Programa de Doctorat en Matemàtica Aplicada

Ph.D. Thesis

A MULTISYMPLECTIC APPROACH TO GRAVITATIONAL THEORIES

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Abstract

The theories of gravity are one of the most important topics in theoretical physics and mathematical physics nowadays. The classical formulation of gravity uses the Hilbert-Einstein Lagrangian, which is a singular second-order Lagrangian; hence it requires a geometric theory for second-order field theories which leads to several difficulties. Another standard formulation is the Einstein-Palatini or Metric-Affine, which uses a singular first order Lagrangian.

Much work has been done with the aim of establishing the suitable geometrical structures for describing classical field theories. In particular, the multisymplectic formulation is the most general of all of them and, in recent years, some works have considered a multisymplectic approach to gravity. This formulation allows us to study and better understand several inherent characteristics of the models of gravity.

The aim of this thesis is to use the multisymplectic formulation for first and second-order field theories in order to obtain a complete covariant description of the Lagrangian and Hamiltonian formalisms for the Einstein-Hilbert and the Metric-Affine models, and explain their characteristics; in particular: order reduction, constraints, symmetries and gauge freedom.

Key words: *$1st$ and $2nd$-order Classical field theories, Jet bundles, Multisymplectic forms, Einstein equations, Hilbert-Einstein action, Einstein-Palatini action and Metric-Affine models, Constraints, Gauge symmetries.*

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Agraïments

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Contents

Introduction

Multisymplectic Formalism for Field Theories

In recent decades, a strong development in the intrinsic study of a wide variety of topics in theoretical physics, control theory and applied mathematics has been done, using methods of differential geometry. Thus, the intrinsic formulation of Lagrangian and Hamiltonian formalisms has been developed for autonomous and non-autonomous mechanics, as well as for field theories.

In particular, much work has been done in order to establish the geometrical structures for describing classical field theories, both in the Lagrangian and the Hamiltonian formalisms. This study has been carried out for first and higher-order field theories, and there are different kinds of geometrical models for making a covariant description of them. In particular, there is the multisymplectic formulation [15, 51, 53, 74, 76, 77, 85], which is the most general way to study geometrically these kinds of theories, and was introduced by Tulczyjew and other authors [42, 50, 68, 69]. They arise from the study of multisymplectic manifolds and their properties (see [9, 10, 28] for general references).

In this formulation, the usual way of working consists in stating their Lagrangian formalism and jet bundles are the appropriate domain for doing so [3, 7, 14, 30, 31, 42, 49, 50, 90]. Then, the Hamiltonian formalism is constructed using bundles of forms; but the choice of the suitable multimomentum bundle is not unique [33, 34], and different kinds of Hamiltonian formalisms can be developed, depending on this choice and on the way of introducing the "Hamiltonian" [27, 59, 60, 81]. In this thesis we have taken one of the most standard ways of defining Hamiltonian systems, which consists in using *Hamiltonian sections* [13] (it can also be done taking *Hamiltonian densities* [13, 49, 89]). The relation between the Lagrangian and the Hamiltonian formalisms is carried out by using the *Legendre map* associated with the Lagrangian system, and it has been studied in the (hyper) regular case [13, 90], and in the singular or, specifically, in the *almost-regular* case [23, 49, 89]. There is also a unified formulation of Lagrangian and Hamiltonian formalisms: it is the so-called the *Lagrangian-Hamiltonian unified formalism* or *Skinner-Rusk formalism* due to the authors' names of the original paper [94], It is a generalization of the Lagrangian and Hamiltonian formalisms that compresses them into a single formalism. It was stated first for autonomous mechanics [94], and later it was generalized for classical field theories [24, 29, 83]. One of the major advantages of this formalism is the natural way in which the Legendre transformation and the Hamiltonian formalism arise. In particular, this is very promising for describing the Einstein-Hilbert model of gravity.

As a particular case, for second-order theories, the phase space of the system is described using third-order jet bundles as the main tool [1, 39, 43, 83, 84, 90]. Nevertheless, in higher-order field theories there are some ambiguities in the definition of the Poincaré-Cartan form; that is, given a Lagrangian density, there are non-equivalent Poincaré-Cartan forms from which we obtain the same Euler-Lagrange equations. Therefore, due to its definition, these ambiguities in the Poincaré-Cartan form are transferred to the Legendre map, thus obtaining different Legendre maps for the same field theory[70, 84, 91]. First order field theories have no ambiguities. For second-order field theories, there is an unambiguous procedure to define these structures (see [90] or also [8, 83] using the unified formalism).

We ought to point out that there are also geometric frameworks for describing the non-covariant or space-time formalism of field theories, where the use of *Cauchy surfaces* is the fundamental tool [26, 52, 54]. Nevertheless we do not consider these topics in this disertation.

In the multisymplectic models, in the Lagrangian, Hamiltonian and unified formalisms, the field equations are written using the multisymplectic form in order to characterize the critical sections which are solutions of the problem. These critical sections can be thought of as being the integral manifolds of distributions which, on its turn, can be characterized by means of certain kinds of integrable multivector fields defined in the bundles where the formalism is developed, and satisfying a suitable geometric equation which is the intrinsic formulation of the systems of partial differential equations locally describing the field [23, 30, 31, 32, 90]. These equations can be derived from a variational principle: the *Hamilton principle* in the Lagrangian formalism, the *Hamilton–Jacobi principle* in the Hamiltonian formulation [3, 27, 30, 34, 42, 50] and a generalization of them in the unified formalism [29, 85].

In all these cases, we have what is generally called a *multisymplectic system*; although in this work, the theories arise from singular Lagrangians, leading to a pre-multisymplectic system. The problem of finding a submanifold where this equations have consistent solutions (if it exists) is solved by applying a constraint algorithm adapted to this premultisymplectic scenario (see, for instance, [22, 23] for a geometric description of these kinds of algorithms).

Another important topic is the study of symmetries and conservation laws. In fact, symmetries, conserved quantities and gauge freedom have always played an essential role in the analysis of physical systems because they help us to solve and understand the field equations. In particular, modern approaches to General Relativity and Quantum Field Theory have in their core the idea of symmetry. They are also very powerful when study the integrability of a system. The geometric treatment of these concepts is lightening and several results have been obtained for the Lagrangian and Hamiltonian formalisms of first and higher-order non-autonomous mechanics [4, 25, 75, 82, 92]. Most of these concepts and results have been generalize for classical Lagrangian field theories of first and higher-order [1, 26, 30, 39, 42, 43, 49, 50, 70, 72, 73, 84, 90], and for their Hamiltonian formalisms [2, 26, 27, 32, 60].

This work is based on results and contributions from several previous papers, such as [13, 23, 24, 29, 30, 31, 32, 34, 59, 81], among others.

Multisymplectic Gravity

General Relativity, the Einstein theory of Gravity, has had important contributions since its origins. In 1915 Hilbert [61] found the so-called Hilbert-Einstein Lagrangian, thus obtaining a variational formulation of General Relativity. Although the Einstein equations have order two, the Lagrangian he obtained is a second order one. For a Lagrangian of order n, the expected Euler-Lagrange equations have order $2n$. The reason for the equations to have a lower order than anticipated is what we refer as the projectability of the theory. In 1925, Einstein propose a new variational approach, the Metric-Affine (or Einstein-Palatini), consisting on taking the components of the connection as additional coordinates [36]. This new Lagrangian is first-order and also lead to the Einstein equations. The first complete solution was given by Schwarschild in 1916 [93]. Nevertheless, the studies were essentially theoretical until the 60s, when exotic objects like quasars and pulsar where observed. Later observations like the background radiation, the accelerating expansion of the universe or gravitational waves, emphasises the importance of the study of General Relativity and its relation with other branches of physics. Several variations and generalizations of Einstein's theory are being proposed to explain these and another phenomena. The interplay between General Relativity, astrophysics and quantum theories is a field of intense study in modern physics. (See [99] as a general reference for all these topics).

The geometric analysis of General Relativity contains particular difficulties because it is a second-order field theory with high degeneracy. In recent years, there is an increasing effort in understanding the covariant description of gravitational theories (General Relativity and other derived from it) using different kinds of geometric frameworks such as the *multisymplectic* or *polysymplectic manifolds*.

An intrinsic approach to General Relativity can be found, for instance, in [72, 73], where the authors study the degeneracy of the Einstein-Hilbert Lagrangian and other aspects of the theory using *Lepage forms*. The same topic is analysed in [16, 86, 87] where, in addition, the reduction of the order of the theory and the projectability of the Poincaré-Cartan form associated with the Einstein-Hilbert action is explained (intrinsically it is defined as a theory where the Poincarè-Cartan form can be pulled-back to a lower order jet bundle). This arise from the fact that the theory is degenerate in a very specific way. In particular, they also show the existence of a first order Lagrangian (different from the metric-affine Lagrangian) which give rise to the Einstein equations, and they also study the existence of natural symmetries. In [53] the multisymplectic description of the *Einstein-Palatini* or *metric-affine model* model is presented as a example and in [12] an exhaustive study of this model is done using a unified formalism. Different geometric formulations of General Relativity and its variational principles are given in [11]. In addition, in [96, 97] a complete study of the vielbein formulation of General Relativity is done using multisymplectic geometry for describing the vielbein (or Cartan) formalism in the Palatini approximation (the Lagrangian and Hamiltonian formalism) and considering different classes of matter sources. Finally, some general features of the gravitational theory following the polysymplectic version of the multisymplectic formalism are described in [49, 89], including the problem of its precanonical quantization [63, 64, 65, 66, 67]. More general aspects of the theory are studied in [6, 11, 17, 18, 19, 37, 47, 95, 88].

General Relativity is a covariant theory; that is, it is invariant under diffeomorphism acting on the base manifold. This property has been partially studied in a geometrical way in [87]. Moreover, the Metric-Affine model contains a gauge symmetry [20, 45]. The conserved quantities for gravitational theories has been studied, for instance, in [95]. Nevertheless, there is open question regarding the symmetries and conserved quantities of General Relativity and its consequences. Thus, in this dissertation we recover some results and expand them, especially the topic of the gauge freedom.

Objectives

In this dissertation we develop essentially two models of Gravity, the *Einstein-Hilbert* and the *Einstein-Palatini* or *Metric-Affine*. We develop a covariant description of both theories using the *multisymplectic framework*, but we emphasize different aspects in each one.

Order Reduction:

There are some models in classical field theories where, as a consequence of the singularity of the Lagrangian, the order of the Euler-Lagrange equations is lower than expected. A geometrical way of understanding this problem is considering the projectability of the higher-order Poincaré-Cartan form onto lower-order jet bundles [16, 44, 73, 86, 87].

We review the conditions for this projectability and study their consequences using the constraint algorithm for the field equations of second order (singular) field theories. Moreover, this analysis is done for the case of higher-order mechanics.

Einstein-Hilbert:

The Einstein-Hilbert model of General Relativity is described by a second-order singular Lagrangian, thus it is a gauge field theory with constraints. Apart of developing the multisymplectic formalism, in this model we are mainly interested in the consequences of the projectability of the Poincaré-Cartan form, the existence of a first-order equivalent Lagrangian and the analysis of the gauge freedom and symmetries.

The use of the unified Lagrangian-Hamiltonian formalism is particularly interesting when it is applied to these kinds of theories, since it simplifies the treatment of them; in particular, the implementation of the constraint algorithm, the retrieval of the Lagrangian description, and the construction of the covariant Hamiltonian formalism. We apply this framework to the Einstein-Hilbert model without and with energy-matter sources.

The framework shows how, as a consequence of the gauge freedom and the constraint algorithm, the Einstein-Hilbert model is equivalent to a first-order regular theory, without gauge freedom. Moreover, we obtain and explain the geometrical and physical meaning of this equivalence, the gauge freedom and the Lagrangian constraints.

In the case of presence of energy-matter sources, we show how some relevant geometrical and physical characteristics of the theory depend on the type of source. In all the cases, we obtain explicitly multivector fields which are solutions to the gravitational field equations. Finally, a brief study of symmetries and conservation laws is done in this context.

Metric-Affine:

The *Metric-Affine* or *Einstein-Palatini* model for General Relativity is described by a first-order affine Lagrangian (in the derivatives of the fields), it is singular and, hence, this is a gauge field theory with constraints. The Einstein-Palatini model has more freedom than the Einstein-Hilbert one, as it was already noticed in the original article by Einstein [36]. In [20], the authors identify this freedom as gauge-like. We aim to use the multisymplectic formalism to characterize this gauge freedom, analyze the constraints of the theory and establish a covariant Hamiltonian formalism.

We develop the Lagrangian and Hamiltonian multisymplectic formulation for the *Einstein-Palatini* model (without energy-matter sources) in chapter 4. A detailed analysis of the unified formalism can be found in [12]. The constraints are obtained after applying a constraint algorithm to the field equations, both in the Lagrangian and the Hamiltonian formalisms. We obtain and explain the geometrical and physical meaning of the Lagrangian constraints and we construct the multimomentum (covariant) Hamiltonian formalism. The gauge freedom of the model is discussed in both formalisms and, from them, the equivalence with the Einstein-Hilbert model is established.

Structure of the dissertation

The dissertation is structured in 5 chapters. In chapter 1 the multisymplectic formalism and the needed geometrical tools are presented. We use a variational approach in the line of [45], and the constraint algorithm used in later chapters is briefly exposed.

Chapter 2 has two parts. Section 2.1 contains a review of symmetries and conserved quantities in field theories. We also present there the concept of gauge vector field and gauge symmetry. This section is based in [45]. In Section 2.2 we explore the consequences of a projectable Poincaré-Cartan form. Following [46], we expand previous results in [16, 44, 73, 86, 87] by analyzing the constraints of such theories. We also prove the results for Higher-order mechanical theories.

In chapter 3 the multisymplectic formalism is applied to the Einstein-Hilbert model, as it is exposed in [47]. First the Lagrangian-Hamiltonian unified formalism for the vacuum case is studied, and using the constraint algorithm we find the final manifold where the field equations have solutions. Then, the Lagrangian and Hamiltonian formalisms are recovered. Two different sets of coordinates are used in the Hamiltonian formalism, and its equivalence with the first-order Lagrangian [16] is shown. Then, we add a matter-source and analyze which results can be recovered, depending on the source. As an example, the Electromagnetic source is considered. The natural symmetries of the theory [87] are revisited in the light of section 2.1. Finally, we compute explicitly all the semi-holonomic multivector fields solution of the field equations.

In chapter 4 the multisymplectic formalism is applied to the Metric-Affine model, as it is exposed in [48]. First the Lagrangian formalism is considered. We find the constraints defining the final submanifold and the natural symmetries. The gauge symmetries are computed and the results are in agreement with [20]. Consequently, we find the general expression for the semi-holonomic multivector fields solution of the field equations. Next, we construct the covariant Hamiltonian formalism and repeat the process, using two different systems of coordinates. For one of them, containing only the connection, a geometric interpretation is provided. Finally, the relation between the Einstein-Hilbert model and the Metric-Affine model is established geometrically.

Finally, the conclusions are presented in chapter 5, together with the list of articles consequence of this work and a list of topics for further research.

Chapter 1

Geometrical setting

This chapter is devoted to present the main structures and mathematical tools needed in the dissertation. We also set here the common notation used along different chapters. Since this is a review chapter, only a handful of proofs are provided. The Einstein-Hilbert model developed in chapter 3 is a second-order field theory, but this dissertation also contains first-order field theories like the Metric-Affine model of chapter 4. Nevertheless, we only present here the second-order formalism. The multisymplectic formalism for first-order first theories can be found, for instance, in [13, 85].

This chapter is structured in three sections. First, in section 1.1 we present the jet bundles and several related concepts. Then, in 1.2 we present the multisymplectic formalism in a general way, based on [45, 58]. Finally, section 1.3 is devoted to particularize this formalism to the Lagrangian, the Hamiltonian and the unified formalism for second-order field theories.

1.1 Geometry of jet bundles

1.1.1 Jet bundles and holonomy

Consider a smooth fiber manifold $\pi : E \to M$, where the base manifold M has dimension m and the total space E has dimension $m + n$. A section of π is an aplication $\phi : M \to E$ such that $\pi \circ \phi = Id_M$. The set of sections of π is denoted $\Gamma(\pi)$. All the manifolds are real, second countable and C^∞. The maps and the structures are C^∞.

In order to write compressed coordinate expressions we will use the following notation. A multi-index I is an element of $\mathbb{Z}^m$ where every component is positive, the ith position of the multi-index is denoted $I(i)$, and $|I| = \displaystyle\sum_{i=1}^{m} I(i)$ is the length of the multi-index. Furthermore, the element $1_i \in \mathbb{Z}^m$ is defined as $1_i(j) = \delta_i^j$ and $n(ij)$ is a combinatorial factor which $n(ij) = 1$ for $i = j$, and $n(ij) = 2$ for $i \neq j$. An expression as $|I| = k$ means that the expression is taken for every multi-index of fixed length k. Equivalently, we will write $\alpha_1 \leq \cdots \leq \alpha_k$. Sum over repeated indices is understood.

Let (x^μ), with $1 \leq \mu \leq m$, be a system of coordinates in M, and (x^μ, u_α), with $1 \leq \alpha \leq n$, a system of coordinates of E adapted to the bundle structure. For a section $\phi \in \Gamma(\pi)$ we denote $\phi_\alpha = u_\alpha \circ \phi$, so that $\phi(x^\mu) = (x^\mu, \phi_\alpha(x^\mu))$. For every point $x \in M$, $\Gamma_x(\pi)$ denotes the set of sections of π defined on a neighbourhood of x. For every integer $k \geq 1$:

Definition 1.1. *Two local sections $\phi, \psi \in \Gamma_x(\pi)$ are* k-equivalent *in x if:*

- $\psi(x) = \phi(x)$
- *All derivatives up to order k coincide in x:*

$$\left.\frac{\partial^{|I|}\psi_\alpha}{\partial x^I}\right|_x = \left.\frac{\partial^{|I|}\phi_\alpha}{\partial x^I}\right|_x ;$$

for every $1 \leq |I| \leq k$ and $1 \leq \alpha \leq n$.

Lemma 1.1. *At every point $x \in M$, the k-equivalence relation in $\Gamma_x(\pi)$ is independent of the choice of coordinate system.*

As a consequence, the k-equivalence relation in x is a well-defined equivalence relation in $\Gamma_x(\pi)$. The equivalence class containing ϕ is called the *k-jet of ϕ at x*, and it is denoted $j^k_x\phi$.

Definition 1.2. *The k-jet manifold is the set:*

$$J^k\pi = \{j^k_x\phi \,|\, x \in M \,,\, \phi \in \Gamma_x(\pi)\}.$$

The k-jet has a natural structure of smooth manifold and we have the following natural projections: if $r \leqslant k$,

$$\begin{array}{rccc} \pi^k_r\colon J^k\pi & \longrightarrow & J^r\pi \\ j^k_x\phi & \longmapsto & j^r_x\phi \end{array} \qquad \begin{array}{rccc} \pi^k\colon J^k\pi & \longrightarrow & E \\ j^k_x\phi & \longmapsto & \phi(x) \end{array} \qquad \begin{array}{rccc} \bar{\pi}^k\colon J^k\pi & \longrightarrow & M \\ j^k_x\phi & \longmapsto & x \end{array} .$$

Observe that $\pi^s_r \circ \pi^k_s = \pi^k_r$, $\pi^k_0 = \pi^k$, $\pi^k_k = \mathrm{Id}_{J^k\pi}$, and $\bar{\pi}^k = \pi \circ \pi^k$.

The local coordinates in $J^k\pi$ associated to (x^μ, u_α) in E are $(x^\mu, u_\alpha, u_{\alpha,I})$ and are defined as follows. Consider a section $\phi \in \Gamma(\pi)$ with coordinate expression $\phi(x^\mu) = (x^\mu, \phi_\alpha(x^\mu))$. Then:

$$x^\mu(j^k_x\phi) = x^\mu, \quad u_\alpha(j^k_x\phi) = \phi_\alpha(x), \quad u_{\alpha,I}(j^k_x\phi) = \frac{\partial^{|I|}\phi_\alpha}{\partial x^I}(x).$$

Using this coordinates, the local expressions of the projections are:

$$\pi^k_r(x^\mu, u_\alpha, u_{\alpha,I}) = (x^\mu, u_\alpha, u_{\alpha,J}); \quad \pi^k(x^\mu, u_\alpha, u_{\alpha,I}) = (x^\mu, u_\alpha); \quad \bar{\pi}^k(x^\mu, u_\alpha, u_{\alpha,I}) = (x^\mu),$$

with $1 \leq |I| \leq k$ and $1 \leq |J| \leq r$.

Definition 1.3. *For a section $\phi \in \Gamma(\pi)$, the* kth prolongation *(or* prolongation *to $J^k\pi$) of a section $\phi \in \Gamma(\pi)$ is the section $j^k\phi \in \Gamma(\bar{\pi}^k)$ defined by :*

$$j^k\phi(x) = j^k_x\phi,$$

for every $x \in M$.

Definition 1.4. *A section $\psi \in \Gamma(\bar{\pi}^k)$ is* holonomic *if any of these equivalent conditions hold:*

- $j^k(\pi^k \circ \phi) = \phi$.
- $\phi^*\omega = 0$*, for every $\omega \in \mathfrak{C}^k$, the Cartan codistribution of $J^k\pi$.*

In coordinates, the kth prolongation is given by:

$$j^k\phi(x) = \left(x^\mu, \phi_\alpha(x), \frac{\partial^{|I|}\phi_\alpha}{\partial x^I}(x)\right),$$

for $1 \leq |I| \leq k$. The condition for a section $\phi \in \Gamma(\bar{\pi}^k)$ with coordinate expression $\phi(x) = (x^\mu, \phi_\alpha(x), \phi_{\alpha,I}(x))$ to be holonomic is that the following system of partial differential equations holds:

$$\phi_{\alpha,I} = \frac{\partial^{|I|}\phi_\alpha}{\partial x^I}, \quad 1 \leq |I| \leq k, \ 1 \leq \alpha \leq n.$$

or, equivalently,

$$\phi_{\alpha,I+1_\mu} = \frac{\partial \phi_{\alpha,I}}{\partial x^\mu}, \quad 1 \leq |I| \leq k-1, \ 1 \leq \mu \leq m, \ 1 \leq \alpha \leq n.$$

Finally, the *coordinate total derivatives* are the vector fields of the form:

$$D_i = \frac{\partial}{\partial x^i} + \sum_{|I|=0}^{k} u_{\alpha,I+1_i} \frac{\partial}{\partial u_{\alpha,I}} .$$

For a function $f \in C^\infty(J^k\pi)$, we write $D_i f \equiv \mathrm{L}_{D_i} f \in C^\infty(J^{k+1}\pi)$. Although in general the total derivatives change the order of the jet where the functions are defined, sometimes we ignore this fact, and write them as an abuse of notation in order to writing compact expressions in coordinates.

1.1.2 Dual and symmetric jet bundles

Definition 1.5. *The* kth-order extended dual jet bundle *is the bundle of m-form over $J^{k-1}\pi$ which vanish under the contraction with two $\bar{\pi}^{k-1}$-vertical vector fields, that is:*

$$\Lambda_2^m(T^*J^{k-1}\pi) := \{\alpha \in \Lambda^m(T^*J^{k-1}\pi) \mid i(V_1)\, i(V_2)\alpha = 0, \, \forall V_1 V_2 \in \mathfrak{X}^{V(\bar{\pi}^{k-1})}(J^{k-1}\pi)\}$$

It has the canonical projections

$$\pi_{J^{k-1}\pi} \colon \Lambda_2^m(T^*J^{k-1}\pi) \to J^{k-1}\pi \quad ; \quad \bar{\pi}_M = \bar{\pi}^1 \circ \pi_{J^{k-1}\pi} \colon \Lambda_2^m(T^*J^{k-1}\pi) \to M \, .$$

Definition 1.6.

- *The* Liouville m-form, *or* tautological *or* canonical m-form, *on $\Lambda_2^m(T^*J^{k-1}\pi)$ is the form $\Theta_1 \in \mathit{\Omega}^m(\Lambda_2^m(T^*J^{k-1}\pi))$ defined as*

  $$\Theta_1(\omega)(X_1, \ldots, X_m) := \omega(T\pi_{J^{k-1}\pi}(X_1), \ldots, T\pi_{J^{k-1}\pi}(X_1)T\pi_{J^{k-1}\pi}(X_m))$$

  where $\omega \in \Lambda_2^m(T^*J^{k-1}\pi)$ and $X_1, \ldots, X_m \in T_\omega\Lambda_2^m(T^*J^{k-1}\pi)$.

- *The* Liouville $(m+1)$-form, *or* canonical multisymplectic $(m+1)$-form, *is the form $\Omega_1 \in \mathit{\Omega}^{m+1}(\Lambda_2^m(T^*J^{k-1}\pi))$ given by*

  $$\Omega_1 = -\mathrm{d}\Theta_1.$$

Ω_1 is a multisymplectic form; that is, it is closed and 1-nondegenerate (see section 1.2.1).

Definition 1.7. *The* canonical pairing *between elements of $J^k\pi$ and elements of $\Lambda_2^m(T^*J^{k-1}\pi)$ is the fibered map over $J^{k-1}\pi$ defined as*

$$\begin{array}{rccc} \mathcal{C} : & J^k\pi \times_{J^{k-1}\pi} \Lambda_2^m(T^*J^{k-1}\pi) & \to & \Lambda_1^m(T^*J^{k-1}\pi) \\ & (j_x^k\psi, \omega) & \mapsto & (j^{k-1}\psi)^*_{j_x^k\psi}\omega \end{array}$$

$\Lambda_2^m(T^*J^{k-1}\pi)$ has too many multimomentum coordinates in order to establish a correspondence between "velocities" and multimomenta in terms of derivatives of the Lagrangian function. This problem to define the Hamiltonian formalism for higher-order field theories can be solved for first and second-order. For the later case:

Induced local coordinates in $\Lambda_2^m(T^*J^1\pi)$ are $(x^i, y^\alpha, u_{\alpha,i}, p, p^{\alpha,i}, p^{\alpha,ij})$. With these coordinates, the local expressions of the Liouville forms are (where $\mathrm{d}^{m-1}x_j = i\left(\dfrac{\partial}{\partial x^j}\right)\mathrm{d}^m x$)

$$\Theta_1 = p\, d^m x + p^{\alpha,i}\,\mathrm{d}u_\alpha \wedge \mathrm{d}^{m-1}x_i + p^{\alpha,ij}\,\mathrm{d}u_{\alpha,i} \wedge \mathrm{d}^{m-1}x_j\,,$$

$$\Omega_1 = -\mathrm{d}p \wedge \mathrm{d}^m x - \mathrm{d}p^{\alpha,i} \wedge \mathrm{d}u_\alpha \wedge \mathrm{d}^{m-1}x_i - \mathrm{d}p^{\alpha,ij} \wedge \mathrm{d}u_{\alpha,i} \wedge \mathrm{d}^{m-1}x_j\,.$$

Now, consider the $\pi_{J^1\pi}$-transverse submanifold $\jmath_s\colon J^2\pi^\dagger \hookrightarrow \Lambda_2^m(T^*J^1\pi)$ defined locally by the constraints $p_\alpha^{ij} = p_\alpha^{ji}$, which is called the *extended* 2*-symmetric multimomentum bundle* (although it is defined using coordinates, this construction is canonical [91]). Let

$$\pi^\dagger_{J^1\pi}\colon J^2\pi^\dagger \to J^1\pi \quad , \quad \bar{\pi}^\dagger_M = \bar{\pi}^1 \circ \pi^\dagger_{J^1\pi}\colon J^2\pi^\dagger \to M$$

be the canonical projections. Natural coordinates in $J^2\pi^\dagger$ are $(x^i, u_\alpha, u_{\alpha,i}, p, p^{\alpha,i}, p^{\alpha,I})$, where $|I| = 2$. Denote $\Theta_1^s = \jmath_s^*\Theta_1 \in \Omega^m(J^2\pi^\dagger)$ and the multisymplectic form $\Omega_1^s = \jmath_s^*\Omega_1 = -d\Theta_1^s \in \Omega^{m+1}(J^2\pi^\dagger)$, which are called *symmetrized Liouville* m *and* $(m+1)$*-forms*, and their coordinate expressions are

$$\Theta_1^s = p\,\mathrm{d}^m x + p^{\alpha,i}\,\mathrm{d}u_\alpha \wedge \mathrm{d}^{m-1}x_i + \frac{1}{n(ij)}\,p^{\alpha,1_i+1_j}\mathrm{d}u_{\alpha,i} \wedge \mathrm{d}^{m-1}x_j\,,$$

$$\Omega_1^s = -\mathrm{d}p \wedge \mathrm{d}^m x - \mathrm{d}p^{\alpha,i} \wedge \mathrm{d}u_\alpha \wedge \mathrm{d}^{m-1}x_i - \frac{1}{n(ij)}\,\mathrm{d}p^{\alpha,1_i+1_j} \wedge \mathrm{d}u_{\alpha,i} \wedge \mathrm{d}^{m-1}x_j\,.$$

Finally, consider the quotient bundle $J^2\pi^\ddagger = J^2\pi^\dagger/\Lambda_1^m(T^*J^1\pi)$, which is called the *restricted* 2*-symmetric multimomentum bundle*, and it is endowed with the natural projections

$$\mu\colon J^2\pi^\dagger \to J^2\pi^\ddagger \quad ; \quad \pi^\ddagger_{J^1\pi}\colon J^2\pi^\ddagger \to J^1\pi \quad , \quad \bar{\pi}^\ddagger_M\colon J^2\pi^\ddagger \to M.$$

Observe that $J^2\pi^\ddagger$ is also the submanifold of $\Lambda_2^m(T^*J^1\pi)/\Lambda_1^m(T^*J^1\pi)$ defined by the local constraints $p_\alpha^{ij} - p_\alpha^{ji} = 0$. Hence, natural coordinates in $J^2\pi^\ddagger$ are $(x^i, u_\alpha, u_{\alpha,i}, p^{\alpha,i}, p^{\alpha,I})$. Observe that $\dim J^2\pi^\ddagger = \dim J^2\pi^\dagger - 1$.

1.1.3 Multivector fields

(See [31] for details).

Definition 1.8.

Let $\kappa\colon \mathcal{M} \to M$ be a fiber bundle.

- *An* m-multivector field *in $\mathcal{M}$ is a skew-symmetric contravariant tensor of order m in $\mathcal{M}$. The set of m-multivector fields in $\mathcal{M}$ is denoted $\mathfrak{X}^m(\mathcal{M})$.*
- *A multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is said to be* locally decomposable *if, for every $p \in \mathcal{M}$, there is an open neighbourhood $U_p \subset \mathcal{M}$ and $X_1, \ldots, X_m \in \mathfrak{X}(U_p)$ such that $\mathbf{X}|_{U_p} = X_1 \wedge \ldots \wedge X_m$.*
- *Locally decomposable m-multivector fields $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ are locally associated with m-dimensional distributions $D \subset \mathrm{T}\mathcal{M}$, and multivector fields associated with the same distribution make an* equivalence class $\{\mathbf{X}\}$ *in the set $\mathfrak{X}^m(\mathcal{M})$. Then, $\mathbf{X}$ is* integrable *if its associated distribution is integrable.*

For every $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$, there exist $X_1, \ldots, X_r \in \mathfrak{X}(U)$ such that

$$\mathbf{X}|_U = \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m} X_{i_1} \wedge \ldots \wedge X_{i_m} ,$$

with $f^{i_1 \ldots i_m} \in C^\infty(U)$, $m \leqslant r \leqslant \dim J^k\pi$. If two multivector fields $\mathbf{X}, \mathbf{X}'$ belong to the same equivalence class $\{\mathbf{X}\}$ then, for every $U \subset \mathcal{M}$, there exists a non-vanishing function $f \in \mathrm{C}^\infty(U)$ such that $\mathbf{X}' = f\mathbf{X}$ on U.

Definition 1.9. *If $\Omega \in \Omega^k(\mathcal{M})$ and $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$, the* contraction *between $\mathbf{X}$ and Ω is defined as the natural contraction between tensor fields; in particular,*

$$i(\mathbf{X})\Omega \mid_U := \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m}\, i(X_{i_1} \wedge \ldots \wedge X_{i_m})\Omega = \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m}\, i(X_{i_1}) \ldots i(X_{i_m})\Omega ,$$

if $k \geq m$, and equal to zero if $k < m$. The Lie derivative *of Ω with respect to $\mathbf{X}$ is defined as the graded bracket (it is an operation of degree $m-1$)*

$$\mathrm{L}(\mathbf{X})\Omega := [\mathrm{d}, i(\mathbf{X})]\Omega = (\mathrm{d}\, i(\mathbf{X}) - (-1)^m\, i(\mathbf{X})\mathrm{d})\Omega .$$

Definition 1.10. *A multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is κ-transverse if, for every $\beta \in \Omega^m(M)$ with $\beta(\bar{\pi}^k(p)) \neq 0$, at every point $p \in \mathcal{M}$, we have that $(i(\mathbf{X})(\kappa^*\beta))_p \neq 0$. If $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is integrable, then it is κ-transverse if, and only if, its integral manifolds are local sections of κ. In this case, if $\psi\colon U \subset M \to \mathcal{M}$ is a local section and $\psi(U)$ is the integral manifold of $\mathbf{X}$ at p, then $T_p(\operatorname{Im}\psi) = \mathcal{D}_p(\mathbf{X})$ and ψ is an* integral section *of* $\mathbf{X}$.

Definition 1.11. *In the case that $\mathcal{M} = J^k\pi$, a multivector field $\mathbf{X} \in \mathfrak{X}^m(J^k\pi)$ is said to be* holonomic *if it is integrable and its integral sections are holonomic sections of $\bar{\pi}^k$ (and hence it is locally decomposable and $\bar{\pi}^k$-transverse).*

For a fiber manifold $\kappa : \mathcal{M} \to M$ with coordinates (x^μ, u_α), a τ-transverse and locally decomposable multivector field $\mathbf{X} \in \mathfrak{X}^m(E)$ is

$$\mathbf{X} = \bigwedge_{\mu=1}^{m} \left(\frac{\partial}{\partial x^\mu} + X_{\alpha,\mu} \frac{\partial}{\partial u_\alpha} \right) .$$

A section of τ, $\psi(x^\mu) = (x^\mu,\, \psi_\alpha(x^\mu))$, is an integral section of $\mathbf{X}$ if its component functions satisfy the following system of partial differential equations

$$\frac{\partial \psi_\alpha}{\partial x^\mu} = X_{\alpha,\mu} \circ \psi . \tag{1.1}$$

1.2 Multisymplectic Formalism

1.2.1 Multisymplectic Systems

Different bundles are used in the multisymplectic formalism. Nevertheless, all share the same basic structure, which can be thought as sort of "multisymplectic dynamics". In this section we will present its general formulation. A lot of properties can be analysed in this framework, in particular the constraint algorithm.

Over a base manifold M with dimension $m \geq 1$, consider the bundle $\bar{\pi} : J \to M$. Inspired by the exterior differential system used in [12, 58], consider a set of forms $\mathcal{I} \subset \Omega^\bullet(J)$, which we will call the *condition set*.

Definition 1.12. • *A section $\psi \in \Gamma(\bar{\pi})$ is* admissible *if $\psi^*\alpha = 0$ for all $\alpha \in \mathcal{I}$. The set of admissible sections will be denoted $\Psi(\mathcal{I})$.*

- *A multivector field $\mathbf{X} \in \mathfrak{X}^m(J)$ is* transverse *if it is $\bar{\pi}$-transverse.*
- *A multivector field $\mathbf{X} \in \mathfrak{X}^m(J)$ is* admisible *if it is transverse, integrable and its integrable sections are admissible.*

Definition 1.13. • *A form $\Omega \in \Omega^{m+1}(J)$ is a* multisymplectic form *if it is closed and 1-nondegenerate, that is, if the map $\flat_\Omega \colon \mathrm{T}J \longrightarrow \Lambda^m \mathrm{T}^*J$, defined by $\flat_\Omega(x, v) = (x, i(v)\Omega_x)$, for every $x \in J$ and $v \in \mathrm{T}_x J$, is injective. Otherwise, the form is said to be a* premultisymplectic form.

- *A (pre)multisymplectic form is* exact *if there exist $\Theta \in \Omega^m(\mathcal{M})$ such that $\Omega = -\mathrm{d}\Theta$.*

We will assume all the forms have constant rank.

Virtually all of the systems that will appear in this dissertation contains constraints which define submanifolds. Consider the submanifold $j : J_f \hookrightarrow J$, and define $\bar{\pi}_f \equiv \bar{\pi} \circ j : J_f \to M$. Frequently J_f is a subset of J, thus we will assimilate it with $j(J_f)$. Some regularity and structure to the submanifolds is imposed:

Assumption. *The constrained spaces are closed smooth manifolds. Thus, the constraints are given locally by a set of functions. The constraints are $\bar{\pi}$-vertical.*

We will not allow to constraint M, because its dimension define the order of Ω. The condition set $\mathcal{I}$ in the constrained system is defined by $\mathcal{I}_f \equiv \{j^*\alpha \,|\, \alpha \in \mathcal{I}\} \subset \Omega^\bullet(J_f)$. This definition is justified because the properties of being admissible derived from both sets are compatible in the following sense:

Lemma 1.2. *$j \circ \Psi(\mathcal{I}_f) = (j \circ \Gamma(\bar{\pi}_f)) \cap \Psi(\mathcal{I})$.*

Proof. It follows from the fact that, for every $\psi \in \Gamma(\bar{\pi}_f)$ and for every form $\alpha \in \Omega^\bullet(J)$,

$$\psi^* j^* \alpha = (j \circ \psi)^* \alpha = 0$$

□

Finally, the variations of the action are represented by an integrable distribution $V = \bigcup_{p \in J_f} V_p$, with $T_p J_f \subset V_p \subset T_p J$. The sections of V will be denoted as $\Gamma(V)$.

Definition 1.14. *A* multisymplectic system *is the 4-tuple $(J, \Omega, V, \mathcal{I})$. If Ω is pre-multisymplectic, then it is a* pre-multisymplectic system.

The systems often have a complex bundle structure, with several projections and intermediate manifolds. For the sake of simplicity, this structure is implicit in the 4-tuple, although it will be clearly established in every case. For the case of the system $(J_f, \Omega, V, \mathcal{I}_f)$ which develops in the submanifold $j : J_f \hookrightarrow J$, the form Ω have support on J_f, but it is an element of $\Omega^{m+1}(J)$. Likewise, the set of variations V is a subdistribution of $\underline{TJ_f} \equiv \bigcup_{p \in J_j} T_p J$. There are two relevant set of variations which arise while performing the constraint algorithm. First there are the pull-backed variations $V = j_* T J_f \subset TJ$, which we sometimes denote TJ_f. Secondly, there are the constrained variations $V = \underline{TJ_f}$.

Definition 1.15. • *The* pull-backed *system is $(J_f, \Omega, TJ_f, \mathcal{I}_f)$*

- *The* constrained *system is* $(J_f, \Omega, \underline{TJ_f}, \mathcal{I}_f)$

Any solution of the second problem is a solution of the first problem, because $TJ_f \subset \underline{TJ_f}$, but not the other way around.

In this dissertation we will consider a particular case of systems, closely related to a certain jet bundle $J^k\pi$.

Definition 1.16. *A system* $(J, \Omega, V, \mathcal{I})$ *is* almost-holonomic *if there exists a projection* $\rho : J \to J^k\pi$ *for some* $k \geq 0$*, (in particular,* $\rho = Id_{J^k\pi}$ *if* $J = J^k\pi$*), and* $\mathcal{I} = \rho^*\mathfrak{C}^k$*, the Cartan codistribution of* $J^k\pi$*.*

The systems we consider in chapters 3 and 4 are almost-holonomic, therefore, we usually use the term *holonomic* instead of admissible. This property is not necessary for the definitions, but it is used to characterize symmetries in section 2.1.3.

1.2.2 Multisymplectic Equations

In this setcion we present the variational problem based on [45, 58].

Let $(J, \Omega, V, \mathcal{I})$ be an exact (pre)multisymplectic system. Let $\Gamma(\bar{\pi})$ be the set of sections of $\bar{\pi}$. Consider the following functional (where the convergence of the integral is assumed)

$$\begin{array}{rcl} \mathcal{F}\colon \Gamma(\bar{\pi}) & \longrightarrow & \mathbb{R} \\ \psi & \longmapsto & \displaystyle\int_M \psi^*\Theta \end{array} .$$

Definition 1.17. *The* generalized variational problem *for* $(J, \Omega, V, \mathcal{I})$ *is the search for the admissible critical (local) sections of the functional* $\mathcal{F}$ *with respect to the variations of* ψ *given by* $\psi_s = \sigma_s \circ \psi$*, where* $\{\sigma_s\}$ *is a local one-parameter group of any compact-supported* $\bar{\pi}$*-vertical vector field in* $\Gamma(V)$*; that is,*

$$\left.\frac{d}{ds}\right|_{s=0} \int_M \psi_s^*\Theta = 0$$

which are an integral section of $\mathcal{I}$*, namely, such that* $\phi^*\alpha = 0$ *for every* $\alpha \in \mathcal{I}$ *.*

Theorem 1.1. *The following assertions on a section* $\psi \in \Gamma(\bar{\pi})$ *are equivalent:*

1. ψ *is a solution to the generalized variational problem.*
2. ψ *is an admissible section solution to the equation*

$$\psi^*\, i(Y)\Omega = 0\,, \quad \textit{for every } Y \in \Gamma(V)\,. \tag{1.2}$$

3. ψ *is an adimissible section solution to the equation*

$$i(Y)\, i(\Lambda^m\psi)(\Omega \circ \psi) = 0\,, \quad \textit{for every } Y \in \Gamma(V)\,. \tag{1.3}$$

4. ψ *is an integral section of an admissible* m*-multivector field contained in a class of* $\bar{\pi}$*-transverse and integrable (and hence locally decomposable)* m*-multivector fields,* $\{\mathbf{X}\} \subset \mathfrak{X}^m(J)$*, satisfying the equation*

$$i(Y)\, i(\mathbf{X})\Omega = 0\,, \quad \textit{for every } Y \in \Gamma(V)\,. \tag{1.4}$$

We will mainly use the problem for multivetor fields (1.4), as it is more operative and, in lesser extend, the problem for sections (1.2). Unfortunately, to check in general that a multivetor field is integrable is tricky, and it is only done in especial cases. On the other hand, the conditions of locally decomposable, admisible and transverse have a definite expression in local coordinates. A multivetor field with a local expression that satisfy these local conditions, but not necessary integrable, is called semi-admisible. However, if such a multivector field admits integral sections, then its integral sections are admisible.

Finally, we introduce the following notation: as it is usual,

$$\ker^m \Omega \equiv \{\mathbf{X} \in \mathfrak{X}^m(J) \mid i(\mathbf{X})\Omega = 0\} \,.$$

We denote by $\ker^m_{\bar{\pi}} \Omega$ the set of locally decomposable and $\bar{\pi}$-transverse multivector fields belonging to $\ker^m \Omega$. Then, $\ker^m_{SH} \Omega$ and $\ker^m_H \Omega$ denote the sets of semi-admissible (semi-holonomic) and the admissible (holonomic) multivector fields belonging to $\ker^m \Omega$, respectively. Obviously we have

$$\ker^m_H \Omega \subset \ker^m_{SH} \Omega \subset \ker^m_{\bar{\pi}} \Omega \subset \ker^m \Omega \,. \tag{1.5}$$

1.2.3 The constraint algorithm

In general, admisible multivector fields $\mathbf{X} \in \mathfrak{X}^m(J)$ which are solutions to (1.4) could not exist. In the best of cases they exist only in some submanifold of J [22]. The aim in this section is to find the constraints that define this submanifold, using an adapted local version of the geometric constraint algorithms [23, 22]. When there is no solution for the problem for multivector fields of the system $(J, \Omega, V, \mathcal{I})$, we consider a weaken version of the problem:

Find submanifolds $j : J_f \hookrightarrow J$ such that the system $(J_f, \Omega, \underline{TJ_f} \cap V, \mathcal{I}_f)$ has a solution.

The algorithm proceeds inductively, by setting $S_0 = J$ and

$S_i := \{p \in S_{i-1} | \exists \mathbf{X} \in \mathfrak{X}^m(S_{i-1})$, semi-admissible and $i(Y)\, i(\mathbf{X})\Omega|_p = 0, \forall Y \in \underline{T_p S_{i-1}} \cap V_p\}$.

We assume the successive subsets are closed submanifolds. Notice that the multivector fields found for the submanifold S_i are tangent to S_{i-1} but not necessarily to S_i, thus the algorithm continues until $S_f = S_{f-1}$ or $S_i = \emptyset$. In the first case, we say that S_f is the *final constraint submanifold*.

The actual computation in coordinates of these sets starts by considering the local expression of a locally decomposable multivector field $\mathbf{X}$. Locally, equations (1.4) and the semi-admissible conditions are equations on the coefficients of $\mathbf{X}$. In the cases we will study, all these equations form a lineal system of equations, at every point of J. S_1 consists on the points where the system is compatible. In the next step, apart of these equations we have the tangency (or consistency) conditions, which arise from imposing that the solutions have to be tangent to S_1. Then we continue the process until the algorithm stops.

Proving the existence of an integrable multivector field is more complicate for field theories. We will present a particular proof for every case.

1.3 Lagrangian, Hamiltonian and unified multisymplectic formalism

1.3.1 Lagrangian-Hamiltonian unified formalism

(See [83, 84, 90] for details). Let $E \xrightarrow{\pi} M$ be a fiber bundle, with $\dim E = m + n$, over an orientable m-dimensional manifold M, whose volume form is denoted $\eta \in \Omega^m(M)$. The 2-*symmetric* *jet-multimomentum bundles* are $\mathcal{W} = J^3\pi \times_{J^1\pi} J^2\pi^\dagger$ and $\mathcal{W}_r = J^3\pi \times_{J^1\pi} J^2\pi^\ddagger$. The coordinates in $\mathcal{W}$ and $\mathcal{W}_r$ are $(x^i, u_\alpha, u_{\alpha,i}, u_{\alpha,I}, u_{\alpha,J}, p, p^{\alpha,i}, p^{\alpha I})$ and $(x^i, u_\alpha, u_{\alpha,i}, u_{\alpha,I}, u_{\alpha,J}, p^{\alpha,i}, p^{\alpha I})$, respectively, with $|I| = 2$ and $|J| = 3$. These bundles are endowed with the canonical projections

$$\begin{array}{c} \rho_1^r \colon \mathcal{W}_r \to J^3\pi \quad , \quad \rho_2^r \colon \mathcal{W}_r \to J^2\pi^\ddagger \quad , \quad \rho_M^r \colon \mathcal{W}_r \to M \\ \rho_1 \colon \mathcal{W} \to J^3\pi \quad , \quad \rho_2 \colon \mathcal{W} \to J^2\pi^\dagger \quad , \quad \rho_M \colon \mathcal{W} \to M \ . \end{array}$$

The *second-order coupling m-form* in $\mathcal{W}$ is the ρ_M-semibasic m-form $\hat{\mathcal{C}} \in \Omega^m(\mathcal{W})$ defined by

$$\hat{\mathcal{C}}(j^3_x\phi, \omega) = \mathcal{C}^s(\pi^3_2(j^3_x\phi), \omega) \quad , \quad (j^3_x\phi, \omega) \in \mathcal{W} \ .$$

Since $\hat{\mathcal{C}}$ is a ρ_M-semibasic m-form, there exists a function $\hat{C} \in C^\infty(\mathcal{W})$ such that $\hat{\mathcal{C}} = \hat{C}\rho_M^*\eta$, and we have the coordinate expression

$$\hat{\mathcal{C}} = \left(p + p^{\alpha,i}u_{\alpha,i} + p^{\alpha,I}u_{\alpha,I}\right) \mathrm{d}^m x \ .$$

Let $\mathcal{L} \in \Omega^m(J^2\pi)$ be a second-order Lagrangian density for this field theory, which is a $\bar{\pi}^2$-semibasic m-form and then $\mathcal{L} = L\,(\bar{\pi}^2)^*\eta \in \Omega^m(J^2\pi)$, where $L \in \mathrm{C}^\infty(J^2\pi)$ is the Lagrangian function. Denoting by $\hat{\mathcal{L}} = (\pi^3_2 \circ \rho_1)^*\mathcal{L} \in \Omega^m(\mathcal{W})$, we can write $\hat{\mathcal{L}} = \hat{L}\,\rho_M^*\eta$, where $\hat{L} = (\pi^3_2 \circ \rho_1)^*L \in C^\infty(\mathcal{W})$. Then, we introduce the Hamiltonian submanifold

$$\mathcal{W}_o = \left\{ w \in \mathcal{W} \colon \hat{\mathcal{L}}(w) = \hat{\mathcal{C}}(w) \right\} \stackrel{\jmath_o}{\hookrightarrow} \mathcal{W} \ ,$$

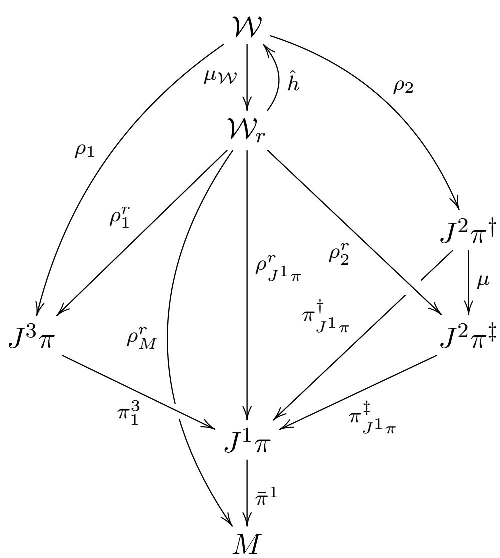
which is defined by the constraint

$$\hat{C} - \hat{L} \equiv p + p^{\alpha,i}u_{\alpha,i} + p^{\alpha,I}u_{\alpha,I} - \hat{L} = 0 \quad , \quad |I| = 2 \ .$$

and it is $\mu_{\mathcal{W}}$-transverse and diffeomorphic to $\mathcal{W}_r$ by $\Phi : \mathcal{W}_0 \to \mathcal{W}_r$. Furthermore, the quotient map $\mu \colon J^2\pi^\dagger \to J^2\pi^\ddagger$ induces a natural submersion $\mu_{\mathcal{W}} \colon \mathcal{W} \to \mathcal{W}_r$. Then, the submanifold $\mathcal{W}_o$ induces a Hamiltonian section $\hat{h} \in \Gamma(\mu_{\mathcal{W}})$ defined as $\hat{h} = \jmath_o \circ \Phi^{-1} \colon \mathcal{W}_r \to \mathcal{W}$, which is specified giving the local Hamiltonian function

$$\hat{H} = p^{\alpha,i}u_{\alpha,i} + p^{\alpha,I}u_{\alpha,I} - \hat{L} \ ; \tag{1.6}$$

that is, $\hat{h}(x^i, u_\alpha, u_{\alpha,i}, u_{\alpha,I}, u_{\alpha,J}, p^{\alpha,i}, p^{\alpha I}) = (x^i, u_\alpha, u_{\alpha,i}, u_{\alpha,I}, u_{\alpha,J}, -\hat{H}, p^{\alpha,i}, p^{\alpha I})$. Hence, we have the diagram:



Now we define the forms $\Theta_r = (\rho_2 \circ \hat{h})^*\Theta \in \Omega^m(\mathcal{W}_r)$ and $\Omega_r = -\mathrm{d}\Theta_r \in \Omega^{m+1}(\mathcal{W}_r)$, with local expressions,

$$\Theta_r = -\hat{H}\mathrm{d}^m x + p^{\alpha,i}\mathrm{d}u_\alpha \wedge \mathrm{d}^{m-1}x_i + \frac{1}{n(ij)}\,p^{\alpha,1_i+1_j}\mathrm{d}u_{\alpha,i} \wedge \mathrm{d}^{m-1}x_j \ ,$$

$$\Omega_r = \mathrm{d}\hat{H} \wedge \mathrm{d}^m x - \mathrm{d}p^{\alpha,i} \wedge \mathrm{d}u_\alpha \wedge \mathrm{d}^{m-1}x_i - \frac{1}{n(ij)}\,\mathrm{d}p^{\alpha,1_i+1_j} \wedge \mathrm{d}u_{\alpha,i} \wedge \mathrm{d}^{m-1}x_j \ .$$

The condition set $\mathcal{I}$ for the multisymplectic unified system is $\mathcal{I} = \rho_1^{r*}\mathfrak{C}^3 \equiv \{\rho_1^{r*}\omega | \omega \in \mathfrak{C}^3\}$, where $\mathfrak{C}^3$ is the Cartan codistribution associated to $J^3\pi$. Therefore, a section $\psi \in \Gamma(\rho_M^r)$ is admissible (or holonomic) if $\rho_1^r \circ \psi \in \Gamma(J_\pi^3)$ is holonomic. An integrable and ρ_M^r-transverse multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{W}_r)$ is *holonomic* if its integral sections are holonomic. The local expression of a semiholonomic multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{W}_r)$ is, in this case,

$$\mathbf{X} = \bigwedge_{j=1}^m \left(\frac{\partial}{\partial x^j} + u_{\alpha,j}\frac{\partial}{\partial u_\alpha} + \sum_{|I|=1}^{2} u_{\alpha,I+1_j}\frac{\partial}{\partial u_{\alpha,I}} + F_{\alpha,J,j}\frac{\partial}{\partial u_{\alpha,J}} + G_j^{\alpha,i}\frac{\partial}{\partial p^{\alpha,i}} + G_j^{\alpha,K}\frac{\partial}{\partial p^{\alpha,K}} \right) , \tag{1.7}$$

with $|K| = 2$ and $|J| = 3$. The second-order Lagrangian-Hamiltonian multisymplectic system is $(\mathcal{W}_r, \Omega_r, T\mathcal{W}_r, \rho_1^{r*}\mathfrak{C}^3)$. The form Ω_r is 1-degenerate because

$$\ker \Omega_r = \{Z \in \mathfrak{X}(\mathcal{W}_r) \mid i(Z)\Omega_r = 0\} = \left\langle \frac{\partial}{\partial u_I^\alpha} \right\rangle \neq \{0\} \quad , \quad \text{for } |I| = 2,3 \, ; \tag{1.8}$$

then $(\mathcal{W}_r, \Omega_r, T\mathcal{W}_r, \rho_1^{r*}\mathfrak{C}^3)$ is a premultisymplectic system and solutions to (1.2) or (1.4) do not exist everywhere in $\mathcal{W}_r$:

Proposition 1.1. *A section $\psi \in \Gamma(\rho_M^r)$ solution to the equation* (1.2) *takes values in a $n(m + m(m+1)/2)$-codimensional submanifold $\jmath_\mathcal{L} \colon \mathcal{W}_\mathcal{L} \hookrightarrow \mathcal{W}_r$ which is identified with the graph of a bundle map $\mathcal{FL} \colon J^3\pi \to J^2\pi^\ddagger$, over $J^1\pi$, defined locally by*

$$\mathcal{FL}^* p^{\alpha,i} = \frac{\partial \hat{L}}{\partial u_{\alpha,i}} - \sum_{j=1}^m \frac{1}{n(ij)} \frac{d}{dx^j}\left(\frac{\partial \hat{L}}{\partial u_{\alpha,1_i+1_j}} \right) \quad ; \quad \mathcal{FL}^* p^{\alpha,I} = \frac{\partial \hat{L}}{\partial u_{\alpha,I}} \, .$$

What is equivalent, the submanifold $\mathcal{W}_\mathcal{L}$ is the graph of a bundle morphism $\widetilde{\mathcal{FL}} \colon J^3\pi \to J^2\pi^\dagger$ over $J^1\pi$ defined locally by

$$\widetilde{\mathcal{FL}}^* p^{\alpha,i} = \frac{\partial \hat{L}}{\partial u_{\alpha,i}} - \sum_{j=1}^m \frac{1}{n(ij)} \frac{d}{dx^j}\left(\frac{\partial \hat{L}}{\partial u_{\alpha,1_i+1_j}} \right) \quad ; \quad \widetilde{\mathcal{FL}}^* p^{\alpha,I} = \frac{\partial \hat{L}}{\partial y_{\alpha,I}} \, ,$$

$$\widetilde{\mathcal{FL}}^* p = \hat{L} - u_{\alpha,i}\left(\frac{\partial \hat{L}}{\partial u_{\alpha,i}} - \sum_{j=1}^m \frac{1}{n(ij)} \frac{d}{dx^j}\left(\frac{\partial \hat{L}}{\partial u_{1_i+1_j,\alpha}} \right) \right) - u_{\alpha,I}\frac{\partial \hat{L}}{\partial u_{\alpha,I}} \, .$$

The maps $\mathcal{FL}$ and $\widetilde{\mathcal{FL}}$ are the *restricted* and the *extended Legendre maps* associated with $\mathcal{L}$, and they satisfy that $\mathcal{FL} = \mu \circ \widetilde{\mathcal{FL}}$. For every $j_x^3\phi \in J^3\pi$, we have that $\operatorname{rank}(\widetilde{\mathcal{FL}}(j_x^3\phi)) = \operatorname{rank}(\mathcal{FL}(j_x^3\phi))$. Then, according to [91], we say that a second-order Lagrangian density $\mathcal{L} \in \Omega^m(J^2\pi)$ is *regular* if

$$\operatorname{rank}(\widetilde{\mathcal{FL}}(j_x^3\phi)) = \operatorname{rank}(\mathcal{FL}(j^3\phi)) = \dim J^2\pi + \dim J^1\pi - \dim E = \dim J^2\pi^\ddagger \, ,$$

otherwise, the Lagrangian density is singular. Regularity is equivalent to demand that $\mathcal{FL} \colon J^3\pi \to J^2\pi^\ddagger$ is a submersion onto $J^2\pi^\ddagger$ and this implies that there exist local sections of $\mathcal{FL}$. If $\mathcal{FL}$ admits a global section $\Upsilon \colon J^2\pi^\ddagger \to J^3\pi$, then the Lagrangian density is said to be *hyperregular*. In a natural chart of $J^3\pi$ the regularity condition for the Lagrangian density $\mathcal{L}$ is equivalent to

$$\det\left(\frac{\partial^2 L}{\partial u_{\beta,I}\partial u_{\alpha,K}} \right)(j_x^3\phi) \neq 0 \, , \quad \text{for every } j_x^3\phi \in J^3\pi \quad ; \quad |I| = |K| = 2 \, .$$

Recall that the regularity of $\mathcal{L}$ determines if the section $\psi \in \Gamma(\rho_M^r)$ solution to the equation (1.2) lies in $\mathcal{W}_\mathcal{L}$ or in a submanifold $\mathcal{W}_f \hookrightarrow \mathcal{W}_\mathcal{L}$ where the section ψ takes values. In order to obtain this *final constraint submanifold*, the best way is to work with the equation (1.4) instead of (1.2). Then, we have:

Proposition 1.2. *A solution* $\mathbf{X} \in \mathfrak{X}^m(\mathcal{W}_r)$ *to equation* (1.4) *exists only on the points of the* compatibility submanifold $\mathcal{W}_c \hookrightarrow \mathcal{W}_r$ *defined by*

$$\begin{aligned}\mathcal{W}_c &= \left\{ w \in \mathcal{W}_r \colon (i(Z)\mathrm{d}\hat{H})(w) = 0\,,\ \textit{for every } Z \in \ker(\Omega_r) \right\} \\ &= \left\{ w \in \mathcal{W}_r \colon (i(Y)\Omega_r)(w) = 0\,,\ \textit{for every } Y \in \mathfrak{X}^{V(\rho_2^r)}(\mathcal{W}_r) \right\}\,.\end{aligned}$$

Bearing in mind (1.8) and that $i\left(\dfrac{\partial}{\partial u_{\alpha,J}}\right)\mathrm{d}\hat{H} = 0$, for $|J| = 3$, the functions locally defining this submanifold have the following coordinate expressions

$$i\left(\frac{\partial}{\partial u_{\alpha,I}}\right) d\hat{H} = p^{\alpha,I} - \frac{\partial \hat{L}}{\partial u_{\alpha,I}} \quad , \quad \text{for } |I| = 2\,. \tag{1.9}$$

Then, the tangency condition for the multivector fields $\mathbf{X}$ which are solutions to (1.4) on $\mathcal{W}_c$ gives rise to mn new constraints

$$p^{\alpha,i} - \frac{\partial \hat{L}}{\partial u_{\alpha,i}} + \sum_{j=1}^{m} \frac{1}{n(ij)} \frac{d}{dx^j} \frac{\partial \hat{L}}{\partial u_{\alpha,1_i+1_j}} = 0\,.$$

which define a submanifold of $\mathcal{W}_c$ that coincides with the submanifold $\mathcal{W}_{\mathcal{L}}$. Now the study of the tangency of $\mathbf{X}$ along $\mathcal{W}_{\mathcal{L}}$ could introduce new constraints depending on the regularity of $\mathcal{L}$, and the algorithm continues until we reach the submanifold $\mathcal{W}_f$.

1.3.2 Lagrangian formalism

(See [83, 84] for details). Let $\Theta_1^s \in \Omega^m(J^2\pi^\dagger)$ and $\Omega_1^s \in \Omega^{m+1}(J^2\pi^\dagger)$ be the symmetrized Liouville forms in $J^2\pi^\dagger$. The *Poincaré-Cartan forms* in $J^3\pi$ are the forms defined as

$$\Theta_{\mathcal{L}} = \widetilde{\mathcal{FL}}^*\Theta_1^s \in \Omega^m(J^3\pi) \quad , \quad \Omega_{\mathcal{L}} = \widetilde{\mathcal{FL}}^*\Omega_1^s = -\mathrm{d}\Theta_{\mathcal{L}} \in \Omega^{m+1}(J^3\pi)\,.$$

These forms coincide with the usual Poincaré-Cartan forms for second-order classical field theories that can be found in the literature [1, 43, 70, 79], and they can also be recovered directly from the unified formalism as follows: if $\Theta = \rho_2^*\Theta_1^s$ and $\Theta_r = \hat{h}^*\Theta$ are the canonical m-forms defined in $\mathcal{W}$ and $\mathcal{W}_r$, respectively, then, the Poincaré-Cartan m-form are $\Theta = \rho_1^*\Theta_{\mathcal{L}}$ and $\Theta_r = (\rho_1^r)^*\Theta_{\mathcal{L}}$, and the same result holds for the Poincaré-Cartan form $\Omega_{\mathcal{L}}$. Using natural coordinates in $J^3\pi$, we have the local expression

$$\begin{aligned}\Theta_{\mathcal{L}} \;=\; & \left(\frac{\partial L}{\partial u_{\alpha,i}} - \sum_{j=1}^{m} \frac{1}{n(ij)} \frac{d}{dx^j} \frac{\partial L}{\partial u_{\alpha,1_i+1_j}} \right)(\mathrm{d}u_\alpha \wedge \mathrm{d}^{m-1}x_i - u_{\alpha,i}\mathrm{d}^m x) \\ & + \frac{1}{n(ij)} \frac{\partial L}{\partial u_{\alpha,1_i+1_j}} (\mathrm{d}u_{\alpha,i} \wedge \mathrm{d}^{m-1}x_j - u_{\alpha,1_i+1_j}\mathrm{d}^m x) + L\,\mathrm{d}^m x\,.\end{aligned} \tag{1.10}$$

The second-order Lagrangian multisymplectic system is $(J^3\pi, \Omega_{\mathcal{L}}, TJ^3\pi, \mathfrak{C}^3)$. Thus, a section is admissible if it is holonomic in $J^3\pi$.

In order to recover the Lagrangian field equations, we have that the map $\rho_1^{\mathcal{L}} = \rho_1^r \circ \jmath_{\mathcal{L}} \colon \mathcal{W}_{\mathcal{L}} \to J^3\pi$ is a diffeomorphism, the Poincaré-Cartan forms defined in $J^3\pi$ satisfy $(\rho_1^{\mathcal{L}})^*\Theta_{\mathcal{L}} = \jmath_{\mathcal{L}}^*\Theta_r$ and $(\rho_1^{\mathcal{L}})^*\Omega_{\mathcal{L}} = \jmath_{\mathcal{L}}^*\Omega_r$, and all of this allows us to prove that:

Proposition 1.3. *If* $\psi \in \Gamma(\rho_M^r)$ *be a holonomic section solution to the equation* (1.2) *for the unified formalism, then the section* $\psi_{\mathcal{L}} = \rho_1^r \circ \psi \in \Gamma(\bar{\pi}^3)$ *is holonomic, and it is a solution to the equation* (1.2) *for the Lagrangian formalism.*

Conversely, if $\psi_{\mathcal{L}} \in \Gamma(\bar{\pi}^3)$ *is a holonomic section solution to the field equation* (1.2) *for the Lagrangian formalism, then the section* $\psi = \jmath_{\mathcal{L}} \circ (\rho_1^{\mathcal{L}})^{-1} \circ \psi_{\mathcal{L}} \in \Gamma(\rho_M^r)$ *is holonomic and it is a solution to the equation* (1.2) *for the unified formalism.*

In local coordinates in $J^3\pi$, the equation for the holonomic section $\psi_{\mathcal{L}} = j^3\phi$ are the Euler-Lagrange equations for a second-order field theory

$$\left.\frac{\partial L}{\partial u_\alpha}\right|_{j^3\phi} - \frac{d}{dx^i}\left.\frac{\partial L}{\partial u_{\alpha,i}}\right|_{j^3\phi} + \sum_{|I|=2} \frac{d^{|I|}}{dx^I}\left.\frac{\partial L}{\partial u_{\alpha,I}}\right|_{j^3\phi} = 0 \quad , \quad 1 \leqslant \alpha \leqslant n \, .$$

Theorem 1.2. *Let $\mathbf{X} \in \mathfrak{X}^m(\mathcal{W}_r)$ be a holonomic multivector field solution to the equation* (1.4) *for the unified formalism, at least on the points of a submanifold $\jmath_f \colon \mathcal{W}_f \subseteq \mathcal{W}_{\mathcal{L}} \hookrightarrow \mathcal{W}_r$, and tangent to $\mathcal{W}_f$. Then there exists a unique holonomic multivector field $\mathbf{X}_{\mathcal{L}} \in \mathfrak{X}^m(J^3\pi)$ solution to the equation* (1.4) *for the Lagrangian formalism, at least on the points of $S_f = \rho_1^{\mathcal{L}}(\mathcal{W}_f)$, and tangent to S_f,*

Conversely, if $\mathbf{X}_{\mathcal{L}} \in \mathfrak{X}^m(J^3\pi)$ is a holonomic multivector field solution to the equation (1.4) *for the Lagrangian formalism, at least on the points of a submanifold $S_f \hookrightarrow J^3\pi$, and tangent to S_f; then there exists a unique holonomic multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{W}_r)$ which is a solution to the equation* (1.4) *for the unified formalism, at least on the points of $\mathcal{W}_f = (\rho_1^{\mathcal{L}})^{-1}(S_f) \hookrightarrow \mathcal{W}_{\mathcal{L}}$, and tangent to $\mathcal{W}_f$.*

The relation between these multivector fields is $\mathbf{X}_{\mathcal{L}} \circ \rho_1^r \circ \jmath_f = \Lambda^m \mathrm{T}\rho_1^r \circ \mathbf{X} \circ \jmath_f$.

$$\begin{tikzcd}
 & & \mathcal{W}_r \arrow[ddll, bend right, "\rho_1^r"'] \arrow[ddrr, bend left, "\rho_2^r"] \arrow[dddrr, bend left=60, "\rho_{\mathcal{P}}^r"] & & \\
 & & \mathcal{W}_{\mathcal{L}} \arrow[u, hook, "\jmath_{\mathcal{L}}"] \arrow[dll, "\rho_1^{\mathcal{L}}"'] \arrow[drr, "\rho_2^{\mathcal{L}}"] \arrow[ddrr, "\rho_{\mathcal{P}}^{\mathcal{L}}"] & & \\
J^3\pi \arrow[rrrr, "\mathcal{FL}"] \arrow[drrrr, "\mathcal{FL}_o"'] & & & & J^2\pi^\ddagger \\
 & & \mathcal{W}_f \arrow[uu, hook] \arrow[dll] \arrow[drr] & & \mathcal{P} \arrow[u, hook, "\jmath"'] \\
S_f \arrow[uu, hook] & & & & P_f \arrow[u, hook]
\end{tikzcd} \tag{1.11}$$

1.3.3 Hamiltonian formalism

For the Hamiltonian formalism, denote $\widetilde{\mathcal{P}} = \widetilde{\mathcal{FL}}(J^3\pi) \overset{\tilde{\jmath}}{\hookrightarrow} J^2\pi^\dagger$ and $\mathcal{P} = \mathcal{FL}(J^3\pi) \overset{\jmath}{\hookrightarrow} J^2\pi^\ddagger$ (we assume they are submanifolds and, if $\mathcal{L}$ is hyperregular, then $\mathcal{P} = J^2\pi^\ddagger$), and let $\bar{\pi}_{\mathcal{P}} \colon \mathcal{P} \to M$ be the natural projection. and $\mathcal{FL}_o$ the map defined by $\mathcal{FL} = \jmath \circ \mathcal{FL}_o$. In order to assure the existence of the Hamiltonian formalism we assume that the Lagrangian density $\mathcal{L} \in \mathit{\Omega}^m(J^2\pi)$ is, at least, *almost-regular*; that is, $\mathcal{P}$ is a closed submanifold of $J^2\pi^\ddagger$, $\mathcal{FL}$ is a submersion onto its image and, for every $j^3_x\phi \in J^3\pi$, the fibers $\mathcal{FL}^{-1}(\mathcal{FL}(j^3_x\phi))$ are connected submanifolds of $J^3\pi$. Then, there exists a diffeomorphism $\widetilde{\mu} = \mu \circ \tilde{\jmath} \colon \widetilde{\mathcal{P}} \to \mathcal{P}$ and we can define a *Hamiltonian μ-section* as $h = \tilde{\jmath} \circ \widetilde{\mu}^{-1}$, which is specified by a local Hamiltonian function $H \in C^\infty(\mathcal{P})$, that is, $h(x^i, u_\alpha, u_{\alpha,i}, p^{\alpha,i}, p^{\alpha,I}) = (x^i, u_\alpha, u_{\alpha,i}, -H, p^{\alpha,i}, p^{\alpha,I})$.

$$\begin{tikzcd}
\widetilde{\mathcal{P}} \arrow[r, "\tilde{\jmath}"] \arrow[d, "\tilde{\mu}"'] & J^2\pi^\dagger \arrow[d, "\mu"] & \mathcal{W} \arrow[l, "\rho_2"'] \arrow[d, "\mu_{\mathcal{W}}"'] \\
\mathcal{P} \arrow[r, "\jmath"'] \arrow[ur, "h"] & J^2\pi^\ddagger & \mathcal{W}_r \arrow[l, "\rho_2^r"] \arrow[u, bend right, "\hat{h}"']
\end{tikzcd}$$

Now, we can define the Hamiltonian forms $\Theta_h := h^*\Theta_1^s \in \mathit{\Omega}^m(\mathcal{P})$ and the condition set is $\mathcal{I} \equiv (\pi^\ddagger_{J^1\pi} \circ \jmath)^*\mathfrak{C}^1$. Then, the second-order Hamiltonian multisymplectic system is $(\mathcal{P}, \Omega_h, T\mathcal{P}, (\pi^\ddagger_{J^1\pi} \circ \jmath)^*\mathfrak{C}^1)$. Then the Hamiltonian formalism is recovered as follows:

Proposition 1.4. *Let $\psi \in \Gamma(\rho^r_M)$ be a solution to the equation* (1.2) *for the unified formalism. Then, the section $\psi_h = \mathcal{FL}_o \circ \rho^r_1 \circ \psi = \mathcal{FL}_o \circ \psi_{\mathcal{L}} \in \Gamma(\bar{\pi}_{\mathcal{P}})$ is a solution to the equation* (1.2) *for the Hamiltonian formalism.*

$$
\begin{tikzcd}
 & \mathcal{W}_r \arrow[dl, "\rho^r_1"'] \arrow[dr, "\rho^r_2"] \arrow[ddr, "\rho^r_{\mathcal{P}}"] \arrow[ddd, "\rho^r_M"] & \\
J^3\pi \arrow[rr, "\mathcal{FL}"] \arrow[drr, "\mathcal{FL}_o"] \arrow[ddr, "\bar{\pi}^3"] & & J^2\pi^\ddagger \\
 & & \mathcal{P} \arrow[u, hook, "\jmath"'] \arrow[dl, "\bar{\pi}_{\mathcal{P}}"] \\
 & M \arrow[uuu, bend left, "\psi"] \arrow[uul, bend left, "\psi_{\mathcal{L}}"] \arrow[ur, bend right, "\psi_h = \mathcal{FL}_o \circ \psi_{\mathcal{L}}"'] &
\end{tikzcd}
$$

Theorem 1.3. *Let $\mathbf{X} \in \mathfrak{X}^m(\mathcal{W}_r)$ be a holonomic multivector field which is a solution to the equation* (1.4) *for the unified formalism, at least on the points of a submanifold $\jmath_f \colon \mathcal{W}_f \subseteq \mathcal{W}_{\mathcal{L}} \hookrightarrow \mathcal{W}_r$, and tangent to $\mathcal{W}_f$. Then there exists a holonomic multivector field $\mathbf{X}_h \in \mathfrak{X}^m(\mathcal{P})$ which is a solution of the equation* (1.4) *for the Hamiltonian formalism, at least on the points of $P_f = \mathcal{FL}(S_f)$, and tangent to P_f.*

Conversely, if $\mathbf{X}_h \in \mathfrak{X}^m(\mathcal{P})$ is a holonomic multivector field which is a solution to the equation (1.4) *for the Hamiltonian formalism, at least on a submanifold $P_f \hookrightarrow \mathcal{P}$, and tangent to P_f; then there exist locally decomposable, ρ^r_M-transverse and integrable multivector fields $\mathbf{X} \in \mathfrak{X}^m(\mathcal{W}_r)$ which are solutions to the equation* (1.4) *for the unified formalism, at least on the points of $\mathcal{W}_f = (\rho^{\mathcal{L}}_2)^{-1}(P_f) \hookrightarrow \mathcal{W}_{\mathcal{L}} \hookrightarrow \mathcal{W}_r$, and tangent to $\mathcal{W}_f$.*

If $\mathbf{X}$ is $\rho^r_{\mathcal{P}}$-projectable (or, what is equivalent, if the multivector field $\mathbf{X}_{\mathcal{L}}$ in Theorem 1.2 is $\mathcal{FL}_o$-projectable), then the relation between these multivector fields is $\mathbf{X}_h \circ \rho^r_{\mathcal{P}} \circ \jmath_f = \Lambda^m \mathrm{T}\rho^r_{\mathcal{P}} \circ \mathbf{X} \circ \jmath_f$.

Chapter 2

Properties of Multisymplectic Systems

The most challenging step in the formalism presented before are equations (1.2), which are a system of PDE in field theories. Accordingly, there has been a lot of effort in finding properties which help solve and understand them. Common examples are symmetries, gauge freedom, conserved quantities or the study of equivalent Lagrangians.

In this chapter we review and extend some properties for multisymplectic systems that we need to study the models of Gravity. In section 2.1 we recover some results for symmetries and conserved quantities, and we present a new analysis of the gauge freedom of field theories. In section 2.2 we review the concept of projectable theories and explore the constraints of this kind of theories.

2.1 Symmetries, Conserved Quantities and Gauge freedom

2.1.1 The challenge in field theories

The concepts of symmetries, gauge freedom and conserved quantities have been developed mainly for mechanical systems and there has been partial success in field theories. During the research we realize that some of the elements, especially gauge symmetries, do not have a clear generalization to field theories. The fundamental problem is that the solution in mechanics are one dimensional distributions (associated to a vector field), while in field theories they are m-dimensional distributions (associated to multivector fields). This simplifies mechanical theories, but also leads to confusion about what is a general property of a physical system and what is a particularity of mechanical systems. This introduction aims to discuss this difficulties.

Roughly speaking, the symmetries of a system are diffeomorphisms which leave the space of solutions invariant. This definition and other related concepts, like the interpretation of infinitesimal symmetries as vector fields, are maintained in field theories. Remarkably, conserved quantities are modelled as $(m-1)$-forms. For mechanical systems $m=1$, thus conserved quantities are functions, which leads to physical magnitudes like energy. The interpretation of a conserved quantity in field theories is the conservation of a flux, the relevance of which was already identified in several physical theories. *Noether's theorem*, which relates infinitesimal symmetries and conserved quantities, also holds for field theories. Through these chapter we will see some of the reasons of its broad validity.

The generalization of the so-called *gauge symmetries* has been more complicated. One can find disparate uses of the term "gauge" in the physics literature. Moreover, some ideas and structures used in the geometrization of gauge symmetries in mechanical theories cannot be generalized to field theories. Commonly, the term "gauge" is used to indicate the presence of different mathematical solutions which correspond to the same physical state. In mechanical systems, the solutions are obtained from an ODE, which, under certain regularity conditions, have a unique solution provided an initial condition. On the other hand, field theories lead to PDE which, in general, also need a boundary condition to have a unique solution. As we discuss in section 2.1.4, a field theory has a kind of multiplicity which is not present in mechanical systems. Motivated by this idea, we have the procedure of *gauge fixing*. This idea has been geometrized for mechanical systems [56], where the key idea is to consider the quotient of the configuration bundle, a process called *gauge reduction*. This mechanism is interesting, but it cannot be generalized to field theories straightforwardly. For instance, in [5, 56] it is shown that the Lie parenthesis of a gauge vector field with the vector field solution of the system is also a gauge vector field. In field theories, the solutions are represented as multivector fields, thus the Lie parenthesis previously mentioned leads to incongruencies. The process of gauge reduction is also associated to the non-regularity of the form Ω.

In order to generalize the concept of "gauge" to field theories, we choose to interpret it as the non-regularity of the form Ω. First, it is easy to apply to field theories. Moreover it appears in a natural way in the description of the space of solution. Finally, it is compatible with the variational principle, as we argue in the following section. Furthermore, this approximation requires an additional analysis of the interaction with the condition set $\mathcal{I}$. It is enlightening to make a separate study of the effects of Ω and $\mathcal{I}$.

2.1.2 Variational Interpretation

We present a general formulation of variational problems at a conceptual level. We aim to show which rol have the different concepts presented before in the variational formalism. The association of the concept of gauge to the non-regularity of the form Ω is justified in the light of this variational interpretation. The precise definition of the elements will be given in the following sections in the framework of the multisymplectic formalism.

Consider the sets Ψ and V, which are called the set of (admissible) sections and the set of variations respectively. Next, consider the action, a map $S : V \times \Psi \to \mathbb{R}$. The triad (S, Ψ, V) represents the variational problem. Then, a section $\psi \in \Psi$ is a solution for the action S if

$$S(v, \psi) = 0, \quad \forall v \in V \,. \tag{2.1}$$

The set of solutions is denoted $Sol(S)$. In this simply general framework we can define several interesting objects:

- A *symmetry*, which is an application $\sigma : \Psi \to \Psi$, such that $\sigma(Sol(S)) \subset Sol(S)$.
- A *critical quantity*, which is an application $q : \Psi \to \mathbb{R}$ such that for all $\psi \in Sol(S)$, $q(\psi) = 0$. We define $\ker(q) := \{\psi \in \Psi | q(\psi) = 0\}$.
- A *gauge variation* (or *geometric gauge*), which is an element of $g \in V$ such that $S(g, \psi) = 0$, for every $\psi \in \Psi$.

These elements help us to find solutions in different ways. In equation (2.1), every element of V imposes a new condition on the sections. But the gauge variations only lead to identities, thus they do not impose any condition. Therefore, we can substract the gauge variations to the total set of variations without changing the solutions. Critical quantities, which are related to conserved quantities, although they are slightly different (see Section 2.1.5), help us to narrow the space of sections. Indeed, since $Sol(S) \subset \ker(q)$ for every critical quantity, we can replace Ψ by $\ker(q)$ without changing the solutions. Symmetries allows us to construct new solutions from previously found ones.

There are some relations between these elements which also appear in the multisymplectic formalism. Let σ, ρ be two symmetries, q a critical quantity, g a gauge variation and $v \in V$. Then:

1. $\sigma \circ \rho$ is a symmetry.
2. $q \circ \sigma$ is a critical quantity.
3. $S(v, \cdot)$ is a critical quantity.
4. $\ker(S(g, \cdot)) = \Psi$.

The third result is central in Noether's theorem, and it is one of the origins of the broad validity of the theorem. The fourth item shows that the critical quantities associated to gauge variations are trivial.

2.1.3 Symmetries

Here we consider multisymplectic systems as presented in chapter 1. The proofs of the results presented here can be found in [45].

Definition 2.1.

- *A* symmetry *of the system $(J, \Omega, V, \mathcal{I})$ is a diffeomorphism $\Phi : J \to J$ such that*

$$\Phi_*(\ker^m_H \Omega) \subset \ker^m_H \Omega \, .$$

  A pre-symmetry *of the system $(J, \Omega, V, \mathcal{I})$ is a diffeomorphism $\Phi : J \to J$ such that*

$$\Phi_*(\ker^m \Omega) \subset \ker^m \Omega \, .$$

- *An* infinitesimal symmetry (pre-symmetry) *of the system $(J, \Omega, V, \mathcal{I})$ is a vector field $Y \in \mathfrak{X}(J)$ whose local flows are local symmetries (pre-symmetries).*

The field equations are EDP's and symmetries are characterized because they transform solutions into solutions. In fact, the following assertion holds:

Theorem 2.1. *Let Φ be a symmetry of a Lagrangian system $(J, \Omega, V, \mathcal{I})$. Then, for every $\mathbf{X} \in \ker^m_H \Omega_{\mathcal{L}}$, the map Φ transforms integral submanifolds of $\mathbf{X}$ into integral submanifolds of $\Phi_*\mathbf{X} \in \ker^m_H \Omega_{\mathcal{L}}$.*

As a straighforward consequence of this result, we obtain that:

Theorem 2.2. *Let $Y \in \mathfrak{X}(J)$ be an infinitesimal symmetry of a Lagrangian system $(J, \Omega, V, \mathcal{I})$, and F_t a local flow of Y. Then, for every $\mathbf{X} \in \ker^m_H \Omega_{\mathcal{L}}$, the map F_t transforms integral submanifolds of $\mathbf{X}$ into integral submanifolds of $F_{t*}\mathbf{X} \in \ker^m_H \Omega_{\mathcal{L}}$.*

This definition of symmetry is not operational, as we need to know the set of solutions a priori. We are interested in sufficient conditions for a pre-symmetry to be a symmetry, at least for the models we study. Consider that the systems $(J, \Omega, V, \mathcal{I})$ is almost-holonomic, with projection $\rho : J \to J^k\pi$ for some $k > 0$, (in particular, $\rho = Id_{J^k\pi}$ if $J = J^k\pi$).

Definition 2.2.

- *Consider a diffeomorphism $\phi : E \to E$, which is π-related to a diffeomorphism $\varphi : M \to M$. Then, the* canonical lift *to $J^k\pi$ is*

$$\begin{array}{rcl} j^k\phi : J^k\pi & \to & J^k\pi \\ j^k_x s & \mapsto & j^k\phi(j^k_x s) \equiv j^k_{\varphi(x)}(\phi \circ s \circ \varphi^{-1}) \, . \end{array}$$

- *A diffeomorphism $\Phi : J^k\pi \to J^k\pi$ is* natural *if it is the the canonical lift of a differormorphism $\phi\colon E \to E$, that is, $\Phi = j^k\phi$. A diffeomorphism $\Phi' : J \to J$ is* natural *if it is ρ-related to a natural diffeomorphism.*

- *A vector field is* natural *if its local flows are natural.*

The following diagram summarizes the diffeomorphisms of the definitions:

$$\begin{tikzcd}
J \arrow[r, "\rho"] \arrow[d, "\Phi'"'] & J^k\pi \arrow[r, "\pi^k"] \arrow[d, "\Phi"'] & E \arrow[r, "\pi"] \arrow[d, "\phi"'] & M \arrow[d, "\varphi"'] \\
J \arrow[r, "\rho"'] & J^k\pi \arrow[r, "\pi^k"'] & E \arrow[r, "\pi"'] & M
\end{tikzcd}$$

For a diffeomorphism $\phi : E \to E$, which is π-related to φ, and whose local expression is $\phi(x^\mu, u_\alpha) = (\varphi^\mu(x^\mu), \phi_\alpha(x^\mu, u_\alpha)$, its canonical lift has the following local expression

$$j^k\phi(x^\mu, u_\alpha, u_{\alpha,I}) = (\varphi^\mu(x^\mu), \phi_\alpha(x^\mu, u_\alpha), v_{\alpha,I}(x^\mu, u_\alpha, u_{\alpha,I})),$$

where

$$v_{\alpha,i} = D_j\phi_\alpha \cdot \left(\frac{\partial \varphi^{-1j}}{\partial x^i} \circ \varphi\right) \ , \quad v_{\alpha,I+1_i} = D_j v_{\alpha,I} \cdot \left(\frac{\partial \varphi^{-1j}}{\partial x^i} \circ \varphi\right) \ , \quad 1 \leq |I| \leq k-1 \ .$$

Proposition 2.1. *If $\Phi : J^k\pi \to J^k\pi$ is natural, then $\Phi^*\mathfrak{C}^k \subset \mathfrak{C}^k$.*

Proof. $\mathfrak{C}^k$ is locally generated by the coordinate contact forms $\theta_{\alpha,I} = \mathrm{d}u_{\alpha,I} - u_{\alpha,I+1_i}\mathrm{d}x^i$, for $0 \leq |I| \leq k-1$. Then:

$$\begin{aligned}
\Phi^*(\mathrm{d}u_{\alpha,I} - u_{\alpha,I+1_i}\mathrm{d}x^i) &= \mathrm{d}v_{\alpha,I} - v_{\alpha,I+1_i}\frac{\partial \varphi^i}{\partial x^j}\mathrm{d}x^j = \mathrm{d}v_{\alpha,I} - D_i v_{\alpha,I}\mathrm{d}x^i \\
&= \frac{\partial v_{\alpha,I}}{\partial x^i}\mathrm{d}x^i + \sum_{|J|=0}^{k-1}\frac{\partial v_{\alpha,I}}{\partial u_{\beta,J}}\mathrm{d}u_{\beta,J} - \left(\frac{\partial v_{\alpha,I}}{\partial x^i} + \sum_{|J|=0}^{k-1}\frac{\partial v_{\alpha,I}}{\partial u_{\beta,J}}u_{\beta,J+1_i}\right)\mathrm{d}x^i \\
&= \sum_{|J|=0}^{k-1}\left(\frac{\partial v_{\alpha,I}}{\partial u_{\beta,J}}\mathrm{d}u_{\beta,J} - \frac{\partial v_{\alpha,I}}{\partial u_{\beta,J}}u_{\beta,J+1_i}\mathrm{d}x^i\right) = \sum_{|J|=0}^{k-1}\frac{\partial v_{\alpha,I}}{\partial u_{\beta,J}}\theta_{\beta,J} \ .
\end{aligned}$$

□

Proposition 2.2. *In almost-holonomic systems, if a pre-symmetry is natural, then it is a symmetry.*

Proof. Given a pre-symmetry Φ', which is ρ-related to a natural diffeomorphism Φ, we only need to check that $(\Phi' \circ \psi)^*(\rho^*\mathfrak{C}^k) = \{0\}$ for every section ψ solution to (1.2). Indeed:

$$(\Phi' \circ \psi)^*(\rho^*\mathfrak{C}^k) = \psi^*((\rho \circ \Phi')^*\mathfrak{C}^k) = \psi^*((\Phi \circ \rho)^*\mathfrak{C}^k) \subset \psi^*(\rho^*\mathfrak{C}^k) = \{0\} \ ,$$

because Φ is natural and ψ is a solution, thus it is holonomic. □

Therefore, to be natural is a sufficient condition for a pre-symmetry to be a symmetry, but it is not necessary. A relevant kind of symmetries are the following:

Definition 2.3.

1. *A* Cartan *or* Noether pre-symmetry *of a system $(J, \Omega, V, \mathcal{I})$ is a diffeomorphism $\Phi \colon J \to J$ such that, $\Phi^*\Omega = \Omega$. If, in addition, $\Phi^*\Theta = \Theta$, then Φ is said to be an* exact Cartan pre-symmetry.
2. *An* infinitesimal Cartan *or* Noether pre-symmetry *of a system $(J, \Omega, V, \mathcal{I})$ is a vector field $Y \in \mathfrak{X}(J)$ satisfying that $\mathrm{L}(Y)\Omega = 0$. If, in addition, $\mathrm{L}(Y)\Theta = 0$, then Φ is said to be an* infinitesimal exact Cartan pre-symmetry.

Proposition 2.3. *Every Cartan pre-symmetry is a pre-symmetry and, as a consequence, every infinitesimal Cartan pre-symmetry is an infinitesimal pre-symmetry.*

It is well known that canonical liftings of diffeomorphisms and vector fields preserve the canonical structures of $J^k\pi$. Nevertheless, the (pre)multisymplectic form $\Omega_{\mathcal{L}}$ of the Lagrangian formalism is not canonical, since it depends on the choice of the Lagrangian density $\mathcal{L}$, and then it is not invariant by these canonical liftings. Thus, given a diffeomorphism $\Phi\colon J^k\pi \to J^k\pi$ or a vector field $Y \in \mathfrak{X}(J^k\pi)$, a sufficient condition to assure this invariance would be to demand that Φ or Y leave the canonical structures of the jet bundle $J^k\pi$ (for instance, Φ and Y being the canonical lifting of a diffeomorphism and a vector field in E), and that the Lagrangian density $\mathcal{L}$ be also invariant. In this way, $\Omega_{\mathcal{L}}$ and hence the Euler-Lagrange equations are invariant by Φ or Y. Therefore, a particular kind of symmetries are those which are *symmetries of the Lagrangian density*. Although those are symmetries particular to Lagrangian systems, it is possible to define them for the unified formalism (see [47] for more details).

Definition 2.4. *Let $(J, \Omega, V, \mathcal{I})$ be a Lagrangian or a unified system, and denote the projection $\rho : J \to J^k\pi$, with $\rho = id_{J^k\pi}$ in the Lagrangian case.*

1. *A* Lagrangian symmetry *of the Lagrangian or unified system is a diffeomorphism $\Phi\colon J \to J$ such that*
   - *(a) Φ leaves the canonical geometric structures of $J^k\pi$ invariant.*
   - *(b) $\Phi^*(\rho^*\mathcal{L}) = \rho^*\mathcal{L}$ (Φ leaves $\mathcal{L}$ invariant).*

   As a particular case, a natural Lagrangian symmetry *of the Lagrangian or unified system is a diffeomorphism $\Phi\colon J \to J$ such that:*
   - *(a) Φ is natural.*
   - *(b) Φ leaves $\mathcal{L}$ invariant.*
2. *An* infinitesimal Lagrangian symmetry *of the Lagrangian or unified system is a vector field $Y \in \mathfrak{X}(J)$ such that:*
   - *(a) The canonical geometric structures of $J^k\pi$ are invariant under the action of Y.*
   - *(b) $\mathrm{L}(Y)\rho^*\mathcal{L} = 0$ (Y leaves $\mathcal{L}$ invariant).*

   As a particular case, an infinitesimal natural Lagrangian symmetry *of the Lagrangian or unified system is a vector field $Y \in \mathfrak{X}(J)$ such that:*
   - *(a) Y is natural.*
   - *(b) Y leaves $\mathcal{L}$ invariant.*

As the canonical lifting of diffeomorphisms and vector fields from E to $J^k\pi$ leave the canonical structures of $J^k\pi$ invariant, it is evident that every (infinitesimal) natural Lagrangian symmetry is also a (infinitesimal) Lagrangian symmetry. In any case, as a direct consequence of these definitions we have:

Proposition 2.4. *Let $(J, \Omega, V, \mathcal{I})$ be a Lagrangian system.*

1. *If $\Phi\colon J^k\pi \to J^k\pi$ is a Lagrangian symmetry of the Lagrangian system, then $\Phi^*\Theta_{\mathcal{L}} = \Theta_{\mathcal{L}}$, and hence it is an exact Cartan symmetry.*
2. *If $Y \in \mathfrak{X}(J^k\pi)$ is an infinitesimal Lagrangian symmetry of the Lagrangian system, then $\mathrm{L}(Y)\Theta_{\mathcal{L}} = 0$, and hence it is an infinitesimal exact Cartan symmetry.*

However, to demand the invariance of $\mathcal{L}$ is a strong condition, since there are Lagrangian densities or, what is equivalent, Lagrangian functions that, being different and even of different order, give rise to the same Euler-Lagrange equations. In mechanics these are the so-called *gauge equivalent Lagrangians*.

2.1.4 Gauge vector fields

The standard use of the term *gauge* in Physics is for describing certain kinds of symmetries which arise as a consequence of the non-regularity of the system and lead to the existence of states that are physically equivalent. This characteristic is known as *gauge freedom*. As we discussed before, there are more interesting properties associated to the gauge symmetries. We will start defining the gauge vector fields, inspired by the variational reasoning, and then we will present what are the extra conditions necessary to recover these properties in field theories. Consider a system $(J_f, \Omega, V, \mathcal{I})$ defined on the submanifold $j : J_f \hookrightarrow J$ with projection $\bar{\pi}_f : J_f \to M$ and denote

$$TJ_f^{\perp} := \bigcup_{p \in J_f} \{v \in V_p \mid \, i(v)\, i(\mathbf{X})\Omega|_p = 0 \quad , \quad \forall \mathbf{X} \in \wedge^m(T_pJ_f)\} \ .$$

Definition 2.5.

- *The* weak gauge distribution *of the system* $(J_f, \Omega, V, \mathcal{I})$ *is*

  $$G_w \equiv \bigcup_{p \in J_f} \{v \in V_p | \, \psi^* \, i(v)\Omega|_{\psi^{-1}(p)} = 0 \ , \ \textit{for all admisible } \psi \in \Gamma(\bar{\pi}_f) \textit{ such that } p \in Im(\psi) \ \}.$$

  A weak gauge vector field *of the system* $(J_f, \Omega, V, \mathcal{I})$ *is a vector field* $Y \in \Gamma(V)$ *such that*

  $$\psi^*(i(Y)\Omega) = 0 \ , \quad \textit{for all admisible } \psi \in \Gamma(\bar{\pi}_f) \ .$$

- *The* strong gauge distribution *of the system* $(J_f, \Omega, V, \mathcal{I})$ *is* $G_s \equiv Ker(T\bar{\pi}_f) \cap TJ_f^{\perp}$. *A* strong gauge vector field *of the system* $(J_f, \Omega, V, \mathcal{I})$ *is a* $\bar{\pi}_f$*-vertical vector field* $Y \in \Gamma(TJ_f^{\perp})$.

- *A* gauge symmetry *of the system* $(J_f, \Omega, V, \mathcal{I})$ *is a strong gauge vector field tangent to* J_f *which is also a symmetry. In particular, if it is natural it is called a* natural gauge symmetry.

Lemma 2.1. *If* Ω *is closed, then* G_w *and* G_s *are involutive distributions on* J_f*.*

Proof. It is a consequence of V being involutive and the properties of the Lie derivative. □

The gauge reduction procedure is centered on quoting the system using gauge distributions. The stronger the conditions, the better behaved the final system is, and narrower is its application.

Consider an involutive subdistribution of the weak gauge distribution $G_w' \subset G_w$ and assume that the quotient manifold $\xi : J_f \to J_f/G_w'$ is well defined. Given a section (gauge fixing) of ξ, we can define $\Omega_\beta \equiv \beta^*\Omega$, but it depends on the section because, in general, Ω is not constant in $\xi^{-1}(p)$ for some $p \in J_f/G_w'$. Nevertheless, they are relevant when analyzing the condition set and the conserved quantities. For instance, the difficulties on defining the Hamilton-Cartan form for the Hamiltonian formalism for higher-order field theories is a consequence that the symetrization of the momenta leads only to weak gauge vector fields.

Consider an involutive subdistribution of the strong gauge distribution $G_s' \subset G_s$ and assume that the quotient manifold $\xi : J_f \to J_f/G_s'$ is well defined. Now, the form can be defined univocally on the quotient space, and we will denote it by Ω_s. Since the strong gauge variations are $\bar{\pi}_f$-vertical, we assure that the base manifold M does not contain gauge equivalent points and then all the gauge degrees of freedom are in the fibres. Therefore, after doing the reduction procedure in order to remove the gauge multiplicity, the base manifold M remains unchanged. Moreover, there exists a projection $\bar{\pi}_\xi : J_f/G_s' \to M$ such that $\bar{\pi}_\xi \circ \xi = \bar{\pi}_f$. Unfortunately, following the same reasoning, the condition set cannot be defined univocally in J_f/G_s', in general. Regardless, consider the system $(J_f/G_s', \Omega_s, V_s \equiv \xi_*V, \{0\})$. Notice that Ω_s is an m-form and the dimension of M is also m. Then

Proposition 2.5. *If the section ψ is a solution of the system $(J_f, \Omega, V, \mathcal{I})$, then $\xi \circ \psi$ is a solution of the system $(J_f/G'_s, \Omega_s, V_s, \{0\})$*

Proof. Consider the section $\beta : J_f/G'_s \to J_f$ such that $\beta \circ \xi \circ \psi = \psi$, (it exists at least locally because $G'_s \subset Ker(T\bar{\pi}_f)$). For every $Y \in \Gamma(V_s)$, we have that $\beta_* Y \in \Gamma(V)$; then:

$$(\xi \circ \psi)^* \, i(Y)\Omega_s = (\xi \circ \psi)^* \, i(Y)\beta^*\Omega = (\beta \circ \xi \circ \psi)^* \, i(\beta_* Y)\Omega = \psi^* \, i(\beta_* Y)\Omega = 0 \, .$$

□

The converse is not true in general. That is, for a solution ψ' of the quotient system and for an arbitrary section β, $\beta \circ \psi'$ is a solution only if it is admissible. Nevertheless, the strong gauge vector fields are interesting as they encode the non-regularity of Ω and are easy to compute. We use them to study the projectability of the systems.

As it is well-known, a regular mechanical system has a unique solution (as a vector field). Conversely, if there are multiply solutions, they are gauge related. Thus, the multiplicity of solutions in mechanical systems is related to the non-regularity of the theory. This close relation between multiplicity of solutions, gauge vector fields and non-regularity is not present in field theories. For instance, consider the regular Lagrangian $L = u_x^2 + u_y^2$ in the bundle $\pi : E \to M$, where the coordinates in M are (x, y) and (x, y, u) in E. The corresponding field equation is the Laplace's equation, which has not a unique solution given an initial condition. Clearly, in field theories there are sources of multiplicity of solutions which are not related to the non-regularity of the theory. In the approach that we have presented here, the gauge vector fields are related to the multiplicity of solutions caused by the non-regularity of Ω.

2.1.5 Conserved Quantities

(See [45] and [32] for the proofs of all the results in this section).

Definition 2.6.
- *A* critical quantity *of the system $(J, \Omega, V, \mathcal{I})$ is an m-form $\beta \in \Omega^m(J)$ such that $i(\mathbf{X})\beta = 0$ for every solution $\mathbf{X} \in \ker^m_H(\Omega)$.*
- *If a critical quantity is exact, there exists an $(m-1)$-form $\xi \in \Omega^{m-1}$ such that $\beta = \mathrm{d}\xi$ and ξ is called a* conserved quantity. *Equivalently, $\xi \in \Omega^{m-1}(J)$ is a* conserved quantity *if $\mathrm{L}(\mathbf{X})\xi := (-1)^{m+1}\, i(\mathbf{X})\mathrm{d}\xi = 0$ for every solution $\mathbf{X} \in \ker^m_H(\Omega)$.*

Proposition 2.6.
- *If $\beta \in \Omega(J)$ is a conserved quantity of the system $(J, \Omega, V, \mathcal{I})$ and $\mathbf{X} \in \ker^m_H \Omega$, then β vanishes on the integral submanifolds of $\mathbf{X}$; that is, if $j_M : M \hookrightarrow J$ is an integral submanifold, then $j^*_M \beta = 0$.*
- *If $\xi \in \Omega^{m-1}(\mathcal{M})$ is a conserved quantity of the system $(J, \Omega, V, \mathcal{I})$ and $\mathbf{X} \in \ker^m_H \Omega$, then ξ is closed on the integral submanifolds of $\mathbf{X}$; that is,if $j_M : M \hookrightarrow J$ is an integral submanifold, then $\mathrm{d}j^*_M \xi = 0$.*

As it was pointed out in 2.1.2, the variations generate critical quantities.

Proposition 2.7. *For every $X \in \Gamma(V)$, $i(X)\Omega$ is a critical quantity.*

Proof. It is straightforward from the definitions of critical quantity and solution to field equations (1.4).

□

We call any critical quantity obtained in this way a *Noether critical quantity*. If $i(X)\Omega$ is exact X is a Hamiltonian vector field, and if $i(X)\Omega$ is closed X is a locally Hamiltonian vector field. We can establish a one-to-one relation between Noether conserved quantities and vector fields modulo strong gauge vector fields.

Lemma 2.2. • *X is a Cartan pre-symmetry if, and only if, $i(X)\Omega$ is closed. Therefore, X is a locally Hamiltonian vector field .*

- *X is an exact Cartan pre-symmetry if, and only if, $i(X)\Omega = \mathrm{d}(-\,i(x)\Theta)$. Therefore, X is a Hamiltonian vector field .*

- *X is a gauge strong vector field if, and only if, $i(X)\Omega = 0$. Therefore, X is a Hamiltonian vector field.*

The critical sections associated with strong gauge vector fields are trivial. These relations, particularized to symmetries and conserved quantities, is the Noether theorem [45]:

Theorem 2.3. (Noether): *Let $Y \in \mathfrak{X}(J)$ be an infinitesimal Cartan symmetry of a Lagrangian system $(J, \Omega, V, \mathcal{I})$, with $i(Y)\Omega = \mathrm{d}\xi_Y$. Then, for every $\mathbf{X} \in \ker^m_H \Omega$ (and hence for every $\mathbf{X} \in \ker^m_H \Omega$), we have that*

$$\mathrm{L}(\mathbf{X})\xi_Y = 0\,;$$

that is, any Hamiltonian $(m-1)$-form ξ_Y associated with Y is a conserved quantity (and, for every integral submanifold ψ of $\mathbf{X}$, the form $\psi^*\xi_Y$, is usually called a Noether current*, in this context).*

And, as a particular case, we have:

Proposition 2.8. *Let $Y \in \mathfrak{X}(J)$ be an infinitesimal Cartan symmetry of a Lagrangian system $(J, \Omega, V, \mathcal{I})$. Then:*

1. *$\mathrm{L}(Y)\Theta_{\mathcal{L}}$ is a closed form, hence, in an open set $U \subset J$, there exist $\zeta_Y \in \Omega^{m-1}(U)$ such that $\mathrm{L}(Y)\Theta_{\mathcal{L}} = \mathrm{d}\zeta_Y$.*

2. *If $i(Y)\Omega_{\mathcal{L}} = \mathrm{d}\xi_Y$, in an open set $U \subset J$, then*

   $$\mathrm{L}(Y)\Theta_{\mathcal{L}} = \mathrm{d}(i(Y)\Theta_{\mathcal{L}} - \xi_Y) = \mathrm{d}\zeta_Y \quad (\text{in } U)\,,$$

   and hence $\xi_Y = i(Y)\Theta_{\mathcal{L}} - \zeta_Y$ (up to a closed $(m-1)$-form).

   As a particular case, if Y is an exact infinitesimal Cartan symmetry, we can take $\xi_Y = i(Y)\Theta_{\mathcal{L}}$.

A conserved quantity can be interpreted as the conservation of a flux. Given $\xi \in \Omega^{m-1}(J)$ and $\mathbf{X} \in \mathfrak{X}^m(J)$, for every integral submanifold $\psi\colon M \to J$ of $\mathbf{X}$, we can construct the so-called *form of flux* associated with the vector field $X_{\psi^*\xi}$ wich is $\psi^*\xi \in \Omega^{m-1}(M)$. Thus we have a unique $X_{\psi^*\xi} \in \mathfrak{X}(M)$ such that $i(X_{\psi^*\xi})\eta = \psi^*\xi$ and, if $\mathrm{div}X_{\psi^*\xi}$ denotes the divergence of $X_{\psi^*\xi}$, we have that $(\mathrm{div}X_{\psi^*\xi})\,\eta = \mathrm{d}\psi^*\xi$. Then, as a consequence of Proposition 2.6, ξ is a conserved quantity if, and only if, $\mathrm{div}X_{\psi^*\xi} = 0$, and hence, by *Stokes theorem*, in every bounded domain $U \subset M$,

$$\int_{\partial U} \psi^*\xi = \int_U (\mathrm{div}X_{\psi^*\xi})\,\eta = \int_U \mathrm{d}\psi^*\xi = 0\,.$$

The form $\psi^*\xi$ is called the *current* associated with the conserved quantity ξ, and this result allows to associate a *conservation law* in M to every conserved quantity in J.

2.2 Order reduction

2.2.1 Order reduction and projectability of the Poincaré-Cartan form

There are some models in classical field theories where, as a consequence of the singularity of the Lagrangian, the order of the Euler-Lagrange equations is lower than expected. A geometrical way of understanding this problem is considering the projectability of the higher-order Poincaré-Cartan form onto lower-order jet bundles [16, 86, 87]. We review the conditions for this projectability and study their consequences using the constraint algorithm for the field equations of second order (singular) field theories, thus enlarging the results stated in previous papers [16, 44, 73, 86, 87].

We analyse the Lagrangian formalism, thus consider the jet $J^k\pi$ of the fiber bundle $\pi\colon E \to M$ over an m-dimensional manifold M and with $\dim E = m + n$. See chapter 1 for more details.

Remember that a form $\omega \in \Omega^s(E)$ is said to be π-*semibasic* if $i(X)\omega = 0$, and π-*basic* or π-*projectable* if $i(X)\omega = 0$ and $\mathrm{L}(X)\omega = 0$, for every π-vertical vector field $X \in \mathfrak{X}^V(\pi)$. As a consequence of Cartan's formula, $\mathrm{L}(X)\omega = i(X)\mathrm{d}\omega + \mathrm{d}\, i(X)\omega$, a form $\omega \in \Omega^n(E)$ is π-basic if, and only if, ω and $\mathrm{d}\omega$ are π-semibasic.

Recall the coordinate total derivatives [83, 90] (or Section 1.1):

$$D_\mu = \frac{\partial}{\partial x^\mu} + \sum_{|I|=0}^{k} u^\alpha_{I+1_\mu} \frac{\partial}{\partial u_{\alpha,I}} \,. \tag{2.2}$$

For every function f, we have $D_\mu f := \mathrm{L}(D_\mu)f$. In addition, we have:

- If $X \in \mathfrak{X}^V(\pi^k_s)$, then $[D_\mu, X] \in \mathfrak{X}^V(\pi^k_{s-1})$.
- For $f \in C^\infty(J^k\pi)$, if f is π^k_s-basic then $D_\mu f$ is π^k_{s+1}-basic.

We show some consequences of the projectability of the Poincaré-Cartan form for second order Lagrangian classical field theories. The *Lagrangian form* that describes the theory is a $\bar{\pi}^2$-semibasic m-form $\mathcal{L} = L\,(\bar{\pi}^2)^*\omega \in \Omega^m(J^2\pi)$, where $L \in \mathrm{C}^\infty(J^2\pi)$ is the Lagrangian function, ω is the volume form in M, and $\bar{\pi}^2\colon J^2\pi \to M$. Natural coordinates of $J^3\pi$ adapted to the fibration are $(x^\mu, u_\alpha, u_{\alpha,\mu}, u_{\alpha,I}, u_{\alpha,J})$, such that $\omega = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x$; $1 \le \mu \le m$, $1 \le \alpha \le n$, and I, J are multiindices with $|I| = 2$, $|J| = 3$, [90].

The Poincaré-Cartan m-form $\Theta_{\mathcal{L}} \in \Omega^m(J^3\pi)$ is locally given by

$$\Theta_{\mathcal{L}} = L^{\alpha,\mu}\mathrm{d}u_\alpha \wedge \mathrm{d}^{m-1}x_\mu + L^{\alpha,\mu\nu}\mathrm{d}u_{\alpha,\mu} \wedge \mathrm{d}^{m-1}x_\nu + \left(L - L^{\alpha,\mu}u_{\alpha,\mu} - L^{\alpha,\mu\nu}u_{\alpha,1_\mu+1_\nu}\right)\mathrm{d}^m x \,,$$

where $\mathrm{d}^{m-1}x_\nu = i\left(\dfrac{\partial}{\partial x^\nu}\right)\mathrm{d}^m x$ and the functions $L^{\alpha,\mu}, L^{\alpha,\mu\nu} \in C^\infty(J^3\pi)$ are

$$L^{\alpha,\mu} = \frac{\partial L}{\partial u_{\alpha,\mu}} - D_\nu L^{\alpha,\mu\nu} \quad ; \quad L^{\alpha,\mu\nu} = \frac{1}{n(\mu\nu)}\frac{\partial L}{\partial u_{\alpha,1_\mu+1_\nu}} \,.$$

Lemma 2.3. *For $s = 1, 2$, the following conditions are equivalent:*

1. *$\Theta_{\mathcal{L}}$ projects onto $J^s\pi$.*
2. *$\mathrm{d}\Theta_{\mathcal{L}}$ is π^3_s-semibasic.*
3. *$\mathrm{L}(X)L^{\alpha,\mu} = 0$ and $\mathrm{L}(X)L^{\alpha,\mu\nu} = 0$; for every $X \in \mathfrak{X}^V(\pi^3_s)$.*

(*Proof*): $(1 \Leftrightarrow 2)$ is a consequence of Cartan's formula.

For $(2 \Leftrightarrow 3)$, in the case $s=2$, we compute the condition 2 in coordinates. It turns to be equivalent to

$$\frac{\partial L^{\alpha,\mu}}{\partial u_{\beta,J}} = 0 \quad , \quad \frac{\partial L^{\alpha,\mu\nu}}{\partial u_{\beta,J}} = 0 \quad , \quad \frac{\partial}{\partial u_{\beta,J}}(L - L^{\alpha,\mu}u_{\alpha,\mu} - L^{\alpha,\mu\nu}u_{\alpha,1_\mu+1_\nu}) = 0 \quad ;$$

(for $|J|=3$, and for every β, α, μ and ν). The last equation is a consequence of the other two (because L does not depend on $u_{\beta,J}$); which are locally equivalent to 3, since $\left\{\frac{\partial}{\partial u_{\beta,J}}\right\}$ generates $\mathfrak{X}^V(\pi^3_s)$. The case $s=1$ can be proved in a similar way. □

Other important results concerning to this topic (that we present here for completeness) are the following [86]:

Proposition 2.9. *If $\Theta_{\mathcal{L}}$ projects onto $J^s\pi$, then the order of the Euler-Lagrange equations is at most $s+1$.*

Proposition 2.10. *If there exist $\mathcal{L}' \in \Omega^m(J^1\pi)$ such that $\Theta_{\mathcal{L}} = (\pi^3_1)^*\Theta_{\mathcal{L}'}$, then $\mathcal{L} = (\pi^3_1)^*\mathcal{L}'$.*

Concerning to the last proposition, the study of the existence of an equivalent lower order Lagrangian $\mathcal{L}' \in \Omega^m(J^1\pi)$ has been analysed in [16, 87].

If the Poincaré-Cartan form $\Theta_{\mathcal{L}}$ projects onto a lower-order jet bundle, it is associated to a highly degenerate Lagrangian (this is just a consequence of the third item in Lemma 2.3). As a consequence of this fact, the field equations could not have admissible solutions everywhere in $J^3\pi$, but in some submanifold of it which can be obtained after applying a suitable constraint algorithm (see, for instance, [22]).

Theorem 2.4. *If $\Theta_{\mathcal{L}}$ projects onto $J^s\pi$, then solutions to the corresponding Euler-Lagrange equations* (1.4) *only exist in the points of a submanifold $\mathcal{S} \hookrightarrow J^3\pi$, where $\mathcal{S}$ is locally defined by the constraint functions given by*

- *$L^\alpha = 0$; if $s=1$.*
- *$L^\alpha = 0$ and $D_\mu L_\alpha = 0$; if $s=2$.*

Where $L^\alpha = \dfrac{\partial L}{\partial u_\alpha} - D_\mu L^{\alpha,\mu} = \dfrac{\partial L}{\partial u_\alpha} - D_\mu \dfrac{\partial L}{\partial u_{\alpha,\mu}} + D_I \dfrac{\partial L}{\partial u_{\alpha,I}}$.

(*Proof*): $\mathbf{X}$ can be written in coordinates as

$$\mathbf{X} = f \bigwedge_{\mu=1}^{m} \left(D_\mu + (F_{\alpha,J,\mu} - u_{\alpha,J+1_\mu}) \frac{\partial}{\partial u_{\alpha,J}} \right) = f \bigwedge_{\mu=1}^{m} X_\mu \; ;$$

for $f, F_{\alpha,J,i} \in C^\infty(J^3\pi)$, ($|J|=3$). Using this expression, equation (1.4) reduces to

$$L^\alpha + (F_{\beta,J,\mu} - u_{\beta,J+1_\mu}) \frac{\partial L^{\alpha,\mu}}{\partial u_{\beta,J}} = 0 \, , \tag{2.3}$$

which are the Euler-Lagrange equations for multivector fields. If $\Theta_{\mathcal{L}}$ projects either onto $J^1\pi$ or $J^2\pi$, by Lemma 2.3 we have $\dfrac{\partial L^{\alpha,\mu}}{\partial u_{\beta,J}} = 0$, and then from (2.3) we get $L^\alpha = 0$. Observe that, as a consequence, we cannot compute any of the functions $F_{\alpha,J,\mu}$. Actually $L^\alpha = 0$ are restrictions for the points of the manifold $J^3\pi$, which we assume that define a submanifold $\mathcal{S}_1 \subset J^3\pi$, where the equation (1.4) have solutions. In order to find $F_{\alpha,J,\mu}$ we use the constraint algorithm (as it is outlined, for instance, in [83]). So we look for the points of $\mathcal{S}_1$ where the multivector fields which are solutions to (1.4) (on $\mathcal{S}_1$) are tangent to $\mathcal{S}_1$. Thus, imposing this *consistency* or *tangency condition* we get

$$0 = \mathrm{L}(X_\mu)L^\alpha = D_\mu L^\alpha + \left(F_{\beta,J,\mu} - u_{\beta,J+1_\mu}\right)\frac{\partial L^\alpha}{\partial u_{\beta,J}} \qquad (\text{on } \mathcal{S}_1)\ .$$

If $\Theta_{\mathcal{L}}$ projects onto $J^1\pi$, then the associated Euler-Lagrange equations are of order at most 2 (by proposition 2.9). This implies that L^α, which are the Euler-Lagrange equations before being evaluatedon sections, are π^3_2-projectable. Thus, $\dfrac{\partial L^\alpha}{\partial u_{\beta,J}} = 0$, and we find new restrictions, $D_\mu L^\alpha = 0$ which are assumed to define a new submanifold $\mathcal{S}_2 \subset \mathcal{S}_1 \subset J^3\pi$ where the solutions to (1.4) are tangent to $\mathcal{S}_1$. □

Notice that, depending on the Lagrangian, we may need to continue the constraint algorithm, so obtaining that

$$D_\nu D_\mu L^\alpha + \left(F_{\beta,J,\nu} - u_{\beta,J+1_\nu}\right)\frac{\partial D_\mu L^\alpha}{\partial u_{\beta,J}} = 0 \qquad (\text{on } \mathcal{S}_2) \quad .$$

This process continues until the new conditions hold identically and we find a *final constraint submanifold* $\mathcal{S}_f$ of $J^3\pi$ where solutions to (1.4) are tangent to $\mathcal{S}_f$.

2.2.2 Application to higher-order mechanics

Now, consider the particular case where $\pi\colon E \to \mathbb{R}$, with $\dim E = n+1$, is the configuration bundle of a higher-order non-autonomous theory. For a theory of order k, we need to use jet bundle up to $J^{2k}\pi$. The (only) total time derivative is

$$D_t = \frac{\partial}{\partial t} + \sum_{i=0}^{k} q^\alpha_{i+1}\frac{\partial}{\partial q^\alpha_i}\ ,$$

which verifies the properties stated in Section 2.2.1. The dynamics is given by a *Lagrangian form* $\mathcal{L} \in \Omega^1(J^k\pi)$, which is a $\bar\pi^k$-semibasic 1-form and it has associated the Lagrangian function $L \in \mathrm{C}^\infty(J^k\pi)$, such that $\mathcal{L} = L\,(\bar\pi^k)^*\mathrm{d}t$, where $\mathrm{d}t$ is the canonical volume form in $\mathbb{R}$ [21]. The Poincaré-Cartan 1-form $\Theta_{\mathcal{L}} \in \Omega^1(J^{2k-1}\pi)$ is given locally by:

$$\Theta_{\mathcal{L}} = \sum_{r=1}^{k} L^r_\alpha \mathrm{d}q^\alpha_{r-1} + \left(L - \sum_{r=1}^{k} L^r_\alpha q^\alpha_r\right)\mathrm{d}t\ ,$$

where the functions $L^r_\alpha \in C^\infty(J^{2k-1}\pi)$ are

$$L^r_\alpha = \sum_{i=0}^{k-r}(-1)^i D^i_t\left(\frac{\partial L}{\partial q^\alpha_{r+i}}\right)\ ,$$

and they can be obtained inductively by setting $L^r_\alpha = 0$, for $r > k$, and

$$L^r_\alpha = \frac{\partial L}{\partial q^\alpha_r} - D_t L^{r+1}_\alpha\ . \tag{2.4}$$

Notice that L^0_α, when evaluated on a section, are the Euler-Lagrange equations. The properties stated in Lemma 2.3 and Propositions 2.9 and 2.10 read:

Lemma 2.4. *For $s \geq k-1$, the following conditions are equivalent:*

1. *$\Theta_{\mathcal{L}}$ projects onto $J^s\pi$.*

2. $\mathrm{d}\Theta_{\mathcal{L}}$ *is* π_s^{2k-1}*-semibasic.*

3. $\mathrm{L}(X)L_\alpha^r = 0$*; for every* $X \in \mathfrak{X}^V(\pi_s^{2k-1})$*, and for* $r = 1, \ldots, k$*,* $\alpha = 1, \ldots, n$*.*

(*Proof*): $(1 \Leftrightarrow 2)$ is a consequence of Cartan's formula. For the equivalence between 2 and 3 we consider two cases:

- If $s \geq k$: The relevant terms of $\mathrm{d}\Theta_{\mathcal{L}}$ are of the form:

$$\frac{\partial L_\alpha^i}{\partial q_r^\beta}\mathrm{d}q_r^\beta \wedge \mathrm{d}q_{i-1}^\alpha \quad , \quad \frac{\partial}{\partial q_r^\beta}\left(L - \sum_{i=1}^{k} L_\alpha^i q_i^\alpha\right)\mathrm{d}q_r^\beta \wedge \mathrm{d}t \quad ; \quad s < r \leq 2k-1 \, .$$

Then, $\mathrm{d}\Theta_{\mathcal{L}}$ is π_s^{2k-1}-semibasic if, and only if, $\dfrac{\partial L_\alpha^i}{\partial q_r^\beta} = 0$, and this is equivalent to $\mathrm{L}(X)L_\alpha^r = 0$, for every $X \in \mathfrak{X}^V(\pi_s^{2k-1})$, since $\left\{\dfrac{\partial}{\partial q_r^\beta}\right\}$ generates $\mathfrak{X}^V(\pi_s^{2k-1})$.

- If $s = k-1$: In this case $\mathrm{d}\Theta_{\mathcal{L}}$ is π_s^{2k-1}-semibasic if, and only if,

$$\frac{\partial L_\alpha^i}{\partial q_r^\beta} = 0 \quad , \quad \frac{\partial L}{\partial q_k^\beta} - L_\beta^k = 0 \quad ;$$

but this last condition is fulfilled by the definition of L_β^k, and the same reasoning above allows us to prove the statement. □

If $\Theta_{\mathcal{L}}$ projects onto $J^s\pi$, with $s < k-1$, then L does not depend on q_j^α, for $j > s+1$, then there exists a function $L' \in C^\infty(J^{s+1}\pi)$ such that $L = (\pi_{s+1}^k)^* L'$ and the theory is not strictly of order k. Furthermore, in the case $s \geq k-1$, a Lagrangian such that $\Theta_{\mathcal{L}}$ projects onto $J^s\pi$ depends on all the variables and thus we have a theory of order k, although the associated Euler-Lagrange equations are of lower order as a system of differential equations.

Proposition 2.11. *$\Theta_{\mathcal{L}}$ projects onto $J^s\pi$, with $s \geq k-1$ if, and only if, L_α^0 is π_{s+1}^k-basic.*

Proof. Note that $L_\alpha^0 \in C^\infty(J^{2k}\pi)$. For $X \in \mathfrak{X}^V(\pi_{s+1}^k)$,

$$\mathrm{L}(X)L_\alpha^0 = \mathrm{L}(X)\frac{\partial L}{\partial q_0^\alpha} - \mathrm{L}(X)(D_t L_\alpha^1) = \mathrm{L}(X)\frac{\partial L}{\partial q_0^\alpha} - D_t(\mathrm{L}(X)L_\alpha^1) - \mathrm{L}([X, D_t])L_\alpha^1 \, .$$

Since $[D_t, X] \in \mathfrak{X}^V(\pi_s^k)$ and L_α^1 and L are π_s^k-basic, then $\mathrm{L}(X)(L_\alpha^0) = 0$. Therefore, L_α^0 is π_{s+1}^k-basic.

The converse holds because $D_t L_\alpha^j$ is π_j^k-basic if, and only if, L_α^j is π_{j-1}^k-basic, for $j > 1$. Indeed, from (2.4), for every $r \geq 1$, if L_α^{r-1} is π_j^k-basic, then L_α^r is π_i^k-basic, with $i \leq \max(k, j) - 1$. By induction, if L_α^0 is π_{s+1}^k-basic, then L_α^r is π_s^k-basic for all r, thus $\Theta_{\mathcal{L}}$ projects onto $J^s\pi$. □

Equating the local expressions of $\Theta_{\mathcal{L}}$ and $\Theta_{\mathcal{L}'}$ the following result holds immediately:

Proposition 2.12. *If there exist $\mathcal{L}' \in \Omega^1(J^{k'}\pi)$ such that $\Theta_{\mathcal{L}} = (\pi_s^{2k-1})^*\Theta_{\mathcal{L}'}$, then $\mathcal{L} = (\pi_s^{2k-1})^*\mathcal{L}'$.*

In particular L is not strictly of order k. Finally, a similar result to theorem 2.4 is the following:

Theorem 2.5. *If $\Theta_{\mathcal{L}}$ projects onto $J^s\pi$, then solutions to the corresponding Euler-Lagrange equations exist only in points of a submanifold $\mathcal{S} \hookrightarrow J^{2k-1}\pi$, where $\mathcal{S}$ is locally defined by the constraint functions given by*

$$D_t^j L_\alpha^0 = 0 \quad ; \quad (j = 0, \ldots, 2k-s-2) \, .$$

(*Proof*): To find a solution to the Euler-Lagrange equations is equivalent to find a holonomic vector field $X \in \mathfrak{X}(J^{2k-1}\pi)$ such that

$$i(X)\mathrm{d}\Theta_{\mathcal{L}} = 0 \,. \tag{2.5}$$

The holonomic vector fields have the local expression:

$$X = D_t + (F^\alpha - q_{2k}^\alpha)\,\frac{\partial}{\partial q_{2k-1}^\alpha}\,,$$

and then equation (2.5) reduces to

$$L_\alpha^0 - (F^\beta - q_{2k}^\beta)\frac{\partial L_\alpha^1}{\partial q_{2k-1}^\beta} = 0\,.$$

If $\Theta_{\mathcal{L}}$ projects onto $J^s\pi$ for $s < 2k-1$, the second term vanishes and $L_\alpha^0 = 0$. Notice that we cannot compute any function F^α. Actually $L_\alpha^0 \in C^\infty(J^{2k-1}\pi)$, thus $L_\alpha^0 = 0$ is just a restriction for the points of the manifold $J^{2k-1}\pi$. Next, following the constraint algorithm, we impose the tangency condition and we get

$$0 = \mathrm{L}(X)L_\alpha^0 = D_t L_\alpha^0 + (F^\alpha - q_{2k}^\alpha)\,\frac{\partial L_\alpha^0}{\partial q_{2k-1}^\alpha}\,.$$

If $\Theta_{\mathcal{L}}$ projects onto $J^s\pi$, then the second term vanishes (Proposition 2.11) and we find another constraint, $D_t L_\alpha^0 = 0$. The algorithm continues until we reach the condition $D_t^{2k-s-2}L_\alpha^0 = 0$. □

As above, depending on the Lagrangian, we may need to continue the constraint algorithm, obtaining that

$$0 = D_t\left(D_t^{2k-s-2}L_\alpha^0\right) + (F^\alpha - q_{2k}^\alpha)\,\frac{\partial}{\partial q_{2k-1}^\alpha}\left(D_t^{2k-s-2}(L_\alpha^0)\right)\,.$$

Chapter 3

Einstein-Hilbert

This chapter is a contribution to the study of the most classical variational model for General Relativity that is, the *Einstein-Hilbert theory* (with and without energy-matter sources), using the *multisymplectic framework* for giving a covariant description of it. As it is well-known, this model is described by a second-order singular Lagrangian, and thus this study presents General Relativity as a second-order premultisymplectic field theory with constraints. Our study is done from a different perspective since we use the unified Lagrangian-Hamiltonian formalism developed for first and second-order multisymplectic field theories [29, 83] (which was stated first by R. Skinner and R. Rusk for autonomous mechanical systems [94]), and is specially interesting for analyzing non-regular constraint theories. Then we derive from it the Lagrangian and multimomentum Hamiltonian formalism.

As a consequence of the singularity of the Lagrangian, the Einstein-Hilbert model exhibits gauge freedom and it can be reduced to a first-order field theory [16, 72, 73, 86, 87]. Then, related to this topic, we analyse also a first-order theory equivalent to Einstein-Hilbert (without matter-energy sources), which is described by a first-order regular Lagrangian, showing, in this way, that General Relativity can be realised as a regular multisymplectic field theory (without constraints). This first-order model is different from the Metric-Affine or Einstein-Palatini approach which is also a first-order but non-regular (gauge) theory. The gauge freedom of the Einstein-Hilbert theory is also discussed, in order to show clearly the relation with the first-order case. In the case of the Einstein-Hilbert model with energy-matter sources, we show how the behaviour of the theory (the constraints arising in the constraint algorithm and the achievement of the multimomentum Hamiltonian formalism) depends on the characteristics of the Lagrangian representing the sources. This study is done in detail for the most standard types of energy-matter sources: those coupled to the metric.

The organization of the chapters is the following: In Section 3.1 the Lagrangian-Hamiltonian formalism for the theory is developed. Then, we recover both the Lagrangian and Hamiltonian formalisms, in the last case we show how this second-order theory can be equivalent to a first-order one. Section 3.2 is devoted to analyse the Einstein-Hilbert Lagrangian with energy-matter sources, following the same procedure as in the previous section. In Section 3.3 we briefly discuss the symmetries for Einstein-Hilbert model. Finally, in the appendices, the calculation of multivector fields which are solutions to the field equations for all these models is explicitly done.

3.1 The Einstein-Hilbert model without energy-matter sources

3.1.1 The Einstein-Hilbert Lagrangian

Fist we consider the Einstein-Hilbert Lagrangian for the Einstein equations of gravity without sources (no matter-energy is present).

The configuration bundle for this system is a fiber bundle $\pi\colon E \to M$, where M is a connected orientable 4-dimensional manifold representing space-time, whose volume form is denoted $\eta \in \Omega^4(M)$. E is the manifold of Lorentz metrics on M; that is, for every $x \in M$, the fiber $\pi^{-1}(x)$ is the set of metrics with signature $(-+++)$ acting on T_xM.

The adapted fiber coordinates in E are $(x^\mu, g_{\alpha\beta})$, $(\mu, 0 \leq \alpha \leq \beta \leq 3)$, such that $\eta = \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^3 \equiv \mathrm{d}^4x$ and where $g_{\alpha\beta}$ are the component functions of the metric. It is usefull to consider also the components $g_{\beta\alpha}$ with $\beta > \alpha$, but we should remember they are not independent because the metric is symmetric, $g_{\alpha\beta} = g_{\beta\alpha}$. Actually there are 10 independent variables, resulting that the dimension of the fibers is 10 and $\dim E = 14$. When we sum over the indices on the fiber and not all the components, we order the indices as $0 \leq \alpha \leq \beta \leq 3$.

In order to state the formalism we need to consider the *kth-order jet bundles* of the projection π, $J^k\pi$, $(k = 1, 2, 3)$. The induced coordinates in $J^3\pi$ are $(x^\mu,\, g_{\alpha\beta},\, g_{\alpha\beta,\mu},\, g_{\alpha\beta,\mu\nu},\, g_{\alpha\beta,\mu\nu\lambda})$, $(0 \leq \mu \leq \nu \leq \lambda \leq 3)$. Again, we will use all the permutations, although only the ordered ones are proper coordinates.

The *coordinate total derivatives* [83, 90], are locally given as

$$D_\tau = \frac{\partial}{\partial x^\tau} + \sum_{\substack{\alpha\leq\beta\\ \mu\leq\nu\leq\lambda}} \left(g_{\alpha\beta,\tau}\frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} + g_{\alpha\beta,\mu\nu\lambda\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} \right) . \quad (3.1)$$

Observe that, if $f \in \mathrm{C}^\infty(J^k\pi)$, then $D_\tau f \in \mathrm{C}^\infty(J^{k+1}\pi)$.

The *Einstein-Hilbert Lagrangian density* is a $\bar\pi^2$-semibasic m-form $\mathcal{L}_{EH} \in \Omega^4(J^2\pi)$, then $\mathcal{L}_{EH} = L_{EH}\,(\bar\pi^2)^*\eta$, where $L_{EH} \in \mathrm{C}^\infty(J^2\pi)$ is the *Einstein-Hilbert Lagrangian function* given by

$$L_{EH} = \varrho R = \varrho g^{\alpha\beta} R_{\alpha\beta} \ ;$$

here $\varrho = \sqrt{|det(g_{\alpha\beta})|}$, R is the *scalar curvature*, $R_{\alpha\beta} = D_\gamma\Gamma^\gamma_{\alpha\beta} - D_\alpha\Gamma^\gamma_{\gamma\beta} + \Gamma^\gamma_{\alpha\beta}\Gamma^\delta_{\delta\gamma} - \Gamma^\gamma_{\delta\beta}\Gamma^\delta_{\alpha\gamma}$ are the components of the *Ricci tensor*, $\Gamma^\lambda_{\mu\nu} = \dfrac{1}{2}\, g^{\lambda\rho}\,(g_{\nu\rho,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho})$ are the *Christoffel symbols of the Levi-Civita connection* of g, and $g^{\alpha\beta}$ denotes the inverse matrix of g, namely: $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$. As the Christoffel symbols depend on first-order derivatives of $g_{\mu\nu}$ and taking into account the expression (3.1) we have that the Lagrangian contains second-order derivatives of the components of the metric and thus this is a second-order field theory.

It is useful to consider the following decomposition [16, 86]:

$$\hat{L} = \sum_{\alpha\leq\beta} \hat{L}^{\alpha\beta,\mu\nu} g_{\alpha\beta,\mu\nu} + \hat{L}_0 \ ,$$

where

$$\hat{L}^{\alpha\beta,\mu\nu} = \frac{1}{n(\mu\nu)}\frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu\nu}} = \frac{n(\alpha\beta)}{2}\varrho(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - 2g^{\alpha\beta}g^{\mu\nu}) \ , \quad (3.2)$$

$$\hat{L}_0 = \varrho g^{\alpha\beta}\{g^{\gamma\delta}(g_{\delta\mu,\beta}\Gamma^\mu_{\alpha\gamma} - g_{\delta\mu,\gamma}\Gamma^\mu_{\alpha\beta}) + \Gamma^\delta_{\alpha\beta}\Gamma^\gamma_{\gamma\delta} - \Gamma^\delta_{\alpha\gamma}\Gamma^\gamma_{\beta\delta}\} \ . \quad (3.3)$$

The point on this decomposition is to isolate the acceleration term, because $\hat{L}^{\alpha\beta,\mu\nu}$ and $\hat{L}_0$ project onto functions $L^{\alpha\beta,\mu\nu} \in C^\infty(E)$ and $L_0 \in C^\infty(J^1\pi)$, respectively. Another useful function is

$$\hat{L}^{\alpha\beta,\mu} = \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3} \frac{1}{n(\mu\nu)} D_\nu \left(\frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu\nu}} \right) = \frac{\partial \hat{L}_0}{\partial g_{\alpha\beta,\mu}} - D_\nu \hat{L}^{\alpha\beta,\mu\nu} \, . \tag{3.4}$$

3.1.2 Langrangian-Hamiltonian Unified Formalism

Langrangian-Hamiltonian Unified System

For the Lagrangian-Hamiltonian unified formalism, we have the *symmetric higher-order jet multimomentum bundle* $\mathcal{W}_r = J^3\pi \times_{J^1\pi} J^2\pi^\ddagger$ (see [83, 84] for details), which have as natural local coordinates

$$(x^\mu,\, g_{\alpha\beta},\, g_{\alpha\beta,\mu},\, g_{\alpha\beta,\mu\nu},\, g_{\alpha\beta,\mu\nu\lambda}, p^{\alpha\beta,\mu}, p^{\alpha\beta,\mu\nu}), \quad (0 \leq \alpha \leq \beta \leq 3; 0 \leq \mu \leq \nu \leq 3) \, .$$

Remember these bundles are endowed with the canonical projections

$$\rho_1^r \colon \mathcal{W}_r \to J^3\pi \, , \quad \rho_2^r \colon \mathcal{W}_r \to J^2\pi^\ddagger \, , \quad \rho_M^r \colon \mathcal{W}_r \to M \, .$$

Hence, we have the diagram:

$$\begin{tikzcd}
 & \mathcal{W}_r \arrow[dl, "\rho_1^r"'] \arrow[dr, "\rho_2^r"] \arrow[dd, "\rho^r_{J^1\pi}"] \arrow[ddd, bend right, "\rho^r_M"'] & \\
J^3\pi \arrow[dr, "\pi_1^3"'] & & J^2\pi^\ddagger \arrow[dl, "\pi^\ddagger_{J^1\pi}"] \\
 & J^1\pi \arrow[d, "\bar{\pi}^1"] & \\
 & M &
\end{tikzcd}$$

Denoting by $\hat{\mathcal{L}} = (\pi_2^3 \circ \rho_1)^* \mathcal{L}_{EH} \in \Omega^4(\mathcal{W})$, we can write $\hat{\mathcal{L}} = \hat{L}\, \rho_M^* \eta$, where $\hat{L} = (\pi_2^3 \circ \rho_1)^* L_{EH} \in C^\infty(\mathcal{W})$. Then, we introduce the *Hamiltonian function*

$$\hat{H} = \sum_{\alpha\leq\beta} p^{\alpha\beta,\mu} g_{\alpha\beta,\mu} + \sum_{\substack{\alpha\leq\beta \\ \mu\leq\nu}} p^{\alpha\beta,\mu\nu} g_{\alpha\beta,\mu\nu} - \hat{L} \, .$$

Now we define the *Liouville forms* in $\mathcal{W}_r$, Θ_r and $\Omega_r = -\mathrm{d}\Theta_r$, with local expressions

$$\Theta_r = -\hat{H}\mathrm{d}^4x + \sum_{\alpha\leq\beta} p^{\alpha\beta,\mu} \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu + \sum_{\alpha\leq\beta} \frac{1}{n(\mu\nu)} p^{\alpha\beta,\mu\nu} \mathrm{d}g_{\alpha\beta,\mu} \wedge \mathrm{d}^3x_\nu \, ,$$
$$\Omega_r = \mathrm{d}\hat{H} \wedge \mathrm{d}^4x - \sum_{\alpha\leq\beta} \mathrm{d}p^{\alpha\beta,\mu} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu - \sum_{\alpha\leq\beta} \frac{1}{n(\mu\nu)} \mathrm{d}p^{\alpha\beta,\mu\nu} \wedge \mathrm{d}g_{\alpha\beta,\mu} \wedge \mathrm{d}^3x_\nu \, .$$

In the following, we commit an abuse of notation denoting also $\hat{L} = (\pi_2^3 \circ \rho_1^r)^* L_{EH} \in C^\infty(\mathcal{W}_r)$. These forms are degenerate; namely,

$$\ker\,\Theta_r = \ker\,\Omega_r = \left\langle \frac{\partial}{\partial g_{\alpha\beta,\mu\nu}}, \frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} \right\rangle_{0\leq\alpha\leq\beta\leq 3;\, 0\leq\mu\leq\nu\leq\lambda\leq 3} . \tag{3.5}$$

In this way, the coordinate vector fields in (3.5) are local strong gauge vector fields. Furthermore, Θ_r is $(\pi_1^3 \circ \rho_1^r)$-projectable.

The condition set for the unified system is $\mathcal{I} = \rho_1^{r*}\mathfrak{C}^3 \equiv \{\rho_1^{r*}\omega | \omega \in \mathfrak{C}^3\}$, where $\mathfrak{C}^3$ is the Cartan codistribution associated to $J^3\pi$. Therefore, a section $\psi \in \Gamma(\rho_M^r)$ is holonomic in $\mathcal{W}_r$ if $\rho_1^r \circ \psi \in \Gamma(\bar{\pi}^3)$ is holonomic in $J^3\pi$, and an integrable and ρ_M^r-transverse multivector field $\mathbf{X} \in \mathfrak{X}^4(\mathcal{W}_r)$ is holonomic if its integral sections are holonomic.

The local expression of a semi-holonomic multivector field $\mathbf{X} \in \mathfrak{X}^4(\mathcal{W}_r)$ is

$$\begin{aligned} \mathbf{X} \;=\; \bigwedge_{\lambda=1}^{4} \sum_{\substack{\alpha\leq\beta \\ \mu\leq\nu\leq\tau}} \Bigg(& \frac{\partial}{\partial x^\lambda} + g_{\alpha\beta,\lambda}\frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\lambda}\frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\lambda}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} + \\ & F_{\alpha\beta,\mu\nu\tau\lambda}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu\tau}} + G_\lambda^{\alpha\beta,\mu}\frac{\partial}{\partial p^{\alpha\beta,\mu}} + G_\lambda^{\alpha\beta,\mu\nu}\frac{\partial}{\partial p^{\alpha\beta,\mu\nu}} \Bigg) . \end{aligned} \tag{3.6}$$

The Lagrangian-Hamiltonian system for the Einstein-Hilbert model without matter sources is $(\mathcal{W}_r, \Omega_r, T\mathcal{W}_r, \rho_1^{r*}\mathfrak{C}^3)$. As the form Ω_r is 1-degenerate we have that actually it is a premultisymplectic system, and solutions to (1.2) or (1.4) do not exist everywhere in $\mathcal{W}_r$. Then [83]:

Proposition 3.1. *A section $\psi \in \Gamma(\rho_M^r)$ solution to the equation (1.2) takes values in a 140-codimensional submanifold $\jmath_\mathcal{L}\colon \mathcal{W}_\mathcal{L} \hookrightarrow \mathcal{W}_r$ which is identified with the graph of a bundle map $\mathcal{FL}\colon J^3\pi \to J^2\pi^\ddagger$, over $J^1\pi$, defined locally by*

$$\mathcal{FL}^* p^{\alpha\beta,\mu} = \frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3} \frac{1}{n(\mu\nu)} D_\nu\left(\frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu\nu}}\right) = \hat{L}^{\alpha\beta,\mu}\ ,\ \mathcal{FL}^* p^{\alpha\beta,\mu\nu} = \frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu\nu}} . \tag{3.7}$$

What is equivalent, the submanifold $\mathcal{W}_\mathcal{L}$ is the graph of a bundle morphism $\widetilde{\mathcal{FL}}\colon J^3\pi \to J^2\pi^\dagger$ over $J^1\pi$ defined locally by

$$\begin{aligned} \widetilde{\mathcal{FL}}^* p^{\alpha\beta,\mu} &= \frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3} \frac{1}{n(\mu\nu)} D_\nu\left(\frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu\nu}}\right) = \hat{L}^{\alpha\beta,\mu} , \\ \widetilde{\mathcal{FL}}^* p^{\alpha\beta,\mu\nu} &= \frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu\nu}} , \\ \widetilde{\mathcal{FL}}^* p &= \hat{L} - \sum_{\alpha\leq\beta} g_{\alpha\beta,\mu}\left(\frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3}\frac{1}{n(\mu\nu)} D_\nu\left(\frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu\nu}}\right)\right) - \sum_{\substack{\alpha\leq\beta \\ \mu\leq\nu}} g_{\alpha\beta,\mu\nu}\frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu\nu}} . \end{aligned}$$

The maps $\mathcal{FL}$ and $\widetilde{\mathcal{FL}}$ are the *restricted* and the *extended Legendre maps* (associated with a Lagrangian density $\mathcal{L}$), and they satisfy that $\mathcal{FL} = \mu \circ \widetilde{\mathcal{FL}}$. For every $j_x^3\phi \in J^3\pi$, we have that $\operatorname{rank}(\widetilde{\mathcal{FL}}(j_x^3\phi)) = \operatorname{rank}(\mathcal{FL}(j_x^3\phi))$. Remember that, according to [91], we say that a second-order Lagrangian density $\mathcal{L} \in \Omega^4(J^2\pi)$ is *regular* if

$$\operatorname{rank}(\widetilde{\mathcal{FL}}(j_x^3\phi)) = \operatorname{rank}(\mathcal{FL}(j^3\phi)) = \dim J^2\pi + \dim J^1\pi - \dim E = \dim J^2\pi^\ddagger ,$$

otherwise, the Lagrangian density is *singular*. Regularity is equivalent to demand that $\mathcal{FL}\colon J^3\pi \to J^2\pi^\ddagger$ is a submersion onto $J^2\pi^\ddagger$ and this implies that there exist local sections of $\mathcal{FL}$. If $\mathcal{FL}$ admits a global section $\Upsilon\colon J^2\pi^\ddagger \to J^3\pi$, then the Lagrangian density is said to be *hyperregular*. Recall that the regularity of $\mathcal{L}$ determines if the section $\psi \in \Gamma(\rho^r_M)$ solution to the equation (1.2) lies in $\mathcal{W}_\mathcal{L}$ or in a submanifold $\mathcal{W}_f \hookrightarrow \mathcal{W}_\mathcal{L}$ where the section ψ takes values. In order to obtain this *final constraint submanifold*, the best way is to work with the equation for multivector fields (1.4) instead of (1.2).

Field equations for multivector fields

First, the premultisymplectic constraint algorithm [22] states that:

Proposition 3.2. *A solution $\mathbf{X} \in \mathfrak{X}^4(\mathcal{W}_r)$ to equation* (1.4) *exists only on the points of the compatibility submanifold $\mathcal{W}_c \hookrightarrow \mathcal{W}_r$ defined by*

$$\begin{aligned}\mathcal{W}_c &= \left\{ w \in \mathcal{W}_r \colon (i(Z)\mathrm{d}\hat{H})(w) = 0\,, \textit{ for every } Z \in \ker(\Omega_r) \right\} \\ &= \left\{ w \in \mathcal{W}_r \colon (i(Y)\Omega_r)(w) = 0\,, \textit{ for every } Y \in \mathfrak{X}^{V(\rho^r_2)}(\mathcal{W}_r) \right\}\,.\end{aligned}$$

Bearing in mind (3.5) and that $i\left(\dfrac{\partial}{\partial g_{\alpha\beta,\mu\nu\tau}}\right)\mathrm{d}\hat{H} = 0$, the functions locally defining this submanifold have the following coordinate expressions

$$i\left(\frac{\partial}{\partial g_{\alpha\beta,\mu\nu\tau}}\right)\mathrm{d}\hat{H} = p^{\alpha\beta,\mu\nu} - \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu\nu}}\,. \tag{3.8}$$

Then, the tangency condition for the multivector fields $\mathbf{X}$ which are solutions to (1.4) on $\mathcal{W}_c$ gives rise to 24 new constraints

$$p^{\alpha\beta,\mu} - \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu}} + \sum_{\nu=0}^{3}\frac{1}{n(\mu\nu)}D_\nu\left(\frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu\nu}}\right) = 0\,.$$

which define a submanifold of $\mathcal{W}_c$ that coincides with the submanifold $\mathcal{W}_\mathcal{L}$. Now the study of the tangency of $\mathbf{X}$ along $\mathcal{W}_\mathcal{L}$ could introduce new constraints depending on the regularity of $\mathcal{L}$, and the algorithm continues until we reach the submanifold $\mathcal{W}_f$. The final result is given in the next theorem:

Theorem 3.1. *Let $\mathcal{W}_f \hookrightarrow \mathcal{W}_r$ be the submanifold defined locally by the constraints*

$$p^{\alpha\beta,\mu\nu} - \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu\nu}} = 0 \quad , \quad p^{\alpha\beta,\mu} - \hat{L}^{\alpha\beta,\mu} = 0 \quad , \quad \hat{L}^{\alpha\beta} = 0 \quad , \quad D_\tau \hat{L}^{\alpha\beta} = 0\,;$$

for $0 \le \alpha \le \beta \le 3$, $0 \le \mu \le \nu \le 3$ and $0 \le \tau \le 3$. Then, there exist classes of holonomic multivector fields $\{\mathbf{X}\} \subset \mathfrak{X}^4(\mathcal{W}_r)$ which are tangent to $\mathcal{W}_f$ and such that

$$i\,(\mathbf{X})\Omega_r|_{\mathcal{W}_f} = 0 \quad , \quad \forall \mathbf{X} \in \{\mathbf{X}\} \subset \mathfrak{X}^4(\mathcal{W}_r)\,. \tag{3.9}$$

Proof. In order to find the final submanifold $\mathcal{W}_f$ we use he constraint algorithm presented in section 1.2.3. Bearing in mind (3.6), the local expression of a representative of a class of a semiholonomic multivector fields, not necessarily integrable, is, in this case,

$$\begin{aligned}\mathbf{X} = \bigwedge_{\tau=0}^{3} X_\tau \;=\; \bigwedge_{\tau=0}^{3}\sum_{\substack{\alpha\le\beta\\ \mu\le\nu\le\lambda}}\Bigg(&\frac{\partial}{\partial x^\tau} + g_{\alpha\beta,\tau}\frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} \\ &F_{\alpha\beta;\mu\nu\lambda,\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} + G^{\alpha\beta,\mu}_\tau\frac{\partial}{\partial p^{\alpha\beta,\mu}} + G^{\alpha\beta,\mu\nu}_\tau\frac{\partial}{\partial p^{\alpha\beta,\mu\nu}}\Bigg)\,,\end{aligned}$$

then, equation (1.4) leads to

$$G_\nu^{\alpha\beta,\nu} - \frac{\partial \hat{L}}{\partial g_{\alpha\beta}} = 0 \, , \tag{3.10}$$

$$\sum_{\nu=0}^{3} \frac{1}{n(\mu\nu)} G_\nu^{\alpha\beta,\mu\nu} - \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu}} + p^{\alpha\beta,\mu} = 0 \, , \tag{3.11}$$

$$p^{\alpha\beta,\mu\nu} - \hat{L}^{\alpha\beta,\mu\nu} = 0 \, . \tag{3.12}$$

Equations (3.12) are what we obtain in Proposition 3.2 (see (3.8)), and they are the constraints defining the compatibility submanifold $\mathcal{W}_c \hookrightarrow \mathcal{W}_r$. The tangency conditions on them,

$$\mathrm{L}(X_\tau)(p^{\alpha\beta,\mu\nu} - \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu\nu}})|_{\mathcal{W}_c} = 0 \, ,$$

allows us to determine some coefficients

$$G_\tau^{\alpha\beta,\mu\nu} = D_\tau \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu\nu}} \quad ; \quad (\text{on } \mathcal{W}_c) \, . \tag{3.13}$$

These new identities are not compatible with (1.4). Indeed, combining them with (3.11) we have:

$$0 = \sum_{\nu=0}^{3} \frac{1}{n(\mu\nu)} D_\nu \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu\nu}} - \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu}} + p^{\alpha\beta,\mu} = p^{\alpha\beta,\mu} - \hat{L}^{\alpha\beta,\mu} \quad ; \quad (\text{on } \mathcal{W}_c) \, . \tag{3.14}$$

These restrictions define the submanifold $\mathcal{W}_\mathcal{L} \hookrightarrow \mathcal{W}_c$. The tangency conditions on these new constraints,

$$\mathrm{L}(X_\tau)(p^{\alpha\beta,\mu} - \hat{L}^{\alpha\beta,\mu})|_{\mathcal{W}_\mathcal{L}} = 0 \, ,$$

lead to

$$G_\tau^{\alpha\beta,\mu} = D_\tau \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu}} - D_\tau D_\sigma \hat{L}^{\alpha\beta,\mu\sigma} \quad ; \quad (\text{on } \mathcal{W}_\mathcal{L}) \, . \tag{3.15}$$

Contracting the indices μ and τ in these restrictions and combining them with (3.10), we obtain the new functions

$$\hat{L}^{\alpha\beta} := \frac{\partial \hat{L}}{\partial g_{\alpha\beta}} - D_\nu \hat{L}^{\alpha\beta,\nu} = \frac{\partial \hat{L}}{\partial g_{\alpha\beta}} - D_\nu \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\nu}} + \sum_{\nu\leq\mu}^{3} D_\nu D_\mu \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\nu\mu}} = 0 \quad ; \quad (\text{on } \mathcal{W}_\mathcal{L}) \, ,$$

which are explicitly

$$\hat{L}^{\alpha\beta} = -\varrho \, n(\alpha\beta) \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) = 0 \quad ; \quad (\text{on } \mathcal{W}_\mathcal{L}) \, . \tag{3.16}$$

These are the Euler-Lagrange equations, and when they are evaluated on sections in $\mathcal{W}_\mathcal{L}$ we recover the Einstein equations $(R_{\alpha\beta} - {}^1\!/\!{}_2 g_{\alpha\beta} R)|_\psi = 0$. From its definition we can see that $\hat{L}^{\alpha\beta}$ do not depend neither on the momenta, nor on higher order velocities than the accelerations of the components of the metric, therefore $\hat{L}^{\alpha\beta}$ project onto $J^2\pi$. The equations (3.16) are algebraic combinations of the coordinates of $\mathcal{W}_\mathcal{L}$ and a solution can only exists on the points where they vanish. Thus, $\hat{L}^{\alpha\beta}$ are new constraints which define locally the submanifold $\mathcal{W}_1 \hookrightarrow \mathcal{W}_\mathcal{L} \hookrightarrow \mathcal{W}_r$. (Note that, as a consequence of the Bianchi identities, these constraints are not independent all of them). Continuing with the constraint algorithm, we consider the tangency conditions on these constraints,

$$\mathrm{L}(X_\tau)\hat{L}^{\alpha\beta}|_{\mathcal{W}_1} = 0 \, ,$$

which lead to

$$D_\tau \hat{L}^{\alpha\beta} = D_\tau \left(-\varrho\, n(\alpha\beta) \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) \right) = 0 \quad ; \quad (\text{on } \mathcal{W}_1)\,. \tag{3.17}$$

These are new constraints again (observe that these functions $D_\tau \hat{L}^{\alpha\beta}$ project onto $J^3\pi$, since they do not depend on the higher-order derivatives and the momenta). They define locally the submanifold $\mathcal{W}_f \hookrightarrow \mathcal{W}_1 \hookrightarrow \mathcal{W}_\mathcal{L} \hookrightarrow \mathcal{W}_r$. Finally, the new tangency conditions,

$$\mathrm{L}(X_\sigma) D_\tau \hat{L}^{\alpha\beta}|_{\mathcal{W}_f} = 0\,,$$

lead to

$$\sum_{\substack{\gamma\leq\lambda \\ \mu\leq\nu\leq\kappa}} \left(\frac{\partial}{\partial x^\sigma} + g_{\gamma\lambda,\sigma} \frac{\partial}{\partial g_{\gamma\lambda}} + g_{\gamma\lambda,\mu\sigma} \frac{\partial}{\partial g_{\gamma\lambda,\mu}} + g_{\gamma\lambda,\mu\nu\sigma} \frac{\partial}{\partial g_{\gamma\lambda,\mu\nu}} + F_{\gamma\lambda;\mu\nu\kappa,\sigma} \frac{\partial}{\partial g_{\gamma\lambda,\mu\nu\kappa}} \right) D_\tau \hat{L}^{\alpha\beta} = 0 \qquad (\text{on } \mathcal{W}_f) \tag{3.18}$$

and these equations allows us to determine some functions $F_{\gamma\lambda;\mu\nu\kappa,\sigma}$. The manifold $\mathcal{W}_f$ is actually the final constraint submanifold because there exist integrable holonomic multivector fields solutions to equations (3.9) on $\mathcal{W}_f$, tangent to $\mathcal{W}_f$, which are (partially) determined by the conditions (3.13), (3.15), and (3.18); that is,

$$\begin{aligned} \mathbf{X} \quad &= \bigwedge_{\tau=0}^{3} \sum_{\substack{\alpha\leq\beta \\ \mu\leq\nu\leq\lambda}} \left(\frac{\partial}{\partial x^\tau} + g_{\alpha\beta,\tau} \frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\tau} \frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\tau} \frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} + \right. \\ &\left. D_\tau D_\lambda (g_{\lambda\sigma}(\Gamma^\lambda_{\nu\alpha}\Gamma^\sigma_{\mu\beta} + \Gamma^\lambda_{\nu\beta}\Gamma^\sigma_{\mu\alpha})) \frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} + D_\tau \hat{L}^{\alpha\beta,\mu} \frac{\partial}{\partial p^{\alpha\beta,\mu}} + D_\tau \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu\nu}} \frac{\partial}{\partial p^{\alpha\beta,\mu\nu}} \right) \end{aligned} \tag{3.19}$$

One can prove (after a long computation) that this is actually an integrable solution (see section 3.A for more details). Finally, we have that the complete set of constraint functions defining the final constraint submanifold $\mathcal{W}_f \hookrightarrow \mathcal{W}_r$ are given by (3.12), (3.14), (3.16) and (3.17); that is,

$$p^{\alpha\beta,\mu\nu} - \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu\nu}} = 0 \quad , \quad p^{\alpha\beta,\mu} - \hat{L}^{\alpha\beta,\mu} = 0 \quad , \quad \hat{L}^{\alpha\beta} = 0 \quad , \quad D_\tau \hat{L}^{\alpha\beta} = 0\,.$$

□

Field equations for sections

Once the holonomic multivector fields which are solutions to equation (1.4) (on $\mathcal{W}_f$) have been obtained, in order to obtain the field equations for sections we can use, either the equations (1.1), or the equivalent equations (1.2) which the integral sections of these multivector fields satisfy. Thus, if these sections are locally given by

$$\psi(x^\lambda) = (x^\lambda,\, \psi_{\alpha\beta}(x^\lambda),\, \psi_{\alpha\beta,\mu}(x^\lambda),\, \psi_{\alpha\beta,\mu\nu}(x^\lambda),\, \psi_{\alpha\beta,\mu\nu\tau}(x^\lambda),\, \psi^{\alpha\beta,\mu}(x^\lambda),\, \psi^{\alpha\beta,\mu\nu}(x^\lambda))\,,$$

the equation (1.4) leads to

$$\begin{aligned}
\frac{\partial\psi^{\alpha\beta,\mu}}{\partial x^{\mu}} - \frac{\partial\hat{L}}{\partial g_{\alpha\beta}} &= 0\,, &(3.20)\\
\frac{\partial\psi^{\alpha\beta,\mu\nu}}{\partial x^{\nu}} + \psi^{\alpha\beta,\mu} - \frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu}} &= 0\,, &(3.21)\\
\psi^{\alpha\beta,\mu\nu} - \hat{L}^{\alpha\beta,\mu\nu} &= 0\,, &(3.22)\\
\psi_{\alpha\beta,\mu} - \frac{\partial\psi_{\alpha\beta}}{\partial x^{\mu}} &= 0\,, &(3.23)\\
\psi_{\alpha\beta,\mu\nu} - \frac{1}{n(\mu\nu)}\left(\frac{\partial\psi_{\alpha\beta,\mu}}{\partial x^{\nu}} + \frac{\partial\psi_{\alpha\beta,\nu}}{\partial x^{\mu}}\right) &= 0\,. &(3.24)
\end{aligned}$$

Equations (3.23) and (3.24) are part of the holonomy conditions. Equations (3.21) and (3.22), as they do not involve the derivatives of the fields higher than 3, are just relations among the coordinates of the points in $\mathcal{W}_r$, which are equivalent to equations (3.12) and (3.11), respectively, and they define the Legendre map introduced in (3.7). They show that, as discussed above, the section ψ take values in the submanifold

$$\mathcal{W}_{\mathcal{L}} = \left\{ w \in \mathcal{W}_r \mid p^{\alpha\beta,\mu\nu} = \frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu\nu}}(w)\,,\ p^{\alpha\beta,\mu} = \hat{L}^{\alpha\beta,\mu}(w) \right\} = \mathrm{graph}\,\mathcal{FL}\,.$$

Finally, combining the equations (3.20) with the local expression of the Legendre map given by the equations (3.21) and (3.22) we obtain

$$\hat{L}^{\alpha\beta}|_{\psi} := \left(\frac{\partial\hat{L}}{\partial g_{\alpha\beta}} - D_{\mu}\frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu}} + \sum_{\mu\leq\nu} D_{\mu}D_{\nu}\frac{\partial\hat{L}}{\partial g_{\alpha\beta,\mu\nu}}\right)\Bigg|_{\psi} = -\varrho\, n(\alpha\beta)\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right)\Bigg|_{\psi} = 0\,. \tag{3.25}$$

These are the Euler-Lagrange equations for a section $\psi \in \Gamma(\rho^r_M)$, which are equivalent to the Einstein equations

$$\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)\Big|_{\psi} = 0\,; \tag{3.26}$$

and, as it is well known, they are of order two.

If ψ is a holonomic section solution to (1.2), the tangency conditions on the Einstein's equations are automatically satisfied. Indeed, the last constraints (3.17) read

$$\left(D_{\tau}\hat{L}^{\alpha\beta}\right)\Big|_{\psi} = \frac{\partial(\hat{L}^{\alpha\beta}\circ\psi)}{\partial x^{\tau}} = 0\,,$$

which is automatically satisfied because ψ, in particular, is a solution to the Einstein equations (3.26) and then (3.25) holds. Using the same reasoning, we can check that (3.18) is also automatically satisfied. These last equations fix the gauge freedom, therefore the gauge symmetry does not show when considering the Einstein's equations for sections.

3.1.3 Recovering the Lagrangian and Hamiltonian formalisms

Lagrangian formalism

The manifold is $J^3\pi$. The *Poincaré-Cartan forms* in $J^3\pi$ are the forms defined as

$$\Theta_{\mathcal{L}} = \widetilde{\mathcal{FL}}^*\Theta_1^s \in \varOmega^4(J^3\pi) \quad , \quad \Omega_{\mathcal{L}} = \widetilde{\mathcal{FL}}^*\Omega_1^s = -\mathrm{d}\Theta_{\mathcal{L}} \in \varOmega^5(J^3\pi)\,.$$

Using natural coordinates in $J^3\pi$, we have the local expression

$$\begin{aligned}\Theta_{\mathcal{L}} &= -\left(\sum_{\alpha\leq\beta} L^{\alpha\beta,\mu} g_{\alpha\beta,\mu} + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu\nu} g_{\alpha\beta,\mu\nu} - L\right) \mathrm{d}^4x \\ &\quad + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu} \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu\nu} \mathrm{d}g_{\alpha\beta,\mu} \wedge \mathrm{d}^3x_\nu \end{aligned} \tag{3.27}$$

Notice that, if

$$H \equiv (\jmath_{\mathcal{L}} \circ (\rho_1^{\mathcal{L}})^{-1})^* \hat{H} = \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu\nu} g_{\alpha\beta,\mu\nu} + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu} g_{\alpha\beta,\mu} - L = \varrho\, g_{\alpha\beta,\mu} g_{kl,\nu} H^{\alpha\beta kl\mu\nu} \ , \tag{3.28}$$

where

$$H^{\alpha\beta kl\mu\nu} = \frac{1}{4} g^{\alpha\beta} g^{kl} g^{\mu\nu} - \frac{1}{4} g^{\alpha k} g^{\beta l} g^{\mu\nu} + \frac{1}{2} g^{\alpha k} g^{l\mu} g^{\beta\nu} - \frac{1}{2} g^{\alpha\beta} g^{l\nu} g^{k\mu} \ , \tag{3.29}$$

then

$$\Omega_{\mathcal{L}} = -\mathrm{d}\Theta_{\mathcal{L}} = \mathrm{d}H \wedge \mathrm{d}^4x - \sum_{\alpha\leq\beta} \mathrm{d}L^{\alpha\beta,\mu} \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^{m-1}x_\mu - \sum_{\alpha\leq\beta} \mathrm{d}L^{\alpha\beta,\mu\nu} \mathrm{d}g_{\alpha\beta,\mu} \wedge \mathrm{d}^{m-1}x_\nu \in \Omega^5(J^3\pi) \ ;$$

where we have denoted $L^{\alpha\beta,\mu\nu} = (\jmath_{\mathcal{L}} \circ (\rho_1^{\mathcal{L}})^{-1})^* \hat{L}^{\alpha\beta,\mu\nu}$, $L^{\alpha\beta,\mu} = (\jmath_{\mathcal{L}} \circ (\rho_1^{\mathcal{L}})^{-1})^* \hat{L}^{\alpha\beta,\mu}$, and $L_0 = (\jmath_{\mathcal{L}} \circ (\rho_1^{\mathcal{L}})^{-1})^* \hat{L}_0$, which have the same coordinate expressions than $\hat{L}^{\alpha\beta,\mu\nu}$, $\hat{L}^{\alpha\beta,\mu}$, and $\hat{L}_0$ given in (3.2), (3.4), and (3.3), respectively. Observe that this is a pre-multisymplectic form since, locally,

$$\ker\, \Omega_{\mathcal{L}} = \left\langle \frac{\partial}{\partial g_{\alpha\beta,\mu\nu}}, \frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} \right\rangle_{0\leq\alpha\leq\beta\leq 3;\, 0\leq\mu\leq\nu\leq\lambda\leq 3} \ .$$

Thus we have the Lagrangian system $(J^3\pi, \Omega_{\mathcal{L}}, TJ^3\pi, \mathfrak{C}^3)$.

In order to recover the Lagrangian field equations, we have that the map $\rho_1^{\mathcal{L}} = \rho_1^r \circ \jmath_{\mathcal{L}} \colon \mathcal{W}_{\mathcal{L}} \to J^3\pi$ is a diffeomorphism, the Poincaré-Cartan forms defined in $J^3\pi$ satisfy $(\rho_1^{\mathcal{L}})^*\Theta_{\mathcal{L}} = \jmath_{\mathcal{L}}^*\Theta_r$ and $(\rho_1^{\mathcal{L}})^*\Omega_{\mathcal{L}} = \jmath_{\mathcal{L}}^*\Omega_r$. Then, the solution to the Lagrangian problem associated with the singular Lagrangian system $(J^3\pi, \Omega_{\mathcal{L}}, TJ^3\pi, \mathfrak{C}^3)$, which is stated in the equations (1.2) and (1.4), is given by the proposition 1.3 and Theorem 1.2:

In local coordinates in $J^3\pi$, the equation for the holonomic section $\psi_{\mathcal{L}} = j^3\phi$ are the Euler-Lagrange equations

$$\left.\left(\frac{\partial L}{\partial g_{\mu\nu}} - D_\mu \frac{\partial L}{\partial g_{\alpha\beta,\mu}} + \sum_{\mu\leq\nu} D_\mu D_\nu \frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}}\right)\right|_{j^3\phi} = 0 \ . \tag{3.30}$$

As we have pointed out before, the equalities (3.12) and (3.14) define the submanifold $\mathcal{W}_{\mathcal{L}}$ which is diffeomorphic with $J^3\pi$, and the constraint functions defining the Lagrangian final constraint submanifold $S_f \hookrightarrow J^3\pi$ are

$$L^{\alpha\beta} = \frac{\partial L}{\partial g_{\alpha\beta}} - D_\mu \frac{\partial L}{\partial g_{\alpha\beta,\mu}} + \sum_{\mu\leq\nu} D_\mu D_\nu \frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}} = -\varrho\, n(\alpha\beta) \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R\right) = 0 \ , \tag{3.31}$$

$$D_\tau L^{\alpha\beta} = 0 \ . \tag{3.32}$$

The local expression of a representative of a class of holonomic multivector fields in $J^3\pi$ is

$$\mathbf{X} = \bigwedge_{\tau=0}^{3} \sum_{\substack{\alpha\leq\beta \\ \mu\leq\nu\leq\lambda}} \left(\frac{\partial}{\partial x^\tau} + g_{\alpha\beta,\tau} \frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\tau} \frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\tau} \frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} + F_{\alpha\beta;\mu\nu\lambda,\tau} \frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}}\right) \ ; \tag{3.33}$$

then, there are holonomic multivector fields which are solutions to the equation (1.4) on S_f, and tangent to S_f. They are obtained from (3.19) using Theorem 1.2:

$$\mathbf{X}_{\mathcal{L}} = \bigwedge_{\tau=0}^{3} \sum_{\substack{\alpha\leq\beta \\ \mu\leq\nu\leq\lambda}} \left(\frac{\partial}{\partial x^\tau} + g_{\alpha\beta,\tau}\frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} + \right.$$
$$\left. (g_{\lambda\sigma}(\Gamma^\lambda_{\nu\alpha}\Gamma^\sigma_{\mu\beta} + \Gamma^\lambda_{\nu\beta}\Gamma^\sigma_{\mu\alpha}))\frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} \right) .$$

Finally, for the equations of the integral sections of these multivector fields (equation (1.2)), from (3.26), we obtain that (3.31), evaluated on the points in the image of holonomic sections $\psi_{\mathcal{L}} = j^3\phi$ in $J^3\pi$ (see Prop 1.3 and (3.30)), are equivalent to the Einstein equations

$$\begin{aligned} L^{\alpha\beta}|_{j^3\phi} &= \left(\frac{\partial L}{\partial g_{\alpha\beta}} - D_\mu \frac{\partial L}{\partial g_{\alpha\beta,\mu}} + \sum_{\mu\leq\nu} D_\mu D_\nu \frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}} \right)\Bigg|_{j^3\phi} \\ &= -\varrho\, n(\alpha\beta) \left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right)\Bigg|_{j^3\phi} = 0 \, . \end{aligned} \tag{3.34}$$

All these results can be also obtained applying the constraint algorithm straightforwardly for the equation (1.2), in the same way as we have done for the unified formalism; then doing a purely Lagrangian analysis. Thus, the Euler-Lagrange equations for an holonomic multivector field like (3.33) (which are obtained from (1.2)) read as

$$\sum_{\rho\leq\sigma,\mu\leq\nu,\lambda\leq\tau} \left(\frac{\partial^2 L}{\partial g_{\alpha\beta,\mu\nu}\partial g_{\rho\sigma,\lambda\tau}} \right) (F_{\rho\sigma;\lambda\tau\mu,\nu} - D_\nu g_{\rho\sigma;\lambda\tau\mu}) + L^{\alpha\beta} = 0 \, ,$$

and, as for the Einstein-Hilbert Lagrangian the Hessian matrix $\left(\dfrac{\partial^2 L}{\partial g_{\alpha\beta,\rho\sigma}\partial g_{\mu\nu,\lambda\tau}} \right)$ vanishes identically, we obtain that $L^{\alpha\beta} = 0$, which are the compatibility conditions for the Euler-Lagrange equations; that is, the primary Lagrangian constraints (3.31). From here, the constraint algorithm continues by requiring the tangency condition, as it is usual (see [46]).

Hamiltonian formalism

Consider the Legendre maps introduced in Proposition 3.1. We have that

$$\mathrm{T}_{j^3_x\phi}\mathcal{F}\mathcal{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \dfrac{\partial \hat{L}^{\alpha\beta,\mu}}{\partial g_{\gamma\delta}} & \dfrac{\partial \hat{L}^{\alpha\beta,\mu}}{\partial g_{\gamma\delta,\tau}} & 0 & 0 \\ 0 & \dfrac{\partial \hat{L}}{\partial g_{\gamma\delta}\partial g_{\alpha\beta,\mu\nu}} & 0 & 0 & 0 \end{pmatrix} ,$$

and we have that $\mathrm{rank}(\mathrm{T}_{j^3_x\phi}\mathcal{F}\mathcal{L}) = 54$. Furthermore, locally we have that

$$\ker\, \mathcal{F}\mathcal{L}_* = \ker\, \Omega_{\mathcal{L}} = \left\langle \frac{\partial}{\partial g_{\alpha\beta,\mu\nu}}, \frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} \right\rangle_{0\leq\alpha\leq\beta\leq3;\, 0\leq\mu\leq\nu\leq\lambda\leq3} , \tag{3.35}$$

and thus $\mathcal{FL}$ is highly degerated.

Denote $\widetilde{\mathcal{P}} = \widetilde{\mathcal{FL}}(J^3\pi) \overset{\tilde{\jmath}}{\hookrightarrow} J^2\pi^\dagger$ and $\mathcal{P} = \mathcal{FL}(J^3\pi) \overset{\jmath}{\hookrightarrow} J^2\pi^\ddagger$, and let $\mathcal{FL}_o$ be the map defined by $\mathcal{FL} = \jmath \circ \mathcal{FL}_o$ and $\bar{\pi}_{\mathcal{P}} \colon \mathcal{P} \to M$ the natural projection. In order to assure the existence of the Hamiltonian formalism we have to assure that the Lagrangian density $\mathcal{L} \in \Omega^4(J^2\pi)$ is, at least, *almost-regular*; that is, $\mathcal{P}$ is a closed submanifold of $J^2\pi^\ddagger$, $\mathcal{FL}$ is a submersion onto its image and, for every $j^3_x\phi \in J^3\pi$, the fibers $\mathcal{FL}^{-1}(\mathcal{FL}(j^3_x\phi))$ are connected submanifolds of $J^3\pi$. Then, the following result allows us to consider the Hamiltonian formalism:

Proposition 3.3. *$\mathcal{L}$ is an almost-regular Lagrangian and $\mathcal{P}$ is diffeomorphic to $J^1\pi$.*

Proof. $\mathcal{P}$ is a closed submanifold of $J^2\pi^\ddagger$ since it is defined by the constraints

$$p^{\alpha\beta,\mu\nu} - \frac{\partial \hat{L}}{\partial g_{\alpha\beta,\mu\nu}} = 0; \quad p^{\alpha\beta,\mu} - \hat{L}^{\alpha\beta,\mu} = 0 \, .$$

The dimension of $\mathcal{P}$ is $4 + 10 + 40 = 54$ and, as $\operatorname{rank}(\mathrm{T}\mathcal{FL}) = 54$ in every point, $\mathrm{T}\mathcal{FL}$ is surjective and $\mathcal{FL}$ is a submersion. Finally, bearing in mind (3.35), we conclude that the fibers of the Legendre map, $\mathcal{FL}^{-1}(\mathcal{FL}(j^3_x\phi))$ (for every $j^3_x\phi \in J^3\pi$), are just the fibers of the projection $\bar{\pi}^3_1$, which are connected submanifolds of $J^3\pi$. Recall that $J^3\pi$ is connected because we are considering metrics with fixed signature. Thus, $\mathcal{L}$ is an almost-regular Lagrangian.

Furthermore, taking any local section ϕ of the projection π^3_1, the map $\Phi = \mathcal{FL} \circ \phi \colon J^1\pi \to \mathcal{P}$ is a local diffeomorphism (which does not depend on the section chosen). Then, using these local sections, from a differentiable structure of $J^1\pi$ we can construct a differentiable structure for $\mathcal{P}$; hence $\mathcal{P}$ and $J^1\pi$ are diffeomorphic.

$$\begin{tikzcd}
J^3\pi \arrow[rr, "\mathcal{FL}_o"] \arrow[dr, "\pi^3_1"] & & \mathcal{P} \subset J^2\pi^\ddagger \\
& J^1\pi \arrow[ul, bend left, "\phi"] \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

□

Then, there exists a diffeomorphism $\widetilde{\mu} = \mu \circ \tilde{\jmath} \colon \widetilde{\mathcal{P}} \to \mathcal{P}$ and we can define a *Hamiltonian μ-section* as $h = \tilde{\jmath} \circ \widetilde{\mu}^{-1}$, which is specified by a local Hamiltonian function $H \in C^\infty(\mathcal{P})$, that is, $h(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, p^{\alpha\beta,\mu}, p^{\alpha\beta,\mu\nu}) = (x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, -H, p^{\alpha\beta,\mu}, p^{\alpha\beta,\mu\nu})$.

$$\begin{tikzcd}
\widetilde{\mathcal{P}} \arrow[r, "\tilde{\jmath}"] \arrow[d, "\tilde{\mu}"] & J^2\pi^\dagger \arrow[d, "\mu"] & \mathcal{W} \arrow[l, "\rho_2"'] \arrow[d, "\mu_{\mathcal{W}}"'] \\
\mathcal{P} \arrow[r, "\jmath"] \arrow[ur, "h"] & J^2\pi^\ddagger & \mathcal{W}_r \arrow[l, "\rho_2^r"'] \arrow[u, bend right, "\hat{h}"']
\end{tikzcd}$$

Now, we can define the Hamiltonian forms

$$\Theta_h := h^*\Theta^s_1 \in \Omega^4(\mathcal{P}) \quad , \quad \Omega_h := -\mathrm{d}\Theta_h = h^*\Omega^s_1 \in \Omega^5(\mathcal{P}) \, .$$

The condition set is $\mathcal{I} = (\pi^\ddagger_{J^1\pi} \circ \jmath)^*\mathfrak{C}^1$. And thus we have the Hamiltonian system $(\mathcal{P}, \Omega_h, T\mathcal{P}, (\pi^\ddagger_{J^1\pi} \circ \jmath)^*\mathfrak{C}^1)$.

Formulation using non multimomentum coordinates.

From the unified formalism, the easiest way to describe locally the Hamiltonian formalism consists in taking $(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu})$ as local coordinates adapted to $\mathcal{P}$. As the function H defined in (3.28) is $\mathcal{FL}_o$-projectable, the Hamiltonian function defined on $\mathcal{P}$ is just

$$H_{\mathcal{P}} = \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu\nu} g_{\alpha\beta,\mu\nu} + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu} g_{\alpha\beta,\mu} - L = \varrho\, g_{\alpha\beta,\mu} g_{kl,\nu} H^{\alpha\beta kl\mu\nu} \ , \tag{3.36}$$

where $H^{\alpha\beta kl\mu\nu}$ is given by (3.29). As $\mathcal{L}$ is almost regular, the Hamiltonian section $h\colon \mathcal{P} \to J^2\pi^\dagger$ exists and its local expression is

$$h(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}) = (x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu}, -H_{\mathcal{P}}, L^{\alpha\beta,\mu}, L^{\alpha\beta,\mu\nu}) \ .$$

Now we define the Hamilton-Cartan forms $\Theta_h = h^*\Theta_1^s \in \Omega^4(\mathcal{P})$ and $\Omega_h = -\mathrm{d}\Theta_h \in \Omega^5(\mathcal{P})$, whose coordinate expressions are

$$\begin{aligned} \Theta_h &= -H_{\mathcal{P}}\,\mathrm{d}^4x + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu} dg_{\alpha\beta}\wedge \mathrm{d}^3x_\mu + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu\nu} dg_{\alpha\beta,\mu}\wedge\mathrm{d}^3x_\nu \ , \\ \Omega_h = -\mathrm{d}\Theta_h &= \mathrm{d}H_{\mathcal{P}}\wedge\mathrm{d}^4x - \sum_{\alpha\leq\beta}\mathrm{d}L^{\alpha\beta,\mu}\wedge\mathrm{d}g_{\alpha\beta}\wedge\mathrm{d}^3x_\mu - \sum_{\alpha\leq\beta}\mathrm{d}L^{\alpha\beta,\mu\nu}\wedge\mathrm{d}g_{\alpha\beta,\mu}\wedge\mathrm{d}^3x_\nu \end{aligned} \tag{3.37}$$

(Observe that, with this choice of coordinates, Θ_h and Ω_h looks locally like $\Theta_{\mathcal{L}}$ and $\Omega_{\mathcal{L}}$). Then, Proposition 1.4 and Theorem 1.3 establish the relation between the solutions to the Hamiltonian and the unified problem.

In this case, first observe that, locally,

$$\ker\,(\pi^r_{\mathcal{P}})_* = \left\langle \frac{\partial}{\partial p^{\alpha\beta,\mu}}, \frac{\partial}{\partial p^{\alpha\beta,\mu\nu}}; \frac{\partial}{\partial g_{\alpha\beta,\mu\nu}}, \frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} \right\rangle_{0\leq\alpha\leq\beta\leq 3;\, 0\leq\mu\leq\nu\leq\lambda\leq 3} ,$$

and as

$$\mathrm{L}\left(\frac{\partial}{\partial g_{\alpha\beta,\mu\nu}}\right)\hat{L}^{\lambda\sigma} \neq 0 \ , \ \mathrm{L}\left(\frac{\partial}{\partial g_{\alpha\beta,\mu\nu}}\right)(D_\tau\hat{L}^{\lambda\sigma}) \neq 0 \ , \ \mathrm{L}\left(\frac{\partial}{\partial g_{\alpha\beta,\mu\nu}}\right)(D_\tau\hat{L}^{\lambda\sigma}) \neq 0 \ ,$$

we have that the constraints (3.16) and (3.17) (which define the final constraint submanifold $\mathcal{W}_f$ as a submanifold of $\mathcal{W}_{\mathcal{L}} = \mathrm{graph}\,\mathcal{FL}$ in the unified formalism) are not $\rho^r_{\mathcal{P}}$-projectable (see diagram (1.11)), and this means that there are no Hamiltonian constraints and the Hamilton equations have solutions everywhere in $\mathcal{P}$. (What is equivalent, the Lagrangian constraints (3.31) and (3.32) are not $\mathcal{FL}_o$-projectable). This is a consequence of the fact that, in the Lagrangian formalism, these constraints really arise as a consequence of demanding the holonomy condition and hence, as it was studied in [21], they are not projectable by the Legendre map. Then:

Proposition 3.4. *An integrable (holonomic) multivector field solution to the equations* (1.4) *is*

$$\mathbf{X}_h = \bigwedge_{\nu=0}^{3}\left(\frac{\partial}{\partial x^\nu} + \sum_{\alpha\leq\beta}\left(g_{\alpha\beta,\nu}\frac{\partial}{\partial g_{\alpha\beta}} + g_{\lambda\sigma}(\Gamma^\lambda_{\nu\alpha}\Gamma^\sigma_{\mu\beta} + \Gamma^\lambda_{\nu\beta}\Gamma^\sigma_{\mu\alpha})\frac{\partial}{\partial g_{\alpha\beta;\mu}}\right)\right) \in \mathfrak{X}^4(\mathcal{P}) \ .$$

Proof. The proof is given in the appendix 3.A. □

For the integral sections of $\mathbf{X}_h$, which are solutions to (1.2), if $\psi(x^\alpha) = (x^\alpha, \psi_{\alpha\beta}(x^\alpha), \psi_{\alpha\beta,\mu}(x^\alpha))$, then the equation (1.2) reads

$$\begin{aligned} \left.\left(D_\mu L^{\alpha\beta,\mu} - \frac{\partial L}{\partial g_{\alpha\beta}}\right)\right|_\psi &= 0 \ , \\ \left.\left(\frac{\partial L^{\alpha\beta,\mu\nu}}{\partial g_{\lambda\sigma}} - \frac{\partial L^{\lambda\sigma,\nu}}{\partial g_{\alpha\beta,\mu}}\right) g_{\lambda\sigma,\nu}\right|_\psi &= \left(\frac{\partial L^{\alpha\beta,\mu\nu}}{\partial g_{\lambda\sigma}} - \frac{\partial L^{\lambda\sigma,\nu}}{\partial g_{\alpha\beta,\mu}}\right)\frac{\partial\psi_{\lambda\sigma}}{\partial x^\nu} \ . \end{aligned}$$

The last equation is equivalent to the holonomy condition, $\dfrac{\partial\psi_{\lambda\sigma}}{\partial x^\nu} = \psi_{\lambda\sigma,\nu}$ (see the appendix 3.A). Writing the first one in terms of the Hamiltonian we obtain

$$\left(\frac{\partial L^{\alpha\beta,\nu}}{\partial g_{ab,\mu}} - \frac{\partial L^{ab,\mu\nu}}{\partial g_{\alpha\beta}}\right)\bigg|_\psi \frac{\partial\psi_{ab,\mu}}{\partial x^\nu} = -\frac{\partial H_{\mathcal{P}}}{\partial g_{\alpha\beta}}\bigg|_\psi - \psi_{ab,\mu}\left(\frac{\partial L^{\alpha\beta,\mu}}{\partial g_{ab}} - \frac{\partial L^{ab,\mu}}{\partial g_{\alpha\beta}}\right)\bigg|_\psi .$$

And rearranging the terms, these equations are equivalent to the Einstein equations (3.34).

Formulation using multimomentum coordinates.

As we have said, the coordinates $(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu})$ arise naturally from the unified formalism. Nevertheless, the standard way to describe locally the Hamiltonian formalism of classical field theory consists in using the natural coordinates in the multimomentum phase spaces; that is, multimomentum coordinates. Then, the first relevant result is:

Proposition 3.5. *The coordinates $p^{\alpha\beta,\mu}$ and $g_{\alpha\beta,\mu}$ are in one-to-one correspondence.*

Proof. The starting point is to consider the constraints $p^{\alpha\beta,\mu} = L^{\alpha\beta,\mu}(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu})$ which define partially the constraint submanifold $\mathcal{W}_{\mathcal{L}}$, and from these relations we can isolate the coordinates $g_{\alpha\beta,\mu}$. Indeed, the functions

$$\begin{aligned} V_{\alpha\beta,\mu}(g_{\alpha\beta}, p^{\alpha\beta,\mu}) &= \frac{p^{\lambda\sigma,\nu}}{3\varrho n(\alpha\beta)}(-2g_{\alpha\lambda}g_{\beta\mu}g_{\sigma\nu} - 2g_{\alpha\mu}g_{\beta\lambda}g_{\sigma\nu} + 6g_{\alpha\lambda}g_{\beta\sigma}g_{\mu\nu} + \\ &\qquad g_{\alpha\nu}g_{\beta\mu}g_{\lambda\sigma} + g_{\alpha\mu}g_{\beta\nu}g_{\lambda\sigma}) \end{aligned}$$

satisfy that

$$g_{\alpha\beta,\mu} = V_{\alpha\beta,\mu}(g_{\alpha\beta}, L^{\lambda\sigma,\nu}(g_{\alpha\beta}, g_{\alpha\beta,\mu})) ,$$

and these relations give the coordinates $g_{\alpha\beta,\mu}$ as functions of $p^{\lambda\sigma,\nu}$ and the other coordinates. □

Thus we can use $(x^\mu, g_{\alpha\beta}, p^{\alpha\beta,\mu})$ as coordinates of $\mathcal{P}$ and then rewrite the Hamiltonian function

$$H_{\mathcal{P}}(x^\mu, g_{\alpha\beta}, p^{\alpha\beta,\mu}) = H_{\mathcal{P}}(x^\mu, g_{\alpha\beta}, V_{\alpha\beta,\mu}(p^{\alpha\beta,\mu}, g_{\alpha\beta})) .$$

The field equations are derived again from (1.4) expressed using the new coordinates. Now, the Hamilton-Cartan form Ω_h has the local expression:

$$\Omega_h = \mathrm{d}H_{\mathcal{P}} \wedge \mathrm{d}^4x - \sum_{\alpha\leq\beta} \mathrm{d}p^{\alpha\beta,\mu} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu - \sum_{\alpha\leq\beta} \mathrm{d}L^{\alpha\beta,\mu\nu} \wedge \mathrm{d}V_{\alpha\beta,\mu} \wedge \mathrm{d}^3x_\nu ,$$

and the local expression of a representative of a class $\{\mathbf{X}_h\}$ of semi-holonomic multivector fields in $\mathcal{P}$ is

$$\mathbf{X}_h = \bigwedge_{i=\nu}^{4}\left(\frac{\partial}{\partial x^\nu} + F_{\alpha\beta,\nu}\frac{\partial}{\partial g_{\alpha\beta}} + G_\nu^{\alpha\beta,\mu}\frac{\partial}{\partial p^{\alpha\beta,\mu}}\right); .$$

with $F_{\alpha\beta,\nu}(x^\mu, g_{\alpha\beta}, p^{\alpha\beta,\mu}), G_\nu^{\alpha\beta,\mu}(x^\mu, g_{\alpha\beta}, p^{\alpha\beta,\mu}) \in C^\infty(\mathcal{P})$. From (1.4) we obtain

$$\begin{aligned} \frac{\partial H_{\mathcal{P}}}{\partial g_{\alpha\beta}} &= -G_\mu^{\alpha\beta,\mu} + G_\nu^{rs,k}\frac{\partial V_{ab,c}}{\partial p^{rs,k}}\frac{\partial L^{ab,c\nu}}{\partial g_{\alpha\beta}} + F_{rs,\nu}\left(\frac{\partial V_{ab,c}}{\partial g_{rs}}\frac{\partial L^{ab,c\nu}}{\partial g_{\alpha\beta}} - \frac{\partial V_{ab,c}}{\partial g_{\alpha\beta}}\frac{\partial L^{ab,c\nu}}{\partial g_{rs}}\right) \\ \frac{\partial H_{\mathcal{P}}}{\partial p^{\alpha\beta,\mu}} &= F_{\alpha\beta,\mu} - F_{rs,\nu}\frac{\partial V_{ab,c}}{\partial p^{\alpha\beta,\mu}}\frac{\partial L^{ab,c\nu}}{\partial g_{rs}} , \end{aligned}$$

which would be the classical Hamilton-De Donder-Weil equations for a first order field theory except by the fact that they contain extra-terms because the Einstein-Hilbert Lagrangian is of second order and $L^{\alpha\beta,\mu\nu} = \dfrac{1}{n(\mu\nu)}\dfrac{\partial L}{\partial g_{\alpha\beta,\mu\nu}}$ does not vanish. A solution to these equations is

$$\mathbf{X}_h = \bigwedge_{i=\nu}^{4}\left(\frac{\partial}{\partial x^\nu} + V_{\alpha\beta,\mu}\frac{\partial}{\partial g_{\alpha\beta}} + g_{rs}(\Gamma^r_{\nu\lambda}\Gamma^s_{\mu\sigma} + \Gamma^r_{\nu\sigma}\Gamma^s_{\mu\lambda})\frac{\partial V_{\alpha\beta\mu}}{\partial g_{\lambda\sigma,\gamma}}\frac{\partial}{\partial p^{\alpha\beta,\mu}}\right) ,$$

where the velocities in the connection are expressed using the momenta, which is a holonomic (i.e., integrable) multivector field in $\mathcal{P}$.

Finally, we consider the equations of the integral sections of $\mathbf{X}_h$. These equations can be obtained from equation (1.2) which, for a section $\psi(x^\alpha) = (x^\alpha, \psi_{\alpha\beta}(x^\alpha), \psi^{\alpha\beta,\mu}(x^\alpha))$, leads to

$$\begin{aligned}
\left.\frac{\partial H_{\mathcal{P}}}{\partial g_{\alpha\beta}}\right|_\psi &= \frac{\partial \psi^{\alpha\beta,\mu}}{\partial x^\mu} + \frac{\partial \psi^{rs,k}}{\partial x^\nu}\left.\left(\frac{\partial V_{ab,c}}{\partial p^{rs,k}}\frac{\partial L^{ab,c\nu}}{\partial g_{\alpha\beta}}\right)\right|_\psi + \frac{\partial \psi_{rs}}{\partial x^\nu}\left.\left(\frac{\partial V_{ab,c}}{\partial g_{rs}}\frac{\partial L^{ab,c\nu}}{\partial g_{\alpha\beta}} - \frac{\partial V_{ab,c}}{\partial g_{\alpha\beta}}\frac{\partial L^{ab,c\nu}}{\partial g_{rs}}\right)\right|_\psi \\
\left.\frac{\partial H_{\mathcal{P}}}{\partial p^{\alpha\beta,\mu}}\right|_\psi &= \frac{\partial \psi_{\alpha\beta}}{\partial x^\mu} - \frac{\partial \psi_{rs}}{\partial x^\nu}\left.\left(\frac{\partial V_{ab,c}}{\partial p^{\alpha\beta,\mu}}\frac{\partial L^{ab,c\nu}}{\partial g_{rs}}\right)\right|_\psi .
\end{aligned}$$

3.1.4 An equivalent first-order Lagrangian to Einstein-Hilbert

There exists a first-order Lagrangian equivalent to the Einstein-Hilbert Lagrangian, which is different to the Einstein-Palatini one [16, 86]. Now we study the Lagrangian and the Hamiltonian formalism of this model, comparing them with the Hamiltonian formulations for the Einstein-Hilbert Lagrangian presented in the above section. As it is a first order Lagrangian, we need to use the multisymplectic formalisms developed for these kind of theories; in particular, those reviewed in [85].

The configuration manifold $\pi : E \to M$, is the same described in Section 3.1.1, and the Lagrangian formalisms takes place in the first jet bundle $J^1\pi$, with coordinates $(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu})$. The first-order Lagrangian density proposed in [86] is $\overline{\mathcal{L}} = \overline{L}\,\mathrm{d}^4x$, where the Lagrangian function is

$$\overline{L} = L_0 - \sum_{\substack{\alpha\leq\beta\\ \lambda\leq\sigma}} g_{\alpha\beta,\mu}g_{\lambda\sigma,\nu}\frac{\partial L^{\alpha\beta,\mu\nu}}{\partial g_{\lambda\sigma}} \in \mathrm{C}^\infty(J^1\pi) . \tag{3.38}$$

The Poincaré-Cartan form for this Lagrangian is

$$\Omega_{\overline{\mathcal{L}}} = \mathrm{d}\overline{L}\wedge \mathrm{d}^4x - \sum_{\alpha\leq\beta}\mathrm{d}\frac{\partial\overline{L}}{\partial g_{\alpha\beta,\mu}}\wedge \mathrm{d}g_{\alpha\beta}\wedge \mathrm{d}^3x_\mu . \tag{3.39}$$

The Lagrangian $\overline{L}$ is regular and hence $\Omega_{\overline{\mathcal{L}}}$ is a multisymplectic form. For the Lagrangian system $(J^1\pi, \Omega_{\overline{\mathcal{L}}}, TJ^1\pi, \mathfrak{C}^1)$ we look for solutions to the equations (1.2) or (1.4) and, as the system is regular, solutions exist everywhere in $J^1\pi$ (there are no Lagrangian constraints). Although it is a first order system, in [86] it is shown how these equations coincide with the Einstein equations.

As $\overline{L}$ is regular, we can state the standard Hamiltonian formalism for first-order regular field theories. Being $J^1\pi^*$ the ("first-order") reduced multimomentum bundle, whose natural coordinates are $(x^\mu, g_{\alpha\beta}, \overline{p}^{\alpha\beta,\mu})$, the corresponding Legendre map $\mathcal{F}\overline{\mathcal{L}}\colon J^1\pi \to J^1\pi^*$ is given by

$$\mathcal{F}\overline{\mathcal{L}}^* x^\mu = x^\mu \quad , \quad \mathcal{F}\overline{\mathcal{L}}^* g_{\alpha\beta} = g_{\alpha\beta} \quad , \quad \mathcal{F}\overline{\mathcal{L}}^*\overline{p}^{\alpha\beta,\mu} = \frac{\partial\overline{L}}{\partial g_{\alpha\beta,\mu}} = L^{\alpha\beta,\mu} - \sum_{\lambda\leq\sigma} g_{\lambda\sigma,\nu}\frac{\partial L^{\lambda\sigma,\nu\mu}}{\partial g_{\alpha\beta}} .$$

Then we have the Hamilton-Cartan form $\Omega_{\overline{h}} := (\mathcal{F}\overline{\mathcal{L}}^{-1})^*\Omega_{\overline{\mathcal{L}}} \in \Omega^4(J^1\pi^*)$. This multisymplectic form can also be obtained introducing the Hamiltonian section $\overline{h}\colon J^1\pi^* \to \Lambda^4_2(E)$ whose local expression is

$$\overline{h}(x^\mu, g_{\alpha\beta}, \overline{p}^{\alpha\beta,\mu}) = (x^\mu, g_{\alpha\beta}, -\overline{H}, \overline{p}^{\alpha\beta,\mu}) \ .$$

where $\overline{H}$ is the Hamiltonian function associated with $\overline{L}$, whose local expression is

$$\overline{H} = \sum_{\alpha\leq\beta} \overline{p}^{\alpha\beta,\mu}(g_{\alpha\beta,\mu} \circ \mathcal{F}\overline{\mathcal{L}}^{-1}) - \overline{L} \circ \mathcal{F}\overline{\mathcal{L}}^{-1} = \overline{L} \circ \mathcal{F}\overline{\mathcal{L}}^{-1} \ .$$

In this way, we have constructed the Hamiltonian system $(J^1\pi^*, \Omega_{\overline{h}}, TJ^1\pi^*, \{0\})$ and the corresponding Hamilton field equations have solutions everywhere in $J^1\pi^*$ (there are no Hamiltonian constraints). Furthermore, as $\mathcal{F}\overline{\mathcal{L}}$ is a diffeomorphism, every solution to the Lagrangian problem stated for the Lagrangian system $(J^1\pi, \Omega_{\overline{\mathcal{L}}}, TJ^1\pi, \mathfrak{C}^1)$ induces a solution to the Hamiltonian problem stated for the Hamiltonian system $(J^1\pi^*, \Omega_{\overline{h}}, TJ^1\pi^*, \{0\})$ via this Legendre map, and conversely.

The following result relates this approach to the one we have presented in the above section.

Proposition 3.6. *$\Phi^* H_{\mathcal{P}} = \overline{L}$ and, as a consequence, $\Phi^*\Omega_h = \Omega_{\overline{\mathcal{L}}}$.*

Proof. In order to prove these equalities, it suffices to prove that, $H_{\mathcal{P}}$ and Ω_h have the same local coordinate expressions than $\overline{L}$ and $\Omega_{\overline{\mathcal{L}}}$, respectively.

First, from (3.36), using (3.38) and taking into account the coordinate expressions stated in (3.2), (3.3), and (3.4), we obtain that

$$\begin{aligned} H_{\mathcal{P}} &= \sum_{\substack{\alpha\leq\beta\\ \mu\leq\nu}} L^{\alpha\beta,\mu\nu} g_{\alpha\beta,\mu\nu} + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu} g_{\alpha\beta,\mu} - L = \sum_{\alpha\leq\beta}\left(\frac{\partial L_0}{\partial g_{\alpha\beta,\mu}} - D_v L^{\alpha\beta,\mu\nu}\right) g_{\alpha\beta,\mu} - L_0 \\ &= 2L_0 - \sum_{\substack{\alpha\leq\beta\\ \lambda\leq\sigma}} g_{\alpha\beta,\mu} g_{\lambda\sigma,\nu} \frac{\partial L^{\alpha\beta,\mu\nu}}{\partial g_{\lambda\sigma}} - L_0 = \overline{L} \ . \end{aligned}$$

We have used that $\frac{\partial L_0}{\partial g_{\alpha\beta,\mu}} g_{\alpha\beta,\mu} = 2L_0$, which is a consequence of L_0 being homogeneous of grade 2 on the velocities. Now we compute

$$\frac{\partial \overline{L}}{\partial g_{\alpha\beta,\mu}} = \frac{\partial L_0}{\partial g_{\alpha\beta,\mu}} - \sum_{\lambda\leq\sigma} g_{\lambda\sigma,\nu}\left(\frac{\partial L^{\alpha\beta,\mu\nu}}{\partial g_{\lambda\sigma}} + \frac{\partial L^{\lambda\sigma,\nu\mu}}{\partial g_{\alpha\beta}}\right) = L^{\alpha\beta,\mu} - g_{\lambda\sigma,\nu}\frac{\partial L^{\lambda\sigma,\nu\mu}}{\partial g_{\alpha\beta}} \ ;$$

then, using these last results and bearing in mind (3.39) and (3.37), we have that

$$\begin{aligned} \Omega_{\overline{\mathcal{L}}} &= \mathrm{d}\overline{L} \wedge \mathrm{d}^4x - \sum_{\alpha\leq\beta} \mathrm{d}\frac{\partial \overline{L}}{\partial g_{\alpha\beta,\mu}} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu \\ &= \mathrm{d}H_{\mathcal{P}} \wedge \mathrm{d}^4x - \sum_{\alpha\leq\beta} \mathrm{d}L^{\alpha\beta,\mu} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu + \sum_{\substack{\alpha\leq\beta\\ \lambda\leq\sigma}} \mathrm{d}\left(g_{\lambda\sigma,\nu}\frac{\partial L^{\lambda\sigma,\nu\mu}}{\partial g_{\alpha\beta}}\right) \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu \\ &= \mathrm{d}H_{\mathcal{P}} \wedge \mathrm{d}^4x - \sum_{\alpha\leq\beta} \mathrm{d}L^{\alpha\beta,\mu} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu + \sum_{\substack{\alpha\leq\beta\\ \lambda\leq\sigma}} \frac{\partial L^{\lambda\sigma,\nu\mu}}{\partial g_{\alpha\beta}} \mathrm{d}g_{\lambda\sigma,\nu} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu \\ &+ \sum_{\alpha\leq\beta} g_{\lambda\sigma,\nu}\frac{\partial^2 L^{\lambda\sigma,\nu\mu}}{\partial g_{\gamma\eta}\partial g_{\alpha\beta}} \mathrm{d}g_{\gamma\eta} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu \ . \end{aligned}$$

The last term vanishes because the coefficient is symmetric under the change of the indices γ, λ by α, β, but the exterior product is skewsymmetric. Finally, notice that $L^{\lambda\sigma,\nu\mu}$ do not contain derivatives of the metric, thus we can write

$$\sum_{\substack{\alpha\leq\beta\\ \lambda\leq\sigma}} \frac{\partial L^{\lambda\sigma,\nu\mu}}{\partial g_{\alpha\beta}} \mathrm{d}g_{\lambda\sigma,\nu} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu = -\sum_{\sigma\leq\lambda} \mathrm{d}L^{\lambda\sigma,\nu\mu} \wedge \mathrm{d}g_{\lambda\sigma,\nu} \wedge \mathrm{d}^3x_\mu \ ,$$

and, therefore, we can conclude that $\Omega_{\overline{\mathcal{L}}}$ and Ω_h have the same local expression. □

As a consequence of this result, the solutions to the Hamiltonian problem stated for the Hamiltonian system $(\mathcal{P}, \Omega_{\mathcal{P}})$ and to the Lagrangian problem stated for the Lagrangian system $(J^1\pi, \Omega_{\overline{\mathcal{L}}})$ are in one-to-one correspondence by the map Φ.

Observe that we have also the diffeomorphism $\Psi = \Phi^{-1} \circ \mathcal{F}\overline{\mathcal{L}} \colon \mathcal{P} \to J^1\pi^*$. Therefore, the solutions to the Hamiltonian problems stated for the Hamiltonian systems $(\mathcal{P}, \Omega_{\mathcal{P}})$ and $(J^1\pi^*, \Omega_{\overline{h}})$ are also one-to-one related by this map.

Summarizing, we have proved that the following formulations are equivalent:

$$(J^1\pi^*, \Omega_{\overline{h}}, TJ^1\pi^*, \{0\}) \xleftarrow{\ \mathcal{F}\overline{\mathcal{L}}\ } (J^1\pi, \Omega_{\overline{\mathcal{L}}}, TJ^1\pi, \mathfrak{C}^1) \xleftarrow{\ \ \Phi\ \ } (\mathcal{P}, \Omega_h, T\mathcal{P}, \{0\})$$

(where, in the last case, we can use the local description using multimomentum coordinates or not). Locally, this equivalence means that all the formulations lead to the same equations (Einstein's equations), up to a change of variables and, hence, every solution in each formalism induces a solution in the others via the appropriate diffeomorphism. The following diagram summarizes all the picture:

$$\begin{tikzcd}
 & & J^2\pi^\dagger \arrow[d, "\mu"] \\
J^3\pi \arrow[rr, "\mathcal{F}\mathcal{L}"] \arrow[rrd, "\mathcal{F}\mathcal{L}_o"] \arrow[ddd, "\pi^3_1"'] & & J^2\pi^\ddagger \\
 & S_f \arrow[lu, hook] \arrow[r] & \mathcal{P} \arrow[u, hook, "\jmath"] \arrow[uu, bend right, "h"'] \arrow[d, "\Psi"] \\
J^1\pi \arrow[rru, "\Phi"] \arrow[rr, "\mathcal{F}\overline{\mathcal{L}}"'] & & J^1\pi^* \arrow[d, bend left, "\overline{h}"] \\
 & & \mathcal{M}\pi \equiv \Lambda^4_2(E) \arrow[u, "\bar{\mu}"]
\end{tikzcd}$$

3.2 The Einstein-Hilbert model with energy-matter sources

3.2.1 The Einstein-Hilbert Langrangian

The Einstein-Hilbert model with energy-matter sources is described by a Lagrangian density $\mathcal{L}_{\mathfrak{G}} = \mathcal{L}_{EH} + \mathcal{L}_{\mathfrak{m}}$, where $\mathcal{L}_{\mathfrak{m}} = L_{\mathfrak{m}}\,(\bar{\pi}^2)^*\eta \in \Omega^4(J^2\pi)$, and $L_{\mathfrak{m}} \in C^\infty(J^2\pi)$ represents the energy-matter source and depends only on the metric and the first and second derivatives of its components. It is related to the *stress-energy-momentum tensor* $\mathrm{T}_{\mu\nu}$ by

$$\mathrm{T}_{\mu\nu} = \frac{c^4}{\varrho n(\mu\nu) 8\pi G} g_{\alpha\mu} g_{\beta\nu} L^{\alpha\beta}_{\mathfrak{m}} \ .$$

(For a geometric study on the stress-energy-momentum tensors see, for instance, [38, 41, 55, 71, 98]). Then, we can write $\mathcal{L}_{\mathfrak{G}} = L_{\mathfrak{G}}\,(\bar{\pi}^2)^*\eta \in \Omega^4(J^2\pi)$, with $L_{\mathfrak{G}} = L_{EH} + L_{\mathfrak{m}} \in C^\infty(J^2\pi)$.

The behaviour of the theory depends on the source. Nevertheless, some qualitative properties can be studied in general, as long as we know the degeneracy of the source.

Definition 3.1. *For a function $f \in C^\infty(J^2\pi)$, consider*

$$f^{\alpha\beta,\mu\nu} := \frac{1}{n(\mu\nu)}\frac{\partial f}{\partial g_{\alpha\beta,\mu\nu}}, \quad f^{\alpha\beta,\mu} := \frac{\partial f}{\partial g_{\alpha\beta,\mu}} - D_\nu f^{\alpha\beta,\mu\nu}, \quad f^{\alpha\beta} = \frac{\partial f}{\partial g_{\alpha\beta}} - D_\mu f^{\alpha\beta,\mu}.$$

(Notice that $f^{\alpha\beta,\mu} \in C^\infty(J^3\pi)$ and $f^{\alpha\beta} \in C^\infty(J^4\pi)$). Then, the degree *of f is the smallest natural number* $\deg(f) = s$ *such that:*

$$\mathrm{L}(X)f^{\alpha\beta,\mu} = \mathrm{L}(X)f^{\alpha\beta,\mu\nu} = 0 \quad ; \quad \text{for every } X \in \mathfrak{X}^V(\pi^4_{s-1})\ ;\ (0 \leq \alpha \leq \beta \leq 3, 0 \leq \mu \leq \nu \leq 3)\ .$$

If $f^{\alpha\beta,\mu} = f^{\alpha\beta,\mu\nu} = 0$, we define $\deg(f) = 0$.

Now, applying the proposition 2.9, we obtain that:

Proposition 3.7. *If $\deg(f) = s$, then $\mathrm{L}(X)f^{\alpha\beta} = 0$; for every $X \in \mathfrak{X}^V(\pi^4_s)$ ($\alpha \leq \beta$), and hence $f^{\alpha\beta}$ are π^4_s-projectable functions.*

The degree of $L_{\mathfrak{G}}$ characterizes partially the behaviour of the theory, as we are going to see in the next paragraphs. For instance, if a Lagrangian is regular it has degree 4, but there are also singular Lagrangians with degree 4. The Einstein-Hilbert Lagrangian in vacuum, L_{EH}, has degree 2. For a source such that $\deg(L_{\mathfrak{m}}) > 2$, we have that $\deg(L_{\mathfrak{G}}) = \deg(L_{\mathfrak{m}})$. The so-called $f(R)$ theories of gravity have $\deg(L_{\mathfrak{G}}) > 2$. For these kinds of systems it is possible to obtain some constraints in the unified and the Lagrangian formalisms but the Hamiltonian formalism depends strongly on the particular energy-matter source. For a source such that $\deg(L_{\mathfrak{m}}) \leq 2$, we have that $\deg(L_{\mathfrak{G}}) \leq 2$), and these theories have a well defined Hamiltonian formalism; in particular, for the case that $\deg(L_{\mathfrak{m}}) \leq 1$ we obtain the general semiholonomic solution. These cases include the energy-matter sources coupled only to the metric; that is, $\deg(L_{\mathfrak{m}}) = 0$, like the electromagnetic source or the perfect fluid. We will present the former as an example.

3.2.2 Langrangian-Hamiltonian Unified formalism

As $L_{\mathfrak{G}} \in C^\infty(J^2\pi)$, we can work with the same manifolds introduced in Section 3.1; that is, the symmetric higher-order jet multimomentum bundles $\mathcal{W} = J^3\pi \times_{J^1\pi} J^2\pi^\ddagger$ and $\mathcal{W}_r = J^3\pi \times_{J^1\pi} J^2\pi^\ddagger$. The pull-back of the Lagrangian to these manifolds is denoted in the same way as above: $\hat{L}_{\mathfrak{G}} = (\pi^3_2 \circ \rho^r_1)^* L_{\mathfrak{G}} \in C^\infty(\mathcal{W}_r)$ (or in $C^\infty(\mathcal{W})$). Then,

$$\hat{H}_{\mathfrak{G}} = \sum_{\alpha\leq\beta} p^{\alpha\beta,\mu} g_{\alpha\beta,\mu} + \sum_{\substack{\alpha\leq\beta\\ \mu\leq\nu}} p^{\alpha\beta,\mu\nu} g_{\alpha\beta,\mu\nu} - \hat{L}_{\mathfrak{G}}\ ;$$

The Liouville forms in $\mathcal{W}_r$, $\Theta_{\mathfrak{G}r}$ and $\Omega_{\mathfrak{G}r}$, are defined likewise and have the local expressions

$$\Theta_{\mathfrak{G}r} = -\hat{H}_{\mathfrak{G}}\mathrm{d}^4x + \sum_{\alpha\leq\beta} p^{\alpha\beta,\mu}\mathrm{d}g_{\alpha\beta}\wedge \mathrm{d}^3x_\mu + \sum_{\alpha\leq\beta}\frac{1}{n(\mu\nu)}p^{\alpha\beta,\mu\nu}\mathrm{d}g_{\alpha\beta,\mu}\wedge\mathrm{d}^3x_\nu\ ,$$

$$\Omega_{\mathfrak{G}r} = \mathrm{d}\hat{H}_{\mathfrak{G}}\wedge\mathrm{d}^4x - \sum_{\alpha\leq\beta}\mathrm{d}p^{\alpha\beta,\mu}\wedge\mathrm{d}g_{\alpha\beta}\wedge\mathrm{d}^3x_\mu - \sum_{\alpha\leq\beta}\frac{1}{n(\mu\nu)}\mathrm{d}p^{\alpha\beta,\mu\nu}\wedge\mathrm{d}g_{\alpha\beta,\mu}\wedge\mathrm{d}^3x_\nu\ .$$

he Lagrangian-Hamiltonian system is $(\mathcal{W}_r, \Omega_{\mathfrak{G}r}, T\mathcal{W}_r, \rho^{r*}_1\mathfrak{C}^3)$.

Proposition 3.1, which defines the Legendre transformation, also holds for $\hat{L}_{\mathfrak{G}}$:

Proposition 3.8. *A section $\psi \in \Gamma(\rho^r_M)$ solution to the equation* (1.2) *takes values in a* 140*-codimensional submanifold $\jmath_{\mathcal{L}_{\mathfrak{G}}} \colon \mathcal{W}_{\mathcal{L}_{\mathfrak{G}}} \hookrightarrow \mathcal{W}_r$ which is identified with the graph of a bundle map $\mathcal{F}\mathcal{L}_{\mathfrak{G}} \colon J^3\pi \to J^2\pi^\ddagger$, over $J^1\pi$, defined locally by*

$$\mathcal{F}\mathcal{L}_{\mathfrak{G}}^* p^{\alpha\beta,\mu} = \frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3} \frac{1}{n(\mu\nu)} D_\nu \left(\frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu\nu}} \right) = \hat{L}_{\mathfrak{G}}^{\alpha\beta,\mu} \ , \ \mathcal{F}\mathcal{L}_{\mathfrak{G}}^* p^{\alpha\beta,\mu\nu} = \frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu\nu}} \ .$$

What is equivalent, the submanifold $\mathcal{W}_{\mathcal{L}_{\mathfrak{G}}}$ is the graph of a bundle morphism $\widetilde{\mathcal{F}\mathcal{L}_{\mathfrak{G}}} \colon J^3\pi \to J^2\pi^\dagger$ over $J^1\pi$ defined locally by

$$\begin{aligned}
\widetilde{\mathcal{F}\mathcal{L}_{\mathfrak{G}}}^* p^{\alpha\beta,\mu} &= \frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3} \frac{1}{n(\mu\nu)} D_\nu \left(\frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu\nu}} \right) = \hat{L}_{\mathfrak{G}}^{\alpha\beta,\mu} \ , \\
\widetilde{\mathcal{F}\mathcal{L}_{\mathfrak{G}}}^* p^{\alpha\beta,\mu\nu} &= \frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu\nu}} \ , \\
\widetilde{\mathcal{F}\mathcal{L}_{\mathfrak{G}}}^* p &= \hat{L}_{\mathfrak{G}} - g_{\alpha\beta,\mu} \left(\frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3} \frac{1}{n(\mu\nu)} D_\nu \left(\frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu\nu}} \right) \right) - g_{\alpha\beta,\mu\nu} \frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu\nu}} \\
&= \hat{L}_{\mathfrak{G}} - \sum_{\alpha\le\beta} p^{\alpha\beta,\mu} g_{\alpha\beta,\mu} - \sum_{\alpha\le\beta,\mu\le\nu} p^{\alpha\beta,\mu\nu} g_{\alpha\beta,\mu\nu} \ .
\end{aligned}$$

As an application of theorem 2.4,

Theorem 3.2. *A solution to the equation* (1.2) *exists only in a submanifold $\mathcal{W}_{\mathfrak{G}} \hookrightarrow \mathcal{W}_r$ wich, depending on the degree of $L_{\mathfrak{m}}$, is locally defined by the following constraints (for $0 \le \alpha \le \beta \le 3$, $0 \le \mu \le \nu \le 3$, $0 \le \tau \le 3$):*

- *If* $\deg(L_{\mathfrak{m}}) = 4$*:* $\quad p^{\alpha\beta,\mu\nu} - \frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu\nu}} = 0, \quad p^{\alpha\beta,\mu} - \hat{L}_{\mathfrak{G}}^{\alpha\beta,\mu} = 0.$
- *If* $\deg(L_{\mathfrak{m}}) = 3$*:* $\quad p^{\alpha\beta,\mu\nu} - \frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu\nu}} = 0, \quad p^{\alpha\beta,\mu} - \hat{L}_{\mathfrak{G}}^{\alpha\beta,\mu} = 0, \quad \hat{L}_{\mathfrak{G}}^{\alpha\beta} = 0.$
- *If* $\deg(L_{\mathfrak{m}}) \le 2$*:* $\quad p^{\alpha\beta,\mu\nu} - \frac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\alpha\beta,\mu\nu}} = 0, \quad p^{\alpha\beta,\mu} - \hat{L}_{\mathfrak{G}}^{\alpha\beta,\mu} = 0, \quad \hat{L}_{\mathfrak{G}}^{\alpha\beta} = 0, \quad D_\tau \hat{L}_{\mathfrak{G}}^{\alpha\beta} = 0.$

Proof. For the case $\deg(L_{\mathfrak{m}}) = 4$, the first two restrictions, which involve the momenta, hold for every second order field theory (Proposition 3.8 and [83]).

If $\deg(L_{\mathfrak{m}}) \le 2$, then $\deg(L_{\mathfrak{G}}) = c \le 2$. Therefore $\Theta_{L_{\mathfrak{G}}}$ is π^4_c-semibasic (in particular π^4_2-semibasic), which implies the other two restrictions. They can also be obtained by a similar procedure as in Section 3.1.2.

Likewise, if $\deg(L_{\mathfrak{m}}) = 3$, then $\deg(L_{\mathfrak{G}}) = 3$, and $\Theta_{L_{\mathfrak{G}}}$ is π^4_3-semibasic, which implies $\hat{L}_{\mathfrak{G}}^{\alpha\beta} = 0$. □

Depending on the energy-matter term, maybe there are not any holonomic solution on $\mathcal{W}_{\mathfrak{G}}$. In this situations, a smaller submanifold has to be considered in order to find a holonomic solution.

3.2.3 Recovering the Lagrangian and Hamiltonian formalisms

In section chapter 1 we have stated how to recover the Lagrangian formalism from the unified formalism for the Einstein-Hilbert Lagrangian with no energy-matter souces. As in that case, now the Lagrangian formalism takes place in $J^3\pi$, and the Poincaré-Cartan forms (3.27) associated with the Einstein-Hilbert Lagrangian with energy-matter sources are

$$\Theta_{\mathcal{L}_{\mathfrak{G}}} \equiv \widetilde{\mathcal{FL}_{\mathfrak{G}}}^*\Theta_1^s \in \Omega^4(J^3\pi) \quad , \quad \Omega_{\mathcal{L}_{\mathfrak{G}}} \equiv \widetilde{\mathcal{FL}_{\mathfrak{G}}}^*\Omega_1^s = -\mathrm{d}\Theta_{\mathcal{L}_{\mathfrak{G}}} \in \Omega^5(J^3\pi) \, ,$$

which have the local expressions

$$\begin{aligned}
\Theta_{\mathcal{L}_{\mathfrak{G}}} &= H_{\mathfrak{G}}\mathrm{d}^4x + \sum_{\alpha\leq\beta} L_{\mathfrak{G}}^{\alpha\beta,\mu}\mathrm{d}g_{\alpha\beta}\wedge\mathrm{d}^3x_\mu + \sum_{\alpha\leq\beta} L_{\mathfrak{G}}^{\alpha\beta,\mu\nu}\mathrm{d}g_{\alpha\beta,\mu}\wedge\mathrm{d}^3x_\nu \, , \\
\Omega_{\mathcal{L}_{\mathfrak{G}}} &= \mathrm{d}H_{\mathfrak{G}}\wedge\mathrm{d}^4x - \sum_{\alpha\leq\beta}\mathrm{d}L_{\mathfrak{G}}^{\alpha\beta,\mu}\mathrm{d}g_{\alpha\beta}\wedge\mathrm{d}^{m-1}x_\mu - \sum_{\alpha\leq\beta}\mathrm{d}L_{\mathfrak{G}}^{\alpha\beta,\mu\nu}\mathrm{d}g_{\alpha\beta,\mu}\wedge\mathrm{d}^{m-1}x_\nu \, ;
\end{aligned}$$

where

$$H_{\mathfrak{G}} \equiv (\jmath_{\mathcal{L}}\circ(\rho_1^{\mathcal{L}})^{-1})^*\hat{H}_{\mathfrak{G}} = \sum_{\alpha\leq\beta} L_{\mathfrak{G}}^{\alpha\beta,\mu\nu}g_{\alpha\beta,\mu\nu} + \sum_{\alpha\leq\beta} L_{\mathfrak{G}}^{\alpha\beta,\mu}g_{\alpha\beta,\mu} - L_{\mathfrak{G}} \, ,$$

and $L_{\mathfrak{G}}^{\alpha\beta,\mu\nu}, L_{\mathfrak{G}}^{\alpha\beta,\mu}$ have the same coordinate expressions than $\hat{L}^{\alpha\beta,\mu\nu}$, $\hat{L}^{\alpha\beta,\mu}$, and $\hat{L}_0$.

The Lagrangian problem associated with the Lagrangian system $(J^3\pi, \Omega_{\mathcal{L}_{\mathfrak{G}}}, TJ^3\pi, \mathfrak{C}^3)$ is stated like in the equations (1.2) and (1.4), but for $\Omega_{\mathcal{L}_{\mathfrak{G}}}$ instead of $\Omega_{\mathcal{L}}$. The solutions are related to the solutions of the unified formalism by Proposition 1.3 and Theorem 1.2.

The Lagrangian counterpart of theorem 3.2 is:

Corollary 3.1. *A solution to the equation* (1.2) *exists only in a submanifold* $S_{\mathfrak{G}} \hookrightarrow J^3\pi$ *wich, depending on the degree of* $L_{\mathfrak{m}}$*, is locally defined by the following constraints (for* $0 \leq \alpha \leq \beta \leq 3$*):*

- *If* $\deg(L_{\mathfrak{m}}) = 3$*:* $\quad L_{\mathfrak{G}}^{\alpha\beta} = 0$.
- *If* $\deg(L_{\mathfrak{m}}) \leq 2$*:* $\quad L_{\mathfrak{G}}^{\alpha\beta} = 0, \quad D_\tau L_{\mathfrak{G}}^{\alpha\beta} = 0$.

The existence of holonomic solutions depends on the energy-mass term. In some cases we must continue the constraint algorithm, together with an integrability algorithm.

Finally, the equations of the integral sections (1.2) can be analyzed in a similar fashion as in Section 3.1.2, and using Proposition 1.3. This leads to the Euler-Lagrange equations

$$L_{\mathfrak{G}}^{\alpha\beta}|_{j^3\phi} = L_{EH}^{\alpha\beta}|_{j^3\phi} + L_{\mathfrak{m}}^{\alpha\beta}|_{j^3\phi} = -\varrho\, n(\alpha\beta)\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R - \frac{1}{\varrho n(\alpha\beta)}L_{\mathfrak{m}}^{\alpha\beta}\right)\bigg|_{j^3\phi} = 0 \, , \qquad (3.40)$$

Introducing the *stress-energy-momentum tensor* as

$$T_{\mu\nu} = \frac{c^4}{8\pi G\varrho n(\alpha\beta)}g_{\alpha\mu}g_{\beta\nu}L_{\mathfrak{m}}^{\alpha\beta} \, .$$

where G as the Newton's gravitational constant and c the speed of light, then

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \, ,$$

and equations (3.40) are equivalent to the Einstein equations with stress-energy-momentum tensor.

All these results can be also obtained applying the constraint algorithm straightforwardly to the equation (1.2), in the same way as we have done for the unified formalism; then doing a purely Lagrangian analysis.

For establishing the multimomentum Hamiltonian formalism we use the Legendre maps $\mathcal{FL}_{\mathfrak{G}}$ and $\widetilde{\mathcal{FL}_{\mathfrak{G}}}$ defined in Proposition 3.8. Now, we denote $\widetilde{\mathcal{P}_{\mathfrak{G}}} = \widetilde{\mathcal{FL}_{\mathfrak{G}}}(J^3\pi) \stackrel{\tilde{\jmath}}{\hookrightarrow} J^2\pi^\dagger$ and $\mathcal{P}_{\mathfrak{G}} = \mathcal{FL}_{\mathfrak{G}}(J^3\pi) \stackrel{\jmath}{\hookrightarrow} J^2\pi^\ddagger$, with the natural projection $\bar{\pi}_{\mathcal{P}_{\mathfrak{G}}} \colon \mathcal{P}_{\mathfrak{G}} \to M$. In order to assure the existence of the Hamiltonian formalism we demand that the Lagrangian density $\mathcal{L}_{\mathfrak{G}} \in \Omega^4(J^2\pi)$ is, at least, *almost-regular*. Then we can define the Hamiltonian forms $\Theta_{h_{\mathfrak{G}}}$ and $\Omega_{h_{\mathfrak{G}}}$, and then we have the Hamiltonian system $(\mathcal{P}_{\mathfrak{G}}, \Omega_{h_{\mathfrak{G}}}, T\mathcal{P}_{\mathfrak{G}}, (\pi^\ddagger_{J^1\pi} \circ \jmath)^*\mathfrak{C}^1))$. This Hamiltonian formalism is recovered from the unified formalism following Proposition 1.4 and Theorem 1.3.

In general, the formalism depends strongly on the singularity of the theory. Nevertheless, if $\deg(L_{\mathfrak{m}}) \leq 2$ (or equivalently $\deg(L_{\mathfrak{G}}) \leq 2$), we have a similar situation as in the vacuum case. In particular:

Proposition 3.9. *If $\deg(L_{\mathfrak{G}}) \leq 2$, then $L_{\mathfrak{G}}$ is an almost-regular Lagrangian and $\mathcal{P}_{\mathfrak{G}}$ is diffeomorphic to $J^1\pi$.*

Proof. If $\deg(L_{\mathfrak{G}}) \leq 2$, we have that

$$\mathrm{T}_{j^3_x\phi}\mathcal{FL}_{\mathfrak{G}} = \begin{pmatrix} Id_4 & 0 & 0 & 0 & 0 \\ 0 & Id_{10} & 0 & 0 & 0 \\ 0 & 0 & Id_{40} & 0 & 0 \\ \dfrac{\partial \hat{L}^{\alpha\beta,\mu}_{\mathfrak{G}}}{\partial x^\tau} & \dfrac{\partial \hat{L}^{\alpha\beta,\mu}_{\mathfrak{G}}}{\partial g_{\gamma\delta}} & \dfrac{\partial \hat{L}^{\alpha\beta,\mu}_{\mathfrak{G}}}{\partial g_{\gamma\delta,\tau}} & 0 & 0 \\ \dfrac{\partial \hat{L}_{\mathfrak{G}}}{\partial x^\tau \partial g_{\alpha\beta,\mu\nu}} & \dfrac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\gamma\delta}\partial g_{\alpha\beta,\mu\nu}} & \dfrac{\partial \hat{L}_{\mathfrak{G}}}{\partial g_{\gamma\delta,\tau}\partial g_{\alpha\beta,\mu\nu}} & 0 & 0 \end{pmatrix}$$

Then we have that $\mathrm{rank}(\mathrm{T}_{j^3_x\phi}\mathcal{FL}) = 54$ at every point $j^3_x\phi \in J^3\pi$. Therefore $\mathrm{T}\mathcal{FL}_{\mathfrak{G}}$ is surjective and $\mathcal{FL}_{\mathfrak{G}}$ is a submersion. From here the proof is the same as in Proposition 3.3. □

In general the functions $\hat{L}^{\alpha\beta,\mu}_{\mathfrak{G}}$ are not invertible, thus we use the non momenta coordinates $(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu})$ as local coordinates adapted to $\mathcal{P}_{\mathfrak{G}}$. The function $H_{\mathcal{P}_{\mathfrak{G}}}$ is defined by

$$H_{\mathcal{P}_{\mathfrak{G}}} = \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu\nu}_{\mathfrak{G}} g_{\alpha\beta,\mu\nu} + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu}_{\mathfrak{G}} g_{\alpha\beta,\mu} - L_{\mathfrak{G}} \ ,$$

and the Hamilton-Cartan form have the coordinate expressions

$$\Omega_{h_{\mathfrak{G}}} = -\mathrm{d}\Theta_{h_{\mathfrak{G}}} = \mathrm{d}H_{\mathcal{P}_{\mathfrak{G}}} \wedge \mathrm{d}^4x - \sum_{\alpha\leq\beta} \mathrm{d}L^{\alpha\beta,\mu}_{\mathfrak{G}} \wedge \mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^3x_\mu - \sum_{\alpha\leq\beta} \mathrm{d}L^{\alpha\beta,\mu\nu}_{\mathfrak{G}} \wedge \mathrm{d}g_{\alpha\beta,\mu} \wedge \mathrm{d}^3x_\nu \ .$$

The resulting Hamiltonian equations for sections (1.2) are

$$\left.\left(\frac{\partial L^{\alpha\beta,\nu}_{\mathfrak{G}}}{\partial g_{ab,\mu}} - \frac{\partial L^{ab,\mu\nu}_{\mathfrak{G}}}{\partial g_{\alpha\beta}}\right)\right|_\psi \frac{\partial \psi_{ab,\mu}}{\partial x^\nu} = -\left.\frac{\partial H_{\mathcal{P}_{\mathfrak{G}}}}{\partial g_{\alpha\beta}}\right|_\psi - \psi_{ab,\mu}\left.\left(\frac{\partial L^{\alpha\beta,\mu}_{\mathfrak{G}}}{\partial g_{ab}} - \frac{\partial L^{ab,\mu}_{\mathfrak{G}}}{\partial g_{\alpha\beta}}\right)\right|_\psi \ ,$$

and rearranging the terms, these equations are locally equivalent to the Einstein equations (3.40).

If $\deg(L_{\mathfrak{G}}) > 2$, then $\mathcal{FL}_{\mathfrak{G}}$ may not be a submersion and, hence, $L_{\mathfrak{G}}$ is not almost-regular. In these cases the construction of the Hamiltonian formalism is more complicated.

3.2.4 Example: Electromagnetic source

Consider the case of a free electromagnetic source with electromagnetic tensor $F^{\mu\nu}$. The corresponding Lagrangian function is

$$L_{\mathfrak{m}} = \sqrt{|det(g_{\alpha\beta})|}\, F_{\mu\nu}F^{\mu\nu} \ ,$$

where the components of the tensor $F_{\mu\nu}$ are functions on the base manifold M. In this case, $\deg(L_{\mathfrak{m}}) = 1$, and the stress-energy-momentum tensor is

$$\mathrm{T}_{\mu\nu} = \frac{c^4}{\varrho n(\mu\nu)8\pi G} g_{\alpha\mu} g_{\beta\nu} L^{\alpha\beta}_{\mathfrak{m}} = \frac{c^4}{\varrho n(\mu\nu)8\pi G} g_{\alpha\mu} g_{\beta\nu} \frac{\partial L_{\mathfrak{m}}}{\partial g_{\alpha\beta}} = \frac{c^4}{4\pi G}\left(\frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}\right) .$$

The corresponding form $\Theta_{\mathfrak{S}^r}$ is π^4_2-projectable, which implies that $\mathfrak{X}^V(\pi^4_2)$ are gauge vector fields. By Theorem 3.2, solutions to the field equations exist on the points of the submanifold defined by

$$p^{\alpha\beta,\mu\nu} - \frac{\hat{L}_{\mathfrak{S}}}{\partial g_{\alpha\beta,\mu\nu}} = 0 \quad , \quad p^{\alpha\beta,\mu} - \hat{L}^{\alpha\beta,\mu}_{\mathfrak{S}} = 0 \quad , \quad \hat{L}^{\alpha\beta}_{\mathfrak{S}} = 0 \quad , \quad D_\tau \hat{L}^{\alpha\beta}_{\mathfrak{S}} = 0 \, .$$

The first two restrictions define the Legendre transformation, and the last two fix the gauge freedom of the higher derivatives. The local expression of any semiholonomic multivetor field solution of (1.4) can be obtained by combining these restrictions, the holonomy conditions, and the solution obtained in the Appendix 3.A.3,

$$\begin{aligned}\mathbf{X}_{LH} \quad &= \bigwedge_{\tau=0}^{3} \sum_{\substack{\alpha\leq\beta \\ \mu\leq\nu\leq\lambda}} \left(\frac{\partial}{\partial x^\tau} + g_{\alpha\beta,\tau}\frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} + \right. \\ &\qquad \left. D_\tau D_\lambda \hat{F}_{\alpha\beta;\mu,\nu}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}} + D_\tau \hat{L}^{\alpha\beta,\mu}_{\mathfrak{S}}\frac{\partial}{\partial p^{\alpha\beta,\mu}} + D_\tau \frac{\hat{L}_{\mathfrak{S}}}{\partial g_{\alpha\beta,\mu\nu}}\frac{\partial}{\partial p^{\alpha\beta,\mu\nu}}\right) ,\end{aligned}$$

where $\hat{F}_{\alpha\beta;\mu,\nu} = (\pi^3_1 \circ \rho^r_1)^* F_{\alpha\beta;\mu,\nu} \in C^\infty(\mathcal{W}_r)$, and

$$F_{\alpha\beta;\mu,\nu} = g_{\lambda\sigma}(\Gamma^\lambda_{\nu\alpha}\Gamma^\sigma_{\mu\beta} + \Gamma^\lambda_{\nu\beta}\Gamma^\sigma_{\mu\alpha}) + \frac{c^4}{4\pi G} g_{\alpha\beta}\left(g^{\lambda\sigma} F_{\mu\lambda} F_{\nu\sigma} - \frac{5}{4} g_{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma}\right) + {F^h}_{\lambda\sigma;\mu,\nu} \in C^\infty(J^1\pi) \, .$$

The Lagrangian formalism takes place in $J^3\pi$, but the Corollary 3.1 states that a solution exists in the submanifold defined by

$$L^{\alpha\beta}_{\mathfrak{S}} = 0 \quad , \quad D_\tau L^{\alpha\beta}_{\mathfrak{S}} = 0 \, .$$

The Euler-Lagrange equations (3.40) are equivalent to the Einstein equations

$$\left.\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)\right|_{j^3\phi} = \frac{c^4}{4\pi G}\left.\left(\frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}\right)\right|_{j^3\phi} ,$$

A section $\psi\colon M \to E$ is a solution to the Einstein equations if, on the points of its image, it is a section of a multivector field with local expression

$$\mathbf{X}_L = \bigwedge_{\tau=0}^{3} \sum_{\substack{\alpha\leq\beta \\ \mu\leq\nu\leq\lambda}} \left(\frac{\partial}{\partial x^\tau} + g_{\alpha\beta,\tau}\frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu}} + g_{\alpha\beta,\mu\nu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu}} + D_\tau D_\lambda {F'}_{\alpha\beta;\mu,\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu\nu\lambda}}\right) ,$$

where ${F'}_{\alpha\beta;\mu,\nu} = \pi^{3*}_1 F_{\alpha\beta;\mu,\nu} \in C^\infty(J^3\pi)$.

For the Hamiltonian formalism, we have the Hamiltonian system $(\mathcal{P}_{\mathfrak{S}}, \Omega_{h_{\mathfrak{S}}}, T\mathcal{P}_{\mathfrak{S}}, \Phi^*\mathfrak{C}^1)$, where $\mathcal{P}_{\mathfrak{S}}$ is diffeomorphic to $J^1\pi$ by the action of Φ, as a consequence of Proposition 3.9, and the Hamiltonian function giving the Hamiltonian section h is

$$H_{\mathcal{P}_{\mathfrak{S}}} = H_{\mathcal{P}} - L_{\mathfrak{m}} \, ,$$

where $H_{\mathcal{P}}$ is the Hamiltonian for the vacuum case (3.36). A semiholonomic multivector field solution to (1.4) has the local expression

$$\mathbf{X}_H = \bigwedge_{\tau=0}^{3} \sum_{\alpha\leq\beta} \left(\frac{\partial}{\partial x^\tau} + g_{\alpha\beta,\tau}\frac{\partial}{\partial g_{\alpha\beta}} + F_{\alpha\beta;\mu,\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu}}\right) .$$

3.3 Symmetries for the Einstein-Hilbert model

Now, consider the Einstein-Hilbert Lagrangian (withouth energy-matter sources).

Definition 3.2. *Let $F\colon M \to M$ be a diffeomorphism. The* canonical lift of ϕ to the bundle of metrics *E is the diffeomorphism $\mathcal{F}\colon E \to E$ defined as follows: for every $(x, g_x) \in E$, then $\mathcal{F}(x, g_x) := (F(x), (F^{-1})^*(g_x))$. (Thus $\pi \circ \mathcal{F} = F \circ \pi$).*

Let $Z \in \mathfrak{X}(M)$. The canonical lift of Z to the bundle of metrics E *is the vector field $Y \in \mathfrak{X}(E)$ whose associated local one-parameter groups of diffeomorphisms $\mathcal{F}_t$ are the canonical lifts to the bundle of metrics E of the local one-parameter groups of diffeomorphisms F_t of Z.*

In coordinates, if $Z = f^\mu(x)\dfrac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$, the canonical lift of Z to the bundle of metrics is

$$Y = f^\mu \frac{\partial}{\partial x^\mu} - \sum_{\alpha\leq\beta}\left(\frac{\partial f^\mu}{\partial x^\alpha} g_{\mu\beta} + \frac{\partial f^\mu}{\partial x^\beta} g_{\mu\alpha}\right)\frac{\partial}{\partial g_{\alpha\beta}} \ ,$$

and then

$$\begin{aligned} j^1 Y &= f^\mu \frac{\partial}{\partial x^\mu} - \sum_{\alpha\leq\beta}\left(\frac{\partial f^\mu}{\partial x^\alpha} g_{\mu\beta} + \frac{\partial f^\mu}{\partial x^\beta} g_{\mu\alpha}\right)\frac{\partial}{\partial g_{\alpha\beta}} \\ &\quad - \sum_{\alpha\leq\beta}\left(\frac{\partial^2 f^\nu}{\partial x^\alpha \partial x^\mu} g_{\nu\beta} + \frac{\partial^2 f^\nu}{\partial x^\beta \partial x^\mu} g_{\alpha\nu} + \frac{\partial f^\nu}{\partial x^\alpha} g_{\nu\beta,\mu} + \frac{\partial f^\nu}{\partial x^\beta} g_{\alpha\nu,\mu} + \frac{\partial f^\nu}{\partial x^\mu} g_{\alpha\beta,\nu}\right)\frac{\partial}{\partial g_{\alpha\beta,\mu}} \\ &\equiv f^\mu \frac{\partial}{\partial x^\mu} + \sum_{\alpha\leq\beta} Y_{\alpha\beta}\frac{\partial}{\partial g_{\alpha\beta}} + \sum_{\alpha\leq\beta} Y_{\alpha\beta\mu}\frac{\partial}{\partial g_{\alpha\beta,\mu}} \ . \end{aligned}$$

For every $Z \in \mathfrak{X}(M)$, as $\mathcal{L}_{\mathfrak{V}}$ is invariant under diffeomorphisms, we have that

$$\mathrm{L}(j^2 Y)\mathcal{L}_{\mathfrak{V}} = \mathrm{L}(j^3 Y)((\pi^3_2)^*\mathcal{L}_{\mathfrak{V}}) = 0 \ ,$$

and $j^3 Y$ it is an exact infinitesimal Cartan symmetry. Its associated conserved quantity is $\xi_Y = i(Y^3)\Theta_{\mathcal{L}_{\mathfrak{V}}}$ and, as $\Theta_{\mathcal{L}_{\mathfrak{V}}}$ is π^3_1-basic, there exists $\Theta^1_{\mathcal{L}_{\mathfrak{V}}} \in \Omega^4(J^1\pi)$ (which has the same coordinate expression) such that $\Theta_{\mathcal{L}_{\mathfrak{V}}} = (\pi^3_1)^*\Theta^1_{\mathcal{L}_{\mathfrak{V}}}$; then

$$\begin{aligned} \xi_Y &= i(j^3 Y)\Theta_{\mathcal{L}_{\mathfrak{V}}} = i(j^1 Y)\Theta^1_{\mathcal{L}_{\mathfrak{V}}} = \left(\sum_{\alpha\leq\beta} Y_{\alpha\beta} L^{\alpha\beta,\mu} + \sum_{\alpha\leq\beta} Y_{\alpha\beta\nu} L^{\alpha\beta,\nu\mu} - f^\mu H\right)\mathrm{d}^3 x_\mu \\ &\quad + \sum_{\alpha\leq\beta}\left(f^\nu L^{\alpha\beta,\mu} - f^\mu L^{\alpha\beta,\nu}\right)\mathrm{d}g_{\alpha\beta}\wedge \mathrm{d}^2 x_{\nu\mu} + \sum_{\alpha\leq\beta}\left(f^\nu L^{\alpha\beta,\lambda\mu} - f^\mu L^{\alpha\beta,\lambda\nu}\right)\mathrm{d}g_{\alpha\beta,\lambda}\wedge \mathrm{d}^2 x_{\nu\mu} \ , \end{aligned}$$

where $\mathrm{d}^2 x_{\mu\nu} = i\left(\dfrac{\partial}{\partial x^\nu}\right) i\left(\dfrac{\partial}{\partial x^\mu}\right)\mathrm{d}^4 x$.

These vector fields are the only natural infinitesimal Lagrangian symmetries [72, 86] for this model.

3.A Solutions to the Hamiltonian equations for the Einstein-Hilbert model

We have seen that the Einstein equations can be stated from different geometrical points of view. In order to solve them, we can use whichever we find more appropriate. Indeed, as it is explained in chapter 1, the solutions can be transported canonically from one formalism to another. In this section we solve the equations for multivector fields in the Hamiltonian formalism.

A solution to the Einstein equations is a metric over the manifold; that is, a section $\psi\colon M \to E$. The multivector fields we find provide system of partial differential equations whose solutions are the sections (1.1). In this sense, finding the multivector fields is only the first step on solving Einstein equations. Nevertheless, this approach leads to new equations, which may be more appealing. For instance, they have a unique solution provided an initial condition: there is no need of boundary conditions.

The relation between sections and multivector fields is explained in chapter 1. Only holonomic multivector fields have associated holonomic integrable sections. Nevertheless, we look first for semiholonomic multivector fields, except in the case of the vacuum case, where we find a particular solution which is a proper holonomic multivector field. It is used in Theorem 3.1 to determine the final submanifold.

Since the equations for multivector fields are lineal, we proceed to find a particular solution and then the homogeneous solutions for the vacuum case. Later, we will consider energy-matter sources.

3.A.1 Particular solution (without energy-matter sources)

The Hamiltonian premultisymplectic system is $(\mathcal{P}, \Omega_h, T\mathcal{P}, (\pi^{\ddagger}_{J^1\pi} \circ \jmath)^*\mathfrak{C}^1))$. The local expression of a representative of a class $\{\mathbf{X}_h\}$ of semiholonomic multivector fields in $\mathcal{P}$ is

$$\mathbf{X}_h = \bigwedge_{\nu=0}^{3} \left(\frac{\partial}{\partial x^\nu} + \sum_{\alpha\leq\beta} \left(F_{\alpha\beta,\nu} \frac{\partial}{\partial g_{\alpha\beta}} + F_{\alpha\beta;\mu,\nu} \frac{\partial}{\partial g_{\alpha\beta;\mu}} \right) \right) .$$

Where the holonomy condition is not explicitly imposed, as it will be recover by the field equations. Equation (1.4) takes the local expression:

$$\frac{\partial H_{\mathcal{P}}}{\partial g_{\alpha\beta,\mu}} + \sum_{\lambda\leq\sigma} F_{\lambda\sigma,\nu} \left(\frac{\partial L^{\alpha\beta,\mu\nu}}{\partial g_{\lambda\sigma}} - \frac{\partial L^{\lambda\sigma,\nu}}{\partial g_{\alpha\beta,\mu}} \right) = 0 , \quad (3.41)$$

$$\frac{\partial H_{\mathcal{P}}}{\partial g_{\alpha\beta}} + \sum_{\lambda\leq\sigma} F_{\lambda\sigma,\mu} \left(\frac{\partial L^{\alpha\beta,\mu}}{\partial g_{\lambda\sigma}} - \frac{\partial L^{\lambda\sigma,\mu}}{\partial g_{\alpha\beta}} \right) + \sum_{\lambda\leq\sigma} F_{\lambda\sigma;\nu,\mu} \left(\frac{\partial L^{\alpha\beta,\mu}}{\partial g_{\lambda\sigma,\nu}} - \frac{\partial L^{\lambda\sigma,\nu\mu}}{\partial g_{\alpha\beta}} \right) = 0 . \quad (3.42)$$

We denote $U^{\alpha\beta,\mu\nu,\lambda\sigma} = \dfrac{\partial L^{\alpha\beta,\mu\nu}}{\partial g_{\lambda\sigma}} - \dfrac{\partial L^{\lambda\sigma,\nu}}{\partial g_{\alpha\beta,\mu}}$, whose explicit expressions are

$$\begin{aligned} U^{\alpha\beta\mu\nu\lambda\sigma} &= \frac{\varrho\, n(\alpha\beta) n(\lambda\sigma)}{4} \Big(-2 g^{\alpha\beta} g^{\lambda\sigma} g^{\mu\nu} + g^{\alpha\lambda} g^{\beta\sigma} g^{\mu\nu} + g^{\beta\lambda} g^{\alpha\sigma} g^{\mu\nu} \\ &+ g^{\alpha\beta} g^{\lambda\mu} g^{\sigma\nu} + g^{\alpha\beta} g^{\sigma\mu} g^{\lambda\nu} + g^{\lambda\sigma} g^{\alpha\nu} g^{\beta\mu} + g^{\lambda\sigma} g^{\beta\nu} g^{\alpha\mu} \\ &- g^{\alpha\nu} g^{\lambda\mu} g^{\beta\sigma} - g^{\beta\nu} g^{\lambda\mu} g^{\alpha\sigma} - g^{\alpha\nu} g^{\sigma\mu} g^{\beta\lambda} - g^{\beta\nu} g^{\sigma\mu} g^{\alpha\lambda} \Big) , \end{aligned} \quad (3.43)$$

and they fulfil the following relations:

$$U^{\alpha\beta,\mu\nu,\lambda\sigma} = U^{\lambda\sigma,\mu\nu,\alpha\beta} = -U^{\alpha\mu,\beta\nu,\lambda\sigma} .$$

The equations are algebraic, in the sense that no derivatives of $F_{\alpha\beta,\mu}$, nor $F_{\alpha\beta,\mu\nu}$ appear. (The indices are symmetrized as usual).

We start by solving equation (3.41). First, we rewrite it as

$$\sum_{\lambda\leq\sigma} (F_{\lambda\sigma,\nu} - g_{\lambda\sigma,\nu}) U^{\alpha\beta,\mu\nu,\lambda\sigma} = 0 .$$

Indeed, since $H_{\mathcal{P}} = \sum_{\lambda\leq\sigma} L^{\lambda\sigma,\nu} g_{\lambda\sigma,\nu} - L_0$,

$$\begin{aligned}
\frac{\partial H_{\mathcal{P}}}{\partial g_{\alpha\beta,\mu}} &= \sum_{\lambda\leq\sigma} \frac{\partial L^{\lambda\sigma,\nu}}{\partial g_{\alpha\beta,\mu}} g_{\lambda\sigma,\nu} + L^{\alpha\beta,\mu} - \frac{\partial L_0}{\partial g_{\alpha\beta,\mu}} \\
&= \sum_{\lambda\leq\sigma} \frac{\partial L^{\lambda\sigma,\nu}}{\partial g_{\alpha\beta,\mu}} g_{\lambda\sigma,\nu} + \frac{\partial L_0}{\partial g_{\alpha\beta,\mu}} - \sum_{\lambda\leq\sigma} \frac{\partial L^{\alpha\beta,\mu\nu}}{\partial g_{\lambda\sigma}} g_{\lambda\sigma,\nu} - \frac{\partial L_0}{\partial g_{\alpha\beta,\mu}} = -\sum_{\lambda\leq\sigma} U^{\alpha\beta,\mu\nu,\lambda\sigma} g_{\lambda\sigma,\nu} \,.
\end{aligned}$$

Now we multiply it by

$$\begin{aligned}
V_{\alpha\beta\mu,abc} = \frac{1}{\varrho\, n(\alpha\beta)} (& g_{\alpha\mu} g_{\beta b} g_{ac} + 2 g_{\alpha\mu} g_{\beta c} g_{ab} + g_{\alpha\beta} g_{b\mu} g_{ac} - g_{\alpha\beta} g_{\mu c} g_{ab} \\
& - 3 g_{\alpha a} g_{\beta c} g_{b\mu} - 3 g_{\alpha b} g_{\beta c} g_{a\mu} + g_{\alpha\mu} g_{\beta a} g_{bc} + g_{\alpha\beta} g_{c\mu} g_{ab}) \,,
\end{aligned}$$

which works as a sort of inverse; then we obtain

$$\begin{aligned}
\sum_{\lambda\leq\sigma} (F_{\lambda\sigma,\nu} - g_{\lambda\sigma,\nu}) U^{\alpha\beta,\mu\nu,\lambda\sigma} V_{\alpha\beta\mu,abc} &= \frac{3}{2} (F_{\lambda\sigma,\nu} - g_{\lambda\sigma,\nu}) (\delta_a^\lambda \delta_b^\sigma \delta_c^\nu + \delta_b^\lambda \delta_a^\sigma \delta_c^\nu) \\
&= 3 (F_{ab,c} - g_{ab,c}) = 0 \,.
\end{aligned}$$

Therefore, $F_{\lambda\sigma,\nu} = g_{\lambda\sigma,\nu}$ and the holonomy condition is recovered. Using this condition, equation (1.4) becomes:

$$\frac{\partial H}{\partial g_{\alpha\beta}} + \sum_{\lambda\leq\sigma} g_{\lambda\sigma,\mu} \left(\frac{\partial L^{\alpha\beta,\mu}}{\partial g_{\lambda\sigma}} - \frac{\partial L^{\lambda\sigma,\mu}}{\partial g_{\alpha\beta}} \right) - \sum_{\lambda\leq\sigma} F_{\lambda\sigma;\mu,\nu} U^{\lambda\sigma,\mu\nu,\alpha\beta} = 0 \,. \tag{3.44}$$

These equations have as particular solution $F^P_{\lambda\sigma;\mu,\nu} = \frac{1}{2} g_{\alpha\beta} (\Gamma^\alpha_{\nu\lambda} \Gamma^\beta_{\mu\sigma} + \Gamma^\alpha_{\nu\sigma} \Gamma^\beta_{\mu\lambda})$, which can be checked after some computation. The multivector field

$$\mathbf{X}^P_h = \bigwedge_{\nu=0}^{3} X^P_\nu = \bigwedge_{\nu=0}^{3} \left(\frac{\partial}{\partial x^\nu} + \sum_{\alpha\leq\beta} \left(g_{\alpha\beta,\nu} \frac{\partial}{\partial g_{\alpha\beta}} + \frac{1}{2} g_{\lambda\sigma} (\Gamma^\lambda_{\nu\alpha} \Gamma^\sigma_{\mu\beta} + \Gamma^\lambda_{\nu\beta} \Gamma^\sigma_{\mu\alpha}) \frac{\partial}{\partial g_{\alpha\beta;\mu}} \right) \right) \,,$$

is semi-holonomic and $\bar{\pi}_{\mathcal{P}}$-transverse, and verifies that $i(\mathbf{X}^P_h)\Omega_h = 0$. The last thing to check is that it is integrable. The Lie bracket for two arbitrary components X^P_γ and X^P_ρ is

$$\begin{aligned}
[X^P_\gamma, X^P_\rho] = & \sum_{\alpha\leq\beta} \left(F^P_{\alpha\beta;\rho,\gamma} - F^P_{\alpha\beta;\gamma,\rho} \right) \frac{\partial}{\partial g_{\alpha\beta}} + \\
& \sum_{\substack{\alpha\leq\beta \\ \lambda\leq\sigma}} \left(g_{\lambda\sigma,\gamma} \frac{\partial F^P_{\alpha\beta;\mu,\rho}}{\partial g_{\lambda\sigma}} + F^P_{\lambda\sigma;\nu,\gamma} \frac{\partial F^P_{\alpha\beta;\mu,\rho}}{\partial g_{\lambda\sigma,\nu}} - g_{\lambda\sigma,\rho} \frac{\partial F^P_{\alpha\beta;\mu,\gamma}}{\partial g_{\lambda\sigma}} - F^P_{\lambda\sigma;\nu,\rho} \frac{\partial F^P_{\alpha\beta;\mu,\gamma}}{\partial g_{\lambda\sigma,\nu}} \right) \frac{\partial}{\partial g_{\alpha\beta,\mu}} \,.
\end{aligned}$$

The vector field $[X^P_\gamma, X^P_\rho]$ is $\bar{\pi}_1$-vertical. Therefore, the integrability condition can only be achieved if $[X^P_\gamma, X^P_\rho] = 0$. Imposing the condition on the coefficient of $\dfrac{\partial}{\partial g_{\alpha\beta}}$, we obtain that $F^P_{\alpha\beta;\rho,\gamma} - F^P_{\alpha\beta;\gamma,\rho} = 0$. These conditions are expected since, for a section, they represent the equality between second order crossed partial derivatives. Clearly the solution proposed fulfils this condition. After a rather long but straightforward computation, we can check that the coefficients of $\dfrac{\partial}{\partial g_{\alpha\beta,\mu}}$ also vanish.

3.A.2 General solution (without energy-matter sources)

The existence of a particular solution $\mathbf{X}^P_h$ to (1.4) is relevant, because it implies that no extra restrictions are needed, as showed in Theorem 3.1. Now, we explore the general behaviour of the solutions of (1.4).

As we have shown before, (1.4) boils down to (3.44), which are linear equations. Therefore, we can split any solution into a particular and a homogeneous part:

$$F_{\lambda\sigma;\mu,\nu} = \frac{1}{2} g_{\alpha\beta}(\Gamma^{\alpha}_{\nu\lambda}\Gamma^{\beta}_{\mu\sigma} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\beta}_{\mu\lambda}) + F^{\mathfrak{h}}_{\lambda\sigma;\mu,\nu} \,.$$

The homogeneous part $F^{\mathfrak{h}}_{\lambda\sigma;\mu,\nu}$ is a set of functions which cancel out when contracted with (3.43), namely

$$\sum_{\lambda\leq\sigma} F^{\mathfrak{h}}_{\lambda\sigma;\mu,\nu} U^{\lambda\sigma,\mu\nu,\alpha\beta} = 0 \,. \tag{3.45}$$

The correspondent multivector field:

$$\mathbf{X}_h = \bigwedge_{\nu=0}^{3}\left(\frac{\partial}{\partial x^\nu} + \sum_{\alpha\leq\beta}\left(F_{\alpha\beta,\nu}\frac{\partial}{\partial g_{\alpha\beta}} + \left(\frac{1}{2}g_{\lambda\sigma}(\Gamma^{\lambda}_{\nu\alpha}\Gamma^{\sigma}_{\mu\beta} + \Gamma^{\lambda}_{\nu\beta}\Gamma^{\sigma}_{\mu\alpha}) + F^{\mathfrak{h}}_{\alpha\beta;\mu,\nu}\right)\frac{\partial}{\partial g_{\alpha\beta;\mu}}\right)\right)$$

is a semiholonomic solution to (1.4). Nevertheless, it may not be integrable. Thus, the integrability of $\mathbf{X}_h$ leads to new constraints on the valid set of functions $T_{\alpha\beta;\mu,\nu}$. Condition (3.45) can be reformulated as follows:

Lemma 3.1. *A set of functions $F^{\mathfrak{h}}_{\alpha\beta;\mu,\nu}$, symmetric under the changes $\alpha \leftrightarrow \beta$ and $\mu \leftrightarrow \nu$, satisfies the condition*

$$\sum_{\lambda\leq\sigma} F^{\mathfrak{h}}_{\lambda\sigma;\mu,\nu} U^{\lambda\sigma,\mu\nu,\alpha\beta} = 0 \tag{3.46}$$

if, and only if,

$$g^{\lambda\sigma}\left(F^{\mathfrak{h}}_{\eta\tau;\lambda,\sigma} + F^{\mathfrak{h}}_{\lambda\sigma;\eta,\tau} - F^{\mathfrak{h}}_{\lambda\eta;\tau,\sigma} - F^{\mathfrak{h}}_{\lambda\tau;\eta,\sigma}\right) = 0 \,. \tag{3.47}$$

Proof. (3.46) can be rewritten as

$$\begin{aligned}\sum_{\lambda\leq\sigma} F^{\mathfrak{h}}_{\lambda\sigma;\mu,\nu} U^{\lambda\sigma,\mu\nu,\alpha\beta} &= \varrho\, n(\alpha\beta) g^{\alpha\beta} g^{\lambda\sigma} g^{\mu\nu}\left(-\frac{1}{2}F^{\mathfrak{h}}_{\lambda\sigma;\mu,\nu} + \frac{1}{2}F^{\mathfrak{h}}_{\lambda\mu;\nu,\sigma}\right) \\ &+ \varrho\, n(\alpha\beta) g^{\lambda\sigma} g^{\alpha\mu} g^{\nu\beta}\left(\frac{1}{2}F^{\mathfrak{h}}_{\mu\nu;\lambda,\sigma} + \frac{1}{2}F^{\mathfrak{h}}_{\lambda\sigma;\mu,\nu} - \frac{1}{2}F^{\mathfrak{h}}_{\lambda\mu;\nu,\sigma} - \frac{1}{2}F^{\mathfrak{h}}_{\lambda\nu;\mu,\sigma}\right)\end{aligned} \tag{3.48}$$

Contracting (3.46) with $g_{\alpha\beta}$, we obtain

$$2\varrho\, n(\alpha\beta) g^{\lambda\sigma} g^{\mu\nu}\left(-\frac{1}{2}F^{\mathfrak{h}}_{\lambda\sigma;\mu,\nu} + \frac{1}{2}F^{\mathfrak{h}}_{\lambda\mu;\nu,\sigma}\right) = 0 \,. \tag{3.49}$$

Therefore the first term in (3.48) vanishes. Contracting the remaining term with $g_{\alpha\eta}g_{\beta\tau}$ we obtain (3.47).

To prove the converse, contract (3.47) with $g^{\eta\tau}$. The resulting expression is equivalent to (3.49) because it is symmetric under the change $(\alpha\beta) \leftrightarrow (\eta\tau)$. Then, (3.46) follows straighforwardly. $\square$

The following theorem summarizes the above results:

Theorem 3.3. *For a class of multivectorfield $\{\mathbf{X}\} \subset \mathfrak{X}^4(\mathcal{P})$, the following conditions are equivalent:*

- $\{\mathbf{X}\}$ *is a solution to the Hamiltonian problem for multivector fields* (1.4) *for the system* $(\mathcal{P}, \Omega_h, T\mathcal{P}, (\pi^{\ddagger}_{J^1\pi} \circ \jmath)^*\mathfrak{C}^1)$.

- *Using the coordinates* $(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu})$, *the local expression of a represesentative of* $\{\mathbf{X}\}$ *is*

$$\mathbf{X}=\bigwedge_{\nu=0}^{3}\left(\frac{\partial}{\partial x^\nu}+\sum_{\alpha\leq\beta}\left(g_{\alpha\beta,\nu}\frac{\partial}{\partial g_{\alpha\beta}}+\left(F^P_{\alpha\beta;\mu,\nu}+F^{\mathfrak{h}}_{\alpha\beta;\mu,\nu}\right)\frac{\partial}{\partial g_{\alpha\beta;\mu}}\right)\right) ,$$

where $F^P_{\alpha\beta;\mu,\nu}=\frac{1}{2}g_{\lambda\sigma}(\Gamma^\lambda_{\nu\alpha}\Gamma^\sigma_{\mu\beta}+\Gamma^\lambda_{\nu\beta}\Gamma^\sigma_{\mu\alpha})$ *and* $F^{\mathfrak{h}}_{\alpha\beta;\mu,\nu}$ *satisfy that:*

1. $F^{\mathfrak{h}}_{\alpha\beta;\mu,\nu}=F^{\mathfrak{h}}_{\beta\alpha;\mu,\nu}=F^{\mathfrak{h}}_{\alpha\beta;\nu,\mu}$.
2. $g^{\alpha\beta}\left(F^{\mathfrak{h}}_{\eta\tau;\alpha,\beta}+F^{\mathfrak{h}}_{\alpha\beta;\eta,\tau}-F^{\mathfrak{h}}_{\alpha\eta;\tau,\beta}-F^{\mathfrak{h}}_{\alpha\tau;\eta,\beta}\right)=0$.
3. *It is a solution to the following differential equations (integrability condition):*

$$\begin{aligned}0 \;=\; & \sum_{\alpha\leq\beta}\left(F^{\mathfrak{h}}_{\alpha\beta;\mu,i}\frac{\partial F^{\mathfrak{h}}_{\lambda\sigma;\nu,j}}{\partial g_{\alpha\beta,\mu}}+\left(F^P_{\alpha\beta;\mu,i}\frac{\partial}{\partial g_{\alpha\beta,\mu}}+g_{\alpha\beta,i}\frac{\partial}{\partial g_{\alpha\beta}}+\frac{\partial}{\partial x^i}\right)F^{\mathfrak{h}}_{\lambda\sigma;\nu,j}+F^{\mathfrak{h}}_{\alpha\beta;\mu,i}\frac{\partial F^P_{\lambda\sigma;\nu,j}}{\partial g_{\alpha\beta,\mu}}\right)\\ - & \sum_{\alpha\leq\beta}\left(F^{\mathfrak{h}}_{\alpha\beta;\mu,j}\frac{\partial F^{\mathfrak{h}}_{\lambda\sigma;\nu,i}}{\partial g_{\alpha\beta,\mu}}+\left(F^P_{\alpha\beta;\mu,j}\frac{\partial}{\partial g_{\alpha\beta,\mu}}+g_{\alpha\beta,j}\frac{\partial}{\partial g_{\alpha\beta}}+\frac{\partial}{\partial x^j}\right)F^{\mathfrak{h}}_{\lambda\sigma;\nu,i}+F^{\mathfrak{h}}_{\alpha\beta;\mu,j}\frac{\partial F^P_{\lambda\sigma;\nu,i}}{\partial g_{\alpha\beta,\mu}}\right)\end{aligned}$$

The equivalent theorem for sections is:

Theorem 3.4. *For a holonomic section* $\psi: M\to\mathcal{P}$, *the following conditions are equivalent:*

1. ψ *is a solution to the Hamiltonian equations for sections* (1.2) *for the system* $(\mathcal{P},\Omega_h,T\mathcal{P},(\pi^\ddagger_{J^1\pi}\circ \jmath)^*\mathfrak{C}^1))$
2. ψ *is holonomic and a solution to the vacuum Einstein's equations*

$$\left.\left(R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R\right)\right|_\psi=0,\quad \alpha,\beta=0,\dots,3.$$

3. ψ *is holonomic and a solution to the differential equations*

$$\frac{\partial^2\psi_{\alpha\beta}}{\partial x^\mu\partial x^\nu}=\left.\left(F^{\mathfrak{h}}_{\alpha\beta;\mu,\nu}+\frac{1}{2}g_{\lambda\sigma}(\Gamma^\lambda_{\nu\alpha}\Gamma^\sigma_{\mu\beta}+\Gamma^\lambda_{\nu\beta}\Gamma^\sigma_{\mu\alpha})\right)\right|_\psi ,$$

for some set of functions $F^{\mathfrak{h}}_{\alpha\beta;\mu,\nu}$ *such that*

$$g^{\alpha\beta}\left(F^{\mathfrak{h}}_{\eta\tau;\alpha,\beta}+F^{\mathfrak{h}}_{\alpha\beta;\eta,\tau}-F^{\mathfrak{h}}_{\alpha\eta;\tau,\beta}-F^{\mathfrak{h}}_{\alpha\tau;\eta,\beta}\right)=0 ,$$

with initial conditions $\psi_{\alpha\beta}(x'^\mu)=g'_{\alpha\beta}$, $\dfrac{\partial\psi_{\alpha\beta}}{\partial x^\mu}(x'^\mu)=g'_{\alpha\beta,\mu}$.

Proof. The equivalence $1\iff 2$ is clear. The implication $1\Rightarrow 3$ comes from Theorem 3.3. To show $3\Rightarrow 2$, we first compute $R_{\alpha\beta}|_\psi$:

$$\begin{aligned}R_{\mu\eta}|_\psi &= g^{\nu\lambda}R_{\lambda\mu,\nu\eta}|_\psi\\ &= -\frac{1}{2}g^{\nu\lambda}\left.\left[\frac{\partial^2\psi_{\lambda\nu}}{\partial x^\mu\partial x^\eta}-\frac{\partial^2\psi_{\mu\nu}}{\partial x^\eta\partial x^\lambda}-\frac{\partial^2\psi_{\lambda\eta}}{\partial x^\mu\partial x^\nu}+\frac{\partial^2\psi_{\mu\eta}}{\partial x^\nu\partial x^\lambda}\right]\right|_\psi+\left.g^{\nu\lambda}g_{\tau\sigma}(\Gamma^\tau_{\eta\lambda}\Gamma^\sigma_{\mu\nu}-\Gamma^\tau_{\nu\lambda}\Gamma^\sigma_{\mu\eta})\right|_\psi\\ &= -\frac{1}{2}g^{\nu\lambda}\left.\left(F^{\mathfrak{h}}_{\eta\mu;\nu,\lambda}+F^{\mathfrak{h}}_{\nu\lambda;\eta,\mu}-F^{\mathfrak{h}}_{\nu\eta;\mu,\lambda}-F^{\mathfrak{h}}_{\nu\mu;\eta,\lambda}\right)\right|_\psi=0 .\end{aligned}$$

Then

$$\left.\left(R^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}R\right)\right|_\psi=\left.\left(g^{\alpha\mu}g^{\beta\eta}-\frac{1}{2}g^{\alpha\beta}g^{\mu\eta}\right)R_{\mu\eta}\right|_\psi=0$$

□

These theorems characterize the solutions to Einstein's equations without sources. The multivector fields solution to (3.3) are described by the set of functions $F^{\mathfrak{h}}_{\alpha\beta;\mu,\nu}$ which have some combinatoric properties. The integral sections of an integrable multivector field are given by (1.1). Every multivector field has one section at every point, therefore, only an initial condition is required to solve these equations. The condition 3 in Theorem 3.3 is the integrability condition. If a multivector field is not integrable, we can still consider (1.1), but we will find out that such equations have no solution everywhere. Thus, the integrability condition is also the condition of existence of solutions to (1.1). Given an initial condition, there is several section solution to the equations: one for every multivector field. Nevertheless, two different multivector fields may lead to the same sections at a given point. These multiple solution are not gauge related, because the multisymplectic form is regular.

3.A.3 General solution (with energy-matter sources)

Theorem 3.5. *Consider an energy-matter term $L_{\mathfrak{m}}$ with degree ≤ 1, and the system $(\mathcal{P}_{\mathfrak{S}}, \Omega_{h_{\mathfrak{S}}}, T\mathcal{P}_{\mathfrak{S}}, (\pi^{\ddagger}_{J^1\pi} \circ \jmath)^*\mathfrak{C}^1)$. For a class of multivector field $\{\mathbf{X}\} \subset \mathfrak{X}^4(\mathcal{P}_{\mathfrak{S}})$, the following conditions are equivalent:*

- $\{\mathbf{X}\}$ *is a class of semiholonomic multivector fields solution to the equation*

$$i(\mathbf{X})\Omega_{h_{\mathfrak{S}}} = 0 \quad , \quad \text{for every } \mathbf{X} \in \{\mathbf{X}\} \; .$$

- *The local expression of a representative $\mathbf{X} \in \{\mathbf{X}\}$ is*

$$\mathbf{X} = \bigwedge_{\nu=0}^{3} \sum_{\alpha\leq\beta} \left(\frac{\partial}{\partial x^\nu} + g_{\alpha\beta,\nu}\frac{\partial}{\partial g_{\alpha\beta}} + F_{\alpha\beta;\mu,\nu}\frac{\partial}{\partial g_{\alpha\beta,\mu}} \right)$$

with

$$F_{\lambda\sigma;\mu,\nu} = \frac{1}{2} g_{\lambda\sigma}(\Gamma^\lambda_{\nu\alpha}\Gamma^\sigma_{\mu\beta} + \Gamma^\lambda_{\nu\beta}\Gamma^\sigma_{\mu\alpha}) + g_{\lambda\sigma}\left(g_{\alpha\beta}g_{\mu\nu} - \frac{1}{3}g_{\alpha\mu}g_{\beta\nu} \right)\frac{{L_{\mathfrak{m}}}^{\alpha\beta}}{\varrho\, n(\alpha\beta)} + {F^{\mathfrak{h}}}_{\lambda\sigma;\mu,\nu}.$$

and where ${F^h}_{\alpha\beta;\mu,\nu}$ satisfies:

1. ${F^{\mathfrak{h}}}_{\alpha\beta;\mu,\nu} = {F^{\mathfrak{h}}}_{\beta\alpha;\mu,\nu} = {F^{\mathfrak{h}}}_{\alpha\beta;\nu,\mu}$.
2. $g^{\alpha\beta}\left({F^{\mathfrak{h}}}_{\eta\tau;\alpha,\beta} + {F^{\mathfrak{h}}}_{\alpha\beta;\eta,\tau} - {F^{\mathfrak{h}}}_{\alpha\eta;\tau,\beta} - {F^{\mathfrak{h}}}_{\alpha\tau;\eta,\beta} \right) = 0$.

Proof. The local expression of the equations is

$$\frac{\partial H_{EH}}{\partial g_{\alpha\beta}} + \sum_{\lambda\leq\sigma} g_{\lambda\sigma,\mu}\left(\frac{\partial L^{\alpha\beta,\mu}_{EH}}{\partial g_{\lambda\sigma}} - \frac{\partial L^{\lambda\sigma,\mu}_{EH}}{\partial g_{\alpha\beta}} \right) - \sum_{\lambda\leq\sigma} F_{\lambda\sigma;\mu,\nu}U^{\lambda\sigma,\mu\nu,\alpha\beta} = -L^{\alpha\beta}_{\mathfrak{m}} \; .$$

Then we split the unknown functions in three parts:

$$F_{\lambda\sigma;\mu,\nu} = {F^R}_{\lambda\sigma;\mu,\nu} + {F^{\mathfrak{m}}}_{\lambda\sigma;\mu,\nu} + {F^{\mathfrak{h}}}_{\lambda\sigma;\mu,\nu}$$

This first term is a solution to the equations at vacuum:

$$\frac{\partial H_{EH}}{\partial g_{\alpha\beta}} + \sum_{\lambda\leq\sigma} g_{\lambda\sigma,\mu}\left(\frac{\partial L^{\alpha\beta,\mu}_{EH}}{\partial g_{\lambda\sigma}} - \frac{\partial L^{\lambda\sigma,\mu}_{EH}}{\partial g_{\alpha\beta}} \right) = \sum_{\lambda\leq\sigma} {F^R}_{\lambda\sigma;\mu,\nu}U^{\lambda\sigma,\mu\nu,\alpha\beta} \; .$$

As we have seen before, we can choose ${F^R}_{\lambda\sigma;\mu,\nu} = \frac{1}{2}g_{\alpha\beta}(\Gamma^\alpha_{\nu\lambda}\Gamma^\beta_{\mu\sigma} + \Gamma^\alpha_{\nu\sigma}\Gamma^\beta_{\mu\lambda})$. The second term is a solution to

$$\sum_{\lambda\leq\sigma} {F^{\mathfrak{m}}}_{\lambda\sigma;\mu,\nu}U^{\lambda\sigma,\mu\nu,\alpha\beta} = {L_{\mathfrak{m}}}^{\alpha\beta}$$

We can choose $F^{\mathfrak{m}}{}_{\lambda\sigma;\mu,\nu} = \frac{1}{\varrho\, n(\tau\gamma)} g_{\lambda\sigma}\left(g_{\tau\mu}g_{\gamma\nu} - \frac{1}{3}g_{\tau\gamma}g_{\mu\nu}\right) L_{\mathfrak{m}}{}^{\tau\gamma}$, which belongs to $C^\infty(J^1\pi)$ because $\deg(L_{\mathfrak{m}}) \leq 1$. Indeed,

$$\sum_{\lambda\leq\sigma} \frac{1}{\varrho\, n(\tau\gamma)} g_{\lambda\sigma}\left(g_{\tau\mu}g_{\gamma\nu} - \frac{1}{3}g_{\tau\gamma}g_{\mu\nu}\right) L_{\mathfrak{m}}{}^{\tau\gamma} U^{\lambda\sigma,\mu\nu,\alpha\beta} =$$

$$\frac{n(\alpha\beta)}{n(\tau\gamma)}\left(g_{\tau\mu}g_{\gamma\nu} - \frac{1}{3}g_{\tau\gamma}g_{\mu\nu}\right) L_{\mathfrak{m}}{}^{\tau\gamma}\left(\frac{1}{2}g^{\alpha\mu}g^{\beta\nu} + \frac{1}{2}g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\mu\nu}\right) =$$

$$\frac{n(\alpha\beta)}{2\, n(\tau\gamma)}(\delta^\alpha_\tau\delta^\beta_\gamma + \delta^\beta_\tau\delta^\alpha_\gamma) L_{\mathfrak{m}}{}^{\tau\gamma} = \frac{1}{2}(L_{\mathfrak{m}}{}^{\alpha\beta} + L_{\mathfrak{m}}{}^{\beta\alpha}) = L_{\mathfrak{m}}{}^{\alpha\beta} \, .$$

Finally, the third term is solution to the homogeneous equation

$$\sum_{\lambda\leq\sigma} F^{\mathfrak{h}}{}_{\lambda\sigma;\mu,\nu} U^{\lambda\sigma,\mu\nu,\alpha\beta} = 0$$

For (3.47), this equation is equivalent to the statement. Notice that any other F^R or $F^{\mathfrak{m}}$ can be obtained from these ones by adding a suitable function of the type $F^{\mathfrak{h}}$. □

It is important to remark that the solution given by this theorem may not be integrable. But any integrable solution follows this structure. The corresponding result for sections is:

Theorem 3.6. *For a holonomic section $\psi : M \to \mathcal{P}_{\mathfrak{S}}$, the following conditions are equivalent:*

1. *ψ is a solution to the Hamiltonian problem for sections* (1.2) *for the system $(\mathcal{P}_{\mathfrak{S}}, \Omega_{h_{\mathfrak{S}}}, T\mathcal{P}_{\mathfrak{S}}, (\pi^{\ddagger}_{J^1\pi} \circ \jmath)^*\mathfrak{C}^1)$.*
2. *ψ is holonomic and solution to the Einstein equations.*
$$\left.\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right)\right|_\psi = -\frac{1}{\varrho n(\alpha\beta)} L^{\alpha\beta}_{\mathfrak{m}}|_\psi \, .$$
3. *ψ is holonomic and solution to the differential equations*
$$\frac{\partial^2\psi_{\alpha\beta}}{\partial x^\mu \partial x^\nu} = \left.\left(F^{\mathfrak{h}}_{\alpha\beta;\mu,\nu} + \frac{1}{2}g_{\lambda\sigma}(\Gamma^\lambda_{\nu\alpha}\Gamma^\sigma_{\mu\beta} + \Gamma^\lambda_{\nu\beta}\Gamma^\sigma_{\mu\alpha}) + g_{\alpha\beta}\left(g_{\tau\mu}g_{\gamma\nu} - \frac{1}{3}g_{\tau\gamma}g_{\mu\nu}\right)\frac{L_{\mathfrak{m}}{}^{\tau\gamma}}{\varrho\, n(\tau\gamma)}\right)\right|_\psi ,$$
for some set of functions $F^{\mathfrak{h}}_{\alpha\beta;\mu,\nu}$ such that
$$g^{\alpha\beta}\left(F^{\mathfrak{h}}_{\eta\tau;\alpha,\beta} + F^{\mathfrak{h}}_{\alpha\beta;\eta,\tau} - F^{\mathfrak{h}}_{\alpha\eta;\tau,\beta} - F^{\mathfrak{h}}_{\alpha\tau;\eta,\beta}\right) = 0 \, ,$$
and with initial conditions $\psi_{\alpha\beta}(x'^\mu) = g'_{\alpha\beta}$, $\dfrac{\partial\psi_{\alpha\beta}}{\partial x^\mu}(x'^\mu) = g'_{\alpha\beta,\mu}$.

Chapter 4

Metric-Affine

The multisymplectic and polysymplectic techniques have been applied to treat the *Einstein-Palatini* or *Metric-Affine model* for General Relativity [11, 12, 62, 78, 80]. In particular, in [12] an exhaustive study of the multisymplectic description of the model has been done, using a unified formalism which joins both the Lagrangian and Hamiltonian formalisms into a single one.

This chapter is another contribution in order to complete the multisymplectic description of the Einstein-Palatini theory (without energy-matter sources). In particular, we are especially interested in the following problem: as a consequence of the degeneracy of the Lagrangian, this is a premultisymplectic field theory and the Lagrangian field equations are incompatible in the jet bundle where the Lagrangian formalism takes place. The problem of finding a submanifold where this equations have consistent solutions (if it exists) is solved by applying a constraint algorithm. Another objective is to construct the Hamiltonian formalism of the theory and, then, apply the corresponding constraint algorithm to solve the incompatibility of the Hamiltonian field equations. In the Hamiltonian formalism, the choice of different kinds of coordinates (which have a clear geometric interpretation) allows us to better understanding several geometrical characteristics of the formalism.

In [20] the authors point-out the existence of gauge symmetry particular for this model. Another objective is to make a geometrical analysis of this gauge freedom and to recover the Einstein-Hilbert model for General Relativity by means of a gauge fixing. A brief discussion on the classical Lagrangian symmetries of the theory and their associated currents is also done.

As a side note, in the literature this model is either denoted Metric-Affine or Palatini. As it is explained in [40], the model does not appear in Palatini's work, although he performed the first steps towards it. The model as is currently understood first appears in the 1925 paper [36] by Einstein. We use both terms indistinguishably.

The chapter is organized as follows: Section 4.1 is devoted to present the Einstein-Palatini Lagrangian and the manifold where it is defined. Next, in Section 4.2, the Lagrangian formalism of this theory is studied in detail and the Lagrangian constraint algorithm is applied by steps. The geometric interpretation of the different kinds of constraints and the gauge and natural Lagrangian symmetries are also discussed here. Second, in Section 4.3 the Hamiltonian formalisms is stated and analysed in an analogous way. Finally, the relation with the Einstein-Hilbert model is established discussed in Section 4.4.

4.1 The Einstein-Palatini Lagrangian

We introduce here the Metric-Affine (or Einstein-Palatini) action for the Einstein equations of gravity without sources (no matter-energy is present).

The configuration bundle for this system is the bundle $\pi\colon \mathrm{E} \to M$, where M is a connected orientable 4-dimensional manifold representing space-time, whose volume form is denoted $\eta \in \Omega^4(M)$, and $\mathrm{E} = \Sigma \times_M C(LM)$, where Σ is the manifold of Lorentzian metrics on M and $C(LM)$ is the bundle of connections on M; that is, linear connections in $\mathrm{T}M$.

Consider a natural system of coordinates (x^μ, v^α) in the tangent space $\tau\colon \mathrm{T}M \to M$, such that $\eta = \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^3 \equiv \mathrm{d}^4x$. We use adapted fiber coordinates in E, denoted $(x^\mu, g_{\alpha\beta}, \Gamma^\nu_{\lambda\gamma})$, (with $0 \leq \alpha \leq \beta \leq 3$, and $\mu, \nu, \gamma, \lambda = 0, 1, 2, 3$). The functions $g_{\alpha\beta}$ are the components of the metric associated to the charts in the base (x^μ), and $\Gamma^\nu_{\lambda\gamma}$ are the Christoffel symbols of the connection (and then the component functions Γ^ν_γ of the linear connection are $\Gamma^\nu_\gamma = \tau^*(-\Gamma^\nu_{\lambda\gamma} v^\lambda)$ [35]). Since g is symmetric, $g_{\alpha\beta} = g_{\beta\alpha}$ and actually there are 10 independent components. We do not assume torsionless connections and hence $\Gamma^\nu_{\lambda\gamma} \neq \Gamma^\nu_{\gamma\lambda}$, in general. Thus $\dim \mathrm{E} = 78$. When we sum over symmetric indices and not over all the components, we order the indices as $0 \leq \alpha \leq \beta \leq 3$.

In order to state the formalism we consider the first-order jet bundle $J^1\pi$, with the natural projections

$$\begin{array}{rcl} \pi^1\colon J^1\pi & \longrightarrow & \mathrm{E} \\ j^1_x\phi & \longmapsto & \phi(x) \end{array} \quad ; \quad \begin{array}{rcl} \bar{\pi}^1\colon J^1\pi & \longrightarrow & M \\ j^1_x\phi & \longmapsto & x \end{array} .$$

Induced coordinates in $J^1\pi$ are denoted $(x^\mu,\, g_{\alpha\beta},\, \Gamma^\nu_{\lambda\gamma},\, g_{\alpha\beta,\mu},\, \Gamma^\nu_{\lambda\gamma,\mu})$, and $\dim J^1\pi = 374$.

A special kind of vector fields are the *coordinate total derivatives* [83, 90], which are locally given as

$$D_\tau = \frac{\partial}{\partial x^\tau} + \sum_{\alpha\leq\beta}\left(g_{\alpha\beta,\tau}\frac{\partial}{\partial g_{\alpha\beta}} + g_{\alpha\beta,\mu\tau}\frac{\partial}{\partial g_{\alpha\beta,\mu}}\right) + \Gamma^\nu_{\alpha\beta,\tau}\frac{\partial}{\partial \Gamma^\nu_{\alpha\beta}} + \Gamma^\nu_{\alpha\beta,\mu\tau}\frac{\partial}{\partial \Gamma^\nu_{\alpha\beta,\mu}} .$$

Observe that, if $f \in \mathrm{C}^\infty(J^1\pi)$, then $D_\tau f \in \mathrm{C}^\infty(J^2\pi)$.

The *Einstein-Palatini* (or *Metric-Affine*) *Lagrangian density* is a $\bar{\pi}^1$-semibasic 4-form $\mathcal{L}_{\mathrm{EP}} \in \Omega^4(J^1\pi)$; then $\mathcal{L}_{\mathrm{EP}} = L_{\mathrm{EP}}\,(\bar{\pi}^1)^*\eta$, where $L_{\mathrm{EP}} \in \mathrm{C}^\infty(J^1\pi)$ is the *Einstein-Palatini Lagrangian function* which, in the above coordinates, is given by

$$L_{\mathrm{EP}} = \sqrt{|\det(g)|}\, g^{\alpha\beta} R_{\alpha\beta} \equiv \varrho g^{\alpha\beta} R_{\alpha\beta} = \varrho\, R ,$$

where $\varrho = \sqrt{|det(g_{\alpha\beta})|}$, $R = g^{\alpha\beta}R_{\alpha\beta}$ is the *scalar curvature*, $R_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha,\gamma} - \Gamma^\gamma_{\gamma\alpha,\beta} + \Gamma^\gamma_{\beta\alpha}\Gamma^\sigma_{\sigma\gamma} - \Gamma^\gamma_{\beta\sigma}\Gamma^\sigma_{\gamma\alpha}$ are the components of the *Ricci tensor*, which depend only on the connection, and $g^{\alpha\beta}$ denotes the inverse matrix of g, namely: $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$. The order of the indexes in the expression of the curvature is crucial. The expression is based in [35]. It is useful to consider the following auxiliary functions:

$$L^{\beta\gamma,\mu}_\alpha := \frac{\partial L_{\mathrm{EP}}}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} = \varrho(\delta^\mu_\alpha g^{\beta\gamma} - \delta^\beta_\alpha g^{\mu\gamma}) , \tag{4.1}$$

4.2 The Metric-Affine model: Lagrangian formalism

4.2.1 Poincaré-Cartan forms and field equations

The Lagrangian formalism of field theories is presented in chapter 1.

$$H := L^{\beta\gamma,\mu}_\alpha \Gamma^\alpha_{\beta\gamma,\mu} - L_{\mathrm{EP}} = \varrho g^{\alpha\beta}\left(\Gamma^\gamma_{\beta\sigma}\Gamma^\sigma_{\gamma\alpha} - \Gamma^\gamma_{\beta\alpha}\Gamma^\sigma_{\sigma\gamma}\right) . \tag{4.2}$$

The Poincaré-Cartan 5-form $\Omega_{\mathcal{L}_{\rm EP}}$ associated with the Einstein-Palatini Lagrangian density is constructed using the canonical structures of the bundle $J^1\pi$ and its expression is

$$\Omega_{\mathcal{L}_{\rm EP}} = {\rm d}H \wedge {\rm d}^4x - {\rm d}L_\alpha^{\beta\gamma,\mu} \wedge {\rm d}\Gamma^\alpha_{\beta\gamma} \wedge {\rm d}^3x_\mu \ . \tag{4.3}$$

Observe that it is a π^1-projectable form. The Lagrangian multisympelctic system for the Einstein-Palatine gravity is $(J^1\pi, \Omega_{EP}, TJ^1\pi, \mathfrak{C}^1)$, where $\mathfrak{C}^1$ is the Cartan codistribution of $J^1\pi$.

Then, for a generic locally decomposable and $\bar{\pi}^1$-transverse multivector field in $J^1\pi$ we have the following local expression $\mathbf{X} = f \bigwedge_{\nu=0}^{3} X_\nu$, with

$$X_\nu = \frac{\partial}{\partial x^\nu} + \sum_{\rho\leq\sigma} \left(f_{\rho\sigma,\nu}\frac{\partial}{\partial g_{\rho\sigma}} + f_{\rho\sigma\mu,\nu}\frac{\partial}{\partial g_{\rho\sigma,\mu}}\right) + f^\alpha_{\beta\gamma,\nu}\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} + f^\alpha_{\beta\gamma\mu,\nu}\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} \ , \tag{4.4}$$

where the coefficients are arbitrary functions of $C^\infty(J^1\pi)$. If the multivector field is semi-holonomic and we set $f=1$, then

$$\mathbf{X} = \bigwedge_{\nu=0}^{3} \left(\frac{\partial}{\partial x^\nu} + \sum_{\rho\leq\sigma} \left(g_{\rho\sigma,\nu}\frac{\partial}{\partial g_{\rho\sigma}} + f_{\rho\sigma\mu,\nu}\frac{\partial}{\partial g_{\rho\sigma,\mu}}\right) + \Gamma^\alpha_{\beta\gamma,\nu}\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} + f^\alpha_{\beta\gamma\mu,\nu}\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} \right) \ . \tag{4.5}$$

Taking (4.4) and (4.3), the equation (1.4) becomes locally

$$0 = i(X_\mu){\rm d}H + f^\alpha_{\beta\gamma,\mu}\, i(X_\nu){\rm d}L_\alpha^{\beta\gamma,\nu} - f^\alpha_{\beta\gamma,\nu}\, i(X_\mu){\rm d}L_\alpha^{\beta\gamma,\nu} \ , \tag{4.6}$$

$$0 = \frac{\partial H}{\partial g_{\sigma\rho}} - f^\alpha_{\beta\gamma,\mu}\frac{\partial L_\alpha^{\beta\gamma,\mu}}{\partial g_{\sigma\rho}} \ , \tag{4.7}$$

$$\begin{aligned} 0 &= \frac{\partial H}{\partial \Gamma^\alpha_{\beta\gamma}} + \sum_{\rho\leq\sigma}\left(f_{\rho\sigma,\mu}\frac{\partial L_\alpha^{\beta\gamma,\mu}}{\partial g_{\rho\sigma}}\right) + f^\tau_{\rho\sigma,\mu}\frac{\partial L_\alpha^{\beta\gamma,\mu}}{\partial \Gamma^\tau_{\rho\sigma}} - f^\tau_{\rho\sigma,\mu}\frac{\partial L_\tau^{\rho\sigma,\mu}}{\partial \Gamma^\alpha_{\beta\gamma}} \\ &= \frac{\partial H}{\partial \Gamma^\alpha_{\beta\gamma}} + \sum_{\rho\leq\sigma} f_{\rho\sigma,\mu}\frac{\partial L_\alpha^{\beta\gamma,\mu}}{\partial g_{\rho\sigma}} \ ; \end{aligned} \tag{4.8}$$

since $\dfrac{\partial L_\alpha^{\beta\gamma,\mu}}{\partial \Gamma^\tau_{\rho\sigma}} = 0$. Equations (4.6) arise from the variations of the coordinates x^μ and they hold as a consequence of (4.7) and (4.8). The equations (4.7) arise from the variations on the components of the metric, and contains the functions $f^\alpha_{\beta\gamma,\mu}$ related to the connection, thus we call them *connection equations*. Finally, the equations (4.8) arise from the variations on the components of the connection, and contain the functions $f_{\sigma\rho,\mu}$, thus they are called *metric equations*.

4.2.2 Compatibility and consistency constraints

In general, $\bar{\pi}^1$-transverse and integrable multivector fields $\mathbf{X} \in \mathfrak{X}^4(J^1\pi)$ which are solutions to (1.4) could not exist. In the best of cases they exist only in some submanifold of $J^1\pi$ [22]. The aim in this section is to find the constraints that define this submanifold, using a local version of the geometric constraint algorithms [23, 22].

First, we introduce the following notation: as it is usual,

$$\ker^4\Omega_{\mathcal{L}_{\rm EP}} := \{\mathbf{X} \in \mathfrak{X}^4(J^1\pi) \mid i(\mathbf{X})\Omega_{\mathcal{L}_{\rm EP}} = 0\} \ .$$

We denote by $\ker^4_{\bar{\pi}^1} \Omega_{\mathcal{L}_{\rm EP}}$ the set of locally decomposable and $\bar{\pi}^1$-transverse multivector fields satisfying equations (1.4) but not being (semi)holonomic necessarily. Then, $\ker^4_{SH} \Omega_{\mathcal{L}_{\rm EP}}$ and $\ker^4_{H} \Omega_{\mathcal{L}_{\rm EP}}$ denote the sets of semi-holonomic and the holonomic multivector fields which are solutions to the equations (1.4), respectively. Obviously we have

$$\ker^4_{H} \Omega_{\mathcal{L}_{\rm EP}} \subset \ker^4_{SH} \Omega_{\mathcal{L}_{\rm EP}} \subset \ker^4_{\bar{\pi}^1} \Omega_{\mathcal{L}_{\rm EP}} \subset \ker^4 \Omega_{\mathcal{L}_{\rm EP}} \ . \tag{4.9}$$

We make the study in several steps, following the next procedure: first we consider the problem of finding locally decomposable and $\bar{\pi}^1$-transverse multivector fields which are solution to (1.4) (that is, the elements of $\ker^4_{\bar{\pi}^1} \Omega_{\mathcal{L}_{\rm EP}}$), then we look for the semi-holonomic multivector fields belonging to $\ker^4_{SH} \Omega_{\mathcal{L}_{\rm EP}}$ and finally, in the next Section, we analyze their integrability (finding the elements of $\ker^4_{H} \Omega_{\mathcal{L}_{\rm EP}}$).

Non-semiholonomic multivector fields (elements of $\ker^4_{\bar{\pi}^1} \Omega_{\mathcal{L}_{\rm EP}}$): compatibility constraints

The set $\ker^4_{\bar{\pi}^1} \Omega_{\mathcal{L}_{\rm EP}}$ consists of multivector fields of the form (4.4) whose coefficients satisfy the connection and metric equations (4.7) and (4.8) respectivelly. But the equations (4.8) are not compatible. In fact:

Proposition 4.1. *The necessary condition for the existence of solutions to the metric equations (4.8) is that the following equalities hold:*

$$A_{\alpha\beta\gamma} \equiv g_{\beta\nu} T^{\nu}_{\alpha\gamma} - g_{\alpha\nu} T^{\nu}_{\beta\gamma} + \tfrac{1}{3} g_{\beta\gamma} T^{\nu}_{\nu\alpha} - \tfrac{1}{3} g_{\alpha\gamma} T^{\nu}_{\nu\beta} = 0 \ , \tag{4.10}$$

where $T^{\alpha}_{\beta\gamma}$ are the components of the torsion tensor which are defined as usual, $T^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta}$.

Proof. We introduce the following functions

$$\beth^{\alpha}_{\beta\gamma,\lambda\zeta\nu} = \frac{1}{\varrho}\left(-\frac{1}{2} g_{\beta\gamma} g_{\lambda\zeta} \delta^{\alpha}_{\nu} + \frac{1}{6} g_{\lambda\zeta} g_{\nu\gamma} \delta^{\alpha}_{\beta} - \frac{1}{3} g_{\lambda\nu} g_{\zeta\gamma} \delta^{\alpha}_{\beta} + g_{\zeta\gamma} g_{\lambda\beta} \delta^{\alpha}_{\nu} \right) \ , \tag{4.11}$$

which satisfy that

$$\frac{\partial L^{\beta\gamma,\mu}_{\alpha}}{\partial g_{\rho\sigma}} \beth^{\alpha}_{\beta\gamma,\lambda\zeta\nu} = \frac{n(\rho\sigma)}{2} (\delta^{\mu}_{\nu} \delta^{\sigma}_{\zeta} \delta^{\rho}_{\lambda} + \delta^{\mu}_{\nu} \delta^{\sigma}_{\lambda} \delta^{\rho}_{\zeta}) \ ;$$

where $n(\rho\sigma)$ is a combinatorial factor such that $n(\rho\sigma) = 1$ for $\rho = \sigma$, and $n(\rho\sigma) = 2$ for $\rho \neq \sigma$. Then, using them in the metric equations (4.8), we obtain

$$0 = \beth^{\alpha}_{\beta\gamma,\lambda\zeta\nu} \left(\frac{\partial H}{\partial \Gamma^{\alpha}_{\beta\gamma}} + \sum_{\rho\leq\sigma} f_{\rho\sigma,\mu} \frac{\partial L^{\beta\gamma,\mu}_{\alpha}}{\partial g_{\rho\sigma}} \right) = \beth^{\alpha}_{\beta\gamma,\lambda\zeta\nu} \frac{\partial H}{\partial \Gamma^{\alpha}_{\beta\gamma}} + \frac{1}{2}(f_{\lambda\zeta,\nu} + f_{\zeta\lambda,\nu}) \ .$$

These are equations for the functions $f_{\lambda\zeta,\nu}$ which, as a consequence of the symmetry of the metric, $g_{\alpha\beta} = g_{\beta\alpha}$, are also symmetric: $f_{\lambda\zeta,\nu} = f_{\zeta\lambda,\nu}$. Nevertheless, the equations are incompatible because they are not symmetric under the change $\lambda \leftrightarrow \zeta$. In fact; we obtain that

$$\beth^{\alpha}_{\beta\gamma,\lambda\zeta\nu} \frac{\partial H}{\partial \Gamma^{\alpha}_{\beta\gamma}} - \beth^{\alpha}_{\beta\gamma,\zeta\lambda\nu} \frac{\partial H}{\partial \Gamma^{\alpha}_{\beta\gamma}} = g_{\lambda\mu} T^{\mu}_{\zeta\nu} - g_{\zeta\mu} T^{\mu}_{\lambda\nu} + \tfrac{1}{3} g_{\lambda\nu} T^{\mu}_{\mu\zeta} - \tfrac{1}{3} g_{\zeta\nu} T^{\mu}_{\mu\lambda} = 0 \ ,$$

and the result follows from here. $\square$

Conditions (4.10) are called *torsion constraints* and they define the submanifold $\mathcal{S}_T \hookrightarrow J^1E$. These torsion constraints are essential in the following discussion, since they impose strong restrictions on the torsion. In fact:

Proposition 4.2. *The torsion constraints* (4.10) *are equivalent to*

$$T^{\alpha}_{\beta\gamma}=\frac{1}{3}\delta^{\alpha}_{\beta}T^{\nu}_{\nu\gamma}-\frac{1}{3}\delta^{\alpha}_{\gamma}T^{\nu}_{\nu\beta}\,. \tag{4.12}$$

Proof. If (4.10) holds, then

$$\begin{aligned}
0 &= \frac{1}{2}g^{\alpha\mu}\left(A_{\beta\mu\gamma}+A_{\beta\gamma\mu}+A_{\mu\gamma\beta}\right)\\
&= \frac{1}{2}g^{\alpha\mu}\big(g_{\mu\nu}T^{\nu}_{\beta\gamma}-g_{\beta\nu}T^{\nu}_{\mu\gamma}+\tfrac{1}{3}g_{\gamma\mu}T^{\nu}_{\nu\beta}-\tfrac{1}{3}g_{\gamma\beta}T^{\nu}_{\nu\mu}+g_{\gamma\nu}T^{\nu}_{\beta\mu}-g_{\beta\nu}T^{\nu}_{\gamma\mu}\\
&\quad+\tfrac{1}{3}g_{\mu\gamma}T^{\nu}_{\nu\beta}-\tfrac{1}{3}g_{\mu\beta}T^{\nu}_{\nu\gamma}+g_{\gamma\nu}T^{\nu}_{\mu\beta}-g_{\mu\nu}T^{\nu}_{\gamma\beta}+\tfrac{1}{3}g_{\beta\gamma}T^{\nu}_{\nu\mu}-\tfrac{1}{3}g_{\beta\mu}T^{\nu}_{\nu\gamma}\big)\\
&= T^{\alpha}_{\beta\gamma}-\frac{1}{3}\delta^{\alpha}_{\beta}T^{\nu}_{\nu\gamma}+\frac{1}{3}\delta^{\alpha}_{\gamma}T^{\nu}_{\nu\beta}\,.
\end{aligned}$$

Conversely, if $T^{\alpha}_{\beta\gamma}=\frac{1}{3}\delta^{\alpha}_{\beta}T^{\nu}_{\nu\gamma}-\frac{1}{3}\delta^{\alpha}_{\gamma}T^{\nu}_{\nu\beta}$, then

$$\begin{aligned}
A_{\alpha\beta\gamma} &= g_{\beta\nu}T^{\nu}_{\alpha\gamma}-g_{\alpha\nu}T^{\nu}_{\beta\gamma}+\tfrac{1}{3}g_{\beta\gamma}T^{\nu}_{\nu\alpha}-\tfrac{1}{3}g_{\alpha\gamma}T^{\nu}_{\nu\beta}\\
&= g_{\beta\nu}\left(\tfrac{1}{3}\delta^{\nu}_{\alpha}T^{\mu}_{\mu\gamma}-\tfrac{1}{3}\delta^{\nu}_{\gamma}T^{\mu}_{\mu\alpha}\right)-g_{\alpha\nu}\left(\tfrac{1}{3}\delta^{\nu}_{\beta}T^{\mu}_{\mu\gamma}-\tfrac{1}{3}\delta^{\nu}_{\gamma}T^{\mu}_{\mu\beta}\right)+\tfrac{1}{3}g_{\beta\gamma}T^{\nu}_{\nu\alpha}-\tfrac{1}{3}g_{\alpha\gamma}T^{\nu}_{\nu\beta}\\
&= \frac{1}{3}\left(g_{\beta\alpha}T^{\mu}_{\mu\gamma}-g_{\beta\gamma}T^{\mu}_{\mu\alpha}-g_{\alpha\beta}T^{\mu}_{\mu\gamma}+g_{\alpha\gamma}T^{\mu}_{\mu\beta}+g_{\beta\gamma}T^{\nu}_{\nu\alpha}-g_{\alpha\gamma}T^{\nu}_{\nu\beta}\right)=0\,.
\end{aligned}$$

□

As a consequence of this result, on $\mathcal{S}_T$ the torsion is determined by its "trace", $tr(T)=T^{\nu}_{\alpha\nu}$.

Proposition 4.3. *On the submanifold $\mathcal{S}_T$, the general solutions to the equations* (4.7) *and* (4.8) *are, respectively,*

$$f^{\alpha}_{\beta\gamma,\mu} = \Gamma^{\lambda}_{\mu\gamma}\Gamma^{\alpha}_{\beta\lambda}+C^{\alpha}_{\beta\gamma,\mu}+K^{\alpha}_{\beta\gamma,\mu}\,, \tag{4.13}$$

$$f_{\sigma\rho,\mu} = g_{\sigma\lambda}\Gamma^{\lambda}_{\mu\rho}+g_{\rho\lambda}\Gamma^{\lambda}_{\mu\sigma}+\frac{2}{3}g_{\sigma\rho}T^{\lambda}_{\lambda\mu}\,; \tag{4.14}$$

for some functions $C^{\alpha}_{\beta\gamma,\mu},K^{\alpha}_{\beta\gamma,\mu}\in C^{\infty}(J^1\pi)$ satisfying that

$$C^{\alpha}_{\beta\gamma,\mu}=C_{\beta\mu}\delta^{\alpha}_{\gamma}\quad,\quad K^{\nu}_{\nu\gamma\mu}=0\quad,\quad K^{\nu}_{\beta\gamma\nu}+K^{\nu}_{\gamma\beta\nu}=0\quad;\quad(\text{on }\mathcal{S}_T)\,.$$

Proof. The metric and connection equations are independent and lineal. Thus we look for particular and homogeneous-general solutions for each one.

It is straightforward to check that (4.14) is a particular solution to the metric equations on $\mathcal{S}_T$. Given two solutions, f^1 and f^2, their difference $h_{\sigma\rho,\mu}=f^1_{\sigma\rho,\mu}-f^2_{\sigma\rho,\mu}$ is a solution to the homogeneous equation

$$\sum_{\rho\leq\sigma}h_{\rho\sigma,\mu}\frac{\partial L^{\beta\gamma,\mu}_{\alpha}}{\partial g_{\rho\sigma}}=0\quad;\quad(\text{on }\mathcal{S}_T)\,.$$

Consider the functions $\daleth^{\alpha}_{\beta\gamma,\lambda\zeta\nu}$ which satisfy (4.11),

$$0=\sum_{\rho\leq\sigma}h_{\rho\sigma,\mu}\frac{\partial L^{\beta\gamma,\mu}_{\alpha}}{\partial g_{\rho\sigma}}\daleth^{\alpha}_{\beta\gamma,\lambda\zeta\nu}=h_{\rho\sigma,\mu}\frac{1}{2}(\delta^{\mu}_{\nu}\delta^{\sigma}_{\lambda}\delta^{\rho}_{\zeta}+\delta^{\mu}_{\nu}\delta^{\sigma}_{\zeta}\delta^{\rho}_{\lambda})=h_{\lambda\zeta\nu}\,.$$

Therefore, $h_{\sigma\rho,\mu}|_{\mathcal{S}_T}=0\Rightarrow f^1(p)=f^2(p)$ on $\mathcal{S}_T$, and the solution is unique. In a similar way,

$$f^{\alpha}_{\beta\gamma,\mu}=\Gamma^{\lambda}_{\mu\gamma}\Gamma^{\alpha}_{\beta\lambda}\quad;\quad(\text{on }\mathcal{S}_T)$$

is a particular solution to the connection equations. The difference between two solutions is a solution to the homogeneous equation:

$$h^{\alpha}_{\beta\gamma,\mu}\frac{\partial L^{\beta\gamma,\mu}_{\alpha}}{\partial g_{\rho\sigma}} = 0 \quad ; \quad (\text{on } \mathcal{S}_T) \ . \tag{4.15}$$

This equation is equivalent to:

$$h^{\lambda}_{\lambda r,s} + h^{\lambda}_{\lambda s,r} - h^{\lambda}_{rs,\lambda} - h^{\lambda}_{sr,\lambda} = 0 \quad ; \quad (\text{on } \mathcal{S}_T) \ .$$

Indeed,

$$\frac{1}{\varrho n(\rho\sigma)}(2g_{r\rho}g_{s\sigma} - g_{\rho\sigma}g_{rs})h^{\alpha}_{\beta\gamma,\mu}\frac{\partial L^{\beta\gamma,\mu}_{\alpha}}{\partial g_{\rho\sigma}} = h^{\lambda}_{\lambda r,s} + h^{\lambda}_{\lambda s,r} - h^{\lambda}_{rs,\lambda} - h^{\lambda}_{sr,\lambda} \quad ; \quad (\text{on } \mathcal{S}_T) \ .$$

$$\frac{\varrho n(\rho\sigma)}{4}(2g^{r\rho}g^{s\sigma} - g^{\rho\sigma}g^{rs})\left(h^{\lambda}_{\lambda r,s} + h^{\lambda}_{\lambda s,r} - h^{\lambda}_{rs,\lambda} - h^{\lambda}_{sr,\lambda}\right) = h^{\alpha}_{\beta\gamma,\mu}\frac{\partial L^{\beta\gamma,\mu}_{\alpha}}{\partial g_{\rho\sigma}} \quad ; \quad (\text{on } \mathcal{S}_T) \ .$$

Some solutions of this equation are the functions of the form

$$h^{\alpha}_{\beta\gamma,\mu} = C_{\beta\mu}\delta^{\alpha}_{\gamma} \quad ; \quad (\text{on } \mathcal{S}_T) \ ,$$

which are called *trace solutions*. for every solution h, consider $K^{\alpha}_{\beta\gamma,\mu} = h^{\alpha}_{\beta\gamma,\mu} - C_{\beta\mu}\delta^{\alpha}_{\gamma}$ with $C_{\beta\mu} = h^{\lambda}_{\lambda\beta\mu}$. It follows that $K^{\lambda}_{\lambda\gamma\mu} = 0$. Since the equation is lineal, these functions must also be solutions. Therefore:

$$0 = K^{\lambda}_{\lambda r,s} + K^{\lambda}_{\lambda s,r} - K^{\lambda}_{rs,\lambda} - K^{\lambda}_{sr,\lambda} = -K^{\lambda}_{rs,\lambda} - K^{\lambda}_{sr,\lambda} \quad ; \quad (\text{on } \mathcal{S}_T) \ .$$

These solutions are called *torsion solutions*. From their definition it is clear that any homogeneous solution is a sum of a trace and a torsion solution. Furthermore, if $K^{\alpha}_{\beta\gamma,\mu} = C^{\alpha}_{\beta\gamma,\mu} = C_{\beta\mu}\delta^{\alpha}_{\gamma}$, then $0 = K^{\lambda}_{\lambda\gamma,\mu} = C_{\gamma\mu}$; on $\mathcal{S}_T$. Thus, the only homogeneous solution which is both trace and torsion is $h^{\alpha}_{\beta\gamma,\mu} = 0$. □

This proposition shows also that:

Corollary 4.1. *The torsion constraints* (4.10) *(or their equivalent expressions* (4.12)*) are sufficient conditions for the existence of solutions to* (4.8).

These constraints could be also obtained in an intrinsic way using the procedure described in [22].

Now we must check the tangency (or consistency) conditions. First, observe that, taking into account (4.4), (4.13), and (4.14), the general solution to the equation (1.4) (before imposing the holonomy condition) are multivector fields of the form

$$\begin{aligned}\mathbf{X} = \bigwedge_{\nu=0}^{3} X_\nu \quad = \quad & \bigwedge_{\nu=0}^{3}\left[\frac{\partial}{\partial x^\nu} + \sum_{\sigma\leq\rho}\left((g_{\sigma\lambda}\Gamma^{\lambda}_{\nu\rho} + g_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma} + \frac{2}{3}g_{\sigma\rho}T^{\lambda}_{\lambda\nu})\frac{\partial}{\partial g_{\sigma\rho}} + f_{\sigma\rho\mu,\nu}\frac{\partial}{\partial g_{\sigma\rho,\mu}}\right)\right.\\ & \left. +(\Gamma^{\lambda}_{\nu\gamma}\Gamma^{\alpha}_{\beta\lambda} + C^{\alpha}_{\beta\gamma,\nu} + K^{\alpha}_{\beta\gamma,\nu})\frac{\partial}{\partial\Gamma^{\alpha}_{\beta\gamma}} + f^{\alpha}_{\beta\gamma\mu,\nu}\frac{\partial}{\partial\Gamma^{\alpha}_{\beta\gamma,\mu}}\right] \ ; \ (\text{on } \mathcal{S}_T) \ .\end{aligned} \tag{4.16}$$

Bearing in mind the conditions on the functions $C^{\alpha}_{\beta\gamma,\mu}, K^{\alpha}_{\beta\gamma,\mu}$ stated in Proposition 4.3, the tangency condition on the torsion constraints (4.12)

$$\mathrm{L}(X_\nu)\left(T^{\alpha}_{\beta\gamma} - \frac{1}{3}\delta^{\alpha}_{\beta}T^{\nu}_{\nu\gamma} + \frac{1}{3}\delta^{\alpha}_{\gamma}T^{\nu}_{\nu\beta}\right) = 0 \quad ; \quad (\text{on } \mathcal{S}_T) \ ,$$

hold on $\mathcal{S}_T$ as long as

$$K^{\alpha}_{[\beta\gamma],\mu} = -\frac{1}{3}\delta^{\alpha}_{[\beta}K^{\nu}_{\gamma]\nu,\mu} - \Gamma^{\lambda}_{\mu[\gamma}\Gamma^{\alpha}_{\beta]\lambda} + \frac{1}{3}\delta^{\alpha}_{[\beta}\Gamma^{\lambda}_{\mu\gamma]}\Gamma^{\nu}_{\nu\lambda} - \frac{1}{3}\delta^{\alpha}_{[\beta}\Gamma^{\lambda}_{\mu\nu}\Gamma^{\nu}_{\gamma]\lambda} \quad ; \quad (\text{on } \mathcal{S}_T) \ .$$

Nevertheless, solutions to equation (1.4) must be holonomic multivector fields. Thus, first we look for semiholonomic solutions, then we analyze their tangency and, finally, we study the existence of holonomic solutions.

Semi-holonomic multivector fields (elements of $\ker^4_{SH}\Omega_{\mathcal{L}_{\rm EP}}$): semiholonomic constraints

If a multivector field is semiholonomic then its local expression is (4.5); that is,

$$f_{\rho\sigma,\mu} = g_{\rho\sigma,\mu} \quad , \quad f^\alpha_{\beta\gamma,\mu} = \Gamma^\alpha_{\beta\gamma,\mu} \ .$$

In this case, there are more constraints which arise from the equations (4.7) and (4.8) and are the Euler-Lagrange equations themselves:

$$\frac{\partial H}{\partial g_{\mu\nu}} - \frac{\partial L^{\beta\gamma,\sigma}_\alpha}{\partial g_{\mu\nu}}\Gamma^\alpha_{\beta\gamma,\sigma} = 0 \ , \tag{4.17}$$

$$\frac{\partial H}{\partial \Gamma^\alpha_{\beta\gamma}} + \sum_{\mu\leq\nu}\frac{\partial L^{\beta\gamma,\sigma}_\alpha}{\partial g_{\mu\nu}} g_{\mu\nu,\sigma} = 0 \ . \tag{4.18}$$

(Geometrically, they are a consequence of the fact that $\Omega_{\mathcal{L}_{Ep}}$ is π^1-projectable 2.2). In this way, the connection and metric equations become *semiholonomic constraints*, which are called *connection* and *metric constrains*, respectively.

In particular, notice that the metric constraints (4.18) arise from the equations (4.8), which lead to the torsion constraints (4.12). Therefore, the metric constraints split into two kinds of conditions: the torsion constraints (4.12) themselves and, according to equation (4.14) (or, equivalently, to (4.16)),

$$g_{\rho\sigma,\mu} = g_{\sigma\lambda}\Gamma^\lambda_{\mu\rho} + g_{\rho\lambda}\Gamma^\lambda_{\mu\sigma} + \frac{2}{3}g_{\rho\sigma}T^\lambda_{\lambda\mu} \ , \tag{4.19}$$

which are called *pre-metricity constraints*. They are closely related to the metricity conditions and the trace of the torsion, as it is proved in the following:

Proposition 4.4. *In the points of the submanifold $\mathcal{S}_m \hookrightarrow J^1\pi$ defined by the metric constraints* (4.18)*, we have that:*

$$\nabla^{\Gamma(p)}g(p) = 0 \iff tr(T^{\Gamma(p)}) = 0 \quad ; \quad p\in\mathcal{S}_m \ .$$

(Here, the notation $\nabla^{\Gamma(p)}$ means the covariant derivative with respect to the connection Γ in the point p, and $T^{\Gamma(p)}$ denotes the torsion tensor associated to this connection).

Proof. In the coordinates of $J^1\pi$ the metricity condition $\nabla^{\Gamma(p)}g(p) = 0$ is

$$\left(\nabla^{\Gamma(p)}g(p)\right)_{\rho\sigma,\mu} = g_{\rho\sigma,\mu} - g_{\sigma\lambda}\Gamma^\lambda_{\mu\rho} - g_{\rho\lambda}\Gamma^\lambda_{\mu\sigma} \ .$$

Therefore, the statement follows immediately since the metric constraints (4.19) can be written as

$$\left(\nabla^{\Gamma(p)}g(p)\right)_{\rho\sigma,\mu} = \frac{2}{3}g_{\rho\sigma}T^\lambda_{\lambda\mu} \ .$$

□

Tangency condition: consistency constraints

Now we check the tangency (or consistency) condition for all the above sets of constraints. A semi-holonomic multivector field $X = \bigwedge_{\nu=0}^{3} X_\nu$ has the local expression (4.5). The tangency condition on the connection constraints (4.17) reads

$$\mathrm{L}(X_\nu)\left(\frac{\partial H}{\partial g_{\rho\sigma}} - \frac{\partial L_\alpha^{\beta\gamma,\mu}}{\partial g_{\rho\sigma}}\Gamma^\alpha_{\beta\gamma,\mu}\right) = D_\nu\frac{\partial H}{\partial g_{\rho\sigma}} - D_\nu\frac{\partial L_\alpha^{\beta\gamma,\mu}}{\partial g_{\rho\sigma}}\Gamma^\alpha_{\beta\gamma,\mu} - \frac{\partial L_\alpha^{\beta\gamma,\mu}}{\partial g_{\rho\sigma}}f^\alpha_{\beta\gamma\mu,\nu} = 0 \quad (\text{on } \mathcal{S}_T)\,, \tag{4.20}$$

and it does not lead to new constraints because they allow to determine the functions $f^\alpha_{\beta\gamma,\sigma\nu}$ (on $\mathcal{S}_T$) . The tangency condition on the pre-metricity constraints (4.19) gives

$$f_{\sigma\rho,\mu\nu} = D_\lambda\left(g_{\sigma\lambda}\Gamma^\lambda_{\mu\rho} + g_{\rho\lambda}\Gamma^\lambda_{\mu\sigma} + \frac{2}{3}g_{\sigma\rho}T^\lambda_{\lambda\mu}\right) \quad ; \quad (\text{on } \mathcal{S}_T)\,, \tag{4.21}$$

and it does not lead either to new constraints. But the tangency condition on the torsion constraints (4.12) does lead to new constraints

$$\mathrm{L}(X_\nu)\left(T^\alpha_{\beta\gamma} - \frac{1}{3}\delta^\alpha_\beta T^\mu_{\mu\gamma} + \frac{1}{3}\delta^\alpha_\gamma T^\mu_{\mu\beta}\right) = T^\alpha_{\beta\gamma,\nu} - \frac{1}{3}\delta^\alpha_\beta T^\mu_{\mu\gamma,\nu} + \frac{1}{3}\delta^\alpha_\gamma T^\mu_{\mu\beta,\nu} = 0 \quad ; \quad (\text{on } \mathcal{S}_T)\,.$$

The tangency condition on these new constraints leads to

$$\mathrm{L}(X_\lambda)\left(T^\alpha_{\beta\gamma,\nu} - \frac{1}{3}\delta^\alpha_\beta T^\mu_{\mu\gamma,\nu} + \frac{1}{3}\delta^\alpha_\gamma T^\mu_{\mu\beta,\nu}\right) = f^\alpha_{\beta\gamma\nu,\tau} - \frac{1}{3}\delta^\alpha_\beta f^\mu_{\mu\gamma\nu,\tau} + \frac{1}{3}\delta^\alpha_\gamma f^\mu_{\mu\beta\nu,\tau} = 0\,;\ (\text{on } \mathcal{S}_{sh})\,,$$

which are not new constraints, but equations for the functions $f^\alpha_{\beta\gamma\mu,\nu}$. Therefore, in the submanifold $\mathcal{S}_{sh} \hookrightarrow \mathcal{S}_T$ defined by these constraints there are semiholonomic multivector fields solutions to the field equations, which are tangent to $\mathcal{S}_{sh}$.

Summarizing, we have proved that:

Theorem 4.1. *There exists a submanifold* $\mathrm{j}_{sh}\colon \mathcal{S}_{sh} \hookrightarrow J^1\pi$ *where there are semi-holonomic multivector fields which are solutions to the field equations* (1.4) *and are tangent to* $\mathcal{S}_{sh}$. *This submanifold is locally defined in* $J^1\pi$ *by the constraints*

$$\begin{aligned}
c^{\mu\nu} &\equiv \frac{\partial H}{\partial g_{\mu\nu}} - \frac{\partial L_\alpha^{\beta\gamma,\sigma}}{\partial g_{\mu\nu}}\Gamma^\alpha_{\beta\gamma,\sigma} = 0\,,\\
m_{\rho\sigma,\mu} &\equiv g_{\rho\sigma,\mu} - g_{\sigma\lambda}\Gamma^\lambda_{\mu\rho} - g_{\rho\lambda}\Gamma^\lambda_{\mu\sigma} - \frac{2}{3}g_{\rho\sigma}T^\lambda_{\lambda\mu} = 0\,,\\
t^\alpha_{\beta\gamma} &\equiv T^\alpha_{\beta\gamma} - \frac{1}{3}\delta^\alpha_\beta T^\mu_{\mu\gamma} + \frac{1}{3}\delta^\alpha_\gamma T^\mu_{\mu\beta} = 0\,,\\
r^\alpha_{\beta\gamma,\nu} &\equiv T^\alpha_{\beta\gamma,\nu} - \frac{1}{3}\delta^\alpha_\beta T^\mu_{\mu\gamma,\nu} + \frac{1}{3}\delta^\alpha_\gamma T^\mu_{\mu\beta,\nu} = 0\,.
\end{aligned}$$

These constraints are not independent all of them. For instance, the pre-metricity constraints $m_{\rho\sigma,\mu}$ are symmetric in the indices σ,ρ and the constraints $t^\alpha_{\beta\gamma}$ and $r^\alpha_{\beta\gamma,\nu}$ are skewsymmetric in the indices β,γ.

Proposition 4.5. *The general expression of the semi-holonomic multivector fields which are solutions to the field equations* (1.4) *on* $\mathcal{S}_{sh}$ *are*

$$\mathbf{X}_\mathcal{L} = \bigwedge_{\nu=0}^{3}\left(\frac{\partial}{\partial x^\nu} + \sum_{\rho\leq\sigma}\left(g_{\rho\sigma,\nu}\frac{\partial}{\partial g_{\rho\sigma}} + f_{\rho\sigma\mu,\nu}\frac{\partial}{\partial g_{\rho\sigma,\mu}}\right) + \Gamma^\alpha_{\beta\gamma,\nu}\frac{\partial}{\partial\Gamma^\alpha_{\beta\gamma}} + f^\alpha_{\beta\gamma\mu,\nu}\frac{\partial}{\partial\Gamma^\alpha_{\beta\gamma,\mu}}\right)\,, \tag{4.22}$$

where, on the points of $\mathcal{S}_{sh}$,

$$\begin{aligned} f_{\rho\sigma\mu,\nu} &= D_\nu\left(g_{\sigma\lambda}\Gamma^\lambda_{\mu\rho}+g_{\rho\lambda}\Gamma^\lambda_{\mu\sigma}+\frac{2}{3}g_{\rho\sigma}T^\lambda_{\lambda\mu}\right) , \\ f^\alpha_{\beta\gamma\mu,\nu} &= \Gamma^\lambda_{\mu\gamma,\nu}\Gamma^\alpha_{\beta\lambda}+\Gamma^\lambda_{\mu\gamma}\Gamma^\alpha_{\beta\lambda,\nu}+C^\alpha_{\beta\gamma,\mu\nu}+K^\alpha_{\beta\gamma,\mu\nu} , \end{aligned}$$

for every $C_{\beta\mu\nu}\in C^\infty(J^1\pi)$ and $K^\alpha_{\beta\gamma,\mu\nu}\in C^\infty(J^1\pi)$ satisfying that, on $\mathcal{S}_{sh}$,

$$\begin{aligned} C^\alpha_{\beta\gamma,\mu\nu} &= C_{\beta\mu\nu}\delta^\alpha_\gamma \quad , \quad K^\lambda_{\lambda\gamma,\mu\nu}=0 \quad , \quad K^\lambda_{\beta\gamma,\lambda\nu}+K^\lambda_{\gamma\beta,\lambda\nu}=0 , \\ K^\alpha_{[\beta\gamma],\mu\nu} &= -\frac{1}{3}\delta^\alpha_{[\beta}K^\lambda_{\gamma]\lambda,\mu\nu}-\Gamma^\lambda_{\mu[\gamma,\nu}\Gamma^\alpha_{\beta]\lambda}-\Gamma^\lambda_{\mu[\gamma}\Gamma^\alpha_{\beta]\lambda,\nu} \\ &+ \frac{1}{3}\delta^\alpha_{[\beta}\Gamma^\lambda_{\mu\gamma],\nu}\Gamma^\rho_{\rho\lambda}+\frac{1}{3}\delta^\alpha_{[\beta}\Gamma^\lambda_{\mu\gamma]}\Gamma^\rho_{\rho\lambda,\nu}-\frac{1}{3}\delta^\alpha_{[\beta}\Gamma^\lambda_{\mu\rho,\nu}\Gamma^\rho_{\gamma]\lambda}-\frac{1}{3}\delta^\alpha_{[\beta}\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\gamma]\lambda,\nu} . \end{aligned}$$

Proof. The functions $f_{\sigma\rho\mu,\nu}$ are given by 4.21. Now, from (4.17) we obtain that

$$\left(\frac{\partial^2 H}{\partial g_{\rho\sigma}\partial g_{\mu\nu}}-\frac{\partial^2 L^{\beta\gamma,\lambda}_\alpha}{\partial g_{\rho\sigma}\partial g_{\mu\nu}}\Gamma^\alpha_{\beta\gamma,\lambda}\right)=0 \quad ; \quad (\text{on } \mathcal{S}_{sh}) ,$$

and therefore (4.20) becomes

$$\left(\Gamma^\alpha_{\beta\gamma,\nu}\frac{\partial^2 H}{\partial\Gamma^\alpha_{\beta\gamma}\partial g_{\rho\sigma}}-\frac{\partial L^{\beta\gamma,\mu}_\alpha}{\partial g_{\rho\sigma}}f^\alpha_{\beta\gamma\mu,\nu}\right)=0 \quad ; \quad (\text{on } \mathcal{S}_{sh}) .$$

A particular solution to these equations is

$$f^\alpha_{\beta\gamma\mu,\nu}=\Gamma^\lambda_{\mu\gamma,\nu}\Gamma^\alpha_{\beta\lambda}+\Gamma^\lambda_{\mu\gamma}\Gamma^\alpha_{\beta\lambda,\nu} \quad ; \quad (\text{on } \mathcal{S}_{sh}) .$$

Now, we need to find a general solution $h^\alpha_{\beta\gamma\mu,\nu}$ to the homogeneous equation, which is just (4.15), but on $\mathcal{S}_{sh}$. Thus, proceeding as in the proof of Proposition 4.3, we obtain that

$$h^\alpha_{\beta\gamma,\mu\nu}=C^\alpha_{\beta\gamma,\mu\nu}+K^\alpha_{\beta\gamma,\mu\nu} \quad ; \quad (\text{on } \mathcal{S}_{sh}) ,$$

for $C_{\beta\mu\nu}\in C^\infty(J^1\pi)$ and $K^\alpha_{\beta\gamma,\mu\nu}\in C^\infty(J^1\pi)$ satisfying that

$$C^\alpha_{\beta\gamma,\mu\nu}=C_{\beta\mu\nu}\delta^\alpha_\gamma \quad , \quad K^\lambda_{\lambda\gamma,\mu\nu}=0 \quad , \quad K^\lambda_{\beta\gamma,\lambda\nu}+K^\lambda_{\gamma\beta,\lambda\nu}=0 \quad ; \quad (\text{on } \mathcal{S}_{sh}) .$$

By construction, the solutions obtained in this way satisfy all the tangent conditions on the constraints given in Theorem 4.1, except

$$\mathrm{L}(X_\nu)r^\alpha_{\beta\gamma,\mu}=0 \quad ; \quad (\text{on } \mathcal{S}_{sh}) ;$$

and these equations lead to the last conditions. □

Comments:

- It is important to point out that, up to the torsion constraints $t^\alpha_{\beta\gamma}$, all the other constraints appear as a consequence of demanding the semiholonomy condition on the multivector fields solution to the field equations (1.4).
- From the constraints $m_{\rho\sigma\,\mu}=0$ and $t^\alpha_{\beta\gamma}=0$ in Theorem 4.1, and Proposition 4.4 we obtain that

  $$T^\alpha_{\beta\alpha}=0 \iff T^\alpha_{\beta\gamma}=0 \iff \nabla^\Gamma g=0 .$$

  Thus, any of these conditions is necessary and sufficient to assure that the connection becomes the Levi-Civita connection. This result completes the already known fact that the vanishing of the trace torsion is sufficient for the connection to be the Levi-Civita connection (see, for instance, [12, 20]).

Holonomic multivector fields (elements of $\ker^4_H \Omega_{\mathcal{L}_{\rm EP}}$): Integrability constraints

The last step is to look for holonomic (i.e., integrable and semiholonomic) multivector fields. Locally, a multivector field is integrable if $[X_\mu, X_\nu] = 0$ for every $\mu, \nu = 0, 1, 2, 3$. In any open of $U \subset \mathcal{S}_f$ where this condition holds, there exist integrable sections for the multivector field defined on $\pi(U)$. In general, integrable multivector fields could only exist in a submanifold $\mathcal{S}_f$ of $\mathcal{S}_{sh}$. In any case, computing where the different multivector fields we have found are integrable is, in general, a complicate task. In this section we outline some guidelines in order to solve this problem.

Consider the following general expression

$$[X_\mu, X_\nu] = F^\epsilon \frac{\partial}{\partial x^\epsilon} + \sum_{\alpha\leq\beta}\left(F_{\alpha\beta}\frac{\partial}{\partial g_{\alpha\beta}} + F_{\alpha\beta,\epsilon}\frac{\partial}{\partial g_{\alpha\beta,\epsilon}}\right) + F^\alpha_{\beta\gamma}\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} + F^\alpha_{\beta\gamma,\epsilon}\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\epsilon}} = 0 \; ; \; (\text{on } \mathcal{S}_{sh}) \; .$$

Next, we have to take into account (4.22). First, the coefficients $F^\epsilon|_{\mathcal{S}_{sh}} = 0$, necessarily (and this is the reason for imposing the vector field to vanish, which is a stronger condition than being inside the distribution). From the conditions $F_{\alpha\beta}|_{\mathcal{S}_{sh}} = 0$, we derive that

$$f_{\rho\sigma\mu,\nu} - f_{\rho\sigma\nu,\mu} = 0 \quad ; \quad (\text{on } \mathcal{S}_{sh}) \; .$$

which are new restrictions on the functions $\Gamma^\alpha_{\beta\gamma,\mu}$, specifically

$$\begin{aligned} i_{\rho\sigma,\mu\nu} &= g_{\rho\gamma}\Gamma^\gamma_{[\nu\lambda}\Gamma^\lambda_{\mu]\sigma} + g_{\sigma\gamma}\Gamma^\gamma_{[\nu\lambda}\Gamma^\lambda_{\mu]\rho} + g_{\rho\lambda}\Gamma^\lambda_{[\mu\sigma,\nu]} + g_{\sigma\lambda}\Gamma^\lambda_{[\mu\rho,\nu]} + \frac{2}{3}g_{\rho\sigma}T^\lambda_{\lambda[\mu,\nu]} \\ &= g_{\rho\lambda}K^\lambda_{[\nu\sigma\mu]} + g_{\sigma\lambda}K^\lambda_{[\nu\rho\mu]} + 2g_{\rho\sigma}T^\lambda_{\mu\nu}\Gamma^\gamma_{\gamma\lambda} = 0 \quad ; \quad (\text{on } \mathcal{S}_{sh}), \end{aligned} \tag{4.23}$$

where the functions $K^\alpha_{\beta\gamma\mu}$ arise from proposition 4.3. (Observe that these constraints are symmetric in the indices ρ, σ and skewsymmetric in the indices μ, ν). In a similar way, from the conditions $F^\alpha_{\beta\gamma}|_{\mathcal{S}_{sh}} = 0$, we obtain that

$$f^\alpha_{\beta\gamma\mu,\nu} - f^\alpha_{\beta\gamma\nu,\mu} = 0 \quad ; \quad (\text{on } \mathcal{S}_{sh}) \; ,$$

which impose some restrictions on the possible solutions, namely:

$$C_{\beta[\mu\nu]} = \Gamma^\lambda_{[\mu\beta,\nu]}\Gamma^\sigma_{\sigma\lambda} + \Gamma^\lambda_{[\mu\beta}\Gamma^\sigma_{\sigma\lambda,\nu]} \quad ; \quad (\text{on } \mathcal{S}_{sh}) \; ,$$

$$K^\alpha_{\beta\gamma,[\mu\nu]} = -\Gamma^\lambda_{[\mu\gamma,\nu]}\Gamma^\alpha_{\beta\lambda} - \Gamma^\lambda_{[\mu\gamma}\Gamma^\alpha_{\beta\lambda,\nu]} - C_{\beta[\mu\nu]}\delta^\alpha_\gamma \quad ; \quad (\text{on } \mathcal{S}_{sh}) \; .$$

The coefficients $F_{\alpha\beta,\gamma}$ vanish automatically on $\mathcal{S}_{sh}$ as long as $(f^\alpha_{\beta\gamma\mu,\nu} - f^\alpha_{\beta\gamma\nu,\mu})|_{\mathcal{S}_{sh}} = 0$. Finally, the conditions $F^\alpha_{\beta\gamma,\epsilon} = 0$ lead to a system of PDE on the functions $C_{\beta\mu\nu}, K^\alpha_{\beta\gamma,\mu\nu}$ which may originate new constraints. The tangency conditions on the constraints $i_{\rho\sigma,\mu\nu}$ give

$$\begin{aligned} g_{\alpha\lambda}K^\lambda_{[\nu\beta\mu],\xi} + g_{\beta\lambda}K^\lambda_{[\nu\alpha\mu],\xi} &= -2g_{\alpha\beta,\xi}T^\lambda_{\mu\nu}\Gamma^\sigma_{\sigma\lambda} - 2g_{\alpha\beta}T^\lambda_{\mu\nu,\xi}\Gamma^\sigma_{\sigma\lambda} - 2g_{\alpha\beta}T^\lambda_{\mu\nu}\Gamma^\sigma_{\sigma\lambda,\xi} \\ &\quad -g_{\alpha\lambda,\xi}K^\lambda_{[\nu\beta\mu]} - g_{\beta\lambda,\xi}K^\lambda_{[\nu\alpha\mu]} \quad ; \quad (\text{on } \mathcal{S}_{sh}) \; . \end{aligned}$$

In what follows, we will denote $\mathrm{j}_f \colon \mathcal{S}_f \hookrightarrow J^1\pi$ the constraint submanifold defined by all the constraints $c^{\mu\nu}$, $m_{\sigma\rho,\mu}$, $t^\alpha_{\beta\gamma}$, $r^\alpha_{\beta\gamma,\nu}$ and $i_{\rho\sigma,\mu\nu}$. This is the submanifold where there exist holonomic multivector fields solution to the field equations which are tangent to $\mathcal{S}_f$, as it is shown in Proposition 4.18. Notice that $\mathcal{S}_f$ is a subbundle of $J^1\pi$ over E and M and, thus, we have the natural submersions

$$\pi^1_f = \pi^1 \circ \mathrm{j}_f \colon \mathcal{S}_f \to \mathrm{E} \quad , \quad \bar{\pi}^1_f = \bar{\pi}^1 \circ \mathrm{j}_f \colon \mathcal{S}_f \to M \; .$$

4.2.3 Symmetries and gauge symmetries

Gauge symmetries of the Einstein-Palatini model

Proposition 4.6. *The natural gauge symmetries for the Einstein-Palatini model are the vector fields $X \in \mathfrak{X}(J^1\pi)$ whose local expressions are*

$$X = C_\beta \delta^\alpha_\gamma \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} + D_\mu C_\beta \delta^\alpha_\gamma \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} \quad , \quad C_\beta \in C^\infty(J^1\pi) \quad ; \quad (\text{on } \mathcal{S}_f) \, .$$

Proof. Consider a vector field

$$X = f^\mu \frac{\partial}{\partial x^\mu} + \sum_{\rho\leq\sigma} \left(f_{\rho\sigma} \frac{\partial}{\partial g_{\rho\sigma}} + f_{\rho\sigma,\mu} \frac{\partial}{\partial g_{\rho\sigma,\mu}} \right) + f^\alpha_{\beta\gamma} \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} + f^\alpha_{\beta\gamma,\mu} \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} \in \mathfrak{X}(J^1\pi) \, .$$

As $\mathcal{S}_f$ is a bundle over M, clearly X is $\bar{\pi}^1_f$-vertical if, and only if, it is $\bar{\pi}^1$-vertical. Therefore $\bar{\pi}^1_* X = 0$ if, and only if, $f^\mu = 0$. Furthermore

$$\begin{aligned} i(X)\Omega_{\mathcal{L}_{\rm EP}} &= \left(\sum_{\rho\leq\sigma} \frac{\partial H}{\partial g_{\rho\sigma}} f_{\rho\sigma} + \frac{\partial H}{\partial \Gamma^\alpha_{\beta\gamma}} f^\alpha_{\beta\gamma} \right) \mathrm{d}^4x - \sum_{\rho\leq\sigma} \frac{\partial L^{\beta\gamma,\mu}_\alpha}{\partial g_{\rho\sigma}} f_{\rho\sigma} \mathrm{d}\Gamma^\alpha_{\beta\gamma} \wedge \mathrm{d}^3x_\mu \\ &\quad - \sum_{\rho\leq\sigma} \frac{\partial L^{\beta\gamma,\mu}_\alpha}{\partial g_{\rho\sigma}} f^\alpha_{\beta\gamma} \mathrm{d}g_{\rho\sigma} \wedge \mathrm{d}^3x_\mu = 0 \, . \end{aligned}$$

After doing the pullback $\mathrm{j}^*_f\, i(X)\Omega_{\mathcal{L}_{\rm EP}}$, we obtain the terms $\mathrm{j}^*_f \mathrm{d}\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\mathrm{d}\Gamma^\alpha_{(\beta\gamma)} + \frac{1}{6}\delta^\alpha_\beta \mathrm{d}T^r_{r\gamma} - \frac{1}{6}\delta^\alpha_\gamma \mathrm{d}T^r_{r\beta}$. As every coefficient must vanish, taking in particular the corresponding to the factor $\mathrm{d}\Gamma^\alpha_{(\beta\gamma)}$, we obtain that $f_{\rho\sigma}|_{\mathcal{S}_f} = 0$. Indeed:

$$0 = \delta^\alpha_\beta (\frac{1}{3} g_{\mu\nu} g_{\gamma\lambda} - \frac{1}{6} g_{\mu\gamma} g_{\nu\lambda}) \sum_{\rho\leq\sigma} f_{\rho\sigma} \frac{\partial L^{(\beta\gamma),\mu}_\alpha}{\partial g_{\rho\sigma}} = \sum_{\nu\leq\lambda} (f_{\nu\lambda} + f_{\lambda\nu}) \Rightarrow f_{\rho\sigma} = 0 \quad ; \quad (\text{on } \mathcal{S}_f) \, .$$

Using these results, the problem is reduced to find $f^\alpha_{\beta\gamma} \in C^\infty(J^1\pi)$ such that

$$f^\alpha_{\beta\gamma} \frac{\partial L^{\beta\gamma,\mu}_\alpha}{\partial g_{\rho\sigma}} = 0 \quad ; \quad (\text{on } \mathcal{S}_f) \, , \tag{4.24}$$

$$f^\alpha_{\beta\gamma} \frac{\partial H}{\partial \Gamma^\alpha_{\beta\gamma}} = 0 \quad ; \quad (\text{on } \mathcal{S}_f) \, . \tag{4.25}$$

Multiplying (4.24) by $g_{\mu\rho} g_{\nu\sigma}$ we obtain:

$$f^\alpha_{\beta\gamma} + f^\alpha_{\gamma\beta} = f^r_{r\beta} \delta^\alpha_\gamma + f^r_{r\beta} \delta^\alpha_\gamma + (f^\alpha_{rs} g^{rs} - f^r_{rs} g^{\alpha s}) g_{\beta\gamma} \quad ; \quad (\text{on } \mathcal{S}_f) \, .$$

This system has two kinds of solutions. First, there are the *trace solutions*, given by $f^\alpha_{\beta\gamma} = C^\alpha_{\beta\gamma} = C_\beta \delta^\alpha_\gamma$, for every arbitrary function $C_\beta \in C^\infty(J^1\pi)$ [20]. Second, for other solutions $f^\alpha_{\beta\gamma}$, we have that $K^\alpha_{\beta\gamma} = f^\alpha_{\beta\gamma} - C^\alpha_{\beta\gamma}$, with $C_\gamma = f^\nu_{\nu\gamma}$. Contracting indices α, β we obtain $K^\alpha_{\alpha\gamma} = 0$. Since (4.24) are lineal, $K^\alpha_{\beta\gamma}$ are also solutions, therefore

$$\begin{aligned} K^\alpha_{\beta\gamma} + K^\alpha_{\gamma\beta} = K^\alpha_{\rho\sigma} g^{\rho\sigma} g_{\beta\gamma} \;\Rightarrow\; K^\alpha_{\beta\gamma} + K^\alpha_{\gamma\beta} = \frac{1}{2}(K^\alpha_{\rho\sigma} + K^\alpha_{\sigma\rho}) g^{\rho\sigma} g_{\beta\gamma} \quad \Rightarrow \\ g^{\beta\gamma}(K^\alpha_{\beta\gamma} + K^\alpha_{\gamma\beta}) = 2(K^\alpha_{\rho\sigma} + K^\alpha_{\sigma\rho}) g^{\rho\sigma} \;\Rightarrow\; -g^{\beta\gamma}(K^\alpha_{\beta\gamma} + K^\alpha_{\gamma\beta}) = 0 \quad ; \quad (\text{on } \mathcal{S}_f) \, , \end{aligned}$$

which implies $K^{\alpha}_{\rho\sigma}g^{\rho\sigma}=0$, thus $K^{\alpha}_{\beta\gamma}+K^{\alpha}_{\gamma\beta}=0$. These are called the *torsion solutions*. Both kinds of solutions fulfil (4.25); in fact,

$$\begin{aligned} C_\beta\delta^\alpha_\gamma\frac{\partial H}{\partial\Gamma^\alpha_{\beta\gamma}} &= \varrho C_\beta\left(g^{\mu\beta}\Gamma^r_{r\mu}+g^{\mu r}\Gamma^\beta_{\mu r}-g^{r\beta}\Gamma^\mu_{\mu r}-g^{\mu\nu}\Gamma^\beta_{\mu\nu}\right)=0 \quad ; \quad (\text{on } \mathcal{S}_f)\ ; \\ K^\alpha_{\beta\gamma}\frac{\partial H}{\partial\Gamma^\alpha_{\beta\gamma}} &= \varrho\left(K^\alpha_{\beta\gamma}(g^{\mu\gamma}\Gamma^\beta_{\alpha\mu}+g^{\mu\beta}\Gamma^\gamma_{\mu\alpha})-K^\alpha_{\beta\gamma}g^{\gamma\beta}\Gamma^\mu_{\mu\alpha}-K^\lambda_{\lambda\gamma}g^{\mu\nu}\Gamma^\gamma_{\mu\nu}\right) \\ &= \varrho K^\alpha_{\beta\gamma}g^{\mu\gamma}T^\beta_{\alpha\mu}=\varrho K^\alpha_{\beta\gamma}g^{\mu\gamma}(\frac{1}{3}\delta^\beta_\alpha T^r_{[r\mu]}-\frac{1}{3}\delta^\beta_\mu T^r_{[r\alpha]})=0 \quad ; \quad (\text{on } \mathcal{S}_f)\ . \end{aligned}$$

Now we impose the tangency condition on the torsion constraints

$$0=\mathrm{L}(X)t^\alpha_{\beta\gamma}=f^\alpha_{[\beta\gamma]}-\frac{1}{3}\delta^\alpha_\beta f^r_{[r\gamma]}+\frac{1}{3}\delta^\alpha_\gamma f^r_{[r\beta]}=2K^\alpha_{\beta\gamma}=0 \quad ; \quad (\text{on } \mathcal{S}_f)\ .$$

The trace solutions are tangent, but the torsion are not. Before checking the other constraints, let us impose the condition of being natural. The local conditions for a $\bar{\pi}^1$-vertical vector field to be natural are that $f_{\rho\sigma}$, $f^\alpha_{\beta\gamma}$ are $\bar{\pi}^1$-projectable, that $f_{\rho\sigma,\mu}=D_\mu f_{\rho\sigma}$, and that $f^\alpha_{\beta\gamma,\sigma}=D_\sigma f^\alpha_{\beta\gamma}$. In our case, these conditions imply that $C_\beta\in C^\infty(J^1\pi)$ are $\bar{\pi}^1$-projectable, that $f^\alpha_{\beta\gamma,\mu}|_{\mathcal{S}_f}=\delta^\alpha_\gamma D_\mu C_\beta$, and that $f_{\rho\sigma,\mu}|_{\mathcal{S}_f}=0$. The tangency condition on the pre-metricity constraints is

$$\begin{aligned} 0 &= \mathrm{L}(X)m_{\rho\sigma,\mu}=L(X)\left(g_{\rho\sigma,\mu}-g_{\sigma\lambda}\Gamma^\lambda_{\mu\rho}-g_{\rho\lambda}\Gamma^\lambda_{\mu\sigma}-\frac{2}{3}g_{\rho\sigma}T^\lambda_{\lambda\mu}\right) \\ &= f_{\rho\sigma,\mu}-g_{\sigma\lambda}\delta^\lambda_\rho C_\mu-g_{\rho\lambda}\delta^\lambda_\sigma C_\mu-\frac{2}{3}g_{\rho\sigma}(C_\lambda\delta^\lambda_\mu-C_\mu\delta^\lambda_\lambda)=0 \quad ; \quad (\text{on } \mathcal{S}_f)\ . \end{aligned}$$

As $f^\alpha_{\beta\gamma}|_{\mathcal{S}_f}=C_\beta\delta^\alpha_\gamma$, then $\dfrac{\partial L^{\beta\gamma,\sigma}_\alpha}{\partial g_{\mu\nu}}f^\alpha_{\beta\gamma,\sigma}=0$ (see 4.3), and hence

$$\mathrm{L}(X)c^{\mu\nu}=\frac{\partial\varrho g^{\alpha\beta}}{\partial g_{\mu\nu}}\left(C_\beta\Gamma^\sigma_{\sigma\alpha}+\Gamma^\gamma_{\beta\alpha}C_\gamma-C_\beta\Gamma^\sigma_{\sigma\alpha}-\Gamma^\gamma_{\beta\alpha}C_\gamma\right)-\frac{\partial L^{\beta\gamma,\sigma}_\alpha}{\partial g_{\mu\nu}}f^\alpha_{\beta\gamma,\sigma}=0 \quad ; \quad (\text{on } \mathcal{S}_f)\ .$$

The tangency condition on $r^\alpha_{\beta\gamma,\nu}$ involves only the functions $f^\alpha_{\beta\gamma,\nu}$:

$$0=\mathrm{L}(X)r^\alpha_{\beta\gamma,\nu}=f^\alpha_{[\beta\gamma],\nu}-\frac{1}{3}\delta^\alpha_\beta f^r_{[r\gamma],\nu}+\frac{1}{3}\delta^\alpha_\gamma f^r_{[r\beta],\nu} \quad ; \quad (\text{on } \mathcal{S}_f)\ .$$

The trace solutions fulfil this condition automatically. Finally, the tangency condition for the integrability constraints (4.23) holds:

$$\begin{aligned} \mathrm{L}(X)i_{\rho\sigma,\mu\nu} &= g_{\rho\gamma}C_{[\nu}\Gamma^\lambda_{\mu]\sigma}+g_{\rho\gamma}\Gamma^\lambda_{[\nu\sigma}C_{\mu]}+g_{\sigma\gamma}C_{[\nu}\Gamma^\lambda_{\mu]\rho}+g_{\sigma\gamma}\Gamma^\lambda_{[\nu\rho}C_{\mu]} \\ &+ g_{\rho\sigma}C_{[\mu\nu]}+g_{\rho\sigma}C_{[\mu\nu]}-2g_{\rho\sigma}C_{[\mu\nu]}=0 \quad ; \quad (\text{on } \mathcal{S}_f)\ . \end{aligned}$$

□

Lagrangian symmetries of the Einstein-Palatini model

Let F be a diffeomorphism in M. For every $x\in M$, if g_x is a metric in T_xM, then $F_*g_x=(F^{-1})^*(g_x)$ is also a metric with the same signature as g_x. In the same way, as a connection Γ_x is a $(1,1)$-tensor in T_xM [35], denoting also by F_* the induced action of F on the tensor algebra, we define:

Definition 4.1. *Let $F\colon M\to M$ be a diffeomorphism. The* canonical lift of F to the bundle E *is the diffeomorphism $\mathcal{F}\colon E\to\mathrm{E}$ defined as follows: for every $(x,g_x,\Gamma_x)\in E$, then $\mathcal{F}(x,g_x,\Gamma_x):=(F(x),F_*g_x,F_*\Gamma_x)$ (Thus $\pi\circ\mathcal{F}=F\circ\pi$).*

Let $Z\in\mathfrak{X}(M)$. The canonical lift of Z to the bundle E *is the vector field $Y_Z\in\mathfrak{X}(\mathrm{E})$ whose associated local one-parameter groups of diffeomorphisms $\mathcal{F}_t$ are the canonical lifts to the bundle E of the local one-parameter groups of diffeomorphisms F_t of Z.*

In coordinates, if $Z = f^\mu(x)\dfrac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$, the canonical lift of Z to the bundle $E \to M$ is

$$
\begin{aligned}
Y_Z \;&=\; f^\mu \frac{\partial}{\partial x^\mu} - \sum_{\alpha\leq\beta}\left(\frac{\partial f^\lambda}{\partial x^\alpha} g_{\lambda\beta} + \frac{\partial f^\lambda}{\partial x^\beta} g_{\lambda\alpha}\right)\frac{\partial}{\partial g_{\alpha\beta}} \\
&\quad + \left(\frac{\partial f^\alpha}{\partial x^\lambda}\Gamma^\lambda_{\beta\gamma} - \frac{\partial f^\lambda}{\partial x^\beta}\Gamma^\alpha_{\lambda\gamma} - \frac{\partial f^\lambda}{\partial x^\gamma}\Gamma^\alpha_{\beta\lambda} - \frac{\partial^2 f^\alpha}{\partial x^\beta \partial x^\gamma}\right)\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} \in \mathfrak{X}(\mathrm{E}) \,.
\end{aligned}
$$

Furthermore, every diffeomorphism in E induces a diffeomorphism in $J^1\pi$. The vector fields generating these transformations are canonical liftings $X = j^1Y$, for $Y \in \mathfrak{X}(E)$. Hence, for the above ones we have

$$
\begin{aligned}
j^1Y_Z \;&=\; f^\mu \frac{\partial}{\partial x^\mu} - \sum_{\alpha\leq\beta}\left(\frac{\partial f^\lambda}{\partial x^\alpha} g_{\lambda\beta} + \frac{\partial f^\lambda}{\partial x^\beta} g_{\lambda\alpha}\right)\frac{\partial}{\partial g_{\alpha\beta}} \\
&\quad - \sum_{\alpha\leq\beta}\left(\frac{\partial^2 f^\nu}{\partial x^\alpha \partial x^\mu} g_{\nu\beta} + \frac{\partial^2 f^\nu}{\partial x^\beta \partial x^\mu} g_{\alpha\nu} + \frac{\partial f^\nu}{\partial x^\alpha} g_{\nu\beta,\mu} + \frac{\partial f^\nu}{\partial x^\beta} g_{\alpha\nu,\mu} + \frac{\partial f^\nu}{\partial x^\mu} g_{\alpha\beta,\nu}\right)\frac{\partial}{\partial g_{\alpha\beta,\mu}} \\
&\quad + \left(\frac{\partial f^\alpha}{\partial x^\lambda}\Gamma^\lambda_{\beta\gamma} - \frac{\partial f^\lambda}{\partial x^\beta}\Gamma^\alpha_{\lambda\gamma} - \frac{\partial f^\lambda}{\partial x^\gamma}\Gamma^\alpha_{\beta\lambda} - \frac{\partial^2 f^\alpha}{\partial x^\beta \partial x^\gamma}\right)\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} \\
&\quad + \left(\frac{\partial f^\alpha}{\partial x^\lambda}\Gamma^\lambda_{\beta\gamma,\mu} - \frac{\partial f^\lambda}{\partial x^\beta}\Gamma^\alpha_{\lambda\gamma,\mu} - \frac{\partial f^\lambda}{\partial x^\gamma}\Gamma^\alpha_{\beta\lambda,\mu} - \frac{\partial f^\lambda}{\partial x^\mu}\Gamma^\alpha_{\beta\gamma,\lambda}\right. \\
&\quad \left. + \frac{\partial^2 f^\alpha}{\partial x^\lambda \partial x^\mu}\Gamma^\lambda_{\beta\gamma} - \frac{\partial^2 f^\lambda}{\partial x^\beta \partial x^\mu}\Gamma^\alpha_{\lambda\gamma} - \frac{\partial^2 f^\lambda}{\partial x^\gamma \partial x^\mu}\Gamma^\alpha_{\beta\lambda} - \frac{\partial^3 f^\alpha}{\partial x^\beta \partial x^\gamma \partial x^\mu}\right)\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} \\
&\equiv\; f^\mu \frac{\partial}{\partial x^\mu} + \sum_{\alpha\leq\beta} Y_{\alpha\beta}\frac{\partial}{\partial g_{\alpha\beta}} + \sum_{\alpha\leq\beta} Y_{\alpha\beta\mu}\frac{\partial}{\partial g_{\alpha\beta,\mu}} + Y^\alpha_{\beta\gamma}\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} + Y^\alpha_{\beta\gamma\mu}\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} \in \mathfrak{X}(J^1\pi) \,.
\end{aligned}
$$

We have that $\mathcal{L}_{\rm EP}$ is invariant under diffeomorphisms (using the constraints $c^{\mu\nu}$). Then, for every $Z \in \mathfrak{X}(M)$, we have that $\mathrm{L}(j^1Y_Z)\mathcal{L}_{\rm EP}|_{\mathcal{S}_f} = 0$. In addition, j^1Y_Z are tangent to $\mathcal{S}_f$. In fact, as they are natural vector fields that leave the Einstein-Palatini Lagrangian invariant, then the corresponding Euler-Lagrange equations are also invariant, and hence for the constraints $c^{\mu\nu}$ we have that

$$
\mathrm{L}(j^1Y_Z)c^{\mu\nu} = -\left(\frac{\partial f^\mu}{\partial x^\rho}\delta^\nu_\sigma + \frac{\partial f^\nu}{\partial x^\sigma}\delta^\mu_\rho\right)\left(\frac{\partial H}{\partial g_{\rho\sigma}} - \frac{\partial L^{\beta\gamma,\lambda}_\alpha}{\partial g_{\rho\sigma}}\Gamma^\alpha_{\beta\gamma,\lambda}\right) = 0 \quad ; \quad (\text{on } \mathcal{S}_f)\,;
$$

while for the other constraints, after a long calculation, we obtain

$$
\begin{aligned}
\mathrm{L}(j^1Y_Z)m_{\rho\sigma,\mu} &= \left(-\frac{\partial f^\alpha}{\partial x^\rho}\delta^\beta_\sigma\delta^\nu_\mu - \frac{\partial f^\beta}{\partial x^\sigma}\delta^\alpha_\rho\delta^\nu_\mu - \frac{\partial f^\nu}{\partial x^\mu}\delta^\alpha_\rho\delta^\beta_\sigma\right) m_{\alpha\beta,\nu} = 0;\ (\text{on } \mathcal{S}_f)\,, \\
\mathrm{L}(j^1Y_Z)t^\alpha_{\beta\gamma} &= \left(\frac{\partial f^\alpha}{\partial x^\lambda}\delta^\rho_\beta\delta^\sigma_\gamma - \frac{\partial f^\rho}{\partial x^\beta}\delta^\alpha_\lambda\delta^\sigma_\gamma - \frac{\partial f^\sigma}{\partial x^\gamma}\delta^\alpha_\lambda\delta^\rho_\beta\right) t^\lambda_{\rho\sigma} = 0;\ (\text{on } \mathcal{S}_f)\,, \\
\mathrm{L}(j^1Y_Z)r^\alpha_{\beta\gamma,\nu} &= \left(\frac{\partial f^\alpha}{\partial x^\lambda}\delta^\rho_\beta\delta^\sigma_\gamma\delta^\tau_\nu - \frac{\partial f^\rho}{\partial x^\beta}\delta^\alpha_\lambda\delta^\sigma_\gamma\delta^\tau_\nu - \frac{\partial f^\sigma}{\partial x^\gamma}\delta^\alpha_\lambda\delta^\rho_\beta\delta^\tau_\nu - \frac{\partial f^\tau}{\partial x^\nu}\delta^\alpha_\lambda\delta^\rho_\beta\delta^\sigma_\gamma\right) r^\lambda_{\rho\sigma,\tau} = 0;\ (\text{on } \mathcal{S}_f)\,, \\
\mathrm{L}(j^1Y_Z)i_{\rho\sigma,\mu\nu} &= \left(-\frac{\partial f^\alpha}{\partial x^\rho}\delta^\beta_\sigma\delta^\lambda_\mu\delta^\gamma_\nu - \frac{\partial f^\beta}{\partial x^\sigma}\delta^\alpha_\rho\delta^\lambda_\mu\delta^\gamma_\nu - \frac{\partial f^\lambda}{\partial x^\mu}\delta^\alpha_\rho\delta^\beta_\sigma\delta^\gamma_\nu - \frac{\partial f^\gamma}{\partial x^\nu}\delta^\alpha_\rho\delta^\beta_\sigma\delta^\lambda_\mu\right) i_{\alpha\beta,\lambda\gamma} = 0;\ (\text{on } \mathcal{S}_f)\,.
\end{aligned}
$$

Thus, these vector fields are natural infinitesimal Lagrangian symmetries and, hence, natural infinitesimal Noether symmetries. Then an associated conserved quantity to each j^1Y_Z is $\xi_{Y_Z} = i(j^1Y_Z)\Theta_{\mathcal{L}_{\rm EP}}$ (see section 2.1.5), which has the local expression:

$$
\xi_{Y_Z} = i(j^1Y)\Theta_{\mathcal{L}_{\rm EP}} = (L^{\beta\gamma,\mu}_\alpha Y^\alpha_{\beta\gamma} - Hf^\mu)\mathrm{d}^3x_\mu + f^\mu L^{\beta\gamma,\nu}_\alpha \mathrm{d}\Gamma^\alpha_{\beta\gamma}\wedge \mathrm{d}^2x_{\mu\nu} \,.
$$

Finally, given a section $\psi_\mathcal{L}$ solution the field equations, the Noether current associated with j^1Y_Z is

$$\psi_\mathcal{L}^*\xi_{Y_Z} = \psi_\mathcal{L}^*(L_\alpha^{\beta\gamma,\mu}(Y_{\beta\gamma}^\alpha - \Gamma_{\beta\gamma,\lambda}^\alpha f^\lambda) - f^\mu L_{\rm EP})\mathrm{d}^3x_\mu \, .$$

Comment: The term "gauge" is also used in physics to refer the invariance of the equations with respect to changes of variables in the base manifold M. Nevertheless, in our geometric formalism, these are actually the natural symmetries that we have studied in this Section, and they are mathematically different from the geometric gauge symmetries that we have analysed in the previous Section.

4.3 The Metric-Affine model: Hamiltonian formalism

4.3.1 Canonical Hamiltonian formalism

The multisymplectic Hamiltonian formalism for second-order field theories is presented in chapter 1 (See, for instance, [23, 32, 85] for the general setting of the multisymplectic Hamiltonian formalism for first-order field theories).

Consider the quotient bundle $J^1\pi^* = \mathcal{M}\pi/\Lambda_1^4(\mathrm{T}^*\mathrm{E})$ (where $\Lambda_1^4(\mathrm{T}^*\mathrm{E})$ is the bundle of π-semibasic 4-forms in E), which is called the *restricted multimomentum bundle* of E, and is endowed with the natural projections

$$\tau\colon\ J^1\pi^* \to \mathrm{E} \qquad , \quad \overline{\tau} = \pi\circ\tau\colon\ J^1\pi^* \to M \qquad , \quad \mu\colon \mathcal{M}\pi \to J^1\pi^* .$$

Induced local coordinates in $J^1\pi^*$ are $(x^\mu,\ g_{\alpha\beta}, \Gamma_{\lambda\gamma}^\nu,\ p^{\alpha\beta,\mu},\ p_\nu^{\lambda\gamma,\mu})$, $(0 \leq \alpha \leq \beta \leq 3)$.

The Legendre transformation $\mathcal{FL}_{\rm EP}\colon J^1\pi \longrightarrow J^1\pi^*$ (see [33] for the definition) is given, for the Einstein-Palatini Lagrangian, by

$$\mathcal{FL}_{\rm EP}{}^*\, x^\mu = x^\mu \quad , \quad \mathcal{FL}_{\rm EP}{}^*\, g_{\alpha\beta} = g_{\alpha\beta} \quad , \quad \mathcal{FL}_{\rm EP}{}^*\, \Gamma_{\beta\gamma}^\alpha = \Gamma_{\beta\gamma}^\alpha$$
$$\mathcal{FL}_{\rm EP}{}^*\, p^{\alpha\beta,\mu} = \frac{\partial L_{\rm EP}}{\partial g_{\alpha\beta,\mu}} = 0 \quad , \quad \mathcal{FL}_{\rm EP}{}^*\, p_\alpha^{\beta\gamma,\mu} = \frac{\partial L_{\rm EP}}{\partial \Gamma_{\beta\gamma,\mu}^\alpha} = L_\alpha^{\beta\gamma,\mu} = \varrho(\delta_\alpha^\mu g^{\beta\gamma} - \delta_\alpha^\beta g^{\mu\gamma}) \, ,$$

and $p^{\alpha\beta,\mu}$ and $p_\alpha^{\beta\gamma,\mu}$ are called the *momentum coordinates of the metric* and *the connection*, respectively. We have that, for every $j_x^1\phi \in J^1\pi$,

$$\mathrm{T}_{j_x^1\phi}\mathcal{FL}_{\rm EP} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\partial^2 L_{\rm EP}}{\partial g_{\nu\lambda}\partial\Gamma_{\beta\gamma,\mu}^\alpha} & 0 & 0 & 0 \end{pmatrix} .$$

Locally we have that

$$\ker\,(\mathcal{FL}_{\rm EP})_* = \left\langle \frac{\partial}{\partial g_{\alpha\beta,\mu}}, \frac{\partial}{\partial \Gamma_{\lambda\gamma,\mu}^\nu} \right\rangle_{0\leq\alpha\leq\beta\leq 3} . \tag{4.26}$$

Proposition 4.7. *$\mathcal{P} \equiv \mathcal{FL}_{\rm EP}(J^1\pi)$ is a closed submanifold of $J^1\pi^*$, which is diffeomorphic to* E.

Proof. From (4.26) we have that $\mathcal{P}$ is locally defined by the constraints

$$p^{\alpha\beta,\mu} = 0 \quad , \quad p_\alpha^{\beta\gamma,\mu} = \varrho(\delta_\alpha^\mu g^{\beta\gamma} - \delta_\alpha^\beta g^{\mu\gamma}) \, , \tag{4.27}$$

which remove the degrees of freedom in the fibers of the projection τ. □

If $\jmath\colon \mathcal{P}\hookrightarrow J^1\pi^*$ is the natural embedding, we denote by

$$\tau_{\mathcal{P}} = \tau\circ\jmath\colon \mathcal{P}\to E \quad , \quad \overline{\tau}_{\mathcal{P}} = \overline{\tau}\circ\jmath\colon \mathcal{P}\to M$$

the restrictions to $\mathcal{P}$ of the natural projections τ and $\overline{\tau}$. Then, this Proposition states that $\tau_{\mathcal{P}}$ is a diffeomorphism.

Proposition 4.8. $\mathcal{L}_{\rm EP}$ *is an almost-regular Lagrangian density.*

Proof. We prove the three conditions that define this concept: First, as we have seen, $\mathcal{P}$ is a closed submanifold of $J^1\pi^*$. Second, as $\dim\mathcal{P} = {\rm rank}({\rm T}_{j^1_x\phi}\mathcal{FL}_{\rm EP}) = 78$, for every $j^1_x\phi\in J^1\pi$, then $\mathcal{FL}_{\rm EP}$ is a submersion onto its image. Finally, taking into account Proposition 4.7, we conclude that the fibers of the Legendre map, $(\mathcal{FL}_{\rm EP})^{-1}(\mathcal{FL}(j^1_x\phi))$, are just the fibers of the projection π^1, and they are connected submanifolds of $J^1\pi$ (recall that $J^1\pi$ is connected because we are considering metrics with fixed signature). □

As a consequence of this Proposition, the existence of the Hamiltonian formalism for this system is assured (see, for instance, [85] for the details on this construction). In particular, let $\mathcal{FL}^o_{\rm EP}$ the map defined by $\mathcal{FL}_{\rm EP} = \jmath\circ\mathcal{FL}^o_{\rm EP}$. Then, tthe Poincaré-Cartan form $\Omega_{\mathcal{L}_{\rm EP}}$ is $\mathcal{FL}^o_{\rm EP}$-projectable and then there exists $\Omega_H\in\Omega^4(\mathcal{P})$ such that $\Omega_H = {\mathcal{FL}^o_{\rm EP}}^*\,\Omega_{\mathcal{L}_{\rm EP}}$, which is called the *Hamilton-Cartan form.*

In this way we have constructed the Hamiltonian system $(\mathcal{P},\Omega_H,T\mathcal{P},\{0\})$. In order to do a local analysis of the Hamiltonian formalism for this system, we can use two kinds of coordinates on $\mathcal{P}$: the so-called *non-momenta* and *pure connection coordinates.*

4.3.2 Non-momenta coordinates

Bearing in mind Proposition 4.7, we can take $(x^\lambda, g_{\rho\sigma}, \Gamma^\alpha_{\beta\gamma})$ as local coordinates in $\mathcal{P}$, with $0\leq\rho\leq\sigma\leq 3$. These are the *non-momenta coordinates* of $\mathcal{P}$. Using them, the local expression of Ω_H is the same as that of $\Omega_{\mathcal{L}_{\rm EP}}$ (see (4.3)). As a consequence, the Hamiltonian analysis of the system is similar to that in the Lagrangian formalism (up to the analysis of the holonomy).

Note that the functions $L^{\beta\gamma,\mu}_\alpha$ and H introduced in (4.1) and (4.2) are also $\mathcal{FL}^o_{\rm EP}$-projectable and, hence, we commit an abuse of notation denoting the corresponding functions of ${\rm C}^\infty(\mathcal{P})$ with the same simbols. Then, for a $\overline{\tau}_{\mathcal{P}}$-transverse multivector field $\mathbf{X}\in\mathfrak{X}^4(\mathcal{P})$, whose local expression in these coordinates is

$$\mathbf{X} = \bigwedge_{\nu=0}^{3} X_\nu = \bigwedge_{\nu=0}^{3}\left(\frac{\partial}{\partial x^\nu} + \sum_{\rho\leq\sigma} f_{\rho\sigma,\nu}\frac{\partial}{\partial g_{\rho\sigma}} + f^\alpha_{\beta\gamma,\nu}\frac{\partial}{\partial\Gamma^\alpha_{\beta\gamma}}\right) ,$$

the local expression of equation for multivector fields (1.4) for the Hamiltonian formalism is

$$\frac{\partial H}{\partial g_{\rho\sigma}} - f^\alpha_{\beta\gamma,\mu}\frac{\partial L^{\beta\gamma,\mu}_\alpha}{\partial g_{\rho\sigma}} = 0, \tag{4.28}$$

$$\frac{\partial H}{\partial\Gamma^\alpha_{\beta\gamma}} + \sum_{\rho\leq\sigma} f_{\rho\sigma,\mu}\frac{\partial L^{\beta\gamma,\mu}_\alpha}{\partial g_{\rho\sigma}} = 0, \tag{4.29}$$

together with other equalities which are consequence of these two sets of equations. This system of equations is the same as (4.7) and 4.8 and, therefore, the analysis made in Section 4.2.2 is valid here.

Proposition 4.9 (Constraints)**.** *A necessary condition for the existence of solutions to the system of equations* (4.28) *and* (4.29) *(and, in particular,* (4.28)*) is that the following equalities hold*

$$T^\alpha_{\beta\gamma} = \frac{1}{3}\delta^\alpha_\beta T^\nu_{\nu\gamma} - \frac{1}{3}\delta^\alpha_\gamma T^\nu_{\nu\beta}\,.$$

These constraints define the submanifold $\jmath_f\colon \mathcal{P}_f \hookrightarrow \mathcal{P}$.

Proof. The proof is the same than for Propositions (4.10) and (4.2). They are also the projections of the torsion constraints by the Legendre map. □

Finally, the tangency conditions of $\mathbf{X}$ for these constraints on $\mathcal{P}_f$ are

$$\mathrm{L}(X_\nu)(T^\alpha_{\beta\gamma} - \frac{1}{3}\delta^\alpha_\beta T^\nu_{\nu\gamma} + \frac{1}{3}\delta^\alpha_\gamma T^\nu_{\nu\beta}) = (f^\alpha_{\beta\gamma,\nu} - \frac{1}{3}\delta^\alpha_\beta f^\nu_{\nu\gamma,\nu} + \frac{1}{3}\delta^\alpha_\gamma f^\nu_{\nu\beta,\nu}) = 0 \quad ; \quad (\text{on } \mathcal{P}_f) \ ,$$

which does not lead to new constraints. Notice that these results about the Hamiltonian constraints are coherent with the comment in Section 4.2.2 about the fact that, up to the torsion constraints $t^\alpha_{\beta\gamma}$, all the other Lagrangian constraints appear as a consequence of demanding the semiholonomy condition for the solutions to the Lagrangian field equations and, hence, they cannot be projectable functions under the Legendre map [21]. In fact, a simple computation shows that

$$\mathrm{L}(X)c^{\mu\nu} \neq 0 \ , \ \mathrm{L}(X)m_{\sigma\rho,\mu} \neq 0 \ , \ \mathrm{L}(X)r^\alpha_{\beta\gamma,\nu} \neq 0 \quad ; \quad \text{for some } X \in \ker(\mathcal{FL}^o_{\mathrm{EP}})_* = \ker(\mathcal{FL}_{\mathrm{EP}})_* \ ,$$

which are the necessary and sufficient conditions for these functions not to be $\mathcal{FL}^o_{\mathrm{EP}}$-projectable. In the same way, the integrability Lagrangian constraints are not $\mathcal{FL}^o_{\mathrm{EP}}$-projectable either.

Proposition 4.10 (Solutions)**.** *The solutions to the Hamiltonian field equations* (4.28) *and* (4.29) *are*

$$\begin{aligned}\mathbf{X}_H = \bigwedge_{\nu=0}^{3} X_\nu \quad &= \quad \bigwedge_{\nu=0}^{3}\Bigg(\frac{\partial}{\partial x^\nu} + (\Gamma^\lambda_{\nu\gamma}\Gamma^\alpha_{\beta\lambda} + C_{\beta,\nu}\delta^\alpha_\gamma + K^\alpha_{\beta\gamma,\nu})\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} \\ &\qquad + \sum_{\rho\leq\sigma}(g_{\sigma\lambda}\Gamma^\lambda_{\mu\rho} + g_{\rho\lambda}\Gamma^\lambda_{\mu\sigma} + \frac{2}{3}g_{\rho\sigma}T^\lambda_{\lambda\mu})\frac{\partial}{\partial g_{\rho\sigma}}\Bigg) \quad ; \quad (\text{on } \mathcal{P}_f) \ ; \end{aligned} \tag{4.30}$$

with $C_{\beta,\nu}$, $K^\alpha_{\beta\gamma,\nu} \in C^\infty(\mathcal{P}_f)$ such that, on the points of $\mathcal{P}_f$, they satisfy

$$K^\mu_{\mu\gamma,\nu} \ = \ 0 \quad , \quad K^\mu_{\beta\gamma,\mu} + K^\mu_{\gamma\beta,\mu} \ = \ 0 \quad , \tag{4.31}$$

$$K^\alpha_{[\beta\gamma],\mu} \ = \ -\frac{1}{3}\delta^\alpha_{[\beta}K^\nu_{\gamma]\nu,\mu} - \Gamma^\lambda_{\mu[\gamma}\Gamma^\alpha_{\beta]\lambda} + \frac{1}{3}\delta^\alpha_{[\beta}\Gamma^\lambda_{\mu\gamma]}\Gamma^\nu_{\nu\lambda} - \frac{1}{3}\delta^\alpha_{[\beta}\Gamma^\lambda_{\mu\nu}\Gamma^\nu_{\gamma]\lambda} \ . \tag{4.32}$$

Proof. From Proposition 4.3 and (4.10), we obtain (4.30) and (4.31), and the tangency conditions on the torsion constraints lead to we obtain (4.32). □

Finally, the integrability condition is $[X_\mu, X_\nu]|_{\mathcal{P}_f} = 0$. The vanishing of the coefficients of $\dfrac{\partial}{\partial g_{\sigma\rho}}$ do not lead to new constraints, but they do impose new restrictions for the possible solutions:

$$g_{\alpha\lambda}K^\lambda_{[\nu\beta\mu]} + g_{\beta\lambda}K^\lambda_{[\nu\alpha\mu]} + 2g_{\alpha\beta}T^\lambda_{\mu\nu}\Gamma^\sigma_{\sigma\lambda} = 0 \quad ; \quad (\text{on } \mathcal{P}_f) \ .$$

The vanishing of the coefficients of $\dfrac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}}$ lead to a system of first order PDE on the functions $C^\alpha_{\beta\gamma\mu}$ and $K^\alpha_{\beta\gamma\mu}$. This system of PDE has solutions everywhere on $\mathcal{P}_f$, as it is shown in Proposition 4.19.

The following diagram summarizes this situation:

$$
\begin{tikzcd}
 & & & & J^1\pi^* \\
J^1\pi \arrow[urrrr, "\mathcal{F}\mathcal{L}_{\mathrm{EP}}"] \arrow[rrrr, "\mathcal{F}\mathcal{L}^o_{\mathrm{EP}}"] \arrow[drr, "\pi^1"] \arrow[dddrr, bend right=60, "\overline{\pi}^1"'] & & & & \mathcal{P} \arrow[u, hook, "\jmath"'] \arrow[dll, "\tau_{\mathcal{P}}"'] \arrow[dddll, bend left=60, "\overline{\tau}_{\mathcal{P}}"] \\
 & & E \arrow[dd, "\pi"] & & \\
S_f \arrow[uu, hook, "\mathrm{j}_f"] \arrow[urr, "\pi^1_f"] \arrow[drr, "\overline{\pi}^1_f"'] & & & & \mathcal{P}_f \arrow[uu, hook, "\jmath_f"'] \arrow[ull, "\tau_f"'] \arrow[dll, "\overline{\tau}_f"] \\
 & & M & &
\end{tikzcd}
\tag{4.33}
$$

The study of the gauge vector fields in the Hamiltonian formalism is simpler than in the Lagrangian one. In fact:

Proposition 4.11 (Gauge symmetries)**.** *The gauge symmetries of the system are*

$$
X = C_\beta \delta^\alpha_\gamma \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} \quad , \quad C_\beta \in C^\infty(\mathcal{P}) \quad ; \quad (\text{on } \mathcal{P}_f) \, .
$$

Proof. A $\overline{\tau}$-vertical vector field has the local expression:

$$
X = \sum_{\rho \leq \sigma} f_{\rho\sigma} \frac{\partial}{\partial g_{\rho\sigma}} + f^\alpha_{\beta\gamma} \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} \, .
$$

The analysis of the equation $i(X)\Omega_H = 0$ is analogous as in Proposition 4.6. We find that $f_{\rho\sigma} = 0$ and $f^\alpha_{\beta\gamma} = C_\beta \delta^\alpha_\gamma + K^\alpha_{\beta\gamma}$, on the points of $\mathcal{P}_f$; that is, they are a combination of a trace and a torsion solution; but the torsion solutions are not tangent to $\mathcal{P}_f$. □

The multiple solutions of the system are given by the functions $C^\alpha_{\beta\gamma,\nu}$ and $K^\alpha_{\beta\gamma,\nu}$ (see (4.30)). The functions $C^\alpha_{\beta\gamma,\nu}$ are related to the gauge freedom, but the former ones $K^\alpha_{\beta\gamma,\nu}$ are not.

4.3.3 Pure-connection coordinates

The non-momenta coordinates arise in a natural way from the structure of the manifolds, but their use turn out to be very similar to the analysis made in the Lagrangian formalism, thus providing little extra understanding about the theory. A more interesting coordinates can be obtained from the second set of constraints in (4.27)

$$
p^{\beta\gamma,\mu}_\alpha = \varrho \left(\delta^\mu_\alpha g^{\beta\gamma} - \delta^\beta_\alpha g^{\mu\gamma} \right) \; ; \tag{4.34}
$$

that is, the momenta of the connection can be obtained from the metric. The converse is also true; in fact:

Lemma 4.1. *Denoting $\mathcal{T} := \sqrt{|\det(p^{\mu\alpha,\beta}_\mu)|}$, we have that*

$$
g^{\alpha\beta} = -\frac{1}{3\varrho} p^{\mu\alpha,\beta}_\mu = -\frac{3}{\mathcal{T}} p^{\mu\alpha,\beta}_\mu \, .
$$

Proof. Contracting the indices α and β on (4.34) we obtain

$$p_\nu^{\nu\gamma,\mu} = -3\varrho g^{\gamma\mu} ,$$

which is the first equality. Now, computing the determinant, as $\varrho = \sqrt{|\det(g_{\gamma\mu})|}$, we obtain that the second equality holds:

$$|\det(p_\nu^{\nu\gamma,\mu})| = 3^4\varrho^4|\det(g_{\gamma\mu})|^{-1} \iff \mathcal{T} = 9\varrho ,$$

□

It is interesting to point out that all the results can be extended to an arbitrary dimension $m > 2$; but $\mathcal{T}$ is proportional to ϱ only for $m = 4$.

Since the degrees of freedom of $g_{\alpha\beta}$ and $p_\alpha^{\beta\gamma,\mu}$ are not equal, equation (4.34) has several implicit restrictions. In fact, using Lemma 4.1 to substitute the metric for momenta in (4.34) we obtain the constraints

$$p_\alpha^{\beta\gamma,\mu} = \frac{1}{3}\delta_\alpha^\beta p_\nu^{\nu\mu,\gamma} - \frac{1}{3}\delta_\alpha^\mu p_\nu^{\nu\beta,\gamma} ,$$

which are very similar to the torsion constraints. Moreover, as $g_{\alpha\beta} = g_{\beta\alpha}$, from Lemma 4.1 we have that $p_\mu^{\mu\alpha,\beta} = p_\mu^{\mu\beta,\alpha}$. Therefore, the only degrees of freedom for the momenta of the connection are the symmetric part of $p_\mu^{\mu\beta,\alpha}$, which equals the degrees of freedom of the metric.

Denoting $p^{\alpha\beta} := p_r^{r\alpha,\beta}$, we can consider the set of coordinates $(x^\mu, \Gamma_{\beta\gamma}^\alpha, p^{\rho\sigma})$ in $\mathcal{P}$, with $0 \leq \rho \leq \sigma \leq 3$, which are called *pure-connection coordinates*. The relation between these coordinates and the non-momenta ones is given by the following map

$$\Psi(x^\lambda, g_{\rho\sigma}, \Gamma_{\beta\gamma}^\alpha) = (x^\mu, \Gamma_{\beta\gamma}^\alpha, p^{\rho\sigma} = \varrho(\delta_\alpha^\mu g^{\beta\gamma} - \delta_\alpha^\beta g^{\mu\gamma})) ,$$

which is invertible, and hence a local diffeomorphism, by Lemma 4.1.

In pure-connection coordinates the Hamiltonian function has the local expression

$$H = -\frac{1}{3}p^{\alpha\beta}\left(\Gamma_{\beta\sigma}^\gamma\Gamma_{\gamma\alpha}^\sigma - \Gamma_{\beta\alpha}^\gamma\Gamma_{\sigma\gamma}^\sigma\right) ,$$

and the Hamilton-Cartan form Ω_H is

$$\begin{aligned}\Omega_H &= \mathrm{d}H \wedge \mathrm{d}^4x + \frac{1}{6}\delta_\alpha^\mu \mathrm{d}p^{\beta\gamma} \wedge \mathrm{d}\Gamma_{(\beta\gamma)}^\alpha \wedge \mathrm{d}^3x_\mu \\ &\quad -\frac{1}{6}\delta_\alpha^\beta \mathrm{d}p^{\mu\gamma} \wedge \mathrm{d}\Gamma_{\beta\mu}^\alpha \wedge \mathrm{d}^3x_\gamma - \frac{1}{6}\delta_\alpha^\beta \mathrm{d}p^{\mu\gamma} \wedge \mathrm{d}\Gamma_{\beta\gamma}^\alpha \wedge \mathrm{d}^3x_\mu .\end{aligned}$$

A general transverse locally decomposable multivector field in $\mathcal{P}$ has the local expression in pure-connection coordinates:

$$\mathbf{X}_H = \bigwedge_{\nu=0}^{3} X_\nu = \bigwedge_{\nu=0}^{3}\left(\frac{\partial}{\partial x^\nu} + f_{\beta\gamma,\nu}^\alpha\frac{\partial}{\partial\Gamma_{\beta\gamma}^\alpha} + \sum_{\alpha\leq\beta} G_\nu^{\alpha\beta}\frac{\partial}{\partial p^{\alpha\beta}}\right) .$$

Then the field equations (1.4) are locally

$$\frac{1}{n(\alpha\beta)}\frac{\partial H}{\partial p^{\alpha\beta}} + \frac{1}{6}f_{(\alpha\beta),\mu}^\mu - \frac{1}{6}f_{\mu(\alpha,\beta)}^\mu = 0 , \tag{4.35}$$

$$\frac{\partial H}{\partial\Gamma_{\beta\gamma}^\alpha} - \frac{1}{3}G_\alpha^{\beta\gamma} + \frac{1}{3}\delta_\alpha^\beta G_\mu^{\mu\gamma} = 0 . \tag{4.36}$$

Next the results previously described in the above Section 4.3.2 are recovered and extended:

The constraints and gauge vector fields are related to the connection, where both the non-momenta and pure-connection coordinates have the same expression. Therefore:

Proposition 4.12 (Constraints)**.** *A necessary condition for the existence of solutions to the system of equations* (4.35) *and* (4.36) *(and, in particular,* (4.36)*) is that the following equalities hold*

$$T^{\alpha}_{\beta\gamma} = \frac{1}{3}\delta^{\alpha}_{\beta}T^{\nu}_{\nu\gamma} - \frac{1}{3}\delta^{\alpha}_{\gamma}T^{\nu}_{\nu\beta} \,.$$

These constraints define the submanifold $\jmath_f\colon \mathcal{P}_f \hookrightarrow \mathcal{P}$.

Proof. They are the projections of the torsion constraints by the Legendre map. Alternatively, they can be deduced from (4.36) imposing that $G^{\beta\gamma}_{\alpha} - G^{\gamma\beta}_{\alpha} = 0$. □

Taking into account the results presented in the above Section 4.3.2, we have:

Proposition 4.13 (Solutions)**.** *The solutions to the Hamiltonian field equation* (1.4) *in the pure-connection coordinates are:*

$$\begin{aligned}\mathbf{X}_H = \bigwedge_{\nu=0}^{3} X_\nu \quad &= \quad \bigwedge_{\nu=0}^{3}\Bigg(\frac{\partial}{\partial x^\nu} + (\Gamma^{\lambda}_{\nu\gamma}\Gamma^{\alpha}_{\beta\lambda} + C_{\beta,\nu}\delta^{\alpha}_{\gamma} + K^{\alpha}_{\beta\gamma,\nu})\frac{\partial}{\partial\Gamma^{\alpha}_{\beta\gamma}} \\ &\quad + \sum_{\alpha\leq\beta}(-p^{\alpha\mu}\Gamma^{\beta}_{\nu\mu} - p^{\beta\mu}\Gamma^{\alpha}_{\nu\mu} - \tfrac{1}{3}p^{\alpha\beta}T^{\mu}_{\mu\nu} + p^{\alpha\beta}\Gamma^{\mu}_{\mu\nu})\frac{\partial}{\partial p^{\alpha\beta}}\Bigg) \quad ; \quad (\text{on } \mathcal{P}_f)\,;\end{aligned}$$

with $C_{\beta,\nu}$, $K^{\alpha}_{\beta\gamma,\nu} \in C^\infty(\mathcal{P}_f)$ such that, on the points of $\mathcal{P}_f$, they satisfy

$$\begin{aligned}K^{\mu}_{\mu\gamma,\nu} \quad &= \quad 0 \quad , \quad K^{\mu}_{\beta\gamma,\mu} + K^{\mu}_{\gamma\beta,\mu} \;=\; 0 \quad , \\ K^{\alpha}_{[\beta\gamma],\mu} \quad &= \quad -\frac{1}{3}\delta^{\alpha}_{[\beta}K^{\nu}_{\gamma]\nu,\mu} - \Gamma^{\lambda}_{\mu[\gamma}\Gamma^{\alpha}_{\beta]\lambda} + \frac{1}{3}\delta^{\alpha}_{[\beta}\Gamma^{\lambda}_{\mu\gamma]}\Gamma^{\nu}_{\nu\lambda} - \frac{1}{3}\delta^{\alpha}_{[\beta}\Gamma^{\lambda}_{\mu\nu}\Gamma^{\nu}_{\gamma]\lambda} \,.\end{aligned}$$

The integrability condition is

$$0 = [X_\nu, X_\mu] = F^{\epsilon}\frac{\partial}{\partial x^\epsilon} + F^{\alpha}_{\beta\gamma}\frac{\partial}{\partial\Gamma^{\alpha}_{\beta\gamma}} + \sum_{\alpha\leq\beta}F_{\alpha\beta}\frac{\partial}{\partial p^{\alpha\beta}} \quad ; \quad (\text{on } \mathcal{P}_f)\,.$$

We have that $F^{\epsilon}|_{\mathcal{P}_f} = 0$, and imposing $F_{\alpha\beta}|_{\mathcal{P}_f} = 0$, we derive the following condition on the possible solutions

$$p^{\alpha\sigma}K^{\beta}_{[\mu\sigma\nu]} + p^{\beta\sigma}K^{\alpha}_{[\mu\sigma\nu]} - \frac{1}{3}p^{\alpha\beta}K^{\sigma}_{[\mu\sigma\nu]} = \frac{2}{3}p^{\alpha\beta}T^{\lambda}_{\nu\mu}\Gamma^{\sigma}_{\sigma\lambda} \quad ; \quad (\text{on } \mathcal{P}_f)\,.$$

The conditions $F^{\alpha}_{\beta\gamma}|_{\mathcal{P}_f} = 0$ lead to a system of PDE on the functions $C_{\alpha,\beta}$ and $K^{\alpha}_{\beta\gamma,\mu}$ which has solutions everywhere on $\mathcal{P}_f$, as it is shown in Proposition 4.19.

Proposition 4.14 (Gauge symmetries)**.** *The gauge symmetries of the system are:*

$$X = C_\beta\delta^{\alpha}_{\gamma}\frac{\partial}{\partial\Gamma^{\alpha}_{\beta\gamma}} \quad , \quad C_\beta \in C^\infty(\mathcal{P}_f) \quad ; \quad (\text{on } \mathcal{P}_f)\,.$$

Proof. For a generic vertical vector field

$$X = f^{\alpha}_{\beta\gamma}\frac{\partial}{\partial\Gamma^{\alpha}_{\beta\gamma}} + \sum_{\alpha\leq\beta} G^{\alpha\beta}\frac{\partial}{\partial p^{\alpha\beta}} \ ,$$

we have that

$$\begin{aligned} i(X)\Omega_H &= \left(\sum_{\alpha\leq\beta}\frac{\partial H}{\partial p^{\alpha\beta}}G^{\alpha\beta} + \frac{\partial H}{\partial\Gamma^{\alpha}_{\beta\gamma}}f^{\alpha}_{\beta\gamma}\right)\mathrm{d}^4x - \left(\frac{1}{3}\delta^{\mu}_{\alpha}G^{\beta\gamma} - \frac{1}{3}\delta^{\beta}_{\alpha}G^{\mu\gamma}\right)\mathrm{d}\Gamma^{\alpha}_{\beta\gamma}\wedge\mathrm{d}^3x_{\mu} \\ &\quad + \left(\frac{1}{6}f^{\mu}_{\alpha\beta} + \frac{1}{6}f^{\mu}_{\beta\alpha} - \frac{1}{6}\delta^{\mu}_{\beta}f^{\nu}_{\nu\alpha} - \frac{1}{6}\delta^{\mu}_{\alpha}f^{\nu}_{\nu\beta}\right)\mathrm{d}g_{\rho\sigma}\wedge\mathrm{d}^3x_{\mu} = 0 \ . \end{aligned}$$

Doing the pullback to $\mathcal{P}_f$, we have that $j^*\mathrm{d}\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\mathrm{d}\Gamma^{\alpha}_{(\beta\gamma)} + \frac{1}{6}\delta^{\alpha}_{\beta}\mathrm{d}T^{r}_{r\gamma} - \frac{1}{6}\delta^{\alpha}_{\gamma}\mathrm{d}T^{r}_{r\beta}$. As every coefficient must vanish, taking in particular the corresponding to the factor $\mathrm{d}\Gamma^{\alpha}_{(\beta\gamma)}$ and contracting with δ^{α}_{μ}, we obtain that $G^{\beta\gamma}=0$. Therefore we have

$$\begin{aligned} \frac{\partial H}{\partial\Gamma^{\alpha}_{\beta\gamma}}f^{\alpha}_{\beta\gamma} &= 0 \quad ; \quad (\text{on } \mathcal{P}_f)\ , \\ -\frac{1}{6}f^{\mu}_{\alpha\beta} + \frac{1}{6}f^{\mu}_{\beta\alpha} + \frac{1}{6}\delta^{\mu}_{\beta}f^{\nu}_{\nu\alpha} + \frac{1}{6}\delta^{\mu}_{\alpha}f^{\nu}_{\nu\beta} &= 0 \quad ; \quad (\text{on } \mathcal{P}_f)\ . \end{aligned}$$

Following the same argument as in 4.6, these equations have two kinds of solutions on $\mathcal{P}_f$: trace solutions, $f^{\alpha}_{\beta\gamma} = C_{\beta}\delta^{\alpha}_{\gamma}$, and torsion solutions, $f^{\alpha}_{\beta\gamma} = k^{\alpha}_{\beta\gamma}$; with $k^{\alpha}_{\beta\gamma} + k^{\alpha}_{\gamma\beta} = 0$ and $k^{\mu}_{\mu\gamma} = 0$. Likewise, only the trace solutions are tangent to $\mathcal{P}_f$. □

4.3.4 Intrinsic interpretation of the pure-connection coordinates

Now we present a fibered manifold and a Hamiltonian function which involve only the connection and we prove that this system is equivalent to the Hamiltonian formalism for the Metric-Affine action.

The configuration bundle for this pure-connection system is the bundle $\pi_\Gamma\colon E_\Gamma \to M$, where M is the connected orientable 4-dimensional manifold representing space-time, as above, and $E_\Gamma = C(LM)$, the bundle of connections on M; that is, linear connections in $\mathrm{T}M$. Then, consider the bundles $\mathcal{M}\pi_\Gamma \equiv \Lambda^4_2(\mathrm{T}^*E_\Gamma)$ and $J^1\pi^*_\Gamma \equiv \mathcal{M}\pi_\Gamma/\Lambda^4_1(\mathrm{T}^*E_\Gamma)$, with local coordinates $(x^\mu, \Gamma^{\alpha}_{\beta\gamma}, p, p^{\beta\gamma,\mu}_{\alpha})$ and $(x^\mu, \Gamma^{\alpha}_{\beta\gamma}, p^{\beta\gamma,\mu}_{\alpha})$ respectively.

Consider a *Hamiltonian section* $h\colon J^1\pi^* \to \mathcal{M}\pi$ of the projection $\mu_\Gamma\colon \mathcal{M}\pi_\Gamma \to J^1\pi^*_\Gamma$. In a local chart of natural coordinates, $U \subset J^1\pi^*_\Gamma$, this Hamiltonian section is specified by a *local Hamiltonian function* $H_\Gamma \in \mathrm{C}^\infty(U)$ such that $h_\Gamma(x^\mu, \Gamma^{\alpha}_{\beta\gamma}, p^{\beta\gamma,\mu}_{\alpha}) = (x^\mu, \Gamma^{\alpha}_{\beta\gamma}, p = -H_\Gamma(x^\nu, \Gamma^{\delta}_{\rho\sigma}, p^{\rho\sigma,\nu}_{\delta}), p^{\beta\gamma,\mu}_{\alpha})$ (see [13, 85]). This Hamiltonian function is

$$H_\Gamma = -\frac{1}{3}p^{\alpha\beta}\left(\Gamma^{\gamma}_{\beta\sigma}\Gamma^{\sigma}_{\gamma\alpha} - \Gamma^{\gamma}_{\beta\alpha}\Gamma^{\sigma}_{\sigma\gamma}\right) \ .$$

The bundle $\mathcal{M}\pi_\Gamma$ is canonically endowed with the corresponding multisymplectic Liouville 5-form $\Omega_\Gamma \in \mathit{\Omega}^5(\mathcal{M}\pi_\Gamma)$. Then, the Hamilton-Cartan form is

$$\Omega_{H_\Gamma} \equiv h^*_\Gamma\Omega_\Gamma = \mathrm{d}H\wedge\mathrm{d}^4x - \mathrm{d}p^{\beta\gamma,\mu}_{\alpha}\wedge\mathrm{d}\Gamma^{\alpha}_{\beta\gamma}\wedge\mathrm{d}^3x_\mu \in \mathit{\Omega}^5(J^1\pi^*_\Gamma).$$

Furthermore, we introduce the following constraints on $J^1\pi^*_\Gamma$:

$$p^{\beta\gamma,\mu}_{\alpha} = \frac{1}{3}\delta^{\beta}_{\alpha}p^{\nu\mu,\gamma}_{\nu} - \frac{1}{3}\delta^{\mu}_{\alpha}p^{\nu\beta,\gamma}_{\nu} \quad , \quad p^{\mu\alpha,\beta}_{\mu} = p^{\mu\beta,\alpha}_{\mu}.$$

Let $\jmath_\Gamma\colon \mathcal{P}_\Gamma \hookrightarrow J^1\pi_\Gamma^*$ be the submanifold locally defined by these constraints. Then we can construct the premultisymplectic form

$$\begin{aligned}\Omega^0_{H_\Gamma} = \jmath_\Gamma^*\Omega_{H_\Gamma} \quad &= \quad \mathrm{d}H\wedge \mathrm{d}^4x + \frac{1}{6}\delta^\mu_\alpha \mathrm{d}p_\nu^{\nu\beta,\gamma}\wedge \mathrm{d}\Gamma^\alpha_{(\beta\gamma)}\wedge \mathrm{d}^3x_\mu \\ &\quad -\frac{1}{6}\delta^\beta_\alpha \mathrm{d}p_\nu^{\nu\mu,\gamma}\wedge \mathrm{d}\Gamma^\alpha_{\beta\mu}\wedge \mathrm{d}^3x_\gamma - \frac{1}{6}\delta^\beta_\alpha \mathrm{d}p_\nu^{\nu\mu,\gamma}\wedge \mathrm{d}\Gamma^\alpha_{\beta\gamma}\wedge \mathrm{d}^3x_\mu \ .\end{aligned}$$

Proposition 4.15. *There exists a diffeomorphism $\zeta\colon \mathcal{P}_\Gamma \to \mathcal{P}$ such that $\Omega_{H_\Gamma} = \zeta^*\Omega_H$ and hence the Hamiltonian systems $(\mathcal{P}_\Gamma, \Omega_{H_\Gamma}, T\mathcal{P}_\Gamma, \{0\})$ and $(\mathcal{P}, \Omega_H, T\mathcal{P}, \{0\})$ are equivalents.*

Proof. Using the pure-connection coordinates in $\mathcal{P}$, the diffeomorphism is locally given by

$$\zeta^* x^\mu = x^\mu \quad , \quad \zeta^*\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} \quad , \quad \zeta^* p^{\gamma\mu} = p_\nu^{\nu\gamma,\mu} \ .$$

Its inverse acting on the momenta is given by

$$\zeta^{-1^*}x^\mu = x^\mu \ , \ \zeta^{-1^*}\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} \ , \ \zeta^{-1^*}p_\alpha^{\beta\gamma,\mu} = \zeta^{-1^*}\left(\frac{1}{3}\delta^\beta_\alpha p_\nu^{\nu\mu,\gamma} - \frac{1}{3}\delta^\mu_\alpha p_\nu^{\nu\beta,\gamma}\right) = \frac{1}{3}\delta^\beta_\alpha p^{\mu\gamma} - \frac{1}{3}\delta^\mu_\alpha p^{\beta\gamma} \ ,$$

and is an exhaustive map because $\mathrm{Im}(\zeta^{-1}) = \mathcal{P}_\Gamma$, as a consequence of the reasoning done before in this paragraph. The equality $\Omega_{H_\Gamma} = \zeta^*\Omega_H$ is obtained straightforwardly from the local expressions of these forms. □

4.4 Relation with the Einstein-Hilbert model

The Einstein-Hilbert model can be recovered from the Einstein-Palatini (metric-affine) model by demanding the connection to be the Levi-Civita connection associated with the metric [20]. In this section we will show this equivalence geometrically.

In order to avoid confusing within the notation of this chapter and the elements of the Eisntein-Hilbert model presented in chapter 3, we denote the bundle of metric as: $\pi_\Sigma\colon \Sigma \to M$. The fibres are spaces of Lorentz metrics on M; that is, for every $x \in M$, the fiber $\pi_\Sigma^{-1}(x)$ is the set of metrics with signature $(-+++)$ acting on T_xM. The adapted fiber coordinates in E are $(x^\mu, g_{\alpha\beta})$.

It is proved [16, 87] that there are first-order (regular) Lagrangians in $J^1\pi_\Sigma$ which are equivalent to the the Einstein-Hilbert Lagrangian. As it is shown in 3.1.4, that allows a description of the Einstein-Hilbert model in $J^1\pi_\Sigma$ (with coordinates $(x^\mu, g_{\alpha\beta}, g_{\alpha\beta,\mu})$ which is geometrically equivalent to the Hamiltonian description of the Einstein-Hilbert model. The first-order Lagrangian density proposed in [86] (see 3.1.4 for more details) is $\overline{\mathcal{L}} = \overline{L}\,\mathrm{d}^4x$, where the Lagrangian function is

$$\begin{aligned}\overline{L} \quad &= \quad L_0 - \sum_{\substack{\alpha\leq\beta\\ \lambda\leq\sigma}} g_{\alpha\beta,\mu}g_{\lambda\sigma,\nu}\frac{\partial L^{\alpha\beta,\mu\nu}}{\partial g_{\lambda\sigma}} \in \mathrm{C}^\infty(J^1\pi_\Sigma) \ ; \\ L^{\alpha\beta,\mu\nu} \quad &= \quad \frac{n(\alpha\beta)}{2}\varrho(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - 2g^{\alpha\beta}g^{\mu\nu}) \ , \\ L_0 \quad &= \quad \varrho g^{\alpha\beta}\{g^{\gamma\delta}(g_{\delta\mu,\beta}\tilde{\Gamma}^\mu_{\alpha\gamma} - g_{\delta\mu,\gamma}\tilde{\Gamma}^\mu_{\alpha\beta}) + \tilde{\Gamma}^\delta_{\alpha\beta}\tilde{\Gamma}^\gamma_{\gamma\delta} - \tilde{\Gamma}^\delta_{\alpha\gamma}\tilde{\Gamma}^\gamma_{\beta\delta}\} \ ,\end{aligned}$$

where $\tilde{\Gamma}^\mu_{\alpha\gamma}$ are the Christoffel symbols of the Levi-Civita connection associated with the metric $g_{\alpha\beta}$. The corresponding Poincaré-Cartan form is

$$\Omega_{\overline{\mathcal{L}}} = \mathrm{d}\overline{L}\wedge \mathrm{d}^4x - \sum_{\alpha\leq\beta}\mathrm{d}\frac{\partial\overline{L}}{\partial g_{\alpha\beta,\mu}}\wedge \mathrm{d}g_{\alpha\beta}\wedge \mathrm{d}^3x_\mu \in \mathit{\Omega}^5(J^1\pi_\Sigma) \ .$$

So we have the Lagrangian system $(J^1\pi_\Sigma, \Omega_{\overline{\mathcal{L}}}, TJ^1\pi_\Sigma, \mathfrak{C}^1)$ and, as the Lagrangian $\overline{L}$ is regular, then $\Omega_{\overline{\mathcal{L}}}$ is a multisymplectic form and the Lagrangian field equations have solutions everywhere in $J^1\pi_\Sigma$.

In addition, the corresponding Legendre map $\mathcal{F}\overline{\mathcal{L}}\colon J^1\pi_\Sigma \to J^1\pi_\Sigma{}^*$ is a diffeomorphism [47]. Then we have the Hamilton-Cartan form $\Omega_{\overline{h}} := ((\mathcal{F}\overline{\mathcal{L}})^{-1})^*\Omega_{\overline{\mathcal{L}}} \in \Omega^5(J^1\pi_\Sigma{}^*)$. So we have the Hamiltonian system $(J^1\pi_\Sigma{}^*, \Omega_{\overline{h}}, TJ^1\pi_\Sigma{}^*, \{0\})$ and the corresponding Hamiltonian field equations have solutions everywhere in $J^1\pi_\Sigma{}^*$. In addition, the solutions to the Lagrangian problem are in one-to-one correspondence with thes solution to the Hamiltonian problem through the Legendre map.

4.4.1 Relation between the Einstein-Hilbert and the Metric-Affine models

The pre-metricity constraints determine the derivatives of the metric in function of the metric and the connection. The converse, which is a similar result to the existence of the Levi-Civita connection, can be formulated as follows:

Proposition 4.16. *Let (M, g) be a (semi)-Riemmanian manifold of dimension $m > 1$ and $C_\alpha \in C^\infty(U)$, $1 \leq \alpha \leq m$, fixed functions defined on a open set $U \subset M$. Then there exists a unique linear connection Γ defined on U such that:*

1. *Pre-metricity:* $(\nabla^\Gamma g)_{\rho\sigma,\mu} = \dfrac{2}{m-1} g_{\rho\sigma} T^\lambda_{\lambda\mu}$.
2. *Torsion:* $T^\alpha_{\beta\gamma} = \dfrac{1}{m-1}\,\delta^\alpha_\beta\, T^\lambda_{\lambda\gamma} - \dfrac{1}{m-1}\,\delta^\alpha_\gamma\, T^\lambda_{\lambda\beta}$
3. *Gauge fixing:* $\Gamma^\lambda_{\alpha\lambda} = C_\alpha$.

Proof. From the pre-metricity conditions we have

$$\begin{aligned}\frac{1}{2}g^{\mu\alpha}(g_{\rho\mu,\sigma} + g_{\sigma\mu,\rho} - g_{\rho\sigma,\mu}) &= \Gamma^\alpha_{\rho\sigma} + \frac{1}{2}(g^{\mu\alpha}g_{\rho\lambda}T^\lambda_{\sigma\mu} + g^{\mu\alpha}g_{\sigma\lambda}T^\lambda_{\rho\mu} - T^\alpha_{\rho\sigma}) \\ &\quad + \frac{1}{m-1}(T^\lambda_{\lambda\sigma}\delta^\alpha_\rho + T^\lambda_{\lambda\rho}\delta^\alpha_\sigma - g^{\alpha\mu}g_{\rho\sigma}T^\lambda_{\lambda\mu})\ .\end{aligned}$$

Using the torsion conditions and the gauge fixing we get

$$\frac{1}{2}\,g^{\mu\alpha}(g_{\rho\mu,\sigma} + g_{\sigma\mu,\rho} - g_{\rho\sigma,\mu}) = \Gamma^\alpha_{\rho\sigma} + \frac{1}{m-1}\Gamma^\lambda_{\lambda\rho}\delta^\alpha_\sigma - \frac{1}{m-1}C_\rho\delta^\alpha_\sigma\ ,$$

and contracting the indices α and ρ and rearranging the terms:

$$\frac{1}{m-1}\,\Gamma^\lambda_{\lambda\sigma} = \frac{1}{2m}g^{\mu\nu}g_{\mu\nu,\sigma} + \frac{1}{m(m-1)}C_\sigma\ .$$

Finally, incorporating this result to the previous equation, we conclude that

$$\Gamma^\alpha_{\rho\sigma} = \frac{1}{2}g^{\mu\alpha}(g_{\rho\mu,\sigma} + g_{\sigma\mu,\rho} - g_{\rho\sigma,\mu}) - \frac{1}{2m}g^{\mu\nu}g_{\mu\nu,\rho}\delta^\alpha_\sigma + \frac{1}{m}C_\rho\delta^\alpha_\sigma\ ,$$

which determines uniquely the connection in U. □

In order to establish the relation between both models, our standpoint is the Hamiltonian formalism of the Einstein-Palatini model developed in Section 4.3.2. So, let $\mathcal{P}_f \hookrightarrow \mathcal{P}$ be the final constraint submanifold for this last model. Then, consider the following local map:

$$\begin{array}{rccl}\xi\colon & \mathcal{P}_f & \to & J^1\pi_\Sigma \\ & (x^\mu, g_{\alpha\beta}, \Gamma^\alpha_{\beta\gamma}) & \mapsto & (x^\mu, g_{\alpha\beta}, \overline{g}_{\alpha\beta,\gamma})\end{array}$$

where $\overline{g}_{\alpha\beta,\gamma} = g_{\alpha\lambda}\Gamma^\lambda_{\mu\beta} + g_{\beta\lambda}\Gamma^\lambda_{\mu\alpha} + \frac{2}{3}g_{\alpha\beta}T^\lambda_{\lambda\mu}$. Notice that $\overline{\tau}_{\mathcal{P}} \circ j = \pi_\Sigma \circ \xi$.

Lemma 4.2. *Denoting by $\mathcal{G}$ the set of gauge symmetries obtained in Proposition 4.11, we have that* $\ker\,\xi_* = \mathcal{G}$.

Proof. Consider a generic vector field $X \in \mathfrak{X}(\mathcal{P})$, tangent to $\mathcal{P}_f$,

$$X = f^\mu \frac{\partial}{\partial x^\mu} + \sum_{\alpha\leq\beta} f_{\alpha\beta}\frac{\partial}{\partial g_{\alpha\beta}} + f^\alpha_{\beta\gamma}\frac{\partial}{\partial\Gamma^\alpha_{\beta\gamma}} \,.$$

If $X \in \ker\,\xi_*$, then $f^\mu = 0$ and $f_{\alpha\beta} = 0$. For the last coefficients we have:

$$0 = \xi_* X = g_{\alpha\lambda} f^\lambda_{\gamma\beta} + g_{\beta\lambda} f^\lambda_{\gamma\alpha} + \frac{2}{3} g_{\alpha\beta}\left(f^\lambda_{\lambda\gamma} - f^\lambda_{\gamma\lambda}\right) \,.$$

For the coefficients of the form $f^\alpha_{\beta\gamma} = C_\beta \delta^\alpha_\gamma$ for $C_\beta \in \mathrm{C}^\infty(\mathcal{P})$, the condition holds. Now, for every solution $f^\alpha_{\beta\gamma}$ to these equations, consider $h^\alpha_{\beta\gamma} = f^\alpha_{\beta\gamma} - f^\lambda_{\lambda\beta}\delta^\alpha_\gamma$, which are also solutions because the equation is lineal. Thus

$$g_{\alpha\lambda} h^\lambda_{\gamma\beta} + g_{\beta\lambda} h^\lambda_{\gamma\alpha} - \frac{2}{3} g_{\alpha\beta} h^\lambda_{\gamma\lambda} = 0 \,. \tag{4.37}$$

Notice that $h^\nu_{\nu\gamma} = 0$. Now, contracting with $g^{\alpha\beta}$, we obtain that $h^\lambda_{\gamma\lambda} = 0$. Furthermore, as we are on the points of $\mathcal{P}_f$, where the torsion constraints hold, this implies that $h^\alpha_{\beta\gamma} - h^\alpha_{\gamma\beta} = 0$, and therefore they are symmetric functions (for the indices $\beta\gamma$). Now, if $S_{\alpha\gamma\beta} := g_{\alpha\lambda} h^\lambda_{\gamma\beta}$; taking into account the symmetry of $h^\alpha_{\beta\gamma}$, we have that $S_{\alpha\gamma\beta} = S_{\alpha\beta\gamma}$, and from (4.37) we obtain $S_{\alpha\gamma\beta} = -S_{\beta\gamma\alpha}$. These two conditions hold simultaneously only if $S_{\alpha\gamma\beta} = 0$. Therefore, $h^\alpha_{\beta\gamma} = 0$, and hence $\ker\,\xi_* = \left\langle C_\beta \delta^\alpha_\gamma \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} \right\rangle = \mathcal{G}$. □

Let $\mathcal{P}'_f$ be the manifold obtained making the quotient of $\mathcal{P}_f$ (which is defined by the torsion constraints) by the gauge vector fields, and let the natural projection $\tau'_f \colon \mathcal{P}_f \to \mathcal{P}'_f$. Then:

Theorem 4.2. *$\mathcal{P}'_f$ is locally diffeomorphic to $J^1\pi_\Sigma$ and hence to $J^1\pi_\Sigma{}^*$.*

Proof. Consider a smooth section ς of τ'_f, and let $\xi' := \xi\circ\varsigma \colon \mathcal{P}'_f \to J^1\pi_\Sigma$. From lemma 4.2, $\ker\,\xi_* \supset \mathcal{G}$; therefore ξ' does not depend on the section chosen. Moreover, $\ker\,\xi_* \subset \mathcal{G}$ and it is injective. Finally, it is exhaustive because for every point of $J^1\pi_\Sigma$, its preimage contains the connection given by proposition 4.16. In conclusion, ξ' is a local diffeomorphism and then $\mathcal{P}'_f$ is (locally) diffeomorphic to $J^1\pi_\Sigma$. □

Then, a simple calculation in coordinates leads to the following result:

Proposition 4.17. $\Omega_H = \xi^*\Omega_{\overline{\mathcal{L}}} = (\mathcal{F}\overline{\mathcal{L}}\circ\xi)^*\Omega_{\overline{h}}$.

Comment: The comparison between the multiplicity of solutions of the Einstein-Hilbert and the metric-affine models can help us to interpret some of the conditions. The multiplicity of the semi-holonomic solutions of the Einstein-Hilbert model appears in the second derivative of the components of the metric (in the Hamiltonian formalism using the non-momentum coordinates). As it is shown in chapter 3, they are of the form (see [47]) $F_{\alpha\beta;\mu,\nu} = \frac{1}{2} g_{\lambda\sigma}(\Gamma^\lambda_{\nu\alpha}\Gamma^\sigma_{\mu\beta} + \Gamma^\lambda_{\nu\beta}\Gamma^\sigma_{\mu\alpha}) + F^\mathfrak{h}_{\alpha\beta;\mu,\nu}$, where

$$F^\mathfrak{h}_{\alpha\beta;\mu,\nu} = F^\mathfrak{h}_{\beta\alpha;\mu,\nu} = F^\mathfrak{h}_{\alpha\beta;\nu,\mu} \quad , \quad g^{\alpha\beta}\left(F^\mathfrak{h}_{\eta\tau;\alpha,\beta} + F^\mathfrak{h}_{\alpha\beta;\eta,\tau} - F^\mathfrak{h}_{\alpha\eta;\tau,\beta} - F^\mathfrak{h}_{\alpha\tau;\eta,\beta}\right) = 0 \,.$$

The map ξ transforms any section ψ solution of the Einstein-Palatini model into a solution $\xi^*\psi$ of the Einstein-Hilbert model. The functions $C^\alpha_{\beta\gamma,\mu}$ in (4.30), corresponding to the gauge variation, get annihilated by the action of ξ. Therefore, we can say that the functions $K^\alpha_{\beta\gamma,\mu}$ (corresponding to ψ) and $F^\mathfrak{h}_{\alpha\beta;\mu,\nu}$ (corresponding to $\xi^*\psi$) are related, as they are in one to one correspondence. Their conditions can be related using this equivalence as it is shown in the following table: supposing that $F^\mathfrak{h}_{\alpha\beta;\mu,\nu}$ and $K^\alpha_{\beta\gamma,\mu}$ are related, we have:

Metric-Affine		Einstein-Hilbert
$K^{\lambda}_{(\eta\tau)\lambda} = 0$	$\Leftrightarrow$	$g^{\alpha\beta}(F^{\mathfrak{h}}_{\eta\tau;\alpha,\beta} + F^{\mathfrak{h}}_{\alpha\beta;\eta,\tau} - F^{\mathfrak{h}}_{\alpha\eta;\tau,\beta} - F^{\mathfrak{h}}_{\alpha\tau;\eta,\beta}) = 0$
$g_{\alpha\lambda}K^{\lambda}_{[\nu\beta\mu]} + g_{\beta\lambda}K^{\lambda}_{[\nu\alpha\mu]} + 2g_{\alpha\beta}T^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\sigma\lambda} = 0$	$\Leftrightarrow$	$F^{\mathfrak{h}}_{\alpha\beta,[\mu\nu]} = 0$
$K^{\lambda}_{\lambda\gamma,\mu} = 0$		for every $F^{\mathfrak{h}}_{\alpha\beta,\mu\nu}$
$K^{\alpha}_{[\beta\gamma],\mu} + \frac{1}{3}\delta^{\alpha}_{[\beta}K^{\nu}_{\gamma]\nu,\mu} + \Gamma^{\lambda}_{\mu[\gamma}\Gamma^{\alpha}_{\beta]\lambda} - \frac{1}{3}\delta^{\alpha}_{[\beta}\Gamma^{\lambda}_{\mu\gamma]}\Gamma^{\nu}_{\nu\lambda}$		
$+\frac{1}{3}\delta^{\alpha}_{[\beta}\Gamma^{\lambda}_{\mu\nu}\Gamma^{\nu}_{\gamma]\lambda} = 0$		for every $F^{\mathfrak{h}}_{\alpha\beta,\mu\nu}$
for every $K^{\alpha}_{\beta\gamma,\mu}$		$F^{\mathfrak{h}}_{[\alpha\beta],\mu\nu} = 0$

4.4.2 Integrability

In the (first-order) Einstein-Hilbert model, every point $p \in J^1\pi_\Sigma$ is in the image of a section solution to the field equations, $\mathrm{Im}(\varphi_p)$. Then $J^1\pi_\Sigma$ is the final manifold for this model. As a consequence of the equivalence between both models, $\mathcal{P}_f$ is also the final constraint submanifold for the Einstein-Palatini model; that is:

Proposition 4.18. *For every $q \in \mathcal{P}_f$, there exists a section ψ_H solution to the Hamiltonian field equations of the Metric-Affine model such that $q \in \mathrm{Im}(\psi_H)$.*

Proof. Consider the solution $\varphi_{\xi(q)}$ in the Einstein-Hilbert Hamiltonian formalism. Moreover, consider $\zeta\colon J^1\pi_\Sigma \to \mathcal{P}_f \subset \mathcal{P}$ a section of ξ such that $\zeta(\xi(q)) = q$ which exists because ξ is exhaustive. Therefore $q \in \mathrm{Im}(\zeta \circ \varphi_{\xi(q)})$ and, in order to check that $\zeta \circ \phi_{\xi(q)}$ is a solution, consider an arbitrary $Y \in \mathfrak{X}(\mathcal{P})$:

$$\begin{aligned}(\zeta \circ \varphi_{\xi(q)})^*(i(Y)\Omega_H) &= (\zeta \circ \varphi_{\xi(q)})^*(i(Y)\xi^*\Omega_{\mathcal{L}_{\mathfrak{V}}}) \\ &= (\xi \circ \zeta \circ \varphi_{\xi(q)})^*(i(\xi_* Y)\Omega_{\mathcal{L}_{\mathfrak{V}}}) = \varphi^*_{\xi(q)}(i(\xi_* Y)\Omega_{\mathcal{L}_{\mathfrak{V}}}) = 0\,.\end{aligned}$$

We have used that $(\xi \circ \zeta)(p) = p$ because it is a section, and that $\varphi_{\xi(q)}$ is a solution. Finally,

$$\overline{\tau}_{\mathcal{P}} \circ \jmath_f \circ \zeta \circ \varphi_{\xi q} = \pi_\Sigma \circ \xi \circ \zeta \circ \varphi_{\xi(q)} = \pi_\Sigma \circ \varphi_{\xi(q)} = \mathrm{Id}_M\,;$$

thus $\psi_H = \zeta \circ \varphi_{\xi(q)}$ is a section of $\overline{\tau}_{\mathcal{P}} \circ \jmath_f = \overline{\tau}_f$, and hence it is a solution. □

The Lagrangian counterpart of this result also holds, although it is not straightforward because we are working with a singular field theory.

Proposition 4.19. *For every $p \in \mathcal{S}_f$, there exists a holonomic section $\psi_{\mathcal{L}}$ solution to the Lagrangian field equations of the Metric-Affine model such that $p \in \mathrm{Im}(\psi_{\mathcal{L}})$.*

Proof. Consider the diffeomorphism $\tau_{\mathcal{P}}\colon \mathcal{P} \to E$ stated in Proposition 4.7 (in particular, it relates the Lagrangian coordinates with the non-momenta coordinates). Then we have that $\tau_{\mathcal{P}}^{-1}(\pi^1_f(p)) \in \mathcal{P}_f$. Furthermore there exists a solution to the Hamiltonian field equations ψ_H such that $\tau_{\mathcal{P}}^{-1}(\pi^1_f(p)) \in \mathrm{Im}(\psi_H)$, as it is shown in the above Proposition. Then, we are going to prove that the holonomic section $\psi_{\mathcal{L}}$ solution in the Lagrangian formalism is $\psi_{\mathcal{L}} = j^1(\tau_{\mathcal{P}} \circ \psi_H)$.

In fact, first observe that, for the Metric-Affine model, the fibers of the Legendre map $\mathcal{F}\mathcal{L}^o_{\mathrm{EP}}$ are the vertical fibers of $\pi^1\colon J^1\pi \to E$ (since $\mathcal{P} = \mathrm{Im}\,\mathcal{F}\mathcal{L}^o_{\mathrm{EP}}$ is diffeomorphic to E), and then, as $\psi_{\mathcal{L}}$ is a canonical lifting to $J^1\pi$ of a section in E, we have that $\mathcal{F}\mathcal{L}^o_{\mathrm{EP}} \circ \psi_{\mathcal{L}} = \psi_H$. Furthermore, $\psi_{\mathcal{L}}$ is a solution to the Lagrangian field equations. Indeed, as $\mathcal{F}\mathcal{L}^o_{\mathrm{EP}}$ is a submersion, we can take a local basis of $\mathfrak{X}(J^1\pi)$ made by vector fields $\{Y_A, Z_a\}$, where Y_A are $\mathcal{F}\mathcal{L}^o_{\mathrm{EP}}$-projectable and $Z_a \in \ker\,(\mathcal{F}\mathcal{L}^o_{\mathrm{EP}})_*$; and then the vector fields $X_A = (\mathcal{FL}^o_{\rm EP})_* Y_A$ are a local basis for $\mathfrak{X}(\mathcal{P})$. Therefore, taking into account that $\mathcal{FL}^o_{\rm EP} \circ \psi_{\mathcal{L}} = \psi_H$ and that ψ_H is a solution to the Hamiltonian field equations,

$$\begin{aligned} \psi_{\mathcal{L}}^* \, i(Y_A)\Omega_{\mathcal{L}_{EP}} &= \psi_{\mathcal{L}}^* \, i(Y_A)({\mathcal{FL}^o_{\rm EP}}^*\Omega_H) = \psi_{\mathcal{L}}^* {\mathcal{FL}^o_{\rm EP}}^* \, i(X_A)\Omega_H \\ &= (\mathcal{FL}^o_{\rm EP} \circ \psi_{\mathcal{L}})^* \, i(X_A)\Omega_H = \psi_H^* \, i(X)\Omega_H = 0 \, ; \end{aligned}$$

and $\psi_{\mathcal{L}}^* \, i(Z_a)\Omega_{\mathcal{L}_{EP}} = 0$ trivially. This allows us to conclude that $\psi_{\mathcal{L}}^* \, i(Y)\Omega_{\mathcal{L}_{EP}} = 0$, for every $Y \in \mathfrak{X}(J^1\pi)$, and hence $\psi_{\mathcal{L}}$ is is a solution to the Lagrangian field equations.

Finally, $\operatorname{Im}\psi_{\mathcal{L}} \subset \mathcal{S}_f$. Indeed, equations (4.28) and (4.29) for ψ_H imply that all the points in $\operatorname{Im}\psi_{\mathcal{L}}$ verify the constraints $c^{\mu\nu}$ and $m_{\rho\sigma,\mu}$. The constraints $r^\alpha_{\beta\gamma,\nu}$ and $i_{\rho\sigma,\mu\nu}$ are also satisfied because they arise from the tangency condition on the semiholonomic constraints (see Section 4.2.2) and the integrability condition respectively; and then they are satisfied for holonomic sections which are solutions to the Lagrangian field equations.

The following diagram summarizes the situation (see also the diagram (4.33)).

$$\begin{tikzcd}
J^1\pi \supset \mathcal{S}_f \arrow[rr, "\mathcal{FL}^o_{\rm EP}"] \arrow[dr, "\pi^1"] & & \mathcal{P}_f \subset \mathcal{P} \arrow[dl, "\tau_{\mathcal{P}}"'] \\
& E & \\
& M \arrow[u, "\phi"] \arrow[uul, "\psi_{\mathcal{L}} = j^1\phi"] \arrow[uur, "\psi_H"'] &
\end{tikzcd}$$

$\square$

Chapter 5

Conclusions and outlook

Summary of results

The study of the models of gravity has required the development of several general properties of multisymplectic systems.

- Following the work of [86, 87], we establish the constraints generated by the projectability of the Poincaré-Cartan form. These constraints are related to the fact that the higher order velocities are strong gauge vector fields. Therefore, in adequate circumstances, the theory is equivalent to a lower order one [16, 87].
- We proposed a new local characterization of the projectability of the Poincaré-Cartan form. Thanks to it, the results have been transported to higher-order mechanical systems. Moreover, this characterization is adequate for the local analysis of the different models of gravity studied.
- The concept of gauge freedom has been analyzed. We propose to use the term "gauge" to refer to the non-regularity of the Poincaré-Cartan form. Therefore, the multiple solutions are characterized by two sources: the gauge related, arising from gauge symmetries and related to the non-regularity; and the non-gauge related, which arise from sources exclusive of field theories.
- The non-regularity of the Poincaré-Cartan form has other consequences, which are related to the weak and strong gauge vector fields. We use this analysis to interpret the covariant Hamiltonian formalism.

We studied in detail two models of Gravity: the Einstein-Hilbert model and the Metric-Affine (or Einstein-Palatini) model. The first one is a singular second order field theory which, as a consequence of its non-regularity, it is equivalent to a regular first order theory. The Metric-Affine model is a singular first order field theory which has a gauge symmetry. When these symmetry is quoted out, both models are equivalent. In both cases, a covariant Hamiltonian multisymplectic formalism has been presented. In every situation, we explicitly write all semi-holonomic multivector fields solutions of the field equations. A more exhaustive presentation of the results follows.

For the Einstein-Hilbert model:

- We have presented a multisymplectic covariant description of the Einstein-Hilbert model of General Relativity using a unified formulation joining both the Lagrangian and Hamiltonian formalisms.

- Our procedure consists in using the constraint algorithm to determine a submanifold of the higher-order jet-multimomentum bundle $\mathcal{W}_r$ where the field equations are compatible. The constraints (3.12) and (3.14), which define $\mathcal{W}_{\mathcal{L}}$, are a natural consequence of the unified formalism and define the Legendre map which allows to state the Hamiltonian formulation and the Hamilton-de Donder-Weyl version of the Einstein equations.

- In the case of no energy-matter sources, among the new constraints, the physical relevant equations are the primary constraints (3.16) which, evaluated on the points of the holonomic sections, are just the Einstein equations. They appear as constraints of the theory as a consequence of the singularity of the Einstein-Hilbert model. The secondary constraints (3.17) contain no physical information: they are of geometrical nature and arise because we are using a manifold prepared for a second-order theory of a Lagrangian which is physically equivalent to a first-order Lagrangian.

- Θ_r is $(\pi_1^3 \circ \rho_1^r)$-projectable and, as a consequence of this, in the Lagrangian formalism,the Poincaré-Cartan form $\Theta_{\mathcal{L}}$ projects onto a form in $J^1\pi$, which is not the Poincaré-Cartan form of any first-order Lagrangian. Nevertheless, there is are first-order regular Lagrangians which are equivalent to the Einstein-Hilbert Lagrangian[16, 62, 72, 73, 86, 87].The Lagrangian and Hamiltonian formalism of one of these Lagrangians have been analyzed in detail. .

- When we recover the Lagrangian formalism from the unified one, as a consequence of the singularity of the Einstein-Hilbert Lagrangian, solutions to the Euler-Lagrange field equations only exist in a constraint submanifold $S_f \hookrightarrow J^3\pi$. Furthermore, in the Lagrangian formalism, the Lagrangian constraints arise as a consequence of demanding the holonomy condition for the solutions to the field equations and the fact that the Hessian matrix of the Einstein-Hilbert Lagrangian with respects to the highest-order coordinates in $J^3\pi$ vanishes identically. Hence these kinds of constraints are not projectable by the Legendre map.

- We construct a covariant multimomentum Hamiltonian formalism for the Einstein Hilbert model. It has not gauge freedom, since the Hamilton-Cartan form is regular and $\mathcal{P}$ is diffeomorphic to $J^1\pi$ and $J^1\pi^*$.

- The Hamiltonian formalism for the Einstein Hilbert is the same than the multimomentum Hamiltonian formalism for the regular $1st$-order equivalent Lagrangian $\overline{\mathcal{L}}$ analysed in Section 3.1.4, thus, proving again the equivalence between this equivalent Lagrangian and the Einstein-Hilbert model.

- When the energy-matter sources are present, some of the geometrical and physical characteristics of the theory depend on the properties of the Lagrangian $L_{\mathfrak{m}}$ representing the source. In particular, the number of constraints arising from the constraint algorithm, the obtention of holonomic multivector fields solution to the Lagrangian field equations, and the construction of the covariant multimomentum formalism. This study has been done in detail for some cases of energy-matter sources (those which we are called "of degree ≤ 2"), which include as a particular case the energy-matter sources coupled to the metric (for instance, the electromagnetic source or the perfect fluid).

- In all the cases, we have obtained explicitly all semiholonomic multivector fields representing integrable distributions whose integral sections are solutions to the field equations.

And the results on the Metric-Affine model:

- We have presented a multisymplectic covariant description of the Lagrangian and Hamiltonian formalisms of the Einstein-Palatini model of General Relativity (without energy-matter sources). It is described by a first-order "metric-affine" Lagrangian which is (highly) degenerate and hence it originates a theory with constraints and gauge content.

- The Lagrangian field equations are expressed in terms of holonomic multivector fields which are associated with distributions whose integral sections are the solutions to the theory. Then, we use a constraint algorithm to determine a submanifold of the jet bundle $J^1\pi$ where, first, there exist semi-holonomic multivector fields which are solution to these equations and are tangent to this submanifold, and second, these multivector field are integrable (i.e., holonomic). The constraints arising from the algorithm determine where the image of the sections may lay.

- In coordinates, the Lagrangian field equations split into two kinds: the *metric* and the *connection equations* (equations (4.6), (4.7), (4.8)). In the same way, the Lagrangian constraints can be classified into three different types. First there are the *torsion constraints*, which impose strict limitations on the torsion of the connection. Then we have the constraints which appear as a consequence of demanding the semi-holonomy condition for the multivector field solutions (Theorem 4.1). In particular, the *Euler-Lagrange equations* themselves (which appear as constraints of the theory as a consequence of the fact that the Poincaré-Cartan form is π^1-projectable and the equations are first-order PDE's), and specially the so-called *pre-metricity constraints*, which are closely related to the metricity condition for the Levi-Civita connection. Only the tangency condition on the torsion constraint lead also to new constraints. Finally, a family of additional *integrability constraints* appear as a consequence of demanding the integrability of the multivector fields which are solutions. Only the initial the torsion constraints are projectable under the Legendre map $\mathcal{FL}_{\mathrm{EP}}$ (because the other ones appear as a consequence of demanding the (semi)holonomy of the solutions), and thus they are the only ones that also appear in the Hamiltonian formalism (see [57] for an analysis of this subject for higher-order dynamical theories). We have obtained explicitly all semiholonomic multivector fields solutions to the field equations (Proposition 4.22).

- We have done also a brief discussion about symmetries and conserved quantities, giving the expression of the natural Lagrangian symmetries, their conserved quantities and the corresponding flows.

- The (covariant) multimomentum Hamiltonian formalism for the Einstein-Palatini model has been also developed. The final constraint submanifold is also obtained in this formalism, and it is defined by the $\mathcal{FL}_{\mathrm{EP}}$-projection of the torsion constraints (Proposition 4.9). The explicit expression of the multivector field solutions is obtained (Proposition 4.13) and their integrability is briefly analysed. The local description is given using two different kinds of coordinates: the *non-momenta coordinates* which, as a consequence of the Legendre map, are the same as in the Lagrangian case, and the *pure-connection coordinates*, where the momenta associated to the connection replace the metric, resulting in metric-free coordinates. An intrinsic interpretation of these last coordinates is also given.

- Analyzing the gauge content of the model, we have obtained the local expression of the natural gauge vector fields, both in the Lagrangian and the Hamiltonian formalisms (Propositions 4.6 and 4.14). We recover the gauge symmetries discussed in [20], showing that there are no more.

- We have used the analysis of gauge freedom and constraints to establish the geometric relation between the Einstein-Palatini and the Einstein-Hilbert models, including the relation between the holonomic solutions in both formalisms. As it is known [12, 20], it is possible to recover the second by a gauge fixing in the first-one, which consists in imposing the trace of the torsion to vanish. This equivalence has been studied in detail (Theorem 4.2 and Propositions 4.16 and 4.17).

- Finally, using this equivalence, we have been able to prove that the constraint submanifolds $\mathcal{S}_f$ and $\mathcal{P}_f$ obtained from the Lagrangian and Hamiltonian constraint algorithms, respectively (where there exist multivector fields tangent to them, satisfying the geometric Lagrangian and Hamiltonian field equations on them) are the (maximal) *final constraint submanifolds* where these multivector fields are integrable; i.e., there are sections solutions to the field equations passing through every point on them (Propositions 4.18 and 4.19).

Further Research

Some lines of research or interesting problems derived from this work are the following:

- The reduction by symmetries is the procedure by which the field equations are reduced or simplified using symmetries and conserved quantities. It generalization to higher-order field theories has several inherent obstacles.
- The concept of gauge vector field for field theories present here can be develop further. It can be compared to other approaches, like Yang-Mills theories. It will be interesting to develop the electromagnetism, as it is the canonical example of gauge theory. Moreover, a geometric formulation of non-vertical gauge vector fields could be investigated.
- The Einstein-Palatini model is only considered in this work without energy-matter sources. An interesting problem is to analyse how the type of source influences the constraints, the gauge freedom and the symmetries of the theory. We have the intention to present this study in a future paper.
- The multisymplectic formalism has shown to be a powerful tool to analyse singular systems, with the presence of constraints, symmetries and gauge freedom. Therefore the formalism could help to understand other complicate models. Thus we want to study other generalized models of Gravity (Lovelock, $f(R)$-theories, etc.), as well as classical versions of other models like string theory, following the same procedures as in this dissertation.

List of publications

Four publications [45, 46, 47] and the preprint [48] derived from this work. Every one of these publications constitutes the core for a section of the dissertation.

Section 2.2. Order Reduction J. Gaset, N. Román-Roy, "Order reduction, projectability and constraints of second-order field theories and higher-order mechanics", *Rep. Math. Phys.* **78**(3) (2016) 327-337. (doi: 10.1063/1.4940047).

Section 2.1. Symmetries, Conserved Quantites and Gauge Variations J. Gaset, P.D. Prieto-Martínez, N. Román-Roy, "Variational principles and symmetries on fibered multisymplectic manifolds", *Comm. in Maths.* **24**(2) (2016) 137-152. (doi: 10.1515/cm-2016-0010).

Chapter 3. Einstein-Hilbert J. Gaset, N. Román-Roy, "Multisymplectic unified formalism for Einstein-Hilbert Gravity", *J. Math. Phys.* **59**(3) (2018) 032502. (doi: 10.1063/1.4998526).

Chapter 4. Metric-Affine J. Gaset, N. Román-Roy, "New multisymplectic approach to the Metric-Affine (Einstein-Palatini) action for gravity", arXiv:1804.06181 [math-ph] (2018).

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