Universitat Autònoma de Barcelona

The Role of Learning in Asset Pricing

Tongbin Zhang

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Autor : Tongbin Zhang

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Director: Prof. Albert Marcet

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Departament d'Economia i d'Història Econòmica

Universitat Autònoma de Barcelona

This thesis is dedicated to my parents Desheng and Zhongping.

ABSTRACT

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There exists many interesting facts or anomalies in asset markets, and this thesis uses "Internal Rationality" learning approach to explain these facts. The first chapter first finds that the co-movement between US stock market and short-term bond market is weak, and the weak co-movement is inconsistent with several rational expectation asset pricing models. Then, we relax rational expectation hypothesis by introducing "Internally Rational" learning agents, so agents' subjective expectations dominantly drive stock price volatility relative to risk-free rate. Quantitative analysis shows the our model can generate data-like co-movement. The second chapter proposes that the high, volatile and persistent AH premium in China stock market is a big challenge for present-value asset pricing models. We show that "Internally Rational" model in which agents have different expectations for capital gains between A-share and H-share is the key to produce AH premium. The third chapter focuses on using "Internal Rationality" approach to explain exchange market puzzles

Abstracte: Hi ha algunes dades interessants o anomalies en els mercats d'actius, i aquesta tesi utilitza enfocament d'aprenentatge "racionalitat interna" per explicar aquests fets. El primer capitol primer determina que el moviment conjunt de la borsa d'Estats Units i el mercat de bons a curt termini es feble, i el feble moviment conjunt es incompatible amb diversos models de valoracio d'actius expectativa racional. Llavors, ens relaxem hipotesi d'expectatives racionals mitjancant la introduccio d'agents d'aprenentatge "Internament racionals", de manera que les expectatives subjectives dels agents dominantment en cotxe preu de les accions volatilitat relativa a la taxa lliure de risc. L'analisi quantitativa mostra el nostre model pot generar co-moviment de dades similar. El segon capitol proposa que la prima d'alta, volatil i persistent AH en el mercat de valors de la Xina es un gran repte per als models de valoraccio d'actius de valor present. Es demostra que el model "Internament racional" en que els agents tenen diferents expectatives per als guanys de capital entre una quota i H-accio la clau per produir prima AH. El tercer capitol se centra en l'us d'enfocament de "racionalitat interna" per explicar els puzles del mercat de canvis.

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Preface

In the field of asset pricing, there exists many interesting asset market facts in addition to the well-known equity premium and stock prices volatility. This thesis has grounded on reconciling empirical facts in asset market and asset pricing models, especially the models with the agents who don't have rational expectations.

The first chapter is my job market paper *Stock Price, Risk-free Rate and Learning.* We study the co-movement between stock price and risk-free rate. The widespread viewpoint or the classical Gordon model with risk-neutral agents predicts that stock price-dividend ratio should negatively correlated with risk-free rate, and risk-free rate should be an important factor in driving stock price volatility. In the consumption-based asset pricing models with rational expectation, risk-free rate is determined by stochastic discount factor (SDF) and the stock price is the discounted sum of future dividend by SDF. When two assets are priced by the same SDF, the model-implied co-movement between stock price and risk-free rate should be strong.

This chapter raises the question "Is the co-movement strong in the data?". This question is often ignored in the literature even though lots of works contribute to explain equity premium and volatility. This question, however, in the first is important for the asset markets participants since both stock and risk-free bond are two typical assets in their portfolios. Understanding the co-movement helps them efficiently invest their wealth. Second, policy makers also concerns about this question. Since the financial crisis, should and how central bank designs monetary policy to stabilize stock market fluctuation has been the subject of a heated debate. The co-movement should be well studied considering the risk-free rate is the channel for conducting monetary policy.

The paper first documents empirical evidences regarding the co-movement between stock price and risk-free rate. It finds a weak correlation between stock price-dividend ratio and risk-free rate. Also using Campbell and Ammer variance decomposition approach the variance of the news about future risk-free rate only contributes a little to the variance of unexpected stock excess return. I then investigate if two rational expectation (RE) asset pricing models–Habit model and Long-run Risk model–can imply the weak co-movement. The reason for choosing these two models is that both them are successful in generating important stock market facts including equity premium, volatility and the mean-reversion. The paper finds that the model-implied correlations between price-dividend ratio and risk-free rate are much stronger than observed empirically primarily because prices of both assets (stock and bond) are influenced by the same set of fundamental variables. Furthermore, both models' variance decomposition results cannot match the data.

Considering the difficulties of RE models in matching data, I propose a simple model by relaxing the RE assumption and allow the existence of "Internally Rational" agents. I extend Adam, Marcet and Nicolini (2016) to introduce an exogenous time-varying risk-free rate. When each agent doesn't know other agents' preferences and information, they don't know the mapping from fundamentals to stock price and stock can no longer be priced as the discounted sum of future dividends. "Internally Rational" agents still optimize their behaviors based on their subjective beliefs about all variables that are beyond their control. Given the subjective beliefs I specify, agents optimally update their expectations about stock price behavior using Kalman filter. Agents' subjective expectations in turn influence equilibrium stock price, and the realized stock price feeds back into agents' beliefs. This self-referential aspect of the model implies that even though risk-free rate is still in the SDF, stock price is dominantly determined by agents' expectations instead of SDF.

Quantitative evaluation of all models utilized in this paper relies on the method of simulated moments (MSM) to test them. The simulation results confirm that my "Internal Rational" model outperforms the above-mentioned RE models in simultaneously matching basic stock market moments and the moments measuring the weak co-movement between stock and short-term bond markets. To explore model's dynamics, I estimate the impulse response of stock price to risk-free rate shock using vector-autoregression analysis. The large confidence band of data impulse response covering from positive to negative territories implies the weak co-movement between stock and short-term bond markets. And our learning model's impulse response is quite close to the data one.

The second chapter jointly with Renbin Zhang, *Understanding AH Premium in China Stock Market* studies AH premium in China stock market. There are 88 companies (AH share) dual-listed in China mainland stock market (A-share) and Hong Kong market (H-share). The market value of AH-share accounts for 20% of total A-share. The stocks of AH-share pay the same dividends to investors holding A-share or H-share. The price difference of AH-share between mainland and Hong Kong markets is called AH premium. Since November 2014, the starting of Shanghai-Hong Kong Stock Connect program makes two previously segmented markets–Shanghai and Hong Kong–connected. According to standard theory, AH premium should be expected to converge. Contrast to theory's prediction, AH premium measured by Hang Seng AH Premium Index continually increased and reached at 150 as the peak, which means price of AH share in Shanghai market is 50% higher than it in Hong Kong.

There exists a lot of works on the price differences of the same stock in different segmented markets (Fernald and Rogers, 2002). But the AH premium problem is an interesting anomaly because two markets now are connected. We then investigate if heterogenous agents asset pricing model can generate AH premium. Agents could have different risk-aversions, different dividend taxes, transaction costs and diverse beliefs on the fundamentals. We arrive at the conclusion that in the complete market or incomplete market risk-aversions and diverse beliefs cannot generate any AH premium, transaction cost is too small to be ignored, and dividend taxes can generate 5%-6% almost constant AH premium.

Given the failure of these asset pricing models, we propose an "Internal Rationality" learning model, in which agents don't know the pricing functions from fundamentals to stock prices and have different subjective beliefs about tomorrow's capital gains between Shanghai and Hong Kong markets. We show that these different beliefs can come from different initial beliefs or different learning speeds, both of them can be supported by the data. This model is able to successfully generate data-like weekly AH premium due to the more optimistic belief in Shanghai market than it in Hong Kong. We also investigate whether the convergence trader can make money with the strategy short-selling in Shanghai and long-buying in Hong Kong. By Monte-Carlo simulation we find that convergence trader will be asked for liquidation with probability from 40% to 90% if not adding security deposit. Even without forced liquidation convergence trader has about 35% probability to lose money in 3, 6, 9 or 12 months.

The third chapter jointly with Prof. Jian Wang and Prof. Jianfeng Yu, *Puzzles in Exchange Market and "Internal Rationality" Approach* is trying to explain two most well-known puzzles in exchange rate market: UIP puzzle and exchange rate disconnect puzzle. These two puzzles are long-standing challenges for exchange rate models, and there are many theories advanced for them. Here we also propose the novel "Internal Rationality" learning approach, in which agents don't know the mapping from economic fundamentals to equilibrium exchange rate. And agents will have their own subjective expectations on future exchange rate change, and update their expectations based on the model they believe. We finally show that this simple exchange rate model, a small deviation from rational expectation, can address UIP puzzle and disconnect puzzle simultaneously.

Engel (2016) finds that there still exists an exchange rate level puzzle, which imposes much more difficulty for matching. We also check whether our model can match this puzzle, but unfortunately quantitatively not enough.

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Introduction

Anomalies in asset markets are always regarded as the great challenges for rational expectation asset pricing models. In this thesis, we focus on these anomalies. The first chapter is about the weak co-movement between US stock and short-term bond markets. The second chapter introduces the AH premium puzzle in China stock market. And the third one concerns on long-standing exchange market puzzles. Towards these anomalies, we make a small deviation from rational expectation hypothesis, and introduce "Internal Rationality" learning agents who don't know the pricing mapping from fundamentals to asset price and have their own subjective beliefs on asset price behavior.

Chapter 1 first shows that the co-movement between stock and short-term bond markets in US data appears weak in terms of the correlation between stock price-dividend ratio and risk-free rate and the variance decomposition of stock excess returns. It is essential to market participants and policy makers to have an asset pricing model consistent with the weak empirical co-movement, especially in light of the fact that several rational expectation asset pricing models that match stock market volatility actually imply a much stronger co-movemnet than empirically observed as shown in our paper. To improve this inconsistency, this chapter presents a small open economy model with "Internally Rational" agents, who optimally update their subjective beliefs on stock prices given their own model. Compared with risk-free rate's variation, agents' subjective beliefs are essential in generating stock price volatility. When testing our model using the method of simulated moments, quantitatively it can simultaneously match moments of the stock and bond markets as well as the weak co-movement between two markets.

Chapter 2 is about China stock market. There are 88 companies (AH-share) dual-listed in both China mainland stock market (A-share) and Hong Kong stock market (H-share) accounted for 20% of total A-share. The 'Shanghai-Hong Kong Stock Connect' program starting at November, 2014 makes previously two segmented markets–Shanghai and Hong Kong stock markets–connected. The price difference of AH-share in Shanghai and Hong Kong stock markets, measured by Hang Seng China AH Premium Index, instead of converging persistently divergences, and even reaches 50% higher in Shanghai market. We have shown that present-value asset pricing models with heterogeneous agents cannot generate any price difference. Transaction cost and different dividend taxes between Shanghai and Hong Kong markets also quantitatively fails to explain such high and volatile AH premium. We, hence, propose an 'Internal Rationality' learning model, in which agents don't know the pricing function from fundamentals to the stock prices and have different subjective beliefs about tomorrow's capital gains in Shanghai and Hong Kong markets. Our learning model can successfully generate data-like weekly AH premium. We also show that convergence traders with strategy short in Shanghai and long in Hong Kong will lose money with 33% probability.

In Chapter 3, we study two old but well-known exchange market puzzles: UIP puzzle and exchange rate disconnect puzzle. There are many theories explaining these two puzzle, but we provide "Internal Rationality" as a novel approach. When agents don't know the pricing mapping from fundamentals to exchange rate, agents' expectation in driving exchange rate volatility plays the key role in matching disconnect puzzle. And the deviation from rational expectation also produces UIP puzzle. But our model quantitatively cannot match a new puzzle called level puzzle proposed in Engel (2016).

CHAPTER 1

Stock Price, Risk-free Rate and Learning

"*There was no historical evidence for a link between interest rates and share prices. You would think that when interest rates are higher people would sell stocks, but the financial world just isn't that simple.*"

–Robert Shiller, Financial Times, 13, September, 2015

1.1. Introduction

This chapter studies the co-movement in prices between stock and short-term bond markets. A variety of basic stock market facts have been extensively studied over the last thirty years, such as the equity premium, the volatility of stock prices and the predictability of long-horizon excess return. There are, however, few studies on the co-movement. Understanding such co-movement is in the first place important for both institutional and individual investors' asset allocation decision when they collect stock and short-term bond into their portfolios. Additionally, after 2007-2009 global financial crisis caused by asset market collapse policy makers concerning the role of policy tools, such as monetary policy and macro-prudential policy, in governing stock market fluctuation would also like to understand this co-movement. In the aspect of monetary policy, having a model that is able to well capture the co-movement should be the first step research before asking what the optimal policy is. And in the aspect of macro-prudential policy, regulatory stress testing for financial stability also requires a framework for modeling the co-movement well.

Then, what should that co-movement be? One popular argument has been that there should be a simple negative co-movement. By the present value models an increase in expected future discount rate should, other things being equal, cause both stock prices to fall and short-term rate to rise; a fall in expected discount rates should have the opposite effect on both. This negative co-movement, however, has not been carefully checked.

This chapter first uses US data to show that co-movement between stock and short-term bond markets is weak along two dimensions. First, the correlation between stock price-dividend ratio and risk-free rate is statistically insignificant from zero. Second, using the variance decomposition approach introduced by Campbell (1991) and Campbell and Ammer (1993) show that the variance of news about future risk-free rate contributes little to the variance of the unexpected excess stock return. In fact and unsurprisingly, the two top components are news about future excess return and news about future dividend growth.

This chapter then investigates whether two rational expectation (RE) asset pricing models is consistent with the weak co-movement: the external habit model (Campbell and Cochrane 1995, Wachter, 2006) and the long-run risk model (Bansal, Kiku and Yaron, 2012). These two models are chosen because both of them are consistent with long-standing empirical puzzles of stock price, such volatility and equity premium. But, we demonstrate that the model-implied correlations between price-dividend ratio and risk-free rate are much stronger than observed empirically primarily because both assets (stock and bond) in two models are priced by the same stochastic discount factors (SDF) as the function of the same set of fundamental variables. Furthermore, both models' variance decomposition results cannot match the data.

The failure of these RE models in matching the co-movement facts motivates the deviation from rational expectation hypothesis that agents have perfect knowledge about how to map from economic fundamentals to equilibrium asset price. We extend Adam, Marcet and Nicolini (2016) into a small open economy model (exogenous risk-free rate process), which introduces "Internally Rational" agents who do not know the fundamental to price mapping and optimize their behaviors based on their subjective beliefs about all variables that are beyond their control. In such circumstance, stock price is no longer the discounted sum of future dividend stream. Given the subjective beliefs we specify, agents optimally update their expectations about stock price behavior using Kalman filter. Agents' subjective expectations in turn influence equilibrium stock price, and the realized stock price feeds back into agents' beliefs. This self-referential aspect of the model implies that agents' endogenous expectations are dominant in generating stock price fluctuation as there is no feedback channel between stock price and risk-free rate.

Our learning model therefore provides a possible resolution to match the weak co-movement between stock and short-term bond markets.

Quantitative evaluation of all models utilized in this chapter relies on the method of simulated moments (MSM) to estimate and test them. The simulation results confirm that our learning model outperforms the above-mentioned RE models in simultaneously matching well-documented stock market moments and the moments measuring the weak co-movement between stock and short-term bond markets. Using t-statistics derived from asymptotic theory we cannot reject the null hypothesis that any of the individual data moments are the same as the moments in the estimated learning model. But, the large t-statistics of co-movement moments in two RE models imply that they are inconsistent with empirical observations.

As an additional measure of the co-movement between stock and short-term bond markets for robustness check, we estimate the impulse response of stock price to risk-free rate shock using vector-autoregression (VAR) analysis following Gali and Gambetti (2015). The VAR analysis also helps us understand the dynamic of stock price to risk-free rate shock. We find that the large confidence band of data impulse response covering from positive to negative territories implies the weak co-movement between stock and short-term bond markets. And our learning model's impulse response is quite close to the data one.

The chapter is organized in the following manner. Section 1.2 discusses related literature. Section 1.3 presents our empirical findings about the co-movement between stock and short-term bond markets. The theoretical model is outlined in the section 1.4. Section 1.5 derives explicit expression for rational expectation equilibrium. The dynamic analysis of the model with "Internally Rational" agents is conducted in section 1.6. Section 1.7 presents the quantitative performance of our model. Section 8 tests the implication of the external habit model and the long-run risk model. Section 1.9 focuses on the impulse response analysis. Finally, section 1.10 concludes.

1.2. Literature Review

Some papers have studied the joint behavior of stock and short-term bond markets. Grossman and Shiller (1981) argues that the stochastic discount factor represented by risk-free rate in the certain economy is not an important driver of stock market volatility since 1950's. Based on the variance decomposition approach, Campbell and Ammer (1993) and Hollifield, Koop and Li (2003) all find that the news on future risk-free rate displays no power in explaining stock market volatility. More recently, Gali and Gambetti (2015) use the impulse response functions from a time-varying VAR model to explore the response of stock price to exogenous monetary policy shock. The most recent theoretical paper in the field is Gali (2014), which challenges the traditional "lean against wind" monetary policy on asset price when allowing the existence of rational bubble. The bubble component in the equilibrium has to grow at the level of risk-free rate.

There are several general equilibrium models containing time-varying risk-free rate which aim at matching stock market facts. Jermann (1998) shows that a model with habit formation and capital adjustment costs can match the historical equity premium and stock market volatility with low dividend growth volatility. Boldrin, Christiano and Fisher (2001) have a model with habit formation and a two-sector technology that can explain the equity premium puzzle and volatility puzzle. It can also generate the low contemporaneous correlation between stock price and output, and the low contemporaneous correlation between risk-free rate and output. Danthine and Donaldson (2002) show that with operating leverage, the incomplete market model also achieves a satisfactory replication of the major stock market stylized facts. However, as mentioned by Guvenen (2009), one drawback of above three models is that all of them generate too high volatility of risk-free rate. Hence, most of stock market volatility is due to extremely volatile risk-free rate in Jermann (1998) and Boldrin, Christiano and Fisher (2001) mentioned in Favilukis and Lin (2015). Guvenen (2009) present a model with two features: limited stock market participation and heterogeneity in the elasticity of intertemporal substitution. His model can have both stock market facts and low volatility of risk-free rate. Even though these dynamic general equilibrium models can match stock market facts and have time-varying risk-free rate, none of them talks about the co-movement between stock and short-term bond markets.

Our paper is also related to the papers studying the correlation between stock price and other variables. Shiller and Beltratti (1992) maintain that the high correlation between real stock return and nominal long-term bond return is a puzzle. Ermolov (2015) reproduces this stock-bond return correlation through a consumption-based asset pricing model with habit utility. Albuquerque, Eichenbaum and Rebelo (2014) present a valuation risk model to replicate the correlation puzzle that is the weak correlation between stock returns and measurable fundamentals.

This paper adds to existing literature by formally studying the weak co-movement between stock and short-term bond markets. We first show that two asset pricing models with rational expectations do not fit the empirical co-movement. Then, we present a learning model that can match basic stock and short-term bond markets facts and the co-movement facts together.

1.3. Stylized Facts

1.3.1. An Illustrative Model

This subsection presents a discreet time partial equilibrium Gordon model to shed some light on the co-movement between stock price and risk-free rate. Consider the economy with risk-neutral agents with rational expectation and an exogenous time-varying risk-free rate R_t. Let P_t denote stock price in period t of an infinite-lived asset, yielding a dividend stream D_t. In the equilibrium the following difference equation must hold

$$P_t R_t = E_t(P_{t+1} + D_{t+1})$$

Rational expectation implies that stock price P_t equals with present value of future dividends discounted by risk-free rate as

$$P_t = E_t \sum_{j=1}^{\infty} \frac{D_{t+j}}{\prod_{k=0}^{j} R_{t+k}}$$

If we model dividend D_t and risk-free rate R_t processes as

$$D_{t+1}/D_t = a\epsilon_t^d$$

$$R_t = R_{t-1} + \epsilon_t^R$$

where ϵ_t^d has mean at 1, and ϵ_t^R has mean at 0. Then, stock price P_t can be expressed as

$$P_t = \frac{a}{R_t - a} D_t$$

This expression obviously tells that there is a strong co-movement between stock price and risk-free rate.

1.3.2. Data

This section reports the stylized facts regarding the stock and short-term bond markets, and the co-movement between them. The quantifiable measures are the correlation between stock price-dividend ratio and risk-free rate, and the variance decomposition statistics introduced by Campbell (1991) and Campbell and Ammer

Statistics	Estimate	SE
Quarterly mean stock return E_{rs}	2.25	0.39
Mean PD ratio E_{PD}	123.91	21.25
Std.dev. stock return σ_{rs}	11.44	2.69
Std.dev. PD ratio σ_{PD}	62.42	17.54
Autocorrel. PD ratio $\rho_{PD,-1}$	0.97	0.02
Excess return reg. coefficient c_5^2	-0.0038	0.0013
R^2 of excess return regression R_5^2	0.1772	0.0828
Mean risk-free rate E_R	0.15	0.19
Std.dev. risk-free rate σ_R	1.27	0.27
Mean dividend growth $E_{\Delta D/D}$	0.41	0.18
Std. dev. dividend growth $\sigma_{\Delta D/D}$	2.88	0.80

Table 1.1. The Statistics Regarding the Stock and Short-term Bond Markets

(1993). The data sample period is from 1927:2 to 2012:2 in quarterly frequency. All of the variables here are in real term, deflated using US CPI.

Table **1.1** contains some of the well-known stock and short-term bond markets facts including the mean and standard deviation of stock return, price-dividend ratio, dividend growth rate, and risk-free rate, the persistence of price-dividend ratio, and the predictability of price-dividend ratio on future five-year's stock excess return. The second column shows the point estimates of these statistics, and the third column has the standard errors of point estimates. These stylized facts are denoted as **Fact 0**. It is well-known that a simple RE asset pricing model has difficulty in matching Fact 0. And, both Campbell and Cochrane (1999) and Adam, Marcet and Nicolini (2016) can match most of the statistics here. But because both models contain constant risk-free rate, they fail in matching the standard deviation of the risk-free rate.

Statistics	Estimate	SE
$corr(PD, R)$	0.069	0.12

Table 1.2. The Correlation between Price-dividend Ratio and Risk-free Rate

According to the illustrative mode in section 1.3.1, stock price-dividend ratio should be highly negatively correlated with risk-free rate. The correlation observed in the data, however, is rather weak as displayed in the table **1.2**. The point estimate of quarterly correlation between price-dividend ratio and risk-free rate is insignificant. The weak correlation between price-dividend ratio and risk-free rate is defined as **Fact 1**.

In addition to the correlation, the statistics of variance decomposition can be an alternative way to measure the co-movement. The variables $\widetilde{e}_d$ in the table **1.3** represents the news about future dividend growth, $\widetilde{e}_r$ represents the news about future risk-free rate, and $\widetilde{e}_e$ represents the news about future excess return. The three statistics in the first column of table **1.3** are the ratios of the variances of the above three variables to the variance of $\widetilde{e}$, where $\widetilde{e}$ is the unexpected excess stock return. Appendix 1.11.2 contains the details of variance decomposition approach. As in Campbell (1991) and Campbell and Ammer (1993), one can interpret the values in the second column of table **1.3** as: variance of news about future dividend growth $\widetilde{e}_d$ accounts for 21% of the variance of unexpected excess stock return $\widetilde{e}$. In comparison, the news about future risk-free rate $\widetilde{e}_r$ only accounts for 4%, while more than half of the variance of unexpected excess return $\widetilde{e}$ can be explained by

Statistics	Estimate	SE
$Var(\widetilde{e}_d)$	21.1%	0.242
$Var(\widetilde{e}_r)$	4.4%	0.026
$Var(\widetilde{e}_e)$	50.8%	0.257

Table 1.3. Variance Decomposition of Excess Stock Return

the news of future excess return $\widetilde{e}_e$ as value in the fourth row, second column. These point estimates are similar to the ones in the Campbell (1991), but the standard deviations are larger in this sample due to a smaller sample size[1]. The variance decomposition results are defined as **Fact 2**. Again, it is also difficult for a simple RE model such as the model in section 1.3.1 to match Fact 2 because this sort of model imply that most of the variance of $\widetilde{e}$ should be explained by $\widetilde{e}_d$ and $\widetilde{e}_r$ instead of $\widetilde{e}_e$.

To summarize, Fact 1 and 2 indicate that co-movement between stock and short-term bond markets is weak.[2]

1.4. The Model

To understand our Fact 0, Fact 1 and Fact 2, we extend Adam, Marcet and Nicolini (2016) asset pricing model with "Internally Rational" agents who hold subjective beliefs about stock price behavior and will be completely rational given their beliefs (Adam and Marcet, 2011). As shown in Adam, Marcet and Nicolini (2016), the presence of such beliefs can generate stock price fluctuation around its

[1]Bernanke and Kuttner (2005) and Balke, Ma and Wohar (2015) also find very large standard errors for the stock price decomposition estimation.
[2]The Appendix 1.11.3 shows the robustness of our Fact 1 and Fact 2.

fundamental value. There are two differences in our model from their model. Our model first is a small open economy with exogenous risk-free rate, then it has one collateral constraint. The exogenous risk-free rate allows us to have time-varying risk-free rate process instead of constant one in Adam, Marcet and Nicolini (2016). And the collateral constraint is important for us to obtain analytical solution for equilibrium stock price.

1.4.1. Model Environment

A unit of stock with dividend claim D_t can be traded in the competitive stock market. In addition to D_t, each agent receives an endowment Y_t of perishable consumption goods. Following traditional setting in asset pricing literature, dividend and endowment growth rates follow i.i.d. lognormal processes

$$\frac{D_t}{D_{t-1}} = a\epsilon_t^d, \log \epsilon_t^d \sim iiN(-\frac{s_d^2}{2}, s_d^2)$$

$$\frac{Y_t}{Y_{t-1}} = a\epsilon_t^y, \log \epsilon_t^y \sim iiN(-\frac{s_y^2}{2}, s_y^2)$$

where endowment and dividend growth rates share the same mean a, and $(\log \epsilon_t^d, \log \epsilon_t^y)$ are joint-normally distributed with correlation between them equaling to $\rho_{y,d} = 0.2$ (Campbell and Cochrane,1999). Since consumption process is considerably less volatile than the dividend process, the parameters' values of standard deviations are chosen as $s_y = \frac{1}{7}s_d$.

The economy is populated by a unit mass of infinite-horizon agents. We model each agent $i \in [0, 1]$ to have the same standard time-separable CRRA utility function and the same subjective beliefs. This fact, however, is not the common knowledge among agents.

The specification of agent i's expected life-time utility function is

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma} \tag{1.1}$$

where C_t^i is the consumption profile of agent i, δ denotes the time discount factor, and γ is the risk-aversion parameter. Instead of the objective probability measure, expectation is formed using the subjective probability measure $\mathcal{P}$ that describes probability distributions for all external variables.

Agent's choices are subjected to standard budget constraint as following

$$C_t^i + R_{t-1} b_{t-1}^i + P_t S_t^i = (P_t + D_t) S_{t-1}^i + b_t^i + Y_t \tag{1.2}$$

where b_t^i is the amount of borrowing at time t, S_t^i the new units of stock agent i buys in period t, and R_{t-1} as exogenous real risk-free rate on maturing loans b_{t-1}^i.

One collateral constraint is imposed. The amount of borrowing is subjected to the collateral constraint as Kiyotaki and Moore (1997) in the form [3]

$$b_t^i \leqq \theta \frac{E_t^{\mathcal{P}}(P_{t+1} + D_{t+1})}{R_t} S_t^i \tag{1.3}$$

Besides transferring income across time, the stock S_t^i plays the role of collateral. The collateral constraint implies that new loans b_t^i should be smaller than a fixed share of expected discounted value of tomorrow's stock. The parameter θ measures the share of stock value that can serve as collateral.

To close the small open economy model, risk-free rate process is specified similar to that of Bianchi (2013) to capture its mean, variance and persistence.

$$R_t = \begin{cases} (1-\rho_R)\overline{R} + \rho_R R_{t-1} + \epsilon_t^R & \text{if } R_t < \frac{1}{\varphi} \\ \frac{1}{\varphi} & \text{otherwise} \end{cases} \tag{1.4}$$

where $\varphi \equiv \delta E_t^{\mathcal{P}}(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}$, $\epsilon_t^R \sim N(0, \sigma_R^2)$ and is orthogonal to dividend and consumption shocks. The upper limit for the risk-free rate can guarantee the binding of collateral constraint to avoid the difficulty of occasionally binding problem, and it matters little for altering the moments of risk-free rate because quantitative analysis confirms that risk-free rate seldom hits the limit in this model.

[3]Following Adam, Pei and Marcet (2011), this specification implicitly assumes that risk-neutral foreigners have the same beliefs as domestic agents

Now we explicitly describes the general joint probability space of the external variables. In the competitive economy, each agent considers the joint process of endowment, dividend, risk-free rate, and stock price $\{Y_t, D_t, R_t, P_t\}$ as exogenous to his decision problem. Rational expectations imply that agents exactly know the mapping from a history of endowment Y_t, dividend D_t, and risk-free rate R_t to equilibrium stock price P_t. Stock price hence just carries redundant information. But if the rational expectation assumption is relaxed, as shown in Adam and Marcet (2011) such that agents do not know such mapping because of the non-existence of common knowledge on agents' identical preferences and beliefs, then equilibrium stock price P_t should be included in the underlying state space. We then define the probability space as $(\mathcal{P}, \mathcal{B}, \Omega)$ with $\mathcal{B}$ denoting the corresponding $\sigma-$Algebra of Borel subsets of Ω and $\mathcal{P}$ denoting the agent's subjective probability measure over $(\mathcal{B}, \Omega)$. The state space Ω of realized exogenous variables is

$$\Omega = \Omega_Y \times \Omega_D \times \Omega_R \times \Omega_P$$

where Ω_X is the space of all possible infinite sequences for the variable $X \in [Y, D, R, P]$. Thereby, a specific element in the set Ω is an infinite sequence $\omega = \{Y_t, D_t, R_t, P_t\}_{t=0}^{\infty}$. The expected utility with probability measure $\mathcal{P}$ is defined as

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma} \equiv \int_{\Omega} \sum_{t=0}^{\infty} \delta^t \frac{C_t^i(\omega)^{1-\gamma}}{1-\gamma} d\mathcal{P}(\omega) \tag{1.5}$$

Agent i makes contingent plans for endogenous variables C_t^i, S_t^i, b_t^i according to the policy function mapping in the following

$$(C_t^i, S_t^i, b_t^i) : \Omega^t \to R^3$$

where Ω^t represents the set of histories from period zero up to period t.

1.4.2. Optimality Conditions

Optimal conditions characterizing agent i's decisions from his maximization problem are derived in this subsection. First order conditions are sufficient and necessary for agent's optimality because of the concavity of objective function and convexity of feasible set.

Agent i should maximize his expected lifetime utility (**1.1**) subject to the budget constraint (**1.2**) and collateral constraint (**1.3**). The Lagrangian of agent's problem can be explicitly written as

$$\max_{\{C_t, S_t, b_t\}} E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \Big(\frac{(C_t^i)^{1-\gamma}}{1-\gamma} - \lambda_t (C_t^i + R_{t-1} b_{t-1}^i + P_t S_t^i - (P_t + D_t) S_{t-1}^i - b_t^i - Y_t)$$
$$+ \eta_t (\theta E_t^{\mathcal{F}} (P_{t+1} + D_{t+1}) S_t^i - R_t b_t^i) \Big)$$

where λ_t and η_t are two Lagrangian multipliers, S_{-1}, b_{-1} as given initial conditions, and agent i is a price-taker for P_t.

The agent i's first order conditions can be expressed as

$$C_t^i : (C_t^i)^{-\gamma} - \lambda_t = 0 \tag{1.6}$$

$$S_t^i : -\lambda_t P_t + \delta E_t^{\mathcal{P}}(\lambda_{t+1}(P_{t+1} + D_{t+1})) + \theta E_t^{\mathcal{P}} \eta_t (P_{t+1} + D_{t+1}) = 0 \tag{1.7}$$

$$b_t^i : \lambda_t = \delta R_t E_t^{\mathcal{P}} \lambda_{t+1} + \eta_t R_t \ \& \ \eta_t(\theta E_t^{\mathcal{P}}(P_{t+1} + D_{t+1}) S_t^i - R_t b_t^i) = 0 \tag{1.8}$$

After substituting λ_t in equation (**1.8**) using the expression in equation (**1.6**), one obtains

$$(C_t^i)^{-\gamma} = \delta R_t E_t^{\mathcal{P}}(C_{t+1}^i)^{-\gamma} + \eta_t R_t \tag{1.9}$$

The binding collateral constraint can lead us to have the non-zero multiplier η_t for all t as

$$\eta_t = \frac{(C_t^i)^{-\gamma} - \delta R_t E_t^{\mathcal{P}}(C_{t+1}^i)^{-\gamma}}{R_t} \tag{1.10}$$

Substitute η_t in equation (**1.10**) back into equation (**1.9**), one obtains

(1.11)

$$-(C_t^i)^{-\gamma} P_t + \delta E_t^{\mathcal{P}}((C_{t+1}^i)^{-\gamma}(P_{t+1} + D_{t+1})) + \theta \frac{(C_t^i)^{-\gamma} - \delta R_t E_t^{\mathcal{P}}(C_{t+1}^i)^{-\gamma}}{R_t} E_t^{\mathcal{P}}(P_{t+1} + D_{t+1}) = 0$$

Finally, the feasibility condition of the economy is

$$C_t = Y_t + D_t + b_t - R_{t-1}b_{t-1} \tag{1.12}$$

where C_t and b_t are aggregate consumption and loan.

1.4.3. Approximation

In order to generate an analytical solution for equilibrium stock price P_t, we rely on several approximations and one assumption. First, aggregate consumption C_t is not necessarily equal to aggregate endowment Y_t according to the feasibility condition (**1.12**). Second, with agent's subjective beliefs we may not have $E_t^{\mathcal{P}}(C_{t+1}^i) \neq E_t^{\mathcal{P}}(C_{t+1})$ even though in the equilibrium $C_{t+1}^i = C_{t+1}$ holds ex-post. To understand the reason, let us consider that $E_t^{\mathcal{P}}(C_{t+1})$ depends on expected stock price only through the channel of b_t. At the same time, apart from the channel of loan b_t^i future stock price can also affect $E_t^{\mathcal{P}}(C_{t+1}^i)$ through capital gains from holding stock. One hence cannot routinely substitute individual consumption C_t^i by aggregate one C_t. We, however, can rely on the approximations as

$$C_t \simeq Y_t \tag{1.13}$$

$$E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}(P_{t+1} + D_{t+1})] \simeq E_t^{\mathcal{P}}[(\frac{C_{t+1}}{C_t})^{-\gamma}(P_{t+1} + D_{t+1})] \tag{1.14}$$

$$E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}] \simeq E_t^{\mathcal{P}}[(\frac{C_{t+1}}{C_t})^{-\gamma}] \tag{1.15}$$

To make these approximations reasonable, the following assumption is made similar to Assumption 1 in Adam, Marcet and Nicolini (2016):

ASSUMPTION 1: We assume that Y_t is sufficiently large and that $E_t^{\mathcal{P}}(P_{t+i} + D_{t+i}) < \overline{M}$ for some $\overline{M} < \infty$ for $i = 1, 2$. Then, expected value from holding stock should be sufficiently small compared to Y_t given finite asset bounds $\overline{S}, \underline{S}$.

The gap between subjective and objective consumption growth can be expressed as

$$\begin{aligned}
& E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})] - E_t[(\frac{C_{t+1}^i}{C_t})] \\
= \quad & E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})] - E_t[(\frac{C_{t+1}}{C_t})] \\
= \quad & E_t^{\mathcal{P}}[\frac{P_{t+1}(1 - S_{t+1}^i) + (b_{t+1} - b_{t+1}^i)}{Y_t + D_t + b_t - R_{t-1}b_{t-1}}]
\end{aligned}$$

Because of the collateral constraint b_t^i is smaller than the expected tomorrow's stock value $E_t^{\mathcal{P}}(P_{t+1} + D_{t+1})S_t^i$. Assumption 1 implies that individual loan b_t^i and aggregate loan b_t are also small enough compared to Y_t. According to equation (**1.12**), when b_t and D_t are small the approximation (**1.13**) holds with sufficient accuracy. Also under this assumption, the approximation (**1.14**) and (**1.15**) hold with sufficient accuracy as the above gap between subjective and objective consumption growth is approximately zero.

After rearranging terms in equation (**1.11**) and substituting related terms using three approximations from equation (**1.13**) to (**1.15**), one obtains the key pricing equation as

$$P_t = E_t^{\mathcal{P}} \eta_t (P_{t+1} + D_{t+1}) \tag{1.16}$$

where $\eta_t \equiv \delta(\frac{Y_{t+1}}{Y_t})^{-\gamma} + \theta(\frac{1}{R_t} - \varphi)$ is the stochastic discount factor (SDF).

1.5. Rational Expectation Equilibrium

This section presents the rational expectation equilibrium of our model and shows that its implications cannot match Fact 1 and 2. This is useful because it confirms that the role of exogenous risk-free rate and collateral constraint cannot contribute to match the weak co-movement, and motivates us to show that how a small departure from RE contributes to it in Section 6. Rational expectation implies that agent's subjective beliefs coincides with the objective ones. Following the routine calculation and imposing the non-bubble condition, we can express the equilibrium stock price in rational expectation from equation (**??**) as

$$P_t^{RE} = [\frac{\delta a^{1-\gamma}\rho_\epsilon}{1-\delta a^{1-\gamma}\rho_\epsilon} + E_t \sum_{j=1}^{\infty} \theta^j a^j \prod_{k=0}^{j-1} (\frac{1}{R_{t+k}} - \varphi)] D_t \tag{1.17}$$

Statistics	**US Data**		**RE**
	Estimate	SE	Statistics
$corr(PD, R)$	0.069	0.12	-1.000
$Var(\widetilde{e_d})$	21.2%	0.242	96.2%
$Var(\widetilde{e_r})$	4.4%	0.026	17.0%
$Var(\widetilde{e_e})$	50.8%	0.257	5.0%

Table 1.4. Simulated Statistics of Rational Expectation Equilibrium

where

$$\begin{aligned} \rho_\epsilon &= E[(\epsilon^y_{t+1})^{-\gamma}\epsilon^d_{t+1}] \\ &= e^{\gamma(1+\gamma)\frac{s_y^2}{2}} e^{-\gamma\rho_{y,d}s_y s_d} \end{aligned}$$

The rational expectation equilibrium first is inconsistent with Fact 0 including equity premium, stock market volatility even though not reported here. Then given the risk-free rate process, we have $E_t[R_{t+k}] = (1-\rho_r^k)\overline{R} + \rho_r^k R_t$ for any k. The analytical solution of price-dividend ratio as equation (**1.17**) directly displays that $\frac{P_t^{RE}}{D_t}$ is highly correlated with R_t since $\frac{P_t^{RE}}{D_t}$ is a function only of the risk-free rate. It is not surprising because stock price is discounted sum of future dividends by SDF η_t, which contains R_t and i.i.d. endowment growth. Hence, the RE equilibrium is likely to miss Fact 1. And the volatility of stock return here mainly comes from the variation of dividend growth and risk-free rate such that the model is also likely to miss Fact 2.

In order to confirm the above shortcomings of the rational expectation equilibrium in matching stylized facts, the model is simulated and the corresponding moments relating to Fact 1 and Fact 2 are calculated. The parameters values here are the same as the ones from the learning model estimation, which are contained in table **1.5** and **1.7**. Table **1.4** presents the simulation results. Column 4 of table **1.4** shows that the rational expectation equilibrium generates the strong co-movement between stock and short-term bond markets. The correlation between price-dividend ratio and risk-free rate is -1, and the news of future dividend growth and risk-free rate instead of excess return contribute too much to the fluctuation of unexpected excess return. The reason of the failure is that stock prices here are only driven by exogenous state variables dividend D_t and risk-free rate R_t.

1.6. Equilibrium Analysis with Learning

1.6.1. Agent's Subjective Beliefs

Now we allow a small deviation from rational expectation assumption such that agents with uncertainty formulate their own joint probability distribution $\mathcal{P}$ different from the objective one. And Adam and Marcet (2011) shows that the joint distribution $\mathcal{P}$ of any agent without common knowledge about other agents' beliefs and preferences could delink stock price from the expected discounted sum of future dividends. The present-value expression of stock price P_t in equation (**1.17**) ceases to hold, leaving only the first-order condition for stock price in equation (**1.16**) intact. Then, agents should have their own beliefs on the behavior of stock price based on subjective distribution $\mathcal{P}$. Specifically, the subjective expectation of risk-adjusted stock price growth β_t can be defined as

$$\beta_t \equiv E_t^{\mathcal{P}}[(\frac{Y_{t+1}}{Y_t})^{-\gamma}\frac{P_{t+1}}{P_t}] \tag{1.18}$$

and subjective expectation of non-adjusted stock price growth m_t as

$$m_t \equiv E_t^{\mathcal{P}}[\frac{P_{t+1}}{P_t}] \tag{1.19}$$

Then, equation (**1.16**) together and these two definitions imply equation (**1.20**) which maps from subjective price beliefs β_t and m_t to realized one P_t[4]

$$P_t = \frac{\delta a^{1-\gamma}\rho_\epsilon + \theta a(\frac{1}{R_t} - \varphi)}{1 - \delta\beta_t - \theta(\frac{1}{R_t} - \varphi)m_t} D_t \tag{1.20}$$

Equation (**1.20**) analytically suggests that learning equilibrium provides a potential resolution to match Fact 1 and Fact 2. Price-dividend ratio in learning equilibrium, in addition to risk-free rate R_t, also depends on agents's subjective beliefs β_t and m_t. If agents have a high subjective expectation on stock price growth, say high β_t and m_t, their increasing holding of stock drives up stock price P_t today. Conversely, P_t will decrease if agents are pessimistic and have low β_t and m_t.

[4]Following Adam, Marcet and Nicolini (2016), we assume that agents know the true process for dividend growth and endowment growth.

1.6.2. Beliefs Updating Rule

This section fully specifies the subjective probability distribution $\mathcal{P}$ and derive the optimal belief updating rule for subjective beliefs β_t and m_t. Similar to the arguments in Adam, Marcet and Nicolini (2016), the true process for risk-adjusted stock price growth can be modeled as the sum of a persistent component and of a transitory component

$$\begin{aligned}(\frac{Y_{t+1}}{Y_t})^{-\gamma}\frac{P_{t+1}}{P_t} &= e_t^\beta + \epsilon_t^\beta,\ \epsilon_t^\beta \sim iiN(0, \sigma^2_{\epsilon,\beta}) \\ e_t^\beta &= e_{t-1}^\beta + \xi_t^\beta,\ \xi_t^\beta \sim iiN(0, \sigma^2_{\xi,\beta})\end{aligned}$$

One way to justify this process is that it is compatible with RE. According to equation (**1.17**), the rational expectation of risk-adjusted price growth is $E_t[(\frac{Y_{t+1}}{Yt})^{-\gamma}\frac{P_{t+1}}{P_t}] = a^{1-\gamma}\rho_\epsilon$ when risk-free rate R_t is at its unconditional mean $\overline{R}$. Hence, the previous setup encompasses the rational expectation equilibrium as a special case when agents believe $\sigma^2_{\xi,\beta} = 0$ and assign probability one to $e_0^\beta = a^{1-\gamma}\rho_\epsilon$.

Then, we allow for a non-zero variance $\sigma^2_{\xi,\beta}$. Agents can only observe the realizations of risk-adjusted growth (the sum of persistent and transitory components), hence the requirement to forecast the persistent components e_t^β calls for a filtering problem. The priors of agents' beliefs can be centered at their rational expectation values and given by

$$e_0^\beta \sim N(a^{1-\gamma}\rho_\epsilon, \sigma^2_{0,\beta})$$

and the variances of prior distributions should be set up to equal with steady state Kalman filter uncertainty about e_t^β

$$\sigma_{0,\beta}^2 = \frac{-\sigma_{\xi,\beta}^2 + \sqrt{\sigma_{\xi,\beta}^4 + 4\sigma_{\xi,\beta}^2\sigma_{\epsilon,\beta}^2}}{2}$$

Then agents' posterior beliefs will be

$$e_t^\beta \sim N(\beta_t, \sigma_{0,\beta}^2)$$

And the optimal updating rule implies that the evolution of β_t is taking the form just as constant gain learning[5]

$$\beta_t = \beta_{t-1} + \frac{1}{\alpha}((\frac{Y_{t-1}}{Y_{t-2}})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}) \tag{1.21}$$

where $\alpha = \frac{\sigma_{\xi,\beta}^2+\sqrt{\sigma_{\xi,\beta}^4+4\sigma_{\xi,\beta}^2\sigma_{\epsilon,\beta}^2}}{2\sigma_{\xi,\beta}^2}$ given by optimal (Kalman) gain. And agents think that non-adjusted stock price growth m_t is uncorrelated with endowment growth. Hence, under agents' knowledge of true endowment growth and subjective expectation of risk-adjusted stock price growth β_t their subjective expectation of non-adjusted stock price growth m_t is pinned down as

$$m_t = \beta_t/(a^{-\gamma}\tau) \tag{1.22}$$

[5]In the appendix 1.11.9 we prove the convergence of least square learning to rational expectation equilibrium.

where $\tau = \exp(\gamma s_y^2/2 + \gamma^2 s_y^2/2)$. [6]

The adaptive learning scheme as equation (**1.21**) and (**1.22**) as well as pricing equation (**1.20**) can generate a high stock markets volatility coming from the feedback channel between stock price P_t and subjective beliefs β_t, m_t. According to equation (**1.20**), a high (low) β_t and m_t will lead to a high (low) realized stock price. This will reinforce the subjective beliefs to induce a even higher (lower) β_{t+1} and m_{t+1} through equation (**1.21**) and (**1.22**) leading to much higher (lower) stock price so on. The self-referential aspect of the model is the key for producing stock market volatility. But there is no feedback channel between stock price P_t and risk-free rate R_t. Even though risk-free rate R_t is still in the stochastic discount factor, stock price having no present value expression and mostly being influenced by agents' beliefs makes the learning model here has the ability to produce the weak co-movement between stock price and risk-free rate as found in the data

Finally, in order to avoid the explosion of stock price P_t agents' subjective belief β_t is replaced by $\omega(\beta_t)$, the projection facilities. [7]

1.7. Quantitative Analysis

This section evaluates the quantitative performance of the learning model. Fact 0, Fact 1 and Fact 2 give the target moments that should be matched. We formally

[6] In the appendix 1.11.4 we consider the case that agents use Kalman filter to update their subjective beliefs of non-adjusted price growth m_t and pin down β_t.

[7] We present the details of projection facilities in appendix 1.11.5.

estimate and test the model using the method of simulated moments (MSM) that provides a natural test on individually matching moments.

1.7.1. MSM Estimation and Statistical Test

In this subsection we outline the MSM approach. Appendix 1.11.6 discusses about the details of it. We first give value to the coefficient of relative risk-aversion γ, and calibrate the collateral ratio θ, the mean and the persistence of risk-free rate $\overline{R}$, ρ_R[8]. Table **1.5** contains the values for these four parameters. Apart from these, there are five free parameters remaining, comprising the discount factor δ, the gain parameter α, the mean and standard deviation of dividend growth a and $\sigma_{\Delta D/D}$, and the standard deviation of risk-free rate σ_R. They can be summarized into parameter vector as

$$\Phi \equiv (\delta, \alpha, a, \sigma_{\Delta D/D}, \sigma_R)$$

[8]Following Adam, Kuang and Marcet (2011), θ is calibrated as the averaged ratio of US current account deficit to the change of US stock market value. θ equals 0.1 using this method. As a robustness check, θ is also calibrated following Bianchi (2013), which relies on the average liabilities-to-asset ratio of US households. The data is from Table B.101, the flow of funds database. The sample is from 1945 to 2006. In this second method, $\theta = 0.115$. $\overline{R}$, ρ_R are calibrated as the sample mean and sample autocorrelation of risk-free rate. The sample is the one in section 1.3.

These five free parameters will be chosen to match all the sample moments describing Fact 0, Fact 1, and Fact 2. The moments are

$$[E_{rs}, E_{PD,}\ \sigma_{rs}, \sigma_{PD}, \rho_{PD,-1}, c_5^2, R_5^2, E_R, \sigma_R, E_{D/D}, \sigma_{D/D,}$$
$$cov(R, PD), var(\widetilde{e}_{d,t+1})/var(\widetilde{e}_{t+1}),$$
$$var(\widetilde{e}_{r,t+1})/var(\widetilde{e}_{t+1}), var(\widetilde{e}_{e,t+1})/var(\widetilde{e}_{t+1})] \tag{1.23}$$

The first eleven moments are Fact 0 moments widely studied in the literature, and the last four moments are Fact 1 and Fact 2 moments. The MSM parameter estimate $\widehat{\Phi}_T$ is defined as

$$\widehat{\Phi}_T \equiv \arg\min_{\Omega} [\widehat{S}_T - \widetilde{S}(\Phi)]'\ \widehat{\Sigma}_{S,T}^{-1}\ [\widehat{S}_T - \widetilde{S}(\Phi)] \tag{1.24}$$

where $\widehat{S}_T$ denotes all of the sample moments in (**1.23**) that will be matched in the estimation, with T the sample size. Furthermore, let $\widetilde{S}(\Phi)$ denote the moments implied by the model for some parameter value Φ. The MSM estimate $\widehat{\Phi}_T$ chooses the model parameters such that the model implied moments $\widetilde{S}(\Phi)$ fit the observed moments $\widehat{S}_T$ as close as possible in terms of a quadratic form with weighting matrix $\widehat{\Sigma}_{S,T}^{-1}$. The optimal weight matrix $\widehat{\Sigma}_{S,T}$ could be estimated from the data in a standard way. According to the standard results of MSM approach (Duffie and Singleton, 1993), the estimate $\widehat{\Phi}_T$ is consistent and efficient.

The MSM estimation approach provides an overall test of the model. Under the null hypothesis that the model is correct, we have

$$\widehat{W}_T \equiv T[\widehat{S}_T - \widetilde{S}(\Phi)]' \, \widehat{\Sigma}_{S,T}^{-1} \, [\widehat{S}_T - \widetilde{S}(\Phi)] \sim \chi^2_{s-5} \text{ as } T \to \infty \tag{1.25}$$

where s is the number of moments in $\widehat{S}_T$ and the convergence is in distribution. We can also obtain the asymptotic distribution for t-statistics that indicate which moment is matched.

1.7.2. Estimation and Simulation Results

Table **1.6** and **1.7** present the estimation outcomes when the value of risk-aversion coefficient is given at $\gamma = 10$. Table **1.6** contains the well-known Fact 0 moments for matching, and table **1.7** displays the results of matching Fact 1 and Fact 2 co-movement moments. In both tables, column 2 and 3 report the values of the moments from US data and the estimated standard errors for each of these moments. Columns 4 and 5 then show the model moments and the t-statistics when estimating the model using all the moments in (23).

The estimated model in the first can quantitatively replicate Fact 0 moments: the volatility of stock return σ_{rs}, the volatility, persistence, and the predictability of price-dividend ratio σ_{PD}, $\rho_{PD,-1}$, c_5^2, and R_5^2, the high stock return E_{rs}, and the low mean and volatility of risk-free rate E_R and σ_R as well as the mean and standard deviation of dividend growth $E_{\Delta D/D}$ and $\sigma_{\Delta D/D}$. All of the t-statistics

Parameters	Value
γ	10
θ	0.1
ρ_R	0.5
$\overline{R}$	1.0015

Table 1.5. Some Parameters Values for Learning Model

in table **1.6** have an absolute value below or close to two. Therefore, this model is consistent with Fact 0 moments and better than Adam, Marcet and Nicolini (2016) in matching the equity premium.

In addition to match Fact 0 moments, this learning model has the ability to generate simultaneously the low co-movement between stock valuations and short-term bond yields. The model correlation between price-dividend ratio and risk-free rate $corr(PD, R)$ is much closer to empirical data compared to those from rational expectation models, and the t-statistics of it is around two. This reflects a match of Fact 1. Furthermore, the three t-statistics, all of which are around 1 in absolute value, for variance decomposition moments confirm the replication of Fact 2. The t-statistics show desirable individual matching of all moments

The p-value for the statistics $\widehat{W_T}$ as the measure for the overall goodness of fit is reported in the last row of table **1.7**. The statistics is computed using equation (**1.25**). The zero p-value implies that the overall fit of the model is rejected, even if all individual moments are matched. Therefore, the overall goodness of fit test is considerably more stringent.

	US data		Model	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	2.08	0.44
E_{PD}	123.91	21.25	88.94	1.65
σ_{rs}	11.44	2.69	12.30	-0.32
σ_{PD}	62.42	17.54	62.64	-0.01
$\rho_{PD,-1}$	0.97	0.02	0.93	1.72
c_5^2	-0.0038	0.0013	-0.0060	1.72
R_5^2	0.1772	0.0828	0.1108	0.80
E_R	0.15	0.19	0.12	0.15
σ_R	1.27	0.27	0.71	2.04
$E_{\Delta D/D}$	0.41	0.18	0.03	2.10
$\sigma_{\Delta D/D}$	2.88	0.80	2.22	0.82

Table 1.6. Basic Stock and Short-term Bond Market Moments from MSM

	US Data		Model	
	Moment	SE	Moment	t-stat
$corr(PD,R)$	0.069	0.12	-0.170	1.92
$Var(\widetilde{e}_d)$	21.1%	0.242	39.7%	-0.77
$Var(\widetilde{e}_r)$	4.4%	0.026	1.7%	1.01
$Var(\widetilde{e}_e)$	50.8%	0.257	56.1%	-0.21
Discount factor $\widehat{\delta}_T$			0.9886	
Gain coefficient $1/\widehat{\alpha}_T$			0.0085	
p-value of $\widehat{W_T}$			0.000%	

Table 1.7. Co-movement Moments from MSM

1.8. Two Asset Pricing Models with Rational Expectations

In this section we replicate two asset pricing models with rational expectations: a variation of the external habit model of Campbell and Cochrane (1999) [9]and the long-run risk model of Bansal, Kiku and Yaron (2012). Their implications on

[9]The risk-free rate is chosen as a constant in Campbell and Cochrane (1999). A time-varying risk-free rate is introduced here according to the same method as their NBER Working paper version (1995) and Wachter (2006).

the joint behavior between stock price and risk-free rate are examined. Section 5 have illustrated that the rational expectation equilibrium of a simple asset pricing model missing Fact 0 is inconsistent with Fact 1 and Fact 2. But two RE models considered here have ability to match Fact 0.

1.8.1. The external habit model

The representative agent maximizes his life-time utility as

$$U = E\sum_{t=0}^{\infty}\delta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}$$

where C_t is consumption at period t and X_t denotes external habit. Instead of modeling the exogenous process for X_t, we can define surplus consumption ratio as

$$S_t = \frac{C_t - X_t}{C_t}$$

The log surplus consumption ratio $s_t \equiv \log(S_t)$ evolves according to a heteroskedastic AR(1) process

$$s_{t+1} = (1-\phi)\overline{s} + \phi s_t + \lambda(s_t)[\Delta c_{t+1} - E(\Delta c_{t+1})]$$

The sensitivity function $\lambda(s_t)$ is specified as

$$\lambda(s_t) = \left\{ \begin{array}{ll} (1/\overline{S})\sqrt{1-2(s_t - s)} - 1, & s_t \leq s_{\max} \\ 0 & , s_t \geq s_{\max} \end{array} \right\}$$

where $\overline{S}$ is set to be

$$\overline{S} = \sigma\sqrt{\frac{\gamma}{1-\phi-B/\gamma}}$$

and

$$s_{\max} = \overline{s} + \frac{1}{2}(1-\overline{S}^2)$$

The growth of consumption and dividend follow lognormal process

$$\Delta c_{t+1} = g + \upsilon_{t+1}$$

$$\Delta d_{t+1} = g + \omega_{t+1}$$

where υ_{t+1} and ω_{t+1} are two i.i.d. normally distributed variables with mean zero and variances σ^2 and σ_ω^2.

Then, the equilibrium price-dividend ratio as the function of state variable s_t satisfies

$$\frac{P_t}{D_t}(s_t) = E_t[M_{t+1}\frac{D_{t+1}}{D_t}[1+\frac{P_t}{D_t}(s_{t+1})]]$$

And the risk-free rate can be calculated as

$$R_t = R^f - B(s_t - \overline{s})$$

where M_{t+1} is stochastic discount factor, R^f and B are parameters.

1.8.2. The long-run risk model

The representative agent with recursive preference maximizes his life-time utility given by

$$V_t = [(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta(E_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}}$$

The variable θ is defined as

$$\theta \equiv \frac{1-\gamma}{1-1/\psi}$$

where the parameters γ and ψ represent relative risk aversion and the elasticity of intertemporal substitution. The consumption and dividend have the following joint dynamics

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \sigma_t \eta_{t+1} \\
x_{t+1} &= \rho x_t + \varphi_e \sigma_t e_{t+1} \\
\sigma_{t+1}^2 &= \overline{\sigma}^2 + \nu(\sigma_t^2 - \overline{\sigma}^2) + \sigma_w w_{t+1} \\
\Delta d_{t+1} &= \mu_d + \phi x_t + \pi \sigma_t^2 \eta_{t+1} + \varphi \sigma_t u_{d,t+1}
\end{aligned}$$

The solutions for price-dividend ratio and risk-free rate are

$$\log(\frac{P_t}{D_t}) = A_{0,d} + A_{1,d} x_t + A_{2,d} \sigma_t^2$$

$$R_t^f = A_{0,f} + A_{1,f}x_t + A_{2,f}\sigma_t^2$$

where $A_{0,d}$, $A_{1,d}$, $A_{0,f,}$ $A_{1,f}$, $A_{2,d}$, $A_{2,f}$ are all the constants as the functions of only deep parameters.

1.8.3. Evaluating the models

To evaluate the quantitative performance of these two RE models and to be consistent with the estimation method of the learning model, the MSM approach is again adopted to estimate models' parameters. The moments chosen for matching are the same as the ones in section 1.7.1. The estimated parameters vector for the external habit model is

$$\Phi^{EH} \equiv (\delta, \phi, g, \sigma)$$

where δ is the discount factor, ϕ is the persistency of surplus consumption, g and σ are the mean and standard deviation of consumption growth. And the risk aversion coefficient γ is fixed at 2 following Campbell and Cochrane (1999). Analogously, the estimated parameters vector for the long-run risk model is

$$\Phi^{LRR} \equiv (\delta, \psi, \mu_d, \varphi_d)$$

where δ is the discount factor, ψ is the intertemporal elasticity of substitution, μ_d is the mean of dividend growth, and φ_d governs the most of standard deviation of dividend growth. We fix other parameters at values set by Bansal, Kiku and Yaron

(2012). Table **1.8** contains the parameter values for the external habit model, and table **1.9** for the long-run risk model .

Both models are simulated at monthly frequency and then aggregated to quarterly frequency. Table **1.10** displays the estimation outcomes for the external habit model, and table **1.11** for the long-run risk model. The fourteenth row in both tables present our Fact 1. The correlations between price-dividend ratio and the risk-free rate in two models are unrealistically high because both of them are the functions of the same exogenous fundamental variables such as s_t in the external habit model and x_t as well as σ_t in the long-run risk model. In contrast, price-dividend ratio in the learning model, in addition to the fundamental variables, is also driven by agent's endogenous subjective beliefs. So the correlation there is weak.

The last three rows in table **1.10** and **1.11** demonstrate that the implications of both models' variance decompositions are inconsistent with the real-life observations. The variance of news about future risk-free rate indeed contributes little to the variance of unexpected excess return in both models. However, the channel is not correct. In the external habit model the variance of news about future excess return contributes considerably larger than that implied by real data, as the risk-aversion there is very volatile and persistent. And in the long-run risk model the variance of news about future's dividend growth can explain about 100% of the variance of unexpected excess return because of the high sensitivity of agent to the long-run risk of fundamentals. However, in the actual data dividend news

Preference	δ	γ	ϕ
	0.9914	2	0.9844
Consumption	g	σ	σ_w
	0.0016	0.0023	0.0161

Table 1.8. Parameters Choices for the External Habit Model

Preference	δ	γ	ψ	
	0.9997	10	1.4980	
Consumption	μ	ρ	ϕ_e	
	0.0015	0.975	0.038	
Dividend	μ_d	ϕ	π	φ_d
	0.0050	2.5	2.6	2.9553
Volatility	σ	υ	σ_w	
	0.0072	0.999	0.0000028	

Table 1.9. Parameters Choices for the Long-Run Risk Model

can only account for 20 percent of the variance of excess return. In summary, both models miss our Fact 1 and Fact 2.

1.9. Vector-Autoregression Analysis

Gali and Gambetti (2015) provide evidence about the response of real stock price to exogenous monetary policy shock using vector-autoregression (VAR) model. Here we use this impulse response from VAR analysis as an additional measure of the co-movement between stock and short-term bond markets. Being different from Gali and Gambetti (2015) we estimate the response of stock price to real risk-free shock instead of nominal risk-free rate shock. If money is neutral, nominal risk-free rate can only influence real stock price through real risk-free rate. As Gali and Gambetti (2015), the state space of our VAR model includes (log) output

	US data		External Habit	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	3.05	-2.06
E_{PD}	123.91	21.25	74.66	2.32
σ_{rs}	11.44	2.69	12.07	-0.23
σ_{PD}	62.42	17.54	26.17	2.07
$\rho_{PD,-1}$	0.97	0.02	0.95	0.85
c_5^2	-0.0038	0.0013	-0.0032	-0.46
R_5^2	0.1772	0.0828	0.4639	-3.46*
E_R	0.15	0.19	0.32	-0.84
σ_R	1.27	0.27	0.26	3.68*
$E_{\Delta D/D}$	0.41	0.18	0.47	-0.32
$\sigma_{\Delta D/D}$	2.88	0.80	2.79	0.11
$corr(PD, R)$	0.069	0.12	-0.956	8.27*
$Var(\widetilde{e}_d)$	21.1%	0.242	18.8%	0.10
$Var(\widetilde{e}_r)$	4.4%	0.026	1.1%	1.25
$Var(\widetilde{e}_e)$	50.8%	0.257	154.5%	-3.99*

Table 1.10. The External Habit Moments from MSM

y_t, (log) dividend d_t, (log) the risk-free rate r_t, and (log) stock price p_t. We define the state space

$$x_t^{VAR} \equiv [\Delta y_t, \Delta d_t, r_t, \Delta p_t]'$$

where Δ means first difference. The VAR model is

$$x_t^{VAR} = A_1 x_{t-1}^{VAR} + A_2 x_{t-2}^{VAR} + A_3 x_{t-3}^{VAR} + A_4 x_{t-4}^{VAR} + u_t$$

The identification strategy is that risk-free shock doesn't affect output and dividend contemporaneously, and risk-free rate doesn't respond contemporaneously to the innovations in stock prices. To facilitate implementation we just use Cholesky decomposition. Figure **1.1** displays the impulse response of stock price to risk-free

	US data		LRR	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	2.45	-0.52
E_{PD}	123.91	21.25	158.09	-1.61
σ_{rs}	11.44	2.69	7.24	1.56
σ_{PD}	62.42	17.54	36.81	1.46
$\rho_{PD,-1}$	0.97	0.02	0.96	0.35
c_5^2	-0.0038	0.0013	-0.0059	1.64
R_5^2	0.1772	0.0828	0.1705	0.08
E_R	0.15	0.19	-0.11	1.36
σ_R	1.27	0.27	0.26	3.68*
$E_{\Delta D/D}$	0.41	0.18	1.57	-6.35*
$\sigma_{\Delta D/D}$	2.88	0.80	3.71	-1.03
$corr(PD, R)$	0.069	0.12	0.608	-4.35*
$Var(\widetilde{e}_d)$	21.1%	0.242	96.6%	-3.12*
$Var(\widetilde{e}_r)$	4.4%	0.026	3.5%	0.33
$Var(\widetilde{e}_e)$	50.8%	0.257	52.7%	-0.08

Table 1.11. The Long-Run Risk Moments from MSM Estimation

rate shock. The red line represents the point estimated response of stock price, and the two blue lines represents 95% confidence bands. The positive risk-free rate shock leads to a slightly increase of stock price in the short-run, and ends up with permanent increase. But the confidence bands are too large to reject the absence of risk-free rate's effect on stock price. The impulse response of stock price to real risk-free rate shock is quite similar to the one to nominal risk-free rate shock in Gali and Gambetti (2015), and confirms the weak co-movement between stock and short-term bond markets.

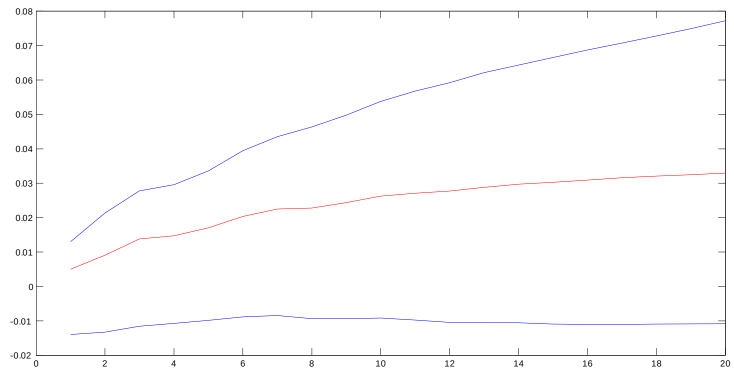

Figure 1.1. The Impulse Response of Stock Prices to Risk-free Rate Shock Using Realized Data.

Then, we replicate the same VAR analysis with simulated data from learning model, habit model and long-run risk model. Figure **1.2** to **1.4** displays the impulse responses of simulated stock price to risk-free rate shock in these models respectively. We can find that the impulse response in figure **1.2** matches the one in figure **1.1** well even quantitatively though we don't choose parameter values to match it. The point estimate of impulse response in habit model is negative consistent with model's negative correlation between PD ratio and risk-free rate. The impulse response in long-run risk model looks like figure **1.1**, but the upper bound is too high compared with data.

1.10. Conclusion

This chapter is an effort to enhance existing understanding on the co-movement between stock and short-term bond markets. Understanding this co-movement is

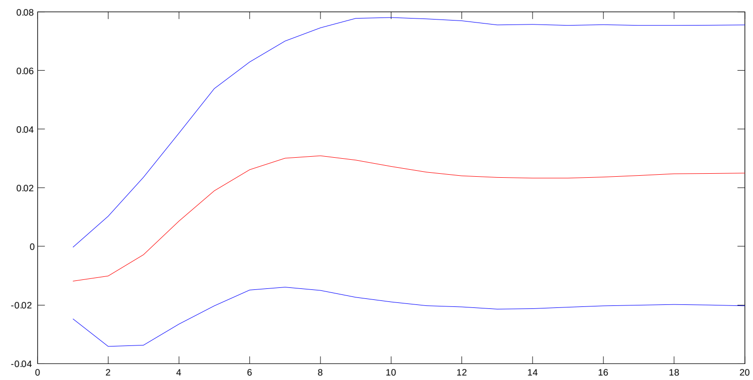

Figure 1.2. The Impulse Response of Stock Prices to Risk-free Rate Shock Using Simulated Data in Learning Model

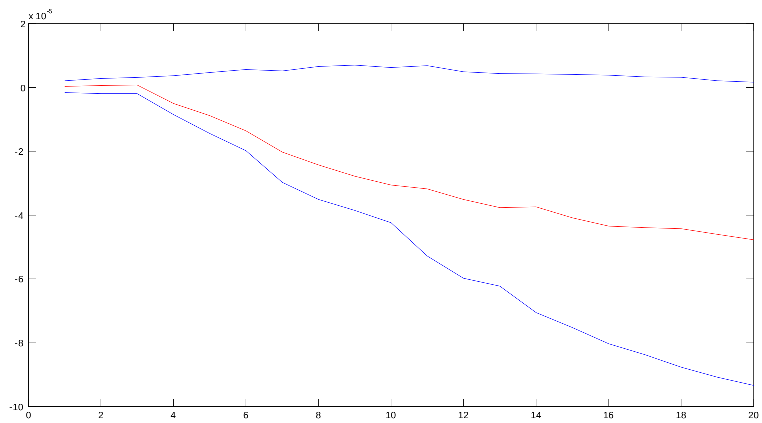

Figure 1.3. The Impulse Response of Stock Prices to Risk-free Rate Shock Using Simulated Data in Habit Model

important for both investors and policy makers. Empirical evidences suggest that the co-movement between these two markets is weak along two dimensions: the weak correlation between stock price-dividend ratio and risk-free rate and the low explanatory power (in terms of variance decomposition) of short-term interest

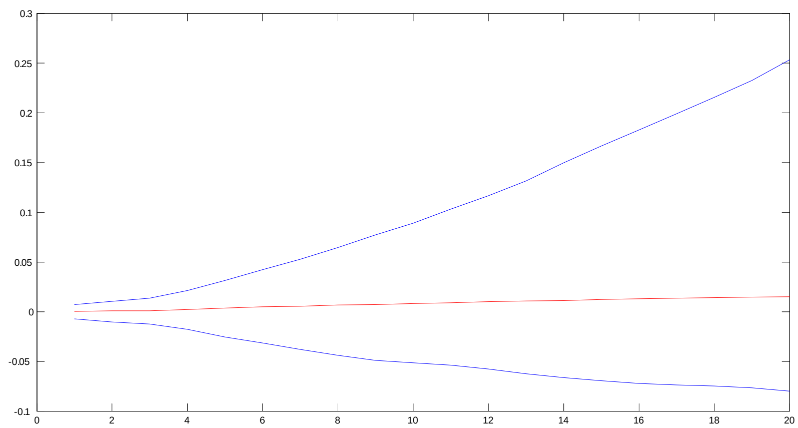

Figure 1.4. The Impulse Response of Stock Prices to Risk-free Rate Shock Using Simulated Data in Long-run Risk Model

rate on unexpected excess stock return. Although the weak co-movement has been observed for a long time, there has been a lack of attempt to find a model explaining this phenomenon. This paper shows that two asset pricing models with rational expectation cannot account for the weak co-movement because stock prices in these models are only driven by fundamental variables. Instead, this paper relaxes the assumption of rational expectation by allowing "Internally Rational" agents, who do not know the mapping from the fundamentals to equilibrium stock price. Agents learn about the stock price from realized outcomes. The self-referential property of this learning model generates the high volatility of stock price without the need for the large risk-free rate variation. The quantitative performance of the learning model based on the method of simulated moments confirms that it can simultaneously match the basic stock market facts and the weak co-movement between stock and short-term bond markets.

The finding that large stock price fluctuation can result from agents' subjective beliefs in addition to risk-free rate is valuable from a policy perspective. It is natural to challenge the effect of monetary policy on governing asset price volatility given that the channel for conducting monetary policy is through altering or influencing risk-free rate, but a detailed discussion of this paper's policy implication are reserved for future research.

1.11. Appendix

1.11.1. Data Sources

The data sample period is from 1927:2 to 2012:2. Since we choose to match the predictability of price-dividend ratio on five-year excess return, the effective sample size is up to 2007:2. The data about stock market behavior is downloaded from Robert Shiller's webpage (http://www.econ.yale.edu/~shiller/data.htm). Stock price is represented by "S&P 500 Composite Price Index". We directly take use of real stock index and real dividend calculated by Shiller and you can also find the details about calculation in the same webpage. The monthly data of stock index are transformed into quarterly by taking the value of the last month of the corresponding quarter. But quarterly dividend is computed as aggregating the dividends of three months of the considered quarter since the dividend is flow variable.

The risk-free rate is using 3-month Treasury Bill deflated by U.S. Consumer Price Index. The method of transforming monthly data into quarterly one is the same as stock index. These data is downloaded from the dataset of Federal Reserve Bank St. Louis.

At the same time, in order to calibrate collateral ratio U.S. current account data is also downloaded from FRB St. Louis. And for the total value of U.S. stock market we use "market capitalization of listed companies", which can be found in database of World Bank (http://data.worldbank.org/). Here we use the annual data and the sample is from 1988 to 2012.

1.11.2. Variance Decomposition

We introduce the approach of variance decomposition adopted in Campbell (1991) and Campbell and Ammer (1993). Theoretically the excess return e_{t+1} of the stock holding from the end of period t to period $t+1$ relative to the return on short bond can be expressed as

$$e_{t+1} - E_t e_{t+1} = (E_{t+1} - E_t)\left\{\sum_{j=0}^{\infty} \rho^j \Delta d_{t+1+j} - \sum_{j=0}^{\infty} \rho^j r_{t+1+j} - \sum_{j=1}^{\infty} \rho^j e_{t+1+j}\right\} \tag{1.26}$$

where e_t is excess return, d_t is dividend and r_t is risk-free rate.

To simplify the notation, equation (**1.26**) can be written as

$$\widetilde{e}_{t+1} = \widetilde{e}_{d,t+1} - \widetilde{e}_{r,t+1} - \widetilde{e}_{e,t+1} \tag{1.27}$$

where $\widetilde{e}_{t+1}$ is the unexpected excess return, $\widetilde{e}_{d,t+1}$ the news about future dividend growth , $\widetilde{e}_{r,t+1}$ news about future risk-free rate and $\widetilde{e}_{e,t+1}$ to be the term representing news about future excess return.

Therefore, the variance of unexpected excess return can be decomposed as

$$\begin{aligned} Var(\widetilde{e}_{t+1}) &= Var(\widetilde{e}_{d,t+1}) + Var(\widetilde{e}_{r,t+1}) \\ &\quad + Var(\widetilde{e}_{e,t+1}) - 2Cov(\widetilde{e}_{d,t+1}, \widetilde{e}_{r,t+1}) \\ &\quad - 2Cov(\widetilde{e}_{d,t+1}, \widetilde{e}_{e,t+1}) + 2Cov(\widetilde{e}_{r,t+1}, \widetilde{e}_{e,t+1}) \end{aligned} \tag{1.28}$$

These variables are directly unobservable but can be discovered from Vector-Autoregression. Write z_t as the state vector containing excess return e_t, risk-free rate r_t and price-dividend ratio $\frac{P_t}{D_t}$[10]

$$z_t = [e_t, r_t, \frac{P_t}{D_t}]'$$

The first-order VAR model is

$$z_{t+1} = Az_t + w_{t+1} \tag{1.29}$$

[10]Being different from six variables in state vector in Campbell (1991) and Campbell and Ammer (1993), only three variables here could be another reason for the high standard errors of statistics in Table 2.

With the VAR system we can compute $\widetilde{e}_{t+1}$, $\widetilde{e}_{r,t+1}$ and $\widetilde{e}_{e,t+1}$

$$\widetilde{e}_{t+1} \equiv e_{t+1} - E_t e_{t+1} = e1' w_{t+1} \tag{1.30}$$

$$\widetilde{e}_{e,t+1} \equiv (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j e_{t+1+j} = e1' \sum_{j=1}^{\infty} \rho^j A^j \epsilon_{t+1} = e1' \rho A (I - \rho A)^{-1} \epsilon_{t+1} \tag{1.31}$$

$$\widetilde{e}_{r,t+1} \equiv (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j r_{t+1+j} = e2' \sum_{j=0}^{\infty} \rho^j A^j \epsilon_{t+1} = e2' (I - \rho A)^{-1} \epsilon_{t+1} \tag{1.32}$$

where $e1$ and $e2$ are the first and second column of 3×3 identity matrix respectively.

Then, $\widetilde{e}_{d,t+1}$ can be treated as residual:

$$\widetilde{e}_{d,t+1} = \widetilde{e}_{t+1} + \widetilde{e}_{r,t+1} + \widetilde{e}_{e,t+1} \tag{1.33}$$

After recovering these unobservable variables, equation (**1.28**) is used to compute results on variance decomposition.

1.11.3. The Robustness of Fact 1 and Fact 2

Table **1.12** shows the statistical results of Fact 1 and Fact 2 using the post-war sample (1953:1 to 2012:2). Table **1.13** shows the results of Fact 1 and Fact 2 using ex-ante risk-free rate. The ex-ante risk-free is computed as subtracting the

Statistics	Data	SE
$corr(PD,R)$	0.026	0.110
$Var(\widetilde{e}_d)$	33.4%	0.266
$Var(\widetilde{e}_r)$	1.5%	0.007
$Var(\widetilde{e}_e)$	61.1%	0.291

Table 1.12. The Fact 1 and Fact 2 using Post-war Sample

Statistics	Data	SE
$corr(PD,R)$	-0.104	0.19
$Var(\widetilde{e}_d)$	14.8%	0.21
$Var(\widetilde{e}_r)$	3.2%	0.01
$Var(\widetilde{e}_e)$	51.2%	0.29

Table 1.13. The Fact 1 and Fact 2 using Ex-ante Risk-free Rate

forecast of inflation (data named "INFPGDP1YR" from the Survey of Professional Forecasts) from nominal rate of 3-month T-Bill. The sample size here is from 1970:2 to 2012:2 due to the availability of survey data. We can find that the results in table **1.12** and **1.13** are similar to the ones in table **1.2** and **1.3**.

1.11.4. The Robustness of Agents' Information

The true process for non-adjusted stock price growth is also modeled as the sum of a persistent component and of a transitory component

$$\begin{aligned}\frac{P_{t+1}}{P_t} &= e^m_{t+1} + \epsilon^m_{t+1},\ \epsilon^m_{t+1} \sim iiN(0, \sigma^2_{\epsilon,m}) \\ e^m_{t+1} &= e^m_t + \xi^m_{t+1},\ \xi^m_{t+1} \sim iiN(0, \sigma^2_{\xi,m})\end{aligned}$$

Agents can only observe the realizations of non-adjusted growth (the sum of persistent and transitory components), hence the requirement to forecast the persistent components e_t^m calls for a filtering problem. The priors of agents' beliefs can be centered at their rational expectation values and given by

$$e_0^m \sim N(a, \sigma_{0,m}^2)$$

and the variances of prior distributions should be set up to equal to the steady state Kalman filter uncertainty about e_t^m

$$\sigma_{0,m}^2 = \frac{-\sigma_{\xi,m}^2 + \sqrt{\sigma_{\xi,m}^4 + 4\sigma_{\xi,m}^2\sigma_{\epsilon,m}^2}}{2}$$

Then agents' posterior beliefs will be

$$e_t^m \sim N(m_t, \sigma_{0,m}^2)$$

And the optimal updating rule implies that the evolution of m_t is taking the form of

$$m_t = m_{t-1} + \frac{1}{\alpha^m}\left(\frac{P_{t-1}}{P_{t-2}} - m_{t-1}\right) \tag{1.34}$$

where $\alpha^m = \frac{\sigma_{\xi,m}^2 + \sqrt{\sigma_{\xi,m}^4 + 4\sigma_{\xi,m}^2\sigma_{\epsilon,m}^2}}{2\sigma_{\xi,m}^2}$ given by optimal (Kalman) gain. And agents think that non-adjusted price growth is uncorrelated with endowment growth.

	US data		Model	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	1.70	1.42
E_{PD}	123.91	21.25	117.89	0.28
σ_{rs}	11.44	2.69	10.69	0.29
σ_{PD}	62.42	17.54	84.65	-1.27
$\rho_{PD,-1}$	0.97	0.02	0.97	-0.18
c_5^2	-0.0038	0.0013	-0.0056	1.41
R_5^2	0.1772	0.0828	0.1301	0.57
E_R	0.15	0.19	0.11	0.19
σ_R	1.27	0.27	0.77	1.87
$E_{\Delta D/D}$	0.41	0.18	0.03	2.09
$\sigma_{\Delta D/D}$	2.88	0.80	2.90	-0.03
$corr(PD, R)$	0.069	0.12	-0.177	1.99
$Var(\widetilde{e}_d)$	21.1%	0.242	38.9%	-0.74
$Var(\widetilde{e}_r)$	4.4%	0.026	2.2%	0.82
$Var(\widetilde{e}_e)$	50.8%	0.257	63.8%	-0.51
$\widehat{\delta}$			0.9883	
$1/\widehat{\alpha}$			0.0071	
γ			10	

Table 1.14. Robustness: Different Learning Model Moments from MSM

Hence, under agents' knowledge of true endowment growth and subjective expectation of non-adjusted stock price growth m_t their subjective expectation of risk-adjusted stock price growth β_t is pinned down as

$$\beta_t = a^{-\gamma}\tau m_t$$

Simulation results are presented using such information set in table **1.14**. Comparing the results to those in table **1.6** and **1.7**, this model's quantitative performance is robust to the agents' information.

1.11.5. Projection Facilities

The projection facilities of agents' subjective beliefs β are

$$\omega(\beta) = \left\{ \begin{array}{cc} \beta & \text{if } x \leq \beta^L \\ \beta^L + \frac{\beta-\beta^L}{\beta+\beta^U-2\beta^L}(\beta^U - \beta^L) & \text{if } \beta^L < x \leq \beta^U \end{array} \right\} \tag{1.35}$$

And we calculate the thresholds β^L and β^U via similar methods utilized by Adam, Marcet and Nicolini (2016). However, the presence of time-varying risk-free rate R_t cannot surely guarantee that the price-dividend ratio will fall within the interval between 0 and 400. Yet, to avoid the rare event that price-dividend ratio jumps out the interval by construction, some constraints are imposed on simulated stock prices.

$$P_t = \left\{ \begin{array}{cc} P_t & \text{if } \frac{P_t}{D_t} < 400 \\ 400 * D_t & \text{if } \frac{P_t}{D_t} \geq 400 \end{array} \right\} \tag{1.36}$$

1.11.6. Simulation Method

We compute simulated model moments following Monte-Carlo procedure. The number of samples is set to $K = 1000$ and each sample has $N = 321$ observations matching stock market data sample from 1927:Q2 to 2007 Q2. In each sample, we first simulate the model to generate artificial data and calculate considered moments. Then, final values of these moments are taking the average of K samples'.

1.11.7. Details of MSM Estimation

1.11.7.1. Optimal Weight Matrix. Let T be the sample size, $(y_1, y_2, ..., y_T)$ the observed data sample, with y_t containing several variables. Define the sample moments as $\widehat{M_T} \equiv \frac{1}{T}\Sigma_{t=1}^{T} h(y_t)$ for a given moment function h. The sample statistics $\widehat{S}_T$ as in (**1.23**) can be written as the function of $\widehat{M_T}$

$$\widehat{S}_T \equiv S(\widehat{M_T})$$

The optimal weighting matrix should be the variance-covariance matrix of $\widehat{S}_T$. The variance-covariance matrix of $\widehat{M_T}$ can be estimated using standard Newey-West method. That is

$$\widehat{S}_{w,T} = \widehat{\Psi}_0 + \Sigma_{j=1}^{ms} w(j, ms)[\widehat{\Psi}_j + \widehat{\Psi}_j'], w(j, m) = 1 - j/(ms + 1) \tag{1.37}$$

where the sample j-th autocovariance $\widehat{\Psi}_j \equiv \Sigma_{t=j+1}^{T}[h(y_t) - \widehat{M_T}][h(y_{t-j}) - \widehat{M_T}]'$. And the Delta-Method implies that the sample variance-covariance matrix of $\widehat{S}_N$ can be calculated as following

$$\widehat{\Sigma}_{S,T} \equiv \frac{\partial S(M)}{\partial M'} \widehat{S}_{w,T} \frac{\partial S(M)'}{\partial M} \tag{1.38}$$

1.11.8. The Statistics, Moment Functions and Their Derivatives

1.11.8.1. The first twelve statistics. Here we first talk about all the statistics except variance decomposition.

The explicit function h^1 for calculating first twelve statistics in (**1.23**) is

$$h^1(y_t) \equiv \begin{bmatrix} rs_t \\ PD_t \\ (rs_t)^2 \\ (PD_t)^2 \\ PD_t PD_{t-1} \\ r^{s,20}_{t-20} \\ (r^{s,20}_{t-20})^2 \\ r^{s,20}_{t-20} PD_{t-20} \\ R_t \\ (R_t)^2 \\ D_t/D_{t-1} \\ (D_t/D_{t-1})^2 \\ R_t PD_t \end{bmatrix}$$

The first twelves statistics can be expressed as follows

$$S(M) \equiv \begin{bmatrix} E(rs_t) \\ E(PD_t) \\ \sigma_{rs} \\ \sigma_{PD} \\ \rho_{PD,-1} \\ c_5^2 \\ R_5^2 \\ E(R) \\ \sigma_R \\ E_{D/D} \\ \sigma_{D/D} \\ cov(R,PD) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ \sqrt{M_3 - (M_1)^2} \\ \sqrt{M_4 - (M_2)^2} \\ \frac{M_5 - (M_2)^2}{M_4 - (M_2)^2} \\ c_2^5(M) \\ R_5^2(M) \\ M_9 \\ \sqrt{M_{10} - (M_9)^2} \\ M_{11} \\ \sqrt{M_{12} - (M_{11})^2} \\ \frac{M_{13} - M_2 M_9}{\sqrt{M_4 - (M_2)^2}\sqrt{M_{10} - (M_9)^2}} \end{bmatrix}$$

where M_i denotes the i-th elements of M. The function $c_2^5(M)$ and $R_5^2(M)$ have the explicit expressions as

$$c^5(M) \equiv \begin{bmatrix} 1 & M_2 \\ M_2 & M_4 \end{bmatrix}^{-1} \begin{bmatrix} M_6 \\ M_8 \end{bmatrix}$$

$$R_5^2(M) \equiv 1 - \frac{M_7 - [M_6,\, M_8] c^5(M)}{M_7 - (M_6)^2}$$

Then, the derivatives of statistics function $S(M)$ with data moments M are

$\frac{\partial S_1}{\partial M_1} = 1$

$\frac{\partial S_2}{\partial M_2} = 1$

$\frac{\partial S_3}{\partial M_1} = \frac{-M_1}{S_3(M)}$ $\frac{\partial S_3}{\partial M_3} = \frac{1}{2S_3(M)}$

$\frac{\partial S_4}{\partial M_2} = \frac{-M_2}{S_4(M)}$ $\frac{\partial S_4}{\partial M_4} = \frac{1}{2S_4(M)}$

$\frac{\partial S_5}{\partial M_2} = \frac{2M_2(M_5-M_4)}{(M_4-M_2^2)^2}$ $\frac{\partial S_5}{\partial M_4} = -\frac{M_5-M_2^2}{(M_4-M_2^2)^2}$ $\frac{\partial S_5}{\partial M_5} = \frac{1}{M_4-M_2^2}$

$\frac{\partial S_6}{\partial M_j} = \frac{\partial c_2^5(M)}{\partial M_j}$ for $j = 2, 4, 6, 8$

$\frac{\partial S_7}{\partial M_2} = \frac{[M_6\ M_8]\frac{\partial c_2^5(M)}{\partial M_2}}{M_7-M_6^2}$ $\frac{\partial S_7}{\partial M_4} = \frac{[M_6\ M_8]\frac{\partial c_2^5(M)}{\partial M_4}}{M_7-M_6^2}$

$\frac{\partial S_7}{\partial M_6} = \frac{[c_1^5(M)+[M_6,M_8]\frac{\partial C^5(M)}{\partial M_6}](M_7-M_6^2)+2M_6[M_6,M_8]c^5(M)-2M_6M_7}{(M_7-M_6^2)^2}$

$\frac{\partial S_7}{\partial M_7} = \frac{M_6^2-[M_6\ M_8]c^5(M)}{(M_7-M_6^2)^2}$ $\frac{\partial S_7}{\partial M_8} = \frac{c_2^5(M)+[M_6\ M_8]\frac{\partial c_2^5(M)}{\partial M_8}}{M_7-M_6^2}$

$\frac{\partial S_8}{\partial M_9} = 1$

$\frac{\partial S_9}{\partial M_9} = \frac{-M_9}{S_9(M)}$ $\frac{\partial S_9}{\partial M_{10}} = \frac{1}{2S_9(M)}$

$\frac{\partial S_{10}}{\partial M_{11}} = 1$

$\frac{\partial S_{11}}{\partial M_{11}} = \frac{-M_{11}}{S_{11}(M)}$ $\frac{\partial S_{11}}{\partial M_{12}} = \frac{1}{2S_{11}(M)}$

$\frac{\partial S_{12}}{\partial M_2} = \frac{-M_9S_4S_9+(M_{13}-M_2M_9)S_9\frac{M_2}{S_4}}{(S_4S_9)^2}$ $\frac{\partial S_{12}}{\partial M_4} = \frac{(M_2M_9-M_{13})S_9\frac{1}{2S_4}}{(S_4S_9)^2}$

$\frac{\partial S_{12}}{\partial M_9} = \frac{-M_2S_4S_9+(M_{13}-M_2M_9)S_4\frac{M_9}{S_9}}{(S_4S_9)^2}$ $\frac{\partial S_{12}}{\partial M_{10}} = \frac{(M_2M_9-M_{13})S_4\frac{1}{2S_9}}{(S_4S_9)^2}$

$\frac{\partial S_{12}}{\partial M_{13}} = \frac{1}{\sqrt{M_4-(M_2)^2}\sqrt{M_{10}-(M_9)^2}}$

1.11.8.2. The statistics for variance decomposition. The three interested statistics are $var(\widetilde{e}_{d,t+1})/var(\widetilde{e}_{t+1})$, $var(\widetilde{e}_{r,t+1})/var(\widetilde{e}_{t+1})$, $var(\widetilde{e}_{e,t+1})/var(\widetilde{e}_{t+1})$.

The unobservable variables $\widetilde{e}_{t+1}, \widetilde{e}_{d,t+1}, \widetilde{e}_{r,t+1}, \widetilde{e}_{e,t+1}$ defined in Campbell and Ammer (1993) are computed from VAR model.

The state vector in VAR is $x_t = [e_t, R_t, PD_t]'$. These variables are demeaned.

The VAR(1) process is expressed as

$$x_{t+1} = Ax_t + \epsilon_{t+1}$$

The SUR representation of this VAR(1) can be stacked as

$$Y = X\Gamma + u$$

where $X = \begin{bmatrix} x_1^{'} \\ x_2^{'} \\ . \\ . \\ x_{T-1}^{'} \end{bmatrix}$, $Y = \begin{bmatrix} x_2^{'} \\ x_3^{'} \\ . \\ . \\ x_T^{'} \end{bmatrix}$, $u = \begin{bmatrix} \epsilon_2^{'} \\ \epsilon_3^{'} \\ . \\ . \\ \epsilon_T^{'} \end{bmatrix}$, $\Gamma = A^{'}$. Hence, we can estimate Γ using OLS method as

$$\Gamma = (\frac{1}{T-1}\sum_{t=1}^{T-1} x_t x_t^{'})^{-1}(\frac{1}{T-1}\sum_{t=1}^{T-1} x_t x_{t+1}^{'})$$

Here in the vector of $h^2(y_t)$ we need the vector data $x_t x_t^{'}$ and $x_t x_{t+1}^{'}$. Then,

$$A(N) = \Gamma^{'} = [N_1^{-1} N_2]^{'}$$

where N_1, N_2 are the sample mean of $x_t x_t^{'}$ and $x_t x_{t+1}^{'}$.

Then, the error term ϵ_{t+1} can be expressed as

$$\epsilon_{t+1} = x_{t+1} - A(N)x_t$$

According to the expression of $\widetilde{e}_{t+1}, \widetilde{e}_{d,t+1}, \widetilde{e}_{r,t+1}$ and $\widetilde{e}_{e,t+1}$,

$$\begin{aligned}\widetilde{e}_{t+1} &= e1'\epsilon_{t+1} \\ &= H_1\epsilon_{t+1}\end{aligned}$$

$$\begin{aligned}\widetilde{e}_{r,t+1} &= e2'(I-\rho A(N))^{-1}\epsilon_{t+1} \\ &= H_2\epsilon_{t+1}\end{aligned}$$

$$\begin{aligned}\widetilde{e}_{e,t+1} &= e1'\rho A(N)(I-\rho A(N))^{-1}\epsilon_{t+1} \\ &= H_3\epsilon_{t+1}\end{aligned}$$

$$\begin{aligned}\widetilde{e}_{d,t+1} &= (e1' + e2'(I-\rho A(N))^{-1} + e1'\rho A(N)(I-\rho A(N))^{-1})\epsilon_{t+1} \\ &= H_4\epsilon_{t+1}\end{aligned}$$

then unconditional $var(\epsilon_{t+1})$

$= E((x_{t+1}-A(N)x_t)(x_{t+1}-A(N)x_t)') - [E(x_{t+1}-A(N)x_t)][E(x_{t+1}-A(N)x_t)]'$

$= E(x_{t+1}x_{t+1}' - x_{t+1}x_t'A(N)' - A(N)x_tx_{t+1}' + A(N)x_tx_t'A(N)') - ((Ex_{t+1})(Ex_{t+1})' - (Ex_{t+1})(Ex_t)'A(N)' - A(N)(Ex_t)(Ex_{t+1})' + A(N)(Ex_t)(Ex_t)'A(N)')$

Since x_t is stationary demeaned variables, the above expression can be simplified into

$var(\epsilon_{t+1}) = E(x_{t+1}x_{t+1}^{'} - x_{t+1}x_t^{'}A(N)' - A(N)x_t x_{t+1}' + A(N)x_t x_t^{'}A(N)')$

Then, the sample variance should be

$$var(\epsilon_{t+1}) = N_1 - N_2^{'}A(N)^{'} - A(N)N_2 + A(N)N_1A(N)'$$

Therefore,

$$var(\widetilde{e}_{t+1}) = H_1 var(\epsilon_{t+1})H_1^{'} \tag{1.39}$$

$$var(\widetilde{e}_{r,t+1}) = H_2 var(\epsilon_{t+1})H_2^{'} \tag{1.40}$$

$$var(\widetilde{e}_{r,t+1}) = H_3 var(\epsilon_{t+1})H_3^{'} \tag{1.41}$$

$$var(\widetilde{e}_{e,t+1}) = H_4 var(\epsilon_{t+1})H_4^{'} \tag{1.42}$$

Write down each element in the vector.

$$h^2(y_t) \equiv \begin{bmatrix} e_{t-1}^2 \\ R_{t-1}^2 \\ PD_{t-1}^2 \\ R_{t-1}e_{t-1} \\ PD_{t-1}e_{t-1} \\ PD_{t-1}R_{t-1} \\ e_{t-1}e_t \\ R_{t-1}R_t \\ PD_{t-1}PD_t \\ R_{t-1}e_t \\ R_te_{t-1} \\ PD_{t-1}e_t \\ PD_te_{t-1} \\ PD_{t-1}R_t \\ PD_tR_{t-1} \end{bmatrix}$$

And $[M_{14}\ M_{15}\ M_{16}, \ldots\ M_{28}]$ are the sample mean of the each element in $h^2(y_t)$.

$$N_1 \equiv \begin{bmatrix} M_{14} & M_{17} & M_{18} \\ M_{17} & M_{15} & M_{19} \\ M_{18} & M_{19} & M_{16} \end{bmatrix} \quad N_2 \equiv \begin{bmatrix} M_{20} & M_{24} & M_{26} \\ M_{23} & M_{21} & M_{28} \\ M_{25} & M_{27} & M_{22} \end{bmatrix}$$

According to (**1.39**) to (**1.42**), though the exact analytical expression is available the partial derivatives of three variance decomposition statistics with respect to sample moments should be extremely complicated. Hence, we use numerical method to approximate these derivatives. The method is called centered differencing and the principle is

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

Take an example to describe this method.

$$\frac{\partial \frac{var(\widetilde{e}_{d,t+1})}{var(\widetilde{e}_{t+1})}}{\partial M_{14}} \approx \frac{\frac{var(\widetilde{e}_{d,t+1})}{var(\widetilde{e}_{t+1})}(M_{14}+h, M_{15}, ..., M_{28}) - \frac{var(\widetilde{e}_{d,t+1})}{var(\widetilde{e}_{t+1})}(M_{14}-h, M_{15}, ..., M_{28})}{2h}$$

1.11.9. Robustness of Parameter Estimation

This section shows that the quantitative performances of the learning model and two RE models are robust to the parameter estimation. Here dividend parameters are calibrated instead of estimated. In particular, it means that we calibrate a, $\sigma_{\Delta D/D}$ in the learning model, g, σ in the external habit model and μ_d, φ_d in the long-run risk model. Then, we estimate the rest of parameters in the parameter vectors Ω, Ω^{EH} and Ω^{LRR}. Table **1.15** contains the quantitative outcomes for the learning model, table **1.16** for the external habit model and table **1.17** the long-run risk model. The results here that are close to the ones in section 1.7 and 1.8, supporting the notion that models' performances are robust to the parameter variations.

	US data		Model ($\delta \leq 1$)	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	2.41	-0.43
E_{PD}	123.91	21.25	92.61	1.47
σ_{rs}	11.44	2.69	12.41	-0.36
σ_{PD}	62.42	17.54	67.64	-0.30
$\rho_{PD,-1}$	0.97	0.02	0.94	1.20
c_5^2	-0.0038	0.0013	-0.0065	-2.05
R_5^2	0.1772	0.0828	0.0991	0.94
E_R	0.15	0.19	0.15	0.04
σ_R	1.27	0.27	0.74	1.95
$E_{\Delta D/D}$	0.41	0.18	0.41	0
$\sigma_{\Delta D/D}$	2.88	0.80	2.88	0
$corr(PD, R)$	0.069	0.12	-0.172	1.95
$Var(\widetilde{e}_d)$	21.1%	0.242	42.4%	-0.88
$Var(\widetilde{e}_r)$	4.4%	0.026	1.8%	0.98
$Var(\widetilde{e}_e)$	50.8%	0.257	55.5%	-0.18
$\widehat{\delta}$			1	
$1/\widehat{\alpha}$			0.0086	
γ			4.5	

Table 1.15. Learning Model Moments from MSM

1.11.10. The Convergence of Least Square Learning to RE

In section 1.6, agents update their beliefs of risk-adjusted stock price growth β_t using constant gain learning. Well known, constant gain learning doesn't converge to RE since E-stability condition isn't satisfied. We here consider that agents use least square learning to update their beliefs and check the convergence of least square learning. Hence, instead of (**1.21**) the belief updating process become

$$\beta_t = \beta_{t-1} + \frac{1}{\alpha_t}((\frac{Y_{t-1}}{Y_{t-2}})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}) \tag{1.43}$$

	US data		External Habit	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	2.88	-1.63
E_{PD}	123.91	21.25	77.06	2.20
σ_{rs}	11.44	2.69	9.88	0.58
σ_{PD}	62.42	17.54	25.91	2.08
$\rho_{PD,-1}$	0.97	0.02	0.96	0.38
c_5^2	-0.0038	0.0013	-0.0025	-1.00
R_5^2	0.1772	0.0828	0.4961	-3.85*
E_R	0.15	0.19	0.34	-0.94
σ_R	1.27	0.27	0.28	3.62*
$E_{\Delta D/D}$	0.41	0.18	0.41	0
$\sigma_{\Delta D/D}$	2.88	0.80	2.88	0
$corr(PD, R)$	0.069	0.12	-0.96	8.30*
$Var(\widetilde{e}_d)$	21.1%	0.242	21.2%	-0.004
$Var(\widetilde{e}_r)$	4.4%	0.026	2.2%	0.85
$Var(\widetilde{e}_e)$	50.8%	0.257	153.9%	-4.00*
$\widehat{g}$			0.0014	
$\widehat{\sigma}$			0.0024	
$\widehat{\phi}$			0.9881	
$\widehat{\delta}$			0.9929	

Table 1.16. The External Habit Moments from MSM

$$\alpha_t = \alpha_{t-1} + 1 \quad t \geq 2 \tag{1.44}$$

$$\alpha_1 \geq 1 \quad \text{given}$$

Since both ϵ_t^y and ϵ_t^d follow log-normal distributions, ϵ_t^y, $\epsilon_t^d \geq 0$. Then, consumption $Y_t \geq 0$ and dividend $D_t \geq 0$ with probability one. We assume the

	US data		**LRR**	
	Moment	SE	Moment	t-stat
E_{rs}	2.25	0.39	1.69	1.44
E_{PD}	123.91	21.25	93.91	1.41
σ_{rs}	11.44	2.69	5.68	2.14
σ_{PD}	62.42	17.54	15.80	2.66*
$\rho_{PD,-1}$	0.97	0.02	0.95	0.68
c_5^2	-0.0038	0.0013	-0.0084	3.56*
R_5^2	0.1772	0.0828	0.1499	0.33
E_R	0.15	0.19	-0.27	2.18
σ_R	1.27	0.27	0.24	3.77*
$E_{\Delta D/D}$	0.41	0.18	0.41	0
$\sigma_{\Delta D/D}$	2.88	0.80	2.89	-0.01
$corr(PD,R)$	0.069	0.12	0.767	-5.63*
$Var(\widetilde{e}_d)$	21.1%	0.242	114.5%	-3.86*
$Var(\widetilde{e}_r)$	4.4%	0.026	4.98%	-0.23
$Var(\widetilde{e}_e)$	50.8%	0.257	47.9%	0.11
$\widehat{\delta}$			1	
$\widehat{\psi}$			1.7111	
$\widehat{\mu}_d$			0.0014	
$\widehat{\varphi}_d$			2.2800	

Table 1.17. The Long-Run Risk Moments from MSM Estimation

existence of some positive bounds for ϵ_t^y, ϵ_t^d such that

$$\Pr((\epsilon_t^y)^{1-\gamma} < U^y) = 1$$

$$\Pr(\epsilon_t^d < U^d) = 1$$

We first show that the projection facility in Appendix A.5 will almost surely cease to be binding after some finite time. The projection facility implies that

(1.45)
$$\Delta\beta_t = \begin{cases} \alpha_t^{-1}[(a(\epsilon_t^y)^{1-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}] & \text{if } \beta_{t-1} + \alpha_t^{-1}[(a(\epsilon_t^y)^{1-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}] < \beta^U \\ 0 & \text{otherwise} \end{cases}$$

We can have that

$$\beta_t \leq \beta_{t-1} + \alpha_t^{-1}[(a(\epsilon_t^y)^{1-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}] \quad (1.46)$$

$$|\beta_t - \beta_{t-1}| \ \leq \alpha_t^{-1}|(a(\epsilon_t^y)^{1-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}| \quad (1.47)$$

hold for all t a.s. because if $\beta_t < \beta^U$ this holds with equality and if $\beta_{t-1} + \alpha_t^{-1}[(a(\epsilon_t^y)^{1-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}] \geq \beta^U$ then $|\beta_t - \beta_{t-1}| = 0$.

Substituting β recursively backwards in (**1.46**) delivers the following expression

$$\begin{aligned} \beta_t &\leq \frac{1}{t-1+\alpha_1}[(\alpha_1-1)\beta_0 + \sum_{j=0}^{t-1}(a\epsilon_t^y)^{-\gamma}\frac{P_j}{P_{j-1}}] \\ &= \underbrace{\frac{t}{t-1+\alpha_1}[\frac{(\alpha_1-1)\beta_0}{t} + \frac{1}{t}\sum_{j=0}^{t-1}a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d]}_{=T_1} + \\ &\quad \underbrace{\frac{1}{t-1+\alpha_1}[\sum_{j=0}^{t-1}\frac{\Pi\Delta\beta_j}{1-\Pi\beta_j}a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d]}_{=T_2} \end{aligned} \quad (1.48)$$

where $\Pi \equiv \delta + \theta(\frac{1}{\overline{R}} - \varphi)/(a^{-\gamma}\tau)$ and the second line follows from equation (**1.20**) and (**1.22**) when R_t holds at unconditional mean $\overline{R}$. Clearly, $T_1 \to 1 * (0 + E(a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d) = a^{1-\gamma}\rho_\epsilon = \beta^{RE}$ as $t \to 0$. Then, we will establish that $|T_2| \to 0$ as $t \to 0$.

$$
\begin{aligned}
|T_2| &\leq \frac{1}{t-1+\alpha_1} \sum_{j=0}^{t-1} \frac{\Pi a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d}{1-\Pi\beta_j} |\Delta\beta_j| \\
&\leq \frac{U^y U^d}{t-1+\alpha_1} \frac{\Pi a^{1-\gamma}}{1-\Pi\beta^U} \sum_{j=0}^{t-1} |\Delta\beta_j|
\end{aligned} \tag{1.49}
$$

where the first inequality comes from the triangle inequality and the second inequality follows from the bounds for ϵ_j^y, ϵ_j^d and β_j. Next, observe that

$$
\begin{aligned}
(a\epsilon_t^y)^{-\gamma} \frac{P_t}{P_{t-1}} &= \frac{1-\Pi\beta_{t-1}}{1-\Pi\beta_t} a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d \\
&< \frac{1}{1-\Pi\beta_t} a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d \\
&< \frac{a^{1-\gamma}U^y U^d}{1-\Pi\beta^U}
\end{aligned} \tag{1.50}
$$

Combining equation (**1.47**) and (**1.50**), we have that

$$
\begin{aligned}
\frac{1}{t-1+\alpha_1} \sum_{j=0}^{t-1} |\Delta\beta_j| &\leq \frac{1}{t-1+\alpha_1} \sum_{j=0}^{t-1} \alpha_j^{-1} \frac{a^{1-\gamma}U^y U^d}{1-\Pi\beta^U} \\
&= \frac{a^{1-\gamma}U^y U^d}{1-\Pi\beta^U} \frac{1}{t-1+\alpha_1} \sum_{j=0}^{t-1} \frac{1}{j-1+\alpha_1}
\end{aligned}
$$

The convergence of the over-harmonic series implies that

$$\frac{1}{t-1+\alpha_1}\sum_{j=0}^{t-1}|\Delta\beta_j| \ \to 0 \text{ for all } t \text{ a.s.}$$

Then, (**1.49**) implies that $|T_2| \to 0$ as $t \to 0$. Taking the lim sup on both side of (**1.48**), it follows from $T_1 \to \beta^{RE}$ and $|T_2| \to 0$ that

$$\lim \sup_{t\to\infty} \beta_t \leq \beta^{RE} < \beta^U$$

Therefore, the projection facility is binding finitely many periods with probability one.

We now proceed to prove that β_t converges to β^{RE} from that time onwards. Consider for a given realization a finite period $\bar{t}$ where the projection facility is not binding for all $t > \bar{t}$. The simple algebra gives

$$\begin{aligned}
\beta_t &= \frac{1}{t-\bar{t}+\alpha_{\bar{t}}}[\alpha_{\bar{t}}\beta_{\bar{t}} + \sum_{j=\bar{t}}^{t-1}(a\epsilon_j^y)^{-\gamma}\frac{P_j}{P_{j-1}}] \\
&= \frac{t-\bar{t}}{t-\bar{t}+\alpha_{\bar{t}}}[\frac{1}{t-\bar{t}}\sum_{j=\bar{t}}^{t-1}a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d \\
&\quad +\frac{1}{t-\bar{t}}\sum_{j=\bar{t}}^{t-1}a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d\frac{\Pi\Delta\beta_j}{1-\Pi\beta_j} + \frac{1}{t-\bar{t}}\alpha_{\bar{t}}\beta_{\bar{t}}] \qquad (1.51)
\end{aligned}$$

for all $t > \bar{t}$. Similar operations as before then deliver

$$\frac{1}{t-\bar{t}}\sum_{j=\bar{t}}^{t-1}a^{1-\gamma}(\epsilon_j^y)^{-\gamma}\epsilon_j^d\frac{\Pi\Delta\beta_j}{1-\Pi\beta_j} \to 0$$

a.s. for $t \to \infty$. Finally, taking the limit on both sides of (**1.51**) establishes

$$\beta_t \to E(a^{1-\gamma}(\epsilon_t^y)^{-\gamma}\epsilon_t^d) = a^{1-\gamma}\rho_\epsilon = \beta^{RE}$$

a.s. as $t \to \infty$. The least square learning thus globally converges to the RE.

CHAPTER 2

Understanding AH Premium in China Stock Market

2.1. Introduction

This chapter studies the stock prices difference named AH premium in the connected China A- and H-share markets, which is an interesting anomaly in asset markets. The stocks listed in China mainland stock exchanges (Shanghai and Shenzhen) are called A-share, and the one listed in Hong Kong exchange are called H-share. There are 88 companies dual-listed in A-share and H-share markets called AH-share, which are identical with respect to shareholder rights, such as voting and profit-sharing. Most of AH-share companies are big ones, especially state-owned enterprises, accounting for 20% of total market value in A-share market. Hang Seng China AH Premium Index plotted in figure **2.1** measures the weighted averaged price difference of AH-share. Index equaling 100 means that A-share are trading at par with H-share, larger than 100 for A-share trading at a premium versus H-share, smaller than 100 for A-share trading at a discount versus H-share.

Figure **2.1** shows that AH-share prices are always different even though they have the same fundamentals in Shanghai and Hong Kong markets. Before November 2014, Shanghai and Hong Kong markets were segmented that mainland investors are not allowed to invest in Hong Kong market and so for foreign investors in Shanghai market. The price difference of dual-listed stocks in the segmented markets is widely studied in the literature. Fernald and Rogers (2002) attribute the discount of China B-share stock (only for foreigners) to A-share stock (only for the domestic) to the fact that Chinese investors have a higher discount rate than foreigners. Chan, Menkveld and Yang (2008) show the evidence that AB-share premium is caused by the fact that foreign investors, who trade B-share, have an informational disadvantage relative to domestic investors, who trade A-share. While, Mei, Scheinkman and Xiong (2009) propose that trading caused by investors' speculative motives can help explain a significant fraction of the price difference between the perfect segmented dual-class AB-share.

The previously segmented Shanghai and Hong Kong markets, however, become connected since the starting of Shanghai-Hong Kong stock connect program in November, 2014. The AH premium index should converge to 100 according to the standard present-value asset pricing theory, but it divergences dramatically to arrive at almost 150 and then fluctuates between 120 and 150. There are very few works on the price difference in the connected markets except Froot and Dabora (1999) focusing on only three twin stocks. This paper studies the price difference in the sample with 88 dual-listed stocks.

This paper first investigates whether the present-value heterogeneous asset pricing models can generate sufficiently high, volatile and persistent AH premium. The heterogeneity could be reflected in agents' different discount factors (Fernald and Rogers, 2002), diverse beliefs caused by asymmetric information (Chan, Menkveld and Yang, 2008), and different transaction costs and dividend taxes (Froot and Dabora, 1999). The model environment could be complete market or incomplete market, and stock prices equalling with the discounted sum of expected future dividends makes agents like fundamental investors. We find that different risk aversions, discount factors, and diverse beliefs cannot produce any AH premium, transaction costs are so small that could be ignored, and dividend taxes are possible to generate constant 6% premium. The generalized model we show in section 2.4 illuminates that prices for A-share and H-share in the connected markets are the same in each period when we only have variations across agents without variations across two shares.

The failure of present-value models in producing AH premium motivates us to propose an 'Internal Rationality' learning model as Adam, Marcet and Nicolini (2016), in which agents who do not know the mapping from the fundamentals to price and optimize their behaviors based on their subjective beliefs about all variables that are beyond their control. Given the subjective beliefs we specify, agents behave as speculators and optimally update their expectations about capital gains using Kalman filter. Agents' subjective expectations in turn influence equilibrium stock prices, and the realized stock prices feed back into agents' beliefs. If agents have initial different beliefs or different learning speeds between A-share and H-share, agents can have different subjective beliefs on them which generate different stock prices.

Figure 2.1. Hang Seng China AH Premium Index

Finally we study the convergence traders' strategy, which relies on the price convergence of similar or identical assets. A typical convergence trade would short sell in AH-share in Shanghai market, and long buy it in the Hong Kong market. But the learning model shows that prices cannot converge in the short-run. Since the longest duration of short-selling tool is one-year, we calculate the distribution of money-making for 3, 6, 9 months and one year. We find that convergence traders have a large probability to lose money.

2.2. Overview of China Stock Market

China mainland stock market is relatively young and started in 1990 with the establishment of two exchanges: the Shanghai Stock Exchange (SSE) and Shenzhen Stock Exchange (SZSE). The number of listed companies was just 13 in the starting time. During the period from 1990 to 2015, the Chinese economy has performed well with averaged annual 10% GDP growth rate. The extraordinary economic growth undoubtedly leads to the rapid growth of financial markets. The market value of China stock market (excluding Hong Kong and Taiwan) reaches $8.4 trillion at the end of 2015 which makes it the second largest one in the world, even though the ratio of market capitalization to GDP is relatively low at about 60%. The number of listed companies also rises to 2827. The main boards of the Shanghai and Shenzhen Stock Exchanges list larger and more mature stocks, like the NYSE in the US. The Shenzhen Stock Exchange also includes two other boards, the Small and Medium Enterprise Board and the ChiNext Board, also known as the Growth Enterprise Board, which provide capital for smaller and high-technology stocks, like the NASDAQ in the US. Mainland stock market has a dual-share system. Before the starting of Shanghai-Hong Kong Stock connect, mainland investors can invest only in A-share, while foreign investors can invest only in B-share.

Figure **2.2** shows the dynamics of stock prices indexes in mainland Shanghai and Shenzhen markets from 1995 to 2015 respectively. Mainland stock price experiences two episodes of obvious boom and bust, one is 2006-2007 and the other is 2014-2015. Stock prices reached the historical peak in 2007 from the bottom in 2005, then quickly busted. Then, from 2008 to 2014 the market generally trended down. Therefore, Allen et al. (2015) thinks that the performance of China stock market has been disappointing, especially compared with the growth of GDP. The

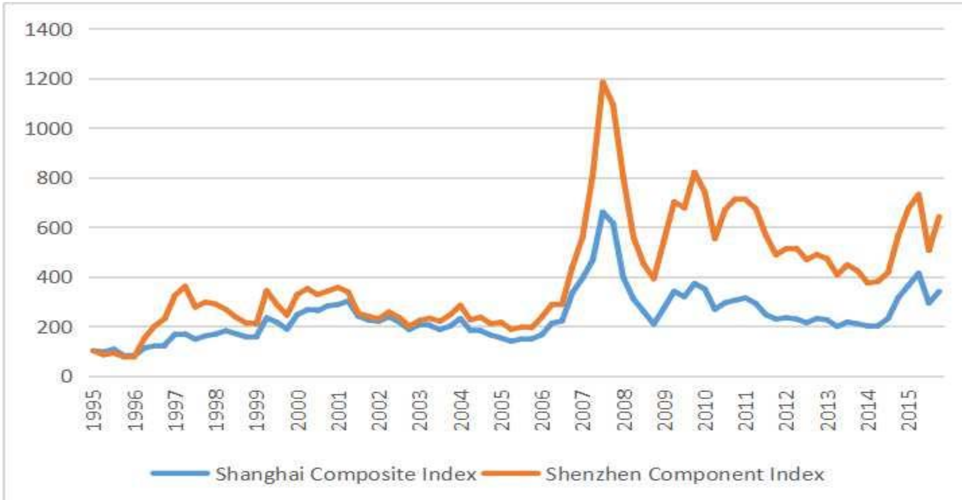

Figure 2.2. China Stock Price Indexes

Statistics	SSE	SZSE	S&P500	CRSP
Std.dev. stock return σ_{rs}	17.06%	22.32%	8.17%	8.69%
Std.dev. stock return σ_{PD}	277.83	167.07	47.26	54.94

Table 2.1. China and US Stock Market Volatility

market price boomed again in the second half of 2014, and almost doubled in the middle of 2015. One distinguished characteristic in China stock market is that stock trading is new to most of participants, 80% of them are individual investors (Mei, Scheinkman and Xiong, 2009). Given the typical Chinese investor's lack of experience, it is reasonable to hypothesize that these investors would often disagree about stock valuation and as a result would behave more like the speculators. The larger volatility in Chinese stock markets than US shown in table **2.1** support this.

2.3. Present-Value Models

In this section, we build present-value models and explore on the potential factors driving the price difference. We consider variations of discount factor, risk aversion, beliefs on fundamental, dividend tax and transaction cost across agents in the complete market and incomplete markets.

2.3.1. Models in Complete Market

Let's describe the economy in the complete market. Basically it is a Lucas tree model with two type of agents.

2.3.1.1. Rational Expectation. In the beginning we endow the agents with objective beliefs i.e. rational expectation. The type i investors in the economy account for a fraction of $\mu^i > 0$ of population $i \in \{1, 2\}$ respectively, where $\mu^1 + \mu^2 = 1$. Type 1 agent stands for mainland investor and 2 for Hong Kong investor. The two types may differ with respect to their degree of risk aversion, discount factor.

Investors' portfolio includes A-share, H-share and contingent bonds. Agents trade A-share and H-share with each other in this economy. $S_t^{1,A}, S_t^{1,H}, S_t^{2,A}, S_t^{2,H}$ are denoted as A-and H-share stocks that agent 1 and agent 2 buy respectively on period t. One unit of A-share and H-share pay investors the same dividend as

$$D_t^A = D_t^H = D_t$$

For convenience and without loss of generality we assume the exogenous dividend process in the complete market economy is $i.i.d$ taking two values of D_h and D_l at each period, where $Prob(D_l) = \pi, Prob(D_h) = 1 - \pi$. We start exploring on complete market with Arrow securities $B_t(D_h)$ and $B_t(D_l)$ that pays 1 unit of consumption if dividend payment on $t + 1$ is high and low respectively.

Commodity goods market clearing condition is

$$2D_t = \mu^1 C_t^1 + \mu^2 C_t^2$$

Arrow securities markets clear conditions are

$$\mu^1 B_t^1(D_j) + \mu^2 B_t^2(D_j) = 0 \ \forall j = h, l$$

A- and H-share market clearing are

$$\mu^1 S_t^{1,Z} + \mu^2 S_t^{2,Z} = 1 \quad \forall Z = A, H$$

We assume utility function is increasing, concave and continuously differentiable. The investors' maximization problem is

$$\max_{\{C_t^i, S_t^{i,A}, S_t^{i,H}, B\}} E_0 \sum_{t=0}^{\infty} (\delta^i)^t u_i(C_t^i)$$

$$s.t. \qquad S_t^{i,A}P_t^A + S_t^{i,H}P_t^H + C_t^i + B_t^i(D_h)Q_t(D_h) + B_t^i(D_l)Q_t(D_l)$$
$$= \quad S_{t-1}^{i,A}(P_t^A + D_t^A) + S_{t-1}^{i,H}(P_t^H + D_t^H) + B_{t-1}^i$$

And we can impose no-short-selling constraints as

$$0 \leq S_t^{i,A}$$
$$0 \leq S_t^{i,H}$$

F.O.Cs lead to

$$\delta^1 E_t \frac{u_c^1(C_{t+1}^1)}{u_c^1(C_t^1)} = \delta^2 E_t \frac{u_c^2(C_{t+1}^2)}{u_c^2(C_t^2)} \tag{2.1}$$

where u_c^i is marginal utility of type i agent.

This result of full insurance features complete market. Although agents could have different discount factors and risk aversions, the property of full insurance gives rise to the same stochastic discount factor (SDF) as equation (**2.1**). Given the stock prices having the present-value expression, the same dividends and SDFs produce no price difference between A-share and H-share. Therefore discount factors and different relative risk aversions across two agents are not able to drive any price difference in the connected markets.

	c_h	c_l	b_h	b_l
Agent 1	0.8736	0.5514	-0.0036	0.0988
Agent 2	1.1264	0.4486	0.0036	-0.0988

Table 2.2. Contingent Bond Holding

Now we are exploring the black box that how agents arrive at full insurance through trading contingent bonds. This is not studied in the literature to our best knowledge. For illustration, we fully solve an exercise where both agents have CRRA utility with same discount factor but agent 1 is more risk averse than agent 2 by approximating expectations in Euler equations with log linear polynomials.

With state contingent bonds, stock A and stock H are indeterminate 'redundant' assets. So we can keep agents' holding of the two assets fixed over time. Intuitively agent 1 prefer smoother consumption than agent 2 does because agent 1 is more relative risk averse. The full insurance is achieved through agent 1 always buying low contingent bond and selling high contingent bond. We confirm this by numerically solving the quantity of bond holdings, and find that agent 1's consumption are relatively smoother across nature states than agent 2's while their consumption varies with endowment. These observations are shown in table **2.2**. The algorithm in detail is in appendix 2.8.1.

Different dividend taxes are suspected to be eligible in driving the price difference. We then investigate whether dividends taxes could generate quantitatively sufficient AH premium. Hence we now add tax into the model.

$$\begin{aligned} & S_t^{i,A} P_t^A + S_t^{i,H} P_t^H + C_t^i + B_t^i(D_h) Q_t(D_h) + B_t^i(D_l) Q_t(D_l) \\ = \; & S_{t-1}^{i,A}(P_t^A + (1-\tau^{i,A}) D_t^A) + S_{t-1}^{i,H}(P_t^H + (1-\tau^{i,H}) D_t^H) + B_{t-1}^i \forall i \end{aligned}$$

In data, $\tau^{1,A}$ is 5%, $\tau^{1,H}$ is 20%, $\tau^{2,A}$ is 10% and $\tau^{2,H}$ is 10%. The agent offering higher price will be the marginal one for the security. Hence type 1 is marginal in Shanghai and type 2 is marginal in Hong Kong as can be seen in the F.O.Cs below. Without loss of generality, we assume that two agents have log utility in the following.

The F.O.Cs in this case become

$$P_t^A = E_t \delta^1 \frac{C_t^1}{C_{t+1}^1} [P_{t+1}^A + (1-\tau^{1,A}) D_{t+1}] \; 0 \leq S_t^1$$

$$P_t^A > E_t \delta^2 \frac{C_t^2}{C_{t+1}^2} [P_{t+1}^A + (1-\tau^{2,A}) D_{t+1}] \; S_t^2 = 0$$

Similarly for H share, we have

$$P_t^H > E_t \delta^1 \frac{C_t^1}{C_{t+1}^1} [P_{t+1}^H + (1-\tau^{1,H}) D_{t+1}] \; S_t^1 = 0$$

$$P_t^H = E_t \delta^2 \frac{C_t^2}{C_{t+1}^2} [P_{t+1}^H + (1-\tau^{2,H}) D_{t+1}] \; 0 \leq S_t^2$$

A-share price is the discounted sum of future dividend by type 1's SDF, so for H-share by type 2's SDF. Hence we obtain

$$P_t^A = E_t[\sum_{j=1}^{\infty}(\delta^1)^j \prod_{k=1}^{j} \frac{C_{t+k-1}^1}{C_{t+k}^1}(1-\tau^{1,A})D_{t+j}] \tag{2.2}$$

$$P_t^H = E_t[\sum_{j=1}^{\infty}(\delta^2)^j \prod_{k=1}^{j} \frac{C_{t+k-1}^2}{C_{t+k}^2}(1-\tau^{2,A})D_{t+j}] \tag{2.3}$$

And since the tax is constant and can be factored out. Dividing (**2.2**) by (**2.3**) leads to

$$\frac{P_t^A}{P_t^H} = \frac{1-\tau^{1,A}}{1-\tau^{2,H}}$$

Hence price ratio is constant over time with approximate ratio of 1.056, which contradicts with the observation that standard deviation of AH premium fluctuates between 1.2 and 1.5. Furthermore it's worthwhile to notice that if $\tau^{1,A} = \tau^{1,H}$ and $\tau^{2,A} = \tau^{2,H}$, then one would notice same price of A and H shares after a quick look at the F.O.Cs. Hence if mainland investors face the same dividend tax of A share and H share and so do the foreign investors, there would be no price difference in this complete market framework even if dividends taxes are not the same across agents.

2.3.2. Diverse Belief

Furthermore, there is popular narrative in the market that says foreign investors are pessimistic about Chinese economy but mainland people have optimistic views on the contrary. These diverse beliefs may be due to imperfect information or other reasons. Dealers and market analysts tend to tell this kind of story to rationalize the AH premium. So let's see what happens when two agents have diverse beliefs on fundamental in this environment while setting the market frictions discussed above aside. Towards this end, we depart a bit from rational expectation model in the way that two agents are endowed with diverse beliefs on fundamental.

Let's assume agent 1 is optimistic while agent 2 is pessimistic. More important let's assume agent 1 is relatively right compared to agent 2. We will see that in the complete market agent 1 will take advantage of his information superiority so that he accumulates assets and consume more goods. Formally let $i \in \{o, p\}$ where o stands for optimistic agent and p stands for pessimistic agent. Optimistic agent perceives $Prob(D_h) = u$ while pessimistic agent perceives $Prob(D_l) = v$ where $u > v$. And the true objective probability follows

$$Prob(D_l) = \pi, Prob(D_h) = 1 - \pi$$

Let $\mathbf{1}(D_h)$ and $\mathbf{1}(D_l)$ be the indicator function that take value 1 if D_h and D_l happen respectively. F.O.Cs lead to

$$\frac{C^p_{t+1}}{C^o_{t+1}} = \frac{C^p_t}{C^o_t}\frac{\delta^p}{\delta^o}\frac{v}{u}\mathbf{1}(D^h_{t+1}) + \frac{C^p_t}{C^o_t}\frac{\delta^p}{\delta^o}\frac{1-v}{1-u}\mathbf{1}(D^l_{t+1}) = [\frac{v}{u}\mathbf{1}(D^h_{t+1}) + \frac{1-v}{1-u}\mathbf{1}(D^l_{t+1})]\frac{\delta^p}{\delta^o}\frac{C^p_t}{C^o_t} \tag{2.4}$$

where $[\frac{v}{u}\mathbf{1}(D^h_{t+1}) + \frac{1-v}{1-u}\mathbf{1}(D^l_{t+1})]$ is denoted as A_{t+1} for simplicity.

The assumption that agent 1 is relatively right leads to $(\frac{v}{u})^\pi(\frac{1-v}{1-u})^{1-\pi} > 1$, which implies that agent 1 will gradually consume the total dividends in the economy consistent with the conclusion in Bloom and Easley (2006). Rearranging (**2.4**) gives

$$\delta^o\frac{C^o_t}{C^o_{t+1}} = A_{t+1}\delta^p\frac{C^p_t}{C^p_{t+1}}$$

which links SDF of optimistic agent with that of pessimistic agent.

$$\begin{aligned}
P^{o,A}_t &= E^o_t[\sum_{j=1}^{\infty}(\delta^o)^j\prod_{k=1}^{j}\frac{C^o_{t+k-1}}{C^o_{t+k}}D_{t+j}] \\
&= E^o_t[\sum_{j=1}^{\infty}(\delta^o)^j\prod_{k=1}^{j}\frac{C^p_{t+k-1}}{C^p_{t+k}}A_{t+j}D_{t+j}] \\
&= \sum_{j=1}^{\infty}[v(\delta^p)^j\prod_{k=1}^{j}\frac{C^p_{t+k-1}}{C^p_{t+k}}D_h + (1-v(\delta^p)^j)\prod_{k=1}^{j}\frac{C^p_{t+k-1}}{C^p_{t+k}}D_l] \\
&= E^p_t[\sum_{j=1}^{\infty}(\delta^p)^j\prod_{k=1}^{j}\frac{C^p_{t+k-1}}{C^p_{t+k}}D_{t+j}] \\
&= P^{p,A}_t
\end{aligned}$$

Above equation shows that there is an unique pricing SDF in the complete market, hence both optimistic and pessimistic are marginal investors. Due to the fact that A-share and H-share release the same amount of dividend at each period, there is no price difference in each period i.e $P_t^{o,A} = P_t^{p,A} = P_t^{o,H} = P_t^{o,H}$. Although the SDFs of the two agents are different and linked by A_{t+1}, the subjectively expected SDFs are still the same. Hence the diverse beliefs on dividends can't give rise to any price difference.

In addition to analytical solution we can also solve this model by approximating expectation in Euler equations with exponentiated log linear polynomial, and find that agents achieve full insurance through contingent bond exchange. The algorithm for this case is on appendix 2.8.2. Agent 1 is more right with respect to the true probability, and so he accumulates assets and consume more while agent 2 accumulates debt and consume less. In the long run agent 1 consumes the total dividend while agent 2 get nothing. This is consistent with Bloom and Easly (2006).

The bond holding converges to steady state value, which is 100 in our setting. The bond holdings are shown on figure **2.3**. However, even if the two agents have diverse beliefs about the economy situation, price difference still remains in silence. Hence the diverse belief for the dividend or economy only leads the two agents to hold opposite amount of bonds but not to regard the stock price differently. We also find that it is the relative rightness of the perceived beliefs that drive the more right agent accumulate bonds and the other agent accumulate debt rather than

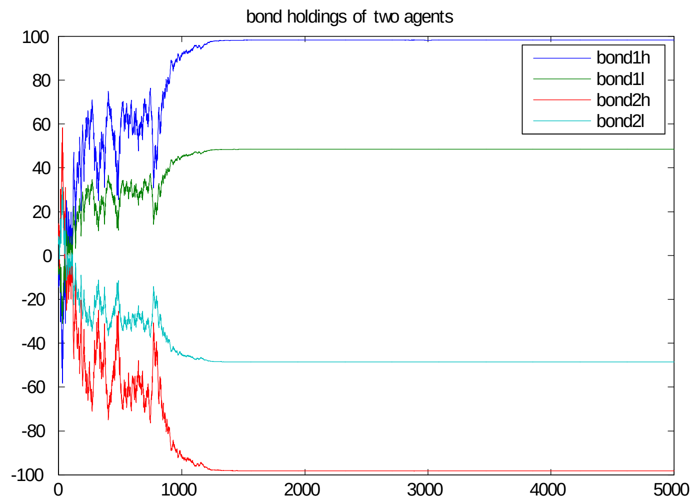

Figure 2.3. Agents' Holding of Contigent Bonds

their degree of optimism. For example, even if agent 1 is relatively optimistic, he will also holding bond because of his information advantage. We also try different degree of risk aversion in order to maintain consistency with rational expectation cases. We find that the results of no price difference remain true, and the bond holdings in the long run converge to another steady state level. If we add dividend tax in this context, the price difference occurs but are not quantitatively desirable in the same way that the rational expectation case does.

2.3.3. Models in Incomplete Market

2.3.3.1. Rational Expectation. Through out the previous section, the assumption of complete market plays a key role in producing one unique SDF which enables us to derive analytical price ratio formula. Under the circumstance of incomplete market, agents can't trade Arrow securities freely to adjust SDF, which seems to be a problem at the first glance. So we suspect that incomplete market perhaps make some difference. Hence in the current section, we turn to investigate on incomplete market.

We then consider an environment without state contingent bond. The simple discrete dividend process is no longer appropriate for incomplete market. We follows the literature and assume a standard dividend process as

$$\frac{D_t}{D_{t-1}} = a\epsilon_t^d$$

where $log\epsilon_t^d \sim i.i.N(-\frac{s_d^2}{2}, s_d^2)$ and $a \geq 1$. And budget constraint in this case for $i = 1, 2$ becomes

$$S_t^{i,A}P_t^A + S_t^{i,H}P_t^H + C_t^i = S_{t-1}^{i,A}(P_t^A + (1-\tau^{i,A})D_t^A) + S_{t-1}^{i,H}(P_t^H + (1-\tau^{i,H})D_t^H)$$

To avoid Ponzi scheme, the standard no-short-selling constraint is assumed

$$S_t^{i,j} \geq 0, \forall i = 1,2 \ \forall j = A, H$$

Typically we don't obtain analytical solutions for price ratio when it comes to incomplete market because we are not equipped with the equation that links the two agents' SDFs. Here to keep parsimonious we assume that two types of agents have the same risk aversions and discount factors, but have different dividend taxes. In the later, we will argue that different risk aversions and discount factors cannot generate any AH premium in the incomplete market. However dividend taxes have to be kept in this economy.

We solve the model numerically using Parameterized Expectation Algorithm (PEA) as Marcet and Singleton (1999). We find that during the dynamics, agent 1 holds more and more A shares and less and less H shares while agent 2 does the opposite. This is attributed to the dividend tax structure with $\tau^{1,A} < \tau^{2,A}$ and $\tau^{2,H} < \tau^{1,H}$. However P_t^A and P_t^H are the same during a long period because the four F.O.Cs all hold with equality and the lower bound is reached only after a long time. Hence the two agents are both marginal agent for a long time.

This observation is illustrated in figure **2.4**. Once the asset holding reaches to the lower bound, as is in the complete market the price difference quantitatively deficient occurs again as a result of dividend tax structure and this case is degenerated to a 'autarky' world in the sense that mainland investors only hold A-share and foreign investors H-share.

2.3.3.2. Diverse Belief. As in the complete market, we also turn to the case with diverse beliefs. Then agents may have subjective beliefs on dividend process.

$$\frac{D_t}{D_{t-1}} = a_i \epsilon_t^d, \forall i = 1, 2$$

where larger a_i is associated with optimism and smaller a_i with pessimism. We solve the two agents two assets incomplete market model with PEA method. We

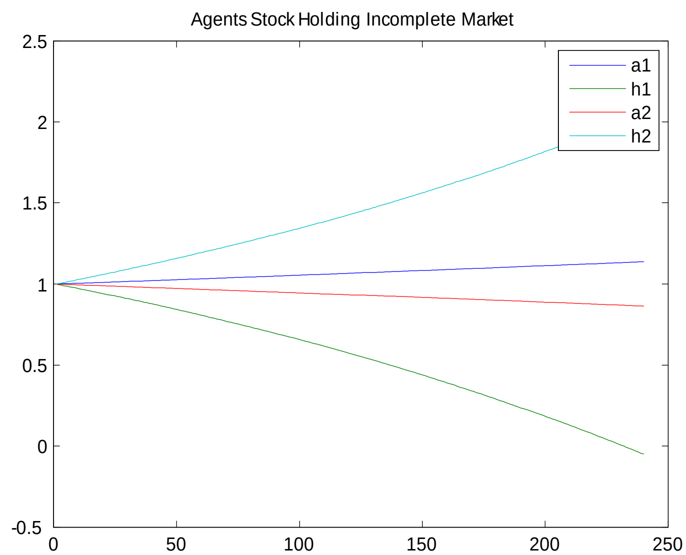

Figure 2.4. Agents' Stock Holding in the Incomplete Market

find that the result in 2.3.2 remains. Hence the diverse belief has nothing to do with price difference even in the incomplete market.

2.4. A General Discussion on Sources of Price Difference under Present-Value Model

We of course have not covered every possible model and it is also impossible to do it. However we give a general discussion here on the criteria determining whether a given factor has the potential to drive AH premium in the connected markets. In this section we will discuss the necessary conditions to generate price difference in a general way nesting all the cases we have showed in the above analysis and other cases we have not covered yet.

2.4.1. Variations across Agents

Proposition 1: Variations across agents has nothing to do with price difference. This proposition means that prices of the two shares with same dividend stream are same in each period if there are only variations across agents but not variations across two shares. If type i agent is marginal investor in A-share market, then from F.O.C we have

$$P_t^A = E_t^i f(SDF_t^i, P_{t+1}^A, D_{t+1}, z_{i,A})$$

where f is a generic function and $z_{i,A}$ represents all possible market specific factors such as transaction cost and market liquidity. E_t^i can capture agent i's expectation on fundamentals. SDF^i could be of any type such as the habit type in Campbell and Cochrane (1999) and the long run risk type in Bansal and Yaron (2004). Functional form of f should be the same for a given agent across shares in equilibrium. Furthermore the functional form could vary with different models so we denote as abstract function f. Therefore f accommodates to any model with variation across agents that we have mentioned or those have not come up with yet.

When we only have variations across agents, then $z_{1,A} = z_{1,H} = z_1$ and $z_{2,A} = z_{2,H} = z_2$ even though $z_1 \neq z_2$. Then agent i is also the marginal investor in H-share market. The above F.O.C also applies to H-share. That is

$$P_t^H = E_t^i f(SDF_t^i, P_{t+1}^H, D_{t+1}, z_{i,H}]$$

Then A-share and H-share entertain exact the same F.O.Cs in this case. We use m_t to denote the marginal investor pricing the asset in period t in equilibrium:

$$m_t = arg\ max_{i \in \{1,2\}} E_t^i f(SDF_t^i, P_{t+1}, D_{t+1}, z_i)$$

By mapping from fundamentals to stock price and assuming no bubbles we obtain

$$P_t^A = E_t^{m_t} g(\{SDF_{t+j-1}^{m_{t+j-1}}\}_{j=1}^{\infty}, \{D_{t+j}\}_{j=1}^{\infty}, \{z_{m_{t+j-1}}\}_{j=1}^{\infty})$$

$$P_t^H = E_t^{m_t} g(\{SDF_{t+j-1}^{m_{t+j-1}}\}_{j=1}^{\infty}, \{D_{t+j}\}_{j=1}^{\infty}, \{z_{m_{t+j-1}}\}_{j=1}^{\infty})$$

Hence prices for A-share and H-share are the same for each period when we only have variations across agents without variations across two shares. Intuitively if there is nothing different across shares, they are the same goods. Then no matter how equilibrium prices are determined in the present-value model, there should not be any price difference. Thus diverse belief, different discount factors and different risk aversions among the two agents do not give rise to the price difference.

2.4.2. Variations across Shares leading to Price Difference

We have to make a given agent regard the A-share and H-share differently even though the same dividend rather than make something different across the two agents, since connection enables two types agents to join into one trading group.

Transaction cost is one of the variations across shares. Generally, the transaction cost here includes financial tax, cost of changing currency and exchange rate change. The financial tax in Hong Kong stock market is 0.118%, in Shanghai is 0.169%. Currency change cost is less than 0.5% through Shanghai-Hong Kong Connect Program. And Hong Kong Dollar is expected to appreciate at mean 1.64% against RMB measured by exchange rate future. So such small transaction cost is impossible to produce desirable quantitative results. Government control can be another variation. Some people hold the long-standing view that Chinese government directly control Shanghai stock market frequently. But this is not true. Since 2000 it only happens one time that when Shanghai stock price bubble burst at the end of June 2015, Chinese government required state-owned investment banks to support stock price by taking long positions to avoid severe financial crisis in the worry of the high leverage held by many Chinese investors. When stock prices was stabilized in August, Chinese governments intervention quickly stepped away.

Another point is about liquidity. The higher liquidity of the stock, the higher price of it. One popular measure of the liquidity is the proportion of no-price-change days of a stock over the sample period (Mei, Scheinkman and Xiong, 2009). Based on daily data for the period 2006-2016, A-share averaged 0.65% of trading days with no price changes, while the corresponding H-share averaged 1.05%. This suggests that A-share is just a little bit more liquid than H-share. We doubt the small difference of liquidities can produce such high and volatile AH premium.

We want to provide a parsimonious way to understand AH premium. When agents don't know the pricing mapping from fundamentals to stock prices and behave like speculators, agents can have different beliefs about capital gains between A-share and H-share markets. Agent's different beliefs could make the agent think A-share and H-share not the same stock, which matches bankers, traders and normal Chinese people's view on the stock market. We are not claiming that we know exactly what are going on in their mind but this sort of story is a dominant view in Chinese market. Hence in the following section we turn to a simple learning model.

2.5. An "Internal Rationality" Learning Model

Section 2.3 and 2.4 have shown that heterogeneous agents present-value asset pricing models in either complete or incomplete markets are not able to generate sufficient AH premium. This section, hence, presents an "Internal Rationality" learning model based on Adam, Marcet and Nicolini (2016) to explain such high and volatile AH premium.

2.5.1. Model Environment

A unit of AH-share stock with dividend claim D_t can be traded in both Shanghai and Hong Kong markets. In addition to D_t, each agent receives an endowment Y_t of perishable consumption goods. So the total supply of the consumption goods in the economy is then given by the feasibility condition $C_t = Y_t + 2D_t$. Following traditional setting in asset pricing literature, dividend and endowment growth rates follow i.i.d. lognormal processes

$$\frac{D_t}{D_{t-1}} = a\epsilon_t^d, \log \epsilon_t^d \sim iiN(-\frac{s_d^2}{2}, s_d^2)$$

$$\frac{C_t}{C_{t-1}} = a\epsilon_t^c, \log \epsilon_t^c \sim iiN(-\frac{s_c^2}{2}, s_c^2)$$

where endowment and dividend growth rates share the same mean a, and $(\log \epsilon_t^d, \log \epsilon_t^y)$ are joint-normally distributed with correlation $\rho_{y,d}$, and s_d and s_c are standard deviations of this joint normal distribution.

The economy is populated by a unit mass of infinite-horizon agents. We model each agent $i \in [0, 1]$ to have the same standard time-separable CRRA utility function and the same subjective beliefs. This fact, however, is not the common knowledge among agents.

The specification of agent i's expected life-time utility function is

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma}$$

where C_t^i is the consumption profile of agent i, δ denotes the time discount factor, and γ is the risk-aversion parameter. Instead of the objective probability measure, expectation is formed using the subjective probability measure $\mathcal{P}$ that describes probability distributions for all external variables. Section 2.5.2 contains more details.

Agent's choices are subjected to standard budget constraint as following

$$C_t^i + P_t^A S_t^{A,i} + P_t^H S_t^{H,i} = (P_t^A + D_t) S_{t-1}^{A,i} + (P_t^H + D_t) S_{t-1}^{H,i} + Y_t$$

where $S_t^{A,i}$, $S_t^{H,i}$, P_t^A and P_t^H are defined as section 2.3. To avoid Ponzi schemes and to insure existence of a maximum the following bounds are assumed to hold

$$\begin{aligned} \underline{S} &\leq S_t^{A,i} \leq \overline{S} \\ \underline{S} &\leq S_t^{H,i} \leq \overline{S} \end{aligned}$$

We only assume the bounds $\underline{S}$ and $\overline{S}$ are finite.

2.5.2. Probability Space

This subsection explicitly describes the general joint probability space of the external variables. In the competitive economy, each agent considers the joint process of endowment, dividend and stock prices $\{Y_t, D_t, P_t^A, P_t^H\}$ as exogenous to his decision problem. Rational expectations imply that agents exactly know the mapping from a history of endowment Y_t and dividend D_t to equilibrium stock price P_t^A and P_t^H. Stock prices hence just carry redundant information. But if the rational expectation assumption is relaxed, as shown in Adam and Marcet (2011) such that agents do not know such mapping because of the non-existence of common knowledge on agents' identical preferences and beliefs, then equilibrium stock price P_t^A and P_t^H should be included in the underlying state space. We then define the probability space as $(\mathcal{P}, \mathcal{B}, \Omega)$ with $\mathcal{B}$ denoting the corresponding $\sigma-$Algebra of

Borel subsets of Ω and $\mathcal{P}$ denoting the agent's subjective probability measure over $(\mathcal{B}, \Omega)$. The state space Ω of realized exogenous variables is

$$\Omega = \Omega_Y \times \Omega_D \times \Omega_{P^A} \times \Omega_{P^H}$$

where Ω_X is the space of all possible infinite sequences for the variable $X \in [Y, D, P^A, P^H]$. Thereby, a specific element in the set Ω is an infinite sequence $\omega = \{Y_t, D_t, P_t^A, P_t^H\}_{t=0}^{\infty}$. The expected utility with probability measure $\mathcal{P}$ is defined as

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma} \equiv \int_{\Omega} \sum_{t=0}^{\infty} \delta^t \frac{C_t^i(\omega)^{1-\gamma}}{1-\gamma} d\mathcal{P}(\omega)$$

Agent i makes contingent plans for endogenous variables C_t^i, $S_t^{A,i}$, $S_t^{H,i}$ according to the policy function mapping in the following

$$(C_t^i, S_t^{A,i}, S_t^{H,i}) : \Omega^t \to R^3$$

where Ω^t represents the set of histories from period zero up to period t.

2.5.3. Optimality Conditions

Since the objective function is concave and the feasible set is convex, the agent's optimal plan is characterized by the first order conditions

$$(C_t^i)^{-\gamma} P_t^A = \delta E_t^{\mathcal{P}}((C_{t+1}^i)^{-\gamma}(P_{t+1}^A + D_{t+1})) \tag{2.5}$$

$$(C_t^i)^{-\gamma} P_t^H = \delta E_t^{\mathcal{P}}((C_{t+1}^i)^{-\gamma}(P_{t+1}^H + D_{t+1})) \tag{2.6}$$

Before exploring why subjective beliefs can explain AH premium, we first briefly review the unique RE solution given by

$$P_t^{A.RE} = \frac{\delta a^{1-\gamma}\rho_\epsilon}{1-\delta a^{1-\gamma}\rho_\epsilon} D_t \tag{2.7}$$

$$P_t^{H.RE} = \frac{\delta a^{1-\gamma}\rho_\epsilon}{1-\delta a^{1-\gamma}\rho_\epsilon} D_t \tag{2.8}$$

where $\rho_\epsilon = E_t^{\mathcal{P}}[(\epsilon_{t+1}^c)^{1-\gamma}\epsilon_{t+1}^d] = e^{\gamma(1+\gamma)\frac{s_c^2}{2}} e^{-\gamma\rho_{c,d}s_c s_d}$. Obviously, RE solutions always generate $P_t^{A.RE} = P_t^{H.RE}$.

We now characterize the equilibrium outcome under learning. According to the arguments in Adam, Marcet and Nicolini (2016), out of strict rational expectations we may have $E_t^{\mathcal{P}}[C_{t+1}^i] \neq E_t^{\mathcal{P}}[C_{t+1}]$ even if in the equilibrium $C_{t+1}^i = C_{t+1}$ holds ex-post. But we can make the same approximations in the following as they do

$$E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}(P_{t+1}^A + D_{t+1})] \simeq E_t^{\mathcal{P}}[(\frac{C_{t+1}}{C_t})^{-\gamma}(P_{t+1}^A + D_{t+1})] \tag{2.9}$$

$$E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}(P_{t+1}^H + D_{t+1})] \simeq E_t^{\mathcal{P}}[(\frac{C_{t+1}}{C_t})^{-\gamma}(P_{t+1}^H + D_{t+1})] \tag{2.10}$$

The following assumption provides sufficient conditions for this to be the case:

Assumption 1 We assume that Y_t is sufficiently large and the $E_t^{\mathcal{P}} P_{t+1}^{A(H)}/D_t < \overline{M}$ for some $\overline{M} < \infty$ so that, given finite asset bounds $\underline{S}$ and $\overline{S}$, the approximations (**2.9**) and (**2.10**) hold with sufficient accuracy.

We then can define the subjective expectations of risk-adjusted stock price growths as

$$\beta_t^A \equiv E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}(P_{t+1}^A)] \tag{2.11}$$

$$\beta_t^H \equiv E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}(P_{t+1}^H)] \tag{2.12}$$

We also assume that agents know the true processes of consumption and dividend growths. The definitions of β_t^A and β_t^H together with two first order conditions (**2.5**) and (**2.6**) give rise to the asset pricing equations

$$P_t^A = \frac{\delta a^{1-\gamma}\rho_\epsilon}{1-\delta\beta_t^A} D_t \tag{2.13}$$

$$P_t^H = \frac{\delta a^{1-\gamma}\rho_\epsilon}{1-\delta\beta_t^H} D_t \tag{2.14}$$

From equation (**2.13**) and (**2.14**), our learning model is possible to generate AH premium if $\beta_t^A \neq \beta_t^H$.

2.5.4. Beliefs Updating Rule

This section fully specifies the subjective probability distribution $\mathcal{P}$, and derives the optimal belief updating rule for subjective beliefs β_t^A and β_t^H. Similar to the arguments in Adam, Marcet and Nicolini (2016), in agents' beliefs the true processes for risk-adjusted stock price growths in both Shanghai and Hong Kong markets can be modeled as the sum of a persistent component and of a transitory component

$$(2.15)\qquad \begin{aligned} (\frac{C_{t+1}}{C_t})^{-\gamma}\frac{P_{t+1}^A}{P_t^A} &= e_t^A + \epsilon_t^A,\ \epsilon_t^A \sim iiN(0, \sigma_{\epsilon,A}^2) \\ e_t^A &= e_{t-1}^A + \xi_t^A,\ \xi_t^A \sim iiN(0, \sigma_{\xi,A}^2) \end{aligned}$$

$$(2.16)\qquad \begin{aligned} (\frac{C_{t+1}}{C_t})^{-\gamma}\frac{P_{t+1}^H}{P_t^H} &= e_t^H + \epsilon_t^H,\ \epsilon_t^H \sim iiN(0, \sigma_{\epsilon,A}^2) \\ e_t^H &= e_{t-1}^H + \xi_t^H,\ \xi_t^H \sim iiN(0, \sigma_{\xi,A}^2) \end{aligned}$$

where e_t^A and e_t^H are persistent components, ϵ_t^A and ϵ_t^H are transitory components. One way to justify these processes is that they are compatible with RE. According to equation (**2.7**) and (**2.8**), the rational expectation of risk-adjusted price growth is $E_t[(\frac{C_{t+1}}{C_t})^{-\gamma}\frac{P_{t+1}^A}{P_t^A}] = E_t[(\frac{C_{t+1}}{C_t})^{-\gamma}\frac{P_{t+1}^H}{P_t^H}] = a^{1-\gamma}\rho_\epsilon$. Hence, the previous setup encompasses the rational expectation equilibrium as a special case when agents believe $\sigma_{\xi,A}^2 = \sigma_{\xi,H}^2 = 0$ and assign probability one to $e_0^A = e_0^H = a^{1-\gamma}\rho_\epsilon$.

Then, we allow for a non-zero variance $\sigma^2_{\xi,A}$ and $\sigma^2_{\xi,H}$. Agents can only observe the realizations of risk-adjusted growths (the sum of persistent and transitory components), hence the requirement to forecast the persistent components e^A_t and e^H_t calls for a filtering problem. The priors of agents' beliefs can be centered at some initial values and given by

$$e^A_0 \sim N(\beta^A_0, \sigma^2_{0,A})$$

$$e^H_0 \sim N(\beta^H_t, \sigma^2_{0,H})$$

and the variances of prior distributions should be set up to equal with steady state Kalman filter uncertainty about e^A_t and e^H_t

$$\sigma^2_{0,A} = \frac{-\sigma^2_{\xi,A} + \sqrt{\sigma^4_{\xi,A} + 4\sigma^2_{\xi,A}\sigma^2_{\epsilon,A}}}{2}$$

$$\sigma^2_{0,H} = \frac{-\sigma^2_{\xi,H} + \sqrt{\sigma^4_{\xi,H} + 4\sigma^2_{\xi,H}\sigma^2_{\epsilon,H}}}{2}$$

Then agents' posterior beliefs will be

$$e^A_t \sim N(\beta^A_t, \sigma^2_{0,A})$$

$$e^H_t \sim N(\beta^H_t, \sigma^2_{0,H})$$

And the optimal updating rule implies that the evolution of β_t^A and β_t^H is taking the form just as constant gain adaptive learning

$$\beta_t^A = \beta_{t-1}^A + \frac{1}{\alpha^A}((\frac{C_{t-1}}{C_{t-2}})^{-\gamma}\frac{P_{t-1}^A}{P_{t-2}^A} - \beta_{t-1}^A) \tag{2.17}$$

$$\beta_t^H = \beta_{t-1}^H + \frac{1}{\alpha^H}((\frac{C_{t-1}}{C_{t-2}})^{-\gamma}\frac{P_{t-1}^H}{P_{t-2}^H} - \beta_{t-1}^H) \tag{2.18}$$

where $\alpha^A = \frac{\sigma_{\xi,A}^2+\sqrt{\sigma_{\xi,A}^2+4\sigma_{\xi,A}^2\sigma_{\epsilon,A}^2}}{2\sigma_{\xi,A}^2}$ and $\alpha^H = \frac{\sigma_{\xi,H}^2+\sqrt{\sigma_{\xi,H}^2+4\sigma_{\xi,H}^2\sigma_{\epsilon,H}^2}}{2\sigma_{\xi,H}^2}$ given by optimal (Kalman) gain.

The adaptive learning schemes as equation (**2.17**) and (**2.18**) as well as pricing equation (**2.13**) and (**2.14**) can generate a high stock markets volatility coming from the feedback channel between stock price $P_t^{A(H)}$ and subjective beliefs $\beta_t^{A(H)}$. According to equation (**2.13**) or (**2.14**), a high (low) $\beta_t^{A(H)}$ will lead to a high (low) realized stock price. This will reinforce the subjective beliefs to induce a even higher (lower) $\beta_{t+1}^{A(H)}$ through equation (**2.17**) or (**2.18**) leading to much higher (lower) stock price so on. The self-referential aspect of the model is the key for producing stock market dynamics. Therefore, a difference of initial beliefs between β^A and β^H or of learning speeds α^A and α^H is promising to generate persistently different prices between A-share and H-share.

Finally, in order to avoid the explosion of stock price $P_t^{A(H)}$ agents' subjective belief $\beta_t^{A(H)}$ is replaced by $\omega(\beta_t^{A(H)})$, the projection facilities as appendix 2.8.4.

2.5.5. Testing for the Rationality of Price Expectation

In this section we use a set of testable restrictions implied by agents' beliefs system developed in Adam, Marcet and Nocolini (2016). These restrictions are listed as follows: Denote $x_t = (e_t, D_t/D_{t-1}, C_t/C_{t-1})$, where $e_t \equiv \Delta(\frac{C_t}{C_{t-1}})^{-\gamma}\frac{P_t}{P_{t-1}}$ with Δ denoting the first difference operator.

Restriction 1: $E(x_{t-i}e_t) = 0$ for all $i \geqslant 2$,

Restriction 2: $E((\frac{D_t}{D_{t-1}} + \frac{D_{t-1}}{D_{t-2}}, \frac{C_t}{C_{t-1}} + \frac{C_{t-1}}{C_{t-2}})e_t) = 0$,

Restriction 3: $b_{DC}^{'} \sum_{DC} b_{DC} + E(e_t e_{t-1}) < 0$,

Restriction 4: $E(e_t) = 0$,

where $\sum_{DC} \equiv var(\frac{D_t}{D_{t-1}}, \frac{C_t}{C_{t-1}})$ and $b_{DC} \equiv \sum_{DC}^{-1} E((\frac{D_t}{D_{t-1}}, \frac{C_t}{C_{t-1}})' e_t)$

These four restrictions are necessary and sufficient conditions for the agents' belief compatible with $\{x_t\}$ in terms of second order moments. Adam, Marcet and Nicolini (2016) proves that under standard assumptions, any process satisfying these testable restrictions can - in terms of its autocovariance function - be generated by the postulated system of beliefs as (**2.15**) and (**2.16**). The set of derived restrictions thus fully characterizes the second-moment implications of the beliefs system. Here we test the derived restrictions against the data to see if the agent's belief system is compatible with the actual data. Table **2.3** reports the test statistics when testing Restrictions 1-4 using actual data. We compute risk-adjusted consumption growth in the data at $\gamma = 5$.

	Test Statistics A (H)	5% Critical Value
Restriction 1 using $\frac{D_t}{D_{t-i-1}}$	2.81 (0.76)	9.48
Restriction 1 using $\frac{C_t}{C_{t-i-1}}$	4.02 (4.77)	9.48
Restriction 1 using $\Delta(\frac{C_{t-i}}{C_{t-i-1}})^{-\gamma}\frac{P_{t-i}}{P_{t-i-1}}$	2.13 (2.55)	9.48
Restriction 2	0.04 (0.15)	5.99
Restriction 3	-3.55 (-3.60)	3.84
Restriction 4	0.002 (0.001)	1.64

Table 2.3. Testing Subjective Beliefs against Actual Data

The 5% critical value of the test statistic is reported in the last column of table **2.3**. The table shows that the test statistic is in all cases below its critical value and often so by a wide margin. It then follows that agents find the observed asset pricing data, in terms of second moments, to be compatible with their belief system. Based on this, we can conclude that the agents' belief system is reasonable: given the behavior of actual data, the belief system is one that agents could have entertained.

2.5.6. Quantitative Performance

This subsection presents the simulation outcomes of our learning model. We simulate our model at weekly frequency. We first give value to the coefficient of relative risk-aversion γ at 5 following Adam, Marcet and Nicolini (2016), then calibrate the mean and standard deviation of dividend growth a, $\sigma_{\Delta D/D}$, the standard deviation of consumption growth $\sigma_{\Delta C/C}$, the correlation between consumption growth and dividend growth $\rho_{c,d}$ using Shanghai stock market data and Chinese consumption

Parameters	**Value**
γ	5
$\sigma_{\Delta D/D}$	0.0204
$\sigma_{\Delta C/C}$	0.0025
a	1.0014
$\rho_{c,d}$	-0.03
δ	0.999
$1/\alpha^A$	0.0030
$1/\alpha^H$	0.0015

Table 2.4. Parameters Values for Learning Model

per capita data. We also calibrate δ to match annual 4% interest rate. Meanwhile, we give values to α^A and α^H such that $\alpha^A < \alpha^H$, which can come from agents' subjective beliefs that $\frac{\sigma_{\xi,A}}{\sigma_{\epsilon,A}} > \frac{\sigma_{\xi,H}}{\sigma_{\epsilon,H}}$. Intuitively, if agents believe that the ratio of standard deviation of persistent component shock to that of transitory component is relative larger in A-share price than H-share price, agents prefer to learn faster for A-share price since only persistent component provides useful information for forecasting. This is not arbitrary setting because the realized data of P_t^A and P_t^H can support this inequality if we use MLE method to estimate related parameters given the data following the processes (**2.15**) and (**2.16**). Table **2.4** contains the parameter values.

We Monte-Carlo simulate the learning model for $K = 10,000$ samples with each sample having $T = 100$ periods to match almost 2 years' sample period since November 2014. Table **2.5** contains the simulation results. Column 2 shows the data moments of AH premium, and column 3 has the 95% interval of model simulated moments. We find that the mean and standard deviation of data locate

Moments	Data	Model
$E(\frac{P_t^A}{P_t^H} * 100)$	130.39	[119.58 197.01]
$\sigma(\frac{P_t^A}{P_t^H} * 100)$	9.53	[7.75 48.69]
$\rho(\frac{P_t^A}{P_t^H} * 100)$	0.78	[0.87 0.98]

Table 2.5. Model Simulated Moments

in the interval, but model generates a little more persistent price difference than data. And figure **2.5** also presents one simulated dynamics of A-share price P_t^A and H-share price P_t^H, and figure **2.6** presents the corresponding simulated AH premium. We set initial conditions $\beta_1^A = \beta_1^H$ and β_2^A a little larger than β_2^H, which are consistent with data observations. Then, a higher learning speed in A-share leads P_t^A to fluctuate more strongly than P_t^H even if two prices dynamics keep the similar shape. Comparing with figure, the model simulated prices display the close shape as data. More importantly, the shape of simulated AH premium captures several important factors of data: 1. starting from around 100; 2. persistently increasing to about 150; and 3. decreasing to about 120 after 2 years. Hence, our learning model does a much better job in generating data-like AH premium than the models in section 2.3.

2.6. Convergence Traders' Strategy

A typical convergence trader is to bet that price difference between two assets with identical, or similar fundamentals will narrow in the future. The convergence trade would hold long positions in one asset he considers undervalues and short

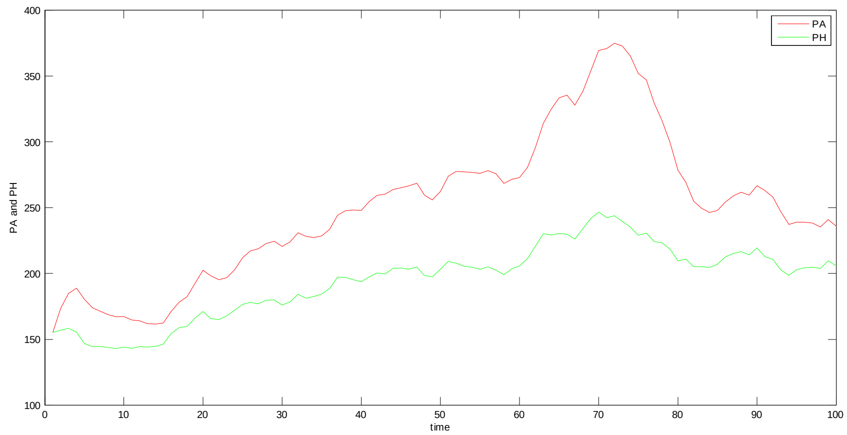

Figure 2.5. Simulated Stock Prices of A-Share and H-Share

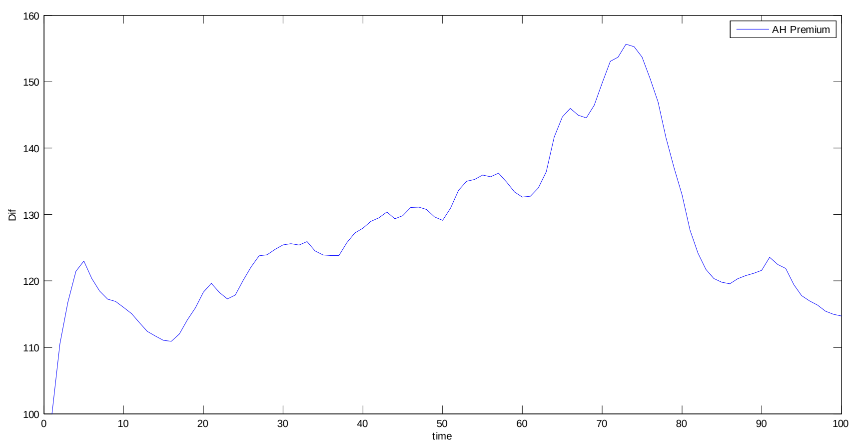

Figure 2.6. Simulated AH Premium

positions in the other asset he considers overvalued. A famous example is that the hedge fund Long-Term Capital Management (LTCM) expected the convergence of bond yields in the emerging market countries and US (Edwards, 1999). They bought emerging markets' bonds and sold short US government bonds. The spread of bond yields, however, widened because of the deterioration of Asian financial crisis and the default of Russian Sovereign debt. The unexpected widening leads to the near-collapse of LTCM. Besides the case of LTCM, Wall Street Journal reported in June 2015 that many convergence traders participated in AH-share market by short selling A-share and long buying H-share, but finally encountered a huge loss.

Xiong (2001) studies the convergence trading strategy in the model which has three types of traders: noise traders, convergence traders and long-term traders. He finds that convergence traders reduces asset price volatility in general, but when an unfavorable shock causes them to suffer substantial capital losses, they liquidate their positions, thereby amplifying the original shock. This section considers the convergence traders differently with zero measure who take the stock prices set by learning agents as given, and studies the probability distribution of profits when they hold convergence trader strategy.

The convergence traders at period 100 expect that AH premium should narrow in the future, hence they short sells 1 unit of A-share stock and use the money from selling to buy H-share stock. To implement short selling in China stock market, convergence traders should have as much money as 50% of short selling value in their account as security deposit. In every period, the guarantee ratio gr_t should

m_t	Mean	Std	$\Pr(m_t < 0)$
3m	0.817	10.657	0.384
6m	0.662	17.345	0.365
9m	0.713	23.231	0.338
1y	0.887	26.659	0.323

Table 2.6. The Distribution of Profits from Convergence Trading Strategy

be calculated as

$$gr_t = \frac{0.5 * P_{100}^A + P_t^H * \frac{P_{100}^A}{P_{100}^H}}{P_t^A}$$

If $gr_t < 130\%$, convergence traders will be asked to add more security deposit to avoid forced liquidation. The maximum duration of short selling is 1 year. We now Monte-Carlo simulate the learning model 10,000 paths with each path representing from 100th period to 152th period. The probability of $gr_t < 130\%$ in every period can reach as high as 13%. We can also calculate the distribution of profits $m_t = P_t^H * \frac{P_{100}^A}{P_{100}^H} - P_t^A$ when $t = 113, 126, 139$ or 152 corresponding to 3 months, 6 months, 9 months and 1 year. Table **??** shows the results. We find the mean of m_t much smaller than the standard deviation of it and a large probability of losing money. Being different from Xiong (2001), our learning model cannot guarantee the convergence of AH premium. Therefore, it is not surprising that convergence traders have large probability to lose money.

2.7. Conclusion

This chapter studies the AH premium, which is an interesting anomaly in asset markets. We have shown that asset pricing models with heterogeneity agents with different risk aversions or diverse beliefs in the complete market and incomplete markets cannot generate any data-like AH premium. Transaction cost and different dividend taxes between Shanghai and Hong Kong markets also fails to explain such high and volatile AH premium. We propose an "Internal Rationality" model, in which agents don't know the pricing function from fundamentals to the stock prices and have different subjective beliefs about tomorrow's capital gains in Shanghai and Hong Kong markets. Our learning model can successfully generate data-like weekly AH premium. Finally we show that convergence traders with strategy short in Shanghai and long in Hong Kong will lose money with 33% probability.

This maybe an evidence that Chinese investors are more speculative, which seems to be related to the higher stock price volatility in China than that in U.S. and the fact that stock price is highly negative correlated with PMI index for economy prospect in China during the year 2015. These topics worth to be explored in the future research.

2.8. Appendix

2.8.1. Algorithm for two agents two shares with rational expectation in complete market

Step 1: Simulate $\{D_t\}$ for a long time. Solve for $\frac{u'(C_0^1)}{u'(C_0^2)}$ by simulating the economy especially $\{\{C_t^{1,n}, C_t^{2,n}\}_{t=0}^T\}_{n=1}^N$ given initial bond holding B_{-1}, since we have one equation of present value budget constraint for B_{-1} and one unknown. Hence we got the equilibrium λ. It could be solved by iterating on λ or for example just use solve the equation directly. The equation is as follows:

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{t=0}^{T}\delta^{1,t}\frac{u^{'}(C_t^1)}{u'(C_0^1)}(C_{t+j}^1 - D_{t+j}) = B_{-1}$$

Here we simulate N times for T period economy. T could be small for example 100.

Step 2: Find $\{C_t^1, C_t^2\}_{t=0}^{T}$ by simulating a very long sequence of D ie. At time t, given $\frac{u^{'}(C_t^1)}{u'(C_t^2)} = \lambda$ and market clear condition, C_t^1 and C_t^2 can be solved. Here we are facing a convex problem. Thus theoretically we should get unique solution though the two conditions lead to a polynomial of C.

Step 3: Solve for the realized present value of primary deficit $\{Dd_t^1\}$. It is useful because the bond holdings are just conditional expectation of Dd

Define $Dd_t^1 = \sum_{j=0}^{\infty}\delta^{1,j}\frac{u^{'}(C_{t+1}^1)}{u^{'}(C_t)}(C_{t+j}^1 - D_{t+j}^1)$ as realized present value of primary deficit. Then we have

$$Dd_t^1 = \delta^1\frac{u^{'}(C_{t+1}^1)}{u'(C_t^1)}Dd_{t+1}^1 + C_t^1 - D_t^1.$$

We impose Dd at the end of the day is 0 to make these two equations equivalent namely $Dd_T^1 = 0$, and we can solve for Dd backwards from $Dd_T^1 = 0$ given that we have got sequence of consumption and dividends in the above steps.

Step 4 : We solve for $\{B_{t-1}^1(D)\}$ in this step by using the equation:

$$B_{t-1}^1(D_l) = E(Dd_t|D_t = D_l)$$

where bond holds are just function of state D. To implement it we use

$$B^1_{t-1}(D_l) = \frac{1}{T}\sum_{t=1}^{T} Dd^1_t I(D_t)$$

where $I(D_t)$ is the indicator function. This could also be regarded as run the regression of Dd to indicator functions, which is the core idea of PEA. Notice that conditional expectation is actually the average over states. However due to the face that we have an i.i.d world which is definitely ergodic, we just use the average over time to estimate the conditional expectation by the property of ergodicity.

Technically speaking we are not using PEA because we are not iterating on parameters, which is not necessary in our case. We are not relying on the approximation of the right hand side of Euler equations as the typical steps do in PEA thanks to the complete market thing gives us the formula to solve for debt and Bd.

2.8.2. Algorithm for two agents two shares with diverse belief in complete market

In this case every steps are same except that λ is not constant any more. We will have a sequence $\{\lambda_t\}$ because of the diverse probability, which follows

$$\alpha_{t-1}(D_t)\lambda_{t-1} = \lambda_t$$

where $\alpha_{t-1}(D_t) = \frac{prob^2_{t-1}(D_t)}{prob^1_{t-1}(D_t)}$. Another difference lies in step 4 because in this case bond holding is not only the function of D but also a function of λ_{t-1}. For example,

$$B^1_{t-1}(D_l) = E(Dd_t|D_t = D_l, \lambda_{t-1})$$

So we need to run the regression of Dd on both D and λ_{t-1}. Actually for the case in 2.8.1 you can also think of λ as a state. But it is neglected in the regression because it is a just constant and has been taken care of by the constant in the regression. Explicitly the best way to write Dd as the function of the two states are as follows:

$$Dd^1_t = (\alpha^h_0 + \alpha^h_1 \lambda_{t-1}) I_{D^h}(D_t) + (\alpha^l_0 + \alpha^l_1 \lambda_{t-1}) I_{D^l}(D_t)$$

Clearly this regression could be run separately both for high and low.

2.8.3. A Learning Model with Diverse Beliefs and Dividend Taxes

This section extends the benchmark learning model with diverse beliefs and dividend taxes. The dividend and consumption growths still follow the same processes as section 2.5. There are two types of agents, one is relative optimistic about fundamental growth and the other is relative pessimistic. Agent i's maximization problem for $i = o$ or p is

$$\max E_0^{\mathcal{P}^i} \sum_{t=0}^{\infty} \delta^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma}$$

$$\text{s.t. } C_t^i + P_t^A S_t^{A,i} + P_t^H S_t^{H,i} = (P_t^A + (1-\tau^{i,A})D_t)S_{t-1}^{A,i} + (P_t^H + (1-\tau^{i,H})D_t)S_{t-1}^{H,i} + Y_t$$

$$0 \leq S_t^{A,i} \ \& \ 0 \leq S_t^{H,i}$$

The subjective belief $\mathcal{P}^i$ is the same as $\mathcal{P}$ in section except that agent i believes fundamental growth at rate of a^i instead of a. The first order conditions are

$$(C_t^i)^{-\gamma} P_t^A \geqq \delta E_t^{\mathcal{P}^i}((C_{t+1}^i)^{-\gamma}(P_{t+1}^A + (1-\tau^{i,A})D_{t+1})) \text{ with equality if } S_t^{A,i} > 0$$

$$(C_t^i)^{-\gamma} P_t^H \geqq \delta E_t^{\mathcal{P}^i}((C_{t+1}^i)^{-\gamma}(P_{t+1}^H + (1-\tau^{i,H})D_{t+1})) \text{ with equality if } S_t^{H,i} > 0$$

We then can define the subjective expectations of risk-adjusted stock price growth as

$$\beta_t^{i,A} \equiv E_t^{\mathcal{P}^i}[(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}(P_{t+1}^A)]$$

$$\beta_t^{i,H} \equiv E_t^{\mathcal{P}^i}[(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}(P_{t+1}^H)]$$

And agent i updates $\beta_t^{i,A}$ and $\beta_t^{i,H}$ according to the same adaptive learning schemes as (**2.17**) and (**2.18**). The consumption good market clearing condition is

$$C_t = C_t^o + C_t^p = 2Y_t + 2D_t$$

Parameters	**Value**
a^o	1.0024
a^p	1.0004
$\tau^{o,A}$	0.05
$\tau^{p,A}$	0.10
$\tau^{o,H}$	0.20
$\tau^{p,H}$	0.10

Table 2.7. Parameters Values for Learning Model

Assumption 1 allows us to have the following approximations

$$E_t^{\mathcal{P}}[(\frac{C_{t+1}^i}{C_t^i})^{-\gamma}] = E_t^{\mathcal{P}}[(\frac{C_{t+1}}{C_t})^{-\gamma}] \text{ for } i = o, p$$

The pricing equations according to Adam and Marcet (2011) are

$$P_t^A = \max_{i \in O,P} \frac{\delta (a^i)^{1-\gamma} \rho_\epsilon}{1 - \delta \beta_t^{i,A}} (1 - \tau^{i,A}) D_t,$$

$$P_t^H = \max_{i \in o,p} \frac{\delta (a^i)^{1-\gamma} \rho_\epsilon}{1 - \delta \beta_t^{i,A}} (1 - \tau^{i,A}) D_t,$$

The new parameter values are given in table **2.7**. The simulated stock prices of A-share and H-share is in figure **2.7**, and the simulate AH premium in figure **2.8**. The similar share of AH premium compared with it in figure **2.6** confirms that different beliefs about capital gains are dominate factor in generating AH premium relative to diverse beliefs on fundamentals and dividend taxes.

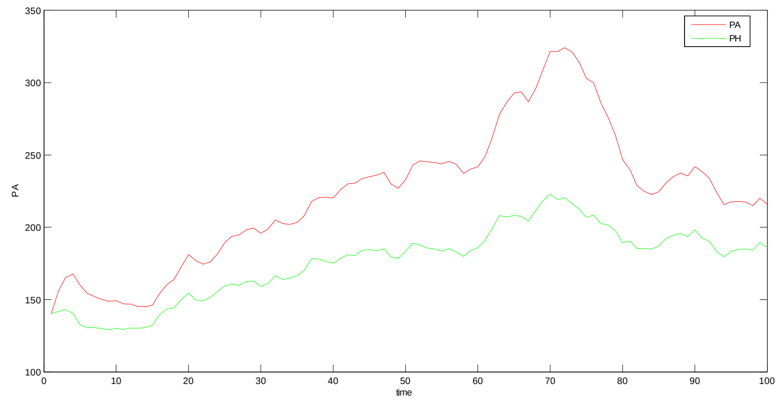

Figure 2.7. Simulated Stock Prices of A-Share and H-Share

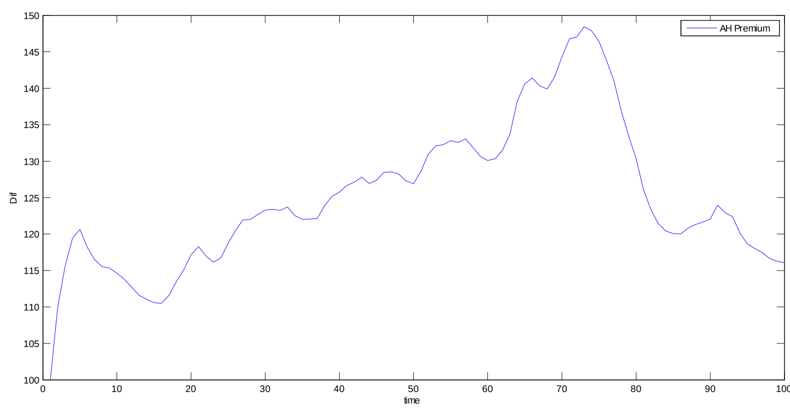

Figure 2.8. Simulated AH Premium

2.8.4. Differentiable Projection Facility

The function ω used in the differentiable projection facility is

$$\omega(\beta) = \left\{ \begin{array}{cc} \beta & \text{if } x \leq \beta^L \\ \beta^L + \frac{\beta-\beta^L}{\beta+\beta^U-2\beta^L}(\beta^U - \beta^L) & \text{if } \beta^L < x \leq \beta^U \end{array} \right\}$$

In our numerical applications, we choose β^U so that the implied PD ratio never exceeds $U^{PD} = 600$ and $\beta^L = \delta^{-1} - 2(\delta^{-1} - \beta^U)$.

2.8.5. Data Sources

Our data set for China stock market price, dividend, Heng Seng China AH premium index, Heng Seng China A index and Heng Seng China H index are downloaded from Wind Financial Database (http://www.wind.com.cn). The daily price series has been transformed into a weekly series by taking the index value of the last day of the considered week.

Our data set for Chinese macro data like consumption and CPI are downloaded from Chung, Chen, Waggoner and Zha (2015) (http://www.tzha.net/code).

CHAPTER 3

Puzzles in Exchange Market and "Internal Rationality" Approach

3.1. Introduction

There are many interesting anomalies in foreign exchange market, similar to other asset markets. The two most well-known ones are uncovered interest parity (UIP) puzzle and exchange rate disconnect puzzle. The UIP puzzle finds that over short time horizons (from a week to a quarter) when the interest rate differential (home country relative to foreign country) is higher than average, the home currency tends to appreciate. This puzzle can be shown clearly using "Fama regression" (Fama, 1984). It is usually reported as a regression of the change of log exchange rate s between time $t+1$ and t on the time t interest differential $i - i^*$

$$s_{t+1} - s_t = \varsigma_s + \beta_s (i_t - i_t^*) + u_{s,t+1} \tag{3.1}$$

Under rational expectation and risk neutrality, β_s should equal with 1. Table **3.1** reports the 90% confidence interval for the regression coefficients of six developed countries against US. For all of the countries except Italy, the point estimates of β_s

are negative. The 90% confidence interval lies below 1 for four (Italy and France being the exceptions). The regression results show that UIP puzzle widely exists.

Regarding the exchange rate disconnect puzzle as one of six major puzzles in international macroeconomics (Obstfeld and Rogoff, 2001), we can directly borrow table **3.2** from Engel and West (2005). m is money supply, y is real output and i is nominal short-term interest rate and variables with a '*' denoting corresponding foreign country ones. We take US as home country. One noteworthy feature is that except Canada the standard deviations of Δs are around twice as large as the standard deviation of fundamentals. And Engel and West (2005) suggests that we always cannot reject the null hypothesis that fundamentals fail to Granger cause Δs. In addition, Djeutem and Kasa (2013) apply traditional volatility tests including Shiller (1981) test, Campbell and Shiller (1987) test, West (1988) test and Engel (2005) test to arrive at the conclusion that exchange rate displays excess volatility unrelated to economic fundamentals.

This chapter proposes a novel approach to explain these two puzzles. We deviate from rational expectation hypothesis by introducing "Internally Rational" agents who don't know the mapping from economic fundamentals to equilibrium exchange rate because of incomplete information (Adam and Marcet, 2011), and optimize their behaviors based on their subjective beliefs about all variables that are beyond their control. In such circumstance, exchange rate is no longer the discounted sum of future economic fundamentals. Given the exchange rate process they believe, agents optimally update their expectations about future exchange rate

Country	$\widehat{\varsigma}_s$	90% c.i.	$\widehat{\beta}_s$	90% c.i.
Canada	-0.045	(-0.250, 0.160)	-1.171	(-2.355, -0.186)
France	-0.028	(-0.346, 0.290)	-0.216	(-1.603, 1.171)
Germany	0.192	(-0.136, 0.520)	-1.091	(-2.583, 0.401)
Italy	0.032	(-0.325, 0.389)	0.661	(-0.359, 1.680)
Japan	0.924	(0.504, 1.343)	-2.713	(-4.036, -1.390)
UK	-0.410	(-0.768, -0.051)	-2.198	(-4.225, -0.170)

Table 3.1. Fama Regression Sample Size: 1979-2009

	Canada	France	Germany	Italy	Japan	UK
Δs						
Mean	-0.44	-0.35	0.15	-1.11	0.76	-0.44
Std	2.20	5.83	6.06	5.79	6.22	5.26
$\Delta(m - m^*)$						
Mean	-0.56	0.03	-0.55	-1.19	-0.39	-1.34
Std	2.59	2.41	2.38	2.24	2.18	1.94
$\Delta(y - y^*)$						
Mean	0.04	0.21	0.21	0.20	0.04	0.19
Std	0.79	0.88	0.88	1.01	1.21	1.06
$i - i^*$						
Mean	-0.92	-1.89	2.02	-4.23	3.64	-2.40
Std	1.72	3.70	3.21	4.25	2.78	2.88

Table 3.2. Summary Statistics

using Kalman filter. Agents' subjective expectations in turn influence equilibrium exchange rate, and the realized one feeds back into agents' beliefs. This self-referential aspect of the model implies that agents' endogenous expectations are dominant in generating exchange rate volatility relative to economic fundamentals. And the mean reversion of realized exchange rate generated from the self-referential property is also the key to produce UIP puzzle.

3.2. Related Literature

There are a large body of literature studying the anomalies in exchange market. We can review the literature into two strands. The first is about UIP puzzle, the second is for exchange rate disconnect puzzle.

Classic early references includes Bilson (1981) and Fama (1984) discover the existence of UIP puzzle. Engel (1996, 2014) surveys the empirical work that establishes this puzzle, and discusses the problem faced by the literature that tries to account for the regularity. Then, recent advances have found that the UIP puzzle can be explained using a risk-based model with non-standard preferences that have been used to account for other asset pricing anomalies. These studies model the ex-ante excess return as a risk premium related to the variances of consumption in the home and foreign country. Verdelhan (2010) builds on the model of external habits of Campbell and Cochrane (1999), and Colacito and Croce (2011, 2013) and Bansal and Shaliastovich (2007, 2013) develop the model of preferences in Epstein and Zin (1989) and Weil (1990) to account for this anomaly. Those studies show how the foreign exchange rate risk premium can be related to the difference in the conditional variance of consumption in the foreign country relative to the home country. A different approach to explain the interest parity puzzle advances an explanation akin to the model of information friction, such as rational inattention model Bacchetta and van Wincoop (2010) and learning model Chakraborty and Evans (2008). Finally, Yu (2013) provides a behavioral model with agents' sentiments on fundamentals to explain UIP puzzle.

In the second strand, Mussa (1986) and Baxter and Stockman (1989) are early studies showing that the transition to floating exchange rate regimes leads to sharp increase in nominal and real exchange rate variability with no corresponding changes in the distributions of fundamental macroeconomic variables. Betts and Devereux (1996, 2000) introduce local currency pricing into the baseline model preventing the exchange rate volatility transferring to macroeconomic fundamentals. Evans and Lyons (2002) show that most of the short-run exchange rate volatility is related to order flow, which also reflects the heterogeneity in investors' expectations. Xu (2010) proposes an exchange rate pricing model to generate this anomaly by introducing noise traders with errorneous stochastic beliefs. Another approach to account for exchange rate disconnect puzzle focuses on deviations from rational expectation including Markiewicz (2012)'s model uncertainty story and Djeutem and Kasa (2013)'s robustness story.

Our paper contributes to the literature by studying the UIP puzzle and disconnect puzzle together, and providing an "Internal Rationality" model as a new perspective.

3.3. The Model

This section presents a simple monetary exchange rate model with agents who hold subjective beliefs about exchange rate behavior. We show that the presence of subjective uncertainty implies that utility-maximizing agents update their beliefs about exchange rate behavior using observed exchange rate realizations. Using a generic updating mechanism, we show that such learning scheme can generate uncovered interest rate puzzle and exchange rate volatility puzzle.

3.3.1. Model Environment

The model is a two-country open economy model following Obstfeld and Rogoff (1996). Domestic variables are denoted without " * ", and the corresponding foreign variables are denoted with a superscript " * ". The (log-linear) domestic money demand equation, which can be derived from money-in-utility model, is

$$m_t - p_t = \lambda y_t - \kappa i_t \tag{3.2}$$

where m_t is the log of money supply, p_t is the log price level, y_t is the log of real output, and i_t is nominal interest rate. One of the key building blocks of this flexible-price monetary model is the assumption of purchasing power parity, which implies that

$$p_t = s_t + p_t^* \tag{3.3}$$

where s_t is nominal exchange rate, home country price of foreign currency. The second building block of this monetary model is the uncovered interest rate parity

(UIP) expressed as

$$i_t = i_t^* + E_t^{\mathcal{P}} s_{t+1} - s_t \tag{3.4}$$

where $\mathcal{P}$ denotes agents' subjective beliefs perhaps being different from objective beliefs.

Deriving from equation (**3.2**), (**3.3**) and (**3.4**), the exchange rate s_t is a convex combination of the fundamental variable f_t and the expected future exchange rate $E_t^{\mathcal{P}} s_{t+1}$ as

$$s_t = (1 - \gamma) f_t + \gamma E_t^{\mathcal{P}} s_{t+1} \tag{3.5}$$

where $f_t \equiv (m_t - m_t^*) - \lambda(y_t - y_t^*)$, and $\gamma \equiv \frac{\kappa}{1+\kappa}$ is the weight on the expectation. Following Markiewcz (2012), the fundamental variable f_t follow an AR(1) process as

$$f_t = \rho_f f_{t-1} + \epsilon_t^f, \ \epsilon_t^f \sim N(0, \sigma_{ef}^2)$$

where ρ_f is the AR(1) coefficient.

3.3.2. Rational Expectation Equilibrium

Under rational expectation $E_t^{\mathcal{P}} = E_t$, agents can forward equation (**3.5**) to yield

$$s_t^{RE} = (1 - \gamma) E_t \sum_{j=0}^{T} \gamma^j f_{t+j} + \gamma^T E_t s_{t+T}^{RE} \tag{3.6}$$

Letting $T \to \infty$ and imposing the no-bubbles condition $\lim_{T\to\infty} \gamma^T E_t s_{t+T}^{RE} = 0$, the present-value expression is

$$s_t^{RE} = (1-\gamma)E_t \sum_{j=0}^{\infty} \gamma^j f_{t+j} = \frac{1-\gamma}{1-\gamma\rho} f_t \tag{3.7}$$

This rational expectation equilibrium in the first place is well-known at odds with UIP puzzle. Let us show the failure in the following

$$s_{t+1} - s_t = \frac{1-\gamma}{1-\gamma\rho}[(\rho-1)f_t + \epsilon_{t+1}^f]$$

And according to UIP equation, we have

$$i_t - i_t^* = \frac{1-\gamma}{1-\gamma\rho}(\rho-1)f_t$$

If we run the regression like

$$s_{t+1} - s_t = \zeta_s + \beta_s(i_t - i_t^*) + u_{s,t+1}$$

the model implied β_s must be 1. And also from equation (**3.7**), the fact that the volatility of s_t^{RE} is completely driven by fundamental f_t misses exchange rate disconnect puzzle.

3.3.3. Equilibrium Analysis with Learning

3.3.3.1. Probability Space. This section explicitly describes the general joint probability space of the external variables. In the competitive economy, each agent as price-taker considers the joint process of fundamental and exchange rate $\{f_t, s_t\}$ as exogenous to his decision problem. Rational expectations imply that agents exactly know the mapping from a history of fundamental f_t to equilibrium exchange rate s_t as section 3.3.2. Exchange rate hence just carries redundant information. But if the rational expectation assumption is relaxed, as shown in Adam and Marcet (2011) such that agents do not know such mapping because of the non-existence of common knowledge on agents' identical preferences and beliefs, then equilibrium exchange rate s_t should be included in the underlying state space. We then define the probability space as $(\mathcal{P}, \mathcal{B}, \Omega)$ with $\mathcal{B}$ denoting the corresponding $\sigma-$Algebra of Borel subsets of Ω and $\mathcal{P}$ denoting the agent's subjective probability measure over $(\mathcal{B}, \Omega)$. The state space Ω of realized exogenous variables is

$$\Omega = \Omega_f \times \Omega_s$$

where Ω_X is the space of all possible infinite sequences for the variable $X \in [s, f]$. Thereby, a specific element in the set Ω is an infinite sequence $\omega = \{s_t, f_t\}_{t=0}^{\infty}$.

3.3.3.2. Agent's Subjective Beliefs. Now we allow a small deviation from rational expectation assumption such that agents with uncertainty formulate their own joint probability distribution $\mathcal{P}$ different from the objective one. In such environment, rational expectation equilibrium (**3.7**) doesn't hold any more leaving only the first-order condition for stock price in equation (**3.5**) intact. Without knowing the mapping from fundamental f_t to exchange rate s_t, agents should have their own beliefs on the behavior of exchange rate based on subjective distribution $\mathcal{P}$. Specifically, the subjective expectation of exchange rate growth β_t^s can be defined as

$$\beta_t^s \equiv E_t^{\mathcal{P}} s_{t+1} - s_t \tag{3.8}$$

Then, definition as equation (**3.8**) together with equation (**3.5**) implies the mapping from agents' subjective beliefs β_t^s to realized exchange rate s_t

$$s_t = f_t + \frac{\gamma}{1-\gamma}\beta_t^s \tag{3.9}$$

Equation (**3.9**) analytically suggests that learning equilibrium provides possible resolution to generate exchange rate disconnect puzzle since in addition to fundamental f_t subjective expectation β_t^s can also drive the volatility of s_t. If agents expect foreign currency to appreciate (depreciate), that is a high (low) s_t, ceteris paribus, an increasing (decreasing) demand in foreign currency drive current s_t up (down).

3.3.3.3. Belief Updating System. Now we can fully specify agents' information set $\mathcal{P}$ to show how they form their expectation β_t^s optimally given the model they believe. Agents think that exchange rate growth is the sum of a persistent component x_t and of a transitory component ϵ_t^s

$$s_t - s_{t-1} = x_t + \epsilon_t^s,\ \epsilon_t^s \sim N(0, \sigma_\epsilon^2) \tag{3.10}$$

$$x_t = x_{t-1} + \eta_t^s,\ \eta_t^s \sim N(0, \sigma_\eta^2)$$

According to equation (**3.7**), rational expectation equilibrium means that $E_t(s_{t+1} - s_t) = 0$ when f_t is at its unconditional mean 0. Hence, the previous setup encompasses the rational expectation equilibrium as a special case when agents believe $\sigma_{\epsilon,s}^2 = 0$ and assign probability one to $x_0 = 0$.

If σ_η^2 is allowed to be non-zero, the setup then gives rise to a filtering problem because agents observe only the realizations of exchange rate growth, but not the persistent and transitory component separately. Agents optimally use Kalman filter to filter out the persistent component x_t. The priors of agents' beliefs can be centered at their rational expectation values and given by

$$x_0 \sim N(0, \sigma_0^2)$$

and the variances of prior distributions should be set up to equal with steady state Kalman filter uncertainty about e_t^β

$$\sigma_0^2 = \frac{-\sigma_\xi^2 + \sqrt{\sigma_\xi^4 + 4\sigma_\xi^2\sigma_\epsilon^2}}{2}$$

Then agents' posterior beliefs will be

$$x_t \sim N(\beta_t^s, \sigma_0^2)$$

And the optimal updating rule implies that the evolution of β_t^s is taking the form just as constant gain learning

$$\beta_t^s = \beta_{t-1}^s + \frac{1}{\alpha^s}(s_{t-1} - s_{t-2} - \beta_{t-1}) \tag{3.11}$$

where $\alpha^s = \frac{\sigma_\xi^2 + \sqrt{\sigma_\xi^4 + 4\sigma_\xi^2\sigma_\epsilon^2}}{2\sigma_\xi^2}$ given by optimal (Kalman) gain.

3.3.3.4. Model's Implication. Equation (**3.9**) and (**3.11**) provide us intuition why this model can generate exchange rate disconnect puzzle. An the low β_t^s (optimistic on home currency) will produce a low s_t (home currency appreciate) as equation (**3.9**), and the realized s_t will feedback into equation (**3.11**) to induce an even lower β_{t+1}^s. This kind of feedback channel is possible to dominate fundamental's effect in driving the volatility of s_t.

In addition, the learning model is also possible to match UIP puzzle. According to (**3.4**) and (**3.8**), $i_t - i_t^*$ is determined by β_t^s. And as shown in Adam, Marcet and Nicolini (2016), learning scheme as (**3.11**) produces a mean-reverting β_t^s, that is the higher β_t^s, the lower $\Delta\beta_t^s$. Then, considering that realized $s_{t+1} - s_t$ is mainly driven by $\beta_{t+1}^s - \beta_t^s$, this channel produces the negative relationship between interest differential $i_t - i_t^*$ and exchange rate change $s_{t+1} - s_t$.

Parameter	Value
ρ_f	0.98
$\sigma_{\epsilon_t^f}$	0.1990
γ	0.984
$1/\alpha$	0.0072

Table 3.3. Parameter Values

3.4. Quantitative Analysis

This section evaluates the quantitative performance of our model. We summarize our model's parameter values in table **3.3** . The persistence of fundamentals is chosen very close to but smaller than 1 to keep stationarity even though it is a little bit larger than 1 as found in Engel and West (2004). The parameter γ is set at 0.98 to capture quarterly interest rate response's to real money supply $\kappa = 60$. The constant gain parameter α has value at 140 as Adam, Marcet and Nicolini (2016).

We simulate our model following Monte-Carlo procedure. The number of samples is set to $K = 10,000$ and each sample has $N = 150$ matching the data sample from Engel (2016). In each sample, we first simulate the model to run the regression (**3.1**). Then, the final values of estimate $\widehat{\beta}_s$ are taking the average of K samples at value -1.7410. This negative number confirms that our learning model is able to produce UIP puzzle. The model correlation between s_t and f_t is 0.278, and the standard deviations of s_t and f_t are 1.83 and 1.01 respectively. Figure **3.1** shows one simulated dynamics of exchange rate s_t and fundamental f_t. Our model has the ability to generate excess exchange rate volatility unrelated to fundamentals.

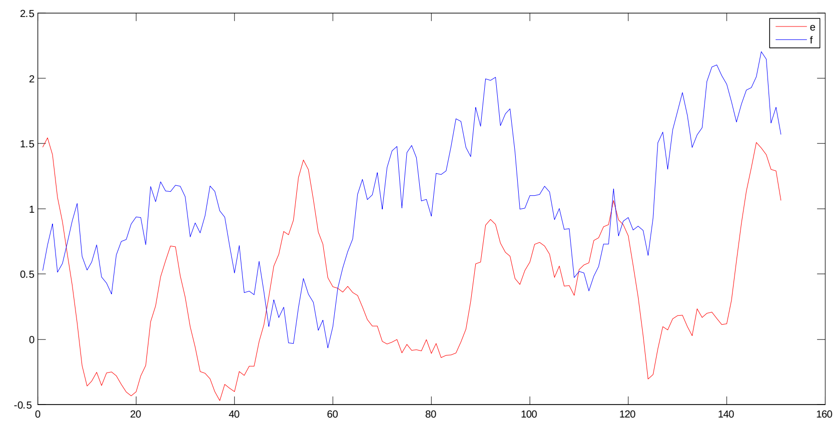

Figure 3.1. The Dynamics of Simulated s_t and f_t

3.5. Engel's Level Puzzle

3.5.1. What is level puzzle?

Engel (2016) first puts forward that the puzzle concerning the level of the exchange rate is particularly difficult to be explained–the explanations advanced for UIP puzzle are completely inadequate for explaining the level puzzle. Let ρ_{t+1} be the difference between the return between period t and $t+1$ on a foreign short-term deposit and the home short-term deposit, inclusive of the return from currency appreciation as

$$\rho_{t+1} \equiv s_{t+1} - s_t + i_t^* - i_t \tag{3.12}$$

UIP puzzle implies that $cov(E_t\rho_{t+1}, i_t^* - i_t) > 0$. The ex-ante excess return on the foreign deposit is positively correlated with the foreign less U.S. interest differential. To measure the relation between the interest differential and the level of the exchange rate, begin by rearranging (**3.12**), substracting off unconditional means, and iterating forward to get

$$s_t^T = s_t^{IP} - E_t \sum_{j=0}^{\infty} (\rho_{t+j+1} - \overline{\rho}) \tag{3.13}$$

where $s_t^T \equiv s_t - \lim_{k\to\infty}(E_t s_{t+k} - k(\overline{s_{+1} - s}))$ and $s_t^{IP} \equiv E_t \sum_{j=0}^{\infty}(i_{t+j}^* - i_{t+j} - (\overline{i^* - i}))$. s_t^{IP} is the level of exchange rate equaling to the infinite sum of the expected nominal interest differentials, and s_t^T as the transitory component should be the same as s_t^{IP} if $E_t\rho_{t+j+1} = 0$ for all $j \geq 0$. And Engel (2016) empirically shows that $cov(E_t \sum_{j=0}^{\infty} \rho_{t+j+1}, i_t^* - i_t) < 0$. From (**3.13**), it follows that there is excess co-movement in the level of the stationary component of the exchange rate, that is

$$cov(s_t^T, i_t^* - i_t) - cov(s_t^{IP}, i_t^* - i_t) = -cov(E_t \sum_{j=0}^{\infty} \rho_{t+j+1}, i_t^* - i_t) > 0$$

It must be the case that while UIP puzzle has $cov(E_t\rho_{t+1}, i_t^* - i_t) > 0$, for some period in the future $cov(E_t\rho_{t+j+1}, i_t^* - i_t) < 0$, the reverse sign. This can be recovered from the regression as

$$\rho_{t+j} = \varsigma_{s,j} + \beta_{s,j}(i_t^* - i_t) + u_t^j \tag{3.14}$$

We repeat figure 4 in Engel (2016) as figure **3.2** here plotting the slope coefficient estimates. The dependent variable here is ex-post excess return instead of ex-ante one. For small values of j, we find $cov(\rho_{t+j}, i_t^* - i_t) > 0$, but as the horizon increases, the sign of the covariance reverses at around period 50. Most of models addressing UIP puzzle fail on the level puzzle (Engel, 2016).

3.5.2. Learning Model's Implication on Level Puzzle

Being similar to section 3.4, we Monte-Carlo simulate our model and run the regression (**3.14**). In each sample, we simulate 1500 periods. To weaken the effect of initial condition, we drop first 200 periods. Figure **3.3** plots the regression results. There is an initially positive slope, and then slope gradually becomes negative, but quantitatively not enough to generate $cov(E_t \sum_{j=0}^{\infty} \rho_{t+j+1}, i_t^* - i_t) < 0$.

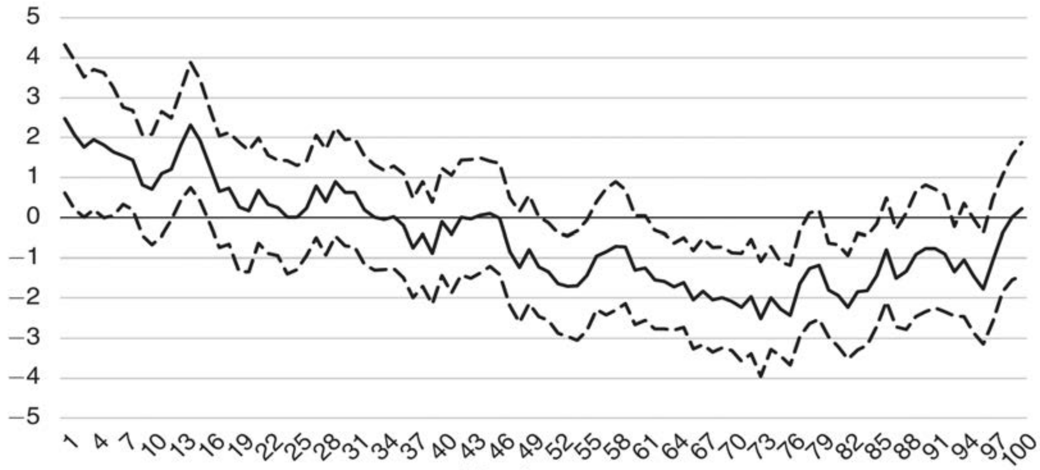

Figure 3.2. Slope Coefficients and Confidence Intervals Using Actual Data

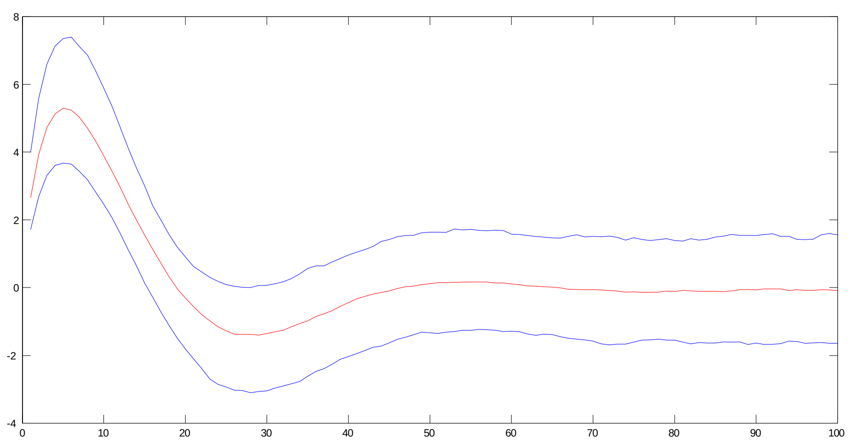

Figure 3.3. Slope Coefficients and Confidence Intervals Using Simulated Data

References

[1] Adam, Klaus, Pei Kuang and Albert Marcet (2012): "House Price Booms and the Current Account", NBER Macroeconomics Annual 2011, University of Chicago Press, vol 26(1), pp 77-122.

[2] Adam, Klaus and Albert Marcet (2011): " Internal Rationality, Imperfect Markets Knowledge and Asset Prices", Journal of Economic Theory, vol 146, pp 1224-1252.

[3] Adam, Klaus, Albert Marcet and Juan Pablo Nicolini (2016): "Stock Markets Volatility and Learning", Journal of Finance, vol 71(1), pp 33-82.

[4] Allen, Franklin, Jun Qian, Susan Chenyu Shan and Julie Lei Zhu (2015): "Explaining the Disconnection between China's Economic Growth and Stock Market Performance", China International Conference in Finance.

[5] Bacchetta, Philippe and Eric van Wincoop (2010): "Infrequent Portfolio Decisions: A Solution to the Forward Discount Puzzle", American Economic Review, vol 100(3), pp 870–904.

[6] Balke, Nathan, Jun Ma and Mark Wohar (2015): "A Bayesian Analysis of Weak Identification in Stock Price Decompositions", Macroeconomic Dynamics, vol 19(4), pp 728-752.

[7] Bansal, Ravi, Dana Kiku and Amir Yaron (2012): "An Empirical Evaluation of the Long-Run Risk Model for Asset Prices", Critical Finance Review, vol 1, pp 183-221.

[8] Bansal, Ravi, and Ivan Shaliastovich (2013): "A Long-Run Risks Explanation of Predictability Puzzles in Bond and Currency Markets", Review of Financial Studies, vol 26(1), pp 1–33.

[9] Bansal, Ravi and Amir Yaron (2004): "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles", Journal of Finance, vol 59(4), pp 1481-1509.

[10] Baxter, Marianne and Alan C. Stockman (1989): "Business Cycles and the Exchange-rate Regime: Some International Evidence", Journal of Monetary Economics, vol 23(3), pp 377-400.

[11] Bernanke, Ben and Kenneth Kuttner (2005): " What Explains the Stock market's Reaction to Federal Reserve Policy?", Journal of Finance, vol 60(3), pp 1221-1257.

[12] Betts, C. and Devereux, M (1996): "The Exchange Rate in A Model of Pricing to Market", European Economic Review, vol (40), pp 1007-1021.

[13] Betts, C. and Devereux, M (2000): "Exchange Rate Dynamics in A model of Pricing-to-Market", Journal of International Economics, vol (50), pp 215-244.

[14] Bianchi, Javier (2016): "Efficient Bailout?", American Economic Review, vol 106(12), pp 3607-3659.

[15] Bilson, John (1981): "The 'Speculative Efficiency' Hypothesis", Journal of Business, vol 54(3), pp 435–451.

[16] Blume, Laurence and David Easley (2006): "If You're So Smart, Why Aren't You Rich? Belief Selection in Complete and Incomplete Markets", Econometrica, vol 74(4), pp 929–966.

[17] Boldrin, M., L. Christiano, and J. Fisher (2001): " Habit Persistence, Asset Returns, and the Business Cycle," American Economic Review, vol 91(1), pp 149-166.

[18] Campbell, John (1991): "A Variance Decomposition for Stock Returns," Economic Journal, vol 101(405), pp 157-79

[19] Campbell, John (1999): "Asset Prices, Consumption, and the Business Cycle," Handbook of Macroeconomics, Edition 1, Volume 1, Chapter 19, pp 1231-1303.

[20] Campbell, John and John Ammer (1993): "What Moves the Stock and Bond markets? A Variance Decomposition for Long-Term Asset Returns", Journal of Finance, vol 48, pp 3-37.

[21] Campbell, John and John Cochrane (1995): "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior", NBER Working Paper No.4995.

[22] Campbell, John and John Cochrane (1999): "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior", Journal of Political Economy, vol 107, pp 205-251.

[23] Campbell, John, Andrew W.Lo and A. Craig MacKinlay (1997): "The Econometrics of Financial Markets", Princeton University Press, Princeton, USA.

[24] Chakraborty, Avik and George W. Evans (2008): "Can Perpetual Learning Explain the Forward-Premium Puzzle?", Journal of Monetary Economics, vol 55(3), pp 477-490.

[25] Chan, Kalok, Albert J Menkveld, and Zhishu Yang (2008): "Information Asymmetry and Asset Prices: Evidence from the China Foreign Share Discount", Journal of Finance, vol 63(1), pp 159–196.

[26] Cochrane, John (2005): "Asset Pricing", Princeton University Press, Princeton, USA.

[27] Cogley, Timothy and Thomas Sargent (2009): "Diverse Beliefs, Survival and the Market Price of Risk", Economic Journal, vol. 119(536), pp 354-376.

[28] Colacito, Riccardo, and Mariano M. Croce (2011): "Risks for the Long Run and the Real Exchange Rate", Journal of Political Economy, vol 119(1), pp 153–82.

[29] Colacito, Riccardo, and Mariano M. Croce (2013): "International Asset Pricing with Recursive Preferences", Journal of Finance, vol 68(6), pp 2651–2686.

[30] Danthine, J.-P., and J. Donalson (2002): "Labor Relations and Asset Returns", Review of Economic Studies, vol 69, pp 41-64.

[31] Djeutem, Edouard and Kenneth Kasa (2013): "Robustness and Exchange Rate Volatility", Journal of International Economics, vol 91(1), pp 27-39.

[32] Duffie, Darrell and Kenneth Singelton (1993): "Simulated Moments Estimation of Markov Models of Asset Prices", Econometrica, vol 61(4), pp 929-952.

[33] Edwards, Franklin R (1999): "Hedge Funds and the Collapse of Long-term Capital Management", Journal of Economic Perspectives vol 13, pp 189-210.

[34] Engel, Charles (1996): "The Forward Discount Anomaly and the Risk Premium: A Survey of Recent Evidence", Journal of Empirical Finance, vol 3(2), pp 123–192.

[35] Engel, Charles (2014): "Exchange Rates and Interest Parity", Handbook of International Economics, Vol. 4, Edited by Gita Gopinath and Elhanan Helpman, pp 453–522. Amsterdam: North-Holland.

[36] Engel, Charles (2016): "Exchange Rates, Interest Rates, and the Risk Premium", American Economic Review, vol 106(2), pp 436-474.

[37] Engel, Charles and Kenneth D. West (2004): "Accounting for Exchange-Rate Variability in Present-Value Models When the Discount Factor is Near 1", American Economic Review, vol 94(2), pp 119-125.

[38] Engel, Charles, and Kenneth D. West (2005): "Exchange rates and fundamentals", Journal of political Economy, vol 113(3), pp 485-517.

[39] Ermolov, Andrey (2015): "Time-varying Risk of Nominal Bonds: How Important Are Macroeconomic Shocks?", Unpublised paper, Columbia University.

[40] Evans, M. and R Lyons (2002): "Order Flow and Exchange Rate Dynamics", Journal of Political Economy, vol 110, pp 170-180.

[41] Fama, Eugene (1984): "Forward and Spot Exchange Rates", Journal of Monetary Economics, vol 14(3), pp 319–38.

[42] Favilukis, Jack and Xiaoji Lin (2015): "Wage Rigidity: A Quantitative Solution to Several Asset Pricing Puzzles", Review of Financial Studies, forthcoming.

[43] Fernald, John and John H Rogers (2002): "Puzzles in the Chinese Stock Market", Review of Economics and Statistics, vol 84(3), pp 416–432.

[44] Froot, Kenneth and Emil M Dabora (1999): "How Are Stock Prices Affected by the Location of Trade?", Journal of Financial economics, vol. 53(2), pp 189–216.

[45] Gali, Jordi (2014): "Monetary Policy and Rational Asset Price Bubbles", American Economic Review, vol 104(3), pp 721-752.

[46] Gali, Jordi and Luca Gambetti (2015): "The Effects of Monetary Policy on Stock Market Bubbles: Some Evidence", American Economic Journal: Macroeconomics, vol 7(1), pp 233-257.

[47] Grossman, Sanford J and Robert Shiller (1981): "The Determinants of the Variability of Stock market Prices", American Economic Review, vol 71(2), pp 222-27.

[48] Guvenen, Faith (2009): "A Parsimonious Macroeconomic Model for Asset Pricing", Econometrica, vol 77(6), pp 1711-1750.

[49] Hollifield, Burton, Gary Koop and Kai Li (2003): "A Bayesian Analysis of a Variance Decomposition for Stock Returns", Journal of Empirical Finance, vol. 10(5), pp 583-601.

[50] Jermann, Urban (1998): "Asset Pricing in Production Economy", Journal of Monetary Economics, vol 41, pp 257-275.

[51] Kiyotaki, N and John Moore (1997): "Credit Cycles", Journal of Political Economy, vol 105, pp 211-248.

[52] Lucas, Robert (1978): "Asset Prices in an Exchange Economy", Econometrica, vol 46, pp 1426-1445.

[53] Marcet, Albert and Thomas Sargent (1989): " Convergence of Least Squares Learning Mechanism in Self Referential Linear Stochastic Models", Journal of Economic Theory, vol 48, pp 337-368.

[54] Marcet, Albert and Kenneth J. Singleton (1999): "Equilibrium Asset Prices and Savings of Heterogeneous Agents in the Presence of Incomplete Markets and Portfolio Constraints." Macroeconomic Dynamics vol 3(2), pp 243-277.

[55] Markiewicz, Agnieszka (2012): "Model Uncertainty and Exchange Rate Volatility", International Economic Review, vol 53(3), pp 815-844.

[56] Mei, Jianping, Jose Scheinkman and Wei Xiong (2009): "Speculative Trading and Stock Prices: Evidence from Chinese A-B Share Premia", Annals of Economics and Finance, vol 10, pp 225-255.

[57] Mussa, Michael (1986): "Nominal Exchange Rate Regimes and the Behavior of Real Exchange Rates: Evidence and Implications", Carnegie-Rochester Conference Series on Public Policy, vol (25), North-Holland.

[58] Obstfeld Maurice and Kenneth Rogoff (1996): "Foundations of International Macroeconomics", Cambridge, MA: MIT Press.

[59] Obstfeld Maurice and Kenneth Rogoff (2001): "The Six Major Puzzles in International Macroeconomics: Is There A Common Cause?". NBER Macroeconomics Annual 2000, vol 15, pp 339-390.

[60] Shiller, Robert (1981): " Do Stock Prices Move Too Much to Be Justified by Subsequent Changes in Dividend?", American Economic Review, vol 71, pp 421-436.

[61] Shiller, Robert (2005): "Irrational Exuberance: Second Edition", Princeton University Press, Princeton, USA.

[62] Shiller, Robert and Andrea Beltratti (1992): "Stock Prices and Bond Yields: Can Their Comovements Be Explained in Terms of Present Value", Journal of Monetary Economics, vol 30, pp 25-46.

[63] Verdelhan, Adrien (2010): "A Habit-Based Explanation of the Exchange Rate Risk Premium", Journal of Finance, vol 65(1), pp 123–46.

[64] Wachter, Jessica (2005): "Solving models with external habit", Financial Research Letters, vol. 2(4), pp 210-226.

[65] Wachter, Jessica (2006): "A Consumption-based Model of Term Structure of Interest Rates", Journal of Financial Economics, vol 79, pp 365-399.

[66] Xiong, Wei (2001): "Convergence Trading with Wealth Effects: An Amplification Mechanism in Financial Markets", Journal of Financial Economics, vol 62, pp 247-292.

[67] Xu, Juanyi (2010): "Noise Traders, Exchange Rate Disconnect Puzzle, and the Tobin Tax", Journal of International Money, and Finance, vol 29(2), pp 336-357.

[68] Yu, Jianfeng (2013): "A Sentiment-based Explanation of the Forward Premium Puzzle", Journal of Monetary Economics, vol 60(4), pp 474-491.