prod_{\omega\in\Lambda_\omega(G)} \left(H\left(\omega\cdot x\right)\right)^{n_{\mathcal{R}}(\omega)} \\ H\left(x\right) = \prod_{n=1}^{\infty} \left(\left(1+\frac{x^2}{n^2}\right)^n e^{-x^2/n}\right) \\ \mathsf{W}_{\mathcal{R}}\left(x\right) = \frac{1}{\dim \mathcal{R}} \text{tr} e^{2\pi x} \end{array}\right\}, \tag{5.11}$$

where (,) denotes the bilinear form obtained from tracing the product in the fundamental representation. It is worth pointing out that the finiteness of the $H\left(x\right)$ factors we obtain for the 1-loop determinant in superconformal theories is a consequence of the artificial introduction of the exponential factors in its definition. These factors cancel each other in the quocient by virtue of equation 5.1.

We now proceed to derive the saddle-point equation for the partition function of these matrix models. Following the standard procedure described in 2.4.3, we bring the Faddeev-Popov and one-loop factors to the exponent. The products over roots and weights become sums of logarithms, and therefore it is convenient to introduce

$$K(x) = -\frac{d\,\log H(x)}{d\,x}\,. \tag{5.12}$$

Since $H(x)$ is an even function under $x \to -x$, $K(x)$ is odd. In the large N limit we will use the continuum approximation (2.76), defined in the interval $\Gamma = [-\mu, \mu]$ and unit normalized.

Notice that the integral saddles obtained from the method described in the previous paragraph will be valid to evaluate $\mathsf{W}_{\mathcal{R}}$ in the large N limit as long as its rank is not comparable to N^2, which is the order that dominates the partition function saddle point equation. This is precisely the rank at which the holographic dual object is expected to become strongly backreacting [147].

It is straightforward to write down an integral equation for the saddle eigenvalue density for each $\mathcal{N}$=2 SCFT. We are now going to argue that all these integral equations can be written in a unified fashion.

Let us first analyse the terms with origin in the Faddeev-Popov determinant. In the continuum limit,

$$\Delta(x) = \begin{cases} \exp\left(\iint_{x\neq y} dx dy \rho(x)\rho(y)\log|x-y|\right) & \text{if } G{=}SU(N) \\ \exp\left(\iint_{x\neq y} dx dy \rho(x)\rho(y)\log|x^2-y^2| + \mathcal{O}(Nx)\right) & \text{if } G{=}SO(N) \text{ or } SP(N) \end{cases} \tag{5.13}$$

Naively, one could expect very different contributions to the integral equation from these two types of determinants. However,

$$\int_{-\mu}^{\mu} dy \rho(y) \frac{2x}{x^2-y^2} = \int_{-\mu}^{\mu} dy\rho(y)\frac{2}{x-y} - \underbrace{\int_{-\mu}^{\mu} dy\rho(y)\frac{2y}{(x-y)(x+y)}}_{\to 0}, \tag{5.14}$$

so the Faddeev-Popov contribution to the Kernel turns out to be the same for all nonexceptional simple algebras, up to a factor of two. A factor of two will be generated as well in the term with λ^{-1} for these cases because the trace in the fundamental representation includes both $\pm a_i$ weights for $SO(N)$ and $SP(N)$.

What about the contributions of the 1-loop determinant? It is easy to keep track of them in the saddle point equation in terms of the $K(x)$ function defined in (5.12). Notice that the $x \leftrightarrow -x$ symmetry in the problem makes $K(x-y)$ and $K(x+y)$ equivalent under $\int dy\rho(y)$ (the parity of the eigenvalue distribution can be argued from physical principles or proven combining the saddle point equation with the one obtained changing $x \leftrightarrow -x$ and redefining $y \leftrightarrow -y$).

A straightforward analysis shows that for all the superconformal field theories of table 5.1 the singular integral equation that determines the eigenvalue distribution is

$$\int_{-\mu}^{\mu} dy\rho(y)\left(\frac{1}{x-y} - \nu K(x-y)\right) = \frac{8\pi^2}{\lambda}x - \nu K(x), \tag{5.15}$$

Let us illustrate the argument for $SU(N)$ with a hypermultiplet in the antisymmetric representation and $N+2$ hypermultiplets in the fundamental one. For this case we obtain

$$\int_{-\mu}^{\mu} dy\rho(y)\left(\frac{1}{x-y} - K(x-y) + \frac{1}{2}K(x+y)\right) = \frac{8\pi^2}{\lambda}x - \frac{1}{2}K(x), \tag{5.16}$$

where the terms inside the parenthesis in the integral come respectively from the Faddeev-Popov determinant, the vector multiplet contribution and the hypermultiplet in the antisymmetric representation. The $K(x)$ term on the right hand side corresponds to the hypermultiplets in the fundamental representation. This equation is consistent with (5.15) with $\nu = 1/2$.

Equation (5.15) has appeared previously in the literature for specific values of ν. For $\nu = 0$ it coincides of course with the integral equation for the Wigner distribution, while for $\nu = 1$ this equation was derived in [175] for the particular case of $\mathcal{N} = 2$ SQCD. To my knowledge, $\nu = \frac{1}{2}$ had not been explored prior to [4]

5.4 Saddle eigenvalue distributions for the partition function

In this section, we will revisit the methods used in [175, 176] to resolve equation (5.15) for $\nu = 1$. We will explicitly extend them to address generic ν, to successfully apply them to $\nu = 1/2$.

5.4.1 Infinite coupling limit

Let us first consider the strict limit $\frac{1}{\lambda} \to 0$, $\mu \to \infty$, where equation (5.15) reduces to

$$\int_{-\infty}^{\infty} dy \rho(y) \left(\frac{1}{x-y} - \nu K(x-y) \right) = -\nu K(x) \,. \tag{5.17}$$

This equation can be solved analytically for $\nu \neq 0$. Taking its Fourier transform we arrive at

$$\hat{\rho}_\infty(p) = \frac{1}{1 + \frac{2}{\nu} \sinh^2 \frac{p}{2}} \,, \tag{5.18}$$

which implies

$$\rho_\infty(x) = \frac{1}{\sqrt{\frac{2}{\nu} - 1}} \frac{\sinh\left((\pi - \theta)x\right)}{\sinh \pi x} \,, \tag{5.19}$$

with

$$\theta = \cos^{-1}(1 - \nu) \,. \tag{5.20}$$

This result is just a slight generalization of the $\nu = 1$ case, already obtained in [175].

5.4.2 Strong coupling

At finite coupling, there is, to my knowledge, no technique that allows to solve generally the saddle-point equation, (5.15). For finite but strong 't Hooft coupling, $\lambda \gg 1$, there are a couple of works in the literature using different approximations to solve this equation. We will follow [175] and also briefly comment on the approximation used in [176].

The first approach we will consider to solve this equation approximately will closely follow [175], and it is based in the Wiener-Hopf method. Our computations will only differ in the treatment of the zero-momentum mode.

Given the integral equation (5.15), one might be tempted to solve it via a Fourier transform, after extending the definition of $\rho(x)$ to be zero outside its support, $[-\mu, \mu]$.

This idea cannot be implemented to (5.15) as it stands, since the Fourier transforms of $K(x)$ and x are divergent. To arrive at an equation amenable to be Fourier transformed, we follow [175] and make use of the integral operator $\mathcal{P}^{-1}_{x\to z}$ we defined in (2.81). Its action onto (5.15) leads to

$$\rho(z) - \frac{8\pi}{\lambda}\sqrt{\mu^2 - z^2} - \nu \int_{-\mu}^{\mu} dy \rho(y) \left(f(y,z) - f(0,z)\right) = 0; \; z \in [-\mu, \mu] \tag{5.21}$$

with

$$f(y,z) \equiv f(y,z) \equiv \mathcal{P}^{-1}_{x\to z}\left[K(x-y)\right] = -\frac{1}{\pi^2} \fint_{-\mu}^{\mu} \frac{dx}{z-x}\sqrt{\frac{\mu^2 - z^2}{\mu^2 - x^2}} K(x-y) \tag{5.22}$$

Observe that the kernel does not only depend on the difference $z - y$ anymore, so the use of Fourier transformation would lead now to more involved integral expressions. We observe nonetheless that by virtue of the symmetry $y \leftrightarrow -y$ the result (5.21)) will remain valid if we use

$$\hat{f}(y,z) \equiv \mathcal{P}^{-1}_{x\to z}\left[\fint_{-\infty}^{\infty} \frac{\omega \coth(\pi\omega)}{x - y - \omega}\right] = (z-y)\coth\left(\pi(z-y)\right) + \delta\hat{f}(y,z) \tag{5.23}$$

in place of $f(y,z)$. The advantage in this replacement is that the Fourier transform of the term $\delta\hat{f}(y,z)$ can be argued to be small, and therefore subdominant in the saddle point equation. This endows us with the possibility of solving the equation iteratively, using at each step the distribution obtained in the previous iteration to improve the estimate on the term that contains $\delta\hat{f}$. For our purposes the first step of the algorithm suffices, where this subleading term is fully neglected.

Once we have reformulated the original equation in this fashion, we are finally ready to apply the Wiener-Hopf method. The first step is to extend the definition of the eigenvalue density $\rho(y)$, outside the interval $[-\mu, \mu]$, by defining $\rho(y) = 0$ outside this interval. This is compatible with analytic methods for $\rho(y)$ as long as it is understood that $\rho(y)$ admits a branch cut outside the domain of integration and we are taking the ill-defined values on it as

$$\rho(|x| > \mu) = \frac{1}{2}\lim_{\epsilon\to 0}\left(\rho(x + i\epsilon) + \rho(x - i\epsilon)\right) . \tag{5.24}$$

Provided that we take the Fourier transform of the eigenvalue density with this prescription, we obtain

$$\begin{cases} \int_{-\infty}^{\infty} e^{-ipz}\left(\hat{\rho}(p)\left(1 + \nu\left(2\sinh^2\frac{p}{2}\right)^{-1}\right) - F(p)\right) = 0; & z \in [-\mu, \mu] \\ \int_{-\infty}^{\infty} e^{-ipz}\hat{\rho}(p) = 0; & z \notin [-\mu, \mu] \end{cases} \tag{5.25}$$

with

$$F(p) \equiv 8\pi^2 \mu \frac{J_1(\mu p)}{\lambda p} + \frac{\nu}{2\sinh^2 \frac{p}{2}} + \dots \,, \tag{5.26}$$

where the dots make reference to the terms coming from $\delta \hat{f}$ that we are neglecting. The general solution for the Fourier transform of the eigenvalue density should consequently be of the form

$$\hat{\rho}(p) \left(1 + \frac{\nu}{2\sinh^2 \frac{p}{2}}\right) = F(p) - \chi_-(p) - \chi_+(p) \,, \tag{5.27}$$

where the functions $\chi_\pm$ in the position space are nonvanishing in the real line only on one side of $|x| > \mu$ each. Their exact expressions can be determined from analiticity constraints in momentum space.

In order to impose those constraints we should pause our calculation for a moment to focus on the analytic structure of

$$1 + \frac{\nu}{2\sinh^2 \frac{p}{2}} \,. \tag{5.28}$$

This function does have double poles at $p = 2\pi n i$ and simple zeroes at $p = 2\pi n i \pm \theta$ with θ defined in eq. (5.20). This analytical structure is reproduced in the upper complex semiplane by the function

$$G_+(p) \equiv \frac{p^2 \Gamma\left(1 + \frac{\theta - ip}{2\pi}\right) \Gamma\left(1 - \frac{\theta + ip}{2\pi}\right)}{(p + i\theta)\, \Gamma^2\left(1 - \frac{ip}{2\pi}\right)} \,. \tag{5.29}$$

It will turn out to be very convenient to split

$$1 + \frac{\nu}{2\sinh^2 \frac{p}{2}} \equiv \frac{1}{G_+(p)\, G_-(p)} \tag{5.30}$$

because the constructions

$$C_+ = \frac{\hat{\rho}(p)\, e^{-ip\mu}}{G_-(p)}; \; C_- = \frac{p^2 \hat{\rho}(p)\, e^{ip\mu}}{G_+(p)}; \tag{5.31}$$

are either totally annihilated or left invariant by the action of $\int_{-\infty}^{\infty} (2\pi i)^{-1} (p - p_0 \pm i\epsilon)^{-1}$ operators. For later reference, we define

$$R_\alpha \equiv \mathrm{Res}\,(G_+, \alpha)\,; \; \tilde{R}_\beta \equiv \mathrm{Res}\,(G_-, \beta) \,. \tag{5.32}$$

We can straightforwardly read expressions for $\chi_\pm$ from the aforementioned projections of (5.31). We obtain

$$\hat{\rho}(p) = \frac{2\sinh^2\frac{p}{2}}{2\sinh^2\frac{p}{2}+\nu}\left(F(p) - \frac{e^{ip\mu}}{G_+(p)}\sum_{\alpha\,\in\text{poles }G_+}\frac{e^{-i\alpha\mu}F(\alpha)\,R_\alpha}{p-\alpha}\right) + \mathcal{O}\left(e^{-ip\mu}\right) \tag{5.33}$$

This expression is only useful to obtain $\rho(x)$ at $x \gg -\mu$, but this covers our needs in this case because of the $x \leftrightarrow -x$ symmetry.

The normalization condition can be applied in the momentum space as

$$1 = 2\int_0^\infty dx\rho(x) = \lim_{\epsilon\to 0}\frac{1}{i\pi}\int_{-\infty}^\infty dp\frac{\hat{\rho}(p)}{p-i\epsilon} \tag{5.34}$$

Using integrals in the $\mathbb{C}$ plane (and in particular comparing how the results change depending on whether the integration circuit is closed by a semicircle of infinite radius above or below the real line), it is straightforward to argue that the term with $F(p)$ in (5.33) already produces the 1 needed to match the left hand side. Then, what remains must evaluate to zero:

$$0 = \sum_{\substack{\alpha\,\in\text{ poles }G_+\\ \beta\,\in\text{ poles }G_-}}\frac{e^{-i(\alpha-\beta)\mu}F(\alpha)\,R_\alpha\tilde{R}_\beta}{\beta-\alpha} \tag{5.35}$$

Observe that $F(\alpha)$ has an exponential contribution that makes all α poles equally important, but the sum in β will be dominated by the pole at $\beta = i\theta$. Keeping only this dominant contribution and using asymptotic expressions for the Bessel functions in $F(p)$ we obtain an equation for the dependence $\mu(\lambda)$, which at large λ can be summarized as

$$\theta\mu = \ln\lambda - \frac{1}{2}\ln\mu + \mathcal{O}(1) \tag{5.36}$$

The expression for the eigenvalue density (in momentum space), eq. (5.33) together with the normalization (5.36) are the main result of this section.

Before we put these results to work, let's briefly comment on a different approximation to solve the saddle-point eq. (5.15). In [176], Bourgine solved (5.15) by truncating the expansion of $K(x)$ and keeping only the first terms in a large x expansion,

$$K(x) \to K_{sc}(x) = 2x\ln|x| + 2\gamma x + \frac{1}{6x} \tag{5.37}$$

This truncation simplifies the computation enormously, compared with the method we just described. As explained in [176], when computing the vev of the Wilson loop, it works remarkably well in capturing the exponent, but not so well with the prefactor. For the sake of comparison, the expressions work out to be the same, with the replacement

$$\theta_B = \sqrt{\frac{2\nu}{1 - \frac{\nu}{6}}} \tag{5.38}$$

Remarkably, this expression differs from $\theta = \cos^{-1}(1-\nu)$ in less than 1,8% in the range $0 \leq \nu \leq 1$. Presumably, keeping further terms in the large x expansion of $K(x)$ would improve the agreement of these two methods. Nevertheless, we will stick to the results obtained by the first method, since they capture exactly the exponent in the power law dependence of $\langle \mathsf{W} \rangle$.

5.5 Bremsstrahlung coefficients: a reprise

In this section we put to use the eigenvalue densities found in the previous section, by computing the quantities h_{W} and f_{W} for fundamental Wilson loops in the corresponding theories. As we anticipated in 3.2, these two coefficients are no longer directly related in generic $\mathcal{N}=2$ SYM theories. The relations relating the Bremsstrahlung coefficient B to h_{W} and κ in 3.1 remain valid, as well as the argument of [4] relating f_W to the dependence of the circular Wilson loop v.e.v. on the coupling that was introduced in 3.2. We can therefore compute f_{W} as

$$f_{\mathsf{W}} = \frac{1}{8\pi^2}\lambda\partial_\lambda \ln \langle \mathsf{W} \rangle \tag{5.39}$$

However, let me reiterate that the first equality in (3.17), namely, the proportionality between $B_{\mathcal{R}}$ and f_W, need not hold anymore for the theories we are now considering.

It turns out, nonetheless, that it is still possible to evaluate h_{W} from the results of the matrix model of localization. It was recently conjectured [93] that for $\mathcal{N}=2$ SCFTs, this coefficient can be related to the vev of a circular Wilson loop in a squashed four-sphere S^4_b [177, 178], since varying the squashing parameter will insert the stress-energy tensor,

$$h_{\mathsf{W}} = \frac{1}{12\pi^2}\partial_b \ln \langle \mathsf{W}_b \rangle \,|_{b=1} \,. \tag{5.40}$$

Furthermore, it was argued in [93] that this computation can be carried out by just inserting W_b in the matrix model for S^4,

$$\langle \mathsf{W}_b \rangle = \int dx\, e^{2\pi b x} \rho(x)\,. \tag{5.41}$$

which is a computation we can readily perform using the results derived in the previous section.

When $\nu = 0$, the eigenvalue density follows the semi-circle law (2.83), and the vev of the Wilson loop displays exponential growth [58, 74]. It is easy to see that this asymptotic behaviour extends to the operator W_b. In turn, the results at $\nu \neq 0$ can be obtained directly from the eigenvalue distribution in momentum space (5.33), using

$$\langle W_b \rangle = \hat{\rho}\left(-2\pi b i\right) \ , \tag{5.42}$$

and the leading term in (5.36). The results obtained both at $\nu = 0$ and $\nu \neq 0$ are summarized in table 5.2.

Notice some interesting features of this result:

- It is amusing that for the two values of ν realized by large N Lagrangian $\mathcal{N}=2$ CFTs, $\nu = 1/2$ and $\nu = 1$, the exponent in the power law dependence of $\langle \mathsf{W} \rangle$

happens to be given by integers. In particular,

$$\frac{2\pi}{\theta} - 1 = \begin{cases} 5; & \nu = 1/2 \\ 3; & \nu = 1 \end{cases} \tag{5.43}$$

I ignore whether there is any deeper reason behind this observation.

- Another suggestive property of these results is the fact that f_W is independent of λ. From its definition (2.48), we can interpret this coefficient as giving the strength of the fields sourced by a static probe; our computation implies that for superconformal theories with matter in the fundamental representation, this strength reaches a limiting value in the large N, large λ limit.
- Notice that for generic $\mathcal{N} = 2$ theories, the λ dependence of the coefficients h_{W} and f_{W} is different. This should not come as a surprise, since for $\mathcal{N} = 2$ theories (unlike what happens in $\mathcal{N} = 4$ SYM) the Lagrangian density and the stress-energy tensor don't belong to the same supermultiplet.

We are now in position of computing the Bremsstrahlung function for fundamental loops in $\nu \neq 0$ theories. Granting that the conjectured relation [91, 93]

$$B = 3h_W\,. \tag{5.44}$$

is true, we conclude that

$$B = \frac{1}{2\pi\theta}\ln\lambda\,. \tag{5.45}$$

One lesson of this result is the following. It has been argued in [179, 180] that a certain class of observables of planar $\mathcal{N} = 2$ superconformal gauge theories can be obtained from the corresponding result of planar $\mathcal{N} = 4$ SYM, by means of replacing the $\mathcal{N} = 4$

Quantity	$\nu = 0$	$\nu \neq 0$
$\langle \mathsf{W} \rangle$	$\sim \lambda^{-3/4} \exp\sqrt{\lambda}$	$\sim \lambda^{\frac{2\pi}{\theta}-1}$
$\ln \langle \mathsf{W} \rangle_b$	$\sim b\sqrt{\lambda}$	$\sim \ln\lambda\left(\frac{2\pi b}{\theta} - 1\right) + \mathcal{O}\left((1-b)^2\right)$
f_{W}	$\frac{\sqrt{\lambda}}{16\pi^2}$	$\frac{1}{8\pi^2}\left(\frac{2\pi}{\theta} - 1\right)$
h_{W}	$\frac{\sqrt{\lambda}}{12\pi^2}$	$\frac{1}{6\pi\theta}\ln\lambda$

Table 5.2: Dependence of W, W_b, and the coefficients h_{W} and f_{W} that we can infer from them for Wilson loops of fundamental representation in $\mathcal{N} = 2$ superconformal Lagrangian theories. The results displayed in this table are only guaranteed to be valid at the large N, large λ limit.

coupling by a single function, universal for a given $\mathcal{N}$=2 SCFT. Comparing the results we have obtained for $\langle W \rangle$ and B for $\mathcal{N}$=4 and $\mathcal{N}$=2 theories, we conclude that this substitution rule does not apply to the computation of B, for theories with a single gauge group.

Finally, we can use our result for h_W to compute the additional entanglement entropy of a spherical region when we add a external probe to the vacuum of the theory. According to [91] it is given by

$$S = \ln \langle W \rangle - 8\pi^2 h_W \,, \tag{5.46}$$

so for the probes we are considering we have

$$S = \left(\frac{2\pi}{3\theta} - 1 \right) \ln \lambda \,. \tag{5.47}$$

Chapter 6

Conclusion

The main purpose of this dissertation has been to put forward new potential uses for the exact predictions of supersymmetric localization, and, more precisely, to determine in which manner they can be helpful in the refinement of the AdS/CFT correspondence. In particular our attention has been centred on a very specific set of quantities: 1-object correlators of half-BPS Wilson loop observables in theories with simple gauge algebras. Even within this narrow set of operators, localization has proven itself a surprisingly versatile tool, which conveys information for holography in numerous manners and covers a widespread range of interests. Let me recollect in what follows the most remarkable findings of this thesis in this regard.

One of the most fertile grounds for learning about holography, at least as far as the comparison to exact results is concerned, is precisely the regime where the duality is better understood, namely, $\mathcal{N} = 4$ SYM theory at large N and λ. The outcome of supersymmetric localization has already been vehemently exploited in the literature to check holographic predictions in this regime. Subleading corrections for the field theoretic observables are known to be efficiently replicated in the dual supergravity construction with the help and D3 brane probes in several examples. The coincidence between the holographic and field theoretic predictions for momentum fluctuation observed in chapter 3 provides an independent new entry to the list of these nontrivial holographic checks. Notably, this is also the regime where the explicit identification of the matrix model combinatorics with the fermionic phase space of the LLM description presented in 4 takes place: the bubbling geometry construction is in principle restricted to regular semiclassical spacetimes.

Another interesting insight comes from the comparison of the results in this regime with those of the equivalent limit in $\mathcal{N} = 2$ Lagrangian field theories without a semiclassical holographic dual. The qualitative differences observed in 5 between these two types of theories suggest that the encoding of semiclassical geometry in field theory is related to efficient spreading of the eigenvalue distribution at the strong coupling regime: The absence of a plausible semiclassical dual happens in these examples whenever the eigenvalue distribution asymptotes a limiting form, with finite density in certain zones. This result is in consonance with several observations made in the context of bubbling geometries explicitly relating the position of certain elements of the theory (in particular giant gravitons) to certain patterns in the dominant eigenvalue saddle (the presence of *vacancies* or *outliers* in the distribution). In addition, the structure at $\lambda \to \infty$ of the eigenvalue distribution repercusses directly in the functional λ dependence of the Wilson loop v.e.v.s; observables that are known to capture important geometrical features of the the holographic dual of the field theory when the latter is semiclassical. We should be cautious around this point, and avoid jumping into precipitated conclusions: correlation does not imply causality and the classification of theories proposed in 5 is certainly too coarse to univoquely flag semiclassical duals[1]. However, in my opinion, it is fair to

[1] In other words, adding a significant amount of fundamental matter is not the only possible way of breaking the semiclassical structure of the holographic dual of a Lagrangian $\mathcal{N} = 2$ SYM theory. None of the theories considered admits a semiclassical dual at finite λ, for instance, but they might still fall into the $\nu = 0$ category.

assess that the saddle eigenvalue distribution of localizable theories is one of the most interesting suspects to displaying an explicit encoding of the geometrical structure arising in their AdS dual.

At finite N and λ, our control over the holographic side of the conjecture drops significantly. An interesting issue in this limit is up to which point the parametric dependences on these variables that the localization technique reveal for certain observables can be extrapolated to less supersymmetric objects. This extension would allow us with the possibility to squeeze information about finite N and λ from the holographic predictions at the limit of semiclassical holography (when it exists). This idea was briefly explored in the last section of 3. Although the discussion is full of subtleties and it is hopeless to expect answers for arbitrary N and λ, our discussion shows that under some considerations this path can provide qualitatively reasonable answers. A complementary question we can ask is what can be said about the structure of a known semiclassical dual when we reduce the rank of the gauge group to a finite N. Two observations made in 4 are very interesting in this regard. On the one hand, the structure observed for the fundamental Wilson loops relates the first subleading ($\sim 1/N$) term in the 't Hooft expansion of the field theory when the gauge group is SO or SP with the planar sector of U. Indeed, when this topological expansion is observed as a perturbative string theory series the results of 4 become of special interest. Although it is conceivable that they could partially be deduced in string theory by symmetry arguments, they open a window into regimes where the worldsheet calculations are currently well out of reach. On the other hand, the fact that the fermionic wavefunction-like structure in the matrix model remains intact at any finite N for localizable observables in $\mathcal{N} = 4$ SYM theories is, in my opinion, highly suggestive: it suggests that certain elements of the bubbling geometry picture remain intact when spacetime becomes a quantum gravity.

Aside from all these considerations, this thesis have also produced interesting results at a more pragmatic level. A central role in this pages has been played by Bremsstrahlung coefficients, which have been considered for a large assortment of particle types in diverse theories. The versatility of these quantities (restricted, I should admit, to the linear response of the particle; they fail to account for the nonabelian broadening of the emitted radiation), has been extensively discussed in 3, where an element was appended to the set of their applications: their relation to the momentum diffusion coefficient for a particle moving through a nontrivial background.

Chapter 7

Summary in Catalan
Resum en Català

La conjectura de Maldacena, anomenada sovint dualitat hologràfica o correspondència AdS/CFT, proposa una equivalència entre teories gravitatòries en un espai hiperbòlic d'una certa dimensió i teories de gauge en un espai de dimensionalitat inferior, que, en un cert sentit, podem pensar que viuen al contorn de les primeres. Aquesta relació ve motivada per l'estudi, en el marc de la teoria de cordes, d'una certa generalització dels forats negres a espais de dimensió més alta que es coneix com a brana negra. En règims diferents, els dos caps que la correspondència connecta apareixen com a descripció física del comportament d'aquest mateix objecte a baixa energia. Hom espera doncs (i nombrosos indicis físics semblen confirmar-ho) que cada una d'aquestes dues teories sigui la descripció efectiva apropiada de l'altra dins el seu règim de validesa. El fet que aquests règims de validesa siguin disjunts fa aquesta dualitat molt útil, però difícil de verificar.

Tan en un sentit com en l'altre, el formalisme descrit al paràgraf anterior podria ser la clau per resoldre un problema clau de la física teòrica. En un cas, aquesta relació connecta teories quàntiques fortament acoblades i de rang de gauge gran (en un límit molt concret que es coneix com a límit de 't Hooft) a descripcions semiclàssiques de teories amb gravetat, permetent així una traducció de càlculs complicats al marc de les primeres a un problema molt més assequible en el marc de la gravetat d'Einstein. En l'altre, certes teories de cordes no semiclàssiques adquireixen una definició i implementació precisa a través d'una teoria de camps. Aquesta relació profunda amb els problemes de les teories de camps no pertorbatives i la gravetat quàntica han fet que aquesta conjectura despertés un ampli interès i li han atorgat un lloc central en la recerca durant les dues darreres dècades.

Malgrat els nombrosos esforços dedicats a la seva caracterització, la comprensió de la dualitat hologràfica es troba, a dia d'avui, majoritàriament restringida a l'entorn dels règims on la gravetat és semiclàssica. L'accés a aquests règims és més senzill des del costat gravitatori, així que l'ús de la correspondència per explicar la gravetat quàntica per mitjà de teories de gauge, tot i el seu gran pes conceptual, ha estat comparativament molt menys explotat que la seva aplicació en sentit invers, consistent en oferir prediccions per teories de camps gravitatòriament.

Una eina clau per tal de resoldre o pal·liar aquesta mancança és l'estudi de quantitats peculiars a la teoria de camps, amb qualitats que les facin accessibles fins i tot a un acoblament fort. Un exemple paradigmàtic d'aquest tipus de quantitats és el de les magnituds protegides de renormalització, per les quals la simetria garanteix que les correccions quàntiques de certes quantitats, almenys a nivell pertorbatiu, seran nul·les, i per tant permet estendre a acoblament arbitrari els resultats obtinguts pertorbativament. El seu estudi ens permet posar a prova la dualitat, refinar-ne la construcció i en última instància estendre'n progressivament l'aplicabilitat a nous règims. Aquesta idea, però, no està restringida als objectes protegits. Diverses tècniques desenvolupades recentment permeten obtenir resultats exactes per a certs sectors d'observables a tot acoblament i així participar d'aquest "apuntalament" de la conjectura hologràfica en el mateix sentit en què ho fan les magnituds protegides.

Aquesta tesi estudia les lliçons que hom pot extreure, des del punt de vista de la dualitat hologràfica, de l'aplicació d'una d'aquestes tècniques, l'anomenada localització supersimètrica, que es basa en la possibilitat de deformar un cert tipus de teoria fins a fer-la analíticament resoluble sense variar en cap moment el valor d'un sector supersimètric d'observables. En particular, es centra en l'aplicació d'aquesta tècnica a un tipus de teoria concret, les teories Lagrangianes de tipus $\mathcal{N} = 2$, i per observables concrets, els circuits de Wilson circulars que sols trenquen la meitat de la supersimetria del problema. L'anàlisi també s'ha restringit a l'estudi dels règims on una de les peces més complexes que involucra el procés de localització, la contribució instantònica, esdevé trivial. Això redueix l'espectre de teories considerades a dues finestres paramètriques: D'una banda les teories maximalment supersimètriques ($\mathcal{N} = 4$ SYM), i de l'altra el límit on el rang del grup de gauge es fa arbitràriament gran.

Aquestes dues situacions no són mútuament excloents i és precisament en la seva intersecció en què ens fixem d'entrada, al capítol 3. Aquest règim conté el límit fortament acoblat de la teoria $\mathcal{N} = 4$ en què el dual hologràfic esdevé semiclàssic. Allà on ambdós mètodes són aplicables, podem comparar-ne les prediccions i verificar les nostres expectatives per les prediccions hologràfiques amb el barem rigorós establert pels resultats supersimètrics, i de fet diversos exemples d'aquesta pràctica ja han estat estudiats anteriorment a la literatura. Un primer resultat de la tesi és refinar la comprovació d'aquesta mena per un observable concret al cas en què els objectes hologràfics considerats són D3-branes supersimètriques, cas en què el càlcul hologràfic captura nombroses correccions no planars a la teoria que fan l'encaix amb els resultats de localització no trivial. L'observable que es considera en aquesta comprovació és el coeficient de difusió pel moment d'una partícula pesada dins la teoria de camps. Aquesta partícula segueix un procés de Langevin, al capdavall. Més enllà dels propòsits d'aquesta tesi, es posa de manifest en aquesta secció que aquest coeficient de difusió es relaciona en general amb l'anomenada funció de Bremsstrahlung de la partícula, la versatilitat de la qual es discuteix àmpliament a les primeres seccions d'aquest capítol.

Al mateix temps, el fet que el rang de validesa dels resultats exactes de localització arribi fins al domini on val la holografia obre una nova perspectiva: la possibilitat d'utilitzar la dependència paramètrica a acoblament i rang arbitrari pels observables que hem localitzat com a guia per estendre les prediccions hologràfiques fora del seu rang de validesa. A la secció 3.4 s'explora l'aplicabilitat d'aquesta idea en termes del coefficient de difusió del moment abans esmentat.

Al capítol 4 es discuteix la dependència paramètrica dels resultats de localització en λ i N per càrregues genèriques inserides en teories maximalment supersimètriques on l'àlgebra de gauge és una àlgebra de Lie clàssica. Una observació notable que s'obté d'aquesta anàlisi és que, almenys per càrregues en representació fonamental, el desenvolupament de 't Hooft té una estructura subjacent que relaciona els sectors amb diferent nombre d'inversions topològiques entre ells. De manera complementària, es constata que els resultats obtinguts pel mètode de localització es poden visualitzar en termes d'una mecànica quàntica de fermions, que encaixa amb l'estructura dels "universos de bombolles" (del tipus proposat per Lin, Lunin i Maldacena) al límit de 't Hooft i es manté a rang de gauge finit.

El darrer marc que s'analitza a la tesi, al capítol 5, és el d'aquelles teories Lagrangianes $\mathcal{N} = 2$ en què, malgrat la simetria conforme i la supersimetria, el límit de 't Hooft no converteix el dual hologràfic en semiclàssic. Això es dóna qual la quantitat de matèria en representació fonamental de la teoria és comparable al rang del grup de gauge. Encara que és possible aplicar la localització supersimètrica de forma universal per ambdues classes de teories, les prediccions per als observables a l'abast d'aquesta tècnica presenten diferències qualitatives que discriminen els dos grups. Certament, aquesta coincidència no és prou sòlida per concloure d'entrada que l'estructura del model de matrius codifica directament l'estructura gravitatòria del dual, però hi ha d'altres resultats a la literatura que aporten pistes en aquest mateix sentit.

Observem, doncs, al llarg d'aquesta tesi, que, tot i haver restringit l'anàlisi a sectors i observables específics, els resultats obtinguts per mitjà de la localització supersimètrica ofereixen una rica gamma de suggeriments sobre l'estructura hologràfica.

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