Diffusion Equations with Nonlinear Boundary Conditions

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The nonlinear parabolic problem

$$\left\{ \begin{array}{rcll} u_t & = & \Delta u + f(u) \ , & \text{a } \Omega \ , \\ u_\nu & = & 0 \ , & \text{a } \partial\Omega \ , \\ u(x,0) & = & u_0(x) \ , & \end{array} \right. \tag{1}$$

in a bounded domain $\Omega \subset R^n$ represents time evolution of a concentration u of a substance in an isolated cotainer, submited to a nonlinear reaction represented by f and with a homogeneous linear diffusion.

Obviously zeroes of f are constant equilibrium solutions for (1). The existence of non constant equilibrium solutions for (1) with $n \geq 2$ is a relevant phenomenous, called usually "morphogenesis" or pattern formation.

Related with this Neumann problem, we present a contribution to the study of pattern formation for nonlinear boundary reactions. We consider next problem:

$$\left\{ \begin{array}{rcll} u_t & = & \Delta u \ , & \text{a } \Omega \ , \\ u_\nu & = & f(u) \ , & \text{a } \partial\Omega \ . \end{array} \right. \tag{2}$$

Let us assume that the domain $\Omega \subset R^n$, $n \geq 2$, is bounded with smooth boundary $\partial\Omega$. In the boundary condition, $\dfrac{\partial}{\partial\nu}$ denotes the outward normal derivative. The solution $u = u(x,t)$ is a fuction from $\overline{\Omega} \times R$ to R and $f(u) : \partial\Omega \to R$.

In a similar way as in (1), se can think (2) as an equation modelling the evolution of a concentration under the effects of a linear homogeneous diffusion in the interior af a container and a nonlinear reaction which happens only on the boundary, represented by f. For example, by the presence of a catalizer.

As in (1), zeroes of f are constant equilibrium solutions of (2). But we ask ourselves by the existence or not of some patterns.

We have, for non connected domain Ω we can build trivial nonconstant stable equilibria fixing in each connected component different constants α_i, such that $f(\alpha_i) = 0$ and $f'(\alpha_i) < 0$. (The stability of them is non obvious and we have for it a principle of stability by linearization for (2), see chapter 2.)

In the thesis we study the abstract functional setting for the problem (chapter 2), some conditions on the boundary nonlinearity to obtain patterns of not (chapter 3), geometric conditions on the domain Ω that give morphogenesis (there are no patterns in balls but we prove existence in dumbbell type domains and in connected domains with disconnected boundary, chapters 4 and 5). The appendix A studies the one dimensinal problem and finally the appendix B gives some numerical results in order to illustrate patterns in dumbbell domains.