Chapter 5

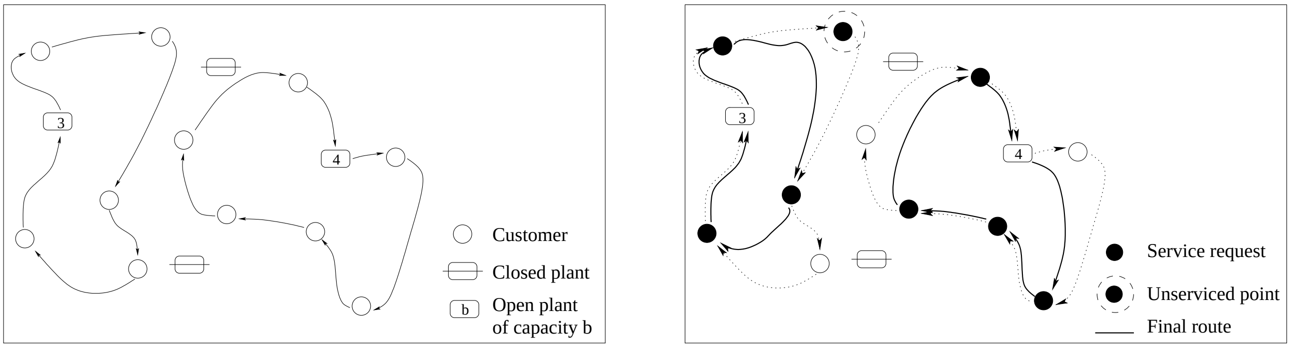
A Stochastic Location-Routing Problem

Introduction

The LRP presented in Chapters 2 and 3 can be used in many real situations when all data are known. It can occur, however, that some of the data of a problem are not known in advance. This is often the case, for instance, when the same problem has to be repeatedly solved (on a daily or weekly basis) but the actual data vary from one instance to the other. The decision maker might want to solve one single problem, and to adapt the solution to the specific instances as they occur. In these situations, instances can differ from each other in the values of the demands of the customers, the actual travel times or even the particular set of customers that request service. When looking for solutions with a good average behavior, the techniques used in the previous chapters are of no use and one has to resort to Stochastic Programming. In this chapter we study a stochastic version of the LRP presented before.

Among the possible sources of stochasticity outlined above we consider the latter case, *i.e.* each of the given customers is assumed to request a service with a known probability. This kind of stochasticity has already been considered in some routing problems, specially in the case of the TSP (Jaillet 1985, 1988, 1993; Laporte, Louveaux, and Mercure 1994). As in those works, we regard this *Stochastic Location-Routing Problem* (SLRP) as a two stage problem. At a first stage, the set of plants to open is decided, and an *a priori* route rooted at each plant is designed, in such a way that each customer belongs to exactly one route. A typical *a priori* solution for a SLRP is shown in Figure 5.1a).

Once the subset of customers that require service is known, the actual routes have to be decided. In our case, this adaptation action is more involved than the one proposed in the works mentioned above, that are either uncapacitated or have capacities on the vehicles but not on the plants. We consider that the allocation of customers to plants decided *a priori* remains unaltered. We also consider that when a plant is overloaded, *i.e.*, the number of customers on its route requesting service exceeds the capacity of the plant, some of these customers are left unserviced at a certain additional cost. This additional cost can be interpreted as the penalty for loosing a customer, or as the cost to provide service by means of acquiring external resources. In this work, the set of customers that are unserviced is randomly chosen as opposite to classical policies that leave unserviced the last customers in the route, or those that require the longest travel times. A reason for such a policy is to avoid priorizing customers that have privileged situations in their respective routes. Once the set of customers to be actually serviced is decided, they are visited in the order defined by the *a priori* routes, as depicted in Figure 5.1b).



a) A priori solution b) Actual routes instance

Figure 5.1: Recourse Action

The goal in this SLRP is to minimize the total cost, that is defined as the sum of the fixed costs for opening the plants plus the expected cost of the recourse action. This expected cost includes the expected penalties to pay for unserviced customers and the expected costs of the actual routes. We further assume that the requests for service of the customers are independent of each other, and that they are identically distributed. This will allow us to find an analytic expression of the recourse function, partially derived from the one presented in Jaillet (1988) for the *Probabilistic Traveling Salesman Problem* (PTSP).

The notation used in this chapter, as well as the analytic expression for the recourse function and the model for the SLRP are presented in Section 5.1. In Section 5.2 we present a heuristic to find good quality solutions for this problem and, in Section 5.3 we give a lower bound that can be used to evaluate the solutions obtained with the heuristic. Computational experiences and conclusions are presented in the last two sections.

5.1 Modeling Issues

The notation in this chapter is basically the same as in the previous ones, although some new parameters must be introduced. An instance of the SLRP under consideration will be given by:

– A set of potential facility locations indexed in I.

– A set of customer locations indexed in J.

– The vector $c = (c_{ij})$ of the travel costs, between customers or between customers and plants.

– The fixed costs f_i, for opening plants at sites $i \in I$.

– The capacities b_i, for each potential plant location $i \in I$.

– The parameter, p, defining the probability distribution of the service request for each customer.

  We assume that customers request for service independently, all with the same probability. So, the demands of the customers $(\xi_j, j \in J)$ can be modeled by means of independent Bernoulli random variables with a common parameter p.

– The penalty cost, P, incurred when the request for service of a customer is disregarded. We assume that this penalty is the same for all customers.

The *a priori* solutions to this SLRP can be characterized as solutions for a deterministic LRP without capacity constraints. Thus, the problem formulation is similar to the one presented in Chapter 2. However, the inclusion of stochasticity requires some modifications:

- The objective function is now the sum of fixed opening costs plus the value of the recourse function.
- In what concerns the constraints, there are two main variations. Capacity constraints must be dropped, since now capacities are not taken into account until the actual demands are known. Note that, for the same reason, it is more convenient to use the disaggregated form of the connectivity constraints.

As in the model for the deterministic LRP, x variables refer to flows through the arcs of the auxiliary network introduced in Chapter 2. The model we obtain is:

$$
\begin{aligned}
\text{(SLRP)} \quad \text{minimize} \quad & \sum_{k \in I} f_i \, x_{sk}^k + \mathcal{Q}(x) && && (5.1)\\
\text{subj. to} \quad & \sum_{k \in I} \sum_{a \in A(s)^+} x_a^k = |I| && && (5.2)\\
& \sum_{a \in A(v)^+} x_a^k = \sum_{a \in A(v)^-} x_a^k && \forall v \in V \backslash \{s,t\}, \forall k \in I && (5.3)\\
& \sum_{k \in I} \sum_{a \in A(t)^-} x_a^k = |I| && && (5.4)\\
& x_{(s,i)}^k = x_{(i',t)}^k && \forall i, k \in I && (5.5)\\
& \sum_{k \in I} \sum_{a \in A(j)^-} x_a^k = 1 && \forall j \in J && (5.6)\\
& \sum_{a \in A(S)} x_a^k \leqslant |S| - 1 && \forall S \subseteq J, \forall k \in I && (5.7)\\
& x_{(s,i)}^k = 0 && \forall i, k \in I, i \neq k && (5.8)\\
& x_a^k \in \{0,1\} && \forall a \in A, \forall k \in I, && (5.9)
\end{aligned}
$$

where the recourse function $\mathcal{Q}(x)$ gives the expected cost of the recourse action. To derive an expression for $\mathcal{Q}(x)$, some extra notation is required. Given an *a priori* solution x, and a realization of the demands vector ξ, for each open plant i we denote:

- $J(i)$, the set of customers allocated to plant i, *i.e.*,

$$J(i) = \left\{ j \in J \;\middle|\; \sum_{a \in A^-(j)} x_a^i = 1 \right\};$$

- n_i the number of customers allocated to plant i; *i.e.*,

$$n_i = |J(i)|;$$

- η_i, the total demand allocated to plant i, *i.e*,

$$\eta_i = \sum_{j \in J(i)} \xi_j .$$

Note that, from the assumption that service requests are *i.i.d.* Bernoulli random variables, it follows that

$$\eta_i \sim \text{Binomial}(n_i,\, p).$$

– $J'(i)$, the set of customers that are actually visited in the route associated with plant i. We can distinguish two cases in the definition of $J'(i)$:

- $\eta_i \leqslant b_i$: In this case, $J'(i) = \{j \in J(i) \,|\, \xi_j = 1\}$.
- $\eta_i > b_i$: Now, $J'(i)$ is a random subset of $\{j \in J(i) \,|\, \xi_j = 1\}$, of cardinality b_i.

– Finally, x'_i denotes the restriction to $J'(i)$ of the *a priori* route associated with i. It is the incidence vector of the arcs in the route obtained from x by skipping the customers not in $J'(i)$.

Using these expressions we can model the recourse function as:

$$\mathcal{Q}(x) = \mathbb{E}_\xi \left[\sum_{i\,|y_i=1} P(\eta_i - b_i)^+ \;+\; \mathbb{E}_{J'(i)}[\sum_{i\,|y_i=1} cx'_i] \right].$$

By linearity of the expectation functional, we can express $\mathcal{Q}(x)$ as:

$$\mathcal{Q}(x) = \mathcal{Q}_P(x) + \sum_{i\,|y_i=1} \mathcal{Q}_r^i(x)$$

where

$$\mathcal{Q}_P(x) = P \sum_{i\,|y_i=1} \mathbb{E}_\xi[(\eta_i - b_i)^+] \quad \text{and} \quad \mathcal{Q}_r^i(x) = \mathbb{E}_\xi\left[\mathbb{E}_{J'(i)}[cx'_i]\right],\; i \in I.$$

Note that $\mathcal{Q}_P(x)$ exactly fits the structure of the recourse function associated with a stochastic problem with simple recourse, extensively studied in the literature (see, for instance, Birge and Louveaux 1997). On the contrary, the functions $\mathcal{Q}_r^i(x)$ do not have a simple structure.

To find an analytical expression of $\mathcal{Q}_r^i(x)$, we need to compute the probabilities of actually using each of the arcs connecting any pair of nodes in $J(i) \cup \{i\}$. Given the assumption that service requests are *i.i.d.* and the definition of the recourse action, the probability of using a certain arc does not depend on its endpoints, but only on the number of customers skipped, according to the order defined by x. For arcs connecting two customers that have $l \geqslant 0$ intermediate customers in the *a priori* route, standard probability computations give raise to the expression

$$P_l = p^2 \left[\sum_{k=2}^{\min\{b_i, n_i - l\}} \binom{n_i - l - 2}{k-2} p^{k-2} q^{n_i-k} + \sum_{k=b_i+1}^{n_i} p^{k-2} q^{n_i-k} \sum_{t=t_0}^{t_1} \binom{l}{t} \binom{n_i - l - 2}{k-t-2} \frac{\binom{k-t-2}{b_i-2}}{\binom{k}{b_i}} \right]. \tag{5.10}$$

The factor p^2 corresponds to the probability that the endpoints of the considered arc have demand, whereas the second factor corresponds to the conditional probability of actually visiting them and servicing them consecutively, given that they have positive demand.

This second probability results from considering all possible values of $k = \eta_i$ (the number of customers that require service). If this number does not exceed the capacity of the plant ($k \leqslant b_i$),

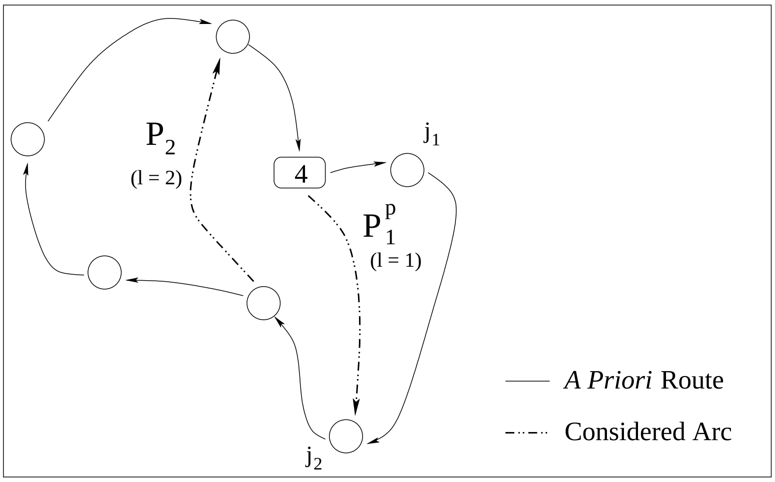


Figure 5.2: Probabilities Associated with Each Arc

then all the customers that have demand will be serviced. The expression $\binom{n_i-l-2}{k-2}$ gives the number of ways of having k demand requests, 2 of them corresponding to the endpoints of the considered arc, and $k-2$ corresponding to customers that are not between them in the *a priori* route. On the other hand, when $k > b_i$, only a subset of the customers requesting service will be actually serviced. In this case, the k customers with demand have been partitioned into *i)* the two endpoints of the considered arc, *ii)* t customers belonging to the group of l customers skipped by the arc, and *iii)* $k-t-2$ customers outside that group. The probability of both endpoints being serviced consecutively is then given by the ratio

$$\frac{\binom{k-t-2}{b_i-2}}{\binom{k}{b_i}},$$

that corresponds to the number of ways of choosing the remaining $b_i - 2$ customers to be serviced from group *iii)*, over all the ways of choosing b_i customers among the k customers with demand.

Reasonings of the same nature give raise to a slightly different expression for arcs incident to a plant, skipping again $l \geqslant 0$ customers:

$$P_l^p = p\left[\sum_{k=1}^{\min\{b_i,n_i-l\}} \binom{n_i-l-1}{k-1} p^{k-1} q^{n_i-k} + \sum_{k=b_i+1}^{n_i} p^{k-1} q^{n_i-k} \sum_{t=t_0}^{t_1} \binom{l}{t} \binom{n_i-l-1}{k-t-1} \frac{\binom{k-t-1}{b_i-1}}{\binom{k}{b_i}}\right]. \tag{5.11}$$

In both expressions,

$$t_0 = \max\{k-n+l, 0\} \quad \text{and} \quad t_1 = \min\{k-b_i, l\}.$$

Let $(j_1, \ldots, j_{n_i})$ be the set of customers of $J(i)$ in the order defined by x (see Figure 5.2). Then, the recourse function $\mathcal{Q}_r^i(x)$ is given by:

$$\mathcal{Q}_r^i(x) = \sum_{k=1}^{n_i} P_{k-1}^p c_{ij_k} + \sum_{k=1}^{n_i-1} \sum_{t=k+1}^{n_i} P_{t-k-1} c_{j_k j_t} + \sum_{k=1}^{n_i} P_{n_i-k}^p c_{j_k i}. \tag{5.12}$$

Approximation of $\mathcal{Q}_r^i$

Evaluations of the function $\mathcal{Q}_r^i$ as described in (5.12), are quite time consuming, given the complexity of the expressions (5.10) and (5.11). Note that this evaluation cost is partially due to the fact that actual visits to different customers in the same *a priori* route are not independent, because of the capacity constraint of the plant. Since the heuristic presented in this work requires a considerable number of evaluations of the recourse function, we propose an approximation of $\mathcal{Q}_r^i$ which is less costly to evaluate. It consists of approximating the probabilities P_l, P_l^p $(l \geqslant 0)$ with the ones we would obtain if actual services to customers were independent random variables. In fact, this is the case when $n_i \leqslant b_i$.

Let $\bar{p}$ be the probability of visiting a given customer. Easy computations give:

$$\bar{p} = p \left(\sum_{k=1}^{b_i} \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} \; + \sum_{r=b_i+1}^{n_i} \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} \frac{b_i}{k} \right).$$

We can approximate the probabilities P_l, P_l^p with

$$\tilde{P}_l = \bar{p}^2(1-\bar{p})^l \quad \text{and} \quad \tilde{P}_l^p = \bar{p}(1-\bar{p})^l,$$

which leads to the following approximation of $\mathcal{Q}_r^i$:

$$\tilde{\mathcal{Q}}_r^i = \sum_{k=1}^{n_i} \bar{p}(1-\bar{p})^{k-1} c_{ij_k} + \sum_{k=1}^{n_i-1} \sum_{t=k+1}^{n_i} \bar{p}^2(1-\bar{p})^{t-k-1} c_{j_k j_t} + \sum_{k=1}^{n_i} \bar{p}(1-\bar{p})^{n_i-k} c_{j_k i}.$$

Note that this expression coincides with the objective function proposed in Jaillet (1988) for the PTSP, for the particular case where all the probabilities of presence of the customers have the same value. The use of approximations of the objective function in the exploration of neighborhoods has been successfully applied in for Stochastic VRP problems (Gendreau, Laporte, and Séguin 1996). In what follows, we will refer to the approximation of the recourse function as $\tilde{\mathcal{Q}}(x)$.

5.2 Heuristic for the Stochastic Location-Routing Problem

For the SLRP presented in the previous section, we propose a two phase heuristic. First, an initial solution is found in the constructive phase which is iteratively improved in a LS phase.

In the constructive phase the problem is splitted into three subproblems that are successively solved to gradually build an initial solution. Standard neighborhoods are then considered in a LS phase to improve this initial solution.

Constructive Phase

In the constructive phase an initial solution is gradually built by splitting the problem in a succession of three small subproblems: *i)* select of the set of plants to open, *ii)* determine the allocation of customers to open plants and *iii)* design a route for each open plant.

Select the Set of Open Plants

The set O of plants to open is chosen in such a way that the overall capacity is enough to satisfy all the service requests within a prespecified probability α:

$$\mathbb{P}\left[\sum_{j\in J}\xi_j \;\leqslant\; \sum_{i\in O} b_i\right] \geqslant \alpha. \tag{5.13}$$

The smallest overall capacity that fulfils (5.13) is denoted by n_α. The value of n_α is easily derived from the probability distribution of the total number of service requests, Binomial(n, p). Once n_α is computed, O is determined by solving the following KP:

$$(\mathrm{KP}_\alpha) \quad \text{minimize} \sum_{i\in I} f_i y_i \tag{5.14}$$

$$\text{subj. to} \sum_{i\in I} b_i y_i \geqslant n_\alpha \tag{5.15}$$

$$y_i \in \{0,1\} \qquad \forall i \in I. \tag{5.16}$$

Observe that the set O obtained this way is the minimum cost set of plants satisfying (5.13).

Allocate Clients to the Open Plants

Once the set of plants is fixed, we regard the allocation subproblem as a SGAP, *i.e.* we solve the allocation problem where the cost function is given by the distances between plants and customers. Thus, distances between customers allocated to the same plant are not taken into account. To find such an allocation, we use Heuristic B in Chapter 4.

The recourse action taken in this SLRP does not consider the possibility of reassigning customers to less busy plants that was taken into account in the SGAP presented in Chapter 4. To model the case without reassignments, we simply set the reassignment cost of all customers at a value higher than the penalty for not servicing them when they have demand.

Build the Set of *a priori* routes

Starting from empty routes, each customer is taken at a time, and inserted in the route of the plant it has been assigned to. The place of the insertion is determined using nearest insertion, with respect to the deterministic route costs. After all customers have been inserted in their respective routes, successive interchanges of two arcs within the same route are performed until all obtained routes are 2-opt, with respect to the objective function $\tilde{\mathcal{Q}}^i_r$.

Improving Phase

Once an initial solution is found in the constructive phase, LS is applied to improve it. All moves are performed using the nearest insertion criterion with respect to $\tilde{\mathcal{Q}}_r$. The considered neighborhoods are the following:

**$N_1(x)$: ** Reassignment of one customer

$N_1(x)$ contains all solutions that can be derived from x by taking a customer from its route and inserting it in a different one.

**$N_2(x)$: ** Interchange of two customers.

$N_2(x)$ contains all solutions that can be obtained from x by taking a pair of customers j_1 and j_2 belonging to different routes i_1 and i_2, respectively, removing them from their routes, and inserting customer j_1 in the route of plant i_2 and j_2 in the route of plant i_1.

**$N_3(x)$: ** Plant interchange

Solutions in $N_3(x)$ are those that can be obtained form x by closing an open plant i_1, opening a closed plant i_2, and assigning the route of i_1 to i_2.

**$N_{2-opt}(x)$: ** Arc interchange

Solutions in $N_{2-opt}(x)$ are obtained from x by interchanging pairs of arcs in the same route.

The heuristic is outlined in Algorithm 5.1, where the order of exploration of the different neighborhoods is shown.

Algorithm 5.1 SLRP_heuri

```
{Constructive Phase}
Select α and find n_α and solve (KP_α) ⟶ O.
Apply Heuristic B from Chapter 4 to find {J(i), i ∈ O}.
for (i ∈ O) do
  for (j ∈ J(i)) do
    Insert j in the route of plant i.
  end for
  Explore N_{2−opt}(x).
end for
{Improving Phase}
StopCriterion ← false.
while (not StopCriterion) do
  repeat
    Explore N2(x) and update x
    Explore N1(x) and update x
  until (no improving moves found)
  repeat
    Explore N_{2−opt}(x) and update x
  until (no improving moves found)
  Explore N3(x) and update x
  Update StopCriterion
end while
```

5.3 Lower Bound for the Stochastic Location-Routing Problem

As in the case of the deterministic LRP, we have split the objective function into two parts and we propose separate bounds for each of them.

1. Fixed costs for opening plants plus expected penalties for unserviced clients: $\underline{z}_{SKP}$

   To bound the expected penalties, we aggregate the capacities of all open plants, without taking into account which plant each client is assigned to. The bound on the overall expected penalty decreases as the total capacity of the open plants increases, until the total capacity reaches the number of customers.

   It is important to bound the expected penalty together with the fixed costs for opening plants, since these two terms are conflicting. Opening a plant increases the fixed costs, but decreases the penalties.

2. Expected cost of the *a posteriori* routes

   To bound the cost of the routes *a posteriori* we use the approach presented in Laporte, Louveaux, and Mercure (1994) for the probabilistic TSP, i.e., we compute a lower bound on the cost of the *a priori* routes, $\bar{z}_{RS}$, and we subtract from it an upper bound on the expected savings for skipping customers, $\underline{z}_{TSP}$.

So, the final lower bound is given by:

$$\underline{z}_{SLRP} = \underline{z}_{SKP} + (\underline{z}_{TSP} - \bar{z}_{RS})$$

Lower bound on the sum of fixed costs plus expected penalties, $\underline{z}_{SKP}$

Given a feasible solution having $S \subseteq I$ as the set of open plants, independently of the assignment of customers to plants, the expected penalty will be bounded from below by:

$$\mathbb{E}_\xi \left[P(\sum_{j \in J} \xi_j - \sum_{i \in S} b_i)^+ \right].$$

So, a lower bound on the sum of fixed costs plus expected penalties is given by

$$\begin{aligned} \underline{z}_{SKP} = \ & \text{minimize} \ \sum_{i \in I} f_i \, y_i \quad + \mathcal{Q}'(y) \\ & \text{subj. to} \ \ y_i \in \{0,1\} \qquad i \in I, \end{aligned}$$

where

$$\mathcal{Q}'(y) = \mathbb{E}_\xi \left[P(\sum_{j \in J} \xi_j - \sum_{i \in I} y_i b_i)^+ \right].$$

The bound $\underline{z}_{SKP}$ is the optimal value of a two stage problem with simple recourse. It can be seen that a deterministic equivalent model (see, for instance, Klein Haneveld and van der Vlerk 2001) of this problem is

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i\in I} f_i\, y_i + \sum_{s=0}^{n+1} \delta^s \; u^s \\
\text{subj. to} \quad & \sum_{i\in I} b_i\, y_i - \sum_{s=0}^{n+1} u^s \quad = np \\
& u^0 \leqslant 1 - np \\
& 0 \leqslant u^{l+1} \leqslant 1 && l = 0, \dots, n \\
& y_i \in \{0,1\} && i \in I,
\end{aligned}
$$

where

$$
\delta^0 = -P,
$$

$$
\delta^{l+1} = P\left(-1 + \mathbb{P}\left[\sum_{j\in J} \xi_j \leqslant l\right]\right), l = 0, \dots, n.
$$

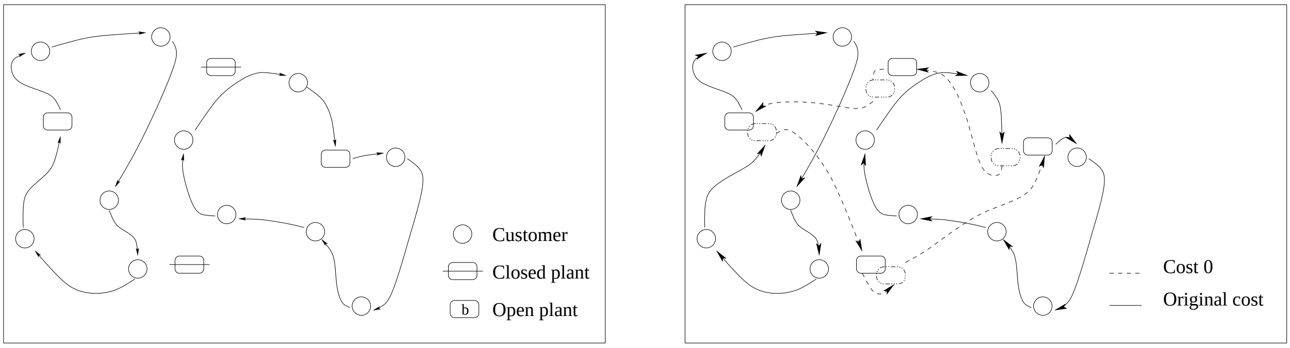
Lower bound on the cost of the a priori *route,* $\underline{z}_{TSP}$

To bound the cost of the *a priori* route, we use a similar approach to the one proposed in Chapter 2 for the deterministic LRP. Again, our bound is the solution to a TSP. This time, however, the set of nodes is slightly different, and we construct a symmetric TSP, instead of an asymmetric one. Additionally we include some side constraints.

We consider a complete graph with nodes $N = I \cup I' \cup J$, where I' is a copy of the set of plants. Node i represents the plant as a starting point of a route, and i' represents the same plant, as an ending point of the route. We define the following costs on the arcs

$$
\tilde{c}_e = \begin{cases} c_e & \text{if } e \in J \times J, e \in I \times J \text{ or } e \in J \times I' \\ 0 & \text{if } e \in I \times I' \text{ or } e = (i', i), i \in I \\ \infty & \text{otherwise.} \end{cases}
$$

According to these costs,in the optimal solution to the TSP, nodes representing plants will only be connected in two special cases. An arc from an ending-plant node to a starting-plant node corresponds to finishing one route in one plant and starting another in the next plant, while an arc from a starting-plant node to its corresponding ending-plant node represents an empty route. In figure 5.3 we show how a feasible solution for this TSP can be derived from any *a priori* set of routes for the SLRP.



a) A priori solution

b) Solution to the TSP

Figure 5.3: Bound on the *a Priori* Route Cost

For this TSP we use a model based on the one proposed in Miller, Tucker, and Zemlin (1960) and reinforced later in Desrochers and Laporte (1991):

$$
\begin{aligned}
\text{(TSPb) minimize } & \sum_{e\in A} c_e x_e && && (5.17)\\
\text{subj. to } & \sum_{e\in\delta^-(j)} x_e = 1 && i \in N && (5.18)\\
& \sum_{e\in\delta^+(j)} x_e = 1 && i \in N && (5.19)\\
& u_i - u_j + (\bar{n}-1)x_{ij} + (\bar{n}-3)x_{ji} \leqslant \bar{n}-2 && i,j>1, i\neq j && (5.20)\\
& x_e \in \{0,1\} && e \in A && (5.21)\\
& 1 \leqslant u_i \leqslant |N|-1 && i>1, && (5.22)
\end{aligned}
$$

where $\bar{n} = |N| = 2m + n$.

In this model, variables x_e indicate if the arc $e \in A$ is used in the solution, and variables u_i give the order in which node i is visited in the tour. In order to tighten the bound provided by this model, some side constraints have been added.

1. The same *a priori* solution for the SLRP can give raise to a variety of solutions for the TSP, depending on the order in which the different plants are considered. Without loss of generality, this awkward symmetry can be avoided by fixing the order of the plants. This can be achieved by adding the constraints
$$u_i - u_{i-1} \geqslant 2, \quad i \in I \setminus \{0\}. \tag{5.23}$$

2. Using the order specified by constraints (5.23), it is easy to force each route to end at the plant it started. We only need to impose
$$u_i - u_{(i-1)'} \geqslant 1, \quad i \in I \setminus \{0\}. \tag{5.24}$$

We denote by $\underline{z}_{TSP}$ the optimal value of (TSPb), reinforced by constraints (5.23) and (5.24).

Upper bound on the expected savings for skipping customers, $\bar{z}_{RS}$

The model presented by Laporte, Louveaux, and Mercure to find an upper bound on the expected savings for the Probabilistic TSP with customers in J , is given by:

$$\text{maximize} \quad \sum_{i,j,k} d_{ikj}\ y_{ikj} \tag{5.25}$$

$$\text{subj. to} \quad \sum_{\substack{i,j \neq k \\ i<j}} y_{ikj} = 1 \qquad k \in J \tag{5.26}$$

$$\sum_{k \neq i,j} y_{ikj} \leqslant 1 \qquad (i,j \in J^*; i<j) \tag{5.27}$$

$$\sum_{\substack{j<i \\ k \neq i,j}} y_{ikj} + \sum_{\substack{j>i \\ k \neq i,j}} y_{ikj} \leqslant 2 \qquad i \in J^* \tag{5.28}$$

$$y_{ikj} \in \{0,1\} \qquad (i,j,k \in J^*; i<j; k \neq i,j), \tag{5.29}$$

where d_{ikj} is an upper bound of the expected saving in the cost of the route for skipping customer k by going directly from i to j. Indices in J^* correspond to any customer or to the depot.

To extend (5.25)-(5.29) to the SLRP some modifications are required:

- Since open plants must belong to different routes, y_{ikj} must be 0 (or, equivalently it is not defined) when i and j are the indices corresponding to two different plants.

- It might be the case that one open plant has one single customer assigned to it *a priori*. To take those cases into consideration, variables y_{iki}, must be defined, for all indices i corresponding to plants.

- For a given plant i and a customer k, if y_{iki} is at value one, plant i cannot be connected to any other customer, so, constraints (5.28) must be modified.

- Any upper bound on the fixed costs for opening plants in the optimal solution can be used to reinforce the model with a knapsack constraint that avoids using sets of plants whose overall fixed opening cost exceeds the value of the bound. To do so, additional variables w_i, $i \in I$, indicating whether plant i is open or not, are needed. The upper bound we use is $n\,p\,P$. Since $n\,p$ is the expected number of customers that have demand, this bound corresponds to the cost of the solution that does not open any plant. Therefore, it is an upper bound on the cost of the optimal solution and, in particular on its fixed costs for opening plants.

- If, in a solution, a plant has one single customer assigned to it, that customer cannot appear in the shortcut associated with any other customer.

To take all this modifications into account, we consider three types of shortcut variables:

A: For each plant i and each customer k, variable y_{iki}.

B: For each plant i and each pair of different customers k, j, variable y_{ikj}.

C: For each group of three different customers k, j_1, j_2, variable $y_{j_1 k j_2}$.

The coefficients in the objective function are as follows. For variables of types B and C,

$$d_{ikj} = q(c_{ik} + c_{kj} - c_{ij}),$$

where $q = 1 - p$ is the probability that customer k need not be visited, and indices i, j correspond to customers or plants. In the case of variables of type A, y_{iki}, the saving does not include the third term since, if plant i is serving customer k exclusively, no arcs will have to be traversed when customer k does not require to be visited. So, the resulting coefficient is $d_{iki} = 2\,q\,c_{ik}$.

If we call, respectively, $A(k), B(k)$ and $C(k)$, the subsets of variables from sets A, B and C corresponding to possible shortcuts for customer k, and $S(k) = A(k) \cup B(k) \cup C(k)$, the resulting model is as follows:

$$\bar{z}_{RS} = \text{ maximize } \sum_{k\in J} \sum_{s\in S(k)} d_s\, y_s \tag{5.30}$$

$$\text{subj. to } \sum_{s\in S(k)} y_s = 1 \qquad k \in J \tag{5.31}$$

$$\sum_{k\in J} y_{iki} \leqslant 1 \qquad i \in I \tag{5.32}$$

$$\sum_{k\in J\setminus\{j\}} y_{ikj} \leqslant 2 \qquad i \in I, j \in J \tag{5.33}$$

$$\sum_{k\in J\setminus\{j_1,j_2\}} y_{j_1kj_2} \leqslant 1 \qquad j1 \neq j2 \in J \tag{5.34}$$

$$\sum_{k\in J\setminus\{j\}} \sum_{i\in I} y_{ikj} + \sum_{\substack{k,j_2\in J\setminus\{j\}\\ k\neq j_2}} y_{jkj_2} \leqslant 2 \qquad j \in J \tag{5.35}$$

$$2\sum_{k\in J} y_{iki} + \sum_{k,j\in J} y_{ikj} \leqslant 2w_i \qquad i \in I \tag{5.36}$$

$$\sum_{i\in I} f_i w_i \leqslant n\,p\,P \tag{5.37}$$

$$2\sum_{s\in A(j)} y_s + \sum_{i\in I} \sum_{\substack{k\in J\\ k\neq j}} y_{ikj} + \sum_{\substack{k,j_2\in J\\ j\neq k\neq j_2\neq j}} y_{jkj_2} \leqslant 2 \qquad j \in J \tag{5.38}$$

$$y_s, w_i \in \{0,1\} \qquad k \in J; s \in S(k), i \in I. \tag{5.39}$$

In this model, constraints (5.31) ensure that exactly one shortcut is selected for each customer, constraints (5.32)-(5.35) are used to avoid the excessive use of any point as an endpoint of a shortcut. Constraints (5.36) prevent the association of shortcuts with non open plants. Constraints (5.37) state an upper bound for the total fixed costs of the open plants, and constraints (5.38) prevent customers from being endpoints of any shortcut if they are associated with a shortcut of type A.

The optimal solution to this model provides us with an upper bound of the expected savings for skipping customers in any set of feasible routes, $\bar{z}_{RS}$.

5.4 Computational Results

The heuristic of Section 5.2 as well as the lower bound of Section 5.3 have been tested on a series of instances. As in Chapters 2 and 3, the instances we use are derived from those in `http:`

`//troubadix.unisg.ch/klose/problems/problems.html\#TSCFLP` for the Two-Stage Capacitated Facility Location Routing, and they are classified according to their dimensions. There are five groups, consisting of 25 instances each. Groups SS1, SS2, and SS3 contain instances with 5 plants and 10, 20 and 30 customers, respectively, while instances with 10 plants and, respectively 20 and 30 customers are contained in groups SM2 and SM3. Within each group instances have been classified according to the ratio between the number of customers and the total capacity. This ratio ranges in the set $\{4.5, 4, 2.5, 2, 1.5\}$, and the resulting subgroups are labelled with a, b, c, d and e. Similarly to the computational experiences of Chapter 2, when the existing instance had more plants and/or clients than we needed, we have ignored the data of the exceeding plants and/or clients. Traveling costs have been computed as the Euclidean distances between points and capacities have been scaled in order to meet the prespecified ratios. Also, the fixed costs for opening the plants (depots in the original instances) have been divided by 20, to obtain instances whose solutions have balanced costs. The penalties for not servicing a customer have been determined according to the relationship:

$$P = 2\max\{c_{ij} \mid i \in I, j \in J\} + \max\left\{\frac{f_i}{c_i} \mid i \in I\right\}$$

Finally, for each set of data, three different values of the probability of demand have been considered: 0.2, 0.5 and 0.8.

Figure 5.4 shows the percent improvement, with respect to the initial solution, of the objective function value ($100(z_{\text{ini}} - z_{\text{best}})/z_{\text{best}}$). Each bar represents the average improvement obtained within one subset of five instances, for a fixed probability of demand. As can be seen, the most significant improvements correspond to high values of the probability of demand. Nevertheless, even in the case of $p = 0.2$ significant improvements have been obtained, up to 57.51%. The same figure shows the effect of the size of the instances on the behavior of the LS. The improvement of this behavior when the size of the instances grows is due to the increase in the number of feasible solutions.

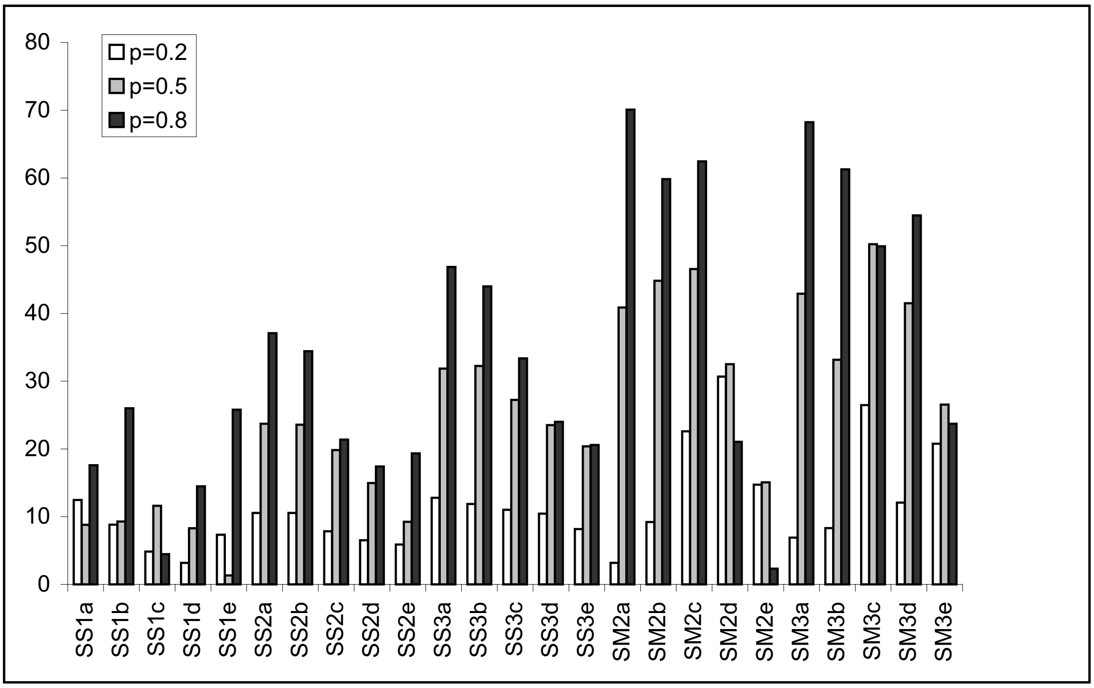


Figure 5.4: Percent Improvements with Respect to the Initial Solution

We next report on the results obtained with the lower bounding procedure. In Table 5.1 we compare $\underline{z}_{SKP}$ with the sum of *term1* (fixed opening costs) plus *term2* (expected penalties) of the objective function. Each entry of this table corresponds to the average, for the instances of same dimension for a given probability, of the percent gap $100(term1 + term2 - \underline{z}_{SKP})/\underline{z}_{SKP}$.

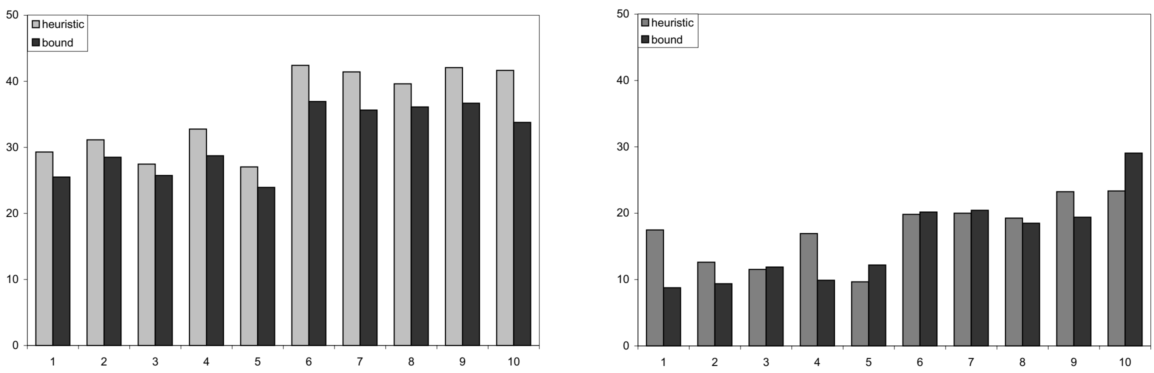
Table 5.1: Average Gaps for Fixed Costs Plus Expected Penalties

group	$p = 0.2$	$p = 0.5$	$p = 0.8$
SS1	0.837	14,490	20,640
SS2	0.038	9,967	12,761
SS3	0.576	7,032	16,404
SM2	0.531	16,203	13,846
SM3	0.485	26,958	16,099

The results obtained for $p = 0.2$ are very satisfactory, since the average gap for each dimension is under the 1%. However, this good behavior is not maintained when the probability of demand increases. We conjecture that this is due to the aggregation of the capacities of the open plants in this lower bound. Since the probability of incurring in penalties for leaving customers unserviced increases with the probability of demand, the effect of this aggregation also becomes stronger. We can observe, however, that the dimension of the problems does not have any influence in the quality of the bounds.

Unfortunately, the results obtained with the lower bound on the expected routing cost, $\underline{z}_{TSP} - \bar{z}_{RS}$ are not satisfactory since in a considerable number of instances, $\underline{z}_{TSP} < \bar{z}_{RS}$. We attribute this behavior to the following facts:

– On the one hand, as stated in Powell, Jaillet, and Odoni (1995) the solution of a deterministic TSP can be arbitrarily bad as a solution for the probabilistic one. To illustrate this, in Figure 5.5, we compare the routes obtained with the heuristic with the routes defined by the solution of the deterministic ATSP that gives the lower bound. In Figure 5.5a) we show their *a priori* costs whereas the expected costs are depicted in Figure 5.5b). In each case, only the instances in groups SS1a and SS2a, with $p = 0.2$ have been considered.



a) *A Priori* Costs

b) Expected Costs

Figure 5.5: Route Costs

It can be observed that in some of the instances, the two costs of the same pair of routes compare differently. For example, in the fifth instance, the set of routes found by the heuristic is better than the one found with the bounding procedure, in terms of their expected costs. However, in terms of their *a priori* costs, the set of routes of the bounding procedure are better.

- On the other hand, routes found by the bounding procedure do not take into account the capacities of the plants and wether they are open or not. This explains why, in some cases, the expected cost of the routes found by the bounding procedure is smaller than the expected cost of the routes found by the heuristic.

- Finally, the fact that in the bounding procedure customers are not clustered but considered all in one single route, favors that in the computation of $\bar{z}_{RS}$, savings take very large values.

We conclude this section with an analysis of the quality of the function $\tilde{\mathcal{Q}}(x)$ as an approximation of $\mathcal{Q}(x)$. It is worth noting that each evaluation of $\mathcal{Q}(x)$ requires the computation of some of the probabilities P_l, P_l^p. In general, this would imply a considerable increase in the computational costs. However, the small sizes of the considered instances allow to compute and to store the values of all the probabilities P_l, P_l^p initially. Thus, it is possible to avoid computing them repeatedly during the process. Proceeding in this way, we have implemented a version of the proposed heuristic that computes $\mathcal{Q}(x)$ exactly. In what follows, we will use EX to denote this version of the heuristic and APP to denote the one that uses the approximation. Very often, both versions gave the same solution but in some cases EX gave better results. However, there are also an important number of instances where results provided by APP are the best ones. Table 5.2 shows the number of instances falling in each category, for different values of p. Column 2 gives the number of instances where EX provided the best results, Column 3 corresponds to instances where both versions gave the same solution, and Column 4 gives the number of instances where APP gave the best solution. The results in this table

Table 5.2: Compariosn of APP with EX

	EX	=	APP
p=0.2	43	80	2
p=0.5	25	80	20
p=0.8	47	54	24

are satisfactory: for more than half of the instances using the approximate objective function did not have any effect on the solution found, and the number of instances where APP gave the best solutions is considerable. In Table 5.3 we show the average percent deviations of the value of the solutions obtained with APP from the ones obtained with EX. Each column corresponds to a value of

Table 5.3: Average Percent Deviations of APP with Respect to EX

	p=0.2	p=0.5	p=0.8
SS1	0.031	0.070	-1.382
SS2	0.000	-0.032	0.356
SS3	0.183	-0.023	0.112
SM2	22.005	2.017	3.864
SM3	20.428	2.936	-5.860

the probability of demand and rows are related to different instance dimensions. It can be observed that, in general, the use of the approximation of the objective function does not affect to the quality of the final solution, the exception being five instances, that give the high values in the last two rows of column "$p = 0.2$". The results in Table 5.3 can be further appreciated in Figure 5.6. Similarly to

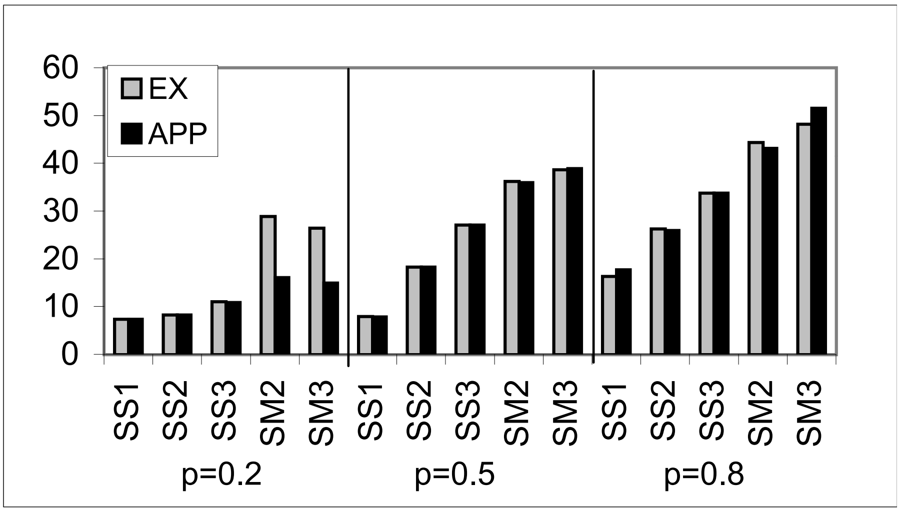


Figure 5.6: Comparison of the Improvements Obtained with EX and APP

.

Figure 5.4, we have computed the average percent improvements of the objective function value, with respect to the initial solution found. Averages are taken over all the instances of the same dimension with the same probability of demand. These values are shown both, for EX (in light grey) and for APP (in black). For a fair comparison, given that both versions start from the same solution, the improvements have been computed relative to the first solution. Again we can observe that both versions behave similarly in most of the groups, with the exception of SM2 and SM3 for $p = 2$.

The CPU times required by the two versions are compared in Figure 5.7. For a great proportion of instances, both versions ended before one second. In general terms, both versions required comparable

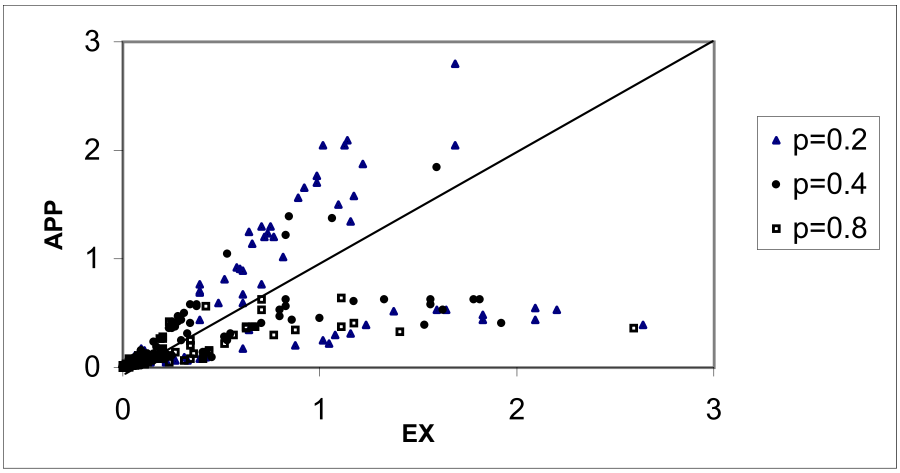


Figure 5.7: CPU Times (in Seconds) Required by the Heuristic

.

CPU times. Despite not having to compute the probabilities P_l, P_l^p repeatedly in EX, the set of instances where APP was faster is considerable. The reason is that in those instances, the LS performed by EX explored a larger set of solutions. We can conclude that the approximation we have proposed for the recourse function seems appropriate in this context. On the one hand, solutions obtained by APP where most often similar in terms of quality. On the other hand, note that this implementation of EX is not extensible to larger instances, since then the storage of the probabilities would not be affordable.

5.5 Conclusions

In this chapter we have presented a capacitated version of the SLRP where customers' demands are modeled as random variables with Bernoulli distribution. For this problem, a two stage model is considered. The set of plants to open and a set of *a priori* routes are chosen at a first stage and, once the values of the demands become available, the routes are modified by skipping some of the customers. For each plant, when its capacity does not allow to service all its allocated clients with demand, a random subset of them is chosen. Penalties are paid for those customers whose request for service is unattended.

We have presented a two phase heuristic to solve this SLRP. It consists of a first constructive phase, that progressively builds a feasible solution by solving a succession of subproblems, and an improving phase, where LS is performed. During this LS phase, an approximation of the recourse function is used whose evaluation requires a smaller computational effort. In the computational experiences, the effect of using this approximation is in general small.

We also propose a lower bound, obtained from bounding separately different parts of the objective function. The results of the part corresponding to the fixed costs for opening plants plus the expected penalties for unserviced customers are quite satisfactory. Unfortunately, the results of the part related to the routes are less encouraging and further research is needed.