CHAPTER 1.
INDUCTION MOTOR MODEL. GENERALITIES.

1.1 - Equations of the induction motor model.

1.1.1 – Introduction.

A dynamic model of the machine subjected to control must be known in order to understand and design vector controlled drives. Due to the fact that every good control has to face any possible change of the plant, it could be said that the dynamic model of the machine could be just a good approximation of the real plant. Nevertheless, the model should incorporate all the important dynamic effects occurring during both steady-state and transient operations. Furthermore, it should be valid for any changes in the inverter's supply such as voltages or currents [ROM 1].

Such a model can be obtained by means of either the space vector phasor theory or two-axis theory of electrical machines. Despite the compactness and the simplicity of the space phasor theory, both methods are actually close and both methods will be explained.

For simplicity, the induction motor considered will have the following assumptions:

- Symmetrical two-pole, three phases windings.
- The slotting effects are neglected.
- The permeability of the iron parts is infinite.
- The flux density is radial in the air gap.
- Iron losses are neglected.
- The stator and the rotor windings are simplified as a single, multi-turn full pitch coil situated on the two sides of the air gap.

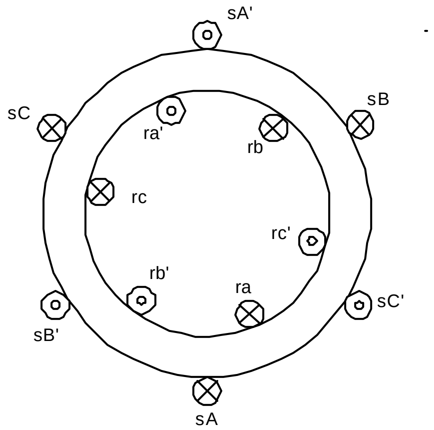


Figure 1.1. Cross-section of an elementary symmetrical three-phase machine.

1.1.2 – Voltage equations.

The stator voltages will be formulated in this section from the motor natural frame, which is the stationary reference frame fixed to the stator. In a similar way, the rotor voltages will be formulated to the rotating frame fixed to the rotor.

In the stationary reference frame, the equations can be expressed as follows:

$$u_{sA}(t) = R_s i_{sA}(t) + \frac{d\psi_{sA}(t)}{dt} \quad (1.1)$$

$$u_{sB}(t) = R_s i_{sB}(t) + \frac{d\psi_{sB}(t)}{dt} \quad (1.2)$$

$$u_{sC}(t) = R_s i_{sC}(t) + \frac{d\psi_{sC}(t)}{dt} \quad (1.3)$$

Similar expressions can be obtained for the rotor:

$$u_{ra}(t) = R_r i_{ra}(t) + \frac{d\psi_{ra}(t)}{dt} \tag{1.4}$$

$$u_{rb}(t) = R_r i_{rb}(t) + \frac{d\psi_{rb}(t)}{dt} \tag{1.5}$$

$$u_{rc}(t) = R_r i_{rc}(t) + \frac{d\psi_{rc}(t)}{dt} \tag{1.6}$$

The instantaneous stator flux linkage values per phase can be expressed as:

$$\psi_{sA} = \overline{L}_s i_{sA} + \overline{M}_s i_{sB} + \overline{M}_s i_{sC} + \overline{M}_{sr}\cos\theta_m i_{ra} + \overline{M}_{sr}\cos(\theta_m + 2\pi/3) i_{rb} + \overline{M}_{sr}\cos(\theta_m + 4\pi/3) i_{rc} \tag{1.7}$$

$$\psi_{sB} = \overline{M}_s i_{sA} + \overline{L}_s i_{sB} + \overline{M}_s i_{sC} + \overline{M}_{sr}\cos(\theta_m + 4\pi/3) i_{ra} + \overline{M}_{sr}\cos\theta_m i_{rb} + \overline{M}_{sr}\cos(\theta_m + 2\pi/3) i_{rc} \tag{1.8}$$

$$\psi_{sC} = \overline{M}_s i_{sA} + \overline{M}_s i_{sB} + \overline{L}_s i_{sC} + \overline{M}_{sr}\cos(\theta_m + 2\pi/3) i_{ra} + \overline{M}_{sr}\cos(\theta_m + 4\pi/3) i_{rb} + \overline{M}_{sr}\cos\theta_m i_{rc} \tag{1.9}$$

In a similar way, the rotor flux linkages can be expressed as follows:

$$\psi_{ra} = \overline{M}_{sr}\cos(-\theta_m) i_{sA} + \overline{M}_{sr}\cos(-\theta_m + 2\pi/3) i_{sB} + \overline{M}_{sr}\cos(-\theta_m + 4\pi/3) i_{sC} + \overline{L}_r i_{ra} + \overline{M}_r i_{rb} + \overline{M}_r i_{rc} \tag{1.10}$$

$$\psi_{rb} = \overline{M}_{sr}\cos(-\theta_m + 4\pi/3) i_{sA} + \overline{M}_{sr}\cos(-\theta_m) i_{sB} + \overline{M}_{sr}\cos(-\theta_m + 2\pi/3) i_{sC} + \overline{M}_r i_{ra} + \overline{L}_r i_{rb} + \overline{M}_r i_{rc} \tag{1.11}$$

$$\psi_{rc} = \overline{M}_{sr}\cos(-\theta_m + 2\pi/3) i_{sA} + \overline{M}_{sr}\cos(-\theta_m + 4\pi/3) i_{sB} + \overline{M}_{sr}\cos(-\theta_m) i_{sC} + \overline{M}_r i_{ra} + \overline{L}_r i_{rb} + \overline{M}_r i_{rc} \tag{1.12}$$

Taking into account all the previous equations, and using the matrix notation in order to compact all the expressions, the following expression is obtained:

$$\begin{bmatrix} u_{sA} \\ u_{sB} \\ u_{sC} \\ u_{ra} \\ u_{rb} \\ u_{rc} \end{bmatrix} = \begin{bmatrix} R_s + p\overline{L}_s & p\overline{M}_s & p\overline{M}_s & p\overline{M}_{sr}\cos\theta_m & p\overline{M}_{sr}\cos\theta_{m1} & p\overline{M}_{sr}\cos\theta_{m2} \\ p\overline{M}_s & R_s + p\overline{L}_s & p\overline{M}_s & p\overline{M}_{sr}\cos\theta_{m2} & p\overline{M}_{sr}\cos\theta_m & p\overline{M}_{sr}\cos\theta_{m1} \\ p\overline{M}_s & p\overline{M}_s & R_s + p\overline{L}_s & p\overline{M}_{sr}\cos\theta_{m1} & p\overline{M}_{sr}\cos\theta_{m2} & p\overline{M}_{sr}\cos\theta_m \\ p\overline{M}_{sr}\cos\theta_m & p\overline{M}_{sr}\cos\theta_{m1} & p\overline{M}_{sr}\cos\theta_{m2} & R_r + p\overline{L}_r & p\overline{M}_r & p\overline{M}_r \\ p\overline{M}_{sr}\cos\theta_{m2} & p\overline{M}_{sr}\cos\theta_m & p\overline{M}_{sr}\cos\theta_{m1} & p\overline{M}_r & R_r + p\overline{L}_r & p\overline{M}_r \\ p\overline{M}_{sr}\cos\theta_{m1} & p\overline{M}_{sr}\cos\theta_{m2} & p\overline{M}_{sr}\cos\theta_m & p\overline{M}_r & p\overline{M}_r & R_r + p\overline{L}_r \end{bmatrix} \cdot \begin{bmatrix} i_{sA} \\ i_{sB} \\ i_{sC} \\ i_{ra} \\ i_{rb} \\ i_{rc} \end{bmatrix} \tag{1.13}$$

1.1.3 – Applying Park's transform.

In order to reduce the expressions of the induction motor equation voltages given in equation 1.1 to equation 1.6 and obtain constant coefficients in the differential equations, the Park's transform will be applied. Physically, it can be understood as transforming the three windings of the induction motor to just two windings, as it is shown in figure 1.2 [VAS 1].

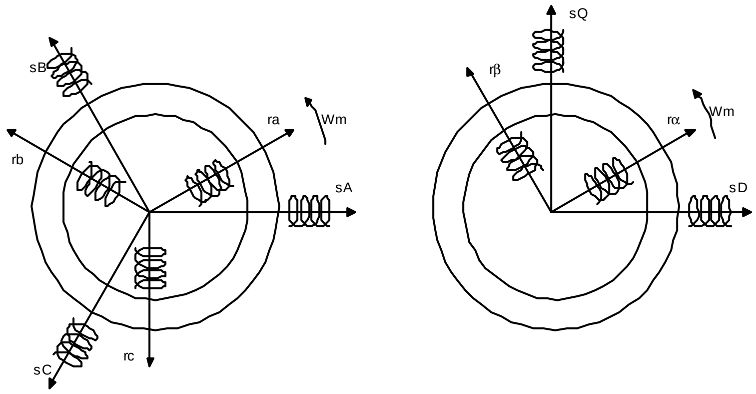


Figure 1.2 Schema of the equivalence physics transformation.

In the symmetrical three-phase machine, the direct- and the quadrature-axis stator magnitudes are fictitious. The equivalencies for these direct (D) and quadrature (Q) magnitudes with the magnitudes per phase are as follows:

$$\begin{bmatrix} u_{s0} \\ u_{sD} \\ u_{sQ} \end{bmatrix} = c \cdot \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos\theta & \cos(\theta - 2\pi/3) & \cos(\theta + 2\pi/3) \\ -\sin\theta & -\sin(\theta - 2\pi/3) & -\sin(\theta + 2\pi/3) \end{bmatrix} \cdot \begin{bmatrix} u_{sA} \\ u_{sB} \\ u_{sC} \end{bmatrix} \tag{1.14}$$

$$\begin{bmatrix} u_{sA} \\ u_{sB} \\ u_{sC} \end{bmatrix} = c \cdot \begin{bmatrix} 1/\sqrt{2} & \cos\theta & -\sin\theta \\ 1/\sqrt{2} & \cos(\theta - 2\pi/3) & -\sin(\theta - 2\pi/3) \\ 1/\sqrt{2} & \cos(\theta + 2\pi/3) & -\sin(\theta + 2\pi/3) \end{bmatrix} \cdot \begin{bmatrix} u_{s0} \\ u_{sD} \\ u_{sQ} \end{bmatrix} \tag{1.15}$$

Where "c" is a constant that can take either the values 2/3 or 1 for the so-called non-power invariant form or the value $\sqrt{2/3}$ for the power-invariant form as it is explained in section 1.3.3. These previous equations can be applied as well for any other magnitudes such as currents and fluxes.

Notice how the expression 1.13 can be simplified into a much smaller expression in 1.16 by means of applying the mentioned Park's transform.

$$\begin{bmatrix} u_{sD} \\ u_{sQ} \\ u_{r\alpha} \\ u_{r\beta} \end{bmatrix} = \begin{bmatrix} R_s + pL_s & -L_s p\theta_s & pL_m & -L_m\left(P\cdot w_m + p\theta_r\right) \\ L_s p\theta_s & R_s + pL_s & L_m\left(P\cdot w_m + p\theta_r\right) & pL_m \\ pL_m & -L_m\left(p\theta_s - P\cdot w_m\right) & R_r + pL_r & -L_r p\theta_r \\ L_m\left(p\theta_s - P\cdot w_m\right) & pL_m & L_r p\theta_r & R_r + pL_r \end{bmatrix} \cdot \begin{bmatrix} i_{sD} \\ i_{sQ} \\ i_{r\alpha} \\ i_{r\beta} \end{bmatrix} \tag{1.16}$$

Where $L_s = \overline{L}_s - \overline{M}_s$, $L_r = \overline{L}_r - \overline{M}_r$ and $L_m = \frac{3}{2}\overline{M}_{sr}$.

1.1.4 – Voltage matrix equations.

If the matrix expression 1.16 is simplified, new matrixes are obtained as shown in equations 1.17, 1.18 and 1.19 [VAS 1].

1.1.4.1 – Fixed to the stator.

It means that ws = 0 and consequently wr = -wm.

$$\begin{bmatrix} u_{sD} \\ u_{sQ} \\ u_{rd} \\ u_{rq} \end{bmatrix} = \begin{bmatrix} R_s + pL_s & 0 & pL_m & 0 \\ 0 & R_s + pL_s & 0 & pL_m \\ pL_m & P\cdot w_m L_m & R_r + pL_r & P\cdot w_m L_r \\ -P\cdot w_m L_m & pL_m & -P\cdot w_m L_r & R_r + pL_r \end{bmatrix} \cdot \begin{bmatrix} i_{sD} \\ i_{sQ} \\ i_{rd} \\ i_{rq} \end{bmatrix} \tag{1.17}$$

1.1.4.2 – Fixed to the rotor.

It means that wr = 0 and consequently ws = wm.

$$\begin{bmatrix} u_{sD} \\ u_{sQ} \\ u_{rd} \\ u_{rq} \end{bmatrix} = \begin{bmatrix} R_s + pL_s & -L_s P\, w_m & pL_m & -L_m P\, w_m \\ L_s P\, w_m & R_s + pL_s & L_m P\, w_m & pL_m \\ pL_m & 0 & R_r + pL_r & 0 \\ 0 & pL_m & 0 & R_r + pL_r \end{bmatrix} \cdot \begin{bmatrix} i_{sD} \\ i_{sQ} \\ i_{rd} \\ i_{rq} \end{bmatrix} \tag{1.18}$$

1.1.4.3 – Fixed to the synchronism.

It means that wr = sws.

$$\begin{bmatrix} u_{sD} \\ u_{sQ} \\ u_{rd} \\ u_{rq} \end{bmatrix} = \begin{bmatrix} R_s + pL_s & -L_s w_s & pL_m & -L_m w_s \\ L_s w_s & R_s + pL_s & L_m w_s & pL_m \\ pL_m & -L_m s w_s & R_r + pL_r & -L_r s w_s \\ L_m s w_s & pL_m & L_r s w_s & R_r + pL_r \end{bmatrix} \cdot \begin{bmatrix} i_{sD} \\ i_{sQ} \\ i_{rd} \\ i_{rq} \end{bmatrix} \tag{1.19}$$

1.2 – Space phasor notation.

1.2.1 – Introduction.

Space phasor notation allows the transformation of the natural instantaneous values of a three-phase system onto a complex plane located in the cross section of the motor. In this plane, the space phasor rotate with an angular speed equal to the angular frequency of the three phase supply system. A space phasor rotating with the same angular speed, for example, can describe the rotating magnetic field. Moreover, in the special case of the steady state, where the supply voltage is sinusoidal and symmetric, the space phasor become equal to three-phase voltage phasors, allowing the analysis in terms of complex algebra. It is shown in figure 1.3 the equivalent schematic for this new model.

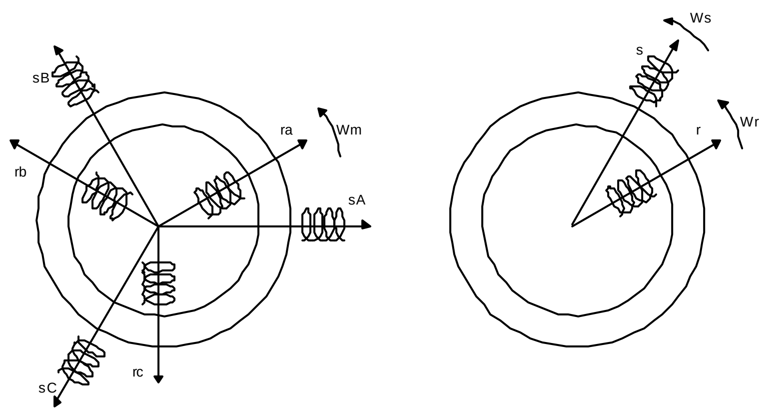


Figure 1.3. On the right the equivalent two rotating windings induction motor.

In order to transform the induction motor model, in natural co-ordinates, into its equivalent space phasor form, the 120º operator is introduced:

$$a = e^{j2\pi/3},\ a^2 = e^{j4\pi/3} \quad (1.20)$$

Thus, the current stator space phasor can be expressed as follows:

$$\bar{i}_s = c \cdot \left[1 \cdot i_{sA}(t) + a \cdot i_{sB}(t) + a^2 \cdot i_{sC}(t)\right] \quad (1.21)$$

The factor "c", takes usually one of two different values either $2/3$ or $\sqrt{2/3}$. The factor $2/3$ makes the amplitude of any space phasor, which represents a three phase balanced system, equal to the amplitudes of one phase of the three-phase system. The factor $\sqrt{2/3}$ may also be used to define the power invariance of a three-phase system with its equivalent two-phase system (see section 1.3.3).

1.2.2 – Current space phasors.

During this section the induction machine assumptions introduced in the section 1.1.1 will be further considered.

It is represented in figure 1.4 the model of the induction machine with two different frames, the D-Q axis which represent the stationary frame fixed to the stator, and the α-β axis which represent rotating frame fixed to the rotor.

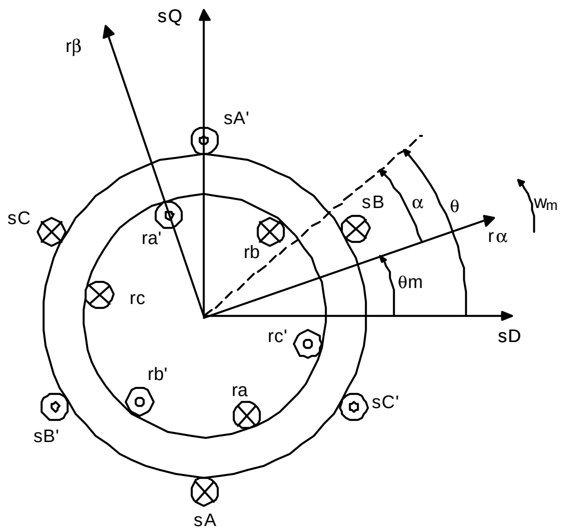


Figure 1.4. Cross-section of an elementary symmetrical three-phase machine, with two different frames, the D-Q axis which represent the stationary frame fixed to the stator, and α-β axis which represent rotating frame fixed to the rotor.

The stator current space phasor can be expressed as follows:

$$\overline{i_s} = \tfrac{2}{3}\left[i_{sA}(t) + a i_{sB}(t) + a^2 i_{sC}(t)\right] = \left|\overline{i_s}\right| e^{j\theta} \tag{1.22}$$

Expressed in the reference frame fixed to the stator, the real-axis of this reference frame is denoted by sD and its imaginary-axis by sQ.

The equivalence between the stator phasor and the D-Q two-axis components is as follows:

$$\overline{i_s} = i_{sD}(t) + j \cdot i_{sQ}(t) \tag{1.23}$$

or:

$$\begin{aligned} Re\left(\overline{i_s}\right) &= Re\left[\tfrac{2}{3}\left(i_{sA} + a i_{sB} + a^2 i_{sC}\right)\right] = i_{sD} \\ Im\left(\overline{i_s}\right) &= Im\left[\tfrac{2}{3}\left(i_{sA} + a i_{sB} + a^2 i_{sC}\right)\right] = i_{sQ} \end{aligned} \tag{1.24}$$

The relationship between the space phasor current and the real stator phase currents can be expressed as follows:

$$\begin{aligned} Re\left(\overline{i_s}\right) &= Re\left[\tfrac{2}{3}\left(i_{sA} + a i_{sB} + a^2 i_{sC}\right)\right] = i_{sA} \\ Re\left(a^2 \overline{i_s}\right) &= Re\left[\tfrac{2}{3}\left(a^2 i_{sA} + i_{sB} + a i_{sC}\right)\right] = i_{sB} \\ Re\left(a \overline{i_s}\right) &= Re\left[\tfrac{2}{3}\left(a i_{sA} + a^2 i_{sB} + i_{sC}\right)\right] = i_{sC} \end{aligned} \tag{1.25}$$

In a similar way, the space phasor of the rotor current can be written as follows:

$$\overline{i_r} = \tfrac{2}{3}\left[i_{ra}(t) + a i_{rb}(t) + a^2 i_{rc}(t)\right] = \left|\overline{i_r}\right| e^{j\alpha} \tag{1.26}$$

Expressed in the reference frame fixed to the rotor, the real-axis of this reference frame is denoted by rα and its imaginary-axis by rβ.

The space phasor of the rotor current expressed in the stationary reference frame fixed to the stator can be expressed as follows:

$$\overline{i_r}' = \left|\overline{i_r}\right| e^{j\theta} = \left|\overline{i_r}\right| e^{j(\alpha + \theta_m)} \tag{1.27}$$

The equivalence between the current rotor space phasor and the α-β two-axis is as follows:

$$\overline{i_r} = i_{r\alpha}(t) + j \cdot i_{r\beta}(t) \tag{1.28}$$

or:

$$\begin{aligned} Re\left(\overline{i_r}\right) &= Re\left[\tfrac{2}{3}\left(i_{ra} + a i_{rb} + a^2 i_{rc}\right)\right] = i_{r\alpha} \\ Im\left(\overline{i_r}\right) &= Im\left[\tfrac{2}{3}\left(i_{ra} + a i_{rb} + a^2 i_{rc}\right)\right] = i_{r\beta} \end{aligned} \tag{1.29}$$

The relationship between the space phasor current and the real stator currents can be expressed as follows:

$$\begin{aligned} Re\left(\overline{i_r}\right) &= Re\left[\tfrac{2}{3}\left(i_{ra} + a i_{rb} + a^2 i_{rc}\right)\right] = i_{ra} \\ Re\left(a^2 \overline{i_r}\right) &= Re\left[\tfrac{2}{3}\left(a^2 i_{ra} + i_{rb} + a i_{rc}\right)\right] = i_{rb} \\ Re\left(a \overline{i_r}\right) &= Re\left[\tfrac{2}{3}\left(a i_{ra} + a^2 i_{rb} + i_{rc}\right)\right] = i_{rc} \end{aligned} \tag{1.30}$$

The magnetising current space-phasor expressed in the stationary reference frame fixed to the stator can be obtained as follows:

$$\overline{i_m} = \overline{i_s} + \left({N_{re}}/{N_{se}}\right)\overline{i_r'} \tag{1.31}$$

1.2.3 – Flux linkage space phasor.

In this section the flux linkages will be formulated in the stator phasor notation according to different reference frames.

1.2.3.1- Stator flux-linkage space phasor in the stationary reference frame fixed to the stator.

Similarly to the definitions of the stator current and rotor current space phasors, it is possible to define a space phasor for the flux linkage as follows:

$$\overline{\psi_s} = \tfrac{2}{3}\left(\psi_{sA} + a\psi_{sB} + a^2\psi_{sC}\right) \tag{1.32}$$

If the flux linkage equations 1.7, 1.8, 1.9 are substituted in equation 1.32, the space phasor for the stator flux linkage can be expressed as follows:

$$\overline{\psi}_s = \tfrac{2}{3}\begin{bmatrix} i_{sA}\left(\overline{L_s} + a\overline{M}_s + a^2\overline{M}_s\right) + i_{sB}\left(\overline{M}_s + a\overline{L_s} + a^2\overline{M}_s\right) + i_{sC}\left(\overline{M}_s + a\overline{M}_s + a^2\overline{L_s}\right) + \\ + i_{ra}\left(\overline{M}_{sr}\cos\theta_m + a\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right) + a^2\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right)\right) + \\ + i_{rb}\left(\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right) + a\overline{M}_{sr}\cos\theta_m + a^2\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right)\right) + \\ + i_{rc}\left(\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right) + a\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right) + a^2\overline{M}_{sr}\cos\theta_m\right) \end{bmatrix} \tag{1.33}$$

Developing the previous expression 1.33, it is obtained the following expression:

$$\overline{\psi}_s = \tfrac{2}{3}\begin{bmatrix} i_{sA}\left(\overline{L_s} + a\overline{M}_s + a^2\overline{M}_s\right) + a\cdot i_{sB}\left(a^2\overline{M}_s + \overline{L_s} + a\overline{M}_s\right) + a^2 i_{sC}\left(a\overline{M}_s + a^2\overline{M}_s + \overline{L_s}\right) + \\ + i_{ra}\left(\overline{M}_{sr}\cos\theta_m + a\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right) + a^2\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right)\right) + \\ + a\cdot i_{rb}\left(a^2\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right) + \overline{M}_{sr}\cos\theta_m + a\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right)\right) + \\ + a^2\cdot i_{rc}\left(a\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right) + a^2\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right) + \overline{M}_{sr}\cos\theta_m\right) \end{bmatrix} \tag{1.34}$$

And finally, expression 1.34 can be represented as follows:

$$\begin{aligned} \overline{\psi}_s &= \left(\overline{L_s} + a\overline{M}_s + a^2\overline{M}_s\right)\bar{i}_s + \left(\overline{M}_{sr}\cos\theta_m + a\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right) + a^2\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right)\right)\bar{i}_r = \\ &= \left(\overline{L_s} - \overline{M}_s\right)\bar{i}_s + 1.5\cos\theta_m\overline{M}_{sr}\bar{i}_r = \left(\overline{L_s} - \overline{M}_s\right)\bar{i}_s + 1.5\overline{M}_{sr}\bar{i}_r e^{j\theta_m} = \left(\overline{L_s} - \overline{M}_s\right)\bar{i}_s + 1.5\overline{M}_{sr}\bar{i}'_r = \\ &= L_s\bar{i}_s + L_m\bar{i}'_r \end{aligned} \tag{1.35}$$

Where Ls is the total three-phase stator inductance and Lm is the so-called three-phase magnetising inductance. Finally, the space phasor of the flux linkage in the stator depends on two components, being the stator currents and the rotor currents.

Once more, the flux linkage magnitude can be expressed in two-axis as follows:

$$\overline{\psi}_s = \psi_{sD} + j\psi_{sQ} \tag{1.36}$$

Where its direct component is equal to:

$$\psi_{sD} = L_s i_{sD} + L_m i_{rd} \tag{1.37}$$

And its quadrature component is expressed as:

$$\psi_{sQ} = L_s i_{sQ} + L_m i_{rq} \tag{1.38}$$

The relationship between the components i_{rd} and $i_{r\alpha}$ and i_{rq} and $i_{r\beta}$ may be introduced as follows:

$$\bar{i}'_r = i_{rd} + j i_{rq} = \bar{i}_r e^{j\theta_m} \tag{1.39}$$

The compactness of the notation in the space phasor nomenclature compared to the two-axis notation in 1.1. is noticeable.

1.2.3.2- Rotor flux-linkage space phasor in the rotating reference frame fixed to the rotor.

The rotor flux linkage space phasor, fixed to the rotor natural frame can be defined as follows:

$$\bar{\psi}_r = \tfrac{2}{3}\left(\psi_{ra} + a\psi_{rb} + a^2\psi_{rc}\right) \tag{1.40}$$

If the flux linkage equations 1.10, 1.11, 1.12 are substituted in equation 1.40, the space phasor for the rotor flux linkage can be expressed as follows:

$$\bar{\psi}_r = \tfrac{2}{3}\begin{bmatrix} i_{ra}\left(\overline{L_r} + a\overline{M}_r + a^2\overline{M}_r\right) + i_{sB}\left(\overline{M}_r + a\overline{L_r} + a^2\overline{M}_r\right) + i_{sc}\left(\overline{M}_r + a\overline{M}_r + a^2\overline{L_r}\right) + \\ + i_{sA}\left(\overline{M}_{sr}\cos\theta_m + a\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right) + a^2\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right)\right) + \\ + i_{sB}\left(\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right) + a\overline{M}_{sr}\cos\theta_m + a^2\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right)\right) + \\ + i_{sC}\left(\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right) + a\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right) + a^2\overline{M}_{sr}\cos\theta_m\right) \end{bmatrix} \tag{1.41}$$

By re-arranging the previous expression 1.41, it can be expressed as:

$$\bar{\psi}_r = \tfrac{2}{3}\begin{bmatrix} i_{ra}\left(\overline{L_r} + a\overline{M}_r + a^2\overline{M}_r\right) + a\cdot i_{rb}\left(a^2\overline{M}_r + \overline{L_r} + a\overline{M}_r\right) + a^2 i_{rc}\left(a\overline{M}_r + a^2\overline{M}_r + \overline{L_r}\right) + \\ + i_{sa}\left(\overline{M}_{sr}\cos\theta_m + a\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right) + a^2\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right)\right) + \\ + a\cdot i_{sb}\left(a^2\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right) + \overline{M}_{sr}\cos\theta_m + a\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right)\right) + \\ + a^2\cdot i_{sc}\left(a\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right) + a^2\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right) + \overline{M}_{sr}\cos\theta_m\right) \end{bmatrix} \tag{1.42}$$

And finally:

$$\begin{aligned} \bar{\psi}_r &= \left(\overline{L_r} + a\overline{M}_r + a^2\overline{M}_r\right)\bar{i}_r + \left(\overline{M}_{sr}\cos\theta_m + a\overline{M}_{sr}\cos\left(\theta_m + 2\pi/3\right) + a^2\overline{M}_{sr}\cos\left(\theta_m + 4\pi/3\right)\right)\bar{i}_s = \\ &= \left(\overline{L_r} - \overline{M}_r\right)\bar{i}_r + 1.5\cos\left(-\theta_m\right)\overline{M}_{sr}\bar{i}_s = \left(\overline{L_r} - \overline{M}_r\right)\bar{i}_r + 1.5\overline{M}_{sr}\bar{i}_s e^{-j\theta_m} = \left(\overline{L_r} - \overline{M}_r\right)\bar{i}_r + 1.5\overline{M}_{sr}\bar{i}'_s = \\ &= L_r\bar{i}_r + L_m\bar{i}'_s \end{aligned} \tag{1.43}$$

Where L_r is the total three-phase rotor inductance and Lm is the so-called three-phase magnetising inductance. $\bar{i}'_s$ is the stator current space phasor expressed in the frame fixed to the rotor.

Once more the flux linkage magnitude can be expressed in the two-axis form as follows:

$$\bar{\psi}_r = \psi_{r\alpha} + j\psi_{r\beta} \tag{1.44}$$

Where its direct component is equal to:

$$\psi_{r\alpha} = L_r i_{r\alpha} + L_m i_{s\alpha} \tag{1.45}$$

And its quadrature component is expressed as:

$$\psi_{r\beta} = L_r i_{r\beta} + L_m i_{s\beta} \tag{1.46}$$

1.2.3.3- Rotor flux-linkage space phasor in the stationary reference frame fixed to the stator.

The rotor flux linkage can also be expressed in the stationary reference frame using the previously introduced transformation $e^{j\theta m}$, and can be written as:

$$\bar{\psi}'_r = \psi_{rd} + j\psi_{rq} = \bar{\psi}_r e^{j\theta_m} = \left(\psi_{r\alpha} + j\psi_{r\beta}\right)e^{j\theta_m} \tag{1.47}$$

The space phasor of the rotor flux linkage can be expressed according to the fixed co-ordinates as follows:

$$\bar{\psi}'_r = L_r \bar{i}'_r + L_m \bar{i}'_s e^{j\theta_m} = L_r \bar{i}'_r + L_m \bar{i}_s \tag{1.48}$$

The relationship between the stator current referred to the stationary frame fixed to the stator and the rotational frame fixed to the rotor is as follows:

$$\begin{aligned} \bar{i}_s &= \bar{i}'_s e^{j\theta_m} \\ \bar{i}_s e^{-j\theta_m} &= \bar{i}'_s \end{aligned} \tag{1.49}$$

Where

$$\begin{aligned} \bar{i}_s &= i_{sD} + j i_{sQ} \\ \bar{i}'_s &= i_{s\alpha} + j i_{s\beta} \end{aligned} \tag{1.50}$$

From figure 1.5, the following equivalencies can be deduced:

$$\begin{aligned} \bar{i}_s &= \left|\bar{i}_s\right| e^{j\theta} \\ \bar{i}'_s &= \left|\bar{i}'_s\right| e^{j\alpha} = \left|\bar{i}'_s\right| e^{j(\theta-\theta_m)} = \bar{i}_s e^{-j\theta_m} \end{aligned} \tag{1.51}$$

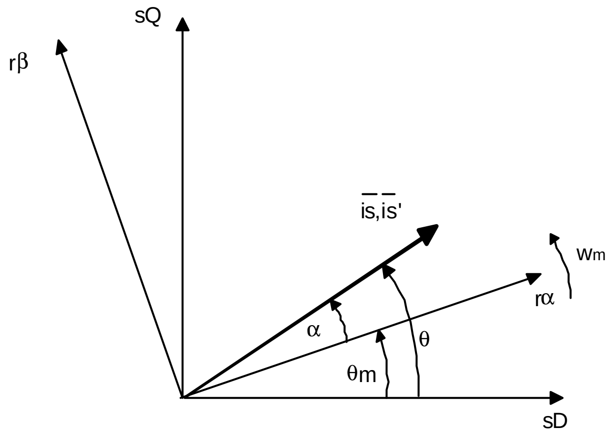


Figure 1.5. Stator-current space phasor expressed in accordance with the rotational frame fixed to the rotor and the stationary frame fixed to the stator.

1.2.3.4- Stator flux-linkage space phasor in the rotating reference frame fixed to the rotor.

Similarly than 1.2.3.3 section, it can be deduced the following expression:

$$\overline{\psi}'_s = \overline{\psi}_s e^{-j\theta_m} = \left(L_s \overline{i}_s + L_m \overline{i}'_r\right) e^{-j\theta_m} = L_s \overline{i}'_s + L_m \overline{i}_r \tag{1.52}$$

1.2.4. – The space phasors of stator and rotor voltages.

The space phasors for the stator and rotor voltages can be defined in a similar way like the one used for other magnitudes.

$$\begin{aligned} \overline{u}_s &= \tfrac{2}{3}\left[u_{sA}(t) + a u_{sB}(t) + a^2 u_{sC}(t)\right] = u_{sD} + j u_{sQ} = \tfrac{2}{3}\left(u_{sA} - \tfrac{1}{2}u_{sB} - \tfrac{1}{2}u_{sC}\right) + j\tfrac{1}{\sqrt{3}}\left(u_{sB} - u_{sC}\right) \\ \overline{u}_r &= \tfrac{2}{3}\left[u_{ra}(t) + a u_{rb}(t) + a^2 u_{rc}(t)\right] = u_{r\alpha} + j u_{r\beta} = \tfrac{2}{3}\left(u_{ra} - \tfrac{1}{2}u_{rb} - \tfrac{1}{2}u_{rc}\right) + j\tfrac{1}{\sqrt{3}}\left(u_{rb} - u_{rc}\right) \end{aligned} \tag{1.53}$$

Where the stator voltage space phasor is referred to the stator stationary frame and the rotor voltage space phasor is referred to the rotating frame fixed to the rotor.

Provided the zero component is zero [VAS 1], it can also be said that:

$$\begin{aligned} u_{sA} &= Re\left(\overline{u}_s\right) \\ u_{sB} &= Re\left(a^2 \overline{u}_s\right) \\ u_{sC} &= Re\left(a\overline{u}_s\right) \end{aligned} \tag{1.54}$$

Equivalent expressions can also be obtained for the rotor.

1.2.5 - Space-phasor form of the motor equations.

The space phasor forms of the voltage equations of the three-phase and quadrature-phase smooth air-gap machines will be presented. Firstly, the equations will be expressed in a general rotating reference frame, which rotates at a general speed w_g, and then to the references frames fixed to the stator, rotor and synchronous speed.

1.2.5.1 - Space-phasor voltage equations in the general reference frame.

If the vector in the figure 1.6 is the stator current, then its formulation in the space phasor form is as follows:

$$\bar{i}_{sg} = \bar{i}_s e^{-j\theta_g} = i_{sx} + j i_{sy} \tag{1.55}$$

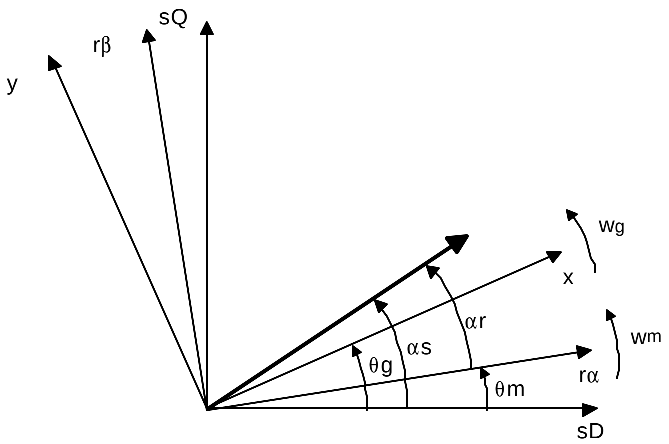


Figure 1.6. It is shown a magnitude represented by means of the vector, and its angle referred to the three different axis. The three different axis are: sD-sQ fixed to the stator, rα-rβ fixed to the rotor whose speed is w_m, and finally the general frame represented by means of the axis x-y whose speed is equal to w_g.

In a similar way and for other magnitudes, it can be written the following equations:

$$\begin{aligned} \bar{u}_{sg} &= \bar{u}_s e^{-j\theta_g} = u_{sx} + j u_{sy} \\ \bar{\psi}_{sg} &= \bar{\psi}_s e^{-j\theta_g} = \psi_{sx} + j\psi_{sy} \end{aligned} \tag{1.56}$$

Where the magnitudes are the voltage space phasor and the stator flux linkage respectively. However, if the magnitude in the figure 1.6 is for instance the rotor current, its space phasor notation will be:

$$\bar{i}_{rg} = \bar{i}_r e^{-j(\theta_g - \theta_m)} = i_{rx} + j i_{ry} \tag{1.57}$$

and for other magnitudes:

$$\bar{u}_{rg} = \bar{u}_r e^{-j(\theta_g - \theta_m)} = u_{rx} + j u_{ry}$$
$$\bar{\psi}_{rg} = \bar{\psi}_r e^{-j(\theta_g - \theta_m)} = \psi_{rx} + j \psi_{ry} \tag{1.58}$$

Manipulating the previous equations yields the following stator and rotor space phasor voltage equations in the general reference frame.

$$\bar{u}_{sg} e^{j\theta_g} = R_s \bar{i}_{sg} e^{j\theta_g} + \frac{d\left(\bar{\psi}_{sg} e^{j\theta_g}\right)}{dt} = R_s \bar{i}_{sg} e^{j\theta_g} + e^{j\theta_g}\frac{d\bar{\psi}_{sg}}{dt} + j e^{j\theta_g} w_g \bar{\psi}_{sg}$$
$$\bar{u}_{rg} e^{j(\theta_g - \theta_m)} = R_r \bar{i}_{rg} e^{j(\theta_g - \theta_m)} + \frac{d\left(\bar{\psi}_r e^{j(\theta_g - \theta_m)}\right)}{dt} = R_r \bar{i}_{rg} e^{j(\theta_g - \theta_m)} + e^{j(\theta_g - \theta_m)}\frac{d\bar{\psi}_{rg}}{dt} + j e^{j(\theta_g - \theta_m)}\left(w_g - P \cdot w_m\right)\bar{\psi}_{rg}$$

(1.59)

Simplifying equation 1.59, it is obtained equation 1.60.

$$\bar{u}_{sg} = R_s \bar{i}_{sg} + \frac{d\bar{\psi}_{sg}}{dt} + j w_g \bar{\psi}_{sg}$$
$$\bar{u}_{rg} = R_r \bar{i}_{rg} + \frac{d\bar{\psi}_{rg}}{dt} + j\left(w_g - P \cdot w_m\right)\bar{\psi}_{rg} \tag{1.60}$$

Where, the flux linkage space phasors are:

$$\bar{\psi}_{sg} = L_s \bar{i}_{sg} + L_m \bar{i}_{rg}$$
$$\bar{\psi}_{rg} = L_r \bar{i}_{rg} + L_m \bar{i}_{sg} \tag{1.61}$$

Using the two-axis notation and the matrix form, the voltage equations can be represented by:

$$\begin{bmatrix} u_{sx} \\ u_{sy} \\ u_{rx} \\ u_{ry} \end{bmatrix} = \begin{bmatrix} R_s + pL_s & -w_g L_s & pL_m & -w_g L_m \\ w_g L_s & R_s + pL_s & w_g L_m & pL_m \\ pL_m & \left(P \cdot w_m - w_g\right) L_m & R_r + pL_r & \left(P \cdot w_m - w_g\right) L_r \\ \left(w_g - P \cdot w_m\right) L_m & pL_m & \left(w_g - P \cdot w_m\right) L_r & R_r + pL_r \end{bmatrix} \cdot \begin{bmatrix} i_{sx} \\ i_{sy} \\ i_{rx} \\ i_{ry} \end{bmatrix} \tag{1.62}$$

1.2.5.2 - Space-phasor voltage equations in the stationary reference frame fixed to the stator.

If wg = 0, the matrix expression obtained is 1.63, being equal to the expression 1.17.

$$\begin{bmatrix} u_{sD} \\ u_{sQ} \\ u_{rd} \\ u_{rq} \end{bmatrix} = \begin{bmatrix} R_s + pL_s & 0 & pL_m & 0 \\ 0 & R_s + pL_s & 0 & pL_m \\ pL_m & P \cdot w_m L_m & R_r + pL_r & P \cdot w_m L_r \\ -P \cdot w_m L_m & pL_m & -P \cdot w_m L_r & R_r + pL_r \end{bmatrix} \cdot \begin{bmatrix} i_{sD} \\ i_{sQ} \\ i_{rd} \\ i_{rq} \end{bmatrix} \tag{1.63}$$

The stator voltage space phasor can be expressed as follows:

$$\bar{u}_s = R_s \bar{i}_s + \frac{d\bar{\psi}_s}{dt} \tag{1.64}$$

The rotor voltage space phasor can be written as:

$$\bar{u}'_r e^{-j\theta_m} = R_r \bar{i}'_r e^{-j\theta_m} + \frac{d\left(\bar{\psi}'_r e^{-j\theta_m}\right)}{dt}$$
$$\bar{u}'_r = R_r \bar{i}'_r + \frac{d\bar{\psi}'_r}{dt} - j \cdot P \cdot w_m \bar{\psi}'_r \tag{1.65}$$

And the flux linkage space phasors can be expressed as follows

$$\bar{\psi}_s = L_s \bar{i}_s + L_m \bar{i}'_r$$
$$\bar{\psi}'_r = L_r \bar{i}'_r + L_m \bar{i}_s \tag{1.66}$$

1.2.5.3 - Space-phasor voltage equations in the rotating reference frame fixed to the rotor.

If wg = wm, the matrix expression obtained is 1.67, being equal to the expression 1.18.

$$\begin{bmatrix} u_{sD} \\ u_{sQ} \\ u_{rd} \\ u_{rq} \end{bmatrix} = \begin{bmatrix} R_s + pL_s & -L_s P\, w_m & pL_m & -L_m P\, w_m \\ L_s P\, w_m & R_s + pL_s & L_m P\, w_m & pL_m \\ pL_m & 0 & R_r + pL_r & 0 \\ 0 & pL_m & 0 & R_r + pL_r \end{bmatrix} \cdot \begin{bmatrix} i_{sD} \\ i_{sQ} \\ i_{rd} \\ i_{rq} \end{bmatrix} \tag{1.67}$$

The stator voltage space phasor can be expressed as follows:

$$\bar{u}'_s = R_s \bar{i}'_s + \frac{d\bar{\psi}'_s}{dt} + j\bar{\psi}'_s \cdot P \cdot w_m \tag{1.68}$$

The rotor voltage space phasor can be written as:

$$\bar{u}_r = R_r \bar{i}_r + \frac{d\bar{\psi}_r}{dt} \tag{1.69}$$

And the flux linkage space phasors can be expressed as follows

$$\bar{\psi}'_s = L_s \bar{i}'_s + L_m \bar{i}_r \tag{1.70}$$
$$\bar{\psi}_r = L_r \bar{i}_r + L_m \bar{i}'_s$$

1.2.5.4 - Space-phasor voltage equations in the rotating reference frame at synchronous speed.

If wg = ws, the matrix expression obtained is 1.71, being equal to expression 1.19.

$$\begin{bmatrix} u_{sD} \\ u_{sQ} \\ u_{rd} \\ u_{rq} \end{bmatrix} = \begin{bmatrix} R_s + pL_s & -L_s w_s & pL_m & -L_m w_s \\ L_s w_s & R_s + pL_s & L_m w_s & pL_m \\ pL_m & -L_m s w_s & R_r + pL_r & -L_r s w_s \\ L_m s w_s & pL_m & L_r s w_s & R_r + pL_r \end{bmatrix} \cdot \begin{bmatrix} i_{sD} \\ i_{sQ} \\ i_{rd} \\ i_{rq} \end{bmatrix} \tag{1.71}$$

The stator voltage space phasor can be expressed as follows:

$$\bar{u}_{sg} = R_s \bar{i}_{sg} + \frac{d\bar{\psi}_{sg}}{dt} + j\bar{\psi}_{sg} w_s \tag{1.72}$$

The rotor voltage space phasor can be written as:

$$\bar{u}_{rg} = R_r \bar{i}_{rg} + \frac{d\bar{\psi}_{rg}}{dt} + j\bar{\psi}_{rg}\left(w_s - P \cdot w_m\right) \tag{1.73}$$

And the flux linkage space phasors can be expressed as follows

$$\begin{aligned} \bar{\psi}_{sg} &= L_s \bar{i}_{sg} + L_m \bar{i}_{rg} \\ \bar{\psi}_{rg} &= L_r \bar{i}_{rg} + L_m \bar{i}_{sg} \end{aligned} \tag{1.74}$$

1.3 – Torque expressions.

1.3.1 - Introduction.

The general expression for the torque is as follows:

$$t_e = c\overline{\psi}_s \times \bar{i}'_r \tag{1.75}$$

Where the "c" is a constant, $\overline{\psi}_s$ and $\bar{i}'_r$ are the space phasors of the stator flux and rotor current respectively, both referred to the stationary reference frame fixed to the stator.

The expression given above can also be expressed as follows:

$$t_e = c\left|\overline{\psi}_s\right| \cdot \left|\bar{i}_r\right| sin\gamma \tag{1.76}$$

Where γ is the angle existing between the stator flux linkage and the rotor current. It follows that when $\gamma=90^o$ the torque obtained is the maximum and its expression is exactly equal to the one for the DC machines. Nevertheless, in DC machines the space distribution of both magnitudes is fixed in space, thus producing the maximum torque for all different magnitude values. Furthermore, both magnitudes can be controlled independently or separately. In an AC machine, however, it is much more difficult to realise this principle because both quantities are coupled and their position in space depends on both the stator and rotor positions. It is a further complication that in squirrel-cage machines, it is not possible to monitor the rotor current, unless the motor is specially prepared for this purpose in a special laboratory. It is impossible to find them in a real application. The search for a simple control scheme similar to the one for DC machines has led to the development of the so-called vector-control schemes, where the point of obtaining two different currents, one for controlling the flux and the other one for the rotor current, is achieved [VAS 1].

1.3.2 - Deduction of the torque expression by means of energy considerations.

Torque equation is being deduced by means of energy considerations. Therefore, the starting equation is as follows:

$$P_{mechanic} = P_{electric} - P_{loss} - P_{field} \tag{1.77}$$

Substituting the previous powers for its values, the equation can be expressed as follows:

$$t_e \cdot w_r = \tfrac{3}{2}\left[\left(Re\left(\bar{u}_s \cdot \bar{i}_s^{*}\right) - R_s\left|i_s\right|^2 - Re\left(\frac{d\bar{\psi}_s}{dt}\bar{i}_s^{*}\right)\right) + \left(Re\left(\bar{u}'_r \cdot \bar{i}'^{*}_r\right) - R_r\left|i'_r\right|^2 - Re\left(\frac{d\bar{\psi}'_r}{dt}\bar{i}'^{*}_r\right)\right)\right] \qquad (1.78)$$

Since in the stationary reference frame, the stator voltage space phasor $\bar{u}_s$ can only be balanced by the stator ohmic drop, plus the rate of change of the stator flux linkage, the previous expression can be expressed as follows:

$$t_e \cdot w_r = \tfrac{3}{2} Re\left(-jw_r\bar{\psi}'_r\bar{i}'^{*}_r\right) = -\tfrac{3}{2} w_r\, Re\left(j\bar{\psi}'_r\bar{i}'^{*}_r\right) = -\tfrac{3}{2} w_r \bar{\psi}'_r \times \bar{i}'_r \qquad (1.79)$$

Expressing the equation in a general way for any number of pair of poles gives:

$$t_e = -\tfrac{3}{2} P\bar{\psi}'_r \times \bar{i}'_r \qquad (1.80)$$

If equations 1.66 and 1.35 are substituted in equation 1.80, it is obtained the following expression for the torque:

$$\mathrm{t_e} = \tfrac{3}{2}\mathrm{P}\bar{\psi}_\mathrm{s} \times \bar{\mathrm{i}}_\mathrm{s} \qquad (1.81)$$

If the product is developed, expression 1.81 is as follows:

$$\mathrm{t_e} = \tfrac{3}{2}\mathrm{P}\left(\psi_\mathrm{sD} \cdot \mathrm{i_{sQ}} - \psi_\mathrm{sQ} \cdot \mathrm{i_{sD}}\right) \qquad (1.82)$$

Finally, different expressions for the torque can be obtained as follows:

$$\mathrm{t_e} = -\tfrac{3}{2}\mathrm{P}\left(\mathrm{L_r}\bar{\mathrm{i}}'_\mathrm{r} + \mathrm{L_m}\bar{\mathrm{i}}_\mathrm{s}\right) \times \bar{\mathrm{i}}'_\mathrm{r} = -\tfrac{3}{2}\mathrm{PL_m}\bar{\mathrm{i}}_\mathrm{s} \times \bar{\mathrm{i}}'_\mathrm{r} = -\tfrac{3}{2}\mathrm{PL_m}\bar{\mathrm{i}}'_\mathrm{s} \times \bar{\mathrm{i}}_\mathrm{r}$$

$$\mathrm{t_e} = -\tfrac{3}{2}\mathrm{P}\frac{\mathrm{L_m}}{\mathrm{L_s}}\left(\mathrm{L_m}\bar{\mathrm{i}}'_\mathrm{r} + \mathrm{L_s}\bar{\mathrm{i}}_\mathrm{s}\right) \times \bar{\mathrm{i}}'_\mathrm{r} = -\mathrm{P}\frac{3\mathrm{L_m}}{2\mathrm{L_s}}\bar{\psi}_\mathrm{s} \times \bar{\mathrm{i}}'_\mathrm{r} = -\frac{3}{2}\mathrm{P}\frac{\mathrm{L_m}}{\mathrm{L_sL_r} - \mathrm{L_m^2}}\bar{\psi}_\mathrm{s} \times \bar{\psi}'_\mathrm{r} \qquad (1.83)$$

1.3.3 – Torque constant.

The value of the torque constant can take two different values. These depend on the constant used in the space phasor. Both possibilities are shown in table I.I.

	Non power invariant		*Power invariant*	
Torque constant	$3/2$		1	
Space phasor constant	*3 ®2*	*2 ®3*	*3 ®2*	*2 ®3*
	$2/3$	1	$\sqrt{2/3}$	$\sqrt{2/3}$

Table I.I. Torque constant values.
"3→2" means the change from three axis to either two axis or space phasor notation, and "2→3" either two axis or space phasor notation to three axis.

1.4 – Simulink model.

1.4.1 - Equations used in the model.

The final expressions used in the implemented models are obtained from all the previously introduced expressions.

All equations have been re-arranged in order to use the operator 1/s instead of the operator p because the “Simulink” deals with the integrator better than with the derivation.

1.4.1.1 – Stator reference.

Stator and rotor fluxes can be expressed as follows:

$$\begin{aligned}
\psi_{sD} &= \tfrac{1}{s}\left(u_{sD} - R_s i_{sD}\right) \\
\psi_{sQ} &= \tfrac{1}{s}\left(u_{sQ} - R_s i_{sQ}\right) \\
\psi'_{rd} &= \tfrac{1}{s}\left(u'_{rd} - R_r i'_{rd} - P\cdot w_m \psi'_{rq}\right) = \tfrac{1}{s}\left(- R_r i'_{rd} - P\cdot w_m \psi'_{rq}\right) \\
\psi'_{rq} &= \tfrac{1}{s}\left(u'_{rq} - R_r i'_{rq} + P\cdot w_m \psi'_{rd}\right) = \tfrac{1}{s}\left(- R_r i'_{rq} + P\cdot w_m \psi'_{rd}\right)
\end{aligned} \tag{1.84}$$

Stator and rotor currents can be expressed as follows:

$$\begin{aligned}
i_{sD} &= \psi_{sD}\frac{L_r}{L_x} - \psi'_{rd}\frac{L_m}{L_x} \\
i_{sQ} &= \psi_{sQ}\frac{L_r}{L_x} - \psi'_{rq}\frac{L_m}{L_x} \\
i'_{rd} &= \psi'_{rd}\frac{L_s}{L_x} - \psi_{sD}\frac{L_m}{L_x} \\
i'_{rq} &= \psi'_{rq}\frac{L_s}{L_x} - \psi_{sQ}\frac{L_m}{L_x} \\
&\text{where } L_x = L_s L_r - L_m^2
\end{aligned} \tag{1.85}$$

1.4.1.2 - Rotor reference.

Stator and rotor fluxes can be expressed as follows:

$$\begin{aligned}
\psi_{rd} &= \tfrac{1}{s}\left(u_{rd} - R_r i_{rd}\right) = 0 \\
\psi_{rq} &= \tfrac{1}{s}\left(u_{rq} - R_r i_{rq}\right) = 0 \\
\psi'_{sd} &= \tfrac{1}{s}\left(u'_{sd} - R_s i'_{sd} + P\cdot w_m \psi'_{sq}\right) \\
\psi'_{sq} &= \tfrac{1}{s}\left(u'_{sq} - R_r i'_{sq} - P\cdot w_m \psi'_{sd}\right)
\end{aligned} \tag{1.86}$$

Stator and rotor currents can be expressed as follows:

$$
\begin{aligned}
i_{sD} &= \psi'_{sD}\frac{L_r}{L_x} - \psi'_{rd}\frac{L_m}{L_x} \\
i_{sQ} &= \psi'_{sQ}\frac{L_r}{L_x} - \psi'_{rq}\frac{L_m}{L_x} \\
i_{rd} &= \psi_{rd}\frac{L_s}{L_x} - \psi'_{sD}\frac{L_m}{L_x} \\
i_{rq} &= \psi_{rq}\frac{L_s}{L_x} - \psi'_{sQ}\frac{L_m}{L_x} \\
&\text{where } L_x = L_s L_r - L_m^2
\end{aligned}
\tag{1.87}
$$

1.4.1.3 - Synchronous reference.

Stator and rotor fluxes can be expressed as follows:

$$
\begin{aligned}
\psi_{sx} &= \tfrac{1}{s}\left(u_{sx} - R_s i_{sx} + w_s \psi_{sy}\right) \\
\psi_{sy} &= \tfrac{1}{s}\left(u_{sy} - R_r i_{sy} - w_s \psi_{sx}\right) \\
\psi_{rx} &= \tfrac{1}{s}\left(u_{rx} - R_r i_{rx} + \psi_{ry}\left(w_s - P\cdot w_m\right)\right) = \tfrac{1}{s}\left(- R_r i_{rx} + \psi_{ry}\left(w_s - P\cdot w_m\right)\right) \\
\psi_{ry} &= \tfrac{1}{s}\left(u_{ry} - R_r i_{ry} - \psi_{rx}\left(w_s - P\cdot w_m\right)\right) = \tfrac{1}{s}\left(- R_r i_{ry} - \psi_{rx}\left(w_s - P\cdot w_m\right)\right)
\end{aligned}
\tag{1.88}
$$

Stator and rotor currents can be expressed as follows:

$$
\begin{aligned}
i_{sx} &= \psi_{sx}\frac{L_r}{L_x} - \psi_{rx}\frac{L_m}{L_x} \\
i_{sy} &= \psi_{sy}\frac{L_r}{L_x} - \psi_{ry}\frac{L_m}{L_x} \\
i_{rx} &= \psi_{rx}\frac{L_s}{L_x} - \psi_{sx}\frac{L_m}{L_x} \\
i_{ry} &= \psi_{ry}\frac{L_s}{L_x} - \psi_{sy}\frac{L_m}{L_x} \\
&\text{where } L_x = L_s L_r - L_m^2
\end{aligned}
\tag{1.89}
$$

1.4.1.4 – Motion equation.

The motion equation is as follows:

$$
\mathrm{t_e} - \mathrm{t_L} = \mathrm{J}\frac{\mathrm{dw_m}}{\mathrm{dt}} + \mathrm{Dw_m} \tag{1.90}
$$

Where, t_e is the electromagnetic torque, t_L is load torque, J is the inertia of the rotor, and finally the D is the damping constant.

Using the torque expressions 1.82, the previous motion equation can be expressed as follows:

$$P \cdot c \cdot \left(\psi_{sD} \cdot i_{sQ} - \psi_{sQ} \cdot i_{sD}\right) = t_L + w_m \left(D + Js\right)$$

$$w_r = \frac{P \cdot c \cdot \left(\psi_{sD} \cdot i_{sQ} - \psi_{sQ} \cdot i_{sD}\right) - t_L}{D + Js} \tag{1.91}$$

Where P is the number of pair of poles and the torque constant take the values either 1 or 2/3 according to the table I.I shown in the previous section 1.3.3.

1.4.2 – Simulated results.

Figures 1.7 and 1.8 show the torque and speed responses obtained from equation 1.84 to equation 1.91. It must be said that all three different references (stator, rotor and synchronous) gave the same simulated results shown in figures 1.7 and 1.8. The validity of the motor model is corroborated.

All simulations are done in Matlab/Simulink. Motor characteristics are listed in section 4.2.

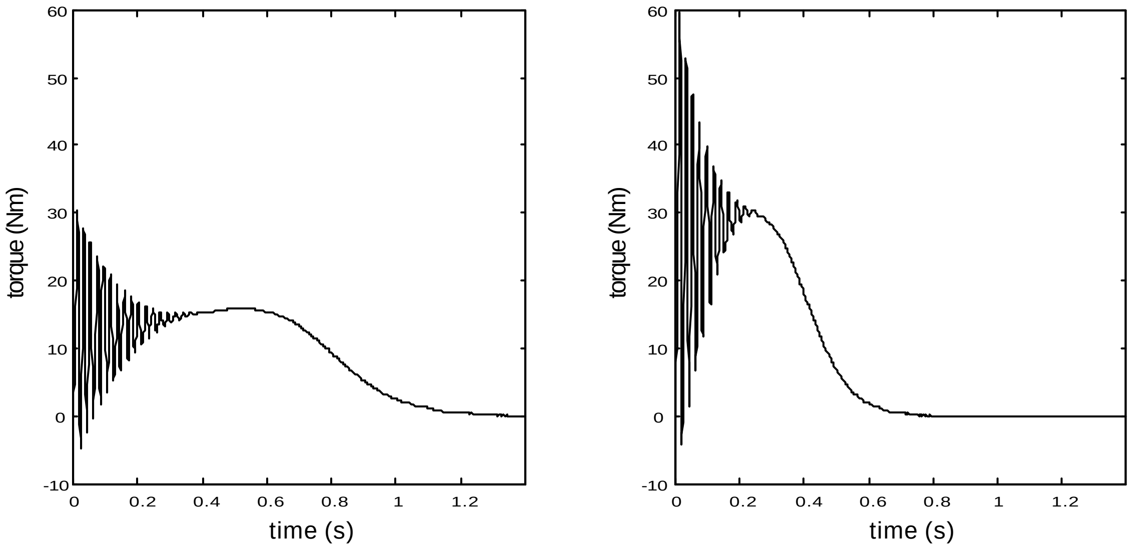
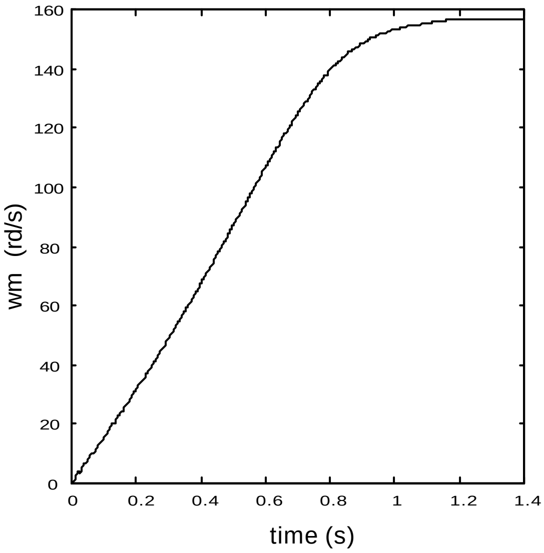


Figure 1.7. Torque response without load. Left: Motor_1kW. Right: Motor_1.5kW. Te= 0Nm, J=0.08Kgm2. Notice the transient at the beginning and the steady state torque value, being 0Nm for this ideal case.



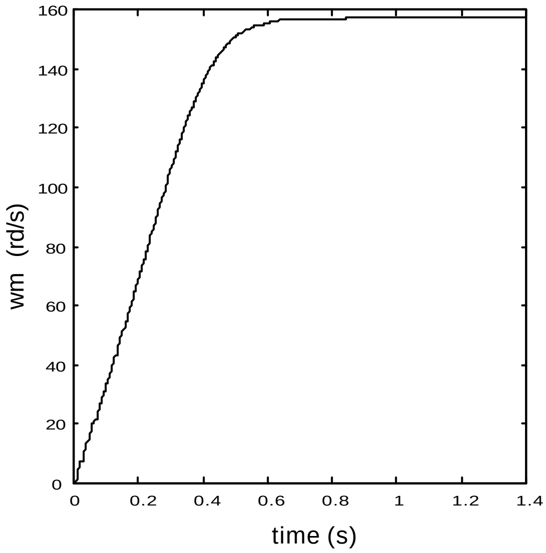


Figure 1.8. Speed response without load. Left: Motor_1kW. Right: Motor_1.5kW. Te=0Nm, J=0.08Kgm2. Notice the small ripple at the beginning due to the transient. The final speed value is 157 rd/s, as expected from this ideal case, where Te=0Nm.

1.5 – Steady state analysis.

1.5.1 - Steady state conditions.

This section deals with the conditions under which, induction motor operates in steady state. When an induction motor operates in steady state and is supplied by symmetrical and sinusoidal waveforms, the space vectors become identical to its phasors. Therefore, the following assumptions expressed in 1.92, can be taken into account:

$$\begin{aligned}
&\bar{u}_s = V_s \\
&\bar{u}'_r = V'_r e^{jq} \\
&\frac{d\bar{i}_s}{dt} = jw_s I_s \\
&\frac{d\bar{i}'_r}{dt} = jw_r I'_r + j \cdot P \cdot w_m I'_r
\end{aligned} \tag{1.92}$$

1.5.2 - Steady state equations.

From the expressions 1.64, 1.65 and 1.66, referred to the stationary reference frame fixed to the stator, can be written the equations 1.93 for the stator and 1.94 for the rotor, valid in both transient and steady state:

$$\bar{u}_s = R_s \bar{i}_s + \left(L_{s1} + \tfrac{3}{2} L_{sm}\right)\frac{d\bar{i}_s}{dt} + \tfrac{3}{2} L_{rm} \frac{d\bar{i}'_r}{dt} \tag{1.93}$$

$$\bar{u}'_r = R_r \bar{i}'_r + \left(L_{r1} + \tfrac{3}{2} L_{rm}\right)\frac{d\bar{i}'_r}{dt} + \tfrac{3}{2} L_{sm} \frac{d\bar{i}_s}{dt} - j \cdot P \cdot w_m \left(\left(L_{r1} + \tfrac{3}{2} L_{rm}\right)\bar{i}'_r + \tfrac{3}{2} L_{sm} \bar{i}_s\right) \tag{1.94}$$

Once the conditions described above in equations 1.92 are applied to the stator 1.93 and rotor 1.94 equations, stator 1.95 and rotor 1.96 steady state equations are obtained:

$$V_s = R_s I_s + jw_s \left(L_{s1} + \tfrac{3}{2} L_{sm}\right) I_s + jw_s \tfrac{3}{2} L_{sm} I'_r \tag{1.95}$$

$$0 = \tfrac{R_r}{s} I'_r + jw_s \left(L_{r1} + \tfrac{3}{2} L_{sm}\right) I'_r + jw_s \tfrac{3}{2} L_{sm} I_s \tag{1.96}$$

1.5.3 - Steady state equivalent circuit.

From the equations 1.95 and 1.96, the well-known equivalent circuit for an induction motor can be drawn as shown in figure 1.9:

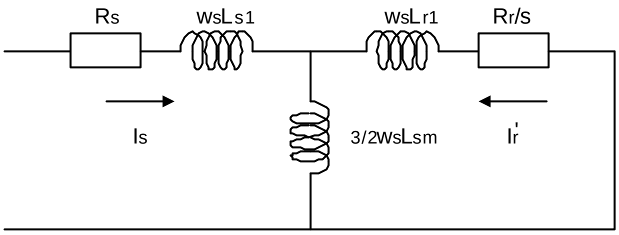


Figure 1.9. Steady state equivalent circuit of Induction motor.

From the previous steady state induction motor model the following expressions for the torque, stator current and stator flux can be obtained:

$$T = 3 \tfrac{P}{2} \tfrac{R_r}{s} \tfrac{1}{w_s} I_r^{'\,2} \tag{1.97}$$

$$I_s = \frac{\tfrac{R_r}{s} + jw_s \left(L_{r1} + \tfrac{3}{2} L_{sm}\right)}{-\, jw_s \tfrac{3}{2} L_{sm}} I_r^{'} \tag{1.98}$$

$$y_s = \left(L_{s1} + \tfrac{3}{2} L_{sm}\right) I_s + \tfrac{3}{2} L_{sm} I_r^{'} \tag{1.99}$$

1.6 – Interim conclusions.

In the present chapter has been deduced the motor model. The model has been formulated by means of the two-axis theory equations and the space phasor notation. Despite the fact that both nomenclatures are valid, it has been proved that the space phasor notation is much more compact and easier to work with. The model has been developed in both nomenclatures for the stator, rotor and synchronous references. In further chapters, the motor model with stator reference, introduced in section 1.4.1.1, will be the one most used.

Different torque expressions have been deduced.

The final concrete equations used in the Matlab/Simulink motor model have been presented by the three different references. Some simulations are shown to prove the validity of the model, being equal for the previously mentioned three references. Two different motors have been used in the model.

Finally the steady state motor analysis has been introduced.