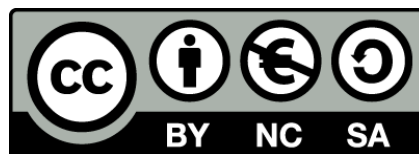




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Contributions to stochastic analysis

Giulia Binotto



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Contributions to stochastic analysis

Giulia Binotto

A Thesis submitted
for the degree of Doctor in Mathematics

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Programa de Doctorat de Matemàtiques.
Departament de Matemàtiques i Informàtica
Universitat de Barcelona

Barcelona, December 2017



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Certifico que la present memòria ha estat
realitzada per la Giulia Binotto
en el Departament de Matemàtiques i Informàtica
sota la meva direcció.

Desembre de 2017, Barcelona.
Carles Rovira Escofet

Alla mamma,
al papà e
a Luca

Agraïments

Aquesta tesi va començar fa quatre anys, quan vaig tornar a la universitat després d'una temporada treballant fora. Són moltes les persones a qui vull donar les gràcies, però no tinc cap dubte que la primera de totes sigui el Carles. Sense ell aquesta tesi no hauria sigut possible. Carles, moltes gràcies no només per ser un excel·lent director de tesi, per tenir sempre idees que em feien avançar i per ajudar-me en qualsevol cosa, sinó perquè també has estat un amic i m'has deixat la llibertat de fer el que més m'agradava. Se'm dibuixa un somriure a la cara quan recordo la teva resposta recurrent: "Fes el que vulguis!", perquè és gràcies a aquesta frase que aquesta experiència ha sigut inoblidable.

Una altra persona sense la qual tot aquest camí no hauria sigut possible és el Xavi. Fa més de nou anys, quan encara era una erasmus despreocupada i emocionada pel nou món que estava coneixent, em vas introduir al món de la recerca dirigint el meu primer treball d'investigació. La teva passió, la paciència i la capacitat de guiar-me em van convèncer a continuar al món acadèmic. El Xavi també va ser qui em va fer conèixer el Marco, che voglio ringraziare per le chiacchierate e le cene fatte insieme, piene di suggerimenti e consigli su come affrontare questi anni. Quan vaig deixar la universitat vaig perdre una mica l'esperança de tornar al món acadèmic, fins que la Marta em va enviar aquell correu que encara conservo dient-me que em donaria una beca FPI. Gràcies per creure en mi i donar-me aquesta oportunitat. La valoro molt.

Durant aquests anys vaig tenir la gran sort de fer tres estades. La primera va ser a Kansas sota la direcció del David Nualart. Moltes gràcies David per l'acollida que vaig rebre i per tota la teva ajuda. Gràcies també per donar-me la possibilitat de treballar, encara que fos a distància, amb l'Ivan Nourdin. Merci à toi aussi Ivan! Working with you both has been an honor. The second research visit was in the École normale supérieure de Lyon with Grégory Miermont. Merci Grégory to be a patient guide in a field that was unexplored for me and to make me feel part of your research group. Finally, in the last months of my thesis, I had the opportunity to work with Ger Koole in the Vrije Universiteit Amsterdam. Thank you Ger to get me into the field of revenue management and make me participate in all your activities.

A Barcelona hi ha una persona que m'ha fet de germana gran: la Mireia. Moltes gràcies per ajudar-me sempre, per treballar amb mi, per donar-me consells i per ser una gran amiga. Ha sigut increïble tenir-te al meu costat.

No podré oblidar mai les meves dues companyes de despatx: la Carme sempre disposada a donar-me un cop de mà i a respondre als meus dubtes, i la Dolors

amb el seu bon humor i la capacitat d'ajudar a tothom. No hauria pogut desitjar unes companyes millors. A elles hi afegeixo l'Eulàlia i la Noèlia, per haver-me inclòs a les nits de noies i compartir amb rialles totes les vicissituds de la vida acadèmica.

Gràcies a totes les persones del departament, en particular al David per les bones paraules que sempre m'ha dedicat i totes les estones que hem passat junts, i a l'Olga, sempre disposada a ajudar-me amb les classes.

I ara toquen els amics! Són aquelles persones que no només m'han fet costat, sinó que han viscut en primera persona les preocupacions, els problemes i tots els moments divertits que un estudiant de doctorat té al seu camí. Encara recordo quan el Carlos em va dir: "Et presentaré el meu grup d'amics, ja veuràs que et cauran bé", i així vaig conèixer a l'Álex, l'Ari, l'Arturo, el Carlos, el Dani, l'Eloi, el Gladston, el Joan, el Jordi, el Marc, la Marta, la Meri, la Nadia, el Narcís, el Roc i el Tommaso! Gràcies a tothom per fer el meu dia a dia tan entretingut. Recordaré especialment el Marc... És més fàcil i divertit començar una aventura així amb algú i tu has estat sempre allà pel que necessitava. Gràcies! Thank you to all friends in Kansas, Lyon and Amsterdam, because you make me feel good even when I was far from my home. In particolare grazie a Luigia per essere una perfetta anfitrione, a Álvaro por ser un excelente compañero de despacho y a Nacho por hacerme siempre reír y por las largas charlas en el parque (espero que seas siempre un canario amarillo tan divertido!) Loïc, à toi je veux dédier un merci particulier. Il y a peu de gens avec qui je me sens aussi lié en si peu de temps. Merci de me faire sentir chez moi, grâce à toi je ne me suis jamais sentie seule.

Gracias también a mi family por todo los momentos juntas, alle tose perché ogni volta che torno è come se non fossi mai andata via e a Paolo per trovare sempre un momento da dedicarmi per raccontarci le nostre vite. Grazie Giorgietta per esserci sempre e per trovare il tempo per le ciaccole di cui ho bisogno.

Aquesta última part la dedico a una persona que des del principi ha estat present per tot el que necessitava. Arnau, no sé què hagués fet sense tu. Sempre que he tingut algun dubte tu eres al meu costat. M'has dedicat tot el temps del món, no t'has cansat mai d'ajudar-me, de donar-me consells, d'obrir-me els ulls i fer-me veure el que realment volia, encara que a vegades fos dur per tu o no ho compartíssis. Tot el que he fet ha sigut també gràcies a tu.

Infine, ma non per ultimi, mando un ricordo speciale alla mia mamma, al mio papà e a Luca. Mi siete sempre stati vicino anche se la distanza ci separa, e mi avete insegnato a non lasciar scappare quello che mi fa felice. Quest'ultimo grazie lo dedico a voi.

Barcelona, desembre 2017
Giulia

Contents

Agraïments	7
Introducció	11
Introduction	15
1 Definitions	19
1.1 The Poisson process	19
1.1.1 Properties of the Poisson process	21
1.2 Brownian motion	22
1.2.1 Properties of the Brownian motion	25
1.3 Complex Brownian motion	30
1.3.1 Properties of the complex Brownian motion	31
1.4 Fractional Brownian motion	33
1.4.1 Properties of the fractional Brownian motion	35
2 Preliminaries	39
2.1 Malliavin calculus	40
2.1.1 Malliavin calculus for fractional Brownian motion	43
2.2 Symmetric, forward and backward integrals	44
2.2.1 Symmetric integrals for the case $H < \frac{1}{2}$	46
2.3 Fractional integrals and derivatives	49
2.3.1 Generalized Stieltjes integrals	52
2.3.2 Compensated fractional integrals and derivatives	54
2.4 Relation between fractional and stochastic calculus	56
3 Strong limit of processes constructed from a Poisson process	59
3.1 Introduction and motivations	60
3.2 Definitions and main ideas	65
3.3 Weak convergence	71
3.4 Strong convergence	79
3.5 Rate of convergence	86
3.6 The case $\theta = \pi$	96

4	Weak symmetric integrals	99
4.1	Introduction and motivations	100
4.2	Main result	103
4.3	Bounds for fractional Brownian motion	106
4.4	Technical lemmas	111
4.5	Proof of the main result	119
5	Delay differential equations	125
5.1	Introduction	126
5.2	Main results	129
5.3	Estimates of the integrals	132
5.4	Estimates of the solutions	136
5.5	Proof of the main result	153
	Bibliography	157
	Index of symbols	165

Introducció

L'anàlisi estocàstica és una branca de les matemàtiques especialitzada en la resolució de problemes que evolucionen al llarg del temps de manera aleatòria. El seu objectiu principal és modelitzar i descriure aquests fenòmens, anomenats processos estocàstics. L'interès per aquesta disciplina va començar als anys setanta del segle passat en els àmbits de la finança i l'economia amb el propòsit de modelitzar l'evolució al llarg del temps dels preus de les accions.

Entre els temes en que se centra l'anàlisi estocàstica hi trobem els processos Gaussians. Es tracta de processos estocàstics tals que tota subcol·lecció finita de variables aleatòries té una distribució gaussiana multivariant, de manera que es poden descriure com a generalitzacions de la distribució de probabilitat normal a dimensió infinita.

El procés Gaussià més conegut és el moviment Brownià, també anomenat procés de Wiener. És un procés estocàstic que comença al zero amb trajectòries quasi segurament contínues i que té increments independents i amb distribució normal. Inicialment va ser implementat per simular el moviment físic de partícules de pol·len observat el 1827 per Brown i descrit al segle XX per Bachelier a [[Bachelier, 1900](#)] i Einstein a [[Einstein, 1905](#)]. Més enllà de les seves aplicacions en els camps de la física, la biologia i l'economia, només per citar-ne alguns, el moviment Brownià juga un paper important tant en matemàtica pura com aplicada, on s'utilitza per definir i estudiar processos estocàstics més complicats.

En aquest sentit, hi trobem el moviment Brownià fraccionari: un procés Gaussià centrat, la covariància del qual és una generalització de la del procés de Wiener. La regularitat de les seves trajectòries i les seves propietats venen determinades per un paràmetre H , anomenat paràmetre d'Hurst, que pren valors a l'interval $(0, 1)$. Aquest procés va ser descrit i estudiat per primer cop per Mandelbrot i Van Ness a [[Mandelbrot and Van Ness, 1968](#)]. Els autors es van inspirar en les idees proposades per Hurst in [[Hurst, 1951](#)] per simular fenòmens que no poden ser modelitzats pel moviment Brownià.

En aquesta tesi es presenten tres treballs relacionats amb els processos Gaussians esmentats. En el primer, treballarem també amb un procés estocàstic que pertany a la família dels processos de salt: el procés de Poisson. Els processos de salt tenen

un nombre numerable d'estats i temps d'arribada aleatoris. Entre ells, el procés de Poisson es caracteritza per començar al zero i tenir increments independents i amb distribució de Poisson.

El nostre objectiu és trobar una aproximació del moviment Brownià complex, que és l'equivalent a \mathbb{C} del procés de Wiener. Per aquest fi estudiem unes generalitzacions dels processos considerats per Kac a [Kac, 1974] per la solució de l'equació telegràfica. La convergència feble d'aquests processos al moviment Brownià estàndard va ser demostrada per primera vegada per Stroock a [Stroock, 1982]. La nostra extensió del resultat de Kac-Stroock es mou en dues direccions: d'una banda, demostrem la convergència en un sentit més fort que la convergència en llei, d'altra banda, afeblim les condicions dels processos aproximadors. En aquest sentit, construïm una família de processos complexos que depenen d'un paràmetre $\theta \in (0, 2\pi)$ i es defineixen a partir d'un únic procés de Poisson i una sèrie de variables aleatòries independents amb distribució de Bernoulli $\text{Ber}(\frac{1}{2})$.

En el cas general quan $\theta \in (0, \pi) \cup (\pi, 2\pi)$, demostrem que aquesta família convergeix en llei a un moviment Brownià complex i trobem realitzacions d'aquests processos que convergeixen quasi segurament a un moviment Brownià complex d -dimensional, uniformement en l'interval de temps $(0, 1)$, per d tan gran com volem. A més, deduïm la velocitat de convergència. La convergència feble s'estableix demostrant que la família de processos és ajustada i identificant la llei de tots els possibles límits febles, mentre que la demostració de la convergència quasi segura es basa en un resultat de Skorokhod i es inspira en el treball de Griego, Heath i Ruiz-Moncayo [Griego et al., 1971]. El càlcul de la velocitat de convergència segueix les idees de Gorostiza i Griego contingudes a [Gorostiza and Griego, 1979] i [Gorostiza and Griego, 1980]. Un dels aspectes més atractius d'aquest estudi és que els processos aproximadors són funcionalment dependents, ja que es construeixen a partir d'un únic procés de Poisson, però, en el límit, s'obtenen processos independents.

En el segon treball presentat en aquesta tesi, considerem la integració respecte a un moviment Brownià fraccionari amb paràmetre d'Hurst $H < \frac{1}{2}$. La integral es defineix com el límit en probabilitat d'una sèrie de sumes de Riemann construïdes a partir d'una mesura simètrica a l'interval $[0, 1]$, que és una mesura de probabilitat invariant respecte al mapa $t \mapsto 1 - t$. En [Gradinaru et al., 2005], Gradinaru, Nourdin, Russo i Vallois demostren que, per valors d' H estrictament més grans que un valor crític dependent de la mesura, aquesta integral existeix. S'ha demostrat que aquesta cota inferior pel paràmetre d'Hurst és òptima.

Estem interessats a estudiar la integral estocàstica quan el paràmetre d'Hurst pren el valor crític. En aquest cas, demostrem que les sumes de Riemann convergeixen en distribució i el límit es pot expressar en termes d'una integral estocàstica respecte a un moviment Brownià independent del moviment Brownià fraccionari. Com a conseqüència, derivem una fórmula de canvi de variable en llei. Aquest fenomen ha sigut estudiat per a determinades mesures simètriques. Per exemple, el cas de les "Midpoint Riemann sums" ha estat considerat per primera vegada per Nourdin i Réveillac a [Nourdin and Réveillac, 2009]. A [Nourdin et al., 2010] i a [Harnett

and Nualart, 2012], s’ha estudiat el cas de les sumes de Riemann corresponents a la ”Trapezoidal rule”, mentre que el cas de les ”Simpson’s rule sums” ha estat investigat per Harnett i Nualart a [Harnett and Nualart, 2015]. Nosaltres trobem un resultat més general que es pot aplicar a tota mesura simètrica que satisfà les condicions requerides pel resultat obtingut a [Gradinaru et al., 2005].

Mitjançant la fórmula de Taylor i les propietats de la mesura simètrica, la integral dirigida pel moviment Brownià fraccionari es pot expressar com suma de tres termes diferents. A [Gradinaru et al., 2005], els autors demostren que dos d’aquests termes convergeixen a zero en probabilitat, de manera uniforme en conjunts compactes, per valors del paràmetre d’Hurst estrictament més grans que el valor crític. Per demostrar el nostre resultat, considerem la mateixa representació de la integral estocàstica. D’una banda, provem que un dels termes convergeix a zero en probabilitat, uniformement en conjunts compactes, també quan H assoleix el valor crític. D’altra banda, demostrem que un dels altres termes convergeix en llei a una integral dirigida per un moviment Brownià independent del moviment Brownià fraccionari. Aquesta última part és la més innovadora i la que més atenció necessita. Per demostrar-la apliquem un mètode basat en ”small blocs / big blocs” i obtenim una extensió d’un lema demostrat per Harnett i Nualart a [Harnett and Nualart, 2015] aplicable al nostre cas. Les tècniques del càlcul fraccionari no són suficients per provar aquest resultat i es requereix l’aplicació de fórmules d’integració per parts derivades del càlcul de Malliavin.

En l’últim treball, estem motivats per l’estudi d’una equació diferencial estocàstica dirigida per un moviment Brownià fraccionari amb paràmetre d’Hurst $H \in (\frac{1}{3}, \frac{1}{2})$. En literatura, trobem molts estudis sobre equacions diferencials estocàstiques dirigides per un moviment Brownià, però les extensions al moviment Brownià fraccionari són escasses. Les tècniques per investigar aquestes equacions són diferents i depenen del valor del paràmetre d’Hurst i de la dimensió de l’equació. Un dels mètodes consisteix en estudiar equacions diferencials deterministes dirigides per una funció Hölder contínua i després aplicar els resultats obtinguts al cas estocàstic. Seguint aquest camí, considerem l’equació diferencial amb retard:

$$\begin{aligned} x_t^r &= \eta_0 + \int_0^t b(u, x_u^r) du + \int_0^t \sigma(x_{u-r}^r) dy_u, & t \in [0, T], \\ x_t^r &= \eta_t, & t \in [-r, 0), \end{aligned}$$

on $\eta : [-r, 0] \rightarrow \mathbb{R}^d$ és una funció contínua i y és una funció Hölder contínua d’ordre $\beta \in (0, 1)$. L’interès per equacions diferencials amb retard sorgeix de la necessitat de modelitzar sistemes on la dinàmica està sotmesa a retard de propagació.

El cas en que $\beta > \frac{1}{2}$ ha estat àmpliament estudiat i s’han aconseguit diversos resultats sobre existència i unicitat de solució i convergència, juntament amb algunes extensions. Nosaltres estem interessats a estudiar el cas en que $\beta \in (\frac{1}{3}, \frac{1}{2})$. Quan β agafa valors més petits o iguals que $\frac{1}{3}$, les notacions incòmodes i les noves dificultats que apareixen fan que treballar amb equacions diferencials sigui més complicat en aquest cas i els resultats són escassos.

Nosaltres demostrem que quan β pren valors a l'interval $(\frac{1}{3}, \frac{1}{2})$, la solució de l'equació diferencial amb retard convergeix quasi segurament en la norma infinit a la solució de l'equació diferencial sense retard

$$x_t = \eta_0 + \int_0^t b(u, x_u) du + \int_0^t \sigma(x_u) dy_u, \quad t \in [0, T],$$

quan el retard tendeix a zero. L'existència i unicitat de solució han estat garantits per Hu i Nualart a [Hu and Nualart, 2009] per l'equació diferencial sense retard, i per Neuenkirch, Nourdin i Tindel a [Neuenkirch et al., 2008] per l'equació diferencial amb retard. Per provar la convergència forta, seguim el mètode utilitzat per Hu i Nualart a [Hu and Nualart, 2009] i treballem amb una fórmula explícita per integrals del tipus $\int_s^t \sigma(x_u) dy_u$ en termes de x , y and $x \otimes y$, on $x \otimes y$ és un funcional multiplicatiu.

Aquesta tesi s'estructura de la següent manera.

Després d'aquesta introducció, al capítol 1, definim els processos estocàstics amb els quals treballem: el procés de Poisson, el moviment Brownià i la seva extensió al pla complex i el moviment Brownià fraccionari. També descriuim les seves propietats principals.

El capítol 2 conté alguns preliminars sobre el càlcul estocàstic. Es dedica a descriure la integració estocàstica respecte al moviment Brownià fraccionari, amb particular atenció al cas en que el paràmetre d'Hurst pren valors a l'interval $(\frac{1}{3}, \frac{1}{2})$. Els tres capítols següents contenen els treballs innovadors d'aquesta tesi. Al capítol 3 trobem aproximacions fortes del moviment Brownià complex donades per una família de processos estocàstics construïts a partir d'un únic procés de Poisson i d'una sèrie de variables aleatòries independents amb distribució de Bernoulli. Part dels resultats d'aquest capítol apareix a [Bardina et al., 2016].

Al capítol 4 establim la convergència feble, a la topologia de l'espai de Skorohod, de les sumes simètriques de Riemann del moviment Brownià fraccionari quan el paràmetre d'Hurst pren un valor crític i derivem una fórmula de canvi de variable en llei. El contingut d'aquest capítol apareix a [Binotto et al.,].

Finalment, al capítol 5 demostrem que la solució d'equacions diferencials amb retard dirigides per una funció Hölder contínua d'ordre $\beta \in (\frac{1}{3}, \frac{1}{2})$ convergeix quasi segurament en la norma infinit a la solució de l'equació diferencial sense retard quan el retard tendeix a zero.

Introduction

Stochastic analysis is a branch of mathematics specified in solving problems which evolve in time according to a random behavior. Its aim is to simulate and describe these phenomena, known as stochastic processes. The interest for this discipline began in the seventies of the last century in the fields of finance and economics to model the evolution in time of stock prices.

Among the various topics on which stochastic analysis is focused, we find the Gaussian processes. They are stochastic processes such that any finite subcollection of random variables has a multivariate Gaussian distribution, so they can be described as generalizations of the normal probability distribution to infinite dimension.

The most known of them is the Brownian motion, also called Wiener process. It is a stochastic process starting at zero with almost surely continuous paths and such that its increments are independent and Gaussian distributed. Firstly it was employed to model the physical movement of particles observed in 1827 by Brown and described in the twentieth century by Bachelier in [Bachelier, 1900] and Einstein in [Einstein, 1905]. Beyond its applications in applied fields, among them physics, biology and economics to mention but a few, Brownian motion plays an important role in pure and applied mathematics, where it is used to define and study more complicated stochastic processes.

In this direction, we come across the fractional Brownian motion: a centered Gaussian process whose covariance function is a generalization of that of the Wiener process. It depends on a parameter $H \in (0, 1)$, called Hurst parameter, that controls the roughness of its paths and also determines its properties. This process was described and studied for the first time by Mandelbrot and Van Ness in [Mandelbrot and Van Ness, 1968]. The authors were inspired by the ideas proposed by Hurst in [Hurst, 1951] to model phenomena that cannot be described by Brownian motion.

In this thesis we present three works related with the Gaussian processes we mentioned.

In the first work, we deal also with the Poisson process, which belongs to a class

of phenomena, called jumps processes, that have a countable number of states and random arrival times. It starts at zero and has independent increments with Poisson distribution.

Our aim is to find an approximation to a complex Brownian motion, that is the equivalent on \mathbb{C} of the Wiener process. We study generalizations of the processes considered by Kac in [Kac, 1974] for the solution of the telegraph equation. The weak convergence of these processes toward a standard Brownian motion was first proved by Stroock in [Stroock, 1982]. Our extension of the Kac-Stroock result moves in two directions: on one hand, we prove the convergence in a stronger sense than the convergence in law, on the other hand, we weaken the conditions of the approximating processes. In this sense, we construct a family of complex processes that depend on a parameter $\theta \in (0, 2\pi)$ and are defined from a unique Poisson process and a sequence of independent random variables with common Bernoulli distribution $\text{Ber}(\frac{1}{2})$.

In the general case when $\theta \in (0, \pi) \cup (\pi, 2\pi)$, we prove that this family converges in law to a complex Brownian motion and we find realizations of these processes that converge almost surely to a d -dimensional complex Brownian motion, uniformly on the unit time interval, for d as large as we want. Moreover, we derive a rate of convergence. The weak convergence is established proving tightness and the identification of the law of all possible weak limits, while the almost sure convergence is based on a result of Skorokhod and inspired by the work of Griego, Heath and Ruiz-Moncayo [Griego et al., 1971]. The computation of the rate of convergence follows the ideas of Gorostiza and Griego contained in [Gorostiza and Griego, 1979] and [Gorostiza and Griego, 1980]. One of the most attractive aspects of this study is that the approximating processes are functionally dependent because they are constructed from a single Poisson process but, in the limit, we obtain independent processes.

The case when the parameter takes the value $\theta = \pi$ is also considered. In this case, the processes are real-valued and we show that there exist realizations of the above process on the same probability space of a standard Brownian motion.

In the second work presented in this thesis, we consider the integration with respect to a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. The integral is defined as the limit in probability of a sequence of Riemann sums built from a symmetric measure in the interval $[0, 1]$, that is a probability measure invariant with respect to the map $t \mapsto 1 - t$. In [Gradinaru et al., 2005], Gradinaru, Nourdin, Russo y Vallois prove that, for values of H strictly bigger than a critical value dependent of the measure, the integral exists. It has been proved that this lower bound for the Hurst parameter is sharp.

We are interested in studying the stochastic integral when the Hurst parameter takes the critical value. In this case, we prove that the Riemann sums converge in distribution and the limit can be expressed in terms of a stochastic integral with respect to a Brownian motion independent of the fractional Brownian motion. As a consequence, we derive a change-of-variable formula in law. This phenomenon has already been studied for particular symmetric measures. For example, the case of

the "Midpoint Riemann sums" has been first considered by Nourdin and Réveillac in [Nourdin and Réveillac, 2009]. In [Nourdin et al., 2010] and [Harnett and Nualart, 2012] the case of the Riemann sums corresponding to the "Trapezoidal rule" has been studied, while the case of the "Simpson's rule sums" has been investigated by Harnett and Nualart in [Harnett and Nualart, 2015]. We find a more general result that can be applied to all symmetric measure that satisfies the conditions required in the result obtained in [Gradinaru et al., 2005].

Using Taylor's formula and the properties of the symmetric measure, the integral driven by the fractional Brownian motion can be expressed as a sum of three different terms. In [Gradinaru et al., 2005], the authors prove that two of them converge to zero in probability, uniformly in compact sets, for values of the Hurst parameter strictly bigger than the critical value. To prove our result, we consider the same representation of the stochastic integral. On one hand, we show that one of the term converges to zero in probability, uniformly in compact sets, also when H reaches the critical value. On the other hand, we prove that another term converges in law to an integral driven by a Brownian motion independent of the fractional Brownian motion. This last part is the most innovative and the one that requires more attention. We apply a method based on "small blocks / big blocks" to obtain an extension of a lemma proved by Harnett y Nualart in [Harnett and Nualart, 2015] and applicable to our case. Its proof cannot be established using fractional calculus techniques and it requires the application of integration-by-parts formulas from Malliavin calculus.

In the last work, we are motivated by the study of a stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$. In literature, there are a lot of references about differential equations driven by a Brownian motion, but the extensions to the fractional Brownian motion are scarce. The approaches to investigate these equations are different and depend on the value of the Hurst parameter and the dimension of the equation. One of them consists in studying deterministic differential equations driven by a Hölder continuous function and then applying the results obtained to the stochastic case. Following this method, we consider the differential equation with delay:

$$\begin{aligned} x_t^r &= \eta_0 + \int_0^t b(u, x_u^r) du + \int_0^t \sigma(x_{u-r}^r) dy_u, & t \in [0, T], \\ x_t^r &= \eta_t, & t \in [-r, 0), \end{aligned}$$

where $\eta : [-r, 0] \rightarrow \mathbb{R}^d$ is a continuous function and y is a Hölder continuous function of order $\beta \in (0, 1)$. The interest for differential equations with delay rises from the need to model systems where the dynamics are subjected to propagation delay.

The case when $\beta > \frac{1}{2}$ has been widely studied and results on existence and uniqueness of solution and convergence, together with some extensions, have been achieved. We are interested in studying the case when $\beta \in (\frac{1}{3}, \frac{1}{2})$. When β takes values less or equal to $\frac{1}{3}$, cumbersome notations and new difficulties make more

complicated to work with differential equations in this case and the results are scarce.

We prove that, when β takes values in the interval $(\frac{1}{3}, \frac{1}{2})$, the solution of the differential equation with delay converges almost surely in the supremum norm to the solution of the differential equation without delay

$$x_t = \eta_0 + \int_0^t b(u, x_u) du + \int_0^t \sigma(x_u) dy_u, \quad t \in [0, T],$$

when the delay tends to zero. The existence and uniqueness of the solution have been ensured by Hu and Nualart in [Hu and Nualart, 2009] for the differential equation without delay, and by Neuenkirch, Nourdin and Tindel in [Neuenkirch et al., 2008] for the differential equation with delay. To prove the strong convergence, we follow the approach of Hu and Nualart in [Hu and Nualart, 2009] and we work with an explicit formula for the integrals of the type $\int_s^t \sigma(x_u) dy_u$ in terms of x , y and $x \otimes y$, where $x \otimes y$ is a multiplicative functional.

This thesis is structured in the following way.

After this introduction, in Chapter 1 we define the stochastic processes we deal with in this dissertation: the Poisson process, the Brownian motion and its extension to the complex plane and the fractional Brownian motion. We also describe their main properties.

Chapter 2 contains some preliminaries on stochastic calculus. It is dedicated to describe the stochastic integration with respect to the fractional Brownian motion, with special attention to the case when the Hurst parameter takes values in the interval $(\frac{1}{3}, \frac{1}{2})$.

The following three chapters contain the innovative works of this dissertation. In Chapter 3 we find strong approximations of the complex Brownian motion given by a family of stochastic processes constructed from a unique Poisson process and a sequence of independent random variables with common Bernoulli distribution. Part of the results of this chapter appears in [Bardina et al., 2016].

In Chapter 4 we establish the weak convergence, in the topology of the Skorohod space, of the symmetric Riemann sums of the fractional Brownian motion when the Hurst parameter takes a critical value and derive a change-of-variable formula in distribution. The contents of this chapter appears in [Binotto et al.,].

Finally, in Chapter 5 we prove that the solution of a delay differential equations driven by a Hölder continuous function of order $\beta \in (\frac{1}{3}, \frac{1}{2})$ converges almost surely in the supremum norm to the solution of the differential equation without delay when the delay tends to zero.

1

Definitions

A stochastic process is a phenomenon which evolves in time in a random way. In everyday life there is a huge variety of these phenomena and stochastic processes have been studied in many disciplines, such as physics, biology, economics and telecommunication, just to name a few. In this thesis we work with some of the most relevant of them: the Poisson process, the standard and complex Brownian motion and the fractional Brownian motion. This chapter is devoted to introduce them and describe their main properties.

1.1 The Poisson process

Many processes involve only a countable number of states and depend on a discrete time parameter, that is, events occur only at fixed epochs $n = 0, 1, \dots$. On the contrary, the Poisson process represents those phenomena where events occur at any time. Some examples are telephone calls, radioactive disintegration of atoms and chromosome breakages. More in general, the Poisson process is applied in various fields such as astronomy, biology, ecology, geology, physics, economics, image processing, and telecommunications.

The underlying physical assumption is that the forces and influences governing the process remain constant so that the probability of any particular event is the same for all time intervals of duration t and is independent of the past development of the process. In any case, all occurrences are assumed to be of the same kind and are represented by points on the time axis.

In mathematical terms, we have the following definition of Poisson process:

Definition 1.1.1. *A stochastic process $\{N_t, t \geq 0\}$ is a **Poisson process** of rate $\lambda > 0$ if it fulfills the following properties:*

- i) $N_0 = 0$*
- ii) For any $n \geq 1$ and for any $0 \leq t_1 < \dots < t_n$, the increments*

$$N_{t_1}, N_{t_2} - N_{t_1}, \dots, N_{t_n} - N_{t_{n-1}}$$

are independent random variables.

iii) For any $0 \leq s < t$, the increment $N_t - N_s$ has a Poisson distribution with parameter $\lambda(t - s)$, that is,

$$P(N_t - N_s = k) = e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^k}{k!}.$$

for $k \in \mathbb{N}$.

Condition i) is a convention: we suppose that at time $t = 0$ no event has occurred. Condition ii) states that the number of events occurred during the sequence of intervals $[t_0, t_1], \dots, [t_{k-2}, t_{k-1}]$ have no influence on the amount of events occurred during $[t_{k-1}, t_k]$.

Condition iii) means that the quantity N_t obeys the Poisson distribution with parameter λt , so

$$\mathbb{E}(N_t) = \text{Var}(N_t) = \lambda t.$$

Thus λ is the expected number of events in an interval of unit length, or in other words, λ is the event rate. On the other hand, the expected time until a new occurrence is $\frac{1}{\lambda}$. In Figure 1.1.1 we see an example of the sample path of a Poisson process.

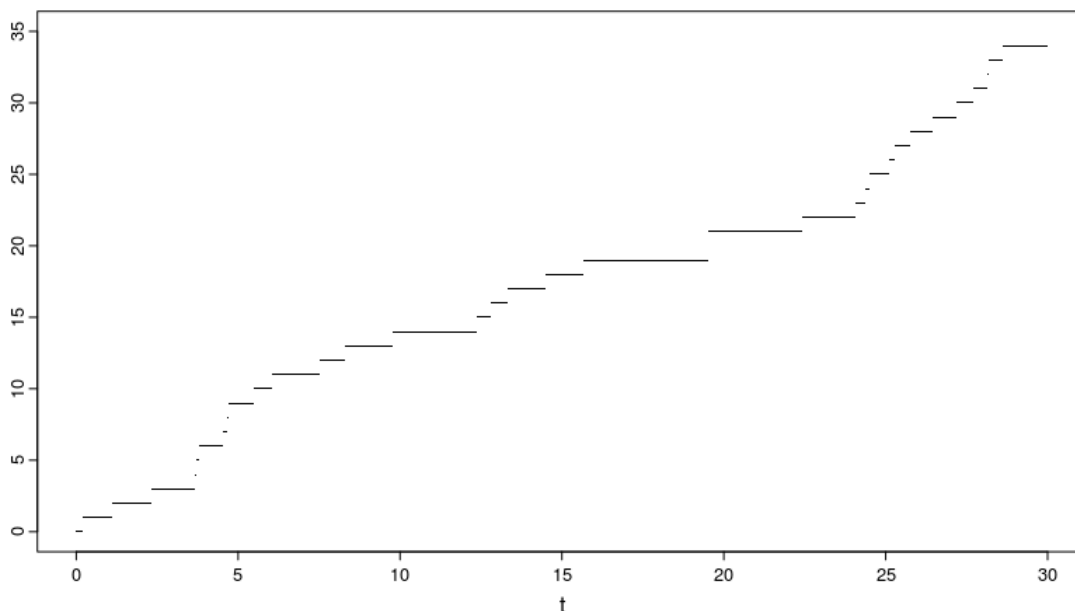


Figure 1.1.1: Sample path of a Poisson process of rate $\lambda = 0.05$.

Observe that the process N_t counts the number of events occurred up to time t starting from 0, while the increment $N_t - N_s$ counts this amount in the interval

(s, t) . Also, its zero term $e^{-\lambda t}$ may be interpreted as the probability that no event occurs within a time interval of fixed length t . But then $e^{-\lambda t}$ is also the probability that the waiting time for the first events exceeds t , and so we are indirectly concerned with a continuous probability distribution on the time axis.

In the following N will denote a Poisson process of rate λ . We give the following definition:

Definition 1.1.2. *Let N be a Poisson process. The **arrival times** for N are the random variables*

$$T_k = \inf\{t > 0 : N_t = k\}$$

for $k \in \mathbb{N} \setminus \{0\}$.

1.1.1 Properties of the Poisson process

Let us present some interesting properties of the Poisson process.

Exponential time differences.

The following proposition states that the time between two consecutive events is exponentially distributed:

Proposition 1.1.3. *Let $T_1, T_2, \dots, T_k, \dots$ be the arrival time for a Poisson process N . Then, for any $k \geq 1$,*

$$T_k - T_{k-1} \sim \text{Exp}(\lambda),$$

that is, for any $x \geq 0$,

$$P(T_k - T_{k-1} \leq x) = 1 - e^{-\lambda x}.$$

Conditioning on the number of arrivals.

In the following proposition we see that, given that in the interval $(0, t)$ the number of arrivals is $N_t = k$, these k arrivals are independently and uniformly distributed in the interval.

Proposition 1.1.4. *Let $T_1, T_2, \dots, T_k, \dots$ be the arrival times for a Poisson process N . Then, the joint distribution of (T_1, \dots, T_k) given $\{N_t = k\}$ is the same of k independent random variables with uniform distribution $U(0, t)$.*

Superposition.

First we give the following definition:

Definition 1.1.5. *Let $\{N_t^1, t \geq 0\}$ and $\{N_t^2, t \geq 0\}$ be two independent Poisson processes with respective rates λ_1 and λ_2 . The process obtained from their sum, $\{N_t^1 + N_t^2, t \geq 0\}$ is called the **superposition** of the processes N^1 and N^2 .*

The following proposition describes the distribution of the superposition of two independent Poisson processes:

Proposition 1.1.6. *Let $\{N_t^1, t \geq 0\}$ and $\{N_t^2, t \geq 0\}$ be two independent Poisson processes with respective rates λ_1 and λ_2 . Then, their superposition $N^1 + N^2$ is a Poisson process of rate $\lambda_1 + \lambda_2$.*

Random selection.

The following result explains what happens when we split randomly a Poisson process into two different processes.

Proposition 1.1.7. *Let N be a Poisson process with rate λ . Let $\{X_k\}_{k \in \mathbb{N}}$ be a sequence of independent random variable with Bernoulli distribution of parameter p , that is, for any $k \in \mathbb{N}$,*

$$P(X_k = 1) = p \quad \text{and} \quad P(X_k = 0) = 1 - p = q.$$

Define the processes

$$M_t^1 = \sum_{k=1}^{N_t} X_k$$

and

$$M_t^2 = \sum_{k=1}^{N_t} (1 - X_k).$$

Then, the processes M^1 and M^2 are two independent Poisson processes with rates λp and λq respectively.

1.2 Brownian motion

Brownian motion is the name given to the random movement of particles of pollen immersed in a liquid observed by the Scottish botanist Robert Brown in 1827. This physical phenomenon remained unexplained for several years. In 1900, in his doctoral thesis "Théorie de la spéculation" ([Bachelier, 1900]), the French mathematician Louis Bachelier worked out a model for the variation of the prices of assets, like stocks and bonds. This is considered the first work which uses advanced mathematical methods to model financial markets. The equations he obtained correspond to the results that in 1905 Albert Einstein, who as far as we know never heard of Bachelier, explained in terms of statistical mechanics in his celebrated paper [Einstein, 1905]. Einstein was the one who began to develop a physical theory of the phenomenon observed by Brown. The mathematical theory of Brownian motion as a stochastic process was later formalized by Norbert Wiener. In 1923, in [Wiener, 1923], he proposed a rigorous mathematical construction using harmonic analysis techniques. For this reason, in mathematical terms, we refer to Brownian motion also as Wiener process.

The range of application of Brownian motion is not limited to the study of microscopic particles in suspension and includes modeling of physical, biological, economic and management systems. It is also applied to quantum mechanics and physical cosmology. Moreover, the Wiener process plays an important role in both pure and applied mathematics. In a pure contest, it is essential in the study of continuous time martingales and diffusion processes, in stochastic calculus and as a key tool to define more complicated stochastic processes. In applied mathematics, it is used in the representation of the integral of a white noise Gaussian process and in the mathematical theory of finance, in particular in the Black–Scholes option pricing model.

The mathematical definition of Brownian motion is the following:

Definition 1.2.1. A **Brownian motion** or **Wiener process** is a stochastic process $\{W_t, t \geq 0\}$, on some probability space (Ω, \mathcal{F}, P) , with these properties:

i) The process starts at 0:

$$P(W_0 = 0) = 1.$$

ii) For all $0 = t_0 < t_1 < \dots < t_n$, the increments

$$W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}$$

are independent.

iii) For $0 \leq s < t$, the increment $W_t - W_s$ has normal distribution $N(0, t - s)$.

iv) The process has continuous trajectories: for each ω , $W_t(\omega)$ is continuous in t and $W_0(\omega) = 0$.

Let briefly describe the physical interpretation of this stochastic process. Imagine a particle suspended in a fluid and bombarded by molecules in thermal motion. The particle will perform a random movement, as the one described by Brown. Consider a single component of this motion, imagine it projected on a vertical axis and denote by W_t , the height at time t of the particle above a fixed horizontal plane.

Condition i) is merely a convention: the particle starts at 0. When this condition is fulfilled we refer to W as **standard Brownian motion**.

Condition ii) reflects a kind of lack of memory. It means that during the intervals $[t_0, t_1], \dots, [t_{k-2}, t_{k-1}]$ the displacements $W_{t_1} - W_{t_0}, \dots, W_{t_{k-1}} - W_{t_{k-2}}$ of the particle do not influence the displacement $W_{t_k} - W_{t_{k-1}}$ occurred during $[t_{k-1}, t_k]$. So, although its future behavior depends on its present position, it does not matter how the particle got there.

In condition iii), the zero mean of the displacement $W_t - W_s$ reflects the fact that the particle is as likely to go up as to go down. The variance grows as the length of the interval $[s, t]$, this means that the particle tends to walk away from its position at time s and it is not forced to restore that position.

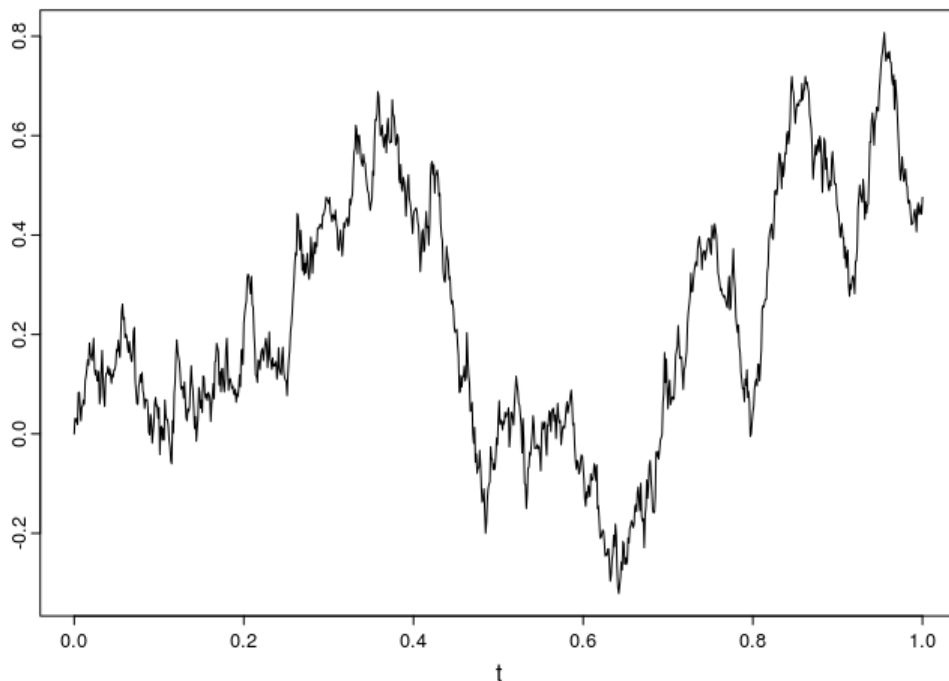


Figure 1.2.1: Sample path of a standard Brownian motion.

Given that the Wiener process wants to represent the motion of a particle, condition iv) is just a natural requirement.

In Figure 1.2.1 we see an example of a sample path of a standard Brownian motion.

Let make some observations about the process we have just described.

Remark 1.2.2. Brownian motion is a Gaussian process. In fact, for $0 = t_0 < t_1 < \dots < t_n$, the joint distribution of the vector $(W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}})$ is the product of the corresponding normal density, because its components are independent and normal. The vector $(W_{t_1}, \dots, W_{t_n})$ is a linear transformation of the vector $(W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}})$, so its joint distribution is also normal.

Remark 1.2.3. The increments of the Brownian motion are stationary in the sense that the distribution of $W_t - W_s$ depends only on the difference $t - s$. Since $W_0 = 0$, the distribution of these increments is described by saying that W_t , is normally distributed with mean 0 and variance t , so

$$\mathbb{E}(W_t) = 0 \quad \text{and} \quad \mathbb{E}(W_t^2) = t.$$

If $0 \leq s \leq t$, by the independence of the increments, we easily obtain the covariance:

$$\mathbb{E}(W_s W_t) = \mathbb{E}(W_s(W_t - W_s)) + \mathbb{E}(W_s^2) = s = \min(s, t).$$

Remark 1.2.4. As the increment $W_t - W_s$ has normal distribution $N(0, t - s)$, for any natural number k , the even moments of the increment are given by

$$\mathbb{E}[(W_t - W_s)^{2k}] = \frac{(2k)!}{2^k k!} (t - s)^k. \quad (1.2.1)$$

1.2.1 Properties of the Brownian motion

Now we present some properties of the Brownian motion, that make it so a meaningful process. In what follow, we denote with $W = \{W_t, t \geq 0\}$ a standard Brownian motion.

Symmetry.

The following property is easy to check:

Proposition 1.2.5. *The process*

$$-W = \{-W_t, t \geq 0\}$$

is also a Brownian motion.

Self-similarity.

Proposition 1.2.6. *For any $c > 0$, the process*

$$\left\{ \frac{1}{c} W_{c^2 t}, t \geq 0 \right\}$$

is a Brownian motion.

Observe that the time scale is contracted by the factor c^2 , but the other scale only by the factor c . The fact that this transformation preserves the properties of Brownian motion implies that the paths, although continuous, must be highly irregular, as we will see below.

Time reversal.

For the next property, we need to restrict the time parameter to a bounded interval of the form $[0, T]$ where $T > 0$:

Proposition 1.2.7. *Define the process*

$$V_t := W_T - W_{T-t}$$

for $t \in [0, T]$. Then, V_t is distributed as W_t for $t \in [0, T]$.

Time inversion.

Proposition 1.2.8. *Consider the transformation*

$$W'_t = \begin{cases} tW_{\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Then, $\{W'_t, t \geq 0\}$ is also a Brownian motion.

Time inversion is a useful tool to relate the properties of Brownian motion in a neighborhood of time $t = 0$ to properties at infinity. The following proposition is a result about the long-term behavior obtained from a trivial statement at the origin:

Proposition 1.2.9. *Almost surely*

$$i) \inf_t W_t = -\infty \quad \text{and} \quad \sup_t W_t = +\infty,$$

$$ii) \lim_{t \rightarrow \infty} \frac{W_t}{t} = 0 \quad [\text{Law of Large Numbers for Brownian motion}]$$

$$iii) \limsup_{t \rightarrow \infty} \frac{W_t}{\sqrt{t}} = +\infty \quad \text{and} \quad \liminf_{t \rightarrow \infty} \frac{W_t}{\sqrt{t}} = -\infty.$$

Using time-inversion, from statement i) we deduce that W' has upper and lower right derivatives of $+\infty$ and $-\infty$ at $t = 0$ and the same must be true for every Brownian motion. This reasoning is useful to prove the nowhere differentiability of Brownian motion, as we will see below.

Statement ii) asserts that Brownian motion grows slower than linearly, while statement iii) shows that the limsup growth of W_t is faster than \sqrt{t} .

Continuity properties.

The definition of Brownian motion requires the sample paths to be continuous almost surely. This implies that on the interval $[0, 1]$ (or any other compact interval) the sample functions are uniformly continuous, that is, there exists a random function ϕ with $\lim_{h \downarrow 0} \phi(h) = 0$ called a **modulus of continuity** of the function $W : [0, 1] \rightarrow \mathbb{R}$ such that

$$\limsup_{h \downarrow 0} \sup_{0 \leq t \leq 1-h} \frac{|W_{t+h} - W_t|}{\phi(h)} \leq 1.$$

For the Brownian motion we can extend that to a nonrandom modulus of continuity. First we find an upper bound and a lower bound for $|W_{t+h} - W_t|$:

Proposition 1.2.10. *There exists a constant $C > 0$ such that, almost surely, for every sufficiently small $h > 0$ and all $t \in [0, 1 - h]$,*

$$|W_{t+h} - W_t| \leq C \sqrt{h \log(1/h)}.$$

Proposition 1.2.11. *For every constant $c < \sqrt{2}$, almost surely, for every $\varepsilon > 0$ there exist $0 < h < \varepsilon$ and $t \in [0, 1 - h]$ such that*

$$|W_{t+h} - W_t| \geq c\sqrt{h \log(1/h)}.$$

It turns out that the value $c = \sqrt{2}$ is optimal. Therefore, we have the following result due to [Lévy, 1937]:

Theorem 1.2.12 (Lévy's modulus of continuity). *Almost surely,*

$$\limsup_{h \downarrow 0} \sup_{0 \leq t \leq 1-h} \frac{|W_{t+h} - W_t|}{\sqrt{2h \log(1/h)}} = 1.$$

Regularity of the trajectories.

Proposition 1.2.13. *Almost surely, the sample paths of the Brownian motion are γ -Hölder continuous with $\gamma \in (0, \frac{1}{2})$.*

In other words, for all $\varepsilon > 0$ there exists a random variable $G_{\varepsilon, T}$ such that

$$|W_t - W_s| \leq G_{\varepsilon, T} |t - s|^{\frac{1}{2} - \varepsilon},$$

for all $s, t \in [0, T]$.

Regularity is consequence of identity (1.2.1) and the Kolmogorov's continuity criterion:

Proposition 1.2.14 (Kolmogorov's continuity criterion). *Let $\{X_t, t \geq 0\}$ be a stochastic process. Suppose that there exist positive constants α , β and C such that*

$$\mathbb{E}(|X_t - X_s|^\alpha) \leq C|t - s|^{1+\beta}.$$

Then, almost surely, the sample paths of the process are γ -Hölder continuous with $\gamma < \frac{\beta}{\alpha}$.

Observe that the exponent $\gamma = \frac{1}{2}$ above is sharp. In fact, fixed $\gamma \in [\frac{1}{2}, 1]$, almost surely the sample paths of $\{W_t, t \geq 0\}$ are nowhere Hölder continuous with exponent γ .

Nowhere differentiability.

Brownian motion is somewhat regular and erratic at the same time. One manifestation of the last aspect is that the paths of Brownian motion have no intervals of monotonicity:

Proposition 1.2.15. *Almost surely, for all $0 < s < t < \infty$, Brownian motion is not monotone on the interval $[s, t]$.*

In order to discuss differentiability of Brownian motion one can make use of the time-inversion trick, which allows relating differentiability at $t = 0$ to a long-term property. The following result was shown by [Paley et al., 1933], and [Dvoretzky et al., 1961]:

Theorem 1.2.16. *Almost surely, Brownian motion is nowhere differentiable.*

A nowhere-differentiable path represents the motion of a particle that at no time has a velocity. Since a function of bounded variation is differentiable almost everywhere, $W(\omega)$ is almost surely of unbounded variation, as we will see in the following paragraph.

Variation and quadratic variation.

Let us define first the variation and the quadratic variation of a real-valued function.

Definition 1.2.17. *Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$. Its **quadratic variation** over the interval $[0, t]$ is*

$$\langle f, f \rangle_t := \lim_{|\pi| \rightarrow 0} \sum_{k=1}^n (f(t_k) - f(t_{k-1}))^2,$$

where $\pi = \{0 = t_0 < t_1 < \dots < t_n = t\}$ is a partition of the interval $[0, t]$ and its norm is defined by $|\pi| = \sup_{1 \leq k \leq n} (t_k - t_{k-1})$.

Definition 1.2.18. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Its **variation** on $[0, t]$ is*

$$V_f(t) := \sup_{\pi} \sum_{k=1}^n |f(t_k) - f(t_{k-1})|,$$

where the supremum is taken over partitions $\pi = \{0 = t_0 < t_1 < \dots < t_n = t\}$. If the supremum is infinite, f is said to be of **unbounded variation**.

For Brownian motion W (or any other stochastic process), we can similarly define its variation and quadratic variation. In such case, they are both random variables.

Definition 1.2.19. *The **quadratic variation of the Brownian motion** W over the interval $[0, t]$ is*

$$\langle W \rangle_t := \lim_{|\pi| \rightarrow 0} \sum_{k=1}^n (W_{t_k} - W_{t_{k-1}})^2, \quad (1.2.2)$$

where $\pi = \{0 = t_0 < t_1 < \dots < t_n = t\}$ is a partition of the interval $[0, t]$, its norm is defined by $|\pi| = \sup_{1 \leq k \leq n} (t_k - t_{k-1})$ and the limit is in $L^2(\Omega)$.

Then, we have the following estimate for this random variable:

Proposition 1.2.20. *The quadratic variation of a standard Brownian motion W is*

$$\langle W \rangle_t = t.$$

In other words, Proposition 1.2.20 states that the sequence $\left\{ \sum_{k=1}^n (W_{t_k} - W_{t_{k-1}})^2, n \geq 1 \right\}$ converges in $L^2(\Omega)$ to t , that is,

$$\lim_{|\pi| \rightarrow 0} \mathbb{E} \left[\left(\sum_{k=1}^n (W_{t_k} - W_{t_{k-1}})^2 - t \right)^2 \right] = 0.$$

The previous proposition means that the Brownian motion accumulates quadratic variation at rate one per unit time.

The limit in the definition of quadratic variation can be taken in sense of any L^p convergence. However, in the sense of almost sure convergence, the limit does not exist unless additional condition on π is assumed, such as, $|\pi| = o(1/\sqrt{\log n})$.

Proposition 1.2.20 together with the continuity of the sample paths of Brownian motion yields the following corollary:

Corollary 1.2.21. *Let W be a standard Brownian motion. Then, almost surely W has infinite variation. That is,*

$$V := \sup_{\pi} \sum_{k=1}^n |W_{t_k} - W_{t_{k-1}}| = \infty \quad a.s.$$

where $\pi = \{0 = t_0 < t_1 < \dots < t_n = t\}$.

Physically, the previous result represents the motion of a particle that in its wanderings back and forth travels an infinite distance in finite time.

Martingale property.

First, we give the definitions of filtration and martingale.

Definition 1.2.22. *A family $\{\mathcal{F}_t, t \geq 0\}$ of sub σ -fields of \mathcal{F} is a **filtration** if*

1. \mathcal{F}_0 contains all the sets of \mathcal{F} of null probability,
2. For any $0 \leq s \leq t$, $\mathcal{F}_s \subset \mathcal{F}_t$.

Moreover, if $\bigcap_{u>t} \mathcal{F}_u = \mathcal{F}_t$ for any $t \geq 0$, the filtration is said to be right-continuous.

Definition 1.2.23. *A stochastic process $\{X_t, t \geq 0\}$ is a **martingale** with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ if each variable belongs to $L^1(\Omega)$ and moreover*

1. X_t is \mathcal{F}_t -measurable for any $t \geq 0$,
2. $\mathbb{E}(X_t | \mathcal{F}_s) = X_s$ for any $0 \leq s \leq t$.

In the last condition, if the equality is replaced by \leq (respectively, \geq), we have a **supermartingale** (respectively, a **submartingale**).

Consider the natural filtration of the Brownian motion, that is,

$$\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\}.$$

We have the following result:

Proposition 1.2.24. *Brownian motion is a martingale with respect to its natural filtration.*

This property is a consequence of the following identity:

$$\mathbb{E}(W_t - W_s | \mathcal{F}_s) = \mathbb{E}(W_t - W_s) = 0,$$

for any $0 \leq s \leq t$.

Moreover, we define a local martingale:

Definition 1.2.25. *Let \mathcal{F} be a filtration and $X = \{X_t, t \geq 0\}$ be a stochastic process adapted to the filtration \mathcal{F} . Then, X is called a **local martingale** with respect to the filtration \mathcal{F} if there exists a sequence of almost surely increasing \mathcal{F} -stopping times $\{\tau_k\}_{k \in \mathbb{N}}$ that diverges almost surely such that the process*

$$X_t^{\tau_k} := X_{t \wedge \tau_k}$$

is a martingale with respect to \mathcal{F} for every k .

We have the following result due to Lévy:

Theorem 1.2.26 (Lévy's characterization theorem). *Let $(M_t)_{t \geq 0}$ be a continuous local martingale such that*

1. $M_0 = 0$,
2. $\forall t \geq 0, \langle M \rangle_t = t$.

Then, the process $(M_t)_{t \geq 0}$ is a standard Brownian motion.

1.3 Complex Brownian motion

Complex Brownian motion is the equivalent on \mathbb{C} of the Brownian motion.

In this context, dimension 2, that is $\mathbb{C} \cong \mathbb{R}^2$, is sometimes considered as critical. Let us explain which is the reason of this terminology. In \mathbb{R} , Brownian motion is strongly recurrent in the sense that it hits every point x almost surely and it also returns to each x infinitely many times. In \mathbb{R}^3 and higher dimensions, Brownian motion is transient. In fact, for each x different from the origin there is positive probability that, for some $\varepsilon > 0$, it does not even hit the ball $\mathcal{B}(x, \varepsilon)$ centered in

x and with radius ε . In $\mathbb{C} \cong \mathbb{R}^2$, it oscillates between transience and recurrence: almost surely it does not hit any given point x , unless it starts there, but the closure of the Brownian path is almost surely the entire complex plane. This unusual mixture makes complex Brownian motion an extraordinary object.

Mathematically speaking, the definition of complex Brownian motion is the following:

Definition 1.3.1. *A complex valued stochastic process $Z = \{Z_t, t \geq 0\}$ is a **complex Brownian motion** if $Z = X + iY$, where $X = \{X_t, t \geq 0\}$ and $Y = \{Y_t, t \geq 0\}$ are independent real Brownian motions. For simplicity, we assume that $Z_0 = 0$ with $0 \in \mathbb{C}$.*

This definition is equivalent to say that the vector (X, Y) is a Brownian motion in \mathbb{R}^2 . An example of a sample path of this process is given in Figure 1.3.1.

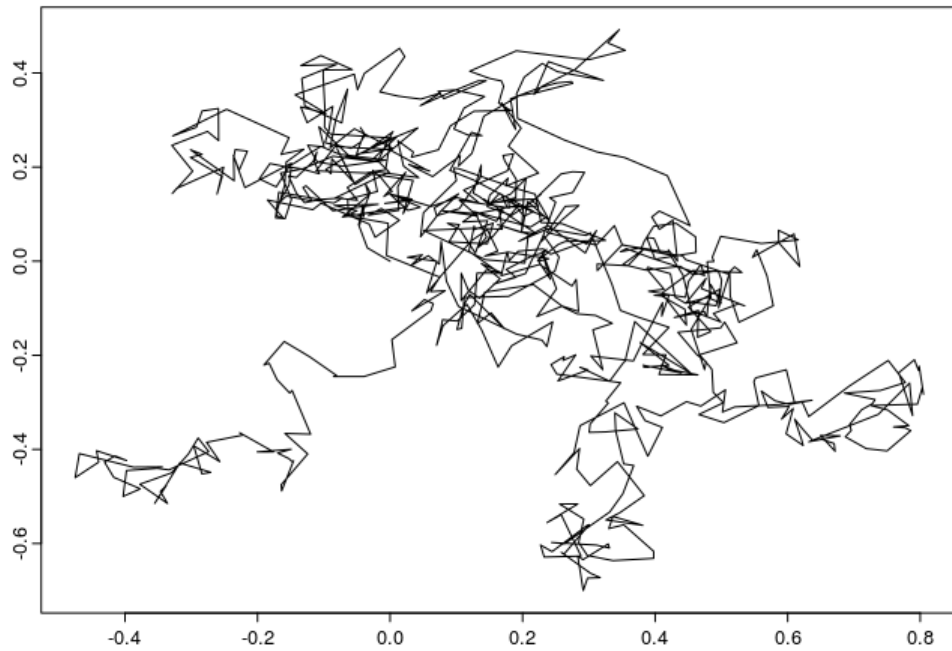


Figure 1.3.1: Sample path of a complex Brownian motion.

Remark 1.3.2. If $z \in \mathbb{C}$ and Z is a complex Brownian motion with $Z_0 = 0$, then $Z' = z + Z$ is a complex Brownian motion with $Z'_0 = z$.

1.3.1 Properties of the complex Brownian motion

On one hand, complex Brownian motion preserves many properties of the Brownian motion, such as, self-similarity, time reversal and time inversion. On the other

hand, it presents other distinguished aspects.

Rotation invariance.

The following property is true for all complex Brownian motions:

Proposition 1.3.3. *For every complex number c such that $|c| = 1$, the process*

$$Z' := \{cZ_t, t \geq 0\}$$

is another complex Brownian motion.

Martingale property.

Let M^1 and M^2 be continuous square integrable martingales. We define the following operator:

$$\langle M^1, M^2 \rangle := \lim_{n \rightarrow \infty} \sum_{k=1}^n (M_{t_k}^1 - M_{t_{k-1}}^1)(M_{t_k}^2 - M_{t_{k-1}}^2)$$

where the limit is in probability and $\pi = \{0 = t_0 < t_1 < \dots < t_n = T\}$ with $|\pi| = \sup_k |t_k - t_{k-1}| \rightarrow 0$ as $n \rightarrow \infty$.

We extend the definition of martingale given in Definition 1.2.23 to the case of a complex-valued process:

Definition 1.3.4. *A complex-valued stochastic process is called a **martingale**, if its real and imaginary parts are martingales.*

Definition 1.3.5. *A continuous complex martingale M_t is a **conformal martingale** if $\langle M, M \rangle_t \equiv 0$.*

Now we can state the following result:

Proposition 1.3.6. *Complex Brownian motion is a conformal martingale.*

Moreover, we have an extension of Lévy's characterization theorem, also due to Lévy:

Proposition 1.3.7. *Let Z_t be a complex valued continuous adapted process. Then, Z_t is a complex Brownian motion if and only if*

1. Z_t is a conformal local martingale
2. $\langle Z, \bar{Z} \rangle_t = 2t$

where \bar{Z} is the conjugate of Z .

Conformal invariance.

Analytic functions are the complex equivalent of what, in real analysis, would be simply called differentiable functions. In fact, it can be shown that analytic functions are smooth, that is, they are infinitely differentiable in the sense exposed in the following definition:

Definition 1.3.8. *Let $D \subseteq \mathbb{C}$ be an open set. A function $f : D \rightarrow \mathbb{C}$ is **analytic** if the limit*

$$f'(z) = \lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z}$$

*exists for all $z \in D$. An analytic function $f : \mathbb{C} \rightarrow \mathbb{C}$ is said to be an **entire** function.*

At first glance analytic functions may appear unrelated to Brownian motion, but it turns out that there is a surprising connection between the two, namely the conformal invariance of complex Brownian motion:

Proposition 1.3.9. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a non-constant entire function and let Z be a conformal local martingale. Suppose that $\langle Z, \bar{Z} \rangle$ is strictly increasing. Then, there exists a strictly increasing time change τ such that $f(Z_{\tau(\cdot)})$ is a complex Brownian motion.*

Recurrence.

As we explained above, the complex Brownian motion is neighborhood-recurrent but does not visit a specific point. This fact is expressed in more technical terms in the following proposition:

Proposition 1.3.10. *Let Z_t be a planar Brownian motion started at $z_0 \in \mathbb{C}$. Then, for all open $U \subset \mathbb{C}$, the set*

$$\{Z_t \in U, t \geq 0\}$$

is almost surely unbounded. Moreover, for all $w \neq z_0$,

$$\{Z_t = w, t \geq 0\} = \emptyset \quad a.s.$$

1.4 Fractional Brownian motion

At the beginning, Brownian motion was employed in the study of physical and hydrological phenomena, such as Einstein's study of particles in a liquid. Kolmogorov and then other mathematicians understood that not all natural phenomena trace random paths which can be represented as Brownian motion trajectories, and that different events could not be explained by the same law.

On this basis, in the middle of twentieth century the British hydrologist Harold Edwin Hurst, who was working in Egypt to determine the intensity of the floods

of the Nile River, obtained an important result according to which the annual variations of the flow of the Nile were not statistically independent as in the model of Bachelier, but that the evolution of hydrogeological phenomena have a positive dependence with what happened in the past.

In 1968, motivated by the research of some applications in hydrology, telecommunications, queueing theory and mathematical finance, Mandelbrot and Van Ness studied a way to describe processes that cannot be explained by standard Brownian motion. In their seminal paper [Mandelbrot and Van Ness, 1968], they define a stochastic process that depends on a parameter $H \in (0, 1)$, called Hurst index as a tribute to the British hydrologist, and coined the term fractional Brownian motion.

Let us explain the reason of this term. The fractional Brownian motion B_t^H can be represented as an integral with respect a to Brownian motion W_t , as follows

$$B_t^H = \int_{-\infty}^0 [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] dW_s + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s, \quad (1.4.1)$$

and this form is reminiscent to the fractional integrals, that are generalizations of the n -fold iterated integral formula

$$\int_0^t dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_2} dt_1 \int_0^{t_1} g(s) ds = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} g(s) ds,$$

for arbitrary values of n .

The mathematical definition of the fractional Brownian motion is the following:

Definition 1.4.1. *A stochastic process $B^H = \{B_t^H, t \leq 0\}$ is a **fractional Brownian motion** (fBM) of Hurst parameter $H \in (0, 1)$ if it is a centered Gaussian process with covariance functions*

$$R_H(t, s) = \mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}). \quad (1.4.2)$$

The Hurst parameter controls the roughness of the paths: the higher H is, the smoother the trajectory will be. It also determines which kind of process the fractional Brownian motion is. Particularly, we will distinguish between three cases: when $0 < H < \frac{1}{2}$, when $H = \frac{1}{2}$ and when $\frac{1}{2} < H < 1$.

In Figure 1.4.1 and Figure 1.4.2 we can compare the sample paths of three fractional Brownian motions whose Hurst parameters take values in the intervals mentioned above. One can easily observe that the process is rougher when the Hurst parameter is smaller.

Observe that, when $H = \frac{1}{2}$, the covariance is

$$R_{\frac{1}{2}}(t, s) = s \wedge t$$

and $B^{\frac{1}{2}}$ is a standard Brownian motion.

In what follows, we use the notation B^H to denote a fractional Brownian motion with Hurst parameter H . As a convention, we assume that $B_0^H = 0$.

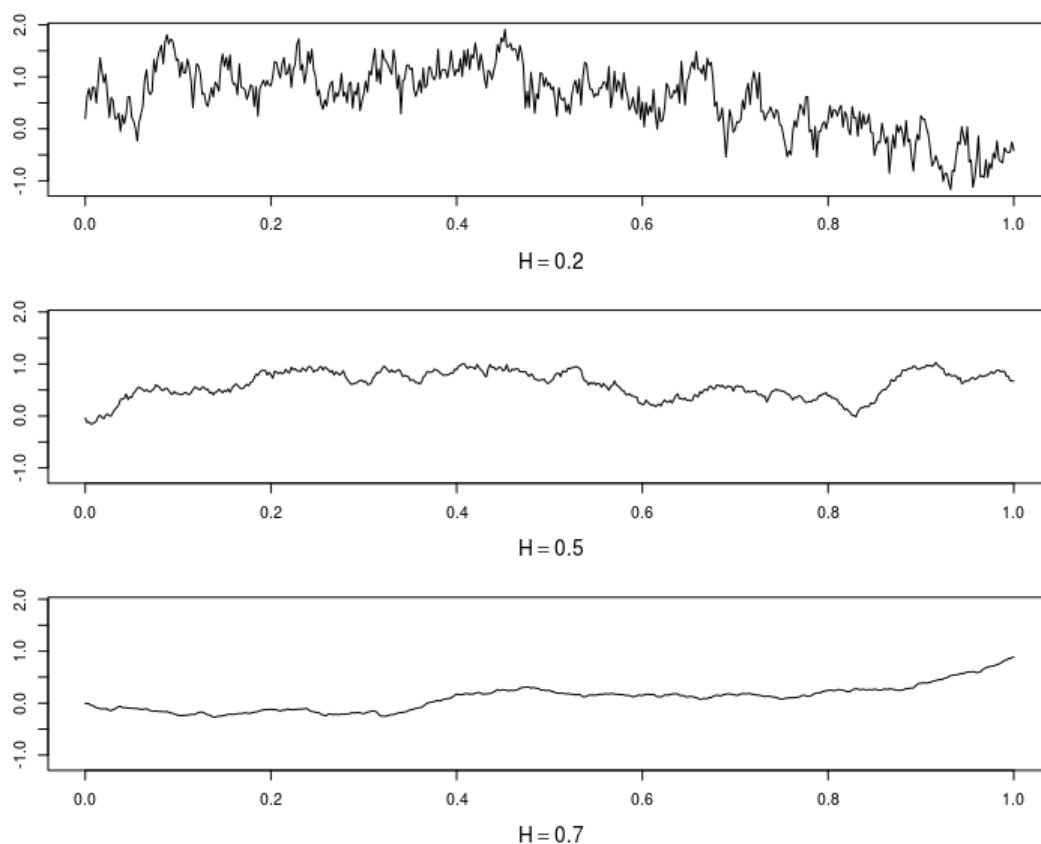


Figure 1.4.1: Sample paths of three different fractional Brownian motions. In the second figure the Hurst parameter is $H = \frac{1}{2}$, so the process is a standard Brownian motion. In the first and in the third figures, the Hurst parameter takes values in $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$, respectively.

1.4.1 Properties of the fractional Brownian motion

In this section we present the principal properties of the fractional Brownian motion.

Self-similarity.

Proposition 1.4.2. *For any constant $a > 0$, the processes $\{a^{-H} B_{at}^H, t \leq 0\}$ and $\{B_t^H, t \leq 0\}$ have the same distribution.*

This property is an immediate consequence of the fact that the covariance function is homogeneous of order $2H$:

$$\mathbb{E}(a^{-H} B_{at}^H a^{-H} B_{as}^H) = \mathbb{E}(a^{-2H} B_{at}^H B_{as}^H) = \mathbb{E}(B_t^H B_s^H).$$

Stationary increments.

From the covariance function we deduce that the variance of an increment of the

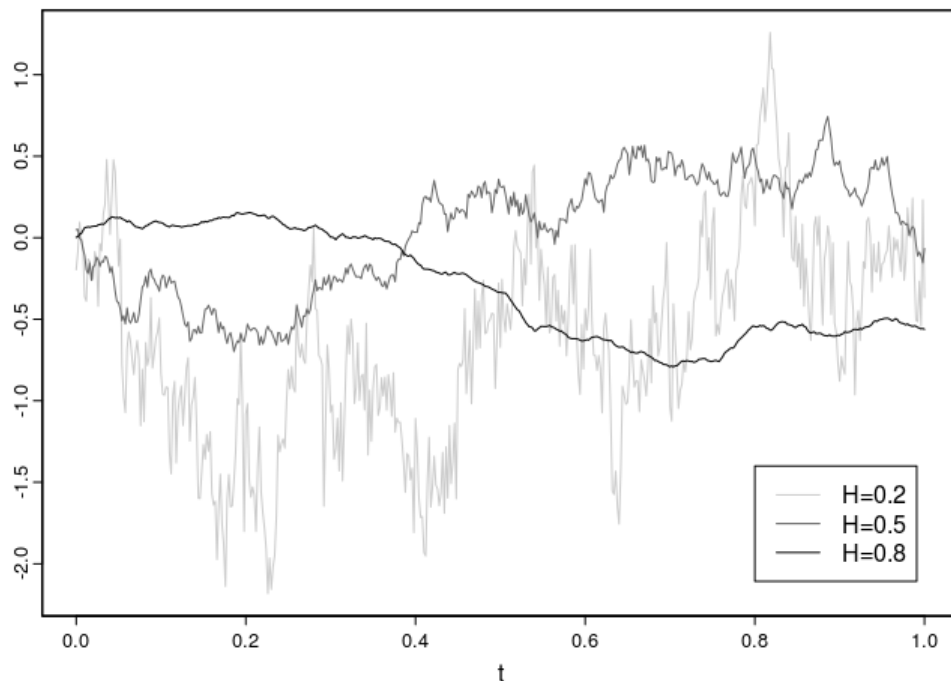


Figure 1.4.2: Sample paths of three different fractional Brownian motions with Hurst parameter $H = 0.2$, $H = 0.5$ (Brownian motion) and $H = 0.8$.

fractional Brownian motion in an interval $[s, t]$ is

$$\text{Var}(|B_t^H - B_s^H|) = |t - s|^{2H}.$$

Then, we can state the following proposition:

Proposition 1.4.3. *Fractional Brownian motion has stationary increments, that is, for any $0 \leq s < t$, the increment $B_t^H - B_s^H$ has the same distribution as B_{t-s}^H .*

Continuity of the trajectories.

Definition 1.4.4. *Let $X = \{X_t, t \geq 0\}$ be a stochastic process. We say that the process $\tilde{X} = \{\tilde{X}_t, t \geq 0\}$ is a **version** of X if, for all $t \geq 0$,*

$$P(X_t = \tilde{X}_t) = 1.$$

We have the following result:

Proposition 1.4.5. *Fractional Brownian motion has a version with continuity trajectories.*

This property is a consequence of the Kolmogorov's continuity criterion (Proposition 1.2.14). For example, when $H > \frac{1}{2}$, given that

$$\mathbb{E}(|B_t^H - B_s^H|^2) = |t - s|^{2H},$$

we deduce that the constants of the Kolmogorov's continuity criterion (Proposition 1.2.14) are $\alpha = 2$, $\beta = 2H - 1$ and $C \geq 1$.

Regularity of the trajectories.

Proposition 1.4.6. *The sample paths of a fractional Brownian motion with Hurst parameter H are γ -Hölder continuous with $\gamma < H$.*

In other words, for every $\varepsilon > 0$ and $T > 0$, there exists a nonnegative random variable $G_{\varepsilon,T}$ such that $\mathbb{E}(|G_{\varepsilon,T}|^p) < \infty$ for all $p \geq 1$, and

$$|B_t^H - B_s^H| \leq G_{\varepsilon,T} |t - s|^{H-\varepsilon},$$

for all $s, t \in [0, T]$.

This property explains how the parameter H controls the regularity of the trajectories and can be deduced from the Garsia-Rodemich-Rumsey Lemma (see [Garsia et al., 1971]).

Nowhere differentiability.

Fractional Brownian motion conserve the following property of the Brownian motion:

Theorem 1.4.7. *Almost surely, sample paths of the fractional Brownian motion are nowhere differentiable.*

Dependence and independence of the increments.

The correlation between the increments of fractional Brownian motion depends on the value of the parameter of Hurst:

Proposition 1.4.8. • *If $H = \frac{1}{2}$, B^H has independent increment.*

- *If $H > \frac{1}{2}$, the increments are positively correlated.*
- *If $H < \frac{1}{2}$, the increments are negatively correlated.*

As we know, if $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ is a standard Brownian motion, so the first statement is trivial. When $H \neq \frac{1}{2}$, the increments of B^H are not independent and their correlation can be easily checked observing that, for $k, n \geq 1$,

$$\mathbb{E}[(B_k^H - B_{k-1}^H)(B_{k+n}^H - B_{k+n-1}^H)] = \frac{1}{2}[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}].$$

(Non) Martingale property.

The following result is another important property that differentiates standard Brownian motion from fractional Brownian motion with $H \neq \frac{1}{2}$. Before giving the statement we introduce some definitions.

Definition 1.4.9. We say that a stochastic process $X = \{X_t, t \geq 0\}$ is **adapted** to the filtration \mathcal{F}_t if, for all t , the random variable X_t is \mathcal{F}_t -measurable.

Definition 1.4.10. A process $X = \{X_t, t \geq 0\}$ is **càdlàg** if its trajectories are right-continuous and have left limits.

Definition 1.4.11. A real valued process $X = \{X_t, t \geq 0\}$ defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ is called a **semimartingale** if it can be decomposed as

$$X_t = M_t + A_t$$

where M is a local martingale and A is a càdlàg adapted process of locally bounded variation.

Therefore, we can state the following result:

Proposition 1.4.12. Fractional Brownian motion is not a semimartingale for $H \neq \frac{1}{2}$.

The result is obtained studying the p -variation of the process over a partition $\pi = \{0 = t_0 < t_1 < \dots < t_n = t\}$ of the interval $[0, t]$:

$$V_p := \lim_{|\pi| \rightarrow 0} \sum_{k=1}^n |B_{t_k}^H - B_{t_{k-1}}^H|^p, \quad (1.4.3)$$

where $|\pi| = \sup_{1 \leq k \leq n} (t_k - t_{k-1})$ is the norm of the partition. Particularly, the quadratic variation on an interval $[s, t]$ equals

$$V_2 = \lim_{|\pi| \rightarrow 0} \sum_{k=1}^n (B_{t_k}^H - B_{t_{k-1}}^H)^2 = \begin{cases} \infty & \text{if } H < 1/2 \\ t - s & \text{if } H = 1/2 \\ 0 & \text{if } H > 1/2 \end{cases}$$

where convergence holds uniformly with probability 1 if $H \neq \frac{1}{2}$ and in the mean squared if $H = \frac{1}{2}$.

2

Preliminaries

This chapter is devoted to give some preliminary results on stochastic calculus that are useful to read the following chapters. In particular, our aim is to describe the stochastic integration with respect to fractional Brownian motion.

In literature the development of the theory of integration with respect to this process moved in several directions: from one side, stochastic differential equations, optimal filtering, financial applications and statistical interference, and from the other side, a lot of theoretical problems and applications.

Semimartingales provide the most general class of stochastic processes for which a stochastic calculus has been developed. Except the case when the Hurst parameter is $H = \frac{1}{2}$, when it is a Wiener process, fractional Brownian motion cannot form a semimartingale so that the classical stochastic integration introduced by Itô does not work. Therefore, we need to construct a stochastic calculus for values of the Hurst parameter different from $\frac{1}{2}$. The Gaussian property together with the Hölder continuity of its trajectories permits us to create an interesting and specific integration theory for this process.

Different approaches have been used in the literature to achieve this aim. This chapter is devoted to explain three of them:

- i) The stochastic calculus of variations, also called Malliavin calculus, for the fractional Brownian motion, exhaustively illustrated in [Nualart, 2006].
- ii) The stochastic calculus concerning symmetric, forward and backward integrals, introduced by Russo and Vallois in [Russo and Vallois, 1993].
- iii) The fractional calculus developed by Zähle in [Zähle, 1998] and meticulously illustrated in [Samko et al., 1993].

In the literature we find several works on stochastic calculus for fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Given that all the innovative results contained in this memory concern the case when $H < \frac{1}{2}$, we focus our attention on the extension of the integration theory to this case. This expansion is not trivial and new difficulties appear.

Before illustrating the different techniques we mentioned above, we introduce some notions that we will use along this chapter.

Let μ_L^n be the Lebesgue measure on \mathbb{R}^n and denote by du the integration with respect to $\mu_L(du)$. Let $a, b \in \mathbb{R}$, with $a < b$. Let $p \geq 1$ and denote by $L^p(a, b)$ the usual space of Lebesgue measurable functions $f : [a, b] \rightarrow \mathbb{R}$ for which $\|f\|_{L^p(a,b)} < \infty$, where

$$\|f\|_{L^p(a,b)} := \begin{cases} \left(\int_a^b |f(u)|^p du \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \text{ess sup}_{t \in [a,b]} |f(t)|, & \text{if } p = \infty \end{cases}$$

where $\text{ess sup}_{t \in [a,b]} |f(t)|$ is the essential supremum of $|f(t)|$ defined by

$$\text{ess sup}_{t \in [a,b]} |f(t)| := \inf \{ M \geq 0 : \mu_L \{ t : |f(t)| > M \} = 0 \}.$$

Fix $\beta \in (0, 1)$. Denote by $C^\beta(a, b)$ be the space of β -Hölder continuous functions on the interval $[a, b]$, that is, the set of continuous functions $f : [a, b] \rightarrow \mathbb{R}$ such that

$$\|f\|_{\beta(a,b)} := \sup_{a < s < t < b} \frac{|f(t) - f(s)|}{(t - s)^\beta} < \infty.$$

Let $\Delta_T := \{(a, b) : 0 \leq a < b \leq T\}$. For any $(a, b) \in \Delta_T$ and for any $g : \Delta_T \rightarrow \mathbb{R}$ we set

$$\|g\|_{\beta(s,t)} := \sup_{s < u < v < t} \frac{|g(u, v)|}{(v - u)^\beta}.$$

2.1 Malliavin calculus

Malliavin calculus is the calculus of variations of finite dimension in the Wiener space, introduced by Malliavin in [Malliavin, 1978]. Since fractional Brownian motion is a Gaussian process, we can develop the Malliavin calculus for this process. We refer to [Nualart, 2006] for an exhaustive and detailed description of this technique.

Let \mathfrak{H} be a real separable infinite-dimensional Hilbert space and denote by $\|\cdot\|_{\mathfrak{H}}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ the norm and the scalar product of \mathfrak{H} , respectively. Let $X = \{X(h) : h \in \mathfrak{H}\}$ be an isonormal Gaussian process over \mathfrak{H} . This means that X is a centered Gaussian family, defined on some probability space (Ω, \mathcal{F}, P) , with a covariance structure given by

$$\mathbb{E}[X(h)X(g)] = \langle h, g \rangle_{\mathfrak{H}}, \quad h, g \in \mathfrak{H}.$$

We assume that \mathcal{F} is the σ -algebra generated by X .

For any integer $q \geq 1$, let $\mathfrak{H}^{\otimes q}$ and $\mathfrak{H}^{\odot q}$ denote, respectively, the q th tensor product and the q th symmetric tensor product of \mathfrak{H} .

Definition 2.1.1. Let $\{e_n, n \geq 1\}$ be a complete orthonormal system in \mathfrak{H} . Given $f \in \mathfrak{H}^{\odot p}$, $g \in \mathfrak{H}^{\odot q}$ and $r \in \{0, \dots, p \wedge q\}$, the **r th-order contraction** of f and g is the element of $\mathfrak{H}^{\otimes(p+q-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1, \dots, i_r=1}^{\infty} \langle f, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \dots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}, \quad (2.1.1)$$

where $f \otimes_0 g = f \otimes g$ and, for $p = q$, $f \otimes_q g = \langle f, g \rangle_{\mathfrak{H}^{\otimes q}}$.

Notice that $f \otimes_r g$ is not necessarily symmetric. We denote its symmetrization by

$$f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(p+q-2r)}.$$

Definition 2.1.2. The **q th Wiener chaos** of X , denoted by \mathcal{H}_q , is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where H_q is the q th Hermite polynomial defined by

$$H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q} (e^{-x^2/2}). \quad (2.1.2)$$

For $q \geq 1$, let $I_q(\cdot)$ the generalized Wiener-Itô multiple stochastic integral. It is known that the map

$$I_q(h^{\otimes q}) = H_q(X(h)) \quad (2.1.3)$$

provides a linear isometry between $\mathfrak{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!} \|\cdot\|_{\mathfrak{H}^{\otimes q}}$) and \mathcal{H}_q (equipped with the $L^2(\Omega)$ norm). For $q = 0$, we set by convention $\mathcal{H}_0 = \mathbb{R}$ and I_0 equal to the identity map.

Let \mathcal{S} be the set of all smooth and cylindrical random variables of the form

$$F = g(X(\phi_1), \dots, X(\phi_n)),$$

where $n \geq 1$, $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is an infinitely differentiable function with compact support, and $\phi_i \in \mathfrak{H}$.

Definition 2.1.3. The **Malliavin derivative** of F with respect to X is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i} (X(\phi_1), \dots, X(\phi_n)) \phi_i.$$

By iteration, we can define the **q th derivative** $D^q F$ for every $q \geq 2$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot q})$.

For $q, p \geq 1$, let $\mathbb{D}^{q,p}$ denote the closure of \mathcal{S} with respect to the norm $\|\cdot\|_{\mathbb{D}^{q,p}}$, defined as

$$\|F\|_{\mathbb{D}^{q,p}}^p = \mathbb{E}[|F|^p] + \sum_{i=1}^q \mathbb{E}(\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p).$$

More generally, for any Hilbert space V , we denote by $\mathbb{D}^{q,p}(V)$ the corresponding Sobolev space of V -valued random variables.

The Malliavin derivative D fulfills the following chain rule. If $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \dots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F) DF_i.$$

We now define the adjoint of the operator D and its multiple version:

Definition 2.1.4. We denote by δ the **Skorohod integral**, also called the **divergence operator**, that is the adjoint of the operator D . Namely, δ is an unbounded operator on $L^2(\Omega, \mathfrak{H})$ with values in $L^2(\mathfrak{H})$ such that:

- i) The domain of δ , denoted by $\text{Dom } \delta$, is the set of \mathfrak{H} -valued square integrable random variables $u \in L^2(\Omega, \mathfrak{H})$ such that, for all $F \in \mathbb{D}^{1,2}$,

$$|\mathbb{E}(\langle DF, u \rangle_{\mathfrak{H}})| \leq c_u \|F\|_{L^2(\Omega)}$$

where c_u is a constant depending on u .

- ii) If $u \in \text{Dom } \delta$, then the random variable $\delta(u) \in L^2(\Omega)$ is defined by the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(\langle DF, u \rangle_{\mathfrak{H}})$$

for any $F \in \mathbb{D}^{1,2}$. This equality is sometimes called the **Malliavin integration-by-part formula**.

Definition 2.1.5. For $q \geq 1$, the **multiple Skorohod integral** is defined iteratively as

$$\delta^q(u) = \delta(\delta^{q-1}(u)),$$

with $\delta^0(u) = u$.

From the last definition, for any $u \in \text{Dom } \delta^q$ and any $F \in \mathbb{D}^{q,2}$, we have

$$\mathbb{E}(F\delta^q(u)) = \mathbb{E}(\langle D^q F, u \rangle_{\mathfrak{H}^{\otimes q}}). \quad (2.1.4)$$

Moreover, for any $h \in \mathfrak{H}^{\otimes q}$,

$$\delta^q(h) = I_q(h).$$

The following results concerning the multiple Skorohod integral were proved in [Nualart, 2006] and [Nourdin and Nualart, 2010].

Lemma 2.1.6 (Meyer inequality). For $p > 1$ and integers $k \geq q \geq 1$, the operator δ^q is continuous from $\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$ to $\mathbb{D}^{k-q,p}$ and, for all $u \in \mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})$,

$$\|\delta^q(u)\|_{\mathbb{D}^{k-q,p}} \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathfrak{H}^{\otimes q})}, \quad (2.1.5)$$

where $c_{k,p}$ is a positive constant.

Lemma 2.1.7. *Let $p, q \geq 1$. Let $u \in \mathfrak{H}^{\odot p}$ and $v \in \mathfrak{H}^{\odot q}$. Then,*

$$\delta^p(u)\delta^q(v) = \sum_{z=0}^{p \wedge q} z! \binom{p}{z} \binom{q}{z} \delta^{p+q-2z}(u \otimes_z v),$$

where \otimes_z is the contraction operator defined in (2.1.1).

2.1.1 Malliavin calculus for fractional Brownian motion

Let $B^H = \{B_t^H, t \geq 0\}$ denote a fractional Brownian motion with Hurst parameter H . Remember that B^H is a centered Gaussian process, defined on a complete probability space (Ω, \mathcal{F}, P) with covariance given by (1.4.2), that is,

$$R_H(t, s) = \mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}).$$

We assume that \mathcal{F} is generated by B^H and we suppose that $H < \frac{1}{2}$.

We denote by \mathcal{E} the set of \mathbb{R} -valued step functions on $[0, \infty)$. Let \mathfrak{H} be the Hilbert space defined as the completion of \mathcal{E} with respect to the scalar product

$$\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle_{\mathfrak{H}} = R(s, t).$$

The mapping $\mathbb{1}_{[0,t]} \rightarrow B_t^H$ can be extended to a linear isometry between the Hilbert space \mathfrak{H} and the Gaussian space spanned by B^H . In this way $\{B^H(h), h \in \mathfrak{H}\}$ is an isonormal Gaussian process as in the previous section. Our goal is to interpret $B^H(h)$ as the Wiener integral of $h \in \mathfrak{H}$ with respect to B^H and to write

$$B^H(h) = \int_0^T h dB^H. \quad (2.1.6)$$

As described in [Nualart, 2006], the fractional Brownian motion has the following integral representation:

$$B_t^H = \int_0^t K_H(t, s) dW_s,$$

where $W = \{W_t, t \geq 0\}$ is an ordinary Wiener process and $K_H(t, s)$ is the Volterra kernel given by

$$K_H(t, s) := c_H \left[\left(\frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2} \right) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right],$$

where c_H is the normalized constant

$$c_H = \sqrt{\frac{(2H - \frac{1}{2}) \Gamma(\frac{1}{2} - H)}{\Gamma(\frac{1}{2} + H) \Gamma(2 - 2H)}},$$

if $s < t$ and $K_H(t, s) = 0$ if $s \geq t$. This kernel satisfies

$$R_H(t, s) = \int_0^{t \wedge s} K_H(t, u) K_H(s, u) du,$$

where the function $R_H(t, s)$ is the covariance of the fractional Brownian motion defined in (1.4.2). Moreover, the linear operator $K_H^* : \mathcal{E} \rightarrow L^2([0, T])$, defined by

$$(K_H^* h)(s) := K_H(T, s)h(s) + \int_s^T (h(u) - h(s)) \frac{\partial K_H}{\partial u}(u, s) du, \quad (2.1.7)$$

satisfies

$$(K_H^* \mathbb{1}_{[0, t]})(s) = K_H(t, s). \quad (2.1.8)$$

Thus, K_H^* is a linear isometry that can be extended to the Hilbert space \mathfrak{H} . In fact, using (2.1.6) and (2.1.8), for any $s, t \in [0, T]$ we have

$$\begin{aligned} \langle K_H^* \mathbb{1}_{[0, t]}, K_H^* \mathbb{1}_{[0, s]} \rangle_{L^2([0, T])} &= \langle K_H(t, \cdot), K_H(s, \cdot) \rangle_{L^2([0, T])} \\ &= \int_0^{t \wedge s} K_H(t, u) K_H(s, u) du \\ &= R_H(t, s) \\ &= \langle \mathbb{1}_{[0, t]}, \mathbb{1}_{[0, s]} \rangle_{\mathfrak{H}}. \end{aligned}$$

As we will see in the following sections, this operator plays a basic role in the construction of a stochastic calculus with respect to B^H .

2.2 Symmetric, forward and backward integrals

A pathwise approach that involves the symmetric, forward and backward integrals was built up by Russo and Vallois. Some of their works are [Russo and Vallois, 1993], [Russo and Vallois, 1995], [Russo and Vallois, 1996] and [Russo and Vallois, 2000]. The aim of the authors was to develop a calculus relatively simple and beyond the barrier of semimartingales, which includes the case of Gaussian processes that have an infinite quadratic variation as fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$.

The concepts and the definitions illustrated in this section are based on the regularity of the sample paths of the processes we consider. For further details we refer to the works [Biagini et al., 2007], [Nualart, 2002] and [Russo and Vallois, 2000].

A natural way to introduce a stochastic integral with respect to the fractional Brownian motion is to consider the so-called Riemann sums:

$$\sum_{k=1}^n f(t_k) (B_{t_{k+1}}^H - B_{t_k}^H),$$

where $\{0 = t_0 < t_1 < \dots < t_n = T\}$ is a partition of the interval $[0, T]$, and then to investigate the conditions on f under which the convergence of this quantity holds

at least in probability. We will begin introducing the notions of symmetric, forward and backward integrals for two generic stochastic processes X and Y . Then, we will consider these integrals with respect to the fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Finally, we will focus on the case $H < \frac{1}{2}$.

Let X be a continuous process and Y be a continuous locally bounded process. We give the following definitions:

Definition 2.2.1. *The **symmetric (Stratonovich) integral** of Y with respect to X is defined as*

$$\int_0^T Y_u d^\circ X_u := \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^T Y_u (X_{u+\varepsilon} - X_{u-\varepsilon}) du, \quad (2.2.1)$$

provided the limit exists uniformly on compacts in probability (ucp).

Definition 2.2.2. *The **forward integral** of Y with respect to X is defined as*

$$\int_0^T Y_u d^- X_u := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^T Y_u (X_{u+\varepsilon} - X_u) du, \quad (2.2.2)$$

provided the limit exists uniformly on compacts in probability.

Definition 2.2.3. *The **backward integral** of Y with respect to X is defined as*

$$\int_0^T Y_u d^+ X_u := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^T Y_u (X_u - X_{u-\varepsilon}) du, \quad (2.2.3)$$

provided the limit exists uniformly on compacts in probability.

In order to clarify the relation between the forward and the symmetric integral, we define the (generalized) covariation:

Definition 2.2.4. *Let X and Y be two continuous processes. Suppose that Y is locally bounded. Their **covariation** is defined as the limit*

$$\langle X, Y \rangle_t := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t (X_{u+\varepsilon} - X_u)(Y_{u+\varepsilon} - Y_u) du$$

if the limit exists uniformly on compacts in probability.

If X and Y are as above, the following relation among the symmetric integral and the forward integral holds:

$$\int_0^t Y_u d^\circ X_u = \int_0^t Y_u d^- X_u + \langle X, Y \rangle_t, \quad (2.2.4)$$

provided that two of these three terms exist. Recall that the quadratic variation is the random variable

$$\langle X, X \rangle_t := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t (X_{u+\varepsilon} - X_u)^2 du.$$

Definition 2.2.5. *If X is such that $\langle X, X \rangle_t$ exists for all t , then X is called **finite quadratic variation process**. Moreover, if $\langle X, X \rangle_t = 0$ for all t , then X will be called **zero quadratic variation process**.*

If X is a finite quadratic variation process and if $f \in \mathcal{C}^2(\mathbb{R})$, then the following Itô's formula holds:

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^- X_u + \frac{1}{2} \langle f'(X), X \rangle_t. \quad (2.2.5)$$

Hence, formulas (2.2.4) and (2.2.5) give the following Itô-Stratonovich formula:

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^o X_u. \quad (2.2.6)$$

We are interested in the case when X is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$, that is $X = B^H$. When $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ is the standard Brownian motion and, as we asserted in Chapter 1.2, its quadratic variation is $\langle B^{\frac{1}{2}}, B^{\frac{1}{2}} \rangle_t = t$. On the other hand, from Proposition 1.2.13 we have that the sample paths of B^H are Hölder continuous of order strictly less than H . This implies that, if $H > \frac{1}{2}$, then B^H is a zero quadratic variation process, as we stated in Chapter 1.4. Hence, Ito-Stratonovich formula (2.2.6) holds for $H \geq \frac{1}{2}$.

Since the quadratic variation is infinite for the fractional Brownian motion when $H < \frac{1}{2}$, a substitution tool is needed. The following paragraph is dedicated to define the pathwise integrals in this case.

2.2.1 Symmetric integrals for the case $H < \frac{1}{2}$

We want to establish when the integrals introduced in the previous section can be defined in the case of a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$. We follow the approach of [Cheridito and Nualart, 2005].

As we said in the previous section, the definition of a pathwise integral for fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$ is rather delicate. For example, the forward integral

$$\int_0^T B_u^H d^- B_u^H$$

does not exist in the sense of Definition 2.2.2 when the limit (2.2.2) is meant in the L^2 -sense, as it is shown in the following example provided in [Nualart, 2002].

Example 2.2.6. Given a partition $t_j = \frac{jT}{n}$ of the interval $[0, T]$, the expectation of the Riemann sums

$$\sum_{j=1}^n B_{t_{j-1}}^H (B_{t_j}^H - B_{t_{j-1}}^H)$$

diverges

$$\begin{aligned} \sum_{j=1}^n \mathbb{E}[B_{t_{j-1}}^H (B_{t_j}^H - B_{t_{j-1}}^H)] &= \frac{1}{2} \sum_{j=1}^n [t_j^{2H} - t_{j-1}^{2H} - (t_j - t_{j-1})^{2H}] \\ &= \frac{1}{2} T^{2H} (1 - n^{1-2H}) \end{aligned}$$

when n goes to infinity if $H < \frac{1}{2}$. However, notice that the symmetric sums

$$\frac{1}{2} \sum_{j=1}^n (B_{t_j}^H - B_{t_{j-1}}^H)^2$$

have finite expectation:

$$\frac{1}{2} \sum_{j=1}^n \mathbb{E}[(B_{t_j}^H - B_{t_{j-1}}^H)(B_{t_j}^H - B_{t_{j-1}}^H)] = \frac{1}{2} \sum_{j=1}^n (t_j - t_{j-1})^{2H} = \frac{1}{2} T^{2H}.$$

In order to establish when the symmetric integral is well-defined for $H < \frac{1}{2}$, recall that the operator K_H^* defined in (2.1.7) induces an isometry between the Hilbert space \mathfrak{H} , introduced in Section 2.1.1, and $L^2([0, T])$. We have that

$$\frac{\partial K_H}{\partial t}(t, s) = c_H \left(H - \frac{1}{2} \right) \left(\frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}},$$

that can be estimated as follows, for $s < t$,

$$\left| \frac{\partial K_H}{\partial t}(t, s) \right| \leq c_H \left(\frac{1}{2} - H \right) (t-s)^{H-\frac{3}{2}}.$$

Consider the following seminorm on the set \mathcal{E} of step functions on $[0, T]$:

$$\|\varphi\|_{K_H}^2 := \int_0^T \varphi^2(u) K(T, u)^2 du + \int_0^T \left(\int_u^T |\varphi(v) - \varphi(u)| (v-u)^{H-\frac{3}{2}} dv \right)^2 du.$$

We denote by \mathfrak{H}_{K_H} the completion of \mathcal{E} with respect to this seminorm. The space \mathfrak{H}_{K_H} is continuously embedded in \mathfrak{H} .

The following proposition, involving the divergence operator δ and the Malliavin derivatives D , defined in Definition 2.1.4 and Definition 2.1.3 respectively, gives sufficient conditions for the existence of the symmetric integral:

Proposition 2.2.7. *Let $v = \{v_t, t \in [0, T]\}$ be a stochastic process in the space $\mathbb{D}^{1,2}(\mathfrak{H}_{K_H})$. Suppose that the **trace**, defined as limit in probability*

$$\text{Tr} Dv := \lim_{\varepsilon \rightarrow 0} \int_0^T \langle Dv_u, \mathbb{1}_{[u-\varepsilon, u+\varepsilon]} \rangle_{\mathfrak{H}} du,$$

exists and

$$\begin{aligned} & \mathbb{E} \left(\int_0^T v_u^2 (u^{2H-1} + (T-u)^{2H-1}) du \right) < \infty, \\ & \mathbb{E} \left(\int_0^T \int_0^T (D_\theta v_u)^2 (u^{2H-1} + (T-u)^{2H-1}) du d\theta \right) < \infty. \end{aligned}$$

Then, the symmetric integral $\int_0^T v_u d^\circ B_u^H$ of v with respect to the fractional Brownian motion B^H defined as the limit in probability

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^T v_u (B_{u+\varepsilon}^H - B_{u-\varepsilon}^H) ds,$$

exists and

$$\int_0^T v_u d^\circ B_u^H = \delta(v) + \text{Tr} D^H v.$$

We can compute the trace in the particular case of the process $v_t = f(B_t^H)$, where $f \in \mathcal{C}^2(\mathbb{R})$ satisfies the growth condition

$$\max\{|f(x)|, |f'(x)|, |f''(x)|\} \leq ce^{\lambda x^2}, \quad (2.2.7)$$

where c and λ are positive constants such that $\lambda < 1/(4T^{2H})$. If $\frac{1}{4} < H < \frac{1}{2}$, then the process $v_t = f(B_t^H)$ belongs to $\mathbb{D}^{1,2}(\mathcal{H}_{K_H})$, the trace $\text{Tr} Dv$ exists and

$$\text{Tr} Dv = H \int_0^T f'(B_u^H) u^{2H-1} du.$$

As a consequence, we get

$$\int_0^T f(B_u^H) d^\circ B_u^H = \delta(f(B^H)) + H \int_0^T f'(B_u^H) u^{2H-1} du.$$

Now we want to consider a wider class of integrands with respect to the one consider in (2.2.7). We recall a result of [Cheridito and Nualart, 2005], in which the authors proved that the symmetric integral of a general smooth function of B^H with respect to B^H exists in L^2 if and only if $H > \frac{1}{6}$.

Note that if $h : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, then

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^T h(u) [h(u+\varepsilon) - h(u-\varepsilon)] du \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \left(\int_0^T h(u) h(u+\varepsilon) du - \int_{-\varepsilon}^{T-\varepsilon} h(u) h(u+\varepsilon) du \right) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \left(\int_{T-\varepsilon}^T h(u) h(u+\varepsilon) du - \int_{-\varepsilon}^0 h(u) h(u+\varepsilon) du \right) \\ &= \frac{1}{2} [h^2(T) - h^2(0)]. \end{aligned}$$

Hence, it follows that, for all $H \in (0, 1)$ and for $s < t$,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_s^t B_u^H (B_{u+\varepsilon}^H - B_{u-\varepsilon}^H) du = \frac{1}{2} [(B_t^H)^2 - (B_s^H)^2]$$

almost surely.

Since for $H \geq \frac{1}{2}$, B^H has finite quadratic variation process, Theorem 2.1 of [Russo and Vallois, 1995] shows that for all $H \geq \frac{1}{2}$ and $g \in \mathcal{C}^1(\mathbb{R})$

$$\int_s^t g(B_u^H) d^\circ B_u^H = G((B_t^H)^2) - G((B_s^H)^2), \quad (2.2.8)$$

where G is given by

$$G(x) := \int_0^x g(y) dy, \quad x \in \mathbb{R}.$$

In Theorem 4.1 of [Russo and Vallois, 1996] it is proved that, for $H = \frac{1}{2}$, formula (2.2.8) even holds if $g \in L_{loc}^2(\mathbb{R})$.

For $H < \frac{1}{2}$, the sample paths of B^H are rougher than the sample paths of Brownian motion. However, it was shown in [Alòs et al., 2001] that, if $\frac{1}{4} < H < \frac{1}{2}$, then formula (2.2.8) is still true for $g \in \mathcal{C}^1(\mathbb{R})$, while in Theorem 4.1 of [Gradinaru et al., 2003] the formula is proved for $H = \frac{1}{4}$ and $g \in \mathcal{C}^3(\mathbb{R})$. The most general result in this direction is contained in Theorem 5.3 of [Cheridito and Nualart, 2005], where they show that for $g \in \mathcal{C}^3(\mathbb{R})$, $H = \frac{1}{6}$ is the critical value for the existence of the symmetric integral in (2.2.8):

Proposition 2.2.8. *Let $g \in \mathcal{C}^3(\mathbb{R})$. Then, the following results hold:*

1. For every $H \in (\frac{1}{6}, \frac{1}{2})$,

$$\int_s^t g(B_u^H) d^\circ B_u^H = G((B_t^H)^2) - G((B_s^H)^2),$$

where $G(x) = \int_0^x g(y) dy$, for $x \in \mathbb{R}$.

2. On the other hand, if $H \in (0, \frac{1}{6}]$, then

$$\int_s^t (B_u^H)^2 d^\circ B_u^H$$

does not exist.

2.3 Fractional integrals and derivatives

Another pathwise approach to define a stochastic integral with respect to a fractional Brownian motion is the fractional calculus. Fractional integrals and derivatives generalize the ideas of integration and differentiation of integer order.

Recall that the n -fold iterated integral formula is the following identity:

$$\int_0^t dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_2} dt_1 \int_0^{t_1} g(s) ds = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} g(s) ds,$$

easily proved by induction. Since $(n-1)! = \Gamma(n)$, it is natural to look for a meaning for arbitrary values of n , not only integer. Moreover, fractional integrals and derivatives arise from certain requirements in applications and have connections with problems of function theory, integral and differential equations and other branches of analysis.

The fractional integrals are connected with the Abel's integral equation for a real function φ :

$$f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varphi(u)}{(t-u)^{1-\alpha}} du,$$

that is solvable if and only if $f_{1-\alpha}$, defined by

$$f_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f(u)}{(t-u)^\alpha} du,$$

is absolutely continuous in the interval $[a, b]$ and $f_{1-\alpha}(a) = 0$. Under these conditions, the solution is unique.

This section is devoted to give some notions of fractional calculus together with the main properties of fractional integrals and derivatives. For a detailed explanation of this theory, we refer to [Samko et al., 1993] and [Zähle, 1998].

The fractional integrals were first considered by Riemann and Liouville, hence the name fractional Riemann-Liouville integrals:

Definition 2.3.1. *Let $f \in L^1(a, b)$ and $\alpha > 0$. The left-sided and right-sided **fractional Riemann-Liouville integrals** of f of order α are defined for almost all $t \in (a, b)$ by*

$$I_{a+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} f(u) ds \quad (2.3.1)$$

and

$$I_{b-}^\alpha f(t) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (u-t)^{\alpha-1} f(u) du \quad (2.3.2)$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$ is the Gamma function.

Let $I_{a+}^\alpha(L^p)$ the image of $L^p(a, b)$ by the operator I_{a+}^α and $I_{b-}^\alpha(L^p)$ the image of $L^p(a, b)$ by the operator I_{b-}^α . The fractional integrals fulfill the following formulas:

1. *Composition formula (semigroup property for fractional integration):* For $\alpha, \beta > 0$ and for all $f \in L^1(a, b)$,

$$I_{a+}^\alpha (I_{a+}^\beta f) = I_{a+}^{\alpha+\beta} f,$$

$$I_{b-}^\alpha (I_{b-}^\beta f) = I_{b-}^{\alpha+\beta} f.$$

2. *First formula for fractional integration by parts:* For $0 \leq \alpha \leq 1$, if $f \in L^p(a, b)$ and $g \in L^q(a, b)$ with $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, we have

$$\int_a^b f(u) I_{a+}^\alpha g(u) du = (-1)^\alpha \int_a^b g(u) I_{b-}^\alpha f(u) du. \quad (2.3.3)$$

It is natural to introduce the fractional differentiation as an operation inverse to fractional integration. To this aim, we define the fractional derivatives:

Definition 2.3.2. Let $f \in I_{a+}^\alpha(L^p)$, respectively $f \in I_{b-}^\alpha(L^p)$, and $0 < \alpha < 1$. Then, the **Weyl derivatives**

$$D_{a+}^\alpha f(x) := \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right) \mathbf{1}_{(a,b)}(x) \quad (2.3.4)$$

and, respectively,

$$D_{b-}^\alpha f(x) := \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} dy \right) \mathbf{1}_{(a,b)}(x) \quad (2.3.5)$$

are defined for almost all $x \in (a, b)$ and the convergence of the integrals at the singularity $y = x$ holds pointwise for almost all $x \in (a, b)$ if $p = 1$ and moreover in L^p -sense if $1 < p < \infty$.

The fractional integrals fulfill the following formulas:

1. *Composition formula for fractional derivatives:* Let $\alpha, \beta > 0$ be such that $\alpha + \beta \leq 1$. For all $f \in I_{a+}^{\alpha+\beta}(L^1)$,

$$D_{a+}^\alpha (D_{a+}^\beta f) = D_{a+}^{\alpha+\beta} f,$$

and for all $f \in I_{b-}^{\alpha+\beta}(L^1)$,

$$D_{b-}^\alpha (D_{b-}^\beta f) = D_{b-}^{\alpha+\beta} f.$$

2. *Second formula for fractional integration by parts:* For $0 \leq \alpha \leq 1$, if $f \in I_{a+}^\alpha(L^p)$ and $g \in I_{b-}^\alpha(L^q)$ with $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, we have

$$\int_a^b f(u) D_{b-}^\alpha g(u) du = (-1)^\alpha \int_a^b g(u) D_{a+}^\alpha f(u) du.$$

The relationship between fractional integrals and fractional derivatives as inverse operations is confirmed by the following inverse formulas:

- For all $f \in I_{a+}^\alpha(L^p)$,

$$I_{a+}^\alpha(D_{a+}^\alpha f) = f,$$
- and for all $f \in I_{b-}^\alpha(L^p)$,

$$I_{b-}^\alpha(D_{b-}^\alpha f) = f.$$
- For $f \in L^1(a, b)$

$$D_{a+}^\alpha(I_{a+}^\alpha f) = f,$$

$$D_{b-}^\alpha(I_{b-}^\alpha f) = f.$$

Moreover, the following statements are true:

- For $\alpha p < 1$ and $q = \frac{p}{1-\alpha p}$,

$$I_{a+}^\alpha(L^p) = I_{b-}^\alpha(L^p) \subset L^q(a, b).$$
- For $\alpha p > 1$,

$$I_{a+}^\alpha(L^p) \cup I_{b-}^\alpha(L^p) \subset C^{\alpha-\frac{1}{p}}(a, b).$$

2.3.1 Generalized Stieltjes integrals

This section is devoted to recall the definitions and the results of Zähle contained in [Zähle, 1998] about the generalization of the classical Lebesgue-Stieltjes integral $\int_a^b f dg$ of real-valued functions on a finite interval (a, b) to a large class of functions of unbounded variation. We work with the composition formulas and the integration-by-part rules for fractional integrals and Weyl derivatives.

Before extending the Lebesgue-Stieltjes integral, we enrich our notation.

Let $f(a+) = \lim_{\varepsilon \searrow 0} f(a + \varepsilon)$ and $g(b-) = \lim_{\varepsilon \searrow 0} g(b - \varepsilon)$. Suppose that the limits exist and are finite. Define

$$f_{a+}(x) = (f(x) - f(a+)) \mathbb{1}_{(a,b)}(x),$$

$$g_{b-}(x) = (g(x) - g(b-)) \mathbb{1}_{(a,b)}(x).$$

The composition formulas for fractional derivatives and the second formula for fractional integration by parts suggest the following notion for the fractional integral:

Definition 2.3.3. *Suppose that f and g are functions such that*

- i) $f(a+)$, $g(a+)$ and $g(b-)$ exist,*
- ii) $f_{a+} \in I_{a+}^\alpha(L^p)$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^q)$ for some $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and for some $0 \leq \alpha \leq 1$.*

Then, the (fractional) integral of f with respect to g is defined by

$$\int_a^b f dg := (-1)^\alpha \int_a^b D_{a+}^\alpha f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x) dx + f(a+)[g(b-) - g(a+)]. \quad (2.3.6)$$

Remark 2.3.4. For $\alpha p < 1$, we have that $f_{a+} \in I_{a+}^\alpha(L^p)$ if and only if $f \in I_{a+}^\alpha(L^p)$, and $f(a+)$ exists. In this case,

$$\begin{aligned} D_{a+}^\alpha f_{a+}(x) &= D_{a+}^\alpha (f - f_{a+} \mathbb{1}_{(a,b)})(x) \\ &= D_{a+}^\alpha f(x) - \frac{1}{\Gamma(1-\alpha)} \frac{f(a+)}{(x-a)^\alpha} \mathbb{1}_{(a,b)}(x) \end{aligned}$$

and (2.3.6) can be written as

$$\int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f(x) D_{b-}^{1-\alpha} g_{b-}(x) dx, \quad (2.3.7)$$

which is determined for general functions $f \in I_{a+}^\alpha(L^p)$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^q)$. For $\alpha = 0$ and $\alpha = 1$ the integral (2.3.7) may be transform into

$$\int_a^b f(x) dg(x) = \int_a^b f(x) g'(x) dx$$

and

$$\int_a^b f(x) dg(x) = - \int_a^b f'(x) g(x) dx + f(b-)g(b-) - f(a+)g(a+)$$

which are the corresponding Lebesgue-Stieltjes integrals, respectively.

In [Zähle, 1998] it was also proved that, in the special case of Hölder continuous functions f and g of summed order greater than 1, the integral defined by (2.3.6) correspond to the Riemann-Stieljes integral:

Theorem 2.3.5. *If $f \in C^\lambda(a, b)$ and $g \in C^\mu(a, b)$ with $\lambda + \mu > 1$, the Riemann-Stieljes integral $\int_a^b f dg$ exists and coincides with the integrals defined in Definition 2.3.3 and Remark 2.3.4.*

The fractional integral defined in (2.3.6) can also be called **forward integral** of f with respect to g . Its construction is controlled by the choice of left-sided derivatives of f and right-sided derivatives of g . In a similar way, we introduce the backward integral:

Definition 2.3.6. *Suppose that $f_{b-} \in I_{b-}^\alpha(L^p)$ and $g_{a+} \in I_{a+}^{1-\alpha}(L^q)$ for some $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ and for some $0 \leq \alpha \leq 1$. Then, the **backward integral** of f with respect to g is defined by*

$$\int_a^b dg(x) f(x) := (-1)^{-\alpha} \int_a^b D_{b-}^\alpha f_{b-}(x) D_{a+}^{1-\alpha} g_{a+}(x) dx + f(b-)[g(b-) - g(a+)]. \quad (2.3.8)$$

One can also prove the backward version of (2.3.7) using arguments analogous to those of the forward case. Moreover, as we see in the following proposition, for indicator functions f or smooth functions f or g the forward and backward integrals agree and we can deduce an integration-by-part formula:

Proposition 2.3.7. *If f and g satisfy the conditions of Definition 2.3.3 and Definition 2.3.6, then we have*

1. *Forward-backward identity:* $\int_a^b f dg = \int_a^b dg f,$
2. *Integration-by-part formula:* $\int_a^b f dg = f(b-)g(b-) - f(a+)g(a+) - \int_a^b g df.$

2.3.2 Compensated fractional integrals and derivatives

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^m$ be a continuously differentiable function. We are interested in defining the integral

$$\int_a^b f(x_u) dy_u = \sum_{i=1}^d \int_a^b f_i(x_u) dy_u^i \quad (2.3.9)$$

when x and y are two vector-valued β -Hölder continuous functions with $\beta \in (\frac{1}{3}, \frac{1}{2})$. Observe that, in this case, the definition of the fractional integral given by (2.3.7) cannot be used to define the integral (2.3.9) because the fractional derivative $D_{a+}^\alpha f(x)$ is not well-defined in this case. For this reason, we use an additional tool and we recall the construction of the integral $\int_a^b f(x_u) dy_u$ given by Hu and Nualart in [Hu and Nualart, 2009] using fractional derivatives.

Fix $\frac{1}{3} < \beta < \frac{1}{2}$. Following [Lyons, 1998] we introduce the following definition:

Definition 2.3.8. *We say that $(x, y, x \otimes y)$ is a (d, m) -dimensional β -Hölder continuous multiplicative functional if:*

1. $x : [0, T] \rightarrow \mathbb{R}^d$ and $y : [0, T] \rightarrow \mathbb{R}^m$ are β -Hölder continuous functions,
2. $x \otimes y : \Delta_T \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ is a continuous function satisfying the following properties:

a) *(Multiplicative property)* For all $s \leq u \leq t$ we have

$$(x \otimes y)_{s,u} + (x \otimes y)_{u,t} + (x_u - x_s) \otimes (y_t - y_u) = (x \otimes y)_{s,t}.$$

b) For all $(s, t) \in \Delta_T$

$$|(x \otimes y)_{s,t}| \leq c|t - s|^{2\beta}.$$

We denote by $M_{d,m}^\beta(0, T)$ the space of (d, m) -dimensional β -Hölder continuous multiplicative functionals. Furthermore, we will denote by $M_{d,m}^\beta(a, b)$ the obvious extension of the definition $M_{d,m}^\beta(0, T)$ to a general interval (a, b) .

Fix a real number α such that $1 - \beta < \alpha < 2\beta$ and $\alpha < \frac{\lambda\beta+1}{2}$. Observe that such a number exists because $3\beta > 1$ and $\frac{\lambda\beta+1}{2} > 1 - \beta$. We need to introduce the following notions.

Definition 2.3.9. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^m$ be a continuously differentiable function such that f' is locally λ -Hölder continuous, where $\lambda > \frac{1}{\beta} - 2$. The **compensated fractional derivative** for $u \in (a, b)$ is defined as

$$\begin{aligned} \widehat{D}_{a+}^\alpha f(x_u) &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x_u)}{(u-a)^\alpha} \right. \\ &\quad \left. + \alpha \int_a^u \frac{f(x_u) - f(x_\theta) - \sum_{i=1}^m \partial_i f(x_\theta)(x_u^i - x_\theta^i)}{(u-\theta)^{\alpha+1}} d\theta \right). \end{aligned} \quad (2.3.10)$$

This derivative is well-defined under our hypotheses, because there exists a constant K such that for all $u, \theta \in (a, b)$, with $\theta < u$, we have

$$\frac{|f(x_u) - f(x_\theta) - \sum_{i=1}^m \partial_i f(x_\theta)(x_u^i - x_\theta^i)|}{(u-\theta)^{\alpha+1}} \leq K(u-\theta)^{(1+\lambda)\beta-\alpha-1}$$

and $(1+\lambda)\beta - \alpha > 0$ since $\alpha < \frac{\lambda\beta+1}{2} < (1+\lambda)\beta$.

We also need to extend the fractional derivative to the multiplicative functional $(x \otimes y)$:

Definition 2.3.10. Let $(x, y, x \otimes y) \in M_{d,m}^\beta(0, T)$. For $u \in (a, b)$, the **extension of the fractional derivative** to $(x \otimes y)$ is defined as

$$\mathcal{D}_{b-}^{1-\alpha}(x \otimes y)(u) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{(x \otimes y)_{u,b}}{(b-u)^{1-\alpha}} + (1-\alpha) \int_u^b \frac{(x \otimes y)_{u,s}}{(s-u)^{2-\alpha}} ds \right). \quad (2.3.11)$$

By Lemma 6.3 of [Hu and Nualart, 2009], for any $0 \leq a \leq b \leq T$, we have that

$$\|\mathcal{D}_{b-}^{1-\alpha}(x \otimes y)\|_{\beta(a,b)} \leq K \Phi_{\beta(a,b)}(x, y)(b-a)^{\beta+\alpha-1},$$

where the constant K depends on α and β and

$$\Phi_{\beta(a,b)}(x, y) = \|x \otimes y\|_{2\beta(a,b)} + \|x\|_{\beta(a,b)} \|y\|_{\beta(a,b)}. \quad (2.3.12)$$

This implicates that the function $\mathcal{D}_{b-}^{1-\alpha}(x \otimes y)(u)$ is β -Hölder continuous.

Now we have all the ingredients to define the integral $\int_a^b f(x_u) dy_u$:

Definition 2.3.11. Let $(x, y, x \otimes y) \in M_{d,m}^\beta(0, T)$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^m \otimes \mathbb{R}^d$ be a continuously differentiable function such that f' is locally λ -Hölder continuous,

where $\lambda > \frac{1}{\beta} - 2$. Fix $\alpha > 0$ such that $1 - \beta < \alpha < 2\beta$, and $\alpha < \frac{\lambda\beta+1}{2}$. Then, for any $0 \leq a < b \leq T$, we define

$$\begin{aligned} \int_a^b f(x_u) dy_u &= (-1)^\alpha \sum_{j=1}^m \int_a^b \widehat{D}_{a+}^\alpha f_j(x_u) D_{b-}^{1-\alpha} y_{b-}^j(u) du \\ &\quad - (-1)^{2\alpha-1} \sum_{i=1}^d \sum_{j=1}^m \int_a^b D_{a+}^{2\alpha-1} \partial_i f_j(x_u) D_{b-}^{1-\alpha} \mathcal{D}_{b-}^{1-\alpha}(x \otimes y)^{i,j}(u) du. \end{aligned} \quad (2.3.13)$$

Observe that the fractional derivatives $D_{b-}^{1-\alpha} y_{b-}^j(u)$ and $D_{b-}^{1-\alpha} \mathcal{D}_{b-}^{1-\alpha}(x \otimes y)^{i,j}(u)$ are well-defined because the functions y^j and $\mathcal{D}_{b-}^{1-\alpha}(x \otimes y)^{i,j}(u)$ are β -Hölder continuous.

2.4 Relation between fractional and stochastic calculus

We have described distinctive ways to define a stochastic integral with respect to a fractional Brownian motion. Now we want to study the relation between the different definitions.

In Section 2.2.1 we show how to use the Malliavin calculus to define the symmetric integral when the Hurst parameter is $H < \frac{1}{2}$. In this section we study the link between the forward integral and the fractional calculus using the approach of [Zähle, 1999].

We introduce the following definition:

Definition 2.4.1. Let $(h_t)_{t \in [0, T]}$ be a stochastic process. We define the **extended forward integral** of h with respect to B^H as

$$\int_0^T h_u d^- B_u^H := \lim_{\varepsilon \rightarrow 0} \frac{1}{\Gamma(\varepsilon)} \int_0^T v^{\varepsilon-1} \int_0^t \frac{h_u (B_{u+v}^H - B_u^H)}{u} du dv, \quad (2.4.1)$$

if the limit exists in ucp as a function of $t \in [0, T]$.

The previous definition provides an extension of the definition of forward integral for fractional Brownian motion given in Definition 2.2.2. In fact, the existence of the limit

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t h_u (B_{u+\varepsilon}^H - B_u^H) du.$$

in uniform convergence in probability implies the existence of the extended forward integral defined in (2.4.1). In what follows, we show that Definition 2.4.1 is the key to describe the link between stochastic and fractional calculus.

Let f and g be two deterministic function on $[0, T]$ satisfying the conditions of Definition 2.3.3. For $0 \leq a \leq b \leq T$, the fractal integral $\int_a^b f dg$ is the one defined

by (2.3.6) and (2.3.7). Then, the following approximation property of the integral holds:

$$\int_a^b f dg = \lim_{\varepsilon \rightarrow 0} \int_a^b I_{a+}^\varepsilon f dg.$$

Recall that $g_{b-}(x) = I_{(a,b)}(g(x) - g(b-))$. By the first formula for fractional integration by parts (2.3.3), we obtain that

$$\int_a^b I_{a+}^\varepsilon f dg = \frac{1}{\Gamma(\varepsilon)} \int_0^T v^{\varepsilon-1} \int_a^b f(u) \frac{g_{b-}(u+v) - g_{b-}(u)}{v} du dv.$$

This formula is valid if the degrees of differentiability of f and g sum up at least $1 - \varepsilon$. By using the previous identity, we extend the definition of fractional integral provided in Definition 2.3.3 as

$$\int_a^b f dg = \lim_{\varepsilon \rightarrow 0} \frac{1}{\Gamma(\varepsilon)} \int_0^T v^{\varepsilon-1} \int_a^b f(u) \frac{g_{b-}(u+v) - g_{b-}(u)}{v} du dv,$$

whenever the limit on the right-hand side is determined. In this way, we obtain the natural extension of fractional calculus to the stochastic case.

3

Strong limit of processes constructed from a Poisson process

The stochastic processes

$$X(t) = v \int_0^t (-1)^{N_a(u)} du,$$

where $a, v > 0$ and $N_a = \{N_a(t), t \geq 0\}$ is a Poisson process of intensity a , were extensively studied in literature, starting with Kac and Stroock. In this chapter we extend the processes $X(t)$ in order to find approximations of the complex Brownian motion. More in detail, we construct a family of processes, from a single Poisson process, that converges in law to a complex Brownian motion. We also find realizations of these processes that converge almost surely to the complex Brownian motion, uniformly on the unit time interval. Finally, we derive the rate of convergence.

The chapter is organized as follows. In the following section we illustrate the path that bring us to study approximations of the complex Brownian motion. In Section 3.2 we define the stochastic processes we deal with and we describe the main results of the chapter. Section 3.3 and Section 3.4 are devoted to prove the weak and strong convergence of the stochastic processes defined in Section 3.2, respectively. In Section 3.5 a rate of convergence is derived. Finally, Section 3.6 complete the chapter with the study of a particular case.

3.1 Introduction and motivations

Consider the equation given by

$$\frac{1}{v} \frac{\partial^2 F}{\partial t^2} = v \frac{\partial^2 F}{\partial x^2} - \frac{2a}{v} \frac{\partial F}{\partial t}, \quad (3.1.1)$$

with $a, v > 0$. It was introduced to describe the propagation of electrical signals traveling along a transmission cable. Adding the initial conditions, we have that a differential equation of the form

$$\begin{aligned} \frac{1}{v} \frac{\partial^2 F}{\partial t^2} &= v \frac{\partial^2 F}{\partial x^2} - \frac{2a}{v} \frac{\partial F}{\partial t}, \\ F(x, 0) &= \varphi(x), \\ \left. \frac{\partial}{\partial t} F(x, t) \right|_{t=0} &= 0, \end{aligned} \quad (3.1.2)$$

where $\varphi(x)$ is an arbitrary function and a is a dissipation coefficient, is denoted as telegraph equation. It is a hyperbolic partial differential equation and, as we will see, is a generalization of the wave equation, where we consider additionally a dissipation term. Moreover, if the first term of the telegraph equation vanishes, we observe the diffusion equation.

A stochastic version of the telegraph equation was illustrated by Kac in [Kac, 1974]. We describe the method he used. Suppose we have a lattice of equidistant points. Consider a particle in the origin $x = 0$. The particle can move either in the positive or in the negative direction, always with speed v . Each step is of duration Δt and covers a distance Δx , so we have the relation $\Delta x = v\Delta t$. Assume that the particle starts moving in the positive direction. Each time we arrive at a lattice point there is a probability of reversal of direction. Assume that $a\Delta t$ is this probability and, of course, $1 - a\Delta t$ is the probability that the direction of motion will be maintained. The particle oscillates from one direction to another. The problem consists in finding the probability that after a certain time t the particle is at a certain interval.

Now suppose that x stands only for abscissas of discrete points, that are the lattice points. We denote by S_n the displacement after n steps, that is the displacement after time $n\Delta t$. This displacement can be constructed as follows. Consider a sequence of independent identically distributed random variables $\epsilon_1, \dots, \epsilon_{n-1}$ with Bernoulli distribution $\text{Ber}(1 - a\Delta t)$, that is, for all k ,

$$\begin{aligned} P(\epsilon_k = 1) &= 1 - a\Delta t, \\ P(\epsilon_k = 0) &= a\Delta t. \end{aligned}$$

For all k , the random variable ϵ_k tells us if at step k we change direction ($\epsilon_k = 1$) or not ($\epsilon_k = 0$). The number of changes of direction after k time steps is given by the random variable

$$N_k = \epsilon_1 + \dots + \epsilon_{k-1}$$

with $N_1 = 0$. The displacement S_n can be derived by the number of changes as follows:

$$S_n = v\Delta t \sum_{k=1}^n (-1)^{N_k}.$$

Consider an arbitrary function $\varphi(x)$ and the average $\langle \varphi(x + S_n) \rangle$.

Example 3.1.1. The simplest case is when $\varphi(x)$ is the characteristic function of a uniform random variable on an interval. In that case, the average $\langle \varphi(x + S_n) \rangle$ is the probability of finding the particle in that interval after n steps if it started at the point x .

To find a solution for the telegraph equation (3.1.2), we have to pass from the discrete case to the differential equation considering the limit $\Delta t \rightarrow 0$. Then, averaging the initial function φ , we obtain the following solution of the equation (3.1.2) for a time $t \neq 0$:

$$F(x, t) = \lim_{n \rightarrow \infty} \left\{ \frac{1}{2} \langle \phi(x + S_n) \rangle + \frac{1}{2} \langle \phi(x - S_n) \rangle \right\}.$$

Given that S_n is the displacement after a time $n\Delta t$, if we want this time to be zero, we get that the solution becomes $F(x, 0) = \varphi(x)$, as required by the initial conditions of (3.1.2).

We observe two particular cases.

When the dissipation coefficient is $a = 0$, the case is extremely easy. The particle starts moving in one direction and never stops, so there are no reversal of direction and no random variables. The differential equation becomes the wave equation:

$$\frac{\partial^2 F}{\partial t^2} = v^2 \frac{\partial^2 F}{\partial x^2}$$

and the solution $F(x, t)$ is given by

$$F(x, t) = \frac{1}{2} [\varphi(x + vt) + \varphi(x - vt)].$$

Another interesting case is when both parameters a and v tend to infinity in such a way that the quotient $\frac{2a}{v^2}$ remains constant and equal to $\frac{1}{D}$. Here the telegraph equation converges to the heat equation:

$$\frac{1}{D} \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2}. \quad (3.1.3)$$

Consider again the general case. In [Kac, 1974], Kac obtained a solution for the telegraph equation in terms of a Poisson process. We describe his result.

Consider a Poisson process N_a of parameter a . Recall from Definition 1.1.1 that N_a is a stochastic process starting at zero, with independent increments and such

that, for $0 \leq s < t$, the increment $N_a(t) - N_a(s)$ has a Poisson distribution of parameter $a(t - s)$. Imagine that my particle is subjected to collisions and, every time it suffers a collision, the particle changes direction. The number of collisions my particle undergoes up to time t is the Poisson process $N_a(t)$.

The velocity of the particle is related to the Poisson process because it changes sign at each collision. So, the velocity at time t is given by the formula

$$v(t) = v(-1)^{N_a(t)}.$$

Notice that, from this definition, after an even number of collisions the particle has its old velocity, while after an odd number it reverses direction. The displacement $X(t)$, that is the continuous version of S_n , is given by the process

$$X(t) = \int_0^t v(u) du = v \int_0^t (-1)^{N_a(u)} du.$$

Then, Kac proved that the solution of the telegraph equation (3.1.2) is given by

$$F(x, t) = \frac{1}{2} \left\langle \varphi \left(x + v \int_0^t (-1)^{N_a(u)} du \right) \right\rangle + \frac{1}{2} \left\langle \varphi \left(x - v \int_0^t (-1)^{N_a(u)} du \right) \right\rangle.$$

Let $X_\varepsilon(t)$ be the processes considered by Kac with $a = \frac{1}{\varepsilon^2}$ and $v = \frac{1}{\varepsilon}$, that is,

$$X_\varepsilon = \left\{ X_\varepsilon(t) := \frac{1}{\varepsilon} \int_0^t (-1)^{N_{\frac{1}{\varepsilon^2}}(u)} du, \quad t \in [0, T] \right\}. \quad (3.1.4)$$

Doing a change of variables, these processes can be written as

$$X_\varepsilon = \left\{ X_\varepsilon(t) := \varepsilon \int_0^{\frac{t}{\varepsilon^2}} (-1)^{N(u)} du, \quad t \in [0, T] \right\},$$

where $\{N(t), t \geq 0\}$ is a Poisson process with parameter 1.

Notice that the parameters a and v satisfy that $\frac{2a}{v^2}$ is constant and $D = \frac{1}{2}$. From (3.1.3), we get the equation

$$2 \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2},$$

whose solution is a standard Brownian motion.

In [Stroock, 1982], Stroock proved that the processes X_ε converge in law to a standard Brownian motion. That is, if we consider (P^ε) the image law of the process X_ε in the Banach space $\mathcal{C}([0, T])$ of continuous functions on $[0, T]$, then (P^ε) converges weakly, when ε tends to zero, towards the Wiener measure.

In the mathematical literature we find generalizations of the Stroock result which can be channeled in three ways:

- i) modifying the processes x_ε in order to obtain approximations of other Gaussian processes,

- ii) proving convergence in a stronger sense than the convergence in law in the space of continuous functions,
- iii) weakening the conditions of the approximating processes.

Following the approach i), a first generalization is also made by Stroock in [Stroock, 1982], who modified the processes X_ε to obtain approximations of stochastic differential equations.

In [Yuqiang and Hongshuai, 2011] an approximation to the fractional Brownian motion by 2-parameter Poisson processes is found. In [Delgado and Jolis, 2000] this approximation is extended to a general class of Gaussian processes: the authors prove that every Gaussian process of the form

$$Y(t) = \int_0^1 K(t, u) dW_u,$$

where W is a standard Brownian motion and K a sufficiently regular deterministic kernel, can be weakly approximated by the family of processes

$$Y^\varepsilon(t) = \frac{1}{\varepsilon} \int_0^1 K(t, u) (-1)^{N_{\frac{u}{\varepsilon^2}}} du.$$

In [Bardina et al., 2010b] a diffusion approximation result is shown for stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$. In [Bardina et al., 2010a] and [Deya et al., 2013], the authors extend the Kac-Strook result to approximations of the stochastic heat equation driven by Gaussian white noise and the Stratonovich heat equation, respectively.

Toward direction ii), in the literature we find studies on the strong convergence of the so called uniform transport processes. Since the approximations always start increasing, we have to consider a modification of the form

$$\tilde{X}_\varepsilon(t) = \varepsilon (-1)^A \int_0^{\frac{t}{\varepsilon^2}} (-1)^{N(u)} du,$$

where A has a Bernoulli distribution $\text{Ber}(\frac{1}{2})$ independent of the Poisson process N .

In [Griego et al., 1971], Griego, Heath and Ruiz-Moncayo show that these processes converge strongly and uniformly on bounded time intervals to Brownian motion. In [Gorostiza and Griego, 1979], Gorostiza and Griego extend the result to diffusions. A rate of convergence is obtained by the same authors in [Gorostiza and Griego, 1980] and by Csörgő and Horváth in [Csörgő and Horváth, 1988].

In [Garzón et al., 2009], Garzón, Gorostiza and León define a sequence of processes that converges strongly to fractional Brownian motion uniformly on bounded intervals, for any Hurst parameter $H \in (0, 1)$, and compute the rate of convergence.

In [Garzón et al., 2011] and [Garzón et al., 2013] the same authors deal with fractional stochastic differential equations and subfractional Brownian motion.

In [Garzón et al., 2012], Garzón, Torres and Tudor give a strong approximation of the Hermite process of order $q = 2$, also called Rosenblatt process, that is, a second iterated integral with respect to a standard Brownian motion of a deterministic function with 2 variables, and derived a rate of convergence.

Finally, in the way iii), given that

$$(-1)^{N(u)} = e^{i\pi N(u)} = \cos(\pi N(u)),$$

the question that if the convergence is also true with other angles appears. In [Bardina, 2001], Bardina show that, if we consider the processes

$$\bar{Z}_\varepsilon^\theta(t) = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} e^{i\theta N_s} ds, \quad (3.1.5)$$

where $\theta \neq 0, \pi$, their laws converge weakly towards the law of a complex Brownian motion, i.e., the laws of the real and imaginary parts

$$\bar{X}_\varepsilon^\theta(t) = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} \cos(\theta N_s) ds$$

and

$$\bar{Y}_\varepsilon^\theta(t) = \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} \sin(\theta N_s) ds,$$

converge weakly towards the law of two independent Brownian motions. The approximating processes are functionally dependent because they are constructed from a single Poisson process but, in the limit, we obtain two independent processes. In [Bardina and Rovira, 2013], Bardina and Rovira prove that, despite using only one Poisson process, the processes

$$(\bar{X}_\varepsilon^{\theta_1}(t), \dots, \bar{X}_\varepsilon^{\theta_\ell}(t), \bar{Y}_\varepsilon^{\theta_{\ell+1}}(t), \dots, \bar{Y}_\varepsilon^{\theta_d}(t)),$$

for different angles θ_k , converge in law towards d independent Brownian motions. In [Bardina and Rovira, 2016], the same authors extend the previous results to a sequence of approximations constructed from a Lévy process instead of a Poisson process.

In this chapter we present an extension of the Kac-Stroock result in the directions ii) and iii). we define modifications of the processes \bar{Z}_ε that depend on a parameter $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and are defined from a unique Poisson process and a sequence of independent random variables with common Bernoulli distribution $\text{Ber}(\frac{1}{2})$. We obtain results concerning weak and strong convergence and a rate of convergence. For simplicity's sake, we only consider $\theta \in (0, \pi) \cup (\pi, 2\pi)$ for which it does not exist any $m \in \mathbb{N}$ such that $\cos(m\theta) = 0$ or $\sin(m\theta) = 0$.

Throughout the chapter K and C will denote any positive constant, not depending on ε , which may change from one expression to another.

3.2 Definitions and main ideas

Let $\{M_t, t \geq 0\}$ be a Poisson process of parameter 2. Recall from Chapter 1.1 that M is a stochastic process starting at zero, with independent increments, such that, for $0 \leq s < t$, the increment $M_t - M_s$ has a Poisson distribution with parameter $2(t - s)$.

We define $\{N_t, t \geq 0\}$ and $\{N'_t, t \geq 0\}$ two other counting processes that, at each jump of M , each of them jumps or does not jump with probability $\frac{1}{2}$, independently of the jumps of the other process and of its past.

Remark 3.2.1. N and N' are Poisson processes of parameter 1 with independent increments on disjoint intervals. We check this statement.

First, we prove that N is a Poisson process of parameter 1, checking that the conditions of Definition 1.1.1 are fulfilled:

- i) Clearly $N_0 = 0$.
- ii) Let $k_j \in \mathbb{N} \cup \{0\}$ for $j = 1, \dots, n$. By the independence of increments of the Poisson process M , for any $n \geq 0$ and for any $0 \leq t_1 < \dots < t_{n+1}$, it holds that

$$\begin{aligned} & P\left(\bigcap_{\ell=1}^n N_{t_{\ell+1}} - N_{t_\ell} = k_\ell \mid \bigcap_{\ell'=1}^n M_{t_{\ell'+1}} - M_{t_{\ell'}} = m_{\ell'}\right) \\ &= \prod_{j=1}^n P(N_{t_{j+1}} - N_{t_j} = k_j \mid M_{t_{j+1}} - M_{t_j} = m_j) \end{aligned}$$

so we have

$$\begin{aligned} & P\left(\bigcap_{\ell=1}^n N_{t_{\ell+1}} - N_{t_\ell} = k_\ell\right) \\ &= \sum_{\substack{m_j=0 \\ j=1, \dots, n}}^{\infty} \left[P\left(\bigcap_{\ell=1}^n N_{t_{\ell+1}} - N_{t_\ell} = k_\ell \mid \bigcap_{\ell'=1}^n M_{t_{\ell'+1}} - M_{t_{\ell'}} = m_{\ell'}\right) \right. \\ & \quad \left. \times P\left(\bigcap_{\ell'=1}^n M_{t_{\ell'+1}} - M_{t_{\ell'}} = m_{\ell'}\right) \right] \\ &= \prod_{j=1}^n P(N_{t_{j+1}} - N_{t_j} = k_j) \end{aligned}$$

Then, the increments $N_{t_2} - N_{t_1}, \dots, N_{t_{n+1}} - N_{t_n}$ are independent random variables.

iii) For any $0 \leq s < t$ and for $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
 P(N_t - N_s = k) &= \sum_{n=k}^{\infty} P(N_t - N_s = k \mid M_t - M_s = n) P(M_t - M_s = n) \\
 &= \sum_{n=k}^{\infty} \binom{n}{k} \frac{1}{2^k} \cdot \frac{1}{2^{n-k}} \cdot \frac{[2(t-s)]^n}{n!} \cdot e^{-2(t-s)} \\
 &= e^{-2(t-s)} \sum_{n=k}^{\infty} \binom{n}{k} \frac{(t-s)^n}{n!} \\
 &= e^{-2(t-s)} \frac{(t-s)^k}{k!} \sum_{n=k}^{\infty} \frac{(t-s)^{n-k}}{(n-k)!} \\
 &= e^{-(t-s)} \frac{(t-s)^k}{k!}
 \end{aligned}$$

Then, the increment $N_t - N_s$ has a Poisson distribution of parameter $t - s$.

Hence, N is a Poisson process with parameter 1. One can prove that N' is also a Poisson process with parameter 1 with identical arguments.

Now we verify that N and N' have independent increments on disjoint intervals. Let $k, j \in \mathbb{N} \cup \{0\}$. For $0 \leq s < t < u < v$,

$$\begin{aligned}
 &P(N_t - N_s = k, N'_v - N'_u = j) \\
 &= \sum_{n=k}^{\infty} \sum_{m=j}^{\infty} P(N_t - N_s = k, N'_v - N'_u = j \mid M_t - M_s = n, M_v - M_u = m) \\
 &\quad \times P(M_t - M_s = n, M_v - M_u = m) \\
 &= \sum_{n=k}^{\infty} \sum_{m=j}^{\infty} P(N_t - N_s = k \mid M_t - M_s = n) P(M_t - M_s = n) \\
 &\quad \times P(N_v - N_u = j \mid M_v - M_u = m) P(M_v - M_u = m) \\
 &= P(N_t - N_s = k) P(N'_v - N'_u = j)
 \end{aligned}$$

as we wanted.

For $\theta \in (0, 2\pi)$, consider the following processes:

$$\left\{ Z_\varepsilon^\theta(t) = (-1)^G \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{N'_u} e^{i\theta N_u} du, \quad t \in [0, T] \right\} \quad (3.2.1)$$

where N and N' are defined above and G is a random variable, independent of N and N' , with Bernoulli distribution of parameter $\frac{1}{2}$, that is, $P(G = 0) = P(G = 1) = \frac{1}{2}$.

We can write the process $Z_\varepsilon^\theta(t)$ as

$$Z_\varepsilon^\theta(t) = X_\varepsilon^\theta(t) + iY_\varepsilon^\theta(t),$$

where

$$X_\varepsilon^\theta(t) := \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{N'_u+G} \cos(\theta N_u) du \quad (3.2.2)$$

and

$$Y_\varepsilon^\theta(t) := \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{N'_u+G} \sin(\theta N_u) du \quad (3.2.3)$$

are the real part and the imaginary part, respectively.

Moreover, notice that with a change of variable we obtain

$$Z_\varepsilon^\theta(t) = \frac{2}{\varepsilon} (-1)^G \int_0^t (-1)^{N'_{2u/\varepsilon^2}} e^{i\theta N_{2u/\varepsilon^2}} du. \quad (3.2.4)$$

Figure 3.2.1 shows the simulation of the trajectories of the real and imaginary parts of Z_ε^θ for a fixed ε and θ , while Figure 3.2.2 and Figure 3.2.3 represent the trajectories of the processes X_ε^θ and Y_ε^θ , respectively, for different values of θ .

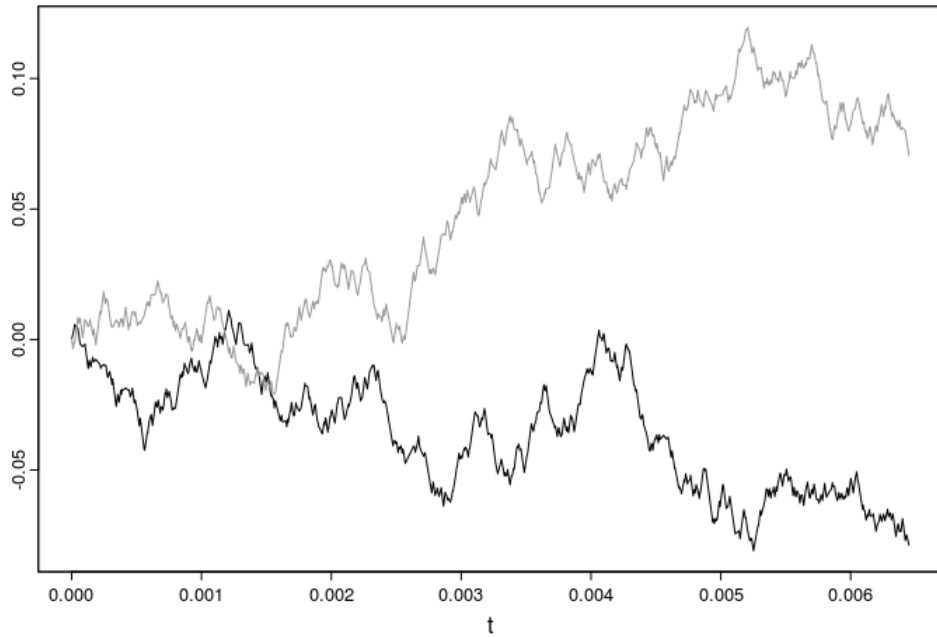


Figure 3.2.1: Trajectories of the processes X_ε^θ and Y_ε^θ for the values of the parameters $\varepsilon = \frac{1}{200}$ and $\theta = 2$.

When $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and it does not exist any $m \in \mathbb{N}$ such that $\cos(m\theta) = 0$ or $\sin(m\theta) = 0$, we are able to construct approximations to d standard independent Brownian motions for d as large as we want and to deduce a rate of convergence. In more detail, in the following sections we deal with results in three directions:

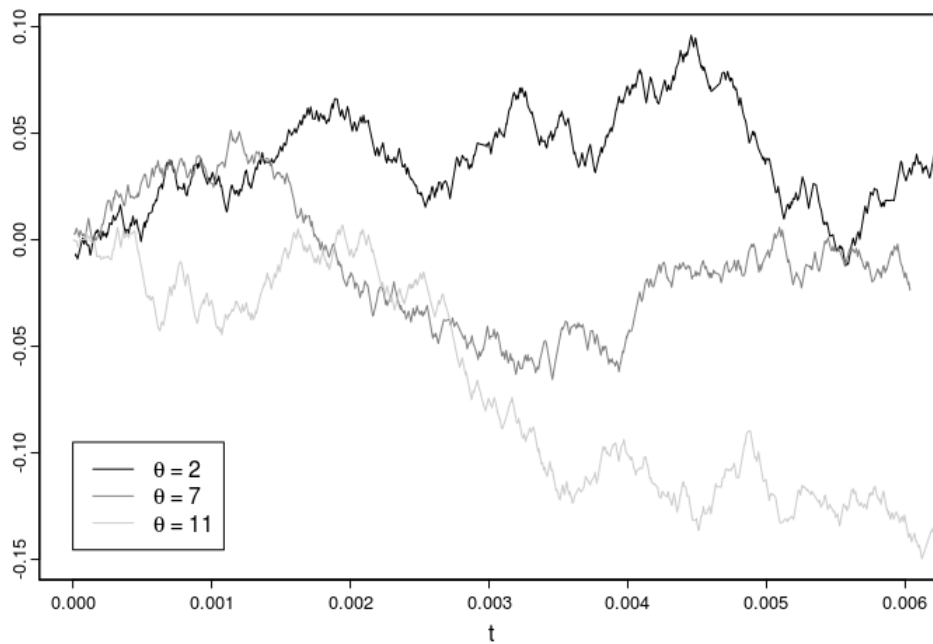


Figure 3.2.2: Trajectories of the processes X_ε^θ for $\varepsilon = \frac{1}{200}$ and different values of θ .

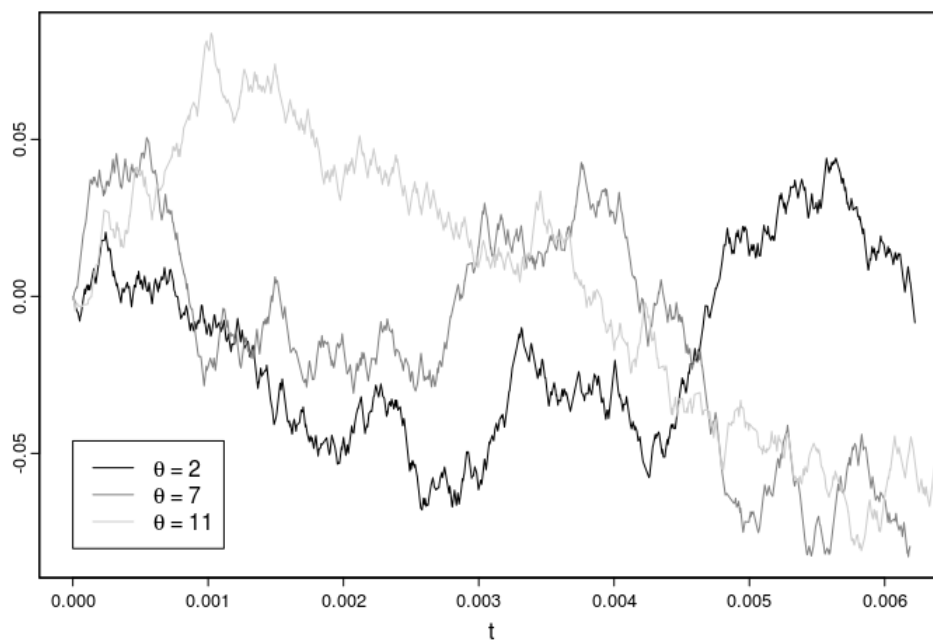


Figure 3.2.3: Trajectories of the processes Y_ε^θ for $\varepsilon = \frac{1}{200}$ and different values of θ .

1. *Weak convergence:* we prove that the process Z_ε^θ converges in law to a complex Brownian motion and that, for a suitable set $\{\theta_1, \theta_2, \dots, \theta_m\}$, the law of $(X_\varepsilon^{\theta_1}, \dots, X_\varepsilon^{\theta_m}, Y_\varepsilon^{\theta_1}, \dots, Y_\varepsilon^{\theta_m})$ converges weakly to the joint law of $2m$ independent Brownian motions.
2. *Strong convergence:* we show that there exist realizations of the process Z_ε^θ on the same probability space of a complex Brownian motion.
3. *Rate of convergence:* we study how fast the realizations of the process Z_ε^θ converge to a complex Brownian motion.

On the other hand, when $\theta = \pi$, the process $Z_\varepsilon^\theta(t)$ is

$$Z_\varepsilon^\pi(t) = (-1)^G \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{N_u + N'_u} du.$$

In the last part of the chapter we prove that there exist realizations of the above process on the same probability space of a standard Brownian motion.

As usual, the weak convergence is established proving tightness and the identification of the law of all possible weak limits. For instance, we refer to [Bardina, 2001] and [Bardina and Rovira, 2013].

The almost sure convergence is inspired by [Griego et al., 1971] and is based on the following result due to Skorokhod:

Theorem 3.2.2 (Skorokhod's Theorem). *Suppose that $\xi_1, \xi_2, \dots, \xi_n$ are independent random variables such that $\mathbb{E}(\xi_k) = 0$ and $\text{Var}(\xi_k) < \infty$, for all k , and that $W(t)$ is a standard Brownian motion. Then, there exist non-negative random variables $\tau_1, \tau_2, \dots, \tau_n$ for which the variables*

$$W(\tau_1), W(\tau_1 + \tau_2) - W(\tau_1), \dots, W\left(\sum_{k=1}^n \tau_k\right) - W\left(\sum_{k=1}^{n-1} \tau_k\right)$$

have the same joint distribution as $\xi_1, \xi_2, \dots, \xi_n$. Moreover,

1. $\mathbb{E}(\tau_k) = \text{Var}(\xi_k)$ for all k .
2. There exists L_m such that $\mathbb{E}[(\tau_k)^m] \leq L_m \mathbb{E}[(\xi_k)^{2m}]$.
3. For $s \in [\sum_{k=1}^n \tau_i, \sum_{k=1}^{n+1} \tau_k]$, if $|\xi_k| \leq h$, then $|W(s) - W(\sum_{k=1}^n \tau_k)| \leq h$.

We refer to [Skorokhod, 1965, Chapter 7] for a complete description of this result.

The computation of the rate of convergence follows the method given in [Gorostiza and Griego, 1979] and [Gorostiza and Griego, 1980].

Along the Chapter we use some well-known results that we state hereunder.

The first statement is a technical result of Feller (see [Feller, 1966, (5.6) page 54]) that gives us information about the law of sums of a random number of independent identically distributed random variables:

Lemma 3.2.3. *Let $\{X_k\}_k$ be a sequence of independent identically distributed random variables with exponential law of parameter α , that is, $X_k \sim \text{Exp}(\alpha)$ for all k . Assume that \widehat{N} is a random variable with distribution $\widehat{N} \sim \text{Geom}(p)$, that is, for all $n \in \mathbb{N}$, $P(\widehat{N} = n) = (1 - p)p^{n-1}$. Consider the sum*

$$S_{\widehat{N}} = X_1 + \cdots + X_{\widehat{N}}$$

with the random number \widehat{N} of terms. Then, its density is

$$f_{S_{\widehat{N}}}(x) = (1 - p)\alpha e^{(1-p)\alpha x}.$$

The following result is a powerful method, due to Skorokhod, of studying the distribution of a partial sum of independent identically distributed random variables with zero mean and finite variance:

Theorem 3.2.4 (Skorokhod's second embedding theorem). *Suppose that $\xi_1, \xi_2, \dots, \xi_n$ are independent and identically distributed random variables with mean 0 and finite variance, and put $S_n = \xi_1 + \cdots + \xi_n$. Let $W(t)$ be a Brownian process. There is a nondecreasing sequence τ_1, τ_2, \dots of stopping times such that the W_{τ_n} has the same joint distribution as S_n and $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \dots$ are independent and identically distributed random variables satisfying*

1. $\mathbb{E}(\tau_n - \tau_{n-1}) = \mathbb{E}(\xi_1^2)$,
2. $\mathbb{E}[(\tau_n - \tau_{n-1})^2] \leq 4\mathbb{E}[(\xi_1)^4]$.

We refer to [Billingsley, 1995, Theorem 37.7] to the proof of this result.

The following lemmas state two inequalities that we find useful in the proof of the strong convergence and the rate of convergence:

Lemma 3.2.5 (Doob's martingale inequality). *Let X_t be a submartingale taking non-negative real values. For every $c > 0$ and every $p \geq 1$,*

$$P\left(\sup_{0 \leq t \leq T} X_t \geq c\right) \leq \frac{1}{c^p} \mathbb{E}(X_T^p).$$

Lemma 3.2.6 (Kolmogorov's inequality). *Let X_1, \dots, X_n be independent random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with expected value $\mathbb{E}(X_k) = 0$ and variance $\text{Var}(X_k) < \infty$ for $k = 1, \dots, n$. Then, for each $m > 0$,*

$$P\left(\max_{1 \leq k \leq n} |S_k| \geq m\right) \leq \frac{1}{m^2} \text{Var}(S_n) = \frac{1}{m^2} \sum_{k=1}^n \text{Var}(X_k),$$

where $S_k = X_1 + \cdots + X_k$.

3.3 Weak convergence

Let $\theta \in (0, \pi) \cup (\pi, 2\pi)$. Consider the processes defined in (3.2.1):

$$Z_\varepsilon^\theta(t) = (-1)^G \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{N'_u} e^{i\theta N_u} du.$$

This section is devoted to illustrate some results on weak convergence of this process. These results are based on the two lemmas hereunder:

Lemma 3.3.1. *Let N and N' the Poisson process defined in Section 3.2. For any $0 \leq x_1 \leq x_2$,*

$$\mathbb{E}[(-1)^{N'_{x_2} - N'_{x_1}} e^{i\theta(N_{x_2} - N_{x_1})}] = e^{-2(x_2 - x_1)}.$$

Proof. From the definition of N and N' it follows that

$$\begin{aligned} & \mathbb{E}[(-1)^{N'_{x_2} - N'_{x_1}} e^{i\theta(N_{x_2} - N_{x_1})}] \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n e^{i\theta m} P(N'_{x_2} - N'_{x_1} = n, N_{x_2} - N_{x_1} = m) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n e^{i\theta m} \sum_{k=n \vee m}^{\infty} P(N'_{x_2} - N'_{x_1} = n, N_{x_2} - N_{x_1} = m | M_{x_2} - M_{x_1} = k) \\ & \quad \times P(M_{x_2} - M_{x_1} = k) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^n e^{i\theta m} \sum_{k=n \vee m}^{\infty} \binom{k}{n} \binom{k}{m} \frac{1}{2^k} \cdot \frac{1}{2^k} \cdot \frac{[2(x_2 - x_1)]^k e^{-2(x_2 - x_1)}}{k!} \\ &= e^{-2(x_2 - x_1)} \sum_{k=0}^{\infty} \left(\frac{x_2 - x_1}{2}\right)^k \frac{1}{k!} \sum_{n=0}^k \binom{k}{n} (-1)^n \sum_{m=0}^k \binom{k}{m} e^{i\theta m}. \end{aligned}$$

By $|e^{i\theta m}| \leq 1$ and $\sum_{n=0}^k \binom{k}{n} = \sum_{m=0}^k \binom{k}{m} = 2^k$, it easy to see that the series is absolutely convergent. Moreover, notice that $\sum_{n=0}^k \binom{k}{n} (-1)^n = 0$ when $k \neq 0$, therefore the above expression is different from zero only when $k = 0$ and, as a consequence, when $n = 0$ and $m = 0$. Hence, as the series is absolutely convergent,

$$\mathbb{E}[(-1)^{N'_{x_2} - N'_{x_1}} e^{i\theta(N_{x_2} - N_{x_1})}] = e^{-2(x_2 - x_1)},$$

as we wanted to prove. □

Using Lemma 3.3.1, we can also get a version of Lemma 3.2 in [Bardina, 2001] well adapted to our processes:

Lemma 3.3.2. *Consider $\{\mathcal{F}_t^{\varepsilon, \theta}\}$ the natural filtration of the processes Z_ε^θ . Then, for any $s < t$ and for any real $\{\mathcal{F}_s^{\varepsilon, \theta}\}$ -measurable and bounded random variable Y , we have that, for any $\theta \in (0, \pi) \cup (\pi, 2\pi)$,*

$$a) \quad \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[(-1)^{N'_{x_2} - N'_{x_1}} e^{i\theta(N_{x_2} - N_{x_1})} \right] dx_1 dx_2 = (t - s) + \frac{\varepsilon^2}{4} (e^{-\frac{4}{\varepsilon^2}(t-s)} - 1)$$

$$b) \quad \lim_{\varepsilon \rightarrow 0} \left| \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[(-1)^{N'_{x_2} + N'_{x_1}} e^{i\theta(N_{x_2} + N_{x_1})} Y \right] dx_1 dx_2 \right| = 0.$$

Proof. The lemma is proved following the same ideas of [Bardina, 2001] and using Lemma 3.3.1.

a) By lemma 3.3.1 and developing the integral, we get

$$\begin{aligned} & \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[(-1)^{N'_{x_2} - N'_{x_1}} \cdot e^{i\theta(N_{x_2} - N'_{x_1})} \right] dx_1 dx_2 \\ &= \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} e^{-2(x_2 - x_1)} dx_1 dx_2 \\ &= (t - s) + \frac{\varepsilon^2}{4} (e^{-\frac{4}{\varepsilon^2}(t-s)} - 1). \end{aligned}$$

b) Recall that $\mathbb{E}[e^{i\theta N_t}] = e^{-t(1-e^{i\theta})}$. By Lemma 3.3.1 and using that Poisson process has independent increments, we have

$$\begin{aligned} & \left| \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[(-1)^{N'_{x_2} + N'_{x_1}} \cdot e^{i\theta(N_{x_2} + N_{x_1})} Y \right] dx_1 dx_2 \right| \\ &= \left| \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[(-1)^{N'_{x_2} - N'_{x_1}} \cdot e^{i\theta(N_{x_2} - N_{x_1})} \right] \mathbb{E} \left[e^{2i\theta(N_{x_1} - N_{2s/\varepsilon^2})} \right] \right. \\ & \quad \left. \times \mathbb{E} \left[Y e^{2i\theta N_{2s/\varepsilon^2}} \right] dx_1 dx_2 \right| \\ &\leq K \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} e^{-2(x_2 - x_1)} \left| e^{-(x_1 - 2s/\varepsilon^2)(1 - e^{2i\theta})} \right| dx_1 dx_2 \\ &= K \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} e^{-2(x_2 - x_1)} e^{-(x_1 - 2s/\varepsilon^2)(1 - \cos(2\theta))} dx_1 dx_2 \\ &\leq \frac{K \varepsilon^2}{2} \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} e^{-(x_1 - 2s/\varepsilon^2)(1 - \cos(2\theta))} dx_1 \\ &\leq \frac{K \varepsilon^2}{2(1 - \cos(2\theta))} \end{aligned}$$

that converges to zero as ε tends to zero.

□

Now we are ready to state our first result on weak convergence to a complex Brownian motion:

Theorem 3.3.3. *Let P_ε^θ be the image law of Z_ε^θ in the Banach space $\mathcal{C}([0, T], \mathbb{C})$ of continuous functions on $[0, T]$. Then P_ε^θ converges weakly when ε tends to zero to the law P^θ on $\mathcal{C}([0, T], \mathbb{C})$ of a complex Brownian motion.*

Proof. The proof consists in checking first that the family P_ε^θ is tight and then that the law of all possible limits of P_ε^θ is the law of a complex Brownian motion.

Tightness. Let $\{X_\varepsilon^\theta(t), t \geq 0\}$ and $\{Y_\varepsilon^\theta(t), t \geq 0\}$ be the processes defined by (3.2.2) and (3.2.2), respectively. We need to prove that the laws corresponding to these two processes are tight. Using Billingsley criterion and that our processes are null in the origin, it is sufficient to check that there exists a constant K such that, for any $s < t$,

$$\sup_\varepsilon \left[\mathbb{E} \left(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N_x+G} \cos(\theta N_x) dx \right)^4 + \mathbb{E} \left(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N_x+G} \sin(\theta N_x) dx \right)^4 \right] \leq K(t-s)^2.$$

We follow the ideas of the proof of lemma 2.1 in [Bardina, 2001]. We denote by \mathcal{I} the 4-dimensional cube $\left[\frac{2s}{\varepsilon^2}, \frac{2t}{\varepsilon^2}\right]^4$, then we have

$$\begin{aligned} & \mathbb{E} \left(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N_x} \cos(\theta N_x) dx \right)^4 + \mathbb{E} \left(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N_x} \sin(\theta N_x) dx \right)^4 \\ &= 24 \varepsilon^2 \mathbb{E} \int_{\mathcal{I}} \mathbf{1}_{\{x_1 \leq x_2 \leq x_3 \leq x_4\}} (-1)^{N_{x_1} + N_{x_2} + N_{x_3} + N_{x_4}} \\ & \quad \times [\cos(\theta N_{x_4}) \cos(\theta N_{x_3}) \cos(\theta N_{x_2}) \cos(\theta N_{x_1}) \\ & \quad + \sin(\theta N_{x_4}) \sin(\theta N_{x_3}) \sin(\theta N_{x_2}) \sin(\theta N_{x_1})] \otimes_{i=1}^4 dx_i \\ &= 12 \varepsilon^4 \mathbb{E} \int_{\mathcal{I}} \mathbf{1}_{\{x_1 \leq x_2 \leq x_3 \leq x_4\}} (-1)^{N_{x_4} - N_{x_3}} (-1)^{N_{x_2} - N_{x_1}} \\ & \quad \times \cos(\theta(N_{x_4} - N_{x_3})) \cos(\theta(N_{x_2} - N_{x_1})) \otimes_{i=1}^4 dx_i \\ & \quad + 12 \varepsilon^4 \mathbb{E} \int_{\mathcal{I}} \mathbf{1}_{\{x_1 \leq x_2 \leq x_3 \leq x_4\}} (-1)^{N_{x_4} + N_{x_3}} (-1)^{N_{x_2} + N_{x_1}} \\ & \quad \times \cos(\theta(N_{x_4} + N_{x_3})) \cos(\theta(N_{x_2} + N_{x_1})) \otimes_{i=1}^4 dx_i \\ &= I_1 + I_2 \end{aligned}$$

Using that the Poisson process has independent increments, we get

$$\begin{aligned}
 I_1 &= 12 \varepsilon^4 \int_{\mathcal{I}} \mathbf{1}_{\{x_1 \leq x_2 \leq x_3 \leq x_4\}} \mathbb{E} \left[(-1)^{N_{x_4} - N_{x_3}} \cos(\theta(N_{x_4} - N_{x_3})) \right] \\
 &\quad \times \mathbb{E} \left[(-1)^{N'_{x_2} - N'_{x_1}} \cos(\theta(N_{x_2} - N_{x_1})) \right] \otimes_{i=1}^4 dx_i \\
 &\leq 12 \left(\varepsilon^2 \int_{\left[\frac{2s}{\varepsilon^2}, \frac{2t}{\varepsilon^2}\right]^2} \mathbf{1}_{\{x_1 \leq x_2\}} \left| \mathbb{E} \left[(-1)^{N'_{x_2} - N'_{x_1}} \cos(\theta(N_{x_2} - N_{x_1})) \right] \right| dx_1 dx_2 \right)^2 \\
 &\leq 12 \left(\varepsilon^2 \int_{\left[\frac{2s}{\varepsilon^2}, \frac{2t}{\varepsilon^2}\right]^2} \mathbf{1}_{\{x_1 \leq x_2\}} \left| \mathbb{E} \left[(-1)^{N'_{x_2} - N'_{x_1}} \cdot e^{i\theta(N_{x_2} - N_{x_1})} \right] \right| dx_1 dx_2 \right)^2 \\
 &\leq 12 \left(\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} e^{-2(x_2 - x_1)} dx_1 dx_2 \right)^2 \\
 &\leq 12(t - s)^2.
 \end{aligned}$$

To prove the bound for I_2 we use again that Poisson process has independent increments, so we obtain:

$$\begin{aligned}
 &\mathbb{E} \left[(-1)^{N'_{x_4} + N'_{x_3}} (-1)^{N'_{x_2} + N'_{x_1}} \cos(\theta(N_{x_4} + N_{x_3})) \cos(\theta(N_{x_2} + N_{x_1})) \right] \\
 &= \mathbb{E} \left[(-1)^{N'_{x_4} + N'_{x_3}} (-1)^{N'_{x_2} + N'_{x_1}} \right. \\
 &\quad \left. \times \cos[\theta((N_{x_4} - N_{x_3}) + 2(N_{x_3} - N_{x_2}) + 2N_{x_2})] \cos[\theta(N_{x_2} + N_{x_1})] \right] \\
 &= \mathbb{E} \left[(-1)^{N'_{x_4} - N'_{x_3}} \cos[\theta((N_{x_4} - N_{x_3}) + 2(N_{x_3} - N_{x_2}))] \right] \\
 &\quad \times \mathbb{E} \left[(-1)^{N'_{x_2} + N'_{x_1}} \cos(2N_{x_2}) \cos[\theta(N_{x_2} + N_{x_1})] \right] \\
 &\quad - \mathbb{E} \left[(-1)^{N'_{x_4} - N'_{x_3}} \sin[\theta((N_{x_4} - N_{x_3}) + 2(N_{x_3} - N_{x_2}))] \right] \\
 &\quad \times \mathbb{E} \left[(-1)^{N'_{x_2} + N'_{x_1}} \sin(2N_{x_2}) \cos[\theta(N_{x_2} + N_{x_1})] \right] \\
 &\leq \left| \mathbb{E} \left[(-1)^{N'_{x_4} - N'_{x_3}} \cos[\theta((N_{x_4} - N_{x_3}) + 2(N_{x_3} - N_{x_2}))] \right] \right| \\
 &\quad + \left| \mathbb{E} \left[(-1)^{N'_{x_4} - N'_{x_3}} \sin[\theta((N_{x_4} - N_{x_3}) + 2(N_{x_3} - N_{x_2}))] \right] \right| \\
 &\leq \left(\left| \mathbb{E} \left[(-1)^{N'_{x_4} - N'_{x_3}} \cos(\theta(N_{x_4} - N_{x_3})) \right] \right| \right. \\
 &\quad \left. + \left| \mathbb{E} \left[(-1)^{N'_{x_4} - N'_{x_3}} \sin(\theta(N_{x_4} - N_{x_3})) \right] \right| \right) \\
 &\quad \times \left(\left| \mathbb{E} \left[\cos(2\theta(N_{x_3} - N_{x_2})) \right] \right| + \left| \mathbb{E} \left[\sin(2\theta(N_{x_3} - N_{x_2})) \right] \right| \right)
 \end{aligned}$$

Integrating with respect to x_2 and x_3 , we obtain the following bound for I_2 :

$$\begin{aligned}
 I_2 &\leq 48 \varepsilon^4 \int_{\mathcal{I}} \mathbf{1}_{\{x_1 \leq x_2 \leq x_3 \leq x_4\}} \left| \mathbb{E} \left[(-1)^{N'_{x_4} - N'_{x_3}} \cdot e^{i\theta(N_{x_4} - N_{x_3})} \right] \right| \\
 &\quad \times \left| \mathbb{E} \left[e^{2i\theta(N_{x_3} - N_{x_2})} \right] \right| dx_1 \dots dx_4 \\
 &\leq 48 \varepsilon^4 \int_{\mathcal{I}} \mathbf{1}_{\{x_1 \leq x_2 \leq x_3 \leq x_4\}} e^{-2(x_4 - x_3)} e^{-(x_3 - x_2)(1 - \cos(2\theta))} dx_1 \dots dx_4 \\
 &\leq \frac{48 \varepsilon^4}{2(1 - \cos(2\theta))} \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_4} dx_1 dx_4 \\
 &= \frac{48 \varepsilon^4}{2(1 - \cos(2\theta))} \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \left(x_4 - \frac{2s}{\varepsilon^2} \right) dx_4 \\
 &= \frac{48(t - s)^2}{1 - \cos(2\theta)}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \sup_{\varepsilon} \left[\mathbb{E} \left(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N'_x + G} \cos(\theta N_x) dx \right)^4 + \mathbb{E} \left(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N'_x + G} \sin(\theta N_x) dx \right)^4 \right] \\
 \leq 12 \left(1 + \frac{4}{1 - \cos(2\theta)} \right) (t - s)^2.
 \end{aligned}$$

So the family of laws is tight.

Identification of the limit law. Consider a subsequence, which we will also denote by $\{P_\varepsilon^\theta\}$, weakly convergent to some probability P^θ . We want to prove that the canonical process $Z^\theta = \{Z^\theta(t), t \geq 0\}$ is a complex Brownian motion under P^θ , that is the real part $Re[Z^\theta]$ and the imaginary part $Im[Z^\theta]$ of this process are two independent Brownian motions. Using Paul Lévy's characterization theorem (Theorem 1.2.26) it is sufficient to prove that, under P^θ , $Re[Z^\theta]$ and $Im[Z^\theta]$ are both martingale with respect to the natural filtration $\{\mathcal{F}_t\}$ with quadratic variation $\langle Re[Z^\theta], Re[Z^\theta] \rangle_{t=0}^t = t$, $\langle Im[Z^\theta], Im[Z^\theta] \rangle_{t=0}^t = t$ and null covariation.

To prove the martingale property with respect to the natural filtration $\{\mathcal{F}_t\}$, following the ideas of Section 3.1 in [Bardina, 2001], it is enough to see that, for any $s_1 \leq s_2 \leq \dots \leq s_n \leq s < t$ and for any bounded continuous function $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$,

$$\left| \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{G + N'_x} e^{i\theta N_x} dx \right] \right|$$

converges to zero as ε tends to zero. We have

$$\begin{aligned}
 & \left| \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{G+N'_x} e^{i\theta N_x} dx \right] \right| \\
 &= \left| \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) (-1)^{G+N'_{2s/\varepsilon^2}} e^{i\theta N_{2s/\varepsilon^2}} \right] \right. \\
 &\quad \left. \times \varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \mathbb{E} \left[(-1)^{N'_x - N'_{2s/\varepsilon^2}} e^{i\theta(N_x - N_{2s/\varepsilon^2})} \right] dx \right| \\
 &\leq K\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} e^{-2(x - \frac{2s}{\varepsilon^2})} dx \\
 &= \frac{K\varepsilon}{2} (1 - e^{-\frac{4}{\varepsilon^2}(t-s)}),
 \end{aligned}$$

that converges to zero as ε tends to zero. Therefore, martingale property is proved. To prove that $\langle \text{Re}[Z^\theta], \text{Re}[Z^\theta] \rangle_t = t$ and $\langle \text{Im}[Z^\theta], \text{Im}[Z^\theta] \rangle_t = t$, we check that, for any $s_1 \leq \dots \leq s_n \leq s < t$ and for any bounded continuous function $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$,

$$\mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) ((X_\varepsilon^\theta(t) - X_\varepsilon^\theta(s))^2 - (t - s)) \right]$$

and

$$\mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) ((Y_\varepsilon^\theta(t) - Y_\varepsilon^\theta(s))^2 - (t - s)) \right]$$

converge to zero as ε tends to zero. Notice that, in our case,

$$\begin{aligned}
 & \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) (X_\varepsilon^\theta(t) - X_\varepsilon^\theta(s))^2 \right] \\
 &= \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \left(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N'_x} \cos(\theta N_x) dx \right)^2 \right] \\
 &= 2\varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \right. \\
 &\quad \left. \times (-1)^{N'_{x_2} + N'_{x_1}} \cos(\theta N_{x_1}) \cos(\theta N_{x_2}) \right] dx_1 dx_2 \\
 &= \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \right] \\
 &\quad \times \mathbb{E} \left[(-1)^{N'_{x_2} - N'_{x_1}} \cos(\theta(N_{x_2} - N_{x_1})) \right] dx_1 dx_2 \\
 &+ \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \right. \\
 &\quad \left. \times (-1)^{N'_{x_2} + N'_{x_1}} \cos(\theta(N_{x_2} + N_{x_1})) \right] dx_1 dx_2.
 \end{aligned}$$

The last expression is equal to

$$\begin{aligned} & \mathbb{E}[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n))] \\ & \quad \times \operatorname{Re} \left\{ \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[(-1)^{N'_{x_2} - N'_{x_1}} \cdot e^{i\theta(N_{x_2} - N_{x_1})} \right] dx_1 dx_2 \right\} \\ & \quad + \operatorname{Re} \left\{ \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \right. \right. \\ & \quad \left. \left. \times (-1)^{N'_{x_2} + N'_{x_1}} \cdot e^{i\theta(N_{x_2} + N_{x_1})} \right] dx_1 dx_2 \right\} \\ & =: F_1 + F_2 \end{aligned}$$

By Lemma 3.3.2 a), we can write F_1 as follows

$$F_1 = \mathbb{E}[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n))] \left((t-s) + \frac{\varepsilon^2}{4} (e^{-\frac{4}{\varepsilon^2}(t-s)} - 1) \right).$$

It clearly converges to $\mathbb{E}[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n))] (t-s)$ when ε tends to zero.

By Lemma 3.3.2 b), F_2 converges to zero when ε tends to zero.

Then, the quadratic variation of the real part is computed. The imaginary part can be done similarly.

Finally, we have to prove that $\langle \operatorname{Re}[Z^\theta], \operatorname{Im}[Z^\theta] \rangle_{t=0} = 0$. It is sufficient to show that, for any $s_1 \leq \dots \leq s_n \leq s < t$ and for any bounded continuous function $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$,

$$\mathbb{E}[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) (X_\varepsilon^\theta(t) - X_\varepsilon^\theta(s)) (Y_\varepsilon^\theta(t) - Y_\varepsilon^\theta(s))]$$

converges to zero as ε tends to zero. We obtain this convergence using similar calculations and statement b) of Lemma 3.3.2. □

We can also get the following extension of Theorem 3.3.3, that is the equivalent of the result obtained in [Bardina and Rovira, 2013] for our processes.

Theorem 3.3.4. *Consider $\theta_1, \theta_2, \dots, \theta_m$ such that for all $i \neq j$, $1 \leq i, j \leq m$, $\theta_i, \theta_j \in (0, \pi) \cup (\pi, 2\pi)$, $\theta_i + \theta_j \neq 2\pi$ and $\theta_i \neq \theta_j$. Then the laws of the processes*

$$(X_\varepsilon^{\theta_1}, \dots, X_\varepsilon^{\theta_m}, Y_\varepsilon^{\theta_1}, \dots, Y_\varepsilon^{\theta_m})$$

converge weakly, in the space of the continuous functions, towards the joint law of $2m$ independent Brownian motions.

Proof. Taking into account the proof of Theorem 3.3.3, it only remains to check that for $i \neq j$, and θ_i, θ_j in the conditions of Theorem 3.3.4, $\langle \operatorname{Re}[Z^{\theta_i}], \operatorname{Re}[Z^{\theta_j}] \rangle_{t=0} = 0$, $\langle \operatorname{Im}[Z^{\theta_i}], \operatorname{Im}[Z^{\theta_j}] \rangle_{t=0} = 0$ and $\langle \operatorname{Im}[Z^{\theta_i}], \operatorname{Re}[Z^{\theta_j}] \rangle_{t=0} = 0$. We prove that

following the proof of Theorem 2 in [Bardina and Rovira, 2013] and taking into account our Lemma 3.3.1.

To prove that $\langle \text{Re}[Z^{\theta_i}], \text{Re}[Z^{\theta_j}] \rangle_t = 0$ it is sufficient to show that for any $s_1 \leq \dots \leq s_n \leq s < t$ and for any bounded continuous function $\varphi : \mathbb{C}^{2k} \rightarrow \mathbb{R}$,

$$\mathbb{E}[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n))(X_\varepsilon^{\theta_1}(t) - X_\varepsilon^{\theta_1}(s))(X_\varepsilon^{\theta_2}(t) - X_\varepsilon^{\theta_2}(s))]$$

converges to zero as ε tends to zero. The last expression can be written as

$$\begin{aligned} & \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \left(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N'_x} \cos(\theta_1 N_x) dx \right) \right. \\ & \quad \left. \times \left(\varepsilon \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} (-1)^{N'_x} \cos(\theta_2 N_x) dx \right) \right] \\ &= \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \right. \\ & \quad \left. \times (-1)^{N'_{x_1} + N'_{x_2}} \sin(\theta_1 N_{x_1}) \sin(\theta_2 N_{x_2}) \right] dx_1 dx_2 \\ & \quad + \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_1} \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \right. \\ & \quad \left. \times (-1)^{N'_{x_1} + N'_{x_2}} \sin(\theta_1 N_{x_1}) \sin(\theta_2 N_{x_2}) \right] dx_1 dx_2 \\ &=: G_1 + G_2 \end{aligned}$$

Using the identity $\sin(a) \sin(b) = (\cos(a - b) - \cos(a + b))/2$, we have

$$\begin{aligned} G_1 &= \frac{1}{2} \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \right. \\ & \quad \left. \times (-1)^{N'_{x_2} - N'_{x_1}} \cos(\theta_1 N_{x_1} - \theta_2 N_{x_2}) \right] dx_1 dx_2 \\ & \quad - \frac{1}{2} \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \right. \\ & \quad \left. \times (-1)^{N'_{x_2} - N'_{x_1}} \cos(\theta_1 N_{x_1} + \theta_2 N_{x_2}) \right] dx_1 dx_2 \\ &=: G_{11} - G_{12} \end{aligned}$$

By Lemma 3.3.1 and using the independence of the increments of the Poisson

process and $\mathbb{E}(e^{i\theta N_t}) = e^{-t(1-e^{i\theta})}$, we obtain a bound for G_{11} :

$$\begin{aligned}
 G_{11} &= \frac{1}{2} \operatorname{Re} \left\{ \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \cdot (-1)^{N'_{x_2} - N'_{x_1}} \cdot e^{i(\theta_1 - \theta_2)N_{2s/\varepsilon^2}} \right. \right. \\
 &\quad \left. \left. \times e^{i(\theta_1 - \theta_2)(N_{x_1} - N_{2s/\varepsilon^2})} \cdot e^{-i\theta_2(N_{x_2} - N_{x_1})} \right] dx_1 dx_2 \right\} \\
 &= \frac{1}{2} \operatorname{Re} \left\{ \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \mathbb{E} \left[\varphi(Z_\varepsilon^\theta(s_1), \dots, Z_\varepsilon^\theta(s_n)) \cdot e^{i(\theta_1 - \theta_2)N_{2s/\varepsilon^2}} \right] \right. \\
 &\quad \left. \times \mathbb{E} \left[e^{i(\theta_1 - \theta_2)(N_{x_1} - N_{2s/\varepsilon^2})} \right] \cdot \mathbb{E} \left[(-1)^{N'_{x_2} - N'_{x_1}} \cdot e^{-i\theta_2(N_{x_2} - N_{x_1})} \right] dx_1 dx_2 \right\} \\
 &\leq K \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} \left| e^{-(x_1 - 2s/\varepsilon^2)(1 - e^{i(\theta_1 - \theta_2)})} \right| \cdot e^{-2(x_2 - x_1)} dx_1 dx_2 \\
 &= K \varepsilon^2 \int_{\frac{2s}{\varepsilon^2}}^{\frac{2t}{\varepsilon^2}} \int_{\frac{2s}{\varepsilon^2}}^{x_2} e^{-(x_1 - 2s/\varepsilon^2)(1 - \cos(\theta_1 - \theta_2))} \cdot e^{-2(x_2 - x_1)} dx_1 dx_2 \\
 &\leq \frac{K \varepsilon^2}{2(1 - \cos(\theta_1 - \theta_2))}.
 \end{aligned}$$

The last bound is obtained as in the proof of the Lemma 3.3.2 *b*). At this point, clearly G_{11} converges to zero when ε tends to zero for $\theta_1 \neq \theta_2$. Replacing θ_2 by $-\theta_2$, we obtain a bound for G_{12} with a similar computation:

$$G_{12} \leq \frac{K \varepsilon^2}{2} \frac{1}{1 - \cos(\theta_1 + \theta_2)}.$$

For θ_1, θ_2 such that $\theta_1 + \theta_2 \neq 2\pi$, it also converges to zero when ε tends to zero. Therefore, the proof for G_1 is completed.

On the other hand, G_2 is equal to G_1 under interchanging the roles of θ_1 and θ_2 , so we have the following bound

$$G_2 \leq \frac{K \varepsilon^2}{2} \left(\frac{1}{1 - \cos(\theta_2 - \theta_1)} + \frac{1}{1 - \cos(\theta_1 + \theta_2)} \right),$$

that also converges to zero when ε tends to zero for $\theta_2 \neq \theta_1$ such that $\theta_1 + \theta_2 \neq 2\pi$.

The proof of $\langle \operatorname{Im}[Z^{\theta_i}], \operatorname{Im}[Z^{\theta_j}] \rangle_{t=0} = 0$ and $\langle \operatorname{Im}[Z^{\theta_i}], \operatorname{Re}[Z^{\theta_j}] \rangle_{t=0} = 0$ is done similarly. □

3.4 Strong convergence

Let $\theta \in (0, \pi) \cup (\pi, 2\pi)$. Consider again the processes defined in (3.2.1):

$$Z_\varepsilon^\theta(t) = (-1)^G \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{N'_u} e^{i\theta N_u} du.$$

This section is devoted to state and prove the following result on strong convergence to a complex Brownian motion:

Theorem 3.4.1. *There exists realizations of the process Z_ε^θ on the same probability space as a complex Brownian motion $\{Z(t), t \geq 0\}$ such that*

$$\lim_{\varepsilon \rightarrow 0} \max_{0 \leq t \leq 1} |Z_\varepsilon^\theta(t) - Z(t)| = 0 \quad a.s. \quad (3.4.1)$$

Proof. Recall that X_ε^θ is the real part of the processes Z_ε^θ , defined by (3.2.2) as

$$X_\varepsilon^\theta(t) = \frac{2}{\varepsilon} (-1)^G \int_0^t (-1)^{\frac{N'_{2r}}{\varepsilon^2}} \cos(\theta N_{\frac{2r}{\varepsilon^2}}) dr.$$

We study the strong convergence of X_ε^θ to a standard Brownian motion $\{X(t), t \in [0, 1]\}$ when ε tends to 0. More precisely, we prove that there exist realizations $\{X_\varepsilon^\theta(t), t \geq 0\}$ of the above process on the same probability space of a standard Brownian motion $\{X(t), t \geq 0\}$ such that

$$\lim_{\varepsilon \rightarrow 0} \max_{0 \leq t \leq 1} |X_\varepsilon^\theta(t) - X(t)| = 0 \quad a.s. \quad (3.4.2)$$

The convergence of the imaginary part of Z_ε to another standard Brownian motion $\{Y(t), t \in [0, 1]\}$, independent of $\{X(t), t \geq 0\}$, follows the same proof. We refer to the method used in [Griego et al., 1971] to prove the strong convergence to a standard Brownian motion. We will divide the proof in five steps.

Step 1: Definition of the processes. Let (Ω, \mathcal{F}, P) be the probability space for a standard Brownian motion $\{X(t), t \geq 0\}$ with $X(0) = 0$ and let us define:

1. for each $\varepsilon > 0$, $\{\epsilon_m^\varepsilon\}_{m \geq 1}$ a sequence of independent identically distributed random variables with law exponential of parameter $\frac{2}{\varepsilon}$, independent of the Brownian motion X ,
2. $\{\eta_m\}_{m \geq 1}$ a sequence of independent identically distributed random variables with law $\text{Ber}(\frac{1}{2})$, independent of X and $\{\epsilon_m^\varepsilon\}_{m \geq 1}$ for all ε .
3. $\{k_m\}_{m \geq 1}$ a sequence of independent identically distributed random variables such that $P(k_1 = 1) = P(k_1 = -1) = \frac{1}{2}$, independent of X , $\{\eta_m\}_{m \geq 1}$ and $\{\epsilon_m^\varepsilon\}_{m \geq 1}$ for all ε .

From these random variables, we are able to introduce the following ones:

1. $\{b_m\}_{m \geq 0}$ such that $b_0 = 0$ and $b_m = \sum_{j=1}^m \eta_j$ for $m \geq 1$. Clearly b_m has a Binomial distribution of parameters $(m, \frac{1}{2})$ and, for all $n \in \{0, 1, \dots, m\}$, $P(b_{m+1} = n | b_m = n) = P(b_{m+1} = n + 1 | b_m = n) = \frac{1}{2}$.
2. $\{\xi_m^{\varepsilon, \theta}\}_{m \geq 1} = \{|\cos(b_{m-1}\theta)| \epsilon_m^\varepsilon\}_{m \geq 1}$. This family of random variables is clearly independent of X .

Let \mathcal{B} be the σ -algebra generated by $\{b_m\}_{m \geq 1}$. The sequence of random variables $\{k_m \xi_m^{\varepsilon, \theta}\}_{m \geq 0}$ satisfies

$$\begin{aligned}\mathbb{E}(k_m \xi_m^{\varepsilon, \theta} | \mathcal{B}) &= 0, \\ \text{Var}(k_m \xi_m^{\varepsilon, \theta} | \mathcal{B}) &= \mathbb{E}[(\xi_m^{\varepsilon, \theta})^2 | \mathcal{B}] = \frac{\varepsilon^2}{2} [\cos(b_{m-1} \theta)]^2.\end{aligned}$$

By Skorokhod's theorem (Theorem 3.2.2), for each $\varepsilon > 0$ there exists a sequence $\{\sigma_1^{\varepsilon, \theta}\}_{m \geq 1}$ of nonnegative random variables on $(\Omega, \mathcal{F}, \mathcal{P})$ so that the sequence $x(\sigma_1^{\varepsilon, \theta}), x(\sigma_1^{\varepsilon, \theta} + \sigma_2^{\varepsilon, \theta}), \dots$, has the same distribution as $k_1 \xi_1^{\varepsilon, \theta}, k_1 \xi_1^{\varepsilon, \theta} + k_2 \xi_2^{\varepsilon, \theta}, \dots$, and, for each m ,

$$\mathbb{E}(\sigma_m^{\varepsilon, \theta} | \mathcal{B}) = \text{Var}(k_m \xi_m^{\varepsilon, \theta} | \mathcal{B}) = \frac{\varepsilon^2}{2} [\cos(b_{m-1} \theta)]^2.$$

For each ε , we define $\gamma_0^{\varepsilon, \theta} \equiv 0$ and, for each m ,

$$\gamma_m^{\varepsilon, \theta} = |\beta_m^{\varepsilon, \theta}|^{-1} \left| X \left(\sum_{j=0}^m \sigma_j^{\varepsilon, \theta} \right) - X \left(\sum_{j=0}^{m-1} \sigma_j^{\varepsilon, \theta} \right) \right|,$$

where $\sigma_0^{\varepsilon, \theta} \equiv 0$ and

$$\beta_m^{\varepsilon, \theta} = \frac{2}{\varepsilon} \cos(b_{m-1} \theta).$$

Then, the random variables $\{\gamma_1^{\varepsilon, \theta}\}_{m \geq 1}$ are independent with common exponential distribution with parameter $\frac{4}{\varepsilon^2}$. Indeed, for any $0 \leq n < m$ and for any $x, y \in \mathbb{R}$,

$$\begin{aligned}P(\gamma_{n+1}^{\varepsilon, \theta} \leq x, \gamma_{m+1}^{\varepsilon, \theta} \leq y) &= \sum_{j=0}^n \sum_{k=j}^{j+(m-n)} P(\gamma_{n+1}^{\varepsilon, \theta} \leq x, \gamma_{m+1}^{\varepsilon, \theta} \leq y | b_n = j, b_m = k) P(b_n = j, b_m = k) \\ &= (1 - e^{-\frac{4x}{\varepsilon^2}})(1 - e^{-\frac{4y}{\varepsilon^2}}) \sum_{j=0}^n \sum_{k=j}^{j+(m-n)} P(b_n = j, b_m = k) \\ &= P(\gamma_{n+1}^{\varepsilon, \theta} \leq x) P(\gamma_{m+1}^{\varepsilon, \theta} \leq y),\end{aligned}$$

using that, when b_n and b_m are known, $\gamma_{n+1}^{\varepsilon, \theta}$ and $\gamma_{m+1}^{\varepsilon, \theta}$ are independent.

Now, we define $X_\varepsilon^\theta(t), t \geq 0$ as a piecewise linear process satisfying

$$X_\varepsilon^\theta \left(\sum_{j=1}^m \gamma_j^{\varepsilon, \theta} \right) = X \left(\sum_{j=1}^m \sigma_j^{\varepsilon, \theta} \right), \quad m \geq 1 \quad (3.4.3)$$

and $X_\varepsilon^\theta(0) \equiv 0$. Observe that the process X_ε^θ has slope $\pm |\beta_m^{\varepsilon, \theta}|$ in the interval $[\sum_{j=1}^{m-1} \gamma_j^{\varepsilon, \theta}, \sum_{j=1}^m \gamma_j^{\varepsilon, \theta}]$.

Let $\rho_m^{\varepsilon, \theta}$ be the time of the m th change of the absolute values of $\beta_j^{\varepsilon, \theta}$'s, i.e. the time when $\beta_j^{\varepsilon, \theta} = \frac{2}{\varepsilon} \cos[(m-1)\theta]$ and $\beta_{j+1}^{\varepsilon, \theta} = \frac{2}{\varepsilon} \cos(m\theta)$, that is when the slope

of $x_\varepsilon^\theta(\cdot)$ changes from $\pm \frac{2}{\varepsilon} |\cos[(m-1)\theta]|$ to $\pm \frac{2}{\varepsilon} |\cos(m\theta)|$. Then the increments $\rho_m^{\varepsilon,\theta} - \rho_{m-1}^{\varepsilon,\theta}$, $m \geq 1$, with $\rho_0^{\varepsilon,\theta} \equiv 0$, are independent and exponentially distributed with a parameter $\frac{2}{\varepsilon^2}$. The proof is similar to the case $\theta = \pi$. Indeed, since

$$P\left(\beta_m^{\varepsilon,\theta} = \frac{2}{\varepsilon} \cos(n\theta) \mid \beta_{m-1}^{\varepsilon,\theta} = \frac{2}{\varepsilon} \cos[(n-1)\theta]\right) = \frac{1}{2}$$

for every $n = 0, 1, \dots, m$, we can write $\rho_1^{\varepsilon,\theta} = \gamma_1^{\varepsilon,\theta} + \dots + \gamma_{\widehat{N}}^{\varepsilon,\theta}$, where $P(\widehat{N} = n) = 2^{-n}$ for $n \in \mathbb{N}$ and \widehat{N} is independent of $\{\gamma_j^{\varepsilon,\theta}\}_{j \geq 1}$. Therefore, by Lemma 3.2.3, $\rho_1^{\varepsilon,\theta}$ has an exponential distribution with parameter $\frac{2}{\varepsilon^2}$. Likewise, each increment $\rho_m^{\varepsilon,\theta} - \rho_{m-1}^{\varepsilon,\theta}$ has an exponential distribution with parameter $\frac{2}{\varepsilon^2}$ and the increments are independent since they are sum of disjoint blocks of the $\gamma_m^{\varepsilon,\theta}$'s.

On the other hand, let τ_m^ε be the time of the m th change of the sign of the slopes. Following the same arguments for the times $\rho_m^{\varepsilon,\theta}$ we get that the increments $\tau_m^\varepsilon - \tau_{m-1}^\varepsilon$, for each m , with $\tau_0^\varepsilon \equiv 0$, are independent and exponentially distributed with a parameter $\frac{2}{\varepsilon^2}$. Moreover, the increments are sum of disjoint blocks of the $\gamma_m^{\varepsilon,\theta}$'s.

Thus, X_ε^θ is a realization of the process defined by (3.2.2).

Step 2 : Decomposition of the convergence. Now we focus on the proof of (3.4.2). Recalling that $\gamma_0^{\varepsilon,\theta} \equiv \sigma_0^{\varepsilon,\theta} \equiv 0$, by (3.4.3) and the uniform continuity of Brownian motion on $[0, 1]$, we have that, almost surely,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \max_{0 \leq t \leq 1} |X_\varepsilon^\theta(t) - X(t)| &= \lim_{\varepsilon \rightarrow 0} \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| X_\varepsilon^\theta \left(\sum_{j=1}^m \gamma_j^{\varepsilon,\theta} \right) - X \left(\sum_{j=1}^m \gamma_j^{\varepsilon,\theta} \right) \right| \\ &= \lim_{\varepsilon \rightarrow 0} \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| X \left(\sum_{j=1}^m \sigma_j^{\varepsilon,\theta} \right) - X \left(\sum_{j=1}^m \gamma_j^{\varepsilon,\theta} \right) \right|. \end{aligned}$$

This reduces the proof to check that

$$\lim_{\varepsilon \rightarrow 0} \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \gamma_1^{\varepsilon,\theta} + \dots + \gamma_m^{\varepsilon,\theta} - m \frac{\varepsilon^2}{4} \right| = 0 \quad a.s., \quad (3.4.4)$$

and that

$$\lim_{\varepsilon \rightarrow 0} \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sigma_1^{\varepsilon,\theta} + \dots + \sigma_m^{\varepsilon,\theta} - m \frac{\varepsilon^2}{4} \right| = 0 \quad a.s. \quad (3.4.5)$$

The first limit (3.4.4) can be obtained easily by Borel-Cantelli lemma since, by Kolmogorov's inequality, for each $\alpha > 0$, we have

$$P\left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \gamma_1^{\varepsilon,\theta} + \dots + \gamma_m^{\varepsilon,\theta} - m \frac{\varepsilon^2}{4} \right| \geq \alpha\right) \leq \frac{1}{\alpha^2} \sum_{m=1}^{\lfloor \frac{4}{\varepsilon^2} \rfloor} \text{Var}(\gamma_m^{\varepsilon,\theta}) \leq \frac{\varepsilon^2}{4\alpha^2}.$$

In order to deal with the second limit (3.4.5), we can use the decomposition

$$\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m \sigma_j^{\varepsilon, \theta} - m \frac{\varepsilon^2}{4} \right| \leq \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m (\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta}) \right| + \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m \alpha_j^{\varepsilon, \theta} - m \frac{\varepsilon^2}{4} \right|, \quad (3.4.6)$$

where $\alpha_j^{\varepsilon, \theta} := \mathbb{E}(\sigma_j^{\varepsilon, \theta} | \mathcal{B}) = \frac{\varepsilon^2}{2} [\cos(b_{j-1}\theta)]^2$. The second term on the right side of the last inequality can be written as:

$$\begin{aligned} \left| \sum_{j=1}^m \alpha_j^{\varepsilon, \theta} - m \frac{\varepsilon^2}{4} \right| &= \left| \sum_{j=1}^m \frac{\varepsilon^2}{2} [\cos(b_{j-1}\theta)]^2 - m \frac{\varepsilon^2}{4} \right| \\ &= \frac{\varepsilon^2}{4} \left| \sum_{j=1}^m [1 + \cos(2b_{j-1}\theta)] - m \right| \\ &= \frac{\varepsilon^2}{4} \left| \sum_{j=1}^m \cos(2b_{j-1}\theta) \right|, \end{aligned} \quad (3.4.7)$$

For a fixed m , consider the random variable $\hat{n} := b_{m-1} - 1$. Then, fixed m , notice that

$$\sum_{j=1}^m \cos(2b_{j-1}\theta) = \sum_{\substack{0 \leq k \leq \hat{n} \\ B_m}} (T_k \cos(2k\theta) + Z_{\hat{n}+1} \cos[2(\hat{n} + 1)\theta]). \quad (3.4.8)$$

where $B_m := \{k \in \{0, \dots, \hat{n}\} \text{ s.t. } T_0 + \dots + T_{\hat{n}} < m, T_0 + \dots + T_{\hat{n}+1} \geq m\}$, T_k are independent identically distributed random variables with $T_k \sim \text{Geom}(\frac{1}{2})$ for each $k = 0, 1, 2, \dots$, that is $P(T_k = j) = 2^{-j}$ for $j \geq 1$ and $\mathbb{E}(T_k) = \text{Var}(T_k) = 2$, and $0 \leq Z_{\hat{n}+1} \leq T_{\hat{n}+1}$. Hence, putting together (3.4.6), (3.4.7) and (3.4.8), it follows that

$$\begin{aligned} &\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m \sigma_j^{\varepsilon, \theta} - m \frac{\varepsilon^2}{4} \right| \\ &\leq \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m (\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta}) \right| + \max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \frac{\varepsilon^2}{4} \left| \sum_{\substack{0 \leq k \leq \hat{n} \\ B_m}} T_k \cos(2k\theta) \right| \\ &\quad + \max_{\substack{1 \leq m \leq \frac{4}{\varepsilon^2} \\ B_m}} \frac{\varepsilon^2}{4} \left| Z_{\hat{n}+1} \cos[2(\hat{n} + 1)\theta] \right| \\ &:= L_1^\varepsilon + L_2^\varepsilon + L_3^\varepsilon, \end{aligned}$$

and reduces the proof of (3.4.5) to check that $\lim_{\varepsilon \rightarrow 0} (L_1^\varepsilon + L_2^\varepsilon + L_3^\varepsilon) = 0$ *a.s.*

Step 3: Study of L_1^ε . Let

$$M_n := \sum_{j=1}^n (\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta}).$$

Let \mathcal{B}_n denote the σ -algebra generated by $\{\mathcal{B}, \sigma_k^{\varepsilon, \theta}; k \leq n\}$. Clearly, M_n is \mathcal{B}_n -measurable and, since $\mathbb{E}(\sigma_n^{\varepsilon, \theta} - \alpha_n^{\varepsilon, \theta} | \mathcal{B}) = 0$, $\mathbb{E}(M_n | \mathcal{B}_{n-1}) = M_{n-1}$. So $|M_n|$ is a submartingale. By Doob's martingale inequality (Lemma 3.2.5), for each $\alpha > 0$

$$P \left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} |M_m| \geq \alpha \right) \leq \frac{1}{\alpha^2} \mathbb{E} \left(|M_{\lfloor \frac{4}{\varepsilon^2} \rfloor}|^2 \right), \quad (3.4.9)$$

where $[v]$ is the integer part of v . On one hand, fixed b_j 's, $\{\sigma_j^{\varepsilon, \theta}\}$'s are independent, and so, for $j \neq k$,

$$\mathbb{E}[(\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta})(\sigma_k^{\varepsilon, \theta} - \alpha_k^{\varepsilon, \theta})] = \mathbb{E}[\mathbb{E}(\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta} | \mathcal{B}) \mathbb{E}(\sigma_k^{\varepsilon, \theta} - \alpha_k^{\varepsilon, \theta} | \mathcal{B})] = 0. \quad (3.4.10)$$

On the other hand

$$\mathbb{E}[\sigma_j^{\varepsilon, \theta} \alpha_j^{\varepsilon, \theta}] = \mathbb{E}[\mathbb{E}(\sigma_j^{\varepsilon, \theta} \alpha_j^{\varepsilon, \theta} | \mathcal{B})] = \mathbb{E}[\alpha_j^{\varepsilon, \theta} \mathbb{E}(\sigma_j^{\varepsilon, \theta} | \mathcal{B})] = \mathbb{E}[(\alpha_j^{\varepsilon, \theta})^2]. \quad (3.4.11)$$

Using (3.4.10) and (3.4.11), we get that

$$\begin{aligned} \mathbb{E} \left(|M_{\lfloor \frac{4}{\varepsilon^2} \rfloor}|^2 \right) &= \sum_{j=1}^{\lfloor \frac{4}{\varepsilon^2} \rfloor} \mathbb{E}[(\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta})^2] + 2 \sum_{j=1}^{\lfloor \frac{4}{\varepsilon^2} \rfloor - 1} \sum_{k=j+1}^{\lfloor \frac{4}{\varepsilon^2} \rfloor} \mathbb{E}[(\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta})(\sigma_k^{\varepsilon, \theta} - \alpha_k^{\varepsilon, \theta})] \\ &= \sum_{j=1}^{\lfloor \frac{4}{\varepsilon^2} \rfloor} \mathbb{E}[(\sigma_j^{\varepsilon, \theta})^2] - \sum_{j=1}^{\lfloor \frac{4}{\varepsilon^2} \rfloor} \mathbb{E}[(\alpha_j^{\varepsilon, \theta})^2]. \end{aligned} \quad (3.4.12)$$

Recalling that, by Skorokhod's theorem (Theorem 3.2.2), there exists a positive constant C_1 such that

$$\mathbb{E}[(\sigma_j^{\varepsilon, \theta})^2 | \mathcal{B}] \leq C_1 \mathbb{E}[(\xi_j^{\varepsilon, \theta})^4 | \mathcal{B}] = C_1 4! \left(\frac{\varepsilon}{2} \right)^4 [\cos(b_{j-1} \theta)]^4,$$

Equation (3.4.12) can be bounded by

$$\begin{aligned} \mathbb{E} \left(|M_{\lfloor \frac{4}{\varepsilon^2} \rfloor}|^2 \right) &\leq C_1 4! \left(\frac{\varepsilon}{2} \right)^4 \sum_{j=1}^{\lfloor \frac{4}{\varepsilon^2} \rfloor} \mathbb{E}[\cos(b_{j-1} \theta)^4] - \left(\frac{\varepsilon^2}{2} \right)^2 \sum_{j=1}^{\lfloor \frac{4}{\varepsilon^2} \rfloor} \mathbb{E}[\cos(b_{j-1} \theta)^4] \\ &= C \varepsilon^4 \sum_{j=1}^{\lfloor \frac{4}{\varepsilon^2} \rfloor} \mathbb{E}[\cos(b_{j-1} \theta)^4] \\ &\leq 4C \varepsilon^2, \end{aligned}$$

where C is a positive constant. So, from (3.4.9) we obtain that, for any α ,

$$P \left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} |M_m| \geq \alpha \right) \leq \frac{4C \varepsilon^2}{\alpha^2},$$

and by Borel-Cantelli lemma it follows that $\lim_{\varepsilon \rightarrow 0} L_1^\varepsilon = 0$ *a.s.*

Step 4: Study of L_2^ε . Since $\hat{n} \leq m - 1$, we have

$$\begin{aligned} L_2^\varepsilon &\leq \max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{4} \left| \sum_{k=0}^n T_k \cos(2k\theta) \right| \\ &\leq \max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{4} \left| \sum_{k=0}^n (T_k - 2) \cos(2k\theta) \right| + \max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{2} \left| \sum_{k=0}^n \cos(2k\theta) \right| \\ &:= L_{21}^\varepsilon + L_{22}^\varepsilon. \end{aligned}$$

We first prove that L_{21}^ε vanishes when ε goes to 0. Let \mathcal{F}^n denote the σ -algebra generated by T_k for $k \leq n$. Define

$$M'_n := \frac{\varepsilon^2}{4} \sum_{k=0}^n (T_k - 2) \cos(2k\theta).$$

It is easy to see that M'_n is a martingale. By Doob's martingale inequality (Lemma 3.2.5), for each $\alpha > 0$,

$$\begin{aligned} P \left(\max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} |M'_n| > \alpha \right) &\leq \frac{1}{\alpha^2} \mathbb{E} \left(|M'_{[\frac{4}{\varepsilon^2}] - 1}|^2 \right) \\ &= \frac{1}{\alpha^2} \mathbb{E} \left(\frac{\varepsilon^4}{16} \left| \sum_{k=0}^{[\frac{4}{\varepsilon^2}] - 1} (T_k - 2) \cos(2k\theta) \right|^2 \right) \\ &= \frac{1}{\alpha^2} \frac{\varepsilon^4}{16} \sum_{k=0}^{[\frac{4}{\varepsilon^2}] - 1} \mathbb{E} [(T_k - 2)^2] \cos^2(2k\theta) \\ &\leq \frac{1}{\alpha^2} \frac{\varepsilon^4}{16} \frac{8}{\varepsilon^2} = \frac{\varepsilon^2}{2\alpha^2}, \end{aligned}$$

where we have used that $(T_k - 2)$'s are independent and centered. Therefore, by Borell-Cantelli lemma $\lim_{\varepsilon \rightarrow 0} L_{21}^\varepsilon = 0$ *a.s.*

On the other hand, since for any n , we get that

$$\begin{aligned} \sum_{k=0}^n \cos(2k\theta) &= \frac{1}{2} \left(\sum_{k=0}^n e^{i2k\theta} + \sum_{k=0}^n e^{-i2k\theta} \right) \\ &= \frac{1}{2} \left(\frac{1 - e^{i2(n+1)\theta}}{1 - e^{i2\theta}} + \frac{1 - e^{-i2(n+1)\theta}}{1 - e^{-i2\theta}} \right) \\ &= \frac{1}{2} \left(1 + \frac{-\cos(2(n+1)\theta) + \cos(2n\theta)}{1 - \cos(2\theta)} \right). \end{aligned}$$

Thus,

$$\left| \sum_{k=0}^n \cos(2k\theta) \right| \leq \frac{1}{2} \left(1 + \frac{2}{1 - \cos(2\theta)} \right), \quad (3.4.13)$$

and it yields that $\lim_{\varepsilon \rightarrow 0} L_{22}^\varepsilon = 0$ a.s.

Step 5: Study of L_3^ε . Since $\hat{n} \leq m - 1$ and $Z_{\hat{n}+1} \leq T_{\hat{n}+1}$,

$$L_3^\varepsilon \leq \max_{\substack{1 \leq m \leq \frac{4}{\varepsilon^2} \\ B_m}} \frac{\varepsilon^2}{4} T_{\hat{n}+1} \leq \max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{4} T_{n+1}. \quad (3.4.14)$$

Thus, it is sufficient to see that

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq n} \frac{T_k}{n} = 0 \quad \text{a.s.}$$

It is enough to observe that

$$\left(\max_{1 \leq k \leq n} \frac{T_k}{n} \right)^2 \leq \frac{1}{n} \frac{T_1^2 + \dots + T_n^2}{n}$$

and apply the strong law of large numbers.

Thus, the proof is completed. □

3.5 Rate of convergence

This section is devoted to prove the rate of convergence of the processes $Z_\varepsilon^\theta(t)$. Before stating our result, for completeness we recall a technical lemma of [Gorostiza and Griego, 1980, (page 298)]:

Lemma 3.5.1. *Let $F(k, n)$ = number of ways of putting k balls into n boxes so that no box contains exactly one ball, i.e.,*

$$F(k, n) = \sum_{\substack{\alpha_1 + \dots + \alpha_n = k \\ \alpha_i \neq 1 \forall i}} \frac{k!}{\alpha_1! \dots \alpha_n!}.$$

Then,

$$F(k, n) \leq 2^k k! n^{k/2}$$

for $k \leq 2 + \log 4 / \log[1 + 2(1/n - 1/n^2)^{1/2}]$.

A rate of convergence of the processes $Z_\varepsilon^\theta(t)$ is given in the following theorem:

Theorem 3.5.2. *For all $q > 0$,*

$$P \left(\max_{0 \leq t \leq 1} |Z_\varepsilon^\theta(t) - Z(t)| > \alpha^* \varepsilon^{\frac{1}{2}} \left(\log \frac{1}{\varepsilon} \right)^{\frac{5}{2}} \right) = o(\varepsilon^q), \quad \text{as } \varepsilon \rightarrow 0$$

where α^* is a positive constant depending on q .

Proof. Although the proof follows the structure of part *b*) of Theorem 1 in [Gorostiza and Griego, 1980], some terms appearing have to be computed in a new way. To prove the theorem it is sufficient to check that, for any $q > 0$,

$$P \left(\max_{0 \leq t \leq 1} |X_\varepsilon^\theta(t) - X(t)| > \alpha \varepsilon^{\frac{1}{2}} \left(\log \frac{1}{\varepsilon} \right)^{\frac{5}{2}} \right) = o(\varepsilon^q) \quad \text{as } \varepsilon \rightarrow 0$$

and

$$P \left(\max_{0 \leq t \leq 1} |Y_\varepsilon^\theta(t) - Y(t)| > \alpha' \varepsilon^{\frac{1}{2}} \left(\log \frac{1}{\varepsilon} \right)^{\frac{5}{2}} \right) = o(\varepsilon^q) \quad \text{as } \varepsilon \rightarrow 0$$

where α and α' are two positive constants depending on q . We will analyze the rate of convergence for the real part. The results for the imaginary part can be obtained by similar computations.

Recall that $\gamma_0^{\varepsilon, \theta} \equiv \sigma_0^{\varepsilon, \theta} \equiv 0$ and define

$$\Gamma_m^{\varepsilon, \theta} = \sum_{j=0}^m \gamma_j^{\varepsilon, \theta} \quad \text{and} \quad \Lambda_m^{\varepsilon, \theta} = \sum_{j=0}^m \sigma_j^{\varepsilon, \theta}.$$

Set

$$J^\varepsilon \equiv \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{0 \leq s \leq \gamma_{m+1}^{\varepsilon, \theta}} |X_\varepsilon^\theta(\Gamma_m^{\varepsilon, \theta} + s) - X(\Gamma_m^{\varepsilon, \theta} + s)|.$$

Since X_ε^θ is piecewise linear and using the definition of $\gamma_m^{\varepsilon, \theta}$, notice that

$$\begin{aligned} X_\varepsilon^\theta(\Gamma_m^{\varepsilon, \theta} + s) &= X_\varepsilon^\theta(\Gamma_m^{\varepsilon, \theta}) + \frac{X_\varepsilon^\theta(\Gamma_{m+1}^{\varepsilon, \theta}) - X_\varepsilon^\theta(\Gamma_m^{\varepsilon, \theta})}{\Gamma_{m+1}^{\varepsilon, \theta} - \Gamma_m^{\varepsilon, \theta}} s \\ &= X(\Lambda_m^{\varepsilon, \theta}) + \frac{X(\Lambda_{m+1}^{\varepsilon, \theta}) - X(\Lambda_m^{\varepsilon, \theta})}{\gamma_{m+1}^{\varepsilon, \theta}} s \\ &= X(\Lambda_m^{\varepsilon, \theta}) + |\beta_{m+1}^{\varepsilon, \theta}| \cdot \operatorname{sgn} \left(X(\Lambda_{m+1}^{\varepsilon, \theta}) - X(\Lambda_m^{\varepsilon, \theta}) \right) s. \end{aligned}$$

Thus,

$$\begin{aligned} J^\varepsilon &\leq \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} |X(\Lambda_m^{\varepsilon, \theta}) - X(\Gamma_m^{\varepsilon, \theta})| + \max_{1 \leq m \leq \frac{4}{\varepsilon^2} + 1} |\beta_m^{\varepsilon, \theta}| \gamma_m^{\varepsilon, \theta} \\ &\quad + \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{0 \leq s \leq \gamma_{m+1}^{\varepsilon, \theta}} |X(\Gamma_m^{\varepsilon, \theta}) - X(\Gamma_m^{\varepsilon, \theta} + s)| \\ &\leq \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| X(\Lambda_m^{\varepsilon, \theta}) - X\left(\frac{m\varepsilon^2}{4}\right) \right| + \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| X(\Gamma_m^{\varepsilon, \theta}) - X\left(\frac{m\varepsilon^2}{4}\right) \right| \\ &\quad + \max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{0 \leq s \leq \gamma_{m+1}^{\varepsilon, \theta}} |X(\Gamma_m^{\varepsilon, \theta}) - X(\Gamma_m^{\varepsilon, \theta} + s)| + \max_{1 \leq m \leq \frac{4}{\varepsilon^2} + 1} |\beta_m^{\varepsilon, \theta}| \gamma_m^{\varepsilon, \theta} \\ &:= J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon + J_4^\varepsilon, \end{aligned}$$

and for any $a_\varepsilon > 0$,

$$P(J^\varepsilon > a_\varepsilon) \leq \sum_{j=1}^4 P\left(J_j^\varepsilon > \frac{a_\varepsilon}{4}\right) := I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon.$$

We will study the four terms separately.

1. *Study of the term I_4^ε .* Since $\beta_m^{\varepsilon,\theta} = \frac{2}{\varepsilon} \cos(b_{m-1}\theta)$ and $\gamma_m^{\varepsilon,\theta}$'s are independent exponentially distributed variables with parameter $\frac{4}{\varepsilon^2}$, we get

$$\begin{aligned} I_4^\varepsilon &= P\left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2}+1} |\beta_m^{\varepsilon,\theta}| \gamma_m^{\varepsilon,\theta} > \frac{a_\varepsilon}{4}\right) \\ &\leq P\left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2}+1} \gamma_m^{\varepsilon,\theta} > \frac{a_\varepsilon \varepsilon}{8}\right) \\ &= 1 - P\left(\gamma_m^{\varepsilon,\theta} \leq \frac{a_\varepsilon \varepsilon}{8}\right)^{\frac{4}{\varepsilon^2}+1} \\ &= 1 - \left(1 - e^{-\frac{a_\varepsilon}{2\varepsilon}}\right)^{\frac{4}{\varepsilon^2}+1}. \end{aligned}$$

We prove that

$$\lim_{\varepsilon \rightarrow 0} \frac{1 - \left(1 - e^{-\frac{a_\varepsilon}{2\varepsilon}}\right)^{\frac{4}{\varepsilon^2}+1}}{\varepsilon^q} = 0 \quad \text{a.s.} \quad (3.5.1)$$

for a_ε of the type $\alpha \varepsilon^{\frac{1}{2}} \left(\log \frac{1}{\varepsilon}\right)^\beta$, where α and β are positive arbitrary fixed constants. If f and g are two functions such that $f(x) \leq g(x)$, then

$$1 - \left(1 - e^{-f(x)}\right)^{\frac{4}{\varepsilon^2}+1} \geq 1 - \left(1 - e^{-g(x)}\right)^{\frac{4}{\varepsilon^2}+1}.$$

Given that, for small ε , $\frac{a_\varepsilon}{2\varepsilon} \geq \alpha^* \varepsilon^{-\frac{1}{2}}$ with $\alpha^* = \frac{\alpha}{2}$, we only have to verify that

$$\lim_{\varepsilon \rightarrow 0} \frac{1 - \left(1 - e^{-\alpha^* \varepsilon^{-\frac{1}{2}}}\right)^{\frac{4}{\varepsilon^2}+1}}{\varepsilon^q} = 0 \quad \text{a.s.}$$

Applying L'Hopital's rule we have

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} \frac{1 - \left(1 - e^{-\alpha^* \varepsilon^{-\frac{1}{2}}}\right)^{\frac{4}{\varepsilon^2}+1}}{\varepsilon^q} \\ &= \frac{1}{q} \lim_{\varepsilon \rightarrow 0} \left(1 - e^{-\alpha^* \varepsilon^{-\frac{1}{2}}}\right)^{\frac{4}{\varepsilon^2}+1} \left[\frac{\alpha^* \left(\frac{4}{\varepsilon^2} + 1\right) \varepsilon^{-q-\frac{1}{2}}}{2 \left(e^{\alpha^* \varepsilon^{-\frac{1}{2}}} - 1\right)} + 8 \varepsilon^{-q-2} \log \left(1 - e^{-\alpha^* \varepsilon^{-\frac{1}{2}}}\right) \right]. \end{aligned}$$

On one hand, applying again L'Hopital's rule we obtain

$$\lim_{\varepsilon \rightarrow 0} \log \left[\left(1 - e^{-\alpha^* \varepsilon^{-\frac{1}{2}}}\right)^{\frac{4}{\varepsilon^2}+1} \right] = 0$$

and then

$$\lim_{\varepsilon \rightarrow 0} (1 - e^{-\alpha^* \varepsilon^{-\frac{1}{2}}})^{\frac{4}{\varepsilon^2} + 1} = 1.$$

On the other hand, if k is a positive constant,

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-k}}{1 - e^{\alpha^* \varepsilon^{-\frac{1}{2}}}} = 0$$

and, by L'Hopital's rule,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-k} \log(1 - e^{\alpha^* \varepsilon^{-\frac{1}{2}}}) = 0.$$

Therefore, the limit (3.5.1) is verified and so $I_4^\varepsilon = o(\varepsilon^q)$. Particularly, the result is true for a_ε of the type $\alpha \varepsilon^{\frac{1}{2}} (\log \frac{1}{\varepsilon})^\beta$ with $\beta = \frac{5}{2}$.

2. *Study of the term I_1^ε .* Recall the random variables we defined in the previous Section:

- $\alpha_j^{\varepsilon, \theta} = \mathbb{E}(\sigma_j^{\varepsilon, \theta} | \mathcal{B}) = \frac{\varepsilon^2}{2} [\cos(b_{j-1} \theta)]^2,$
- $\hat{n} := b_{m-1} - 1,$
- T_k are independent identically distributed random variables with $T_k \sim \text{Geom}(\frac{1}{2})$ for each $k = 0, 1, 2, \dots,$
- $0 \leq Z_{\hat{n}+1} \leq T_{\hat{n}+1},$
- $B_m := \{k \in \{0, \dots, \hat{n}\} \text{ s.t. } T_0 + \dots + T_{\hat{n}} < m, T_0 + \dots + T_{\hat{n}+1} \geq m\}.$

Let $\delta_\varepsilon > 0$. From the definition of I_1^ε we have

$$\begin{aligned} I_1^\varepsilon \leq & P \left(\max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{|s| \leq \delta_\varepsilon} \left| X \left(\frac{m\varepsilon^2}{4} + s \right) - X \left(\frac{m\varepsilon^2}{4} \right) \right| > \frac{a_\varepsilon}{4} \right) \\ & + P \left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \Lambda_m^{\varepsilon, \theta} - \frac{m\varepsilon^2}{4} \right| > \delta_\varepsilon \right). \end{aligned}$$

Using the same decomposition that in the previous Section (see (3.4.7) and (3.4.8)),

we can write

$$\begin{aligned}
 I_1^\varepsilon &\leq P \left(\max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{|s| \leq \delta_\varepsilon} \left| X \left(\frac{m\varepsilon^2}{4} + s \right) - X \left(\frac{m\varepsilon^2}{4} \right) \right| > \frac{a_\varepsilon}{4} \right) \\
 &\quad + P \left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m (\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta}) \right| > \frac{\delta_\varepsilon}{2} \right) \\
 &\quad + P \left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \frac{\varepsilon^2}{4} \left| \sum_{\substack{0 \leq k \leq \hat{n} \\ B_m}} T_k \cos(2k\theta) \right| > \frac{\delta_\varepsilon}{4} \right) \\
 &\quad + P \left(\max_{\substack{1 \leq m \leq \frac{4}{\varepsilon^2} \\ B_m}} \frac{\varepsilon^2}{4} \left| Z_{\hat{n}+1} \cos[2(\hat{n}+1)\theta] \right| > \frac{\delta_\varepsilon}{4} \right) \\
 &=: I_{11}^\varepsilon + I_{12}^\varepsilon + I_{13}^\varepsilon + I_{14}^\varepsilon,
 \end{aligned}$$

We study again the four terms separately.

2.1. Study of the term I_{12}^ε . In Section 3.4 we have seen that $|M_n| = \left| \sum_{j=1}^m (\sigma_j^{\varepsilon, \theta} - \alpha_j^{\varepsilon, \theta}) \right|$ is a submartingale, so by Doob's martingale inequality (Lemma 3.2.5),

$$\begin{aligned}
 I_{12}^\varepsilon &= P \left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=1}^m \left(\frac{4}{\varepsilon^2} \sigma_j^{\varepsilon, \theta} - 2[\cos(b_{j-1}\theta)]^2 \right) \right| > \frac{2\delta_\varepsilon}{\varepsilon^2} \right) \\
 &\leq \left(\frac{\varepsilon^2}{2\delta_\varepsilon} \right)^{2p} \mathbb{E} \left[\left(\sum_{m=1}^{\lfloor 4/\varepsilon^2 \rfloor} \left(\frac{4}{\varepsilon^2} \sigma_m^{\varepsilon, \theta} - 2[\cos(b_{m-1}\theta)]^2 \right) \right)^{2p} \right], \quad (3.5.2)
 \end{aligned}$$

for any $p \geq 1$.

Set

$$Y_m := \frac{4}{\varepsilon^2} \sigma_m^{\varepsilon, \theta} - 2[\cos(b_{m-1}\theta)]^2.$$

Using Hölder's inequality, we obtain

$$\begin{aligned}
 \mathbb{E} \left[\left(\sum_{m=1}^{\lfloor 4/\varepsilon^2 \rfloor} Y_m \right)^{2p} \right] &= \sum_{\substack{|u|=2p \\ u_m \neq 1 \forall m}} \binom{2p}{u} \mathbb{E} \left(Y_1^{u_1} \cdots Y_{\lfloor 4/\varepsilon^2 \rfloor}^{u_{\lfloor 4/\varepsilon^2 \rfloor}} \right) \\
 &\leq \sum_{\substack{|u|=2p \\ u_m \neq 1 \forall m}} \binom{2p}{u} [\mathbb{E}(Y_1^{2p})]^{u_1/2p} \cdots [\mathbb{E}(Y_{\lfloor 4/\varepsilon^2 \rfloor}^{2p})]^{u_{\lfloor 4/\varepsilon^2 \rfloor}/2p}.
 \end{aligned} \quad (3.5.3)$$

where $u = (u_1, \dots, u_{\lfloor 4/\varepsilon^2 \rfloor})$ with $|u| = u_1 + \cdots + u_{\lfloor 4/\varepsilon^2 \rfloor}$ and

$$\binom{2p}{u} = \frac{(2p)!}{u_1! \cdots u_{\lfloor 4/\varepsilon^2 \rfloor}!}.$$

Notice that in the first equality we have used that if $u_m = 1$ for any m , then $\mathbb{E}(Y_1^{u_1} \cdots Y_{[4/\varepsilon^2]}^{u_{[4/\varepsilon^2]}}) = 0$. Indeed, assume that $u_m = 1$, then

$$\begin{aligned} \mathbb{E}(Y_1^{u_1} \cdots Y_{[4/\varepsilon^2]}^{u_{[4/\varepsilon^2]}}) &= \mathbb{E}\left[\mathbb{E}(Y_1^{u_1} \cdots Y_{[4/\varepsilon^2]}^{u_{[4/\varepsilon^2]}} \mid \mathcal{B})\right] \\ &= \mathbb{E}\left[\mathbb{E}(Y_1^{u_1} \mid \mathcal{B}) \cdots \mathbb{E}(Y_{m-1}^{u_{m-1}} \mid \mathcal{B}) \mathbb{E}(Y_m \mid \mathcal{B}) \mathbb{E}(Y_{m+1}^{u_{m+1}} \mid \mathcal{B}) \cdots \mathbb{E}(Y_{[4/\varepsilon^2]}^{u_{[4/\varepsilon^2]}} \mid \mathcal{B})\right] \end{aligned}$$

that is clearly zero since we have used that fixed $\{b_j\}$'s, $\{\sigma_j\}$'s are independent and $\mathbb{E}(Y_m \mid \mathcal{B}) = 0$. On the other hand, by Skorokhod's theorem (Theorem 3.2.2), we have

$$\begin{aligned} \mathbb{E}[(\sigma_m^{\varepsilon, \theta})^{2p}] &= \mathbb{E}\left[\mathbb{E}[(\sigma_m^{\varepsilon, \theta})^{2p} \mid \mathcal{B}]\right] \\ &\leq \mathbb{E}\left[2(2p)! \mathbb{E}[(k_i \xi_m^{\varepsilon, \theta})^{4p} \mid \mathcal{B}]\right] \\ &\leq 2(2p)! \mathbb{E}\left[(4p)! \left(\frac{\varepsilon}{2}\right)^{4p} (\cos(b_{m-1}\theta))^{4p}\right] \\ &\leq 2(2p)! (4p)! \left(\frac{\varepsilon}{2}\right)^{4p}. \end{aligned}$$

So, using the inequality $|a + b|^{2p} \leq 2^{2p}(|a|^{2p} + |b|^{2p})$, we obtain

$$\begin{aligned} \mathbb{E}(Y_m^{2p}) &\leq 2^{2p} \left[\left(\frac{4}{\varepsilon^2}\right)^{2p} \mathbb{E}[(\sigma_m^{\varepsilon, \theta})^{2p}] + 2^{2p} \mathbb{E}[(\cos(b_{m-1}\theta))^{4p}] \right] \\ &\leq 2^{2p} \left[\left(\frac{4}{\varepsilon^2}\right)^{2p} 2(2p)! (4p)! \left(\frac{\varepsilon}{2}\right)^{4p} + 2^{2p} \right] \\ &\leq 2^{2p} [2(2p)! (4p)! + 2^{2p}] \\ &\leq 2^{2p+1} \cdot 2(2p)! (4p)! \\ &= 4 \cdot 2^{2p} (2p)! (4p)! \end{aligned} \tag{3.5.4}$$

Finally, Lemma 3.5.1 yields that, for $p \leq 1 + \frac{\log 2}{\log [1 + \varepsilon(1 - \varepsilon^2/4)^{\frac{1}{2}}]}$,

$$\sum_{\substack{|u|=2p \\ u_i \neq 1 \forall i}} \binom{2p}{u} \leq 2^{2p} (2p)! \left(\frac{4}{\varepsilon^2}\right)^p. \tag{3.5.5}$$

Therefore, for p as above, putting together (3.5.2), (3.5.3), (3.5.4) and (3.5.5) and applying Stirling formula, $k! = \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k} e^{\frac{a}{12k}}$, with $0 < a < 1$, we obtain

$$\begin{aligned} I_{12}^\varepsilon &\leq 4(\delta_\varepsilon)^{-2p} \varepsilon^{2p} 2^{4p} ((2p)!)^2 (4p)! \\ &\leq 4(\delta_\varepsilon)^{-2p} \varepsilon^{2p} 2^{4p} \left[\sqrt{2\pi} (2p)^{2p+\frac{1}{2}} e^{-2p} e^{\frac{a}{24p}} \right]^2 \left[\sqrt{2\pi} (4p)^{4p+\frac{1}{2}} e^{-4p} e^{\frac{a}{48p}} \right] \\ &= (\delta_\varepsilon)^{-2p} \varepsilon^{2p} 2^{16p+4} (2\pi)^{\frac{3}{2}} e^{-8p+\frac{a}{12p}+\frac{a}{48p}} p^{8p+\frac{3}{2}} \\ &\leq K_1^p (\delta_\varepsilon)^{-2p} \varepsilon^{2p} p^{8p+3/2} \end{aligned}$$

where K_1 is a constant.

Now we impose $K_1^p (\delta_\varepsilon)^{-2p} \varepsilon^{2p} p^{8p+3/2} = \varepsilon^{2q}$ and $p = \lceil \log \frac{1}{\varepsilon} \rceil$ is the integer part of $\log \frac{1}{\varepsilon}$. Observe that $p = \lceil \log \frac{1}{\varepsilon} \rceil$ fulfills the condition on p of inequality (3.5.5). We get

$$\delta_\varepsilon = K_2 \varepsilon^{1-q/\lceil \log 1/\varepsilon \rceil} \left[\log \frac{1}{\varepsilon} \right]^{4+3/(4\lceil \log 1/\varepsilon \rceil)}, \quad (3.5.6)$$

where $K_2 = \sqrt{K_1}$ is a constant. Clearly, with this δ_ε , it follows that $I_{12}^\varepsilon = o(\varepsilon^q)$.

2.2. *Study of the term I_{13}^ε .* As in Section 3.4, since $\hat{n} \leq m - 1$, we can write

$$\begin{aligned} I_{13}^\varepsilon &\leq P \left(\max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{4} \left| \sum_{k=0}^n (T_k - 2) \cos(2k\theta) \right| > \frac{\delta_\varepsilon}{8} \right) \\ &\quad + P \left(\max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{2} \left| \sum_{k=0}^n \cos(2k\theta) \right| > \frac{\delta_\varepsilon}{8} \right) \\ &\leq P \left(\max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{4} \left| \sum_{k=0}^n (T_k - 2) \cos(2k\theta) \right| > \frac{\delta_\varepsilon}{8} \right) \\ &\quad + \mathbb{1}_{\left\{ \max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{2} \left| \sum_{k=0}^n \cos(2k\theta) \right| > \frac{\delta_\varepsilon}{8} \right\}} \\ &:= I_{131}^\varepsilon + I_{132}^\varepsilon. \end{aligned}$$

We begin studying I_{132}^ε . By (3.5.6) and (3.4.13),

$$\begin{aligned} I_{132}^\varepsilon &\leq \mathbb{1}_{\left\{ \frac{\varepsilon^2}{2} K > \frac{\delta_\varepsilon}{8} \right\}} \\ &= \mathbb{1}_{\left\{ \frac{\varepsilon^2}{2} K > \frac{K_2}{8} \varepsilon^{1-q/\lceil \log 1/\varepsilon \rceil} \left[\log \frac{1}{\varepsilon} \right]^{4+3/(4\lceil \log 1/\varepsilon \rceil)} \right\}} \\ &= \mathbb{1}_{\left\{ \varepsilon^{-1-q/\lceil \log 1/\varepsilon \rceil} \left[\log \frac{1}{\varepsilon} \right]^{4+3/(4\lceil \log 1/\varepsilon \rceil)} < \frac{4K}{K_2} \right\}}, \end{aligned}$$

and clearly $I_{132}^\varepsilon = 0$ for small ε .

On the other hand, since $M'_n := \frac{\varepsilon^2}{4} \sum_{k=0}^n (T_k - 2) \cos(2k\theta)$ is a martingale, as we see in Section 3.4, by Doob's martingale inequality (Lemma 3.2.5),

$$\begin{aligned} I_{131}^\varepsilon &= P \left(\max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \left| \sum_{k=0}^n (T_k - 2) \cos(2k\theta) \right| > \frac{\delta_\varepsilon}{2\varepsilon^2} \right) \\ &\leq \left(\frac{2\varepsilon^2}{\delta_\varepsilon} \right)^{2p} \mathbb{E} \left[\left(\sum_{k=0}^{\lceil 4/\varepsilon^2 \rceil - 1} (T_k - 2) \cos(2k\theta) \right)^{2p} \right]. \end{aligned}$$

Set $U_k := (T_k - 2) \cos(2k\theta)$. Clearly, U_k 's are independent and centered random variables. Moreover, by Hölder's inequality, we have

$$\mathbb{E} \left[\left(\sum_{k=0}^{\lceil 4/\varepsilon^2 \rceil - 1} U_k \right)^{2p} \right] \leq \sum_{\substack{|u|=2p \\ u_i \neq 1 \forall i}} \binom{2p}{u} [\mathbb{E}(U_0^{2p})]^{u_0/2p} \cdots [\mathbb{E}(U_{\lceil 4/\varepsilon^2 \rceil - 1}^{2p})]^{u_{\lceil 4/\varepsilon^2 \rceil - 1}/2p}.$$

Let us recall that $T_k \sim \text{Geom}(\frac{1}{2})$. It is well-known that $T_k = [\tilde{T}_k] + 1$ where $\tilde{T}_k \sim \text{Exp}(\log 2)$. Hence

$$\mathbb{E}(T_k^{2p}) \leq \mathbb{E}[(\tilde{T}_k + 1)^{2p}] \leq 2^{2p} [\mathbb{E}(\tilde{T}_k^{2p}) + 1] \leq 2^{2p} \left[\frac{(2p)!}{(\log 2)^{2p}} + 1 \right] \leq 2(2p)!(4p)!,$$

and it follows that

$$\begin{aligned} \mathbb{E}(U_k^{2p}) &\leq 2^{2p} \left[\mathbb{E}[(T_k \cos(2k\theta))^{2p}] + 2^{2p} [\cos(2k\theta)]^{2p} \right] \\ &\leq 2^{2p} [\cos(2k\theta)]^{2p} (\mathbb{E}(T_k^{2p}) + 2^{2p}) \\ &\leq 4 \cdot 2^{2p} (2p)!(4p)! \end{aligned}$$

Some inequalities are very crude, but they are helpful since we get the same bounds that in the study of I_{12}^ε . Thus, with δ_ε as in (3.5.6) we get that $I_{131}^\varepsilon = o(\varepsilon^q)$.

2.3. Study of the term I_{14}^ε . Since T_k 's are independent identically distributed random variables with $T_k \sim \text{Geom}(\frac{1}{2})$ for each k , by (3.4.14),

$$\begin{aligned} I_{14}^\varepsilon &\leq P \left(\max_{0 \leq n \leq \frac{4}{\varepsilon^2} - 1} \frac{\varepsilon^2}{4} T_{n+1} > \frac{\delta_\varepsilon}{4} \right) = 1 - P \left(\max_{1 \leq n \leq \frac{4}{\varepsilon^2}} T_n \leq \frac{\delta_\varepsilon}{\varepsilon^2} \right) \\ &= 1 - P \left[\left(T_n \leq \frac{\delta_\varepsilon}{\varepsilon^2} \right) \right]^{4/\varepsilon^2} = 1 - \left(\sum_{k=1}^{[\delta_\varepsilon/\varepsilon^2]} \frac{1}{2^k} \right)^{4/\varepsilon^2} \\ &= 1 - \left(1 - \left(\frac{1}{2} \right)^{[\delta_\varepsilon/\varepsilon^2]} \right)^{4/\varepsilon^2} = 1 - \left(1 - e^{-\log 2 [\delta_\varepsilon/\varepsilon^2]} \right)^{4/\varepsilon^2} \\ &\leq 1 - \left(1 - e^{-\frac{a_\varepsilon}{2\varepsilon}} \right)^{4/\varepsilon^2 + 1}. \end{aligned}$$

Notice that in the last inequality we have used that $\frac{a_\varepsilon}{2\varepsilon} \leq \log 2 [\frac{\delta_\varepsilon}{\varepsilon^2}]$. The bound for I_{14}^ε is the same that we get in the study of I_4^ε . So, by the same arguments, we can conclude that $I_{14}^\varepsilon = o(\varepsilon^q)$.

2.4. Study of the term I_{11}^ε . As in Theorem 1 in [Gorostiza and Griego, 1980], for small ε and using a Doob's martingale inequality for Brownian motion (Lemma 3.2.5), we get

$$\begin{aligned} I_{11}^\varepsilon &\leq \sum_{m=0}^{[4/\varepsilon^2]} P \left(\max_{|s| \leq \delta_\varepsilon} \left| X \left(\frac{m\varepsilon^2}{4} + s \right) - X \left(\frac{m\varepsilon^2}{4} \right) \right| > \frac{a_\varepsilon}{4} \right) \\ &= \left(\frac{4}{\varepsilon^2} + 1 \right) P \left(\max_{|s| \leq \delta_\varepsilon} |X(s)| > \frac{a_\varepsilon}{4} \right) \\ &\leq \frac{32}{\varepsilon^2} P \left(\max_{0 \leq s \leq \delta_\varepsilon} X(s) > \frac{a_\varepsilon}{4} \right) \\ &\leq \frac{32}{\varepsilon^2} \exp \left(- \left(\frac{a_\varepsilon}{4} \right)^2 \frac{1}{2\delta_\varepsilon} \right) \\ &= \frac{32}{\varepsilon^2} \exp \left(- \frac{(a_\varepsilon)^2}{32\delta_\varepsilon} \right). \end{aligned}$$

Condition $\frac{32}{\varepsilon^2} \exp\left(-\frac{(a_\varepsilon)^2}{32\delta_\varepsilon}\right) \leq 32 \varepsilon^{2q}$ yields that

$$a_\varepsilon \geq K_3 \varepsilon^{1/2} \varepsilon^{-q/2 \lceil \log 1/\varepsilon \rceil} \left[\log \frac{1}{\varepsilon} \right]^{2+3/8 \lceil \log 1/\varepsilon \rceil} \left(\log \frac{1}{\varepsilon} \right)^{1/2},$$

where K_3 is a constant depending on q . Notice that, for small ε ,

$$a_\varepsilon = \alpha \varepsilon^{1/2} \left(\log \frac{1}{\varepsilon} \right)^{5/2},$$

where α is a constant that depends on q , satisfies such a condition. Thus, with δ_ε as in (3.5.6), it follows that $I_{11}^\varepsilon = o(\varepsilon^q)$.

3. *Study of the term I_2^ε .* For our $\delta_\varepsilon > 0$, we have

$$\begin{aligned} I_2^\varepsilon &\leq P \left(\max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{|s| \leq \delta_\varepsilon} \left| X\left(\frac{m\varepsilon^2}{4} + s\right) - X\left(\frac{m\varepsilon^2}{4}\right) \right| > \frac{a_\varepsilon}{4} \right) \\ &\quad + P \left(\max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| \Gamma_m^{\varepsilon, \theta} - \frac{m\varepsilon^2}{4} \right| > \delta_\varepsilon \right) \\ &=: I_{21}^\varepsilon + I_{22}^\varepsilon. \end{aligned}$$

On one hand, observe that $I_{21}^\varepsilon = I_{11}^\varepsilon$, thus $I_{21}^\varepsilon = o(\varepsilon^q)$.

On the other hand, it is easy to check that $\sum_{j=0}^m (\gamma_j^{\varepsilon, \theta} - \frac{\varepsilon^2}{4})$ is a martingale. So, applying Doob's martingale inequality (Lemma 3.2.5),

$$\begin{aligned} I_{22}^\varepsilon &= P \left(\max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| \sum_{j=0}^m \left(\frac{4}{\varepsilon^2} \gamma_j^{\varepsilon, \theta} - 1 \right) \right| > \frac{4\delta_\varepsilon}{\varepsilon^2} \right) \\ &\leq \left(\frac{\varepsilon^2}{4\delta_\varepsilon} \right)^{2p} \mathbb{E} \left[\left(\sum_{m=1}^{4/\varepsilon^2} \left(\frac{4}{\varepsilon^2} \gamma_m^{\varepsilon, \theta} - 1 \right) \right)^{2p} \right]. \end{aligned}$$

Set $V_m := \frac{4}{\varepsilon^2} \gamma_m^{\varepsilon, \theta} - 1$. Notice that V_m 's are independent and centered random variables with

$$\begin{aligned} \mathbb{E}(V_m^{2p}) &\leq 2^{2p} \left(\left(\frac{4}{\varepsilon^2} \right)^{2p} \mathbb{E}[(\gamma_m^{\varepsilon, \theta})^{2p}] + 1 \right) \\ &\leq 2^{2p} ((2p)! + 1) \\ &\leq 2^{2p+1} (2p)! \\ &\leq 4 \cdot 2^{2p} (2p)! (4p)!. \end{aligned}$$

Then, using an inequality of the type of (3.5.3) and following the same arguments that in the study of I_{12}^ε , we get that $I_{22}^\varepsilon = o(\varepsilon^q)$.

4. *Study of the term I_3^ε .* For $\delta_\varepsilon > 0$ defined in (3.5.6) and a_ε of the type $\alpha \varepsilon^{\frac{1}{2}} (\log \frac{1}{\varepsilon})^{\frac{5}{2}}$, we have

$$\begin{aligned} I_3^\varepsilon &\leq P \left(\max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{|r| \leq \delta_\varepsilon} |X(\Gamma_m^{\varepsilon, \theta}) - X(\Gamma_m^{\varepsilon, \theta} + r)| > \frac{a_\varepsilon}{4} \right) \\ &\quad + P \left(\max_{1 \leq m \leq \frac{4}{\varepsilon^2} + 1} \gamma_m^{\varepsilon, \theta} > \delta_\varepsilon \right) \\ &=: I_{31}^\varepsilon + I_{32}^\varepsilon. \end{aligned}$$

On one hand, $I_{31}^\varepsilon = o(\varepsilon^q)$ is proved in the same way as I_{11}^ε .
On the other hand,

$$I_{32}^\varepsilon = 1 - [P(\gamma_m^{\varepsilon, \theta} \leq \delta_\varepsilon)]^{(4/\varepsilon^2)+1} = 1 - \left(1 - e^{-4\delta_\varepsilon/\varepsilon^2}\right)^{(4/\varepsilon^2)+1}.$$

Thus $I_{32}^\varepsilon = o(\varepsilon^q)$, similarly as we have proved for I_4^ε .

We have checked that all the terms in our decomposition are of order ε^q . More precisely, we have proved that, for $a_\varepsilon = \alpha \varepsilon^{\frac{1}{2}} (\log \frac{1}{\varepsilon})^{\frac{5}{2}}$ and $q > 0$,

$$E := P \left(\max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \max_{0 \leq r \leq \gamma_{m+1}^{\varepsilon, \theta}} |X_\varepsilon^\theta(\Gamma_m^{\varepsilon, \theta} + r) - X(\Gamma_m^{\varepsilon, \theta} + r)| > a_\varepsilon \right) = o(\varepsilon^q).$$

Fix $0 < v < 1$. Then,

$$\begin{aligned} &P \left(\max_{0 \leq t \leq 1-v} |X_\varepsilon^\theta(t) - X(t)| > a_\varepsilon \right) \\ &\leq P \left(\max_{0 \leq t \leq 1-v} |X_\varepsilon^\theta(t) - X(t)| > a_\varepsilon, \Gamma_{4/\varepsilon^2}^{\varepsilon, \theta} \geq 1-v \right) + P \left(\Gamma_{4/\varepsilon^2}^{\varepsilon, \theta} < 1-v \right) \\ &=: E_1 + E_2. \end{aligned}$$

On one hand, since $E_1 \leq E$, we get that $E_1 = o(\varepsilon)$. On the other hand, for small ε ,

$$E_2 \leq P \left(\max_{0 \leq m \leq \frac{4}{\varepsilon^2}} \left| \Gamma_m^{\varepsilon, \theta} - \frac{m\varepsilon^2}{4} \right| > v \right) \leq I_{22}^\varepsilon.$$

Thus, $E_2 = o(\varepsilon^q)$.

We have given a rate of convergence in the interval $[0, 1-v]$, but we can extend the argument for any compact interval. So, the proof of Theorem 3.5.2 is completed. \square

3.6 The case $\theta = \pi$

In the previous sections we studied the general case when $\theta \in (0, \pi) \cup (\pi, 2\pi)$. For completeness, now we take $\theta = \pi$ and we study realizations of the process $Z_\varepsilon^\theta(t)$ in this case.

When $\theta = \pi$, the process $Z_\varepsilon^\theta(t)$ is defined by

$$Z_\varepsilon^\pi(t) = (-1)^G \varepsilon \int_0^{\frac{2t}{\varepsilon^2}} (-1)^{N'_u} du, \quad (3.6.1)$$

where $N'' = N + N'$ is the superposition of the Poisson processes N and N' and, as we claimed in Section 1.1.1, it is a Poisson process of rate 2.

With a change of variable, we obtain

$$Z_\varepsilon^\pi(t) = (-1)^G \frac{2}{\varepsilon} \int_0^t (-1)^{N''_{\frac{2u}{\varepsilon^2}}} du. \quad (3.6.2)$$

Notice that $Z_\varepsilon^\pi(t)$ is a process that switches between uniform velocity $+\frac{2}{\varepsilon}$ and $-\frac{2}{\varepsilon}$ at the jump times of a Poisson process with intensity $\frac{4}{\varepsilon^2}$.

The process $Z_\varepsilon^\pi(t)$ is a real-valued process. It is well-known that it converges in law to a standard Brownian motion. In the following result we prove that the strong convergence is also verified:

Theorem 3.6.1. *There exist realizations of the process $\{Z_\varepsilon^\pi(t), t \geq 0\}$ defined by (3.6.1) and (3.6.2) on the same probability space of a standard Brownian motion $\{W(t), t \geq 0\}$, so that we have*

$$\lim_{\varepsilon \rightarrow 0} \max_{0 \leq t \leq 1} |Z_\varepsilon^\pi(t) - W(t)| = 0 \quad a.s.$$

Proof. The proof is an adaptation of the one given by Griego, Heath and Ruiz-Moncayo in [Griego et al., 1971]. It consists of two steps.

Step 1: Definition of the processes. Let (Ω, \mathcal{F}, P) be the probability space of a standard Brownian motion $\{W(t), t \geq 0\}$. Define

1. for each $\varepsilon > 0$, $\{\xi_m^\varepsilon\}_{m \geq 1}$ a sequence of independent identically distributed random variables with an exponential distribution with parameter $\frac{8}{\varepsilon}$, independent of the Brownian motion W ,
2. $\{k_m\}_{m \geq 1}$ a sequence of independent identically distributed random variables such that $P(k_m = 1) = P(k_m = -1) = \frac{1}{2}$ for each $m \geq 1$, independent of W and $\{\xi_m^\varepsilon\}_{m \geq 1}$ for all ε .

Consider the sequence of the independent, identically distributed random variables $\{k_m \xi_m^\varepsilon\}_{m \geq 1}$. It is easy to see that

$$\begin{aligned}\mathbb{E}(k_m \xi_m^\varepsilon) &= 0 \\ \text{Var}(k_m \xi_m^\varepsilon) &= \mathbb{E}[(k_m \xi_m^\varepsilon)^2] = \frac{\varepsilon^2}{8}.\end{aligned}$$

By Skorokhod's theorem (Theorem 3.2.2), for each $\varepsilon > 0$ there exists a sequence $\{\sigma_m^\varepsilon\}_{m \geq 1}$ of nonnegative independent identically distributed random variables on (Ω, \mathcal{F}, P) so that the sequence $W(\sigma_1^\varepsilon), W(\sigma_1^\varepsilon + \sigma_2^\varepsilon), \dots$, has the same distribution as $k_1 \xi_1^\varepsilon, k_1 \xi_1^\varepsilon + k_2 \xi_2^\varepsilon, \dots$, and, for each m ,

$$\mathbb{E}(\sigma_m^\varepsilon) = \text{Var}(k_m \xi_m^\varepsilon) = \frac{\varepsilon^2}{8}.$$

For each ε , we define $\gamma_0^\varepsilon \equiv 0$ and, for each m ,

$$\gamma_m^\varepsilon = \frac{\varepsilon}{2} \left| W \left(\sum_{j=0}^m \sigma_j^\varepsilon \right) - W \left(\sum_{j=0}^{m-1} \sigma_j^\varepsilon \right) \right|$$

where $\sigma_0^\varepsilon \equiv 0$. The random variables $\{\gamma_m^\varepsilon\}_{m \geq 1}$ are independent identically distributed with exponential law of parameter $\frac{8}{\varepsilon^2}$. In fact, by Skorokhod's theorem (Theorem 3.2.2), for each m , γ_m^ε has the same distribution as

$$\frac{\varepsilon}{2} \left| \sum_{j=0}^m k_j \xi_j^\varepsilon - \sum_{j=0}^{m-1} k_j \xi_j^\varepsilon \right| = \frac{\varepsilon}{2} |k_m \xi_m^\varepsilon| = \frac{\varepsilon}{2} \xi_m^\varepsilon \sim \text{Exp} \left(\frac{8}{\varepsilon^2} \right).$$

Now, we define $\{Z_\varepsilon^\pi(t), t \geq 0\}$ as a piecewise linear process satisfying

$$Z_\varepsilon^\pi \left(\sum_{j=1}^m \gamma_j^\varepsilon \right) = W \left(\sum_{j=1}^m \sigma_j^\varepsilon \right) \quad (3.6.3)$$

and $Z_\varepsilon^\pi(0) \equiv 0$. Observe that $Z_\varepsilon^\pi(\cdot)$ has slope $+\frac{2}{\varepsilon}$ or $-\frac{2}{\varepsilon}$, as desired. In fact,

$$\frac{Z_\varepsilon \left(\sum_{j=1}^m \gamma_j^\varepsilon \right) - Z_\varepsilon \left(\sum_{j=1}^{m-1} \gamma_j^\varepsilon \right)}{\sum_{j=1}^m \gamma_j^\varepsilon - \sum_{j=1}^{m-1} \gamma_j^\varepsilon} = \frac{W \left(\sum_{j=0}^m \sigma_j^\varepsilon \right) - W \left(\sum_{j=0}^{m-1} \sigma_j^\varepsilon \right)}{\frac{\varepsilon}{2} \left| W \left(\sum_{j=0}^m \sigma_j^\varepsilon \right) - W \left(\sum_{j=0}^{m-1} \sigma_j^\varepsilon \right) \right|} = \pm \frac{2}{\varepsilon}.$$

We denote by τ_m^ε the time of the m th discontinuity of the right-hand derivative of $Z_\varepsilon^\pi(\cdot)$ and we assume that $\tau_0^\varepsilon \equiv 0$. To claim that the process Z_ε^π that we have constructed is a realization of the process defined by (3.6.1) and (3.6.2), we only have to check that, for each m , the increments $\tau_m^\varepsilon - \tau_{m-1}^\varepsilon$ are independent and identically distributed with exponential law of parameter $\frac{4}{\varepsilon^2}$. Observe that the probability that $W(\sum_{j=0}^m \sigma_j^\varepsilon) - W(\sum_{j=0}^{m-1} \sigma_j^\varepsilon)$ is positive is $\frac{1}{2}$, independently of the

past up to time $\sum_{j=0}^{m-1} \sigma_j^\varepsilon$. Therefore, we can write the random variable τ_1^ε as a partial sum of $\{\gamma_m^\varepsilon\}_{m \geq 1}$ in this way:

$$\tau_1^\varepsilon = \gamma_1^\varepsilon + \cdots + \gamma_{\widehat{N}}^\varepsilon,$$

where $P(\widehat{N} = m) = 2^{-m}$, for each m . By Lemma 3.2.3, τ_1^ε has an exponential distribution with parameter $\frac{4}{\varepsilon^2}$. In the same way, each increment $\tau_m^\varepsilon - \tau_{m-1}^\varepsilon$ is identically distributed as τ_1^ε . The increments are independent since they are sum of disjoint blocks of the γ_m^ε 's.

Thus, we obtained a realization of the process defined by (3.6.1) and (3.6.2).

Step 2: Decomposition of the convergence. Recalling that $\gamma_0^\varepsilon \equiv \sigma_0^\varepsilon \equiv 0$, by (3.6.3) and the uniform continuity of Brownian motion on $[0, 1]$, we have that, almost surely,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \max_{0 \leq t \leq 1} |Z_\varepsilon^\pi(t) - W(t)| &= \lim_{\varepsilon \rightarrow 0} \max_{0 \leq m \leq \frac{8}{\varepsilon^2}} \left| Z_\varepsilon^\pi \left(\sum_{j=1}^m \gamma_j^\varepsilon \right) - W \left(\sum_{j=1}^m \gamma_j^\varepsilon \right) \right| \\ &= \lim_{\varepsilon \rightarrow 0} \max_{0 \leq m \leq \frac{8}{\varepsilon^2}} \left| W \left(\sum_{j=1}^m \sigma_j^\varepsilon \right) - W \left(\sum_{j=1}^m \gamma_j^\varepsilon \right) \right|. \end{aligned}$$

To conclude the proof, we only have to check that

$$\lim_{\varepsilon \rightarrow 0} \max_{1 \leq m \leq \frac{8}{\varepsilon^2}} \left| \gamma_1^\varepsilon + \cdots + \gamma_m^\varepsilon - m \frac{\varepsilon^2}{8} \right| = 0 \quad a.s. \quad (3.6.4)$$

and

$$\lim_{\varepsilon \rightarrow 0} \max_{1 \leq m \leq \frac{8}{\varepsilon^2}} \left| \sigma_1^\varepsilon + \cdots + \sigma_m^\varepsilon - m \frac{\varepsilon^2}{8} \right| = 0 \quad a.s. \quad (3.6.5)$$

On one hand, by Kolmogorov's inequality (Lemma 3.2.6), for each $\alpha > 0$, we have

$$P \left(\max_{1 \leq m \leq \frac{8}{\varepsilon^2}} \left| \gamma_1^\varepsilon + \cdots + \gamma_m^\varepsilon - m \frac{\varepsilon^2}{8} \right| \geq \alpha \right) \leq \frac{1}{\alpha^2} \sum_{m=1}^{\lfloor \frac{8}{\varepsilon^2} \rfloor} \text{Var}(\gamma_m^\varepsilon) \leq \frac{\varepsilon^2}{8\alpha^2},$$

and, by Borel-Cantelli lemma, the limit (3.6.4) vanishes.

On the other hand, by Kolmogorov's inequality (Lemma 3.2.6) and Skorokhod's second embedding theorem (Theorem 3.2.4), for each $\alpha > 0$, we have

$$P \left(\max_{1 \leq m \leq \frac{8}{\varepsilon^2}} \left| \sigma_1^\varepsilon + \cdots + \sigma_m^\varepsilon - m \frac{\varepsilon^2}{8} \right| \geq \alpha \right) \leq \frac{1}{\alpha^2} \sum_{m=1}^{\lfloor \frac{8}{\varepsilon^2} \rfloor} \text{Var}(\sigma_m^\varepsilon) \leq \frac{23\varepsilon^2}{8\alpha^2},$$

and again, by Borel-Cantelli lemma, the limit (3.6.5) vanishes.

Therefore, the proof is completed. □

4

Weak symmetric integrals

Consider a symmetric probability measure ν , that is, ν is invariant with respect to the map $t \mapsto 1 - t$, and a continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$. The ν -symmetric Riemann sums for the function g of the fractional Brownian motion in the interval $[0, t]$ are given by

$$S_n^\nu(g, t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} (B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H) \int_0^1 g\left(B_{\frac{j}{n}}^H + \alpha(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H)\right) \nu(d\alpha).$$

It is proved by Gradinaru, Nourdin, Russo and Vallois in [[Gradinaru et al., 2005](#)] that the limit in probability of these ν -symmetric Riemann sums exists for values of the Hurst parameter strictly bigger than the critical value $H = (4\ell + 2)^{-1}$, where $\ell = \ell(\nu) \geq 1$ is the largest natural number satisfying

$$\int_0^1 \alpha^{2j} \nu(d\alpha) = \frac{1}{2j + 1}$$

for all $j = 0, \dots, \ell - 1$.

This chapter is devoted to establish the weak convergence, in the topology of the Skorohod space, of these ν -symmetric Riemann sums when the Hurst parameter takes the critical value $H = (4\ell + 2)^{-1}$. As a consequence, we derive a change-of-variable formula in distribution, where the correction term is a stochastic integral with respect to a Brownian motion that is independent of the fractional Brownian motion.

The chapter is organized as follows. In the following section we give the motivations that bring us to the study of weak symmetric integrals. In Section 4.2 we state the main result of the chapter. In the two sections that follow we present some preliminary lemmas and some technical results obtained using the properties of fractional Brownian motion and the Malliavin calculus. Finally, in Section 4.5 we prove the main result.

4.1 Introduction and motivations

Suppose that X is a finite quadratic variation process and $f \in \mathcal{C}^2(\mathbb{R})$. In Section 2.2, according to the definitions given by Russo and Vallois, we have defined the forward integral

$$\int_0^T f'(X_u) d^- X_u$$

and the symmetric (Stratonovich) integral

$$\int_0^T f'(X_u) d^\circ X_u$$

(see Definition 2.2.1 and Definition 2.2.2, respectively). We have also seen that the fundamental equality (2.2.4) between this two integrals and the Itô's formula (2.2.5) give the Itô-Stratonovich formula (2.2.6):

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) d^\circ X_u.$$

This formula does not apply for a fractional Brownian motion B^H with Hurst parameter $H < \frac{1}{2}$, because the quadratic variation of the process B^H is not defined, so we need to find a substitution tool. For this reason we introduce the notation of symmetric measure and symmetric integral:

Definition 4.1.1. *A probability measure ν on $[0, 1]$ is called **symmetric** if $\nu(A) = \nu(1 - A)$ for any Borel set $A \subset [0, 1]$.*

Let denote by $\ell(\nu) \geq 1$ the largest positive natural number such that

$$\int_0^1 \alpha^{2j} \nu(d\alpha) = \frac{1}{2j+1} \quad \forall j = 0, 1, \dots, \ell(\nu) - 1. \quad (4.1.1)$$

The definition of ν -integral for a probability measure is given for the first time by Yor in [Yor, 1977, page 521]:

Definition 4.1.2. *Let ν be a probability measure. Suppose that $g : \mathbb{R} \rightarrow \mathbb{R}$ is a locally bounded function. The **ν -integral** of $g(B^H)$ with respect to B^H*

$$\int_0^t g(B_u^H) d^\nu B_u^H$$

is the limit in probability, provided it exists, of the following sums:

$$\sum_{j=0}^{n-1} (B_{t_{j+1}}^H - B_{t_j}^H) \int_0^1 g\left(B_{t_j}^H + \alpha(B_{t_{j+1}}^H - B_{t_j}^H)\right) \nu(d\alpha).$$

In [Gradinaru et al., 2005, Proposition 3.5] it is proved, for example, that if ν is the Lebesgue measure on $[0, 1]$ and if $g = f'$ with $f \in \mathcal{C}^1(\mathbb{R})$, then the integral $\int_0^t g(B_u^H) d^\nu B_u^H$ exists and we have:

$$f(B_t^H) = f(B_0^H) + \int_0^t g(B_u^H) d^\nu B_u^H.$$

Actually the result is more general, since it is proved for any continuous process, not only for fractional Brownian motion.

In [Gradinaru et al., 2005], Gradinaru, Nourdin, Russo and Vallois prove that the Itô-Stratonovich formula (2.2.6) can be extended to a fractional Brownian motion with $H > \frac{1}{6}$. Moreover, if $H \leq \frac{1}{6}$, it is still possible to expand $f(B_t^H)$ through a pathwise type Itô formula. Their result is the following theorem:

Theorem 4.1.3. *1. If $H > \frac{1}{6}$ and $f \in \mathcal{C}^6(\mathbb{R})$, then the integral $\int_0^t f'(B_u^H) d^\nu B_u^H$ exists for any symmetric probability measure ν and we have*

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H) d^\nu B_u^H.$$

2. Let $r \geq 2$ be an integer. If $(2r + 1)H > \frac{1}{2}$ and $f \in \mathcal{C}^{4r+2}(\mathbb{R})$, then the integral $\int_0^t f'(B_u^H) d^\nu B_u^H$ exists for any symmetric probability measure ν verifying

$$m_{2j} := \int_0^1 \alpha^{2j} \nu(d\alpha) = \frac{1}{2j+1} \quad \forall j = 0, 1, \dots, r-1.$$

Moreover, we have

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H) d^\nu B_u^H.$$

Observe that the symmetric probability measure $\frac{1}{2}(\delta_0 + \delta_1)$ satisfies $m_{2j} = \frac{1}{2}$ for any integer $j \geq 1$. Consequently, the second part of theorem does not apply. However, by the first part, we have, for $H > \frac{1}{6}$ and $f \in \mathcal{C}^6(\mathbb{R})$,

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_u^H) d^\circ B_u^H.$$

This explains why $H = \frac{1}{6}$ is a sharp barrier for the validity of Itô-Stratonovich formula.

The proof of the second part of Theorem 4.1.3 consists first in establishing the following identity:

$$\begin{aligned} f(B_T) &= f(0) + \sum_{j=0}^{n-1} (B_{t_{j+1}}^H - B_{t_j}^H) \int_0^1 f'(B_{t_j}^h + \alpha(B_{t_{j+1}}^H - B_{t_j}^H)) \nu(d\alpha) \\ &\quad + \sum_{j=0}^{n-1} \sum_{h=\ell(\nu)}^{m-1} k_{\nu,h} f^{(2h+1)} \left(\frac{B_{t_j}^H + B_{t_{j+1}}^H}{2} \right) (B_{t_{j+1}}^H - B_{t_j}^H)^{2h+1} \\ &\quad + \sum_{j=0}^{n-1} C(B_{t_j}^H, B_{t_{j+1}}^H) (B_{t_{j+1}}^H - B_{t_j}^H)^{2m}, \end{aligned} \quad (4.1.2)$$

where $\ell(\nu)$ is defined by (4.1.1), $k_{\nu,h}$ are suitable constants, $C \in \mathcal{C}(\mathbb{R}^2)$ verifies $C(a, a) = 0$ and m is a positive natural number bigger than $\ell(\nu)$.

The proof follows providing that the last two terms converge in probability uniformly on bounded intervals (cup) to zero, for some $m > \ell(\nu)$, namely

$$\sum_{j=0}^{n-1} C(B_{t_j}^H, B_{t_{j+1}}^H) (B_{t_{j+1}}^H - B_{t_j}^H)^{2m} \xrightarrow[n \rightarrow \infty]{} 0 \quad (\text{cup})$$

and

$$\sum_{j=0}^{n-1} \sum_{h=\ell(\nu)}^{m-1} k_{\nu,h} f^{(2h+1)} \left(\frac{1}{2} (B_{t_j}^H + B_{t_{j+1}}^H) \right) (B_{t_{j+1}}^H - B_{t_j}^H)^{2h+1} \xrightarrow[n \rightarrow \infty]{} 0 \quad (\text{cup}).$$

The results obtained by Gradinaru, Nourdin, Russo and Vallois in [Gradinaru et al., 2005] bring us to a natural question: what happens when the Hurst parameter takes the critical value $H = \frac{1}{2(2\ell(\nu)+1)}$? It is shown in the same paper and also by Cheridito and Nualart in [Cheridito and Nualart, 2005] that in this case the ν -symmetric integral does not converge in probability for $f(x) = x^2$. In literature, for some particular probability measures, the ν -symmetric integral has been proved to be a limit in law of suitable sums when the Hurst parameter takes the critical value $H = \frac{1}{2(2\ell(\nu)+1)}$. Here there are some examples.

Example 4.1.4. If $\nu = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, consider the trapezoidal Riemann sums:

$$S_n^\nu(f', t) = \frac{1}{2} \sum_{j=0}^{\lfloor nt \rfloor - 1} (B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H) \left[f' \left(B_{\frac{j+1}{n}}^H \right) + f' \left(B_{\frac{j}{n}}^H \right) \right].$$

In this case $\ell(\nu) = 1$ and ν -symmetric integrals exist for $H > \frac{1}{6}$. When the Hurst parameter is the critical value $H = \frac{1}{6}$, Nourdin, Réveillac and Swanson prove in [Nourdin et al., 2010] that, if $f \in C^\infty(\mathbb{R})$,

$$S_n^\nu(f', t) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} f(B_t^H) - f(0) + K_1 \int_0^t f'''(B_s^H) dW_s,$$

where $W = \{W_t, t \geq 0\}$ is a Brownian motion independent of B^H , K_1 is a suitable constant and the convergence holds in the topology of the Skorohod space $D([0, \infty))$.

In [Harnett and Nualart, 2012], Harnett and Nualart extend these results to a general class of Gaussian processes.

Example 4.1.5. In the case $\nu = \frac{1}{6}\delta_0 + \frac{2}{3}\delta_{1/2} + \frac{1}{6}\delta_1$, consider the Simpson Riemann sums:

$$S_n^\nu(f', t) = \frac{1}{6} \sum_{j=0}^{\lfloor nt \rfloor - 1} (B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H) \left[f' \left(B_{\frac{j+1}{n}}^H \right) + 4f' \left(\frac{1}{2} B_{\frac{j+1}{n}}^H + \frac{1}{2} B_{\frac{j}{n}}^H \right) + f' \left(B_{\frac{j}{n}}^H \right) \right].$$

It is easy to check that $\ell(\nu) = 2$, so the results contained in [Gradinaru et al., 2005] state that the ν -symmetric integrals exist for $H > \frac{1}{10}$. When the Hurst parameter is the critical value $H = \frac{1}{10}$, Harnett and Nualart prove in [Harnett and Nualart, 2015] that under suitable conditions on smoothness and boundness of f and for $t \geq 0$,

$$S_n^\nu(f', t) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} f(B_t^H) - f(0) + K_2 \int_0^t f^{(5)}(B_s^H) dW_s,$$

where $W = \{W_t, t \geq 0\}$ is a Brownian motion independent of B^H , K_2 is a suitable constant and the convergence holds in the topology of the Skorohod space $D([0, \infty))$.

Example 4.1.6. If $\nu = \delta_{\frac{1}{2}}$, we deal with the midpoint Riemann sums:

$$S_n^\nu(f', t) = \frac{1}{2} \sum_{j=0}^{\lfloor nt \rfloor - 1} \left(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H \right) f' \left(\frac{B_{(j+1)/n}^H + B_{j/n}^H}{2} \right).$$

In this case the critical value of the Hurst parameter is $H = \frac{1}{4}$. The case was first studied by Nourdin and Réveillac in [Nourdin and Réveillac, 2009], who prove that, when f is sufficiently smooth,

$$S_n^\nu(f', t) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} f(B_t^H) - f(0) + K_3 \int_0^t f''(B_s^H) dW_s,$$

where $W = \{W_t, t \geq 0\}$ is a Brownian motion independent of B^H , K_3 is a suitable constant and the convergence holds in the topology of the Skorohod space $D([0, \infty))$.

These results are extended to a family of processes with fourth order local scaling properties by Burdzy and Swanson in [Burdzy and Swanson, 2010], to a family of Gaussian stochastic processes under certain conditions on the covariance function by Harnett and Nualart in [Harnett and Nualart, 2013] and to the case of a 2D fractional Brownian motion by Nourdin in [Nourdin, 2009].

Looking at these examples, we can observe that they have a pattern in common. This gave us the ideas to study the ν -symmetric integral when $H = \frac{1}{2(2\ell(\nu)+1)}$ for a generic symmetric measure ν and obtain a more general result. As we will show in the rest of the chapter, the ν -symmetric integral satisfies a change of variable formula with a correction term that is a standard Itô integral with respect to Brownian motion independent of the fractional Brownian motion B^H .

4.2 Main result

Let $B^H = \{B_t^H, t \geq 0\}$ be a fractional Brownian motion with Hurst parameter H and assume that $H < \frac{1}{2}$. Consider a continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$.

Definition 4.2.1. We say that a function $g : \mathbb{R} \rightarrow \mathbb{R}$ has **moderate growth** if there exist positive constants A, B and $\alpha < 2$ such that $|g(x)| \leq Ae^{B|x|^\alpha}$ for all $x \in \mathbb{R}$.

Suppose that ν is a symmetric probability measure on $[0, 1]$, meaning that it is invariant with respect to the map $t \mapsto 1 - t$ (see Definition 4.1.1). Recall that $\ell(\nu) \geq 1$ is the largest positive natural number such that

$$\int_0^1 \alpha^{2j} \nu(d\alpha) = \frac{1}{2j+1} \quad \forall j = 0, 1, \dots, \ell(\nu) - 1.$$

Let $t \in [0, T]$. We consider the partition $\{0 = t_0 \leq t_1 \leq \dots \leq t_{\lfloor nt \rfloor - 1} \leq t_{\lfloor nt \rfloor} = t\}$ of the interval $[0, t]$ with $t_j = \frac{j}{n}$ for $j = 0, 1, \dots, \lfloor nt \rfloor - 1$, where $n \geq 1$ is an integer and $\lfloor x \rfloor$ denotes the integer part of x for any $x \geq 0$.

Definition 4.2.2. Given a continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$, the **ν -symmetric Riemann sums** of $g(B_s^H)$ in the interval $[0, t]$ are the sums defined by

$$S_n^\nu(g, t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} (B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H) \int_0^1 g\left(B_{\frac{j}{n}}^H + \alpha(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H)\right) \nu(d\alpha).$$

For the sake of simplify, we introduce the following notation:

$$\Delta_j^n B^H = B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H. \quad (4.2.1)$$

So the ν -symmetric Riemann sums are given by

$$S_n^\nu(g, t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} \Delta_j^n B^H \int_0^1 g\left(B_{\frac{j}{n}}^H + \alpha \Delta_j^n B^H\right) \nu(d\alpha).$$

As we have seen in the previous section, when $H < \frac{1}{2}$, in [Gradinaru et al., 2005] Gradinaru, Nourdin, Russo and Vallois consider the ν -symmetric integral

$$\int_0^t g(B_u^H) d^\nu B_u^H$$

defined in Definition 4.1.2 and prove that this integral is the limit in probability of the ν -symmetric Riemann sums as n tends to infinity, namely,

$$\int_0^t g(B_s^H) d^\nu B_s^H = \lim_{n \rightarrow \infty} S_n^\nu(g, t).$$

They show that this integral exists for $g = f'$ with $f \in \mathcal{C}^{4\ell(\nu)+2}(\mathbb{R})$, if the Hurst parameter satisfies $H > \frac{1}{4\ell(\nu)+2}$. Moreover, in this case the integral $\int_0^t f'(B_s^H) d^\nu B_s^H$ satisfies the chain rule

$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H) d^\nu B_s^H.$$

As we have mentioned above, the lower bound $\frac{1}{4\ell(\nu)+2}$ for the Hurst parameter is sharp and we are interested in studying this case. The following is the main result of this chapter. It proves that, when $H = \frac{1}{4\ell(\nu)+2}$, the ν -symmetric Riemann sums converge in distribution and a change-of-variable formula in law is derived:

Theorem 4.2.3. *Fix a symmetric probability measure ν on $[0, 1]$ with $\ell := \ell(\nu) < \infty$ defined in (4.1.1). Let $B^H = \{B_t^H, t \geq 0\}$ be a fractional Brownian motion with Hurst parameter $H = \frac{1}{4\ell+2}$. Consider a function $f \in C^{20\ell+5}(\mathbb{R})$ such that f and its derivatives up to the order $20\ell + 5$ have moderate growth. Then,*

$$S_n^\nu(f', t) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} f(B_t^H) - f(0) - c_\nu \int_0^t f^{(2\ell+1)}(B_s^H) dW_s, \quad (4.2.2)$$

where $W = \{W_t, t \geq 0\}$ is a Brownian motion independent of B^H , c_ν is a constant depending only on ν , and the convergence holds in the topology of the Skorohod space $D([0, \infty))$.

The value of the constant c_ν in (4.2.2) is $c_\nu = k_{\nu, \ell} \sigma_\ell$, where

$$k_{\nu, \ell} = \frac{1}{(2\ell)!} \left[\frac{1}{(2\ell+1)4^\ell} - \int_0^1 \left(\alpha - \frac{1}{2} \right)^{2\ell} \nu(d\alpha) \right] \quad (4.2.3)$$

and

$$\sigma_\ell^2 = \mathbb{E}[(B_1^H)^{4\ell+2}] + 2 \sum_{j=1}^{\infty} \mathbb{E} \left[(B_1^H (B_{j+1}^H - B_j^H))^{2\ell+1} \right]. \quad (4.2.4)$$

The statement of Theorem 4.2.3 can be interpreted as a change-of-variable formula in law. Indeed, although the sequence of ν -symmetric Riemann sums $S_n^\nu(f', t)$ fails in general to converge in probability and the ν -symmetric integral $\int_0^t f'(B_s^H) d^\nu B_s^H$ does not exist in the sense introduced above, this sequence converges in law and we can still call the limit (which is defined only in law) the ν -symmetric integral, and denote it by $\int_0^t f'(B_s^H) d^\nu B_s^H$. In this way, we can write

$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H) d^\nu B_s^H + c_\nu \int_0^t f^{(2\ell+1)}(B_s^H) dW_s,$$

where this formula has to be understood in the sense that the random variables $\int_0^t f'(B_s^H) d^\nu B_s^H$ and $f(B_t^H) - f(0) - c_\nu \int_0^t f^{(2\ell+1)}(B_s^H) dW_s$ have the same law.

Before of giving the proof of Theorem 4.2.3, in the following sections we prove some bounds for the fractional Brownian motion and some technical lemmas.

Throughout the chapter C_T and C will denote any positive constants depending or not on T respectively; they may change from one expression to another.

4.3 Bounds for fractional Brownian motion

Recall the definition of $\Delta_j^n B^H$ given by (4.2.1)

$$\Delta_j^n B^H = B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H$$

and define

$$\tilde{B}_{\frac{j}{n}}^H = \frac{1}{2} (B_{\frac{j}{n}}^H + B_{\frac{j+1}{n}}^H). \quad (4.3.1)$$

Moreover, set

$$\begin{aligned} \partial_{\frac{j}{n}} &= \mathbf{1}_{[\frac{j}{n}, \frac{j+1}{n}]}, \\ \varepsilon_t &= \mathbf{1}_{[0, t]} \end{aligned}$$

and

$$\tilde{\varepsilon}_{\frac{j}{n}} = \frac{1}{2} (\varepsilon_{\frac{j}{n}} + \varepsilon_{\frac{j+1}{n}}) = \frac{1}{2} (\mathbf{1}_{[0, \frac{j}{n}]} + \mathbf{1}_{[0, \frac{j+1}{n}]}).$$

The fractional Brownian motion with Hurst parameter H satisfies

$$\mathbb{E}[(\Delta_j^n B^H)^2] = \langle \partial_{\frac{j}{n}}, \partial_{\frac{j}{n}} \rangle_{\mathfrak{H}} = n^{-2H}. \quad (4.3.2)$$

Moreover, using the fact that the function $x \mapsto x^{2H}$ is concave for $H < \frac{1}{2}$, for any $t \geq 0$ and any integer $j \geq 0$, we obtain

$$|\mathbb{E}[(\Delta_j^n B^H) B_t^H]| = |\langle \partial_{\frac{j}{n}}, \varepsilon_t \rangle_{\mathfrak{H}}| \leq n^{-2H}. \quad (4.3.3)$$

The following lemma is proved in [Harnett and Nualart, 2015, Lemma 2.6].

Lemma 4.3.1. *Let $H < \frac{1}{2}$ and let $n \geq 2$ be an integer. Then, there exists a constant C not depending on T such that:*

a) *For any $t \in [0, T]$,*

$$\sum_{j=0}^{\lfloor nT \rfloor - 1} |\langle \partial_{\frac{j}{n}}, \varepsilon_t \rangle_{\mathfrak{H}}| \leq C \lfloor nT \rfloor^{2H} n^{-2H}.$$

b) *For any integers $r \geq 1$ and $0 \leq i \leq \lfloor nT \rfloor - 1$,*

$$\sum_{j=0}^{\lfloor nT \rfloor - 1} |\langle \partial_{\frac{j}{n}}, \partial_{\frac{i}{n}} \rangle_{\mathfrak{H}}^r| \leq C n^{-2rH}, \quad (4.3.4)$$

and consequently

$$\sum_{j, i=0}^{\lfloor nT \rfloor - 1} |\langle \partial_{\frac{j}{n}}, \partial_{\frac{i}{n}} \rangle_{\mathfrak{H}}^r| \leq C \lfloor nT \rfloor n^{-2rH}. \quad (4.3.5)$$

The next result provides useful estimates when we compare two partitions. Its proof is based on computing telescopic sums.

Lemma 4.3.2. *We fix two integers $n > m \geq 2$, and for any $j \geq 0$, we define $k := k(j) = \sup\{i \geq 0 : \frac{i}{m} \leq \frac{j}{n}\}$. The following inequalities hold true for some constant C_T depending only on T :*

$$\sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \langle \partial_{\frac{j}{n}}, \varepsilon_{\frac{k(j)}{m}} \rangle_{\mathfrak{H}} \right| \leq C_T m^{1-2H}, \quad (4.3.6)$$

$$\sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \langle \partial_{\frac{j}{n}}, \tilde{\varepsilon}_{\frac{j}{n}} - \varepsilon_{\frac{k(j)}{m}} \rangle_{\mathfrak{H}} \right| \leq C_T m^{1-2H} \quad (4.3.7)$$

and, for any $0 \leq i \leq \lfloor nT \rfloor - 1$,

$$\sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \langle \partial_{\frac{i}{n}}, \tilde{\varepsilon}_{\frac{i}{n}} - \varepsilon_{\frac{k(i)}{m}} \rangle_{\mathfrak{H}} \right| \leq C_T m^{-2H}. \quad (4.3.8)$$

Proof. Let us first show (4.3.6). We can write

$$\begin{aligned} \sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \langle \partial_{\frac{j}{n}}, \varepsilon_{\frac{k(j)}{m}} \rangle_{\mathfrak{H}} \right| &= \sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \mathbb{E}[(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H) B_{\frac{k(j)}{m}}^H] \right| \\ &= \frac{1}{2} \sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \left(\left(\frac{j+1}{n} \right)^{2H} - \left(\frac{j}{n} \right)^{2H} - \left| \frac{j+1}{n} - \frac{k(j)}{m} \right|^{2H} + \left| \frac{j}{n} - \frac{k(j)}{m} \right|^{2H} \right) \right| \\ &\leq \frac{1}{2} n^{-2H} \sum_{j=0}^{\lfloor nT \rfloor - 1} \left[(j+1)^{2H} - j^{2H} \right] \\ &\quad + \frac{1}{2} \sum_{j=0}^{\lfloor nT \rfloor - 1} \left[\left(\frac{j+1}{n} - \frac{k(j)}{m} \right)^{2H} - \left(\frac{j}{n} - \frac{k(j)}{m} \right)^{2H} \right]. \end{aligned}$$

The first term is a telescopic sum and it is easy to show that

$$\frac{1}{2} n^{-2H} \sum_{j=0}^{\lfloor nT \rfloor - 1} [(j+1)^{2H} - j^{2H}] = \frac{1}{2} n^{-2H} (\lfloor nT \rfloor)^{2H} \leq C_T \leq C_T m^{1-2H}.$$

For the second term, observe that, for a fixed $k = 0, \dots, \lfloor mT \rfloor + 1$, the sum of the terms for which $k(j) = k$ is telescopic and is bounded by a constant times m^{-2H} . Summing over all possible values of k , we obtain the desired bound $C_T m^{1-2H}$.

The inequality (4.3.7) is an immediate consequence of (4.3.6) and the following

easy fact:

$$\begin{aligned}
 \sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \langle \partial_{\frac{j}{n}}, \tilde{\varepsilon}_{\frac{j}{n}} \rangle_{\mathfrak{H}} \right| &= \frac{1}{2} \sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \mathbb{E} \left[(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H) (B_{\frac{j+1}{n}}^H + B_{\frac{j}{n}}^H) \right] \right| \\
 &= \frac{n^{-2H}}{2} \sum_{j=0}^{\lfloor nT \rfloor - 1} [(j+1)^{2H} - j^{2H}] \\
 &= \frac{1}{2} n^{-2H} (\lfloor nT \rfloor)^{2H} \\
 &\leq C_T \\
 &\leq C_T m^{1-2H}.
 \end{aligned}$$

Let us now proceed with the proof of (4.3.8). We can write

$$\begin{aligned}
 \sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \langle \partial_{\frac{j}{n}}, \tilde{\varepsilon}_{\frac{j}{n}} - \varepsilon_{\frac{k(i)}{m}} \rangle_{\mathfrak{H}} \right| &= \frac{1}{2} \sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \mathbb{E} \left[(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H) (B_{\frac{j}{n}}^H + B_{\frac{j+1}{n}}^H - 2B_{\frac{k(i)}{m}}^H) \right] \right| \\
 &\leq \frac{1}{2} \sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \mathbb{E} \left[(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H) (B_{\frac{j}{n}}^H - B_{\frac{k(i)}{m}}^H) \right] \right| \\
 &\quad + \frac{1}{2} \sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \mathbb{E} \left[(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H) (B_{\frac{j+1}{n}}^H - B_{\frac{k(i)}{m}}^H) \right] \right| \\
 &=: A_1 + A_2.
 \end{aligned}$$

Let us first consider the term A_1 . The main idea to estimate this term is to use the fact that the covariance between the increments $B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H$ and $B_{\frac{j}{n}}^H - B_{\frac{k(i)}{m}}^H$ is nonpositive if $j \geq i$ or $j \leq j_0$, for some index j_0 depending on i . Then the sums with $j \geq i$ or $j \leq j_0$ are telescopic and can be easily estimated. Finally, it suffices to consider the remaining summands. Proceeding in this way, we write

$$A_1 = \frac{1}{2} \sum_{j=0}^{\lfloor nT \rfloor - 1} |H_j|,$$

where

$$H_j = \left| \frac{i}{n} - \frac{j}{n} \right|^{2H} - \left| \frac{i}{n} - \frac{j+1}{n} \right|^{2H} + \left| \frac{k(i)}{m} - \frac{j+1}{n} \right|^{2H} - \left| \frac{k(i)}{m} - \frac{j}{n} \right|^{2H}.$$

Taking into account that $\frac{k(i)}{m} \leq \frac{i}{n}$, it follows that, for $j \geq i$

$$\begin{aligned} H_j &= \left(\frac{j}{n} - \frac{i}{n}\right)^{2H} - \left(\frac{j+1}{n} - \frac{i}{n}\right)^{2H} + \left(\frac{j+1}{n} - \frac{k(i)}{m}\right)^{2H} - \left(\frac{j}{n} - \frac{k(i)}{m}\right)^{2H} \\ &= 2H \int_0^{\frac{1}{n}} \left[\left(\frac{j}{n} + x - \frac{k(i)}{m}\right)^{2H-1} - \left(\frac{j}{n} + x - \frac{i}{n}\right)^{2H-1} \right] dx \leq 0. \end{aligned}$$

On the other hand, if j_0 is the largest integer $j \geq 0$ such that $\frac{j+1}{n} \leq \frac{k(i)}{m}$, then, for $j \leq j_0$,

$$\begin{aligned} H_j &= \left(\frac{i}{n} - \frac{j}{n}\right)^{2H} - \left(\frac{i}{n} - \frac{j+1}{n}\right)^{2H} + \left(\frac{k(i)}{m} - \frac{j+1}{n}\right)^{2H} - \left(\frac{k(i)}{m} - \frac{j}{n}\right)^{2H} \\ &= -2H \int_0^{\frac{1}{n}} \left[\left(\frac{k(i)}{m} - \frac{j}{n} - x\right)^{2H-1} - \left(\frac{i}{n} - \frac{j}{n} - x\right)^{2H-1} \right] dx \leq 0. \end{aligned}$$

Consider the decomposition

$$A_1 = \frac{1}{2} \left(\sum_{j=0}^{j_0} |H_j| + \sum_{j=j_0+1}^{i-1} |H_j| + \sum_{j=i}^{\lfloor nT \rfloor - 1} |H_j| \right) =: \frac{1}{2} (A_{11} + A_{12} + A_{13}).$$

For the terms A_{11} and A_{13} , we obtain, respectively

$$\begin{aligned} A_{11} &= \sum_{j=0}^{j_0} (-H_j) \\ &= \left(\frac{i}{n} - \frac{j_0+1}{n}\right)^{2H} - \left(\frac{k(i)}{m} - \frac{j_0+1}{n}\right)^{2H} - \left(\frac{i}{n}\right)^{2H} + \left(\frac{k(i)}{m}\right)^{2H} \\ &\leq 2 \left(\frac{i}{n} - \frac{k(i)}{m}\right)^{2H} \\ &\leq Cm^{-2H} \end{aligned}$$

and

$$\begin{aligned} A_{13} &= \sum_{j=i}^{\lfloor nT \rfloor - 1} (-H_j) \\ &= \left(\frac{\lfloor nT \rfloor}{n} - \frac{i}{n}\right)^{2H} - \left(\frac{\lfloor nT \rfloor}{n} - \frac{k(i)}{m}\right)^{2H} + \left(\frac{i}{n} - \frac{k(i)}{m}\right)^{2H} \\ &\leq \left(\frac{i}{n} - \frac{k(i)}{m}\right)^{2H} \\ &\leq m^{-2H}. \end{aligned}$$

Finally, for the term A_{12} , we have

$$\begin{aligned} A_{12} &\leq \sum_{j=j_0+1}^{i-1} \left| \left(\frac{i}{n} - \frac{j}{n} \right)^{2H} - \left(\frac{i}{n} - \frac{j+1}{n} \right)^{2H} \right| \\ &\quad + \sum_{j=j_0+1}^{i-1} \left| \left(\frac{j+1}{n} - \frac{k(i)}{m} \right)^{2H} - \left(\frac{j}{n} - \frac{k(i)}{m} \right)^{2H} \right| \\ &=: A_{121} + A_{122}. \end{aligned}$$

The term A_{121} is a telescopic sum which produces a contribution of the form

$$\left(\frac{i - j_0 - 1}{n} \right)^{2H} \leq C_T m^{-2H}$$

and the term A_{122} can be bounded as follows

$$\begin{aligned} A_{122} &\leq \left| \frac{j_0+2}{n} - \frac{k(i)}{m} \right|^{2H} + \left| \frac{j_0+1}{n} - \frac{k(i)}{m} \right|^{2H} \\ &\quad + \sum_{j=j_0+2}^{i-1} \left[\left(\frac{j+1}{n} - \frac{k(i)}{m} \right)^{2H} - \left(\frac{j}{n} - \frac{k(i)}{m} \right)^{2H} \right] \\ &\leq C_T m^{-2H} + \left(\frac{i}{n} - \frac{k(i)}{m} \right)^{2H} - \left(\frac{j_0+2}{n} - \frac{k(i)}{m} \right)^{2H} \\ &\leq C_T m^{-2H}. \end{aligned}$$

The term A_2 can be treated in a similar way. This completes the proof. \square

For the following result we use Malliavin calculus and we recall some definitions and identities introduced in Section 2.1.

Let $I_q(\cdot)$ be the generalized Wiener-Itô multiple stochastic integral. We will use the following lemma.

Lemma 4.3.3. *For any odd integer $r \geq 1$, we have*

$$\left(\Delta_j^n B^H \right)^r = \sum_{u=0}^{\lfloor \frac{r}{2} \rfloor} C_{r,u} n^{-2uH} I_{r-2u}(\partial_{\frac{j}{n}}^{\otimes r-2u}),$$

where $C_{r,u}$ are some integers.

Proof. By (4.3.2), we have $\|\Delta_j^n B^H\|_{L^2(\Omega)} = n^{-H}$. For any integer $q \geq 1$, let $H_q(x)$ denotes the Hermite polynomial of degree q defined by (2.1.2) in Section 2.1. Using an inductive argument coming from the relation $H_{q+1}(x) = xH_q(x) - qH_{q-1}(x)$, it follows that

$$x^r = \sum_{u=0}^{\lfloor \frac{r}{2} \rfloor} C_{r,u} H_{r-2u}(x), \quad (4.3.9)$$

where $C_{r,u}$ is an integer. Recall identity (2.1.3):

$$I_q(h^{\otimes q}) = H_q(X(h)).$$

Applying this identity to $h = n^H \partial_{\frac{i}{n}}$, that is,

$$X(h) = \frac{\Delta_j^n B^H}{\|\Delta_j^n B^H\|_{L^2(\Omega)}} = n^H \Delta_j^n B^H,$$

we can write

$$H_r(n^H \Delta_j^n B^H) = I_r(n^{rH} \partial_{\frac{i}{n}}^{\otimes r}). \quad (4.3.10)$$

Substituting (4.3.10) into (4.3.9), yields

$$n^{rH} (\Delta_j^n B^H)^r = \sum_{u=0}^{\lfloor \frac{r}{2} \rfloor} C_{r,u} I_{r-2u} (n^{(r-2u)H} \partial_{\frac{i}{n}}^{\otimes r-2u}),$$

which implies the desired result. □

4.4 Technical lemmas

This section is devoted to state and prove a couple of technical lemmas. Let first introduce the notation we will use.

Definition 4.4.1. *Let $h \in \mathbb{N}$ and assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a \mathcal{C}^{2h+1} function. We define the process*

$$\Phi_n^h = \left\{ \Phi_n^h(t) := \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(2h+1)}(\tilde{B}_{\frac{j}{n}}^H) (\Delta_j^n B^H)^{2h+1}, t \in [0, T] \right\},$$

where $\tilde{B}_{\frac{j}{n}}^H$ and $\Delta_j^n B^H$ have been defined in (4.2.1) and (4.3.1) respectively.

The following lemma gives an estimation of the fourth moment of the increments of the process Φ_n^h :

Lemma 4.4.2. *Consider the process Φ_n^h defined in Definition 4.4.1. Then, for any $0 \leq s < t \leq T$ we have*

$$\mathbb{E}[|\Phi_n^h(t) - \Phi_n^h(s)|^4] \leq C_T \sum_{N=2}^4 ([nt] - [ns])^N n^{-2NH(2h+1)},$$

where the constant C_T depends only on T .

Proof. The proof of this lemma consists in expressing the product of increments $\prod_{i=1}^4 (\Delta_{j_i}^n B^H)^{2h+1}$ as a linear combination of multiple stochastic integrals and applying the duality relationship between multiple stochastic integrals and the iterated Malliavin derivative.

For any $0 \leq s < t \leq T$ we can write

$$\mathbb{E}[|\Phi_n^h(t) - \Phi_n^h(s)|^4] = \sum_{j_1, j_2, j_3, j_4 = \lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \mathbb{E} \left[\prod_{i=1}^4 \left(f^{(2h+1)}(\tilde{B}_{\frac{j_i}{n}}^H) (\Delta_{j_i}^n B^H)^{2h+1} \right) \right].$$

By Lemma 4.3.3 we obtain

$$(\Delta_{j_i}^n B^H)^{2h+1} = \sum_{u=0}^h C_{2h+1, u} n^{-2uH} I_{2h+1-2u} \left(\partial_{\frac{j_i}{n}}^{\otimes (2h+1-2u)} \right),$$

which leads to

$$\prod_{i=1}^4 (\Delta_{j_i}^n B^H)^{2h+1} = \sum_{u_1, u_2, u_3, u_4=0}^h C_{h, \mathbf{u}} n^{-2|\mathbf{u}|H} \prod_{i=1}^4 \left(I_{2h+1-2u_i} \left(\partial_{\frac{j_i}{n}}^{\otimes (2h+1-2u_i)} \right) \right),$$

where $C_{h, \mathbf{u}}$ is a constant depending on h and the vector $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and we use the notation $|\mathbf{u}| = u_1 + u_2 + u_3 + u_4$. To simplify the notation we write $2h + 1 - u_i = v_i$ for $i = 1, 2, 3, 4$.

Recall the product formula for multiple stochastic integrals introduced in Definition 2.1.1:

$$I_p(f)I_q(g) = \sum_{z=0}^{p \wedge q} z! \binom{p}{z} \binom{q}{z} I_{p+q-2z}(f \tilde{\otimes}_z g), \quad (4.4.1)$$

where $p, q \geq 1$ and $\tilde{\otimes}_z$ is the contraction operator. This formula allows us to write

$$\prod_{i=1}^4 \left(I_{v_i} \left(\partial_{\frac{j_i}{n}}^{\otimes v_i} \right) \right) = \sum_{\alpha \in \Lambda} C_\alpha \prod_{1 \leq i < k \leq 4} \langle \partial_{\frac{j_i}{n}}, \partial_{\frac{j_k}{n}} \rangle_{\mathcal{H}}^{\alpha_{ik}} \times I_{|\mathbf{v}| - 2|\alpha|} \left(\partial_{\frac{j_1}{n}}^{\otimes v_1 - \alpha_{12} - \alpha_{13} - \alpha_{14}} \right. \\ \left. \otimes \partial_{\frac{j_2}{n}}^{\otimes v_2 - \alpha_{12} - \alpha_{23} - \alpha_{24}} \otimes \partial_{\frac{j_3}{n}}^{\otimes v_3 - \alpha_{13} - \alpha_{23} - \alpha_{34}} \otimes \partial_{\frac{j_4}{n}}^{\otimes v_4 - \alpha_{14} - \alpha_{24} - \alpha_{34}} \right),$$

where $|\mathbf{v}| = v_1 + v_2 + v_3 + v_4 = 8h + 4 - |\mathbf{u}|$, Λ is the set of all multiindices $\alpha = (\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34})$ with $\alpha_{ik} \geq 0$, such that

$$\begin{aligned} \alpha_{12} + \alpha_{13} + \alpha_{14} &\leq v_1 \\ \alpha_{12} + \alpha_{23} + \alpha_{24} &\leq v_2 \\ \alpha_{13} + \alpha_{23} + \alpha_{34} &\leq v_3 \\ \alpha_{14} + \alpha_{24} + \alpha_{34} &\leq v_4. \end{aligned}$$

For any $\mathbf{j} = (j_1, j_2, j_3, j_4)$, $\lfloor ns \rfloor \leq j_i \leq \lfloor nt \rfloor - 1$, we set

$$Y_{\mathbf{j}} = \prod_{i=1}^4 f^{(2h+1)} \left(\tilde{B}_{\frac{j_i}{n}}^H \right),$$

and

$$h_{\mathbf{j},\alpha,\mathbf{v}} = \partial_{\frac{j_1}{n}}^{\otimes v_1 - \alpha_{12} - \alpha_{13} - \alpha_{14}} \otimes \partial_{\frac{j_2}{n}}^{\otimes v_2 - \alpha_{12} - \alpha_{23} - \alpha_{24}} \otimes \partial_{\frac{j_3}{n}}^{\otimes v_3 - \alpha_{13} - \alpha_{23} - \alpha_{34}} \otimes \partial_{\frac{j_4}{n}}^{\otimes v_4 - \alpha_{14} - \alpha_{24} - \alpha_{34}}.$$

Applying the duality formula defined in (2.1.4),

$$\mathbb{E}(F\delta^q(u)) = \mathbb{E}(\langle D^q F, u \rangle_{\mathfrak{F}^{\otimes q}}),$$

to $F = Y_{\mathbf{j}}$, $u = h_{\mathbf{j},\alpha,\mathbf{v}}$ and $q = |\mathbf{v}| - 2|\alpha|$, we obtain

$$\mathbb{E}[Y_{\mathbf{j}} I_{|\mathbf{v}| - 2|\alpha|}(h_{\mathbf{j},\alpha,\mathbf{v}})] = \mathbb{E}[\langle D^{|\mathbf{v}| - 2|\alpha|} Y_{\mathbf{j}}, h_{\mathbf{j},\alpha,\mathbf{v}} \rangle_{\mathcal{H}^{\otimes |\mathbf{v}| - 2|\alpha|}}].$$

Therefore, we have shown the following formula

$$\begin{aligned} \mathbb{E}[|\Phi_n^h(t) - \Phi_n^h(s)|^4] &= \sum_{\mathbf{j}} \sum_{\mathbf{u}} C_{h,\mathbf{u}} n^{-2|\mathbf{u}|H} \sum_{\alpha \in \Lambda} C_{\alpha} \\ &\quad \times \left(\prod_{1 \leq i < k \leq 4} \langle \partial_{\frac{j_i}{n}}, \partial_{\frac{j_k}{n}} \rangle_{\mathcal{H}}^{\alpha_{ik}} \right) \mathbb{E}[\langle D^{|\mathbf{v}| - 2|\alpha|} Y_{\mathbf{j}}, h_{\mathbf{j},\alpha,\mathbf{v}} \rangle_{\mathcal{H}^{\otimes |\mathbf{v}| - 2|\alpha|}}], \end{aligned}$$

where the components of \mathbf{j} satisfy $\lfloor ns \rfloor \leq j_i \leq \lfloor nt \rfloor - 1$ and $0 \leq u_i \leq h$. Finally, the inner product $\langle D^{|\mathbf{v}| - 2|\alpha|} Y_{\mathbf{j}}, h_{\mathbf{j},\alpha,\mathbf{v}} \rangle_{\mathcal{H}^{\otimes |\mathbf{v}| - 2|\alpha|}}$ can be expressed in the form

$$\sum_{\beta \in \Gamma} \Phi_{\beta} \prod_{1 \leq i, k \leq 4} \langle \partial_{\frac{j_i}{n}}, \tilde{\varepsilon}_{\frac{j_k}{n}} \rangle_{\mathcal{H}}^{\beta_{ik}},$$

where the random variables Φ_{β} are linear combinations of products of the form $\prod_{i=1}^4 f^{(w_i)}(B_{\tilde{\varepsilon}_{\frac{j_i}{n}}}^H)$, with $2h + 1 \leq w_i \leq 2h + 1 + |\mathbf{v}| - 2|\alpha|$ and $\beta = (\beta_{ik})_{1 \leq i, k \leq 4}$ is a matrix with nonnegative entries such that

$$\begin{aligned} \sum_{k=1}^4 \beta_{1k} &= v_1 - \alpha_{12} - \alpha_{13} - \alpha_{14} \\ \sum_{k=1}^4 \beta_{2k} &= v_2 - \alpha_{12} - \alpha_{23} - \alpha_{24} \\ \sum_{k=1}^4 \beta_{3k} &= v_3 - \alpha_{13} - \alpha_{23} - \alpha_{34} \\ \sum_{k=1}^4 \beta_{4k} &= v_4 - \alpha_{14} - \alpha_{24} - \alpha_{34}. \end{aligned}$$

Notice that $|\beta| = \sum_{i,k=1}^4 \beta_{ik} = |\mathbf{v}| - 2|\alpha|$. This leads to the following estimate

$$\begin{aligned} &\mathbb{E}[|\Phi_n^h(t) - \Phi_n^h(s)|^4] \\ &\leq C_T \sum_{\mathbf{j}} \sum_{\mathbf{u}} n^{-2|\mathbf{u}|H} \sum_{\alpha \in \Lambda} \sum_{\beta \in \Gamma} \prod_{1 \leq i < k \leq 4} \left| \langle \partial_{\frac{j_i}{n}}, \partial_{\frac{j_k}{n}} \rangle_{\mathcal{H}}^{\alpha_{ik}} \right| \prod_{1 \leq i, k \leq 4} \left| \langle \partial_{\frac{j_i}{n}}, \tilde{\varepsilon}_{\frac{j_k}{n}} \rangle_{\mathcal{H}}^{\beta_{ik}} \right|. \end{aligned}$$

Consider the decomposition of the above sum as follows

$$\mathbb{E}[|\Phi_n(t) - \Phi_n(s)|^4] \leq C_T (A_n^{(1)} + A_n^{(2)} + A_n^{(3)}),$$

where $A_n^{(1)}$ contains all the terms such that at least two components of α are nonzero, $A_n^{(2)}$ contains all the terms such that one component of α is nonzero and the others vanish, and $A_n^{(3)}$ contains all the terms such that all the components of α are zero.

Step 1. Let us first estimate $A_n^{(1)}$. Without any loss of generality, we can assume that $\alpha_{12} \geq 1$ and $\alpha_{13} \geq 1$. From inequality (4.3.4) in Lemma 4.3.1 with $r = 1$, we obtain

$$\sum_{j_1=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \left| \langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_2}{n}} \rangle_{\mathcal{H}} \right| \leq Cn^{-2H} \quad (4.4.2)$$

and

$$\sum_{j_3=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \left| \langle \partial_{\frac{j_1}{n}}, \partial_{\frac{j_3}{n}} \rangle_{\mathcal{H}} \right| \leq Cn^{-2H}.$$

We estimate each of the remaining factors by n^{-2H} . In this way, we obtain a bound of the form

$$A_n^{(1)} \leq C(\lfloor nt \rfloor - \lfloor ns \rfloor)^2 n^{-2H(|\mathbf{u}| + |\alpha| + |\beta|)}.$$

Taking into account that $|\alpha| \leq \frac{1}{2}|\mathbf{v}|$, we can write

$$\begin{aligned} |\mathbf{u}| + |\alpha| + |\beta| &= |\mathbf{u}| + |\alpha| + |\mathbf{v}| - 2|\alpha| \\ &= |\mathbf{u}| + |\mathbf{v}| - |\alpha| \\ &\geq |\mathbf{u}| + \frac{|\mathbf{v}|}{2} \\ &= 4h + 2 - \frac{|\mathbf{u}|}{2} \\ &\geq 4h + 2, \end{aligned}$$

and, as a consequence,

$$A_n^{(1)} \leq C(\lfloor nt \rfloor - \lfloor ns \rfloor)^2 n^{-4H(2h+1)}. \quad (4.4.3)$$

Step 2. For the term $A_n^{(2)}$, we can assume that $\alpha_{12} \geq 1$ and all the other components of α vanish. In this case, we still have the inequality (4.4.2). Then, we estimate each of the remaining factors by n^{-2H} . In this way, we obtain a bound of the form

$$A_n^{(2)} \leq C(\lfloor nt \rfloor - \lfloor ns \rfloor)^3 n^{-2H(|\mathbf{u}| + |\alpha| + |\beta|)}.$$

Taking into account that $|\alpha| = \alpha_{12} \leq v_1 = 2h + 1 - u_1 \leq 2h + 1$, we can write

$$\begin{aligned} |\mathbf{u}| + |\alpha| + |\beta| &= |\mathbf{u}| + |\alpha| + |\mathbf{v}| - 2|\alpha| \\ &= |\mathbf{u}| + |\mathbf{v}| - |\alpha| \\ &\geq |\mathbf{u}| + |\mathbf{v}| - 2h - 1 \\ &= 6h + 3, \end{aligned}$$

and, as a consequence, we obtain

$$A_n^{(2)} \leq C([\!|nt|] - [\!|ns|])^3 n^{-6H(2h+1)}. \quad (4.4.4)$$

Step 3. For the last term, estimating all terms by n^{-2H} , we get

$$A_n^{(3)} \leq C([\!|nt|] - [\!|ns|])^4 n^{-2H(|\mathbf{u}|+|\beta|)}.$$

We have

$$|\mathbf{u}| + |\beta| = |\mathbf{u}| + |\mathbf{v}| = 8h + 4,$$

and, as a consequence, we obtain

$$A_n^{(3)} \leq C([\!|nt|] - [\!|ns|])^4 n^{-8H(2h+1)}. \quad (4.4.5)$$

In conclusion, from (4.4.3), (4.4.4) and (4.4.5), we obtain the desired estimate. This completes the proof of the lemma. \square

The following lemma is a variation of [Harnett and Nualart, 2015, Lemma 3.2]. Its proof is based on the techniques of Malliavin calculus and the application of the small blocks/big blocks technique.

Let $n > m \geq 2$ be two integers. As in Section 4.3, for any $j \geq 0$, we define

$$k := k(j) = \sup \left\{ i \geq 0 : \frac{i}{m} \leq \frac{j}{n} \right\}.$$

Lemma 4.4.3. *Let $r = 1, 3, 5, \dots$ and $n > m \geq 2$ be two integers. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a \mathcal{C}^{2r} function such that ϕ and all derivatives up to order $2r$ have moderate growth. Then, for any $q > 2$ and any $T > 0$,*

$$\sup_{t \in [0, T]} \mathbb{E} \left[\left(\sum_{j=0}^{[\!|nt|]-1} \left(\phi(\tilde{B}_{\frac{j}{n}}^H) - \phi(B_{\frac{k(j)}{m}}^H) \right) (\Delta_j^n B^H)^r \right)^2 \right] \leq C_T \Gamma_{m,n} n^{1-2rH},$$

where C_T is a positive constant depending on q, r, H and T , and

$$\begin{aligned} \Gamma_{m,n} &:= \sup_{0 \leq w \leq 2r} \sup_{0 \leq j \leq [nT]-1} \left\| \phi^{(w)}(\tilde{B}_{\frac{j}{n}}^H) - \phi^{(w)}(B_{\frac{k(j)}{m}}^H) \right\|_{L^q(\Omega)}^2 \\ &+ \sup_{0 \leq w \leq 2r} \sup_{0 \leq j \leq [nT]-1} \left\| \phi^{(w)}(\tilde{B}_{\frac{j}{n}}^H) \right\|_{L^2(\Omega)}^2 (m^{-2H} + n^{2H-1} m^{2-4H}) \\ &+ \sup_{0 \leq w \leq 2r} \sup_{0 \leq i, j \leq [nT]-1} \left\| \phi^{(w)}(\tilde{B}_{\frac{i}{n}}^H) \right\|_{L^2(\Omega)} \left\| \phi^{(w)}(\tilde{B}_{\frac{j}{n}}^H) - \phi^{(w)}(B_{\frac{k(j)}{m}}^H) \right\|_{L^2(\Omega)} \\ &\quad \times (1 + n^{2H-1} m^{2-4H}). \end{aligned}$$

Proof. The proof is based in the methodology used to show Lemma 3.2 in [Harnett and Nualart, 2015]. To simplify notation, let $Y_j(\phi) := \phi(\tilde{B}_{\frac{j}{n}}^H) - \phi(B_{\frac{k(j)}{m}}^H)$, and set

$$I_t := \mathbb{E} \left[\left(\sum_{j=0}^{\lfloor nt \rfloor - 1} Y_j(\phi) (\Delta_j^n B^H)^r \right)^2 \right].$$

From Lemma 4.3.3 we obtain

$$\begin{aligned} I_t &= \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} C_{r,u} C_{r,v} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nt \rfloor - 1} \mathbb{E} \left[Y_i(\phi) Y_j(\phi) I_{r-2u}(\partial_{\frac{i}{n}}^{\otimes r-2u}) I_{r-2v}(\partial_{\frac{j}{n}}^{\otimes r-2v}) \right] \\ &\leq C \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nT \rfloor - 1} \left| \mathbb{E} \left[Y_i(\phi) Y_j(\phi) I_{r-2u}(\partial_{\frac{i}{n}}^{\otimes r-2u}) I_{r-2v}(\partial_{\frac{j}{n}}^{\otimes r-2v}) \right] \right|. \end{aligned}$$

Recall again the product formula for multiple stochastic integrals (2.1.1):

$$I_p(f) I_q(g) = \sum_{z=0}^{p \wedge q} z! \binom{p}{z} \binom{q}{z} I_{p+q-2z}(f \tilde{\otimes}_z g), \quad (4.4.6)$$

where $p, q \geq 1$ and $\tilde{\otimes}_z$ is the contraction operator. We apply this formula in order to develop the product of two multiple stochastic integrals and we end up with

$$\begin{aligned} I_t &\leq C \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} \sum_{z=0}^{(r-2u) \wedge (r-2v)} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nT \rfloor - 1} \left| \mathbb{E} \left[Y_i(\phi) Y_j(\phi) \right. \right. \\ &\quad \left. \left. \times I_{2r-2(u+v)-2z} \left(\partial_{\frac{i}{n}}^{\otimes r-2u-z} \tilde{\otimes} \partial_{\frac{j}{n}}^{\otimes r-2v-z} \right) \langle \partial_{\frac{i}{n}}, \partial_{\frac{j}{n}} \rangle_{\mathfrak{H}}^z \right] \right| \\ &= C \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nT \rfloor - 1} \left| \mathbb{E} \left[Y_i(\phi) Y_j(\phi) I_{2r-2(u+v)} \left(\partial_{\frac{i}{n}}^{\otimes r-2u} \tilde{\otimes} \partial_{\frac{j}{n}}^{\otimes r-2v} \right) \right] \right| \\ &\quad + C \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} \sum_{z=1}^{(r-2u) \wedge (r-2v)} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nT \rfloor - 1} \left| \mathbb{E} \left[Y_i(\phi) Y_j(\phi) \right. \right. \\ &\quad \left. \left. \times I_{2r-2(u+v)-2z} \left(\partial_{\frac{i}{n}}^{\otimes r-2u-z} \tilde{\otimes} \partial_{\frac{j}{n}}^{\otimes r-2v-z} \right) \langle \partial_{\frac{i}{n}}, \partial_{\frac{j}{n}} \rangle_{\mathfrak{H}}^z \right] \right| \\ &=: C(D_1 + D_2). \end{aligned}$$

We first study term D_2 , that is when $z \geq 1$. On one hand, by Lemma 2.1.7 we get

$$\begin{aligned} \left\| I_{2r-2(u+v)-2z} \left(\partial_{\frac{i}{n}}^{\otimes r-2u-z} \tilde{\otimes} \partial_{\frac{j}{n}}^{\otimes r-2v-z} \right) \right\|_{L^{q/(q-2)}(\Omega)} &\leq C \left(\|\partial_{\frac{i}{n}}\|_{\mathfrak{H}}^{r-2u-z} \|\partial_{\frac{j}{n}}\|_{\mathfrak{H}}^{r-2v-z} \right) \\ &= C \|\partial_{\frac{i}{n}}\|_{\mathfrak{H}}^{2r-2(u+v)-2z} \\ &= C n^{-2H(r-u-v-z)}. \end{aligned} \quad (4.4.7)$$

On the other hand, using (4.3.5), we obtain

$$\sum_{i,j=0}^{\lfloor nT \rfloor - 1} \left| \langle \partial_{\frac{i}{n}}, \partial_{\frac{j}{n}} \rangle_{\mathfrak{H}}^z \right| \leq C n^{1-2zH}. \quad (4.4.8)$$

Thus, from (4.4.7) and (4.4.8) and using Hölder's inequality, we deduce that the term D_2 is bounded by

$$\begin{aligned} D_2 &\leq C \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} \sum_{z=1}^{(r-2u) \wedge (r-2v)} n^{-2H(u+v)} \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \|Y_j(\phi)\|_{L^q(\Omega)}^2 n^{-2H(r-u-v-z)} n^{1-2zH} \\ &\leq C \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \|Y_j(\phi)\|_{L^q(\Omega)}^2 n^{1-2rH}. \end{aligned}$$

Now, let us study term D_1 , that is when $z = 0$. Applying the duality formula (2.1.4), that is

$$\mathbb{E}(F\delta^q(u)) = \mathbb{E}(\langle D^q F, u \rangle_{\mathfrak{H}^{\otimes q}}),$$

to our case, we obtain

$$\begin{aligned} &\left| \mathbb{E} \left[Y_i(\phi) Y_j(\phi) I_{2(r-u-v)} \left(\partial_{\frac{i}{n}}^{\otimes r-2u} \tilde{\otimes} \partial_{\frac{j}{n}}^{\otimes r-2v} \right) \right] \right| \\ &= \left| \mathbb{E} \left[\langle D^{2(r-u-v)}(Y_i(\phi) Y_j(\phi)), \partial_{\frac{i}{n}}^{\otimes r-2u} \tilde{\otimes} \partial_{\frac{j}{n}}^{\otimes r-2v} \rangle_{\mathfrak{H}^{\otimes 2(r-u-v)}} \right] \right|. \end{aligned}$$

Write $s = 2(r - u - v)$. By definition of Malliavin derivative and Leibniz rule, $D_{u_1, \dots, u_s}^s(Y_i(\phi) Y_j(\phi))$ consists of terms of the form $D_{\mathbf{u}_J}^{|J|}(Y_i(\phi)) D_{\mathbf{u}_J^c}^{s-|J|}(Y_j(\phi))$, where J is a subset of $\{1, \dots, s\}$, $|J|$ denotes the cardinality of J and $\mathbf{u}_J = (u_i)_{i \in J}$. Without loss of generality, we may fix J and assume that $a = |J| \geq 1$. By our assumptions on ϕ and the definition of Malliavin derivative, we know that

$$\begin{aligned} D^a(Y_i(\phi)) &= \phi^{(a)} \left(\tilde{B}_{\frac{i}{n}}^H \right) \tilde{\varepsilon}_{\frac{i}{n}}^{\otimes a} - \phi^{(a)} \left(B_{\frac{k(i)}{m}}^H \right) \varepsilon_{\frac{k(i)}{m}}^{\otimes a} \\ &= Y_i(\phi^{(a)}) \varepsilon_{\frac{k(i)}{m}}^{\otimes a} + \phi^{(a)} \left(\tilde{B}_{\frac{i}{n}}^H \right) \left(\tilde{\varepsilon}_{\frac{i}{n}}^{\otimes a} - \varepsilon_{\frac{k(i)}{m}}^{\otimes a} \right), \end{aligned}$$

where we recall that $k = k(i) = \sup\{j : \frac{j}{m} \leq \frac{i}{n}\}$, and, for each $a \leq 2r$, we have $D^a(Y_i(\phi)) \in L^2(\Omega; \mathfrak{H}^{\otimes a})$. Setting $b = s - |J| = s - a$ and with a slight abuse of

notation, it follows that

$$\begin{aligned}
 & \mathbb{E} \left[\left\langle D_{\mathbf{u}_J}^a(Y_i(\phi)) D_{\mathbf{u}_{J^c}}^b(Y_j(\phi)), \partial_{\frac{i}{n}}^{\otimes r-2u} \otimes \partial_{\frac{j}{n}}^{\otimes r-2v} \right\rangle_{\mathfrak{H}^{\otimes 2r-2(u+v)}} \right] \\
 & \leq \|Y_i(\phi^{(a)})\|_{L^2(\Omega)} \|Y_j(\phi^{(b)})\|_{L^2(\Omega)} \left| \left\langle \varepsilon_{\frac{k(i)}{m}}^{\otimes a}(\mathbf{u}_J) \otimes \varepsilon_{\frac{k(j)}{m}}^{\otimes b}(\mathbf{u}_{J^c}), \partial_{\frac{i}{n}}^{\otimes r-2u} \otimes \partial_{\frac{j}{n}}^{\otimes r-2v} \right\rangle_{\mathfrak{H}^{\otimes s}} \right| \\
 & \quad + \|Y_i(\phi^{(a)})\|_{L^2(\Omega)} \|\phi^{(b)}(\tilde{B}_{\frac{j}{n}}^H)\|_{L^2(\Omega)} \\
 & \quad \times \left| \left\langle \varepsilon_{\frac{k(i)}{m}}^{\otimes a}(\mathbf{u}_J) \otimes (\tilde{\varepsilon}_{\frac{j}{n}}^{\otimes b} - \varepsilon_{\frac{k(j)}{m}}^{\otimes b})(\mathbf{u}_{J^c}), \partial_{\frac{i}{n}}^{\otimes r-2u} \otimes \partial_{\frac{j}{n}}^{\otimes r-2v} \right\rangle_{\mathfrak{H}^{\otimes s}} \right| \\
 & \quad + \|\phi^{(a)}(\tilde{B}_{\frac{i}{n}}^H)\|_{L^2(\Omega)} \|Y_j(\phi^{(b)})\|_{L^2(\Omega)} \\
 & \quad \times \left| \left\langle (\tilde{\varepsilon}_{\frac{i}{n}}^{\otimes a} - \varepsilon_{\frac{k(i)}{m}}^{\otimes a})(\mathbf{u}_J) \otimes \varepsilon_{\frac{k(j)}{m}}^{\otimes b}(\mathbf{u}_{J^c}), \partial_{\frac{i}{n}}^{\otimes r-2u} \otimes \partial_{\frac{j}{n}}^{\otimes r-2v} \right\rangle_{\mathfrak{H}^{\otimes s}} \right| \\
 & \quad + \|\phi^{(a)}(\tilde{B}_{\frac{i}{n}}^H)\|_{L^2(\Omega)} \|\phi^{(b)}(\tilde{B}_{\frac{j}{n}}^H)\|_{L^2(\Omega)} \\
 & \quad \times \left| \left\langle (\tilde{\varepsilon}_{\frac{i}{n}}^{\otimes a} - \varepsilon_{\frac{k(i)}{m}}^{\otimes a})(\mathbf{u}_J) \otimes (\tilde{\varepsilon}_{\frac{j}{n}}^{\otimes b} - \varepsilon_{\frac{k(j)}{m}}^{\otimes b})(\mathbf{u}_{J^c}), \partial_{\frac{i}{n}}^{\otimes r-2u} \otimes \partial_{\frac{j}{n}}^{\otimes r-2v} \right\rangle_{\mathfrak{H}^{\otimes s}} \right| \\
 & := D_{11} + D_{12} + D_{13} + D_{14}.
 \end{aligned}$$

Consider first the term D_{11} . By (4.3.3), we have either

$$D_{11} \leq C \left| \left\langle \varepsilon_{\frac{k(i)}{m}} \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| n^{-2H(a+b-1)} \sup_{0 \leq w \leq 2r} \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \|Y_j(\phi^{(w)})\|_{L^2(\Omega)}^2$$

or

$$D_{11} \leq C \left| \left\langle \varepsilon_{\frac{k(i)}{m}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| n^{-2H(a+b-1)} \sup_{0 \leq w \leq 2r} \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \|Y_j(\phi^{(w)})\|_{L^2(\Omega)}^2.$$

By Lemma 4.3.1 a)

$$\sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \left\langle \varepsilon_{\frac{k(i)}{m}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| \leq C \quad (4.4.9)$$

and by (4.3.6),

$$\sum_{i=0}^{\lfloor nT \rfloor - 1} \sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \left\langle \varepsilon_{\frac{k(i)}{m}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| \left| \left\langle \varepsilon_{\frac{k(j)}{m}}, \partial_{\frac{j}{n}} \right\rangle_{\mathfrak{H}} \right| \leq C_T m^{2-4H}. \quad (4.4.10)$$

As a consequence, inequalities (4.4.9) and (4.4.10) imply

$$\begin{aligned}
 & \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nT \rfloor - 1} D_{11} \\
 & \leq C \sup_{0 \leq w \leq 2r} \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \|Y_j(\phi^{(w)})\|_{L^2(\Omega)}^2 \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} \sum_{i=0}^{\lfloor nT \rfloor - 1} n^{-2H(u+v+a+b-1)} \\
 & \quad + C_T \sup_{0 \leq w \leq 2r} \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \|Y_j(\phi^{(w)})\|_{L^2(\Omega)}^2 \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} n^{-2H(u+v+a+b-1)} n^{2H} m^{2-4H} \\
 & \leq C_T \sup_{0 \leq w \leq 2r} \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \|Y_j(\phi^{(w)})\|_{L^2(\Omega)}^2 (1 + n^{2H-1} m^{2-4H}) n^{1-2rH},
 \end{aligned}$$

where we used that $u + v + a + b - 1 = 2r - (u + v) - 1 \geq r$, since $u + v + 1 \leq 2 \lfloor \frac{r}{2} \rfloor + 1 = r$ for any odd integer r .

We apply the same calculation to D_{12} and D_{13} , and we similarly obtain that

$$\begin{aligned} & \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nT \rfloor - 1} (D_{12} + D_{13}) \\ & \leq C_T \sup_{0 \leq w \leq 2r} \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \|\phi^{(w)}(\tilde{B}_{\frac{j}{n}})\|_{L^2(\Omega)} \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \|Y_j(\phi^{(w)})\|_{L^2(\Omega)} \\ & \quad \times (1 + n^{1-2H} m^{2-4H}) n^{1-2rH}. \end{aligned}$$

Now we study term D_{14} . Inequalities (4.3.7) and (4.3.8) state that

$$\sum_{j=0}^{\lfloor nT \rfloor - 1} \left| \langle \tilde{\varepsilon}_{\frac{j}{n}} - \varepsilon_{\frac{k(i)}{m}}, \partial_{\frac{j}{n}} \rangle_{\mathfrak{H}} \right| \leq C_T m^{-2H}$$

and

$$\sum_{i=0}^{\lfloor nT \rfloor - 1} \left| \langle \tilde{\varepsilon}_{\frac{i}{n}} - \varepsilon_{\frac{k(i)}{m}}, \partial_{\frac{i}{n}} \rangle_{\mathfrak{H}} \right| \leq C_T m^{1-2H}.$$

Then, with the same arguments as those used for D_{11} , we obtain

$$\begin{aligned} & \sum_{u,v=0}^{\lfloor \frac{r}{2} \rfloor} n^{-2H(u+v)} \sum_{i,j=0}^{\lfloor nT \rfloor - 1} D_{14} \\ & \leq C_T \sup_{0 \leq w \leq 2r} \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \|\phi^{(w)}(\tilde{B}_{\frac{j}{n}}^H)\|_{L^2(\Omega)}^2 (m^{-2H} + n^{2H-1} m^{2-4H}) n^{1-2rH}. \end{aligned}$$

The proof is now concluded. □

4.5 Proof of the main result

This section is devoted to prove the main result of the chapter. Recall that, for $t \in [0, T]$,

$$\Phi_n^h(t) = \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(2h+1)}(\tilde{B}_{\frac{j}{n}}^H) (\Delta_j^n B^H)^{2h+1}$$

where $\tilde{B}_{\frac{j}{n}}^H$ and $\Delta_j^n B^H$ have been defined in (4.2.1) and (4.3.1) respectively.

In order to give the proof of Theorem 4.2.3, we first prove the following proposition:

Proposition 4.5.1. *Let $\ell < \infty$ be a natural number and $B^H = \{B_t^H, t \geq 0\}$ be a fractional Brownian motion with Hurst parameter $H = \frac{1}{4\ell+2}$. Consider a function*

$f \in \mathcal{C}^{20\ell+5}(\mathbb{R})$ such that f and its derivatives up to the order $20\ell+5$ have moderate growth. Then,

$$\Phi_n^\ell(t) = \frac{\mathcal{L}}{n \rightarrow \infty} \sigma_\ell \int_0^t f^{(2\ell+1)}(B_s^H) dW_s, \quad (4.5.1)$$

where $W = \{W_t, t \geq 0\}$ is a Brownian motion independent of B^H , σ_ℓ is the constant defined in (4.2.4), and the convergence holds in the topology of the Skorohod space $D([0, \infty))$.

Observe that the process Φ_n^ℓ is a weighted sum of the odd powers $(\Delta_j^n B^H)^{2\ell+1}$. When $H = \frac{1}{4\ell+2}$, the sums of these odd powers converge in law to a Gaussian random variable. More precisely, the following stable convergence holds

$$\left(\sum_{j=0}^{\lfloor nt \rfloor - 1} (\Delta_j^n B^H)^{2\ell+1}, B_t^H, t \geq 0 \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} (\sigma_\ell W_t, B_t^H, t \geq 0), \quad (4.5.2)$$

The proof of the convergence for a fixed t follows from the Breuer-Major Theorem, which shown that the central limit theorem holds for some non-linear functionals of stationary Gaussian fields if the correlation function of the underlying field tends fast enough to zero (see [Breuer and Major, 1983]). Nourdin and Peccati in [Nourdin and Peccati, 2012, Chapter 7] and Corcuera, Nualart and Woerner in [Corcuera et al., 2006] proved this result using the Fourth Moment theorem, proved by Nualart and Peccati in [Nualart and Peccati, 2005], which states that the convergence in law of a normalized sequence of multiple Wiener-Itô integrals towards the Gaussian law $\mathcal{N}(0, \sigma^2)$ is equivalent to the convergence of just the fourth moment to $3\sigma^4$ (see also [Nourdin, 2013]).

Then the convergence of Φ_n^ℓ follows from the methodology of small blocks/big blocks used, for instance in the works [Corcuera et al., 2014] and [Corcuera et al., 2006]. However, unlike the above references, the convergence cannot be established using fractional calculus techniques because $H < \frac{1}{2}$, and it requires the application of integration-by-parts formulas from Malliavin calculus.

Proof of Proposition 4.5.1. In order to show Proposition 4.5.1, we will first prove that the sequence of processes $\{\Phi_n^\ell(t), t \geq 0\}$ is tight in $D([0, \infty))$, and then that their finite dimensional distributions converge to those of

$$\left\{ \sigma_\ell \int_0^t f^{(2\ell+1)}(B_s^H) dW_s, t \geq 0 \right\}.$$

Notice that the tightness of the sequence Φ_n^ℓ is a consequence of Lemma 4.4.2. Indeed, this lemma implies that for any $0 \leq s < t \leq T$, there exist a constant C_T depending on T , such that

$$\mathbb{E} [|\Phi_n^\ell(t) - \Phi_n^\ell(s)|^4] \leq C_T \sum_{N=2}^4 \left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n} \right)^N.$$

It remains to show the convergence of the finite-dimensional distributions. Fix a finite set of points $0 \leq t_1 < \dots \leq t_d \leq T$. We want to show the following convergence in law, as n tends to infinity:

$$(\Phi_n^\ell(t_1), \dots, \Phi_n^\ell(t_d)) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} (Y_1, \dots, Y_d), \quad (4.5.3)$$

where

$$Y_i = \sigma_\ell \int_0^{t_i} f^{(2\ell+1)}(B_s^H) dW_s, \quad i = 1, \dots, d,$$

$W = \{W_t, t \geq 0\}$ is a Brownian motion independent of B^H and σ_ℓ is the constant defined in (4.2.4).

Taking into account the convergence (4.5.2), the main ingredient in the proof of the convergence (4.5.3) is the methodology based on the small blocks/big blocks. This method consists in considering two integers $2 \leq m < n$ and let first n tend to infinity and later m tend to infinity. For any $k \geq 0$ we define the set

$$I_k = \left\{ j \in \{0, \dots, \lfloor nt_i \rfloor - 1\} : \frac{k}{m} \leq \frac{j}{n} < \frac{k+1}{m} \right\}.$$

The basic ingredient in this approach is the decomposition

$$\begin{aligned} \Phi_n^\ell(t_i) &= \sum_{k=0}^{\lfloor mt_i \rfloor} \sum_{j \in I_k} f^{(2\ell+1)}(B_{\frac{k}{m}}^H) (\Delta_j^n B^H)^{2\ell+1} \\ &\quad + \sum_{k=0}^{\lfloor mt_i \rfloor} \sum_{j \in I_k} \left[f^{(2\ell+1)}(\tilde{B}_{\frac{j}{n}}^H) - f^{(2\ell+1)}(B_{\frac{k}{m}}^H) \right] (\Delta_j^n B^H)^{2\ell+1} \\ &=: A_{n,m}^{(1,i)} + A_{n,m}^{(2,i)}. \end{aligned}$$

From Lemma 4.4.3 with $r = 2\ell + 1$ and $\phi = f^{(2\ell+1)}$, we can write, for any $q > 2$,

$$\begin{aligned} &\mathbb{E}[(A_{n,m}^{(2,i)})^2] \\ &\leq C_T \sup_{0 \leq w \leq 3(2\ell+1)} \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \left\| f^{(w)}(\tilde{B}_{\frac{j}{n}}^H) - f^{(w)}(B_{\frac{k(j)}{m}}^H) \right\|_{L^q(\Omega)}^2 \\ &\quad + C_T \sup_{0 \leq w \leq 3(2\ell+1)} \sup_{0 \leq j \leq \lfloor nT \rfloor - 1} \left\| f^{(w)}(\tilde{B}_{\frac{j}{n}}^H) \right\|_{L^2(\Omega)}^2 (m^{-2H} + n^{2H-1} m^{2-4H}) \\ &\quad + C_T \sup_{0 \leq w \leq 3(2\ell+1)} \sup_{0 \leq i, j \leq \lfloor nT \rfloor - 1} \left\| f^{(w)}(\tilde{B}_{\frac{i}{n}}^H) \right\|_{L^2(\Omega)} \left\| f^{(w)}(\tilde{B}_{\frac{j}{n}}^H) - f^{(w)}(B_{\frac{k(j)}{m}}^H) \right\|_{L^2(\Omega)} \\ &\quad \times (1 + n^{2H-1} m^{2-4H}) \\ &\leq C_T \left[m^{-2H} + n^{2H-1} m^{2-4H} \right. \\ &\quad \left. \times \left(1 + \sup_{0 \leq w \leq 3(2\ell+1)} \sup_{\substack{s, t \in [0, T] \\ |t-s| \leq \frac{1}{m}}} \left\| f^{(w)}\left(\frac{B_s^H + B_{s+\frac{1}{n}}^H}{2}\right) - f^{(w)}(B_t^H) \right\|_{L^q(\Omega)}^2 \right) \right], \end{aligned}$$

where $k := k(j) = \sup\{i \geq 0 : \frac{i}{m} \leq \frac{j}{n}\}$. This implies

$$\limsup_{n \rightarrow \infty} \mathbb{E}[(A_{n,m}^{(2,i)})^2] \leq C_T \left(m^{-2H} + \sup_{0 \leq w \leq 3(2\ell+1)} \sup_{\substack{s, t \in [0, T] \\ |t-s| \leq \frac{1}{m}}} \|f^{(w)}(B_s^H) - f^{(w)}(B_t^H)\|_{L^2(\Omega)}^2 \right),$$

which converges to zero as m tends to infinity.

On the other hand, from (4.5.2) we deduce that the vector $(A_{n,m}^{(1,1)}, \dots, A_{n,m}^{(1,d)})$ converges in law, as n tends to infinity, to the vector with components

$$\sigma_\ell \sum_{k=0}^{\lfloor mt_i \rfloor} f^{(2\ell+1)}\left(B_{\frac{k}{m}}^H\right) (W_{\frac{k+1}{m}} - W_{\frac{k}{m}}),$$

$i = 1, \dots, d$, where W is a Brownian motion independent of B^H . Each of these components converges in $L^2(\Omega)$ to the stochastic integral $\sigma_\ell \int_0^{t_i} f^{(2\ell+1)}(B_s^H) dW_s$, as m tends to infinity. This completes the proof. \square

Now we have all the elements to prove the main result of the chapter.

Proof of Theorem 4.2.3. The proof consists of two steps. First, using Taylor's formula and the properties of the symmetric measure ν , we determine a decomposition for the ν -symmetric Riemann sums. Secondly, we study the convergence of the terms of the decomposition when n tends to infinity.

Step 1. The first ingredient of the proof is the following expansion, established in [Gradinaru et al., 2005], based on Taylor's formula and the properties of the measure ν :

$$\begin{aligned} f(b) &= f(a) + (b-a) \int_0^1 f'(a + \alpha(b-a)) \nu(d\alpha) \\ &\quad + \sum_{h=\ell}^{2\ell} k_{\nu,h} f^{(2h+1)}\left(\frac{a+b}{2}\right) (b-a)^{2h+1} + (b-a)^{4\ell+2} \mathcal{C}(a,b), \end{aligned} \tag{4.5.4}$$

where $a, b \in \mathbb{R}$ and $\mathcal{C}(a, b)$ is a continuous function such that $\mathcal{C}(a, a) = 0$. The constants $k_{\nu,h}$ are given by

$$k_{\nu,h} = \frac{1}{(2h)!} \left[\frac{1}{(2h+1)4^h} - \int_0^1 \left(\alpha - \frac{1}{2}\right)^{2h} \nu(d\alpha) \right].$$

Observe that these constants are the same ones defined in (4.2.3).

Applying identity (4.5.4) to $a = B_{\frac{j}{n}}^H$ and $b = B_{\frac{j+1}{n}}^H$, it yields

$$\begin{aligned} f(B_{[nt]}^H) - f(0) &= \sum_{j=0}^{[nt]-1} \Delta_j^n B^H \int_0^1 f'(B_{\frac{j}{n}}^H + \alpha \Delta_j^n B^H) \nu(d\alpha) \\ &\quad + \sum_{h=\ell}^{2\ell} \sum_{j=0}^{[nt]-1} k_{\nu,h} f^{(2h+1)}(\tilde{B}_{\frac{j}{n}}^H) (\Delta_j^n B^H)^{2h+1} \\ &\quad + \sum_{j=0}^{[nt]-1} \mathcal{C}(B_{\frac{j}{n}}^H, B_{\frac{j+1}{n}}^H) (\Delta_j^n B^H)^{4\ell+2}, \end{aligned}$$

where $\tilde{B}_{\frac{j}{n}}^H$ and $\Delta_j^n B^H$ have been defined in (4.2.1) and (4.3.1) respectively. The last decomposition can be written as

$$f(B_{[nt]}^H) - f(0) = S_n^\nu(f', t) + \sum_{h=\ell}^{2\ell} k_{\nu,h} \Phi_n^h(t) + R_n(t), \quad (4.5.5)$$

where, for each $h = \ell, \dots, 2\ell$,

$$\Phi_n^h(t) = \sum_{j=0}^{[nt]-1} f^{(2h+1)}(\tilde{B}_{\frac{j}{n}}^H) (\Delta_j^n B^H)^{2h+1} \quad (4.5.6)$$

is the process defined in Definition 4.4.1 and

$$R_n(t) = \sum_{j=0}^{[nt]-1} \mathcal{C}(B_{\frac{j}{n}}^H, B_{\frac{j+1}{n}}^H) (\Delta_j^n B^H)^{4\ell+2}$$

is the residual term.

Step 2. Let us consider the convergence of the terms in the decomposition (4.5.5).

Convergence of $R_n(t)$. The residual term $R_n(t)$ is a weighted sum of the powers $(\Delta_j^n B^H)^{4\ell+2}$ with coefficients that converge to zero as n tends to infinity. Taking into account that $H = \frac{1}{4\ell+2}$, it is not difficult to show that $R_n(t)$ converges to zero in probability, uniformly in compact sets. Indeed, for any $T > 0$ and for any $K, \epsilon > 0$, we can write

$$\begin{aligned} &P \left(\sup_{0 \leq t \leq T} |R_n(t)| > \epsilon \right) \\ &\leq P \left(\sup_{\substack{s, t \in [0, T] \\ |t-s| \leq \frac{1}{n}}} |\mathcal{C}(B_s^H, B_t^H)| > \frac{1}{K} \right) + P \left(\sum_{j=0}^{[nT]-1} (\Delta_j^n B^H)^{4\ell+2} > K\epsilon \right). \quad (4.5.7) \end{aligned}$$

Since $H = \frac{1}{4\ell+2}$ and using (4.3.2), we can write

$$P\left(\sum_{j=0}^{\lfloor nT \rfloor - 1} (\Delta_j^n B^H)^{4\ell+2} > K\epsilon\right) \leq \frac{\mu_{4\ell+2}}{K\epsilon} \frac{\lfloor nT \rfloor}{n} \leq \frac{T\mu_{4\ell+2}}{K\epsilon}, \quad (4.5.8)$$

where μ_k denote the k th moment of the standard Gaussian distribution. From (4.5.7) and (4.5.8), letting first $n \rightarrow \infty$ and then $K \rightarrow \infty$ it follows that for any $\epsilon > 0$ and $T > 0$,

$$\lim_{n \rightarrow \infty} P\left(\sup_{0 \leq t \leq T} |R_n(t)| > \epsilon\right) = 0.$$

Convergence of Φ_n^h . In Lemma 4.4.2, we have shown that for any $h = \ell, \dots, 2\ell$, the moment of order four $\mathbb{E}[|\Phi_n^h(t) - \Phi_n^h(s)|^4]$ can be estimated by

$$C_T \sum_{N=2}^4 n^{\frac{2(\ell-h)N}{2\ell+1}} \left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)^N,$$

for any $0 \leq s \leq t \leq T$. As a consequence, the terms Φ_n^h with $h = \ell + 1, \dots, 2\ell$ converge to zero in the topology of $D([0, \infty))$ and do not contribute to the limit. Therefore, the only nonzero contribution to the limit in law of the ν -symmetric Riemann sums $S_n^\nu(f', t)$ is the term

$$k_{\nu, \ell} \Phi_n^\ell(t) = k_{\nu, \ell} \sum_{j=0}^{\lfloor nt \rfloor - 1} f^{(2\ell+1)}(\tilde{B}_{\frac{j}{n}}^H) (\Delta_j^n B^H)^{2\ell+1}.$$

Proposition 4.5.1 proved that this term converges in law in the topology of the Skorohod space $D([0, \infty))$, as n tends to infinity, to

$$c_\nu \int_0^t f^{(2\ell+1)}(B_s^H) dW_s,$$

and this completes the proof. □

5

Delay differential equations driven by a Hölder continuous function of order $\beta \in (\frac{1}{3}, \frac{1}{2})$.

Consider the following differential equation with delay:

$$\begin{aligned}x_t^r &= \eta_0 + \int_0^t b(u, x_u^r) du + \int_0^t \sigma(x_{u-r}^r) dy_u, & t \in [0, T], \\x_t^r &= \eta_t, & t \in [-r, 0),\end{aligned}$$

where $\eta : [-r, 0] \rightarrow \mathbb{R}^d$ is a smooth function and y is a Hölder continuous function of order $\beta \in (\frac{1}{3}, \frac{1}{2})$. It is proved in [Hu and Nualart, 2009] that there exists a unique solution of this equation. In this chapter we prove that it converges almost surely in the supremum norm to the solution of the differential equation without delay

$$x_t = \eta_0 + \int_0^t b(u, x_u) du + \int_0^t \sigma(x_u) dy_u, \quad t \in [0, T],$$

when the delay tends to zero. Our approach is based on the techniques of the classical fractional calculus and it is inspired by [Hu and Nualart, 2009].

In a future work we will apply these results to some stochastic process as the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$.

The chapter is organized as follows. The following section is devoted to contextualize our result and explain the results that have been achieved in the study of differential equations with and without delay driven by a Hölder continuous function. In Section 5.2 we describe our main result. Section 5.3 contains technical estimates for the study of the integrals. In Section 5.4 we define the equations and the solutions we work with and we give some estimations. Finally, the last section is dedicated to the proof of the main result.

5.1 Introduction

The classical integration theory in stochastic analysis introduced by Itô is focused mainly on semimartingales. In Chapter 1 we see that fractional Brownian motion is not a semimartingale except when the Hurst parameter is $H = \frac{1}{2}$, that is, when the process is a Brownian motion. Starting from that, great efforts have been made to develop a stochastic calculus for fractional Brownian motion when $H \neq \frac{1}{2}$. In Chapter 2 we describe some methods to establish the stochastic integration with respect to this process. After these studies, stochastic differential equations driven by fractional Brownian motion have been considered and, depending on the value of the Hurst parameter and the dimension of the equation, different approaches have been adopted. An efficient method to investigate these stochastic differential equations consists in studying deterministic differential equations driven by a Hölder continuous function and then applying the results obtained to the stochastic case.

Consider the differential equation

$$x_t = x_0 + \int_0^t b(u, x) du + \int_0^t \sigma(x_u) dy_u, \quad t \in (0, T], \quad (5.1.1)$$

where y is a β -Hölder continuous function for some $\beta \in (0, 1)$ and the hereditary term $b(u, x)$ depends on the path $\{x_s, 0 \leq s \leq u\}$.

In [Nualart and Răşcanu, 2002] Nualart and Răşcanu study the dynamical systems

$$dx_t = f(x_t) dy_t,$$

where $f(x)$ and y are Hölder continuous of order larger than $\frac{1}{2}$, and the Riemann-Stieltjes integral $\int_0^t f(x_u) dy_u$ can be expressed as a Lebesgue integral using fractional derivatives. They prove existence and uniqueness of solution using a contraction principle and they also show that the solution has finite moments. As an application, they extend the results to a stochastic differential equation driven by a fractional Brownian motion B^H with Hurst parameter $H > \frac{1}{2}$. They use a pathwise approach and define the integral with respect to B^H as a pathwise Riemann-Stieltjes integral, thanks to the results given by Young in [Young, 1936] and Zähle in [Zähle, 1998].

When y is a β -Hölder continuous function with $\beta < \frac{1}{2}$, important results are obtained by Lyons in [Lyons, 1998]. He proves that the integral equations

$$x_t = x_0 + \sum_{j=1}^m \int_0^t f^j(x_u) dy_u^j, \quad t \in [0, T],$$

where y^j are continuous functions with bounded p -variation on $[0, T]$ for some $p \in [1, 2)$ and with a Hölder continuous derivative of order $\alpha > p - 1$, have a unique solution in the space of continuous functions of bounded p -variation.

In [Hu and Nualart, 2009] Hu and Nualart establish the existence and uniqueness of solution for the dynamical system

$$dx_t = f(x_t) dy_t,$$

where y is a Hölder continuous function of order $\beta \in (\frac{1}{3}, \frac{1}{2})$. Recall from Definition 2.3.8 that, given $\Delta_T := \{(a, b) : 0 \leq a < b \leq T\}$, the tern $(x, y, x \otimes y)$ is called a (d, m) -dimensional β -Hölder continuous multiplicative functional, and we write $(x, y, x \otimes y) \in M_{d,m}^\beta(0, T)$, if:

1. $x : [0, T] \rightarrow \mathbb{R}^d$ and $y : [0, T] \rightarrow \mathbb{R}^m$ are β -Hölder continuous functions,
2. $x \otimes y : \Delta_T \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ is a continuous function satisfying the following properties:

- a) (Multiplicative property) For all $s \leq u \leq t$, we have

$$(x \otimes y)_{s,u} + (x \otimes y)_{u,t} + (x_u - x_s) \otimes (y_t - y_u) = (x \otimes y)_{s,t}.$$

- b) For all $(s, t) \in \Delta_T$,

$$|(x \otimes y)_{s,t}| \leq c|t - s|^{2\beta}.$$

Hu and Nualart prove an explicit formula for integrals of the form $\int_a^b f(x_u) dy_u$ in terms of x , y and $x \otimes y$, transform the dynamical system $dx_t = f(x_t) dy_t$ into a closed system of equations involving only x , $x \otimes y$ and $x \otimes (y \otimes y)$ and solve it using a classical fixed point argument. Finally, they apply these results to solve stochastic differential equations driven by a multidimensional Brownian motion. Given that the fractional Brownian motion B^H has locally bounded p -variation for $p > 1/H$, in [Coutin and Qian, 2002] Coutin and Qian follow the approach of Lyons to establish the existence of strong solutions for stochastic differential equations driven by a fractional Brownian motion with parameter $H > \frac{1}{4}$.

Now consider the differential equations with delay

$$\begin{aligned} x_t^r &= \eta_0 + \int_0^t b(u, x^r) du + \int_0^t \sigma(x_{u-r}^r) dy_u, & t \in (0, T], \\ x_t^r &= \eta_t, & t \in [-r, 0], \end{aligned} \tag{5.1.2}$$

where r denotes a strictly positive time delay and $\eta : [-r, 0] \rightarrow \mathbb{R}^d$ is a smooth function. Here again y is β -Hölder continuous for some $\beta \in (0, 1)$ and the hereditary term $b(u, x)$ depends on the path $\{x_s, 0 \leq s \leq u\}$.

Delay differential equations arise from the need to study models that behave more like the real processes. They find their applications in dynamical systems with aftereffects or when the dynamics are subjected to propagation delay. Some examples are epidemiological models with incubation periods that postpone the transmission of disease, or neuronal models where the spatial distribution of neurons can

cause a delay in the transmission of the impulse. Sometimes the delay avoids some usual problems, but in general it adds difficulties and cumbersome notations.

The existence and uniqueness of solution in the case when $\beta > \frac{1}{2}$ is proved by Ferrante and Rovira in [Ferrante and Rovira, 2010]. They extend the results of Nualart and Răşcanu contained in [Nualart and Răşcanu, 2002] to deterministic equations with delay and then they easily apply the results pathwise to stochastic differential equations with delay driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Moreover, they prove that the solution is bounded.

With a different approach based on a slight variation of the Young integration theory, called algebraic integration, León and Tindel prove in [León and Tindel, 2012] the existence of a unique solution for a general class of delay differential equations driven by a Hölder continuous function with parameter greater than $\frac{1}{2}$. They obtain some estimates of the solution which allow to show that the solution of a delay differential equation driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ has a \mathcal{C}^∞ -density.

In the case when $\beta < \frac{1}{2}$ more difficulties appear and in literature we find results only up to the value $\beta > \frac{1}{3}$, eventually extended to $\beta > \frac{1}{4}$.

In [Neuenkirch et al., 2008], the authors consider a finite sequence of discrete delays satisfying $0 < r_1 < \dots < r_k < \infty$ and study the following differential equation:

$$\begin{aligned} x_t^r &= \eta_0 + \int_0^t \sigma(x_u^r, x_{u-r_1}^r, \dots, x_{u-r_k}^r) dy_u, & t \in (0, T], \\ x_t^r &= \eta_t, & t \in [-r_k, 0], \end{aligned}$$

where y is a β -Hölder continuous function with $\beta > \frac{1}{3}$ and η is a weakly controlled path on y . The hereditary term vanishes to avoid cumbersome notations. The authors show the existence of a unique solution for these equations under suitable hypothesis. Then, they apply these results to a delay differential equations driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{3}$. These results are extended by Tindel and Torrecilla in [Tindel and Torrecilla, 2012] to the deterministic case of order $\beta > \frac{1}{4}$ and the corresponding stochastic case with Hurst parameter $H > \frac{1}{4}$.

In literature variations of the differential equation (5.1.2) are also considered. We take as example the differential equations with delay and positivity constraints on \mathbb{R}^d of the form:

$$\begin{aligned} x_t^r &= \eta_0 + \int_0^t b(u, x^r) du + \int_0^t \sigma(x_{u-r}^r) dy_u + z_t, & t \in (0, T], \\ x_t^r &= \eta_t, & t \in [-r, 0], \end{aligned} \tag{5.1.3}$$

where $\eta : [-r, 0] \rightarrow \mathbb{R}^d$ is a deterministic nonnegative smooth function and z is a vector-valued non-decreasing function which ensures that the non-negativity constraints on x^r are enforced.

In [Besalú and Rovira, 2012], Besalú and Rovira prove that, when y is a β -Hölder continuous function with $\beta > \frac{1}{2}$, equations (5.1.3) admit a unique solution with bounded moments. The authors follow the ideas contained in [Nualart and Răşcanu, 2002] and [Ferrante and Rovira, 2010].

In [Besalú et al., 2014], Besalú, Márquez-Carreras and Rovira extend the previous results to the case $\beta \in (\frac{1}{3}, \frac{1}{2})$, following the methodology introduced in [Hu and Nualart, 2009].

In both papers, as an application the authors study stochastic differential equations with delay and nonnegative constraints driven by a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $H \in (\frac{1}{3}, \frac{1}{2})$, respectively.

In this chapter we consider the following deterministic differential equation with delay

$$\begin{aligned} x_t^r &= \eta_0 + \int_0^t b(u, x_u^r) du + \int_0^t \sigma(x_{u-r}^r) dy_u, & t \in [0, T], \\ x_t^r &= \eta_t, & t \in [-r, 0), \end{aligned}$$

where y is a Hölder continuous function of order $\beta \in (\frac{1}{3}, \frac{1}{2})$. We are interested in studying the strong convergence of the solution when the delay tends to zero.

The case when $\beta > \frac{1}{2}$ is studied by Ferrante and Rovira in [Ferrante and Rovira, 2010]. They prove that the solution of the delay equation converges, almost surely and in L^p , to the solution of the equation without delay and then apply the result pathwise to fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

5.2 Main results

Let $\beta \in (\frac{1}{3}, \frac{1}{2})$. Consider the following hypothesis:

- (H1) $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^m$ is a bounded and continuously twice differentiable function such that σ' and σ'' are bounded and λ -Hölder continuous for $\lambda > \frac{1}{\beta} - 2$.
- (H2) $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a measurable function such that there exists $b_0 \in L^\rho(0, T; \mathbb{R}^d)$ with $\rho \geq 2$ and $\forall N \geq 0$ there exists $L_N > 0$ such that:
 - (1) $|b(t, x_t) - b(t, y_t)| \leq L_N |x_t - y_t|$, $\forall x, y$ such that $|x_t| \leq N$, $|y_t| \leq N$ $\forall t \in [0, T]$,
 - (2) $|b(t, x_t)| \leq L_0 |x_t| + b_0(t)$, $\forall t \in [0, T]$.
- (H3) σ and b are bounded functions.

Consider the following differential equation on \mathbb{R}^d with delay:

$$\begin{aligned} x_t^r &= \eta_0 + \int_0^t b(u, x_u^r) du + \int_0^t \sigma(x_{u-r}^r) dy_u, & t \in [0, T], \\ x_t^r &= \eta_t, & t \in [-r, 0), \end{aligned} \tag{5.2.1}$$

where x and y are Hölder continuous functions of order $\beta \in (\frac{1}{3}, \frac{1}{2})$, η is a continuous function and r denotes a strictly positive time delay.

As we will see, conditions **(H1)** and **(H2)** are a particular case of the hypothesis for the proof of existence and uniqueness of solution of the delay equation (5.2.1), while condition **(H3)** is necessary to prove that the solution is bounded.

On one hand, in [Neuenkirch et al., 2008], the authors consider the equation

$$\begin{aligned} x_t^r &= \eta_0 + \int_0^t \sigma(x_u^r, x_{u-r_1}^r, \dots, x_{u-r_k}^r) dy_u, & t \in (0, T], \\ x_t^r &= \eta_t, & t \in [-r_k, 0], \end{aligned}$$

where $0 < r_1 < \dots < r_k < \infty$ is a finite sequence of discrete delays and σ is a bounded smooth function, and prove that there exists a unique solution $(x^r, y, x^r \otimes y) \in M_{d,m}^\beta(0, T)$. In their paper, the hereditary term vanishes to avoid cumbersome notations.

On the other hand, following the ideas contained in [Besalú et al., 2014], it is easy to show that there exists a unique solution $(x^r, y, x^r \otimes y) \in M_{d,m}^\beta(0, T)$ of the delay equation (5.2.1) when the hereditary term does not disappear. This is proved assuming that σ and b satisfy the hypothesis **(H1)** and **(H2)**, respectively, with $\rho \geq \frac{1}{1-\beta}$, $(\eta_{-r}, y, \eta_{-r} \otimes y) \in M_{d,m}^\beta(0, r)$ and $(y_{-r}, y, y_{-r} \otimes y) \in M_{m,m}^\beta(r, T)$. Assuming also that hypothesis **(H3)** is satisfied, we obtain that the solution is bounded. Although Besalú, Márquez-Carrera and Rovira study a differential equation with positive constraints, their ideas can be easily applied to show these results also for our equation. Actually, the previous results are obtained under more general conditions for σ and b and hypothesis **(H1)** and **(H2)** are none other than a particular case.

On the other side, we denote by $(x, y, x \otimes y) \in M_{d,m}^\beta(0, T)$ the solution of the stochastic differential equation on \mathbb{R}^d without delay:

$$x_t = \eta_0 + \int_0^t b(u, x_u) du + \int_0^t \sigma(x_u) dy_u, \quad t \in [0, T]. \quad (5.2.2)$$

In [Hu and Nualart, 2009], Hu and Nualart prove under the assumptions that $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^m$ is a continuously differentiable function such that σ' is λ -Hölder continuous, where $\lambda > \frac{1}{\beta} - 2$, σ and σ' are bounded, and $(y, y, y \otimes y) \in M_{m,m}^\beta(0, T)$, that there exists a bounded solution $(x, y, x \otimes y) \in M_{d,m}^\beta(0, T)$ for the differential equation

$$x_t = \eta_0 + \int_0^t \sigma(x_u) dy_u, \quad t \in [0, T].$$

Moreover, if σ is twice continuously differentiable with bounded derivatives and σ'' is λ -Hölder continuous, where $\lambda > \frac{1}{\beta} - 2$, the solution is unique. Here again the authors consider the equation without the hereditary term, but the results can be easily extended to the case when the hereditary term does not vanish.

Before stating the main result of the chapter, we introduce the following functional on $M_{d,m}^\beta(0, T)$ for $a, b \in \Delta_T$:

$$\Phi_{\beta(a,b)}(x, y) = \|x \otimes y\|_{2\beta(a,b)} + \|x\|_{\beta(a,b)} \|y\|_{\beta(a,b)}. \quad (5.2.3)$$

Moreover, if $(x, y, x \otimes y)$ and $(y, z, y \otimes z)$ belongs to $M_{d,m}^\beta(0, T)$ we define

$$\begin{aligned} \Phi_{\beta(a,b)}(x, y, z) &= \|x\|_{\beta(a,b)} \|y\|_{\beta(a,b)} \|z\|_{\beta(a,b)} + \|z\|_{\beta(a,b)} \|x \otimes y\|_{2\beta(a,b)} \\ &\quad + \|x\|_{\beta(a,b)} \|y \otimes z\|_{2\beta(a,b)}. \end{aligned} \quad (5.2.4)$$

From these definitions it follows that

$$\|(x \otimes y)_{\cdot, b}\|_{\beta(a,b)} \leq \Phi_{\beta(a,b)}(x, y)(b - a)^\beta \quad (5.2.5)$$

and

$$\|x \otimes (y \otimes z)_{\cdot, b}\|_{2\beta(a,b)} \leq K \Phi_{\beta(a,b)}(x, y, z)(b - a)^\beta \quad (5.2.6)$$

that are equations (3.29) and (3.30) of [Hu and Nualart, 2009] respectively. We refer to [Hu and Nualart, 2009] and [Lyons, 1998] for a more detailed presentation on β -Hölder continuous multiplicative functionals.

Let $(x, y, x \otimes y)$ and $(x^r, y, x^r \otimes y)$ be the solutions of equations (5.2.2) and (5.2.1) respectively. Let $\beta \in (\frac{1}{3}, \frac{1}{2})$ and set $\beta' = \beta - \varepsilon$, where $\varepsilon > 0$ is such that $\beta - 2\varepsilon > 0$ and $\lambda > \frac{1}{\beta - \varepsilon} - 2$. The main result of the chapter is the following theorem:

Theorem 5.2.1. *Suppose that $(x, y, x \otimes y)$, $(x^r, y, x^r \otimes y)$ and $(x_{\cdot-r}, y, x_{\cdot-r} \otimes y)$ belong to $M_{d,m}^\beta(0, T)$, $(y, y, y \otimes y)$ belongs to $M_{d,m}^\beta(0, T)$ and $(y_{\cdot-r}, y, y_{\cdot-r} \otimes y)$ belongs to $M_{d,m}^\beta(r, T)$. Assume that σ and b satisfy **(H1)** and **(H2)** respectively, and both satisfy **(H3)**. Assume also that $\|\eta\|_{\beta(-r_0, 0)} < \infty$ and $\sup_{r \leq r_0} \Phi_{\beta(0,r)}(\eta_{\cdot-r}, y) < \infty$ and suppose that $\|(y - y_{\cdot-r}) \otimes y\|_{2\beta'(r,T)} \rightarrow 0$ and $\|y_{\cdot-r} \otimes (y - y_{\cdot-r})\|_{2\beta'(r,T)} \rightarrow 0$ when r tends to zero. Then,*

$$\lim_{r \rightarrow 0} \|x - x^r\|_\infty = 0 \quad a.s.$$

and

$$\lim_{r \rightarrow 0} \|(x \otimes y) - (x^r \otimes y)\|_\infty = 0 \quad a.s.$$

To define the integral $\int_a^b f(x_u) dy_u$ where $0 \leq a < b \leq T$ and f satisfies hypothesis **(H1)**, we recall the construction of the integral given by Hu and Nualart in [Hu and Nualart, 2009]. They are inspired by the work of Zälhe [Zälhe, 1998] and use fractional derivatives. Fix $\alpha > 0$ such that $1 - \beta < \alpha < 2\beta$, and $\alpha < \frac{\lambda\beta+1}{2}$. Then,

$$\begin{aligned} \int_a^b f(x_u) dy_u &= (-1)^\alpha \sum_{j=1}^m \int_a^b \widehat{D}_{a+}^\alpha f_j(x_u) D_{b-}^{1-\alpha} y_{b-}^j(u) du \\ &\quad - (-1)^{2\alpha-1} \sum_{i=1}^d \sum_{j=1}^m \int_a^b D_{a+}^{2\alpha-1} \partial_i f_j(x_u) D_{b-}^{1-\alpha} \mathcal{D}_{b-}^{1-\alpha} (x \otimes y)^{i,j}(u) du, \end{aligned} \quad (5.2.7)$$

where

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right) \mathbf{1}_{(a,b)}(x)$$

and

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} dy \right) \mathbf{1}_{(a,b)}(x)$$

are the Weyl derivatives introduced in Definition 2.3.2,

$$\begin{aligned} \widehat{D}_{a+}^{\alpha} f(x_u) &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x_u)}{(u-a)^{\alpha}} \right. \\ &\quad \left. + \alpha \int_a^u \frac{f(x_u) - f(x_{\theta}) - \sum_{i=1}^m \partial_i f(x_{\theta})(x_u^i - x_{\theta}^i)}{(u-\theta)^{\alpha+1}} d\theta \right) \end{aligned}$$

is the compensated fractional derivative defined in Definition 2.3.9 and

$$\mathcal{D}_{b-}^{1-\alpha}(x \otimes y)(u) = \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{(x \otimes y)_{u,b}}{(b-u)^{1-\alpha}} + (1-\alpha) \int_u^b \frac{(x \otimes y)_{u,s}}{(s-u)^{2-\alpha}} ds \right)$$

is the extension of the fractional derivative described in Definition 2.3.10. This definition of the integral is Definition 2.3.11 given in Chapter 2. We refer to Section 2.3 of this dissertation for a better explanation.

In the sequel, K denotes a generic constant that may depend on the parameters β, α, λ and T and vary from line to line.

5.3 Estimates of the integrals

We begin this section recalling Propositions 3.4 and Proposition 3.9 from [Hu and Nualart, 2009].

Proposition 5.3.1. *Let $(x, y, x \otimes y)$ be in $M_{d,m}^{\beta}(0, T)$. Assume that $f : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is a continuous differentiable function such that f' is bounded and λ -Hölder continuous, where $\lambda > \frac{1}{\beta} - 2$. Then, for any $0 \leq a < b \leq T$, we have*

$$\begin{aligned} \left| \int_a^b f(x_u) dy_u \right| &\leq K |f(x_a)| \|y\|_{\beta(a,b)} (b-a)^{\beta} + K \Phi_{\beta(a,b)}(x, y) \\ &\quad \times (\|f'\|_{\infty} + \|f'\|_{\lambda} \|x\|_{\beta(a,b)}^{\lambda} (b-a)^{\lambda\beta}) (b-a)^{2\beta}, \end{aligned}$$

where $\Phi_{\beta(a,b)}(x, y)$ is defined in (5.2.3).

Proposition 5.3.2. *Suppose that $(x, y, x \otimes y)$ and $(y, z, y \otimes z)$ belong to $M_{d,m}^{\beta}(0, T)$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^m$ be a continuously differentiable function such that f' is λ -Hölder*

continuous and bounded, where $\lambda > \frac{1}{\beta} - 2$. Then, the following estimate holds:

$$\begin{aligned} & \left| \int_a^b f(x_u) d_{u(y \otimes z), b} \right| \\ & \leq K |f(x_a)| \Phi_{\beta(a,b)}(y, z) (b-a)^{2\beta} \\ & \quad + K (\|f'\|_\infty + \|f'\|_\lambda \|x\|_{\beta(a,b)}^\lambda) \Phi_{\beta(a,b)}(x, y, z) (b-a)^{3\beta}, \end{aligned}$$

where $\Phi_{\beta(a,b)}(x, y, z)$ is defined in (5.2.4).

The following propositions give some estimates useful for the proof of Theorem 5.2.1.

First we give a result for a function b that fulfills conditions **(H2)**.

Proposition 5.3.3. *Assume that b satisfies **(H2)**. Let $x, \tilde{x} \in C(0, T; \mathbb{R}^d)$ such that $\|x\|_\infty \leq N$ and $\|\tilde{x}\|_\infty \leq N$. Then, for $0 \leq a < b \leq T$,*

$$\left| \int_a^b [b(u, x_u) - b(u, \tilde{x}_u)] du \right| \leq L_N (b-a) \|x - \tilde{x}\|_{\infty(a,b)}.$$

Proof. For sake of simplicity, we assume $d = 1$. Using the hypothesis **(H2)**, we obtain

$$\begin{aligned} \left| \int_a^b [b(u, x_u) - b(u, \tilde{x}_u)] du \right| & \leq L_N \int_a^b |x_u - \tilde{x}_u| du \\ & \leq L_N (b-a) \|x - \tilde{x}\|_{\infty(a,b)}. \end{aligned}$$

□

Now we give some results for a function f under conditions **(H1)**. The first result is Proposition 6.4 of [Hu and Nualart, 2009]:

Proposition 5.3.4. *Suppose that $(x, y, x \otimes y)$ and $(\tilde{x}, y, \tilde{x} \otimes y)$ belong to $M_{d,m}^\beta(0, T)$. Assume that f satisfies **(H1)**. Then, for $0 \leq a < b \leq T$,*

$$\begin{aligned} \left| \int_a^b [f(x_u) - f(\tilde{x}_u)] dy_u \right| & \leq G_{\beta(a,b)}^1(f, x, \tilde{x}, y) (b-a)^{2\beta} \|x - \tilde{x}\|_{\infty(a,b)} \\ & \quad + G_{\beta(a,b)}^2(f, x, \tilde{x}, y) (b-a)^{2\beta} \|x - \tilde{x}\|_{\beta(a,b)} \\ & \quad + G_{\beta(a,b)}^3(f, \tilde{x}) (b-a)^{2\beta} \|(x - \tilde{x}) \otimes y\|_{2\beta(a,b)}, \end{aligned}$$

where

$$\begin{aligned} G_{\beta(a,b)}^1(f, x, \tilde{x}, y) & = K \left[\|y\|_\beta \|f'\|_\infty + (\|f''\|_\infty + \|f''\|_\lambda (\|x\|_{\beta(a,b)}^\lambda + \|\tilde{x}\|_{\beta(a,b)}^\lambda)) (b-a)^{\lambda\beta} \right. \\ & \quad \left. \times (\Phi_{\beta(a,b)}(x, y) + \|y\|_\beta \|\tilde{x}\|_{\beta(a,b)}) \right], \\ G_{\beta(a,b)}^2(f, x, \tilde{x}, y) & = K \left[\|y\|_\beta \|f'\|_\infty + \|f''\|_\infty (\Phi_{\beta(a,b)}(x, y) + \|y\|_\beta \|\tilde{x}\|_{\beta(a,b)}) (b-a)^\beta \right], \\ G_{\beta(a,b)}^3(f, \tilde{x}) & = K \left[\|f'\|_\infty + \|f''\|_\infty \|\tilde{x}\|_{\beta(a,b)} (b-a)^\beta \right]. \end{aligned}$$

We can also deduce the following estimate:

Proposition 5.3.5. *Suppose that $(x, y, x \otimes y)$ and $(x_{-r}, y, x_{-r} \otimes y)$ belong to $M_{d,m}^\beta(0, T)$. Assume that f satisfies **(H1)**. Then, for $0 \leq a < b \leq T$,*

$$\begin{aligned} \left| \int_a^b [f(x_u) - f(x_{u-r})] dy_u \right| &\leq G_{\beta(a,b)}^1(f, x, x_{-r}, y) (b-a)^{2\beta} \|x - x_{-r}\|_{\infty(a,b)} \\ &\quad + G_{\beta(a,b)}^2(f, x, x_{-r}, y) (b-a)^{2\beta} \|x - x_{-r}\|_{\beta(a,b)} \\ &\quad + G_{\beta(a,b)}^3(f, x_{-r}) (b-a)^{2\beta} \|(x - x_{-r}) \otimes y\|_{2\beta(a,b)}, \end{aligned} \quad (5.3.1)$$

where $G_{\beta(a,b)}^1(f, x, x_{-r}, y)$, $G_{\beta(a,b)}^2(f, x, x_{-r}, y)$ and $G_{\beta(a,b)}^3(f, x_{-r})$ are defined in Proposition 5.3.4.

Proof. The proposition is a particular case of Proposition 5.3.4 with $\tilde{x} \equiv x_{-r}$. \square

From the previous results it is possible to prove the following two propositions:

Proposition 5.3.6. *Suppose that $(x, y, x \otimes y)$, $(\tilde{x}, y, \tilde{x} \otimes y)$ and $(y, z, y \otimes z)$ belong to $M_{d,m}^\beta(0, T)$. Assume that f satisfies **(H1)**. Then, for every $0 \leq a < b \leq T$,*

$$\begin{aligned} \left| \int_a^b [f(x_u) - f(\tilde{x}_u)] d_u(y \otimes z) \right| & \quad (5.3.2) \\ &\leq G_{\beta(a,b)}^4(f, x, \tilde{x}, y, z) (b-a)^{3\beta} \|x - \tilde{x}\|_{\infty(a,b)} \\ &\quad + G_{\beta(a,b)}^5(f, x, \tilde{x}, y, z) (b-a)^{3\beta} \|x - \tilde{x}\|_{\beta(a,b)} \\ &\quad + G_{\beta(a,b)}^6(f, \tilde{x}, z) (b-a)^{3\beta} \|(x - \tilde{x}) \otimes y\|_{2\beta(a,b)} \end{aligned} \quad (5.3.3)$$

where

$$\begin{aligned} G_{\beta(a,b)}^4(f, x, \tilde{x}, y, z) &= K \left[\|f'\|_{\infty} \Phi_{\beta(a,b)}(y, z) \right. \\ &\quad \left. + (\|f''\|_{\infty} + \|f''\|_{\lambda} (\|x\|_{\beta(a,b)}^{\lambda} + \|\tilde{x}\|_{\beta(a,b)}^{\lambda})) (b-a)^{\lambda\beta} \right. \\ &\quad \left. \times (\Phi_{\beta(a,b)}(x, y, z) + \|\tilde{x}\|_{\beta(a,b)} \Phi_{\beta(a,b)}(y, z)) \right] \\ G_{\beta(a,b)}^5(f, x, \tilde{x}, y, z) &= K \left[(\|f'\|_{\infty} + \|f''\|_{\infty} \|\tilde{x}\|_{\beta(a,b)} (b-a)^{\beta}) \Phi_{\beta(a,b)}(y, z) \right. \\ &\quad \left. + \|f''\|_{\infty} \Phi_{\beta(a,b)}(x, y, z) (b-a)^{\beta} \right] \\ G_{\beta(a,b)}^6(f, \tilde{x}, z) &= K G_{\beta(a,b)}^3(f, \tilde{x}) \|z\|_{\beta(a,b)} \end{aligned}$$

for $G_{\beta(a,b)}^3(f, \tilde{x})$ defined in Proposition 5.3.4.

Proof. To simplify the proof we will assume $d = m = 1$. Observe that from inequalities (5.2.5) and (5.2.6) we obtain

$$\Phi_{\beta(a,b)}(x, y \otimes z) \leq K \Phi_{\beta(a,b)}(x, y, z)(b-a)^\beta \quad (5.3.4)$$

and

$$\begin{aligned} \|(x - \tilde{x}) \otimes (y \otimes z)_{\cdot,b}\|_{2\beta(a,b)} &\leq K \Phi_{\beta(a,b)}(y, z)(b-a)^\beta \|x - \tilde{x}\|_{\beta(a,b)} \\ &\quad + K \|z\|_{\beta(a,b)}(b-a)^\beta \|(x - \tilde{x}) \otimes y\|_{2\beta(a,b)}. \end{aligned} \quad (5.3.5)$$

The proof of the proposition is obtained applying Proposition 5.3.4 and using inequalities (5.2.5), (5.2.6), (5.3.4), (5.3.5). \square

Proposition 5.3.7. *Suppose that $(x, y, x \otimes y)$, $(x_{\cdot-r}, y, x_{\cdot-r} \otimes y)$ and $(y, z, y \otimes z)$ belong to $M_{d,m}^\beta(0, T)$. Assume that f satisfies **(H1)**. Then, for $0 \leq a < b \leq T$,*

$$\begin{aligned} \left| \int_a^b [f(x_u) - f(x_{u-r})] d_u(y \otimes z)_{\cdot,b} \right| &\quad (5.3.6) \\ &\leq G_{\beta(a,b)}^4(f, x, x_{\cdot-r}, y, z) (b-a)^{3\beta} \|x - x_{\cdot-r}\|_{\infty(a,b)} \\ &\quad + G_{\beta(a,b)}^5(f, x, x_{\cdot-r}, y, z) (b-a)^{3\beta} \|x - x_{\cdot-r}\|_{\beta(a,b)} \\ &\quad + G_{\beta(a,b)}^6(f, x_{\cdot-r}, z) (b-a)^{3\beta} \|(x - x_{\cdot-r}) \otimes y\|_{2\beta(a,b)} \end{aligned} \quad (5.3.7)$$

where $G_{\beta(a,b)}^4(f, x, x_{\cdot-r}, y, z)$, $G_{\beta(a,b)}^5(f, x, x_{\cdot-r}, y, z)$ and $G_{\beta(a,b)}^6(f, x_{\cdot-r}, z)$ are defined in Proposition 5.3.6.

Proof. The proposition is a particular case of Proposition 5.3.6 with $\tilde{x} \equiv x_{\cdot-r}$. \square

We conclude this section with a general result on β -Hölder functions:

Lemma 5.3.8. *Let $y : [0, T] \rightarrow \mathbb{R}^m$ be a β -Hölder continuous function and $\beta' = \beta - \varepsilon$ for $\varepsilon > 0$, then*

$$\|y - y_{\cdot-r}\|_{\infty(r,T)} \leq \|y\|_\beta r^\beta, \quad (5.3.8)$$

$$\|y - y_{\cdot-r}\|_{\beta'(r,T)} \leq 2\|y\|_\beta r^\varepsilon. \quad (5.3.9)$$

Proof. On one hand,

$$\|y - y_{\cdot-r}\|_{\infty(r,T)} = \sup_{t \in [r, T]} \frac{|y_t - y_{t-r}|}{r^\beta} \cdot r^\beta \leq \|y\|_\beta r^\beta.$$

On the other hand, we have

$$\begin{aligned}
 & \sup_{\substack{s < t \in [r, T] \\ t-s \leq r}} \frac{|(y - y_{\cdot-r})_t - (y - y_{\cdot-r})_s|}{(t-s)^{\beta'}} \\
 & \leq \sup_{\substack{s < t \in [r, T] \\ t-s \leq r}} \frac{|y_t - y_s|}{(t-s)^\beta} \cdot \frac{(t-s)^\beta}{(t-s)^{\beta'}} + \sup_{\substack{s < t \in [r, T] \\ t-s \leq r}} \frac{|y_{t-r} - y_{s-r}|}{(t-s)^\beta} \cdot \frac{(t-s)^\beta}{(t-s)^{\beta'}} \\
 & \leq 2\|y\|_\beta r^\varepsilon
 \end{aligned}$$

and

$$\begin{aligned}
 & \sup_{\substack{s < t \in [r, T] \\ t-s \geq r}} \frac{|(y - y_{\cdot-r})_t - (y - y_{\cdot-r})_s|}{(t-s)^{\beta'}} \\
 & \leq \sup_{\substack{s < t \in [r, T] \\ t-s \geq r}} \frac{|y_t - y_{t-r}|}{r^\beta} \cdot \frac{r^\beta}{(t-s)^{\beta'}} + \sup_{\substack{s < t \in [r, T] \\ t-s \geq r}} \frac{|y_s - y_{s-r}|}{r^\beta} \cdot \frac{r^\beta}{(t-s)^{\beta'}} \\
 & \leq 2\|y\|_\beta r^\varepsilon,
 \end{aligned}$$

so

$$\begin{aligned}
 \|y - y_{\cdot-r}\|_{\beta'(r, T)} & \leq \max \left(\sup_{\substack{s < t \in [r, T] \\ t-s \leq r}} \frac{|(y - y_{\cdot-r})_t - (y - y_{\cdot-r})_s|}{(t-s)^{\beta'}}, \right. \\
 & \quad \left. \sup_{\substack{s < t \in [r, T] \\ t-s \geq r}} \frac{|(y - y_{\cdot-r})_t - (y - y_{\cdot-r})_s|}{(t-s)^{\beta'}} \right) \\
 & \leq 2\|y\|_\beta r^\varepsilon.
 \end{aligned}$$

□

5.4 Estimates of the solutions

Let η be a non-negative bounded function such that $(\eta_{\cdot-r}, y, \eta_{\cdot-r} \otimes y) \in M_{d,m}^\beta(0, r)$. Recall the differential equation (5.2.2):

$$x_t = \eta_0 + \int_0^t b(u, x_u) du + \int_0^t \sigma(x_u) dy_u, \quad t \in [0, T].$$

and consider its solution. If $(y, y, y \otimes y) \in M_{m,m}^\beta(0, T)$, the main idea is to consider the $(x, y, x \otimes y)$, where

$$(x \otimes y)_{s,t} = \int_s^t (y_t - y_u) b(u, x_u) du + \int_s^t \sigma(x_u) d_u(y \otimes y)_{\cdot,t}. \quad (5.4.1)$$

Indeed, by definition, a solution of equation (5.2.2) is an element of $M_{d,m}^\beta(0, T)$ such that (5.2.2) and (5.4.1) hold.

Analogously, recall the differential equation (5.2.1):

$$\begin{aligned} x_t^r &= \eta_0 + \int_0^t b(u, x_u^r) du + \int_0^t \sigma(x_{u-r}^r) dy_u, & t \in [0, T], \\ x_t^r &= \eta_t, & t \in [-r, 0). \end{aligned}$$

Let $(x^r, y, x^r \otimes y) \in M_{d,m}^\beta(-r, T)$ be its solution, where $(x^r \otimes y)_{s,t}$ is defined as follows:

- for $s < t \in [-r, 0)$,

$$(x^r \otimes y)_{s,t} = (\eta \otimes y)_{s,t} = \int_s^t (y_t - y_u) d\eta_u, \quad (5.4.2)$$

- for $s \in [-r, 0)$ and $t \in [0, T]$,

$$\begin{aligned} (x^r \otimes y)_{s,t} &= (\eta \otimes y)_{s,0} + \int_0^t (y_t - y_u) b(u, x_u^r) du \\ &\quad + \int_0^t \sigma(x_{u-r}^r) d_u(y \otimes y)_{\cdot,t} + (\eta_0 - \eta_s) \otimes (y_t - y_0), \end{aligned} \quad (5.4.3)$$

- for $s < t \in [0, T]$,

$$(x^r \otimes y)_{s,t} = \int_s^t (y_t - y_u) b(u, x_u^r) du + \int_s^t \sigma(x_{u-r}^r) d_u(y \otimes y)_{\cdot,t}. \quad (5.4.4)$$

For $t \in [0, T]$, by definition

$$(x - x^r)_t = \int_0^t [b(u, x_u) - b(u, x_u^r)] du + \int_0^t [\sigma(x_u) - \sigma(x_{u-r}^r)] dy_u, \quad (5.4.5)$$

and we can express $(x - x^r)_t$ in this way

$$\begin{aligned} (x - x^r)_t &= \int_0^t [b(u, x_u) - b(u, x_u^r)] du + \int_0^t [\sigma(x_u) - \sigma(x_u^r)] dy_u \\ &\quad + \int_0^t [\sigma(x_u^r) - \sigma(x_{u-r}^r)] dy_u. \end{aligned} \quad (5.4.6)$$

Following the ideas in Section 4 of [Besalú et al., 2014], let us define $((x - x^r) \otimes y)_{s,t}$ for $s, t \in [0, T]$:

$$\begin{aligned} ((x - x^r) \otimes y)_{s,t} &= \int_s^t (y_t - y_u) [b(u, x_u) - b(u, x_u^r)] du \\ &\quad + \int_s^t [\sigma(x_u) - \sigma(x_u^r)] d_u(y \otimes y)_{\cdot,t} \\ &\quad + \int_s^t [\sigma(x_u^r) - \sigma(x_{u-r}^r)] d_u(y \otimes y)_{\cdot,t}. \end{aligned} \quad (5.4.7)$$

Moreover, if x^r is the solution of (5.2.1), then we define $\widehat{x}_t^r = x_{t-r}^r$, that is,

$$\begin{aligned} \widehat{x}_t^r &= \eta_0 + \int_0^{t-r} b(u, x_u^r) du + \int_0^{t-r} \sigma(x_{u-r}^r) dy_u, \quad t \in [r, T], \\ \widehat{x}_t^r &= \eta_{t-r}, \quad t \in [0, r). \end{aligned} \quad (5.4.8)$$

We define $(\widehat{x}^r \otimes y)_{s,t}$ as follows:

- for $s < t \in [0, r)$,

$$(\widehat{x}^r \otimes y)_{s,t} = (\eta_{-r} \otimes y)_{s,t} = \int_s^t (y_t - y_u) d\eta_{u-r}, \quad (5.4.9)$$

- for $s \in [0, r)$ and $t \in [r, T]$,

$$\begin{aligned} (\widehat{x}^r \otimes y)_{s,t} &= (\eta_{-r} \otimes y)_{s,r} + \int_r^t (y_t - y_u) b(u - r, \widehat{x}_u^r) du \\ &\quad + \int_r^t \sigma(\widehat{x}_{u-r}^r) d_u(y_{\cdot-r} \otimes y)_{\cdot,t} + (\eta_0 - \eta_{s-r}) \otimes (y_t - y_r), \end{aligned} \quad (5.4.10)$$

- for $s < t \in [r, T]$,

$$(\widehat{x}^r \otimes y)_{s,t} = \int_s^t (y_t - y_u) b(u - r, \widehat{x}_u^r) du + \int_s^t \sigma(\widehat{x}_{u-r}^r) d_u(y_{\cdot-r} \otimes y)_{\cdot,t}. \quad (5.4.11)$$

Now we need to distinguish two cases to define $(x^r - \widehat{x}^r)_t$:

- for $t \in [0, r)$,

$$(x^r - \widehat{x}^r)_t = \eta_0 - \eta_{t-r} + \int_0^t b(u, x_u^r) du + \int_0^t \sigma(\eta_{u-r}) dy_u, \quad (5.4.12)$$

- for $t \in [r, T]$,

$$(x^r - \widehat{x}^r)_t = \int_{t-r}^t b(u, x_u^r) du + \int_{t-r}^t \sigma(x_{u-r}^r) dy_u. \quad (5.4.13)$$

As a consequence, we define

- for $s < t \in [0, r)$,

$$(x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s = \eta_{s-r} - \eta_{t-r} + \int_s^t b(u, x_u^r) du + \int_s^t \sigma(\eta_{u-r}) dy_u, \quad (5.4.14)$$

- for $s \in [0, r)$ and $t \in [r, T]$,

$$\begin{aligned} (x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s &= \eta_{s-r} - \eta_0 + \int_{t-r}^t b(u, x_u^r) du - \int_0^s b(u, x_u^r) du \\ &\quad + \int_{t-r}^t \sigma(x_{u-r}^r) dy_u - \int_0^s \sigma(\eta_{u-r}) dy_u, \end{aligned} \quad (5.4.15)$$

- for $s < t \in [r, T]$,

$$\begin{aligned} (x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s &= \int_{t-r}^t b(u, x_u^r) du - \int_{s-r}^s b(u, x_u^r) du \\ &\quad + \int_{t-r}^t \sigma(x_{u-r}^r) dy_u - \int_{s-r}^s \sigma(x_{u-r}^r) dy_u \end{aligned} \quad (5.4.16)$$

$$\begin{aligned} &= \int_s^t b(u, x_u^r) du - \int_{s-r}^{t-r} b(u, x_u^r) du \\ &\quad + \int_s^t \sigma(x_{u-r}^r) dy_u - \int_{s-r}^{t-r} \sigma(x_{u-r}^r) dy_u \end{aligned} \quad (5.4.17)$$

Finally, following the ideas in Section 4 of [Besalú et al., 2014], we define

$$((x^r - \widehat{x}^r) \otimes y)_{s,t} := (x^r \otimes y)_{s,t} - (\widehat{x}^r \otimes y)_{s,t},$$

that is:

- for $s < t \in [0, r)$,

$$\begin{aligned} ((x^r - \widehat{x}^r) \otimes y)_{s,t} &= \int_s^t (y_u - y_t) d\eta_{u-r} + \int_s^t (y_t - y_u) b(u, x_u^r) du \\ &\quad + \int_s^t \sigma(\eta_{u-r}) d_u(y \otimes y)_{.,t}, \end{aligned} \quad (5.4.18)$$

- for $s \in [0, r)$ and $t \in [r, T]$,

$$\begin{aligned} ((x^r - \widehat{x}^r) \otimes y)_{s,t} &= \int_s^t (y_t - y_u) b(u, x_u^r) du + \int_s^t \sigma(x_{u-r}^r) d_u(y \otimes y)_{.,t} \\ &\quad - (\eta_{.-r} \otimes y)_{s,r} - \int_r^t (y_t - y_u) b(u-r, \widehat{x}_u^r) du \\ &\quad - \int_r^t \sigma(\widehat{x}_{u-r}^r) d_u(y_{.-r} \otimes y)_{.,t} - (\eta_0 - \eta_{s-r}) \otimes (y_t - y_r), \end{aligned} \quad (5.4.19)$$

- for $s < t \in [r, T]$,

$$\begin{aligned}
 ((x^r - \widehat{x}^r) \otimes y)_{s,t} &= \int_s^t (y_t - y_u) b(u, x_u^r) du + \int_s^t \sigma(x_{u-r}^r) d_u(y \otimes y)_{\cdot,t} \\
 &\quad - \int_s^t (y_t - y_u) b(u-r, \widehat{x}_u^r) du - \int_s^t \sigma(\widehat{x}_{u-r}^r) d_u(y_{\cdot-r} \otimes y)_{\cdot,t} \\
 &= \int_s^t (y_{u+r} - y_u) b(u, x_u^r) du \\
 &\quad + \int_s^t \sigma(x_{u-r}^r) d_u((y - y_{\cdot-r}) \otimes y)_{\cdot,t} \\
 &\quad + \int_s^t [\sigma(x_{u-r}^r) - \sigma(\widehat{x}_{u-r}^r)] d_u(y_{\cdot-r} \otimes y)_{\cdot,t}
 \end{aligned} \tag{5.4.20}$$

Let $\varepsilon > 0$ be such that $\beta - 2\varepsilon > 0$ and $\lambda > \frac{1}{\beta - \varepsilon} - 2$. Recall that $\beta' = \beta - \varepsilon$.

Before giving further results, we will prove that the norms $\|\widehat{x}^r\|_{\beta'}$ and $\|\widehat{x}^r \otimes y\|_{2\beta'}$ are bounded and their upper bound does not depend on r . To this aim, the following lemma will be useful:

Lemma 5.4.1. *Let $(\eta_{\cdot-r}, y, \eta_{\cdot-r} \otimes y) \in M_{d,m}^\beta(0, r)$ and $(y_{\cdot-r}, y, y_{\cdot-r} \otimes y) \in M_{d,d}^\beta(r, T)$. Let $(x^r, y, x^r \otimes y) \in M_{d,m}^\beta(0, T)$ be the solution of the equation (5.2.1). Then,*

$$\|\widehat{x}^r\|_{\beta'} \leq \|\widehat{x}^r\|_{\beta'(0,r)} + \|\widehat{x}^r\|_{\beta'(r,T)} \tag{5.4.21}$$

and

$$\|\widehat{x}^r \otimes y\|_{2\beta'} \leq \|\widehat{x}^r \otimes y\|_{2\beta'(0,r)} + \|\widehat{x}^r \otimes y\|_{2\beta'(r,T)} + \|\eta\|_{\beta'(-r,0)} \|y\|_{\beta'}. \tag{5.4.22}$$

Proof. On one hand, observe that

$$\|\widehat{x}^r\|_{\beta'} \leq \max \left(\sup_{0 \leq s < t < r} \frac{|\widehat{x}_t^r - \widehat{x}_s^r|}{(t-s)^{\beta'}}, \sup_{0 \leq s < r \leq t \leq T} \frac{|\widehat{x}_t^r - \widehat{x}_s^r|}{(t-s)^{\beta'}}, \sup_{r \leq s < t \leq T} \frac{|\widehat{x}_t^r - \widehat{x}_s^r|}{(t-s)^{\beta'}} \right),$$

and

$$\begin{aligned}
 \sup_{0 \leq s < r \leq t \leq T} \frac{|\widehat{x}_t^r - \widehat{x}_s^r|}{(t-s)^{\beta'}} &\leq \sup_{0 \leq s < r \leq t \leq T} \frac{|\widehat{x}_t^r - \widehat{x}_r^r|}{(t-s)^{\beta'}} + \sup_{0 \leq s < r \leq t \leq T} \frac{|\widehat{x}_r^r - \widehat{x}_s^r|}{(t-s)^{\beta'}} \\
 &\leq \sup_{r \leq t \leq T} \frac{|\widehat{x}_t^r - \widehat{x}_r^r|}{(t-r)^{\beta'}} + \sup_{0 \leq s < r} \frac{|\widehat{x}_r^r - \widehat{x}_s^r|}{(r-s)^{\beta'}} \\
 &\leq \sup_{r \leq s < t \leq T} \frac{|\widehat{x}_t^r - \widehat{x}_s^r|}{(t-s)^{\beta'}} + \sup_{0 \leq s < t < r} \frac{|\widehat{x}_t^r - \widehat{x}_s^r|}{(t-s)^{\beta'}}.
 \end{aligned}$$

So we easily get (5.4.21).

On the other hand, observe that from the multiplicative property we obtain

$$\sup_{0 \leq s < r \leq t \leq T} \frac{|(\widehat{x}^r \otimes y)_{s,t}|}{(t-s)^{2\beta'}} \leq \sup_{0 \leq s < r \leq t \leq T} \left[\frac{|(\widehat{x}^r \otimes y)_{s,r}|}{(t-s)^{2\beta'}} + \frac{|(\widehat{x}^r \otimes y)_{r,t}|}{(t-s)^{2\beta'}} + \frac{|(\widehat{x}_r^r - \widehat{x}_s^r) \otimes (y_t - y_r)|}{(t-s)^{2\beta'}} \right],$$

and using the same argument as before (5.4.22) follows easily. \square

Now we can give the following result:

Proposition 5.4.2. *Let $(\eta_{-r}, y, \eta_{-r} \otimes y) \in M_{d,m}^\beta(0, r)$ and $(y_{-r}, y, y_{-r} \otimes y) \in M_{d,d}^\beta(r, T)$. Assume that σ and b satisfy **(H1)** and **(H2)** respectively, and both satisfy **(H3)**. Let $(x^r, y, x^r \otimes y) \in M_{d,m}^\beta(0, T)$ be the solution of the equation (5.2.1). Assume also that $\|\eta\|_{\beta(-r_0, 0)} < \infty$ and $\sup_{r \leq r_0} \|\eta_{-r} \otimes y\|_{2\beta(0, r)} < \infty$. Then, for $r \leq r_0$, we have the following estimates:*

$$\|\widehat{x}^r\|_{\infty(0, T+r)} \leq M_{\eta, y}, \quad (5.4.23)$$

$$\|\widehat{x}^r\|_{\beta'(0, T+r)} \leq K \rho_{\eta, b, \sigma} \Lambda_y (1 + 2M_{\eta, y}), \quad (5.4.24)$$

$$\|\widehat{x}^r \otimes y\|_{2\beta'(0, T+r)} \leq K \rho_{\eta, b, \sigma} \Lambda_y (2 + (T + r_0)(K \rho_{\eta, b, \sigma} \Lambda_y)^{\frac{1}{\beta}}), \quad (5.4.25)$$

where $K \geq 1$ and

$$\rho_{\eta, b, \sigma} := 2\|\eta\|_{\beta(-r_0, 0)} + \|b\|_{\infty} T^{1-\beta} + \|\sigma\|_{\infty} + \|\sigma'\|_{\infty} + \|\sigma'\|_{\lambda}, \quad (5.4.26)$$

$$\Lambda_y := \|y\|_{\beta} + \max(1, \|y\|_{\beta}^2 + \|y \otimes y\|_{2\beta}), \quad (5.4.27)$$

and

$$M_{\eta, y} := |\eta_0| + (T + r_0)(K \rho_{\eta, b, \sigma} \Lambda_y)^{\frac{1}{\beta}} + 1. \quad (5.4.28)$$

Proof. To simplify the proof we will assume $d = m = 1$. Assume also that $r \leq r_0$.

First we observe that, if $\|\eta\|_{\beta(-r_0, 0)} < C$ and $\sup_{r \leq r_0} \|\eta_{-r} \otimes y\|_{2\beta(0, r)} < C'$, with C and C' two positive constants, then $\|\eta\|_{\beta'(-r_0, 0)} < C r_0^\varepsilon$ and $\sup_{r \leq r_0} \|\eta_{-r} \otimes y\|_{2\beta'(0, r)} < C' r_0^{2\varepsilon}$.

Secondly, notice that by definition (5.4.9)

$$\|\eta_{-r} \otimes y\|_{2\beta(0, r)} = \sup_{s, t \in [0, r]} \frac{|\int_s^t (y_t - y_u) d\eta_{u-r}|}{(t-s)^{2\beta}} \leq \|\eta_{-r}\|_{\beta(0, r)} \|y\|_{\beta} \leq \|\eta\|_{\beta(-r_0, 0)} \|y\|_{\beta}.$$

If η is differentiable and monotone and $\|\eta\|_{\beta(-r_0, 0)} < \infty$, then

$$\|\eta_{-r} \otimes y\|_{2\beta(0, r)} < \infty.$$

Moreover, for all $\beta' < \beta$, it is true that $\|\eta\|_{\beta'(-r_0, 0)} < \infty$ and

$$\|\eta_{-r} \otimes y\|_{2\beta'(0, r)} \leq \|\eta\|_{\beta'(-r_0, 0)} \|y\|_{\beta'} < \infty.$$

To prove the result we will follow the ideas of Theorem 4.1 of [Besalú and Nualart, 2011]. Consider the mapping $J : M_{1,1}^\beta(0, T+r) \rightarrow M_{1,1}^\beta(0, T+r)$ given by $J(\widehat{x}^r, y, \widehat{x}^r \otimes y) = (J_1, y, J_2)$ where J_1 and J_2 are the right-hand sides of the definition of \widehat{x}^r (see equation (5.4.8)) and $(\widehat{x}^r \otimes y)$ (see equations from (5.4.9) to (5.4.11)) respectively:

$$J_1(\widehat{x}^r, y, \widehat{x}^r \otimes y)(t) = \begin{cases} \eta_{t-r}, & 0 \leq t < r \\ \eta_0 + \int_0^{t-r} b(u, \widehat{x}_{u+r}^r) du + \int_0^{t-r} \sigma(\widehat{x}_u^r) dy_u, & r \leq t \leq T \end{cases} \quad (5.4.29)$$

$$J_2(\widehat{x}^r, y, \widehat{x}^r \otimes y)(s, t) = \begin{cases} \int_s^t (y_t - y_u) d\eta_{u-r}, & 0 \leq s \leq t < r \\ (\eta_0 - \eta_{s-r}) \otimes (y_t - y_r) + \int_s^r (y_r - y_u) d\eta_{u-r} \\ \quad + \int_s^t (y_t - y_u) b(u-r, \widehat{x}_u^r) du \\ \quad + \int_r^t \sigma(\widehat{x}_{u-r}^r) d_u(y_{-r} \otimes y)_{.,t}, & 0 \leq s < r \leq t \leq T \\ \int_s^t (y_t - y_u) b(u-r, \widehat{x}_u^r) du \\ \quad + \int_s^t \sigma(\widehat{x}_{u-r}^r) d_u(y_{-r} \otimes y)_{.,t} & r \leq s \leq t \leq T \end{cases} \quad (5.4.30)$$

Remark that this mapping is well-defined because (J_1, y, J_2) is a real-valued β -Hölder continuous multiplicative functional for each $(\widehat{x}^r, y, \widehat{x}^r \otimes y) \in M_{1,1}^\beta(0, T)$.

Now we bound the Hölder norms of J_1 and J_2 using Proposition 5.3.1 and Proposition 5.3.2. Let $s < t \in [0, T]$, we have

- for $s < t \in [0, r)$

$$\|J_1\|_{\beta(s,t)} \leq \|\eta\|_{\beta(-r_0,0)}, \quad (5.4.31)$$

$$\|J_2\|_{2\beta(s,t)} \leq \|\eta\|_{\beta(-r_0,0)} \|y\|_{\beta}, \quad (5.4.32)$$

- for $s < t \in [r, T]$

$$\begin{aligned} \|J_1\|_{\beta(s,t)} &\leq \|b\|_\infty (t-s)^{1-\beta} + K\|\sigma\|_\infty \|y\|_\beta \\ &\quad + K\left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}_{-r}^r\|_{\beta(s,t)}^\lambda (t-s)^{\lambda\beta}\right) \\ &\quad \times \Phi_{\beta(s,t)}(\widehat{x}_{-r}^r, y_{-r})(t-s)^\beta, \end{aligned} \quad (5.4.33)$$

$$\begin{aligned} \|J_2\|_{2\beta(s,t)} &\leq \|b\|_\infty \|y\|_\beta (t-s)^{1-\beta} + K\|\sigma\|_\infty \Phi_{\beta(s,t)}(y_{-r}, y) \\ &\quad + K\left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}_{-r}^r\|_{\beta(s,t)}^\lambda (t-s)^{\lambda\beta}\right) \\ &\quad \times \Phi_{\beta(s,t)}(\widehat{x}_{-r}^r, y_{-r}, y)(t-s)^\beta, \end{aligned} \quad (5.4.34)$$

- for $s \in [0, r)$ and $t \in [r, T]$

$$\begin{aligned} \|J_1\|_{\beta(s,t)} &\leq \|J_1\|_{\beta(s,r)} + \|J_1\|_{\beta(r,t)} \\ &\leq \|\eta\|_{\beta(-r_0,0)} + \|b\|_\infty (t-r)^{1-\beta} + K\|\sigma\|_\infty \|y\|_\beta \\ &\quad + K\left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}_{-r}^r\|_{\beta(r,t)}^\lambda (t-r)^{\lambda\beta}\right) \\ &\quad \times \Phi_{\beta(r,t)}(\widehat{x}_{-r}^r, y_{-r})(t-r)^\beta, \end{aligned} \quad (5.4.35)$$

$$\begin{aligned} \|J_2\|_{2\beta(s,t)} &\leq \|J_2\|_{2\beta(s,r)} + \|J_2\|_{2\beta(r,t)} \\ &\leq 2\|\eta\|_{\beta(-r_0,0)} \|y\|_\beta + \|b\|_\infty \|y\|_\beta (t-r)^{1-\beta} + K\|\sigma\|_\infty \Phi_{\beta(r,t)}(y_{-r}, y) \\ &\quad + K\left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}_{-r}^r\|_{\beta(r,t)}^\lambda (t-r)^{\lambda\beta}\right) \\ &\quad \times \Phi_{\beta(r,t)}(\widehat{x}_{-r}^r, y_{-r}, y)(t-r)^\beta. \end{aligned} \quad (5.4.36)$$

For $s < t \in [r, T]$, we set

$$(\widehat{x}_{-r}^r \otimes y_{-r})_{s,t} := (\widehat{x} \otimes y)_{s-r, t-r}. \quad (5.4.37)$$

In Section 5 of [Besalú et al., 2014] it is proved that it is a β -Hölder continuous multiplicative functional.

We proceed dividing the proof in two steps.

Step 1: We will find a set C^y of elements $(\widehat{x}^r, y, \widehat{x}^r \otimes y) \in M_{1,1}^\beta(0, T)$ such that $\overline{J(C^y)} \subset C^y$. Recall definitions of $\rho_{\eta,b,\sigma}$ and Λ_y from (5.4.26) and (5.4.27), respectively, and set

$$\widetilde{\Delta}_y := (K\rho_{\eta,b,\sigma}\Lambda_y)^{-\frac{1}{\beta}}. \quad (5.4.38)$$

Let C^y be the set of elements $(\widehat{x}^r, y, \widehat{x}^r \otimes y) \in M_{1,1}^\beta(0, T)$ satisfying the following conditions:

$$\|\widehat{x}^r\|_\infty \leq M_{\eta,y}, \quad (5.4.39)$$

$$\sup_{0 < t-s \leq \widetilde{\Delta}_y} \|\widehat{x}^r\|_{\beta(s,t)} \leq K\rho_{\eta,b,\sigma}(\|y\|_\beta + 1), \quad (5.4.40)$$

$$\sup_{0 < t-s \leq \widetilde{\Delta}_y} \|\widehat{x}^r \otimes y\|_{2\beta(s,t)} \leq K\rho_{\eta,b,\sigma}(\|y\|_\beta + \|y\|_\beta^2 + \|y_{-r} \otimes y\|_{2\beta}). \quad (5.4.41)$$

We take $s, t \in [0, T]$ such that

$$0 < t - s \leq \tilde{\Delta}_y, \quad (5.4.42)$$

and then we have

$$(t - s)^\beta \leq \tilde{\Delta}_y^\beta \leq \frac{1}{K\rho_{\eta,b,\sigma}(\|y\|_\beta + 1)} \quad (5.4.43)$$

and

$$(t - s)^\beta \leq \tilde{\Delta}_y^\beta \leq \frac{1}{K\rho_{\eta,b,\sigma}(\|y\|_\beta + \|y\|_\beta^2 + \|y_{-r} \otimes y\|_{2\beta})}. \quad (5.4.44)$$

Suppose that $(\hat{x}^r, y, \hat{x}^r \otimes y) \in C^y$, then using (5.4.40), (5.4.43) and (5.4.41), (5.4.44) respectively, we have

$$(t - s)^\beta \|\hat{x}^r\|_{\beta(s,t)} \leq 1, \quad (5.4.45)$$

$$(t - s)^\beta \|\hat{x}^r \otimes y\|_{2\beta(s,t)} \leq 1. \quad (5.4.46)$$

Now observe that, if $s, t \in [r, T]$ satisfy (5.4.42), then $s - r, t - r \in [0, T]$ also satisfy this condition. As a consequence,

$$(t - s)^\beta \|\hat{x}_{-r}^r\|_{\beta(s,t)} \leq 1 \quad (5.4.47)$$

and

$$(t - s)^\beta \|\hat{x}_{-r}^r \otimes y_{-r}\|_{2\beta(s,t)} \leq 1. \quad (5.4.48)$$

From the last inequality it easily follows that

$$\begin{aligned} & \Phi_{\beta(s,t)}(\hat{x}_{-r}^r, y_{-r}, y)(t - s)^\beta \\ &= \left[\|\hat{x}_{-r}^r\|_{\beta(s,t)} \|y_{-r}\|_{\beta(s,t)} \|y\|_{\beta(s,t)} + \|y\|_{\beta(s,t)} \|\hat{x}_{-r}^r \otimes y_{-r}\|_{2\beta(s,t)} \right. \\ & \quad \left. + \|\hat{x}_{-r}^r\|_{\beta(s,t)} \|y_{-r} \otimes y\|_{2\beta(s,t)} \right] (t - s)^\beta \\ & \leq \|y\|_\beta + \|y\|_\beta^2 + \|y_{-r} \otimes y\|_{2\beta}. \end{aligned} \quad (5.4.49)$$

Also, observe that if $s \in [0, r)$ and $t \in [r, T]$ satisfy (5.4.42), then $t - r \leq \tilde{\Delta}_y$ and all the previous inequality are satisfied if we change the interval (s, t) to the interval (r, t) for $t \in [r, T]$.

By expressions from (5.4.31) to (5.4.34) and from (5.4.45) to (5.4.49) we easily get that

$$\begin{aligned} \|J_1\|_{\beta(s,t)} & \leq \|\eta\|_{\beta(-r_0,0)} + \|b\|_\infty T^{1-\beta} + K\|\sigma\|_\infty \|y\|_\beta \\ & \quad + K(\|\sigma'\|_\infty + \|\sigma'\|_\lambda)(\|y\|_\beta + 1) \\ & \leq K\rho_{\eta,b,\sigma}(\|y\|_\beta + 1) \end{aligned} \quad (5.4.50)$$

and

$$\begin{aligned} \|J_2\|_{2\beta(s,t)} & \leq 2\|\eta\|_{\beta(-r_0,0)}\|y\|_\beta + \|b\|_\infty \|y\|_\beta T^{1-\beta} + K\|\sigma\|_\infty(\|y\|_\beta^2 + \|y_{-r} \otimes y\|_{2\beta}) \\ & \quad + K(\|\sigma'\|_\infty + \|\sigma'\|_\lambda)(\|y\|_\beta + \|y\|_\beta^2 + \|y_{-r} \otimes y\|_{2\beta}) \\ & \leq K\rho_{\eta,b,\sigma}(\|y\|_\beta + \|y\|_\beta^2 + \|y_{-r} \otimes y\|_{2\beta}) \end{aligned} \quad (5.4.51)$$

where $K \geq 1$.

It only remains to prove that $\|J_1\|_\infty \leq M_{\eta,y}$. Set $N = \lceil (T+r)\tilde{\Delta}_y^{-1} \rceil + 1$ and define the partition $t_0 = 0 < t_1 < \dots < t_N = T+r$, where $t_i = i\tilde{\Delta}_y$ for $i = 0, \dots, N-1$. The estimates (5.4.43) and (5.4.50) imply

$$\sup_{u \in [t_{i-1}, t_i]} |(J_1)_u| \leq |(J_1)_{t_{i-1}}| + (t_i - t_{i-1})^\beta \|J_1\|_{\beta(t_{i-1}, t_i)} \leq |(J_1)_{t_{i-1}}| + 1.$$

Moreover,

$$\sup_{u \in [0, t_i]} |(J_1)_u| \leq \sup_{u \in [0, t_{i-1}]} |(J_1)_u| + 1,$$

and iterating we finally get that

$$\sup_{u \in [0, T]} |(J_1)_u| \leq |\eta_0| + N \leq |\eta_0| + T\tilde{\Delta}_y^{-1} + 1 = M_{\eta,y}.$$

Hence, $(J_1, y, J_2) \in C^y$.

Step 2: We find a bound for the Hölder norms of \hat{x}^r and $(\hat{x}^r \otimes y)$.

We can construct a sequence of functions $\hat{x}^{r(n)}$ and $(\hat{x}^r \otimes y)^{(n)}$ such that,

$$\hat{x}^{r(0)} = \eta_0 \quad \text{and} \quad (\hat{x}^r \otimes y)^{(0)} = 0$$

and

$$\begin{aligned} \hat{x}^{r(n)} &= J_1 \left(\hat{x}^{r(n-1)}, y, (\hat{x}^r \otimes y)^{(n-1)} \right), \\ (\hat{x}^r \otimes y)^{(n)} &= J_2 \left(\hat{x}^{r(n-1)}, y, (\hat{x}^r \otimes y)^{(n-1)} \right). \end{aligned}$$

Notice that $(\hat{x}^{r(0)}, y, (\hat{x}^r \otimes y)^{(0)}) \in C^y$ and, since we have proved in Step 1 that $J(C^y) \subset C^y$, we have that $(\hat{x}^{r(n)}, y, (\hat{x}^r \otimes y)^{(n)}) \in C^y$ for each n . We estimate $\|\hat{x}^{r(n)}\|_\beta$ as follows:

$$\begin{aligned} \|\hat{x}^{r(n)}\|_\beta &\leq \sup_{\substack{0 \leq s < t \leq T \\ t-s \leq \tilde{\Delta}_y}} \frac{|\hat{x}_t^{r(n)} - \hat{x}_s^{r(n)}|}{(t-s)^\beta} + \sup_{\substack{0 \leq s < t \leq T \\ t-s \geq \tilde{\Delta}_y}} \frac{|\hat{x}_t^{r(n)} - \hat{x}_s^{r(n)}|}{(t-s)^\beta} \\ &\leq K\rho_{\eta,b,\sigma}(\|y\|_\beta + 1) + 2\tilde{\Delta}_y^{-\beta} \|\hat{x}^{r(n)}\|_\infty \\ &\leq K\rho_{\eta,b,\sigma}(\|y\|_\beta + 1) + 2\tilde{\Delta}_y^{-\beta} M_{\eta,y} \\ &\leq K\rho_{\eta,b,\sigma}\Lambda_y(1 + 2M_{\eta,y}). \end{aligned} \tag{5.4.52}$$

This implies that the sequence of functions $\hat{x}^{r(n)}$ is equicontinuous and bounded in $C^\beta(0, T)$ and the upper bound does not depend on r . So, there exists a subsequence which converges in the β' -Hölder norm if $\beta' < \beta$ and such that the upper bound of the β' -Hölder norm does not depend on r .

In a similar way we obtain the same result for $(\widehat{x}^r \otimes y)^{(n)}$. From inequality (5.4.46) we obtain that

$$\begin{aligned} \sup_{t_{i-1} \leq s < t \leq t_i} |(\widehat{x}^r \otimes y)_{s,t}^{(n)}| &\leq \|(\widehat{x}^r \otimes y)^{(n)}\|_{2\beta(t_{i-1}, t_i)} (t_i - t_{i-1})^{2\beta} \\ &\leq (t_i - t_{i-1})^\beta \leq \widetilde{\Delta}_y^\beta \end{aligned}$$

and

$$\sup_{0 \leq s < t \leq T} |(\widehat{x}^r \otimes y)_{s,t}^{(n)}| \leq N \widetilde{\Delta}_y^\beta \leq T \widetilde{\Delta}_y^{\beta-1} + \widetilde{\Delta}_y^\beta.$$

As for (5.4.52), we estimate $\|(\widehat{x}^r \otimes y)^{(n)}\|_{2\beta}$ as follows:

$$\begin{aligned} \|(\widehat{x}^r \otimes y)^{(n)}\|_{2\beta} &\leq K \rho_{\eta, b, \sigma} (\|y\|_\beta^2 + \|y\|_\beta + \|y_{\cdot-r} \otimes y\|_{2\beta}) + T \widetilde{\Delta}_y^{-\beta-1} + \widetilde{\Delta}_y^{-\beta} \\ &\leq K \rho_{\eta, b, \sigma} \Lambda_y (2 + (T + r_0) (K \rho_{\eta, b, \sigma} \Lambda_y)^{\frac{1}{\beta}}). \end{aligned} \quad (5.4.53)$$

This implies that the sequence of functions $(\widehat{x}^r \otimes y)^{(n)}$ is bounded and equicontinuous in the set of functions 2β -Hölder continuous on Δ_T , and the upper bound does not depend on r . So, there exists a subsequence which converges in the β' -Hölder norm if $\beta' < \beta$ and such that the upper bound of the β' -Hölder norm does not depend on r .

Now as n tends to infinity it is easy to see that the limit is a solution, and the limit defines a β -Hölder continuous multiplicative functional $(\widehat{x}^r, y, \widehat{x}^r \otimes y)$ and this functional satisfies estimates (5.4.23), (5.4.24) and (5.4.25). \square

Remark 5.4.3. In Proposition 5.4.2 it is proved that $\|\widehat{x}^r\|_{\beta'(0, T)} \leq K \rho_{\eta, b, \sigma} \Lambda_y (1 + 2M_{\eta, y})$, so we have the same bound for $\|x^r\|_{\beta'(r)}$. Moreover, using the ideas in the proof of Proposition 5.4.2 it is possible to prove that $\|x^r \otimes y\|_{2\beta'}$ is bounded and its bound does not depend on r .

The following proposition gives us a result about the behavior of $(x^r - \widehat{x}^r)$ when r tends to zero.

Proposition 5.4.4. *Let $\beta' = \beta - \varepsilon$, where $\varepsilon > 0$ is such that $\beta - 2\varepsilon > 0$ and $\lambda > \frac{1}{\beta - \varepsilon} - 2$. Suppose that $(x, y, x \otimes y)$, $(x^r, y, x^r \otimes y)$, $(\widehat{x}^r, y, \widehat{x}^r \otimes y)$ and $(y, y, y \otimes y)$ belong to $M_{d, m}^\beta(0, T)$. Assume that σ and b satisfy **(H1)** and **(H2)** respectively, and both satisfy **(H3)**. Assume also that $\|\eta\|_{\beta(-r_0, 0)} < \infty$ and $\sup_{r \leq r_0} \Phi_{\beta(0, r)}(\eta_{\cdot-r}, y) < \infty$ and suppose that $\|(y - y_{\cdot-r}) \otimes y\|_{2\beta'(r, T)} \rightarrow 0$ and $\|y_{\cdot-r} \otimes (y - y_{\cdot-r})\|_{2\beta'(r, T)} \rightarrow 0$ when r tends to zero. Then*

$$\begin{aligned} \|x^r - \widehat{x}^r\|_\infty &\leq K \rho \Lambda r^{\beta'} \\ \|x^r - \widehat{x}^r\|_{\beta'} &\leq K \rho \Lambda r^\varepsilon \\ \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'} &\leq K M \rho^3 \Lambda^3 r^\varepsilon + K M \rho^3 \Lambda^2 \Lambda_r \end{aligned}$$

where $K \geq 1$, $M \geq 1$ are constants depending on $\beta, \beta', r_0, T, \sigma, y$ and

$$\begin{aligned} \rho &= \left(1 + 3\|b\|_\infty T^{1-\beta'} + 3\|\sigma\|_\infty(1 + T^{\beta'}) + 2\|\sigma'\|_\infty(1 + T^{\beta'}) + 3\|\sigma'\|_\infty T^{\beta'-\varepsilon} \right. \\ &\quad \left. + \|\sigma'\|_\lambda \left(2 \sup_{r \leq r_0} \|x^r\|_{\beta'}^\lambda + \|\eta\|_{\beta'(-r_0,0)}^\lambda\right) T^{(\lambda+1)\beta'-\varepsilon} + \|\sigma''\|_\infty T^{\beta'}(1 + T^{\beta'}) \right. \\ &\quad \left. + 2\|\sigma''\|_\lambda \sup_{r \leq r_0} \|\widehat{x}^r\|_{\beta'}^\lambda T^{(\lambda+1)\beta'}\right) (1 + T^\varepsilon), \\ \Lambda &= \max \left(1, \|\eta\|_{\beta(-r_0,0)}, \sup_{r \leq r_0} \Phi_{\beta'(0,r)}(\eta_{-r}, y), \Phi_{\beta(0,T)}(y, y), \sup_{r \leq r_0} \Phi_{\beta'(r,T)}(y_{-r}, y), \right. \\ &\quad \left. \sup_{r \leq r_0} \Phi_{\beta'(0,r)}(\eta_{-r}, y, y), \sup_{r \leq r_0} \Phi_{\beta'(0,T)}(x^r, y), \sup_{r \leq r_0} \Phi_{\beta'(0,T)}(\widehat{x}^r, y)\right) \\ &\quad \times \left(1 + \sup_{r \leq r_0} \|x^r\|_{\beta'(r)}\right) (1 + \|y\|_\beta), \\ \Lambda_r &= \max \left(1, \sup_{r \leq r_0} \|x^r\|_{\beta'}\right) \left(\|(y - y_{-r}) \otimes y\|_{2\beta'(r,T)} + \|y_{-r} \otimes (y - y_{-r})\|_{2\beta'(r,T)}\right). \end{aligned}$$

Remark 5.4.5. Thanks to Proposition 5.4.2, ρ and Λ are finite and, by hypothesis, Λ_r converges to zero when r tends to zero. Hence, the proposition states that

$$\begin{aligned} \|x^r - \widehat{x}^r\|_\infty &\xrightarrow{r \downarrow 0} 0, \\ \|x^r - \widehat{x}^r\|_{\beta'} &\xrightarrow{r \downarrow 0} 0, \\ \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'} &\xrightarrow{r \downarrow 0} 0. \end{aligned}$$

Proof. We start studying the supremum norm. Observe that

$$\|x^r - \widehat{x}^r\|_\infty \leq \|x^r - \widehat{x}^r\|_{\infty(0,r)} + \|x^r - \widehat{x}^r\|_{\infty(r,T)}.$$

On one hand, by definition (5.4.12) and using Proposition 5.3.1, for $r \leq r_0$, we obtain

$$\begin{aligned} \|x^r - \widehat{x}^r\|_{\infty(0,r)} &\leq \|\eta\|_{\beta'(-r,0)} r^{\beta'} + \|b\|_\infty r + K\|\sigma\|_\infty \|y\|_{\beta'} r^{\beta'} \\ &\quad + K\Phi_{\beta'(0,r)}(\eta_{-r}, y) (\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\eta_{-r}\|_{\beta'(0,r)}^\lambda r^{\lambda\beta'}) r^{2\beta'} \\ &\leq \left[\|\eta\|_{\beta(-r_0,0)} T^\varepsilon + \|b\|_\infty T^{1-\beta'} + K\|\sigma\|_\infty \|y\|_\beta T^\varepsilon \right. \\ &\quad \left. + K\Phi_{\beta'(0,r)}(\eta_{-r}, y) (\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\eta\|_{\beta'(-r_0,0)}^\lambda T^{\lambda\beta'}) T^{\beta'} \right] r^{\beta'} \end{aligned}$$

where we used that $\|\eta\|_{\beta'(-r,0)} \leq \|\eta\|_{\beta(-r_0,0)} T^\varepsilon$ and $\|y\|_{\beta'} \leq \|y\|_\beta T^\varepsilon$.

On the other hand, by definition (5.4.13) and using Proposition 5.3.1 we obtain

$$\begin{aligned} \|x^r - \widehat{x}^r\|_{\infty(r,T)} &\leq \|b\|_\infty r + K\|\sigma\|_\infty \|y\|_{\beta'} r^{\beta'} \\ &\quad + K\Phi_{\beta'(0,T)}(\widehat{x}^r, y) (\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r\|_{\beta'(r,T)}^\lambda r^{\lambda\beta'}) r^{2\beta'} \\ &\leq \left[\|b\|_\infty T^{1-\beta'} + K\|\sigma\|_\infty \|y\|_\beta T^\varepsilon \right. \\ &\quad \left. + K\Phi_{\beta'(0,T)}(\widehat{x}^r, y) (\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|x^r\|_{\beta'}^\lambda T^{\lambda\beta'}) T^{\beta'} \right] r^{\beta'}. \end{aligned}$$

Hence, we have that

$$\|x^r - \widehat{x}^r\|_\infty \leq K\rho\Lambda r^{\beta'}, \quad (5.4.54)$$

as we wanted.

Now we study the Hölder norms. Following the proof of Lemma 5.4.1 we easily obtain that

$$\|x^r - \widehat{x}^r\|_{\beta'} \leq \|x^r - \widehat{x}^r\|_{\beta'(0,r)} + \|x^r - \widehat{x}^r\|_{\beta'(r,T)} \quad (5.4.55)$$

and

$$\begin{aligned} \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'} &\leq \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(0,r)} + \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(r,T)} \\ &\quad + \|x^r - \widehat{x}^r\|_{\beta'(0,r)} \|y\|_{\beta'}. \end{aligned} \quad (5.4.56)$$

So we can study the Hölder norms independently in the intervals $[0, r)$ and $[r, T]$. We study the Hölder norm of $(x^r - \widehat{x}^r)$. In the interval $[0, r)$ by definition (5.4.14) and Proposition 5.3.1 we have

$$\begin{aligned} \|x^r - \widehat{x}^r\|_{\beta'(0,r)} &\leq \|\eta\|_{\beta'(-r,0)} + \|b\|_\infty r^{1-\beta'} + K\|\sigma\|_\infty \|y\|_{\beta'(0,r)} \\ &\quad + K\Phi_{\beta'(0,r)}(\eta_{-r}, y) (\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\eta_{-r}\|_{\beta'(0,r)}^\lambda r^{\lambda\beta'}) r^{\beta'} \\ &\leq \left[\|\eta\|_{\beta(-r_0,0)} + \|b\|_\infty T^{1-\beta} + K\|\sigma\|_\infty \|y\|_\beta \right. \\ &\quad \left. + K\Phi_{\beta'(0,r)}(\eta_{-r}, y) (\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\eta\|_{\beta'(-r_0,0)}^\lambda T^{\lambda\beta'}) T^{\beta'-\varepsilon} \right] r^\varepsilon. \end{aligned} \quad (5.4.57)$$

In the interval $[r, T]$, observe that

$$\begin{aligned} &\|x^r - \widehat{x}^r\|_{\beta'(r,T)} \\ &\leq \max \left(\sup_{\substack{s < t \in [r, T] \\ t-s \leq r}} \frac{|(x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s|}{(t-s)^{\beta'}}, \sup_{\substack{s < t \in [r, T] \\ t-s \geq r}} \frac{|(x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s|}{(t-s)^{\beta'}} \right). \end{aligned} \quad (5.4.58)$$

On one hand, by definition (5.4.17) and Proposition 5.3.1 we have

$$\begin{aligned} &\sup_{\substack{s < t \in [r, T] \\ t-s \leq r}} \frac{|(x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s|}{(t-s)^{\beta'}} \\ &\leq 2\|b\|_\infty r^{1-\beta'} + 2K\|\sigma\|_\infty \|y\|_\beta r^\varepsilon \\ &\quad + 2K\Phi_{\beta'(r,T)}(\widehat{x}^r, y) (\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r\|_{\beta'(r,T)}^\lambda r^{\lambda\beta'}) r^{\beta'} \\ &\leq \left[2\|b\|_\infty T^{1-\beta} + 2K\|\sigma\|_\infty \|y\|_\beta \right. \\ &\quad \left. + 2K\Phi_{\beta'(0,T)}(\widehat{x}^r, y) (\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|x^r\|_{\beta'}^\lambda T^{\lambda\beta'}) T^{\beta'-\varepsilon} \right] r^\varepsilon. \end{aligned} \quad (5.4.59)$$

where we used that $\sup_{\substack{s < t \in [r, T] \\ t-s \leq r}} \|y\|_{\beta'(s,t)} \leq \|y\|_{\beta} r^{\varepsilon}$.

On the other hand, with a similar computation, by definition (5.4.16) and Proposition 5.3.1 we have

$$\begin{aligned} & \sup_{\substack{s < t \in [r, T] \\ t-s \geq r}} \frac{|(x^r - \widehat{x}^r)_t - (x^r - \widehat{x}^r)_s|}{(t-s)^{\beta'}} \\ & \leq 2\|b\|_{\infty} r^{1-\beta'} + 2K\|\sigma\|_{\infty} \|y\|_{\beta} T^{\varepsilon} r^{\varepsilon} \\ & \quad + 2K\Phi_{\beta'(r,T)}(\widehat{x}^r, y) (\|\sigma'\|_{\infty} + \|\sigma'\|_{\lambda} \|\widehat{x}^r\|_{\beta'(r,T)}^{\lambda} r^{\lambda\beta'}) r^{\beta'} \\ & \leq \left[2\|b\|_{\infty} T^{1-\beta} + 2K\|\sigma\|_{\infty} \|y\|_{\beta} T^{\varepsilon} \right. \\ & \quad \left. + 2K\Phi_{\beta'(0,T)}(\widehat{x}^r, y) (\|\sigma'\|_{\infty} + \|\sigma'\|_{\lambda} \|x^r\|_{\beta'}^{\lambda} T^{\lambda\beta'}) T^{\beta'-\varepsilon} \right] r^{\varepsilon} \end{aligned} \quad (5.4.60)$$

where we used that $\sup_{t \in [r, T]} \|y\|_{\beta'(t-r,t)} \leq \|y\|_{\beta} r^{\varepsilon}$.

Then, by inequality (5.4.55) and using (5.4.57), (5.4.58), (5.4.59) and (5.4.60) it follows that

$$\|x^r - \widehat{x}^r\|_{\beta'} \leq K\rho\Lambda r^{\varepsilon}. \quad (5.4.61)$$

Finally, we study the Hölder norm $\|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'}$. By definition (5.4.18) and Proposition 5.3.2 we have

$$\begin{aligned} & \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(0,r)} \\ & \leq \|\eta_{\cdot-r}\|_{\beta'(0,r)} \|y\|_{\beta'(0,r)} + \|y\|_{\beta'(0,r)} \|b\|_{\infty} r^{1-\beta'} + K\|\sigma\|_{\infty} \Phi_{\beta'(0,r)}(y, y) \\ & \quad + K(\|\sigma'\|_{\infty} + \|\sigma'\|_{\lambda} \|\eta_{\cdot-r}\|_{\beta'(0,r)}^{\lambda} r^{\lambda\beta'}) \Phi_{\beta'(0,r)}(\eta_{\cdot-r}, y, y) r^{\beta'} \\ & \leq \left[\|\eta\|_{\beta(-r_0,0)} \|y\|_{\beta} T^{\varepsilon} + \|y\|_{\beta} \|b\|_{\infty} T^{1-\beta'} + K\|\sigma\|_{\infty} \Phi_{\beta(0,T)}(y, y) T^{\varepsilon} \right. \\ & \quad \left. + K(\|\sigma'\|_{\infty} + \|\sigma'\|_{\lambda} \|\eta\|_{\beta'(-r_0,0)}^{\lambda} T^{\lambda\beta'}) \Phi_{\beta'(0,r)}(\eta_{\cdot-r}, y, y) T^{\beta'-\varepsilon} \right] r^{\varepsilon} \\ & \leq K\rho\Lambda r^{\varepsilon} \end{aligned} \quad (5.4.62)$$

where we used that $\Phi_{\beta'(0,r)}(y, y) \leq \Phi_{\beta(0,T)}(y, y) r^{2\varepsilon}$.

Now we study the Hölder norm in the interval $[r, T]$. Let $a < b \in [r, T]$. By definition (5.4.20)

$$\begin{aligned} \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a,b)} & \leq \sup_{s < t \in [a,b]} \frac{|\int_s^t (y_{u+r} - y_u) b(u, x_u^r) du|}{(t-s)^{2\beta'}} \\ & \quad + \sup_{s < t \in [a,b]} \frac{|\int_s^t \sigma(x_{u-r}^r) du ((y - y_{\cdot-r}) \otimes y)_{\cdot,t}|}{(t-s)^{2\beta'}} \\ & \quad + \sup_{s < t \in [a,b]} \frac{|\int_s^t [\sigma(x_{u-r}^r) - \sigma(\widehat{x}_{u-r}^r)] du (y_{\cdot-r} \otimes y)_{\cdot,t}|}{(t-s)^{2\beta'}} \\ & = A_1 + A_2 + A_3. \end{aligned} \quad (5.4.63)$$

It is easy to see that

$$A_1 \leq \|y\|_\beta \|b\|_\infty T^{1-\beta'} \leq K\rho\Lambda r^\varepsilon. \quad (5.4.64)$$

By Proposition 5.3.2 we have

$$\begin{aligned} A_2 &\leq K\|\sigma\|_\infty \Phi_{\beta'(a,b)}(y - y_{\cdot-r}, y) \\ &\quad + K \left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r\|_{\beta'(a,b)}^\lambda T^{\lambda\beta'} \right) \Phi_{\beta'(a,b)}(\widehat{x}^r, y - y_{\cdot-r}, y) T^{\beta'} \\ &= K\|y\|_{\beta'} \left(\|\sigma\|_\infty + \left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r\|_{\beta'}^\lambda T^{\lambda\beta'} \right) \|\widehat{x}^r\|_{\beta'} T^{\beta'} \right) \|y - y_{\cdot-r}\|_{\beta'(r,T)} \\ &\quad + K \left(\|\sigma\|_\infty + \left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r\|_{\beta'}^\lambda T^{\lambda\beta'} \right) \|\widehat{x}^r\|_{\beta'} T^{\beta'} \right) \|(y - y_{\cdot-r}) \otimes y\|_{2\beta'(r,T)} \\ &\quad + K\|y\|_{\beta'} T^{\beta'} \left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r\|_{\beta'}^\lambda T^{\lambda\beta'} \right) \|\widehat{x}^r \otimes (y - y_{\cdot-r})\|_{2\beta'(a,b)}. \end{aligned} \quad (5.4.65)$$

Now we will estimate the norm $\|\widehat{x}^r \otimes (y - y_{\cdot-r})\|_{2\beta'(a,b)}$. By definition (5.4.11), for $s < t \in [a, b]$,

$$\begin{aligned} (\widehat{x}^r \otimes (y - y_{\cdot-r}))_{s,t} &= \int_s^t (y_t - y_u) b(u - r, \widehat{x}_u^r) du - \int_s^t (y_{t-r} - y_{u-r}) b(u - r, \widehat{x}_u^r) du \\ &\quad + \int_s^t \sigma(\widehat{x}_{u-r}^r) d_u(y_{\cdot-r} \otimes y)_{\cdot,t} - \int_s^t \sigma(\widehat{x}_{u-r}^r) d_u(y_{\cdot-r} \otimes y_{\cdot-r})_{\cdot,t} \\ &= \int_s^t (y_t - y_{t-r} - y_u + y_{u-r}) b(u - r, \widehat{x}_u^r) du \\ &\quad + \int_s^t \sigma(\widehat{x}_{u-r}^r) d_u(y_{\cdot-r} \otimes (y - y_{\cdot-r}))_{\cdot,t}. \end{aligned}$$

So by Proposition 5.3.2 and Lemma 5.3.8 we have

$$\begin{aligned} &\|\widehat{x}^r \otimes (y - y_{\cdot-r})\|_{2\beta'(a,b)} \\ &\leq 2\|b\|_\infty \|y\|_\beta T^{1-\beta'} r^\varepsilon + K\|\sigma\|_\infty \Phi_{\beta'(a,b)}(y_{\cdot-r}, y - y_{\cdot-r}) \\ &\quad + K \left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r\|_{\beta'(a,b)}^\lambda T^{\lambda\beta'} \right) \Phi_{\beta'(a,b)}(\widehat{x}^r, y_{\cdot-r}, y - y_{\cdot-r}) T^{\beta'} \\ &= K \left[\left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r\|_{\beta'}^\lambda T^{\lambda\beta'} \right) \left(\|\widehat{x}^r\|_{\beta'} \|y\|_{\beta'} + \|\widehat{x}^r \otimes y\|_{2\beta'} \right) T^{\beta'} \right. \\ &\quad \left. + \|b\|_\infty T^{1-\beta'} + \|\sigma\|_\infty \|y\|_{\beta'} \right] \|y\|_\beta r^\varepsilon \\ &\quad + K \left[\|\sigma\|_\infty + \left(\|\sigma'\|_\infty + \|\sigma'\|_\lambda \|\widehat{x}^r\|_{\beta'}^\lambda T^{\lambda\beta'} \right) \|\widehat{x}^r\|_{\beta'} T^{\beta'} \right] \\ &\quad \times \|y_{\cdot-r} \otimes (y - y_{\cdot-r})\|_{2\beta'(r,T)}. \end{aligned} \quad (5.4.66)$$

Now we put together (5.4.65) and (5.4.66). Also, we apply inequality (5.3.9) and we use the notation of Λ , ρ and Λ_r appearing in the beginning of the proof. With all of that we obtain that

$$\begin{aligned} A_2 &\leq K\rho\Lambda r^\varepsilon + K\rho^2\Lambda^2 r^\varepsilon + K\rho^2\Lambda\Lambda_r + K\rho\Lambda_r \\ &\leq K\rho^2\Lambda^2 r^\varepsilon + K\rho^2\Lambda\Lambda_r, \end{aligned} \quad (5.4.67)$$

where we used that $1 \leq \rho \leq \rho^2$ and $1 \leq \Lambda \leq \Lambda^2$.

Finally, by Proposition 5.3.7 and inequalities (5.4.54) and (5.4.61) we have

$$\begin{aligned}
 A_3 &\leq G_{\beta'(a,b)}^4(\sigma, \widehat{x}^r, \widehat{x}_{-r}^r, y_{-r}, y)(b-a)^{\beta'} \|\widehat{x}^r - \widehat{x}_{-r}^r\|_{\infty(a,b)} \\
 &\quad + G_{\beta'(a,b)}^5(\sigma, \widehat{x}^r, \widehat{x}_{-r}^r, y_{-r}, y)(b-a)^{\beta'} \|\widehat{x}^r - \widehat{x}_{-r}^r\|_{\beta'(a,b)} \\
 &\quad + G_{\beta'(a,b)}^6(\sigma, \widehat{x}_{-r}^r, y)(b-a)^{\beta'} \|(\widehat{x}^r - \widehat{x}_{-r}^r) \otimes y_{-r}\|_{2\beta'(a,b)} \\
 &\leq G_{\beta'(r,T)}^4(\sigma, \widehat{x}^r, \widehat{x}_{-r}^r, y_{-r}, y) T^{\beta'} \|x^r - \widehat{x}^r\|_{\infty} \\
 &\quad + G_{\beta'(r,T)}^5(\sigma, \widehat{x}^r, \widehat{x}_{-r}^r, y_{-r}, y) T^{\beta'} \|x^r - \widehat{x}^r\|_{\beta'} \\
 &\quad + G_{\beta'(r,T)}^6(\sigma, \widehat{x}_{-r}^r, y)(b-a)^{\beta'} \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a-r,b-r)} \\
 &\leq K\rho^2\Lambda^2 r^\varepsilon + G_{\beta'(r,T)}^6(\sigma, \widehat{x}_{-r}^r, y)(b-a)^{\beta'} \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a-r,b-r)}
 \end{aligned} \tag{5.4.68}$$

where we used that $G_{\beta'(r,T)}^i(\sigma, \widehat{x}^r, \widehat{x}_{-r}^r, y_{-r}, y) T^{\beta'} \leq K\rho\Lambda$ for $i = 4, 5$.

Applying the multiplicative property, it is easy to see that

$$\begin{aligned}
 &\|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a-r,b-r)} \\
 &\leq \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a-r,a)} + \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a,b)} + \|x^r - \widehat{x}^r\|_{\beta'} \|y\|_{\beta'}.
 \end{aligned}$$

On one hand, by (5.4.61)

$$\|x^r - \widehat{x}^r\|_{\beta'} \|y\|_{\beta'} \leq \|x^r - \widehat{x}^r\|_{\beta'} \|y\|_{\beta} T^\varepsilon \leq K\rho^2\Lambda^2 r^\varepsilon.$$

On the other hand, we have that

$$\|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a-r,a)} \leq K\rho^2\Lambda^2 r^\varepsilon + K\rho^2\Lambda\Lambda_r,$$

where the result is obtained considering separately the two cases $a \in [r, 2r)$ and $a \in [2r, T]$ and applying multiplicative property, inequalities (5.4.62), (5.4.63), (5.4.64), (5.4.67), (5.4.68) and $G_{\beta'(r,T)}^6(\sigma, \widehat{x}_{-r}^r, y) T^{\beta'} \leq K\rho\Lambda$. Therefore,

$$\|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a-r,b-r)} \leq K\rho^2\Lambda^2 r^\varepsilon + K\rho^2\Lambda\Lambda_r + \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a,b)}.$$

It follows that

$$A_3 \leq G_{\beta'(r,T)}^6(\sigma, \widehat{x}_{-r}^r, y)(b-a)^{\beta'} \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a,b)} + K\rho^3\Lambda^3 r^\varepsilon + K\rho^3\Lambda^2\Lambda_r, \tag{5.4.69}$$

where we used again that $G_{\beta'(r,T)}^6(\sigma, \widehat{x}_{-r}^r, y) T^{\beta'} \leq K\rho\Lambda$. From inequalities (5.4.63), (5.4.64), (5.4.67) and (5.4.69) we have that

$$\begin{aligned}
 \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a,b)} &\leq G_{\beta'(r,T)}^6(\sigma, \widehat{x}_{-r}^r, y)(b-a)^{\beta'} \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a,b)} \\
 &\quad + K\rho^3\Lambda^3 r^\varepsilon + K\rho^3\Lambda^2\Lambda_r,
 \end{aligned}$$

where we used that $\rho^n \leq \rho^{n+1}$ and $\Lambda^n \leq \Lambda^{n+1}$ for $n \in \mathbb{N}$.

Set

$$\tilde{\Delta} := \left(2 \sup_{r \leq r_0} G_{\beta'(r,T)}^6(\sigma, \hat{x}_{-r}^r, y) \right)^{-\frac{1}{\beta'}}. \quad (5.4.70)$$

Observe that, if a, b are such that $(b - a) \leq \tilde{\Delta}$, then

$$\|(x^r - \hat{x}^r) \otimes y\|_{2\beta'(a,b)} \leq K\rho^3\Lambda^3 r^\varepsilon + K\rho^3\Lambda^2\Lambda_r. \quad (5.4.71)$$

Now consider a partition $r = t_0 < \dots < t_M = T$ such that $(t_{i+1} - t_i) \leq \tilde{\Delta}$ for $i = 0, \dots, M-1$. Then, using the multiplicative property iteratively, we have

$$\|(x^r - \hat{x}^r) \otimes y\|_{2\beta'(r,T)} \leq \sum_{i=0}^{M-1} \|(x^r - \hat{x}^r) \otimes y\|_{2\beta'(t_i, t_{i+1})} + (M-1)\|x^r - \hat{x}^r\|_{\beta'}\|y\|_{\beta'}.$$

Applying (5.4.61) and (5.4.71), we obtain

$$\begin{aligned} \|(x^r - \hat{x}^r) \otimes y\|_{2\beta'(r,T)} &\leq KM\rho^3\Lambda^3 r^\varepsilon + KM\rho^3\Lambda^2\Lambda_r + K(M-1)\rho^2\Lambda^2 r^\varepsilon \\ &\leq KM\rho^3\Lambda^3 r^\varepsilon + KM\rho^3\Lambda^2\Lambda_r. \end{aligned} \quad (5.4.72)$$

Finally, by (5.4.56), (5.4.61), (5.4.62) and (5.4.72)

$$\|(x^r - \hat{x}^r) \otimes y\|_{2\beta'} \leq KM\rho^3\Lambda^3 r^\varepsilon + KM\rho^3\Lambda^2\Lambda_r. \quad (5.4.73)$$

So the proof is complete. \square

The following definitions will be useful in the next results:

$$\bar{G}_{\beta'}^i := \sup_{r \leq r_0} G_{\beta'(0,T)}^i(\sigma, x, x^r, y) \quad i = 1, 2 \quad (5.4.74)$$

$$\bar{G}_{\beta'}^3 := \sup_{r \leq r_0} G_{\beta'(0,T)}^3(\sigma, x^r) \quad (5.4.75)$$

$$\bar{G}_{\beta'}^j := \sup_{r \leq r_0} G_{\beta'(0,T)}^j(\sigma, x, x^r, y, y) \quad j = 4, 5 \quad (5.4.76)$$

$$\bar{G}_{\beta'}^6 := \sup_{r \leq r_0} G_{\beta'(0,T)}^6(\sigma, x^r, y). \quad (5.4.77)$$

The following result gives as a bound for $\|(x - x^r) \otimes y\|_{2\beta'(a,b)}$ when the interval (a, b) is sufficiently small. Define $\Delta_{\beta'}^1$ as follows:

$$\Delta_{\beta'}^1 = (2\bar{G}_{\beta'}^6)^{-\frac{1}{\beta'}}.$$

We state the following proposition:

Proposition 5.4.6. *Suppose that $(x, y, x \otimes y)$, $(x^r, y, x^r \otimes y)$ and $(\widehat{x}^r, y, \widehat{x}^r \otimes y)$ belong to $M_{d,m}^\beta(0, T)$, $(y, y, y \otimes y)$ belongs to $M_{d,m}^\beta(0, T)$ and $(y_{\cdot-r}, y, y_{\cdot-r} \otimes y)$ belongs to $M_{d,m}^\beta(r, T)$. Assume that σ and b satisfy **(H1)** and **(H2)** respectively. Then, for all $0 \leq a < b \leq T$ such that $(b - a) \leq \Delta_{\beta'}^1$,*

$$\begin{aligned} & \| (x - x^r) \otimes y \|_{2\beta'(a,b)} \\ & \leq 2 [L_N \|y\|_{\beta'} (b - a)^{1-2\beta'} + G_{\beta'(a,b)}^4(\sigma, x, x^r, y, y)] (b - a)^{\beta'} \|x - x^r\|_{\infty(a,b)} \\ & \quad + 2 G_{\beta'(a,b)}^5(\sigma, x, x^r, y, y) (b - a)^{\beta'} \|x - x^r\|_{\beta'(a,b)} \\ & \quad + 2K\rho\Lambda G_{\beta'(a,b)}^4(\sigma, x^r, \widehat{x}^r, y, y) (b - a)^{\beta'} r^{\beta'} \\ & \quad + 2K\rho\Lambda [G_{\beta'(a,b)}^5(\sigma, x^r, \widehat{x}^r, y, y) + M\rho^2\Lambda^2 G_{\beta'(a,b)}^6(\sigma, \widehat{x}^r, y)] (b - a)^{\beta'} r^\varepsilon \\ & \quad + 2KM\rho^3\Lambda^2 G_{\beta'(a,b)}^6(\sigma, \widehat{x}^r, y) (b - a)^{\beta'} \Lambda_r \end{aligned}$$

where functions $G_{\beta'(a,b)}^i$ with $i = 4, 5, 6$ are defined in Proposition 5.3.6 and ρ, Λ, Λ_r are defined in the proof of Proposition 5.4.4.

Proof. The proposition is proved applying first Proposition 5.3.3, Proposition 5.3.6 and Proposition 5.3.7 to definition (5.4.7) and then Proposition 5.4.4, and observing that for $a < b$ such that $(b - a) \leq \Delta_{\beta'}^1$,

$$G_{\beta'(a,b)}^6(\sigma, x^r, y) (b - a)^{\beta'} \leq \frac{1}{2}.$$

□

5.5 Proof of the main result

We start studying $\lim_{r \rightarrow 0} \|x - x^r\|_\infty$.

As in Lemma 5.4.1, it is easy to see that

$$\|x - x^r\|_{\beta'} \leq \|x - x^r\|_{\beta'(0,r)} + \|x - x^r\|_{\beta'(r,T)}. \quad (5.5.1)$$

First we study the norm in the interval $[0, r)$. We apply Proposition 5.3.3 and Proposition 5.3.4 to definition (5.4.6) and we obtain

$$\begin{aligned} \|x - x^r\|_{\beta(0,r)} & \leq L_N r^{1-\beta} \|x - x^r\|_{\infty(0,r)} + G_{\beta(0,r)}^1(\sigma, x, \eta_{\cdot-r}, y) r^\beta \|x - \eta_{\cdot-r}\|_{\infty(0,r)} \\ & \quad + G_{\beta(0,r)}^2(\sigma, x, \eta_{\cdot-r}, y) r^\beta \|x - \eta_{\cdot-r}\|_{\beta(0,r)} \\ & \quad + G_{\beta(0,r)}^3(\sigma, \eta_{\cdot-r}) r^\beta \|(x - \eta_{\cdot-r}) \otimes y\|_{2\beta(0,r)}. \end{aligned}$$

Using that the supremum norm of x is bounded and the bound does not depend on r , we easily see that $\sup_{r \leq r_0} G_{\beta(0,r)}^i(\sigma, x, \eta_{\cdot-r}, y) < \infty$ for $i = 1, 2$ and $\sup_{r \leq r_0} G_{\beta(0,r)}^3(\sigma, \eta_{\cdot-r}) < \infty$. So last expression clearly goes to zero when r tends to zero.

Now we work on the interval $[r, T]$. Let $r \leq a < b \leq T$. Applying Proposition 5.3.3, Proposition 5.3.4, Proposition 5.3.5 and Proposition 5.4.4, we obtain

$$\begin{aligned}
 \|x - x^r\|_{\beta'(a,b)} &\leq [L_N(b-a)^{1-2\beta'} + G_{\beta'(a,b)}^1(\sigma, x, x^r, y)] (b-a)^{\beta'} \|x - x^r\|_{\infty(a,b)} \\
 &\quad + G_{\beta'(a,b)}^2(\sigma, x, x^r, y) (b-a)^{\beta'} \|x - x^r\|_{\beta'(a,b)} \\
 &\quad + G_{\beta'(a,b)}^3(\sigma, x^r) (b-a)^{\beta'} \|(x - x^r) \otimes y\|_{2\beta'(a,b)} \\
 &\quad + G_{\beta'(a,b)}^1(\sigma, x^r, \widehat{x}^r, y) (b-a)^{\beta'} \|x^r - \widehat{x}^r\|_{\infty(a,b)} \\
 &\quad + G_{\beta'(a,b)}^2(\sigma, x^r, \widehat{x}^r, y) (b-a)^{\beta'} \|x^r - \widehat{x}^r\|_{\beta'(a,b)} \\
 &\quad + G_{\beta'(a,b)}^3(\sigma, \widehat{x}^r) (b-a)^{\beta'} \|(x^r - \widehat{x}^r) \otimes y\|_{2\beta'(a,b)} \\
 &\leq [L_N(b-a)^{1-2\beta'} + G_{\beta'(a,b)}^1(\sigma, x, x^r, y)] (b-a)^{\beta'} \|x - x^r\|_{\infty(a,b)} \\
 &\quad + G_{\beta'(a,b)}^2(\sigma, x, x^r, y) (b-a)^{\beta'} \|x - x^r\|_{\beta'(a,b)} \\
 &\quad + G_{\beta'(a,b)}^3(\sigma, x^r) (b-a)^{\beta'} \|(x - x^r) \otimes y\|_{2\beta'(a,b)} \\
 &\quad + K\rho\Lambda G_{\beta'(a,b)}^1(\sigma, x^r, \widehat{x}^r, y) (b-a)^{\beta'} r^{\beta'} \\
 &\quad + [K\rho\Lambda G_{\beta'(a,b)}^2(\sigma, x^r, \widehat{x}^r, y) + K\rho^3\Lambda^3 G_{\beta'(a,b)}^3(\sigma, \widehat{x}^r)] (b-a)^{\beta'} r^\varepsilon \\
 &\quad + K\rho^3\Lambda^2\Lambda_r G_{\beta'(a,b)}^3(\sigma, \widehat{x}^r) (b-a)^{\beta'}.
 \end{aligned}$$

We take a and b such that

$$(b-a) \leq \Delta_{\beta'}^1 \quad (5.5.2)$$

and apply Proposition 5.4.6, so we have

$$\begin{aligned}
 &\|x - x^r\|_{\beta'(a,b)} \\
 &\leq \left[L_N(2\|y\|_{\beta'} G_{\beta'(a,b)}^3(\sigma, x^r) (b-a)^{\beta'} + 1) (b-a)^{1-2\beta'} \right. \\
 &\quad \left. + G_{\beta'(a,b)}^1(\sigma, x, x^r, y) + 2G_{\beta'(a,b)}^3(\sigma, x^r) G_{\beta'(a,b)}^4(\sigma, x, x^r, y, y) (b-a)^{\beta'} \right] \\
 &\quad \times (b-a)^{\beta'} \|x - x^r\|_{\infty(a,b)} \\
 &\quad + \left[G_{\beta'(a,b)}^2(\sigma, x, x^r, y) + 2G_{\beta'(a,b)}^3(\sigma, x^r) G_{\beta'(a,b)}^5(\sigma, x, x^r, y, y) (b-a)^{\beta'} \right] \\
 &\quad \times (b-a)^{\beta'} \|x - x^r\|_{\beta'(a,b)} \\
 &\quad + K\rho\Lambda (G_{\beta'(a,b)}^1(\sigma, x^r, \widehat{x}^r, y) + 2G_{\beta'(a,b)}^3(\sigma, x^r) G_{\beta'(a,b)}^4(\sigma, x^r, \widehat{x}^r, y, y) (b-a)^{\beta'}) \\
 &\quad \times (b-a)^{\beta'} r^{\beta'} \\
 &\quad + \left[K\rho\Lambda (G_{\beta'(a,b)}^2(\sigma, x^r, \widehat{x}^r, y) + 2G_{\beta'(a,b)}^3(\sigma, x^r) G_{\beta'(a,b)}^5(\sigma, x^r, \widehat{x}^r, y, y) (b-a)^{\beta'}) \right. \\
 &\quad \left. + K\rho^3\Lambda^3 (G_{\beta'(a,b)}^3(\sigma, \widehat{x}^r) + 2G_{\beta'(a,b)}^3(\sigma, x^r) G_{\beta'(a,b)}^6(\sigma, \widehat{x}^r, y) (b-a)^{\beta'}) \right] \\
 &\quad \times (b-a)^{\beta'} r^\varepsilon \\
 &\quad + K\rho^3\Lambda^2 (G_{\beta'(a,b)}^3(\sigma, \widehat{x}^r) + 2G_{\beta'(a,b)}^3(\sigma, x^r) G_{\beta'(a,b)}^6(\sigma, \widehat{x}^r, y) (b-a)^{\beta'}) (b-a)^{\beta'} \Lambda_r.
 \end{aligned}$$

To simplify the notation, let define

$$\begin{aligned}
 H_r &:= K\rho\Lambda(G_{\beta'(0,T)}^1(\sigma, x^r, \widehat{x}^r, y) + 2G_{\beta'(0,T)}^3(\sigma, x^r)G_{\beta'(0,T)}^4(\sigma, x^r, \widehat{x}^r, y, y)T^{\beta'})T^{\beta'}r^{\beta'} \\
 &+ \left[K\rho\Lambda(G_{\beta'(0,T)}^2(\sigma, x^r, \widehat{x}^r, y) + 2G_{\beta'(0,T)}^3(\sigma, x^r)G_{\beta'(0,T)}^5(\sigma, x^r, \widehat{x}^r, y, y)T^{\beta'}) \right. \\
 &\quad \left. + K\rho^3\Lambda^3(G_{\beta'(0,T)}^3(\sigma, \widehat{x}^r) + 2G_{\beta'(0,T)}^3(\sigma, x^r)G_{\beta'(0,T)}^6(\sigma, \widehat{x}^r, y)T^{\beta'}) \right] T^{\beta'}r^\varepsilon \\
 &+ K\rho^3\Lambda^2(G_{\beta'(0,T)}^3(\sigma, \widehat{x}^r) + 2G_{\beta'(0,T)}^3(\sigma, x^r)G_{\beta'(0,T)}^6(\sigma, \widehat{x}^r, y)T^{\beta'}) \left. \right] T^{\beta'}\Lambda_r. \quad (5.5.3)
 \end{aligned}$$

Observe that H_r converges to zero when r tends to zero. Hence, we can write

$$\begin{aligned}
 \|x - x^r\|_{\beta'(a,b)} &\leq \left[L_N(2\|y\|_{\beta'}G_{\beta'(a,b)}^3(\sigma, x^r)T^{\beta'} + 1)T^{1-2\beta'} \right. \\
 &\quad \left. + G_{\beta'(a,b)}^1(\sigma, x, x^r, y) + 2G_{\beta'(a,b)}^3(\sigma, x^r)G_{\beta'(a,b)}^4(\sigma, x, x^r, y, y)T^{\beta'} \right] \\
 &\quad \times (b-a)^{\beta'}\|x - x^r\|_{\infty(a,b)} \\
 &+ \left[G_{\beta'(a,b)}^2(\sigma, x, x^r, y) + 2G_{\beta'(a,b)}^3(\sigma, x^r)G_{\beta'(a,b)}^5(\sigma, x, x^r, y, y)T^{\beta'} \right] \\
 &\quad \times (b-a)^{\beta'}\|x - x^r\|_{\beta'(a,b)} \\
 &+ H_r.
 \end{aligned}$$

We take a and b such that

$$\left[G_{\beta'(a,b)}^2(\sigma, x, x^r, y) + 2G_{\beta'(a,b)}^3(\sigma, x^r)G_{\beta'(a,b)}^5(\sigma, x, x^r, y, y)T^{\beta'} \right] (b-a)^{\beta'} \leq \frac{1}{2}. \quad (5.5.4)$$

In this way we have

$$\begin{aligned}
 \|x - x^r\|_{\beta'(a,b)} &\leq 2 \left[L_N(2\|y\|_{\beta'}G_{\beta'(a,b)}^3(\sigma, x^r)T^{\beta'} + 1)T^{1-2\beta'} \right. \\
 &\quad \left. + G_{\beta'(a,b)}^1(\sigma, x, x^r, y) + 2G_{\beta'(a,b)}^3(\sigma, x^r)G_{\beta'(a,b)}^4(\sigma, x, x^r, y, y)T^{\beta'} \right] \\
 &\quad \times (b-a)^{\beta'}\|x - x^r\|_{\infty(a,b)} + 2H_r. \quad (5.5.5)
 \end{aligned}$$

On the other hand,

$$\|x - x^r\|_{\infty(a,b)} \leq |x_a - x_a^r| + (b-a)^{\beta'}\|x - x^r\|_{\beta'(a,b)},$$

and replacing inequality (5.5.5) we obtain

$$\begin{aligned}
 \|x - x^r\|_{\infty(a,b)} &\leq |x_a - x_a^r| + 2 \left[L_N(2\|y\|_{\beta'}G_{\beta'(a,b)}^3(\sigma, x^r)T^{\beta'} + 1)T^{1-2\beta'} \right. \\
 &\quad \left. + G_{\beta'(a,b)}^1(\sigma, x, x^r, y) + 2G_{\beta'(a,b)}^3(\sigma, x^r)G_{\beta'(a,b)}^4(\sigma, x, x^r, y, y)T^{\beta'} \right] \\
 &\quad \times (b-a)^{2\beta'}\|x - x^r\|_{\infty(a,b)} + 2T^{\beta'}H_r. \quad (5.5.6)
 \end{aligned}$$

Now, we take a and b such that

$$2 \left[L_N (2 \|y\|_{\beta'} G_{\beta'(a,b)}^3(\sigma, x^r) T^{\beta'} + 1) T^{1-2\beta'} + G_{\beta'(a,b)}^1(\sigma, x, x^r, y) + 2 G_{\beta'(a,b)}^3(\sigma, x^r) G_{\beta'(a,b)}^4(\sigma, x, x^r, y, y) T^{\beta'} \right] T^{\beta'} (b-a)^{\beta'} \leq \frac{1}{2} \quad (5.5.7)$$

and we obtain

$$\|x - x^r\|_{\infty(a,b)} \leq 2|x_a - x_a^r| + 4T^{\beta'} H_r.$$

We can observe that

$$\sup_{0 \leq t \leq b} |x_t - x_t^r| \leq 2 \sup_{0 \leq t \leq a} |x_t - x_t^r| + 4T^{\beta'} H_r. \quad (5.5.8)$$

We define $\Delta_{\beta'}$ such that all a, b with $(b-a) \leq \Delta_{\beta'}$ fulfill the following conditions (5.5.2), (5.5.4) and (5.5.7):

$$\Delta_{\beta'} := \left(16L_N T^{1-\beta'} + 16\bar{G}_{\beta'}^1 T^{\beta'} + 4\bar{G}_{\beta'}^2 + 8\bar{G}_{\beta'}^3 [4L_N \|y\|_{\beta'} T + 4\bar{G}_{\beta'}^4 T^{2\beta'} + \bar{G}_{\beta'}^5 T^{\beta'}] + 2\bar{G}_{\beta'}^6 \right)^{-\frac{1}{\beta'}}. \quad (5.5.9)$$

Then, it is clear that (5.5.8) holds for all a and b such that $b-a \leq \Delta_{\beta'}$.

Now, we take a partition $0 = t_0 < t_1 < \dots < t_M = T$ of the interval $[0, T]$ such that $(t_{i+1} - t_i) \leq \Delta_{\beta'}$. Then,

$$\sup_{0 \leq t \leq t_M = T} |x_t - x_t^r| \leq 2 \sup_{0 \leq t \leq t_{M-1}} |x_t - x_t^r| + 4T^{\beta'} H_r. \quad (5.5.10)$$

Repeating the process M times we obtain

$$\sup_{0 \leq t \leq T} |x_t - x_t^r| \leq 2^M |x_0 - x_0^r| + \left(\sum_{k=0}^{M-1} 2^k \right) 4T^{\beta'} H_r = 4(2^M - 1)T^{\beta'} H_r$$

that clearly converges to zero when r tends to zero.

The proof that the limit

$$\lim_{r \rightarrow 0} \|(x \otimes y) - (x^r \otimes y)\|_{\infty} = \lim_{r \rightarrow 0} \|(x - x^r) \otimes y\|_{\infty}$$

vanishes follows easily using the same ideas. □

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Index of symbols

B^H	Fractional Brownian motion with Hurst parameter H .
\mathbb{C}	The complex plane.
\mathcal{C}	Space of continuous functions.
\mathcal{C}^k	Space of functions with k continuous derivatives.
\mathcal{C}^∞	Space of infinitely often differentiable functions.
C^β	Space of β -Hölder continuous functions.
$D_{a+}^\alpha f, D_{b-}^\alpha f$	Weyl derivatives.
$\widehat{D}_{a+}^\alpha f$	Compensated fractional derivative.
$\mathcal{D}_{b-}^{1-\alpha}(x \otimes y)$	Extension of the fractional derivative.
\mathbb{E}	The expected value.
\mathcal{F}	Some filtration.
$I_{a+}^\alpha f, I_{b-}^\alpha f$	Fractional Riemann-Liouville integrals.
L^p	Lebesgue spaces.
N	Poisson process.
\mathbb{N}	Natural numbers.
(Ω, \mathcal{F}, P)	Some probability space.
P	Probability.
π	Partition of an interval.
$ \pi $	Mesh of the partition π .
\mathbb{R}	The real line.
ν	Some probability measure.
W	Brownian motion.
Z	Complex Brownian motion.
$\ \cdot\ _\infty$	Supremum norm.
$\ \cdot\ _\beta$	β -Hölder norm.