Normal Forms around Lower Dimensional Tori of Hamiltonian Systems

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Introduction and previous results

Normal forms are a standard tool in Hamiltonian mechanics to study the dynamics in a neighbourhood of invariant objects, like equilibrium points, periodic orbits or invariant tori. Usually, these normal forms are obtained as divergent series, but their asymptotic character is what makes them useful. From a theoretical point of view, they provide nonlinear approximations to the dynamics in a neigbourhood of the invariant object, that allows to obtain information about the real solutions of the system by taking the normal form up to a suitable finite order. In this work we will restrict ourselves to analytic Hamiltonian systems. In this case, it is well known that under certain (generic) nonresonance conditions, the remainder of this finite normal form turns out to be exponentially small with respect to some parameters. This is the basis to derive the classical Nekhoroshev estimates (see Section 0.1.2). Moreover, although those series are usually divergent on open sets, it is still possible in some cases to prove convergence on certain sets with empty interior (Cantor-like sets) by replacing the standard linear normal form scheme by a quadratical one. This is the basis of KAM theory (see Section 0.1.1).

From a more practical point of view, normal forms can be used as a computational method to obtain very accurate approximations to the dynamics in a neighbourhood of the selected invariant object, by neglecting the remainder. They have been applied, for example, to compute invariant manifols (see [31]) or invariat tori (see [26], [71]). To do that, it is necessary to compute the explicit expression of the normal form and of the (canonical) transformation that put the Hamiltonian into this reduced form. A context where this computational formulation has special interest is in some celestial mechanics models, that can be used to approximate the dynamics of some real world problems (see Section 3.4).

This computational methodology has also been used to produce estimates on the diffusion time near an elliptic fixed point of a Hamiltonian system. Let us introduce the problem, that can be considered one of the main motivations of this work.

Let us consider an analytic Hamiltonian H with ℓ degrees of freedom, having an elliptic equilibrium point. If $\ell = 1$, the Hamiltonian is integrable, and hence, any trajectory close to the point takes place on a periodic orbit. This implies that the point is (Lyapunov) stable. If $\ell \geq 2$, the system is generically non-integrable but, under generic conditions of nonresonance and nondegeneracy, KAM theory (see Section 0.1.1) ensures that there is plenty of ℓ-dimensional invariant tori around the point. If ℓ is 2 this also implies stability, as the 2-dimensional tori split the 3-dimensional energy levels $H = h$ in two connected components. If $\ell > 2$ the same argument does not hold, as the ℓ-dimensional tori do not separe the energy levels ($(2\ell - 1)$-dimensional manifolds) in two connected components. Hence, it is widely accepted that for a generic Hamiltonian system with 3 or more degrees of freedom, some diffusion can take place.

In this case, the use of normal forms allows to produce lower bounds on this diffusion time, that are exponentially big with the distance to the origin (see Section 0.1.2). This gives rise to the so-called effective stability, that is, even in the cases when the system is not stable it looks like it were (i.e., the time needed to observe the unstability is very long, usually longer than the expected lifetime of the studied physical system).

The practical application of the theoretical formulation of these results to concrete examples encountres many difficulties, and can also produce very disappointing results. At first, in most of the cases we are not able to check in the concrete example the non-resonance conditions needed (this problem can be solved taking a formulation of the result that only uses Diophantine estimates up to some finite order), and moreover, if we assume that these conditions are fullfiled and we try to apply the result (with a proof that holds for a generic system) to our case, probably we will obtain a very poor result: we will only prove effective stability on a very small neighbourhood of the fixed point, much smaller than the real region of slow diffusion.

So, if we want more realistic estimates for concrete problems (of course not optimal, but at least more relevant), it is very convenient to compute numerically the normal form around the point up to some finite order. This allows to derive much better estimates than the ones obtained applying purely theoretical results, by transforming the most significant estimates in the proof (that are the ones in which the first order terms are involved) in explicit computations. This methodology avoids the pesimistic estimates of the general formulation.

In this work both formulations, the theoretical and the computational one are taken into account. As a central problem, we consider the description of the dynamics around a non-degenerate lower dimensional reducible torus of an analytic Hamiltonian system (see Section 0.1.3). To be more precise, we specially focus in the case in which the torus has (some) elliptic directions. For such a torus, KAM and Nekhoroshev approaches are considered.

Hence, in Chapter 1 we use these normal forms to obtain lower bounds for the effective stability time around an elliptic lower dimensional torus, and to give estimates on the amount of lower dimensional tori around the initial one. We do not restrict to tori of the same dimension: we also consider the ones obtained by adding the excitation coming from the different elliptic directions of the torus. In both cases, the estimates on the diffusion velocity and on the proximity to 1 of the density of invariant tori (in a suitable space), turns out to be exponentially small with respect to the distance to the initial torus. We remark that these results are formulated in order to describe, in an unified way, all the possible cases according to the dimension, from a fixed point (dimension zero) to a maximal dimensional torus. They can be considered extensions of previous results for the two limit cases (see Sections 0.1.1 and 0.1.2).

Moreover, these statements are also given in a way than can be applied to help in the description of the dynamics of some celestial mechanics problems. To illustrate that, in Chapter 2 we perform a computational application of the results of Chapter 1 to the case of a linearly stable orbit of the RTBP. Hence, in the first part of Chapter 2 we present, from a formal point of view, a methodology to compute the normal form around some periodic orbits of an analytic Hamiltonian system. The formulation is done around a non-degenerate elliptic orbit, but can be extended to other cases with minor changes. Then, in the second part of Chapter 2, we use this methodology on a certain elliptic orbit of the RTBP, for certain value of the mass parameter μ. The orbit has been chosen to show (numerically) the existence of regions of long time stability near the Lagrangian points of the RTBP for a value of μ bigger than the Routh critical value, for which both Lagrangian points become unstable (see section 2.3.1).

In Chapter 3 we consider the important role played by the quasiperiodic time dependent perturbations of autonomous Hamiltonian systems (see section 0.1.4), that appear when dealing with realistic models of the Solar system (see Section 3.4). Hence, we have extended the results contained in [35] on quasiperiodic perturbations of elliptic fixed points of systems of differential equations, and of maximal dimensional tori of autonomous Hamiltonian systems, to the case of quasiperiodic perturbations of reducible lower dimensional tori of Hamiltonian systems. The estimates for the measure of parameters for which we do not have the expected invariant torus, are also exponentially small with respect to the size of the allowed set of perturbative parameters. Finally, results on Chapter 3 are applied to some celestial mechanics models (see Section 3.4).

In Appendix A we present a result that is a joint work with Rafael Ramírez-Ros, in which the reducibility (in the aim of the Floquet theory) of quasiperiodic linear equations close to constant coefficients is considered. There it is proved that (under generic hypotheses of non-resonance) these kind of equations can be reduced to constants coefficients by means of a quasiperiodic linear change of variables (having the same basic frequencies as the perturbation), except an exponentially small remainder with respect to the size of the initial perturbation.

The connection of this result with the ones of Chapter 3 and the computational methodology of Chapter 2 is clear: if one wants to extend the explicit normal form computations done in Chapter 2 to the case of an invariant torus of a Hamiltonian system depending on time in a quasiperiodic way, it is necessary to perform a quasiperiodic Floquet reduction of the normal variational equations of the initial torus. Note that an (approximate) torus obtained by means of a perturbative procedure (for example an asymptotic Lindstedt-Poincaré method, see for instance [26], [71] or [30]), does not need to be reducible. Then, the result of Appendix A is suitable to ensure the effective reducibility of these variational equations, except a very small remainder. A numerical example showing this effective reducibility is also given at the end of Appendix A.

We remark that this last result can be considered a practical version of [33], where it is shown total reducibiliy on a Cantor set for the perturbative parameter, with small measure for the complementary. In [35], exponentially small upper bounds for this measure are given.

Finally, to end this part of the Introduction, let us mention that the contents of the different chapters are organized in a way that every of them can be considered a self-contained work. Hence, every chapter contains not only the formulation of the results contained therein and its proof, but also the notations used to formulate an prove these results, as well the statements and proofs of the auxiliar results needed. We know this can produce unnecessary repetitions along the work, but we think it helps on the readibility of the parts. Moreover, we notice on the difficult to find a common notation, suitable to deal with all the different contents of this work (theoretical results and practical ones, excitation frequencies and quasiperiodic perturbative ones, ...).

Let us remark that the results contained in some chapters can also be found in [38] (Chapter 1), [36] and [39] (Chapter 2), [37] (Chapter 3), and [32] (Appendix A).

0.1 Previous results

In this section we have included some classical results that can be found in the literature, related to the ones contained here. They can also help to understand the approach we have taken.

0.1.1 KAM theory

The study of small perturbations of integrable Hamiltonian systems is, according to Poincaré (see [59]), the fundamental problem of the dynamics. As the phase space of an integrable Hamiltonian system with ℓ-degrees of freedom (that is, there exist ℓ independent and univaluated first integrals in involution) is foliated (under hypothesis of bounded motion) by ℓ-dimensional invariant tori, it is a classical subject in dynamical systems to study the persistence of these invariant tori in the near-integrable case. In a suitable system of coordinates (action-angle coordinates, see [2]), a near-integrable Hamiltonian can be writen in the following form:

$$H = H_0(I) + H_1(\theta, I), \quad \theta = (\theta_1, \ldots, \theta_\ell), \;\; I = (I_1, \ldots, I_\ell),$$

where θ (positions) ranges in $\mathbb{T}^\ell$, and I (the conjugate momenta) ranges in an open set of $\mathbb{R}^\ell$. Here, $H_1 \equiv \mathcal{O}(\varepsilon)$ is the small perturbation. If we neglect H_1, each solution takes place on a torus $I = I^*$ with quasiperiodic motion. The frequencies of the motion are given by $\omega(I^*) \equiv \nabla H_0(I^*)$. Kolmogorov showed (see [40] and [41], or [5] for a more recent proof) that under standard hypotheses of non-degeneracy and non-resonance, a concrete torus, characterized by its vector of intrinsic frequencies, is only slightly deformed but not destroyed by the perturbation, at least if ε is small enough (of course, the smallness of ε depends on the concrete torus). These usual hypotheses are:

a) Nonresonance. The frequencies of the torus must satisfy a Diophantine condition:

$$|k^\top \omega(I^*)| \geq \frac{\mu}{|k|_1^\gamma}, \qquad \gamma > \ell - 1,$$

where $k^\top \omega(I^*)$ denotes the scalar product of k with the gradient of H_0, and $|k|_1 = |k_1| + \cdots + |k_\ell|$.

b) Nondegeneracy. The frequencies must depend on the actions:

$$\det\left(\frac{\partial^2 H_0}{\partial I^2}(I^*)\right) \neq 0.$$

The necessity of the first hypothesis comes from the fact that, during the proof, we obtain the divisors $k^\top \nabla H_0(I^*)$. Hence, if they are too small it is not possible to prove the convergence of the series that appear in the proof.

The persistence of these invariant tori from a global point of view is a more difficult problem. In this case, KAM theorem states that, if the Hamiltonian H_0 is non-degenerate (global validity of item b) in the whole initial domain), the invariant tori of H_0 survive (only slightly deformed) in certain Cantorian set with empty interior, but that fills the phase space except for a set of small (Lebesgue) measure. This measure is bounded by $\mathcal{O}(\sqrt{\varepsilon})$. In fact, the tori for which persistence is proved, are the initial ones for which a) holds with μ bounded away from zero by a quantity that is $\mathcal{O}(\sqrt{\varepsilon})$. In this assertion, we identify (it is always possible under condition b) by means of the inverse function theorem) invariant tori by the vector of intrinsic frequencies. This is a usual tool in this work (see Chapters 1 and 3).

This result was initially proved in [1]. Analogous results were obtained in [49] for area-preserving maps of the plane in the differentiable case. A later result worth to mention is [60], where the regularity of these tori in the differentiable case is studied (by means of the so-called Whitney smoothness).

The nondegeneracy condition b) can be relaxed or replaced by other different conditions. For instance, let us mention two classical examples: the proper degeneracy case and the isoenergetic non-degenerate case. The proper degeneracy case happens when the unperturbed Hamiltonian H_0 depends on less than ℓ actions. In this case condition b) is not fullfiled (in fact the determinant vanishes identically). Nevertheless, if taking into account the first order contribution of the perturbation εH_1 it is possible to remove this degeneracy, then it is possible to show the persistence of mostly of the quasiperiodic motions (but having some slow frequencies). For instance, this situation happens in the planetary problem assuming the masses of the planets small enough with respect to the mass of the Sun. This proper degeneracy result was proved at first time in [2], where it was also applied to the case of the planar planetary problem. Recently, in [64], this was proved for the case of the spatial three-body problem.

In the isoenergetic case, condition b) is replaced by assumptions on the diffeomorphic character of the map

$$I \mapsto \left(\frac{\omega_2}{\omega_1}, \ldots, \frac{\omega_\ell}{\omega_1}, H_0\right),$$

assuming, of course, that ω_1 does not vanish. Under this hypothesis, it is not possible in general to show persistence of a torus with the same vector of intrinsic frequencies as in the unperturbed system, but it is possible to find an invariant torus of H with the same energy level and with a proportional frequency. This is a remarkable case, because is the classical example where condition b) can not hold for H_0, and hence the identification of tori with vectors of intrinsic frequencies is not possible, but most (in the sense of Lebesgue measure) of the invariant tori of H_0, survive to the perturbation if ε is small enough. For the proof, see [51], [7] and [13].

The upper estimates $\mathcal{O}(\sqrt{\varepsilon})$ for the measure of destroyed tori can be improved if one consider only "local versions" of KAM theorem. Let us explain this idea. Let us take a Diophantine torus of H_0, and let us assume that it is non-degenerate in the sense that b) holds. Then, one can try to estimate the density of invariant tori of H nearby, using the distance to the torus as a perturbative parameter. This formulation has been considered in [48]. Here it is shown that this density turns to be close to 1 except an exponentially small quantity with respect to this distance. The key point to interpretate this improvement, is that, near a Diophantine vector of frequencies, there are no vectors with low order resonances. For a clearer explanation of this fact, we refer to Sections 3.2.3 and A.2.1, or to [35].

A point worth to mention with this local formulation, is that the hypothesis of near-integrability for the unperturbed Hamiltonian H_0 is not required. It can be deduced in a strongly form (exponentially small estimates for the non-integrable remainder with respect to several parameters, as the size of the perturbation of the distance to the invariant torus) around every Diophantine torus of the unperturbed system. For instance, let us mention [14], where, under the isoenergetic hypothesis, these exponentially small bounds are shown for the measure of the set not filled up by quasiperiodic motions around an elliptic fixed point, with a Diophantine vector of normal frequencies. This is suitable to be applied to systems coming from some celestial mechanics models (see Section 3.4), not necessarily close to a integrable one. Hence, this formulation is also taken into account in this work.

The previous paragraph can be sintetized in the following assertion: every invariant torus of an analytic Hamiltonian system for which condition a) holds, organizes a dynamics that is close to integrable, except by a perturbation exponentially with respect to the distance to the torus.

0.1.2 Nekhroshev estimates

As it has been mentioned before, for a near-integrable Hamiltonian system with 3 or more degrees of freedom, KAM theorem does not imply perpetual stability. A mechanism to explain this unstability was initially described in [3]. It is the so-called Arnol'd diffusion.

Nevertheless, if the unperturbed Hamiltonian H_0 satisfies certain steepness conditions (for instance, to be quasiconvex, that can be described as the transversality of the energy levels of H_0 with respect to the manifolds of resonant tori), then it is possible to obtain lower bounds for the stability time, exponentially big with respect to the size of the perturbation, that is

$$|I(t) - I(0)| \le c_1 \varepsilon^b, \text{ for } |t| \le c_2 \exp\left(c_3/\varepsilon^a\right),$$

for every initial condition. The exponents a and b are called stability exponents. They are positive and related to ℓ. A result of this type was first proved in [55]. Later results ([44], [61], [13], [57], among others) improve the stability exponents to reach $a = b = \frac{1}{2\ell}$. The proofs of these results are based in the computation of (suitable) resonant normal forms.

As in Section 0.1.1, let us focus on the local formulation of the Nekhoroshev-like results. In this context, it is showed in [58] that, without any steepness hypothesis, a Diophantine KAM torus of a Hamiltonian system is very sticky, that is, the time to double the initial distance to the torus is exponentially big with respect to this initial distance (an analogous result can be obtained around an elliptic fixed point, using the bounds of [14]). If we also assume local quasiconvexity arund the torus, this time becomes superexponential (see [48]).

0.1.3 Lower dimensional tori of Hamiltonian systems

Now, let us assume that H has the origin as an elliptic equilibrium point. If we take the linearization at this point as a first approximation to the dynamics, we see that all the solutions are quasiperiodic and can be described as the product of ℓ linear oscillators. The solutions of each oscillator can be parametrized by the amplitude of the orbits.

When the nonlinear part is added, each oscillator becomes a one parametric family of periodic orbits (usually called Lyapunov orbits), that can be still parametrized by the amplitude, at least near the origin (see [68]). Generically, the frequency of these orbits varies with their amplitude.

To obtain a more accurate description of the dynamics of H near the origin, one can apply some steps of (Birkhoff) normal form to H, to write it as an integrable Hamiltonian H_0 plus a non-integrable remainder. This can be achieved for instance, if the normal frequencies at the origin satisfy a standard Diophantine condition. As it has been mentioned before, the order of the normal form can be selected such that this remainder turns out to be exponentially small with respect to the distance to the origin.

By the dynamics associated to H_0, the phase space is, of course, foliated by ℓ-dimensional tori, that can be seen as the combination of the ℓ oscillators associated to the linearized system. We notice that (under generic conditions of nondegeneracy for the nonlinear part of H_0), the exponentially smalls bounds on the remainder imply exponentially small bounds for the measure of destroyed tori.

Moreover, there are other families of quasiperiodic motions that come from the Hamiltonian in normal form H_0. By combining some of the elliptic directions associated to the fixed point, that is, by considering the product of some of the oscillators of the linearization, we obtain families of invariant tori of dimension ranging from 1 to $\ell - 1$. They are generically nonresonant (for the conditions of non-resonance needed to deal with lower dimensional tori, we refer to Chapters 1 and 3), and some of them also survive when we add the nonitegrable perturbation (see [17] and [8]). They are the generalization of the periodic Lyapunov orbits to higher dimensional tori and hence, they can be called Lyapunov tori.

We have considered this simple example to introduce the study of the lower dimensional tori.

This is not the only way to construct lower dimensional tori. Returning to the near-integrable Hamiltonian H of Section 0.1.1, we consider the case when, for certain action I^*, $k^\top \nabla H_0(I^*)$ is exactly zero for some $k \in \mathbb{Z}^\ell$. This implies that, since the frequencies are rationally dependent, the flow on the torus $I = I^*$ is not dense. More precisely, if one has ℓ_i independent frequencies, the torus $I = I^*$ contains an $(\ell - \ell_i)$-family of ℓ_i-dimensional invariant tori, and each of these tori is densely filled up by the flow. Here, the natural problem is also to study the persistence of these lower dimensional invariant tori when the nonintegrable part H_1 is taken into account. Generically, some of these tori survive but their normal behaviour can be either elliptic or hyperbolic (see [74], [43], [19], [43], [30] and [28]). The study of these tori is a classical subject in Hamiltonian dynamics as the invariant manifolds associated to their hyperbolic directions (usually called "whiskers") seem to be the skeleton that organizes the diffusion (see [3]).

In this work we will focus on every kind of nondegenerate reducible lower dimensional torus of analytic Hamiltonian systems, in the sense that its normal behaviour only contains elliptic or hyperbolic directions but not degenerate ones. This implies that the torus is not contained in a (resonant) higher dimensional invariant torus. We also restrict to the case of isotropic tori (this is, the canonical 2-form of $\mathbb{C}^{2\ell}$ restricted to the tangent bundle of the torus vanishes everywhere). This fact (that is always true for a periodic orbit) is not a strong assumption for a torus, because all the tori obtained by applying KAM techniques to near-integrable Hamiltonian systems are isotropic.

Let r be the dimension of the torus, and let ω be its vector of basic frequencies. Then, from the previous hypotheses, we assume that we can introduce (with a canonical change of coordinates) r angular variables θ describing the initial torus. Hence, the Hamiltonian takes the form

$$H(\theta, x, I, y) = \omega^\top I + H_1,$$

where x, y, θ and I are complex vectors, x and y elements of $\mathbb{C}^r$ and θ and I elements of $\mathbb{C}^s$, with $r + m = \ell$. Here, θ and x are the positions and I and y are the conjugate momenta. We assume H is an analytic function with 2π-periodic dependence on θ. Then, if H has an invariant r-dimensional torus with vector of basic frequencies ω, given by $I = 0$ and $x = y = 0$, then the Taylor expansion of H_1 must begin with terms of second order in the variables x, y and I.

Let us consider the variational flow around one of the quasiperiodic orbits of the initial r-dimensional invariant torus of H. The variational equations are a linear system with quasiperiodic time dependence, with vector of basic frequencies ω. When the torus is a periodic orbit, the well know Floquet theorem states that we can reduce this periodic system to constant coefficients via a linear periodic change of variables (with the same period of the system). This change can be selected to be canonical if the equations are Hamiltonian. So, the reduced matrix has a pair of zero eigenvalues (one associated to the tangent direction to the periodic orbit and a second one from the symplectic character of the monodromy matrix of the periodic orbit) plus eigenvalues that describe the linear normal behaviour around the torus. We will assume that these eigenvalues are all different (this condition implies, from the canonical character of the system, that they are also non-zero). Usually, the imaginary parts of these eigenvalues are called normal frequencies, and ω is called the vector of intrinsic frequencies of the torus.

The quasiperiodic case ($r > 1$) is more complex, because we cannot guarantee in general the reducibility to constant coefficients of the variational equations with a linear quasiperiodic change of variables with the same basic frequencies as the initial system. The question of reducibility of linear quasiperiodic systems (proved in some cases, see [29], [11], [18], [33], [35], [30], and [42], among others) remains open in the general case. However, we can say that if this reduction is possible, we have $2r$ zero eigenvalues (related to the r tangent vectors to the torus).

We restrict to the case when such reduction is possible for the initial torus. It is usual to call such a torus reducble or Floquet. This reducibility is a standard assumption to deal with KAM theory, where in general the lower dimensional tori we can construct (except for the normal hyperbolic case, see the discussion below) is reducible.

We remark that, if this initial torus comes from an autonomous perturbation of a resonant torus of an integrable Hamiltonian, this hypothesis is not very strong. To justify this assertion, we mention the following fact: let us write the Hamiltonian as

$H = \mathcal{H}(I) + \varepsilon\hat{\mathcal{H}}(\theta, I)$, and let $\mathcal{T}_0$ be a low dimensional invariant tori of the integrable Hamiltonian $\mathcal{H}(I)$ that survives to the perturbation $\varepsilon\hat{\mathcal{H}}(\theta, I)$. Then, under generic hypothesis of nondegeneracy and nonresonance, this low dimensional torus exists and its normal flow is also reducible for a Cantor set of values of ε. The Lebesgue measure of the complementary of this set in $[0, \varepsilon_0]$ is exponentially small with ε_0. This fact is proved in [30] for symplectic diffeomorphisms of $\mathbb{R}^4$, but it is immediate to extend to other cases.

If we have that the initial torus is reducible, we can assume that H takes the following form:

$$H(\theta, x, I, y) = \omega^\top I + \frac{1}{2} z^\top \mathcal{B} z + H_2,$$

where $z^\top = (x^\top, y^\top)$, and when $\mathcal{B}$ is a symmetric $2m$-dimensional matrix (with complex coefficients). If we also assume that the quadratic terms of H_2 in the z variables vanish, then the normal variational equations are given by the matrix $J_m\mathcal{B}$, where J_m is the canonical 2-form of $\mathbb{C}^{2m}$.

KAM-like results can be considered around a lower dimensional torus. From a classical point of view, and following the same lines as in the near-integrable case, in the literature it is usual to work with perturbations of Hamiltonians having a r-dimensional familiy of r-dimensional reducible tori (for instance, this can be achieved if the system has r first integrals independent and in involution, see [56]). Hence, the "unperturbed" Hamiltonian takes the form:

$$H(\theta, x, I, y) = \Omega(I) + \frac{1}{2} z^\top \mathcal{B}(I) z + H_1,$$

where $H_1 \equiv \mathcal{O}_3(z)$. Classical results in this general case can be found in [17] and [62].

The most difficult case is when the normal behaviour of the torus contains some elliptic directions, because the (small) divisors obtained contain combinations of the intrinsic frequencies with the normal ones. Under suitable nondegeneracy conditions, it is not difficult to control the value of the intrinsic frequencies but then, we have no control (in principle) on the corresponding normal ones. Hence, we can not ensure if they are going to satisfy suitable Diophantine condition. This is equivalent to say that we can not select a torus with given both intrinsic and normal frequencies, because there are not enough available parameters (see [50]).

When the initial torus is normally hyperbolic we do not need to control the eigenvalues in the normal direction, as the divisors involved in this case can be controled by means of Diophantine assumptions on the intrinsic frequencies and from the non-vanishing character of the real parts of the normal eigenvalues. Hence, we do not have to deal with the lack of parameters. In this case, it is also remarkable that the invariant tori of the perturbed system can be constructed without asking for reducibility. The only condition needed is that the variational flow of the initial torus can be written as a quasiperiodic perturbation of an autonomous (hyperbolic) matrix.

In the genral case, one possibility is to add enough (extra) parameters to the system to overcome this lack of parameters problem (see [9]). This methodology can be used to derive, under suitable nondegeneracy hypotheses (without any extra parameters), estimates on the measure of surviving tori (see [67] and [8]). In this context, the measure of the set filled up by these tori (in a suitable space) tends to the measure of the initial set of tori as the perturbation goes to zero.

As it has been mentioned before, one of main contribution of Chapters 1 and 3, is an improvement of these results on measure of surviving tori by considering local formulations of the problem.

Moreover, this improvement is also done in the estimates of invariant tori when the excitation coming from the elliptic normal directions is considered. These tori can be seen as a generalization of the different families of Lyapunov tori around an elliptic (or partially elliptic) fixed point, to the case of a partially elliptic lower dimensional torus (See Section 1.2.1 for a more precise explanation). For previous results in this case, see [17] and [8].

Let us mention that the Nekhoroshev bounds obtained in Chapter 1 for elliptic tori, are obtained following the same line as [58], without any steepness condition.

0.1.4 Quasiperiodic perturbations of Hamiltonian systems

Let us introduce the standard formulation to work with quasiperiodic time dependent perturbations of Hamiltonian systems. For instance, let us consider a Hamiltonian $H_0(x, y)$, where x and y are the canonical variables. If we add a quasiperiodic perturbation depending on time with s basic frequencies, the Hamiltonian is of the form

$$H(x, \theta, y, I) = \omega^\top I + H_0(x, y) + \varepsilon H_1(x, \theta, y, I), \tag{0.1}$$

where θ and I are new canonical variables introduced to put the Hamiltonian in autonomous form. Here H_1 is 2π-periodic on θ. As it is usual here, we restrict ourselves to the case in which H is an analytic function.

The problem of the preservation of maximal dimension tori of Hamiltonians like (0.1) has been considered in [35]. There it is proved that most (in the usual measure sense) of the tori of the unperturbed system survive to the perturbation, but adding the perturbing frequencies to the ones they already have.

In Chapter 3, we will consider the persistence of lower dimensional tori under quasiperiodic perturbations. We will show that, under some standard hypotheses of nondegeneracy and nonresonance, a r-dimensional invariant torus of H_0 can be continued with respect to ε to an $(r+s)$-dimensional torus of H, except for a small set for this parameter (of exponentially small measure with respect to ε_0 when intersecting with $[0, \varepsilon_0]$). Moreover, we are also going to show that, if ε is fixed and small enough, there are plenty of $(r+s)$-dimensional tori near the initial one. These tori have vectors of basic frequencies contains r frequencies close to the frequencies of the intial torus and the s of the perturbation.

The reason to consider those kind of perturbations is that they appear in a natural way in several problems of celestial mechanics: for instance, to study the dynamics of a small particle (an asteroid or spacecraft) near the equilateral libration points ([72]) of the Earth–Moon system, one can take the Earth–Moon system as a restricted three body problem (that can be written as an autonomous Hamiltonian) plus perturbations coming for the real motion of Earth and Moon and the presence of the Sun. As these perturbations can be very well approximated by quasiperiodic functions (at least for moderate time spans), it is usual to do so. Hence, one ends up with an autonomous model perturbed with a function that depends on time in a quasiperiodic way. Details on these models and their applications can be found in [15], [22], [24], [26] and [12].

Chapter 1

Normal Behaviour of Partially Elliptic Lower Dimensional Tori

1.1 Introduction

The study of the solutions close to an invariant object is a classical subject in Dynamical Systems. Here we will address the problem of describing the phase space near an invariant torus of a Hamiltonian system. To fix the notation, let us call H to a real analytic Hamiltonian with ℓ degrees of freedom, and let us assume it has an invariant r-dimensional torus, $0 \leq r \leq \ell$. Note that we are including the two limit cases, that is, when it is an equilibrium point and when it is a maximal dimensional torus.

Let us assume that the torus has some elliptic directions, this is, that the linearized normal flow contains some harmonic oscillators. A natural question is if these oscillations persist when the nonlinear part of the Hamiltonian is added. If the torus is totally elliptic, another natural problem is the (nonlinear) stability around this torus.

In this work we will consider these problems for a lower dimensional torus. The two limit cases ($r = 0$ and $r = \ell$) are included, and the results obtained can be summarized as follows: for a totally elliptic torus, we have obtained lower bounds for the diffusion time. They agree with the bounds of [48] in the case $r = \ell$ but, for the case $r = 0$, they are better than the ones directly derived from [14]. Moreover, we show the existence of quasiperiodic solutions that generalize the linear oscillations of the normal flow to the complete system. If the torus has normal behaviour of the kind "some centres"$\times$"some saddles" we obtain, for any combination of centres, a Cantor family of invant tori around the initial one, by adding to the initial set of frequencies new ones that come from the nonlinear oscillations associated to the chosen centres. Those invariant tori have the same normal behaviour as the initial one (of course, skipping the centres that give rise to the family). This result is a sort of "Cantorian central manifold" theorem, in which we obtain an invariant manifold parametrized on a Cantor set and completely filled up by invariant tori. We note that we obtain a Cantorian central "submanifold" for each combination of centres, and that it is uniquely defined.

The proofs are based on the construction of suitable normal forms. The estimates on the difussion time are obtained bounding the remainder of this normal form, while the existence of families of lower dimensional tori is proved by applying a KAM scheme to this remainder.

This chapter has been organized in the following way: Section 1.2 summarizes the main ideas and results contained in the work. Section 1.3 contains the details concerning the normal form and the bounds on the diffusion time. Section 1.4 is devoted to the existence of families of tori near the initial one and, finally, in Section 1.5, we have included some basic lemmas used along this chapter.

1.2 Summary

Here we have included a technical description of the problem, the methodology used in the proofs and the results obtained. We have ommitted the technical details of the proofs in order to simplify the reading.

1.2.1 Notation and formulation of the problem

Let H be a real analytic Hamiltonian system of ℓ degrees of freedom, having an invariant r-dimensional reducible and isotropic torus, $0 \le r \le \ell$, with a quasiperiodic flow given by a vector of basic frequencies $\hat{\omega}^{(0)} \in \mathbb{R}^r$.

In this chapter, we restrict to the case when this reduction can be done by means of a real change of variables. This is not possible, in general, for any reducible torus, but is an standard assumption to keep the real character of the Hamiltonian system after we rewrite it in the reduced form. To show that this restriction, that always hold for a periodic orbit (doubling the period if necessary), is not so strong as it seems, let us remark that a linear differential system corresponding to a real analytic quasiperiodic perturbation of a constant real matrix, is reducible to real coefficients under very general hypothesis, except for a set of very small measure (exponentially small) for the perturbative parameter. (see [35]).

Then, we assume, from the isotropic character of the torus, that we can introduce (with a canonical change of coordinates) r angular variables $\hat{\theta}$ describing the initial torus. Moreover, as the normal variational flow of the torus is reducible to constant coefficiens, we also assume that such variables are chosen in such a way that these variational equations are already reduced to constants coefficients. Hence, the Hamiltonian in these coordinates takes the form

$$H(\hat{\theta}, x, \hat{I}, y) = \hat{\omega}^{(0)\top} \hat{I} + \frac{1}{2} z^\top \mathcal{B} z + H_1(\hat{\theta}, x, \hat{I}, y), \tag{1.1}$$

where $z^\top = (x^\top, y^\top)$. Here, x, y are m-dimensional real vectors, and $\hat{\theta}$, $\hat{I}$ belong to $\mathbb{R}^r$, $r+m = \ell$. Of course, $\hat{\theta}$, x are the positions and $\hat{I}$, y the respective conjugate momenta. As $\hat{\theta}$ is an angular variable, we assume that H depends on it in a 2π-periodic way. Moreover, we will use $u^\top v$ to denote the scalar product of two vectors.

We also suppose that the Hamiltonian H can be extended to a real analytic function defined on the set $\mathcal{D}_{r,m}(\rho_0, R_0)$ given by

$$\mathcal{D}_{r,m}(\rho_0, R_0) = \{(\hat{\theta}, x, \hat{I}, y) \in \mathbb{C}^r \times \mathbb{C}^m \times \mathbb{C}^r \times \mathbb{C}^m \,:\, |\mathrm{Im}\hat{\theta}| \le \rho_0,\ |z| \le R_0,\ |\hat{I}| \le R_0^2\}, \tag{1.2}$$

where $|.|$ denotes the infinity norm of a complex vector (we will use the same notation for the matrix norm induced). The different scaling for the variables z and $\hat{I}$ in $\mathcal{D}_{r,m}(\rho_0, R_0)$

is motivated by the definition of degree for a monomial of the Taylor expansion (with respect to z and $\hat{I}$, see (1.10)) used along this chapter:

$$\deg\left(h_{l,s}(\hat{\theta})z^l\hat{I}^s\right) = |l|_1 + 2|s|_1, \tag{1.3}$$

with $l \in \mathbb{N}^{2m}$, $s \in \mathbb{N}^r$, and where $|k|_1$ is defined as $\sum_j |k_j|$. The reason for counting twice the exponent s will be clear later (it is motivated, basically, by the properties of the Poisson bracket).

We assume the initial invariant torus is given by $z = 0$ and $\hat{I} = 0$. Hence, we can take $\mathcal{B}$ as a symmetric $2m$-dimensional real matrix. Moreover, the Taylor expansion of H_1 around $z = 0$, $\hat{I} = 0$ begins with terms of degree at least three.

Linear normal behaviour of the torus

We also assume that the matrix $J_m\mathcal{B}$ has different eigenvalues, given by the complex vector $\lambda \in \mathbb{C}^{2m}$, that takes the form $\lambda^\top = (\lambda_1, \dots, \lambda_m, -\lambda_1, \dots, -\lambda_m)$ (this structure comes from the canonical character of the system). We note that in this case, different eigenvalues also means nonzero eigenvalues. We will refer to those eigenvalues as the normal eigenvalues of the torus. We remark that if $\lambda_j = i\beta$ (with $\beta \in \mathbb{R}\setminus\{0\}$ and $i = \sqrt{-1}$) is an eigenvalue, then $\lambda_{j+m} = -i\beta$. The vectors of $\mathbb{R}^{2m}$ that are combination of eigenvectors corresponding to (couples of) eigenvalues of this form are called the elliptic directions of the torus.

The study of the behaviour of the initial torus in those directions is the main issue in this chapter. Moreover, there may be other eigenvalues with real part different from zero, that define the hyperbolic directions of the torus. They can be grouped in one of these two following forms:

1. if $\lambda_j = \alpha \in \mathbb{R}\setminus\{0\}$, then $\lambda_{j+m} = -\alpha$,
2. if $\lambda_j = \alpha + i\beta$ (with $\alpha, \beta \in \mathbb{R}\setminus\{0\}$), then, from the real character of the matrix $\mathcal{B}$, we can take $\lambda_{j+1} = \alpha - i\beta$, and hence, $\lambda_{j+m} = -\alpha - i\beta$ and $\lambda_{j+m+1} = -\alpha + i\beta$.

The imaginary parts of the eigenvalues are usually called normal frequencies of the torus.

For reasons that will be clear later, it is very convenient to put the matrix $J_m\mathcal{B}$ in diagonal form. This is possible with a complex canonical change of basis, that transforms the initial real Hamiltonian system into a complex one. Thus, the complexified Hamiltonian has some symmetries because it comes from a real one. As this symmetries are preserved by the transformations used along the proofs, the final Hamiltonian can be realified. In fact, complexification is not necessary, but it simplifies the proofs. Nevertheless, in the proofs we have not written explicitly the preservation of those symmetries. This is because the details are very tedious and cumbersome and, on the other hand, the interested reader should not have problems in writting them (it is a very standard methodology). For further uses, we denote by $Z^\top = (X^\top, Y^\top)$ those complex (canonical) variables, and by $\mathcal{B}^*$ the complex symmetric matrix such that $J_m\mathcal{B}^* = \mathrm{diag}(\lambda)$.

Seminormal form: formal description

Now we take a subbundle of $\mathbb{R}^{2m}$, $\mathcal{G} \subset \mathbb{R}^{2m}$, invariant by the action of the matrix $J_m\mathcal{B}$, and such that it only contains eigenvectors of elliptic type. We put $2m_1 = \dim(\mathcal{G})$

(we recall that this dimension is always even) and we call $\tilde{\omega}^{(0)} \in \mathbb{R}^{m_1}$ to the vector of normal frequencies associated to this subbundle. As $\mathcal{G}$ will be fixed along this chapter, we introduce some notation related to it. First, we assume that the first m_1 eigenvalues of λ are the ones associated to $\mathcal{G}$, that is, $\lambda_j = i\tilde{\omega}_j^{(0)}$, $j = 1, \dots, m_1$. We also denote by $\hat{\lambda} \in \mathbb{C}^{2(m-m_1)}$ the vector obtained skipping from λ the $2m_1$ eigenvalues associated to $\mathcal{G}$. This introduces in a natural way the decomposition $X^\top = (\tilde{X}^\top, \hat{X}^\top)$, $Y^\top = (\tilde{Y}^\top, \hat{Y}^\top)$, obtained taking apart the first m_1 components from the last $m - m_1$. Moreover, we define $\tilde{Z}^\top = (\tilde{X}^\top, \tilde{Y}^\top)$ and $\hat{Z}^\top = (\hat{X}^\top, \hat{Y}^\top)$. A similar notation can be used for any vector $l \in \mathbb{N}^{2m}$, splitting $l^\top = (l_X^\top, l_Y^\top)$, where l_X and l_Y are the exponents of X and Y in the monomial Z^l ($Z^l = X^{l_X} Y^{l_Y}$). Then, we introduce $\omega^{(0)} \in \mathbb{R}^{r+m_1}$ as $\omega^{(0)\top} = (\hat{\omega}^{(0)\top}, \tilde{\omega}^{(0)\top})$, and we ask for a Diophantine condition of the following form,[1]

$$|ik^\top \omega^{(0)} + l^\top \hat{\lambda}| \geq \frac{\mu}{|k|_1^\gamma}, \quad k \in \mathbb{Z}^{r+m_1} \setminus \{0\}, \quad l \in \mathbb{N}^{2(m-m_1)}, \quad 0 \leq |l|_1 \leq 2, \tag{1.4}$$

being $\mu > 0$ and $\gamma > r + m_1 - 1$. This nonresonance condition allows to construct (formally) a seminormal form related to the chosen $\mathcal{G}$. If we express the Hamiltonian in terms of the variables Z, this seminormal form is done by removing from H the monomials of the following form (see (1.11) for the notations):

$$h_{l,s,k} \exp(ik^\top \hat{\theta}) Z^l \hat{I}^s, \quad l \in \mathbb{N}^{2m}, \quad s \in \mathbb{N}^r, \quad k \in \mathbb{Z}^r, \quad |k|_1 + |l_X - l_Y|_1 \neq 0, \quad |\hat{l}|_1 \leq 2, \tag{1.5}$$

where $\hat{l}$ is the part of l that corresponds to $\hat{\lambda}$. After this normal form process, using the preservation of the symmetries that come from the complexification, we can rewrite this (formal) seminormal form, in terms of suitable real variables, in the following form:

$$H(\hat{\theta}, x, \hat{I}, y) = \omega^{(0)\top} I + \frac{1}{2} \hat{z}^\top \hat{\mathcal{B}} \hat{z} + \mathcal{F}(I) + \frac{1}{2} \hat{z}^\top \mathcal{Q}(I) \hat{z} + O_3(\hat{z}), \tag{1.6}$$

where, for simplicity, we do not change the name of the Hamiltonian, and where we extend the decomposition introduced above to the variables (x, y). Here, the matrix $\hat{\mathcal{B}}$ is a real symmetric matrix obtained by projecting $\mathcal{B}$ on the directions given by the eigenvalues corresponding to the eigenvectors $\hat{\lambda}$. I is a compact notation for $I^\top = (\hat{I}^\top, \tilde{I}^\top)$, where the actions $\tilde{I}$ can be taken as $\tilde{I}_j = \frac{1}{2}(x_j^2 + y_j^2)$, $j = 1, \dots, m_1$, if we choose the real normal form variables (x, y) associated to the considered elliptic directions in adequate (and standard) way (see (1.44) in the proof of Theorem 1.2). Of course, $\mathcal{F} = O_2(I)$ and $\mathcal{Q} = O_1(I)$.

Now, we proceed to describe the normal behaviour of the torus derived from this seminormal form. It is not difficult to check that we have the following (formal) quasiperiodic solutions for the canonical equations of (1.6):

$$\begin{aligned}
\hat{\theta}(t) &= \left(\hat{\omega}^{(0)} + \frac{\partial \mathcal{F}}{\partial \hat{I}}(I(0)) \right) t + \hat{\theta}(0), \\
\hat{I}(t) &= \hat{I}(0),
\end{aligned}$$

[1] The Diophantine condition can be relaxed when $|l|_1 = 2$ and $l^\top \hat{\lambda}$ only involves hyperbolic eigenvalues. In this case, the results are proved using a combined method based on a fixed point scheme for the hyperbolic directions and a Newton method for the remaining ones. This technique allows to have multiple hyperbolic eigenvalues.

$$\begin{aligned}\tilde{x}_j(t) &= \sqrt{2\tilde{I}_j(0)}\sin\left(\left(\tilde{\omega}^{(0)}+\frac{\partial\mathcal{F}}{\partial\tilde{I}_j}(I(0))\right)t+\tilde{\theta}_j(0)\right),\\ \tilde{y}_j(t) &= \sqrt{2\tilde{I}_j(0)}\cos\left(\left(\tilde{\omega}^{(0)}+\frac{\partial\mathcal{F}}{\partial\tilde{I}_j}(I(0))\right)t+\tilde{\theta}_j(0)\right),\\ \hat{z}(t) &= 0.\end{aligned} \tag{1.7}$$

That is, we obtain a $2(r+m_1)$-dimensional invariant manifold ($\hat{z}=0$) foliated by a continuous $(r+m_1)$-dimensional family of $(r+m_1)$-dimensional invariant reducible tori, parametrized by $I(0)$. The selection of the parameter $I(0)$ is natural, as $I_1,\ldots,I_{r+m_1}$ are first integrals of the Hamiltonian (1.6) restricted to the invariant manifold $\hat{z}=0$. We remark that the tori of the family collapse to lower dimensional ones when any of the $\tilde{I}_j(0)$ become zero. In particular, if we take $I(0)=0$ we recover the initial r-dimensional one. In fact, for every $0\le m_2\le m_1$ we have, for this seminormal form, $\binom{m_1}{m_2}$ different $(r+m_2)$-dimensional families of $(r+m_2)$-dimensional invariant tori. They are associated to every invariant real subbundle contained in $\mathcal{G}$. The skeleton of these families comes from the natural r-dimensional family of r-dimensional tori containing the initial one. This family is associated to the neutral directions of the torus (the neutral directions are conjugated to the tangent ones), an it is obtained taking $\mathcal{G}=0$ in our notation. Moreover, we also remark that in (1.7) we only have real tori when all the $\tilde{I}_j\ge 0$. This comes directly from the definition of $\tilde{I}$ as a function of the real normal form variables. To explain this fact let us give the classical example of a 1-dimensional pendulum near the elliptic equilibrium point, $\ddot{x}+\sin(x)=0$. The linear (normal) frequency at the equilibrium point is 1. Moving the energy level in the real phase space we obtain periodic orbits with frequency smaller than 1. If one wants periodic orbits with frequency bigger than 1, one is forced to extend the phase space from $\mathbb{R}^2$ to $\mathbb{C}^2$, keeping the time in $\mathbb{R}$. This same phenomenon happens when we study the normal elliptic directions of a torus. It is important to note that, for us, a s-dimensional complex torus is a map from $\mathbb{T}^s$ to $\mathbb{C}^{2\ell}$. Hence, we will use the word "dimension" to refer to the real dimension.

1.2.2 Results and main ideas

A basic result in this chapter is the quantitative version of the seminormal form, if we only kill the monomials like (1.5) up to some finite order. From the estimates on this seminormal form, we deduce (under certain nondegeneracy conditions) that the normal behaviour of the initial torus described in Section 1.2.1 is "correct" in the sense of the classical KAM ideas: the "majority" of these tori really exist (but slightly deformed) in the initial Hamiltonian system. Moreover, we also deduce the long time effective stability of any real trajectory close to a totally elliptic torus. In the following sections we present the explicit description of those results, and we explain the main ideas used in the proofs.

Seminormal form: bounds on the remainder

We start with the Hamiltonian (1.1), where the normal flow is reduced to constant coefficients. Then, we perform a finite number of (semi)normal form steps, by using suitable canonical transformations that remove the monomials (1.5) up to a finite degree. This allows to show the convergence of the process on the set $\mathcal{D}_{r,m}(\rho_1, R)$, where ρ_1 is independent from R and R is small enough. By selecting the order up to which the seminormal form is done as a suitable function of R, it is possible to obtain a remainder for the seminormal form which is exponentially small with R. This is contained in Theorem 1.1.

Elliptic tori are very sticky

Now let us assume that the inital torus has all the normal directions of elliptic type. In this case we can take $\mathcal{G} = \mathbb{R}^{2m}$, the whole set of normal directions.

Then, using the normal form explained above, one can write the initial Hamiltonian as an integrable one plus an exponentially small perturbation. Hence, it is very natural to obtain exponentially big estimates for the diffusion time: the time needed for a real trajectory to go away from the set $\mathcal{D}_{r,m}(\rho, R)$ (for a precise definition of "going away" see Theorem 1.2) is bigger than

$$T(R) = const. \exp\left(const. \left(\frac{1}{R}\right)^{\frac{2}{\gamma+1}}\right), \tag{1.8}$$

being the constants on the definition of $T(R)$ independent from R. As usual, we call the exponent $\frac{2}{\gamma+1}$ the stability exponent.

Let us compare this result with previous ones. In the case in which the initial torus is of maximal dimension, note that the normal variables $z^\top = (x^\top, y^\top)$ are missing everywhere. So, the set $\mathcal{D}_{r,m}(\rho, R)$ (see (1.2)) reads

$$\mathcal{D}_{\ell,0}(\rho, R) = \{(\hat{\theta}, \hat{I}) \in \mathbb{C}^\ell \times \mathbb{C}^\ell : \ |\mathrm{Im}\hat{\theta}| \le \rho, \ , \ |\hat{I}| \le R^2\}.$$

To compare with [48] we must redefine R^2 as R, in order to have the same units. Then, the stability exponent in (1.8) coincides with [48].

If the initial torus is an equilibrium point, the variables $\hat{\theta}$ and $\hat{I}$ are the ones that are missing. Hence, $\mathcal{D}_{r,m}(\rho, R)$ becomes

$$\mathcal{D}_{0,\ell}(0, R) = \{(x, y) \in \mathbb{C}^\ell \times \mathbb{C}^\ell : \ |(x, y)| \le R\}.$$

Hence, no rescaling is necessary to compare the diffusion time of (1.8) with the one derived from [14]: the improvement is that the exponent $\frac{1}{\gamma+1}$ in [14] here becomes $\frac{2}{\gamma+1}$. We note that this improvement is not only on the diffusion time, but also on the measure of the destroyed tori (see Remark 1.10).

Cantor families of invariant tori

It is clear from Section 1.2.1 that computing the seminormal formal form associated to $\mathcal{G}$ around the initial torus, up to finite order, and skipping the non-integrable remainder, those elliptic directions define a unique $(r+m_1)$-dimensional family of $(r+m_1)$-dimensional tori around the initial r-dimensional one. When we approach the inital torus, the intrinsic frequencies of the tori of the family can be selected such that they tend to $\omega^{(0)}$.

In this case we will show that when we add the remainder of the seminormal form most of these tori still persist in the complete system H, having also reducible normal flow. The normal eigenvalues of these tori are close to the eigenvalues $\hat{\lambda}_j$ (that are the ones not related with $\mathcal{G}$). Of course, due to the different small divisors involved in the problem, we can not prove the persistence of all the invariant tori predicted by the normal form.

The hypotheses needed are usual in KAM methods. The first one is a non-resonance condition involving the frequencies $\hat{\omega}^{(0)}$ and the normal ones λ, that depends on the concrete selection of $\mathcal{G}$ and it is explicitly given in (1.4). The second hypothesis is a nondegeneracy condition, asking that all the frequencies vary with the actions. Note that, in general, we have more frequencies $(r+m)$ than actions $(r+m_1,\ m_1 \le m)$. This introduces the classical lack-of-parameters problem when working with lower dimensional torus, that needs a special treatement (for related results, see [9], [67] and [8]). The idea that we have used here is to choose a suitable $(r+m_1)$-dimensional set of parameters, and to ask for the existence of lower dimensional tori associated to some of the values of these parameters. Here, the natural parameter is the vector of intrinsic frequencies $\omega \in \mathbb{R}^{r+m_1}$ of the invariant tori. To use this parametrization we need a typical nondegeneracy condition on the frequency map from I to ω, this is, that this map be a (local) diffeomorphism around $I=0$. This condition can be explicitly formulated computing the normal form of Section 1.2.1 up to degree 4 and it is given in (1.50). The control of the remaining $m-m_1$ normal frequencies (normal to the $(r+m_1)$-dimensional family of tori) is more difficult, since there are no free parameters to control them. Note that those frequencies are functions of the intrinsic ones. Then, the idea is to eliminate all the frequencies for which the Diophantine conditions needed to construct invariant tori are not satisfied. This will lead us to eliminate values of ω to: a) control the intrinsic frequencies ω and b) control the normal ones as a function of the intrinsic ones. To control the measure of the set of intrisic frequencies for which the associated normal ones are close to resonance, we ask for a extra set of nondegeneracy conditions for the dependence of these normal frequencies with respect to the intrinsic ones. Those conditions are given in (1.54). They have already been considered in [47] and [17].

With the formulation given above, the result is that the measure of the complementary of the preserved tori is exponentially small: we introduce

$$\mathcal{U}(A) = \left\{ \omega \in \mathbb{R}^{r+m_1} \,:\, |\omega - \omega^{(0)}| \le A \right\}, \quad A > 0, \tag{1.9}$$

and let us define $\mathcal{A}(A)$ as the set of frequencies of $\mathcal{U}(A)$ for which we have reducible invariant tori. Then, if A is small enough, we have

$$\frac{\mathrm{mes}(\mathcal{U}(A) \setminus \mathcal{A}(A))}{\mathrm{mes}(\mathcal{U}(A))} \le const. \exp\left(-const. \left(\frac{1}{A}\right)^{\frac{1}{\gamma+1}} \right),$$

where $\mathrm{mes}(\cdot)$ denotes the Lebesgue measure of $\mathbb{R}^{r+m_1}$, γ is the exponent of the Diophantine condition (1.4), and the constants that appear in this bound are positive and independent from A. This result is formulated in Theorem 1.3. Nevertheless, as we have noted in Section 1.2.1, some of the frequencies of $\mathcal{A}(A)$ give rise to complex tori. If one wants to ensure that the obtained tori are real tori, one can look at the formulation of Theorem 1.4.

Let us describe how those result are proved. For this purpose, we start from the seminormal form provided by Theorem 1.1, and we assume that the reader is familiar with the standard KAM techniques (see [4] and references therein).

Initially, we have the seminormal form tori parametrized by the vector of "actions" $I \in \mathbb{R}^{r+m_1}$ (see (1.7)). By using the nondegeneracy condition of (1.50), we can replace this parameter by the $(r+m_1)$-dimensional vector of frequencies (see Lemma 1.3 for the details).

The main issue is to kill, for a given frequency, the part of the remainder that obstructs the existence of the corresponding invariant torus in the complete Hamiltonian. This will be done by a standard iterative Newton method. As usual, we need to have some control on the combinations of intrinsic frequencies and normal eigenvalues that appear in the divisors of the series used to keep them satisfying a suitable Diophantine condition (like (1.4)). This control can be done using the nondegeneracy conditions of (1.54). As we will start these iterations from an integrable Hamiltonian (at least in the direction $\mathcal{G}$) with an exponentially small perturbation, we can take the μ in (1.4) of the same order. This produces convergence except for a set of "bad frequencies" with exponentially small measure.

Now, we use Poincaré variables (see (1.57) and (1.58)) to introduce extra m_1 angular variables to describe the invariant $(r+m_1)$-dimensional tori of the seminormal form. When we introduce those variables, there is also another source for degeneracy that, essentially, is due to the fact that the family of $(r+m_1)$-dimensional tori comes from an r-dimensional one. It causes the Poincaré variables to become singular when some of the $\tilde{I}_j$ are zero.[2] We remark that this degeneracy corresponds, in the seminormal form (1.6), to the families of invariant tori of dimension between r and $r+m_1-1$ (assuming $m_1 \geq 1$). If we ask for real invariant tori, this degeneracy also corresponds to the transition manifold from real to complex tori. We remark that we have an exact knowledge of the manifold of degenerate frequencies for the seminormal form, but the exponentially small remainder makes that we only know the set of degenerate frequencies up to an exponentially small error. This is the main reason that forces to refine the seminormal form as we approach to the initial torus. Moreover, the same remarks apply when we look for real tori: we know the boundary of the set of frequencies that are candidate to give a real torus in the complete Hamiltonian, with an exponentially small error. To remove the degeneracy, we will take out a neighbourhood of the frequencies corresponding to the transition manifold. As the terms of the remainder are exponentially small with R, this neighbourhood can be selected with exponentially small measure with respect to R.

Finally, we note that the application of the results mentioned above show that, around the initial torus, there exists (Cantorian) families of tori of dimensions between r and m_e (we recall m_e is the number of elliptic directions of the initial torus), under generic conditions of non-resonance and nondegeneracy. For previous results in this context, we refer to [17] and [8].

[2]This is the same problem that appears when we put action-angle variables around an elliptic equilibrium point of a one degree of freedom Hamiltonian system. A neighbourhood of the origin has to be excluded since the change of variables is singular there.

1.3 Normal form and effective stability

This section contains the technical details of the seminormal form process with rigorous bounds on the remainder, as well as bounds on the diffusion time around an elliptic torus.

1.3.1 Notation

First, let us introduce some notation. We will consider analytic functions $h(\hat{\theta}, x, \hat{I}, y)$ defined on $\mathcal{D}_{r,m}(\rho, R)$, for some $\rho > 0$ and $R > 0$, and 2π-periodic with respect to $\hat{\theta}$. We denote the Taylor series of h as

$$h = \sum_{(l,s)\in\mathbb{N}^{2m}\times\mathbb{N}^r} h_{l,s}(\hat{\theta}) z^l \hat{I}^s. \tag{1.10}$$

Moreover, the coefficients $h_{l,s}$ will be expanded in Fourier series,

$$h_{l,s}(\hat{\theta}) = \sum_{k\in\mathbb{Z}^m} h_{l,s,k} \exp\left(ik^\top \hat{\theta}\right). \tag{1.11}$$

We will denote by $\bar{h}_{l,s} = h_{l,s,0}$ the average of $h_{l,s}(\hat{\theta})$, and let us define $\tilde{h}_{l,s}(\hat{\theta}) = h_{l,s}(\hat{\theta}) - \bar{h}_{l,s}$. Then, we use the expressions (1.10) and (1.11) to introduce the following norms:

$$|h_{l,s}|_\rho = \sum_{k\in\mathbb{Z}^m} |h_{l,s,k}| \exp\left(|k|_1 \rho\right), \tag{1.12}$$

$$|h|_{\rho,R} = \sum_{(l,s)\in\mathbb{N}^{2m}\times\mathbb{N}^r} |h_{l,s}|_\rho R^{|l|_1+2|s|_1}. \tag{1.13}$$

Some basic properties of these norms are given in Section 1.5. Here we only note that, if those norms are convergent, they are bounds for the supremum norms of $h_{l,s}(\hat{\theta})$ (on the complex strip of width $\rho > 0$) and of h (on $\mathcal{D}_{r,m}(\rho, R)$).

Let us recall the definition of Poisson bracket of two functions depending on $(\hat{\theta}, x, \hat{I}, y)$:

$$\{f, g\} = \frac{\partial f}{\partial \hat{\theta}} \left(\frac{\partial g}{\partial \hat{I}}\right)^\top - \frac{\partial f}{\partial \hat{I}} \left(\frac{\partial g}{\partial \hat{\theta}}\right)^\top + \frac{\partial f}{\partial z} J_m \left(\frac{\partial g}{\partial z}\right)^\top.$$

We use a similar definition when f and g depend on $(\hat{\theta}, X, \hat{I}, Y)$. Note that, with our definition of degree (see (1.3)), if f and g are homogeneous polynomials and $\{f, g\} \neq 0$, one has

$$\deg\left(\{f, g\}\right) = \deg(f) + \deg(g) - 2. \tag{1.14}$$

To introduce more notation, let us define $\mathcal{N} = \{(l, s) \in \mathbb{N}^{2m} \times \mathbb{N}^r \,:\, |l|_1 + 2|s|_1 \geq 3\}$, and let $\mathcal{S}$ be a subset of $\mathcal{N}$. We will say that $h \in \mathcal{M}(\mathcal{S})$ if $h_{l,s} = 0$ when $(l, s) \notin \mathcal{S}$. We will also use the following decomposition: given $h \in \mathcal{M}(\mathcal{N})$, we write $h = \mathcal{S}(h) + (\mathcal{N} \setminus \mathcal{S})(h)$, where $\mathcal{S}(h) \in \mathcal{M}(\mathcal{S})$ and $(\mathcal{N} \setminus \mathcal{S})(h) \in \mathcal{M}(\mathcal{N} \setminus \mathcal{S})$.

Let us split $l^\top = (l_x^\top, l_y^\top)$, with $l_x, l_y \in \mathbb{N}^m$. Now, given $h \in \mathcal{M}(\mathcal{S})$, we say that $h \in \overline{\mathcal{M}}(\mathcal{S})$ when $h_{l,s} = 0$ for all $(l, s) \in \mathcal{S}$ such that $l_x \neq l_y$, and $h_{l,s} = \bar{h}_{l,s}$ if $l_x = l_y$. We say that $h \in \widetilde{\mathcal{M}}(\mathcal{S})$ if $\bar{h}_{l,s} = 0$ for all $(l, s) \in \mathcal{S}$ such that $l_x = l_y$. Note that, for any $h \in \mathcal{M}(\mathcal{S})$, we have $h = \overline{\mathcal{S}}(h) + \tilde{\mathcal{S}}(h)$, with $\overline{\mathcal{S}}(h) \in \overline{\mathcal{M}}(\mathcal{S})$ and $\tilde{\mathcal{S}}(h) \in \widetilde{\mathcal{M}}(\mathcal{S})$. We remark that the functions in $\overline{\mathcal{M}}(\mathcal{S})$ only depend on $\hat{I}$ and on the products $x_j y_j$, $j = 1, \ldots, m$.

1.3.2 Bounding the remainder of the normal form

We introduce $\mathcal{S} \subset \mathcal{N}$ in the following form: we recall that the first m_1 components of λ are eigenvalues associated to $\mathcal{G}$, and then, we put $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, with

$$\mathcal{S}_1 = \{(l,s) \in \mathcal{N} : \ |l_{m_1+1}| + \ldots + |l_m| + |l_{m+m_1+1}| + \ldots + |l_{2m}| \leq 1\}, \quad (1.15)$$
$$\mathcal{S}_2 = \{(l,s) \in \mathcal{N} : \ |l_{m_1+1}| + \ldots + |l_m| + |l_{m+m_1+1}| + \ldots + |l_{2m}| = 2\}. \quad (1.16)$$

This splitting of $\mathcal{S}$ will be used during the proof of Lemma 1.1, to identify in a precise form the contribution to $\mathcal{M}(\mathcal{S})$ from the Poisson brackets involving monomials of $\mathcal{M}(\mathcal{S})$ (see the bounds (1.31) and (1.32)). This is essential to obtain the estimates of Lemma 1.2.

We take the Hamiltonian $H(\hat{\theta}, x, \hat{I}, y)$ of (1.1), we write it in the variables Z (introduced at the end of Section 1.2.1) and we decompose it in the following form:

$$H(\hat{\theta}, X, \hat{I}, Y) = \hat{\omega}^{(0)\top}\hat{I} + \frac{1}{2}Z^\top \mathcal{B}^* Z + N(X, \hat{I}, Y) + S(\hat{\theta}, X, \hat{I}, Y) + T(\hat{\theta}, X, \hat{I}, Y), \quad (1.17)$$

with $N \in \overline{\mathcal{M}}(\mathcal{S})$, $S \in \widetilde{\mathcal{M}}(\mathcal{S})$ and $T \in \mathcal{M}(\mathcal{N} \setminus \mathcal{S})$. We also define $S_1 := \mathcal{S}_1(S)$ and $S_2 := \mathcal{S}_2(S)$. With this formulation, we say that N is in normal form with respect to $\mathcal{S}$, that S contains the terms of H that are not in normal form with respect to $\mathcal{S}$, and that T contains the terms of the Taylor expansion of H not associated to $\mathcal{S}$. We will show that, assuming the Diophantine conditions of (1.4), we can put H in normal form with respect to $\mathcal{S}$, with a canonical transformation defined around the initial r-dimensional torus, $Z = 0$ and $\hat{I} = 0$, leaving a small remainder of non-resonant terms. This remainder will be exponentially small with respect to R on the set $\mathcal{D}_{r,m}(\rho_1, R)$, provided that R be small enough, and for certain $\rho_1 > 0$ independent from R. This is done with a finite iterative scheme, with general step described in the following lemma:

Lemma 1.1 *We consider the Hamiltonian H given in* (1.17). *We assume that it is defined on $\mathcal{D}_{r,m}(\rho, R)$, with $0 < \rho < 1$ and $0 < R < 1$, and that there exists $\mu_0 > 0$ and $\gamma > r + m_1 - 1$ such that*

$$|ik^\top \hat{\omega}^{(0)} + l^\top \lambda| \geq \frac{\mu_0}{(|k|_1 + |l_x - l_y|_1)^\gamma} \quad \forall (l,s) \in \mathcal{S} \ \forall k \in \mathbb{Z}^r \text{ with } |l_x - l_y|_1 + |k|_1 \neq 0,$$

and, given $\delta > 0$, let us introduce $\rho_j = \rho - j\delta$ and $R_j = R\exp(-j\delta)$. Then, we can construct an analytic function $G(\hat{\theta}, X, \hat{I}, Y) \in \widetilde{\mathcal{M}}(\mathcal{S})$, such that for any $0 < \delta \leq \rho/8$ we have the following properties :

1. *G is defined on $\mathcal{D}_{r,m}(\rho_1, R_1)$, and if we decompose $G = G_1 + G_2$, being $G_1 = \tilde{\mathcal{S}}_1(G)$ and $G_2 = \tilde{\mathcal{S}}_2(G)$, the bounds for $|G|_{\rho_1,R_1}$, $|G_1|_{\rho_1,R_1}$ and $|G_2|_{\rho_1,R_1}$ are given in* (1.24).

2. *Let us denote by Ψ_t^G the flow at time t of the Hamiltonian system G. Then, if*

$$\Delta \frac{|S|_{\rho,R}}{\delta^{\gamma+2} R^2} \leq 1, \quad (1.18)$$

where Δ depends only on γ and μ_0, we have

$$\Psi_1^G, \Psi_{-1}^G : \mathcal{D}_{r,m}(\rho_4, R_4) \longrightarrow \mathcal{D}_{r,m}(\rho_3, R_3).$$

3. *If we take* $(\hat{\theta}, X, \hat{I}, Y) \in \mathcal{D}_{r,m}(\rho_4, R_4)$*, and we put* $(\hat{\theta}^*, X^*, \hat{I}^*, Y^*) = \Psi_1^G(\hat{\theta}, X, \hat{I}, Y)$*, then we have* $|\hat{\theta}^* - \hat{\theta}| \le \delta$*,* $|Z^* - Z| \le R\delta \exp(-1/2)/2$*,* $|\hat{I}^* - \hat{I}| \le R^2\delta\exp(-1)$*. The same bounds also hold for* Ψ_{-1}^G*.*

4. Ψ_1^G *transforms*

$$H^{(1)} := H \circ \Psi_1^G = \hat{\omega}^{(0)\top}\hat{I} + \frac{1}{2}Z^\top \mathcal{B}^* Z + N^{(1)} + S^{(1)} + T^{(1)}, \tag{1.19}$$

decomposition analogous to (1.17)*, with the bounds* (1.29)*–*(1.33)*.*

Remark 1.1 *In the Diophantine condition of the statement of the lemma, we remark that if we write* $\lambda^\top = (\underline{\lambda}^\top, -\underline{\lambda}^\top)$*, then, for any* $l \in \mathbb{N}^{2m}$*, we have* $l^\top\lambda = (l_x - l_y)^\top\underline{\lambda}$*. Hence, this condition is equivalent to the one formulated in* (1.4)*, and one can take* μ_0 *as the minimum of* μ *and* $\min\{|\hat{l}^\top\hat{\lambda}|\}$*, where this last expression is taken on the* $\hat{l} \in \mathbb{Z}^{2(m-m_1)}$ *with* $0 < |\hat{l}|_1 \le 2$ *and* $\hat{l}_x - \hat{l}_y \ne 0$*. Moreover, we remark that if we take* $\gamma > r + m_1 - 1$*, the set of vectors* $\hat{\omega}^{(0)}$ *and* λ *for which any Diophantine condition of this kind is not satisfied, has zero measure.*

Remark 1.2 *The canonical transformation generated by* G *has been chosen to remove the term* S *in the decomposition* (1.17)*, formulating the homological equation in terms of the monomials of degree* 2 *of the Hamiltonian. This is, in fact, a classical (and linearly convergent) normal form scheme.*

Remark 1.3 *The bounds on* $H^{(1)}$ *given by Lemma* 1.1 *are not very concrete. This is because we will use this lemma in iterative form, but the estimates used in the first steps will be different from the ones used in a general step of the iterative process. A description of a general step is given in Lemma* 1.2*.*

Proof: We look for a generating function $G \in \widetilde{\mathcal{M}}(\mathcal{S})$, such that

$$S + \{\hat{\omega}^{(0)\top}\hat{I} + \frac{1}{2}Z^\top\mathcal{B}^*Z, G\} = 0.$$

From the definition of the Poisson bracket, we have

$$S + \left(-\frac{\partial G}{\partial\hat{\theta}}\hat{\omega}^{(0)} + Z^\top\mathcal{B}^* J_m\left(\frac{\partial G}{\partial Z}\right)^\top\right) = 0.$$

Expanding G and S, we obtain

$$G_{l,s,k} = \frac{S_{l,s,k}}{ik^\top\hat{\omega}^{(0)} + (l_x - l_y)^\top\underline{\lambda}}, \tag{1.20}$$

for the admissible scripts (l, s, k) in the expansion of S (otherwise $G_{l,s,k}$ is defined as 0). Then, from the definition of G, we have

$$H^* := H \circ \Psi_1^G - (\hat{\omega}^{(0)\top}\hat{I} + \frac{1}{2}Z^\top\mathcal{B}^*Z + N + T + \{N + T, G\}) =$$

$$
\begin{aligned}
&= \int_0^1 \frac{d}{dt}\left(tH + (1-t)(\hat{\omega}^{(0)\top}\hat{I} + \frac{1}{2}Z^\top \mathcal{B}^* Z + N + T + \{N+T,G\})\right)\circ \Psi_t^G dt = \\
&= \int_0^1 \{tH + (1-t)(\hat{\omega}^{(0)\top}\hat{I} + \frac{1}{2}Z^\top \mathcal{B}^* Z + N + T + \{N+T,G\}), G\}\circ \Psi_t^G dt + \\
&\quad + \int_0^1 (H - \hat{\omega}^{(0)\top}\hat{I} - \frac{1}{2}Z^\top \mathcal{B}^* Z - N - T - \{N+T,G\})\circ \Psi_t^G dt = \\
&= \int_0^1 \{tS + (1-t)\{N+T,G\}, G\}\circ \Psi_t^G dt. \qquad (1.21)
\end{aligned}
$$

Hence, we have $N^{(1)} = N + \overline{\mathcal{S}}(\{N+T,G\} + H^*)$, $S_1^{(1)} = \tilde{\mathcal{S}}_1(\{N,G_1\} + H^*)$, $S_2^{(1)} = \tilde{\mathcal{S}}_2(\{N,G\} + \{T,G_1\} + H^*)$ and $T^{(1)} = T + (\mathcal{N}\setminus\mathcal{S})(\{N+T,G\} + H^*)$. To give the expressions of $S_1^{(1)}$ and $S_2^{(1)}$, we remark that $\{T,G_1\}\in\mathcal{M}(\mathcal{N}\setminus\mathcal{S}_1)$, $\{T,G_2\}\in\mathcal{M}(\mathcal{N}\setminus\mathcal{S})$ and $\{N,G_2\}\in\mathcal{M}(\mathcal{N}\setminus\mathcal{S}_1)$. Those facts are consequence of the definition of $\mathcal{S}_1$, $\mathcal{S}_2$, the structure of $N\in\overline{\mathcal{M}}(\mathcal{S})$, and the properties of the Poisson bracket.

We proceed to describe the effect of the transformation Ψ_1^G and to bound the transformed Hamiltonian $H\circ\Psi_1^G$. For this purpose, we take a fixed value of δ, $0<\delta\le\rho/8$. Then, for any $(l,s)\in\mathcal{S}$, $k\in\mathbb{Z}^r$, with $|l_x-l_y|_1+|k|_1\neq 0$, we have from (1.20)

$$
\begin{aligned}
&\left|\frac{S_{l,s,k}Z^l\exp(ik^\top\hat{\theta})}{ik^\top\hat{\omega}^{(0)} + (l_x-l_y)^\top\underline{\lambda}}\right|_{\rho_1,R_1} \le \\
&\quad\le \frac{(|k|_1+|l_x-l_y|_1)^\gamma}{\mu_0}\exp(-\delta|k|_1-\delta|l|_1)|S_{l,s,k}|R^{|l|_1}\exp(|k|_1\rho) \le \\
&\quad\le \sup_{\alpha\ge 1}\{\alpha^\gamma\exp(-\delta\alpha)\}\frac{|S_{l,s,k}|}{\mu_0}R^{|l|_1}\exp(|k|_1\rho). \qquad (1.22)
\end{aligned}
$$

Now, using that for any $\gamma>0$ and $\delta>0$,

$$
\sup_{\alpha\ge 1}\{\alpha^\gamma\exp(-\delta\alpha)\}\le\left(\frac{\gamma}{\delta\exp(1)}\right)^\gamma, \qquad (1.23)
$$

we deduce from (1.22),

$$
|G|_{\rho_1,R_1}\le\left(\frac{\gamma}{\delta\exp(1)}\right)^\gamma\frac{|S|_{\rho,R}}{\mu_0}. \qquad (1.24)
$$

Moreover, the same bounds hold for $G_1=\mathcal{S}_1(G)$ and $G_2=\mathcal{S}_2(G)$, if one adds the subscripts "1" or "2" to G and S in (1.24). Hence, using Lemma 1.5, we have

$$
\left|\frac{\partial G}{\partial\hat{I}}\right|_{\rho_2,R_2}\le\frac{|G|_{\rho_1,R_1}}{R^2\exp(-2\delta)(1-\exp(-2\delta))}\le\frac{|G|_{\rho_1,R_1}}{R^2\delta\exp(-2\delta)}, \qquad (1.25)
$$

$$
\left|\frac{\partial G}{\partial Z}\right|_{\rho_2,R_2}\le\frac{|G|_{\rho_1,R_1}}{R\exp(-\delta)(1-\exp(-\delta))}\le\frac{2|G|_{\rho_1,R_1}}{R\delta\exp(-\delta)}, \qquad (1.26)
$$

$$
\left|\frac{\partial G}{\partial\hat{\theta}}\right|_{\rho_2,R_2}\le\frac{|G|_{\rho_1,R_1}}{\delta\exp(1)},
$$

where in (1.25) and (1.26) we have used that, if $0 < \alpha \le 1$, then

$$\frac{\alpha}{2} \le 1 - \exp(-\alpha). \tag{1.27}$$

Now, to check the bounds

$$\left|\frac{\partial G}{\partial \hat{I}}\right|_{\rho_2,R_2} \le \delta, \qquad \left|\frac{\partial G}{\partial Z}\right|_{\rho_2,R_2} \le \frac{R\delta\exp(-1/2)}{2}, \qquad \left|\frac{\partial G}{\partial \hat{\theta}}\right|_{\rho_2,R_2} \le R^2\delta\exp(-1),$$

we use (1.18) with the following Δ:

$$\Delta(\gamma,\mu_0) = \left(\frac{\gamma}{\exp(1)}\right)^{\gamma}\frac{4\exp(1)}{\mu_0}.$$

If we use the notation $\Psi_t^G - Id = (\hat{\Theta}_t^G, \mathcal{X}_t^G, \hat{\mathcal{I}}_t^G, \mathcal{Y}_t^G)$ (see Lemma 1.9), then we obtain,

$$\begin{array}{rclcrcl} |\hat{\mathcal{I}}_t^G|_{\rho_2,R_2} & \le & R^2\delta\exp(-1), & & |\hat{\Theta}_t^G|_{\rho_2,R_2} & \le & \delta, \\ |\mathcal{Z}_t^G|_{\rho_2,R_2} & \le & R\delta\exp(-1/2)/2, & & & & \end{array} \tag{1.28}$$

for any $-1 \le t \le 1$, being $\mathcal{Z}_t^G = (\mathcal{X}_t^G, \mathcal{Y}_t^G)$. From the bounds of (1.28), and using the inequality (1.27), we can also deduce that the transformations Ψ_1^G and Ψ_{-1}^G act as we describe in the statement. Moreover, (1.28) and Lemma 1.7 allows to bound (1.21) as

$$|H^*|_{\rho_4,R_4} \le |\{S,G\}|_{\rho_2,R_2} + |\{\{T,G\},G\}|_{\rho_3,R_3} + |\{\{N,G\},G\}|_{\rho_3,R_3}. \tag{1.29}$$

Finally, the same arguments hold to bound the terms of $H^{(1)}$ in (1.19) by

$$|N^{(1)} - N|_{\rho_4,R_4} \le |\{N,G\}|_{\rho_2,R_2} + |\{T,G\}|_{\rho_2,R_2} + |H^*|_{\rho_4,R_4}, \tag{1.30}$$
$$|S_1^{(1)}|_{\rho_4,R_4} \le |\{N,G_1\}|_{\rho_2,R_2} + |H^*|_{\rho_4,R_4}, \tag{1.31}$$
$$|S_2^{(1)}|_{\rho_4,R_4} \le |\{N,G\}|_{\rho_2,R_2} + |\tilde{S}_2(\{T,G_1\})|_{\rho_2,R_2} + |H^*|_{\rho_4,R_4}, \tag{1.32}$$
$$|T^{(1)} - T|_{\rho_4,R_4} \le |\{N,G\}|_{\rho_2,R_2} + |\{T,G\}|_{\rho_2,R_2} + |H^*|_{\rho_4,R_4}. \tag{1.33}$$

■

Before giving more concrete estimates on the bounds of Lemma 1.1, we assume that H is in normal form up to certain order p, to be determined later (the reduction of H to this finite normal form will be described in the proof of Theorem 1.1). Then, taking advantage of this fact, the bounds of Lemma 1.1 produce better estimates on the different steps of the normal form process (this is done in Lemma 1.2). This allows to produce a very accurate bound on the final remainder. We want to stress that these bounds are not so good if the initial Hamiltonian is not in normal form up to degree p.

Let us introduce now the following notation: we break the Hamiltonian (1.17) as

$$N = N_4 + N^*, \quad T = T_3 + T^*, \tag{1.34}$$

where N_4 contains the monomials of N of degree 4 and T_3 contains the monomials of degree 3 of T. Then, We assume that, for R small enough, we have the bounds

$$\begin{array}{rclrclrcl} |S_1|_{\rho,R} & \le & \hat{S}R^{p+1}, & |S_2|_{\rho,R} & \le & \hat{S}R^p, & |S|_{\rho,R} & \le & \hat{S}R^p, \\ |N_4|_{\rho,R} & \le & \hat{N}_4R^4, & |N^*|_{\rho,R} & \le & \hat{N}^*R^6, & & & \\ |T_3|_{\rho,R} & \le & \hat{T}_3R^3, & |T^*|_{\rho,R} & \le & \hat{T}^*R^4, & & & \end{array} \tag{1.35}$$

being $\hat{S}$, $\hat{N}_4$, $\hat{N}^*$, $\hat{T}_3$ and $\hat{T}^*$ positive constants. Here, $p \in \mathbb{N}$, $p \geq 6$, is the order of the previous normal form and will be chosen later.

Lemma 1.2 *Let us consider the Hamiltonian H of* (1.17)*, with the same hypotheses as in Lemma* 1.1*. We use the notations* (1.34)*, and we assume* (1.35)*. We also assume that $\hat{S} \leq \hat{S}^*$, $\hat{N}^* \leq \hat{N}^{**}$ and $\hat{T}^* \leq \hat{T}^{**}$, for some $\hat{S}^*$, $\hat{N}^{**}$ and $\hat{T}^{**}$. Let G be the generating function obtained in Lemma* 1.1*, and let δ, $0 < \delta \leq \rho/8$, be such that*

$$\Delta \frac{\hat{S} R^{p-2}}{\delta^{\gamma+2}} \leq 1,$$

(Δ is given by Lemma 1.1*).*

Then, there exists a constant Π, depending only on r, m, γ, μ_0, $\hat{N}_4$, $\hat{N}^{**}$, $\hat{S}^*$, $\hat{T}_3$ and $\hat{T}^{**}$, such that the following bounds hold for the transformed Hamiltonian $H \circ \Psi_1^G$,

$$\begin{aligned}
|N^{(1)} - N|_{\rho_4, R_4} &\leq \Pi \hat{S} \left(\frac{R^{p+1}}{\delta^{\gamma+2}} + \frac{R^{2p-1}}{\delta^{2(\gamma+2)}} \right), \\
|S_1^{(1)}|_{\rho_4, R_4} &\leq \Pi \hat{S} R^{p+1} \left(\frac{R^2}{\delta^{\gamma+1}} + \frac{R^4}{\delta^{\gamma+2}} + \frac{R^{p-3}}{\delta^{\gamma+2}} + \frac{R^{p-2}}{\delta^{2(\gamma+2)}} \right), \\
|S_2^{(1)}|_{\rho_4, R_4} &\leq \Pi \hat{S} R^{p} \left(\frac{R^2}{\delta^{\gamma+1}} + \frac{R^3}{\delta^{\gamma+2}} + \frac{R^{p-1}}{\delta^{2(\gamma+2)}} \right), \\
|T^{(1)} - T|_{\rho_4, R_4} &\leq \Pi \hat{S} \left(\frac{R^{p+1}}{\delta^{\gamma+2}} + \frac{R^{2p-1}}{\delta^{2(\gamma+2)}} \right).
\end{aligned}$$

Remark 1.4 *(A very important one) If p is big enough and $\delta > R$, the dominant term in the bounds of $S_1^{(1)}$ and $S_2^{(1)}$ is given by the factor $R^2/\delta^{\gamma+1}$. This will be the factor of decreasing of those terms during the normal form process and it allows to take δ of order $R^{2/(\gamma+1)}$, that will produce the exponent $2/(\gamma+1)$ in* (1.36)*. As we have $2/(\gamma+1) < 1$, we can deduce that an adequate selection for p is $p = 8$. This allows to keep bounds like* (1.35) *during all the iterative process.*

If we start with a "raw" Hamiltonian (without any previous step of normal form) the decreasing factor obtained is of order $R/\delta^{\gamma+1}$, that forces us to select δ of order $R^{1/(\gamma+1)}$. This produces a worse exponent $1/(\gamma+1)$ in (1.36)*. For instance, let us assume that the normal form has been done around an elliptic equilibrium point. Here the important issue is to note that the bounds obtained when killing degree* 3 *are much worse than the bounds obtained for the other degrees (this has been observed numerically in* [69]*). Hence, to apply the same bounds to all the degrees results in poor estimates.*

Remark 1.5 *The exponent $2/(\gamma+1)$ in Remark* 1.4 *can be improved in some very degenerate cases. For instance, let us consider a totally elliptic torus, and we take $\mathcal{G} = \mathbb{R}^{2m}$. Let q be the lowest degree of the monomials of N corresponding to the (formal) normal form of H around the torus (of course, $q \geq 4$). Then, δ can be taken of order $R^{(q-2)/(\gamma+1)}$, that produces the exponent $(q-2)/(\gamma+1)$ in* (1.36)*.*

Proof: During this proof we will use different constants Π_j, $j \geq 0$, that will depend only on the same parameters as the final constant Π of the statement of the lemma. First, from the bound (1.24) of Lemma 1.1, we have that

$$|G_1|_{\rho_1,R_1} \leq \Pi_0 \frac{\hat{S}R^{p+1}}{\delta^\gamma}, \quad |G_2|_{\rho_1,R_1} \leq \Pi_0 \frac{\hat{S}R^p}{\delta^\gamma}, \quad |G|_{\rho_1,R_1} \leq \Pi_0 \frac{\hat{S}R^p}{\delta^\gamma},$$

where, as in Lemma 1.1, $\rho_j = \rho - j\delta$ and $R_j = R\exp(-j\delta)$. Then, to obtain the bounds for the different terms of the transformed Hamiltonian, we only need to bound the Poisson brackets that appear in (1.29)–(1.33).

To obtain precise estimates, we will look carefully into the critical bounds of the different partial derivatives involved, that is, the ones associated to N_4 and T_3. So, we estimate, separately, the contribution of N_4, N^*, T_3 and T^*, taking into account that N does not depend on $\hat{\theta}$, N_4 is a polynomial of degree 4, and T_3 only contains terms of degree 3. Moreover, to bound $\tilde{\mathcal{S}}_2(\{T, G_1\})$ we note that (from the definition of $\mathcal{S}_1$ and $\mathcal{S}_2$) it only contains terms corresponding to $\frac{\partial}{\partial \hat{Z}}$, and not to $\frac{\partial}{\partial \hat{\theta}}$ or $\frac{\partial}{\partial \hat{I}}$. Thus, using the bounds on the Poisson bracket provided by Lemma 1.6 (see Remark 1.11 for the case in which one of the terms has finite degree), we have

$$\begin{aligned}
|\tilde{\mathcal{S}}_2(\{T, G_1\})|_{\rho_2,R_2} &\leq \Pi_1 \hat{S} R^p \left(\frac{R^2}{\delta^{\gamma+1}} + \frac{R^3}{\delta^{\gamma+2}} \right), \\
|\{T, G\}|_{\rho_2,R_2} &\leq \Pi_2 \hat{S} \frac{R^{p+1}}{\delta^{\gamma+2}}, \\
|\{N, G_1\}|_{\rho_2,R_2} &\leq \Pi_3 \hat{S} R^{p+1} \left(\frac{R^2}{\delta^{\gamma+1}} + \frac{R^4}{\delta^{\gamma+2}} \right), \\
|\{N, G\}|_{\rho_2,R_2} &\leq \Pi_4 \hat{S} R^p \left(\frac{R^2}{\delta^{\gamma+1}} + \frac{R^4}{\delta^{\gamma+2}} \right), \\
|\{S, G\}|_{\rho_2,R_2} &\leq \Pi_5 \hat{S} \frac{R^{2p-2}}{\delta^{\gamma+2}}, \\
|\{\{T, G\}, G\}|_{\rho_3,R_3} &\leq \Pi_6 \hat{S} \frac{R^{2p-1}}{\delta^{2(\gamma+2)}}, \\
|\{\{N, G\}, G\}|_{\rho_3,R_3} &\leq \Pi_7 \hat{S} \left(\frac{R^{2p}}{\delta^{2\gamma+3}} + \frac{R^{2p+2}}{\delta^{2(\gamma+2)}} \right),
\end{aligned}$$

and finally

$$|H^*|_{\rho_4,R_4} \leq \Pi_8 \hat{S} \left(\frac{R^{2p-2}}{\delta^{\gamma+2}} + \frac{R^{2p-1}}{\delta^{2(\gamma+2)}} \right).$$

From that, with a suitable definition of Π as a function of Π_0–Π_8, the bounds of the statement of the lemma are clear, if we recall that we have taken $p \geq 6$. ■

Now, we are in conditions to formulate a quantitative result about "partial reduction to seminormal form" of the initial Hamiltonian. For this purpose, we consider the Hamiltonian H of (1.1), written as in (1.17) in terms of the Z variables. We assume that H is defined on $\mathcal{D}_{r,m}(\rho_0, R_0)$, for some $0 < \rho_0 < 1$ and $0 < R_0 < 1$, with the following bounds: $|N|_{\rho_0,R} \leq \hat{N}R^4$, $|S|_{\rho_0,R} \leq \hat{S}R^3$ and $|T|_{\rho_0,R} \leq \hat{T}R^3$, for any $0 < R \leq R_0$, being $\hat{N}$, $\hat{S}$ and $\hat{T}$, positive constants (independent from R). Then, we prove the following result:

Theorem 1.1 *We consider the Hamiltonian H of (1.17), with the hypotheses previously described. We suppose that there exists $\mu_0 > 0$ and $\gamma > r + m_1 - 1$ such that*

$$|ik^\top \hat{\omega}^{(0)} + l^\top \lambda| \geq \frac{\mu_0}{(|k|_1 + |l_x - l_y|_1)^\gamma} \quad \forall (l,s) \in \mathcal{S}\ \forall k \in \mathbb{Z}^r \text{ with } |l_x - l_y|_1 + |k|_1 \neq 0.$$

Then, for any $R > 0$ small enough (this condition on R depends only on r, m, γ, μ_0, ρ_0, R_0, $\hat{N}$, $\hat{S}$ and $\hat{T}$), there exists an analytical canonical transformation Ψ^R such that

1. *$\Psi^R - Id$ and $(\Psi^R)^{-1} - Id$ are 2π-periodic on $\hat{\theta}$.*

2. $$\Psi^R : \mathcal{D}_{r,m}(3\rho_0/4, R\exp(-\rho_0/4)) \longrightarrow \mathcal{D}_{r,m}(\rho_0, R),$$
   and
   $$(\Psi^R)^{-1} : \mathcal{D}_{r,m}(11\rho_0/16, R\exp(-5\rho_0/16)) \longrightarrow \mathcal{D}_{r,m}(\rho_0, R).$$

3. *If we take $(\hat{\theta}, X, \hat{I}, Y) \in \mathcal{D}_{r,m}(3\rho_0/4, R\exp(-\rho_0/4))$ and we define $(\hat{\theta}^*, X^*, \hat{I}^*, Y^*) = \Psi^R(\hat{\theta}, X, \hat{I}, Y)$, then $|\hat{\theta}^* - \hat{\theta}| \leq \rho_0/16$, $|Z^* - Z| \leq R\rho_0 \exp(-1/2)/32$, $|\hat{I}^* - \hat{I}| \leq R^2\rho_0 \exp(-1)/16$. Moreover, the same bounds hold for $(\Psi^R)^{-1}$ if $(\hat{\theta}, X, \hat{I}, Y) \in \mathcal{D}_{r,m}(11\rho_0/16, R\exp(-5\rho_0/16))$.*

4. *Ψ^R transforms*
   $$H^R := H \circ \Psi^R = \hat{\omega}^{(0)\top}\hat{I} + \frac{1}{2}Z^\top \mathcal{B}^* Z + N^R + S^R + T^R,$$
   decomposition analogous to (1.17), with the bounds: $|N^R - N_4|_{3\rho_0/4, R\exp(-\rho_0/4)} \leq const. R^6$, $|T^R - T_3|_{3\rho_0/4, R\exp(-\rho_0/4)} \leq const. R^4$, where N_4 and T_3 were introduced in (1.34), and can be computed with a normal form with respect to $\mathcal{S}$ up to degree 4, and
   $$|S^R|_{3\rho_0/4, R\exp(-\rho_0/4)} \leq const. \exp\left(-const. \left(\frac{1}{R}\right)^{\frac{2}{\gamma+1}}\right) R^8, \tag{1.36}$$
   being the constats that appear in the bounds of N^R, T^R and S^R, positive and independent from R. Moreover, for any R for which the result holds, H^R is in normal form with respect to $\mathcal{S}$, at least up to degree 8.

Remark 1.6 *The dependence of Ψ^R on R is not continuous but piecewise analytic.*

Remark 1.7 *From the bounds provided by Lemma 1.2 for the iterative normal form procedure described in Lemma 1.1, this exponentially small bound seems to be the best that one can obtains by using this linearly convergent scheme.*

Proof: The proof is done simultaneously for any $0 < R \leq R_0$. The bounds where R is not written explicitly are independent from R. All these bounds and the different conditions on the smallness of R will depend only on the fixed parameters of the statement. The main idea of this proof is to use Lemma 1.1 recursively, and to iterate the bounds provided by Lemma 1.2 for $p = 8$ (see Remark 1.4). Hence, to use this lemma, we need to put the initial Hamiltonian in normal form with respect to $\mathcal{S}$, up to degree at least 8. For this purpose, we construct recursively the generating functions $G^{(0)}$, $G^{(1)}$,..., $G^{(5)}$, provided by Lemma 1.1. Putting $H^{(0)} = H$, we can define

$$H^{(n+1)} := H^{(n)} \circ \Psi_1^{G^{(n)}} = H^{(n)} + \{H^{(n)}, G^{(n)}\} + \frac{1}{2!}\{\{H^{(n)}, G^{(n)}\}, G^{(n)}\} + \cdots, \qquad (1.37)$$

for $n = 0, \dots, 5$. Let us consider first the expression (1.37) as a formal transformation. From the property (1.14) for the Poisson bracket, and from the way in which the different $G^{(j)}$ are selected in Lemma 1.1 (see Remark 1.2), we can ensure that the non-resonant terms associated to $\mathcal{S}$ that remain in $H^{(6)}$ are of degree at least 9. To show that this construction is not only formal, we are going to prove the well defined character of the transformations $\Psi_1^{G^{(n)}}$, $n = 0, \dots, 5$, and to bound $H^{(n)}$, $n = 1, \dots, 6$. For this purpose, we expand $H^{(n)}$ as in (1.17), but adding the superscript "(n)" to N, S and T. We define $\delta_0 = \frac{\rho_0}{192}$, to introduce $\rho^{(0)} = \rho_0$, $R^{(0)} = R$, and $\rho^{(n)} = \rho^{(n-1)} - 4\delta_0$, $R^{(n)} = R^{(n-1)} \exp(-4\delta_0)$, $n = 1, \dots, 6$. Then, we are going to show that taking $\delta \equiv \delta_0$ in Lemma 1.1, we have for $n = 0, \dots, 6$, that, if R is small enough,

$$\begin{aligned} |N^{(n)}|_{\rho^{(n)},R^{(n)}} &\leq \hat{N}^{(n)}(R^{(n)})^4, \quad |S^{(n)}|_{\rho^{(n)},R^{(n)}} \leq \hat{S}^{(n)}(R^{(n)})^{n+3}, \\ |T^{(n)}|_{\rho^{(n)},R^{(n)}} &\leq \hat{T}^{(n)}(R^{(n)})^3. \end{aligned} \qquad (1.38)$$

This is proved by (finite) induction: assuming that (1.38) holds for some n ($0 \leq n \leq 5$) and using that, if R is sufficiently small,

$$\Delta \frac{\hat{S}^{(n)}(R^{(n)})^{n+1}}{\delta_0^{\gamma+2}} \leq 1 \qquad (1.39)$$

(Δ is provided by Lemma 1.1), we have

$$\Psi_1^{G^{(n)}},\ \Psi_{-1}^{G^{(n)}} : \mathcal{D}_{r,m}(\rho^{(n+1)}, R^{(n+1)}) \longrightarrow \mathcal{D}_{r,m}(\rho^{(n)} - 3\delta_0, R^{(n)} \exp(-3\delta_0)). \qquad (1.40)$$

Then, the fact that the successive steps increase at least by one the degree of the normal form with respect to $\mathcal{S}$, makes evident the estimates of (1.38) for $n+1$. For more details one can rewrite, with minor changes, the proof of Lemma 1.2, using (1.38) instead of (1.35). Here, the different R-independent constants $\hat{N}^{(n)}$, $\hat{T}^{(n)}$ and $\hat{S}^{(n)}$, defined recursively for $n = 0, \dots, 6$, depend only on the same parameters involved in the formulation of the Theorem. We remark that condition (1.39) for $n = 0, \dots, 5$, imposes only a finite number of restrictions on R. Let R_0^* the biggest value of R for which they hold.

The next step is to continue with the iterative normal form process, but using Lemma 1.2 (with $p = 8$) to bound $H^{(n)}$, $6 \leq n \leq L+1$ (L will be determined below). This will be done in an inductive way, showing that bounds like (1.35) hold for each $H^{(n)}$, $n \geq 6$. Hence, we add in (1.35) the superscript "(n)" to S, S_1, S_2, N^* and T^*, and we replace $\hat{S}$, $\hat{N}^*$ and $\hat{T}^*$ by $\hat{S}^{(n)}$, $\hat{N}^{(n)*}$ and $\hat{T}^{(n)*}$. All these bounds have been taken on $\mathcal{D}_{r,m}(\rho^{(n)}, R^{(n)})$, for some $\rho^{(n)}$, $R^{(n)}$ that will be determined below. Initially, for $n = 6$, we can take for instance $\hat{N}_4^{(6)} = \hat{N}^{(6)}$ and $\hat{N}^{(6)*} = \hat{N}^{(6)}/(R_0^*)^2$. The definition of the other super-(6) constants can be done similarly. Before continuing the iterative procedure, we remark that, as the following steps only afect high order terms, N_4 and T_3 remain invariant during all the normal form procedure. Then, Remark 1.4 suggests the definition $\delta \equiv \delta(R) = (AR)^{2/(\gamma+1)}$, where $A \geq 1$ will be determined later (independently from R). From this value of δ we define, recursively, $\rho^{(n+1)} = \rho^{(n)} - 4\delta$, $R^{(n+1)} = R^{(n)} \exp(-4\delta)$, for $n \geq 6$. To preserve the positiveness of $\rho^{(n)}$, we restrict $n \leq L(R)$, being $L(R)$ the greatest integer for which we have $4(L-5)\delta \leq \rho_0/8$. This implies the following restriction on L:

$$L \leq 5 + \frac{\rho_0}{32}\left(\frac{1}{AR}\right)^{\frac{2}{\gamma+1}}. \tag{1.41}$$

Hence, we take as L the integer part of (1.41). This implies $R\exp(-\rho_0/4) \leq R^{(n)} \leq R$ if $6 \leq n \leq L+1$. To apply Lemma 1.2, we assume that for the current Hamiltonian $H^{(n)}$, $6 \leq n \leq L$, we have $\hat{S}^{(n)} \leq \hat{S}^{(6)}$, $\hat{N}^{(n)*} \leq \hat{N}^{**}$ and $\hat{T}^{(n)*} \leq \hat{T}^{**}$, for some $\hat{N}^{**}$ and $\hat{T}^{**}$ to be precised later (those bounds are necessary to define Π in Lemma 1.2, independent from n). If for the current value of n we have

$$\Delta \frac{\hat{S}^{(n)}(R^{(n)})^6}{\delta^{\gamma+2}} \leq 1, \tag{1.42}$$

then, the canonical transformation $\Psi_1^{G^{(n)}}$ given by Lemma 1.1 acts like (1.40), replacing δ_0 by δ. Therefore, using Lemma 1.2, and recalling that $\frac{2}{\gamma+1} < 1$, $A \geq 1$ and $R^{(n)} < R < 1$, one obtains the following bounds for the transformed Hamiltonian:

$$|N^{(n+1)} - N^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leq \Pi\hat{S}^{(n)}\left(\frac{(R^{(n)})^9}{\delta A^2R^2} + \frac{(R^{(n)})^{15}}{\delta^2A^4R^4}\right) \leq \frac{2\Pi\hat{S}^{(n)}}{A^2}(R^{(n)})^6,$$

$$\begin{aligned} |S_1^{(n+1)}|_{\rho^{(n+1)},R^{(n+1)}} &\leq \Pi\hat{S}^{(n)}(R^{(n)})^9\left(\frac{(R^{(n)})^2}{A^2R^2} + \frac{(R^{(n)})^4}{\delta A^2R^2} + \frac{(R^{(n)})^5}{\delta A^2R^2} + \frac{(R^{(n)})^6}{\delta^2A^4R^4}\right) \leq \\ &\leq \frac{4\Pi\hat{S}^{(n)}}{A^2}(R^{(n)})^9, \end{aligned}$$

$$|S_2^{(n+1)}|_{\rho^{(n+1)},R^{(n+1)}} \leq \Pi\hat{S}^{(n)}(R^{(n)})^8\left(\frac{(R^{(n)})^2}{A^2R^2} + \frac{(R^{(n)})^3}{\delta A^2R^2} + \frac{(R^{(n)})^7}{\delta^2A^4R^4}\right) \leq \frac{3\Pi\hat{S}^{(n)}}{A^2}(R^{(n)})^8,$$

$$|T^{(n+1)} - T^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leq \Pi\hat{S}^{(n)}\left(\frac{(R^{(n)})^9}{\delta A^2R^2} + \frac{(R^{(n)})^{15}}{\delta^2A^4R^4}\right) \leq \frac{2\Pi\hat{S}^{(n)}}{A^2}(R^{(n)})^4.$$

We take $A = \max\left\{1, \sqrt{8\Pi\exp(1)}\right\}$, and then, recalling that $R^{(n+1)} = R^{(n)}\exp(-4\delta)$, we can define inductively ($n \geq 6$),

$$\begin{aligned} \hat{S}^{(n+1)} &= \frac{(\exp(4\delta))^9}{\exp(1)}\hat{S}^{(n)}, \\ \hat{N}^{(n+1)*} &= (\exp(4\delta))^9\left(\hat{N}^{(n)*} + \frac{1}{\exp(1)}\hat{S}^{(n)}\right), \\ \hat{T}^{(n+1)*} &= (\exp(4\delta))^9\left(\hat{T}^{(n)*} + \frac{1}{\exp(1)}\hat{S}^{(n)}\right). \end{aligned}$$

Assuming R small enough such that $\delta \leq 1/72$, we obtain

$$\begin{aligned} \hat{S}^{(n)} &= \hat{S}^{(6)} \exp\left((6-n)(1-36\delta)\right) \leq \hat{S}^{(6)} \exp\left((6-n)/2\right), \\ \hat{N}^{(n)*} &\leq \exp\left(36\delta(n-6)\right) \left(\hat{N}^{(6)*} + \hat{S}^{(6)} \frac{1}{(\exp(1)-1)}\right), \\ \hat{T}^{(n)*} &\leq \exp\left(36\delta(n-6)\right) \left(\hat{T}^{(6)*} + \hat{S}^{(6)} \frac{1}{(\exp(1)-1)}\right). \end{aligned} \tag{1.43}$$

As we are only intrerested in those bounds for $n \leq L+1$, from the restriction on L in (1.41) we can easily introduce n-independent bounds $\hat{N}^{**}$ and $\hat{T}^{**}$ for $\hat{N}^{(n)*}$ and $\hat{T}^{(n)*}$. Now, assuming that all the steps are well defined, if one puts $n \equiv L(R)+1$ in (1.43), we obtain the exponentially small bound of the statement for $\hat{S}^{(L+1)}$. To justify that we can reach this value, we note that (1.42) holds for all the previous n, if we restrict R with $\Delta \hat{S}^{(6)} R^3 \leq 1$.

Then, to prove the Theorem, we only have to introduce $\Psi^R = \Psi_1^{G^{(0)}} \circ \ldots \circ \Psi_1^{G^{(L)}}$, and hence, $(\Psi^R)^{-1} = \Psi_{-1}^{G^{(L)}} \circ \ldots \circ \Psi_{-1}^{G^{(0)}}$. If those transformations act as it has been said in the statement, the proof is finished. First, and from the domains of definition of the different canonical transformations $\Psi_1^{G^{(n)}}$ (see (1.40), replacing δ_0 by δ if $n \geq 6$), we deduce that Ψ^R is defined on the domain given in the statement. Moreover, from the bounds for the different components of $\Psi_1^{G^{(n)}} - Id$ given by Lemma 1.1, and remarking that $6\delta_0 + (L-5)\delta \leq \rho_0/16$, the final bounds for $\Psi^R - Id$ follow immediatly. We consider now $(\Psi^R)^{-1}$. In this case, and using the same arguments on $\Psi_{-1}^{G^{(n)}}$, one can check that if we define $\rho_n = 11\rho_0/16 + n\delta_0$, $R_n = R\exp\left(-5\rho_0/16 + n\delta_0\right)$ for $n = 0, \ldots, 6$, and $\rho_n = 11\rho_0/16 + 6\delta_0 + (n-6)\delta$, $R_n = R\exp\left(-5\rho_0/16 + 6\delta_0 + (n-6)\delta\right)$ for $n = 6, \ldots, L+1$, then we have

$$\Psi_{-1}^{G^{(n)}} : \mathcal{D}_{r,m}(\rho_n, R_n) \longrightarrow \mathcal{D}_{r,m}(\rho_{n+1}, R_{n+1}),$$

for $0 \leq n \leq L$. The proof of this fact can be done by combining the bounds on $\Psi_{-1}^{G^{(n)}} - Id$ with the inequality (1.27). Moreover, this allows to estimate $(\Psi^R)^{-1} - Id$ as it has been done with the case of Ψ^R. ■

1.3.3 Effective stability

An immediate consequence of Theorem 1.1 is that we can bound the diffusion speed around a linearly stable torus of a Hamiltonian system. In this case, we take $\mathcal{G} = \mathbb{R}^{2m}$, and hence, $\mathcal{S} = \mathcal{N}$. Then, we apply Theorem 1.1, without taking into account the term T of the decomposition (1.17). In fact, in this case one can rewrite the proofs of Lemmas 1.1, 1.2, and Theorem 1.1, in a simpler form (although the actual formulation also holds in this particular case), to obtain exponentially small bounds for the remainder S^R.

Theorem 1.2 *We consider the real analytic Hamiltonian* (1.1) *defined on* $\mathcal{D}_{r,m}(\rho_0, R_0)$ *for some* $0 < \rho_0 < 1$ *and* $R_0 > 0$*. We also assume that all the eigenvalues of* $J_m\mathcal{B}$ *are of elliptic type, and that there exists* $\mu_0 > 0$ *and* $\gamma > \ell - 1$ *such that*

$$|ik^\top \hat{\omega}^{(0)} + l^\top \lambda| \geq \frac{\mu_0}{(|k|_1 + |l_x - l_y|_1)^\gamma} \quad \forall l \in \mathbb{N}^{2m}\ \forall k \in \mathbb{Z}^r \text{ with } |l_x - l_y|_1 + |k|_1 \neq 0.$$

Let $R \in (0, R_0)$, and let us take real initial conditions at $t = 0$ contained in $\mathcal{D}_{r,m}(0, R)$. Then, we can define $\alpha > 2$ such that, if R is small enough, the corresponding trajectories belong to $\mathcal{D}_{r,m}(0, \alpha R)$ for any time $0 \le t \le T(R)$, with

$$T(R) = const. \exp\left(const. \left(\frac{1}{R}\right)^{\frac{2}{\gamma+1}} \right),$$

being the constants in the definition of $T(R)$ independent from R.

Remark 1.8 *In the proof, and only for technical reasons, α depends on ρ_0. Nevertheless, we can take α as close as we want to 2 (by taking an initial ρ_0 small enough, see the proof for details), but this implies a reduction on the set of allowed R, and on the constants of the stability time.*

The reason that forces to take $\alpha > 2$ is the norm used for the normal variables. If one takes the Euclidean norm instead of the supremum norm, the condition $\alpha > 2$ is replaced by $\alpha > 1$.

Proof: In order to simplify the proof, we assume that the initial real variables (x, y) of (1.1) correspond to the ones that put $\mathcal{B}$ in canonical real form, that is,

$$z^\top \mathcal{B} z = \sum_{j=1}^{m} \alpha_j (x_j^2 + y_j^2),$$

with $\lambda_j = i\alpha_j$, $j = 1, \ldots, m$. Moreover, we assume that $R_0 < 1$. We introduce (X, Y) to denote the complexified variables

$$x_j = \frac{X_j + iY_j}{\sqrt{2}}, \quad y_j = \frac{iX_j + Y_j}{\sqrt{2}}, \quad j = 1, \ldots, m, \tag{1.44}$$

that put the matrix $J_m\mathcal{B}$ in the diagonal form $J_m\mathcal{B}^*$. Then, we can write the Hamiltonian in these variables as

$$\mathcal{H} = \hat{\omega}^{(0)\top} \hat{I} + \frac{1}{2} Z^\top \mathcal{B}^* Z + N(X, \hat{I}, Y) + S(\hat{\theta}, X, \hat{I}, Y),$$

where N can be rewritten as a function of I, $I^\top = (\hat{I}^\top, \tilde{I}^\top)$, with $\tilde{I}_j = iX_jY_j = \frac{1}{2}(x_j^2 + y_j^2)$, and S verifies $\overline{\mathcal{N}}(S) = 0$. This corresponds to the decomposition (1.17) if one puts $\mathcal{S} = \mathcal{N}$. $\mathcal{H}$ is defined on $\mathcal{D}_{r,m}(\rho_0, R_0/\sqrt{2})$, with bounds of the following form: $|N|_{\rho_0,R} \le \hat{N} R^4$ and $|S|_{\rho_0,R} \le \hat{S} R^3$, for any $0 < R \le R_0/\sqrt{2}$. Now, we apply Theorem 1.1 and we obtain, for any R small enough, a canonical change Ψ^R such that in the new coordinate system $(\hat{\theta}, X, \hat{I}, Y) = \Psi^R(\hat{\theta}^R, X^R, \hat{I}^R, Y^R)$, we have

$$\mathcal{H}^R := \mathcal{H} \circ \Psi^R = \hat{\omega}^{(0)\top} \hat{I}^R + \frac{1}{2} (Z^R)^\top \mathcal{B}^* Z^R + N^R(X^R, \hat{I}^R, Y^R) + S^R(\hat{\theta}^R, X^R, \hat{I}^R, Y^R),$$

being N^R a function of $(I^R)^\top = ((\hat{I}^R)^\top, (\tilde{I}^R)^\top)$, with $\tilde{I}_j^R = iX_j^R Y_j^R$. $\mathcal{H}^R$ is defined on $\mathcal{D}_{r,m}(3\rho_0/4, R\exp(-\rho_0/4))$, with

$$|S^R|_{3\rho_0/4, R\exp(-\rho_0/4)} \le const. \exp\left(-const. \left(\frac{1}{R}\right)^{\frac{2}{\gamma+1}} \right) R^8 := M(R).$$

The canonical equations for $(X^R, \hat{I}^R, Y^R)$ are

$$\begin{aligned} \dot{X}_j^R &= \frac{\partial \mathcal{H}^R}{\partial Y_j^R}, & \dot{Y}_j^R &= -\frac{\partial \mathcal{H}^R}{\partial X_j^R}, \quad j = 1, \ldots, m, \\ \dot{I}_j^R &= \frac{\partial \mathcal{H}^R}{\partial \hat{\theta}_j^R} = \frac{\partial S^R}{\partial \hat{\theta}_j^R}, & & \quad j = 1, \ldots, r. \end{aligned} \tag{1.45}$$

From this, one obtains (using that N^R is in fact only a function of I, and recalling $\ell = r + m$),

$$\dot{I}_j^R = i\frac{\partial \mathcal{H}^R}{\partial Y_j^R} Y_j^R - i\frac{\partial \mathcal{H}^R}{\partial X_j^R} X_j^R = i\frac{\partial S^R}{\partial Y_j^R} Y_j^R - i\frac{\partial S^R}{\partial X_j^R} X_j^R, \quad j = r+1, \ldots, \ell.$$

We put $\mathcal{I}_j^R(\hat{\theta}^R, X^R, \hat{I}^R, Y^R)$ for the expressions on the right-hand side of $\dot{I}_j^R$, $j = 1, \ldots, \ell$. We use Lemma 1.5 to bound these expressions. Then, for $j = 1, \ldots, r$ one has

$$|\mathcal{I}_j^R|_{0, R\exp(-\rho_0/2)} \leq \frac{4M(R)}{3\rho_0 \exp(1)}. \tag{1.46}$$

If we combine Lemma 1.5 with the inequality (1.27), one obtains for $j = r+1, \ldots, \ell$ that,

$$|\mathcal{I}_j^R|_{0, R\exp(-\rho_0/2)} \leq \frac{2M(R)\exp(-\rho_0/2)}{\exp(-\rho_0/4)(1 - \exp(-\rho_0/4))} \leq \frac{16M(R)\exp(-\rho_0/4)}{\rho_0}. \tag{1.47}$$

To continue the proof, we put (x^R, y^R) for the variables that come from the "realification" of (X^R, Y^R), that is, $X_j^R = (x_j^R - iy_j^R)/\sqrt{2}$, $Y_j^R = (y_j^R - ix_j^R)/\sqrt{2}$. In fact, as Ψ^R preserves the symmetries of $\mathcal{H}$ (due to the complexification of a real Hamiltonian, see Section 1.2.1), we have that the Hamiltonian in the variables (x^R, y^R) is real analytic. To work with those different representations of the variables, we give the following remarks: (*i*) the set of real variables (x_j, y_j) such that $|x_j|, |y_j| \leq A$ is contained in the set of complex (X_j, Y_j) such that $|X_j|, |Y_j| \leq A$, (*ii*) the set of complex (X_j, Y_j) such that $|X_j|, |Y_j| \leq A$, is contained in the complex set for (x_j, y_j) such that $|x_j|, |y_j| \leq \sqrt{2}A$ (this property has been used to say that $\mathcal{H}$ is defined in $\mathcal{D}_{r,m}(\rho_0, R_0/\sqrt{2})$), (*iii*) the set of real (x_j, y_j) such that $I_j \leq A^2$, is contained in the set of real (x_j, y_j) such that $|x_j|, |y_j| \leq \sqrt{2}A$. Those remarks are used when working with these different kind of variables, and one wants to control the size of the corresponding domains, when we change the variable representation.

Now, we take real values for $(\hat{\theta}^R, x^R, \hat{I}^R, y^R)$ as initial conditions at $t = 0$. To prove the lower bound for the stability time, we consider a fixed $0 < \beta < 1$, and we restrict to initial conditions such that, when expressed in terms of $(\hat{\theta}^R, X^R, \hat{I}^R, Y^R)$, they belong to $\mathcal{D}_{r,m}(0, R\beta\exp(-\rho_0/2)/\sqrt{2})$. Then, we have that the corresponding initial actions I^R are bounded by $|I_j^R(0)| \leq R^2\beta^2\exp(-\rho_0)/2$, $j = 1, \ldots, \ell$. Using this, we deduce from the bounds (1.46) and (1.47) that, for the trajectories of the Hamiltonian equations (1.45), we have $|I_j^R(t)| \leq R^2\exp(-\rho_0)/2$ for $0 \leq t \leq T(R)$, where we can take

$$T(R) = \frac{R^2\rho_0\exp(-3\rho_0/4)(1 - \beta^2)}{32M(R)}.$$

This bound comes from (1.47), that is the worst case. This is the expression for the stability time of the statement of the Theorem. To use the bounds (1.46) and (1.47) for $\mathcal{I}_j^R$, we need that these trajectories expressed in terms of $(\hat{\theta}^R, X^R, \hat{I}^R, Y^R)$ belong to $\mathcal{D}_{r,m}(0, R\exp(-\rho_0/2))$ up to time $T(R)$. As we have $|I_j^R(t)| \leq R^2\exp(-\rho_0)/2$, this follows from remarks (iii) and (i). From that we deduce, using the bounds for $(\Psi^R)^{-1} - Id$ provided by Theorem 1.1 and remark (ii), that the corresponding real trajectories in terms of $(\hat{\theta}, x, \hat{I}, y)$ are contained in $\mathcal{D}_{r,m}(0, R_1)$, being R_1 defined by

$$R_1 = \max\left\{\sqrt{2}\left(R\exp\left(-\frac{\rho_0}{2}\right) + \frac{R\rho_0\exp(-1/2)}{32}\right), \sqrt{R^2\exp(-\rho_0) + \frac{R^2\rho_0\exp(-1)}{16}}\right\}.$$

Then, if we give for $(\hat{\theta}, x, \hat{I}, y)$ a real set of points such that, expressed in terms of $(\hat{\theta}, X, \hat{I}, Y)$, they belong to the domain $(\Psi^R)^{-1}(\mathcal{D}_{r,m}(0, R\beta\exp(-\rho_0/2)/\sqrt{2}))$, then, the trajectories of H with initial conditions in this set remain in $\mathcal{D}_{r,m}(0, R_1)$ for a time span $T(R)$. With similar arguments as the ones used to define R_1 (using now remark (ii)), one can check that this domain can be taken as $\mathcal{D}_{r,m}(0, R_2)$, being R_2 defined by

$$R_2 = \min\left\{\frac{R\beta\exp(-\rho_0/2)}{\sqrt{2}} - \frac{R\rho_0\exp(-1/2)}{32}, \sqrt{\frac{R^2\beta^2\exp(-\rho_0)}{2} - \frac{R^2\rho_0\exp(-1)}{16}}\right\}.$$

If one considers the flow Ψ_t^H defined from $\mathcal{D}_{r,m}(0, R_2) \cap \mathbb{R}^{2\ell}$ to $\mathcal{D}_{r,m}(0, R_1) \cap \mathbb{R}^{2\ell}$, for $0 \leq t \leq T(R)$, then, putting $R \equiv R_2$ in the statement, and taking an R-independent value of β close enough to 1 such that $R_2 > 0$, we can define $\alpha = R_1/R_2$. ■

1.4 Estimates on the families of lower dimensional tori

Let us consider the real analytic reduced Hamiltonian H of (1.1) and a fixed subbundle $\mathcal{G}$ of elliptic directions of $J_m\mathcal{B}$. In Theorem 1.1 we have proved that, under standard Diophantine conditions, one can put H in normal form with respect to the set $\mathcal{S}$ (see (1.15) and (1.16) for the definition), with an exponentially small remainder. If we write this seminormal form in terms of the complexified variables Z, and without changing the name of the Hamiltonian, one has

$$H = \omega^{(0)\top} I + \frac{1}{2}\hat{Z}^\top \hat{\mathcal{B}}^* \hat{Z} + \mathcal{F}(I) + \frac{1}{2}\hat{Z}^\top \mathcal{Q}(I)\hat{Z} + \mathcal{T}(\hat{\theta}, X, \hat{I}, Y) + \mathcal{R}(\hat{\theta}, X, \hat{I}, Y). \quad (1.48)$$

To explain the notation used, let us recall that the different resonant terms depend only on $\hat{I}$ and on the products X_jY_j, $j = 1, \ldots, m$, but, from the structure of $\mathcal{S}$, not all the possible combinations of those monomials take place in $\overline{\mathcal{M}}(\mathcal{S})$. Then, we introduce $I^\top = (\hat{I}^\top, \tilde{I}^\top)$, with $\tilde{I}_j = iX_jY_j$, $j = 1, \ldots, m_1$, and with this definition (1.48) can be described as follows: the symmetric matrix $\hat{\mathcal{B}}^*$ is defined from $\mathcal{B}^*$ skipping the $2m_1$ eigenvalues associated to $\mathcal{G}$, $J_{m-m_1}\hat{\mathcal{B}}^* = \operatorname{diag}(\hat{\lambda})$. $\mathcal{F}$ and $\mathcal{Q}$ correspond to the normal form with respect to $\mathcal{S}$, with the expansion of $\mathcal{F}$ starting at second order with respect to I, and with $\mathcal{Q}(0) = 0$. It is not difficult to check that by choosing the variables $\tilde{Z}$ in suitable form (as it has been done in the proof of Theorem 1.2), $\mathcal{F}$ is real analytic. Moreover, $\mathcal{Q}$ is a symmetric matrix such that $J_{m-m_1}\mathcal{Q}$ is diagonal, $\mathcal{T} \in \mathcal{M}(\mathcal{N} \setminus \mathcal{S})$ (so $\mathcal{T} \equiv O_3(\hat{Z})$) and $\mathcal{R} \in \widetilde{\mathcal{M}}(\mathcal{S})$.

We assume that this normal form has been done for a given (and small enough) R, as in the formulation of Theorem 1.1. We only consider the R-dependence when we give the bounds of the different terms of (1.48). To obtain these bounds, let us define $\rho_1 = 3\rho_0/4$, where we recall that ρ_0 is the width of the strip of analiticity, with respect to $\hat{\theta}$, for the initial Hamiltonian. Then, Theorem 1.1 implies that, for any R small enough, we have

$$\begin{array}{rclrcl} |\mathcal{F}|_{0,R} & \leq & \hat{\mathcal{F}}R^4, & |\mathcal{F}_3|_{0,R} & \leq & \hat{\mathcal{F}}_3 R^6, \\ |\mathcal{Q}|_{0,R} & \leq & \hat{\mathcal{Q}}R^2, & |\mathcal{Q}_2|_{0,R} & \leq & \hat{\mathcal{Q}}_2 R^4, \\ |\mathcal{T}|_{\rho_1,R} & \leq & \hat{\mathcal{T}}R^3, & |\mathcal{R}|_{\rho_1,R} & \leq & const.\exp\left(-const.\left(\frac{1}{R}\right)^{\frac{2}{\gamma+1}}\right)R^8, \end{array} \tag{1.49}$$

To derive these bounds on $\mathcal{D}_{r,m}(0,R)$, we have considered the functions that depend on $\tilde{I}$ as functions of $\tilde{Z}$. Here, we have split $\mathcal{F} = \mathcal{F}_2 + \mathcal{F}_3$ and $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$. $\mathcal{F}_2$ and the components of $\mathcal{Q}_1$ are polynomials on I of degrees 2 and 1 respectively. $\mathcal{F}_3$ and $\mathcal{Q}_2$ contain the remaining terms. We note that the definition of $\mathcal{F}_2$ and $\mathcal{Q}_1$ does not depend on the order of the seminormal form.

This seminormal form has been formally explained in Section 1.2.1, and we will use the notation related to (1.7) to represent the normal form tori.

The main purpose of this section is to study the persistence of those tori when we add the remainder $\mathcal{R}$. We note that, as $|\mathcal{R}|$ is exponentially small with R, we can expect that the tori of (1.7) will survive, except the ones corresponding to a set of parameters ($I(0)$) of exponentially small measure with respect to R. We will show that this assertion holds, assuming certain standard nondegeneracy conditions on this family of tori, that have been explained in Section 1.2.2 (conditions that, as we will see, can be checked by computing a normal form up to degree 4, that is, from $\mathcal{F}_2$ and $\mathcal{Q}_1$). As it is a more natural parameter, the results will be formulated in terms of frequencies instead of actions.

1.4.1 Nondegeneracy conditions

Before the rigorous formulation of the results, let us give in explicit form these nondegeneracy conditions.

Nondegeneracy of the intrinsic frequencies

The first one is a standard nondegeneracy condition on the dependence of the frequencies with respect to the actions: we require

$$\det \mathcal{C} \neq 0, \quad \mathcal{C} = \frac{\partial^2 \mathcal{F}_2}{\partial I^2}(0). \tag{1.50}$$

This allows to parametrize the tori of the family by their vector of intrinsic frequencies (instead of $I(0)$). Of course, we have to be close enough to the initial r-dimensional torus. This assertion is justified by the following lemma:

Lemma 1.3 *Let us assume that* $\det \mathcal{C} \neq 0$. *Then, if* R *is small enough, there exists a real analytic vectorial function* $\mathcal{I}(\omega)$, *defined on the set*

$$\left\{ \omega \in \mathbb{C}^{r+m_1} \, : \, |\omega - \omega^{(0)}| \leq \frac{1}{8}(|\mathcal{C}^{-1}|)^{-1} R^2 \right\}, \tag{1.51}$$

such that

$$\frac{\partial \mathcal{F}}{\partial I}(\mathcal{I}(\omega)) = \omega - \omega^{(0)},$$

with $\mathcal{I}(\omega^{(0)}) = 0$. *Moreover, we have* $|\mathcal{I}(\omega)| \leq \frac{1}{4}R^2$ *for any* ω *in the set* (1.51), *and if* $\omega^{(1)}$, $\omega^{(2)}$ *belong in* (1.51), *then*

$$|\mathcal{I}(\omega^{(1)}) - \mathcal{I}(\omega^{(2)})| \leq 2|\mathcal{C}^{-1}||\omega^{(1)} - \omega^{(2)}|.$$

Of course, we are still using the notation of Section 1.4.

Proof: We have $\mathcal{F}(I) = \frac{1}{2} I^\top \mathcal{C} I + \mathcal{F}_3$. Then, we take a fixed ω in the set (1.51), and we want to solve the equation:

$$\mathcal{I}(\omega) = \mathcal{C}^{-1} \left(\omega - \omega^{(0)} - \frac{\partial \mathcal{F}_3}{\partial I}(\mathcal{I}(\omega)) \right). \tag{1.52}$$

Putting the superscripts "$(k+1)$" and "(k)" to $\mathcal{I}(\omega)$ in (1.52), we can consider this expression as an iterative procedure, using $\mathcal{I}^{(0)}(\omega) = 0$ as the seed. If we assume $|\mathcal{I}^{(k)}(\omega)| \leq \frac{1}{4}R^2$, then, using Cauchy inequalities, we have for R small enough,

$$|\mathcal{I}^{(k+1)}(\omega)| \leq |\mathcal{C}^{-1}| \left(\frac{1}{8}(|\mathcal{C}^{-1}|)^{-1} R^2 + \frac{\hat{\mathcal{F}}_3 R^6}{\frac{3}{4}R^2} \right) \leq \frac{1}{4}R^2,$$

where we have used the bounds of (1.49) for $\mathcal{F}_3$, remarking that $|\mathcal{F}_3|_{0,R}$ is a bound for the supremum norm of $\mathcal{F}_3(I)$ if $|I| \leq R^2$. Moreover, to ensure convergence, we remark that using the main value theorem one has,

$$|\mathcal{I}^{(k+1)}(\omega) - \mathcal{I}^{(k)}(\omega)| \leq (r+m_1)\frac{\hat{\mathcal{F}}_3 R^6}{\left(\frac{3}{8}\right)^2 R^4} |\mathcal{I}^{(k)}(\omega) - \mathcal{I}^{(k-1)}(\omega)| \leq \frac{1}{2}|\mathcal{I}^{(k)}(\omega) - \mathcal{I}^{(k-1)}(\omega)|,$$

if R is small enough. Clearly, the limit function is analytic with respect to ω, and from the real analytic character of $\mathcal{F}$, $\mathcal{I}$ is in fact real analytic. Taking $\omega^{(1)}$, $\omega^{(2)}$ in the set (1.51), one has

$$\mathcal{I}(\omega^{(1)}) - \mathcal{I}(\omega^{(2)}) = \mathcal{C}^{-1}(\omega^{(1)} - \omega^{(2)}) + \mathcal{C}^{-1} \left(\frac{\partial \mathcal{F}_3}{\partial I}(\mathcal{I}(\omega^{(2)})) - \frac{\partial \mathcal{F}_3}{\partial I}(\mathcal{I}(\omega^{(1)})) \right),$$

and with the same arguments previously used, we obtain for R small enough

$$|\mathcal{I}(\omega^{(1)}) - \mathcal{I}(\omega^{(2)})| \leq 2|\mathcal{C}^{-1}||\omega^{(1)} - \omega^{(2)}|.$$

■

Nondegeneracy of the normal frequencies

The other nondegeneracy condition considered refers to the normal eigenvalues. Skipping again the remainder $\mathcal{R}$, for every invariant tori parametrized by $I(0)$ in (1.7), the corresponding normal eigenvalues are the ones of the diagonal matrix $J_{m-m_1}(\hat{\mathcal{B}}^* + \mathcal{Q}(I(0)))$. Using the parametrization $I \equiv \mathcal{I}(\omega)$ provided by Lemma 1.3, we can consider those eigenvalues as functions of ω instead of functions of $I(0)$.

$$J_{m-m_1}(\hat{\mathcal{B}}^* + \mathcal{Q}(\mathcal{I}(\omega))) \equiv \mathrm{diag}(\hat{\lambda}^{(0)}(\omega)), \tag{1.53}$$

we ask for the condition

$$\mathrm{Im}\left(\frac{\partial}{\partial\omega}(l^\top\hat{\lambda}^{(0)})(\omega^{(0)})\right) \notin \mathbb{Z}^{r+m_1}, \;\; l \in \mathbb{Z}^{2(m-m1)}, \; 0 < |l|_1 \leq 2, \;\; l_{\hat{X}} \neq l_{\hat{Y}}, \tag{1.54}$$

where we have used the notation $l^\top = (l_{\hat{X}}^\top, l_{\hat{Y}}^\top)$. To check this condition, we only need to know the first order approximation to $\mathcal{Q}$, remarking that from (1.53) one has

$$\hat{\lambda}_j^{(0)}(\omega) = \hat{\lambda}_j + \left(\frac{\partial}{\partial I}\mathcal{Q}_{1,j,j+m-m_1}(0)\right)\mathcal{C}^{-1}(\omega-\omega^{(0)}) + O_2(\omega-\omega^{(0)}), \tag{1.55}$$

$j = 1, \dots, m - m_1$, and that $\hat{\lambda}_{j+m-m_1}^{(0)} = -\hat{\lambda}_j^{(0)}$. With those assumptions and notations, we can formulate the following results.

1.4.2 Main theorems

Theorem 1.3 *We consider the real analytic Hamiltonian H of* (1.1)*, defined on $\mathcal{D}_{r,m}(\rho_0, R_0)$ (for some $0 < \rho_0 < 1$ and $R_0 > 0$), and such that the first m_1 components ($0 \leq m_1 \leq m$) of the vector λ of eigenvalues of $J_m\mathcal{B}$ are of elliptic type. Let $\mathcal{S} \subset \mathcal{N}$ be the set introduced in* (1.15) *and* (1.16)*. Then, we also assume that there exists $\mu_0 > 0$ and $\gamma > r + m_1$ such that*

$$|ik^\top\hat{\omega}^{(0)} + l^\top\lambda| \geq \frac{\mu_0}{(|k|_1 + |l_x - l_y|_1)^\gamma} \quad \forall (l,s) \in \mathcal{S}\, \forall k \in \mathbb{Z}^r \text{ with } |l_x - l_y|_1 + |k|_1 \neq 0.$$

This allows to put H in seminormal form with respect to $\mathcal{S}$ up to finite degree. We assume that this seminormal form up to degree 4 is nondegenerate, in the sense that the two nondegeneracy conditions given in (1.50) *and* (1.54) *hold. Then, there exist a Cantor subset $\mathcal{A} \subset \mathbb{R}^{r+m_1}$ such that, for any $\omega \in \mathcal{A}$, the Hamiltonian system H has an invariant $(r+m_1)$-dimensional (complex) torus with ω as a vector of basic frequencies, with reducible normal flow. Moreover, if $\mathcal{A}(A) = \mathcal{U}(A) \cap \mathcal{A}$ (being $\mathcal{U}(A)$ the set defined in* (1.9)*), then,*

$$\mathrm{mes}(\mathcal{U}(A) \setminus \mathcal{A}(A)) \leq const.\exp\left(-const.\left(\frac{1}{A}\right)^{\frac{1}{\gamma+1}}\right),$$

where the constants in this bound are independent from A.

The key to prove Theorem 1.3 is the parametrization $\mathcal{I}(\omega)$ of the invariant tori of the normal form given by Lemma 1.3. To construct this function, we take R of order $\sqrt{A}$ in Theorem 1.1, and we obtain a Hamiltonian in normal form with respect to $\mathcal{S}$ (as the one of (1.48)), with exponentially small bounds for $\mathcal{R}$ as a function of A, of the same order of the measure of destroyed tori in Theorem 1.3. Using Lemma 1.3 on the function $\mathcal{F}$ of (1.48), we can construct for any frequency A-close to $\omega^{(0)}$ the corresponding action, $I \equiv \mathcal{I}(\omega)$, that gives in (1.7) the invariant torus of the seminormal form having this concrete vector of intrinsic frequencies. Nevertheless, as the action I can have some of the $\tilde{I}_j < 0$, the corresponding torus in (1.7) can be complex. This is not an obstruction to construct an invariant (and complex) torus for the complete Hamiltonian (1.48), but the final torus and its reduced variational normal flow can be in $\mathbb{C}$. If we want to have real tori in (1.7), we need to take $\omega \in \mathcal{W}(A)$,

$$\mathcal{W}(A) = \left\{ \omega \in \mathcal{U}(A) : \; \omega = \omega^{(0)} + \frac{\partial \mathcal{F}}{\partial I}(I), \text{ with } \tilde{I}_j \geq 0, \; j = 1, \dots, m_1 \right\}. \tag{1.56}$$

Note that the degenerate (transition) tori have frequencies whose corresponding actions satisfy $\tilde{I}_j = 0$, for some j. Then, we are forced to remove actions in a tiny slice around the hyperplanes $\tilde{I}_j = 0$, that implies to take out in $\mathcal{W}(A)$ the corresponding frequencies. Unfortunately, $\mathcal{F}$ changes with A as $A \to 0$ by increasing the order up to which this seminormal form is done. This is necessary because the successive approximations to $\mathcal{F}$ given by Theorem 1.1 do not converge in general, and hence, as we want to eliminate only an exponentially small set of frequencies, we need to know this map with an exponentially small precision.

In this context, we have the following result about the existence of real invariant tori:

Theorem 1.4 *With the same hypoteses as in Theorem* 1.3*, there exist a Cantor subset* $\mathcal{A} \subset \mathbb{R}^{r+m_1}$ *such that, for any* $\omega \in \mathcal{A}$*, the Hamiltonian system* H *has an invariant* $(r+m_1)$*-dimensional real torus with vector of basic frequencies given by* ω*. The normal flow of this torus can be reduced to constant coefficients by means of a real change of variables.*

$\mathcal{A}$ can be caracterized in the following form: for any $R > 0$ small enough, there exists a convergent (partial) seminormal form with respect to $\mathcal{S}$ (it takes the form (1.48)*) defined on $\mathcal{D}_{r,m}(\rho_1, R)$ (being ρ_1 independent from R), and such that if we put $\mathcal{A}(R^2) = \mathcal{W}(R^2) \cap \mathcal{A}$ (see* (1.56) *for the definition of $\mathcal{W}(A)$), then, we have*

$$mes(\mathcal{W}(R^2) \setminus \mathcal{A}(R^2)) \leq const. \exp\left(-const. \left(\frac{1}{R}\right)^{\frac{2}{\gamma+1}} \right),$$

being the constants in this bound independent from R.

Remark 1.9 *Let us explain how the transition set (in the frequency space) from real to complex tori can be constructed independently from the seminormal form. For any tori of dimension s, $r \leq s < r + m_1$, obtained applying Theorem* 1.4 *to an invariant subbundle $\mathcal{G}_1 \subset \mathcal{G}$, with $\dim \mathcal{G}_1 < m_1$, we consider the $(r+m_1)$-dimensional vector of frequencies obtained jointly the intrinsic frequencies of the tori, with the the normal frequencies that* *generalize the ones associated to $\mathcal{G}$ not added as new intrinsic frequencies. We note that those normal frequencies are well defined from the reducible character of the normal variational flow of the constructed tori. Then, as union of those vectors, we obtain a Cantor set that acts as a transition set, with an exponentially small error (as $R \to 0$), to separate the real and complex tori.*

Remark 1.10 *In the case of maximal dimensional tori, this result can also be compared with* [48] *and* [14]*. We remark that as we work with $m_1 \leq m$, the estimates on the measure of invariant tori in Theorem* 1.4 *are given in the frequency space and not in the phase space, where the total measure filled by those tori is zero when $m_1 < m$. In the case of maximal dimensional tori, it is not difficult to check (for example, if one uses a formulation of the proof in terms of the actions I instead of the frequencies ω, see Section* 1.4.3 *for more details) that the measure of the complementary of the set that the real tori fill in $\mathcal{D}_{r,m}(0,R) \cap \mathbb{R}^{2\ell}$ is of order const.* $\exp\left(-const.\left(\frac{1}{R}\right)^{\frac{2}{\gamma+1}}\right)$*. This result coincides with* [48], *if we look for maximal dimensional invariant tori around a given maximal dimensional one (we note that we need to done the same rescaling as in Section* 1.2.2 *in order to compare the results), but it improves the estimates on the measure of invariant tori around an elliptic fixed point of* [14].

In what follows, we will only prove Theorem 1.4. The proof of Theorem 1.3 is very similar, and it is done by splitting the set $\mathcal{U}(A)$ as the union of the sets of frequencies defined like (1.56), but taking into account all the possible combinations of conditions $\tilde{I}_j \geq 0$ and $\tilde{I}_j \leq 0$, $j = 1, \ldots, m_1$.

1.4.3 Proof of Theorem 1.4

Before starting with the details, let us mention a technical problem that will appear during the proof. At each step of the iterative KAM process, we will have to deal with the classical small divisors problem. This will lead us to eliminate the set of resonant frequencies, to avoid convergence problems. To bound the measure of the eliminated set of frequencies, we need some kind of regularity. In a problem depending on parameters, it is usual to ask for smooth dependence on the parameters, but note that at every step of the inductive process we need to remove a dense set of frequencies, and this does not allow to keep, in principle, any kind of smooth dependence for the Hamiltonian with respect to this parameter (because now the parameters move on a set with empty interior).

Fortunately, there are many solutions to this problem. One possibility is to work, at every step of the inductive procedure, with a finite number of terms in the different Fourier expansion with respect to θ.[3] This is based on dropping the harmonics f_k such that $|k|$ is bigger than some quantity $O(2^n)$ (n is the step). Hence, we can use that the remainder of the truncated expressions is exponentially small with the order of truncation to show the convergence of the sequence of changes involved in the iterative scheme on a suitable set of parameters. Then, since we only need to deal with a finite number of resonances at every step, we can work on open sets with respect to the frequencies, and to

[3]This is the so called "ultraviolet cut".

keep the smooth parametric dependences in those sets. This smooth dependence allows to bound the measure of the resonances. Those ideas are for example used in [1], [13] or [30].

Another possibility is to consider Lipschitz parametric dependence instead of a smooth one (this has already been done in [33] or [35]). We can see that it is possible to keep this Lipschitz dependence at every step (the control of this kind of dependence on Cantor sets is analogous to the control of differentiable dependence on open sets), and this kind of dependence suffices to bound the measure of the resonant sets. In this work we have chosen this Lipschitz formulation. This implies that the invariant tori obtained will depend on the parameters in a Lipschitz way.

If one is interested in obtaining C^∞ Whitney smoothness (see [60]) for the dependence of the invariant tori with respect to the parameters, the standard procedure is to work with a finite number of harmonics in the Fourier expansions, so that at every step we keep the analytic character of the Hamiltonian (see [65]).

Another different approach, that also allows to obtain Whitney regularity, is to add external (auxiliar) parameters to the Hamiltonian, in order to have enough parameters to control the intrinsic and normal frequencies (to avoid the lack of parameters problem), and such that for every Diophantine vector of intrinsic and normal eigenvalues, we have the corresponding invariant tous for a suitable value of the (enlarged) parameters. This can be done with a Whitney smooth foliation (see [9] and [8]). Then, if we consider the value of the external parameters for which we recover the initial family of Hamiltonians, we only have to study which of the Diophantine tori constructed correspond to this value of the extra parameter. This can be done under very weak nondegeneracy hypotheses by using the theory of Diophantine approximations on submanifolds (see for instance, [67] and [8], or [76] for the case of volume preserving diffeomorphisms).

To work with the Lipschitz dependence, we introduce some notations and definitions. Given $f(\varphi)$ a function defined for $\varphi \in \mathcal{E}$, $\mathcal{E} \subset \mathbb{R}^n$ for some n, and with values in $\mathbb{C}$, $\mathbb{C}^{n_1}$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$, we define the Lipschitz constant of f on $\mathcal{E}$ as

$$\mathcal{L}_{\mathcal{E}}\{f\} = \sup_{\substack{\varphi_1,\varphi_2\in\mathcal{E}\\ \varphi_1\neq\varphi_2}} \frac{|f(\varphi_2)-f(\varphi_1)|}{|\varphi_2-\varphi_1|}.$$

If $\mathcal{L}_{\mathcal{E}}\{f\} < +\infty$, we say that f is Lipschitz on $\mathcal{E}$, with respect to the norm $|.|$. We also define $\|f\|_{\mathcal{E}} = \sup_{\varphi\in\mathcal{E}} |f(\varphi)|$. The same definitions can be extended to analytic functions depending on the parameter φ. Hence, if $f(\theta,\varphi)$ or $f(\theta,x,I,y,\varphi)$ are, for every $\varphi \in \mathcal{E}$, analytic, 2π-periodic on θ, and defined on $\{\theta \in \mathbb{C}^r \,:\, |\mathrm{Im}\theta| \leq \rho\}$ or $\mathcal{D}_{r,m}(\rho,R)$, respectively, we can introduce $\mathcal{L}_{\mathcal{E},\rho}\{f\}$, $\|f\|_{\mathcal{E},\rho}$, or $\mathcal{L}_{\mathcal{E},\rho,R}\{f\}$ and $\|f\|_{\mathcal{E},\rho,R}$, taking the supremum on the norms $|.|_\rho$ or $|.|_{\rho,R}$, respectively. Some basic results related to those kind of dependence are given in Section 1.5.

Preliminaries

First, we take a fixed value of R, small enough, and we use Theorem 1.1 to put the initial Hamiltonian H in (semi)normal form with respect to the set $\mathcal{S}$, except a remainder $\mathcal{R}$ of exponentially small size with respect to R. Hence, we can work with the Hamiltonian H of (1.48) as if it were the initial one. As the order of this normal form depends on R, we have to write explicitly this R-dependence in all the bounds.

We take the analytic function $\mathcal{F}$ of (1.48) and we construct the parametrization $\mathcal{I}(\omega)$ provided by Lemma 1.3, that is well defined if R is small enough. Then, as we want to work with $(r+m_1)$-dimensional tori, we introduce new m_1 angular variables $\tilde{\theta}$ conjugated to the actions $\tilde{I}$ previously added to describe this family of invariant tori. That is, for any frequency $\omega \in \mathbb{R}^{r+m_1}$ close to $\omega^{(0)}$, we replace the real (semi)normal form variables $(\tilde{x}_j, \tilde{y}_j)$, $j = 1, \ldots, m_1$, by the new canonical variables $(\tilde{\theta}_j, \tilde{I}_j)$ defined as

$$\tilde{x}_j = \sqrt{2(\tilde{I}_j + \tilde{\mathcal{I}}_j(\omega))}\sin(\tilde{\theta}_j), \quad \tilde{y}_j = \sqrt{2(\tilde{I}_j + \tilde{\mathcal{I}}_j(\omega))}\cos(\tilde{\theta}_j), \tag{1.57}$$

or, in terms of the complexified variables $(\tilde{X}_j, \tilde{Y}_j)$ (see (1.44)),

$$\tilde{X}_j = -i\sqrt{\tilde{I}_j + \tilde{\mathcal{I}}_j(\omega)}\exp(i\tilde{\theta}_j), \quad \tilde{Y}_j = \sqrt{\tilde{I}_j + \tilde{\mathcal{I}}_j(\omega)}\exp(-i\tilde{\theta}_j). \tag{1.58}$$

We remark that those $\tilde{I}$ are not exactly the same ones used to parametrize the tori (1.7), because they differ in a translation by $\tilde{\mathcal{I}}(\omega)$. This is done to put the ω-invariant torus, with respect to the seminormal form, in $\tilde{I} = 0$. Hence, we extend this translation to the whole set of actions I, doing the transformation

$$\hat{I}_j \to \hat{I}_j + \hat{\mathcal{I}}_j(\omega), \; j = 1, \ldots, r. \tag{1.59}$$

Moreover, we denote by $\theta^\top = (\hat{\theta}^\top, \tilde{\theta}^\top)$ the vector of all the angular variables. Then, we have constructed, for each ω, a new canonical system of coordinates (with $r+m_1$ angular variables) that put the corresponding seminormal form torus in $I = 0$. If we insert those new variables in the Hamiltonian (1.48), we obtain a ω-depending family of Hamiltonians $H_\omega^{(0)}$, that we simply denote by $H^{(0)} \equiv H^{(0)}(\theta, \hat{X}, I, \hat{Y}, \omega)$,

$$H^{(0)} = \omega^{(0)\top} I + \frac{1}{2}\hat{Z}^\top \hat{\mathcal{B}}^* \hat{Z} + \mathcal{F}(I + \mathcal{I}(\omega)) + \frac{1}{2}\hat{Z}^\top \mathcal{Q}(I + \mathcal{I}(\omega))\hat{Z} + \mathcal{T}^* + \mathcal{R}^*,$$

where $\mathcal{T}^*$ and $\mathcal{R}^*$ are $\mathcal{T}$ and $\mathcal{R}$ expressed in terms of $\tilde{\theta}$ and $\tilde{I}$, and composed with the translation (1.59). We cast $H^{(0)}$ into the following form:

$$H^{(0)} = \phi^{(0)}(\omega) + \omega^\top I + \frac{1}{2}\hat{Z}^\top \hat{\mathcal{B}}^{(0)*}(\omega)\hat{Z} + \frac{1}{2} I^\top \mathcal{C}^{(0)}(\omega) I + H_*^{(0)} + \hat{H}^{(0)}, \tag{1.60}$$

where $\phi^{(0)} = \mathcal{F}(\mathcal{I}(\omega))$, $\hat{\mathcal{B}}^{(0)*} = \hat{\mathcal{B}}^* + \mathcal{Q}(\mathcal{I}(\omega))$, $\mathcal{C}^{(0)} = \frac{\partial^2 \mathcal{F}}{\partial I^2}(\mathcal{I}(\omega))$, $\hat{H}^{(0)} = \mathcal{R}^*$ and

$$\begin{aligned} H_*^{(0)} \;\; = \;\; & \mathcal{F}(I + \mathcal{I}(\omega)) - \mathcal{F}(\mathcal{I}(\omega)) - \frac{\partial \mathcal{F}}{\partial I}(\mathcal{I}(\omega))I - \frac{1}{2} I^\top \frac{\partial^2 \mathcal{F}}{\partial I^2}(\mathcal{I}(\omega)) I + \\ & + \frac{1}{2}\hat{Z}^\top (\mathcal{Q}(I + \mathcal{I}(\omega)) - \mathcal{Q}(\mathcal{I}(\omega)))\hat{Z} + \mathcal{T}^*, \end{aligned}$$

where we have used the properties of $\mathcal{I}(\omega)$ (see Lemma 1.3). We remark that, if one considers $H_\omega^{(0)}$ for a fixed ω, and one skips the term $\hat{H}^{(0)}$ in (1.60), then, $I = 0$ and $\hat{Z} = 0$ correspond to an invariant $(r+m_1)$-dimensional torus with vector of basic frequencies ω, with reducible normal variational flow given by the (complex) diagonal matrix $J_{m-m_1}\hat{\mathcal{B}}^{(0)*}$. Moreover, in this case the variables I and $\hat{Z}$ are uncoupled, at least up to first order.

Nevertheless, the coordinates of (1.58) become singular for $I=0$ when we take frequencies ω with some $\tilde{I}_j \equiv \tilde{I}_j(\omega)=0$. Thus, we have to eliminate a neighbourhood of the set of those critical frequencies to ensure that the change is well defined.

Let $M(R)R^8$ be the expression (with M exponentially small in R) bounding $|\mathcal{R}|_{\rho_1,R}$ in (1.49). We consider a fixed value of R, small enough, such that $M(R)<1$, and we take a fixed number $0<\alpha<1$, to be precised later. Then, as we are only interested in real tori, we use definition (1.56) to introduce the following set:

$$\mathcal{E}^{(0)}(R)=\mathcal{W}\left(\frac{1}{8}(|\mathcal{C}^{-1}|)^{-1}R^2\right)\setminus\left(\mathcal{I}^{-1}\left(\mathcal{V}\left(16(M(R))^{2\alpha}\right)\right)\right),$$

being $\mathcal{V}(A)=\{I\in\mathbb{R}^{r+m_1}\;:\;|\tilde{I}_j|\le A,\text{ for any } j=1,\ldots,m_1\}$. That is, we eliminate from $\mathcal{W}(\frac{1}{8}(|\mathcal{C}^{-1}|)^{-1}R^2)$ the frequencies corresponding to actions $\tilde{I}$ close to any of the hyperplanes $\tilde{I}_j=0$. As from Lemma 1.3 we have $|\mathcal{I}(\omega)|\le\frac{1}{4}R^2$ for any $\omega\in\mathcal{W}\left(\frac{1}{8}(|\mathcal{C}^{-1}|)^{-1}R^2\right)$, the measure of $\tilde{\mathcal{V}}(R):=\mathcal{V}\left(16M^{2\alpha}\right)\cap\mathcal{I}\left(\mathcal{W}\left(\frac{1}{8}(|\mathcal{C}^{-1}|)^{-1}R^2\right)\right)$ is of order $(M(R))^{2\alpha}$. Then, to control the measure that this set fills in $\mathcal{W}(\frac{1}{8}(|\mathcal{C}^{-1}|)^{-1}R^2)$, we put $\widetilde{\mathcal{W}}(R)=\mathcal{I}^{-1}\left(\mathcal{V}\left(16M^{2\alpha}\right)\right)\cap\mathcal{W}(\frac{1}{8}(|\mathcal{C}^{-1}|)^{-1}R^2)$, and then we have

$$\operatorname{mes}(\widetilde{\mathcal{W}}(R))=\int_{\widetilde{\mathcal{W}}(R)}d\omega=\int_{\tilde{\mathcal{V}}(R)}\left|\det D\mathcal{I}^{-1}(I)\right|dI, \tag{1.61}$$

where, from the definition of $\mathcal{I}$ in Lemma 1.3, $\mathcal{I}^{-1}(I)=\omega^{(0)}+\frac{\partial\mathcal{F}}{\partial I}(I)$. Then, the bounds on $\mathcal{F}$ in (1.49) suffices to justify that $\operatorname{mes}(\widetilde{\mathcal{W}}(R))$ is also of order $(M(R))^{2\alpha}$.

Now, we are going to see that $H^{(0)}$ is defined on a small neighbourhood of $I=0$ and $\hat{Z}=0$, with positive (and bounded from below) distance to the critical set of frequencies. To do that, and as we will work with functions of $(\theta,\hat{X},I,\hat{Y})$, we will take the different norms on domains of the form $\mathcal{D}_{r+m_1,m-m_1}(.,.)$. More concretely, we will see that, for any $\omega\in\mathcal{E}^{(0)}(R)$, $H^{(0)}$ is well defined on $\mathcal{D}_{r+m_1,m-m_1}(\rho^{(0)},R^{(0)})$, being $\rho^{(0)}=\rho_0/2$ (this is smaller than the width of analyticity for the $\hat{\theta}$ variables given by Theorem 1.1 for the seminormal form) and $R^{(0)}=2M^{\alpha}$. Let us check that.

First, we remark that as $\mathcal{F}$ only depends on I, we deduce from (1.49) that $|\mathcal{F}|_{0,R}\le\hat{\mathcal{F}}R^4$. Then, assuming $\frac{1}{2}R^2+4M^{2\alpha}\le R^2$, and using Lemmas 1.7 and 1.8, we obtain

$$\|\mathcal{F}(\mathcal{I}(\omega)+I)\|_{\mathcal{E}^{(0)},0,R^{(0)}}\le\hat{\mathcal{F}}R^4,\quad \mathcal{L}_{\mathcal{E}^{(0)},0,R^{(0)}}\{\mathcal{F}(\mathcal{I}(\omega)+I)\}\le(r+m_1)\frac{\hat{\mathcal{F}}R^4}{\frac{1}{4}R^2}2|\mathcal{C}^{-1}|, \tag{1.62}$$

where we have used that $\mathcal{L}_{\mathcal{E}^{(0)}}\{\mathcal{I}\}\le 2|\mathcal{C}^{-1}|$ (see Lemma 1.3). Similar bounds can be derived for $\mathcal{F}_3$, $\mathcal{Q}$ and $\mathcal{Q}_1$.

To bound $\mathcal{T}^*$ and $\mathcal{R}^*$, we need to study the well defined character of the transformation (1.58). Using Lemma 1.10 (we recall that, from the definition of $\mathcal{E}^{(0)}$, one has $\tilde{\mathcal{I}}_j(\omega)\ge 16M^{2\alpha}$ for any $\omega\in\mathcal{E}^{(0)}$), and using that $\rho_0\le 1$, it is not difficult to check that if we consider $(\tilde{X}_j,\tilde{Y}_j)$ in (1.58) as a function of $(\theta,\hat{X},I,\hat{Y})$ and ω, we have for $j=1,\ldots,m_1$:

$$\max\left\{\|\hat{X}_j\|_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}},\|\hat{Y}_j\|_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\right\}\le\frac{1}{2}R\left(2-\sqrt{3/4}\right)\exp\left(1/2\right)\le\frac{15}{16}R,$$

and

$$\max\left\{\mathcal{L}_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\{\hat{X}_j\},\mathcal{L}_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\{\hat{Y}_j\}\right\}\le\frac{2|\mathcal{C}^{-1}|\exp(1/2)}{8M^{\alpha}\sqrt{3/4}}\le\frac{|\mathcal{C}^{-1}|}{2M^{\alpha}}.$$

Assuming R small enough such that $4M^{2\alpha}+\frac{1}{4}R^2\le\left(\frac{15}{16}\right)^2R^2$ (this is used to control the transformation (1.59)) one has, using Lemmas 1.7 and 1.8 on the bounds (1.49), that

$$\|\mathcal{T}^*\|_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\le\hat{\mathcal{T}}R^3,\quad \|\mathcal{R}^*\|_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\le MR^8, \tag{1.63}$$

and

$$\mathcal{L}_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\{\mathcal{T}^*\}\ \le\ r\frac{\hat{\mathcal{T}}R^3}{\left(1-\left(\frac{15}{16}\right)^2\right)R^2}2|\mathcal{C}^{-1}|+2m_1\frac{\hat{\mathcal{T}}R^2 2M^{\alpha}}{\frac{1}{16}R}\frac{|\mathcal{C}^{-1}|}{2M^{\alpha}}, \tag{1.64}$$

$$\mathcal{L}_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\{\mathcal{R}^*\}\ \le\ r\frac{MR^8}{\left(1-\left(\frac{15}{16}\right)^2\right)R^2}2|\mathcal{C}^{-1}|+2m_1\frac{MR^8}{\frac{1}{16}R}\frac{|\mathcal{C}^{-1}|}{2M^{\alpha}}, \tag{1.65}$$

where, to bound $\mathcal{T}^*$, we have used that it is $O_3(\hat{Z})$.

Now, we can bound the different terms of $H^{(0)}$ in (1.60). First, one has $\|\phi^{(0)}\|_{\mathcal{E}^{(0)}}\le\hat{\mathcal{F}}R^4$. We do not care about its Lipschitz constant because this term can be eliminated without changing the canonical equations (so, we only need to worry about its bounded character). To bound $\mathcal{C}^{(0)}$ we remark that $\mathcal{C}^{(0)}=\mathcal{C}+\frac{\partial^2\mathcal{F}_3}{\partial I^2}(\mathcal{I})$ with $\det\mathcal{C}\neq 0$. Using the bounds on $\mathcal{F}_3$ (that can be obtained in a similar form as the ones on $\mathcal{F}$ in (1.62)) one has that $\det\mathcal{C}^{(0)}\neq 0$ on $\mathcal{E}^{(0)}$, if R is small enough. In quantitative form, it means that $\|(\bar{\mathcal{C}}^{(0)})^{-1}\|_{\mathcal{E}^{(0)}}\le\bar{m}^{(0)}$, for certain R-independent constant $\bar{m}^{(0)}$. Moreover, we also have $\|\mathcal{C}^{(0)}\|_{\mathcal{E}^{(0)},\rho^{(0)}}\le\hat{m}^{(0)}$, $\mathcal{L}_{\mathcal{E}^{(0)},\rho^{(0)}}\{\mathcal{C}^{(0)}\}\le\tilde{m}^{(0)}$. We have used the norm $\|.\|_{\mathcal{E}^{(0)},\rho^{(0)}}$ for the constant matrix $\mathcal{C}^{(0)}$ because, in the succesive steps of the inductive procedure, the matrices replacing $\mathcal{C}^{(0)}$ will depend on θ. To control the normal eigenvalues, we remark that from the expression (1.55), we can write $\hat{\lambda}_j^{(0)}$, $j=1,\dots,2(m-m_1)$, in the following form:

$$\hat{\lambda}_j^{(0)}(\omega)=\hat{\lambda}_j+iv_j^{\top}(\omega-\omega^{(0)})+\breve{\lambda}_j^{(0)}(\omega), \tag{1.66}$$

where $v_j\in\mathbb{C}^{r+m_1}$ and $\breve{\lambda}_j^{(0)}\equiv O_2(\omega-\omega^{(0)})$. From the non-degeneracy hypotesis of (1.54), we have that $\mathrm{Re}(v_j)\notin\mathbb{Z}^{r+m_1}$ and, if we put $v_{j,l}=v_j-v_l$, then $\mathrm{Re}(v_{j,l})\notin\mathbb{Z}^{r+m_1}$, for $j\neq l$. Moreover, using the expression (1.55) and the fact that the eigenvalues of $J_{m-m_1}\hat{\mathcal{B}}^*$ are all different, it is also easy to check the existence of R-independent constants $0<\alpha_1^{(0)}<\alpha_2^{(0)}$, $\beta_1^{(0)}>0$, such that $0<\alpha_1^{(0)}\le|\hat{\lambda}_j^{(0)}(\omega)-\hat{\lambda}_l^{(0)}(\omega)|$, $\alpha_1^{(0)}/2\le|\hat{\lambda}_j^{(0)}(\omega)|\le\alpha_2^{(0)}/2$, for any $\omega\in\mathcal{E}^{(0)}$, $j\neq l$, and $\mathcal{L}_{\mathcal{E}^{(0)}}\{\hat{\lambda}_j^{(0)}\}\le\beta_1^{(0)}$. We do not give here explicit bounds on the $\breve{\lambda}_j^{(0)}$, as those functions do not appear in the iterative process, but we remark that one has that $\mathcal{L}_{\mathcal{E}^{(0)}}\{\breve{\lambda}_j^{(0)}\}$ is of order R. This will be used in Section 1.4.3. Moreover, if one uses bounds like (1.62) for $\mathcal{F}$ and $\mathcal{Q}$, and the ones of (1.63) and (1.64) for $\mathcal{T}^*$, it is not difficult to check that for certain positive R-independent constants $\hat{\nu}^{(0)}$ and $\tilde{\nu}^{(0)}$, one has $\|H_*^{(0)}\|_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\le\hat{\nu}^{(0)}$ and $\mathcal{L}_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\{H_*^{(0)}\}\le\tilde{\nu}^{(0)}$. Finally, using the bounds of (1.63) and (1.65) for $\mathcal{R}^*$, one can bound the size of the perturbative term $\hat{H}^{(0)}$ by $\|\hat{H}^{(0)}\|_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\le M$ and $\mathcal{L}_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\{\hat{H}^{(0)}\}\le M^{1-\alpha}$. Some of these bounds are far from optimal, but they suffice for our purposes.

The iterative scheme

Now, we can describe the iterative procedure used to construct invariant $(r+m_1)$-dimensional tori. This process is given by a sequence of canonical changes of variables, constructed as the time one flow of a suitable generating function S_ω. The changes are constructed to kill the terms that obstructs the existence of an invariant reduced torus with vector of basic frequencies given by ω. As usual (to overcome the effect of the small divisors), the changes are chosen to produce a quadratically convergent scheme, instead of the linear one of Lemma 1.1.

First, we describe a generic step of this iterative process. For this purpose, we expand the Hamiltonian $H^{(0)}$ in the following form

$$H^{(0)} = a(\theta)+b(\theta)^\top\hat{Z}+c(\theta)^\top I+\frac{1}{2}\hat{Z}^\top B(\theta)\hat{Z}+I^\top E(\theta)\hat{Z}+\frac{1}{2}I^\top C(\theta)I+\Omega(\theta,\hat{X},I,\hat{Y}), \quad (1.67)$$

where we do not write explicitly the ω-dependence and where we have skipped the superscript "(0)" in the different parts of the Hamiltonian. From this expansion, we introduce the following notations: $[H^{(0)}]_{(\hat{Z},\hat{Z})}=B$, $[H^{(0)}]_{(I,\hat{Z})}=E$ and $< H^{(0)} >= H^{(0)}-\Omega$. From the bounds on the terms of the decomposition (1.60), we have that $\tilde{a}$, b, $c-\omega$, $B-\hat{\mathcal{B}}^{(0)*}$, $C-\mathcal{C}^{(0)}$ and E are all $O(\hat{H}^{(0)})$. Note that if we are able to kill the terms $\tilde{a}$, b and $c-\omega$, we will obtain an invariant torus with intrinsic frequency ω. Nevertheless, as we want to have simple equations at every step of the iterative scheme (this is, linear equations with constant coefficients), we are forced to kill something more. Then, we ask the final torus to have reducible normal flow given by a diagonal matrix. This is, we want that the new matrix B verifies $B=\mathcal{J}_{m-m_1}(B)$ where, for a $(2s)$-dimensional matrix $A(\theta)$ depending 2π-periodically on θ, we define $\mathcal{J}_s(A)=-J_s\mathrm{dp}(J_s\bar{A})$. Here, $\mathrm{dp}(A)$ denotes the diagonal matrix obtained taking the diagonal entries of A. Moreover, we have to eliminate E to uncouple the "neutral" and the normal directions of the torus up to first order. Thus, for each step of the iterative process, we use a canonical change of variables similar to one the used in [5] to prove the Kolmogorov theorem. The generating function is of the form

$$S(\theta,\hat{X},I,\hat{Y})=\xi^\top\theta+d(\theta)+e(\theta)^\top\hat{Z}+f(\theta)^\top I+\frac{1}{2}\hat{Z}^\top G(\theta)\hat{Z}+I^\top F(\theta)\hat{Z},$$

where $\xi\in\mathbb{C}^{r+m_1}$, $\bar{d}=0$, $\bar{f}=0$ and G is a symmetric matrix, with $\mathcal{J}_{m-m_1}(G)=0$. The transformed Hamiltonian is $H^{(1)}=H^{(0)}\circ\Psi_1^S$. We expand $H^{(1)}$ in the same way as $H^{(0)}$ in (1.67), keeping the same name for the new variables, but adding the superscript "(1)" to a, b, c, B, C, E and Ω. Then, we ask $\tilde{a}^{(1)}=0$, $b^{(1)}=0$, $c^{(1)}-\omega=0$, $E^{(1)}=0$ and $B^{(1)}=\mathcal{J}_{m-m_1}(B^{(1)})$. We will show that this can be achieved up to first order in the size of $\hat{H}^{(0)}$. For this purpose, we write those conditions in terms of the initial Hamiltonian and the generating function, and then, we obtain the following equations:

(eq_1) $\tilde{a}-\frac{\partial d}{\partial\theta}\omega=0,$

(eq_2) $b-\frac{\partial e}{\partial\theta}\omega+\hat{\mathcal{B}}^{(0)*}J_{m-m_1}e=0,$

(eq_3) $c-\omega-\frac{\partial f}{\partial\theta}\omega-\mathcal{C}^{(0)}\left(\xi+\left(\frac{\partial d}{\partial\theta}\right)^\top\right)=0,$

(eq_4) $B^* - \mathcal{J}_{m-m_1}(B^*) - \frac{\partial G}{\partial \theta}\omega + \hat{\mathcal{B}}^{(0)*} J_{m-m_1} G - G J_{m-m_1} \hat{\mathcal{B}}^{(0)*} = 0,$

(eq_5) $E^* - \frac{\partial F}{\partial \theta}\omega - F J_{m-m_1} \hat{\mathcal{B}}^{(0)*} = 0,$

being

$$\begin{aligned} B^* &= B - \left[\frac{\partial H_*^{(0)}}{\partial I} \left(\xi + \left(\frac{\partial d}{\partial \theta} \right)^\top \right) - \frac{\partial H_*^{(0)}}{\partial \hat{Z}} J_{m-m_1} e \right]_{(\hat{Z},\hat{Z})}, \\ E^* &= E - \mathcal{C}^{(0)} \left(\frac{\partial e}{\partial \theta} \right)^\top - \left[\frac{\partial H_*^{(0)}}{\partial I} \left(\xi + \left(\frac{\partial d}{\partial \theta} \right)^\top \right) - \frac{\partial H_*^{(0)}}{\partial \hat{Z}} J_{m-m_1} e \right]_{(I,\hat{Z})}. \end{aligned}$$

To solve those homological equations, we expand them in Fourier series and we equate the corresponding coefficients, obtaining the formal solutions. The next step is to derive bounds on those solutions. As we will use these bounds in iterative form, we want to make clear which expressions change from one step to another, and which ones can be bounded independently from the step. For this purpose, we take fixed positive constants $\bar{m}$, $\hat{m}$, $\tilde{m}$, α_2, β_1, $\hat{\nu}$, $\tilde{\nu}$ defined as twice the corresponding inital values $\bar{m}^{(0)}$, $\hat{m}^{(0)}$, $\tilde{m}^{(0)}$, $\alpha_2^{(0)}$, $\beta_1^{(0)}$, $\hat{\nu}^{(0)}$, $\tilde{\nu}^{(0)}$ and a fixed α_1, $0 < \alpha_1 < \alpha_1^{(0)}$. In what follows, $\hat{N}$ will denote an expression depending only on $\bar{m}$, $\hat{m}$, α_1, α_2, $\hat{\nu}$, the different dimensions r, m, m_1, plus γ and ρ_0. $\hat{N}$ will be redefined during the description of the iterative scheme to meet a finite number of conditions. The idea is to perform the bounds on the iterative scheme putting the superscript "(0)" on the terms that change at every iteration. Hence, we write the bounds on $\hat{H}^{(0)}$ as $\|\hat{H}^{(0)}\|_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}} \le M^{(0)}$ and $\mathcal{L}_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}}\{\hat{H}^{(0)}\} \le L^{(0)}$, with $M^{(0)}(R) \equiv M(R)$ and $L^{(0)}(R) \equiv (M(R))^{1-\alpha}$. Hence, using Lemma 1.5,

$$\begin{aligned} \|a - \phi^{(0)}\|_{\mathcal{E}^{(0)},\rho^{(0)}} &\le M^{(0)}, & \|E\|_{\mathcal{E}^{(0)},\rho^{(0)}} &\le \tfrac{2(m-m_1)M^{(0)}}{(R^{(0)})^3}, \\ \|c - \omega\|_{\mathcal{E}^{(0)},\rho^{(0)}} &\le \tfrac{M^{(0)}}{(R^{(0)})^2}, & \|B - \hat{\mathcal{B}}^{(0)*}\|_{\mathcal{E}^{(0)},\rho^{(0)}} &\le \tfrac{(2(m-m_1)+1)M^{(0)}}{(R^{(0)})^2}, \\ \|b\|_{\mathcal{E}^{(0)},\rho^{(0)}} &\le \tfrac{M^{(0)}}{R^{(0)}}, & \|C - \mathcal{C}^{(0)}\|_{\mathcal{E}^{(0)},\rho^{(0)}} &\le \tfrac{(2(r+m_1)+1)M^{(0)}}{(R^{(0)})^4}, \\ \|\Omega\|_{\mathcal{E}^{(0)},\rho^{(0)},R^{(0)}} &\le \hat{\nu}^{(0)} + M^{(0)}. \end{aligned} \tag{1.68}$$

Moreover, we can use Lemma 1.11 to deduce that the same bounds hold for their Lipschitz constants on $\mathcal{E}^{(0)}$, replacing $M^{(0)}$ by $L^{(0)}$, and $\hat{\nu}^{(0)}$ by $\tilde{\nu}^{(0)}$. Then, to prove the convergence of the expansion of S, we need some kind of control on the different small divisors involved. For this purpose, we restrict the parameter ω to the subset $\mathcal{E}^{(1)}(R) \subset \mathcal{E}^{(0)}(R)$ for which the following Diophantine estimates hold: we say that $\omega \in \mathcal{E}^{(1)}$, if $\omega \in \mathcal{E}^{(0)}$, and

$$|ik^\top \omega + l^\top \hat{\lambda}^{(0)}(\omega)| \ge \frac{\mu^{(0)}(R)}{|k|_1^\gamma}, \quad k \in \mathbb{Z}^{r+m_1} \setminus \{0\}, \quad l \in \mathbb{N}^{2(m-m_1)}, \quad 0 < |l|_1 \le 2, \tag{1.69}$$

for certain $\mu^{(0)} > 0$. We expect the measure of $\mathcal{E}^{(0)} \setminus \mathcal{E}^{(1)}$ to be of order $\mu^{(0)}$ and, hence, as we want to have exponentially small bounds for this measure, we take $\mu^{(0)} \equiv (M^{(0)})^\alpha$. Then, we proceed to bound the solutions of the different homological equations. For this purpose, we use Lemma 1.4. More precisely, we define $\delta^{(0)} = (M^{(0)})^\alpha$, and we take $\delta^{(0)}$ as a value for δ to use the different estimates provided by this lemma. In order to simplify the proofs, we assume $\rho^{(0)} - N\delta^{(0)} \ge \rho_0/4$, where $N \in \mathbb{N}$ will be a fixed integer that will be determined before the description of the iterative scheme. Moreover, we also assume that $(M^{(0)})^\alpha \le R^{(0)} \le 1$. Then, one can solve $(eq_1) - (eq_5)$ as follows:

(eq_1) For d, we have

$$d(\theta) = \sum_{k\in\mathbb{Z}^{r+m_1}\setminus\{0\}} \frac{a_k}{ik^\top\omega} \exp(ik^\top\theta),$$

that implies,

$$\|d\|_{\mathcal{E}^{(1)},\rho^{(0)}-\delta^{(0)}} \le \left(\frac{\gamma}{\delta^{(0)}\exp(1)}\right)^\gamma \frac{\|\tilde{a}\|_{\mathcal{E}^{(1)},\rho^{(0)}}}{\mu^{(0)}} \le \hat{N}(M^{(0)})^{1-\alpha-\alpha\gamma}.$$

(eq_2) For any j, $1\le j\le 2(m-m_1)$, we have

$$e_j(\theta) = \sum_{k\in\mathbb{Z}^{r+m_1}} \frac{b_{j,k}}{ik^\top\omega + \hat{\lambda}_j^{(0)}} \exp(ik^\top\theta),$$

and hence,

$$\|e\|_{\mathcal{E}^{(1)},\rho^{(0)}-\delta^{(0)}} \le \left(\frac{2}{\alpha_1} + \left(\frac{\gamma}{\delta^{(0)}\exp(1)}\right)^\gamma \frac{1}{\mu^{(0)}}\right) \|b\|_{\mathcal{E}^{(1)},\rho^{(0)}} \le \hat{N}(M^{(0)})^{1-2\alpha-\alpha\gamma}.$$

(eq_3) Taking average with respect to θ, we obtain

$$\xi = (\bar{\mathcal{C}}^{(0)})^{-1}\left(\bar{c} - \omega - \overline{\mathcal{C}^{(0)}\left(\frac{\partial d}{\partial\theta}\right)^\top}\right).$$

Thus,

$$\begin{aligned} \|\xi\|_{\mathcal{E}^{(1)}} &= \|(\bar{\mathcal{C}}^{(0)})^{-1}\bar{\mathcal{C}}^{(0)}\xi\|_{\mathcal{E}^{(1)}} \le \|(\bar{\mathcal{C}}^{(0)})^{-1}\|_{\mathcal{E}^{(1)}}\|\bar{\mathcal{C}}^{(0)}\xi\|_{\mathcal{E}^{(1)}} \le \\ &\le \bar{m}\left(\|\bar{c}-\omega\|_{\mathcal{E}^{(1)},0} + \left\|\mathcal{C}^{(0)}\left(\frac{\partial d}{\partial\theta}\right)^\top\right\|_{\mathcal{E}^{(1)},0}\right) \le \\ &\le \bar{m}\left(\|c-\omega\|_{\mathcal{E}^{(1)},\rho^{(0)}} + \hat{m}\frac{\|d\|_{\mathcal{E}^{(1)},\rho^{(0)}-\delta^{(0)}}}{(\rho^{(0)}-\delta^{(0)})\exp(1)}\right) \le \hat{N}(M^{(0)})^{1-\alpha-\alpha\gamma}. \end{aligned}$$

To solve the equation for f, we define

$$c^* = \tilde{c} - \tilde{\mathcal{C}}^{(0)}\xi - \mathcal{C}^{(0)}\left(\frac{\partial d}{\partial\theta}\right)^\top + \overline{\mathcal{C}^{(0)}\left(\frac{\partial d}{\partial\theta}\right)^\top},$$

and then, for any $1\le j\le r+m_1$, we have

$$f_j(\theta) = \sum_{k\in\mathbb{Z}^{r+m_1}\setminus\{0\}} \frac{c^*_{j,k}}{ik^\top\omega}\exp(ik^\top\theta).$$

To bound f, first we have that

$$\begin{aligned} \|c^*\|_{\mathcal{E}^{(1)},\rho^{(0)}-2\delta^{(0)}} &\le \|\tilde{c}\|_{\mathcal{E}^{(1)},\rho^{(0)}} + \|\mathcal{C}^{(0)}\|_{\mathcal{E}^{(1)},\rho^{(0)}}\left(\|\xi\|_{\mathcal{E}^{(1)}} + \frac{\|d\|_{\mathcal{E}^{(1)},\rho^{(0)}-\delta^{(0)}}}{\delta^{(0)}\exp(1)}\right) \le \\ &\le \hat{N}(M^{(0)})^{1-2\alpha-\alpha\gamma}, \end{aligned}$$

and from here

$$\|f\|_{\mathcal{E}^{(1)},\rho^{(0)}-3\delta^{(0)}} \le \left(\frac{\gamma}{\delta^{(0)}\exp(1)}\right)^\gamma \frac{\|c^*\|_{\mathcal{E}^{(1)},\rho^{(0)}-2\delta^{(0)}}}{\mu^{(0)}} \le \hat{N}(M^{(0)})^{1-3\alpha-2\alpha\gamma}.$$

(eq_4) We define $B^{**} = B^* - \mathcal{J}_{m-m_1}(B^*)$, and then, if $G = (G_{j,l})$, $1 \le j,l \le 2(m-m_1)$, we have

$$G_{j,l}(\theta) = \sum_{k \in \mathbb{Z}^{r+m_1}} \frac{B^{**}_{j,l,k}}{ik^\top \omega + \hat{\lambda}^{(0)}_j + \hat{\lambda}^{(0)}_l} \exp(ik^\top \theta).$$

In this sum we have to avoid the indices (j,l,k) for which $|j-l| = m-m_1$ and $k=0$. In these cases we have trivial zero divisors, but also the coefficient $B^{**}_{j,l,0}$ is 0. Moreover, we remark that the matrix G is symmetric. Then, to bound G, we have to bound B^{**}. First, we have

$$\begin{aligned} \|B^* - \hat{\mathcal{B}}^{(0)*}\|_{\mathcal{E}^{(1)},\rho^{(0)}-2\delta^{(0)}} &\le \|B - \hat{\mathcal{B}}^{(0)*}\|_{\mathcal{E}^{(1)},\rho^{(0)}-2\delta^{(0)}} + \\ &+ (2(m-m_1)+1)(r+m_1)\frac{\|H^{(0)}_*\|_{\mathcal{E}^{(1)},\rho^{(0)},R^{(0)}}}{(R^{(0)})^4}\left(\|\xi\|_{\mathcal{E}^{(1)}} + \frac{\|d\|_{\mathcal{E}^{(1)},\rho^{(0)}-\delta^{(0)}}}{\delta^{(0)}\exp(1)}\right) + \\ &+ 24(m-m_1)^2\frac{\|H^{(0)}_*\|_{\mathcal{E}^{(1)},\rho^{(0)},R^{(0)}}}{(R^{(0)})^3}\|e\|_{\mathcal{E}^{(1)},\rho^{(0)}-\delta^{(0)}} \le \hat{N}(M^{(0)})^{1-6\alpha-\alpha\gamma}, \end{aligned}$$

and from the definition of B^{**} and the norm used, the same bound holds for B^{**}. Then,

$$\begin{aligned} \|G\|_{\mathcal{E}^{(1)},\rho^{(0)}-3\delta^{(0)}} &\le \left(\frac{1}{\alpha_1} + \left(\frac{\gamma}{\delta^{(0)}\exp(1)}\right)^\gamma \frac{1}{\mu^{(0)}}\right) 2(m-m_1)\,\|B^{**}\|_{\mathcal{E}^{(1)},\rho^{(0)}-2\delta^{(0)}} \le \\ &\le \hat{N}(M^{(0)})^{1-7\alpha-2\alpha\gamma}. \end{aligned}$$

(eq_5) The different components of F are given by

$$F_{j,l}(\theta) = \sum_{k \in \mathbb{Z}^{r+m_1}} \frac{E^*_{j,l,k}}{ik^\top \omega + \hat{\lambda}^{(0)}_l} \exp(ik^\top \theta),$$

for $j = 1, \ldots, r+m_1$ and $l = 1, \ldots, 2(m-m_1)$. Thus,

$$\begin{aligned} \|E^*\|_{\mathcal{E}^{(1)},\rho^{(0)}-2\delta^{(0)}} &\le \|E\|_{\mathcal{E}^{(1)},\rho^{(0)}} + 2(m-m_1)\|\mathcal{C}^{(0)}\|_{\mathcal{E}^{(1)},\rho^{(0)}} \frac{\|e\|_{\mathcal{E}^{(1)},\rho^{(0)}-\delta^{(0)}}}{\delta^{(0)}\exp(1)} + \\ &+ 4(m-m_1)(r+m_1)\frac{\|H^{(0)}_*\|_{\mathcal{E}^{(1)},\rho^{(0)},R^{(0)}}}{(R^{(0)})^5}\left(\|\xi\|_{\mathcal{E}^{(1)}} + \frac{\|d\|_{\mathcal{E}^{(1)},\rho^{(0)}-\delta^{(0)}}}{\delta^{(0)}\exp(1)}\right) + \\ &+ 8(m-m_1)^2\frac{\|H^{(0)}_*\|_{\mathcal{E}^{(1)},\rho^{(0)},R^{(0)}}}{(R^{(0)})^4}\|e\|_{\mathcal{E}^{(1)},\rho^{(0)}-\delta^{(0)}} \le \hat{N}(M^{(0)})^{1-7\alpha-\alpha\gamma}, \end{aligned}$$

and, hence,

$$\begin{aligned} \|F\|_{\mathcal{E}^{(1)},\rho^{(0)}-3\delta^{(0)}} &\le \left(\frac{2}{\alpha_1} + \left(\frac{\gamma}{\delta^{(0)}\exp(1)}\right)^\gamma \frac{1}{\mu^{(0)}}\right) 2(m-m_1)\,\|E^*\|_{\mathcal{E}^{(1)},\rho^{(0)}-2\delta^{(0)}} \le \\ &\le \hat{N}(M^{(0)})^{1-8\alpha-2\alpha\gamma}. \end{aligned}$$

We use these estimates to bound the transformed Hamiltonian $H^{(1)}$. For this purpose, we define $H^{(0)*} := \{H^{(0)}, S\} = H_1^{(0)*} + H_2^{(0)*}$, with

$$H_1^{(0)*} = \left\{\omega^\top I + \frac{1}{2}\hat{Z}^\top \hat{\mathcal{B}}^{(0)*}\hat{Z} + \frac{1}{2}I^\top \mathcal{C}^{(0)} I + H_*^{(0)}, S\right\},$$

and $H_2^{(0)*} = \{\hat{H}^{(0)}, S\}$. Note that we are splitting the contributions that are $O_1(\hat{H}^{(0)})$ and $O_2(\hat{H}^{(0)})$. Then, by construction of S, one has

$$H^{(0)} + H_1^{(0)*} = \phi^{(1)} + \omega^\top I + \frac{1}{2}\hat{Z}^\top \hat{\mathcal{B}}^{(1)*}\hat{Z} + \frac{1}{2}I^\top \mathcal{C}^{(1)}(\theta) I + H_*^{(1)},$$

with $\hat{\mathcal{B}}^{(1)*} = \mathcal{J}_{m-m_1}(\hat{\mathcal{B}}^{(1)*})$ and $< H_*^{(1)} >= 0$. Hence, $H^{(1)}$ takes the same form as $H^{(0)}$ in (1.60) if we define

$$\hat{H}^{(1)} = H^{(0)} \circ \Psi_1^S - H^{(0)} - H_1^{(0)*} = \int_0^1 \left(H_2^{(0)*} + (1-t)\{H_1^{(0)*}, S\}\right) \circ \Psi_t^S dt. \tag{1.70}$$

To bound the different terms of $H^{(1)}$, we use Lemma 1.6 to bound the Poisson brackets involved in the previous expressions:

$$\begin{aligned} \|H_1^{(0)*}\|_{\mathcal{E}^{(1)},\rho^{(0)}-4\delta^{(0)},R^{(0)}\exp(-\delta^{(0)})} &\le \hat{N}(M^{(0)})^{1-12\alpha-2\alpha\gamma}, \\ \|\{H_1^{(0)*}, S\}\|_{\mathcal{E}^{(1)},\rho^{(0)}-5\delta^{(0)},R^{(0)}\exp(-2\delta^{(0)})} &\le \hat{N}(M^{(0)})^{2-24\alpha-4\alpha\gamma}, \\ \|H_2^{(0)*}\|_{\mathcal{E}^{(1)},\rho^{(0)}-4\delta^{(0)},R^{(0)}\exp(-\delta^{(0)})} &\le \hat{N}(M^{(0)})^{2-12\alpha-2\alpha\gamma}. \end{aligned}$$

Hence, to bound $\hat{H}^{(1)}$ one only needs to control the effect of Ψ_t^S. To this end, we remark that from the bounds on the solutions of $(eq_1) - (eq_5)$, one has

$$\|\nabla S\|_{\mathcal{E}^{(1)},\rho^{(0)}-4\delta^{(0)},R^{(0)}} \le \hat{N}(M^{(0)})^{1-9\alpha-2\alpha\gamma}, \tag{1.71}$$

where ∇S is taken with respect to $(\theta, \hat{X}, I, \hat{Y})$. If we assume that

$$\|\nabla S\|_{\mathcal{E}^{(1)},\rho^{(0)}-4\delta^{(0)},R^{(0)}} \le (R^{(0)})^2\delta^{(0)}\exp(-1)/2, \tag{1.72}$$

then, Ψ_t^S is well defined from $\mathcal{D}_{r+m_1,m-m_1}(\rho^{(0)}-5\delta^{(0)}, R^{(0)}\exp(-\delta^{(0)}))$ to $\mathcal{D}_{r+m_1,m-m_1}(\rho^{(0)}-4\delta^{(0)}, R^{(0)})$, for any $-1 \le t \le 1$, and for any $\omega \in \mathcal{E}^{(1)}$ (this follows from Lemma 1.9 and (1.27)). More precisely, we have that

$$\|\Psi_t^S - Id\|_{\mathcal{E}^{(1)},\rho^{(0)}-5\delta^{(0)},R^{(0)}\exp(-\delta^{(0)})} \le \|\nabla S\|_{\mathcal{E}^{(1)},\rho^{(0)}-4\delta^{(0)},R^{(0)}}, \tag{1.73}$$

for any $-1 \le t \le 1$. From (1.71) we have that (1.72) holds if $\hat{N}(M^{(0)})^{1-12\alpha-2\alpha\gamma} \le 1$, condition that will follow immediately from the inductive restrictions. Applying the bounds (1.71), (1.71) and (1.73) to (1.70) and using Lemma 1.7, we deduce

$$\|\hat{H}^{(1)}\|_{\mathcal{E}^{(1)},\rho^{(0)}-6\delta^{(0)},R^{(0)}\exp(-3\delta^{(0)})} \le \hat{N}(M^{(0)})^{2-24\alpha-4\alpha\gamma}. \tag{1.74}$$

Moreover, the bound on $H_1^{(0)*}$ produces

$$\begin{array}{rcl} \|\phi^{(1)}-\phi^{(0)}\|_{\mathcal{E}^{(1)}} & \leq & \hat{N}(M^{(0)})^{1-12\alpha-2\alpha\gamma}, \\ \|\hat{\mathcal{B}}^{(1)*}-\hat{\mathcal{B}}^{(0)*}\|_{\mathcal{E}^{(1)}} & \leq & \hat{N}(M^{(0)})^{1-14\alpha-2\alpha\gamma}, \\ \|\mathcal{C}^{(1)}-\mathcal{C}^{(0)}\|_{\mathcal{E}^{(1)},\rho^{(0)}-4\delta^{(0)}} & \leq & \hat{N}(M^{(0)})^{1-16\alpha-2\alpha\gamma}, \\ \|H_*^{(1)}-H_*^{(0)}\|_{\mathcal{E}^{(1)},\rho^{(0)}-4\delta^{(0)},R^{(0)}\exp(-\delta^{(0)})} & \leq & \hat{N}(M^{(0)})^{1-12\alpha-2\alpha\gamma}. \end{array} \tag{1.75}$$

We take $N \geq 6$, and we define $\rho^{(1)} = \rho^{(0)} - N\delta^{(0)}$, and $R^{(1)} = R^{(0)}\exp(-(N-3)\delta^{(0)})$. Then, it is not difficult to rewrite the bounds on $H^{(1)}$ as the ones on $H^{(0)}$, but now on $\mathcal{D}_{r+m_1,m-m_1}(\rho^{(1)}, R^{(1)})$. To iterate this scheme, we only need to check that the bounds assumed on $H^{(0)}$ to define $\hat{N}$ still hold on $H^{(1)}$. This is done in the next section.

Convergence of the iterative scheme

Looking at the bounds of the previous section, we take $\alpha > 0$ small enough such that, for $s = 2(1-16\alpha-2\alpha\gamma)$, we have $s > 1$. Then, assuming $\hat{N} \geq 1$, we define $M^{(1)} = (\hat{N}M^{(0)})^s$ (note that this is a bound for the norm of $\hat{H}^{(1)}$ in (1.74)). If the hypotheses needed to iterate hold, we obtain recursively $M^{(n)} = (\hat{N}M^{(0)})^{s^n}$, and hence, for R small enough, we have $\lim_{n\to\infty} M^{(n)} = 0$. Let us define $\mathcal{E}^*(R)$ as the set of parameters ω for which all the steps are well defined. We assume that, for any $\omega \in \mathcal{E}^*(R)$, the composition of canonical transformations $\Psi^* = \Psi_1^{S^{(0)}} \circ \Psi_1^{S^{(1)}} \circ \ldots$ (being $S^{(n)}$ the generating function used at the n-step of the iterative procedure) is convergent. Then, the limit Hamiltonian $H^* = H^{(0)} \circ \Psi^*$ takes the form:

$$H^* = \phi^*(\omega) + \omega^\top I + \frac{1}{2}\hat{Z}^\top \hat{\mathcal{B}}^{**}(\omega)\hat{Z} + \frac{1}{2} I^\top \mathcal{C}^*(\theta,\omega) I + H_*^*(\theta,\hat{X},I,\hat{Y},\omega),$$

with $< H_*^* >= 0$. This is, we obtain for any $\omega \in \mathcal{E}^*$ a Hamiltonian with an $(r+m_1)$-dimensional reducible torus, with linear quasiperiodic flow given by ω.

Let us prove that the inductive bounds hold. First, we check that we can define, recursively, constants $\bar{m}^{(n)}$, $\hat{m}^{(n)}$, $\alpha_1^{(n)}$, $\alpha_2^{(n)}$ and $\hat{\nu}^{(n)}$, replacing the initial super-“(0)” ones, such that they are also bounded by $\bar{m}$, $\hat{m}$, α_1, α_2 and $\hat{\nu}$, respectively. To prove that, we note that the expressions in the right-hand side of (1.75) can be bounded by $(\hat{N}M^{(0)})^{s/2}$ (we remark that the same bound holds for (1.71)). Hence, iterating this bounds, we only need to use that the sum

$$\sum_{n\geq 0} \left(\hat{N}M^{(0)}\right)^{\frac{s^{n+1}}{2}}, \tag{1.76}$$

is convergent for R small enough (and in fact, that it goes to zero when R does), to justify these n-independent bounds. The same arguments can be used to prove that $\|\phi^*\|_{\mathcal{E}^*} < +\infty$. Here, we only check the bound $\bar{m}^{(n)} \leq \bar{m}$, because is the only one that does not follow directly: note that one can define

$$\bar{m}^{(1)} = \frac{\bar{m}^{(0)}}{1-\bar{m}^{(0)}(\hat{N}M^{(0)})^{s/2}},$$

and then, taking R small enough, we have $\bar{m}^{(1)} \le \bar{m}$. Hence, iterating this definition and assuming $\bar{m}^{(n)} \le \bar{m}$ by induction, we have

$$\bar{m}^{(n)} \le \bar{m}^{(0)} \prod_{j=0}^{n-1} \frac{1}{1 - \bar{m}(\hat{N} M^{(0)})^{s^{n+1}/2}}.$$

Under this inductive hypotesis, one can bound $\bar{m}^{(n)}$ by an infinite product that it is convergent because (1.76) does. From here, the bound $\bar{m}^{(n)} \le \bar{m}$ follows immediately for R small enough. Finally, with the inductive definitions $\rho^{(n+1)} = \rho^{(n)} - N\delta^{(n)}$ and $R^{(n+1)} = R^{(n)} \exp\left(-(N-3)\delta^{(n)}\right)$, $n \ge 0$, we need to check that $\rho^{(n)} \ge \rho_0/4$ and $R^{(n)} \ge M^{(n)}$. We remark that, as we take $\delta^{(n)} = (M^{(n)})^{\alpha}$, we have,

$$\sum_{n\ge 0} \delta^{(n)} \le (M^{(0)})^{\alpha} + \sum_{n \ge 1} (\hat{N} M^{(0)})^{s^n \alpha} \le 2(M^{(0)})^{\alpha}, \tag{1.77}$$

at least for R small enough. Then, as N will be a fixed number, the bound on $\rho^{(n)}$ is clear, taking R small enough. Moreover, we also have $R^{(n)} \ge R^{(0)} \exp\left(-\rho_0/4\right) > R^{(0)}/2 = M^{(0)} \ge M^{(n)}$. To justify this last inequality, we only need to take R small enough such that $M^{(1)} \le M^{(0)}$. Under this assumption, the sequence $\{M^{(n)}\}_{n\ge 0}$ is clearly decreasing.

Finally, to prove the well defined character of the limit Hamiltonian, it only remains to check the convergence of Ψ^*. To do that we write, for simplicity, $\Psi^{(n)} = \Psi_1^{S^{(n)}}$ and we define $\breve{\Psi}^{(n)} = \Psi^{(0)} \circ \ldots \circ \Psi^{(n)}$, for $n \ge 0$. We also put $\rho'_n = \rho^{(n)} - \rho_0/8$ and $R'_n = R^{(n)} \exp\left(-\rho_0/8\right)$, $n \ge 1$. Then, using in inductive form the bounds (1.73), (1.71) and (1.72), it is not difficult to check that from Lemma 1.8 we have

$$\begin{aligned} &\|\breve{\Psi}^{(n+1)} - \breve{\Psi}^{(n)}\|_{\mathcal{E}^*, \rho'_{n+2}, R'_{n+2}} \le \\ &\quad \le (1 + \hat{\Delta}(\hat{N} M^{(0)})^{\frac{s}{2} - 2\alpha}) \|\Psi^{(1)} \circ \ldots \circ \Psi^{(n+1)} - \Psi^{(1)} \circ \ldots \circ \Psi^{(n)}\|_{\mathcal{E}^*, \rho'_{n+2}, R'_{n+2}}, \end{aligned}$$

where $\hat{\Delta}$ only depends on r, m, m_1, ρ_0 and $\hat{N}$. Iterating this bound and taking α small enough, one obtains for R small enough

$$\|\breve{\Psi}^{(n+1)} - \breve{\Psi}^{(n)}\|_{\mathcal{E}^*, \rho'_{n+2}, R'_{n+2}} \le \prod_{j=0}^{n} (1 + \hat{\Delta}(\hat{N} M^{(0)})^{\frac{s^{j+1}}{2} - 2\alpha}) (\hat{N} M^{(0)})^{\frac{s^{n+2}}{2}} \le 2 (\hat{N} M^{(0)})^{\frac{s^{n+2}}{2}},$$

where we have used again the convergent character of the sum (1.76). From this bound, it is clear that if $p > q \ge 0$, then

$$\|\breve{\Psi}^{(p)} - \breve{\Psi}^{(q)}\|_{\mathcal{E}^*, \rho_0/8, R^{(0)} \exp\left(-3\rho_0/8\right)} \le \sum_{j \ge q} 2 (\hat{N} M^{(0)})^{\frac{s^{n+2}}{2}},$$

bound that goes to zero as $p, q \to +\infty$. This allows to check that the limit canonical transformation Ψ^* goes from $\mathcal{D}_{r+m_1, m-m_1}(\rho_0/8, R^{(0)} \exp\left(-3\rho_0/8\right))$ to $\mathcal{D}_{r+m_1, m-m_1}(\rho^{(0)}, R^{(0)})$.

Bounds on the measure

Then, we have shown the existence of real invariant reducible tori for a set of parameters $\omega \in \mathcal{E}^*$. It only remains to bound the measure of $\mathcal{E}^*$ or, equivalently, the measure of the complementary set. To do that, we start recalling how $\mathcal{E}^*$ is constructed. Iterating the definition of $\mathcal{E}^{(1)}$ from $\mathcal{E}^{(0)}$, we define $\mathcal{E}^{(n+1)}$ from $\mathcal{E}^{(n)}$ in the same way as it has been done in (1.69), replacing $\mu^{(0)} \equiv (M^{(0)})^\alpha$ by $\mu^{(n)} \equiv (M^{(n)})^\alpha$. Then, we have $\mathcal{E}^* = \cap_{n\geq 1}\mathcal{E}^{(n)}$. This is, $\mathcal{E}^*$ is constructed by taking out, in recursive form, the set of parameters ω for which the Diophantine conditions (1.69), formulated on the eigenvalues of the previous step and depending on the size of the remaining perturbative terms, do not hold. Then, the set of removed parameters can be obtained as union of sets for which one of those conditions is not satisfied at some step of the iterative process.

To estimate the size of the removed sets, we will use a Lipschitz condition with respect to ω for the different eigenvalues $\hat{\lambda}_j^{(n)}$ of $\mathcal{B}^{(n)*}$, for $n \geq 0$. To this end, we will prove that this kind of regularity holds for the successive transformed Hamiltonians. As this condition holds for the initial one, we have to check, by induction, that the canonical transformations used preserve this kind of dependence. The key point is to bound the Lipschitz constants of the different solutions of $(eq_1) - (eq_5)$. To do it, we recall that we have bounds like the ones of (1.68) for the Lipschitz constants of the different terms of the decomposition (1.60) of $H^{(0)}$. Then, we only have to prove that those bounds for the Lipschitz constants, can be iterated in the same way as the bounds on the norms. To see that, we can use the different results given in item (a) of Lemma 1.11 to bound the Lipschitz constants of the solutions of $(eq_1)-(eq_5)$. We remark that, for the denominators that appear solving these equations, we have

$$\mathcal{L}_{\mathcal{E}^{(0)}}\{ik^\top\omega + l^\top\hat{\lambda}^{(0)}\} \leq |k|_1 + \beta_1^{(0)}|l|_1.$$

Then, combining Lemma 1.11 with standard inequalities to bound the Lipschitz constants of sums and products, it is not difficult to check that one can iterate bounds of the following form:

$$\begin{aligned}
\mathcal{L}_{\mathcal{E}^{(1)},\rho^{(1)},R^{(1)}}\{\hat{H}^{(1)}\} &\leq \tilde{N}(M^{(0)})^{2s_1},\\
\mathcal{L}_{\mathcal{E}^{(1)},\rho^{(1)},R^{(1)}}\{\hat{\mathcal{B}}^{(1)*} - \hat{\mathcal{B}}^{(0)*}\} &\leq \tilde{N}(M^{(0)})^{s_1},\\
\mathcal{L}_{\mathcal{E}^{(1)},\rho^{(1)},R^{(1)}}\{\mathcal{C}^{(1)} - \mathcal{C}^{(0)}\} &\leq \tilde{N}(M^{(0)})^{s_1},\\
\mathcal{L}_{\mathcal{E}^{(1)},\rho^{(1)},R^{(1)}}\{H_*^{(1)} - H_*^{(0)}\} &\leq \tilde{N}(M^{(0)})^{s_1},
\end{aligned}$$

that are analogous to the ones of (1.74) and (1.75). $\tilde{N} \geq 1$ depends on the same parameters as $\hat{N}$, plus $\tilde{m}$, $\tilde{\nu}$ and β_1. Moreover, taking α small enough, we have $2s_1 > 1$. Here, the selection of N (used to define $\rho^{(1)}$ and $R^{(1)}$) is done depending on the number of times that we need to use Cauchy estimates to bound the different norms and Lipschitz constants. Iterating those expressions, it is not difficult to check (by induction) that we can define inductively $\tilde{m}^{(n)}$, $\tilde{\nu}^{(n)}$ and $\beta_1^{(n)}$ for which the assumed n-independent bounds hold. The deduction of those Lipschitz bounds is tedious but it only involves simple inequalities. For full details in a very similar context, we refer the lector to chapter 3 or to [35].

Let us particularize those bounds on the eigenvalues of $\mathcal{B}^{(n)*}$. If we expand $\hat{\lambda}_j^{(n)}$, $j = 1, \ldots, 2(m - m_1)$, $n \geq 0$, as in (1.66), replacing only the superscript "(0)" by "(n)", we have that $\mathcal{L}_{\mathcal{E}^{(n)}}\{\breve{\lambda}_j^{(n)}\} \leq \breve{N}R$, being $\breve{N}$ a positive constant independent from R, j and n. To justify this assertion, we note that it holds for $n = 0$, and that the contributions that come from the next steps are exponentially small with R.

Those bounds on the Lipschitz constants of $\lambda_j^{(n)}$ plus the nondegeneracy conditions (1.66) are the key to control the measure of $\mathcal{E}^{(n)} \setminus \mathcal{E}^{(n+1)}$. We consider the decomposition

$$\mathcal{E}^{(n)} \setminus \mathcal{E}^{(n+1)} = \bigcup_{\substack{l \in \mathbb{Z}^{2(m-m_1)} \\ 0<|l|_1\leq 2 \\ l_{\hat{X}} \neq l_{\hat{Y}}}} \bigcup_{k\in\mathbb{Z}^{r+m_1}\setminus\{0\}} \mathcal{R}_{l,k}^{(n)},$$

with

$$\mathcal{R}_{l,k}^{(n)}(R) = \left\{ \omega \in \mathcal{E}^{(n)}(R) : \; |ik^\top\omega + l^\top \hat{\lambda}^{(n)}(\omega)| < \frac{\mu^{(n)}(R)}{|k|_1^\gamma} \right\}.$$

To estimate the measure of $\mathcal{R}_{l,k}^{(n)}$, we take $\omega^{(1)}$ and $\omega^{(2)}$ in this set and then, we have

$$|ik^\top(\omega^{(1)} - \omega^{(2)}) + l^\top(\hat{\lambda}^{(n)}(\omega^{(1)}) - \hat{\lambda}^{(n)}(\omega^{(2)}))| < \frac{2\mu^{(n)}}{|k|_1^\gamma}.$$

Let us start with the case $|l|_1 = 1$. Then, $l^\top\hat{\lambda}^{(n)} = \hat{\lambda}_j^{(n)}$ for some $j = 1, \dots, 2(m-m_1)$. Hence, the previous expression can be rewritten as

$$|i(k+v_j)^\top(\omega^{(1)} - \omega^{(2)}) + \breve{\lambda}_j^{(n)}(\omega^{(1)}) - \breve{\lambda}_j^{(n)}(\omega^{(2)})| < \frac{2\mu^{(n)}}{|k|_1^\gamma}.$$

Assuming that $\omega^{(1)} - \omega^{(2)}$ is parallel to $k + \mathrm{Re}(v_j)$, we have

$$\begin{aligned} |\omega^{(1)} - \omega^{(2)}|_2 &= \frac{|(k+\mathrm{Re}(v_j))^\top(\omega^{(1)} - \omega^{(2)})|}{|k+\mathrm{Re}(v_j)|_2} \leq \frac{|(k+v_j)^\top(\omega^{(1)} - \omega^{(2)})|}{|k+\mathrm{Re}(v_j)|_2} \leq \\ &\leq \frac{1}{|k+\mathrm{Re}(v_j)|_2}\left(|\breve{\lambda}_j^{(n)}(\omega^{(1)}) - \breve{\lambda}_j^{(n)}(\omega^{(2)})| + \frac{2\mu^{(n)}}{|k|_1^\gamma} \right) \leq \\ &\leq \frac{1}{|k+\mathrm{Re}(v_j)|_2}\left(\breve{N}R|\omega^{(1)} - \omega^{(2)}| + \frac{2\mu^{(n)}}{|k|_1^\gamma} \right). \end{aligned}$$

being $|.|_2$ the Euclidean norm of a real vector. Using that $\mathrm{Re}(v_j) \neq 0$ (see (1.66)), we obtain that there exists a positive constant Π_1, independent from j, k and n, such that

$$|\omega^{(1)} - \omega^{(2)}|_2 \leq \Pi_1 \frac{\mu^{(n)}}{|k|_1^{\gamma+1}},$$

for R small enough. In fact, this bound can be extended to the case $|l|_1 = 2$, $l_x \neq l_y$, using that $\mathrm{Re}(v_{j_1,j_2}) \neq 0$ if $j_1 \neq j_2$. This is a bound for the width of a section of $\mathcal{R}_{l,k}^{(n)}$ by a line in the direction $k + \mathrm{Re}(v_j)$. Then, the measure of $\mathcal{R}_{l,k}^{(n)}$ can be bounded by

$$\mathrm{mes}(\mathcal{R}_{l,k}) \leq \Pi_1 \frac{\mu^{(n)}}{|k|_1^{\gamma+1}} \left(\sqrt{r+m_1}\frac{1}{4}(|\mathcal{C}^{-1}|)^{-1}R^2 \right)^{r+m_1-1},$$

where $2\sqrt{r+m_1}\frac{1}{8}(|\mathcal{C}^{-1}|)^{-1}R^2$ is a bound for the diameter of $\mathcal{E}^{(0)}(R)$. Then, we have

$$\mathrm{mes}(\mathcal{E}^{(n)} \setminus \mathcal{E}^{(n+1)}) \le \Pi_2 R^{2(r+m_1-1)} \mu^{(n)} \sum_{k\in\mathbb{Z}^{r+m_1}\setminus\{0\}} \frac{1}{|k|_1^{\gamma+1}},$$

where Π_2 does not depend on n and R. Using that $\#\{k \in \mathbb{Z}^{r+m_1} : \ |k|_1 = j\} \le 2(r+m_1)j^{r+m_1-1}$ and that $\gamma > r+m_1-1$ we obtain

$$\mathrm{mes}(\mathcal{E}^{(n)} \setminus \mathcal{E}^{(n+1)}) \le \Pi_2 R^{2(r+m_1-1)} \mu^{(n)} \sum_{j\ge 1} 2(r+m_1) j^{r+m_1-2-\gamma} \le \Pi_3 R^{2(r+m_1-1)} \mu^{(n)},$$

being Π_3 also independent from n and R. As $\mu^{(n)} = (M^{(n)})^\alpha$, we deduce, using (1.77), that for $R \le 1$ small enough,

$$\mathrm{mes}(\mathcal{E}^{(0)} \setminus \mathcal{E}^{*}) \le \Pi_3 R^{2(r+m_1-1)} \left((M^{(0)})^\alpha + \sum_{n\ge 1} (\hat{N} M^{(0)})^{s^n \alpha} \right) \le 2\Pi_3 (M^{(0)})^\alpha.$$

Taking into account the bound on the measure of $\mathcal{W}\left(\frac{1}{8}(|\mathcal{C}^{-1}|)^{-1}R^2\right) \setminus \mathcal{E}^{(0)}$ (we have shown, from (1.61), that it is of order $(M^{(0)})^{2\alpha}$), one obtains the exponentially small bounds on the measure of destroyed tori. To finish the proof, we define $\mathcal{A}$ as $\cup_{0<R\le R^*}\mathcal{E}^*(R)$, where R^* is the maximum value of R for which the iterative scheme converges.

1.5 Basic lemmas

In this section, we give some basics results used to bound the norms (1.12) and (1.13) and the related Lipschitz constants, as well as the expressions and transformations involved in the different proofs. Similar lemmas are used in Chapter 3.

Lemma 1.4 *Let $f(\theta)$ and $g(\theta)$ be analytic functions of r complex arguments defined on a strip of width $\rho > 0$, 2π-periodic on θ, and taking values in $\mathbb{C}$. Let us denote by f_k the Fourier coefficients of f, $f = \sum_{k\in\mathbb{Z}^r} f_k \exp(ik^\top\theta)$. Then, we have:*

(i) $|f_k| \le |f|_\rho \exp(-|k|_1\rho)$.

(ii) $|fg|_\rho \le |f|_\rho |g|_\rho$.

(iii) For every $0 < \delta < \rho$,

$$\left|\frac{\partial f}{\partial \theta_j}\right|_{\rho-\delta} \le \frac{|f|_\rho}{\delta \exp(1)}, \quad j = 1, \dots, r.$$

(iv) Let $\{d_k\}_{k\in\mathbb{Z}^r\setminus\{0\}} \subset \mathbb{C}$, with $|d_k| \ge \frac{\mu}{|k|_1^\gamma}$, for some $\mu > 0$ and $\gamma \ge 0$. If we assume that $\bar{f} = 0$, then, for any $0 < \delta < \rho$, we have that the function g defined as

$$g(\theta) = \sum_{k\in\mathbb{Z}^r\setminus\{0\}} \frac{f_k}{d_k} \exp(ik^\top\theta),$$

satisfies the bound

$$|g|_{\rho-\delta} \le \left(\frac{\gamma}{\delta \exp(1)}\right)^\gamma \frac{|f|_\rho}{\mu}.$$

All these bounds can be extended to the case in which f and g take values in $\mathbb{C}^{n_1}$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$.

Proof: Items *(i)* and *(ii)* are easily verified. Proofs of *(iii)* and *(iv)* follows immediately using (1.23). ■

Lemma 1.5 *Let $f(\theta,x,I,y)$ and $g(\theta,x,I,y)$ be analytic functions on $\mathcal{D}_{r,m}(\rho,R)$, and 2π-periodic on θ. Then,*

(i) If $f=\sum_{(l,s)\in\mathbb{N}^{2m}\times\mathbb{N}^r} f_{l,s}(\theta)z^l\hat{I}^s$, we have $|f_{l,s}|_\rho \le \frac{|f|_{\rho,R}}{R^{|l|_1+2|s|_1}}$.

(ii) $|fg|_{\rho,R}\le |f|_{\rho,R}|g|_{\rho,R}$.

(iii) For every $0<\delta<\rho$ and $0<\chi<1$, we have for $j=1,\dots,r$ and $k=1,\dots,2m$:

$$\left|\frac{\partial f}{\partial \theta_j}\right|_{\rho-\delta,R} \le \frac{|f|_{\rho,R}}{\delta\exp(1)}, \qquad \left|\frac{\partial f}{\partial I_j}\right|_{\rho,R\chi} \le \frac{|f|_{\rho,R}}{(1-\chi^2)R^2}, \qquad \left|\frac{\partial f}{\partial z_k}\right|_{\rho,R\chi} \le \frac{|f|_{\rho,R}}{(1-\chi)R}.$$

As in Lemma 1.4, all the bounds hold if f and g take values in $\mathbb{C}^{n_1}$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$.

Proof: The proof of *(i)* and *(ii)* is straightforward. *(iii)* is proved using item *(iii)* of Lemma 1.4 and applying Cauchy estimates to the function $\sum_{(l,s)\in\mathbb{N}^{2m}\times\mathbb{N}^r}|f_{l,s}|_\rho z^l\hat{I}^s$. ■

Lemma 1.6 *Let us consider $f(\theta,x,I,y)$ and $g(\theta,x,I,y)$ complex-valued functions, such that f and ∇g are analytic functions defined on $\mathcal{D}_{r,m}(\rho,R)$, 2π-periodic on θ. Then, for every $0<\delta<\rho$ and $0<\chi<1$, we have:*

$$|\{f,g\}|_{\rho-\delta,R\chi} \le \frac{r|f|_{\rho,R}}{\delta\exp(1)}\left|\frac{\partial g}{\partial I}\right|_{\rho-\delta,R\chi} + \frac{r|f|_{\rho,R}}{R^2(1-\chi^2)}\left|\frac{\partial g}{\partial \theta}\right|_{\rho-\delta,R\chi} + \frac{2m|f|_{\rho,R}}{R(1-\chi)}\left|\frac{\partial g}{\partial z}\right|_{\rho-\delta,R\chi}.$$

Remark 1.11 *If f has a finite Taylor expansion with respect to (I,z), the expressions in the bound of $|\{f,g\}|_{\rho-\delta,R\chi}$ that come from the Cauchy estimates on the derivatives of f with respect to I or z, can be replaced by bounds on the degree of the different Taylor expansions. Moreover, if f does not depend on θ, the first term on the bound can be eliminated. Similar comments can be extended to ∇g. This remark has been used in the proof of Lemma 1.2.*

Proof: It follows from Lemma 1.5. ■

Lemma 1.7 *Let us take $0<\rho_0<\rho$ and $0<R_0<R$, and let us consider analytic functions Θ, $\mathcal{I}$ with values in $\mathbb{C}^{r'}$, and $\mathcal{X}$, $\mathcal{Y}$ with values in $\mathbb{C}^{m'}$, all defined for $(\theta,x,I,y)\in\mathcal{D}_{r,m}(\rho_0,R_0)$, and 2π-periodic on θ. We assume that $|\Theta|_{\rho_0,R_0}\le\rho-\rho_0$, $|\mathcal{I}|_{\rho_0,R_0}\le R^2$, and that $|\mathcal{X}|_{\rho_0,R_0}$, $|\mathcal{Y}|_{\rho_0,R_0}$ are both bounded by R. Let $f(\theta^*,x^*,I^*,y^*)$ be a given (2π-periodic on θ^*) analytic function, defined on $\mathcal{D}_{r',m'}(\rho,R)$. If we introduce:*

$$F(\theta,x,I,y)=f(\theta+\Theta,\mathcal{X},\mathcal{I},\mathcal{Y}),$$

then, $|F|_{\rho_0,R_0}\le|f|_{\rho,R}$.

Proof: It can be directly checked expanding f in Taylor series as in Lemma 1.5, and using item *(ii)* of Lemmas 1.4 and 1.5. ■

Lemma 1.8 *Let us consider $\Theta^{(j)}$, $\mathcal{I}^{(j)}$, $\mathcal{X}^{(j)}$ and $\mathcal{Y}^{(j)}$, $j=1,2$, in the same conditions of the ones of Lemma 1.7, but with the following bounds: $|\Theta^{(j)}|_{\rho_0,R_0} \le \rho-\rho_0-\delta$, $|\mathcal{I}^{(j)}|_{\rho_0,R_0} \le R^2-\sigma$, and with $|\mathcal{X}^{(j)}|_{\rho_0,R_0}$, $|\mathcal{Y}^{(j)}|_{\rho_0,R_0}$ bounded by $R-\chi$, with $0<\delta<\rho-\rho_0$, $0<\sigma<R^2$ and $0<\chi<R$. Then, if one takes the function f of Lemma 1.7 to define*

$$F^{(j)}(\theta,x,I,y) = f(\theta+\Theta^{(j)},\mathcal{X}^{(j)},\mathcal{I}^{(j)},\mathcal{Y}^{(j)}),\ j=1,2,$$

one has $|F^{(1)}-F^{(2)}|_{\rho_0,R_0} \le \mathcal{K}|f|_{\rho,R}$, where if we put $\mathcal{Z}^\top = (\mathcal{X}^\top,\mathcal{Y}^\top)$, then

$$\mathcal{K} \equiv \frac{|\Theta^{(1)}-\Theta^{(2)}|_{\rho_0,R_0}}{\delta\exp(1)} + r'\frac{|\mathcal{I}^{(1)}-\mathcal{I}^{(2)}|_{\rho_0,R_0}}{\sigma} + \frac{1}{\chi}\sum_{j=1}^{2m'}|\mathcal{Z}_j^{(1)}-\mathcal{Z}_j^{(2)}|_{\rho_0,R_0}.$$

Proof: It follows from the same ideas used to prove Lemma 1.7. ■

Lemma 1.9 *Let $S(\theta,x,I,y)$ be a function defined on $\mathcal{D}_{r,m}(\rho,R)$, with $\rho>0$ and $R>0$, being ∇S analytic on $\mathcal{D}_{r,m}(\rho,R)$ and 2π-periodic on θ. If we assume that*

$$\left|\frac{\partial S}{\partial\theta}\right|_{\rho,R} \le R^2(1-\chi^2), \qquad \left|\frac{\partial S}{\partial I}\right|_{\rho,R} \le \delta, \qquad \left|\frac{\partial S}{\partial z}\right|_{\rho,R} \le R(1-\chi),$$

for certain $0<\chi<1$ and $0<\delta<\rho$, then one has

(a) $\Psi_t^S : \mathcal{D}_{r,m}(\rho-\delta,R\chi) \longrightarrow \mathcal{D}_{r,m}(\rho,R)$, for every $-1\le t\le 1$, where Ψ_t^S is the flow time t of the Hamiltonian system given by S.

(b) If one writes $\Psi_t^S - Id = (\Theta_t^S,\mathcal{X}_t^S,\mathcal{I}_t^S,\mathcal{Y}_t^S)$, then, for every $-1\le t\le 1$, we have that Θ_t^S, $\mathcal{Y}_t^S$ and $\mathcal{Z}_t^S = (\mathcal{X}_t^S,\mathcal{Y}_t^S)$ are analytic functions on $\mathcal{D}_{r,m}(\rho-\delta,R\chi)$, 2π-periodic on θ. Moreover, the following bounds hold:

$$\left|\Theta_t^S\right|_{\rho-\delta,R\chi} \le \left|\frac{\partial S}{\partial I}\right|_{\rho,R}, \qquad \left|\mathcal{I}_t^S\right|_{\rho-\delta,R\chi} \le \left|\frac{\partial S}{\partial\theta}\right|_{\rho,R}, \qquad \left|\mathcal{Z}_t^S\right|_{\rho-\delta,R\chi} \le \left|\frac{\partial S}{\partial z}\right|_{\rho,R}.$$

Proof: A similar result can be found in [13], where it is proved working with the supremum norm. The ideas are basically the same, but here we use Lemma 1.7 to bound the composition of functions. ■

Lemma 1.10 *Let $I^{(0)},I^{(1)}\in\mathbb{R}$, $L^2\le I^{(0)},I^{(1)}$, with $L>0$, and let us consider the functions $f_{I^{(j)}}(I)=\sqrt{I^{(j)}+I}$, $j=0,1$. Then, for every $0<M<L$, one has*

$$|f_{I^{(0)}}|_{0,M} \le \sqrt{I^{(0)}}\left(2-\sqrt{1-M^2/L^2}\right), \qquad |f_{I^{(0)}}-f_{I^{(1)}}|_{0,M} \le \frac{|I^{(0)}-I^{(1)}|}{2L\sqrt{1-M^2/L^2}},$$

where the different norms are taken on $\mathcal{D}_{1,0}(0,M)$.

Proof: As $f_{I^{(j)}}(I) = \sum_{k\geq 0} \binom{1/2}{k} (I^{(j)})^{-k+1/2} I^k$, one has

$$
\begin{aligned}
|f_{I^{(0)}}|_{0,M} &\leq \sqrt{I^{(0)}} \sum_{k\geq 0} \left|\binom{1/2}{k}\right| \left(\frac{M^2}{L^2}\right)^k = \sqrt{I^{(0)}} \left(1 - \sum_{k\geq 1} \binom{1/2}{k} (-1)^k \left(\frac{M^2}{L^2}\right)^k\right) = \\
&= \sqrt{I^{(0)}} \left(2 - \sqrt{1 - M^2/L^2}\right).
\end{aligned}
$$

Moreover, as $f_{I^{(0)}}(I) - f_{I^{(1)}}(I) = \sum_{k\geq 0} \binom{1/2}{k} \left((I^{(0)})^{-k+1/2} - (I^{(1)})^{-k+1/2}\right) I^k$, one obtains

$$
\begin{aligned}
|f_{I^{(0)}} - f_{I^{(1)}}|_{0,M} &\leq \sum_{k\geq 0} \left|\binom{1/2}{k}\right| \left|\frac{1}{2} - k\right| (L^2)^{-k-1/2} (M^2)^k |I^{(0)} - I^{(1)}| = \\
&= \frac{1}{L} \sum_{k\geq 1} k \binom{1/2}{k} (-1)^{k-1} \left(\frac{M^2}{L^2}\right)^{k-1} |I^{(0)} - I^{(1)}| = \\
&= \frac{-1}{L} \frac{d}{ds} \left(\sqrt{1-s}\right)\Big|_{s=\frac{M^2}{L^2}} |I^{(0)} - I^{(1)}|.
\end{aligned}
$$

■

In the following lemma we describe how to control the Lipschitz dependences related to the norms introduced in (1.12) and (1.13). For this purpose, we consider a fixed subset $\mathcal{E} \subset \mathbb{R}^n$ (for some n) and functions defined on $\mathcal{E}$.

Lemma 1.11 *We assume that $f(\theta, \varphi)$ and $g(\theta, x, I, y, \varphi)$ are, for any φ, analytic with respect to (θ, x, I, y) and 2π-periodic on the r complex arguments θ. We assume that for every $\varphi \in \mathcal{E}$, f is defined on a strip of width ρ and g is defined on $\mathcal{D}_{r,m}(\rho, R)$, respectively. Then, one has the following results:*

(a) (i) *If $f = \sum_{k\in\mathbb{Z}^r} f_k(\varphi) \exp(ik^\top\theta)$, then $\mathcal{L}_{\mathcal{E}}\{f_k\} \leq \mathcal{L}_{\mathcal{E},\rho}\{f\} \exp(-|k|_1\rho)$.*

(ii) *For every $0 < \delta < \rho$,*

$$
\mathcal{L}_{\mathcal{E},\rho-\delta}\left\{\frac{\partial f}{\partial \theta_j}\right\} \leq \frac{\mathcal{L}_{\mathcal{E},\rho}\{f\}}{\delta \exp(1)}, \; j = 1, \ldots, r.
$$

(iii) *Let $\{d_k(\varphi)\}_{k\in\mathbb{Z}^r\setminus\{0\}}$ be a set of complex-valued functions defined for $\varphi \in \mathcal{E}$. We assume that the following bounds hold:*

$$
|d_k(\varphi)| \geq \frac{\mu}{|k|_1^\gamma}, \quad \mathcal{L}_{\mathcal{E}}\{d_k\} \leq A + B|k|_1,
$$

for some $\mu > 0$, $\gamma \geq 0$, $A \geq 0$ and $B \geq 0$. We assume $\bar{f} = 0$ for every $\varphi \in \mathcal{E}$, and we consider the function $g(\theta, \varphi)$ defined from f and $\{d_k(\varphi)\}$ as in the item (iv) of Lemma 1.4. Then, for any $0 < \delta < \rho$, we have:

$$
\begin{aligned}
\mathcal{L}_{\mathcal{E},\rho-\delta}\{g\} &\leq \left(\frac{\gamma}{\delta \exp(1)}\right)^\gamma \frac{\mathcal{L}_{\mathcal{E},\rho}\{f\}}{\mu} + \left(\frac{2\gamma+1}{\delta \exp(1)}\right)^{2\gamma+1} \frac{\|f\|_{\mathcal{E},\rho}}{\mu^2} B + \\
&+ \left(\frac{2\gamma}{\delta \exp(1)}\right)^{2\gamma} \frac{\|f\|_{\mathcal{E},\rho}}{\mu^2} A.
\end{aligned}
$$

(b) (i) *If* $g = \sum_{(l,s)\in\mathbb{N}^{2m}\times\mathbb{N}^r} g_{l,s}(\theta,\varphi)z^l I^s$, *then* $\mathcal{L}_{\mathcal{E},\rho}\{g_{l,s}\} \leq \frac{\mathcal{L}_{\mathcal{E},\rho,R}\{g\}}{R^{|l|_1+2|s|_1}}$.

(ii) *For every* $0<\delta<\rho$ *and* $0<\chi<1$, *we have for* $j=1,\ldots,r$ *and* $k=1,\ldots,2m$:

$$\mathcal{L}_{\mathcal{E},\rho-\delta}\left\{\frac{\partial g}{\partial \theta_j}\right\} \leq \frac{\mathcal{L}_{\mathcal{E},\rho,R}\{g\}}{\delta \exp{(1)}}, \quad \mathcal{L}_{\mathcal{E},\rho,R\chi}\left\{\frac{\partial g}{\partial I_j}\right\} \leq \frac{\mathcal{L}_{\mathcal{E},\rho,R}\{g\}}{(1-\chi^2)R^2},$$

$$\mathcal{L}_{\mathcal{E},\rho,R\chi}\left\{\frac{\partial g}{\partial z_k}\right\} \leq \frac{\mathcal{L}_{\mathcal{E},\rho,R}\{g\}}{(1-\chi)R}.$$

Proof: It can be immediately verified, using the same ideas as in Lemmas 1.4 and 1.5, plus standard inequalities for the Lipschitz dependence. ■

Chapter 2

Numerical Computation of Normal Forms around Periodic Orbits of the RTBP

2.1 Introduction

In this chapter will focus on the problem of computing the normal form around a linearly stable periodic orbit of an autonomous analytic Hamiltonian system with 3 degrees of freedom. This numerical approach can be considered an application of the results of Chapter 1. The main objective is to describe a methodology that allows to bound the diffusion velocity around the orbit, to generalize the standard computation of the Birkhoff normal form around an elliptic fixed point to an elliptic orbit of a Hamiltonian system.

The stability of the Trojan asteroids is a classical example of this kind. A first model for this problem is provided by the Restricted Three Body Problem (RTBP), where the problem boils down to estimate the speed of diffusion around an elliptic equilibrium point of a 3 degrees of freedom autonomous Hamiltonian system. In order to produce good estimates, it is necessary to compute numerically the normal form around the point, up to some finite order (see [69] and also [10] for a slightly different approach). This allows to derive much better estimates than the ones obtained by only using purely theoretical methods. This has also been extended to consider time-dependent periodic perturbations, in a very natural way: in a first step one computes the periodic orbit that replaces the equilibrium point and, by means of a translation, one puts it at the origin. Now, a single linear (and periodic) change of variables removes the time dependence at first order, and then, the methodology above can be extended without major problems (see [26], [34], [71]).

One can think that in the case of a periodic orbit the problem can be solved in the same way as for periodically perturbed Hamiltonian systems, that is, to bring the orbit at the origin and to apply a Floquet transformation. The main difficulty of this method comes from the following fact: due to the symplectic structure of the problem, the monodromy matrix around the periodic orbit has, at least, two eigenvalues equal to 1. This implies that the reduced Floquet matrix is going to have two zero eigenvalues, and this does not allow to continue with the normal form process.

For this reason, here we have taken a different approach. As we are going to work around a non-degenerate elliptic periodic orbit, we expect that the monodromy matrix has the following structure (may be after a linear change of variables): a 2-dimensional Jordan box with two eigenvalues 1, plus two couples of conjugate eigenvalues of modulus 1, all different. Under some generical conditions of non-resonance and non-degeneracy, we have that a) the Jordan box expands, in the complete system, a one parametric family of periodic orbits, b) each couple of conjugate eigenvalues expand a Cantorian family of 2-dimensional invariant tori, and c) if we consider the excitation coming from both elliptic directions, we obtain a Cantorian family of 3-dimensional invariant tori (see Section 1.4, [17] or [8], for the proofs). Hence, we will use suitable coordinates for this structure: we will introduce an angular variable (θ) as coordinate along the initial orbit, and a symplectically conjugate action variable (I). For the normal directions we will simply apply the procedure used for the examples mentioned above: we will translate the orbit to the origin and we will perform a complex Floquet change to remove the dependence on the angle of the normal variational equations, and to put them in diagonal form (by means of a complex change of coordinates). Denoting by ω_0 the frequency of the selected periodic orbit and by $\omega_{1,2}$ the two normal frequencies, the Hamiltonian will take the form

$$H(\theta, q, I, p) = \omega_0 I + i\omega_1 q_1 p_1 + i\omega_2 q_2 p_2 + \cdots,$$

suitable to start the normal form process.

These ideas have been applied to a concrete example coming from the Restricted Three Body Problem. The selected periodic orbit belongs to the Lyapunov family associated to the vertical oscillation of the equilibrium point L_5. The mass parameter μ is chosen big enough such that L_5 is unstable, but not too big in order to have the selected orbit normally elliptic (see Section 2.3.2). The first changes of variables are computed taking advantage of the particularities of this concrete model, but they can be extended to similar problems. The normal form is then computed by a standard recurrent procedure (based on Lie series) up to order 16. Bounding the remainder of this approximate normal form allows to derive bounds on the diffusion time around the orbit, and from the normal form we can easily obtain (approximate) periodic orbits (belonging to the previously mentioned Lyapunov family) as well as invariant tori of dimensions 2 and 3, that as has been mentioned before, generalize the linear oscillations around the orbit.

The computations have been done using formal expansions for the involved series, but with numerical coefficients. The algebraic manipulators needed have been written from scratch by the authors, using C.

To end this section, we comment how this chapter is organized. In Section 2.2 we present a general (and formal) formulation of the normal form methodology, that is directly adapted to the case of a linearly stable periodic orbit of a Hamiltonian system with three degrees of freedom, but that can be used (slighty modificated) in some other different contexts (as the study of systems with more than three degrees of freedom, or around periodic orbits with some hyperbolic directions). This formulation has as a reference point the objective to obtain bounds for the diffusion speed around the orbit. Section 2.3 contains the application to the RTBP.

2.2 Methodology

We consider a real analytic Hamiltonian system with three degrees of freedom given by

$$H(X,Y,Z,P_X,P_Y,P_Z) \tag{2.1}$$

where X, Y and Z are the positions and P_X, P_Y and P_Z the conjugate momenta. Let $(f(\theta),g(\theta))$ be a 2π-periodic parametrization of an elliptic periodic orbit of the system, with period $\frac{2\pi}{\omega_0}$. Here, $f=(f_1,f_2,f_3)$ and $g=(g_1,g_2,g_3)$ are real analytic functions with the normalization $f_3(0)=0$. Now we will assume the following condition (that will be satisfied by the selected example):

Condition **C**: the projection of the orbit into the coordinates (Z,P_Z) is a simple curve close to a circle.

Note that this may not be directly satisfied by a generic exmple. The reason we have use it is that it really simplifies the computations. In cases when it is not satisfied one should try to introduce changes of variables in order to obtain such condition. For instance, this is always possible if we are dealing with Lyapunov orbits not too far from an equilibrium point. Condition **C** implies that one can write

$$f_3(\theta)=A\sin(\theta)+\hat{f}_3(\theta), \quad g_3(\theta)=A\cos(\theta)+\hat{g}_3(\theta), \tag{2.2}$$

where $|\hat{f}_3(\theta)|$ and $|\hat{g}_3(\theta)|$ are small on the set $|\mathrm{Im}(\theta)|\le\rho$, for some $\rho>0$ (this will be stated rigorously in Section 2.2.2). Moreover, without loss of generality, we can assume $A>0$. Then, the function

$$\Delta(\theta)=(f_3'(\theta))^2+(g_3'(\theta))^2, \tag{2.3}$$

is always positive and "close" to the non-zero constant A^2. The non-vanishing character of Δ is necessary for technical reasons. It is used in Section 2.2.1 to define the (canonical) transformation (2.16). Thus, condition **C** is needed to guarantee the diffeomorphic character of this transformation.

Some notations

Before to continue with the formal description of the methodology, let us give some notation to be used in the next sections. As it has been mentioned before, we will introduce a new set of variables (θ,q,I,p) to describe a neighbourhood of the periodic orbit. For functions depending on these variables, we will use the following notations.

If $f(\theta,q,I,p)$ is an analytic function, we expand it as

$$f(\theta,q,I,p)=\sum_{k,l,m} f_{k,l,m}(\theta)I^k q^l p^m, \tag{2.4}$$

being $f_{k,l,m}(\theta)$ an analytic 2π-periodic function, that can also be expanded as

$$f_{k,l,m}(\theta)=\sum_{s} f_{k,l,m,s}\exp(is\theta), \tag{2.5}$$

where $i = \sqrt{-1}$. In those sums, the indices k, l, m and s range on $\mathbb{N}$, $\mathbb{N}^2$, $\mathbb{N}^2$ and $\mathbb{Z}$ respectively. We also introduce the following non-standard definition of degree for a monomial $f_{k,l,m}(\theta)I^k q^l p^m$, to be used along the chapter:

$$\deg\,(I^k q^l p^m) = 2k + |l|_1 + |m|_1, \tag{2.6}$$

where $|a|_1 = \sum_j |a_j|$. The reason for counting twice the degree of I will be explained in Section 2.2.5.

Let us now assume that $f(\theta, q, I, p)$ is defined on the complex domain

$$\mathcal{D}(\rho, R) = \{(\theta, q, I, p) \in \mathbb{C}^6 \,:\, |\mathrm{Im}(\theta)| \le \rho, |I| \le R_0, |q_j| \le R_j, |p_j| \le R_{j+2}, j = 1, 2\}, \tag{2.7}$$

where $R = (R_0, R_1, R_2, R_3, R_4)$. Then, we introduce the norms

$$\|f_{k,l,m}\|_\rho = \sup_{|\mathrm{Im}(\theta)|\le\delta} |f_{k,l,m}(\theta)|, \tag{2.8}$$

and

$$\|f\|_{\rho,R} = \sup_{\mathcal{D}(\rho,R)} |f(\theta, q, I, p)|. \tag{2.9}$$

We note that the explicit computation of these norms for a given function can be difficult, but they can be bounded by the following norms,

$$|f_{k,l,m}|_\rho = \sum_s |f_{k,l,m,s}| \exp\,(|s|\rho), \tag{2.10}$$

and

$$|f|_{\rho,R} = \sum_{k,l,m} |f_{k,l,m}|_\rho R_0^k R_1^{l_1} R_2^{l_2} R_3^{m_1} R_4^{m_2}, \tag{2.11}$$

that are easier to control. Moreover, using standard Cauchy estimates on f, we have for the coefficients of the expansion (2.4) that,

$$\|f_{k,l,m}\|_\rho \le \frac{\|f\|_{\rho,R}}{R_0^k R_1^{l_1} R_2^{l_2} R_3^{m_1} R_4^{m_2}}, \tag{2.12}$$

and, if $0 < \rho_0 < \rho$, that

$$\left\|\frac{\partial f_{k,l,m}}{\partial\theta}\right\|_{\rho-\rho_0} \le \frac{\|f_{k,l,m}\|_\rho}{\rho_0}. \tag{2.13}$$

2.2.1 Adapted coordinates

The initial system of (Cartesian) coordinates (X, Y, Z, P_X, P_Y, P_Z) is not a suitable system of reference to describe the dynamics around the periodic orbit. As it has been mentioned in the Introduction, the natural system of reference should contain an angular variable describing the orbit. Hence, we want to replace the coordinates of (2.1) by a new system of canonical coordinates $(\theta, q, I, p) = (\theta, q_1, q_2, I, p_1, p_2)$, with a real analytic transformation, depending on θ in a 2π-periodic way. The change has to satisfy that the periodic orbit must correspond to the set $q = p = 0$ and $I = 0$.

To construct this change, we take advantage on the hypothesis that Δ is different from zero. To give this change explicitly, let us start by defining the function

$$\sigma(\theta) = \frac{g_3'(\theta) f_3''(\theta) - g_3''(\theta) f_3'(\theta)}{\Delta(\theta)^2},$$

and let $\alpha(\theta, s)$ be the only solution of

$$s = \alpha(\theta, s) + \frac{1}{2}\sigma(\theta)\alpha(\theta, s)^2, \tag{2.14}$$

such that $\alpha(\theta, 0) = 0$. Then, if we denote by F the function

$$F(q, \theta, p, I) = I + \sum_{j=1}^{2} (g_j'(\theta) q_j - f_j'(\theta) p_j), \tag{2.15}$$

the change is given by

$$\begin{array}{rclrcl} X & = & f_1(\theta) + q_1, & P_X & = & g_1(\theta) + p_1, \\ Y & = & f_2(\theta) + q_2, & P_Y & = & g_2(\theta) + p_2, \\ Z & = & f_3(\theta) - \frac{g_3'(\theta)}{\Delta(\theta)}\alpha(\theta, F(q, \theta, p, I)), & P_Z & = & g_3(\theta) + \frac{f_3'(\theta)}{\Delta(\theta)}\alpha(\theta, F(q, \theta, p, I)). \end{array} \tag{2.16}$$

Now we are going to prove that this is a canonical transformation and, in Section 2.2.2, we will show that if $|\hat{f}_3|$ and $|\hat{g}_3|$ are small enough (see (2.2), then (2.16) is a diffeomorphism from a complex domain in the variables (θ, q, I, p) to a (complex) neighbourhood of the periodic orbit.

Lemma 2.1 *The transformation* (2.16) *is symplectic.*

Proof: Let us consider the (formal) generating function $S(\theta, q_1, q_2, P_X, P_Y, P_Z)$ given by

$$\begin{array}{rcl} S & = & S_0(\theta) + (f_1(\theta) + q_1)(P_X - g_1(\theta)) + (f_2(\theta) + q_2)(P_Y - g_2(\theta)) + \\ & & + S_1(\theta)(P_Z - g_3(\theta)) + S_2(\theta)(P_Z - g_3(\theta))^2, \end{array}$$

with

$$\begin{array}{rcl} S_0'(\theta) & = & f_1(\theta) g_1'(\theta) + f_2(\theta) g_2'(\theta) + f_3(\theta) g_3'(\theta), \\ S_1(\theta) & = & f_3(\theta), \\ S_2(\theta) & = & -\dfrac{g_3'(\theta)}{2 f_3'(\theta)}. \end{array}$$

Let us see that (2.16) is obtained from the relations

$$I = \frac{\partial S}{\partial \theta}, \quad p_1 = \frac{\partial S}{\partial q_1}, \quad p_2 = \frac{\partial S}{\partial q_2}, \quad X = \frac{\partial S}{\partial P_X}, \quad Y = \frac{\partial S}{\partial P_Y}, \quad Z = \frac{\partial S}{\partial P_Z}.$$

From $p_1 = \frac{\partial S}{\partial q_1}$, $p_2 = \frac{\partial S}{\partial q_2}$, $X = \frac{\partial S}{\partial P_X}$ and $Y = \frac{\partial S}{\partial P_Y}$, we easily obtain the expressions for X, Y, P_X and P_Y in (2.16). Moreover, putting $I = \frac{\partial S}{\partial \theta}$ in (2.15) one obtains

$$F(q, \theta, p, I) = f_3'(P_Z - g_3) - \frac{g_3'' f_3' - g_3' f_3''}{2(f_3')^2}(P_Z - g_3)^2 + \frac{(g_3')^2}{f_3'}(P_Z - g_3),$$

and from $Z = \frac{\partial S}{\partial P_Z}$ we derive

$$f_3'(Z - f_3) = -g_3'(P_Z - g_3).$$

Finally, from the last two equations it is not difficult to obtain the expressions of the change for Z and P_Z.

In principle, this is only a formal construction (note that $S_2(\theta)$ has singularities), but we remark that the transformation (2.16) is well defined around the whole periodic orbit. A more rigorous (and much more tedious) verification, without any singularity, can be obtained by directly checking the symplectic character of the differential of the change. ■

For further uses, let us denote by $\mathcal{F}(q, p, \theta, I)$ the transformation

$$\mathcal{F} : (q, p, \theta, I) \to (X, Y, Z, P_X, P_Y, P_Z). \tag{2.17}$$

2.2.2 Bounds on the domain of definition of the adapted coordinates

Now we are going to give conditions to ensure the diffeomorphic character of the canonical transformation (2.16) in a neighbourhood of the periodic orbit. These conditions have to be explicit enough to be applied in a practical example. More concretely, what we want to show is that if $\hat{f}_3$ and $\hat{g}_3$ are small enough, then the transformation $\mathcal{F}$ of (2.17) is invertible from a complex neighbourhood of $\mathrm{Im}(\theta) = 0$, $I = 0$, $q = 0$ and $p = 0$, to a complex neighbourhood of the periodic orbit.

The main difficulty is the presence of the angular variable θ. It turns the study of the injectivity of (2.16) into a "global" problem (we want injectivity around the whole periodic orbit), instead of the classical local formulation (injectivity around a fixed point).

To check this global injectivity, we will use the following construction. First, we note that for a fixed θ the correspondence $(q, p) \to (X, Y, P_X, P_Y)$ is clearly bijective. Hence, to check the diffeomorphic character of $\mathcal{F}$, what we have to prove is that the correspondence $(\theta, F) \to (Z, P_Z)$ is also injective. For this purpose, we consider an auxiliar transformation of the form

$$(\theta, \beta) \to (x, y), \quad x = f(\theta) - g'(\theta)\beta, \quad y = g(\theta) + f'(\theta)\beta, \tag{2.18}$$

with $f(\theta) = A\sin(\theta) + \hat{f}(\theta)$ and $g(\theta) = A\cos(\theta) + \hat{g}(\theta)$, where A is a positive number, and $\hat{f}$, $\hat{g}$ are arbitrary real analytic functions, 2π-periodic on θ. We note that replacing $\frac{\alpha}{\Delta}$ by β in the expressions of Z and P_Z, we have that the correspondence $(\theta, \beta) \to (Z, P_Z)$ is analogous to (2.18). We will show that, if $\hat{f}$ and $\hat{g}$ are small (enough) functions, and we consider values of θ in a complex neighbourhood of $\mathbb{T}^1$ and of β in a complex neighbourhood of $\beta = 0$, both small enough, then (2.18) is injective. This result is contained in the next proposition.

Proposition 2.1 *With the notations given above, if we consider a fixed* $0 < R_\beta < 1/2$, *and we take* $0 < \rho < 1$ *such that the* $|.|_\rho$ *norms (see* (2.10)*) of* $\hat{f}$, $\hat{g}$, $\hat{f}'$, $\hat{g}'$, $\hat{f}''$ *and* $\hat{g}''$ *are small enough (condition depending only on* A *and* R_β*), then there exist positive values* $\delta_0(R_\beta, \rho)$ *and* $\delta_1(R_\beta, \rho)$*, such that if* $\delta_0 > \delta_1$*, then the transformation* (2.18) *is injective on* $|\mathrm{Im}\theta| \le \rho$ *and* $|\beta| \le R_\beta$.

The proof of this proposition is contained in Section 2.4.

Remark 2.1 *Admissible values of δ_0 and δ_1 are explicitly constructed during the proof of the proposition. They verify that $\delta_1 \to 0$ and δ_0 is bounded away from zero, when the $|.|_\rho$ norms displayed above go to zero.*

Remark 2.2 *Of course, Proposition 2.1 is which motivates condition* **C** *on the initial periodic orbit. Without this assumption, to invert (2.18) can be very difficult.*

Then, assuming $\hat{f}_3$ and $\hat{g}_3$ small enough, we deduce from this result that the correspondence $(\theta, \beta) \to (Z, P_Z)$ is injective if $|\mathrm{Im}(\theta)| \leq \rho$ and $|\beta| \leq R_\beta$, for some small $\rho > 0$ and $R_\beta > 0$.

The next step is to invert the correspondence $(\theta, .) \to \alpha(\theta, .)$ of (2.14). We note that α is explicitly given by

$$\alpha(\theta, s) = \sum_{k \geq 1} 2^k \binom{1/2}{k} (\sigma(\theta) s)^{k-1} s,$$

and hence, it is well defined if (θ, s) verifies $|s| < \frac{1}{2|\sigma(\theta)|}$. Moreover, if we take a fixed θ, and values $s_0 \neq s_1$ with the previous restriction, we deduce from (2.14) that $\alpha(\theta, s_0) \neq \alpha(\theta, s_1)$. As we are also interested in the size of α, we remark that if $|s| < \frac{1}{2|\sigma(\theta)|}$, then we can write $|\alpha(\theta, s)| \leq \frac{1}{|\sigma(\theta)|} \left(1 - (1 - 2|s||\sigma(\theta)|)^{1/2}\right)$. This bound is an increasing function of $|\sigma(\theta)|$ and $|s|$.

Finally, from all these remarks, we deduce that if we consider values for (θ, q, I, p) belonging in $\mathcal{D}(\rho, R)$, with all the components of the vector R small enough such that

$$|F(q, \theta, p, I)| < \sup_{|\mathrm{Im}(\theta)| \leq \rho} \left\{ \frac{1}{2|\sigma(\theta)|} \right\}, \tag{2.19}$$

and

$$\frac{|\alpha(\theta, F(q, \theta, p, I))|}{|\Delta(\theta)|} \leq R_\beta, \tag{2.20}$$

then, we can guarantee that the transformation $\mathcal{F}$ of (2.17) is a diffeomorphism from this domain to a neighbourhood of the periodic orbit.

2.2.3 Floquet transformation

If we rewrite the Hamiltonian (2.1) in the adapted coordinates (2.16), it takes the form:

$$H(\theta, q, I, p) = h_0 + \omega_0 I + \frac{1}{2} \left(q^\top, p^\top\right) A(\theta) \begin{pmatrix} q \\ p \end{pmatrix} + \sum_{j \geq 3} H_j(\theta, q, I, p), \tag{2.21}$$

where we keep, for simplicity, the name H for the transformed Hamiltonian. Here, h_0 is the energy level of the periodic orbit, $A(\theta)$ is a symmetric matrix (2π-periodic on θ) and the terms H_j are homogeneous polynomials of degree j (see (2.6)). The next step is to remove the angular dependence of A on θ, that is, to reduce the normal variational equations of the orbit to constant coefficients. So, we will perform a canonical change of variables, linear with respect to (q,p) and depending 2π-periodically on θ, such that it reduces A to constant coefficients (this is, a Floquet transformation). As the initial Hamiltonian is real, we would like to use a real Floquet transformation. This is not possible in general (it is well known that one can be forced to double the period to obtain a real change) but it can be done in some particular situations. In the case we are considering (reducibility around a periodic orbit of a Hamiltonian system), the change can be selected to be real if, for instance, the projection of the monodromy matrix associated to the orbit into their normal directions diagonalize without any negative eigenvalue. Note that this hold on any elliptic periodic orbit under the assumption of different normal eigenvalues.

The variational flow

Let $\Psi(t)$ be the variational matrix, $\Psi(0) = Id_6$, of the periodic orbit for the initial Hamiltonian system (2.1). Then, $\Phi(t) = (D\mathcal{F}(0,0,\omega_0 t,0))^{-1}\Psi(t)D\mathcal{F}(0,0,0,0)$ is the variational matrix of the orbit for the Hamiltonian system (2.21), that is, for the system expressed in the variables (q,p,θ,I). We note that the variational equations in these variables are given by:

$$\dot{\Phi} = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$J_4 A$}} & 0 & \partial^2_{pI}H \\ & & 0 & -\partial^2_{qI}H \\ \partial^2_{Iq}H & \partial^2_{Ip}H & 0 & \partial^2_{II}H \\ 0 & 0 & 0 & 0 \end{pmatrix} \Phi, \qquad (2.22)$$

where the matrix of this linear system is evaluated on the periodic orbit, $\theta(t) = \omega_0 t$, $I(t) = 0$ and $p(t) = q(t) = 0$. Here, $\partial^2_{uv}H$ denotes the matrix of partial derivatives $\left(\frac{\partial^2 H}{\partial u_j \partial v_k}\right)_{j,k}$, and J_4 is the matrix of the canonical 2-form of $\mathbb{C}^4$. Let $\tilde{\Phi}(t)$ be the 4-dimensional matrix obtained by taking the first 4 rows and colums of $\Phi(t)$ (it corresponds to the variational flow in the normal directions of the periodic orbit). If we use the notation $\Phi = \frac{\partial(q,p,\theta,I)}{\partial(q^0,p^0,\theta^0,I^0)}$, we can identify $\tilde{\Phi} = \frac{\partial(q,p)}{\partial(q^0,p^0)}$. Then, we obtain from (2.22) that

$$\dot{\tilde{\Phi}} = J_4 A(\omega_0 t)\tilde{\Phi} + \left.\begin{pmatrix} \partial^2_{pI}H \\ -\partial^2_{qI}H \end{pmatrix}\right|_{(\omega_0 t,0,0,0)} \frac{\partial I}{\partial(q^0,p^0)}.$$

Moreover, from the last row of the matrix of (2.22), we have that $\frac{d}{dt}\left(\frac{\partial I}{\partial(q^0,p^0)}\right) = 0$, and hence, from the initial conditions for Φ at $t=0$, we deduce that $\left(\frac{\partial I}{\partial(q^0,p^0)}\right)(t) = 0$. So, the matrix $\tilde{\Phi}$ is the solution of the linear periodic system

$$\dot{\tilde{\Phi}} = J_4 A(\omega_0 t)\tilde{\Phi}, \quad \tilde{\Phi}(0) = Id_4. \qquad (2.23)$$

As we are interested in a numerical implementation of this method, we remark that it is not difficult to check that

$$DF(0,0,\omega_0 t,0) = \begin{pmatrix} 1 & 0 & 0 & 0 & f_1' & 0 \\ 0 & 1 & 0 & 0 & f_2' & 0 \\ -\frac{g_1' g_3'}{\Delta} & -\frac{g_2' g_3'}{\Delta} & \frac{f_1' g_3'}{\Delta} & \frac{f_2' g_3'}{\Delta} & f_3' & -\frac{g_3'}{\Delta} \\ 0 & 0 & 1 & 0 & g_1' & 0 \\ 0 & 0 & 0 & 1 & g_2' & 0 \\ \frac{g_1' f_3'}{\Delta} & \frac{g_2' f_3'}{\Delta} & -\frac{f_1' f_3'}{\Delta} & -\frac{f_2' f_3'}{\Delta} & g_3' & \frac{f_3'}{\Delta} \end{pmatrix},$$

being all the components of the matrix evaluated on $\theta = \omega_0 t$. Moreover, we also remark that, to compute the matrix $(D\mathcal{F}(0,0,\omega_0 t,0))^{-1}$, we can use that it is symplectic with respect to the 2-form $\widetilde{J}_6$,

$$\widetilde{J}_6 = \begin{pmatrix} J_4 & 0 \\ 0 & J_2 \end{pmatrix}.$$

The change of variables

Now, let us introduce $\mathcal{C} = \widetilde{\Phi}\left(\frac{2\pi}{\omega_0}\right)$, the monodromy matrix of the normal variational equations (2.23). From the assumed linearly stable character of the initial orbit, we have that $\mathcal{C}$ has four different eigenvalues of modulus 1, that is, eigenvalues of the form $\exp(i\alpha_j)$ and $\exp(-i\alpha_j)$, for $j = 1, 2$, with $\alpha_j \in \mathbb{R}$. To compute them, we can use that (as $\mathcal{C}$ is a symplectic matrix) the characteristic polynomial of $\mathcal{C}$ takes the form $Q(\lambda) = \lambda^4 - a\lambda^3 + b\lambda^2 - a\lambda + 1$, being $a = tr_1(\mathcal{C})$ (the trace of $\mathcal{C}$), and $b = tr_2(\mathcal{C})$ (that is, the sum of the main minors of order 2 of $\mathcal{C}$). From these expressions, we obtain the following relations:

$$\begin{aligned} a &= 2\cos(\alpha_1) + 2\cos(\alpha_2), \\ b &= 2 + 4\cos(\alpha_1)\cos(\alpha_2). \end{aligned}$$

Hence, $\cos(\alpha_1)$ and $\cos(\alpha_2)$ are the solutions for c of the quadratic equation $4c^2 - 2ac + b - 2 = 0$. Let $w^{(j)} = u^{(j)} + iv^{(j)}$ be non-zero eigenvectors of $\mathcal{C}$ associated to the eigenvalues $\exp(i\alpha_j)$, for $j = 1, 2$. From the symplectic character of $\mathcal{C}$ with respect to J_4, we have that $w^{(1)\top} J_4 w^{(2)} = w^{(1)\top} J_4 \bar{w}^{(2)} = 0$, where the bar denotes the complex conjugation. Moreover, from the non-degenerate character of J_4, we also have that $w^{(1)\top} J_4 \bar{w}^{(1)} \neq 0$ (in fact, this is a purely imaginary number). Hence, if we introduce C the matrix that has as columns the vectors $u^{(1)}$, $u^{(2)}$, $v^{(1)}$ and $v^{(2)}$, we have that $C^\top J_4 C$ takes the form

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

with $D = \mathrm{diag}(d_1, d_2)$, $d_j = u^{(j)\top} J_4 v^{(j)}$, $j = 1, 2$, both different from zero. We can assume $d_j > 0$ (otherwise we only have to change α_j by $-\alpha_j$, which means that $v^{(j)}$ is replaced by $-v^{(j)}$, changing the sign of d_j). So, we can replace $u^{(j)}$ and $v^{(j)}$ by $u^{(j)}/\sqrt{d_j}$ and $v^{(j)}/\sqrt{d_j}$, and then one has that $C^\top J_4 C = J_4$ (that is, C is a symplectic matrix).

Before continuing, let us introduce the following matrices:

$$\Omega_1(\theta) = \begin{pmatrix} \cos\left(\frac{\alpha_1}{2\pi}\theta\right) & 0 \\ 0 & \cos\left(\frac{\alpha_2}{2\pi}\theta\right) \end{pmatrix}, \quad \Omega_2(\theta) = \begin{pmatrix} \sin\left(\frac{\alpha_1}{2\pi}\theta\right) & 0 \\ 0 & \sin\left(\frac{\alpha_2}{2\pi}\theta\right) \end{pmatrix},$$

and

$$\Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \\ -\Omega_2 & \Omega_1 \end{pmatrix}.$$

Then, using this notation, what we have by construction is that $C^{-1}\mathcal{C}C = \Omega(2\pi) \equiv \Omega^*$. From here we deduce that, taking $\mathcal{M} = C\tilde{\Omega}C^{-1}$, with

$$\tilde{\Omega} = \begin{pmatrix} 0 & \tilde{\Omega}^* \\ -\tilde{\Omega}^* & 0 \end{pmatrix}, \quad \tilde{\Omega}^* = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix},$$

we have $\exp(\mathcal{M}) = \mathcal{C}$. A direct verification shows, from the symplectic character of C, that $J_4\mathcal{M}$ is a symmetic matrix. Then, we have that the matrix $B(\theta)$ defined as

$$B(\theta) = C^{-1}\exp\left(\frac{1}{2\pi}\mathcal{M}\theta\right)\tilde{\Phi}\left(\frac{\theta}{\omega_0}\right)^{-1}, \tag{2.24}$$

is symplectic and 2π-periodic on θ. The symplectic character of B is clear, and the fact that it is 2π-periodic can be checked showing that B verifies the linear differential system

$$B' = C^{-1}\frac{\mathcal{M}}{2\pi}B - \frac{1}{\omega_0}BJ_4A,$$

and that by construction we have $B(0) = B(2\pi)$. Then, we use the matrix B to define the following canonical transformation,

$$\begin{aligned} \begin{pmatrix} q \\ p \end{pmatrix} &= B(\theta)^{-1}\begin{pmatrix} x \\ y \end{pmatrix}, \\ I &= \xi - \tfrac{1}{2}(x^\top, y^\top)J_4B'(\theta)B(\theta)^{-1}\begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned} \tag{2.25}$$

The canonical character of (2.25) is equivalent to the following equalities: if we put $\zeta^\top = (q^\top, p^\top)$ and $z^\top = (x^\top, y^\top)$, then we have to check that $\{\zeta, \zeta\} = J_4$, $\{\zeta, I\} = 0$, $\{\zeta, \theta\} = 0$ and $\{\theta, I\} = 1$, where if $f(\theta, x, \xi, y)$ and $g(\theta, x, \xi, y)$ are functions taking vectorial values. We define the matrix of Poisson brackets $\{f, g\}$ as

$$\{f, g\} = \frac{\partial f}{\partial z}J_4\left(\frac{\partial g}{\partial z}\right)^\top + \frac{\partial f}{\partial \theta}\left(\frac{\partial g}{\partial \xi}\right)^\top - \frac{\partial f}{\partial \xi}\left(\frac{\partial g}{\partial \theta}\right)^\top.$$

These equalities are simple to verify using the symplectic character of $B(\theta)$ on the explicit expressions $\frac{\partial \zeta}{\partial z} = B(\theta)^{-1}$, $\frac{\partial \zeta}{\partial \theta} = -B(\theta)^{-1}B'(\theta)B(\theta)^{-1}z$, $\frac{\partial \zeta}{\partial \xi} = 0$, $\frac{\partial I}{\partial z} = -z^\top J_4B'(\theta)B(\theta)^{-1}$ (this symplectic character implies that $JB'(\theta)B(\theta)^{-1}$ is a symmetric matrix) and $\frac{\partial I}{\partial \xi} = 1$, that are computed from (2.25).

The reduced Hamiltonian

If we insert this in the Hamiltonian (2.21), it takes the form (we keep the name H):

$$H(\theta,x,\xi,y)=h_0+\omega_0\xi+\frac{\omega_1}{2}\left(x_1^2+y_1^2\right)+\frac{\omega_2}{2}\left(x_2^2+y_2^2\right)+\sum_{j\geq 3}H_j(\theta,x,\xi,y), \qquad (2.26)$$

being $\omega_j=\alpha_j/T$, $j=1,2$. Note that after this change the quadratic part of (2.26) is reduced to constant coefficients.

2.2.4 Complexification of the Hamiltonian

With the Hamiltonian (2.26) we have a good system of coordinates to start computing the normal form. Nevertheless, to solve in a simpler form the different homological equations that will appear, it is much better to put the quadratic part of the Hamiltonian in diagonal form. For this purpose, we introduce new (complex) variables

$$x_j=(Q_j+iP_j)/\sqrt{2},\quad y_j=(iQ_j+P_j)/\sqrt{2},\quad j=1,2. \qquad (2.27)$$

We note that these relations define a canonical change that transforms the Hamiltonian (2.26) into

$$H(\theta,Q,\xi,P)=h_0+\omega_0\xi+i\omega_1Q_1P_1+i\omega_2Q_2P_2+\sum_{j\geq 3}H_j(\theta,Q,\xi,P), \qquad (2.28)$$

keeping again the name for H. This is the expression of the Hamiltonian that we will use to start the nonlinear part of the normal form. Note that the image of the real domain for x_j and y_j in the complex variables Q_j and P_j, is given by the relation $\bar{P}_j=iQ_j$. Hence, if this property is preserved during the normal form computation, we will be able to return to a real analytic Hamiltonian by means of the inverse of the change (2.27),

$$Q_j=(x_j-iy_j)/\sqrt{2},\quad P_j=(y_j-ix_j)/\sqrt{2},\quad j=1,2, \qquad (2.29)$$

In this context, the variables (Q,P) in (2.29) denote the current variables obtained after the different normal form transformations, and then, (x,y) are the transformed variables of the initial real ones.

Returning to the complexified Hamiltonian (2.28), we change the previous notation to a more suitable one to describe the normal form. We write (2.28) as

$$H^{(0)}(\theta,q,I,p)=h_0+\omega_0I+i\omega_1q_1p_1+i\omega_2q_2p_2+\sum_{j\geq 3}H_j^{(0)}(\theta,q,I,p), \qquad (2.30)$$

We note that this Hamiltonian has the following symmetry coming from the complexification: if we expand $H^{(0)}$ as f in (2.4) and (2.5), we have

$$\bar{h}_{j,k,l,m,s}^{(0)}i^{|l|_1+|m|_1}=h_{j,k,m,l,-s}^{(0)}. \qquad (2.31)$$

2.2.5 Computing the normal form

The objective of this section is to put the Hamiltonian (2.30) in normal form up to finite order, by using a canonical change of variables (2π-periodic on θ). We will construct this change as a composition of time one flows associated to suitable Hamiltonians (generating functions) G_j. They are selected to remove, in recursive form, the non-integrable terms of degree j. So, we will compute G_3, G_4, …, G_n, such that:

$$\begin{aligned} H^{(n-2)} \equiv\ & H^{(0)} \circ \Phi_{t=1}^{G_3} \circ \cdots \circ \Phi_{t=1}^{G_n}(\theta, q, I, p) = h_0 + \omega_0 I + i\omega_1 q_1 p_1 + i\omega_2 q_2 p_2 + \\ & + N^{(n)}(I, iq_1p_1, iq_2p_2) + H_{n+1}^{(n-2)}(\theta, q, I, p) + H_{n+2}^{(n-2)}(\theta, q, I, p) + \cdots, \end{aligned} \quad (2.32)$$

where Φ_t^G means the flow time t associated to the Hamiltonian system G. Here $N^{(n)}$ is in normal form up to order n, that is, only contains exact resonant terms of degree not bigger than n.

Before describing how to perform this normal form, let us mention two important properites of the Poisson bracket.

1. If f and g only contain monomials of degree r and s respectively, then $\{f, g\}$ only contains monomials of degree $r + s - 2$.

2. If the expansions of f and g verify the symmetry (2.31), $\{f, g\}$ too.

The last property will guarantee that, after the normal form process, the change (2.29) will transform the final Hamiltonian into a real analytic one.

A general step

Let us describe one step of this normal form process. For this purpose, we take the Hamiltonian (2.32), and we compute G_{n+1} by imposing that the expression

$$\{\omega_0 I + i\omega_1 q_1 p_1 + i\omega_2 q_2 p_2, G_{n+1}\} + H_{n+1}^{(n-2)},$$

only contains exact resonant terms. Then, doing for G_{n+1} and $H_{n+1}^{(n-2)}$ the same expansions as in (2.4) and (2.5), we formally obtain

$$g_{n+1,k,l,m,s} = \frac{h_{n+1,k,l,m,s}^{(n-2)}}{i(l-m)^\top \omega + is\omega_0},$$

being $\omega^\top = (\omega_1, \omega_2)$, provided that the denominators do not vanish. The well-defined character of G_{n+1} is ensured asking the denominators that do not correspond to trivial resonances to satisfy a suitable Diphantine condition (see Section 1.3), to guarantee convergence. The exactly resonant monomials correspond to $m = l$ and $s = 0$, and they can not be removed (they are the only ones present in the normal form). Moreover, we remark that the coefficients $g_{n+1,k,l,m,s}$ satisfy the symmetry (2.31). Finally, as the selection of the coefficients $g_{n+1,k,l,l,0}$ is free, in a practical implementation we will take $g_{n+1,k,l,l,0} = 0$, to have minimal norm for G_{n+1} and to keep the symmetry (2.31). Then, applying the transformation $\Phi_{t=1}^{G_{n+1}}$ on the Hamiltonian $H^{(n-2)}$, we obtain

$$H^{(n-1)} \equiv H^{(n-2)} \circ \Phi_{t=1}^{G_{n+1}} = H^{(n-2)} + \{H^{(n-2)}, G_{n+1}\} + \frac{1}{2!}\{\{H^{(n-2)}, G_{n+1}\}, G_{n+1}\} + \cdots,$$

that is in normal form up to order $n+1$, and it also satisfies the symmetry (2.31). Then, given a fixed $N \geq 3$, if we finish the process after $N-3$ normal form steps, we obtain a Hamiltonian of the form:

$$\mathcal{H}(\theta, q, I, p) \equiv H^{(N-3)} = \mathcal{N}(I, iq_1p_1, iq_2p_2) + \mathcal{R}(\theta, q, I, p), \tag{2.33}$$

where $\mathcal{N}$ is in normal form up to order $N-1$, and $\mathcal{R}$ is a remainder of order N.

Changes of variables

In practical computations one is usually interested in obtaining an explicit expression of the transformation that brings Hamiltonian (2.30) into its normal form (2.33),

$$\Psi^{(N-3)} \equiv \Phi_{t=1}^{G_3} \circ \cdots \circ \Phi_{t=1}^{G_{N-1}}(\theta, q, I, p), \tag{2.34}$$

as well as its inverse transformation

$$(\Psi^{(N-3)})^{-1} = \Phi_{t=-1}^{G_{N-1}} \circ \cdots \circ \Phi_{t=-1}^{G_3} = \Phi_{t=1}^{-G_{N-1}} \circ \cdots \circ \Phi_{t=1}^{-G_3}.$$

To this end, we consider a generic (analytic) generating function $G(\theta, q, I, p)$, 2π-periodic on θ, with a Taylor expansion starting with monomials of degree 3, and a function $F(\theta, q, I, p)$ taking one of the following forms:

$$F = q_1 + f(\theta, q, I, p), \tag{2.35}$$

$$F = I + f(\theta, q, I, p), \tag{2.36}$$

$$F = \theta + f(\theta, q, I, p), \tag{2.37}$$

being f an analytic function, 2π-periodic on θ, and with a Taylor expansion starting with terms of degree 2 in (2.35), 3 in (2.36) and 1 in (2.37), respectively. Now we will describe how to compute $F \circ \Phi_{t=1}^{G}$ using the Lie series method and, hence, putting initially $f \equiv 0$ in (2.35), (2.36) and (2.37), the different components of $\Psi^{(N-3)}$ are obtained by computing recursively $F \circ \Phi_{t=1}^{G_3} \circ \cdots \circ \Phi_{t=1}^{G_{N-1}}$. Note that to compute the transformation for q_2, p_1 or p_2, one only has to replace in (2.35) q_1 by one of these variables.

Let us start, for instance, with expression (2.35). We put $F_1 \equiv F \circ \Phi_{t=1}^{G} = q_1 + f_1$ and then, one has

$$f_1 = f + \{F, G\} + \frac{1}{2!}\{\{F, G\}, G\} + \cdots.$$

If G begins with terms of degree 3 and f of degree 2 we have, from the homogeneity of $\{.,.\}$ with respect to the adapted definition of degree, that f_1 also begins with degree 2. An analogous remark holds if we compute f_1 for (2.36). In this case, we have that f_1 begins with terms of order at least 3. Finally, the transformation of (2.37) is a little bit different. To do it, we define f^* as

$$f^* = \{f, G\} + \frac{\partial G}{\partial I}.$$

Note that it is of degree at least 1. Then

$$f_1 = f + f^* + \frac{1}{2!}\{f^*, G\} + \frac{1}{3!}\{\{f^*, G\}, G\} + \cdots,$$

that it is still of order 1. Similar ideas are used to compute $(\Psi^{(N-3)})^{-1}$.

2.2.6 The normal form

Let us consider the normal form $\mathcal{N}$ of (2.33). In order to go back to real coordinates we apply the change

$$q_j = (x_j - iy_j)/\sqrt{2}, \;\; p_j = (y_j - ix_j)/\sqrt{2}, \; j = 1, 2, \tag{2.38}$$

as it has mentioned in (2.29). Then, $\mathcal{N}$ takes the form $\mathcal{N}(I_0, I_1, I_2)$, where the actions

$$I_0 = I, \;\; I_j = iq_j p_j = \frac{x_j^2 + y_j^2}{2}, \; j = 1, 2, \tag{2.39}$$

are first integrals for the Hamiltonian equations of $\mathcal{N}$. Moreover, as the changes of variables used for the normal form computation preserve the symmetry (2.31), we have that $\mathcal{N}(I_0, I_1, I_2)$ is a real analytic function (in fact, $\mathcal{N}$ is a polynomial of degree $[N/2]$). Then, if we put $\omega_j(I_0, I_1, I_2) \equiv \frac{\partial \mathcal{N}}{\partial I_j}$, $j = 0, 1, 2$, the solutions of the Hamiltonian equations of $\mathcal{N}$ are explicitly given by

$$\begin{array}{rclrcl} \theta(t) & = & \omega_0^0 t + \theta^0, & x_j(t) & = & \sqrt{2I_j^0} \sin(\omega_j^0 t + \theta_j^0), \\ I_0(t) & = & I_0^0, & y_j(t) & = & \sqrt{2I_j^0} \cos(\omega_j^0 t + \theta_j^0), \end{array} \tag{2.40}$$

for $j = 1, 2$, where $\omega_j^0 \equiv \omega_j(I_0^0, I_1^0, I_2^0)$.

Invariant tori

From the normal form obtained in Section 2.2.6, it is easy to produce approximations to periodic orbits and invariant tori of dimensions 2 and 3. They are obtained neglecting the remainder of the Hamiltonian and selecting values for the actions in a sufficiently small neighbourhood of the periodic orbit.

More concretely, if we put $I_j^0 = 0$, $j = 1, 2$, in (2.40), we have parameterized by I_0^0 a 1-parameter family of periodic orbits (that contains the initial one for $I_0^0 = 0$). By putting $I_1^0 = 0$, we obtain a 2-parameter family of 2-dimensional tori, parameterized by I_0^0 and I_2^0. We have a symmetric situation swapping I_1^0 by I_2^0. If we use the three parameters simultaneously, I_j^0 $j = 0, 1, 2$, we describe in (2.40) a 3-parameter family of 3-dimensional tori.

If we send these tori along the different normal form transformations (see Section 2.2.5), complexified coordinates (Section 2.2.4), the Floquet transformation (Section 2.2.3) and the adapted coordinates (Section 2.2.1), we obtain approximations of periodic orbits and invariant tori for the initial system (2.1).

Similar ideas can be used, for example, in the case of periodic orbits with some hyperbolic directions to compute approximations of hyperbolic tori and the corresponding stable and unstable manifolds. The main difference from the case which we are actually dealing, appears in the Floquet transformation, where it is necessary to take into account the hyperbolic eigenvalues. We recall (see Section 2.2.3) that in some cases (presence of negative eigenvalues) these hyperbolic directions can be an obstruction for the determination of a real Floquet transformation.

Effective stability

Normal form computations are also useful to derive bounds on the diffusion speed near some invariant objects. It is well known that, in Hamiltonian systems with more than 2 degrees of freedom, linear stability does not imply stability (see, for instance, [3] for a first description of a model for this unstability), but accurate bounds on the diffusion velocity show that it must be very slow (see references in the Introduction). This leads to the introduction of the concept of effective stability ([21]): An object is called σ-stable ($\sigma > 1$) up to time T, if there exists $\tau > 0$ such that any solution starting at distance τ of the invariant object remains at distance not great than $\sigma\tau$ up to, at least, time T.

This kind of stability can be obtained from the normal form we have constructed here. One only needs to derive good estimates for the remainder of the normal form (this is the part of the Hamiltonian that produces the diffusion) to derive the desired estimates. Next sections are devoted to describe how these estimates can be obtained.

2.2.7 Bounds on the domain of convergence of the normal form

Now, we will estimate the size of the region of effective stability around the periodic orbit. To determine this region we use the following criteria: we identify the region of slow diffusion with the domain around the orbit where we can prove that the normal form up to order "big enough" (in a practical implementation, this is usually the biggest order one can reach within the computer limitations) is convergent with a sufficiently small remainder.

To implement the previous approach, we will give a method to bound the domain where the changes that introduce the normal form coordinates (see Section 2.2.5) are convergent. For this purpose we consider, as in Section 2.2.5, a generic (analytic) generating function $G(\theta, q, I, p)$. Then, if one puts $\Phi_t^G(\theta(0), q(0), I(0), p(0)) = (\theta(t), q(t), I(t), p(t))$ for the time t flow associated to the Hamiltonian equations of G, one can write

$$\theta(t) = \theta(0) + \int_0^t \frac{\partial G}{\partial I}(\theta(s), q(s), I(s), p(s))ds, \tag{2.41}$$

$$I(t) = I(0) - \int_0^t \frac{\partial G}{\partial \theta}(\theta(s), q(s), I(s), p(s))ds, \tag{2.42}$$

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} + J_4 \int_0^t \left(\frac{\partial G}{\partial (q,p)} \right)^\top (\theta(s), q(s), I(s), p(s))ds. \tag{2.43}$$

To estimate the region where the transformation $\Phi_{t=1}^G$ is defined, we use the norm $\|.\|_{\rho,R}$ introduced in (2.9). Hence, it is not difficult to deduce from the integral expressions (2.41), (2.42) and (2.43), that if one puts

$$\rho_0 = \left\| \frac{\partial G}{\partial I} \right\|_{\rho^{(0)}, R^{(0)}}, \quad \delta_0 = \left\| \frac{\partial G}{\partial \theta} \right\|_{\rho^{(0)}, R^{(0)}}, \quad \delta_j = \left\| \frac{\partial G}{\partial p_j} \right\|_{\rho^{(0)}, R^{(0)}}, \quad \delta_{j+2} = \left\| \frac{\partial G}{\partial q_j} \right\|_{\rho^{(0)}, R^{(0)}}, \tag{2.44}$$

for $j = 1, 2$, and one assumes $\rho_0 < \rho^{(0)}$ and $\delta_j < R_j^{(0)}$, $j = 0, \ldots, 4$, then $\Phi_{t=1}^G$ is well defined from $\mathcal{D}(\rho^{(0)} - \rho_0, R^{(0)} - \delta)$ to $\mathcal{D}(\rho^{(0)}, R^{(0)})$, where $\delta = (\delta_0, \ldots, \delta_4)$ (see (2.7) for the definition of $\mathcal{D}(.,.)$). For the proof see [13].

2.2.8 Bounds on the diffusion speed

To apply the ideas described along Section 2.2 to a practical example, we modify the normal form method introduced in Section 2.2.5, to adapt it to the standard implementation of normal forms in a computer, where it is usual to work with (truncated) power series stored using the standard definition of degree of a monomial, instead of the adapted one made in (2.6).

To avoid confusions, in what follows the word "degree" will refer to the adapted degree defined in (2.6), while "standard degree" will refer to the usual degree for monomials. Hence, instead of using generating functions G_j that are homogeneous polynomials of degre j, we will use generating functions that are homogeneous polynomials of standard degree j. With this formulation the remainder $\mathcal{R}$ of (2.33) begins with terms of standard degree N. As the normal form is independent of the process used to compute it, the only difficult that we find if we do not use the adapted definition of degree is of technical character: if we work with the standard degree, and we perform the Poisson bracket of two monomial of degree r and s, we loose the homogeneity of the Poisson bracket, and the result contains terms of degree $r+s-1$ and $r+s-2$. Note that, although this is not very nice for theoretical purposes, it is not a problem for a computational scheme. [1]

We use the criteria given in Section 2.2.7 to compute the effective stability region: we are interested in obtaining a domain where the canonical transformation $\Psi^{(N-3)}$ of (2.34) is convergent. To this end, we define (for technical reasons) $G \equiv G_3 + G_4 + \cdots + G_{N-3}$. Then, given an initial domain $\mathcal{D}(\rho^{(0)}, R^{(0)})$ (small enough) where we expect $\Psi^{(N-3)}$ to be convergent, we compute ρ_0 and the vector δ given by (2.44), using the definition of G previously done, but replacing the norm $\|.\|_{\rho^{(0)},R^{(0)}}$ by $|.|_{\rho^{(0)},R^{(0)}}$. It is not difficult to check (using the bounds given in Section 2.2.7 on any transformation G_j, $j = N-3, N-2, \dots, 3$) that $\Psi^{(N-3)}$ is convergent from $\mathcal{D}(\rho, R)$ to $\mathcal{D}(\rho^{(0)}, R^{(0)})$, where $\rho = \rho^{(0)} - \rho_0$ and $R = R^{(0)} - \delta$, provided that the initial domain is small enough such that $\rho > 0$ and $R_j > 0$, $j = 0, \dots, 4$.

To bound the diffusion speed on $\mathcal{D}(\rho, R)$, we assume that we know $M \geq 0$ such that $\|H^{(0)}\|_{\rho^{(0)},R^{(0)}} \leq M$ (that is, a bound for the norm of the Hamiltonian used to begin the normal form computations). So, for the Hamiltonian $\mathcal{H}$ of (2.33) we also have that $\|\mathcal{H}\|_{\rho,R} \leq M$.

Then, what we are going to do is to take arbitrary initial data in $\mathcal{D}(0, R)$, with the restriction that these points correspond to a representation of real points expressed in the complexified variables introduced in (2.27), and to estimate the time that the solution of the Hamiltonian equations corresponding to $\mathcal{H}$ needs to increase the distance to the initial periodic orbit in a given amount, at least, until this solution leaves $\mathcal{D}(0, R)$. In fact, we are going to produce bounds for this time as a function of the initial and final distance to the periodic orbit.

For this purpose, we consider the canonical equations for (I, q, p) related to $\mathcal{H}$. From (2.33), and using the notation for I_0, I_1 and I_2 introduced in (2.39), we have:

$$\dot{q}_j = \frac{\partial \mathcal{N}}{\partial I_j} i q_j + \frac{\partial \mathcal{R}}{\partial p_j}, \qquad (2.45)$$

[1] In fact, it is also possible to perform all the computations using the adapted degree, with a similar amount of work.

$$\dot{p}_j = -\frac{\partial \mathcal{N}}{\partial I_j} i p_j - \frac{\partial \mathcal{R}}{\partial q_j}, \tag{2.46}$$

for $j = 1, 2$, and

$$\dot{I}_0 = -\frac{\partial \mathcal{N}}{\partial \theta} - \frac{\partial \mathcal{R}}{\partial \theta} = -\frac{\partial \mathcal{R}}{\partial \theta}, \tag{2.47}$$

where we recall that $\mathcal{N} \equiv \mathcal{N}(I_0, iq_1p_1, iq_2p_2)$. We do not consider the "diffusion" in the θ-direction, as it does not increase the distance from the initial periodic orbit. From (2.45), (2.46) and (2.47), one obtains

$$\dot{I}_0 = -\frac{\partial \mathcal{R}}{\partial \theta} \equiv \hat{I}_0(\theta, q, I, p), \tag{2.48}$$

$$\dot{I}_j = i\left(\frac{\partial \mathcal{R}}{\partial p_j} p_j - \frac{\partial \mathcal{R}}{\partial q_j} q_j\right) \equiv \hat{I}_j(\theta, q, I, p), \quad j = 1, 2. \tag{2.49}$$

We remark that, to bound (2.48) and (2.49), we do not have explicit bounds for the remainder $\mathcal{R}$, but we recall that it begins with terms of standard degree N. As we know a bound M for the transformed Hamiltonian $\mathcal{H}$ in the domain $\mathcal{D}(\rho, R)$, we can compute (using Cauchy estimates) a bound for the remainder of its Taylor series corresponding to standard degree N, that is, a bound for $\mathcal{R}$.

We use this idea to bound the right-hand sides of (2.48) and (2.49). To do that, we define $R_0^* = R_0$, $R_1^* = \min\{R_1, R_3\}$ and $R_2^* = \min\{R_2, R_4\}$, and we consider real points (θ, x, I, y), using the variables (x_j, y_j) introduced in (2.38), such that $|I_0| < R_0^*$ and $I_j < (R_j^*)^2$. From definition (2.38), we note that the set of real points (x_j, y_j) such that $I_j \leq I_j^{(0)}$, is contained in the complex set $\{|q_j|, |p_j| \leq \sqrt{I_j^{(0)}}\}$, $j = 1, 2$. Then, applying the Cauchy estimates given in (2.12) and (2.13), to the coefficients of the Taylor expansion of $\hat{I}_0$ and $\hat{I}_j$, $j = 1, 2$, we deduce the bounds

$$|\hat{I}_0| \leq \frac{1}{\rho} \sum_{m \in \mathbb{N}^5, |m|_1 \geq N} \frac{M |I_0|^{m_0} (\sqrt{I_1})^{m_1+m_3} (\sqrt{I_2})^{m_2+m_4}}{(R_0^*)^{m_0} (R_1^*)^{m_1+m_3} (R_2^*)^{m_2+m_4}}, \tag{2.50}$$

$$\begin{aligned} |\hat{I}_j| &\leq \sum_{m \in \mathbb{N}^5, |m|_1 \geq N} (m_j + m_{j+2}) \frac{M |I_0|^{m_0} (\sqrt{I_1})^{m_1+m_3} (\sqrt{I_2})^{m_2+m_4}}{(R_0^*)^{m_0} (R_1^*)^{m_1+m_3} (R_2^*)^{m_2+m_4}} = \\ &= \sum_{m \in \mathbb{N}^5, |m|_1 \geq N} 2m_1 \frac{M |I_0|^{m_0} (\sqrt{I_1})^{m_1+m_3} (\sqrt{I_2})^{m_2+m_4}}{(R_0^*)^{m_0} (R_1^*)^{m_1+m_3} (R_2^*)^{m_2+m_4}}, \quad j = 1, 2\ , \end{aligned} \tag{2.51}$$

being $m = (m_0, m_1, \ldots, m_5)$ in these sums. To bound (2.50) and (2.51) uniformily, we define

$$\delta = \delta(I_0, I_1, I_2) \equiv \max\left\{\frac{|I_0|}{R_0^*}, \frac{\sqrt{I_1}}{R_1^*}, \frac{\sqrt{I_2}}{R_2^*}\right\}. \tag{2.52}$$

We note that δ is an estimate for the distance to the initial periodic orbit. When I_0, I_1 and I_2 move as a function of t, we can also consider $\delta \equiv \delta(t)$. Then, from the definition of δ, we can bound, for $\delta \leq 1$,

$$|\hat{I}_0| \leq \frac{M}{\rho} \sum_{m \in \mathbb{N}^5, |m|_1 \geq N} \delta^{|m|_1} \leq \frac{M}{\rho} \binom{4+N}{4} \frac{\delta^N}{(1-\delta)^{N+5}}, \tag{2.53}$$

and

$$|\hat{I}_1| \leq 2M \sum_{m\in\mathbb{N}^5, |m|_1\geq N} m_1 \delta^{|m|_1} \leq 2M \binom{4+N}{5} \frac{\delta^N}{(1-\delta)^{N+5}}, \quad j=1,2. \tag{2.54}$$

To bound the sums (2.53) and (2.54), we have used Lemma 2.2, that is given at the end of this section. Then, if we assume $|I_0|$, I_1 and I_2 to be increasing functions of t (this is the worst case to bound the diffusion), we have:

$$\frac{d}{dt}\left|\frac{I_0^2}{(R_0^*)^2}\right| \leq \frac{2|I_0||\dot{I}_0|}{(R_0^*)^2} \leq \frac{2\delta}{R_0^*}\frac{M}{\rho}\binom{4+N}{4}\frac{\delta^N}{(1-\delta)^{N+5}} \equiv A\frac{\delta^{N+1}}{(1-\delta)^{N+5}},$$

where

$$A = \frac{2}{R_0^*}\frac{M}{\rho}\binom{4+N}{4},$$

and

$$\frac{d}{dt}\left|\frac{I_j}{(R_j^*)^2}\right| \leq \frac{2M}{(R_j^*)^2}\binom{4+N}{5}\frac{\delta^N}{(1-\delta)^{N+5}} \equiv B_j\frac{\delta^N}{(1-\delta)^{N+5}},$$

with

$$B_j = \frac{2M}{(R_j^*)^2}\binom{4+N}{5}.$$

Putting $B=\max\{B_1,B_2\}$, we deduce the following bound for the speed of δ:

$$\frac{d}{dt}(\delta^2) \leq \max\{A\delta, B\}\frac{\delta^N}{(1-\delta)^{N+5}},$$

for $0<\delta<1$. Then, the problem of bounding the stability time can be solved in the following form: we take initial data corresponding to real coordinate points (θ,x,I,y), such that the corresponding I_j are bounded by $I_j^{(0)}$, $j=0,1,2$. Let us assume that $\delta_0 \equiv \delta(I_0^{(0)}, I_1^{(0)}, I_2^{(0)}) < 1$. Given δ_1, $\delta_0<\delta_1\leq 1$, we want a lower bound for the time $T(\delta_0,\delta_1)$ needed for the value of δ along the trajectory to go from δ_0 to δ_1. For this purpose, we take as initial condition $\delta(0)=\delta_0$, and we integrate

$$\dot{\delta} = \frac{A}{2}\frac{\delta^N}{(1-\delta)^{N+5}}, \tag{2.55}$$

or

$$\dot{\delta} = \frac{B}{2}\frac{\delta^{N-1}}{(1-\delta)^{N+5}}, \tag{2.56}$$

depending on the current value of δ. Let us explain the method. We define $\delta^*=B/A$, and hence, if $\delta_0\geq\delta^*$, we use equation (2.55) for $\dot{\delta}$. If $\delta_0<\delta^*$, we use (2.56) for $0<t\leq T^*$, where T^* is the value of t for which the bound of $\delta(t)$ obtained integrating (2.56) reachs δ^*. Then, for $t>T^*$, we use the bound (2.55) (of course, it can happen that $\delta^*>1$). For

(2.55), the necessary time Δt to move from δ_i to δ_f is given by:

$$\begin{aligned}\Delta t &= \frac{2}{A}\int_{\delta_i}^{\delta_f}\frac{(1-\delta)^{N+5}}{\delta^N}d\delta = \frac{2}{A}\sum_{\substack{j=0\\ j\neq N-1}}^{N+5}\binom{N+5}{j}(-1)^j\left.\frac{\delta^{j-N+1}}{j-N+1}\right|_{\delta_i}^{\delta_f}+\\ &+\frac{2}{A}\binom{N+5}{N-1}(-1)^{N-1}\log(\delta)\Big|_{\delta_i}^{\delta_f},\end{aligned}$$

and using (2.56), this time is

$$\begin{aligned}\Delta t &= \frac{2}{B}\int_{\delta_i}^{\delta_f}\frac{(1-\delta)^{N+5}}{\delta^{N-1}}d\delta = \frac{2}{B}\sum_{\substack{j=0\\ j\neq N-2}}^{N+5}\binom{N+5}{j}(-1)^j\left.\frac{\delta^{j-N+2}}{j-N+2}\right|_{\delta_i}^{\delta_f}+\\ &+\frac{2}{B}\binom{N+5}{N-2}(-1)^{N-2}\log(\delta)\Big|_{\delta_i}^{\delta_f}.\end{aligned}$$

To end this section, we formulate and prove the result used to sum the bounds for the diffusion speed in (2.53) and (2.54). For this purpose, we define $c_{n,l} = \#\{m \in \mathbb{N}^n : |m|_1 = l\}$, that is, the number of monomials in n variables of degree l. This number is given by

$$c_{n,l} = \binom{n+l-1}{n-1}.$$

Lemma 2.2 *For any $0 \le R < 1$, we have the following bounds:*

(*i*)

$$\sum_{m\in\mathbb{N}^n,|m|_1\ge N} R^{|m|_1} = \sum_{l\ge N} c_{n,l}R^l \le \binom{n+N-1}{n-1}\frac{R^N}{(1-R)^{N+n}},$$

(*ii*)

$$\sum_{m\in\mathbb{N}^n,|m|_1\ge N} m_1 R^{|m|_1} = \sum_{l\ge N}\sum_{j=0}^{l} j c_{n-1,l-j}R^l \le \binom{n+N-1}{n}\frac{R^N}{(1-R)^{N+n}},$$

being, in both cases, $m = (m_1, \dots, m_n)$.

Proof: : We define $f(x) = (1-x)^{-n} = \sum_{l\ge 0} c_{n,l}x^l$. Then, part ($i$) is obtained bounding the remainder of the Taylor expansion up to degree $N-1$ of f around $x = 0$, evaluated at $x = R$. To do that, we remark that differentiating f j times, we have $f^{(j)}(x) = n(n+1)\cdots(n+j-1)(1-x)^{-n-j}$ and, hence,

$$\frac{f^{(N)}(x)}{N!} \le \binom{n+N-1}{N}(1-R)^{-N-n},$$

for every $0 < x \le R$. To prove (ii), we use that

$$\sum_{j=0}^{l} j c_{n-1,l-j} = \binom{n+l-1}{n} = c_{n+1,l-1},$$

that can be proved by induction. Finally, using the bound (i), we obtain:

$$\sum_{l\geq N}\sum_{j=0}^{l} j c_{n-1,l-j} R^l = \sum_{l\geq N} c_{n+1,l-1} R^l = R \sum_{l\geq N-1} c_{n+1,l} R^l \leq \binom{n+N-1}{n} \frac{R^N}{(1-R)^{N+n}}.$$

■

2.3 Application to the spatial RTBP

Here we present an application of the methods of this chapter to a concrete example coming from the Restricted Three Body Problem (RTBP). As it will be explained in the following sections, we have taken an elliptic periodic orbit of the RTBP, we have computed (numerically) the normal form (up to order 16) and we have bounded the corresponding remainder. This normal form has been used to compute invariant tori (of dimensions 1, 2 and 3) near the periodic orbit, and the bounds on the remainder have been used to estimate the corresponding rate of diffusion.

2.3.1 The Restricted Three Body Problem

Let us consider two bodies (usually called primaries) revolving in circular orbits around their common centre of masses, by means of the Newton's law. With this, we can write the equations of motion of a third infinitessimal particle moving under the gravitational attraction of the primaries, but without affecting them. The study of the motion of this third particle is the so-called Restricted Three Body Problem (see [72]). To simplify the equations, the units of length, time and mass are chosen such that the sum of masses of the primaries, the distance between them and the gravitational constant are all equal to one. With these normalized units, the angular velocity of the primaries around their centre of masses is also equal to one. A usual system of reference (called synodical system) is the following: the origin is taken at the centre of mass of the two primaries, the x axis is given by the line defined by the two primaries and oriented from the smaller primary to the biggest one, the z axis has the direction of the angular momentum of the motion of the primaries and the y axis is taken in order to have a positively oriented system of reference. If we suppose that the masses of the primaries are μ and $1-\mu$, with $0<\mu\leq 1/2$, we have that the primaries are located (in the synodic system) at the points $(\mu-1,0,0)$ and $(\mu,0,0)$, respectively. In this reference, the Hamiltonian for the motion of the third particle is

$$H(x,y,z,p_x,p_y,p_z) = \frac{1}{2}(p_x^2+p_y^2+p_z^2) + yp_x - xp_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \tag{2.57}$$

being the momenta $p_x=\dot{x}-y$, $p_y=\dot{y}+x$ and $p_z=\dot{z}$, with $r_1^2=(x-\mu)^2+y^2+z^2$ and $r_2^2=(x-\mu+1)^2+y^2+z^2$. The parameter μ is usually called the mass parameter of the system. Note that the (x,y) plane is invariant by the flow. The restriction of this Hamiltonian to this plane is the so-called planar RTBP, while (2.57) is usually called spatial RTBP. From now on, we will simply use RTBP to refer to the spatial problem.

This system has five equilibrium points: three of them are on the x axis (usually called collinear points, or L_1, L_2 and L_3) and the other two are forming an equilateral triangle with the primaries (and are usually called triangular points, or L_4 and L_5). The points L_4 and L_5 are given in the phase space by $(-\frac{1}{2}+\mu, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{1}{2}+\mu, 0)$ and $(-\frac{1}{2}+\mu, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{1}{2}+\mu, 0)$.

2.3.2 The vertical family of periodic orbits of L_5

In what follows we will focus on the L_5 point, but the results are also valid for L_4, due to the symmetries of the problem.

The linearized system around L_5 always has a vertical oscillation with frequency 1. Then, the vertical family of periodic orbits is the Lyapunov family associated to this normal frequency (for a proof of the existence of these families, see [68]). To study the linear stability of these orbits, we recall that the eigenvalues of the linearized vectorfield at L_5 are given by $\pm i$ (the ones responsible of the vertical oscillation), and by

$$\pm\sqrt{-\frac{1}{2}\pm\frac{1}{2}\sqrt{1-27\mu(1-\mu)}},$$

that are the ones of the planar RTBP. All of them are purely imaginary and different if $0<\mu<\mu_R\equiv\frac{1}{2}(1-\sqrt{23/27})\approx 0.03852$ (this is the so-called Routh critical value). For $\mu=\mu_R$, the planar frequencies collide and this fact produces a bifurcation in the linear stability and L_5 becomes unstable for $\mu_R\leq\mu\leq 1/2$. Note that, if $\mu\neq\mu_R$, the linear character of the vertical family is the same as L_5, at least for small vertical amplitudes.

Now, we want to continue the vertical family of periodic orbits in the vertical direction, by increasing the vertical amplitude. To do that, we identify the point L_5 with the periodic orbit of zero amplitude and period 2π. Hence, the monodromy matrix of this orbit is given by the exponential matrix of 2π times the differential matrix of the RTBP vectorfield at L_5. For any periodic orbit of the vertical family, its monodromy matrix has, of course, a pair of eigenvalues 1, plus other four eigenvalues that generalize the planar ones of L_5 to the vertical periodic orbits. As it has been mentioned in Section 2.2.3, the linear stability condition for these periodic orbits is that these four eigenvalues are all different and of modulus 1. Returning to the case $\mu=\mu_R$, we have for the orbit of zero amplitude that these four eigenvalues collapse to two in the complex unity circle. This resonance can be continued (numerically) with respect to μ and the amplitude of the orbit (in fact, it can be continuated with respect to any regular parameter in the family). The curve corresponding to this resonance is displayed in Figure 2.1. The parameters plotted are μ and the vertical velocity ($\dot{z}$) of the orbit when it cuts the hyperplane $z=0$ in the positive direction. Note that, for values of μ slightly larger than μ_R, L_5 is unstable but, if we "go up" in the vertical family, we find (linearly) stable orbits after crossing the above-mentioned bifurcation.

The selected periodic orbit

For the application of the methods exposed, we have selected a periodic orbit of the vertical family of L_5 for the mass parameter $\mu=0.04$ (that is bigger than μ_R), and with

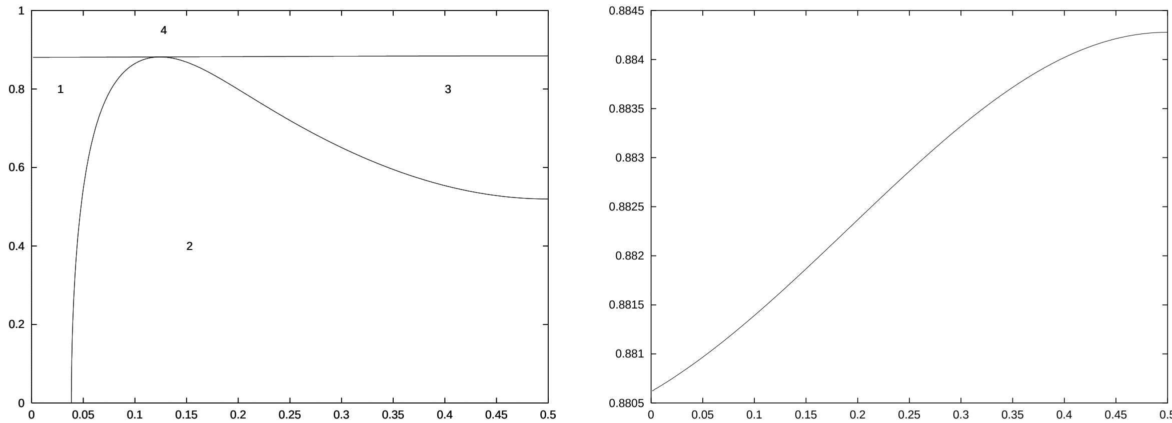

Figure 2.1: Left: Some curves of change of the linear character of the orbits of the vertical family of L_5. The parameters are μ and the value of $\dot{z}$ when $z = 0$. Excluding the bifurcation curves, the normal eigenvalues of the periodic orbit are all different and can be described as follows: 1. two couples of conjugate eigenvalues of modulus 1, 2. two conjugate eigenvalues ouside $\mathbb{S}^1$ and the corresponding inverse ones, 3. two couples of positive eigenvalues γ, $1/\gamma$, 4. two conjugate eigenvalues of modulus 1 and a couple of positive eigenvalues γ, $1/\gamma$, at least for moderate values of $\dot{z}$. Right: The upper curve with a more suitable scale for the $\dot{z}$ variable.

$\dot{z} = 0.2499997395037823$. This is a linearly stable orbit (the corresponding pair $(\mu, \dot{z})$ belongs in region 1 in Figure 2.1, see Section 2.3.6 for more details), but its proximity to resonance will produce small domains of convergence for the normal form.

The reasons for selecting this orbit are the following. The vertical family of the RTBP has its own interest, since it is the skeleton that organizes the dynamics of some physically relevant problems ([26], [71]), and the tools used here can be useful to deal with those problems. On the other hand, this example allows to show (numerically) the existence of regions of effective stability near $L_{4,5}$ for $\mu > \mu_R$. Finally, the example has not been "cooked" to simplify computations so it is a good problem to test the effectivity of these techniques.

2.3.3 Expansion of the Hamiltonian of the RTBP

Let us denote by $\frac{2\pi}{\omega_0}$ the period of the selected orbit. The next step is to perform the different changes of coordinates introduced in Section 2.2, and to compute the explicit expansion of the Hamiltonian expressed in these adapted coordinates (to obtain the Hamiltonian $H^{(0)}$ of (2.30), to start the computation of the normal form).

For this purpose, we proceed in the following form. First, we write (X, Y, Z, P_X, P_Y, P_Z) for the initial coordinates with origin at L_5: $X = x + \frac{1}{2} - \mu$, $Y = y - \frac{\sqrt{3}}{2}$, $Z = z$, $P_X = p_x + \frac{\sqrt{3}}{2}$, $P_Y = p_y + \frac{1}{2} - \mu$ and $P_Z = p_z$. Then, we assume that $(f(\theta), g(\theta))$ is a 2π-periodic parametrization of the periodic orbit, expressed in these coordinates. We note that, for orbits of this family with moderate amplitudes, the expression $(f_3')^2 + (\frac{g_3'}{\omega_0})^2$ is close to a constant. More concretely, if we introduce the new coordinates $\zeta = \sqrt{\omega_0} Z$ and $P_\zeta = \frac{P_Z}{\sqrt{\omega_0}}$, the expressions for the new f_3 and g_3 take the form (2.2), with the function Δ

of (2.3) close to constant. If we we rewrite the Hamiltonian (2.57) in these new variables, but keeping for simplicity the notation Z, P_Z, then we can expand the Hamiltonian (2.57) as:

$$\begin{aligned} H &= \frac{1}{2}(P_X^2 + P_Y^2 + \omega_0 P_Z^2) + YP_X - XP_Y + \frac{X^2}{8} - \frac{5}{8}Y^2 - aXY + \frac{1}{2}\frac{Z^2}{\omega_0} - \\ &- \sum_{k\geq 3} r_0^k P_k\left(\frac{X-\sqrt{3}Y}{2r_0}\right)(1-\mu) - \sum_{k\geq 3} r_0^k P_k\left(\frac{-X-\sqrt{3}Y}{2r_0}\right)\mu, \end{aligned} \tag{2.58}$$

where we have skipped the constant term of this expansion, $(-3+\mu-\mu^2)/2$, that corresponds to the energy level of L_5. Here, $a = -\frac{3\sqrt{3}}{4}(1-2\mu)$, $r_0^2 = X^2 + Y^2 + \frac{Z^2}{\omega_0}$ and P_k denotes the Legendre polynomial of degree k. This expression comes from the expansion of the Hamiltonian of the RTBP around the point L_5. To compute this expansion we can use the recurrence of the Legendre polynomial. This recurrence is given by $P_0(x) = 1$, $P_1(x) = x$ and $P_{k+1}(x) = \frac{2k+1}{k+1}xP_k(x) - \frac{k}{k+1}P_{k-1}(x)$, for $k \geq 1$, and hence, if we define:

$$R_k^{(0)} = r_0^k P_k\left(\tfrac{X-\sqrt{3}Y}{2r_0}\right), \quad R_k^{(1)} = r_0^k P_k\left(\tfrac{-X-\sqrt{3}Y}{2r_0}\right),$$

we have $R_0^{(0)} = 1$, $R_1^{(0)} = \frac{X-\sqrt{3}Y}{2}$, and $R_0^{(1)} = 1$, $R_1^{(1)} = \frac{-X-\sqrt{3}Y}{2}$. Thus,

$$R_{k+1}^{(j)} = \frac{2k+1}{k+1}R_1^{(j)}R_k^{(j)} - \frac{k}{k+1}R_{k-1}^{(j)}r_0^2, \tag{2.59}$$

for $k \geq 1$ and $j = 0, 1$. Then, a method to expand the Hamiltonian (2.58) expressed in the adapted variables introduced to perform the normal form, is to compute the composition of the change (2.16) adapted to the periodic orbit (and that we assume well defined for our concrete orbit, fact that can be tested using the methodology described in Section 2.2.2), with the Floquet transformation (2.25) and the complexification (2.27), and then, to insert this change into the recurrences (2.59). This method seems to be an efficient way to obtain the desired expansion of the Hamiltonian (2.58). We remark that, to compute the changes (2.16) and (2.25) we only need to know the explicit expression of the periodic orbit and the variational matrix of the orbit for the initial Hamiltonian (2.57).

2.3.4 Bounds on the norm of the Hamiltonian

First note that the expansion of the Hamiltonian in (2.58) is done around L_5, and not around the periodic orbit. Hence, it is easy to check that this expansion only converges if $r_0 < 1$ (the distance from L_5 to the primaries, that both are singularities of the Hamiltonian (2.57)). It implies that, when we introduce the adapted system of coordinates, and we replace X, Y and Z by their expressions in terms of the new coordinates, we need to control the value of r_0 as a function of the allowed range for the new variables, not only to ensure convergence of (2.58), but also to bound the supremum norm of H. The control of this norm, that is necessary to obtain the estimates for the diffusion time provided by Section 2.2.8, can be done by looking at the explicit expressions of the adapted coordinates of (2.16), the Floquet transformation (2.25), and the complexification (2.27), and then, computing the norms of the different expansions of the change, as a function of the size of the given domain for the new variables. Then, we can bound the norm of the Hamiltonian using the following lemma

Lemma 2.3 *Let $A > 0$ and let $\{P_k\}_{k\geq 0}$ be a sequence of positive numbers that verify $P_{k+1} \leq \frac{2k+1}{k+1} P_k P_1 + \frac{k}{k+1} P_{k-1} A$, for $k \geq 1$. We assume that for certain N we have $|P_j| \leq \tilde{P}_j$ for $j = 1, N-1, N$, and we define $h = \max\left\{ \frac{\tilde{P}_N}{\tilde{P}_{N-1}}, \tilde{P}_1 + \sqrt{\tilde{P}_1^2 + A} \right\}$. Then, if $h < 1$ we have*

$$\sum_{k\geq N+1} P_k \leq \sum_{k\geq N+1} h^{k-N+1} \tilde{P}_{N-1} \leq \tilde{P}_{N-1} \frac{h^2}{1-h}.$$

Proof: See [10]. ∎

We remark that the recurrence for the $\{P_k\}_{k\geq 0}$ in Lemma 2.3 is the same obtained taking norms on the $\{R_k^{(j)}\}_{k\geq 0}$ in the recurrence (2.59). Hence, with this lemma we can bound the norm of the Hamiltonian (using the expansion (2.58)) as well as the remainder of the expansion when we deal with a finite number of terms.

2.3.5 Numerical implementation

In this section, we describe the algorithm used to perform a computer implementation of the methodology introduced in Section 2.2 to the case of the RTBP.

Of course, computer assisted works have many inconvenients from a theoretical point of view. First, we have the obvious problem that the arithmetic is not exact, that is, we can only store finite decimal representations for the numerical coefficients, with errors that are propagated with the successive operations. Moreover, we can only deal with truncated expansions for the Taylor and Fourier series. Without losing the formal approach, these problems can be solved, for example, using intervalar arithmetic for the numerical coefficients, and storing for every truncated expansion a bound for the remainder. This methodology allows to do a rigorous computer assisted proof.

If one is only interested in obtaining numerical estimations for the region of effective stability, but based in a rigorous approach, one only needs to look at the most significative terms in the control of this diffusion, ignoring the errors on the computer arithmetic as well as the higher order truncations. Nevertheless, the final result is, if we really work with all the significative terms, "the same" as in the rigorous approach. As we mentioned in the Introduction, this is the approach taken in this chapter.

Then, in our software we select certain degree N, and we only work with coefficients (that are 2π-periodic functions on θ) of monomials of standard degree less than or equal to N. It means that we store for every monomial $q^l p^m I^k$, with $k + |l|_1 + |m|_1 \leq N$, a truncated Fourier expansion on θ for the 2π-periodic coefficient. For each (complex) Fourier coefficient we store a finite decimal approximation, using the standard double precision of the computer. As it has been mentioned before, we have used this software with $N = 16$, and taking the biggest order in the Fourier expansion as 18.

When working with finite approximations, we remark that in some cases the Taylor truncations can be done such that they involve terms with order bigger that the one of the normal form, and also, from the analytic character of the Hamiltonian, we have that the coefficients of the Fourier expansions decrease exponentially fast with the order of of the harmonic. So, if we take a "sufficiently" large number of terms, they give the most significative contribution.

With this formulation, we can follow all the theoretical steps of Section 2.2, bounding (when necessary) the supremum norms (2.8) and (2.9) by the norms (2.10) and (2.11), evaluated using the truncated expansions.

Hence, after computing a periodic orbit of the Hamiltonian system (2.57), computation that can be done with high precision, we perform a Fourier analysis of this orbit and of the matrix $B(\theta)$ (see (2.24)). With these Fourier analysis we compute the Floquet transformation (2.25), that composed with the complexification (2.27) allows to compute the function F of (2.15) expressed in the complexified Floquet variables. Then we solve, up to degree N, the equation (2.14) for $\alpha(\theta, s)$ after substituting $s = F$. This can be done by means of an iterative scheme.

To continue with the computations, we take this expression for the change as a function of the complexified variables as "exact" up to degree N. Thus, if we insert this transformation in the Hamiltonian (2.58), we obtain a Hamiltonian like (2.30), suitable to compute the normal form. To do this expansion, we use recurrences (2.59) up to some finite "big" order. We remark that, if we compute a reduced number of terms in these recurrences, we can not say that the remainder of this expansion contains only terms of "higher order" of the Taylor expansion around the orbit (we recall this Taylor expansion is done around L_5, not around the orbit). This fact implies that, if one wants to justify that working close to the periodic orbit one has small remainder, one needs to take a sufficiently large number of terms in recurrences (2.59). As we commented in Section 2.3.4, we can use Lemma 2.3 to estimate the error in this truncation. For instance, in our concrete application we have considered the recurrences for $R_k^{(0)}$ and $R_k^{(1)}$ for $k \leq 30$.

At this point, we have an approximation to the Hamiltonian (expressed in the Floquet complexified variables) given by a polynomial of degree N, with 2π-periodic coefficients on θ. Those coefficients are given by a trigonometric polynomial of certain finite degree.

We apply to this Hamiltonian the normal form scheme of Section 2.2.5. To this end, we choose the formulation explained in Section 2.2.8, that is, we remove in an increasing form the non-integrable terms of standard degree $3, 4, \ldots, N-1$. Then, the final product of these computations is an explicit expression of the normal form up to syandard degree $N-1$, and of the generating functions used to put the Hamiltonian in this reduced form. As in the practical implementation we do not take into consideration the errors due to the arithmetic or to the truncated expansions, we take this normal form and the generating functions as correct up to standard degree N. Note that the use of standard degree instead of the adapted one forces us to remove some extra monomials. This does not affect the final results since this does not introduce extra small divisors.

2.3.6 Results in a concrete example

We start computing the vertical family of periodic orbits for $\mu = 0.04$. To do that, we look for fixed points of the return map generated by the Poincaré section $z = 0$, and we obtain a curve of fixed points in this hyperplane. Hence, after we pass the stability bifurcation plotted in Figure 2.1, and as it has been mentioned in Section 2.3.2, we take the orbit with $\dot{z} = 0.2499997395037823$. The initial conditions, period and normal frequencies of this orbit, are given in Table 2.1.

Then, we implement the normal form methodology of Section 2.2 for a generic linearly

$x =$-4.669907803550619e-01	$p_x =$-8.347975347250963e-01
$y =$ 8.616112997374481e-01	$p_y =$-4.524543662847003e-01
$z =$ 0.000000000000000e+00	$p_z =$ 2.499997395037823e-01
$T =$ 6.286004008046562e+00	
$\alpha_1 =$-1.590665653770649e+00	
$\alpha_2 =$ 2.082743361412927e+00	

Table 2.1: Initial conditions of the chosen periodic orbit. T is the period, and the non-trivial eigenvalues of the monodromy matrix are $\exp(\pm\alpha_j i)$, $j = 1, 2$.

stable orbit of the RTBP around L_5, and we particularize the computations on the orbit previously chosen. For this purpose, we work with truncated power series, up to standard degree 16, and with trigonometric polynomials of degree 18. Hence, after we write the Hamiltonian in the adapted Floquet complexified variables (that is, it takes the form (2.30)), the normal form is computed up to standard degree 16 as a composition of time one flows associated to generating functions of degrees ranging from 2 to 16.

The following sections are devoted to show some of the results obtained.

Explicit normal form

To illustrate the results obtained, we begin given the first terms of the normal form. To do that, we write this normal form as

$$\mathcal{N}(I_0, I_1, I_2) = \sum_{n\in\mathbb{N}^3} c_n I_0^{n_0} I_1^{n_1} I_2^{n_2},$$

where $n = (n_0, n_1, n_2)$, being I_0 the conjugate action of the angular variable, and I_1, I_2 the actions related to the normal directions. Then, the coefficients c_n are displayed in Table 2.2 up to $|n|_1 \leq 5$.

The term $c_{0,0,0}$ corresponds to the energy level of the orbit in the RTBP. $c_{1,0,0}$ is the frequency of the periodic orbit, $\omega_0 = 2\pi/T$, and we recall (see Section 2.2.3) that $c_{0,1,0} = \alpha_1/T$, $c_{0,0,1} = \alpha_2/T$. With respect the other terms, we specialy focus on $c_{2,0,0}$. This coefficient is responsible, at first order, of the variation of the intrinsic frequency of the periodic orbits of the vertical family around the initial one, with respect to the action I_0. We notice that this coefficient is non-zero, but very small. It implies that this family is close to degenerate, and hence, these orbits are very sensible to external perturbations.

Effective stability estimates

Next step is to derive a domain where this normal form is convergent. To this end, we take $\mathcal{D}(\rho^{(0)}, R^{(0)})$ as initial domain for the complexified Floquet adapted variables (the ones corresponding to Hamiltonian (2.30)), with $\rho^{(0)} = 5\times 10^{-2}$, $R_0^{(0)} = 6.5\times 10^{-6}$ and $R_j^{(0)} = 4\times 10^{-4}$, $j = 1, 2, 3, 4$ (these are suitable values for the following computations). Then, we transform this domain, by means of the complexification (2.27) and the Floquet change (2.25), to the variables (θ, q, I, p) introduced by the canonical transformation (2.16), and

n_0	n_1	n_2	c_n	n_0	n_1	n_2	c_n
0	0	0	3.171173123282592e-02	1	1	2	1.410592146185009e+10
1	0	0	9.995515909847708e-01	1	0	3	1.911519314101390e+10
0	1	0	-2.530487813457447e-01	0	4	0	3.101951889171696e+08
0	0	1	3.313302630330585e-01	0	3	1	3.971949034520016e+09
2	0	0	-7.085969782537171e-03	0	2	2	5.199795716604182e+11
1	1	0	2.347808932329671e+00	0	1	3	1.372427816791639e+12
1	0	1	2.498732579493477e+00	0	0	4	9.569190343277255e+11
0	2	0	3.419196867088647e+01	5	0	0	2.915956534456797e-04
0	1	1	1.797153868665541e+02	4	1	0	-3.648221760506484e+05
0	0	2	1.074668694374247e+02	4	0	1	-3.648221364966501e+05
3	0	0	-2.981856450722603e-04	3	2	0	-7.705208342798983e+07
2	1	0	-7.583799076892481e+01	3	1	1	-3.324615700347268e+08
2	0	1	-7.607125470486629e+01	3	0	2	1.282415913249085e+11
1	2	0	-4.767712514746165e+03	2	3	0	-5.464683024337851e+09
1	1	1	-2.142185039061398e+04	2	2	1	-4.502428761893904e+10
1	0	2	6.751575625159690e+04	2	1	2	2.706294371464132e+13
0	3	0	-7.465962311103106e+04	2	0	3	4.376991351234328e+13
0	2	1	-6.317236766614448e+05	1	4	0	-1.604851084130283e+11
0	1	2	4.698477549792072e+06	1	3	1	-2.024480139039151e+12
0	0	3	5.964521815470135e+06	1	2	2	1.941981055326712e+15
4	0	0	1.645707771255077e-03	1	1	3	6.218155508134416e+15
3	1	0	4.702235456552206e+03	1	0	4	5.001397263004861e+15
3	0	1	4.702324042854812e+03	0	5	0	-1.682040578296126e+12
2	2	0	6.109486945253862e+05	0	4	1	-2.916743601544909e+13
2	1	1	2.678982801363627e+06	0	3	2	4.747246259201865e+16
2	0	2	9.788138881558010e+07	0	2	3	2.245654381035990e+17
1	3	0	2.441103645749133e+07	0	1	4	3.575855436974813e+17
1	2	1	2.041332816688016e+08	0	0	5	1.904536783690966e+17

Table 2.2: Coefficients of the normal form around the chosen orbit up to degree 5.

we derive (by bounding the components of the transformations) the upper bounds $R_0^{(1)} = 9.860 \times 10^{-6}$, $R_1^{(1)} = 1.405 \times 10^{-2}$, $R_2^{(1)} = 9.681 \times 10^{-3}$, $R_3^{(1)} = 6.804 \times 10^{-3}$ and $R_4^{(1)} = 9.856 \times 10^{-3}$ for this new domain. On this domain, it is not difficult to check that (2.19) holds, and to obtain the bound $R_\beta = 2.243700 \times 10^{-2}$ for the expression (2.20). Then, if we implement numerically the proof of Proposition 2.1, we obtain, for the current values of R_β and $\rho^{(0)}$, the estimate $\delta_0 = 0.9534070$ and $\delta_1 = 9.268077 \times 10^{-4}$. As $\delta_0 > \delta_1$, we can ensure that the adapted coordinates introduced by (2.16) are well defined on $\mathcal{D}(\rho^{(0)}, R^{(1)})$. Moreover, we can also obtain (by bounding the components of the transformation (2.16)) bounds on the size of the initial (complex) domain expressed in the (synodical) coordinates of the RTBP. For instance, we have a bound for r_0 (the distance to L_5, see Section 2.3.3) of order 0.274 (that is smaller than 1). Moreover, if we apply Lemma 2.3 to these estimates, with $N = 30$, we deduce that a bound for the remainder of expansion (2.58) is 6.82×10^{-17}. The value $N = 30$ has been selected because this is the number of Legendre polynomials taken to perform a numerical implementation of recurrences (2.59).

Now, we use the method described in Section 2.2.7 to deduce a domain where we can prove convergence of the normal form transformation up to degrees 10, 12, 14 and 16. Of course, if we increase the order of the normal form, this domain shrinks, but, as we do not find a very strong resonance for these orders, it remains practically constant. It is given by $\mathcal{D}(\rho, R)$, with $\rho = 4.038 \times 10^{-2}$, $R_0 = 6.238 \times 10^{-6}$, $R_1 = R_3 = 1.793 \times 10^{-4}$ and $R_2 = R_4 = 1.349 \times 10^{-4}$. Tables on the time needed to leave this domain are plotted in Figure 2.3. To compute those estimates, we use again Lemma 2.3 to bound the norm of the Hamiltonian (2.58) in the considered domain, and we obtain the value 9×10^{-2}.

It is interesting to compare these results with the stability region obtained using direct numerical integration. Thus, we have taken the Poincaré section $z = 0$, and we have taken a mesh of points for the x and y variables. Then, we have used as initial condition for a numerical integration the points given by the (x, y) values of the mesh and the values $\dot{x}$, $\dot{y}$ and $\dot{z}$ corresponding to the selected periodic orbit (see Table 2.1). If, after 10000 units of time the orbit does not go away, we consider that the initial point is inside the region of stability. The criteria to decide if a point goes away is to check if, at some moment, y becomes negative (this heuristic criteria has been previously used in [45], [73] and [26]). The points corresponding to initial conditions of stable orbits have been plotted in Figure 2.2. Moreover, we have tried to use the normal form computation to determine which points of the mesh above correspond to the effectively stable region. To this end, we have send all the initial conditions through the changes of variables to reach the normal form coordinates (of course, if a point in the mesh is outside the convergence domain of these transformations we assume that it is unstable). Then, it is easy to check if this point is inside the domain of effective stability. So, we have also plotted those points in Figure 2.2.

Invariant tori gallery

Here (Figures 2.4, 2.5, 2.6 and 2.7) we present some invariant tori of the truncated normal form (see Section 2.2.6) translated by the different changes of variables, and plotted in the initial coordinates of the RTBP. We also give numerical values for the normal and intrinsic frequencies of the computed tori.

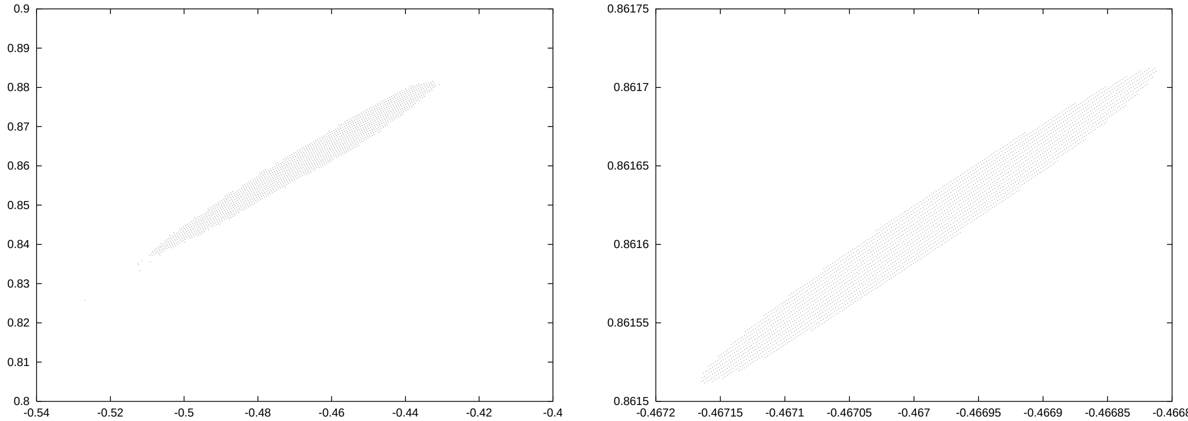

Figure 2.2: Left: Surviving points of the numerical integration. Right: Points of the table used to perform the numerical integration for which the coordinates of the normal form up to order 16 belong in $\mathcal{D}(\rho, R)$.

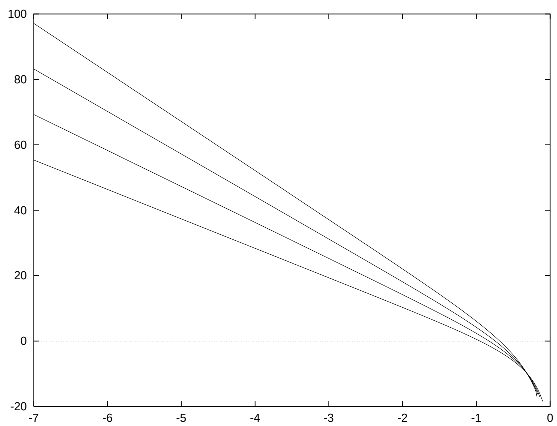

Figure 2.3: Estimates on the time needed to leave the domain $\mathcal{D}(0, R)$ (see Section 2.3.6). These curves correspond to normal forms of orders 10, 12, 14 and 16. The values plotted are $\log_{10} \delta_0$ (see (2.52)) for the initial condition, and a lower bound for the time needed to reach $\delta = 1$, also in $\log_{10}$ scale.

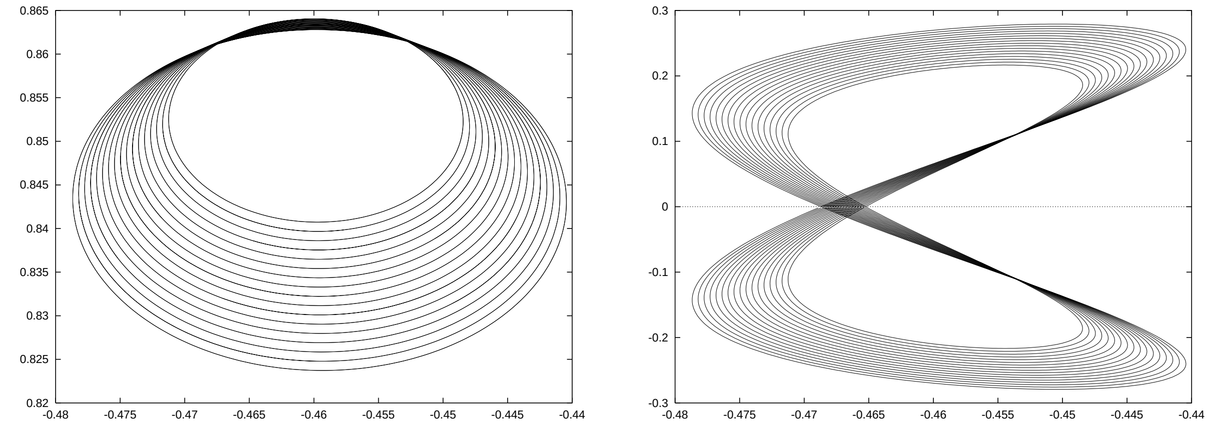

Figure 2.4: Periodic orbits of the vertical family obtained from the truncated normal form. They can be easily computed by using a standard continuation method, but by the normal form are trivial to obtain putting $I_1 = I_2 = 0$, and using I_0 as a parameter in the family. Here, we plot the projections (x, y) and (x, z) of the orbits corresponding to I_0 from -8×10^{-3} to 8×10^{-3} with step 10^{-3}. We recall that the orbit with $I_0 = 0$ is the initial one.

Finally, we comment a method to estimate the error on the determination of these tori. If we neglect the errors on the different compositions and of the numerical integrator, we can assume that it is enterely due to the truncated normal form. If we take an initial condition expressed in the coordinates of the truncated normal form, we can explicitly compute the intrinsic frequencies of the corresponding invariant torus, and hence, it can be easily integrated up to time T by the flow of this truncated normal form. Then, if we send the initial and final points to the coorresponding coordinates of the RTBP, and we transform the initial one by the flow of the RTBP up to time T_f, we can compare both the final points. If there were no error in the determination of the torus, both points should coincide. Their difference is an estimate for the error in the determination of the torus.

Note that if we perform this numerical integration for very long time spans, we will have an extra source of error, coming from the finite precision in the intrinsic frequencies: when we integrate the torus in the normal form coordinates, the product of these frequencies with T_f modulus 2π is considered. This operation acts as a Bernoulli shift on the significative digits of the frequencies as we increase T, despite on the precision of the initial condition on the torus.

2.3.7 Software

The software used has been developed by the authors in C language, and it is specially adapted to the problem. It consist, roughly speaking, in an algebraic manipulator to perform the basic operations (sums, products, Poisson brackets, ...) for homogeneous polynomials in 5 variables, having as coefficients trigonometric polynomials of some finite (and fixed) order. This strategy improves, in several orders of magnitude, the efficiency (both in speed and memory) obtained by using commercial algebraic manipulators.

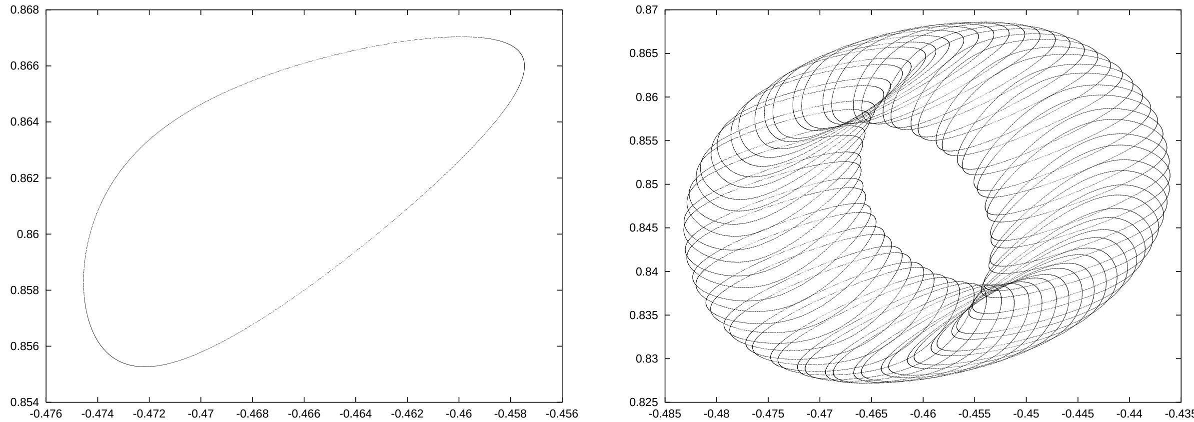

Figure 2.5: These figures correspond to a 2-D torus near the periodic orbit, given by $I_0 = I_1 = 0$ and $I_2 = 1 \times 10^{-6}$. Left: (x, y) projection of the Poincaré section $z = 0$ of the torus (points plotted up to time 10000). Right: the same projection without the Poicaré section (also plotted up to time 5000). The intrinsic frequencies of this torus (with the determination given by the truncated normal form) are $\omega_0 = 0.9995542$ and $\omega_2 = 0.3315682$. The normal one is $\omega_1 = -0.2528625$.

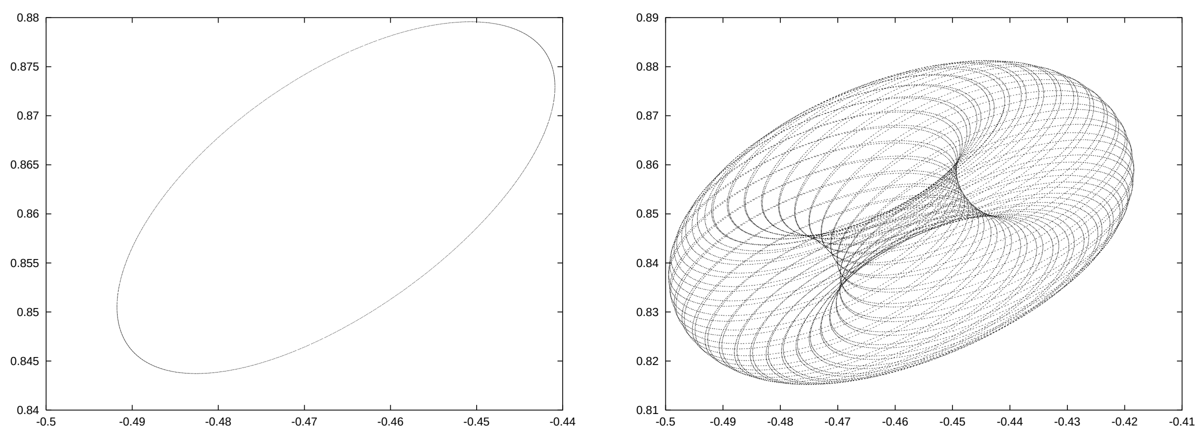

Figure 2.6: Like Figure 2.5, but for a torus in the other family around the initial orbit: $I_0 = I_2 = 0$, and $I_1 = 1 \times 10^{-5}$. In this case, the first figure is plotted up to time 10000 and the second one up to time 5000. The intrinsic frequencies of this torus are $\omega_0 = 0.9995746$ and $\omega_1 = -0.2523862$, and the normal one is $\omega_2 = 0.3330679$.

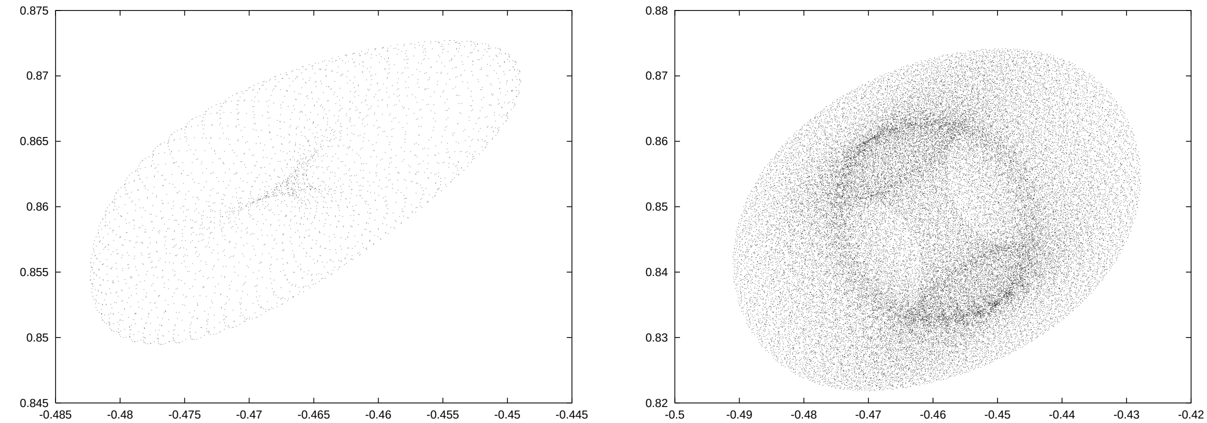

Figure 2.7: Like Figure 2.5, but now for a 3-D torus. It is obtained putting $I_0 = 0$ and $I_1 = I_2 = 1 \times 10^{-6}$. The first figure is obtained integrating up to time 20000. The second one up to time 5000. The intrinsic frequencies of this torus are $\omega_0 = 0.9995565$, $\omega_1 = -0.2527935$ and $\omega_2 = 0.3317656$.

2.4 Proof of Proposition 2.1

In this section we prove Proposition 2.1 (see Section 2.2.2) to bound the domain of definition of the adapted system of coordinates introduced by (2.16).

To this end, to work with the transformation (2.18) we introduce

$$F(\theta,\beta) = \hat{g}(\theta) + i\hat{f}(\theta) + (\hat{f}'(\theta) - i\hat{g}'(\theta))\beta \equiv F_0(\theta) + F_1(\theta)\beta,$$

$$G(\theta,\beta) = \hat{g}(\theta) - i\hat{f}(\theta) + (\hat{f}'(\theta) + i\hat{g}'(\theta))\beta \equiv G_0(\theta) + G_1(\theta)\beta.$$

With these notations, we can write the relations

$$\begin{aligned} y + ix &= A\exp(i\theta)(1+\beta) + F(\theta,\beta), \\ y - ix &= A\exp(-i\theta)(1+\beta) + G(\theta,\beta). \end{aligned} \tag{2.60}$$

Before continue the proof, let us introduce some additional notations. To deal with the different 2π-periodic functions on θ, we introduce a new variable $\zeta = \exp(i\theta)$, that overcomes the difficulty of the identification modulus 2π of the variable θ. Hence, given $h(\theta)$, a 2π-periodic analytic function on θ, we have that the Fourier expansion of h,

$$h(\theta) = \sum_{k\in\mathbb{Z}} h_k \exp(ik\theta),$$

becomes in this new variable a Laurent expansion,

$$\hbar(\zeta) = \sum_{k\in\mathbb{Z}} h_k \zeta^k.$$

Moreover, if h converges in a complex strip of width ρ, then $\hbar$ converges in the complex annulus $\mathcal{A}(R_\zeta) \equiv \{\zeta \in \mathbb{C} : \ (R_\zeta)^{-1} \le |\zeta| \le R_\zeta\}$, with $R_\zeta = \exp(\rho)$. To work with this Laurent expansion, we introduce the norm:

$$|\hbar|_R = \sum_{k\in\mathbb{Z}} |h_k| R^{|k|},$$

The notation $|.|_R$ for this norm can be confusing, because is the same one used for the norm $|.|_\rho$ defined in (2.10), but we remark that clearly $|\hbar|_{R_\zeta} = |h|_\rho$. For further uses of this norm, we remark its multiplicative character: $|\hbar^{(1)}\hbar^{(0)}|_R \le |\hbar^{(1)}|_R|\hbar^{(0)}|_R$. Moreover, we have the following bounds related to the $|.|_R$ norm:

Lemma 2.4 *With the notations given above, and for any $\zeta_1, \zeta_2 \in \mathcal{A}(R)$, we have:*

(*i*) $|\hbar(\zeta_2) - \hbar(\zeta_1)| \le |\hbar'|_R|\zeta_2 - \zeta_1|$.

(*ii*) $|\hbar(\zeta_2) - \hbar(\zeta_1) - \hbar'(\zeta_1)(\zeta_2 - \zeta_1)| \le \frac{1}{2}|\hbar''|_R|\zeta_2 - \zeta_1|^2$.

Proof:

(*i*) We take $k \in \mathbb{Z}\backslash\{0\}$. If $k > 0$:

$$\zeta_2^k - \zeta_1^k = (\zeta_2 - \zeta_1)(\zeta_2^{k-1} + \zeta_2^{k-2}\zeta_1 + \cdots + \zeta_1^{k-1}),$$

and hence, $|\zeta_2^k - \zeta_1^k| \le kR^{k-1}|\zeta_2 - \zeta_1|$. Analogously, if $k < 0$:

$$\begin{aligned} \zeta_2^k - \zeta_1^k &= \zeta_2^k\zeta_1^k(\zeta_1^{-k} - \zeta_2^{-k}) = \zeta_2^k\zeta_1^k(\zeta_1 - \zeta_2)(\zeta_1^{-k-1} + \zeta_1^{-k-2}\zeta_2 + \cdots + \zeta_2^{-k-1}) = \\ &= (\zeta_1 - \zeta_2)(\zeta_2^k\zeta_1^{-1} + \zeta_2^{k+1}\zeta_1^{-2} + \cdots + \zeta_2^{-1}\zeta_1^{-k}), \end{aligned}$$

and then we have $|\zeta_2^k - \zeta_1^k| \le |k|R^{|k|+1}|\zeta_2 - \zeta_1|$. From here the result follows immediately.

(*ii*) In this case, for $k > 1$, we have

$$\begin{aligned} \zeta_2^k - \zeta_1^k - k\zeta_1^{k-1}(\zeta_2 - \zeta_1) &= \zeta_2^k - k\zeta_1^{k-1}\zeta_2 + (k-1)\zeta_1^k = \\ &= (\zeta_2 - \zeta_1)(\zeta_2^{k-1} + \zeta_2^{k-2}\zeta_1 + \cdots + \zeta_2\zeta_1^{k-2} + (1-k)\zeta_1^{k-1}) = \\ &= (\zeta_2 - \zeta_1)^2(\zeta_2^{k-2} + 2\zeta_2^{k-3}\zeta_1 + \cdots + (k-1)\zeta_1^{k-2}), \end{aligned}$$

and then, $|\zeta_2^k - \zeta_1^k - k\zeta_1^{k-1}(\zeta_2 - \zeta_1)| \le (1 + 2 + \cdots + (k-1))R^{k-2} = \frac{(k-1)k}{2}R^{k-2}$. If $k < 0$, we have

$$\begin{aligned} \zeta_2^k - \zeta_1^k - k\zeta_1^{k-1}(\zeta_2 - \zeta_1) &= \zeta_1^{k-1}\zeta_2^k(\zeta_1^{-k+1} - k\zeta_2^{-k+1} + (k-1)\zeta_1\zeta_2^{-k}) = \\ &= \zeta_1^{k-1}\zeta_2^k(\zeta_1 - \zeta_2)(\zeta_1^{-k} + \zeta_2\zeta_1^{-k-1} + \cdots + \zeta_2^{-k-1}\zeta_1 + k\zeta_2^{-k}) = \\ &= \zeta_1^{k-1}\zeta_2^k(\zeta_1 - \zeta_2)^2(\zeta_1^{-k-1} + 2\zeta_2\zeta_1^{-k-2} + \cdots + (-k)\zeta_2^{-k-1}) = \\ &= (\zeta_1 - \zeta_2)^2(\zeta_1^{-2}\zeta_2^k + 2\zeta_1^{-3}\zeta_2^{k+1} + \cdots + (-k)\zeta_1^{k-1}\zeta_2^{-1}), \end{aligned}$$

and then, $|\zeta_2^k - \zeta_1^k - k\zeta_1^{k-1}(\zeta_2 - \zeta_1)| \le \frac{(-k)(-k+1)}{2}R^{|k|+2}|\zeta_1 - \zeta_2|^2 = \frac{k(k-1)}{2}R^{|k|+2}|\zeta_1 - \zeta_2|^2$. Hence, the bound is proved.

■

To use this new notation, we introduce the function $\hat{F}(\zeta, \beta)$ by the identity $\hat{F}(\zeta, \beta) \equiv F(\theta, \beta)$. The same holds to define $\hat{F}_0$, $\hat{F}_1$, $\hat{G}$, $\hat{G}_0$ and $\hat{G}_1$ as a functions of ζ and β.

Then, to study the injectivity of (2.18), we take points (ζ_0, β_0) and (ζ_1, β_1) for which we assume we have the same image for x and y by the action of the transformation

$$(\zeta, \beta) \mapsto (x, y), \tag{2.61}$$

induced by (2.18). We will prove that, if $\hat{f}$ and $\hat{g}$ are small enough, this is only possible if $(\zeta_0,\beta_0)=(\zeta_1,\beta_1)$, at least if we take complex values for ζ and β close enough to $|\zeta|=1$ and $\beta=0$. To check that, we deduce from (2.60) the equalities

$$\begin{aligned} A\zeta_0(1+\beta_0)+\hat{F}(\zeta_0,\beta_0) &= A\zeta_1(1+\beta_1)+\hat{F}(\zeta_1,\beta_1),\\ A\zeta_0^{-1}(1+\beta_0)+\hat{G}(\zeta_0,\beta_0) &= A\zeta_1^{-1}(1+\beta_1)+\hat{G}(\zeta_1,\beta_1), \end{aligned} \tag{2.62}$$

and from here, we have

$$A(1+\beta_0)(\zeta_0-\zeta_1)+A\zeta_0(\beta_0-\beta_1)=A(\zeta_1-\zeta_0)(\beta_1-\beta_0)+\hat{F}_0(\zeta_1)-\hat{F}_0(\zeta_0)+\hat{F}_1(\zeta_1)\beta_1-\hat{F}_1(\zeta_0)\beta_0,$$

$$A(1+\beta_0)(\zeta_1-\zeta_0)+A\zeta_0(\beta_0-\beta_1)=\zeta_0\zeta_1(\hat{G}_0(\zeta_1)-\hat{G}_0(\zeta_0)+\hat{G}_1(\zeta_1)\beta_1-\hat{G}_1(\zeta_0)\beta_0),$$

expressions that can be written in the following form:

$$\begin{aligned} &\begin{pmatrix} A(1+\beta_0) & A\zeta_0\\ -A(1+\beta_0) & A\zeta_0\end{pmatrix}\begin{pmatrix}\zeta_0-\zeta_1\\ \beta_0-\beta_1\end{pmatrix}=\\ &=\begin{pmatrix}\hat{F}_0'(\zeta_0)(\zeta_1-\zeta_0)+\hat{F}_1'(\zeta_0)\beta_0(\zeta_1-\zeta_0)+\hat{F}_1(\zeta_0)(\beta_1-\beta_0)\\ \zeta_0^2\hat{G}_0'(\zeta_0)(\zeta_1-\zeta_0)+\zeta_0^2(\hat{G}_1(\zeta_0)(\beta_1-\beta_0)+\hat{G}_1'(\zeta_0)(\zeta_1-\zeta_0)\beta_0)\end{pmatrix}+\\ &\quad+\begin{pmatrix}\hat{R}_{\hat{F}}+A(\beta_1-\beta_0)(\zeta_1-\zeta_0)\\ \hat{R}_{\hat{G}}\end{pmatrix}, \end{aligned}\tag{2.63}$$

being

$$\begin{aligned}\hat{R}_{\hat{F}} &= \hat{F}_0(\zeta_1)-\hat{F}_0(\zeta_0)-\hat{F}_0'(\zeta_0)(\zeta_1-\zeta_0)+(\hat{F}_1(\zeta_1)-\hat{F}_1(\zeta_0)-\hat{F}_1'(\zeta_0)(\zeta_1-\zeta_0))\beta_0+\\ &\quad+(\hat{F}_1(\zeta_1)-\hat{F}_1(\zeta_0))(\beta_1-\beta_0),\end{aligned}\tag{2.64}$$

and

$$\begin{aligned}\hat{R}_{\hat{G}} &= \zeta_0(\zeta_1-\zeta_0)^2\hat{G}_0'(\zeta_0)+\zeta_0(\zeta_1-\zeta_0)(\hat{G}_1(\zeta_1)(\beta_1-\beta_0)+\hat{G}_1'(\zeta_0)(\zeta_1-\zeta_0)\beta_0)+\\ &\quad+\zeta_0^2(\hat{G}_1(\zeta_1)-\hat{G}_1(\zeta_0))(\beta_1-\beta_0)+\zeta_0\zeta_1(\hat{G}_0(\zeta_1)-\hat{G}_0(\zeta_0)-\hat{G}_0'(\zeta_0)(\zeta_1-\zeta_0))+\\ &\quad+\zeta_0\zeta_1(\hat{G}_1(\zeta_1)-\hat{G}_1(\zeta_0)-\hat{G}_1'(\zeta_0)(\zeta_1-\zeta_0))\beta_0.\end{aligned}\tag{2.65}$$

From (2.63), we obtain

$$\begin{aligned} &\begin{pmatrix}\zeta_0-\zeta_1\\ \beta_0-\beta_1\end{pmatrix}=\\ &=\begin{pmatrix}\frac{1}{2A(1+\beta_0)} & -\frac{1}{2A(1+\beta_0)}\\ \frac{1}{2A\zeta_0} & \frac{1}{2A\zeta_0}\end{pmatrix}\begin{pmatrix}\hat{F}_0'(\zeta_0)+\hat{F}_1'(\zeta_0)\beta_0 & \hat{F}_1(\zeta_0)\\ \zeta_0^2\hat{G}_0'(\zeta_0)+\zeta_0^2\hat{G}_1'(\zeta_0)\beta_0 & \zeta_0^2\hat{G}_1(\zeta_0)\end{pmatrix}\begin{pmatrix}\zeta_1-\zeta_0\\ \beta_1-\beta_0\end{pmatrix}+\\ &\quad+\begin{pmatrix}\frac{1}{2A(1+\beta_0)} & -\frac{1}{2A(1+\beta_0)}\\ \frac{1}{2A\zeta_0} & \frac{1}{2A\zeta_0}\end{pmatrix}\begin{pmatrix}\hat{R}_{\hat{F}}+A(\beta_1-\beta_0)(\zeta_1-\zeta_0)\\ \hat{R}_{\hat{G}}\end{pmatrix}\equiv\\ &\equiv M\begin{pmatrix}\zeta_1-\zeta_0\\ \beta_1-\beta_0\end{pmatrix}+N.\end{aligned}\tag{2.66}$$

At this point, we proceed taking the domain for ζ and β given in the statement of the proposition, where we expect the transformation (2.61) to be injective. In what follows we describe how to test if this assumption holds. We recall that in this statement we are assuming $\hat{f}$ and $\hat{g}$ well defined in a complex strip of width ρ, for certain $\rho > 0$, that is, if $|\mathrm{Im}(\theta)| \leq \rho$. In consequence, $\hat{F}_0$, $\hat{G}_0$, $\hat{F}_1$ and $\hat{G}_1$ are analytic in $\mathcal{A}(R_\zeta)$. With respect to β, we consider complex values with $|\beta| \leq R_\beta$. Then, we want to check the injectivity of the transformation in the domain generated by fixed values of δ and R_β.

For this purpose, we use Lemma 2.4 to bound the expressions $\hat{R}_{\hat{F}}$ and $\hat{R}_{\hat{G}}$ of (2.64) and (2.65) in the annulus $\mathcal{A}(R_\zeta)$. It can be done bounding the $|.|_{R_\zeta}$ norm of the ζ-functions $\hat{F}_0$, $\hat{F}_0'$, $\hat{F}_0''$, $\hat{G}_0$, $\hat{G}_0'$ and $\hat{G}_0''$. Then, we obtain bounds for the components of N, $N^\top = (N_1, N_2)$, of the form:

$$|N_j| \leq N_{j,0}|\zeta_1 - \zeta_0|^2 + 2N_{j,1}|\zeta_1 - \zeta_0||\beta_1 - \beta_0|, \ j = 1, 2,$$

that holds if $\zeta_0, \zeta_1 \in \mathcal{A}(R_\zeta)$ and $|\beta_0|, |\beta_1| \leq R_\beta$. Using these estimates, we obtain from (2.66),

$$\begin{aligned} &|\zeta_0 - \zeta_1| + |\beta_0 - \beta_1| \leq \\ &\leq \frac{1}{1 - M_*}\Big((N_{1,0} + N_{2,0})|\zeta_1 - \zeta_0|^2 + 2(N_{1,1} + N_{2,1})|\zeta_1 - \zeta_0||\beta_1 - \beta_0|\Big), \end{aligned} \tag{2.67}$$

where M_*, that we assume verifies $M_* < 1$, is a bound (in the considered domain) for the matrix norm of M induced by the vectorial norm of $|.|_1$ of $\mathbb{C}^2$ (see Section 2.2). Then, for the admissible values of (ζ_j, β_j), $j = 0, 1$, for which (2.67) holds, we have that either $(\zeta_0, \beta_0) = (\zeta_1, \beta_1)$ or

$$1 \leq \frac{|\zeta_1 - \zeta_0|}{1 - M_*} \max\{N_{1,0} + N_{2,0}, 2(N_{1,1} + N_{2,1})\}. \tag{2.68}$$

From (2.68) we deduce the local injectivity of (2.61). The maximum value allowed for $|\zeta_1 - \zeta_0|$ defines $\delta_0(R_\beta, \rho)$ on the statement of Proposition 2.1. To deduce global injectivity, we write (2.62) in the following form

$$\begin{aligned} &\begin{pmatrix} \zeta_0 - \zeta_1 - \frac{1}{2(1+\beta_0)}(\beta_1 - \beta_0)(\zeta_1 - \zeta_0) \\ \beta_0 - \beta_1 - \frac{1}{2\zeta_0}(\beta_1 - \beta_0)(\zeta_1 - \zeta_0) \end{pmatrix} = \\ &= \begin{pmatrix} \frac{1}{2A(1+\beta_0)} & -\frac{1}{2A(1+\beta_0)} \\ \frac{1}{2A\zeta_0} & \frac{1}{2A\zeta_0} \end{pmatrix} \begin{pmatrix} \hat{F}(\zeta_1, \beta_1) - \hat{F}(\zeta_0, \beta_0) \\ \zeta_0\zeta_1(\hat{G}(\zeta_1, \beta_1) - \hat{G}(\zeta_0, \beta_0)) \end{pmatrix} \equiv S. \end{aligned} \tag{2.69}$$

Assuming that if $\zeta_0, \zeta_1 \in \mathcal{A}(R_\zeta)$ and $|\beta_0|, |\beta_1| \leq R_\beta$, then the components of S are bounded by S_1 and S_2, we have that:

$$\begin{aligned} |\zeta_0 - \zeta_1|\left(1 - \frac{1}{2(1 - R_\beta)}|\beta_1 - \beta_0|\right) &\leq S_1, \\ |\beta_1 - \beta_0|\left(1 - \frac{R_\zeta}{2}|\zeta_1 - \zeta_0|\right) &\leq S_2, \end{aligned} \tag{2.70}$$

From that, we deduce that the pairs (ζ_0, β_0) and (ζ_1, β_1) in the considered domain, with the same image by (2.61), are necessarily close. As we have $|\beta_1 - \beta_0| \leq 2R_\beta$, we deduce from (2.70) that it is necessary to take

$$|\zeta_0 - \zeta_1| \leq \frac{S_1}{1 - \frac{R_\beta}{1 - R_\beta}} \equiv \delta_1(R_\beta, \rho), \tag{2.71}$$

where if $R_\beta < 1/2$, we have $\delta_1 > 0$. Assuming δ_1 smaller than the value of δ_0 that guarantee local injectivity in (2.68), we have global injectivity for $\zeta \in \mathcal{A}(R_\zeta)$ and $|\beta| \le R_\beta$, that is, in terms of θ, if $|\mathrm{Im}(\theta)| \le \rho$.

Chapter 3

Persistence of Lower Dimensional Tori under Quasiperiodic Perturbations

3.1 Introduction

In this chapter, we will develop a perturbation theory for lower dimensional torus, focussing on the case in which the perturbation is analytic and also depends on time in a quasiperiodic way, with s basic frequencies. The Hamiltonian is of the form

$$H(\theta, x, I, y) = \tilde{\omega}^{(0)\top}\tilde{I} + H_0(\hat{\theta}, x, \hat{I}, y) + H_1(\hat{\theta}, \tilde{\theta}, x, \hat{I}, y), \tag{3.1}$$

with respect to the symplectic form $d\hat{\theta} \wedge d\hat{I} + d\tilde{\theta} \wedge d\tilde{I} + dx \wedge dy$. Here, $\hat{\theta}$ are the angular variables that describe an initial r-dimensional torus of H_0, x and y are the normal directions to the torus, $\tilde{\theta}$ are the angular variables that denotes the time, $\tilde{I}$ are the corresponding momenta (that has only been added to put the Hamiltonian in autonomous form) and $\tilde{\omega}^{(0)} \in \mathbb{R}^s$ is the frequency associated to time.

Then, the estimates on the measure of invariant tori have been obtained for two different formulations of the problem. In the first one, we study the persistence of a single invariant torus of the initial Hamiltonian, under a quasiperiodic time-dependent perturbation, using as a parameter the size (ε) of this perturbation. Our results show that this torus can be continued for a Cantor set of values of ε, adding the perturbing frequencies to the ones it already have. Moreover, if $\varepsilon \in [0, \varepsilon_0]$, the measure of the complementary of that Cantor set is exponentially small with ε_0. If the perturbation is autonomous this result is already contained in [30] but for 4-D symplectic maps.

The second approach is to fix the size of the pertubation to a given (and small enough) value. Then it is possible that the latter result can not be applied because ε can be in the complementary of the above-defined Cantor set. In this case, it is still possible to prove the existence of invariant tori with $r + s$ basic frequencies. The first of which are close to those of the unperturbed torus and the last ones those of the perturbation. These tori are a Cantor family parametrized (for instance) by the frequencies of the unperturbed problem. Again, the measure of the complementary of this Cantor set is exponentially small with the distance to the frequencies of the initial torus.

We note that, when the perturbation is autonomous and the size of the perturbation is fixed, we are proving, for the perturbed Hamiltonian, the existence of a Cantor family of invariant tori near the initial one, with measure for the set of frequencies for which the construction is not possible bounded by exponentially small with the distance to the initial tori. This result also follows from Chapter 1 (Theorems 1.3 and 1.4).

The main trick in the proofs, to overcome the lack of parameters of the problem (see section 0.1.3), is to assume that the normal frequencies move as a function of ε (then we derive the existence of the torus for a Cantor set of ε) or as a function of the intrinsic frequencies (then we obtain the existence of the above-mentioned family of tori, close to the initial one). In the last case, this is the same idea used in Chapter 1 to derive analogous estimates on the families of invariant tori around a given partially elliptic one.

When the initial torus is normally hyperbolic we do not need to control the eigenvalues in the normal direction and, hence, we do not have to deal with the lack of parameters. Of course, in this case the results are much better and the proofs can be seen as simplifications of the ones contained here. Hence, this case is not explicitly considered.

We have also included examples where the application of these results helps to understand the dynamics of concrete models of celestial mechanics.

The chapter has been organized as follows: section 3.2 contains the main ideas used to derive these results. Section 3.3 contains the rigorous statement of the results. The applications of these results to some concrete problems can be found in section 3.4 and, finally, section 3.5 contains the technical details of the proofs.

3.2 Main ideas

Let $\mathcal{H}$ be an autonomous analytic Hamiltonian system of ℓ degrees of freedom in $\mathbb{C}^{2\ell}$ having an invariant r-dimensional torus, $0 \le r \le \ell$, with a quasiperiodic flow given by the vector of basic frequencies $\hat{\omega}^{(0)} \in \mathbb{R}^r$. Let us consider the (perturbed) Hamiltonian system $H = \mathcal{H} + \varepsilon\hat{\mathcal{H}}$, where $\hat{\mathcal{H}}$ is also analytic. As it has been mentioned before, we do not restrict ourselves to the case of autonomous perturbations, but we will assume that $\hat{\mathcal{H}}$ depends on time in a quasiperiodic way, with vector of basic frequencies given by $\tilde{\omega}^{(0)} \in \mathbb{R}^s$.

As in Chapter 1, we will assume that the torus is isotropic and reducible. Moreover, we will assume that the eigenvalues of the reduced matrix are all different (this condition implies, from the canonical character of the system, that they are also non-zero).

In what follows, we will always assume that vectors are "matrices with one column", so the scalar product between two vectors u and v will be denoted by $u^\top v$.

Then, we use the reducible and isotropic character of the torus in the same way than Chapter 1 (see Section 1.2.1): we assume that we can introduce (with a canonical change of coordinates) r angular variables $\hat{\theta}$ describing the initial torus such that the Hamiltonian takes the form

$$\mathcal{H}(\hat{\theta}, x, \hat{I}, y) = \hat{\omega}^{(0)\top}\hat{I} + \frac{1}{2}z^\top \mathcal{B} z + \mathcal{H}_*(\hat{\theta}, x, \hat{I}, y),$$

where $z^\top = (x^\top, y^\top)$, being z, $\hat{\theta}$ and $\hat{I}$ complex vectors, x and y elements of $\mathbb{C}^r$ and $\hat{\theta}$ and $\hat{I}$ elements of $\mathbb{C}^s$, with $r + m = \ell$. Here, $\hat{\theta}$ and x are the positions and $\hat{I}$ and y

are the conjugate momenta. In this notation, $\mathcal{B}$ is a symmetric $2m$-dimensional matrix (with complex coefficients). Moreover, $\mathcal{H}_*$ is an analytic function (with respect to all its arguments) with 2π-periodic dependence on $\hat{\theta}$. More concretely, we will assume that it is analytic on a neighbourhood of $z = 0$, $\hat{I} = 0$, and on a complex strip of positive width ρ for the variable $\hat{\theta}$, that is, if $|\mathrm{Im}\hat{\theta}_j| \leq \rho$, for all $j = 1, \dots, r$. Then, if we assume that $\mathcal{H}$ has an invariant r-dimensional torus with vector of basic frequencies $\hat{\omega}^{(0)}$, given by $\hat{I} = 0$ and $z = 0$, this implies that the Taylor expansion of $\mathcal{H}_*$ must begin with terms of second order in the variables $\hat{I}$ and z. As we have that the normal variational flow around this torus can be reduced to constant coefficients, we can assume that the quadratic terms of $\mathcal{H}_*$ in the z variables vanish. Hence, the normal variational equations are given by the matrix $J_m\mathcal{B}$, where J_m is the canonical 2-form of $\mathbb{C}^{2m}$. We also assume that the matrix $J_m\mathcal{B}$ is in diagonal form with different eigenvalues $\lambda^\top = (\lambda_1, \dots, \lambda_m, -\lambda_1, \dots, -\lambda_m)$.

A point worth to comment is the real or complex character of the matrix $\mathcal{B}$. In this chapter we work, in principle, with complex analytic Hamiltonian systems, but the most interesting case happens when we deal with real analytic ones, and when the initial torus is also real. In this case, to guarantee that the perturbative scheme preserves the real character of the tori, we want that the initial reduced matrix $\mathcal{B}$ comes from a real matrix. We note that this is equivalent to assume that if λ is an eigenvalue of $J_m\mathcal{B}$, then $\bar{\lambda}$ is also an eigenvalue. The fact that $\mathcal{B}$ is real guarantees that all the tori obtained in this chapter also real. To see that, we note that we can use the same proof but putting $J_m\mathcal{B}$ in real normal form instead of diagonal form, and this makes that all the steps of the proof are also real. However, the technical details in this case are a little more tedious and, hence, we have prefered to work with a diagonal $J_m\mathcal{B}$.

3.2.1 Normal form around the initial torus

The first step is to rearrange the initial Hamiltonian $\mathcal{H}^{(0)} \equiv \mathcal{H}$ in a suitable form to apply an inductive procedure.

In what follows, we will define degree of a monomial $z^l\hat{I}^j$ as $|l|_1+2|j|_1$. This definition is motivated below. Let us expand $\mathcal{H}_*^{(0)}$ in power series with respect to z and $\hat{I}$ around the origin:

$$\mathcal{H}_*^{(0)} = \sum_{d\geq 2} \mathcal{H}_d^{(0)},$$

where $\mathcal{H}_d^{(0)}$ are homogeneous polynomials of degree d, that is,

$$\mathcal{H}_d^{(0)} = \sum_{\substack{l\in\mathbb{N}^{2m},\, j\in\mathbb{N}^r \\ |l|_1+2|j|_1=d}} h_{l,j}^{(0)}(\hat{\theta}) z^l \hat{I}^j.$$

We also expand the (periodic) coefficients in Fourier series:

$$h_{l,j}^{(0)}(\hat{\theta}) = \sum_{k\in\mathbb{Z}^r} h_{l,j,k}^{(0)} \exp{(ik^\top\hat{\theta})}, \tag{3.2}$$

being $i = \sqrt{-1}$. The definition of degree for a monomial $z^l\hat{I}^j$ counting twice the contribution of the variable $\hat{I}$ is motivated by the definition of the Poisson bracket of two functions depending on $(\hat\theta, x, \hat I, y)$:

$$\{f,g\} = \frac{\partial f}{\partial \hat\theta}\left(\frac{\partial g}{\partial \hat I}\right)^\top - \frac{\partial f}{\partial \hat I}\left(\frac{\partial g}{\partial \hat\theta}\right)^\top + \frac{\partial f}{\partial z} J_m \left(\frac{\partial g}{\partial z}\right)^\top.$$

Note that, if f is an homogeneous polynomial of degree d_1 and g is an homogeneous polynomial of degree d_2 then $\{f,g\}$ is an homogeneous polynomial of degree d_1+d_2-2. This property shows that if we try to construct canonical changes using the Lie series method, a convenient way to put $\mathcal{H}^{(0)}$ in normal form is to remove in an increasing order the terms of degree 3, 4, ..., with a suitable generating function.

To introduce some of the parameters (see section 3.2.4), it is very convenient that the initial Hamiltonian has the following properties:

P1 The coefficients of the monomials $(z,\hat I)$ (degree 3) and $(z,\hat I,\hat I)$ (degree 5) are zero.

P2 The coefficients of the monomials $(z,z,\hat I)$ (degree 4) and $(\hat I,\hat I)$ (degree 4) do not depend on $\hat\theta$ and, in the case of $(z,z,\hat I)$, they vanish except for the coefficients of the trivial resonant terms.

Here, we have used the following notation: for instance, by the terms of order $(z,z,\hat I)$ we denote the monomials $z^l \hat I^j$, with $|l|_1 = 2$ and $|j|_1 = 1$, with the corresponding coefficients. We will apply three steps of a normal form procedure in order to achieve these conditions. Each step is done using a generating function of the following type:

$$S^{(n)}(\hat\theta, x, \hat I, y) = \sum_{\substack{l\in\mathbb{N}^{2m},\, j\in\mathbb{N}^r \\ |l|_1+2|j|_1=n}} s_{l,j}^{(n)}(\hat\theta) z^l \hat I^j,$$

for $n=3$, 4 and 5. Then, if we denote by $\Psi^{S^{(n)}}$ the flow at time one of the Hamiltonian system associated to $S^{(n)}$, we transform the initial Hamiltonian into

$$\begin{aligned}
\mathcal{H}^{(n-2)} &= \mathcal{H}^{(n-3)} \circ \Psi^{S^{(n)}} = \\
&= \mathcal{H}^{(n-3)} + \{\mathcal{H}^{(n-3)}, S^{(n)}\} + \frac{1}{2!}\{\{\mathcal{H}^{(n-3)}, S^{(n)}\}, S^{(n)}\} + O_{n+1} = \\
&= \hat\omega^{(0)\top}\hat I + \frac{1}{2} z^\top \mathcal{B} z + \mathcal{H}_*^{(n-3)} + \{\hat\omega^{(0)\top}\hat I + \frac{1}{2} z^\top \mathcal{B} z, S^{(n)}\} + O_{n+1},
\end{aligned}$$

for $n=3,4,5$. In each step, we take $S^{(n)}$ such that $\mathcal{H}_n^{(n-3)} + \{\hat\omega^{(0)\top}\hat I + \frac{1}{2} z^\top \mathcal{B} z, S^{(n)}\}$ satisfies conditions **P1** and **P2** for the monomials of degree n ($n=3,4,5$). To compute $S^{(n)}$ we expand $\mathcal{H}_n^{(n-3)}$ and we find (formally) an expansion for $S^{(n)}$:

$$s_{l,j,k}^{(n)} = \frac{h_{l,j,k}^{(n-3)}}{ik^\top \hat\omega^{(0)} + l^\top \lambda},$$

where the indices have the same meaning as in (3.2). If we split $l=(l_x,l_y)$ ($z^l = x^{l_x} y^{l_y}$), the exactly resonant terms correspond to $k=0$ and $l_x = l_y$ (we recall that $\lambda^\top = (\lambda_1,\ldots,\lambda_m,-\lambda_1,\ldots,-\lambda_m)$). Hence, it would be possible to formally compute a normal form depending only on $\hat{I}$ and the products $x_j y_j$, $j = 1, \ldots, m$. As it has been mentioned before, our purpose is much more modest. To kill the monomials mentioned above (in conditions **P1** and **P2**) with a convergent change of variables, one needs a condition on the smallness of $|ik^\top \hat{\omega}^{(0)} + l^\top \lambda|$, $k \in \mathbb{Z}^r \setminus \{0\}$, $l \in \mathbb{N}^{2m}$ and $|l|_1 \leq 2$. We have used the usual one

$$|ik^\top \hat{\omega}^{(0)} + l^\top \lambda| \geq \frac{\mu_0}{|k|_1^\gamma},$$

that we will assume true in the statement of the results. We notice that with these conditions we can construct convergent expressions for the different generating functions $S^{(n)}$, $n = 3, 4, 5$, to achieve conditions **P1** and **P2**. We can call this process a seminormal form construction.

Then, the final form for the Hamiltonian is

$$\mathcal{H} = \hat{\omega}^{(0)\top} \hat{I} + \frac{1}{2} z^\top \mathcal{B} z + \frac{1}{2} \hat{I}^\top \mathcal{C} \hat{I} + H_*(\hat{\theta}, x, \hat{I}, y), \tag{3.3}$$

for which conditions **P1** and **P2** holds. Here, $\mathcal{C}$ is a symmetric constant matrix and we will assume the standard nondegeneracy condition,

NDC1 $\det \mathcal{C} \neq 0$.

Now let us introduce the quasiperiodic time-dependent perturbation. To simplify the notation, we write this perturbation in the normal form variables, and we add this perturbation to (3.3). We call H to the new Hamiltonian:

$$H(\theta, x, I, y, \varepsilon) = \omega^{(0)\top} I + \frac{1}{2} z^\top \mathcal{B} z + \frac{1}{2} \hat{I}^\top \mathcal{C} \hat{I} + H_*(\hat{\theta}, x, \hat{I}, y) + \varepsilon \hat{\mathcal{H}}(\theta, x, \hat{I}, y, \varepsilon), \tag{3.4}$$

for a fixed $\omega^{(0)\top} = (\hat{\omega}^{(0)\top}, \tilde{\omega}^{(0)\top})$, $\omega^{(0)} \in \mathbb{R}^{r+s}$, where $\theta^\top = (\hat{\theta}^\top, \tilde{\theta}^\top)$, $I^\top = (\hat{I}^\top, \tilde{I}^\top)$ and $z^\top = (x^\top, y^\top)$, being $\tilde{\theta}$, $\tilde{I}$ (s-dimensional complex vectors) the new positions and momenta added to put in autonomous form the quasiperiodic perturbation. Hence, H is 2π-periodic in θ. Moreover, ε is a small positive parameter. This is the Hamiltonian that we consider in the formulation of the results.

3.2.2 The iterative scheme

Before the explicit formulation of the results, let us describe a generic step of the iterative method used in the proof. So, let us consider a Hamiltonian of the form:

$$H(\theta, x, I, y) = \omega^{(0)\top} I + \frac{1}{2} z^\top \mathcal{B} z + \frac{1}{2} \hat{I}^\top \mathcal{C}(\theta) \hat{I} + H_*(\theta, x, \hat{I}, y) + \varepsilon \hat{H}(\theta, x, \hat{I}, y), \tag{3.5}$$

with the same notations of (3.4), where we assume that skipping the term $\varepsilon \hat{H}$, we have that $z = 0$, $\hat{I} = 0$ is a reducible $(r+s)$-dimensional torus with vector of basic frequencies $\omega^{(0)}$, such that the variational normal flow is given by $J_m \mathcal{B} = \operatorname{diag}(\lambda_1, \ldots, \lambda_m, -\lambda_1, \ldots, -\lambda_m)$, and that $\det \bar{\mathcal{C}} \neq 0$, where $\bar{\mathcal{C}}$ means the average of $\mathcal{C}$ with respect to its angular variables (although initially $\mathcal{C}$ does not depend on θ, during the iterative scheme it will). Moreover, we suppose that in H_* the terms of order $(\hat{I}, z)$ vanish (that is, we suppose that the "central" and "normal" directions of the unperturbed torus have been uncoupled up to first order). Here we only use the parameter ε to show that the perturbation $\varepsilon\hat{H}$ is of $O(\varepsilon)$.

We expand $\hat{H}$ in power series around $\hat{I}=0$, $z=0$ and we add these terms to the previous expansion of the unperturbed Hamiltonian. This makes that the initial torus is not longer invariant. Hence, the expression of the Hamiltonian must be (without writting explicitly the dependence on ε):

$$H(\theta,x,I,y)=\tilde{\omega}^{(0)\top}\tilde{I}+H^*(\theta,x,\hat{I},y), \tag{3.6}$$

where

$$H^*=a(\theta)+b(\theta)^\top z+c(\theta)^\top\hat{I}+\frac{1}{2}z^\top B(\theta)z+\hat{I}^\top E(\theta)z+\frac{1}{2}\hat{I}^\top C(\theta)\hat{I}+\Omega(\theta,x,\hat{I},y),$$

being Ω the remainder of the expansion. Looking at this expression, we introduce the notation $[H^*]_{(z,z)}=B$, $[H^*]_{(\hat{I},\hat{I})}=C$, $[H^*]_{(\hat{I},z)}=E$ and $<H^*>=H^*-\Omega$.

We have that $\tilde{a}$, b, $c-\hat{\omega}^{(0)}$, $B-\mathcal{B}$, $C-\mathcal{C}$ and E are $O(\varepsilon)$, where if $f(\theta)$ is a periodic function on θ, $\tilde{f}=f-\bar{f}$.

Note that if we are able to kill the terms $\tilde{a}$, b and $c-\hat{\omega}^{(0)}$ we obtain a lower dimensional invariant torus with intrinsic frequency $\omega^{(0)}$. We will try to do that by using a quadratically convergent scheme. As it is usual in this kind of Newton methods, it is very convenient to kill something more. Before continuing, let us introduce the following notation: if A is a $n\times n$ matrix, $\mathrm{dp}(A)$ denotes the diagonal part of A, that is, $\mathrm{dp}(A)=\mathrm{diag}(a_{1,1},\dots,a_{n,n})^\top$, where $a_{i,i}$ are the diagonal entries of A. Here, we want that the new matrix B verifies $B=\mathcal{J}_m(B)$, where we define $\mathcal{J}_m(B)=-J_m\mathrm{dp}(J_m\bar{B})$ (this is, we ask the normal flow to the torus to be reducible and given by a diagonal matrix like for the unperturbed torus) and to eliminate E (to uncouple the "central" and the normal directions of the torus up to first order in ε). Hence, the torus we will obtain has also these two properties. This is a very usual technique (see [17], [30]).

At each step of the iterative procedure, we use a canonical change of variables analogous to the one used in the iterative scheme of Chapter 1. The generating function is of the form

$$S(\theta,x,\hat{I},y)=\xi^\top\hat{\theta}+d(\theta)+e(\theta)^\top z+f(\theta)^\top\hat{I}+\frac{1}{2}z^\top G(\theta)z+\hat{I}^\top F(\theta)z, \tag{3.7}$$

where $\xi\in\mathbb{C}^r$, $\bar{d}=0$, $\bar{f}=0$ and G is a symmetric matrix with $\mathcal{J}_m(G)=0$. Keeping the same name for the new variables, the transformed Hamiltonian is

$$H^{(1)}=H\circ\Psi^S=\tilde{\omega}^{(0)\top}\tilde{I}+H^{(1)*}(\theta,x,\hat{I},y),$$

being

$$\begin{aligned}H^{(1)*}(\theta,x,\hat{I},y) &= a^{(1)}(\theta)+b^{(1)}(\theta)^\top z+c^{(1)}(\theta)^\top\hat{I}+\frac{1}{2}z^\top B^{(1)}(\theta)z+\\ &\quad+\hat{I}^\top E^{(1)}(\theta)z+\frac{1}{2}\hat{I}^\top C^{(1)}(\theta)\hat{I}+\Omega^{(1)}(\theta,x,\hat{I},y).\end{aligned}$$

We want $\tilde{a}^{(1)}=0$, $b^{(1)}=0$, $c^{(1)}-\hat{\omega}^{(0)}=0$, $E^{(1)}=0$ and $J_m B^{(1)}$ to be a constant diagonal matrix. We will show that this can be achieved up to first order in ε. So, we write those conditions in terms of the initial Hamiltonian and the generating function. Skipping terms of $O_2(\varepsilon)$, we obtain:

(eq_1) $\tilde{a}-\frac{\partial d}{\partial\theta}\omega^{(0)}=0,$

(eq_2) $b-\frac{\partial e}{\partial\theta}\omega^{(0)}+\mathcal{B}J_m e=0,$

(eq_3) $c-\hat{\omega}^{(0)}-\frac{\partial f}{\partial\theta}\omega^{(0)}-\mathcal{C}\left(\xi+\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}\right)=0,$

(eq_4) $B^*-\mathcal{J}_m(B^*)-\frac{\partial G}{\partial\theta}\omega^{(0)}+\mathcal{B}J_m G-GJ_m\mathcal{B}=0,$

(eq_5) $E^*-\frac{\partial F}{\partial\theta}\omega^{(0)}-FJ_m\mathcal{B}=0,$

where

$$B^*=B-\left[\frac{\partial H_*}{\partial\hat{I}}\left(\xi+\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}\right)-\frac{\partial H_*}{\partial z}J_m e\right]_{(z,z)} \tag{3.8}$$

and

$$E^*=E-\mathcal{C}\left(\frac{\partial e}{\partial\hat{\theta}}\right)^{\top}-\left[\frac{\partial H_*}{\partial\hat{I}}\left(\xi+\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}\right)-\frac{\partial H_*}{\partial z}J_m e\right]_{(\hat{I},z)}. \tag{3.9}$$

Here we denote by $\frac{\partial}{\partial q}$ the matrix of partial derivatives with respect to the variables q and, for instance, $\frac{\partial G}{\partial\theta}\omega^{(0)}$ means $\sum_{j=1}^{r+s}\frac{\partial G}{\partial\theta_j}\omega_j^{(0)}$. These equations are solved formally by expanding in Fourier series and equating the corresponding coefficients. This leads us to the following expressions for S:

(eq_1)

$$d(\theta)=\sum_{k\in\mathbb{Z}^{r+s}\setminus\{0\}}\frac{a_k}{ik^{\top}\omega^{(0)}}\exp(ik^{\top}\theta).$$

(eq_2) If we put $e^{\top}=(e_1,\dots,e_{2m})$,

$$e_j(\theta)=\sum_{k\in\mathbb{Z}^{r+s}}\frac{b_{j,k}}{ik^{\top}\omega^{(0)}+\lambda_j}\exp(ik^{\top}\theta).$$

(eq_3)

$$\xi=(\bar{\mathcal{C}})^{-1}\left(\bar{c}-\hat{\omega}^{(0)}-\overline{\mathcal{C}\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}}\right),$$

and if we define

$$c^*=\tilde{c}-\tilde{\mathcal{C}}\xi-\mathcal{C}\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}+\overline{\mathcal{C}\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}},$$

we have for $f^{\top}=(f_1,\dots,f_r)$:

$$f_j(\theta)=\sum_{k\in\mathbb{Z}^{r+s}\setminus\{0\}}\frac{c^*_{j,k}}{ik^{\top}\omega^{(0)}}\exp(ik^{\top}\theta).$$

(eq_4) If we define

$$B^{**} = B^{*} - \mathcal{J}_m(B^{*}), \tag{3.10}$$

then we have for $G = (G_{j,l})$, $1 \le j, l \le 2m$,

$$G_{j,l}(\theta) = \sum_{k\in\mathbb{Z}^{r+s}} \frac{B^{**}_{j,l,k}}{ik^\top\omega^{(0)} + \lambda_j + \lambda_l} \exp(ik^\top\theta), \quad j,l = 1,\dots,2m.$$

In the definition of $G_{j,l}$, we notice that we have trivial zero divisors when $|j-l| = m$ and $k = 0$, but from the expression of B^{**}, in these cases the coefficient $B^{**}_{j,l,0}$ is 0. Moreover, the matrix G is symmetric.

(eq_5) If $F = (F_{j,l})$, $j = 1,\dots,r$ and $l = 1,\dots,2m$, then

$$F_{j,l}(\theta) = \sum_{k\in\mathbb{Z}^{r+s}} \frac{E^{*}_{j,l,k}}{ik^\top\omega^{(0)} + \lambda_l} \exp(ik^\top\theta).$$

Note that if we have Diophantine hypothesis on the small divisors of these expressions,

$$|ik^\top\omega^{(0)} + l^\top\lambda| \ge \frac{\mu_0}{|k|_1^\gamma}, \quad k\in\mathbb{Z}^{r+s}\setminus\{0\}, \quad l\in\mathbb{N}^{2m}, \quad |l|_1 \le 2, \quad \gamma > r+s-1, \tag{3.11}$$

we can guarantee the convergence of the expansion of S. We assume that they hold in the first step, and we want to have similar conditions after each step of the process, to be able to iterate. As the frequencies $\tilde\omega^{(0)}$ are fixed in all the process and $\hat\omega^{(0)}$ can be preserved by the nondegeneracy and the kind of generating function we are using (this is done by the ξ term), we will be able to recover the Diophantine properties on them. The main problem are the eigenvalues λ, because, in principle, we can not preserve their value. Hence, we will control the way they vary, to try to ensure they are still satisfying a good Diophantine condition. Our first approach is to consider λ as a function of ε (the size of the perturbation). This leads us to eliminate a Cantor set of values of these parameters in order to have all the time good (in a Diophantine sense) values of λ. Another possibility is to consider λ as a function of the (frequencies of the) torus. This leads us to eliminate a Cantor set of those tori. Both procedures require some non-degeneracy conditions.

3.2.3 Estimates on the measure of preserved tori

The technique we are going to apply to produce exponentially small estimates has already been used in [35]. It is based on working at every step n of the iterative procedure with values of ε for which we have Diophantine conditions of the type

$$|ik^\top\omega^{(0)} + \lambda_l^{(n)}(\varepsilon)| \ge \frac{\mu_n}{|k|_1^\gamma} \exp(-\delta_n|k|_1), \quad k\in\mathbb{Z}^{r+s}\setminus\{0\}, \tag{3.12}$$

where $\lambda_l^{(n)}(\varepsilon)$ denotes the eigevalues of $J_m\mathcal{B}^{(n)}(\varepsilon)$, being $\mathcal{B}^{(n)}(\varepsilon)$ the matrix that replaces $\mathcal{B}$ after n steps of the iterative process. Of course, we ask for the same condition for the sum of eigenvalues $\lambda_j^{(n)} + \lambda_l^{(n)}$. We will see that, if we take a suitable sequence of δ_n, the exponential term in (3.12) is not an obstruction to the convergence of the scheme.

This condition will be used to obtain exponentially small estimates for the measure of the values of ε for which we do not have invariant tori of frequency $\omega^{(0)}$ in the perturbed system. The key idea can be described as follows: for the values of ε for which we can prove convergence, we obviously have that, if ε is small enough, $|\lambda_l^{(n)}(\varepsilon)-\lambda_l| \leq a\varepsilon$, at every step n. Now, if we assume that $\mu_n \leq \mu_0/2$, from the Diophantine bounds on $ik^\top\omega^{(0)}+\lambda_l$ in (3.11), we only need to worry about the resonances corresponding to values of k such that

$$|k|_1 \geq \left(\frac{\mu_0}{2a\varepsilon}\right)^{1/\gamma} \equiv K(\varepsilon).$$

This is equivalent to say that we do not have low order resonances nearby, hence we only have to eliminate higher order ones. When we eliminate the values of ε for which the Diophantine condition is not fullfiled for some k, we only need to worry about controlling the measure of the "resonant" sets associated to $|k|_1 \geq K(\varepsilon)$. From that, and from the exponential in $|k|_1$ for the admissible small divisors, we obtain exponentially small estimates for the set of values of ε for which we can not prove the existence of invariant tori. If $\varepsilon \in]0,\varepsilon_0]$ this measure is of order $\exp(-1/\varepsilon_0^c)$, for any $0<c<1/\gamma$.

Note that we have used ε because is a natural parameter of the perturbed problem, but this technique can also be applied to other parameters. We will do this in the next section.

3.2.4 Other parameters: families of lower dimensional tori

Let us consider the following truncation of the Hamiltonian (3.3):

$$\hat{\omega}^{(0)\top}\hat{I} + \frac{1}{2}z^\top\mathcal{B}z + \frac{1}{2}\hat{I}^\top\mathcal{C}\hat{I}. \tag{3.13}$$

Note that, for this truncation, there exists a r parametric family of r-dimensional invariant tori around the initial torus. One can ask what happens to this family when the nonintegrable part (including the quasiperiodic perturbations) is added. The natural parameters in this case are the frequencies $\hat{\omega}$ of the tori of the family. We will work with this parameter as follows: if for every $\hat{\omega}$ we perform the canonical transformation

$$\hat{I} \to \hat{I} + \mathcal{C}^{-1}(\hat{\omega}-\hat{\omega}^{(0)}), \tag{3.14}$$

on the Hamiltonian (3.13), we obtain (skipping the constant term) a Hamiltonian like (3.13), replacing $\hat{\omega}^{(0)}$ by $\hat{\omega}$. So, we have that $\hat{I}=0$, $z=0$ is a r-dimensional reducible torus but now with vector of basic frequencies given by $\hat{\omega}$. If we consider the Hamiltonian (3.3), and we perform the transformation (3.14), it is not difficult to see from conditions **P1** and **P2** that the $\hat{\omega}$-torus obtained from the truncated normal form remains as an invariant reducible torus for the Hamiltonian (3.3), plus an error of $O_2(\hat{\omega}-\hat{\omega}^{(0)})$. Then, the idea is to consider $\hat{\omega}$ as a new perturbative parameter (in fact the small parameter is $\hat{\omega}-\hat{\omega}^{(0)}$). In this case we can apply the same technique as in section 3.2.3 to control the measure of the destroyed tori. It turns out that this measure is exponentially small with the distance to the initial torus.

In fact, the proof has been done working simultaneously with both parameters ε and $\hat{\omega}$. This allows to derive all the results mentioned before in an unified way. To bound the measure of the eliminated set of parameters, we have chosen to work with Lipschitz regularity. Hence, we are going to follow the same methodology used in Chaper 1 to estimate this measure. For a more precise formulation of this Lipschitz dependence, we refer to section 3.5.1.

3.3 Statement of results

Now, we can state precisely the main result of this chapter, whose proof we have sketched above.

Theorem 3.1 *Let us consider a Hamiltonian of the form* (3.4)*, satisfying the following hypotheses:*

(i) H_* *and* $\hat{\mathcal{H}}$ *are analytic with respect to* $(\theta, x, \hat{I}, y)$ *around* $z = 0$ *and* $\hat{I} = 0$*, with* 2π*-periodic dependence on* θ*, for any* $\varepsilon \in \mathcal{I}_0 \equiv [0, \varepsilon_0]$*, in a domain that is independent on* ε*. The dependence on* ε *is assumed to be* C^2*, and the derivatives of the Hamiltonian* $\hat{\mathcal{H}}$ *with respect to* ε *are also analytic in* $(\theta, x, \hat{I}, y)$ *on the same domain.*

(ii) $\mathcal{B}$ *is a symmetric constant matrix such that* $J_m\mathcal{B}$ *is diagonal with different eigenvalues* $\lambda^\top = (\lambda_1, \dots, \lambda_m, -\lambda_1, \dots, -\lambda_m)$*.*

(iii) $\mathcal{C}$ *is a symmetric constant matrix with* $\det \mathcal{C} \neq 0$ *(this is the assumption* **NDC1** *above).*

(iv) For certain $\mu_0 > 0$ *and* $\gamma > r + s - 1$*, the following Diophantine conditions hold:*

$$|ik^\top \omega^{(0)} + l^\top \lambda| \geq \frac{\mu_0}{|k|_1^\gamma}, \quad k \in \mathbb{Z}^{r+s} \setminus \{0\}, \quad l \in \mathbb{N}^{2m}, \quad |l|_1 \leq 2.$$

Then, under certain generic nondegeneracy conditions for the Hamiltonian H *(that are given explicitly in* **NDC2** *at the end of section* 3.5.4*), the following assertions hold:*

(a) There exists a Cantor set $\mathcal{I}_* \subset \mathcal{I}_0$*, such that for every* $\varepsilon \in \mathcal{I}_*$ *the Hamiltonian* H *has a reducible* $(r+s)$*-dimensional invariant torus with vector of basic frequencies* $\omega^{(0)}$*. Moreover, for every* $0 < \sigma < 1$*:*

$$mes([0, \bar{\varepsilon}] \setminus \bar{\mathcal{I}}_*) \leq \exp\left(-(1/\bar{\varepsilon})^{\frac{\sigma}{\gamma}}\right),$$

if $\bar{\varepsilon}$ *is small enough (depending on* σ*) where, for every* $\bar{\varepsilon}$*,* $\bar{\mathcal{I}}_* \equiv \bar{\mathcal{I}}_*(\bar{\varepsilon}) = [0, \bar{\varepsilon}] \cap \mathcal{I}_*$*.*

(b) Given $R_0 > 0$ *small enough and a fixed* $0 \leq \varepsilon \leq R_0^{\frac{\gamma}{\gamma+1}}$*, there exists a Cantor set* $\mathcal{W}_*(\varepsilon, R_0) \subset \{\hat{\omega} \in \mathbb{R}^r : |\hat{\omega} - \hat{\omega}^{(0)}| \leq R_0\} \equiv \mathcal{V}(R_0)$*, such that for every* $\hat{\omega} \in \mathcal{W}_*(\varepsilon, R_0)$ *the Hamiltonian* H *corresponding to this fixed value of* ε*, has a reducible* $(r+s)$*-dimensional invariant torus with vector of basic frequencies* ω*,* $\omega^\top = (\hat{\omega}^\top, \tilde{\omega}^{(0)\top})$*. Moreover, for every* $0 < \sigma < 1$*, if* R_0 *is small enough (depending on* σ*):*

$$mes(\mathcal{V}(R_0) \setminus \mathcal{W}_*(\varepsilon, R_0)) \leq \exp\left(-(1/R_0)^{\frac{\sigma}{\gamma+1}}\right).$$

Here, mes(A) denotes the Lebesgue measure of the set A.

Note: Condition **NDC2** is an standard nondegeneracy condition on the normal frequencies of the initial torus. Essentially, it asks that the normal frequencies depend on ε and on the intrinsic frequencies of the basic family of tori (this family has been introduced in section 3.2.4). In order to formulate **NDC2** in an explicit form one has to perform first one step of the normal form with respect to ε (see section 3.5.4). This is the reason why we have prefered to keep this hipothesis inside the proof, where it arises naturally.

3.3.1 Remarks

The result (b) has special interest if we take $\varepsilon = 0$. It shows that for the unperturbed system, around the initial r-dimensional reducible torus there exist an r-dimensional family (with Cantor structure) of r-dimensional reducible tori parametrized by $\hat{\omega} \in \mathcal{W}_*(0, R_0)$, with relative measure for the complementary of the Cantor set exponentially small with R_0, for values of $\hat{\omega}$ R_0-close to $\hat{\omega}^{(0)}$. There are previous results on the existence of these lower dimensional tori (see the references), but the estimates on the measure of preserved tori close to a given one are not so good as the ones presented here.

Moreover, we have the same result around every $(r+s)$-dimensional torus that we can obtain for the perturbed system for some $\varepsilon \neq 0$ small enough, if we assume that their intrinsic and normal frequencies verify the same kind of Diophantine bounds as the frequencies of the unperturbed torus. In this case, for every R_0 small enough we have a (Cantor) family of $(r+s)$-dimensional reducible tori parametrized by $\hat{\omega} \in \mathcal{W}_*(\varepsilon, R_0)$, with the same kind of exponentially small measure with respect to R_0 on the complementary of this set. To prove it, we remark that we can reduce to the case $\varepsilon = 0$ if we note that it is easy to see that Theorem 3.1 also holds if the unperturbed Hamiltonian depends on θ and not only on $\hat{\theta}$ (that is, if the initial torus is $(r+s)$-dimensional).

If the initial torus is normally hyperbolic, the problem is easier. For instance, it is possible to prove the existence of invariant tori without using reducibility conditions. Then, in case (a), one obtains an open set of values of ε for which the torus exists, although its normal flow could not be reducible. The reason is that the intrinsic frequencies of the torus are fixed with respect to ε and the normal eigenvalues (that depend on ε) do not produce extra small divisors if we consider only equations $(eq_1) - (eq_3)$ of the iterative scheme described in section 3.2.2 (we take $G = 0$ and $F = 0$ in equation (3.7)). Note that now we can solve (eq_2) using a fixed point method, because the matrices $J_m\mathcal{B}^{(n)}$ are ε-close to the initial hyperbolic matrix $J_m\mathcal{B}$ (that is supposed to be reducible). This makes unnecessary to consider (eq_4) and (eq_5). Of course, the tori produced in this way are not necessarily reducible. If one wants to ensure reducibility, it is necessary to use the normal eigenvalues and this can produce (depending on some conditions on those eigenvalues, see [35]) a Cantor set of ε of the same measure as the one in (a). If we consider the case (b) when the normal behaviour is hyperbolic, the results do not change with respect to the normally elliptic case. As we are "moving" the intrinsic frequencies, we have to take out the corresponding resonances. The order (when R_0 goes to zero) of the measure of these resonances is still exponentially small with R_0 (see lemma 3.15).

Finally, let us recall that the Diophantine condition (iv) is satisfied for all the frequencies $\omega^{(0)}$ and eigenvalues λ, except for a set of zero measure.

3.4 Applications

In this section we are going to illustrate the possible applications of these results to some concrete problems of celestial mechanics. We have not included a formal verification of the several hypotheses of the theorems, but we want to make some remarks.

The nondegeneracy conditions can be checked numerically, observing if the frequencies involved depend on the parameters. As the applications we will deal with are perturbations of families of periodic orbits, this hypothesis can be easily verified computing the variation of the period along the family as well as the the eigenvalues of the monodromy matrix. The numerical verification of the Diophantine condition is more difficult, but note that the nondegeneracy condition ensures that most of the orbits of the family are going to satisfy it. Let us also note that these conditions are generic, and that the numerical behaviour observed in those examples (see below) corresponds to the one obtained from our results.

3.4.1 The bicircular model near $L_{4,5}$

The bicircular problem is a first approximation to study the motion of a small particle in the Earth–Moon system, including perturbations coming for the Sun. In this model it is assumed that Earth and Moon revolve in a circular orbits around their centre of masses, and that this centre of masses moves in circular orbit around the Sun. Usually, in order to simplify the equations, the units of lenght, time and mass are chosen such that the angular velocity of rotation of Earth and Moon (around their centre of masses), the sum of masses of Earth and Moon and the gravitational constant are all equal to one. With these normalized units, the Earth–Moon distance is also one. The system of reference is defined as follows: the origin is taken at the centre of mass of the Earth–Moon system, the X axis is given by the line that goes from Moon to Earth, the Z axis has the direction of the angular momentum of Earth and Moon and the Y axis is taken such that the system is orthogonal and positive-oriented. Note that, in this (non-inertial) frame, called synodic system, Earth and Moon have fixed positions and the Sun is rotating around the barycentre of the Earth-Moon system. If we define momenta $P_X = \dot{X} - Y$, $P_Y = \dot{Y} + X$ and $P_Z = \dot{Z}$, in these coordinates, the motion of a infinitessimal particle moving under the gravitational attraction of Earth, Moon and Sun is given by the Hamiltonian

$$H = \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + YP_X - XP_Y - \frac{1-\mu}{r_{PE}} - \frac{\mu}{r_{PM}} - \frac{m_s}{r_{PS}} - \frac{m_s}{a_s^2}(Y\sin\theta - X\cos\theta),$$

where $\theta = w_S t$, being w_S the mean angular velocity of the Sun in synodic coordinates, μ the mass parameter for the Earth–Moon system, a_s the semimajor axis of the Sun, m_s the Sun mass, and r_{PE}, r_{PM}, r_{PS} are defined in the following form:

$$\begin{aligned} r_{PE}^2 &= (X-\mu)^2 + Y^2 + Z^2, \\ r_{PM}^2 &= (X-\mu+1)^2 + Y^2 + Z^2, \\ r_{PS}^2 &= (X-X_s)^2 + (Y-Y_s)^2 + Z^2, \end{aligned}$$

where $X_s = a_s\cos\theta$ and $Y_s = -a_s\sin\theta$.

Note that one can look at this model as a time-periodic perturbation of an autonomous system, the Restricted Three Body Problem (usually called RTBP, see [72] for definition and basic properties). Hence, the Hamiltonian is of the form

$$H = H_0(x, y) + \varepsilon H_1(x, y, t),$$

where ε is a parameter such that $\varepsilon = 0$ corresponds to the unperturbed RTBP and $\varepsilon = 1$ to the bicircular model with the actual values for the perturbation.

Note that the bicircular model is not dynamically consistent, because the motion of Earth, Moon and Sun does not follow a true orbit of the system (we are not taking into account the interaction between the Sun and the Earth–Moon system). Nevertheless, numerical simulation shows that, in some regions of the phase space, this model gives the same qualitative behaviour as the real system and this makes it worth to study (see [71]).

We are going to focus in the dynamics near the equilateral points $L_{4,5}$ of the Earth–Moon system. These points are linearly stable for the unperturbed problem ($\varepsilon = 0$), so we can associate three families of periodic (Lyapounov) orbits to them: the short period family, the long period family and the vertical family of periodic orbits. Classical results about these families can be found in [72].

When the perturbation is added the points $L_{4,5}$ become (stable) periodic orbits with the same period as the perturbation. These orbits become unstable for the actual value of the perturbation ($\varepsilon = 1$ in the notation above). In this last case, numerical simulation shows the existence of a region of stability not very close to the orbit and outside of the plane of motion of Earth and Moon. This region seems to be centered around some of the (Lyapounov) periodic orbits of the vertical family. See [26] or [71] for more details.

Let us consider the dynamics near $L_{4,5}$ for ε small. In this case, the equilibrium point has been replaced by a small periodic orbit. Our results imply that the three families of Lyapounov periodic orbits become three cantorian families of 2-D invariant tori, adding the perturbing frequency to the one of the periodic orbit. Moreover, the Lyapounov tori (the 2-D invariant tori of the unperturbed problem that are obtained by "product" of two families of periodic orbits) become 3-D invariant tori, provided they are nonresonant with the perturbation. Finally, the maximal dimension (3-D) invariant tori of the unperturbed problem become 4-D tori, adding the frequency of the Sun to the ones they already had (this last result is already contained in [35]).

Now let us consider $\varepsilon = 1$. This value of ε is too big to apply these results. In particular, ε is big enough to cause a change of stability in the periodic orbit that replaces the equilibrium point. Hence, if one wants to apply the results of this chapter to this case, it is necessary to start by putting the Hamiltonian in a suitable form. To describe the dynamics near the unstable periodic orbit that replaces the equilibrium point, we can perform some steps of a normal form procedure to write the Hamiltonian as an autonomous (and integrable) Hamiltonian plus a small time dependent periodic perturbation (see [26], [34] or [71] for more details about these kind of computations). Then, if we are close enough to the periodic orbit, Theorem 3.1 applies and we have invariant tori of dimensions 1, 2 and 3. They are in the "central" directions of the periodic orbit.

The application to the stable region that is in the vertical direction is more difficult. A possiblity is to compute (numerically) an approximation to a 2-D invariant torus of the vertical family (note that its existence has not already been proved rigorously) and to perform some steps of a normal form procedure, in order to write the problem as an integrable autonomous Hamiltonian plus a time dependent periodic perturbation. Then, if the (approximate) torus satisfies the equations within a small enough error, it should be possible to show the existence of a torus nearby, and to establish that it is stable and surrounded by invariant tori of dimensions 1 to 4. Numerical experiments suggest (see [26] or [71]) that this is what happens in this case.

Extensions

In fact, the bicircular model is only the first step in the study of the dynamics near the libration points of Earth-Moon system. One can construct better models taking into account the non-circular motion of Earth and Moon (see [15], [24], [26]). Our results can be applied to these models in the same way it has been done in the bicircular case. The main difference is that now the equilibrium point is replaced by a quasiperiodic solution that, due to the resonances, does not exist for all values of ε but only for a Cantor set of them (see [35]).

3.4.2 Halo orbits

Let us consider the Earth and Sun as a RTBP, and let us focus in the dynamics near the equilibrium point that it is in between (the so called L_1 point). It is well known the existence of a family of periodic orbits (called Halo orbits, see [63]) such that, when one looks at them from the Earth, they seem to describe an halo around the solar disc. These orbits are a very suitable place to put a spacecraft to study the Sun: from that place, the Sun is always visible and it is always possible to send data back to Earth (because the probe does not cross the solar disc, otherwise the noise coming from the Sun would make communications impossible). These orbits have been used by missions ISEE-C (from 1978 to 1982) and SOHO (launched in 1995).

In the RTBP, Halo orbits are a one parameter family of periodic orbits with a normal behaviour of the type centre$\times$saddle. Unfortunately, the RTBP is too simple to produce good approximations to the dynamics. If one wants to have a cheap station keeping it is necessary to compute the nominal orbit with a very accurate model (see [23], [24], [25] and [26]).

The usual analytic models for this problem are written as an autonomous Hamiltonian (the RTBP) plus the effect coming from the real motion of Earth and Moon, the effect of Venus, etc. All these effects can be modelled very accurately using quasiperiodic functions that depend on time in a quasiperiodic way. Hence, we end up with an autonomous Hamiltonian plus a quasiperiodic time dependent perturbation with $r > 0$ frequencies. As usual, we add a parameter ε in front of this perturbation.

Then, Theorem 3.1 implies that, if ε is small enough, the Halo orbits become a cantorian family of $(r+1)$-D invariant tori. The normal behaviour of these tori is also of the type centre$\times$saddle.

To study the case $\varepsilon = 1$ we refer to the remarks for the case of the bicircular problem.

3.5 Proofs

This section contains the proof of Theorem 3.1. It has been split in several parts to simplify the reading. Section 3.5.1 introduces the basic notation used along the proof. In section 3.5.2 we give the basic lemmas needed during the proof. Section 3.5.3 gives quantitative estimates on one step of the iterative scheme and section 3.5.4 contains the technical details of the proof.

3.5.1 Notations

Here we introduce some of the notations used to prove the different results.

Norms and Lipschitz constants

As usual we denote by $|v|$ the absolute value of $v \in \mathbb{C}$, and we use the same notation to refer to the (maximum) vectorial or matrix norm on $\mathbb{C}^n$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$.

Let us denote by f an analytic function defined on a complex strip of width $\rho > 0$, having r arguments and being 2π-periodic in all of them. The range of this function can be in $\mathbb{C}$, $\mathbb{C}^n$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$. If we write its Fourier expansion as

$$f(\theta) = \sum_{k\in\mathbb{Z}^r} f_k \exp{(ik^\top\theta)},$$

we can introduce the norm

$$|f|_\rho = \sum_{k\in\mathbb{Z}^r} |f_k| \exp{(|k|_1\rho)}.$$

Let $f(\theta,q)$ be a 2π-periodic function on θ, and analytic on the domain

$$\mathcal{U}^{r,m}_{\rho,R} = \{(\theta,q) \in \mathbb{C}^r \times \mathbb{C}^m \,:\, |\mathrm{Im}\theta| \le \rho,\ |q| \le R\}.$$

If we write its Taylor expansion around $q = 0$ as:

$$f(\theta,q) = \sum_{l\in\mathbb{N}^m} f_l(\theta) q^l,$$

then, from this expansion we define the norm:

$$|f|_{\rho,R} = \sum_{l\in\mathbb{N}^m} |f_l|_\rho R^{|l|_1}.$$

If f takes values in $\mathbb{C}$, we put ∇f to denote the gradient of f with respect to (θ,q).

Now, we introduce the kind of Lipschitz dependence considered. Assume that $f(\varphi)$ is a function defined for $\varphi \in \mathcal{E}$, $\mathcal{E} \subset \mathbb{R}^j$ for some j, and with values in $\mathbb{C}$, $\mathbb{C}^n$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$. We call f a Lipschitz function with respect to φ on the set $\mathcal{E}$ if:

$$\mathcal{L}_{\mathcal{E}}\{f\} = \sup_{\substack{\varphi_1,\varphi_2\in\mathcal{E}\\ \varphi_1\neq\varphi_2}} \frac{|f(\varphi_2)-f(\varphi_1)|}{|\varphi_2-\varphi_1|} < +\infty.$$

The value $\mathcal{L}_{\mathcal{E}}\{f\}$ is called the Lipschitz constant of f on $\mathcal{E}$. For these kind of functions we define $\|f\|_{\mathcal{E}} = \sup_{\varphi\in\mathcal{E}} |f(\varphi)|$.

Similarly, if $f(\theta,\varphi)$ is a 2π-periodic analytic function on θ for every $\varphi\in\mathcal{E}$, we denote:

$$\mathcal{L}_{\mathcal{E},\rho}\{f\} = \sup_{\substack{\varphi_1,\varphi_2\in\mathcal{E}\\ \varphi_1\neq\varphi_2}} \frac{|f(.,\varphi_2)-f(.,\varphi_1)|_\rho}{|\varphi_2-\varphi_1|}.$$

In the same way we can introduce $\mathcal{L}_{\mathcal{E},\rho,R}\{f\}$, if we work with $f(\theta,q,\varphi)$ and the norm $|.|_{\rho,R}$. We can also extend $\|.\|_{\mathcal{E}}$ to both cases to define $\|.\|_{\mathcal{E},\rho}$ and $\|.\|_{\mathcal{E},\rho,R}$.

Canonical transformations

In this chapter, we will perform changes of variables using the Lie series method. We want to keep the quasiperiodic time dependence (after each transformation) with the same vector of basic frequencies $\tilde{\omega}^{(0)}$ as the initial one. This is achieved when the generating function does not depend on $\tilde{I}$.

Let us consider a generating function $S(\theta,x,\hat{I},y)$ such that ∇S depends analytically on $(\theta,x,\hat{I},y)$ and it is 2π-periodic in θ. The equations related to the Hamiltonian function S are

$$\dot{\hat{\theta}} = \left(\frac{\partial S}{\partial \hat{I}}\right)^\top, \ \dot{\tilde{\theta}} = \left(\frac{\partial S}{\partial \tilde{I}}\right)^\top = 0, \ \dot{\hat{I}} = -\left(\frac{\partial S}{\partial \hat{\theta}}\right)^\top, \ \dot{\tilde{I}} = -\left(\frac{\partial S}{\partial \tilde{\theta}}\right)^\top, \ \dot{z} = J_m\left(\frac{\partial S}{\partial z}\right)^\top.$$

We denote by $\Psi_t^S(\theta,x,I,y)$ the flow at time t of S with initial conditions (θ,x,I,y) when $t=0$. We note that Ψ_t^S is (for a fixed t) a canonical change of variables that acts in a trivial way on $\tilde{\theta}$. If we put $(\theta(t),x(t),I(t),y(t)) = \Psi_t^S(\theta(0),x(0),I(0),y(0))$, we can express Ψ_t^S as:

$$\begin{aligned}
\hat{\theta}(t) &= \hat{\theta}(0) + \int_0^t \left(\frac{\partial S}{\partial \hat{I}}(\theta(\tau),x(\tau),\hat{I}(\tau),y(\tau))\right)^\top d\tau,\\
I(t) &= I(0) - \int_0^t \left(\frac{\partial S}{\partial \theta}(\theta(\tau),x(\tau),\hat{I}(\tau),y(\tau))\right)^\top d\tau,\\
z(t) &= z(0) + J_m\int_0^t \left(\frac{\partial S}{\partial z}(\theta(\tau),x(\tau),\hat{I}(\tau),y(\tau))\right)^\top d\tau,
\end{aligned}$$

and $\tilde{\theta}(t)=\tilde{\theta}(0)$. We note that the function $\Psi_t^S - Id$ does not depend on the auxiliar variables $\tilde{I}$. Then, we put $\theta(0)=\theta$, $\hat{I}(0)=\hat{I}$ and $z(0)=z$ to introduce the transformations $\hat{\Psi}_t^S$ and $\hat{\Phi}_t^S$, defined as $\hat{\Psi}_t^S(\theta,x,\hat{I},y) = (\theta(t),x(t),\hat{I}(t),y(t))$ and $\hat{\Phi}_t^S = \hat{\Psi}_t^S - Id$. It is not difficult to check that $\hat{\Phi}_t^S(\theta,x,\hat{I},y)$ is (for a fixed t) 2π-periodic in θ.

If we consider the Hamiltonian function H of (3.6), and we put

$$H^{**} = \{H^*,S\} - \frac{\partial S}{\partial \tilde{\theta}}\tilde{\omega}^{(0)}, \tag{3.15}$$

Ψ_t^S transforms the Hamiltonian H into

$$H\circ\Psi_t^S(\theta,x,I,y) = \tilde{\omega}^{(0)\top}\tilde{I} + H^*(\theta,x,\hat{I},y) + tH^{**}(\theta,x,\hat{I},y) + \Sigma_t(H^{**},S)(\theta,x,\hat{I},y),$$

where

$$\Sigma_t(H^{**}, S) = \sum_{j\geq 2} \frac{t^j}{j!} L_S^{j-1}(H^{**}), \tag{3.16}$$

with $L_S^0(H^{**}) = H^{**}$ and $L_S^j(H^{**}) = \{L_S^{j-1}(H^{**}), S\}$, for $j \geq 1$.

Now, by controlling $\hat{\Phi}_t^S$ we will show that $H \circ \Psi_t^S$ is defined in a domain slightly smaller than H.

Finally, as the change of variables is selected as the flow at time one of a Hamiltonian S, in what follows we will omit the subscript t and we will assume that it means $t = 1$.

3.5.2 Basic lemmas

Lemmas on norms and Lipschitz constants

In this section we state some bounds used when working with the norms and Lipschitz constants introduced in section 3.5.1. We follow here the same notations of section 3.5.1 for the different analytic functions used in the lemmas.

Lemma 3.1 *Let $f(\theta)$ and $g(\theta)$ be analytic functions on a strip of width $\rho > 0$, 2π-periodic in θ and taking values in $\mathbb{C}$. Let us denote by f_k the Fourier coefficients of f, $f(\theta) = \sum_{k\in\mathbb{Z}^r} f_k \exp(ik^\top\theta)$. Then we have:*

(i) $|f_k| \leq |f|_\rho \exp(-|k|_1\rho)$.

(ii) $|fg|_\rho \leq |f|_\rho |g|_\rho$.

(iii) *For every* $0 < \rho_0 < \rho$

$$\left|\frac{\partial f}{\partial \theta_j}\right|_{\rho-\rho_0} \leq \frac{|f|_\rho}{\rho_0 \exp(1)}, \quad j = 1, \dots, r.$$

(iv) *Let* $\{d_k\}_{k\in\mathbb{Z}^r\setminus\{0\}} \subset \mathbb{C}$ *satisfy the following bounds:*

$$|d_k| \geq \frac{\mu}{|k|_1^\gamma} \exp(-\delta|k|_1),$$

for some $\mu > 0$, $\gamma \geq 0$, $0 \leq \delta < \rho$. If we assume that $\bar{f} = 0$, then the function g defined as

$$g(\theta) = \sum_{k\in\mathbb{Z}^r\setminus\{0\}} \frac{f_k}{d_k} \exp(ik^\top\theta),$$

satisfies the bound

$$|g|_{\rho-\rho_0} \leq \left(\frac{\gamma}{(\rho_0 - \delta)\exp(1)}\right)^\gamma \frac{|f|_\rho}{\mu},$$

for every $\rho_0 \in]\delta, \rho[$.

All these bounds can be extended to the case when f and g take values in $\mathbb{C}^n$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$. Of course, in the matrix case, in (ii) it is necessary that the product fg be well defined.

Proof: Items (i) and (ii) are easily verified. Proofs of (iii) and (iv) are essentially contained in [35], but working with the supremum norm.

Lemma 3.2 *Let $f(\theta,q)$ and $g(\theta,q)$ be analytic functions on a domain $\mathcal{U}^{r,m}_{\rho,R}$ and 2π-periodic in θ. Then we have:*

(i) *If we expand $f(\theta,q)=\sum_{l\in\mathbb{N}^m} f_l(\theta)q^l$, then $|f_l|_\rho \le \frac{|f|_{\rho,R}}{R^{|l|_1}}$.*

(ii) $|fg|_{\rho,R}\le |f|_{\rho,R}|g|_{\rho,R}$.

(iii) *For every $0<\rho_0<\rho$ and $0<R_0<R$, we have:*

$$\left|\frac{\partial f}{\partial \theta_j}\right|_{\rho-\rho_0,R} \le \frac{|f|_{\rho,R}}{\rho_0\exp(1)},\ j=1,\dots,r,$$

and

$$\left|\frac{\partial f}{\partial q_j}\right|_{\rho,R-R_0} \le \frac{|f|_{\rho,R}}{R_0},\ j=1,\dots,m.$$

As in lemma 3.1, all the bounds hold if f and g take values in $\mathbb{C}^n$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$.

Proof: Items (i) and (ii) are straightforward. The first part of (iii) is a consequence of lemma 3.1. The second part is obtained applying standard Cauchy estimates to the function $F(q)=\sum_{l\in\mathbb{N}^m}|f_l|_\rho q^l$.

Lemma 3.3 *Let us take $0<\rho_0<\rho$ and $0<R_0<R$, and let us consider analytic functions $\Theta(\theta,q)$ (with values in $\mathbb{C}^r$) and $X(\theta,q)$ (with values in $\mathbb{C}^m$), both 2π-periodic on θ, and such that $|\Theta|_{\rho_0,R_0}\le \rho-\rho_0$ and $|X|_{\rho_0,R_0}\le R-R_0$. Let $f(\theta,q)$ be a given (2π-periodic on θ) analytic function. If we define:*

$$F(\theta,q)=f(\theta+\Theta(\theta,q),q+X(\theta,q)),$$

then, $|F|_{\rho_0,R_0}\le |f|_{\rho,R}$.

Proof: Expanding f in Taylor series (as (i) in lemma 3.2) one obtains the expansion of F as a function of Θ and X. Then the bound is a consequence of (ii) in 3.2.

Lemma 3.4 *Let us consider $\Theta^{(j)}$ and $X^{(j)}$, $j=1,2$, with the same conditions as Θ and X lemma 3.3, but with the following bounds: $|\Theta^{(j)}|_{\rho_0,R_0}\le \rho-\rho_0-\delta$ and $|X^{(j)}|_{\rho_0,R_0}\le R-R_0-\chi$, with $0<\delta<\rho-\rho_0$ and $0<\chi<R-R_0$. Then, if we define*

$$F^{(j)}(\theta,q)=f(\theta+\Theta^{(j)}(\theta,q),q+X^{(j)}(\theta,q)),\ j=1,2,$$

one has

$$|F^{(1)}-F^{(2)}|_{\rho_0,R_0}\le \left(\frac{|\Theta^{(1)}-\Theta^{(2)}|_{\rho_0,R_0}}{\exp(1)\delta}+m\frac{|X^{(1)}-X^{(2)}|_{\rho_0,R_0}}{\chi}\right)|f|_{\rho,R}.$$

Proof: We can use here the same ideas as in lemma 3.3, combined with the ones used to prove lemmas 3.1 and 3.2.

Now we give some basic results related to the Lipschitz dependences introduced in section 3.5.1. For that purpose, we work with a parameter φ on the set $\mathcal{E} \subset \mathbb{R}^j$, for some $j \geq 1$.

Lemma 3.5 *We consider Lipschitz functions $f(\varphi)$ and $g(\varphi)$ defined for $\varphi \in \mathcal{E}$ with values in $\mathbb{C}$, then:*

(i) $\mathcal{L}_{\mathcal{E}}\{f+g\} \leq \mathcal{L}_{\mathcal{E}}\{f\} + \mathcal{L}_{\mathcal{E}}\{g\}$.

(ii) $\mathcal{L}_{\mathcal{E}}\{fg\} \leq \|f\|_{\mathcal{E}}\mathcal{L}_{\mathcal{E}}\{g\} + \|g\|_{\mathcal{E}}\mathcal{L}_{\mathcal{E}}\{f\}$.

(iii) $\mathcal{L}_{\mathcal{E}}\{1/f\} \leq \|1/f\|_{\mathcal{E}}^2\mathcal{L}_{\mathcal{E}}\{f\}$, *if f does not vanish.*

Moreover, (i) holds if f and g take values in $\mathbb{C}^n$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$, and (ii) and (iii) also hold when f and g are matrix-valued functions such that the matrix product fg is defined (for case (ii)) and that f is invertible (for case (iii)).

Proof: It is straightforward.

Remark 3.1 *In lemma 3.5 we obtain analogous results if we work with functions of the form $f(\theta,\varphi)$ or $f(\theta,q,\varphi)$, defined for $\varphi \in \mathcal{E}$ and analytical with respect to the variables (θ,q) and the norms $|.|_\rho$, $|.|_{\rho,R}$.*

Lemma 3.6 *We assume that $f(\theta,\varphi)$ is, for every $\varphi \in \mathcal{E}$, an analytic 2π-periodic function in θ on a strip of width $\rho > 0$, with Lipschitz dependence with respect to φ. Let us expand $f(\theta,\varphi) = \sum_{k\in\mathbb{Z}^r} f_k(\varphi)\exp(ik^\top\theta)$. Then, we have:*

(i) $\mathcal{L}_{\mathcal{E}}\{f_k\} \leq \mathcal{L}_{\mathcal{E},\rho}\{f\}\exp(-|k|_1\rho)$.

(ii) For every $0 < \rho_0 < \rho$

$$\mathcal{L}_{\mathcal{E},\rho-\rho_0}\left\{\frac{\partial f}{\partial\theta_j}\right\} \leq \frac{\mathcal{L}_{\mathcal{E},\rho}\{f\}}{\rho_0\exp(1)}, \; j = 1,\ldots,r.$$

(iii) Let $\{d_k(\varphi)\}_{k\in\mathbb{Z}^r\setminus\{0\}}$ be a set of complex-valued functions defined for $\varphi \in \mathcal{E}$, with the following bounds:

$$|d_k(\varphi)| \geq \frac{\mu}{|k|_1^\gamma}\exp(-\delta|k|_1),$$

and

$$\mathcal{L}_{\mathcal{E}}\{d_k\} \leq A + B|k|_1,$$

for some $\mu > 0$, $\gamma \geq 0$, $0 \leq 2\delta < \rho$, $A \geq 0$ and $B \geq 0$. As in lemma 3.1 we assume $\bar{f} = 0$ for every $\varphi \in \mathcal{E}$. If

$$g(\theta,\varphi) = \sum_{k\in\mathbb{Z}^r\setminus\{0\}} \frac{f_k(\varphi)}{d_k(\varphi)}\exp(ik^\top\theta),$$

then, for every δ_0, $2\delta < \delta_0 < \rho$, we have:

$$\mathcal{L}_{\mathcal{E},\rho-\delta_0}\{g\} \le \left(\frac{\gamma}{(\delta_0-\delta)\exp(1)}\right)^{\gamma}\frac{\mathcal{L}_{\mathcal{E},\rho}\{f\}}{\mu} + \left(\frac{2\gamma+1}{(\delta_0-2\delta)\exp(1)}\right)^{2\gamma+1}\frac{\|f\|_{\mathcal{E},\rho}}{\mu^2}B + \\ + \left(\frac{2\gamma}{(\delta_0-2\delta)\exp(1)}\right)^{2\gamma}\frac{\|f\|_{\mathcal{E},\rho}}{\mu^2}A.$$

Proof: It is analogous to lemma 3.1, using also the results of lemma 3.5.

Lemma 3.7 *We assume that $f(\theta,q,\varphi)$ is, for every $\varphi\in\mathcal{E}$, an analytic function on $\mathcal{U}^{r,m}_{\rho,R}$ and 2π-periodic in θ. Then we have:*

(i) *If we write $f(\theta,q,\varphi)=\sum_{l\in\mathbb{N}^m} f_l(\theta,\varphi)q^l$, then $\mathcal{L}_{\mathcal{E},\rho}\{f_l\}\le\frac{\mathcal{L}_{\mathcal{E},\rho,R}\{f\}}{R^{|l|_1}}$.*

(ii) *For every $0<\rho_0<\rho$ and $0<R_0<R$, we have:*

$$\mathcal{L}_{\mathcal{E},\rho-\rho_0,R}\left\{\frac{\partial f}{\partial\theta_j}\right\}\le\frac{\mathcal{L}_{\mathcal{E},\rho,R}\{f\}}{\rho_0\exp(1)},\ j=1,\dots,r,$$

and

$$\mathcal{L}_{\mathcal{E},\rho,R-R_0}\left\{\frac{\partial f}{\partial q_j}\right\}\le\frac{\mathcal{L}_{\mathcal{E},\rho,R}\{f\}}{R_0},\ j=1,\dots,m.$$

Proof: As in lemma 3.6, but using now the same ideas as in lemma 3.2.

Lemmas on canonical transformations

In this section we establish some lemmas that we will use to work with the canonical transformations that we have introduced in section 3.5.1. The purpose is to bound the changes as well as the transformed Hamiltonian. We also take into account the possibility that the generating function depends on a parameter $\varphi\in\mathcal{E}$ in a Lipschitz way.

To simplify the notations in the lemmas of this section, we define

$$\Delta_{\rho_0,R_0}=\frac{r}{\rho_0\exp(1)}+\frac{r+2m}{R_0},\tag{3.17}$$

and we will use (without explicit mention) the notations introduced in section 3.5.1.

The proofs of lemmas 3.8, 3.9, 3.10 and 3.11 can be obtained from the bounds of lemmas of section 3.5.2. The proof of lemma 3.9 is essentially contained in [13]. The proof of 3.11 is similar. The proof of lemma 3.12 can also be found in [13], where it is proved working with the supremum norm. In our case the proof is analogous from the explicit expressions for the transformation $\hat\Psi^S$ given in section 3.5.1, using the result of lemma 3.3 to bound the compositions.

Lemma 3.8 *Let us consider $f(\theta,x,\hat I,y)$ and $g(\theta,x,\hat I,y)$ complex-valued functions such that f and ∇g are analytic functions defined on $\mathcal{U}^{r+s,r+2m}_{\rho,R}$, 2π-periodic on θ. Then, for every $0<\rho_0<\rho$ and $0<R_0<R$, we have:*

$$\left|\{f,g\}\right|_{\rho-\rho_0,R-R_0}\le\Delta_{\rho_0,R_0}\left|\nabla g\right|_{\rho,R}\left|f\right|_{\rho,R}.$$

Lemma 3.9 *With the same hypotheses of lemma 3.8 we have, for the expression* $\Sigma(f,g)$ *introduced in* (3.16),

$$|\Sigma(f,g)|_{\rho-\rho_0,R-R_0} \leq \sum_{j\geq 1} \frac{1}{j+1} \left(\Delta_{\rho_0,R_0} \exp{(1)} |\nabla g|_{\rho,R}\right)^j |f|_{\rho,R}.$$

Lemma 3.10 *Assume that the complex-valued functions* $f(\theta,x,\hat{I},y,\varphi)$ *and* $g(\theta,x,\hat{I},y,\varphi)$ *verify that, for every* $\varphi \in \mathcal{E}$, f *and* ∇g *are analytic functions on* $\mathcal{U}^{r+s,r+2m}_{\rho,R}$, 2π*-periodic on* θ*, with Lipschitz dependence on* φ*. Then, if* $\|f\|_{\mathcal{E},\rho,R} \leq F_1$, $\|\nabla g\|_{\mathcal{E},\rho,R} \leq F_2$, $\mathcal{L}_{\mathcal{E},\rho,R}\{f\} \leq L_1$ *and* $\mathcal{L}_{\mathcal{E},\rho,R}\{\nabla g\} \leq L_2$*, we have that, for every* $0 < \rho_0 < \rho$ *and* $0 < R_0 < R$,

$$\mathcal{L}_{\mathcal{E},\rho-\rho_0,R-R_0}\{\{f,g\}\} \leq \Delta_{\rho_0,R_0}\left(F_1 L_2 + F_2 L_1\right).$$

Lemma 3.11 *With the same hypotheses of lemma 3.10, we have:*

$$\mathcal{L}_{\mathcal{E},\rho-\rho_0,R-R_0}\{\Sigma(f,g)\} \leq \sum_{j\geq 1} \left(\frac{1}{j+1} \left(\Delta_{\rho_0,R_0} \exp{(1)}\right)^j F_2^{j-1}(jL_2F_1 + L_1F_2)\right).$$

Lemma 3.12 *We assume that the generating function* $S(\theta,x,\hat{I},y)$ *of section 3.5.1 verifies that* ∇S *is analytic on* $\mathcal{U}^{r+s,r+2m}_{\rho,R}$, 2π*-periodic in* θ*, with* $|\nabla S|_{\rho,R} \leq \delta$*, where* $\delta < \min\{\rho,R\}$*. Then, with the notations of section 3.5.1, we have:*

(*i*) $|\hat{\Phi}^S|_{\rho-\delta,R-\delta} \leq |\nabla S|_{\rho,R}$.

(*ii*) $\hat{\Psi}^S : \mathcal{U}^{r+s,2m+r}_{\rho-\delta,R-\delta} \longrightarrow \mathcal{U}^{r+s,2m+r}_{\rho,R}$.

Convergence lemma

We will use the following lemma during the proof of Theorem 3.1, to relate the bounds on the Hamiltonian after n steps of the iterative scheme as a function of bounds for the initial Hamiltonian.

Lemma 3.13 *Let* $\{K_n\}_{n\geq 1}$ *be a sequence of positive numbers with* $K_{n+1} \leq a n^b K_n^2 \exp{(\varrho^n c)}$ *if* $n \geq 1$*, being* $a > 0$, $b \geq 0$, $c > 0$ *and* $1 < \varrho < 2$*. Then:*

$$K_{n+1} \leq \frac{1}{a}\left(\left(\frac{5}{3}\right)^b a K_1 \exp\left(\frac{c\varrho}{2-\varrho}\right)\right)^{2^n}.$$

Proof: The proof is a direct combination of the proofs of lemma 5 in [33] and lemma 2.14 in [35].

Estimates on measures in parameter space

In the following lemmas, we consider a fixed $\omega^{(0)\top} = (\hat{\omega}^{(0)\top}, \tilde{\omega}^{(0)\top})$, with $\hat{\omega}^{(0)} \in \mathbb{R}^r$ and $\tilde{\omega}^{(0)} \in \mathbb{R}^s$. Let $\lambda(\varphi)$ be a function defined on $\mathcal{E} \subset \mathbb{R}^{r+1}$ with range in $\mathbb{C}$, where $\varphi^\top = (\hat{\omega}^\top, \varepsilon)$, with $\hat{\omega} \in \mathbb{R}^r$ and $\varepsilon \in \mathbb{R}$. We assume that λ takes the form:

$$\lambda(\varphi) = \lambda_0 + iu\varepsilon + iv^\top(\hat{\omega} - \hat{\omega}^{(0)}) + \tilde{\lambda}(\varphi),$$

where λ_0, $u \in \mathbb{C}$, $v \in \mathbb{C}^r$ and, if we denote by $\bar{\mathcal{E}} \equiv \bar{\mathcal{E}}(\bar{\vartheta}) = \left\{\varphi \in \mathcal{E} \,:\, |\varphi - \varphi^{(0)}| \leq \bar{\vartheta}\right\}$, $\varphi^{(0)\top} = (\hat{\omega}^{(0)\top}, 0)$, then we have that $\mathcal{L}_{\bar{\mathcal{E}}}\{\tilde{\lambda}\} \leq L\bar{\vartheta}$ for certain $L \geq 0$, for all $0 \leq \bar{\vartheta} \leq \vartheta_0$. Note that this Lipschitz condition on $\tilde{\lambda}$ would imply, if $\tilde{\lambda}$ were of class C^2, that $\tilde{\lambda}$ is of $O_2(\varphi - \varphi^{(0)})$. We also assume that $|\lambda(\varphi) - \lambda_0| \leq M|\varphi - \varphi^{(0)}|$ for all $\varphi \in \bar{\mathcal{E}}(\vartheta_0)$.

Now, we take $\mu > 0$, $\gamma > r+s-1$ and $0 < \delta \leq 1$ to define from λ and $\mathcal{E}$ the following "resonant" sets:

$$\begin{aligned}\mathcal{R}(\varepsilon_0, R_0) \;=\; & \Big\{\hat{\omega} \in \mathbb{R}^r \,:\, |\hat{\omega} - \hat{\omega}^{(0)}| \leq R_0,\; (\hat{\omega}^\top, \varepsilon_0)^\top = \varphi \in \mathcal{E} \text{ and} \\ & \quad \exists k \in \mathbb{Z}^{r+s} \setminus \{0\} \text{ such that } |ik^\top\omega + \lambda(\varphi)| < \frac{\mu}{|k|_1^\gamma}\exp\left(-\delta|k|_1\right)\Big\},\end{aligned}$$

for every $\varepsilon_0 \geq 0$ and $R_0 \geq 0$, and

$$\begin{aligned}\mathcal{A}(\varepsilon_0, \hat{\omega}) \;=\; & \Big\{\varepsilon \in [0, \varepsilon_0] \,:\, (\hat{\omega}^\top, \varepsilon)^\top = \varphi \in \mathcal{E} \text{ and} \\ & \quad \exists k \in \mathbb{Z}^{r+s} \setminus \{0\} \text{ such that } |ik^\top\omega + \lambda(\varphi)| < \frac{\mu}{|k|_1^\gamma}\exp\left(-\delta|k|_1\right)\Big\},\end{aligned}$$

for every $\hat{\omega} \in \mathbb{R}^r$ and $\varepsilon_0 > 0$, where in both cases $\omega \in \mathbb{R}^{r+s}$ is defined from $\varphi^\top = (\hat{\omega}^\top, \varepsilon)$ as $\omega^\top = (\hat{\omega}^\top, \tilde{\omega}^{(0)\top})$. Note that these sets depend on δ and μ.

As the purpose of this section is to deal with the measure of these resonant sets, we will always assume we are in the worst case: $\mathrm{Re}\lambda_0 = 0$. When this is not true (this is, when there are no resonances) it is not difficult to see that the sets $\mathcal{R}$ and $\mathcal{A}$ are empty if we are close enough to $\varphi^{(0)}$ (the value of the parameter for the unperturbed system). We want to remark that we are not making any assumption on the values $\mathrm{Im}u$ and $\mathrm{Im}v$. According to the size of the resonant sets the worst case happens when $\mathrm{Im}u = 0$ and/or $\mathrm{Im}v = 0$. Hence, the proof will be valid in this case, although it is possible to improve the measure estimates in the case that $\mathrm{Im}u \neq 0$ and $\mathrm{Im}v \neq 0$.

Lemma 3.14 *If we assume that* $|ik^\top\omega^{(0)} + \lambda_0| \geq \frac{\mu_0}{|k|_1^\gamma}$, $\gamma > r+s-1$, *for all* $k \in \mathbb{Z}^{r+s} \setminus \{0\}$, *for certain* $\mu_0 \geq 2\mu$, *then, if* K *is the only positive solution of* $\frac{\mu_0}{2K^\gamma} = M\max\{R_0, \varepsilon_0\} + KR_0$, *we have:*

(*i*) *If* $\mathrm{Re}v \notin \mathbb{Z}^r$, *and* ε_0, R_0 *are small enough (condition that depends only on* v, L, ϑ_0, γ, μ_0 *and* M*), then,*

$$mes(\mathcal{R}(\varepsilon_0, R_0)) \leq 16\mu(2\sqrt{r}R_0)^{r-1}(r+s)\hat{K}(v)K^{r+s-1-\gamma}\frac{\exp\left(-\delta K\right)}{\delta},$$

where $\hat{K}(v) = \sup_{\hat{k}\in\mathbb{Z}^r}\left\{\frac{1}{|\hat{k}+\mathrm{Re}v|_2}\right\}$, *being* $|.|_2$ *the Euclidean norm of* $\mathbb{R}^r$.

(*ii*) *If* $u \neq 0$, *and* ε_0, $R_0 = |\hat{\omega} - \hat{\omega}^{(0)}|$ *are small enough (condition that depends only on* u, L, ϑ_0, γ, μ_0 *and* M*), then*

$$mes(\mathcal{A}(\varepsilon_0, \hat{\omega})) \leq \frac{16\mu}{|u|}(r+s)K^{r+s-1-\gamma}\frac{\exp(-\delta K)}{\delta}.$$

Proof: We prove part (i). Similar ideas can be used to prove (ii). To study the measure of $\mathcal{R}(\varepsilon_0, R_0)$, we consider the following decomposition:

$$\mathcal{R}(\varepsilon_0, R_0) = \bigcup_{k \in \mathbb{Z}^{r+s} \setminus \{0\}} \mathcal{R}_k(\varepsilon_0, R_0),$$

where $\mathcal{R}_k(\varepsilon_0, R_0)$ is defined as:

$$\begin{aligned} \mathcal{R}_k(\varepsilon_0, R_0) &= \Big\{ \hat{\omega} \in \mathbb{R}^r \,:\, |\hat{\omega} - \hat{\omega}^{(0)}| \leq R_0,\ (\hat{\omega}^\top, \varepsilon_0)^\top = \varphi \in \mathcal{E} \\ &\qquad \text{and } |ik^\top \omega + \lambda(\varphi)| < \frac{\mu}{|k|_1^\gamma} \exp(-\delta |k|_1) \Big\}. \end{aligned}$$

To compute the measure of these sets, we take $\hat{\omega}^{(1)}, \hat{\omega}^{(2)} \in \mathcal{R}_k(\varepsilon_0, R_0)$, and we put $\varphi^{(j)\top} = (\hat{\omega}^{(j)\top}, \varepsilon_0)$ and $\omega^{(j)\top} = (\hat{\omega}^{(j)\top}, \tilde{\omega}^{(0)\top})$. Then, from $|ik^\top \omega^{(j)} + \lambda(\varphi^{(j)})| < \frac{\mu}{|k|_1^\gamma} \exp(-\delta|k|_1)$, we clearly have that $|i\hat{k}^\top(\hat{\omega}^{(1)} - \hat{\omega}^{(2)}) + \lambda(\varphi^{(1)}) - \lambda(\varphi^{(2)})| < \frac{2\mu}{|k|_1^\gamma} \exp(-\delta|k|_1)$, where we have split $k^\top = (\hat{k}^\top, \tilde{k}^\top)$, with $\hat{k} \in \mathbb{Z}^r$ and $\tilde{k} \in \mathbb{Z}^s$. From that, and using the definition of $\lambda(\varphi)$ one obtains:

$$|i(\hat{k} + v)^\top (\hat{\omega}^{(1)} - \hat{\omega}^{(2)}) + \tilde{\lambda}(\varphi^{(1)}) - \tilde{\lambda}(\varphi^{(2)})| < \frac{2\mu}{|k|_1^\gamma} \exp(-\delta|k|_1).$$

Note that the set $\mathcal{R}_k$ is a slice of the set of $\hat{\omega}$ such that $|\hat{\omega} - \hat{\omega}^{(0)}| \leq R_0$. To estimate its measure, we are going to take the values $\hat{\omega}^{(1)}$ and $\hat{\omega}^{(2)}$ such that $\hat{\omega}^{(1)} - \hat{\omega}^{(2)}$ is (approximately) perpendicular to the slice, that is, parallel to the vector $\hat{k} + \mathrm{Re}v$. Then, mes($\mathcal{R}_k$) can be bounded by the product of a bound of the value $|\hat{\omega}^{(1)} - \hat{\omega}^{(2)}|$ by (a bound of) the measure of the worst (biggest) section of an hyperplane (of codimension 1) with the set $|\hat{\omega} - \hat{\omega}^{(0)}| \leq R_0$.

Hence, assuming now that $\hat{\omega}^{(1)} - \hat{\omega}^{(2)}$ is parallel to the vector $\hat{k} + \mathrm{Re}v$, we have:

$$\begin{aligned} |\hat{\omega}^{(1)} - \hat{\omega}^{(2)}|_2 &= \frac{|(\hat{k} + \mathrm{Re}v)^\top (\hat{\omega}^{(1)} - \hat{\omega}^{(2)})|}{|\hat{k} + \mathrm{Re}v|_2} \leq \frac{|(\hat{k} + v)^\top (\hat{\omega}^{(1)} - \hat{\omega}^{(2)})|}{|\hat{k} + \mathrm{Re}v|_2} \leq \\ &\leq \frac{1}{|\hat{k} + \mathrm{Re}v|_2} \left(L \max\{R_0, \varepsilon_0\} |\hat{\omega}^{(1)} - \hat{\omega}^{(2)}| + \frac{2\mu}{|k|_1^\gamma} \exp(-\delta|k|_1) \right). \end{aligned}$$

In consequence:

$$\left(1 - \frac{L\max\{R_0, \varepsilon_0\}}{|\hat{k} + \mathrm{Re}v|_2}\right) |\hat{\omega}^{(1)} - \hat{\omega}^{(2)}|_2 \leq \frac{2\mu}{|k|_1^\gamma} \exp(-\delta|k|_1)) \frac{1}{|\hat{k} + \mathrm{Re}v|_2}.$$

So, if ε_0 and R_0 are small enough (independent on k) we can bound:

$$|\hat{\omega}^{(1)} - \hat{\omega}^{(2)}|_2 \leq \frac{4\mu}{|k|_1^\gamma} \exp(-\delta|k|_1)\hat{K},$$

where we put $\hat{K}=\hat{K}(v)$. From that:

$$\mathrm{mes}(\mathcal{R}_k(\varepsilon_0,R_0)) \le \frac{4\mu}{|k|_1^\gamma}\exp\left(-\delta|k|_1\right)(2\sqrt{r}R_0)^{r-1}\hat{K},$$

where $2\sqrt{r}R_0$ is a bound for the diameter of the set $\{\hat{\omega}\in\mathbb{R}^r:\ |\hat{\omega}-\hat{\omega}^{(0)}|\le R_0\}$. Then, we have:

$$\mathrm{mes}(\mathcal{R}(\varepsilon_0,R_0)) \le \sum_{k\in\mathbb{Z}^{r+s}\setminus\{0\}} \frac{4\mu}{|k|_1^\gamma}\exp\left(-\delta|k|_1\right)(2\sqrt{r}R_0)^{r-1}\hat{K}. \tag{3.18}$$

In fact, in this sum we only need to consider $k\in\mathbb{Z}^{r+s}\setminus\{0\}$ such that $\mathcal{R}_k(\varepsilon_0,R_0)\ne\emptyset$. Now, let us see that $\mathcal{R}_k(\varepsilon_0,R_0)$ is empty if $|k|_1$ is less than some critical value K.

Let $\varphi\in\mathcal{R}_k(\varepsilon_0,R_0)$, then we can write:

$$\begin{aligned}
\frac{\mu_0}{|k|_1^\gamma} &\le |ik^\top\omega^{(0)}+\lambda_0| \le |ik^\top\omega+\lambda(\varphi)|+|\lambda(\varphi)-\lambda_0|+|\hat{k}^\top(\hat{\omega}-\hat{\omega}^{(0)})| \le \\
&\le \frac{\mu}{|k|_1^\gamma}\exp\left(-\delta|k|_1\right)+M\max\{R_0,\varepsilon_0\}+|\hat{k}|_1|\hat{\omega}-\hat{\omega}^{(0)}| \le \\
&\le \frac{\mu}{|k|_1^\gamma}+M\max\{R_0,\varepsilon_0\}+|\hat{k}|_1R_0,
\end{aligned}$$

and then:

$$\frac{\mu_0}{2|k|_1^\gamma}\le M\max\{R_0,\varepsilon_0\}+|k|_1R_0.$$

So, in the sum (3.18), we only need to consider $k\in\mathbb{Z}^{r+s}\setminus\{0\}$ such that $|k|_1\ge K$, where K (that depends on R_0 and ε_0) is defined in the statement of the lemma. We assume R_0 and ε_0 small enough such that $K\ge1$. Now, using that $\#\{k\in\mathbb{Z}^{r+s}:\ |k|_1=j\}\le 2(r+s)j^{r+s-1}$ and that $\gamma>r+s-1$, we have:

$$\begin{aligned}
\mathrm{mes}(\mathcal{R}(\varepsilon_0,R_0)) &\le 4\mu(2\sqrt{r}R_0)^{r-1}\hat{K}\sum_{\substack{k\in\mathbb{Z}^{r+s}\setminus\{0\}\\ |k|_1\ge K}}\frac{\exp\left(-\delta|k|_1\right)}{|k|_1^\gamma}\le \\
&\le 4\mu(2\sqrt{r}R_0)^{r-1}\hat{K}\sum_{j\ge K}2(r+s)j^{r+s-1}\frac{\exp\left(-\delta j\right)}{j^\gamma}\le \\
&\le 8\mu(2\sqrt{r}R_0)^{r-1}(r+s)\hat{K}K^{r+s-1-\gamma}\sum_{j\ge K}\exp\left(-\delta j\right)= \\
&= 8\mu(2\sqrt{r}R_0)^{r-1}(r+s)\hat{K}K^{r+s-1-\gamma}\frac{\exp\left(-\delta K\right)}{1-\exp\left(-\delta\right)}\le \\
&\le 16\mu(2\sqrt{r}R_0)^{r-1}(r+s)\hat{K}K^{r+s-1-\gamma}\frac{\exp\left(-\delta K\right)}{\delta},
\end{aligned}$$

where we used that $\frac{1}{1-\exp(-\delta)}\le\frac{2}{\delta}$, if $0<\delta\le1$.

Lemma 3.15 *With the previous notations, we introduce the set*

$$\begin{aligned}
\mathcal{D}(R_0) &= \Big\{\hat{\omega}\in\mathbb{R}^r:\ |\hat{\omega}-\hat{\omega}^{(0)}|\le R_0 \ \textit{and} \\
&\qquad \exists k\in\mathbb{Z}^{r+s}\setminus\{0\}\ \textit{such that}\ |k^\top\omega|<\frac{\mu}{|k|_1^\gamma}\exp\left(-\delta|k|_1\right)\Big\}.
\end{aligned}$$

Let us assume $|k^\top \omega^{(0)}| \geq \frac{\mu_0}{|k|_1^\gamma}$ *for all* $k \in \mathbb{Z}^{r+s} \setminus \{0\}$, *for certain* $\mu_0 \geq 2\mu$. *Then, if* R_0 *is small enough (depending only on* γ *and* μ_0*), one has:*

$$mes(\mathcal{D}(R_0)) \leq 8\mu(2\sqrt{r}R_0)^{r-1}(r+s)K^{r+s-1-\gamma}\frac{\exp(-\delta K)}{\delta},$$

where $K = \left(\frac{\mu_0}{2R_0}\right)^{\frac{1}{\gamma+1}}$.

Proof: It is similar to the one of lemma 3.14.

Lemma 3.16 *Let* $\mu_0 > 0$, $\rho > 0$ *and* $1 < \varrho < 2$ *fixed. We put* $\sigma_n = \frac{6}{\pi^2 n^2}$, $\delta_n = \frac{\sigma_n \rho}{18}$ *and* $\mu_n = \mu_0 \exp(-\varrho^n)$, *for all* $n \geq 1$. *Then, for every* $0 < \sigma < 1$, *we have that, if* K *is big enough (depending only on* ϱ, ρ, μ_0 *and* σ*),*

$$\sum_{n\geq 1} \mu_n \frac{\exp(-\delta_n K)}{\delta_n} \leq \exp(-K^\sigma).$$

Proof: Let $n_*(K) = \frac{\ln K}{\ln \varrho}$. We remark that $n_*(K) \to +\infty$ as $K \to +\infty$. Then, if K is big enough one has that, for all $n \geq n_*(K)$,

$$\frac{(n+1)^2 \exp(-\varrho^{n+1})}{n^2 \exp(-\varrho^n)} = \left(\frac{n+1}{n}\right)^2 \exp(-\varrho^n(\varrho - 1)) \leq \exp\left(-\frac{\varrho-1}{2}\right),$$

that allows to bound

$$\sum_{n\geq 1} n^2 \exp(-\varrho^n) \leq \frac{n_*^2 \exp(-\varrho^n)}{1 - \exp\left(-\frac{\varrho-1}{2}\right)}.$$

Hence,

$$\begin{aligned}
\sum_{n\geq 1} \mu_n \frac{\exp(-\delta_n K)}{\delta_n} &= \sum_{n\geq 1} \frac{3\mu_0\pi^2}{\rho} n^2 \exp\left(-\varrho^n - \frac{K\rho}{3\pi^2 n^2}\right) = \\
&= \frac{3\mu_0\pi^2}{\rho}\left(\sum_{1\leq n<n_*} + \sum_{n\geq n_*}\right) n^2 \exp\left(-\varrho^n - \frac{K\rho}{3\pi^2 n^2}\right) \leq \\
&\leq \frac{3\mu_0\pi^2}{\rho} n_*^2 \exp\left(-\frac{K\rho}{3\pi^2 n_*^2}\right) \sum_{n\geq 1} \exp(-\varrho^n) + \frac{3\mu_0\pi^2}{\rho} \sum_{n\geq n_*} n^2 \exp(-\varrho^n) \leq \\
&\leq \frac{3\mu_0\pi^2}{\rho} n_*^2 \left(\exp(-\varrho)\exp\left(-\frac{K\rho}{3\pi^2 n_*^2}\right) + \exp(-\varrho^{n_*})\right) \frac{1}{1 - \exp\left(-\frac{\varrho-1}{2}\right)} \leq \\
&\leq \exp(-K^\sigma),
\end{aligned}$$

for any $0 < \sigma < 1$ if K is big enough.

3.5.3 Iterative lemma

Here we give the details of a step of the iterative process used to prove Theorem 3.1. For that purpose, let us consider a Hamiltonian $H = H(\theta, x, I, y, \varphi)$ of the form:

$$H = \phi(\varphi) + \omega^\top I + \frac{1}{2} z^\top \mathcal{B}(\varphi) z + \frac{1}{2} \hat{I}^\top \mathcal{C}(\theta, \varphi) \hat{I} + H_*(\theta, x, \hat{I}, y, \varphi) + \hat{H}(\theta, x, \hat{I}, y, \varphi), \quad (3.19)$$

with the same notations as (3.5) and φ was introduced in section 3.5.2. Moreover, given $\varphi \in \mathcal{E}$ we recall the definition of $\omega \in \mathbb{R}^{r+s}$ as $\omega^\top = (\hat{\omega}^\top, \tilde{\omega}^{(0)\top})$, where $\hat{\omega}$ comes from the first r components of φ and $\tilde{\omega}^{(0)} \in \mathbb{R}^s$ is given by the quasiperiodic time dependence. Let us write

$$H(\theta, x, I, y, \varphi) = \tilde{\omega}^{(0)\top} \tilde{I} + H^*(\theta, x, \hat{I}, y, \varphi),$$

where we assume that H^* depends on $(\theta, x, \hat{I}, y)$ in an analytic form, it is 2π-periodic in θ, it depends on $\varphi \in \mathcal{E}$ in a Lipschitz way and, moreover, $< H_* >= 0$ (see section 3.2.2 for the definition) for all φ. This implies that, skipping the term $\hat{H}$ (this is the small perturbation), we have for every $\varphi \in \mathcal{E}$ an invariant $(r+s)$-dimensional reducible torus with basic frequencies ω. Moreover, we also assume that $\mathcal{B}$ and $\mathcal{C}$ are symmetric matrices with $\mathcal{J}_m(\mathcal{B}) = \mathcal{B}$ and $\det \bar{\mathcal{C}} \neq 0$. Hence, $J_m \mathcal{B}$ is a diagonal matrix, with eigenvalues $\lambda(\varphi)^\top = (\lambda_1(\varphi), \dots, \lambda_m(\varphi), -\lambda_1(\varphi), \dots, -\lambda_m(\varphi))$, that we asume all different, that gives the normal behaviour around the unpertubed invariant torus.

More concretely, let us assume that the following bounds hold: for the unperturbed part, for every $j, l = 1, \dots, 2m$ with $j \neq l$, we have $0 < \alpha_1 \leq |\lambda_j(\varphi) - \lambda_l(\varphi)|$, $\alpha_1/2 \leq |\lambda_j(\varphi)| \leq \alpha_2/2$ for all $\varphi \in \mathcal{E}$, and that $\mathcal{L}_\mathcal{E}\{\lambda_j\} \leq \beta_2/2$. Moreover, $\|(\bar{\mathcal{C}})^{-1}\|_\mathcal{E} \leq \bar{m}$ and, for certain $\rho > 0$ and $R > 0$, $\|\mathcal{C}\|_{\mathcal{E},\rho} \leq \hat{m}$, $\mathcal{L}_{\mathcal{E},\rho}\{\mathcal{C}\} \leq \tilde{m}$, $\|H_*\|_{\mathcal{E},\rho,R} \leq \hat{\nu}$ and $\mathcal{L}_{\mathcal{E},\rho,R}\{H_*\} \leq \tilde{\nu}$. Finally, we bound the size of the perturbation $\hat{H}$ by $\|\hat{H}\|_{\mathcal{E},\rho,R} \leq M$ and $\mathcal{L}_{\mathcal{E},\rho,R}\{\hat{H}\} \leq L$. To simplify the bounds, we will assume that $M \leq L$.

Lemma 3.17 (Iterative lemma) *Let us consider a Hamiltonian H as the one we have just described above. We assume that we can bound ρ, R, α_2, β_2, $\bar{m}$, $\hat{m}$, $\tilde{m}$, $\hat{\nu}$, $\tilde{\nu}$, M and L by certain fixed absolute constants ρ_0, R_0, α_2^*, β_2^*, $\bar{m}^*$, $\hat{m}^*$, $\tilde{m}^*$, $\hat{\nu}^*$, $\tilde{\nu}^*$, M_0 and L_0, and that for some fixed $R^* > 0$ and $\alpha_1^* > 0$ we have $R^* \leq R$ and $\alpha_1^* \leq \alpha_1$. We assume that for every $\varphi \in \mathcal{E}$, the corresponding ω verifies $|\omega| \leq \kappa^*$, for some fixed $\kappa^* > 0$. Finally, we also consider fixed $\tilde{\delta}_0 > 0$, $\gamma > r+s-1$ and $\mu_0 > 0$.*

In these conditions, there exists a constant $\hat{N}$, depending only on the constants above and on r, s and m, such that for every $\delta > 0$, $\hat{\delta} > 0$ and $0 < \mu \leq \mu_0$ for which the following three conditions hold,

a) $0 < 9\delta < \rho$, $0 < 9\hat{\delta} < \rho$ *and* $\delta/\hat{\delta} \leq \tilde{\delta}_0$,

b) *for every* $\varphi \in \mathcal{E}$,

$$|ik^\top \omega + l^\top \lambda(\varphi)| \geq \frac{\mu}{|k|_1^\gamma} \exp(-\delta |k|_1), \quad (3.20)$$

for all $k \in \mathbb{Z}^{r+s} \setminus \{0\}$ *and for all* $l \in \mathbb{N}^{2m}$ *with* $|l|_1 \leq 2$.

c) $\Theta = \hat{N} \frac{M}{\delta^{2\gamma+3} \mu^2} \leq 1/2$, *we have that there exists a function $S(\theta,x,\hat{I},y,\varphi)$, defined for every $\varphi\in\mathcal{E}$, with ∇S an analytic function with respect to $(\theta,x,\hat{I},y)$ on $\mathcal{U}^{r+s,2m+r}_{\rho-8\delta,R-8\hat{\delta}}$, 2π-periodic on θ and with Lipschitz dependence on $\varphi\in\mathcal{E}$, such that $\|\nabla S\|_{\mathcal{E},\rho-8\delta,R-8\hat{\delta}}\leq\min\{\delta,\hat{\delta}\}$. Moreover, following the notations of section 3.5.1, the canonical change of variables Ψ^S is well defined for every $\varphi\in\mathcal{E}$,*

$$\hat{\Psi}^S:\mathcal{U}^{r+s,2m+r}_{\rho-9\delta,R-9\hat{\delta}}\longrightarrow\mathcal{U}^{r+s,2m+r}_{\rho-8\delta,R-8\hat{\delta}},\tag{3.21}$$

and transforms H into

$$H^{(1)}(\theta,x,I,y,\varphi)\equiv H\circ\Psi^S(\theta,x,I,y),$$

where

$$H^{(1)}=\phi^{(1)}(\varphi)+\omega^\top I+\frac{1}{2}z^\top\mathcal{B}^{(1)}(\varphi)z+\frac{1}{2}\hat{I}^\top\mathcal{C}^{(1)}(\theta,\varphi)\hat{I}+H^{(1)}_*+\hat{H}^{(1)},$$

with $<H^{(1)}_*>=0$, and where $\mathcal{B}^{(1)}$ and $\mathcal{C}^{(1)}$ are symmetric matrices with $\mathcal{J}_m(\mathcal{B}^{(1)})=\mathcal{B}^{(1)}$. Moreover if we put $R^{(1)}=R-9\hat{\delta}$ and $\rho^{(1)}=R-9\delta$, we have the following bounds:

$$\begin{array}{rclrcl}
\|\nabla S\|_{\mathcal{E},\rho-8\delta,R-8\hat{\delta}} & \leq & \hat{N}\frac{M}{\delta^{2+2\gamma}\mu^2}, & \|\phi^{(1)}-\phi\|_{\mathcal{E}} & \leq & \hat{N}\frac{M}{\delta^{1+\gamma}\mu},\\
\|\mathcal{B}^{(1)}-\mathcal{B}\|_{\mathcal{E}} & \leq & \hat{N}\frac{M}{\delta^{1+\gamma}\mu}, & \mathcal{L}_{\mathcal{E}}\{\mathcal{B}^{(1)}-\mathcal{B}\} & \leq & \hat{N}\frac{L}{\delta^{2+2\gamma}\mu^2},\\
\|\mathcal{C}^{(1)}-\mathcal{C}\|_{\mathcal{E},\rho^{(1)}} & \leq & \hat{N}\frac{M}{\delta^{2+2\gamma}\mu^2}, & \mathcal{L}_{\mathcal{E},\rho^{(1)}}\{\mathcal{C}^{(1)}-\mathcal{C}\} & \leq & \hat{N}\frac{L}{\delta^{3+3\gamma}\mu^3},\\
\|H^{(1)}_*-H_*\|_{\mathcal{E},\rho^{(1)},R^{(1)}} & \leq & \hat{N}\frac{M}{\delta^{3+2\gamma}\mu^2}, & \mathcal{L}_{\mathcal{E},\rho^{(1)},R^{(1)}}\{H^{(1)}_*-H_*\} & \leq & \hat{N}\frac{L}{\delta^{4+3\gamma}\mu^3},\\
\|\hat{H}^{(1)}\|_{\mathcal{E},\rho^{(1)},R^{(1)}} & \leq & \hat{N}\frac{M^2}{\delta^{6+4\gamma}\mu^4}, & \mathcal{L}_{\mathcal{E},\rho^{(1)},R^{(1)}}\{\hat{H}^{(1)}\} & \leq & \hat{N}\frac{ML}{\delta^{7+5\gamma}\mu^5}.
\end{array}$$

Proof: The idea is to use the scheme described in section 3.2.2, to remove the perturbative terms that are an obstruction for the existence of a (reducible) torus with vector of basic frequencies ω up to first order in the size of the perturbation. Hence, as we described in section 3.2.2 we expand $\hat{H}$ in power series around $\hat{I}=0$, $z=0$ to obtain $H=\tilde{\omega}^{(0)\top}\tilde{I}+H^*$, where

$$H^*=a(\theta)+b(\theta)^\top z+c(\theta)^\top\hat{I}+\frac{1}{2}z^\top B(\theta)z+\hat{I}^\top E(\theta)z+\frac{1}{2}\hat{I}^\top C(\theta)\hat{I}+\Omega(\theta,x,\hat{I},y),$$

with $<\Omega>=0$, where we have not written explicitly the dependence on φ. We look for a generating function S,

$$S(\theta,x,\hat{I},y)=\xi^\top\hat{\theta}+d(\theta)+e(\theta)^\top z+f(\theta)^\top\hat{I}+\frac{1}{2}z^\top G(\theta)z+\hat{I}^\top F(\theta)z,$$

with the same properties as the one given in (3.7). If we want to obtain the transformed Hamiltonian $H^{(1)}$ we need to compute (see section 3.5.1):

$$H^{**}=\{H^*,S\}-\frac{\partial S}{\partial\tilde{\theta}}\tilde{\omega}^{(0)}.$$

We introduce the decomposition $H^{**}=H^{**}_1+H^{**}_2$, with

$$H^{**}_1=\{\omega^\top I+\frac{1}{2}z^\top\mathcal{B}z+\frac{1}{2}\hat{I}^\top\mathcal{C}\hat{I}+H_*,S\}-\frac{\partial S}{\partial\tilde{\theta}}\tilde{\omega}^{(0)},$$

and $H_2^{**}=\{\hat{H},S\}$. Then, we want to select S such that $H+H_1^{**}$ takes the form

$$H+H_1^{**}=\phi^{(1)}(\varphi)+\omega^\top I+\frac{1}{2}z^\top\mathcal{B}^{(1)}(\varphi)z+\frac{1}{2}\hat{I}^\top\mathcal{C}^{(1)}(\theta,\varphi)\hat{I}+H_*^{(1)}(\theta,x,\hat{I},y,\varphi), \quad (3.22)$$

and hence, $\hat{H}^{(1)}=H_2^{**}+\Sigma(H^{**},S)$. We can explicitly compute H_1^{**}:

$$\begin{aligned} H_1^{**} &= \left(\frac{1}{2}\frac{\partial}{\partial\hat{\theta}}\left(\hat{I}^\top\mathcal{C}\hat{I}\right)+\frac{\partial H_*}{\partial\theta}\right)(f+Fz)-\left(\hat{I}^\top\mathcal{C}+\frac{\partial H_*}{\partial\hat{I}}\right)\left(\frac{\partial S}{\partial\hat{\theta}}\right)^\top+ \\ &\quad +\left(z^\top\mathcal{B}+\frac{\partial H_*}{\partial z}\right)J_m(e+Gz+F^\top\hat{I})-\frac{\partial S}{\partial\theta}\omega. \end{aligned} \quad (3.23)$$

Then, it is not difficult to see that equation (3.22) leads to equations $(eq_1)-(eq_5)$ given in section 3.2.2, replacing $\omega^{(0)}$ by ω, and that

$$\phi^{(1)}=\phi-\hat{\omega}^\top\xi, \quad (3.24)$$

$$\mathcal{B}^{(1)}=\mathcal{J}_m(B^*), \quad (3.25)$$

$$\mathcal{C}^{(1)}=\mathcal{C}+\left[\hat{I}^\top\left(\frac{1}{2}\frac{\partial\mathcal{C}}{\partial\theta}f-\mathcal{C}\left(\frac{\partial f}{\partial\hat{\theta}}\right)^\top\right)\hat{I}-\frac{\partial H_*}{\partial\hat{I}}\left(\xi+\left(\frac{\partial d}{\partial\hat{\theta}}\right)^\top\right)+\frac{\partial H_*}{\partial z}J_m e\right]_{(\hat{I},\hat{I})}, \quad (3.26)$$

$$H_*^{(1)}=\Omega+H_1^{**}-<H_1^{**}>. \quad (3.27)$$

We will prove that, from the Diophantine bounds of (3.20), it is possible to construct a convergent expression for S, and to obtain suitable bounds for the transformed Hamiltonian.

The first step is to bound the solutions of $(eq_1)-(eq_5)$, using lemmas of section 3.5.2. In what follows, $\hat{N}$ denotes a constant that bounds all the expressions depending on all the fixed constants of the statement of the lemma, and its value is redefined several times during the proof in order to simplify the notation. Moreover, sometimes we do not write explicitly the dependence on φ, but all the bounds hold for all $\varphi\in\mathcal{E}$. First, we remark that using the bounds on $\hat{H}$, and from lemmas 3.2 and 3.7 we can bound $\|.\|_{\mathcal{E},\rho}$ and $\mathcal{L}_{\mathcal{E},\rho}\{.\}$ of $a-\phi$, b, $c-\hat{\omega}$, $B-\mathcal{B}$, E and $C-\mathcal{C}$ by $\hat{N}M$ and $\hat{N}L$ respectively, with an $\hat{N}$ that only depends on R^*, r and m. We recall that from the expressions of section 3.2.2 the solutions of $(eq_1)-(eq_5)$ are unique, and for them we have (working here for a fixed $\varphi\in\mathcal{E}$):

(eq_1) From the expression of d as a function of the coefficients of the Fourier expansion of a, it is clear that if we use the bounds on the denominators given by the Diophantine conditions of (3.20) and lemma 3.1 we can see that:

$$|d|_{\rho-\chi}\le\left(\frac{\gamma}{(\chi-\delta)\exp(1)}\right)^\gamma\frac{|\tilde{a}|_\rho}{\mu},$$

if $\rho>\chi>\delta$, and then using that $|\tilde{a}|_\rho\le|a-\phi|_\rho$, we can write:

$$|d|_{\rho-\chi}\le\hat{N}\frac{M}{(\chi-\delta)^\gamma\mu}.$$

(eq_2) We have for e

$$|e|_{\rho-\chi} \le \left(\frac{2}{\alpha_1^*} + \left(\frac{\gamma}{(\chi-\delta)\exp(1)}\right)^{\gamma}\frac{1}{\mu}\right)|b|_{\rho},$$

for all $\rho > \chi > \delta$. Consequently:

$$|e|_{\rho-\chi} \le \hat{N}\frac{M}{(\chi-\delta)^{\gamma}\mu}.$$

(eq_3) First we bound ξ:

$$\begin{aligned} |\xi| &= |(\bar{\mathcal{C}})^{-1}\bar{\mathcal{C}}\xi| \le |(\bar{\mathcal{C}})^{-1}||\bar{\mathcal{C}}\xi| \le \bar{m}^*\left|\bar{c}-\hat{\omega}-\overline{\mathcal{C}\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}}\right|_{\rho-\chi} \le \\ &\le \bar{m}^*\left(|c-\hat{\omega}|_{\rho} + \left|\mathcal{C}\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}\right|_{\rho-\chi}\right) \le \bar{m}^*\left(\hat{N}M + |\mathcal{C}|_{\rho}\frac{2|d|_{\rho-\chi/2}}{\chi\exp(1)}\right), \end{aligned}$$

where $\rho \ge \chi > 2\delta$. Hence,

$$|\xi| \le \hat{N}\frac{M}{(\chi-2\delta)^{\gamma+1}\mu},$$

for all $\rho \ge \chi > 2\delta$. Then, for c^* we have:

$$\begin{aligned} |c^*|_{\rho-\chi} &\le \left|\tilde{c}-\tilde{\mathcal{C}}\xi-\mathcal{C}\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}+\overline{\mathcal{C}\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}}\right|_{\rho-\chi} \le \\ &\le |\tilde{c}|_{\rho} + |\mathcal{C}|_{\rho}\left(|\xi| + \left|\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}\right|_{\rho-\chi}\right) \le \\ &\le |c-\hat{\omega}|_{\rho} + \hat{m}^*\left(|\xi| + \frac{2|d|_{\rho-\chi/2}}{\chi\exp(1)}\right) \le \hat{N}\frac{M}{(\chi-2\delta)^{\gamma+1}\mu}. \end{aligned}$$

Hence, if $\rho > \chi > 3\delta$,

$$|f|_{\rho-\chi} \le \left(\frac{3\gamma}{(\chi-3\delta)\exp(1)}\right)^{\gamma}\frac{|c^*|_{\rho-2\chi/3}}{\mu} \le \hat{N}\frac{M}{(\chi-3\delta)^{2\gamma+1}\mu^2}.$$

(eq_4) From the definition of B^* given in (3.8), we have

$$\begin{aligned} |B^*-\mathcal{B}|_{\rho-\chi} &\le |B-\mathcal{B}|_{\rho-\chi} + \\ &+ \left|\left[\frac{\partial H_*}{\partial\hat{I}}\left(\xi+\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}\right)\right]_{(z,z)}\right|_{\rho-\chi} + \left|\left[\frac{\partial H_*}{\partial z}J_m e\right]_{(z,z)}\right|_{\rho-\chi} \le \\ &\le \hat{N}M + (2m+1)r\frac{|H_*|_{\rho,R}}{(R^*)^3}\left(|\xi| + \frac{2|d|_{\rho-\chi/2}}{\chi\exp(1)}\right) + 24m^2\frac{|H_*|_{\rho,R}}{(R^*)^3}|e|_{\rho-\chi}, \end{aligned}$$

and then

$$|B^* - \mathcal{B}|_{\rho-\chi} \le \hat{N}\frac{M}{(\chi-2\delta)^{\gamma+1}\mu},$$

if $\rho > \chi > 2\delta$, and the same bound holds for $|B^{**}|_{\rho-\chi}$ (see (3.10)). Lemma 3.1 allows to bound

$$|G|_{\rho-\chi} \le \left(\frac{1}{\alpha_1^*} + \left(\frac{3\gamma}{(\chi-3\delta)\exp(1)}\right)^{\gamma}\frac{1}{\mu}\right) 2m\,|B^{**}|_{\rho-2\chi/3},$$

with $\rho > \chi > 3\delta$. Hence,

$$|G|_{\rho-\chi} \le \hat{N}\frac{M}{(\chi-3\delta)^{2\gamma+1}\mu^2}.$$

(eq_5) If $\rho > \chi > 2\delta$, we have for E^* defined in (3.9):

$$\begin{aligned}
|E^*|_{\rho-\chi} \le \; & |E|_{\rho-\chi} + \left|\mathcal{C}\left(\frac{\partial e}{\partial\hat{\theta}}\right)^{\top}\right|_{\rho-\chi} + \left|\left[\frac{\partial H_*}{\partial \hat{I}}\left(\xi + \left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}\right)\right]_{(\hat{I},z)}\right|_{\rho-\chi} + \\
& + \left|\left[\frac{\partial H_*}{\partial z}J_m e\right]_{(\hat{I},z)}\right|_{\rho-\chi} \le \hat{N}M + |\mathcal{C}|_\rho 2m\frac{2|e|_{\rho-\chi/2}}{\chi\exp(1)} + \\
& + 4mr\frac{|H_*|_{\rho,R}}{(R^*)^3}\left(|\xi| + \frac{2|d|_{\rho-\chi/2}}{\chi\exp(1)}\right) + 8m^2\frac{|H_*|_{\rho,R}}{(R^*)^3}|e|_{\rho-\chi}.
\end{aligned}$$

Then,

$$|E^*|_{\rho-\chi} \le \hat{N}\frac{M}{(\chi-2\delta)^{\gamma+1}\mu}.$$

Now, if $\rho > \chi > 3\delta$,

$$|F|_{\rho-\chi} \le 2m\left(\frac{2}{\alpha_1^*} + \left(\frac{3\gamma}{(\chi-3\delta)\exp(1)}\right)^{\gamma}\frac{1}{\mu}\right)|E^*|_{\rho-2\chi/3},$$

that implies

$$|F|_{\rho-\chi} \le \hat{N}\frac{M}{(\chi-3\delta)^{2\gamma+1}\mu^2}.$$

Now, we repeat the same process to bound the Lipschitz constants for the solutions of these equations. For that purpose, we will also need the results of lemmas 3.6 and 3.7 to work with the different Lipschitz dependences. We remark that, for the different denominators, we can bound:

$$\mathcal{L}_{\mathcal{E}}\{ik^\top\omega + l^\top\lambda\} \le |k|_1 + \frac{\beta_2^*}{2}|l|_1,$$

for every $k \in \mathbb{Z}^{r+s}$, $l \in \mathbb{N}^{2m}$, $|l|_1 \le 2$. Moreover, we will also use the hypothesis $M \le L$ to simplify the bounds. Then we have:

(eq_1) We need to take into account the φ dependence for all the functions, and so for d we have

$$d(\theta,\varphi)=\sum_{k\in\mathbb{Z}^{r+s}\setminus\{0\}}\frac{a_k(\varphi)}{ik^\top\omega}\exp(ik^\top\theta).$$

Then, using lemma 3.6 and $\mathcal{L}_{\mathcal{E},\rho}\{\tilde{a}\}\le\mathcal{L}_{\mathcal{E},\rho}\{a-\phi\}$, one obtains

$$\begin{aligned}\mathcal{L}_{\mathcal{E},\rho-\chi}\{d\} &\le \left(\frac{\gamma}{(\chi-\delta)\exp(1)}\right)^{\gamma}\frac{\mathcal{L}_{\mathcal{E},\rho}\{\tilde{a}\}}{\mu}+\left(\frac{2\gamma+1}{(\chi-2\delta)\exp(1)}\right)^{2\gamma+1}\frac{\|\tilde{a}\|_{\mathcal{E},\rho}}{\mu^2}\le\\ &\le \hat{N}\frac{L}{(\chi-2\delta)^{2\gamma+1}\mu^2},\end{aligned}$$

for every $\rho>\chi>2\delta$.

(eq_2)

$$\begin{aligned}\mathcal{L}_{\mathcal{E},\rho-\chi}\{e\} &\le \left(\frac{\gamma}{(\chi-\delta)\exp(1)}\right)^{\gamma}\frac{\mathcal{L}_{\mathcal{E},\rho}\{b\}}{\mu}+\left(\frac{2\gamma+1}{(\chi-2\delta)\exp(1)}\right)^{2\gamma+1}\frac{\|b\|_{\mathcal{E},\rho}}{\mu^2}+\\ &+\left(\frac{2\gamma}{(\chi-2\delta)\exp(1)}\right)^{2\gamma}\frac{\|b\|_{\mathcal{E},\rho}}{\mu^2}\frac{\beta_2^*}{2}+\frac{2}{\alpha_1^*}\mathcal{L}_{\mathcal{E},\rho}\{b\}+\frac{4}{(\alpha_1^*)^2}\|b\|_{\mathcal{E},\rho}\frac{\beta_2^*}{2}\le\\ &\le \hat{N}\frac{L}{(\chi-2\delta)^{2\gamma+1}\mu^2},\end{aligned}$$

if $\rho>\chi>2\delta$.

(eq_3) If $\rho\ge\chi>3\delta$, we have:

$$\begin{aligned}\mathcal{L}_{\mathcal{E}}\{\xi\} &\le \mathcal{L}_{\mathcal{E}}\left\{(\bar{\mathcal{C}})^{-1}\right\}\left\|\bar{c}-\hat{\omega}-\overline{\mathcal{C}\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}}\right\|_{\mathcal{E},\rho-\chi}+\\ &+\|(\bar{\mathcal{C}})^{-1}\|_{\mathcal{E}}\mathcal{L}_{\mathcal{E},\rho-\chi}\left\{\bar{c}-\hat{\omega}-\overline{\mathcal{C}\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}}\right\}\le\hat{N}\frac{L}{(\chi-3\delta)^{2\gamma+2}\mu^2},\end{aligned}$$

where we have used that, from lemma 3.5,

$$\mathcal{L}_{\mathcal{E}}\{(\bar{\mathcal{C}})^{-1}\}\le\|(\bar{\mathcal{C}})^{-1}\|_{\mathcal{E}}^2\mathcal{L}_{\mathcal{E}}\{\bar{\mathcal{C}}\}\le(\bar{m}^*)^2\mathcal{L}_{\mathcal{E},\rho}\{\mathcal{C}\},$$

and also that

$$\mathcal{L}_{\mathcal{E},\rho-\chi}\left\{\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}\right\}\le\frac{3}{\chi\exp(1)}\mathcal{L}_{\mathcal{E},\rho-2\chi/3}\{d\},$$

and $\mathcal{L}_{\mathcal{E},\rho-\chi}\{\bar{c}-\hat{\omega}\}\le\mathcal{L}_{\mathcal{E},\rho}\{c-\hat{\omega}\}$. Then, if $\rho>\chi>3\delta$, using that $\mathcal{L}_{\mathcal{E},\rho-\chi}\{\tilde{c}\}\le\mathcal{L}_{\mathcal{E},\rho-\chi}\{c-\hat{\omega}\}$, one has

$$\mathcal{L}_{\mathcal{E},\rho-\chi}\{c^*\}\le\mathcal{L}_{\mathcal{E},\rho-\chi}\{\tilde{c}\}+\mathcal{L}_{\mathcal{E},\rho-\chi}\left\{\tilde{\mathcal{C}}\xi\right\}+\mathcal{L}_{\mathcal{E},\rho-\chi}\left\{\mathcal{C}\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}\right\}\le\hat{N}\frac{L}{(\chi-3\delta)^{2\gamma+2}\mu^2}.$$

Hence,

$$\mathcal{L}_{\mathcal{E},\rho-\chi}\{f\} \leq \left(\frac{3\gamma}{(\chi-3\delta)\exp(1)}\right)^{\gamma}\frac{\mathcal{L}_{\mathcal{E},\rho-2\chi/3}\{c^*\}}{\mu}+ + \left(\frac{2(2\gamma+1)}{(\chi-6\delta)\exp(1)}\right)^{2\gamma+1}\frac{\|c^*\|_{\mathcal{E},\rho-2\chi/3}}{\mu^2} \leq \hat{N}\frac{L}{(\chi-6\delta)^{3\gamma+2}\mu^3},$$

if $\rho > \chi > 6\delta$.

(eq_4) We first bound:

$$\mathcal{L}_{\mathcal{E},\rho-\chi}\{B^*-\mathcal{B}\} \leq \mathcal{L}_{\mathcal{E},\rho-\chi}\{B-\mathcal{B}\}+\mathcal{L}_{\mathcal{E},\rho-\chi}\left\{\left[\frac{\partial H_*}{\partial \hat{I}}\left(\xi+\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}\right)\right]_{(z,z)}\right\}+ +\mathcal{L}_{\mathcal{E},\rho-\chi}\left\{\left[\frac{\partial H_*}{\partial z}J_m e\right]_{(z,z)}\right\} \leq \hat{N}\frac{L}{(\chi-3\delta)^{2\gamma+2}\mu^2},$$

if $\rho > \chi > 3\delta$, and the same bound holds for $\mathcal{L}_{\mathcal{E},\rho-\chi}\{B^{**}\}$. This implies

$$\begin{aligned}\mathcal{L}_{\mathcal{E},\rho-\chi}\{G\} \leq\ & (2m-1)\frac{1}{\alpha_1^*}\mathcal{L}_{\mathcal{E},\rho-\chi}\{B^{**}\}+(2m-1)\frac{1}{(\alpha_1^*)^2}\|B^{**}\|_{\mathcal{E},\rho-\chi}\beta_2^*+\\ &+2m\left(\frac{3\gamma}{(\chi-3\delta)\exp(1)}\right)^{\gamma}\frac{\mathcal{L}_{\mathcal{E},\rho-2\chi/3}\{B^{**}\}}{\mu}+\\ &+2m\left(\frac{3(2\gamma+1)}{(\chi-6\delta)\exp(1)}\right)^{2\gamma+1}\frac{\|B^{**}\|_{\mathcal{E},\rho-2\chi/3}}{\mu^2}+\\ &+2m\left(\frac{6\gamma}{(\chi-6\delta)\exp(1)}\right)^{2\gamma}\frac{\|B^{**}\|_{\mathcal{E},\rho-2\chi/3}}{\mu^2}\beta_2^*\leq\\ \leq\ & \hat{N}\frac{L}{(\chi-6\delta)^{3\gamma+2}\mu^3},\end{aligned}$$

if $\rho > \chi > 6\delta$.

(eq_5) From the definition of E^*,

$$\begin{aligned}\mathcal{L}_{\mathcal{E},\rho-\chi}\{E^*\} \leq\ & \mathcal{L}_{\mathcal{E},\rho-\chi}\{E\}+\mathcal{L}_{\mathcal{E},\rho-\chi}\left\{\left[\frac{\partial H_*}{\partial \hat{I}}\left(\xi+\left(\frac{\partial d}{\partial\hat{\theta}}\right)^{\top}\right)\right]_{(\hat{I},z)}\right\}+\\ &+\mathcal{L}_{\mathcal{E},\rho-\chi}\left\{\mathcal{C}\left(\frac{\partial e}{\partial\hat{\theta}}\right)^{\top}\right\}+\mathcal{L}_{\mathcal{E},\rho-\chi}\left\{\left[\frac{\partial H_*}{\partial z}J_m e\right]_{(\hat{I},z)}\right\}\leq\\ \leq\ & \hat{N}\frac{L}{(\chi-4\delta)^{2\gamma+2}\mu^2},\end{aligned}$$

if $\rho > \chi > 4\delta$. Hence, if now $\rho > \chi > 6\delta$, we can bound:

$$\mathcal{L}_{\mathcal{E},\rho-\chi}\{F\} \leq 2m\frac{2}{\alpha_1^*}\mathcal{L}_{\mathcal{E},\rho-\chi}\{E^*\}+2m\frac{4}{(\alpha_1^*)^2}\|E^*\|_{\mathcal{E},\rho-\chi}\frac{\beta_2^*}{2}+$$

$$+2m\left(\frac{3\gamma}{(\chi-3\delta)\exp(1)}\right)^{\gamma}\frac{\mathcal{L}_{\mathcal{E},\rho-2\chi/3}\{E^*\}}{\mu}+$$
$$+2m\left(\frac{3(2\gamma+1)}{(\chi-6\delta)\exp(1)}\right)^{2\gamma+1}\frac{\|E^*\|_{\mathcal{E},\rho-2\chi/3}}{\mu^2}+$$
$$+2m\left(\frac{6\gamma}{(\chi-6\delta)\exp(1)}\right)^{2\gamma}\frac{\|E^*\|_{\mathcal{E},\rho-2\chi/3}}{\mu^2}\frac{\beta_2^*}{2}\leq$$
$$\leq\ \hat{N}\frac{L}{(\chi-6\delta)^{3\gamma+2}\mu^3}.$$

Before bounding the transformed Hamiltonian, let us check that the change given by the generating function S is well defined. First, we have that:

$$\|\nabla S\|_{\mathcal{E},\rho-\chi,R}\leq\hat{N}\frac{M}{(\chi-4\delta)^{2\gamma+2}\mu^2},\tag{3.28}$$

and that

$$\mathcal{L}_{\mathcal{E},\rho-\chi,R}\{\nabla S\}\leq\hat{N}\frac{L}{(\chi-7\delta)^{3\gamma+3}\mu^3},$$

provided that $\rho>\chi>7\delta$. If we select $\chi=8\delta$, and if we consider (3.28), we have a bound of the type:

$$\|\nabla S\|_{\mathcal{E},\rho-8\delta,R-8\hat{\delta}}\leq\hat{N}\frac{M}{\delta^{2\gamma+2}\mu^2}.$$

Before continuing, let us ask to the quantity

$$\left(r+(r+2m)\exp(1)\max\{1,\tilde{\delta}_0\}\right)\hat{N}\frac{M}{\delta^{2\gamma+3}\mu^2},\tag{3.29}$$

to be bounded by $1/2$ (this will be used in (3.30) and (3.31)). This can be achieved redefining $\hat{N}$ such that (3.29) is bounded by $\Theta=\hat{N}\frac{M}{\delta^{2\gamma+3}\mu^2}$. Hence, the condition we are asking for is $\Theta\leq 1/2$.

From this last bound one obtains,

$$\|\nabla S\|_{\mathcal{E},\rho-8\delta,R-8\hat{\delta}}\leq\frac{\Theta\delta}{\max\{1,\tilde{\delta}_0\}}\leq\min\{\delta,\delta/\tilde{\delta}_0\}\leq\min\{\delta,\hat{\delta}\},\tag{3.30}$$

and

$$\Delta_{\delta,\hat{\delta}}\exp(1)\|\nabla S\|_{\mathcal{E},\rho-8\delta,R-8\hat{\delta}}\leq\Theta,\tag{3.31}$$

where we use the definition of $\Delta_{\delta,\hat{\delta}}$ given in (3.17).

From (3.30) and lemma 3.12 we have that Ψ^S is well defined (for every $\varphi\in\mathcal{E}$), according to (3.21). From (3.31) and lemma 3.9 we can bound the expression of $\Sigma(H^{**},S)$ that appears in the transformed Hamiltonian,

$$\|\Sigma(H^{**},S)\|_{\mathcal{E},\rho^{(1)},R^{(1)}}\leq\left(\sum_{j\geq 1}\frac{1}{j+1}\left(\frac{1}{2}\right)^{j-1}\right)\Theta\|H^{**}\|_{\mathcal{E},\rho-8\delta,R-8\hat{\delta}}.$$

and, similarly, for the Lipschitz constant we can use lemma 3.11 to produce

$$\mathcal{L}_{\mathcal{E},\rho^{(1)},R^{(1)}}\left\{\Sigma(H^{**},S)\right\} \leq \sum_{j\geq 1}\left(\frac{1}{j+1}\left(\Delta_{\delta,\hat{\delta}}\exp(1)\right)^j \hat{F}_2^{j-1}(j\hat{L}_2\hat{F}_1+\hat{L}_1\hat{F}_2)\right),$$

with $\hat{F}_1 = \|H^{**}\|_{\mathcal{E},\rho-8\delta,R-8\hat{\delta}}$, $\hat{F}_2 = \|\nabla S\|_{\mathcal{E},\rho-8\delta,R-8\hat{\delta}}$, $\hat{L}_1 = \mathcal{L}_{\mathcal{E},\rho-8\delta,R-8\hat{\delta}}\{H^{**}\}$ and $\hat{L}_2 = \mathcal{L}_{\mathcal{E},\rho-8\delta,R-8\hat{\delta}}\{\nabla S\}$. Then

$$\begin{aligned}\mathcal{L}_{\mathcal{E},\rho^{(1)},R^{(1)}}\left\{\Sigma(H^{**},S)\right\} \leq & \left(\sum_{j\geq 1}\frac{j}{j+1}\left(\frac{1}{2}\right)^{j-1}\right)\Delta_{\delta,\hat{\delta}}\exp(1)\hat{L}_2\hat{F}_1 + \\ & + \left(\sum_{j\geq 1}\frac{1}{j+1}\left(\frac{1}{2}\right)^{j-1}\right)\Delta_{\delta,\hat{\delta}}\exp(1)\hat{L}_1\hat{F}_2.\end{aligned}$$

With those expressions, to bound $\Sigma(H^{**},S)$ is reduced to bound H^{**}, with the only remark that the sums $\sum_{j\geq 1}\frac{1}{j+1}\nu^{j-1} = -\frac{\ln(1-\nu)+\nu}{\nu^2}$ and $\sum_{j\geq 1}\frac{j}{j+1}\nu^{j-1} = \frac{(1-\nu)\ln(1-\nu)+\nu}{\nu^2(1-\nu)}$ are well defined for $\nu = 1/2$.

Now, we can bound the transformed Hamiltonian. From the bounds that come from the solutions of $(eq_1)-(eq_5)$ we have:

$$\|H_1^{**}\|_{\mathcal{E},\rho-\chi,R-\eta} \leq \hat{N}\frac{M}{(\chi-4\delta)^{2\gamma+3}\mu^2}\max\{1,\chi/\eta\},$$

and

$$\mathcal{L}_{\mathcal{E},\rho-\chi,R-\eta}\{H_1^{**}\} \leq \hat{N}\frac{L}{(\chi-7\delta)^{3\gamma+4}\mu^3}\max\{1,\chi/\eta\}.$$

To obtain these bounds, we use the explicit expression of H_1^{**} given in (3.23), and lemmas 3.1, 3.2, 3.6 and 3.7 to bound the different partial derivatives. We remark that here we need to use that $|\omega| \leq \kappa^*$ for any $\varphi \in \mathcal{E}$. Moreover, from the bound for the Poisson brackets given in lemmas 3.8 and 3.10 we have, for H_2^{**},

$$\|H_2^{**}\|_{\mathcal{E},\rho-\chi,R-\eta} \leq \hat{N}\frac{M^2}{(\chi-4\delta)^{2\gamma+3}\mu^2}\max\{1,\chi/\eta\},$$

and

$$\mathcal{L}_{\mathcal{E},\rho-\chi,R-\eta}\{H_2^{**}\} \leq \hat{N}\frac{LM}{(\chi-7\delta)^{3\gamma+4}\mu^3}\max\{1,\chi/\eta\}.$$

The techniques that we use to control the reduction in the different domains when we use Cauchy estimates, are analogous to the ones used in all the previous bounds. Hence, it is clear that we can estimate H^{**} with an analogous bounds as the ones for H_1^{**}.

Finally, using all those bounds and from the explicit expressions of $\phi^{(1)}$, $\mathcal{B}^{(1)}$, $\mathcal{C}^{(1)}$, $H_1^{(1)}$ and $\hat{H}^{(1)}$ in (3.24)–(3.27) it is not difficult to obtain the final $\hat{N}$ such that all the bounds in the statements of the lemma hold.

3.5.4 Proof of the theorem

We split the proof of the theorem in several parts: in the first one we use one step of the iterative method described in section 3.2.2 as a linear scheme to reduce the size of the perturbation. Then, we introduce $\hat{\omega}$ as a new parameter to describe the family of lower dimensional tori near the initial one. The next step is to apply the bounds of the iterative scheme given by lemma 3.17, and we prove the convergence of this scheme for a suitable set of parameters. Finally, we obtain the different estimates on the measure of this set.

Linear scheme with respect to ε

We consider the initial Hamiltonian given in the formulation of Theorem 3.1, and we apply one step of the iterative method described in section 3.2.2. We remark that from the Diophantine bounds in the statements of the theorem, we can guarantee that this step is possible for small enough values of ε, and that it keeps the initial C^2 differentiability with respect to ε on the transformed Hamiltonian. We put $H^{(0)}$ for this Hamiltonian which, up to constant terms that are irrelevant is

$$H^{(0)} = \omega^{(0)\top} I + \frac{1}{2} z^\top \mathcal{B}^{(0)}(\varepsilon) z + \frac{1}{2} \hat{I}^\top \mathcal{C}^{(0)}(\theta,\varepsilon)\hat{I} + H_*^{(0)}(\theta,x,\hat{I},y,\varepsilon) + \varepsilon^2 \hat{\mathcal{H}}^{(0)}(\theta,x,\hat{I},y,\varepsilon), \tag{3.32}$$

with the same kind of analytic properties with respect to $(\theta,x,\hat{I},y)$ as the initial one, in a new domain that is independent on ε (small enough). We remark that the new matrices $\mathcal{B}^{(0)}$ and $\mathcal{C}^{(0)}$ depend on ε, and that $\mathcal{C}^{(0)}$ depends also on θ. Moreover, for H_* we do not have the semi-normal form conditions given in **P1** and **P2**. As this step comes from a perturbative (linear) method, we have that $\mathcal{B}^{(0)} - \mathcal{B}$, $\mathcal{C}^{(0)} - \mathcal{C}$ and $H_*^{(0)} - H_*$ are of $O(\varepsilon)$.

Our aim is to repeat the same iterative scheme. We remark that in the next step and in the ones that follows, we can not guarantee good Diophantine properties for the new eigenvalues of $J_m\mathcal{B}^{(0)}$ because this matrix changes at each step of the process. This is the reason that forces us to use parameters to control these eigenvalues. So, we can only work in the set of parameters for which certain Diophantine bounds hold. But before that, we want to introduce a new parameter.

Introduction of the vector of frequencies as a parameter

We consider values of $\hat{\omega} \in \mathbb{R}^r$ close to $\hat{\omega}^{(0)}$, and for any of these values we perform the change given in (3.14). So, putting $\varphi^\top = (\hat{\omega}^\top, \varepsilon)$ we obtain the following family of Hamiltonians,

$$\begin{aligned} H^{(1)}(\theta,x,I,y,\varphi) &= \tilde{\omega}^{(0)\top}\tilde{I} + \hat{\omega}^{(0)\top}(\hat{I} + \mathcal{C}^{-1}(\hat{\omega} - \hat{\omega}^{(0)})) + \frac{1}{2} z^\top \mathcal{B}^{(0)}(\varepsilon) z + \\ &\quad + \frac{1}{2}(\hat{I} + \mathcal{C}^{-1}(\hat{\omega} - \hat{\omega}^{(0)}))^\top \mathcal{C}^{(0)}(\theta,\varepsilon)(\hat{I} + \mathcal{C}^{-1}(\hat{\omega} - \hat{\omega}^{(0)})) + \\ &\quad + H_*^{(0)}(\theta,x,\hat{I} + \mathcal{C}^{-1}(\hat{\omega} - \hat{\omega}^{(0)}),y,\varepsilon) + \\ &\quad + \varepsilon^2 \mathcal{H}^{(0)}(\theta,x,\hat{I} + \mathcal{C}^{-1}(\hat{\omega} - \hat{\omega}^{(0)}),y,\varepsilon). \end{aligned}$$

Now, we use the semi-normal form structure that we have for H_* and the fact that $H_*^{(0)}$ is ε-close to H_* to expand

$$H^{(1)} = \phi^{(1)}(\varphi)+\omega^\top I+\frac{1}{2}z^\top \mathcal{B}^{(1)}(\varphi)z+\frac{1}{2}\hat{I}^\top \mathcal{C}^{(1)}(\theta,\varphi)\hat{I}+H_*^{(1)}(\theta,x,\hat{I},y,\varphi)+\hat{H}^{(1)}(\theta,x,\hat{I},y,\varphi),$$

where $\hat{H}^{(1)}$ contains all the terms that are of $O_2(\varphi-\varphi^{(0)})$, $\varphi^{(0)\top}=(\hat{\omega}^{(0)\top},0)$ and $\omega^\top = (\hat{\omega}^\top,\tilde{\omega}^{(0)\top})$. Note that this Hamiltonian takes the same form as (3.19) in section 3.5.3. We remark that we have differentiable dependence of this Hamiltonian with respect to φ (in fact it is analytic with respect to $\hat{\omega}$) but, as it has been mentioned at the end of section 3.2.4, we replace the differentiable dependence by a Lipschitz one (in the sense given in section 3.5.1). To quantify all these facts, we take $0<\rho\leq 1$, $0<R\leq 1$ and $0<\vartheta_1\leq 1$, such that if we put $\rho^{(1)}=\rho$ and $R^{(1)}=R$, then we have analogous bounds as the ones described in section 3.5.3 for (3.19), given by $\rho^{(1)}$, $R^{(1)}$, and some positive constants $\alpha_1^{(1)}$, $\alpha_2^{(1)}$, $\beta_2^{(1)}$, $\bar{m}^{(1)}$, $\hat{m}^{(1)}$, $\tilde{m}^{(1)}$, $\hat{\nu}^{(1)}$ and $\tilde{\nu}^{(1)}$ on the set $\mathcal{E}^{(1)}=\{\varphi\in\mathbb{R}^{r+1}\;:|\varphi-\varphi^{(0)}|\leq\vartheta_1\}$, with respect to the "unperturbed part". For the perturbation $\hat{H}^{(1)}$, if we work with sets of the form $\bar{\mathcal{E}}^{(1)}\equiv\bar{\mathcal{E}}^{(1)}(\bar{\vartheta})=\left\{\varphi\in\mathcal{E}^{(1)}\;:|\varphi-\varphi^{(0)}|\leq\bar{\vartheta}\right\}$, for all $0\leq\bar{\vartheta}\leq\vartheta_1$, we can replace M and L by $N_1\bar{\vartheta}^2$ and $N_1\bar{\vartheta}$, for some $N_1>0$. To simplify the following bounds we assume, without loss of generality, that $N_1\geq 1$.

Finally, we finish this part with an explicit formulation of the nondegeneracy hypothesis of the normal eigenvalues with respect to the parameters. Let us consider $\mathcal{B}^{(1)}$. By construction, we have that $J_m\mathcal{B}^{(1)}$ is a diagonal matrix. Then, using the C^2 differentiability with respect to φ, we can write its eigenvalues as:

$$\lambda_j^{(1)}(\varphi)=\lambda_j+iu_j\varepsilon+iv_j^\top(\hat{\omega}-\hat{\omega}^{(0)})+\tilde{\lambda}_j^{(1)}(\varphi), \tag{3.33}$$

for $j=1,\ldots,2m$, with $u_j\in\mathbb{C}$ and $v_j\in\mathbb{C}^r$, and where the Lipschitz constant of $\tilde{\lambda}_j^{(1)}$ on $\bar{\mathcal{E}}^{(1)}$ is of $O(\bar{\vartheta})$. Then, those generic nondegeneracy conditions are:

NDC2 For any j such that $\mathrm{Re}\lambda_j=0$, we have $u_j\neq 0$ and $Re(v_j)\notin\mathbb{Z}^r$. Moreover, if we define $u_{j,l}=u_j-u_l$ and $v_{j,l}=v_j-v_l$, we have these same conditions for $u_{j,l}$ and $v_{j,l}$ for any $j\neq l$ such that $\mathrm{Re}(\lambda_j-\lambda_l)=0$.

Note that we have used the C^2 dependence on φ to ensure that the Lipschitz constant of $\tilde{\lambda}_j^{(1)}$ on $\bar{\mathcal{E}}^{(1)}$ is $O(\bar{\vartheta})$. If the dependence is C^1 we can only say that this constant is $o(\bar{\vartheta})$. Nevertheless, it is still possible in this case to derive the same results as in the C^2 case, but the details are more tedious.

The nondegeneracy conditions with respect to ε are the same ones used in [35] to study the quasiperiodic perturbations of elliptic fixed points, and the nondegeneracy conditions with respect to the $\hat{\omega}$-dependence are the same used in Chapter 1. They are analogous to the ones appeared in [47] and [17], but in those cases they were formulated for an unperturbed system having an r-dimensional analytic family of r-dimensional reducible elliptic tori.

Inductive part

We want to apply here the iterative lemma in an inductive form. For this purpose, we define $\sigma_n=\frac{6}{\pi^2 n^2}$ for every $n\geq 1$, and we note that $\sum_{n\geq 1}\sigma_n=1$. From this definition, we put $\delta_n = \frac{\sigma_n \rho}{18}$, $\hat{\delta}_n = \frac{\sigma_n R}{18}$ and we introduce $\rho^{(n+1)} = \rho^{(n)} - 9\delta_n$ and $R^{(n+1)} = R^{(n)} - 9\hat{\delta}_n$ for every $n \geq 1$. We also consider a fixed $1 < \varrho < 2$, to define $\mu_n = \exp(-\varrho^n)\mu_0$.

We suppose that, at step n, we have a Hamiltonian $H^{(n)}$ like $H^{(1)}$ defined for φ in a set $\mathcal{E}^{(n)} \subset \mathcal{E}^{(1)}$, with analogous bounds as $H^{(1)}$, replacing the superscript (1) by (n) in the unperturbed part, and with bounds for the perturbation given by $\bar{M}_n = M_n(\bar{\vartheta}) = N_n \bar{\vartheta}^{2^n}$ and $\bar{L}_n = L_n(\bar{\vartheta}) = N_n \bar{\vartheta}^{2^n-1}$, in every set of the form $\bar{\mathcal{E}}^{(n)}(\bar{\vartheta})$, for all $0 \leq \bar{\vartheta} \leq \vartheta_1$, being N_n independent on $\bar{\vartheta}$. We will show that this is possible if ϑ_1 is small enough, with conditions on ϑ_1 that are independent on the actual step.

At this point, we define the new set $\mathcal{E}^{(n+1)}$ of good parameters from $\mathcal{E}^{(n)}$ looking at the new Diophantine conditions. We have that $\varphi \in \mathcal{E}^{(n+1)}$ if $\varphi \in \mathcal{E}^{(n)}$ and the following conditions hold:

$$|k^\top \omega + l^\top \lambda^{(n)}(\varphi)| \geq \frac{\mu_n}{|k|_1^\gamma} \exp(-\delta_n |k|_1), \tag{3.34}$$

for all $k \in \mathbb{Z}^{r+s} \setminus \{0\}$, $l \in \mathbb{N}^{2m}$, $|l|_1 \leq 2$.

Now, we use the iterative lemma for $\varphi \in \mathcal{E}^{(n+1)}$. First we remark that at every step we have $\rho^{(n)} \leq \rho$, $R/2 \leq R^{(n)} \leq R$, $\delta_n/\hat{\delta}_n = \rho/R$, $\mu_n \leq \mu_0$ and, as $\vartheta_1 \leq 1$, we have that for every $\varphi \in \mathcal{E}^{(1)}$, $|\omega| \leq \max\{|\tilde{\omega}^{(0)}|, |\hat{\omega}^{(0)}| + 1\}$. Moreover, we assume that we can bound $\alpha_1^{(1)}/2 \leq \alpha_1^{(n)}$, $\alpha_2^{(n)} \leq 2\alpha_2^{(1)}$, $\beta_2^{(n)} \leq 2\beta_2^{(1)}$, $\bar{m}^{(n)} \leq 2\bar{m}^{(1)}$, $\hat{m}^{(n)} \leq 2\hat{m}^{(1)}$, $\tilde{m}^{(n)} \leq 2\tilde{m}^{(1)}$, $\hat{\nu}^{(n)} \leq 2\hat{\nu}^{(1)}$ $\tilde{\nu}^{(n)} \leq 2\tilde{\nu}^{(1)}$ and $N_n \vartheta_1^{2^n-2} \leq N_1$. We remark that all those bounds hold for $n = 1$. Then we consider the constant $\hat{N}$, given by the iterative lemma, corresponding to these bounds.

If we assume that in the actual step we have for $\Theta_n = \hat{N} \frac{N_n \vartheta_1^{2^n}}{\delta_n^{2\gamma+3} \mu_n^2}$, $\Theta_n \leq 1/2$, then we can apply the iterative lemma to obtain the generating function $S^{(n)}(\theta, x, \hat{I}, y, \varphi)$, with $\|\nabla S^{(n)}\|_{\mathcal{E}^{(n+1)}, \rho^{(n)} - 8\delta_n, R^{(n)} - 8\hat{\delta}_n} \leq \min\{\delta_n, \hat{\delta}_n\}$. So, in this case we have for $\Psi^{S^{(n)}}$

$$\hat{\Psi}^{S^{(n)}} : \mathcal{U}^{r+s, 2m+r}_{\rho^{(n+1)}, R^{(n+1)}} \longrightarrow \mathcal{U}^{r+s, 2m+r}_{\rho^{(n)} - 8\delta_n, R^{(n)} - 8\hat{\delta}_n}.$$

The next step is to bound the transformed Hamiltonian $H^{(n+1)} = H^{(n)} \circ \Psi^{S^{(n)}}$. We work in a set of the form $\bar{\mathcal{E}}^{(n+1)}$, for all $0 < \bar{\vartheta} \leq \vartheta_1$. From the bounds of the iterative lemma, and the explicit expressions of σ_n, δ_n and μ_n, we can deduce that there exists $\tilde{N}$ (we can assume $\tilde{N} \geq 1$) depending on the same constants as $\hat{N}$, such that

$$\begin{aligned}
\|\nabla S^{(n)}\|_{\bar{\mathcal{E}}^{(n+1)}, \rho^{(n)} - 8\delta_n, R^{(n)} - 8\hat{\delta}_n} &\leq \tilde{N} n^{4+4\gamma} (\exp(\varrho^n))^2 N_n \bar{\vartheta}^{2^n}, \\
\|\phi^{(n+1)} - \phi^{(n)}\|_{\bar{\mathcal{E}}^{(n+1)}} &\leq \tilde{N} n^{2+2\gamma} \exp(\varrho^n) N_n \bar{\vartheta}^{2^n}, \\
\|\mathcal{B}^{(n+1)} - \mathcal{B}^{(n)}\|_{\bar{\mathcal{E}}^{(n+1)}} &\leq \tilde{N} n^{2+2\gamma} \exp(\varrho^n) N_n \bar{\vartheta}^{2^n}, \\
\mathcal{L}_{\bar{\mathcal{E}}^{(n+1)}}\{\mathcal{B}^{(n+1)} - \mathcal{B}^{(n)}\} &\leq \tilde{N} n^{4+4\gamma} (\exp(\varrho^n))^2 N_n \bar{\vartheta}^{2^n-1}, \\
\|\mathcal{C}^{(n+1)} - \mathcal{C}^{(n)}\|_{\bar{\mathcal{E}}^{(n+1)}, \rho^{(n+1)}} &\leq \tilde{N} n^{4+4\gamma} (\exp(\varrho^n))^2 N_n \bar{\vartheta}^{2^n}, \\
\mathcal{L}_{\bar{\mathcal{E}}^{(n+1)}, \rho^{(n+1)}}\{\mathcal{C}^{(n+1)} - \mathcal{C}^{(n)}\} &\leq \tilde{N} n^{6+6\gamma} (\exp(\varrho^n))^3 N_n \bar{\vartheta}^{2^n-1}, \\
\|H_*^{(n+1)} - H_*^{(n)}\|_{\bar{\mathcal{E}}^{(n+1)}, \rho^{(n+1)}, R^{(n+1)}} &\leq \tilde{N} n^{6+4\gamma} (\exp(\varrho^n))^2 N_n \bar{\vartheta}^{2^n}, \\
\mathcal{L}_{\bar{\mathcal{E}}(n+1), \rho^{(n+1)}, R^{(n+1)}}\{H_*^{(n+1)} - H_*^{(n)}\} &\leq \tilde{N} n^{8+6\gamma} (\exp(\varrho^n))^3 N_n \bar{\vartheta}^{2^n-1}, \\
\|\hat{H}^{(n+1)}\|_{\bar{\mathcal{E}}^{(n+1)}, \rho^{(n+1)}, R^{(n+1)}} &\leq \tilde{N} n^{12+8\gamma} (\exp(\varrho^n))^4 N_n^2 \bar{\vartheta}^{2^{n+1}}, \\
\mathcal{L}_{\bar{\mathcal{E}}^{(n+1)}, \rho^{(n+1)}, R^{(n+1)}}\{\hat{H}^{(1)}\} &\leq \tilde{N} n^{14+10\gamma} (\exp(\varrho^n))^5 N_n^2 \bar{\vartheta}^{2^{n+1}-1}.
\end{aligned}$$

Moreover, we assume that we can bound $\Theta_n \leq \tilde{N} n^{6+4\gamma} (\exp(\varrho^n))^2 N_n \vartheta_1^{2^n}$, with the same constant $\tilde{N}$. Then, we use all these expressions as a motivation to define $N_{n+1} = \tilde{N} n^{14+10\gamma} (\exp(\varrho^n))^5 N_n^2$, for $n \geq 1$. To bound how fast N_{n+1} grows with n and N_1 we use lemma 3.13:

$$N_n \leq \frac{1}{\tilde{N}} \left(\left(\frac{5}{3} \right)^{14+10\gamma} \tilde{N} N_1 \exp\left(\frac{5\varrho}{2-\varrho} \right) \right)^{2^{n-1}},$$

if $n \geq 1$. If we also define $\tilde{N}_{n+1} = \tilde{N} n^{8+6\gamma} (\exp(\varrho^n))^5 N_n$, for $n \geq 1$, we clearly have, using that $N_1 \geq 1$ and $\tilde{N} \geq 1$, that $\tilde{N}_n \leq N_n$ for $n \geq 2$.

Now, we have to justify that we can use the iterative lemma in this inductive form when $n \geq 2$. To this end we need to see that the bounds that we have assumed at the step n (to define $\hat{N}$ and to use the iterative lemma) hold at every step if ϑ_1 is small enough. So, we note that if ϑ_1 is small enough, the following sum:

$$\sum_{n\geq 1} N_{n+1} \vartheta_1^{2^n-2}, \tag{3.35}$$

is bounded by $\hat{N}^*$ that depends on ϱ and the same constants as $\hat{N}$. This bound is not difficult to obtain if we look at how fast $\tilde{N}_n$ grows. Moreover the same ideas can be used to prove that $N_n \vartheta_1^{2^n-2} \leq N_1$, if $n \geq 1$ and ϑ_1 is small enough.

Then, we can define $\alpha_1^{(n+1)} = \alpha_1^{(n)} - 2N_{n+1}\vartheta_1^{2^n}$, $\alpha_2^{(n+1)} = \alpha_2^{(n)} + 2N_{n+1}\vartheta_1^{2^n}$, $\beta_2^{(n+1)} = \beta_2^{(n)} + N_{n+1}\vartheta_1^{2^n-1}$, $\hat{m}^{(n+1)} = \hat{m}^{(n)} + N_{n+1}\vartheta_1^{2^n}$, $\tilde{m}^{(n+1)} = \tilde{m}^{(n)} + N_{n+1}\vartheta_1^{2^n-1}$, $\hat{\nu}^{(n+1)} = \hat{\nu}^{(n)} + N_{n+1}\vartheta_1^{2^n}$ and $\tilde{\nu}^{(n+1)} = \tilde{\nu}^{(n)} + N_{n+1}\vartheta_1^{2^n-1}$, that from the convergence of (3.35) allows to apply another step of the iterative scheme, at least for sufficiently small values of ϑ_1. Moreover, it is clear that $\Theta_n \leq N_{n+1}\vartheta_1^{2^n} \leq \hat{N}^*\vartheta_1^2 \leq 1/2$ taken ϑ_1 small enough. Then, it only reamains to bound $\bar{m}^{(n+1)}$. For that purpose we first consider the bound $\|\bar{\mathcal{C}}^{(n+1)} - \bar{\mathcal{C}}^{(n)}\|_{\mathcal{E}^{(n+1)},\rho^{(n+1)}} \leq N_{n+1}\vartheta_1^{2^n}$, and then, if we work with a fixed value of $\varphi \in \bar{\mathcal{E}}^{(n+1)}(\vartheta_1)$, we have for any $W \in \mathbb{C}^r$:

$$|\bar{C}^{(n+1)} W| \geq |\bar{C}^{(n)} W| - \left| \left(\bar{C}^{(n+1)} - \bar{C}^{(n)} \right) W \right| \geq \left(\left(\bar{m}^{(n)} \right)^{-1} - N_{n+1}\vartheta_1^{2^n} \right) |W|.$$

We note that, from the equivalence $| \left(\bar{C}^{(n)} \right)^{-1} | \leq \bar{m}^{(n)} \iff |\bar{C}^{(n)} W| \geq \left(\bar{m}^{(n)} \right)^{-1} |W|$, for any $W \in \mathbb{C}^r$, we can take $\bar{m}^{(n+1)} = \frac{\bar{m}^{(n)}}{1-\bar{m}^{(n)} N_{n+1}\vartheta_1^{2^n}}$, provided that $\bar{m}^{(n)} N_{n+1}\vartheta_1^{2^n} < 1$. Then, using this expression we can see that $\bar{m}^{(n)} \leq 2\bar{m}^{(1)}$ for any $n \geq 1$, if ϑ_1 is small enough: if we assume that it holds for n, when we compute $\bar{m}^{(n+1)}$ we have that

$$\bar{m}^{(n)} N_{n+1}\vartheta_1^{2^n} \leq 2\bar{m}^{(1)} N_{n+1}\vartheta_1^{2^n} \leq \frac{1}{2},$$

if ϑ_1 is small enough. Moreover, we have by induction that

$$\bar{m}^{(n+1)} \leq \bar{m}^{(1)} \prod_{j=1}^{n} \frac{1}{1 - 2\bar{m}^{(1)} N_{j+1}\vartheta_1^{2^j}}.$$

So, it is clear that, if ϑ_1 is small enough,

$$\sum_{j\geq 1} 2\bar{m}^{(1)} N_{j+1}\vartheta_1^{2^j} \leq 2\bar{m}^{(1)} \hat{N}^* \vartheta_1^2 \leq \frac{1}{2}\ln(2),$$

and hence, if we note that when $0 \leq X \leq 1/2$,

$$\ln\left(\frac{1}{1-X}\right) = \ln\left(1 + \frac{X}{1-X}\right) \leq \frac{X}{1-X} \leq 2X,$$

we can bound $\ln\left(\bar{m}^{(n+1)}\right) \leq \ln\left(\bar{m}^{(1)}\right) + \ln(2)$, that proves $\bar{m}^{(n+1)} \leq 2\bar{m}^{(1)}$.

Convergence of the changes of variables

Now, we are going to prove the convergence of the composition of changes of variables. Let $\mathcal{E}^* = \cap_{n\geq 1}\mathcal{E}^{(n)}$ be the set of φ where all the transformations are well defined. We consider a fixed $\varphi \in \mathcal{E}^*$, but in fact, the results will hold in the whole set $\mathcal{E}^*$ provided that ϑ_1 is small enough.

We put $\breve{\Psi}^{(n)} = \hat{\Psi}^{(1)} \circ \ldots \circ \hat{\Psi}^{(n)}$ for $n \geq 1$, that goes from $\mathcal{U}^{r+s,2m+r}_{\rho^{(n+1)},R^{(n+1)}}$ to $\mathcal{U}^{r+s,2m+r}_{\rho,R}$, where $\hat{\Psi}^{(n)}$ means $\hat{\Psi}^{S^{(n)}}$. Then, if $p > q \geq 1$, we have

$$\breve{\Psi}^{(p)} - \breve{\Psi}^{(q)} = \sum_{j=q}^{p-1}\left(\breve{\Psi}^{(j+1)} - \breve{\Psi}^{(j)}\right).$$

To bound $\breve{\Psi}^{(j+1)} - \breve{\Psi}^{(j)}$, we define $\rho'_j = \rho^{(j)} - \rho/4$ and $R'_j = R^{(j)} - R/4$, and we put $\hat{\Delta}_{\rho,R} = \frac{1}{\exp(1)\rho} + \frac{r+2m}{R}$. Now, let us see that

$$\begin{aligned} &|\breve{\Psi}^{(j+1)} - \breve{\Psi}^{(j)}|_{\rho'_{j+2},R'_{j+2}} = |\hat{\Psi}^{(1)} \circ \ldots \circ \hat{\Psi}^{(j+1)} - \hat{\Psi}^{(1)} \circ \ldots \circ \hat{\Psi}^{(j)}|_{\rho'_{j+2},R'_{j+2}} \leq \\ \leq\ & \left(1 + 4\hat{\Delta}_{\rho,R}|\hat{\Phi}^{(1)}|_{\rho^{(2)},R^{(2)}}\right)|\hat{\Psi}^{(2)} \circ \ldots \circ \hat{\Psi}^{(j+1)} - \hat{\Psi}^{(2)} \circ \ldots \circ \hat{\Psi}^{(j)}|_{\rho'_{j+2},R'_{j+2}}, \qquad (3.36) \end{aligned}$$

where we note $\hat{\Psi}^{(n)} - Id = \hat{\Phi}^{S^{(n)}} \equiv \hat{\Phi}^{(n)}$, if $n \geq 1$. To prove it, we write $\hat{\Psi}^{(1)} \circ \ldots \circ \hat{\Psi}^{(j)} = \hat{\Psi}^{(2)} \circ \ldots \circ \hat{\Psi}^{(j)} + \hat{\Phi}^{(1)} \circ \hat{\Psi}^{(2)} \circ \ldots \circ \hat{\Psi}^{(j)}$, for every $j \geq 1$, and then we note that can bound $|\hat{\Psi}^{(2)} \circ \ldots \circ \hat{\Psi}^{(j+1)} - Id|_{\rho'_{j+2},R'_{j+2}}$ and $|\hat{\Psi}^{(2)} \circ \ldots \circ \hat{\Psi}^{(j)} - Id|_{\rho'_{j+2},R'_{j+2}}$ by $\min\{\rho^{(2)} - \rho'_{j+2} - \rho/4, R^{(2)} - R'_{j+2} - R/4\}$. We prove the first bound, the second is analogous. To prove this, we have:

$$\begin{aligned} &|\hat{\Psi}^{(2)} \circ \ldots \circ \hat{\Psi}^{(j+1)} - Id|_{\rho'_{j+2},R'_{j+2}} \leq |\hat{\Psi}^{(2)} \circ \ldots \circ \hat{\Psi}^{(j+1)} - Id|_{\rho^{(j+2)},R^{(j+2)}} \leq \\ &\leq \sum_{l=2}^{j}|\hat{\Psi}^{(l)} \circ \ldots \circ \hat{\Psi}^{(j+1)} - \hat{\Psi}^{(l+1)} \circ \ldots \circ \hat{\Psi}^{(j+1)}|_{\rho^{(j+2)},R^{(j+2)}} + \\ &+|\hat{\Psi}^{(j+1)} - Id|_{\rho^{(j+2)},R^{(j+2)}} \leq \sum_{l=2}^{j+1}|\hat{\Phi}^{(l)}|_{\rho^{(l+1)},R^{(l+1)}} = \sum_{l=2}^{j+1}\min\{\delta_l, \tilde{\delta}_l\} = \\ =\ & \min\{\rho^{(2)} - \rho^{(j+2)}, R^{(2)} - R^{(j+2)}\} = \min\{\rho^{(2)} - \rho'_{j+2} - \rho/4, R^{(2)} - R'_{j+2} - R/4\}, \end{aligned}$$

where we have used lemma 3.3 to bound the norms of the compositions. From that, we can prove (3.36) from lemma 3.4. Now, if we iterate (3.36) using similar ideas at every step, we produce the bound

$$|\breve{\Psi}^{(j+1)} - \breve{\Psi}^{(j)}|_{\rho'_{j+2},R'_{j+2}} \leq \prod_{l=1}^{j}\left(1 + 4\hat{\Delta}_{\rho,R}|\hat{\Phi}^{(l)}|_{\rho^{(l+1)},R^{(l+1)}}\right)|\hat{\Phi}^{(j+1)}|_{\rho^{(j+2)},R^{(j+2)}}.$$

So, from lemma 3.12,

$$|\hat{\Phi}^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leq |\nabla S^{(n)}|_{\rho^{(n)}-8\delta_n,R^{(n)}-8\hat{\delta}_n} \leq N_{n+1}\vartheta_1^{2^n},$$

and if we assume

$$4\hat{\Delta}_{\rho,R}\sum_{l\geq 1} N_{l+1}\vartheta_1^{2^l} \leq 4\hat{\Delta}_{\rho,R}\hat{N}^*\vartheta_1^2 \leq \ln 2,$$

we have that:

$$\prod_{l=1}^{j}\left(1+4\hat{\Delta}_{\rho,R}N_{l+1}\vartheta_1^{2^l}\right) \leq 2, \text{ for every } j\geq 1.$$

Hence, using the convergent character of the sum (3.35), one obtains

$$|\breve{\Psi}^{(p)}-\breve{\Psi}^{(q)}|_{\rho/4,R/4} \leq \sum_{j\geq q} 2N_{j+2}\vartheta_1^{2^{j+1}} \to 0, \text{ as } p,q\to+\infty.$$

This fact allows to define

$$\hat{\Psi}^{(*)} = \lim_{n\to+\infty}\breve{\Psi}^{(n)},$$

that maps $\mathcal{U}_{\rho/4,R/4}^{r+s,2m+r}$ into $\mathcal{U}_{\rho,R}^{r+s,2m+r}$.

So, as we remark in section 3.5.1 for this kind of canonical transformations, we only need to show that the final Hamiltonian is well defined to obtain the convergence of the final canonical change $\Psi^{(*)}$, defined as the composition of all the $\Psi^{S^{(n)}}$. This follows immediately from the different bounds for the terms of $H^{(n)}$.

Hence, the limit Hamiltonian $H^{(*)}$ takes the form:

$$H^{(*)}(\theta,x,I,y,\varphi) = \phi^{(*)}(\varphi)+\omega I+\frac{1}{2}z^\top\mathcal{B}^{(*)}(\varphi)z+\frac{1}{2}z^\top\mathcal{C}^{(*)}(\theta,\varphi)z+H_*^{(*)}(\theta,x,\hat{I},y,\varphi),$$

with $< H_*^{(*)} >= 0$, that is, for every $\varphi\in\mathcal{E}^*$, we have a $(r+s)$-dimensional reducible torus.

Control of the measure

To prove the assumptions (a) and (b) of Theorem 3.1, we only need to control the measure of the set of parameters for which we can prove convergence of the scheme or, in an equivalent form, which is the measure of the different sets that we remove at each step of the iterative method: the key idea is to study the characteritzation of these sets given by the Diophantine conditions of (3.34). Hence, we only need to look at the eigenvalues of $\mathcal{B}^{(n)}$. From the bounds of the inductive scheme, we have that $\|\lambda_j^{(n)}-\lambda_j^{(1)}\|_{\bar{\mathcal{E}}^{(n)}} = O_2(\bar{\vartheta})$ and $\mathcal{L}_{\bar{\mathcal{E}}^{(n)}}\left\{\lambda_j^{(n)}-\lambda_j^{(1)}\right\} = O(\bar{\vartheta})$ for every $j=1,\dots,2m$ and $n\geq 2$, provided that $0<\bar{\vartheta}\leq\vartheta_1$, where the constants that give the different $O_2(\bar{\vartheta})$ and $O(\bar{\vartheta})$ are independent on n and j. Then, from expression (3.33) we can write:

$$\lambda_j^{(n)}(\varphi) = \lambda_j + iu_j\varepsilon + iv_j^\top(\hat{\omega}-\hat{\omega}^{(0)}) + \tilde{\lambda}_j^{(n)}(\varphi),$$

with $\mathcal{L}_{\bar{\mathcal{E}}^{(n)}}\{\tilde{\lambda}_j^{(n)}\} \leq L\bar{\vartheta}$ and $|\lambda_j^{(n)}(\varphi)-\lambda_j| \leq M|\varphi-\varphi^{(0)}|$, for certain L and M positive. Then, if we use the nondegeneracy conditions of **NDC2** plus the Diophantine assumptions for the frequencies and eigenvalues of the initial torus, the results of (a) and (b) are consequence of lemmas of section 3.5.2. Here we skip any kind of "hyperbolicity" and we assume that we are always in the worst case, that is, we assume all the normal directions to be of elliptic type.

(a) From the bound for the measure of the set $\mathcal{A}$ in lemma 3.14, we cleary have that if we put in this lemma $\lambda \equiv \lambda_j^{(n)}$ or $\lambda \equiv \lambda_j^{(n)} - \lambda_l^{(n)}$, $j \neq l$, in the set $\mathcal{E} \equiv \mathcal{E}^{(n)}$ with $\vartheta_0 \equiv \vartheta_1$, $\mu \equiv \mu_n$ and $\delta \equiv \delta_n$, we can bound

$$\mathrm{mes}(\bar{\mathcal{I}}^{(n)} \setminus \bar{\mathcal{I}}^{(n+1)}) = O\left(K^{r+s-1-\gamma}\mu_n \frac{\exp(-\delta_n K)}{\delta_n}\right),$$

with $\bar{\mathcal{I}}^{(n)} = \{\varepsilon \in [0,\bar{\varepsilon}] \, : \, (\hat{\omega}^{(0)\top}, \varepsilon)^\top = \varphi \in \mathcal{E}^{(n)}\}$ ($\bar{\varepsilon} > 0$ small enough) and we can take K such that $\frac{\mu_0}{2K^\gamma} = M\bar{\varepsilon}$, that is, $K \equiv K(\bar{\varepsilon}) = \left(\frac{\mu_0}{2M\bar{\varepsilon}}\right)^{\frac{1}{\gamma}}$. Then, if we put $\bar{\mathcal{I}}_* = \cap_{n\geq 1}\bar{\mathcal{I}}^{(n)}$ we have from the bounds of lemma 3.16 that for every $0 < \sigma < 1$, if $K(\bar{\varepsilon})$ is big enough (that is implied by taking $\bar{\varepsilon}$ small enough) depending on σ:

$$\mathrm{mes}([0,\bar{\varepsilon}] \setminus \bar{\mathcal{I}}_*) \leq \exp(-(1/\bar{\varepsilon})^{\frac{\sigma}{\gamma}}).$$

(b) Now we can use the result (i) of lemma 3.14 plus lemma 3.15, working on sets of the form $\mathcal{W}^{(n)}(\varepsilon_0, R_0) = \{\hat{\omega} \in \mathbb{R}^r \, : \, (\hat{\omega}^\top, \varepsilon_0)^\top = \varphi \in \mathcal{E}^{(n)}\}$. We have

$$\mathrm{mes}(\mathcal{W}^{(n)} \setminus \mathcal{W}^{(n+1)}) = O\left(R_0^{r-1}K^{r+s-1-\gamma}\mu_n \frac{\exp(-\delta_n K)}{\delta_n}\right),$$

where for K we can take (depending on ε_0 and R_0), $K = \min\{K_1, K_2\}$, with $\frac{\mu_0}{2K_1^\gamma} = M\max\{R_0, \varepsilon_0\} + K_1 R_0$, condition that comes from lemma 3.14, and $K_2 = \left(\frac{\mu_0}{2R_0}\right)^{\frac{1}{\gamma+1}}$, that comes from lemma 3.15. Then, if we take a fixed $0 \leq \varepsilon_0 \leq R_0^{\frac{\gamma}{\gamma+1}}$ (we recall $R_0 \leq 1$) we can obtain a lower bound for K of $O\left(R_0^{-\frac{1}{\gamma+1}}\right)$, where the constant that give this order depends only on μ_0, γ and M. So, if we use lemma 3.16 we have the desired bound for the measure of the set $\mathcal{W}_*(\varepsilon_0, R_0) = \cap_{n\geq 1}\mathcal{W}^{(n)}(\varepsilon_0, R_0)$:

$$\mathrm{mes}(\mathcal{V}(R_0) \setminus \mathcal{W}_*(\varepsilon, R_0)) \leq \exp(-(1/R_0)^{\frac{\sigma}{\gamma+1}}),$$

for every $0 < \sigma < 1$, if R_0 is small enough depending on σ.

Appendix A

Effective Reducibility of Quasiperiodic Linear Equations close to Constant Coefficients

A.1 Introduction

The well-known Floquet theorem states that any linear periodic system, $\dot{x} = A(t)x$, can be reduced to constant coefficients, $\dot{y} = By$, by means of a periodic change of variables. Moreover, this change of variables can be taken, over $\mathbb{C}$, with the same period than $A(t)$.

A natural extension is to consider the case in which the matrix $A(t)$ depends on time in a quasiperiodic way. Before starting the discussion of this issue, let us recall the definition and basic properties of quasiperiodic functions.

Definition A.1 *A function f is a quasiperiodic function with vector of basic frequencies $\omega = (\omega_1, \dots, \omega_r)$ if $f(t) = F(\theta_1, \dots, \theta_r)$, where F is 2π periodic in all its arguments and $\theta_j = \omega_j t$ for $j = 1, \dots, r$. Moreover, f is called analytic on a strip of width ρ if F is analytical on an open set containing $|\mathrm{Im}\ \theta_j| \le \rho$ for $j = 1, \dots, r$.*

It is also known that an analytic quasiperiodic function $f(t)$ on a strip of width ρ has Fourier coefficients defined by

$$f_k = \frac{1}{(2\pi)^r} \int_{\mathbb{T}^r} F(\theta_1, \dots, \theta_r) e^{-(k,\theta)\sqrt{-1}}\, d\theta,$$

such that f can be expanded as

$$f(t) = \sum_{k \in \mathbb{Z}^r} f_k e^{(k,\omega)\sqrt{-1}t},$$

for all t such that $|\mathrm{Im}\ t| \le \rho / \|\omega\|_\infty$. We denote by $\|f\|_\rho$ the norm

$$\|f\|_\rho = \sum_{k \in \mathbb{Z}^r} |f_k| e^{|k|\rho},$$

and it is not difficult to check that it is well defined for any analytical quasiperiodic function defined on a strip of width ρ. Finally, to define an analytic quasiperiodic matrix, we note that all these definitions hold when f is a matrix-valued function. In this case, to define $\|f\|_\rho$ we use the infinity norm (that will be denoted by $|\cdot|_\infty$) for the matrices f_k.

After those definitions and properties, let us return to the problem of the reducibility of a linear quasiperiodic equation, $\dot{x} = \widehat{A}(t)x$, to constant coefficients. The approach of this work is to assume that the system is close to constant coefficients, that is, $\widehat{A}(t) = A + \varepsilon Q(t,\varepsilon)$, where ε is small. This case has already been considered in many papers (see [6], [33] and [35] among others), and the results can be summarized as follows: let λ_i be the eigenvalues of A, and $\alpha_{ij} = \lambda_i - \lambda_j$, for $i \neq j$. Then, if all the values Re α_{ij} are different from zero, the reduction can be performed for $|\varepsilon| < \varepsilon_0$, ε_0 sufficiently small (see [6]). If some of the Re α_{ij} are zero (this happens, for instance, if A is elliptic, that is, if all the λ_i are on the imaginary axis) more hypothesis are needed. The usual one is a diophantine condition involving the α_{ij} and the basic frequencies of $Q(t,\varepsilon)$, and to assume a nondegeneracy condition with respect to ε on the corresponding $\alpha_{ij}(\varepsilon)$ of the matrix $A + \varepsilon\overline{Q}(\varepsilon)$ ($\overline{Q}(\varepsilon)$ denotes the average of $Q(t,\varepsilon)$). This allows to prove (see [35] for the details) that there exists a Cantorian set $\mathcal{E}$ such that the reduction can be performed for all $\varepsilon \in \mathcal{E}$. Moreover, the relative measure of the set $[0,\, \varepsilon_0] \setminus \mathcal{E}$ in $[0,\, \varepsilon_0]$ is exponentially small in ε_0.

Our purpose here is a little bit different: instead of looking for a total reduction to constant coefficients (this seems to lead us to eliminate a dense set of values of ε, see [33] or [35]), we try to minimize the quasiperiodic part, without taking out any value of ε. The result obtained is that the quasiperiodic part can be made exponentially small. As all the proof is constructive (and it can be carried out with a finite number of steps), it can be applied to practical examples in order to do an "effective" reduction: if ε is small enough, the remainder will be so small that, for practical purposes, it can be taken equal to zero. The error produced with this dropping can be bounded easily, by means of the Gronwall lemma. Finally, we want to stress that we have also eliminated the nondegeneracy hypothesis of previous papers ([33], [35]).

Before finishing this introduction, we want to mention some similar results obtained when the dynamics of the system is slow: $\dot{x} = \varepsilon(A + \varepsilon Q(t,\varepsilon))x$. This case is contained in [70], which is an extension of [54]. The result obtained is also that the quasiperiodic part can be made exponentially small in ε. Total reducibility has been also considered in this case: in [75] is stated that the reduction can be performed except for a set of values of ε of measure exponentially small.

There are many other results for the reducibility problem. For instance, in the case of the Schrödinger equation with quasiperiodic potential we can mention [11], [16], [18], [52], [53] and [66]. Another classical and remarkable paper is [29], where the general case (that is, without asking to be close to constant coefficients) is considered. Finally, the classical results for quasiperiodic systems can be found in [20].

In order to simplify the reading, this chapter has been divided in sections as follows: Section A.2 contains the exposition (without technical details) of the main ideas and methodology, Section A.3 contains the main theorem, Sections A.4 and A.5 are devoted to the proofs and, finally, Section A.6 contains an example to show how these results can be applied to a concrete problem.

A.2 The method

The method used is based on the same inductive scheme that [33]. Let us write our equation as

$$\dot{x} = (A + \varepsilon Q(t, \varepsilon))x, \tag{A.1}$$

where A is an elliptic $d \times d$ matrix and $Q(t, \varepsilon)$ is quasiperiodic with $\omega = (\omega_1, \dots, \omega_r)$ as vector of basic frequencies, and analytic on a strip of width ρ. First of all, let us rewrite this equation as

$$\dot{x} = (A_0(\varepsilon) + \varepsilon \tilde{Q}(t, \varepsilon))x,$$

where $A_0(\varepsilon) = A + \overline{Q}(\varepsilon)$ and $\tilde{Q}(t, \varepsilon) = Q(t, \varepsilon) - \overline{Q}(\varepsilon)$. Now let us assume that we are able to find a quasiperiodic $d \times d$ matrix P (with the same basic frequencies than Q) verifying

$$\dot{P} = A_0(\varepsilon)P - PA_0(\varepsilon) + \tilde{Q}(t, \varepsilon), \tag{A.2}$$

such that $\|\varepsilon P(t, \varepsilon)\|_\sigma < 1$, for some $\sigma > 0$. In this case, it is not difficult to check that the change of variables $x = (I + \varepsilon P(t, \varepsilon))y$ transforms equation (A.1) into

$$\dot{y} = (A_0(\varepsilon) + \varepsilon^2 (I + \varepsilon P(t, \varepsilon))^{-1} \tilde{Q}(t, \varepsilon) P(t, \varepsilon))y. \tag{A.3}$$

As this equation is like (A.1) but with ε^2 instead of ε, the inductive scheme seems clear: to average the quasiperiodic part of (A.3) and to restart this process. The main difficulty that appear in this process comes from equation (A.2), because the solution contains the denominators $\lambda_i(\varepsilon) - \lambda_j(\varepsilon) + \sqrt{-1}(k, \omega)$, $1 \le i, j \le d$, where $\lambda_i(\varepsilon)$ are the eigenvalues of $A_0(\varepsilon)$ (this is shown inside the proof of Lemma A.2). This divisor appears in the kth Fourier coefficient of P. Note that if the values $\lambda_i(\varepsilon) - \lambda_j(\varepsilon)$ are outside the imaginary axis, the (modulus of the) divisor can be bounded from below, being easy to prove the convergence. On the other hand, the value $\lambda_i(\varepsilon) - \lambda_j(\varepsilon) + \sqrt{-1}(k, \omega)$ can be arbitrarily small giving rise to convergence problems.

A.2.1 Avoiding the small divisors

Let us start assuming that the eigenvalues λ_i of the original unperturbed matrix A (see equation (A.1)) and the basic frequencies of Q satisfy the diophantine condition

$$|\lambda_i - \lambda_j + \sqrt{-1}(k, \omega)| \ge \frac{c}{|k|^\gamma}, \quad \forall k \in \mathbb{Z}^r \setminus \{0\}. \tag{A.4}$$

where $|k| = |k_1| + \dots + |k_r|$. Note that, in principle, we can not guarantee that in equation (A.2) this condition holds, because the eigenvalues of $A_0(\varepsilon)$ have been changed with respect to the ones of A (in an amount of $\mathcal{O}(\varepsilon)$) and some of the divisors can be very small or even zero.

The key point is to realize that, as the eigenvalues of A move in an amount of $\mathcal{O}(\varepsilon)$ at most, the quantities $\lambda_i(\varepsilon) - \lambda_j(\varepsilon)$ are contained in a (complex) ball $B_{i,j}(\varepsilon)$ centered in $\lambda_i - \lambda_j$ and with radius $\mathcal{O}(\varepsilon)$. As the centre of the ball satisfies condition (A.4), the values (k, ω) can not be inside that ball if $|k|$ is less than some value $M(\varepsilon)$. This implies that it is possible to cancel all the harmonics such that $0 < |k| < M(\varepsilon)$, because they do not produce small divisors (note that we can only have resonances when (k,ω) is inside $B_{i,j}(\varepsilon)$). The harmonics with $|k| \geq M(\varepsilon)$ are exponentially small in $M(\varepsilon)$ (when $M(\varepsilon) \to \infty$), this is, exponentially small in ε (when $\varepsilon \to 0$), so we do not need to eliminate them.

The idea of considering only frequencies less than some threshold M has already been applied before in other contexts (see, for instance, [1]).

A.2.2 The iterative scheme

To apply the considerations above we define, as before, $A_0(\varepsilon) = A + \varepsilon\overline{Q}(\varepsilon)$, $\tilde{Q}(t,\varepsilon) = Q(t,\varepsilon) - \overline{Q}(\varepsilon)$ and we split $\tilde{Q}(t,\varepsilon)$ in the sum of two matrices $Q_0(t,\varepsilon)$, $R_0(t,\varepsilon)$: $Q_0(t,\varepsilon)$ contains the harmonics $Q_k e^{(k,\omega)\sqrt{-1}t}$ with $|k| < M(\varepsilon)$ and $R_0(t,\varepsilon)$ the ones with $|k| \geq M(\varepsilon)$. So, (A.1) can be rewritted as

$$\dot{x} = (A_0(\varepsilon) + \varepsilon Q_0(t,\varepsilon) + \varepsilon R_0(t,\varepsilon))x, \tag{A.5}$$

Now the idea is to cancel $Q_0(t,\varepsilon)$ and to leave $R_0(t,\varepsilon)$ (it is already exponentially small with ε). So, we compute P_0 such that

$$\dot{P}_0 = A_0(\varepsilon)P_0 - P_0A_0(\varepsilon) + Q_0(t,\varepsilon).$$

Then, the change $x = (I + \varepsilon P_0(t,\varepsilon))y$ gives

$$\dot{y} = \left[A_0 + \varepsilon^2(I+\varepsilon P_0)^{-1}Q_0P_0 + \varepsilon(I+\varepsilon P_0)^{-1}R_0(I+\varepsilon P_0)\right] y.$$

This equation can be rewritten to be like (A.5) to repeat the process. Note that the size of the harmonics with $0 < |k| < M(\varepsilon)$ has been squared. As we will see in the proofs, this is enough to guarantee convergence of those terms to zero. Thus, the final equation has a purely quasiperiodic part exponentially small with ε.

A.2.3 Remarks

It is interesting to note that it is enough to apply a finite number of steps of the inductive process: we do not need to cancel completely the harmonics with $0 < |k| < M(\varepsilon)$ but we can stop the process when they are of the same size of the ones of R (from the proof it can be seen that the number of steps needed to achieve this is of order $|\ln|\varepsilon||$). This allows to apply (with the help of a computer) this procedure on a practical example.

Another remarkable point is about the diophantine condition: note that we only need the condition up to a finite order ($M(\varepsilon)$, that is of order $(1/|\varepsilon|)^{1/\gamma}$, as we shall see in the proofs). This means that, in a practical example when the perturbing frequencies are known with finite precision, the diophantine condition can be checked easily.

A.3 The Theorem

In what follows, $\mathcal{Q}_d(\rho,\omega)$ states for the set of the analytic quasiperiodic $d \times d$ matrices on a strip of width ρ and having ω as vector of basic frequencies. Moreover, i will denote $\sqrt{-1}$.

Theorem A.1 *Consider the equation* $\dot{x} = (A + \varepsilon Q(t,\varepsilon))x$, $|\varepsilon| \le \varepsilon_0$, *and* $x \in \mathbb{R}^d$, *where*

1. *A is a constant $d \times d$ matrix with different eigenvalues $\lambda_1, \dots, \lambda_d$.*
2. *$Q(\cdot,\varepsilon) \in \mathcal{Q}_d(\rho,\omega)$ with $\|Q(\cdot,\varepsilon)\|_\rho \le q$, $\forall\, |\varepsilon| \le \varepsilon_0$, for some $\omega \in \mathbb{R}^r$, and $q, \rho > 0$.*
3. *The vector ω satisfies the diophantine conditions*

$$|\lambda_j - \lambda_\ell + i(k,\omega)| \ge \frac{c}{|k|^\gamma}, \quad \forall\, k \in \mathbb{Z}^r \setminus \{0\}, \quad \forall\, j, \ell \in \{1,\dots,d\}, \tag{A.6}$$

for some constants $c > 0$, $\gamma > r-1$. As usual, $|k| = |k_1| + \cdots + |k_r|$.

Then there exist positive constants ε^*, a^*, r^* and m such that for all ε, $|\varepsilon| \le \varepsilon^*$, the initial equation can be transformed into

$$\dot{y} = (A^*(\varepsilon) + \varepsilon R^*(t,\varepsilon))y\,, \tag{A.7}$$

where:

1. *A^* is a constant matrix with $|A^*(\varepsilon) - A|_\infty \le a^*|\varepsilon|$.*
2. *$R^*(\cdot,\varepsilon) \in \mathcal{Q}_d(\rho,\omega)$ and $\|R^*(\cdot,\varepsilon)\|_{\rho-\delta} \le r^* \exp\left(-\left(\frac{m}{|\varepsilon|}\right)^{1/\gamma}\delta\right)$, $\forall\, \delta \in]0,\rho]$.*

Furthermore the quasiperiodic change of variables that performs this transformation is also an element of $\mathcal{Q}_d(\rho,\omega)$. Finally, a general explicit computation of ε^*, a^*, r^* and m is possible:

$$\varepsilon^* = \min\left(\varepsilon_0, \frac{\alpha}{eq\beta(3d-1)}\right), \quad a^* = \frac{eq\beta^2}{e-1}, \quad r^* = ea^*, \quad m = \frac{c}{10eq\beta}$$

where $e = \exp(1)$, $\alpha = \min_{j\ne\ell}(|\lambda_j - \lambda_\ell|)$ and β is the condition number of a regular matrix S such that $S^{-1}AS$ is diagonal, that is, $\beta = C(S) = |S^{-1}|_\infty |S|_\infty$.

Remark A.1 *For fixed values of $\lambda_1, \dots, \lambda_d$ and γ hypothesis 3 is not satisfied for any $c > 0$ only for a set of values of ω of zero measure if $\gamma > r-1$.*

Remark A.2 *In case that the eigenvalues of the perturbed matrices move on balls of radius $\mathcal{O}(\varepsilon^p)$ (that is, if the nondegeneracy hypothesis needed in [33] or [35] is not satisfied), it is not difficult to show that the bound of the exponential can be improved: $\|R^*(\cdot,\varepsilon)\|_{\rho-\delta} \le r^* \exp(-(m/|\varepsilon|)^{p/\gamma}\delta)$. The proof is very similar, but using $M(\varepsilon) = (m/|\varepsilon|)^{p/\gamma}$ instead of $(m/|\varepsilon|)^{1/\gamma}$.*

This last remark seems to show that this nondegeneracy hypothesis is not necessary, and it is only used for technical reasons. In fact, the results seem to be better when this hypothesis is not satisfied.

Remark A.3 *If the unperturbed matrix A has multiple eigenvalues (that is, if hypothesis 1 is not satisfied) the theorem is still true, but the exponent of ε in the exponential of the remainder is slightly worse. This happens because the (small) divisors are now raised to a power that increases with the multiplicity of the eigenvalues. The proof is not included, since it does not introduce new ideas and the technical details are rather tedious.*

Remark A.4 *The values of ε^*, a^*, r^* and m given in the theorem are rather pessimistic. In the proof, we have preferred to use simple (but rough) bounds instead of cumbersome but more accurate ones. If one is interested in realistic bounds for a given problem, the best thing to do is to rewrite the proof for that particular case. We have done this in Section A.6 where, with the help of a computer program, we have applied some steps of the method to an example. This allows not only to obtain better bounds, but also to obtain (numerically) the reduced matrix as well as the corresponding change of variables.*

A.4 Lemmas

We will use some lemmas to simplify the proof of the theorem.

A.4.1 Basic lemmas

Lemma A.1 *Let $Q(t) = \sum_{k\in\mathbb{Z}^r} Q_k e^{i(k,\omega)t}$ be an element of $\mathcal{Q}_d(\rho,\omega)$ and $M > 0$. Let us define $\overline{Q} = Q_0$, $\tilde{Q}(t) = Q(t) - Q_0$,*

$$Q_{\geq M}(t) = \sum_{\substack{k\in\mathbb{Z}^r \\ |k|\geq M}} Q_k e^{i(k,\omega)t},$$

and $\tilde{Q}_{<M} = \tilde{Q} - Q_{\geq M}$. Then we have the bounds

1. $|\overline{Q}|_\infty$, $\|\tilde{Q}\|_\rho$, $\|\tilde{Q}_{<M}\|_\rho \leq \|Q\|_\rho$.
2. $\|Q_{\geq M}\|_{\rho-\delta} \leq \|Q\|_\rho e^{-M\delta}$, $\forall\, \delta \in]0,\rho]$.

Proof: It is an immediate check. ■

The next lemma is used to control the variation of the eigenvalues of a perturbed diagonal matrix.

Lemma A.2 *Let D be a $d\times d$ diagonal matrix with different eigenvalues $\lambda_1,\ldots,\lambda_d$ and $\alpha = \min_{j\neq\ell}(|\lambda_j - \lambda_\ell|)$. Then if A verifies $|A-D|_\infty \leq b \leq \frac{\alpha}{3d-1}$, the following conditions hold:*

1. *A has different eigenvalues $\mu_1,\ldots,\mu_d$ and $|\lambda_j - \mu_j| \leq b$ if $j = 1,\ldots,d$.*
2. *There exists a regular matrix S such that $S^{-1}AS = D^* = \mathrm{diag}\,(\mu_1,\ldots,\mu_d)$ satisfying $C(S) \leq 2$.*

Proof: It is contained in [33]. ■

Lemma A.3 *Let $(q_n)_n$, $(a_n)_n$ and $(r_n)_n$ be sequences defined by*

$$q_{n+1} = q_n^2, \quad a_{n+1} = a_n + q_{n+1}, \quad r_{n+1} = \frac{2+q_n}{2-q_n} r_n + q_{n+1} \,.$$

with initial values $q_0 = a_0 = r_0 = e^{-1}$. Then $(q_n)_n$ is decreasing to zero and $(a_n)_n$, $(r_n)_n$ are increasing and convergent to some values a_∞ and r_∞ respectively, with $a_\infty < \frac{1}{e-1}$, $r_\infty < \frac{e}{e-1}$.

Proof: It is immediate that q_n goes to zero quadratically and this implies that a_n is convergent to the value a_∞:

$$a_\infty = \sum_{j=0}^{\infty} q_j < \sum_{j=1}^{\infty} e^{-j} = \frac{1}{e-1}.$$

Then

$$r_n \le p \left(r_0 + \sum_{j=1}^{n} q_j \right) \le p a_\infty,$$

where $p = \prod_{j=0}^{\infty} \frac{2+q_j}{2-q_j}$. This product is convergent, in fact:

$$\ln p = \sum_{j=0}^{\infty} [\ln(1+q_j/2) - \ln(1-q_j/2)] \le \frac{3}{2} a_\infty \le \frac{3}{2(e-1)} < 1,$$

and so $p < e$, where we have used that $\ln(1+x) \le x$ and $-\ln(1-x) \le 2x$, for $x \in (0, 1/2)$.
■

A.4.2 The inductive lemma

The next lemma is used to do a step of the inductive procedure.

Before stating the result, let us introduce some notation. Let D and α be like in Lemma A.2 and let ε^*, q^*, L and $M(\varepsilon)$ be positive constants. We consider the equation at the step n of the iterative process:

$$\dot{x}_n = (A_n(\varepsilon) + \varepsilon Q_n(t,\varepsilon) + \varepsilon R_n(t,\varepsilon)) x_n, \;\; |\varepsilon| \le \varepsilon^*, \tag{A.8}$$

where $Q_n(\cdot,\varepsilon)$, $R_n(\cdot,\varepsilon) \in \mathcal{Q}_d(\rho,\omega)$ and $\overline{Q}_n(\varepsilon) = Q_n(\cdot,\varepsilon)_{\ge M(\varepsilon)} = 0$. We assume that for some a_n, q_n, $r_n \ge 0$ and $|\varepsilon| < \varepsilon^*$ the following bounds hold:

$$|A_n(\varepsilon) - D| \le q^* a_n |\varepsilon|, \quad \|Q_n(\cdot,\varepsilon)\|_\rho \le q^* q_n, \quad \|R_n(\cdot,\varepsilon)\|_{\rho-\delta} \le q^* r_n e^{-M(\varepsilon)\delta},$$

where δ is such that $0 < \delta \le \rho$ (the constant q^* has been introduced to simplify, later, the proof of the theorem). We want to see if it is possible to apply a step of the iterative process to equation (A.8) to obtain

$$\dot{x}_{n+1} = (A_{n+1}(\varepsilon) + \varepsilon Q_{n+1}(t,\varepsilon) + \varepsilon R_{n+1}(t,\varepsilon)) x_{n+1}, \quad |\varepsilon| \le \varepsilon^*, \tag{A.9}$$

such that $Q_{n+1}(\cdot,\varepsilon)$, $R_{n+1}(\cdot,\varepsilon) \in \mathcal{Q}_d(\rho,\omega)$, $\overline{Q}_{n+1}(\varepsilon) = Q_{n+1}(\cdot,\varepsilon)_{\ge M(\varepsilon)} = 0$. We also want to relate the bounds a_{n+1}, q_{n+1} and r_{n+1} of the terms of this equation with the corresponding bounds of equation (A.8).

Lemma A.4 *Let $\lambda_1^{(n)}(\varepsilon), \dots, \lambda_d^{(n)}(\varepsilon)$ be the eigenvalues of $A_n(\varepsilon)$. Under the previous notations, if*

1. $L \geq 8q^*$, $\varepsilon^* \leq \frac{\alpha}{q^*(3d-1)}$,
2. $a_n \leq 1$, $q_n \leq e^{-1}$,
3. *the condition*
$$|\lambda_j^{(n)}(\varepsilon) - \lambda_\ell^{(n)}(\varepsilon) + i(k,\omega)| \geq L|\varepsilon|, \quad |\varepsilon| \leq \varepsilon^*,$$
is satisfied for all j, ℓ and for all $k \in \mathbb{Z}^r$ such that $0 < |k| < M(\varepsilon)$,

then, equation (A.8) can be transformed into (A.9) and:

$$q_{n+1} = q_n^2, \quad a_{n+1} = a_n + q_{n+1}, \quad r_{n+1} = \frac{2+q_n}{2-q_n} r_n + q_{n+1}\,.$$

The quasiperiodic change of variables that performs this transformation is

$$x_n = (I + \varepsilon P_n(t,\varepsilon)) x_{n+1}, \tag{A.10}$$

where $P_n(\cdot,\varepsilon)$ is the (only) solution of

$$\dot{P}_n = A_n(\varepsilon) P_n - P_n A_n(\varepsilon) + Q_n(t,\varepsilon), \quad \overline{P}_n = 0, \tag{A.11}$$

that belongs to $\mathcal{Q}_d(\rho,\omega)$. Moreover, $\|\varepsilon P_n(\cdot,\varepsilon)\|_\rho \leq q_n/2 < 1/2$.

Remark A.5 *A_n, Q_n, R_n, P_n, M and $\lambda_j^{(n)}$ depend on ε but, for simplicity, we will not write this explicitely.*

Proof: Let us start studying the solutions of (A.11). Let S_n be the matrix found in Lemma A.2 with $S_n^{-1} A_n S_n = D_n = \mathrm{diag}\,(\lambda_1^{(n)}, \dots, \lambda_d^{(n)})$, $C(S_n) \leq 2$. This lemma can be applied because

$$|A_n - D|_\infty \leq q^* a_n |\varepsilon| \leq q^* \varepsilon^* \leq \frac{\alpha}{3d-1} \quad \text{for all} \quad |\varepsilon| \leq \varepsilon^*.$$

Making the change of variables $P_n = S_n X_n S_n^{-1}$ and defining $Y_n = S_n^{-1} Q_n S_n$, equation (A.11) becomes

$$\dot{X}_n = D_n X_n - X_n D_n + Y_n, \quad \overline{Y}_n = 0.$$

As D_n is a diagonal matrix we can handle this equation as d^2 unidimensional equations, that can be solved easily by expanding in Fourier series. If $X_n = (x_{\ell j,n})$, $Y_n = (y_{\ell j,n})$, with

$$x_{\ell j,n}(t) = \sum_{\substack{k \in \mathbb{Z}^r \\ 0<|k|<M}} x_{\ell j,n}^k e^{i(k,\omega)t}, \qquad y_{\ell j,n}(t) = \sum_{\substack{k \in \mathbb{Z}^r \\ 0<|k|<M}} y_{\ell j,n}^k e^{i(k,\omega)t},$$

the coefficients must be

$$x_{\ell j,n}^k = \frac{y_{\ell j,n}^k}{\lambda_j^{(n)} - \lambda_\ell^{(n)} + i(k,\omega)},$$

and, by hypothesis 3 they can be bounded by $|x^k_{\ell j,n}| \leq (L|\varepsilon|)^{-1}|y^k_{\ell j,n}|$, and this implies

$$\begin{aligned} \|P_n\|_\rho &\leq C(S_n)\|X_n\|_\rho \leq C(S_n)(L|\varepsilon|)^{-1}\|Y_n\|_\rho \leq C(S_n)^2(L|\varepsilon|)^{-1}\|Q_n\|_\rho \leq \\ &\leq 4(L|\varepsilon|)^{-1}q^*q_n \leq |\varepsilon|^{-1}\frac{q_n}{2}. \end{aligned}$$

Hence, $\|\varepsilon P_n\|_\rho \leq q_n/2 < 1/2$. Thus $I + \varepsilon P_n$ is invertible and

$$\|(I+\varepsilon P_n)^{-1}\|_\rho \leq \frac{1}{1 - \|\varepsilon P_n\|_\rho} < 2.$$

Now, applying the change (A.10) to (A.8) and defining $Q^*_n = \varepsilon(I+\varepsilon P_n)^{-1}Q_nP_n$, $A_{n+1} = A_n + \varepsilon\overline{Q^*_n}$, $Q_{n+1} = (\widetilde{Q^*_n})_{<M}$ and $R_{n+1} = (I+\varepsilon P_n)^{-1}R_n(I+\varepsilon P_n) + (Q^*_n)_{\geq M}$, it is easy to derive equation (A.9). Finally we use Lemma A.1 to bound the terms of this equation:

$$\begin{aligned} \|Q^*_n\|_\rho &\leq \|(I+\varepsilon P_n)^{-1}\|_\rho\|Q_n\|_\rho\|\varepsilon P_n\|_\rho \leq \|Q_n\|_\rho q_n \leq q^*q_n^2 = q^*q_{n+1} \\ \|Q_{n+1}\|_\rho &\leq \|Q^*_n\|_\rho \leq q^*q_{n+1} \\ |A_{n+1} - D|_\infty &\leq |A_n - D|_\infty + |\varepsilon\overline{Q^*_n}|_\infty \leq q^*(a_n + q_{n+1})|\varepsilon| = q^*a_{n+1}|\varepsilon| \\ \|R_{n+1}\|_{\rho-\delta} &\leq \frac{1+\|\varepsilon P_n\|_\rho}{1-\|\varepsilon P_n\|_\rho}\|R_n\|_{\rho-\delta} + \|(Q^*_n)_{\geq M}\|_{\rho-\delta} \leq \\ &\leq \left(\frac{1+q_n/2}{1-q_n/2}r_n + q_{n+1}\right)q^*e^{-M\delta} = q^*r_{n+1}e^{-M\delta}, \quad \forall\, \delta \in]0,\rho]. \end{aligned}$$

■

A.5 Proof of Theorem

Let S be a regular matrix such that $S^{-1}AS = D = \text{diag}\,(\lambda_1, \ldots, \lambda_d)$. We define ε^*, α, β and m as in the statement of Theorem A.1. We also define $q^* = e\beta q$, $M = M(\varepsilon) = \left(\frac{m}{|\varepsilon|}\right)^{1/\gamma}$ and $L = 8q^*$.

The (constant) change $x = Sx_0$ transforms the initial equation into

$$\dot{x}_0 = (D + \varepsilon Q^*(t,\varepsilon))x_0 \tag{A.12}$$

where $Q^* = S^{-1}QS$ and so $\|Q^*\|_\rho \leq e^{-1}q^*$ for $|\varepsilon| \leq \varepsilon^*$. We split equation (A.12) as

$$\dot{x} = (A_0 + \varepsilon Q_0(t) + \varepsilon R_0(t))x_0$$

where $A_0 = D + \varepsilon\overline{Q^*}$, $Q_0 = \widetilde{Q^*}_{<M}$ and $R_0 = Q^*_{\geq M}$. Using Lemma A.1 it is easy to see that

$$|A_0 - D|_\infty \leq q^*a_0|\varepsilon|, \qquad \|Q_0\|_\rho \leq q^*q_0, \qquad \|R_0\|_{\rho-\delta} \leq q^*r_0e^{-M\delta},$$

$\forall\, \delta \in]0,\rho]$, $|\varepsilon| \leq \varepsilon^*$, if $a_0 = q_0 = r_0 = e^{-1}$.

We are going to show that in all the steps the hypothesis of Lemma A.4 are satisfied. As hypothesis 1 and 2 are easy to check, we focus on hypothesis 3.

Now since $a_n \leq 1$ and $|\varepsilon| \leq \varepsilon^*$, $|A_n - D|_\infty \leq q^*|\varepsilon| \leq \frac{\alpha}{3d-1}$, Lemma A.2 gives that

$$|\alpha^{(n)}_{j\ell} - \alpha_{j\ell}| < 2q^*|\varepsilon| \quad \text{for all } j,\ \ell,\ |\varepsilon| \leq \varepsilon^*,$$

where $\alpha_{j\ell} = \lambda_j - \lambda_\ell$, $\alpha_{j\ell}^{(n)} = \lambda_j^{(n)} - \lambda_\ell^{(n)}$ being $\lambda_1^{(n)}, \ldots, \lambda_d^{(n)}$ the eigenvalues of $A_n(\varepsilon)$.

Using hypothesis 3 of the Theorem we obtain that, if $k \in \mathbb{Z}^r$ and $0 < |k| < M(\varepsilon)$,

$$\begin{aligned} |\alpha_{j\ell}^{(n)} + i(k,\omega)| &\geq |\alpha_{j\ell} + i(k,\omega)| - |\alpha_{j\ell}^{(n)} - \alpha_{j\ell}| > \frac{c}{|k|^\gamma} - 2q^*|\varepsilon| > \\ &> \left(\frac{c}{m} - 2q^*\right)|\varepsilon| = L|\varepsilon|, \end{aligned}$$

and hypothesis 3 of Lemma A.4 is verified.

In consequence the iterative process can be carried out and Lemma A.3 ensures the convergence of the process. The composition of all the changes $I + \varepsilon P_n$ is convergent because $\|I + \varepsilon P_n\|_\rho \leq 1 + q_n/2$. Then the final equation is

$$\dot{x}_\infty = (A_\infty(\varepsilon) + \varepsilon R_\infty(t,\varepsilon))x_\infty, \quad |\varepsilon| \leq \varepsilon^*, \tag{A.13}$$

where $|A_\infty(\varepsilon) - D|_\infty \leq q^* a_\infty |\varepsilon| \leq \frac{e\beta}{e-1} q|\varepsilon|$, and

$$\|R_\infty(\cdot,\varepsilon)\|_{\rho-\delta} \leq q^* r_\infty e^{-M(\varepsilon)\delta} \leq \frac{e^2\beta}{e-1} q \exp\left\{-\left(\frac{m}{|\varepsilon|}\right)^{1/\gamma}\delta\right\}, \ \forall\, \delta \in]0,\rho].$$

To end up the proof, the change $x_\infty = S^{-1}y$ transforms equation (A.13) into equation (A.7) with the bounds that we were looking for.

A.6 An example

The results of this chapter can be applied in many ways, according to the kind of problem we are interested in. Let us illustrate this with the help of an example.

Let us consider the equation

$$\ddot{x} + (1 + \varepsilon q(t))x = 0, \tag{A.14}$$

where $q(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$, being $\omega_1 = \sqrt{2}$ and $\omega_2 = \sqrt{3}$. Defining y as $\dot{x}$ we can rewrite (A.14) as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 \\ -q(t) & 0 \end{pmatrix}\right] \begin{pmatrix} x \\ y \end{pmatrix}. \tag{A.15}$$

As $\lambda_{1,2} = \pm i$, the diophantine condition (A.6) is satisfied for $\gamma = 1$ (because the frequencies are quadratic irrationals). The value of c will be discussed later. For the sake of simplicity, let us take $\rho = 2$ and $\delta = 1$. This implies that $q = \|Q\|_\rho = 2e^2$. It is not difficult to derive $\beta = 2$ and, finally, $\varepsilon^* = 4.9787\ldots \times 10^{-3}$ and $r^* = 2.5419\ldots \times 10^2$.

The value of c might be calculated for all $k = (k_1, k_2)$, but better (bigger) values can be used since we only need to consider $|k|$ up to a finite order. For instance, an easy computation shows that for $|k| \leq 125$ c is 0.149. If $|k| = 126$, then c must be 0.013 at most, due to the quasiresonance produced by $k = (70, -56)$. In the range $126 \leq |k| \leq 10^5$ there are no more relevant resonances, so the value $c = 0.013$ suffices.

To start the discussion, let us suppose that the value of ε in (A.15) is $\varepsilon = 2 \times 10^{-6}$. If we take $c = 0.149$ we obtain that $m = 1.8545\ldots \times 10^{-4}$ and $M = 93$ (recall that the process cancels frequencies such that $|k| < M(\varepsilon)$). If the value of M had been bigger than 125, we should have used the value $c = 0.013$ instead. So, we can reduce the system to constant coefficients with a remainder R^* such that $\|R^*\|_{\rho-1} < 10^{-37}$.

If the given value of ε is smaller, for instance $\varepsilon = 10^{-7}$, the computed value of M if $c = 0.149$ is 1855, so $c = 0.013$ must be used. This produces $M = 162$ and $\|R^*\|_{\rho-1} < 10^{-67}$. A value of $\varepsilon = 5 \times 10^{-8}$ implies $M = 324$ and $\|R^*\|_{\rho-1} < 10^{-138}$. The computation of the reduced matrix, as well as the quasiperiodic change of variables will be discussed below.

Another interesting problem is to study the reducibility for a value of ε bigger than the ε^* given above. Let us continue working with the same equation but selecting, as an example, $\varepsilon = 0.1$.

To increase the value of ε^*, one may try to rewrite the proof, using optimal bounds at each step. This has not been done here in order to get an easy, clean and short proof. Instead of doing this, we think that it is much better to rewrite the proof for our example, using no bounds but exact values. This will produce the best results for this problem.

For that purpose, we have implemented the algorithm used in the proof of the theorem as a C program, for a (given) fixed value of ε. The program computes and performs a finite number of the changes of variables used to prove the theorem. As a result, the reduced system (including the remainder) as well as the final change of variables are written.

To simplify and make the program more efficient, all the coefficients have been stored as double precision variables. During all the operations, all the coefficients less than 10^{-20} have been dropped, in order to control the size of the Fourier series appearing during the process. Of course, this introduces some (small) numerical error in the results.[1]

After four changes of variables, (A.15) is transformed into

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left[\begin{pmatrix} 0.0 & b_{12} \\ b_{21} & 0.0 \end{pmatrix} + R(t) \right] \begin{pmatrix} x \\ y \end{pmatrix}, \tag{A.16}$$

where $b_{12} = 1.000000366251255$ and $b_{21} = -0.992421151834871$. The remainder R is very small: the biggest coefficient it contains is less than 10^{-16}. Note that the accuracy (relative error) of this remainder is very poor, due to the use of double precision arithmetic (15–16 digits) for the coefficients. During the computations, M has not been given a value. Instead, we have tried to cancel all the frequencies with amplitude bigger than 10^{-16} (it turns out from the computations that all these frequencies satisfy $|k| \le 20$). It is also possible to obtain a better accuracy in the result, using a multiple-precision arithmetic.

Finally, to check the software, we have tabulated a solution of (A.16) for a time span of 10 time units. We have transformed this table by means of the (quasiperiodic) change of variables given by the program. Then, we have taken the first point of the transformed table as initial condition of (A.15), to produce (by means of numerical integration) a new table. The differences between these two tables are less than 10^{-13}, as expected.

[1] If one wants to control that error, it is possible to use intervalar arithmetic for the coefficients and to carry a bound of the remainder for each Fourier series.

So, for practical purposes, this is an "effective" Floquet Theorem in the sense that it allows to compute the reduced matrix as well as the change of variables, with the usual accuracy used in numerical computations.

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