Appendix C

Solution of Contact Problems using Lagrange Multipliers

C.1 Introduction

In this Appendix the solution of the equations of motion with contact conditions using Lagrange multipliers is described. At first, the restrained equation of motion is formulated. Next, the Lagrange multipliers are introduced. Then, a step-by-step procedure to solve the new equations of motion is presented. At the end, the ADINA program, which is based on a modification of the method of Lagrange multipliers, is introduced.

C.2 Restrained Equation of Motion

The equation of motion expressed in general form is

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{Q}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{R} \tag{C.1}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{x}$ is the displacement vector of the degrees of freedom, $\dot{\mathbf{x}}$ is the velocity vector, $\ddot{\mathbf{x}}$ is the acceleration vector, $\mathbf{Q}$ is the vector of internal forces and $\mathbf{R}$ is the vector of external forces. It is assumed that the solution of Eq. (C.1) is subjected to the imposed boundary conditions, as well as to the displacement restraints on the contact surfaces.

In Fig. C.1 the case of bi-dimensional contact between two bodies spatially discretized using low-order continuous finite elements can be seen. The nodes of the contactor are distinguished by a 'C' and the nodes of the target by a 'T'. Displacement restraints have been imposed to prevent the penetration of the contactor nodes into the target domain and to control the tangential sliding of the contactor nodes through the target surface. These restraints can be expressed as

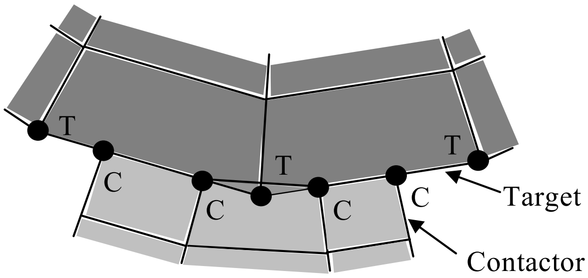

Figure C.1 Contact for bi-dimensional finite elements

$$\mathbf{G}[\mathbf{x} + \mathbf{D}] = \mathbf{0} \tag{C.2}$$

where $\mathbf{D}$ is the vector of the material coordinates; the sum of $\mathbf{x}$ and $\mathbf{D}$ is the vector of spatial coordinates and $\mathbf{G}$ is the restrained displacements matrix on the surface contact.

The components of $\mathbf{G}$ are unknown *a priori* and generally change when displacement and deformation occur. Starting with a configuration in which the surfaces are separated, the motion of the contactor and target must be tracked so the restrained displacement components can be introduced into $\mathbf{G}$ when the contact occurs. During the contact the components of $\mathbf{G}$ can be changed with the time to assure that the reaction forces associated with contact satisfy the conditions of the contact forces. Similarly, if a tangential component of a force approaches to a condition of limit friction force, then the restriction associated with the tangential displacement must be relaxed to allow sliding. The components of $\mathbf{G}$ change with time when the sliding takes place.

It is convenient to consider equations (C.2) with known linear restrictions that don't change during an integration through a time increment. This assumption simplifies the initial discussion on the method of Lagrange multipliers that follows.

C.3 Method of Lagrange Multipliers

The Lagrange multipliers can be introduced into Eq. (C.1) to give

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{Q}(\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{G}^T\boldsymbol{\lambda} = \mathbf{R} \tag{C.3}$$

where vector components of the Lagrange multiplier, $\boldsymbol{\lambda}$, are the forces on the contact surfaces. The method of the Lagrange multipliers starts by treating $\boldsymbol{\lambda}$ as unknown and solving equations (C.2) and (C.3) simultaneously.

C.4 Numerical Solution of the Equations of Motion

For an elementary problem of small displacements in which the internal forces are independent of the deformation and proportional to the displacements, the restrained equation of motion referenced at time t_{k+1} is

$$\mathbf{M}\ddot{\mathbf{x}}_{k+1} + \mathbf{C}\dot{\mathbf{x}}_{k+1} + \mathbf{K}\mathbf{x}_{k+1} + \mathbf{G}_{k+1}^T \boldsymbol{\lambda}_{k+1} = \mathbf{R}_{k+1} \tag{C.4a}$$

$$\mathbf{G}_{k+1}[\mathbf{x}_{k+1} + \mathbf{D}] = \mathbf{0} \tag{C.4b}$$

Equations (C.4a) and (C.4b) can be solved by means of methods of direct integration [48, 59]. Now, considering a second order integration through time, the following incremental equation of motion can be obtained:

$$\begin{bmatrix} b_2\mathbf{M} + b_1\mathbf{C} + b_0\mathbf{K} & \mathbf{G}_{k+1}^T \\ b_0\mathbf{G}_{k+1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_{k+1} \\ \boldsymbol{\lambda}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{k+1} - [\mathbf{M}\mathbf{q}_2 + \mathbf{C}\mathbf{q}_1 + \mathbf{K}\mathbf{q}_0] \\ -\mathbf{G}_{k+1}[\mathbf{q}_0 + \mathbf{D}] \end{bmatrix} \tag{C.5}$$

where $\mathbf{q}_0$, $\mathbf{q}_1$ and $\mathbf{q}_2$ are differential operators of second order:

$$\mathbf{q}_0 = \mathbf{x}_k + \Delta t\dot{\mathbf{x}}_k + \tfrac{1}{2}(\Delta t)^2\ddot{\mathbf{x}}_k$$
$$\mathbf{q}_1 = \dot{\mathbf{x}}_k + \Delta t\ddot{\mathbf{x}}_k$$
$$\mathbf{q}_2 = \ddot{\mathbf{x}}_k$$

and

$$b_0 = \tfrac{1}{2}(\Delta t)^2\beta_0$$
$$b_1 = \Delta t\beta_1$$
$$b_2 = 1$$
$$\Delta t = t_{k+1} - t_k$$

This time discretization is equivalent to the well-known *Newmark's method* [48]. However, the operational form shown here corresponds to the *Beta-2 method*, which is a subset of the *generalized Beta-m method* developed in [76]. Three well-known Beta-2 methods are the following:

1. *Linear acceleration method*, in which $\beta_0 = \frac{1}{3}$ and $\beta_1 = \frac{1}{2}$.
2. *Constant acceleration method,* also known as *the trapezoidal rule*, in which $\beta_0 = \beta_1 = \frac{1}{2}$.
3. *Central difference method*, in which $\beta_0 = 0$ and $\beta_1 = \frac{1}{2}$.

C.5 The ADINA Program

Among the various existing commercial computer programs for the solution of problems of structural and mechanical engineering, the **A**utomatic **D**ynamic **I**ncremental **N**on-linear **A**nalysis (ADINA) seems well oriented to solve contact problems. This program introduces the *restrained function method* [48, 77], which is a modification of the method of Lagrange multipliers above mentioned.

In the restrained function method a system of equations similar to (C.5) is solved for each iteration. The stability and accuracy of this method are good for slow static and transient problems. However, the method behaves badly when the inertial forces are relatively big [58].

The comparison of results is presented in Sections 3.7 and 4.5.