# Appendix B

# Some Aspects of Structural Dynamics

This Appendix deals with some aspects of the dynamic behavior of SDOF and MDOF. It starts with the formulation of the equation of motion of SDOF systems. Next, the effects that viscous and Coulomb damping have on the dynamic behavior of SDOF are studied. The solution of equations of motion by means of the linear acceleration method (Newmark's method) is what follows. The energy formulation finishes with the study of SDOF systems. Then, similar studies are carried out for MDOF systems.

Since Chapters 3 and 4 deal basically with the concepts included in this appendix, other important topics (e.g., response of viscously damped systems to harmonic input, modal analysis or dynamics of distributed-parameter systems) are not included in this appendix.

## B.1 Single-Degree-of-Freedom (SDOF) Systems

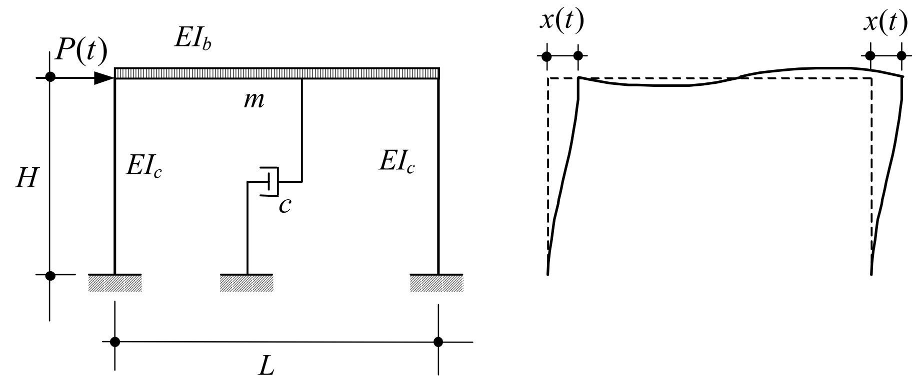
### B.1.1 General formulation

Fig. B.1a shows a typical single-story building (SSB). Making the usual static and cinematic condensations, it can be assumed that the lateral dynamic behavior of this structure can be described by a single linear degree of freedom (DOF) model. This DOF is the horizontal displacement $x$, as illustrated in Fig. B.1b.

### B.1.2 Equation of motion with viscous damping

The SDOF mechanical model for the frame of Fig. B.1a is shown in Fig. B.2a.

From the free body diagram of Fig. B.2b, in which the forces acting on the system of Fig. B.2a are shown, the equation of dynamic equilibrium —equation of motion— can be expressed as




(a) Single-degree-of-freedom (SDOF) frame

(b) Assumed motion of the frame

**Figure B.1** Frame modelled as a single-degre-of-freedom (SDOF) system

$$F_I + F_D + F_S = P(t) \tag{B.1}$$

where $F_I$ is the inertia force ($m\ddot{x}$), $F_D$ is the force due to the viscous damping ($c\dot{x}$), and $F_S$ is the force due to the deformation of the spring ($kx$), $P(t)$ represents the external time-varying force acting on the structure and $t$ is the time. The equation (B.1) can also be written as

$$m\ddot{x} + c\dot{x} + kx = P(t) \tag{B.2}$$

where $m$ is the mass of the structure, $c$ is the damping coefficient, $k$ is the stiffness coefficient and $x$ is the displacement, as a function of time $t$, of the structure. Dots above $x$ represent the derivatives with respect the time; hence $\dot{x}$ represents the velocity and $\ddot{x}$ represents the acceleration.

For the particular case of the structure depicted in Fig. B.1 the stiffness coefficient $k$ can be calculated with the following expression:

$$k = \frac{12(6I_bI_cH + I_c^2L)E}{H^3(3I_bH + 2I_cL)} \tag{B.3}$$

where $E$ is the young modulus of the material, $I_b$ and $I_c$ are the moments of inertia of the beam and the column, respectively, and $L$ and $H$ are the length and the height of the beam and the column, respectively.

On the other hand, the damping coefficient $c$ plays an important role in the response of the system shown in Fig. B.2a. The convention widely accepted is that this coefficient can

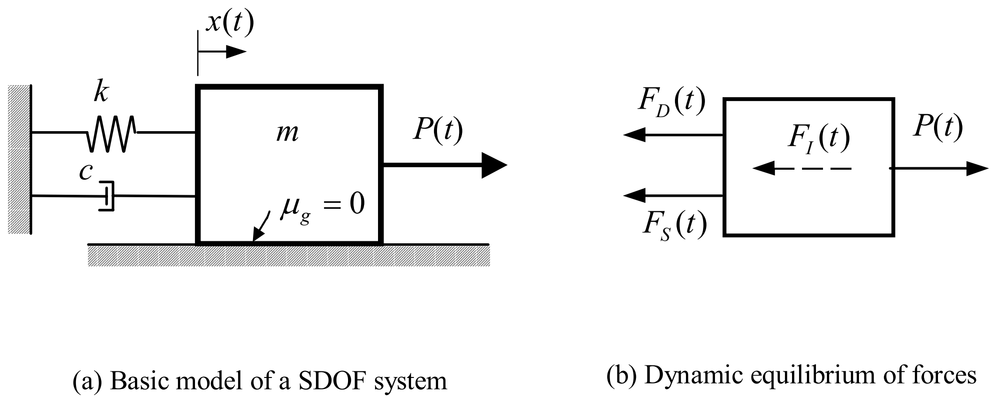



**Figure B.2** Mechanical model of a SDOF with viscous damping

be considered as a *visco-elastic damping*, which, when multiplied by the velocity $\dot{x}$, yields the damping force $F_D$.

The effects of viscous and friction damping on the dynamic response of the system of Fig. B.2a are studied next.

### B.1.3 Free vibration with viscous damping

With damping added, as shown in Fig. B.2a, the free vibration (i.e., $P(t) = 0$) of a SDOF system can be described for the following differential equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{B.4}$$

and the solution is given by

$$x = e^{\lambda t} \tag{B.5}$$

Since Eq. (B.4) is an ordinary second-order differential equation, two valid values of $\lambda$ are to be expected. Differentiating Eq. (B.5) and substituting the resultant values of $x$, $\dot{x}$ and $\ddot{x}$ into Eq. (B.4) yields the following characteristic equation:

$$(m\lambda^2 + c\lambda + k)e^{\lambda t} = 0 \tag{B.6}$$

The exponential term in Eq. (B.6) cannot be zero, therefore the expression inside the parenthesis must be so. Then,

$$m\lambda^2 + c\lambda + k = 0 \tag{B.7}$$

or

$$\lambda = \frac{-c \pm (c^2 - 4mk)^{1/2}}{2m} \tag{B.8}$$

This last equation can written in the form

$$\lambda = -\frac{c}{2(km)^{1/2}}\left(\frac{k}{m}\right)^{1/2} \pm \left(\frac{k}{m}\right)^{1/2}\left(\frac{c}{4km} - 1\right)^{1/2} \tag{B.9}$$

Let be

$$\omega = (k/m)^{1/2} = \text{undamped natural frequency} \tag{B.10}$$

and

$$\xi = \frac{c}{2(km)^{1/2}} = \frac{c}{2m\omega} \tag{B.11}$$

where $\xi$ is the critic damping rate. The equation of motion (B.1.6) can be rewritten as

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0 \tag{B.12}$$

and Eq. (B.9) becomes

$$\lambda = -\xi\omega \pm \omega(\xi^2 - 1)^{1/2} \tag{B.13}$$

The solution of Eq. (B.12) is

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{B.14}$$

where $\lambda_1$ and $\lambda_2$ are the values of $\lambda$ of Eq. (B.13). The form of the solution depends on the value of $\xi$.

If $\xi > 1$ (overdamped system), then $(\xi^2 - 1)^{1/2}$ is real and the solution is

$$x = e^{-\xi\omega t}\left\{A \exp\left[\omega(\xi^2 - 1)^{1/2} t\right] + B \exp\left[-\omega(\xi^2 - 1)^{1/2} t\right]\right\} \tag{B.15}$$

If $\xi = 1$ (critically damped system), then $(\xi^2 - 1)^{1/2}$ is zero (i.e., both values of $\lambda$ are equal) and Eq. (B.14) is reduced to

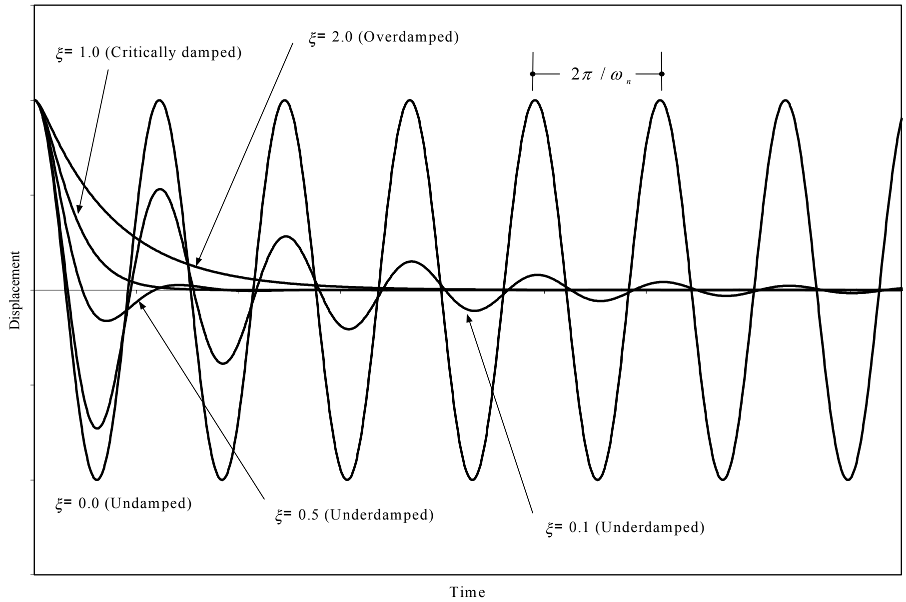



**Figure B.3** Free vibration of a SDOF system for different values of the critic damping rate $\xi$.

$$x = e^{-\omega t}(A + Bt) \tag{B.16}$$

If $\xi < 1$ (underdamped system), then $(\xi^2 - 1)^{1/2}$ is imaginary and is convenient to turn the exponentials to the equivalent trigonometric functions. This way Eq. (B.14) becomes

$$x = e^{-\xi\omega t}\left\{A\cos\left[\omega(1-\xi^2)^{1/2}t\right] + B\ \sin\left[-\omega(1-\xi^2)^{1/2}t\right]\right\} \tag{B.17}$$

In any case, constants $A$ and $B$ are determined from the initial values of $x$ and $\dot{x}$.

Fig. B.3 shows different types of time-history responses for some values of $\xi$.

If $\xi = 1$, the system turns back to its equilibrium position without any oscillation. This is the lowest value of parameter $\xi$ that restricts the oscillation completely, and it is called as the damping critic case. That is why parameter $\xi$ is known as the *critic damping rate.*

If $\xi > 1$, it is the case of over damping, and the system returns to its equilibrium position without any oscillation, but slower than the case of critic damping.

If $0 < \xi < 1$, the system oscillates with a decaying amplitude and a constant frequency equal to $\omega(1-\xi^2)^{1/2}$, which is little smaller than the oscillation frequency without damping, $\omega$. This is the case of damping smaller than the critic (under damping), and it is the case usually found. For this case, it is defined the *natural damped frequency*, in radians per second, which is given by

$$\omega_d = \omega(1 - \xi^2)^{1/2} \tag{B.18}$$

and the *natural damped period*, in seconds, is

$$T_d = \frac{2\pi}{\omega_d} \tag{B.19}$$

The free response in the case of damping smaller than the critic, in terms of initial conditions, can be written as

$$x = e^{-\xi\omega t}\left[x_0 \cos\omega_d t + \left(\frac{\dot{x}_0 + \xi\omega x_0}{\omega_d}\right)\sin\omega_d t\right] \tag{B.20}$$

### B.1.4 Free vibration with Coulomb damping

It has been told that damping in actual structures is due to many dissipative mechanisms that act simultaneously, and a convenient approach is to idealize them by means of an equivalent viscous damping. Although this approach is sufficient for the analysis of the majority of structures, can be inappropriate when some special devices of friction have been incorporated to a building to reduce its vibrations during an earthquake. In this section the free vibration of systems under the presence of Coulomb friction forces is going to be analyzed.

Coulomb damping results of the friction generated by the relative sliding of two dry surfaces. The maximum friction force is equal to $F = \mu N$, where $\mu$ denotes the static and dynamic friction coefficients, here considered equal, and $N$ is the normal force between the sliding surfaces (see Section 2.2). The friction force is assumed to be independent from velocity once the motion is started. The direction of the friction force is always opposite to motion, therefore its sign will change as the motion direction changes. For this reason, it is necessary to establish and to solve two differential equations, one equation valid for motion in one direction and the other equation valid when motion reverts.

Fig. B.4 shows a mass-spring system with the mass sliding on a dry surface, and the free-body diagram for the mass which includes the inertia force. The equation that rules the motion, from right to left, is

$$F_I + F_S = F \tag{B.21}$$

or

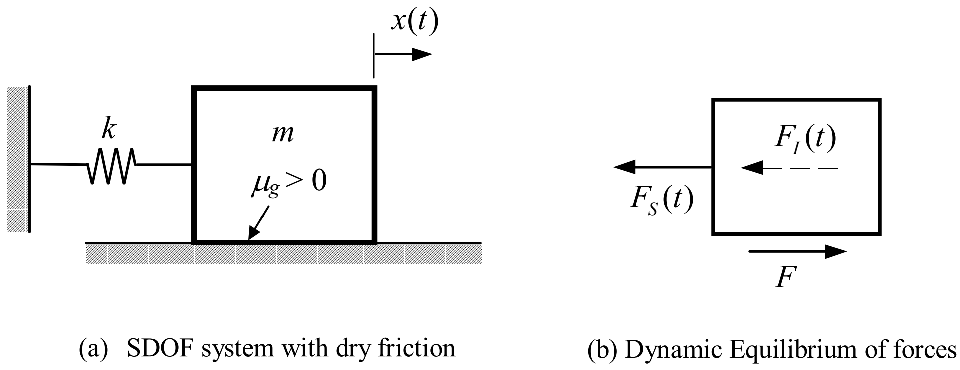



**Figure B.4** Mechanical model of a SDOF with Coulomb damping

$$m\ddot{x} + kx = F \tag{B.22}$$

and its solution is

$$x(t) = A_1 \cos \omega t + B_1 \sin \omega t + x_F \tag{B.23}$$

where $x_F = F/k$. For the mass motion from left to right the ruling equation is

$$m\ddot{x} + kx = -F \tag{B.24}$$

and its solution is

$$x(t) = A_2 \cos \omega t + B_2 \sin \omega t - x_F \tag{B.25}$$

Constants $A_1$, $B_1$, $A_2$ and $B_2$ depend on the initial conditions of each half-cycle of successive motion; $\omega = \sqrt{k/m}$ and the constant $x_F$ can be interpreted as the static deformation of the spring due to the friction force $F$. Each differential equation is linear, but the whole problem is non-linear because the ruling equation changes each half-cycle of motion.

Considering now the motion of the system of Fig. B.4 it will be started with some initial conditions and will be continued until the motion stops. At time $t = 0$, the mass moves a distance $x(0)$ towards the right direction and it is set free from rest, such that $\dot{x}(0) = 0$. For the first half-cycle of motion, Eq. (B.23) applies with constants $A_1$ and $B_1$ determined from the initial conditions at $t = 0$:

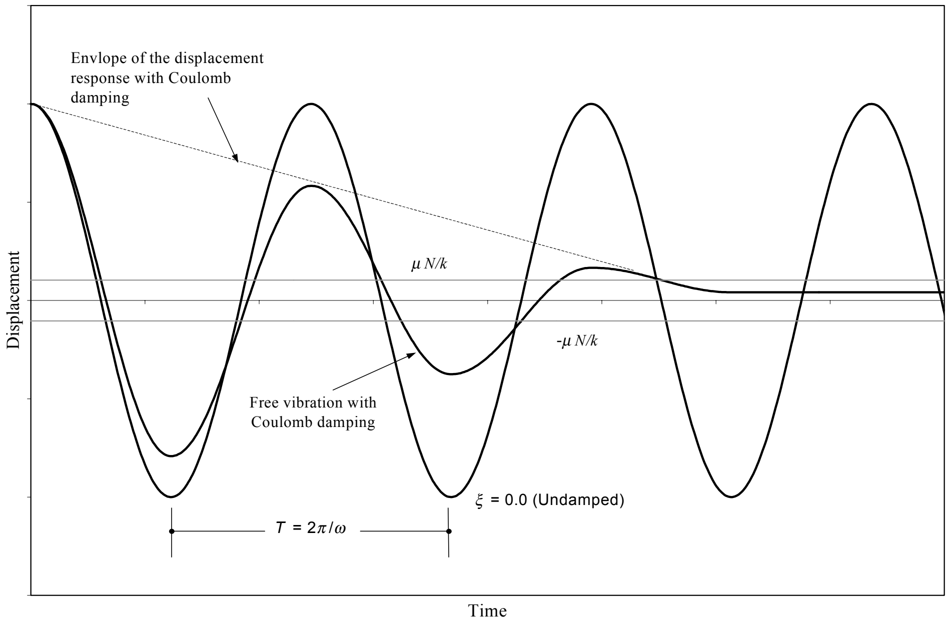



**Figure B.5** Free vibration of a SDOF system with Coulomb dry friction

$$A_1 = x(0) - x_F \qquad\qquad B_1 = 0$$

Substituting these values into Eq. (B.23) yields

$$x(t) = [x(0) - x_F]\cos\omega t + x_F \qquad\qquad 0 \le t \le \pi/\omega \tag{B.26}$$

This is plotted in Fig. B.5; it is a cosine function with an amplitude $= x(0) - x_F$ and shifted in the positive $x$ direction by $x_F$. Eq. (B.26) is valid until the velocity becomes zero again at $t = \pi/\omega = T/2$ (see Fig. B.5); at this instant $x = -x(0) + 2x_F$.

Starting now in this position on the left end, the mass now moves to the right with its motion described by Eq. (B.25). Constants $A_2$ and $B_2$ are determined from the established conditions at the beginning of this half-cycle:

$$A_2 = x(0) - 3x_F \qquad\qquad B_2 = 0$$

Substituting these into Eq. (B.25) yields

$$x(t) = [x(0) - 3x_F]\cos\omega t - x_F \qquad\qquad \pi/\omega \le t \le 2\pi/\omega \tag{B.27}$$

This is plotted in Fig. (B.5); it is a cosine function with a reduced amplitude = $x(0)-3x_F$ and shifted in the negative $x$ direction by $x_F$. Eq. (B.27) is valid until the velocity becomes zero again at $t = 2\pi/\omega = T$ (see Fig. B.5); at this time $x = x(0) - 4x_F$.

At $t = 2\pi/\omega$ the motion reverses and it is described by Eq. (B.23), which, after evaluating constants $A_1$ and $B_1$, becomes

$$x(t) = [x(0) - 5x_F] \cos \omega t + x_F \qquad 2\pi/\omega \leq t \leq 3\pi/\omega \tag{B.28}$$

This is a cosine function with a reduced amplitude = $x(0) - 5x_F$ and shifted, as before, in the positive direction $x$ by $x_F$.

The time taken for each half-cycle is $\pi/\omega$ and the duration of a full cycle, the natural period of vibration , is

$$T = \frac{2\pi}{\omega} \tag{B.29}$$

It can be observed that the natural period of a system with Coulomb damping is the same as for the system without any damping. On the contrary, the viscous damping has the effect of lengthening the natural period [Eqs. (B.18) and (B.19)].

At each cycle of motion, the amplitude is reduced by $4x_F$; i.e., the displacements $x_k$ and $x_{k+1}$ at two successive maxima are related by

$$x_{k+1} = x_k - 4x_F \tag{B.30}$$

Therefore the envelopes of the displacement-time curves are straight lines, as shown in Fig. B.5, instead of the exponential functions for systems with viscous damping.

If at end of any half-cycle the motion has an amplitude lesser than $x_F$, this means that in this point the spring force acting on the mass is less than the friction force, $kx < F$, and motion ceases. In Fig. B.5 this occurs at the end of the second and a half cycle. The final rest position of the mass is shifted from its original equilibrium position and represents a permanent deformation in which the friction force and the spring force are locked in. If the system is shaken, this will usually jar it sufficiently to restore equilibrium .

Damping in real structures is due partially to Coulomb friction [71]. This is because only this mechanism can stop completely the free vibration motion. If damping were purely viscous, theoretically the motion will keep on forever, although it had infinitesimally small amplitudes. This is merely an academic point of view, but it is basic for a full understanding of damping mechanisms.

The diverse damping mechanisms that prevail in real structures are rarely modeled individually. In particular, the Coulomb frictional forces that must exist are not considered explicitly unless frictional devices have been incorporated in the structure itself. Even when these devices are used, it is possible to use an equivalent viscous damping to obtain approximately the dynamic response [61, 72]. This equivalent coefficient is determined by equalizing the areas of the hysteresis loops (dissipated energy).

### B.1.5 Steady-state response to a harmonic input with Coulomb damping

In this subsection the steady-state response of a single-degree-of-freedom model with Coulomb damping and subjected to a harmonic driving force $P\,(t) = P_0 \cos \bar{\omega}\, t$ is obtained. A closed-form expression for the amplitude of the harmonic response is provided. This case and the one considered in the previous subsection (free vibration) are the only ones (with Coulomb damping) where analytical (closed-form) solutions can be derived. The analysis presented here relies on the assumption that there are no stops, hence, it can be mostly applied to vibrations occurring near resonance.

During the steady-state motion there are no stops and, hence, the absolute value of the friction force is equal to $\mu\, N$. The equation of motion is given by

$$m\, \ddot{x} + (-1)^{\operatorname{sgn} \dot{x}}\, \mu\, N + k\, x = P_0 \cos \bar{\omega}\, t \tag{B.31}$$

In this equation $P_0$ and $\bar{\omega}$ are the amplitude and the frequency of the harmonic input.

Provided that only the steady-state response is of interest, to analyze half a cycle is enough. During such interval, if $\dot{x} < 0$, the above equation can be written as

$$m\, \ddot{x} + k\, x = P_0 \cos \bar{\omega}\, t + \mu\, N \tag{B.32}$$

This is a second-order, linear, ordinary differential equation. Its general solution can be written as the sum of the general solution of the homogeneous equation plus any particular solution of the complete equation:

$$x\,(t) = C_1 \sin \bar{\omega}\, t + C_2 \cos \bar{\omega}\, t + \frac{P_0}{k} \frac{1}{1 - \beta^2} \cos\left(\bar{\omega}\, t - \phi\right) + \frac{\mu\, N}{k} \tag{B.33}$$

Where $\beta$ is the ratio between the input frequency and the natural one ($\beta = \frac{\bar{\omega}}{\omega}$, where $\omega = \sqrt{\frac{k}{m}}$ is the natural frequency of the undamped system) and $\phi$ is the phase angle. It is remarkable that both parts of the response do not damp out, i.e., no transient component exists.

There are four unknowns that can be obtained by imposing the conditions at the beginning and at the end of the half cycle. Such unknowns are the constants $C_1$ and $C_2$, the phase angle $\phi$ and the amplitude $x_0$. The initial and final conditions are:

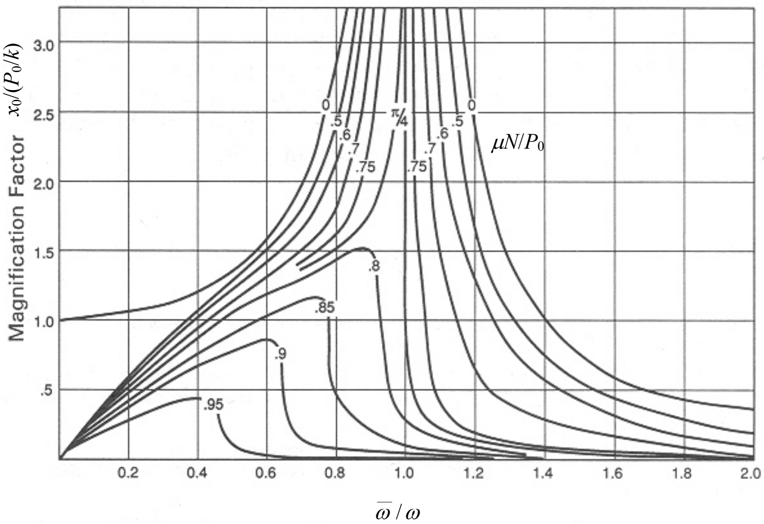



**Figure B.6** Amplification of motion of the sliding system in the region of resonance

$$x\left(t_0\right)=x_0, \qquad \dot{x}\left(t_0\right)=0, \qquad x\left(t_0+\frac{\pi}{\bar{\omega}}\right)=-x_0, \qquad \dot{x}\left(t_0+\frac{\pi}{\bar{\omega}}\right)=0 \tag{B.34}$$

Since these conditions are symmetric, it implies that the friction force has to reverse its sign at the end of the half cycle.

By substituting the above conditions in the general solution of the equation of motion during the half cycle, the following closed-form solution for the sought amplitude $x_0$ is obtained:

$$x_0=\frac{P_0}{k}\sqrt{\frac{1}{\left(1-\beta^2\right)^2}-\left(\frac{\mu N}{P_0}\right)^2\left(\frac{\frac{1}{\beta}\sin\frac{\pi}{\beta}}{1+\cos\frac{\pi}{\beta}}\right)^2} \tag{B.35}$$

As $\frac{P_0}{k}$ can be considered as the static response, $D=\sqrt{\frac{1}{\left(1-\beta^2\right)^2}-\left(\frac{\mu N}{P_0}\right)^2\left(\frac{\frac{1}{\beta}\sin\frac{\pi}{\beta}}{1+\cos\frac{\pi}{\beta}}\right)^2}$ is the dynamic magnification factor: $D=x_0/(P_0/k)$. The ratio $\frac{\mu N}{P_0}$ indicates the amount of Coulomb damping compared to the amplitude of the excitation. The bigger the sliding force $\mu N$, the smaller the amplitude of $D$. Fig. B.6 displays the plots of $D$ vs. $\beta$ for different values of the ratio $\frac{\mu N}{P_0}$.

If $\frac{\mu N}{P_0}\leq\frac{\pi}{4}$, the amplitude of peak at resonance ($\beta=1$, $\phi=\frac{\pi}{2}$) becomes infinitely large. This conclusion can be confirmed (and explained too) by energy balance relations: if the energy $E_I$ fuelled into the structure by the input is bigger than the energy $E_F$ dissipated by friction (and the difference $E_I - E_F$ keeps constant with time or, at least, does not decreases too fast), the amplitude grows indefinitely.

The input energy is equal to the work done by the driving force $P(t)$:

$$E_I = \int_0^{\frac{2\pi}{\bar{\omega}}} P(t)\ \dot{x}(t)\ dt = \int_0^{\frac{2\pi}{\bar{\omega}}} P_0 \cos(\bar{\omega}\, t)\ x_0 \bar{\omega} \cos(\bar{\omega}\, t)\ dt = P_0\, x_0\, \pi \tag{B.36}$$

As the hysteresis loop is rectangular (see Fig. 2.2) the energy dissipated by friction is $E_F = 4\, x_0\, \mu\, N$. Comparison between the values of $E_I$ and of $E_F$ shows that $E_I > E_F$ if $\frac{\mu N}{P_0} < \frac{\pi}{4}$. This is due to the fact that the energy dissipated by friction is proportional to $x_0$ instead of to $x_0^2$ as for viscous damping [73, 74].

In practical cases $\frac{\mu N}{P_0} < \frac{\pi}{4}$, hence, friction dissipators or isolators are not useful by themselves to cut the resonance peak. Some viscous damped should be also considered for this purpose. This is particularly relevant for narrow-band inputs.

### B.1.6 Solution of the equation of motion

Eq. (B.1.6) can be solved by means of many procedures, but generally a numerical solution is preferred; this way the external force $P(t)$ can be of any type.

#### B.1.6.1 Direct analysis in time domain for non-linear systems

The analysis in time domain consists in the solution for Eq. (B.1.6) by means of diverse algorithms. Among the most utilized are the step-by-step numerical integration methods [1, 61, 75]. One of these procedures is described next.

#### B.1.6.2 Incremental equation of dynamic equilibrium

The structure to be considered in this discussion is the SDOF system shown in Fig. B.2a. The forces acting on the mass of the system are indicated in Fig. B.2b, and the general nonlinear characteristics of the spring and damping forces are shown in Fig. B.7a and B.7b, respectively, while an arbitrary applied loading is sketched in Fig. B.7c.

At any time instant $t$ the dynamic equilibrium of the forces acting on the mass $m$ requires

$$F_I(t) + F_D(t) + F_S(t) = P(t) \tag{B.37}$$

while a short time $\Delta t$ later the equation would be

$$F_I(t + \Delta t) + F_D(t + \Delta t) + F_S(t + \Delta t) = P(t + \Delta t) \tag{B.38}$$

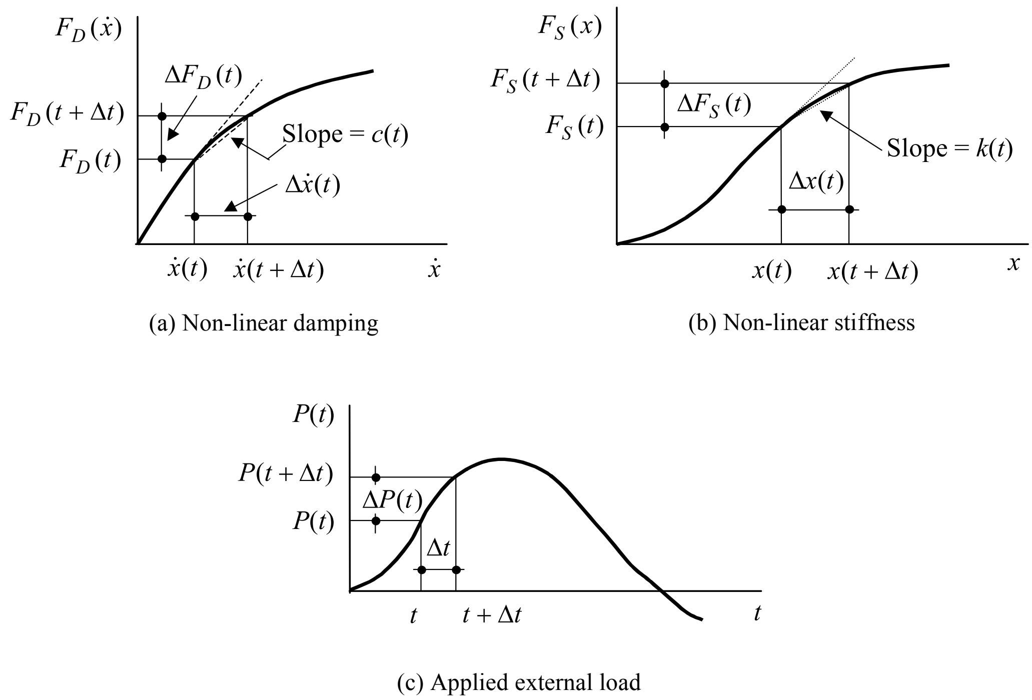



**Figure B.7** Stiffness, damping and driving forces for a non-linear system

Subtracting Eq. (B.37) from Eq. (B.38) then yields the incremental form of the equation of motion for the time interval $\Delta t$:

$$\Delta F_I(t) + \Delta F_D(t) + \Delta F_S(t) = \Delta P(t) \tag{B.39}$$

The incremental forces in this equation may be expressed as follows:

$$\begin{aligned}
\Delta F_I(t) &= F_I(t+\Delta t) - F_I(t) = m\Delta\ddot{x}(t) \\
\Delta F_D(t) &= F_D(t+\Delta t) - F_D(t) = c(t)\Delta\dot{x}(t) \\
\Delta F_S(t) &= F_S(t+\Delta t) - F_S(t) = k(t)\Delta x(t) \\
\Delta P(t) &= P(t+\Delta t) - P(t)
\end{aligned} \tag{B.40}$$

in which it is tacitly assumed that the mass remains constant, and where the terms $c(t)$ and $k(t)$ represent the damping and stiffness properties corresponding to the velocity and displacement existing during the time interval, as indicated in Figs. B.7a and B.7b, respectively. In practice, the secant slopes indicated could be evaluated only by iteration because the velocity and displacement at the end of the time increment depend on these properties;

for this reason the tangent slopes defined at the beginning of the time intervals are frequently used instead:

$$c(t) \approx \left(\frac{dF_D}{d\dot{x}}\right)_t \qquad\qquad k(t) \approx \left(\frac{dF_S}{dx}\right)_t \tag{B.41}$$

Substituting the force expressions of Eqs. (B.40) into Eq. (B.39) leads to the final form of the incremental equilibrium equations for time $t$:

$$m\Delta\ddot{x}(t) + c(t)\Delta\dot{x}(t) + k(t)\Delta x(t) = \Delta P(t) \tag{B.42}$$

The material properties considered in this type of analysis may include any form of nonlinearity. Thus, there is no need for the spring force $F_S$ to be dependent only on the displacement, as in a nonlinear elastic material. A nonlinear hysteretic material may equally be well specified, in which the force depends on the past history of deformation as well as the current value of displacement. The only requirement is that the stiffness properties be completely defined by the past as well as the current state of deformation. Moreover, it is evident that the implicit assumption of a constant mass is arbitrary; it also could be represented as a time-varying quantity.

#### B.1.6.3 Step-by-step integration

As said before, a number of procedures is available for the numerical integration of Eq. (B.42). The technique employed here is simple in concept but has been found to yield excellent results with relatively little computational effort. The basic assumption of the process is that the acceleration varies linearly during each time increment while the properties of the system remain constant during this interval. The motion of the mass during the time interval is indicated in graphical form in Fig. B.8, together with equations for the assumed linear variation of the acceleration an the corresponding quadratic and cubic variations of the velocity and displacement, respectively. Evaluating these latter expressions at the end of the interval ($\tau = \Delta t$) leads to the following equations for the increments of velocity and displacement:

$$\Delta\dot{x}(t) = \ddot{x}(t)\Delta t + \Delta\ddot{x}(t)\frac{\Delta t}{2} \tag{B.43a}$$

$$\Delta x(t) = \dot{x}(t)\Delta t + \ddot{x}(t)\frac{\Delta t^2}{2} + \Delta\ddot{x}(t)\frac{\Delta t^2}{6} \tag{B.43b}$$

Now it will be convenient to use the incremental displacement as the basic variable of the analysis; hence Eq. (B.43a) is solved for the incremental acceleration, and this expression is substituted into Eq. (B.43b) to obtain

$$\Delta\ddot{x}(t) = \frac{6}{\Delta t^2}\Delta x(t) - \frac{6}{\Delta t}\dot{x}(t) - 3\ddot{x}(t) \tag{B.44a}$$

$$\Delta\dot{x}(t) = \frac{3}{\Delta t}\Delta x(t) - 3\dot{x}(t) - \frac{\Delta t}{2}\ddot{x}(t) \tag{B.44b}$$

Substituting Eqs. (B.44) into Eq. (B.42) leads to the following form of the equation of motion:

$$m\left[\frac{6}{\Delta t^2}\Delta x(t) - \frac{6}{\Delta t}\dot{x}(t) - 3\ddot{x}(t)\right] + c(t)\left[\frac{3}{\Delta t}\Delta x(t) - 3\dot{x}(t) - \frac{\Delta t}{2}\ddot{x}(t)\right] + k(t)\Delta x(t) = \Delta P(t)$$

Finally, transferring all terms associated with the known initial conditions to the right-hand side gives

$$\tilde{k}(t)\Delta x(t) = \Delta\tilde{P}(t) \tag{B.45}$$

in which

$$\tilde{k}(t) = k(t) + \frac{6}{\Delta t^2}m + \frac{3}{\Delta t}c(t) \tag{B.46a}$$

$$\Delta\tilde{P}(t) = \Delta P(t) - m\Delta\ddot{x}_g(t) + m\left[\frac{6}{\Delta t}\dot{x}(t) + 3\ddot{x}(t)\right] + c(t)\left[3\dot{x}(t) + \frac{\Delta t}{2}\ddot{x}(t)\right] \tag{B.46b}$$

It will be noted that Eq. (B.45) is equivalent to a static incremental-equilibrium relationship, and may be solved for the incremental displacement by dividing the incremental load by the stiffness. The dynamic behavior is accounted for by the inclusion of inertial and damping effects in the effective-load and stiffness terms. After Eq. (B.45) is solved for the displacement increment, this value is substituted into Eq. (B.44) to obtain the incremental velocity. Then, initial conditions for the next time step result from the addition of these incremental values to the velocity and displacement at the beginning of the time step.

This numerical procedure includes two significant approximations: (1) the acceleration varies linearly and (2) the damping and stiffness properties remain constant during the time step. In general, neither of these assumptions is entirely correct, even though the errors are small if the time step is short. Therefore, errors generally will arise in the incremental-equilibrium relationship which might tend to accumulate from step to step, and this accumulation should be avoided by imposing the total-equilibrium condition at each step of the analysis. This may be accomplished conveniently by expressing the accelerations at the beginning of the time step in terms of the total external load minus the total damping and spring forces (Eq. (B.47)).

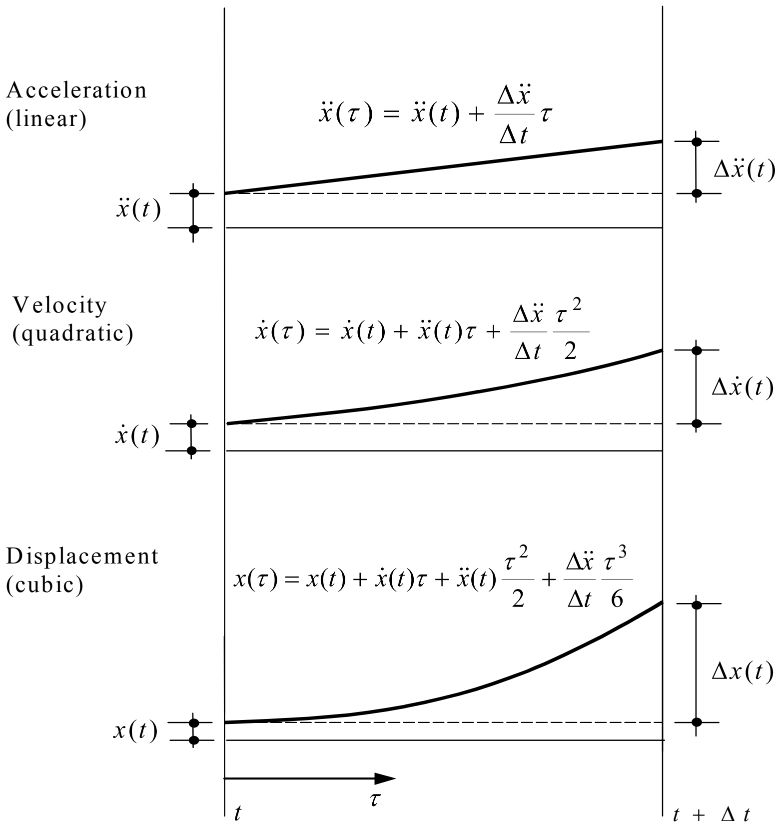



**Figure B.8** Schematic diagram of the method of linear acceleration

#### B.1.6.4 Summary of the procedure

For any given time increment, the analysis procedure consists of the following operations:

(1) Initial velocity and displacement values $\dot{x}(t)$ and $x(t)$ are known, either from values at the end of the preceding increment or as initial conditions of the problem.

(2) With these values and the specified nonlinear properties of the structure, the damping $c(t)$ and the stiffness $k(t)$ for the interval, as well as current values of the damping $F_D(t)$ and strain $F_S(t)$ forces are found, e.g., from Figs. B.7a and b.

(3) The initial acceleration is given by

$$\ddot{x}(t) = \frac{1}{m}\left[P(t) - F_D(t) - F_S(t)\right] \tag{B.47}$$

This is merely a rearrangement of the equation of equilibrium (B.37) for time $t$.

(4) The effective load increment $\Delta\tilde{P}(t)$ and effective stiffness $\tilde{k}(t)$ are computed from Eqs. (B.46).

(5) The increment of displacement is given by Eq. (B.45), and with it the increment of velocity is found from Eq. (B.44b).

(6) Finally the velocity and displacement at the end of the increment are obtained from

$$\dot{x}(t+\Delta t) = \dot{x}(t) + \Delta\dot{x}(t) \tag{B.48a}$$

$$x(t+\Delta t) = x(t) + \Delta x(t) \tag{B.48b}$$

When step 6 has been completed, the analysis for this time increment is finished, and the entire process may be repeated for the next time interval. Obviously the process can be carried out consecutively for any desired number of time increments; thus the complete response history can be evaluated for any SDOF having any prescribed nonlinear properties. Linear systems can also be treated by the same process, of course; in this case the damping and stiffness properties remain constant so the analysis procedure is somewhat simpler.

It is interesting to notice that no iterative calculations are required during the process.

### B.1.7 Energy formulation

Multiplying Eq. (B.1.6) by $\dot{x}dt$ and integrating through time leads to the following energy balance expression

$$E_K + E_D + E_S = E_I \tag{B.49}$$

where

$$\begin{aligned} E_K &= \int_0^t m\ddot{x}\dot{x}dt = \frac{1}{2}m\dot{x}^2 \\ E_D &= \int_0^t c\dot{x}\dot{x}dt = \int_0^t c\dot{x}^2 dt \\ E_S &= \int_0^t kx\dot{x}dt = \frac{1}{2}kx^2 \\ E_I &= \int_0^t P(t)\dot{x}dt \end{aligned} \qquad \text{(B.50)}$$

The terms included in the left member of Eq. (B.49) represent the kinetic energy ($E_K$), the energy dissipated by means of the viscous damping ($E_D$) and the strain energy ($E_S$). The addition of these terms must be equal to the input energy introduced to the structure ($E_I$) due to any lateral force.

## B.2 Multi-Degree-of-Freedom (MDOF) Systems

### B.2.1 General formulation for buildings

Fig. B.9a represents a typical frame belonging to a symmetric multi-story building (MSB). For simplicity, the contributions of partitions, installations and nonstructural elements to the lateral stiffness of the building are neglected and, moreover, this frame will be treated as if the floor and the ceiling were infinitely rigid (for bending) compared to the columns. This is the usual approximation of the so-called 'shear buildings'. Also, the axial deformation of members will be neglected, as well as the second order effect of the axial force over the columns stiffness. Hence, the lateral stiffness of the building is due entirely to the bending stiffness of columns. Fig. B.9b shows the forces acting on a bent column. The degrees of freedom of these types of models are equal to the number of floors, $N$, thus the motion of each floor is defined by its horizontal displacement. However, if the building were not symmetric (e.g., if there were eccentricities between the centers of mass and rigidity), its dynamic behavior could not be described by a 2D model (one DOF per floor); but rather by a 3D model with 3 DOF per floor (two horizontal displacements and one twist angle).

The mass is assumed to be distributed through the building, but here it will be treated as if it were concentrated on each floor —'lumped mass' assumption. The sum of all masses compresses the total mass of the building from the half height of the upper floor until the half height of the lower floor, including nonstructural components and a percentage of the variable loads. This way the building will be treated as if it were a weightless frame with lumped masses attached to it on every floor, as shown in Fig. B.10. Masses are constrained to move horizontally only, and each one is rigidly joined to the frame so the horizontal inertia

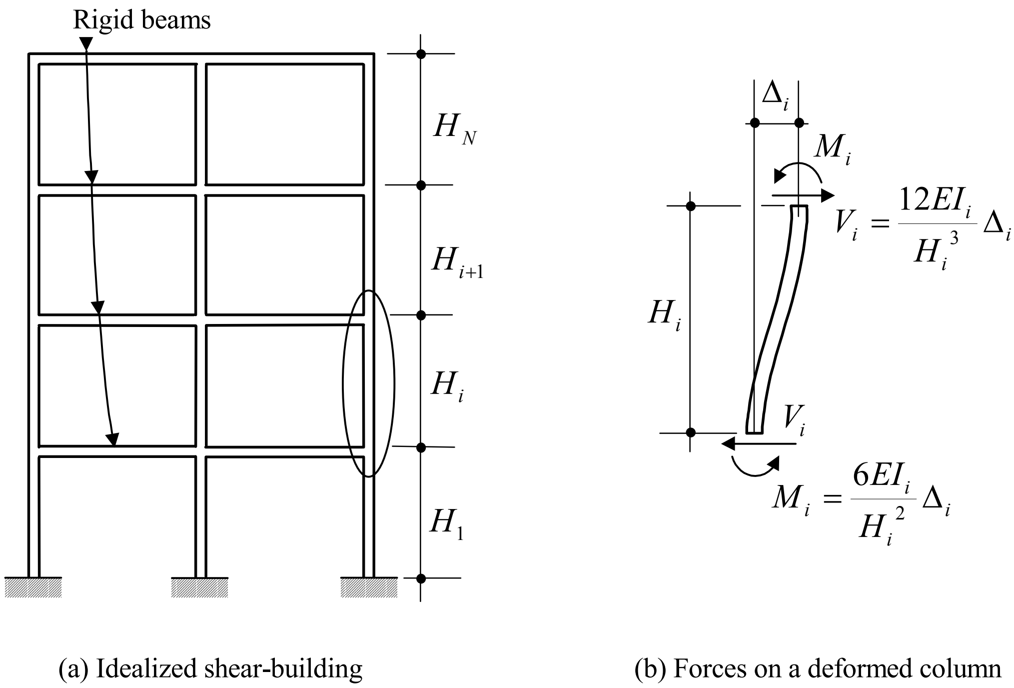



**Figure B.9** Multi-story building (MSB) idealized as a shear building

force enters into the equations of dynamic equilibrium, but there is no vertical motion of the mass.

On the other hand, the damping is difficult to evaluate, here and in almost all dynamic systems. In the case of SDOF systems the damping was considered as proportional to the velocity, i.e., viscous damping, and it is mathematically convenient because leads to simple, consistent solutions in vibration problems of real systems affected by small amplitude vibrations. The damping force will be considered as the force exerted by a valve attached to the mass, producing a force proportional to the mass and opposite to the direction of motion. The multi-degree-of-freedom system complicates the concept, since the damping force proportional to the velocity exerted on a mass could be taken as proportional to the velocity of this mass, relative to a fixed reference frame, 'absolute damping', or relative to the other masses of the system, 'relative damping'. This work deals with the relative damping, which will be analyzed by the effects that valves exert on each floor, each one generating a damping force proportional to the velocity of the horizontal motion between the floors.

### B.2.2 Equations of motion

With the assumptions made in the previous subsection, it is possible to derive the equations of motion considering the free-body diagrams of the masses and writing the equations of

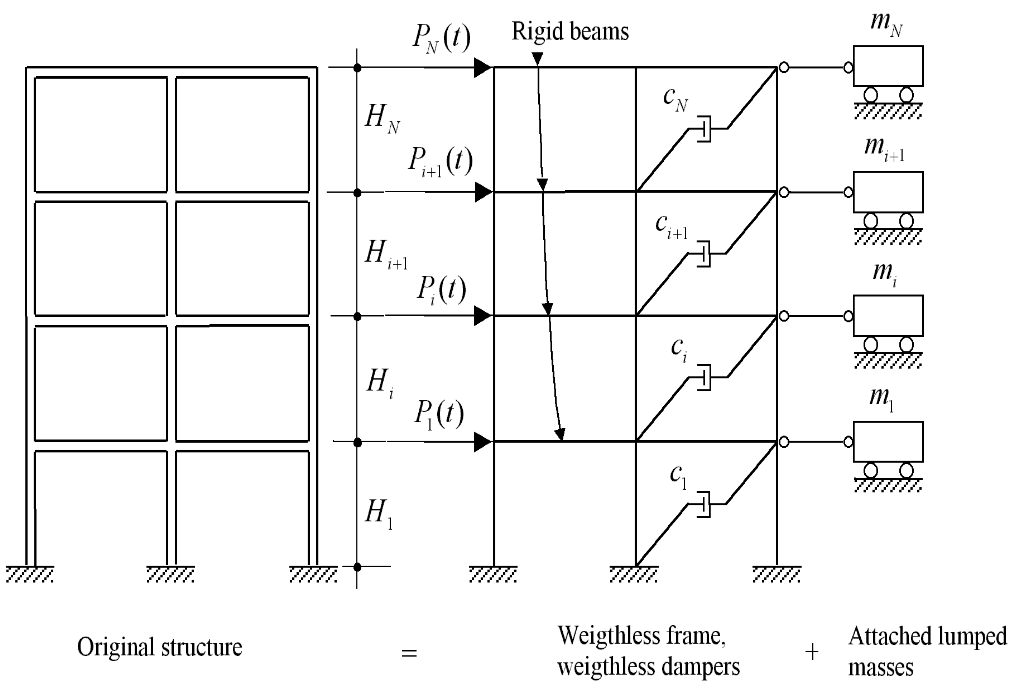



**Figure B.10** Multi-story building (MSB) modelled as a shear building with lumped masses

dynamic equilibrium for each of them. From Fig. B.11, let $m_1$, $m_2$ and $m_3$ be the masses designated from bottom to top; let $c_1$, $c_2$ and $c_3$ be the three damping coefficients; and let $k_1$, $k_2$ and $k_3$ be the stiffness of each floor, each one representing the sum of the stiffnesses of all columns in each floor. Besides, let $P_1(t)$, $P_2(t)$ and $P_3(t)$ be the lateral dynamic forces acting on the masses. Fig. B.12 shows the mechanical model of the frame of Fig. B.11 and Fig. B.13 shows the free-body diagrams of the three masses in dynamic equilibrium.

Writing the equation of motion for each mass, leads to a system of three linear differential equations of motion:

$$\begin{aligned} &P_3(t) - m_3\ddot{x}_3 - c_3(\dot{x}_3 - \dot{x}_2) \\ &\qquad -k_3(x_3 - x_2) = 0 \\[2ex] &P_2(t) - m_2\ddot{x}_2 + c_3(\dot{x}_3 - \dot{x}_2) - c_2(\dot{x}_2 - \dot{x}_1) \\ &\qquad +k_3(x_3 - x_2) - k_2(x_2 - x_1) = 0 \\[2ex] &P_1(t) - m_1\ddot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1) \\ &\qquad -c_1\dot{x}_1 + k_2(x_2 - x_1) - k_1x_1 = 0 \end{aligned} \tag{B.51}$$

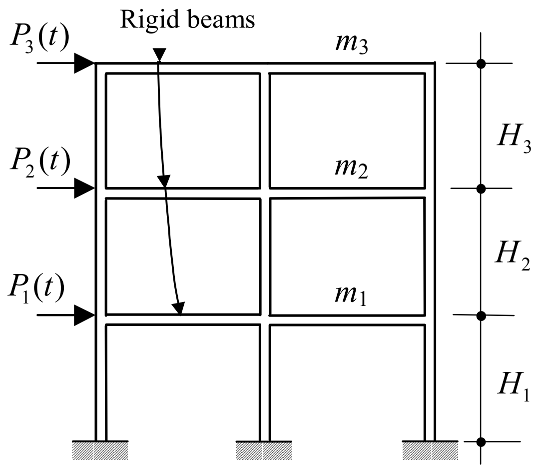



**Figure B.11** 3-story building with lumped masses

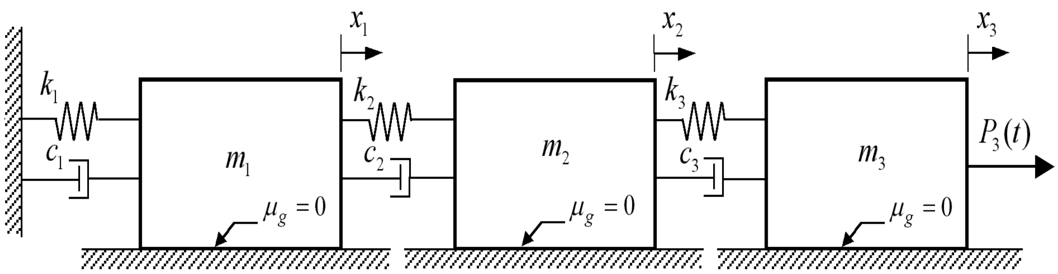



**Figure B.12** Mechanical model for a 3-story shear building

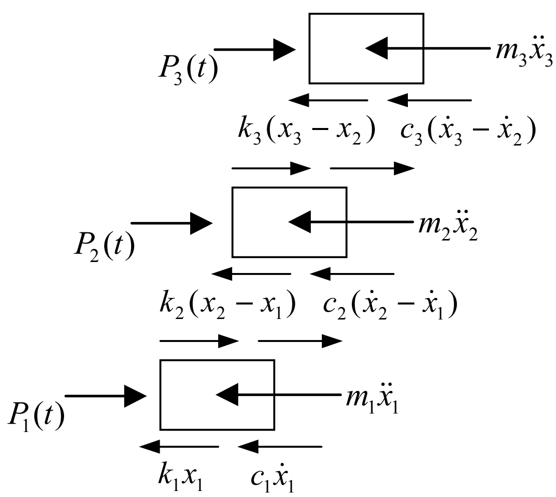



**Figure B.13** Dynamic forces acting on the three masses

These equations can be rewritten as

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 \\ -k_2 x_2 = P_1(t)$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - c_3\dot{x}_3 - k_2 x_1 \\ +(k_2 + k_3)x_2 - k_3 x_3 = P_2(t) \tag{B.52}$$

$$m_3\ddot{x}_3 - c_3\dot{x}_2 + c_3\dot{x}_3 - k_3 x_2 + k_3 x_3 = P_3(t)$$

And these can be rewritten in matrix form as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x}(t) = \mathbf{P}(t) \tag{B.53}$$

where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \text{inertia (mass) matrix}$$

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} = \text{damping matrix}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = \text{stiffness matrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \text{displacement vector}$$

$$\mathbf{P}(t) = \begin{bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{bmatrix} = \text{external force vector}$$

### B.2.3 Properties of system matrices

Some properties of system matrices **M**, **C** and **K** can be observed, and it is useful to take it into account to avoid potential errors and to take advantages in the calculation process.

First, **M**, **C** and **K** are symmetric matrices. In this case, **M** is a diagonal matrix, but this it is not always true. The symmetry of matrix **K** comes from Maxwell's reciprocity theorem, which is based on the superposition principle and on stress-energy considerations. Comparable arguments using dissipated energy in viscous damping instead of strain energy can be utilized to establish the symmetry of **C**, and considerations based on the kinetic energy can lead to the symmetry of **M**.

Second, diagonal elements of **M** and **K** are always positive, due to the defined positive character of strain and kinetic energies. In a similar way, diagonal elements of **C** are always positive because any velocity must be accompanied by energy dissipation in the viscous damping, or, in a better case, for no change in the energy —the damping cannot increase the energy of the system. The possibility that a part of the system does not have any damping makes possible that some diagonal elements are zero.

Third, the biggest element of **M** is always in the diagonal, as well as the biggest element of **K**. Some non-diagonal elements can be bigger than some diagonal elements, but the biggest element of the whole matrix will always be in the diagonal of these matrices.

Fourth, for building structures that are not modelled as shear buildings, the stiffness matrix **K** can have non-zero elements even in the farthest corners. However, the solution presented next applies still.

### B.2.4 Solution of the equations of motion using the linear acceleration method

A number of numerical methods can be employed to solve the differential equations of motion of MDOF systems. The procedure is relatively simple, but since the equations are generally coupled, as in this case, they must be solved simultaneously.

For example, considering a multi-story frame with $N$ stories, the equation of motion (B.53) at any time $t_k$ can be written as

$$\ddot{\mathbf{x}}_k = \mathbf{M}^{-1}\left[\mathbf{P}(t_k) - \mathbf{C}\dot{\mathbf{x}}_k - \mathbf{K}\mathbf{x}_k\right] \tag{B.54}$$

Since the inertia matrix is diagonal, Eq. (B.54) can also be written as

$$\ddot{x}_i = \left(P_i - \sum_j c_{ij}\dot{x}_j - \sum_j k_{ij}x_j\right)/m_i \qquad i = 1, 2, \ldots, N \tag{B.55}$$

for the $i$-th floor at time $t_k$.

If $\mathbf{M}$ were not diagonal, the following more general form could be used:

$$\ddot{x}_i = \sum_j m_{ij}^{-1} \left( P_i - \sum_l c_{jl} \dot{x}_l - \sum_l k_{jl} x_l \right) \qquad i = 1, 2, \ldots, N \tag{B.56}$$

In this last equation the term $m_{ij}^{-1}$ refers to the element of file $i$ and column $j$ of matrix $\mathbf{M}^{-1}$, i.e., $m_{ij}^{-1}$ is not equal, generally, to $1/m_{ij}$.

Knowing the displacement $\mathbf{x}_k$ and the velocity $\dot{\mathbf{x}}_k$ vectors at a time $t_k$ (normally, the initial conditions would be the starting point), $\ddot{\mathbf{x}}_k$ is calculated from Eq. (B.54). Now, a time interval $\Delta t$ is chosen and then:

1. Assume a set of values for $\ddot{\mathbf{x}}_{k+1}^*$, the acceleration vector at the end of the interval. The initial acceleration $\ddot{\mathbf{x}}_k$ could be the first estimate of $\ddot{\mathbf{x}}_{k+1}^*$.

2. Calculate

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \dot{\mathbf{x}}_k + \frac{\Delta t^2}{6} (2\ddot{\mathbf{x}}_k + \ddot{\mathbf{x}}_{k+1}^*) \tag{B.57}$$

3. Calculate

$$\dot{\mathbf{x}}_{k+1} = \dot{\mathbf{x}}_k + \frac{\Delta t}{2} (\ddot{\mathbf{x}}_k + \ddot{\mathbf{x}}_{k+1}^*) \tag{B.58}$$

4. Calculate

$$\ddot{\mathbf{x}}_{k+1} = \mathbf{M}^{-1} \left[ \mathbf{P}(t_{k+1}) - \mathbf{C}\dot{\mathbf{x}}_{k+1} - \mathbf{K}\mathbf{x}_{k+1} \right] \tag{B.59}$$

5. (a) If $\ddot{\mathbf{x}}_{k+1} \neq \ddot{\mathbf{x}}_{k+1}^*$, the calculated values of $\ddot{\mathbf{x}}_{k+1}$ must be taken as an improved estimate of $\ddot{\mathbf{x}}_{k+1}^*$. Go back to step 2.

   (b) If $\ddot{\mathbf{x}}_{k+1} = \ddot{\mathbf{x}}_{k+1}^*$ (with a prescribed tolerance), the iteration has converged. Go to the next time interval, starting from step 1.

Note that the procedure is applicable for linear and nonlinear systems.

### B.2.5 Energy formulation

Pre-multiplying and post-multiplying Eq. (B.53) by $\dot{\mathbf{x}}^T$ and $dt$, respectively, and integrating through time, the following energy balance equation is obtained:

$$E_K + E_D + E_S = E_I \tag{B.60}$$

where

$$
\begin{aligned}
E_K &= \int_0^t \dot{\mathbf{x}}^T \mathbf{M}\ddot{\mathbf{x}}\, dt = \frac{1}{2}\dot{\mathbf{x}}^T\, \mathbf{M}\dot{\mathbf{x}} \\
E_D &= \int_0^t \dot{\mathbf{x}}^T \mathbf{C}\dot{\mathbf{x}}\, dt \\
E_S &= \int_0^t \dot{\mathbf{x}}^T\, \mathbf{K}\mathbf{x}\, dt = \frac{1}{2}\mathbf{x}^T\, \mathbf{K}\mathbf{x} \\
E_I &= \int_0^t \dot{\mathbf{x}}^T\, \mathbf{P}(t)\, dt
\end{aligned}
\qquad \text{(B.61)}
$$

The terms included in the left members of Eqs. (B.61) represent the kinetic energy ($E_K$), the dissipated energy due to the viscous damping ($E_D$) and the strain energy (elastic or inelastic) ($E_S$). The addition of these terms must be equal to the input energy ($E_I$) introduced to the structure due to any lateral load.