Universitat Autònoma de Barcelona

Singular integral operators and rectifiability

Petr Chunaev

Universitat Autònoma de Barcelona
Departament de Matemàtiques

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This thesis is presented to obtain the degree of PhD in Mathematics

Petr Chunaev
May 2018

We certify that this thesis was written by Petr Chunaev under our supervision.

Advisor Juan Eugenio Mateu Bennassar   Advisor and tutor Xavier Tolsa Domènech

Barcelona, May 2018

Acknowledgements

First of all, I would like to thank my advisors, Joan Mateu and Xavier Tolsa, not only for their wise and patient mentoring but also for their readiness to help me in overcoming numerous obstacles that a foreigner and a young mathematician can face. Moreover, I am very grateful for the financial support associated with the European Research Council Advanced Grant 320501 "Geometric Analysis in the Euclidean Space" (Principal investigator Xavier Tolsa) that made possible my research and life in Barcelona and visiting numerous scientific events all over the world.

Also, I am very thankful to the Departament de Matemàtiques for providing wonderful condition for my work and always friendly attitude towards me.

Additionally, I would like to express my sincere gratitude to Pertti Mattila for the possibility of my research stay at the University of Helsinki from January to April, 2018, and for the financial support provided by the Academy of Finland.

Furthermore, special thanks are to my mentors and colleagues Vladimir Danchenko, Lyudmila Morozova and Sergey Tikhonov who helped and influenced me greatly at earlier stages and who continue supporting me in different instances within and beyond my scientific work now.

I am also grateful to many young colleagues whom I met along the way and who became role models of a productive and passionate mathematician for me, in particular, to Jonas Azzam, Murilo Rodolfo Candido, Mihalis Mourgoglou, Martí Prats, Tuomas Orponen and Malik Younsi.

What is more, I would like to thank my friends around the world for all their encouragement, enthusiasm and faithfulness. Among others, I thank Kirill Angel, Marta Bergamo, Viktoria Bolshakova, Murilo Rodolfo Candido, Clara Darnet, Mariya Fabia, Elisa Giambra, Maria Celeste Gritti, Ainur Jumabayeva, Anton Lehnen, Roman di Liukshini, Mohammad Mahdavipour, Sergey Margolin, Nathália Oliveira, Paweł Rachel, Claudio Scalia, Mikkel Thøgersen and Aleksander Tyshchenko for the days spent together in the Vila Universitària of the Universitat Autònoma de Barcelona.

Last but not the least, I must express my profound gratitude to my mother, Svetlana, and my wife, Kseniya, for providing me with unfailing support, continuous encouragement and unconditional love throughout my years of research and living abroad (and other times of course). This accomplishment would not have been possible without them. Finally, I am grateful to my late father, Vladimir, and brother, Anton, who would be proud of who I have become and whose invisible support helps me to move forward.

Contents

Introduction

The problems that we study in this thesis lie in the area of Harmonic Analysis and Geometric Measure Theory. More precisely, we consider the connection between the analytic properties of singular integral operators defined in $L^2(\mu)$ and the geometric properties of the measure μ. In the forthcoming several pages we will make a historical review of the topic, give necessary definitions and briefly discuss the results that we obtained. Let us mention that the exposition in this thesis is based on the papers [Chu, CMT1, CMT2].

We start with necessary notation and background facts. Note that we work mostly in the plane and therefore usually skip dimension markers in definitions.

Let $E \subset \mathbb{C}$ be a Borel set and $B(z,r)$ be an open disc with center $z \in \mathbb{C}$ and radius $r > 0$. We denote by $\mathcal{H}^1(E)$ the (1-dimensional) *Hausdorff measure* of E, i.e. *length*, and call E a 1-*set* if $0 < \mathcal{H}^1(E) < \infty$. A set E is called *rectifiable* if it is contained in a countable union of Lipschitz graphs, up to a set of $\mathcal{H}^1$-measure zero. A set E is called *purely unrectifiable* if it intersects any Lipschitz graph in a set of $\mathcal{H}^1$-measure zero.

By a *measure* often denoted by μ we mean a *positive locally finite Borel measure on* $\mathbb{C}$.

We consider *Calderón-Zygmund* (*CZ*) *kernels* $K : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with the following properties: there exist constants $C > 0$ and $\eta \in (0, 1]$ such that $|K(z)| \leqslant C|z|^{-1}$ for all $z \in \mathbb{C} \setminus \{0\}$, and moreover

$$|K(z) - K(z+\zeta)| \leqslant C\frac{|\zeta|^\eta}{|z|^{1+\eta}} \quad \text{if} \quad |\zeta| \leqslant \tfrac{1}{2}|z|, \qquad z, \zeta \in \mathbb{C}. \tag{0.1}$$

We will mostly deal with -1-homogeneous CZ kernels K.

Given a measure μ, a CZ kernel K and an $f \in L^1(\mu)$, we define a truncated *singular integral operator* (*SIO*) as

$$T_{K,\varepsilon} f(z) := \int_{E \setminus B(z,\varepsilon)} f(\zeta) K(z-\zeta)\, d\mu(\zeta), \qquad \text{where } E = \operatorname{spt}\mu \text{ and } \varepsilon > 0. \tag{0.2}$$

We do not define the SIO T_K explicitly because several delicate problems such as the existence of the *principal value* (p.v.), i.e. $\lim_{\varepsilon\to 0} T_{K,\varepsilon} f(z)$, might arise. On the contrary, the integral in (0.2) always converges absolutely. Nevertheless, we say that T_K is $L^2(\mu)$-*bounded* if the operators $T_{K,\varepsilon}$ are $L^2(\mu)$-bounded uniformly on ε.

How to relate the $L^2(\mu)$-boundedness of a certain SIO to the geometric properties of the support of μ is an old problem in Harmonic Analysis. It stems from Calderón's paper [Cal] where it is proved that the Cauchy transform, i.e. the SIO T_K with $K(z) = 1/z$, is $L^2(\mathcal{H}^1\lfloor E)$-bounded if E is a Lipschitz graph with small slope. Later on, Coifman, McIntosh and Meyer [CMM] removed the small Lipschitz constant assumption and thus extended the result to all Lipschitz graphs. Furthermore, in [Dav1] David fully characterized rectifiable *curves* Γ for which the Cauchy transform is $L^2(\mathcal{H}^1\lfloor\Gamma)$-bounded. Namely, they have to satisfy the *linear growth* condition:

$$\mathcal{H}^1(\Gamma \cap B(z,r)) \leqslant Cr \quad \text{for all } r > 0 \text{ and } z \in \mathbb{C} \text{ and some } C > 0. \tag{0.3}$$

These results led to further development of tools for understanding the above-mentioned problem (see also the corresponding parts of [Chr, DS2, Mat1, Mat2, Mat3, Tol5]).

A new quantitative characterization of rectifiability in terms of the so-called *β-numbers* introduced by Jones [Jon1] and the concept of *uniform rectifiability* proposed by David and Semmes [DS1, DS2] are among these tools. Several related definitions for the plane are in order. (We refer the reader to [DS1, DS2] for definitions and results in the multidimensional case). A measure μ is called *Ahlfors-David regular* (or *AD-regular*, for short) if

$$C^{-1}r \leqslant \mu(B(z,r)) \leqslant Cr \quad \text{for all } z \in \operatorname{spt}\mu \text{ and } r \in (0, \operatorname{diam}(\operatorname{spt}\mu)), \tag{0.4}$$

where $C > 0$ is a fixed constant. Moreover, μ is called *uniformly rectifiable* if it is AD-regular and $\operatorname{spt}\mu$ is contained in a curve satisfying (0.3).

The well-known *David-Semmes problem* is stated in the plane as follows: *does the $L^2(\mu)$-boundedness of the Cauchy transform is sufficient for the uniform rectifiability of the AD-regular measure μ?* This problem was settled by Mattila, Melnikov and Verdera:

Theorem A [MMV]. *Let μ be an AD-regular measure. The measure μ is uniformly rectifiable if and only if the Cauchy transform is $L^2(\mu)$-bounded.*

Note that an analogous problem in higher dimensions is still unsolved except for the case of codimension 1 settled by Nazarov, Tolsa and Volberg in [NTV] not long ago.

The proof of Theorem A relied on the so-called *curvature* (or *symmetrisation*) *method* that was new at that time but soon became very influential in solving many long-standing problems related to the Cauchy transform and analytic capacity, for example, Painlevé's problem on metric/geometric description of removable singularities for bounded analytic functions, Vitushkin's conjecture and the semiadditivity of analytic capacity (see [Dav2, MMV, Mel, Tol1] and especially historical remarks in [Tol5]). For our purposes it is more convenient to describe a generalised version of the curvature method and so do we.

Let $(z_1, z_2, z_3) \in \mathbb{C}^3$. For a kernel K, consider the following *permutations*:

$$P_K(z_1, z_2, z_3) := \sum_{s \in \mathfrak{S}_3} K(z_{s_2} - z_{s_1})\overline{K(z_{s_3} - z_{s_1})}, \tag{0.5}$$

where $\mathfrak{S}_3$ is the group of permutations of the three elements $\{1, 2, 3\}$. Supposing that μ_1, μ_2 and μ_3 are measures, set

$$P_K(\mu_1, \mu_2, \mu_3) := \iiint P_K(z_1, z_2, z_3)\, d\mu_1(z_1)\, d\mu_2(z_2)\, d\mu_3(z_3). \tag{0.6}$$

We write $P_K(\mu) := P_K(\mu, \mu, \mu)$ for short and call it *permutation of the measure μ*. Moreover, in what follows $P_{K,\varepsilon}(\mu_1, \mu_2, \mu_3)$ stands for the integral in the right hand side of (0.6) defined over the set

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_k - z_j| \geqslant \varepsilon > 0, \quad 1 \leqslant k, j \leqslant 3, \quad j \neq k\}, \tag{0.7}$$

and $P_{K,\varepsilon}(\mu) := P_{K,\varepsilon}(\mu, \mu, \mu)$.

The identities (0.5) and (0.6) were first considered by Melnikov [Mel] in the case of the Cauchy kernel. He showed that

$$P_K(z_1, z_2, z_3) = c(z_1, z_2, z_3)^2, \tag{0.8}$$

where $K(z) = 1/z$,

$$c(z_1, z_2, z_3) := \frac{1}{R(z_1, z_2, z_3)}$$

is the so called *Menger curvature* and $R(z_1,z_2,z_3)$ stands for the radius of the circle passing through z_1, z_2 and z_3. Moreover, points z_1, z_2 and z_3 are collinear if and only if $c(z_1,z_2,z_3)=0$ (and $R(z_1,z_2,z_3)=\infty$). From what is said it is clear that

$$c(z_1,z_2,z_3)\geqslant 0 \qquad \text{for any } (z_1,z_2,z_3)\in\mathbb{C}^3, \tag{0.9}$$

which is very important in applications. Additionally, it is easily seen that Menger curvature can be calculated in geometrical terms in different ways, e.g.

$$c(z_1,z_2,z_3)=\frac{4S(z_1,z_2,z_3)}{|z_1-z_2||z_1-z_3||z_2-z_3|}=\frac{2\sin\widehat{z_1z_2z_3}}{|z_1-z_3|}, \tag{0.10}$$

where $S(z_1,z_2,z_3)$ stands for the area of the triangle (z_1,z_2,z_3) and $\widehat{z_1z_2z_3}$ is the angle of this triangle opposite to the side z_1z_3. One can find the proof of these and other formulas, e.g. in [Tol5, Section 3.2].

Melnikov also introduced in [Mel] the notion of *curvature of measure* μ:

$$c^2(\mu):=P_K(\mu), \quad \text{where } K(z)=1/z. \tag{0.11}$$

One can also define $c^2_\varepsilon(\mu)$ in an obvious way over the set (0.7).

Permutations (0.5) and (0.6) for more general kernels K were considered later by Chousionis, Mateu, Prat and Tolsa in [CMPT1] (see also [CMPT2]).

Now let K be a -1-homogeneous CZ kernel, see before (0.1). Suppose that the permutations (0.5) for K are non-negative for any $(z_1,z_2,z_3)\in\mathbb{C}^3$. If μ is a finite measure with C_**-linear growth*, i.e. there exists a constant $C_*>0$ such that

$$\mu(B(z,r))\leqslant C_*r \qquad \text{for all } r>0 \text{ and } z\in\operatorname{spt}\mu, \tag{0.12}$$

then the following relation between truncated versions of $P_K(\mu)$ and T_K1 holds:

$$\|T_{K,\varepsilon}1\|^2_{L^2(\mu)}=\tfrac{1}{6}P_{K,\varepsilon}(\mu)+\mathcal{R}_{K,\varepsilon}(\mu), \qquad |\mathcal{R}_{K,\varepsilon}(\mu)|\leqslant c\,C_*^2\mu(\mathbb{C}), \tag{0.13}$$

with some $c>0$ independent of ε. We call (0.13) *generalised Melnikov-Verdera's identity.* Actually, it was first proved for the Cauchy kernel in the seminal paper [MV] by Melnikov and Verdera where they gave a new geometric proof of the above-mentioned result about the L^2-boundedness of the Cauchy transform on Lipschitz graphs. It turns out that one can follow Melnikov-Verdera's proof for the Cauchy kernel in order to obtain (0.13) as stated, see e.g. [CMPT2, Section 2].

The formulas (0.8) and (0.13), generating the *curvature method*, are remarkable in the sense that they relate an analytic notion (the SIO T_K, in particular, the Cauchy transform) with a metric-geometric one (permutations, in particular, curvature).

Later on, Theorem A was pushed even further by David and Léger [Leg]. They used the curvature method and, in particular, the property (0.9) to prove the following deep result (see a brief exposition of the proof in Section 1.5).

Theorem B [Leg]. *Let E be a 1-set. If $c^2(\mathcal{H}^1\lfloor E)<\infty$, then E is rectifiable. Moreover, if the Cauchy transform is $L^2(\mathcal{H}^1\lfloor E)$–bounded, then E is rectifiable.*

Note that the $L^2(\mathcal{H}^1\lfloor E)$–boundedness of the Cauchy transform and the identity (0.13) imply that $c^2(\mathcal{H}^1\lfloor E)<\infty$. Consequently, to prove Theorem B it is enough to prove just the first statement and this was actually done in [Leg]. Let us mention that Theorem B is originally stated in [Leg] in a more general form.

For a long time very few results analogous to Theorem B were known for kernels different from $1/z$. For instance, it was clear [MMV, CMPT1] that the same result is true for the coordinate parts of the Cauchy kernel, in particular, for the real part that we denote by

$$K_{\mathsf{DL}}(z) := \frac{\operatorname{Re} z}{|z|^2} \equiv \operatorname{Re} \frac{1}{z}. \tag{0.14}$$

Indeed, considering the permutations (0.5), it is not difficult to show that

$$P_{K_{\mathsf{DL}}}(z_1, z_2, z_3) = \tfrac{1}{2} c(z_1, z_2, z_3)^2. \tag{0.15}$$

This fact was a motivation point of the paper [CMPT1] by Chousionis, Mateu, Prat and Tolsa, where analogues of Theorems A and B were proved for the kernels

$$\kappa_n(z) := \frac{(\operatorname{Re} z)^{2n-1}}{|z|^{2n}}, \qquad n \in \mathbb{N}. \tag{0.16}$$

(We also refer the reader to the paper [CP] where the kernels (0.16) and other related ones were studied.) Namely, it is shown in [CMPT1] that

$$P_{\kappa_n}(z_1, z_2, z_3) \geqslant 0 \qquad \text{for any } (z_1, z_2, z_3) \in \mathbb{C}^3 \tag{0.17}$$

and $P_{\kappa_n}(z_1, z_2, z_3) = 0$ if and only if points z_1, z_2 and z_3 are collinear. This is an analogue of (0.9). Moreover, it is proved that the permutations $P_{\kappa_n}(z_1, z_2, z_3)$ behave similarly to $c(z_1, z_2, z_3)^2$ for triangles (z_1, z_2, z_3) with comparable sides such that one side makes a big angle with the vertical line. This fact enables the authors of [CMPT1] to adapt the curvature method from [Leg] to the kernels κ_n. This adaptation however requires several essential modifications in crucial points, where the curvature must be exchanged for the permutations P_{κ_n}. In particular, new arguments are provided in [CMPT1] when Léger's scheme does not work (see Section 1.5 for more details). As a result, the following statements were proved.

Theorem C [CMPT1]. *Let $n \geqslant 1$ and μ be an AD-regular measure. The measure μ is uniformly rectifiable if and only if T_{κ_n} is $L^2(\mu)$-bounded.*

Theorem D [CMPT1]. *Let $n \geqslant 1$ and E be a 1-set. If $P_{\kappa_n}(\mathcal{H}^1\lfloor E) < \infty$, then E is rectifiable. Moreover, if the SIO T_{κ_n} is $L^2(\mathcal{H}^1\lfloor E)$–bounded, then E is rectifiable.*

Obviously, for $n = 1$ one gets the case of (0.14) (or equivalently the case of the Cauchy kernel due to (0.15)) and thus Theorems A and B. However, for $n \geqslant 2$ these are the first examples of SIOs that are not directly related to the Cauchy transform and whose $L^2(\mu)$-boundedness implies certain rectifiability properties of μ. In what follows, we will typically deal with the case $n = 2$ in (0.16) and therefore we set

$$K_{\mathsf{CMPT}}(z) := \frac{(\operatorname{Re} z)^3}{|z|^4}. \tag{0.18}$$

There are two more results which are actually counterexamples to the property that L^2-boundedness implies rectifiability. The first one is due to Huovinen.

Theorem E [Huo]. *Let K belong to the class $\mathscr{H}$ of odd kernels satisfying*

$$|K(x-y) - K(x-z)| \leqslant \frac{C\,|y-z|}{|x-y||x-z|}, \qquad |K(z)| \leqslant \frac{C}{|z|}, \qquad x, y, z \in \mathbb{C},$$
$$K(r) = 0, \qquad K(z) = -K(-\overline{z}), \qquad r \in \mathbb{R}, \qquad C = \text{const}.$$

Then there exists a purely unrectifiable 1-set E such that the operator T_K, associated with the kernel K, is bounded on $L^2(\mathcal{H}^1\lfloor E)$ and, moreover, p.v. T_K *exists $\mathcal{H}^1$-a.e. and is finite.*

As a typical example from $\mathscr{H}$ one can take the kernel

$$K_{\mathsf{H}}(z) := \frac{(\operatorname{Re} z)^3}{|z|^4} - \frac{\operatorname{Re} z}{|z|^2}. \tag{0.19}$$

Another counterexample is due to Jaye and Nazarov. We formulate it here in a slightly different form than it was done originally.

Theorem F [JN]. *There exists a purely unrectifiable* 1*-set* E *such that the operator* T_K*, associated with the kernel* $K(z) = \overline{z}/z^2$*, is bounded on* $L^2(\mathcal{H}^1\lfloor E)$ *and, moreover,* p.v. T_k *fails to exist* $\mathcal{H}^1$*-a.e.*

For further exposition it is important to notice that the same statement holds for the (renormalised) real part of $K(z) = \overline{z}/z^2$, namely, for the kernel

$$K_{\mathsf{JN}}(z) := \frac{1}{4}\frac{\overline{z}}{z^2} = \frac{(\operatorname{Re} z)^3}{|z|^4} - \frac{3}{4}\frac{\operatorname{Re} z}{|z|^2}. \tag{0.20}$$

Take into account that the examples of purely unrectifiable sets from Theorems E and F are quite intricate, essentially use analytic properties of the kernels and avoid the curvature method (as it is actually unavailable as we will see below).

Note that up to now we meant only -1-homogeneous CZ kernels in the plane which we are mostly interested in. However, there are other interesting examples of kernels in different settings with a certain connection between the $L^2(\mu)$–boundedness of the corresponding operator and the geometry of μ. For example, David [Dav3] constructed odd and even CZ kernels in the plane that define $L^2(\mathcal{H}^1\lfloor E)$–bounded SIOs, where E is a purely unrectifiable set (namely, the corner quarters Cantor set). Unfortunately, David's kernels are *not* -1*-homogeneous.* Chousionis [Cho] had similar results for a larger class of kernels for some s-dimensional fractals with $s < 1$. Recently Chousionis and Li [CL] gave examples of -1-homogeneous kernels in the *Heisenberg group* with a direct connection between the $L^2(\mu)$–boundedness of the corresponding singular integral operators and the rectifiability of μ. Unlike the Euclidean case, where most of kernels related to rectifiability are odd, Chousionis-Li's kernels defined in the Heisenberg group are even and non-negative.

Now we are going to summarise the above-mentioned results for -1-homogeneous kernels in the plane (defined in (0.14) and (0.18)–(0.20)) and formulate more precisely the questions that we consider in this thesis. From what is said we conclude that an affirmative answer to the question *does the* $L^2(\mathcal{H}^1\lfloor E)$*–boundedness of* T_K *implies that the* 1*-set* E *is rectifiable* is given by curvature-like methods essentially only for the SIOs associated with the kernels K_{DL} and K_{CMPT}. On the other hand, only two kernels, $K = K_{\mathsf{H}}$ and $K = K_{\mathsf{JN}}$, are known to be such that T_K is still $L^2(\mathcal{H}^1\lfloor E)$–bounded for some purely unrectifiable 1-sets E (the curvature method is not available in both cases). Thus there is a natural problem consisting in finding other -1-homogeneous CZ kernels in the plane and corresponding SIOs whose $L^2(\mu)$–boundedness does or does not imply that μ is rectifiable (uniformly rectifiable in the AD-regular case). Of course, we also want to partly move forward in characterizing the class of such kernels but this seems to be a much more ambitious and difficult objective so we will make just a few remarks on it. Another related problem that we will briefly discuss is whether $L^2(\mu)$-boundedness of a certain SIO (besides the Cauchy transform) implies that all other "reasonable" SIOs in the plane are $L^2(\mu)$–bounded.

To move in achieving the above-mentioned objectives, in Chapter 1 we introduce the following t-parametric family of kernels:

$$k_t(z) := \frac{(\operatorname{Re} z)^3}{|z|^4} + t\cdot\frac{\operatorname{Re} z}{|z|^2}, \quad \text{where } t \in \mathbb{R}, \quad \text{and} \quad k_\infty(z) := \frac{\operatorname{Re} z}{|z|^2}. \tag{0.21}$$

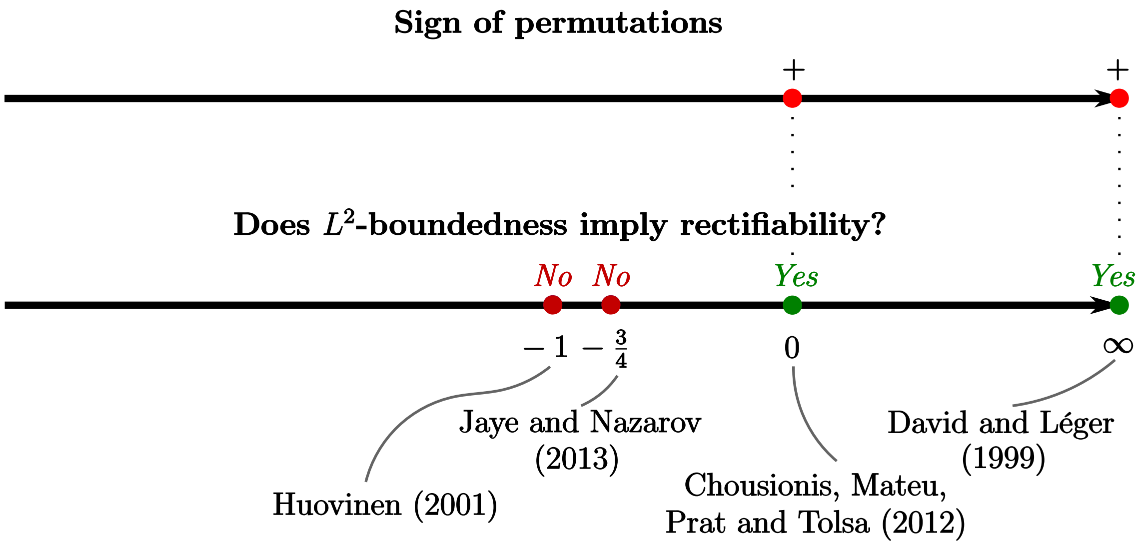

Figure 0.1: The overall picture of previously known results for the kernels k_t.

One can immediately notice that this is a reasonable generalisation of the kernels (0.14) and (0.18)–(0.20), see Figure 0.1. Indeed,

$$K_{\mathsf{DL}} \equiv k_\infty, \qquad K_{\mathsf{CMPT}} \equiv k_0, \qquad K_{\mathsf{H}} \equiv k_{-1} \qquad \text{and} \qquad K_{\mathsf{JN}} \equiv k_{-3/4}.$$

Thus studying SIOs associated with the kernels (0.21) for different t may provide useful settings to further understanding the connection between the L^2–boundedness and rectifiability. Now we give an overall picture of the results that we obtained in this direction. To avoid repetitions, we do not number the statements here but indicate the corresponding results in further chapters.

In Chapter 1 we study the permutations associated with the kernels (0.21) and prove the following result that complements the ones for $t = 0$ and $t = \infty$ (see Theorem 1.1 and Remark 1.1).

Theorem. *Let k_t be a kernel of the form* (0.21)*, where $t \in (-\infty, -2] \cup (0, +\infty)$. Then*

$$P_{k_t}(z_1, z_2, z_3) \geqslant 0 \quad \textit{for all} \quad (z_1, z_2, z_3) \in \mathbb{C}^3.$$

Furthermore, the range of the parameter t is sharp in the sense that there are triples (z_1, z_2, z_3) such that $P_{k_t}(z_1, z_2, z_3)$ change sign if $t \in (-2, 0)$.

This theorem implies that the curvature method is not available directly if $t \in (-2, 0)$, and surprisingly exactly this interval contains the known counterexamples from Theorems E and F. However, by adapting a curvature-type method from [Leg, CMPT1] we will prove the following result for the rest values of t that provides a big family of SIOs whose L^2-boundedness implies rectifiability (see Theorem 1.2).

Theorem. *Let k_t be a kernel of the form* (0.21)*, where $t \in (-\infty, -2] \cup (0, +\infty)$, and E a 1-set. If $P_{k_t}(\mathcal{H}^1 \lfloor E) < \infty$, then E is rectifiable. Moreover, if the SIO T_{k_t} is $L^2(\mathcal{H}^1 \lfloor E)$–bounded, then E is rectifiable.*

What is more, the situation in the plane where $t \in (-2, 0)$ is somehow similar to the one in higher dimensions where curvature-type methods are *likely unavailable* (see [NTV, Far]) and one has to come up with other tools instead. As for our case, in Chapter 2 we propose a *perturbation method* that enables us to prove the following result, being the first example in the plane when the curvature method fails but it is still possible to prove that L^2-boundedness implies rectifiability (see Theorem 2.2).

Theorem. *Let k_t be a kernel of the form* (0.21)*, where $t \in (-2, -\sqrt{2})$, and E a* 1*-set. If the SIO T_{k_t} is $L^2(\mathcal{H}^1\lfloor E)$–bounded, then E is rectifiable.*

The proof of this result is based on the following (see Lemma 2.2 and Remark 2.2).

Lemma. *It holds that*

$$P_{k_0}(z_1, z_2, z_3) \leqslant 2P_{k_\infty}(z_1, z_2, z_3) \qquad \text{for all} \qquad (z_1, z_2, z_3) \in \mathbb{C}^3.$$

Additionally, the constant 2 *is sharp in the sense that for any small $\varepsilon > 0$ there are triples (z_1, z_2, z_3) such that $P_{k_0}(z_1, z_2, z_3)/P_{k_\infty}(z_1, z_2, z_3) \geqslant 2 - \varepsilon$.*

Note that this lemma with an implicit constant is contained in [CMPT2, Lemma 7]. Nevertheless, the explicitness of our constant is essential here and actually enables us to obtain the required result by the perturbation method in Chapter 2.

The latter lemma yields that $P_{k_0,\varepsilon}(\mu) \leqslant 2P_{k_\infty,\varepsilon}(\mu)$ for any measure μ. This and (0.13) immediately imply that the L^2-norm of $T_{k_0}1$ is controlled by the L^2-norm of $T_{k_\infty}1$ (with an explicit constant). The triangle inequality and a proper version of $T1$-theorem which form the perturbation method then give the required theorem.

Unfortunately, an inequality of the form $P_{k_\infty}(z_1, z_2, z_3) \leqslant C \cdot P_{k_0}(z_1, z_2, z_3)$ with an absolute constant $C > 0$ hardly exists and this makes the study of kernels (0.21) for small negative t more difficult. Nevertheless, the following lemma holds as shown in Chapter 3 (see Inequality (3.4)).

Lemma. *There exist absolute constants $t_0 > 0$ and $c > 0$ such that for any finite measure μ with C_*-linear growth it holds that*

$$P_{k_\infty}(\mu) \leqslant t_0^{-2} P_{k_0}(\mu) + cC_*^2\mu(\mathbb{C}).$$

In a similar manner as in Chapter 2, from the latter lemma we get that the L^2-norm of $T_{k_\infty}1$ (and thus of the Cauchy transform of measure) is controlled by the L^2-norm of $T_{k_0}1$ (see Theorem 3.1). Applying the perturbation method yields the following theorem using the same absolute constant t_0 as above (see Theorem 3.2).

Theorem. *Let k_t be a kernel of the form* (0.21)*, where $t \in (-t_0, 0)$, and E a* 1*-set. If the SIO T_{k_t} is $L^2(\mathcal{H}^1\lfloor E)$-bounded, then E is rectifiable.*

To prove the latter Lemma and Theorem, we use the David-Mattila dyadic lattice from [DM] and a corona decomposition that is similar to the one in [AT]. More precisely, we split the lattice into some collections of cubes, which are called *trees*, where the density of μ does not oscillate too much and most of the measure is concentrated close to the graph of a Lipschitz function. To construct this function, we use a variant of the Whitney extension theorem adapted to the David-Mattila dyadic lattice. Further, we show that the family of trees of the corona decomposition satisfies a packing condition by arguments inspired by some of the techniques used in [AT] and earlier in [Tol3] to prove the bilipschitz "invariance" of analytic capacity.

It is worth mentioning that the structure of our trees is more complicated than in [AT]. This is because we deal with permutations which are not pointwise comparable to curvature in general and this leads to additional technical difficulties.

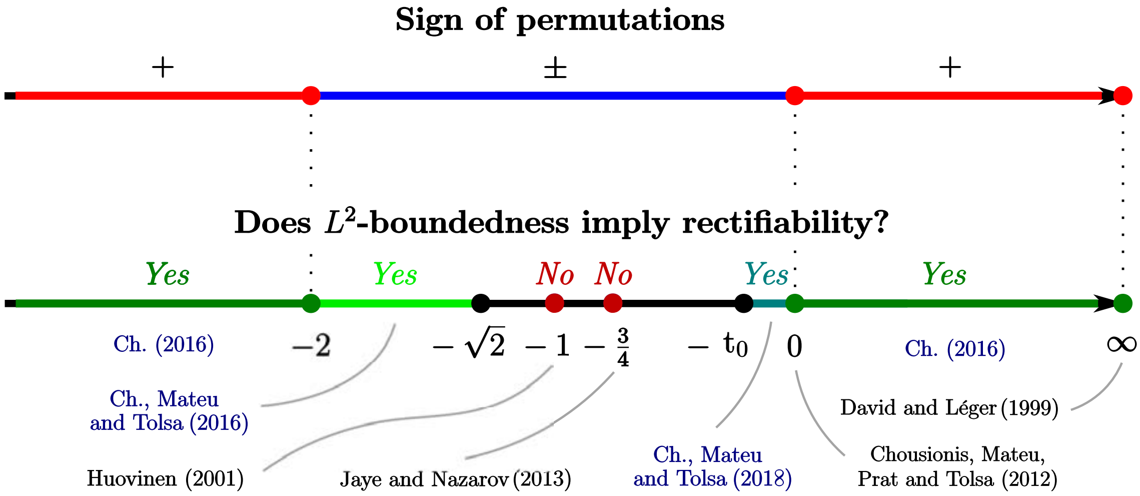

Figure 0.2: The overall picture of known results for the kernels k_t.

Finally, we summarise all the results known for the kernels k_t in Figure 0.2, cf. Figure 0.1.

Let us also mention that our results in Chapters 1–3 (more precisely, Lemmas 1.5, 1.6 and 2.3 and Theorem 3.1) together with the obvious case $t = \infty$ imply the following necessary and sufficient condition.

Theorem. *Let μ be a measure with linear growth and*

$$t \in (-\infty, -\sqrt{2}) \cup (-t_0, \infty]. \tag{0.22}$$

The Cauchy transform is $L^2(\mu)$-bounded if and only if so is the SIO T_{k_t}.

This has further consequences. For example, one can characterise uniformly rectifiable measures via L^2–bounded SIOs T_{k_t} (see Theorems 2.1, 2.3 and Corollary 3.1).

Corollary. *Let μ be an AD-regular measure and k_t a kernel of the form* (0.21) *with t as in* (0.22)*. The measure μ is uniformly rectifiable if and only if the SIO T_{k_t} is $L^2(\mu)$-bounded.*

The part of this Corollary for $t = \infty$, i.e. in fact for the Cauchy transform, was proved in [MMV] (see Theorem A) and for $t = 0$ in [CMPT1] (see Theorem C).

The following result is from Section 2.5 and Corollary 3.2.

Corollary. *Let μ be a measure with linear growth and k_t a kernel of the form* (0.21) *with t as in* (0.22)*. If the SIO T_{k_t} is $L^2(\mu)$-bounded, then so are all* 1*-dimensional SIOs associated with a wide class of kernels.*

By "a wide class of kernels" we mean the class of kernels $K = K(z)$, where $z \in \mathbb{C} \setminus \{0\}$, which are odd $\mathcal{C}^2$–functions satisfying

$$\left|\nabla^j K(z)\right| \leqslant \frac{C(j)}{|z|^{1+j}} \qquad \text{for all } z \in \mathbb{C} \setminus \{0\} \text{ and } j \in \{0, 1, 2\}. \tag{0.23}$$

It is easily seen that (0.23) implies that K is a CZ kernel with the properties indicated before (0.1), where $\eta = 1$.

The latter Corollary for $t = \infty$, i.e. actually for the Cauchy transform, was earlier proved in [Tol2, Tol4], see also [Gir].

It is worth mentioning that some of our results stated above are actually proved in Chapters 1–3 not just for k_t but for more general t-parametric kernels:

$$K_t(z) := \kappa_N(z) + t \cdot \kappa_n(z), \;\; t \in \mathbb{R}, \quad K_\infty(z) := \kappa_n(z), \quad n \leqslant N, \;\; n, N \in \mathbb{N}, \tag{0.24}$$

i.e. linear combinations of the kernels (0.16) of different order. Clearly, $K_t(z) \equiv k_t(z)$ for $(n, N) = (1, 2)$. For the sake of simplicity, we do not state the corresponding results in Introduction but thoroughly discuss them in the subsequent chapters (see Theorems 1.1, 1.2, 2.3, 3.3, 3.4 and 3.5).

A few words about notation. We use the letters c and C to denote constants which may change their values at different occurrences. On the other hand, constants with subscripts such as A_0 or c_1 do not change their values throughout *each chapter*. In a majority of cases constants depend on some parameters which are usually indicated explicitly, for instance, we write $C(\varepsilon)$ or C_ε if C depends on ε. If there is a constant C such that $A \leqslant C\,B$, we write $A \lesssim B$. Furthermore, $A \approx B$ is equivalent to saying that $A \lesssim B \lesssim A$, possible with different implicit constants. If the implicit constant in expressions with "$\lesssim$" or "$\approx$" depends on some positive parameter, say α, we write $A \lesssim_\alpha B$ or $A \approx_\alpha B$.

Equations, theorems, lemmas and other statements are numbered within a chapter. Introduction is considered as Chapter 0 for numbering.

Chapter 1

A new parametric family of singular integral operators whose L^2-boundedness implies rectifiability by a curvature-like method

1.1 Introduction

The exposition in this chapter is based on [Chu]. Below, we consider the SIOs associated with the kernels (0.24), i.e. with

$$K_t(z) = \frac{(\operatorname{Re} z)^{2N-1}}{|z|^{2N}} + t \cdot \frac{(\operatorname{Re} z)^{2n-1}}{|z|^{2n}} \quad \text{and} \quad K_\infty(z) = \frac{(\operatorname{Re} z)^{2n-1}}{|z|^{2n}}, \tag{1.1}$$

where n and N are positive integer numbers such that $N \geqslant n$, and $t \in \mathbb{R}$. We prove that the $L^2(\mathcal{H}^1\lfloor E)$-boundedness of these operators implies that E is rectifiable for $t \in \mathbb{R} \setminus (t_1, t_2)$ with certain t_1, t_2 depending only on n and N. We use the curvature method for this so all the definitions related to curvature and permutations given in Introduction will be used in this chapter. We nevertheless remind some of them and introduce new ones now.

For a CZ kernel K, we introduced the following *permutations* in (0.5):

$$P_K(z_1, z_2, z_3) = \sum_{s \in \mathfrak{S}_3} K(z_{s_2} - z_{s_1})\overline{K(z_{s_3} - z_{s_1})},$$

where $\mathfrak{S}_3$ is the group of permutations of three elements $\{1, 2, 3\}$. Note that the kernels (1.1) that we deal with in this chapter are real-valued so the bar in the latter sum may be skipped. For this reason it is more convenient for us to consider the permutations

$$\begin{aligned} &p_K(z_1, z_2, z_3) \\ &:= K(z_1 - z_2)K(z_1 - z_3) + K(z_2 - z_1)K(z_2 - z_3) + K(z_3 - z_1)K(z_3 - z_2) \\ &= \tfrac{1}{2} P_K(z_1, z_2, z_3), \end{aligned} \tag{1.2}$$

where K is an odd and real-valued kernel.

Taking into account the definitions (0.6) and (0.7) and supposing that μ, μ_1, μ_2 and μ_3 are measures, we also set

$$p_K(\mu_1, \mu_2, \mu_3) := \tfrac{1}{2} P_K(\mu_1, \mu_2, \mu_3), \qquad p_{K,\varepsilon}(\mu_1, \mu_2, \mu_3) := \tfrac{1}{2} P_{K,\varepsilon}(\mu_1, \mu_2, \mu_3) \tag{1.3}$$

and

$$p_K(\mu) := \tfrac{1}{2} p_K(\mu,\mu,\mu), \qquad p_{K,\varepsilon}(\mu) := \tfrac{1}{2} P_{K,\varepsilon}(\mu,\mu,\mu). \tag{1.4}$$

We will use (1.3) and (1.4) many times below without additional mentioning.

Recall that by a *measure* we mean a *positive locally finite Borel measure on* $\mathbb{C}$.

Bear in mind that (1.1) can be written in the form (0.24), i.e.

$$K_t(z) = \kappa_N(z) + t\cdot \kappa_n(z), \quad t \in \mathbb{R}, \qquad K_\infty(z) = \kappa_n(z), \quad n \leqslant N, \;\; n, N \in \mathbb{N},$$

where, according to (0.16),

$$\kappa_m(z) = \frac{(\operatorname{Re} z)^{2m-1}}{|z|^{2m}}, \qquad m \in \mathbb{N}.$$

Note also that in our new terms it follows from (0.17) and (1.3) that

$$p_{\kappa_m}(z_1, z_2, z_3) \geqslant 0 \qquad \text{for any} \quad (z_1,z_2,z_3) \in \mathbb{C}^3 \quad \text{and } m \in \mathbb{N},$$

and $p_{\kappa_m}(z_1,z_2,z_3) = 0$ if and only if z_1, z_2 and z_3 are collinear.

In the forthcoming sections, in order to find values of t such that a result analogous to Theorems B and D is valid for the kernels (1.1), we first study the sign of the permutations (1.2) and then, for the case when these permutations are non-negative, adapt the scheme from [CMPT1] to our situation.

1.2 Main results

First of all let us mention that the case $t = 0$ in the theorems below agrees with the inequality (0.17) and Theorem D, proved in [CMPT1]. We now indicate the values of t such that the permutations $p_{K_t}(z_1,z_2,z_3)$ are non-negative for all triples (z_1,z_2,z_3).

Theorem 1.1. *Let K_t be a kernel of the form* (1.1) *with $t = 0$ or*

$$t \in \mathbb{R} \setminus \left(-\tfrac{1}{2}\left(3 + \sqrt{9 - 4\tfrac{N}{n}}\right), 2 - \tfrac{N}{n}\right), \qquad n < N \leqslant 2n, \tag{1.5}$$

$$t \in \mathbb{R} \setminus \left(-\tfrac{1}{2}\left(3 + \rho_{n,N} + \sqrt{(3+\rho_{n,N})^2 - 4\tfrac{N}{n}}\right), \rho_{n,N}\right), \qquad N \geqslant 2n, \tag{1.6}$$

where $\rho_{n,N} := (\frac{N}{n} - 2)\sqrt{N - 2n}$. In particular, $t \in \mathbb{R} \setminus (-2; 0)$ for $N = 2n$. Then

$$p_{K_t}(z_1,z_2,z_3) \geqslant 0 \quad \textit{for all} \quad (z_1,z_2,z_3) \in \mathbb{C}^3.$$

Furthermore, the range of the parameter t in the case $N = 2n$ is sharp.

Remark 1.1. The conditions (1.5) and (1.6), guaranteeing that $p_{K_t}(z_1,z_2,z_3) \geqslant 0$, cannot be weakened much in the following sense. As we will show in Section 1.4, there are triples (z_1,z_2,z_3) such that $p_{K_t}(z_1,z_2,z_3)$ can change sign if

$$\begin{aligned} &t \in (-\tfrac{N}{n}, 0) && \text{for all } n \text{ and } N, \\ &t \in \left(0, \tfrac{2}{e^{3/2}}\tfrac{N}{n}\right) && \text{for } N \gg n \quad \text{(i.e. } N \text{ is large enough with respect to } n). \end{aligned} \tag{1.7}$$

Surprisingly, in this context the case $t = 0$ is an isolated point with non-negative permutations. Thus the curvature method, requiring the permutations to be non-negative, cannot be applied directly for t indicated in (1.7).

From the aforesaid it follows that (1.5) and (1.6) are indeed sharp for $N = 2n$.

Figure 1.1 illustrates Theorem 1.1 and this remark for $n = 3$ and different $N > 3$ (consider the horizontal line passing through a fixed *positive integer* N in order to determine the corresponding t). The green area represents the values of t, indicated in (1.5) and (1.6), i.e. those where p_{K_t} are guaranteed to be non-negative (the boundaries are included). In the blue area (the boundaries are not included), obtained by computer-based exhaustive search, the permutations can change sign. Note that the part of the blue area for $t < 0$ is exactly the former in (1.7). Moreover, the part for $t > 0$ quite agrees with the latter in (1.7). The white area is not covered by our results and, generally speaking, we can say nothing about the sign of p_{K_t} therein. However, computer experiments suggest that the permutations p_{K_t} are non-negative everywhere except the blue area and thus (1.7) seems to give likely boundaries for t, whose corresponding permutations can change sign.

Relying deeply on Theorem 1.1, we will prove the following result.

Theorem 1.2. *Let K_t be a kernel of the form* (1.1) *with t as in Theorem 1.1 and E a 1-set. If $p_{K_t}(\mathcal{H}^1\lfloor E) < \infty$, then E is rectifiable. Moreover, if the operator T_{K_t} is $L^2(\mathcal{H}^1\lfloor E)$-bounded, then E is rectifiable.*

Remark 1.2. It is known that for $t = -1$ (see the red line in Figure 1.1), which belongs to the area, where the permutations can change sign, the statement of Theorem 1.2 is not valid anymore, i.e. L^2-boundedness does not imply rectifiability. Indeed, it is a corollary of Theorem E since all the kernels K_t of the form (1.1) with $t = -1$ belong to Huovinen's class $\mathscr{H}$ as can be easily checked.

For the subfamily (0.21) of the kernels (1.1) with $(n, N) = (1, 2)$, i.e. for

$$k_t(z) = \frac{(\operatorname{Re} z)^3}{|z|^4} + t \cdot \frac{\operatorname{Re} z}{|z|^2},$$

even more is known as we have already mentioned in Introduction. Namely, for $t = -3/4$, whose corresponding p_{k_t} change sign, there also exists a purely unrectifiable set E such that T_{k_t} is $L^2(\mathcal{H}^1\lfloor E)$-bounded as follows from Theorem F.

Recall Figure 0.2 that illustrates the known results for the kernels k_t. By Theorems 1.1 and 1.2, if $t \in \mathbb{R} \setminus (-2, 0)$, then the permutations p_{k_t} are always non-negative and the $L^2(\mathcal{H}^1\lfloor E)$-boundedness of T_{k_t} implies the rectifiability of E. By the arguments from Remarks 1.1 and 1.2, the permutations p_{k_t} for $t \in (-2, 0)$ change sign and there are two values of t such that the operator T_{k_t} is $L^2(\mathcal{H}^1\lfloor E)$-bounded but E is not rectifiable.

1.3 Auxiliary results

This section is devoted to several auxiliary lemmas, which will be used to prove Theorems 1.1 and 1.2 in Section 1.5.

If a kernel K is real and odd, then one can show that the permutations (1.2) are invariant under translations. This can be done, e.g. by the substitutions $u = z_1 - z_2$ and $v = z_1 - z_3$. Consequently, one point can be always fixed and it is enough to consider only permutations of the form

$$p_K(0, u, v) = K(u)K(v) + K(u)K(u - v) + K(v)K(v - u), \tag{1.8}$$

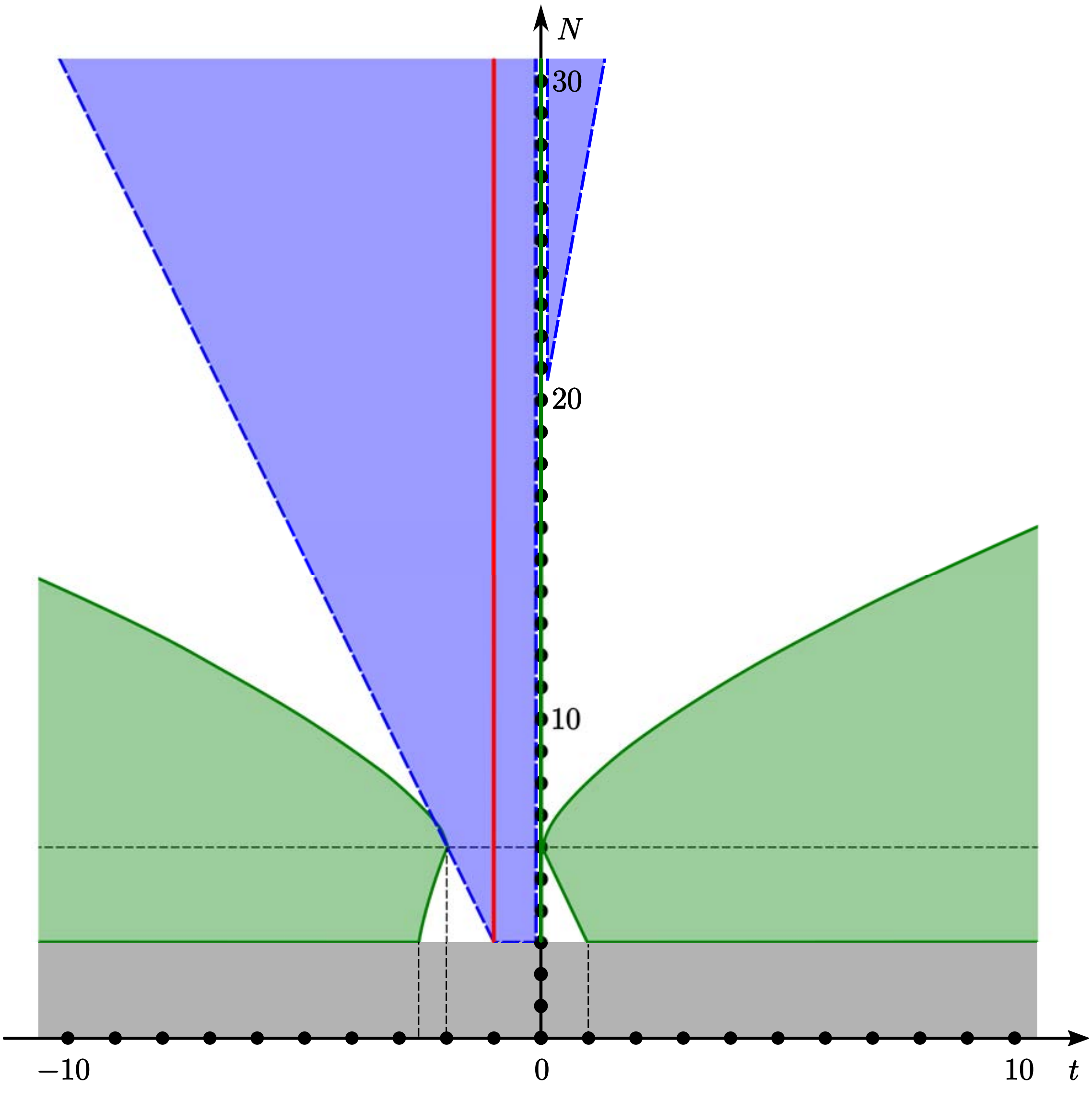

Figure 1.1: Theorem 1.1 and Remark 1.1 for $n = 3$ and different $N > 3$.

where $u, v \in \mathbb{C} \setminus \{0\}$ are distinct points. The kernels (0.16) and (1.1) that we study are real and odd and hence we can use (1.8) instead of (1.2). Furthermore, the case of collinear points u and v is trivial as then $p_K(0,u,v) \equiv 0$ and thus we can skip it.

We will use the following lemma many times below. Note that it can be easily generalised for another couple of kernels instead of κ_n and κ_N.

Lemma 1.1. *Given K_t of the form* (1.1),

$$p_{K_t}(0,u,v) = p_{\kappa_N}(0,u,v) + \varphi_{n,N}(0,u,v)\,t + p_{\kappa_n}(0,u,v)\,t^2, \tag{1.9}$$

where

$$\begin{aligned}\varphi_{n,N}(0,u,v) := \quad &\kappa_N(u)(\kappa_n(v)+\kappa_n(u-v)) \\ &+\kappa_N(v)(\kappa_n(u)+\kappa_n(v-u))+\kappa_N(u-v)(\kappa_n(u)-\kappa_n(v)).\end{aligned} \tag{1.10}$$

Proof. We substitute (1.1) into (1.8) and get

$$\begin{aligned}&p_{K_t}(0,u,v) \\ &= (\kappa_N(u)+t\,\kappa_n(u))(\kappa_N(v)+t\,\kappa_n(v)) \\ &\ +(\kappa_N(u)+t\,\kappa_n(u))(\kappa_N(u-v)+t\,\kappa_n(u-v)) \\ &\ +(\kappa_N(v)+t\,\kappa_n(v))(\kappa_N(v-u)+t\,\kappa_n(v-u)) \\ &= \kappa_N(u)\kappa_N(v)+(\kappa_N(u)\kappa_n(v)+\kappa_n(u)\kappa_N(v))\,t+\kappa_n(u)\kappa_n(v)\,t^2 \\ &+\kappa_N(u)\kappa_N(u-v)+(\kappa_N(u)\kappa_n(u-v)+\kappa_n(u)\kappa_N(u-v))\,t+\kappa_n(u)\kappa_n(u-v)\,t^2 \\ &+\kappa_N(v)\kappa_N(v-u)+(\kappa_N(v)\kappa_n(v-u)+\kappa_n(v)\kappa_N(v-u))\,t+\kappa_n(v)\kappa_n(v-u)\,t^2.\end{aligned}$$

To finish the proof it is enough to group the terms and take into account (1.8). ■

It is important that the leading coefficient of the quadratic polynomial (1.9) (with respect to t) is always non-negative by (0.17) and (1.2).

From now on, in order to simplify formulas we skip $(0,u,v)$ in permutations and other expressions if there is no confusion. For example, we write p_K instead of $p_K(0,u,v)$. In addition, we use the following notations:

$$\lambda_1 := \frac{\operatorname{Re} u}{|u|}, \qquad \lambda_2 := \frac{\operatorname{Re} v}{|v|}, \qquad \lambda_3 := \frac{\operatorname{Re}(u-v)}{|u-v|}, \qquad \Lambda := \lambda_1\lambda_2\lambda_3, \tag{1.11}$$

where the denominators do not vanish as the points u and v are assumed to be distinct and non-collinear. Note that in these terms,

$$p_{\kappa_n} = \frac{(\lambda_1\lambda_2)^{2n-1}}{|u||v|} + \frac{(\lambda_1\lambda_3)^{2n-1}}{|u||u-v|} - \frac{(\lambda_2\lambda_3)^{2n-1}}{|v||v-u|} \tag{1.12}$$

and

$$\begin{aligned}\varphi_{n,N} =& \frac{\lambda_1^{2N-1}}{|u|}\left(\frac{\lambda_2^{2n-1}}{|v|}+\frac{\lambda_3^{2n-1}}{|u-v|}\right) \\ &+\frac{\lambda_2^{2N-1}}{|v|}\left(\frac{\lambda_1^{2n-1}}{|u|}-\frac{\lambda_3^{2n-1}}{|v-u|}\right)+\frac{\lambda_3^{2N-1}}{|u-v|}\left(\frac{\lambda_1^{2n-1}}{|u|}-\frac{\lambda_2^{2n-1}}{|v|}\right).\end{aligned} \tag{1.13}$$

What is more, another representation of $\varphi_{n,N}$ is valid.

Lemma 1.2. *In terms of* (1.11) *it holds that*

$$\varphi_{n,N} = \tau_1 p_{\kappa_n} - \tau_2, \tag{1.14}$$

where

$$\tau_1 := \lambda_1^{2(N-n)} + \lambda_2^{2(N-n)} + \lambda_3^{2(N-n)}, \qquad 0 \leqslant \tau_1 \leqslant 3, \tag{1.15}$$

and

$$\tau_2 := \Lambda^{2(N-n)} \left(\frac{(\lambda_1\lambda_2)^{2(2n-N)-1}}{|u||v|} + \frac{(\lambda_1\lambda_3)^{2(2n-N)-1}}{|u||u-v|} - \frac{(\lambda_2\lambda_3)^{2(2n-N)-1}}{|v||v-u|} \right). \tag{1.16}$$

In particular, $\tau_2 \equiv 0$ *if* $N = 2n$.

Proof. Direct multiplication of τ_1 by p_{κ_n} gives

$$\begin{aligned}
&\left(\lambda_1^{2(N-n)} + \lambda_2^{2(N-n)} + \lambda_3^{2(N-n)}\right) \cdot \left(\frac{(\lambda_1\lambda_2)^{2n-1}}{|u||v|} + \frac{(\lambda_1\lambda_3)^{2n-1}}{|u||u-v|} - \frac{(\lambda_2\lambda_3)^{2n-1}}{|v||v-u|} \right) \\
&= \left(\frac{\lambda_3^{2(N-n)}(\lambda_1\lambda_2)^{2n-1}}{|u||v|} + \frac{\lambda_2^{2(N-n)}(\lambda_1\lambda_3)^{2n-1}}{|u||u-v|} - \frac{\lambda_1^{2(N-n)}(\lambda_2\lambda_3)^{2n-1}}{|v||v-u|} \right) \\
&+ \frac{\lambda_1^{2N-1}}{|u|}\left(\frac{\lambda_2^{2n-1}}{|v|} + \frac{\lambda_3^{2n-1}}{|u-v|}\right) + \frac{\lambda_2^{2N-1}}{|v|}\left(\frac{\lambda_1^{2n-1}}{|u|} - \frac{\lambda_3^{2n-1}}{|v-u|}\right) + \frac{\lambda_3^{2N-1}}{|u-v|}\left(\frac{\lambda_1^{2n-1}}{|u|} - \frac{\lambda_2^{2n-1}}{|v|}\right),
\end{aligned}$$

which is exactly $\tau_2 + \varphi_{n,N}$ by (1.13) and (1.16). ■

Lemma 1.3. *Given* κ_n *and* κ_N *of the form* (0.16),

$$\frac{N}{n} \cdot \Lambda^{2(N-n)} \cdot p_{\kappa_n} \leqslant p_{\kappa_N}, \qquad 1 \leqslant n \leqslant N. \tag{1.17}$$

Note that this inequality for $n = 1$ was obtained in [CMPT1, Proof of Lemma 2.3]. We will use the following lemma from there in order to prove the general form.

Lemma 1.4 (Proof of Proposition 2.1 in [CMPT1])**.** *One has the representation*

$$p_{\kappa_m} = \sum_{k=1}^{m} \binom{m}{k} \Lambda^{2(m-k)} h_k(u,v),$$

where $h_k(u,v) \geqslant 0$ *and are defined as follows:*

$$\begin{aligned}
h_k(u,v) =& (|u||v||u-v|)^{-2k} \Big((\operatorname{Re} u \operatorname{Re} v)^{2k-1} (\operatorname{Im}(u-v))^{2k} \\
&+ (\operatorname{Re} u \operatorname{Re}(u-v))^{2k-1} (\operatorname{Im} v)^{2k} + (\operatorname{Re} v \operatorname{Re}(v-u))^{2k-1} (\operatorname{Im} u)^{2k} \Big).
\end{aligned}$$

Proof. Within the settings of Lemma 1.4,

$$\Lambda^{2(N-n)} \frac{p_{\kappa_n}}{p_{\kappa_N}} = \frac{\sum_{k=1}^{n} \binom{n}{k} \binom{N}{k}^{-1} H_k(u,v)}{\sum_{k=1}^{N} H_k(u,v)},$$

where $H_k(u,v) := \binom{N}{k} \Lambda^{2(N-k)} h_k(u,v) \geqslant 0$. Furthermore,

$$\binom{n}{k}\binom{N}{k}^{-1} = \frac{n!}{(n-k)!} \frac{(N-k)!}{N!} = \frac{(n-k+1)\cdots n}{(N-k+1)\cdots N} \leqslant \frac{n}{N}, \qquad 1 \leqslant k \leqslant n,$$

and finally

$$\Lambda^{2(N-n)} \frac{p_{\kappa_n}}{p_{\kappa_N}} \leqslant \frac{n}{N} \cdot \frac{\sum_{k=1}^{n} H_k(u,v)}{\sum_{k=1}^{N} H_k(u,v)} \leqslant \frac{n}{N}, \qquad n \leqslant N,$$

which is the desired result. ∎

Lemmas 1.1, 1.2 and 1.3 enable us to obtain lower pointwise estimates for the permutations p_{K_t} via the permutations p_{κ_n} for some t. To do so, we will use (1.14) and (1.17) to estimate the coefficients of the quadratic polynomial (1.9). Let us start with the case $n < N \leqslant 2n$.

Lemma 1.5. *Given K_t of the form* (1.1) *with $n < N \leqslant 2n$, if*

$$t \in \mathbb{R} \setminus \left[-\tfrac{1}{2} \left(3 + \sqrt{9 - 4\tfrac{N}{n}} \right), 2 - \tfrac{N}{n} \right],$$

then $p_{K_t} \geqslant C(t) \cdot p_{\kappa_n}$ with some $C(t) > 0$.

Proof. To get the required estimate, we first look at the expression for τ_2 in (1.16) for our case. Since $n < N \leqslant 2n$, from (1.12) and (1.16) we immediately get

$$\tau_2 = \Lambda^{2(N-n)} \cdot p_{\kappa_{2n-N}}, \qquad 0 \leqslant 2n - N \leqslant n - 1,$$

with $\tau_2 \equiv 0$ if $N = 2n$. Consequently, by (1.9) and (1.14),

$$p_{K_t} = p_{\kappa_N} + (\tau_1 p_{\kappa_n} - \Lambda^{2(N-n)} p_{\kappa_{2n-N}})\, t + p_{\kappa_n} t^2. \tag{1.18}$$

Now we show that the right hand side of (1.18) for t mentioned in the lemma is bounded from below by p_{κ_n}, multiplied by a positive constant, depending only on t.

Applying the inequality (1.17) to p_{κ_N} and $p_{\kappa_{2n-N}}$ in (1.18) for $t \geqslant 0$ gives

$$p_{K_t} \geqslant \left(\tfrac{N}{n}\, \Lambda^{2(N-n)} + (\tau_1 - 2 + \tfrac{N}{n})\, t + t^2 \right) \cdot p_{\kappa_n} = f(\xi_1, \xi_2, \xi_3) \cdot p_{\kappa_n}, \tag{1.19}$$

where $\xi_j := \lambda_j^{2(N-n)} \in [0, 1]$, $j = 1, 2, 3$, and

$$f(\xi_1, \xi_2, \xi_3) := \tfrac{N}{n}\, \xi_1 \xi_2 \xi_3 + (\xi_1 + \xi_2 + \xi_3 - 2 + \tfrac{N}{n})\, t + t^2. \tag{1.20}$$

Analysis of $\partial f / \partial \xi_j$ shows that f is non-decreasing for $t \geqslant 0$ with respect to each $\xi_j \in [0, 1]$. Consequently,

$$f(\xi_1, \xi_2, \xi_3) \geqslant f(0, 0, 0) = t(t - 2 + \tfrac{N}{n}),$$

which is strictly positive for $t > 2 - \frac{N}{n} \geqslant 0$.

For $t \leqslant 0$ we apply (1.17) to p_{κ_N} and use that $p_{\kappa_{2n-N}} \geqslant 0$ (see (0.17)). This yields

$$p_{K_t} \geqslant \left(\tfrac{N}{n}\, \Lambda^{2(N-n)} + \tau_1\, t + t^2 \right) \cdot p_{\kappa_n} = F(\xi_1, \xi_2, \xi_3) \cdot p_{\kappa_n}, \tag{1.21}$$

where the function

$$F(\xi_1, \xi_2, \xi_3) := \tfrac{N}{n}\, \xi_1 \xi_2 \xi_3 + (\xi_1 + \xi_2 + \xi_3)\, t + t^2 \tag{1.22}$$

is non-increasing for $t \leqslant -\frac{N}{n}$ with respect to each $\xi_j \in [0, 1]$. Consequently,

$$F(\xi_1, \xi_2, \xi_3) \geqslant F(1, 1, 1) = \tfrac{N}{n} + 3\, t + t^2,$$

where the latter expression is positive for $t < -\frac{1}{2} \left(3 + \sqrt{9 - 4\frac{N}{n}} \right) \leqslant -\frac{N}{n}$. ∎

Now let $N > 2n$. Note that the following lemma coincides with the previous one if we put $N = 2n$.

Lemma 1.6. *Given K_t of the form* (1.1) *with $N > 2n$, if*

$$t \in \mathbb{R} \setminus \left[-\tfrac{1}{2} \left(3 + \rho_{n,N} + \sqrt{(3+\rho_{n,N})^2 - 4\tfrac{N}{n}} \right), \rho_{n,N} \right], \quad \rho_{n,N} = \left(\tfrac{N}{n} - 2 \right) \sqrt{N - 2n},$$

then $p_{K_t} \geqslant C(t) \cdot p_{\kappa_n}$ with some $C(t) > 0$.

Proof. We will again estimate the coefficients of the polynomial (1.9) in terms of p_{κ_n}. At first, we will estimate $|\phi_{n,N}|$. By (1.14), this will only need to estimate $|\tau_2|$.

As we have already mentioned before Lemma 1.1, the permutations p_{K_t} and p_{κ_n} are invariant under translations. Therefore we can assume without loss of generality that all triangles $(0, u, v)$ that we consider belong to the half plane $\operatorname{Re} z \geqslant 0$. This will be necessary in the further analysis of angles of these triangles.

From now on, we use the following notation additionally to (1.11):

$$\sin \alpha_j := \lambda_j, \qquad \lambda_j \in [-1; 1], \qquad j = 1, 2, 3. \tag{1.23}$$

We also suppose that λ_j^2 are pairwise distinct. One can get the other case by passage to a limit below. For the geometrical interpretation of α_j see Figures 1.2 and 1.3.

Now we aim to represent τ_2 from (1.16) in terms of the curvature written in the form (0.10). For this purpose we will segregate the area squared $S(0, u, v)^2$ in the numerator and $|u|^2|v|^2|u-v|^2$ in the denominator of τ_2. First, from (1.16), taking into account (1.11), we obtain

$$\begin{aligned}
\tau_2 &= \frac{\Lambda^{2n-1}}{|u||v||u-v|} \frac{(\lambda_1\lambda_2)^{2(2n-N)-1}|u-v| + (\lambda_1\lambda_3)^{2(2n-N)-1}|v| - (\lambda_2\lambda_3)^{2(2n-N)-1}|u|}{\Lambda^{2(2n-N)-1}} \\
&= \frac{\Lambda^{2n-1}}{|u||v||u-v|} \left(\lambda_3^{2(N-2n)} \operatorname{Re}(u-v) + \lambda_2^{2(N-2n)} \operatorname{Re} v - \lambda_1^{2(N-2n)} \operatorname{Re} u \right) \\
&= \frac{\Lambda^{2n-1}}{|u||v||u-v|} \left(\operatorname{Re} u \left(\lambda_3^{2(N-2n)} - \lambda_1^{2(N-2n)} \right) - \operatorname{Re} v \left(\lambda_3^{2(N-2n)} - \lambda_2^{2(N-2n)} \right) \right) \\
&= \frac{\Lambda^{2n-1}}{|u||v||u-v|} \left(\lambda_1 |u| \left(\lambda_3^2 - \lambda_1^2 \right) A_1(u,v) - \lambda_2 |v| \left(\lambda_3^2 - \lambda_2^2 \right) A_2(u,v) \right),
\end{aligned}$$

where

$$A_1(u,v) := \frac{\lambda_3^{2(N-2n)} - \lambda_1^{2(N-2n)}}{\lambda_3^2 - \lambda_1^2} \quad \text{and} \quad A_2(u,v) := \frac{\lambda_3^{2(N-2n)} - \lambda_2^{2(N-2n)}}{\lambda_3^2 - \lambda_2^2}. \tag{1.24}$$

Finally, we can rewrite τ_2 as

$$\tau_2 = \frac{\Lambda^{2n-1}}{|u|^2|v|^2|u-v|^2} \cdot A(u,v), \tag{1.25}$$

where $A(u,v) := |u||v||u-v| \left(\lambda_1 \left(\lambda_3^2 - \lambda_1^2 \right) |u| A_1(u,v) - \lambda_2 \left(\lambda_3^2 - \lambda_2^2 \right) |v| A_2(u,v) \right)$.

By (1.23) and the formulas for the sum of sines and the sine of a double angle,

$$\begin{aligned}
\lambda_3^2 - \lambda_1^2 &= (\sin \alpha_3 + \sin \alpha_1)(\sin \alpha_3 - \sin \alpha_1) \\
&= 2 \sin \frac{\alpha_3 + \alpha_1}{2} \cos \frac{\alpha_3 - \alpha_1}{2} \cdot 2 \sin \frac{\alpha_3 - \alpha_1}{2} \cos \frac{\alpha_3 + \alpha_1}{2} \\
&= \sin(\alpha_3 + \alpha_1) \sin(\alpha_3 - \alpha_1).
\end{aligned}$$

Analogously, $\lambda_3^2 - \lambda_2^2 = \sin(\alpha_3 + \alpha_2) \sin(\alpha_3 - \alpha_2)$. Thus

$$\begin{aligned}
A(u,v) = \; & |u|^2|v||u-v| \sin(\alpha_3 + \alpha_1) \sin(\alpha_3 - \alpha_1) \lambda_1 A_1(u,v) \\
& - |u||v|^2|u-v| \sin(\alpha_3 + \alpha_2) \sin(\alpha_3 - \alpha_2) \lambda_2 A_2(u,v).
\end{aligned} \tag{1.26}$$

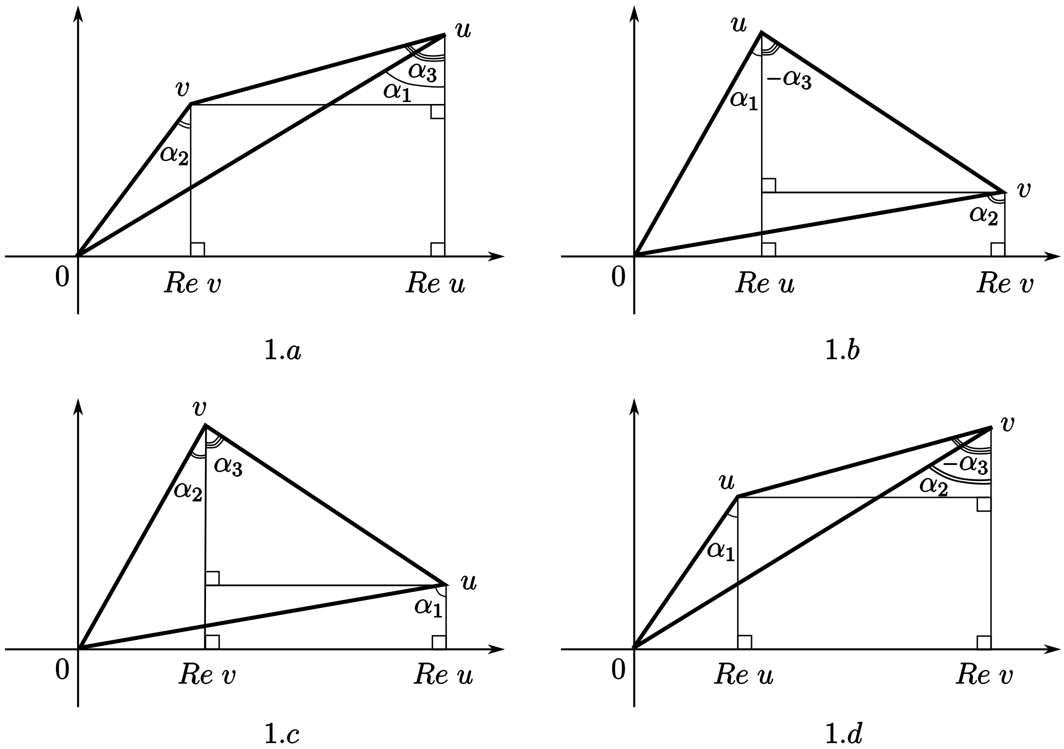

Figure 1.2: Triangles in the case 1 (the proof of Lemma 1.6).

Now let us see how one can calculate the angles $\angle(u,0,v)$, $\angle(0,u,v)$ and $\angle(0,v,u)$ of the triangle $(0,u,v)$, using the angles α_j, $j=1,2,3$. Recall that the triangles $(0,u,v)$ belong to the half plane $\operatorname{Re} z \geqslant 0$. Thus only two cases are possible:

1. The vertexes u and v both lie in the same (first or forth) quarter of the plane.

2. The vertexes u and v lie in different quarters of the plane.

One can check that four options are realizable in the case 1 (see the examples in Figure 1.2; several other situations are possible but they produce the same cases):

1a. $\angle(u,0,v) = \quad \alpha_1-\alpha_2$, $\angle(0,u,v) = -(\alpha_1-\alpha_3)$, $\angle(0,v,u) = \pi+(\alpha_2-\alpha_3)$;

1b. $\angle(u,0,v) = -(\alpha_1-\alpha_2)$, $\angle(0,u,v) = \quad \alpha_1-\alpha_3$, $\angle(0,v,u) = \pi-(\alpha_2-\alpha_3)$;

1c. $\angle(u,0,v) = \quad \alpha_1-\alpha_2$, $\angle(0,u,v) = \pi-(\alpha_1+\alpha_3)$, $\angle(0,v,u) = \quad \alpha_2+\alpha_3$;

1d. $\angle(u,0,v) = -(\alpha_1-\alpha_2)$, $\angle(0,u,v) = \pi+(\alpha_1+\alpha_3)$, $\angle(0,v,u) = -(\alpha_2+\alpha_3)$.

In the case 2 (see Figure 1.3) one always has

$$\angle(u,0,v) = \pi-(\alpha_1+\alpha_2), \quad \angle(0,u,v) = \alpha_1-\alpha_3, \quad \angle(0,v,u) = \alpha_2+\alpha_3.$$

Consequently, taking into account the formulas

$$S(0,u,v) = \tfrac{1}{2}|u||v|\sin\angle(u,0,v) = \tfrac{1}{2}|u||u-v|\sin\angle(0,u,v) = \tfrac{1}{2}|v||u-v|\sin\angle(0,v,u),$$

we conclude from (1.26) that,

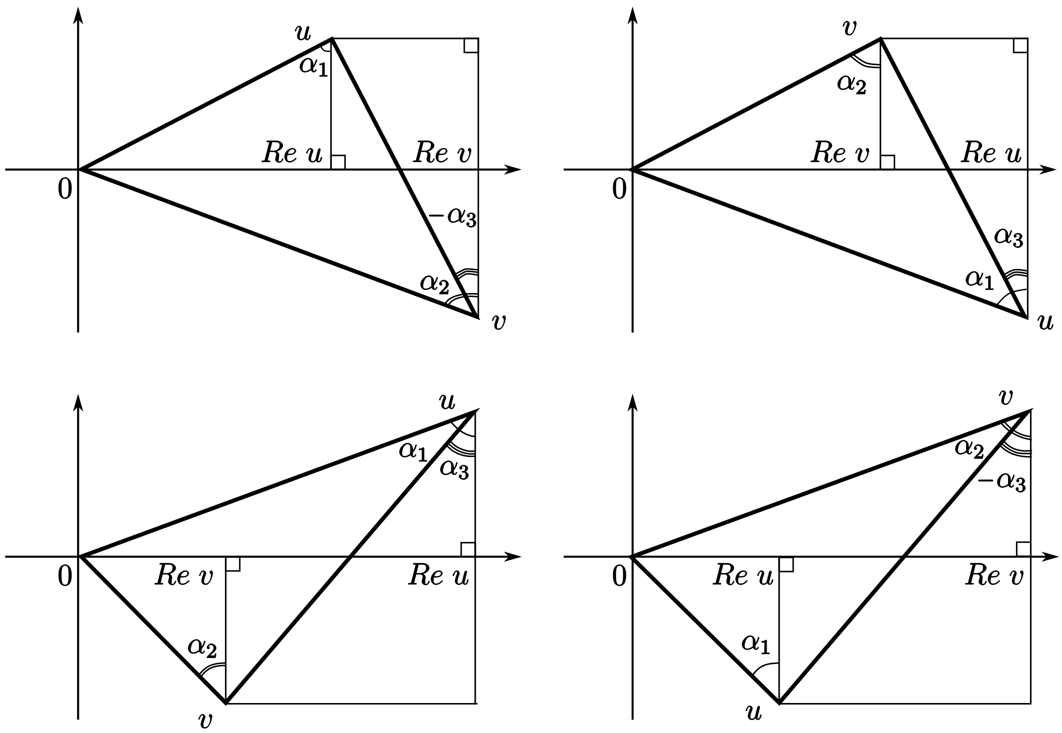

Figure 1.3: Triangles in the case 2 (the proof of Lemma 1.6).

- in the cases 1.a and 1.b:

$$\begin{aligned}
A(u,v) &= |u||v|\sin(\pm(\alpha_2-\alpha_1))|u||u-v|\sin(\pm(\alpha_1-\alpha_3))\frac{\sin(\alpha_3+\alpha_1)}{\sin(\alpha_1-\alpha_2)}\lambda_1 A_1(u,v) \\
&\quad - |u||v|\sin(\pm(\alpha_2-\alpha_1))|v||u-v|\sin(\pi\pm(\alpha_3-\alpha_2))\frac{\sin(\alpha_3+\alpha_2)}{\sin(\alpha_1-\alpha_2)}\lambda_2 A_2(u,v) \\
&= 4S(0,u,v)^2\,\frac{\sin(\alpha_3+\alpha_1)\lambda_1 A_1(u,v)-\sin(\alpha_3+\alpha_2)\lambda_2 A_2(u,v)}{\sin(\alpha_1-\alpha_2)};
\end{aligned}$$

- in the cases 1.c and 1.d:

$$\begin{aligned}
A(u,v) &= |u||v|\sin(\pm(\alpha_1-\alpha_2))|u||u-v|\sin(\pi\mp(\alpha_1+\alpha_3))\frac{\sin(\alpha_3-\alpha_1)}{\sin(\alpha_1-\alpha_2)}\lambda_1 A_1(u,v) \\
&\quad - |u||v|\sin(\pm(\alpha_1-\alpha_2))|v||u-v|\sin(\pm(\alpha_2+\alpha_3))\frac{\sin(\alpha_3-\alpha_2)}{\sin(\alpha_1-\alpha_2)}\lambda_2 A_2(u,v) \\
&= 4S(0,u,v)^2\,\frac{\sin(\alpha_3-\alpha_1)\lambda_1 A_1(u,v)-\sin(\alpha_3-\alpha_2)\lambda_2 A_2(u,v)}{\sin(\alpha_1-\alpha_2)};
\end{aligned}$$

• in the case 2:

$$\begin{aligned}A(u,v) &= -|u||v|\sin(\alpha_1+\alpha_2)|u||u-v|\sin(\alpha_1-\alpha_3)\frac{\sin(\alpha_3+\alpha_1)}{\sin(\alpha_1+\alpha_2)}\lambda_1 A_1(u,v)\\ &\quad -|u||v|\sin(\alpha_1+\alpha_2)|v||u-v|\sin(\alpha_2+\alpha_3)\frac{\sin(\alpha_3-\alpha_2)}{\sin(\alpha_1+\alpha_2)}\lambda_2 A_2(u,v)\\ &= -4S(0,u,v)^2\,\frac{\sin(\alpha_3+\alpha_1)\lambda_1 A_1(u,v)+\sin(\alpha_3-\alpha_2)\lambda_2 A_2(u,v)}{\sin(\alpha_1+\alpha_2)}.\end{aligned}$$

Note that the substitutions $\alpha_1 \mapsto -\alpha_1$, $\alpha_2 \mapsto -\alpha_2$ ($\lambda_1 \mapsto -\lambda_1$, $\lambda_2 \mapsto -\lambda_2$) in the expression for $A(u,v)$ for the case 1.a and 1.b give $A(u,v)$ in the cases 1.c and 1.d. Moreover, the substitution $\alpha_2 \mapsto -\alpha_2$ ($\lambda_2 \mapsto -\lambda_2$) in $A(u,v)$ for the case 1.a and 1.b gives $-A(u,v)$ in the case 2. In what follows, this allows us to consider only one expression for $A(u,v)$, say, the one corresponding to 1.a and 1.b, instead of the three. This reduction will not affect the final result. By this reason, let

$$A(u,v) = 4S(0,u,v)^2\cdot\frac{V(u,v)}{\sin(\alpha_1-\alpha_2)},$$

where

$$V(u,v) := \sin(\alpha_3+\alpha_1)\lambda_1 A_1(u,v) - \sin(\alpha_3+\alpha_2)\lambda_2 A_2(u,v). \tag{1.27}$$

From this and (1.25) by the formula (0.10), connecting the curvature $c(0,u,v)$ and the area $S(0,u,v)$, we get

$$\tau_2 = \frac{4S(0,u,v)^2}{|u|^2|v|^2|u-v|^2}\cdot\Lambda^{2n-1}\cdot\frac{V(u,v)}{\sin(\alpha_1-\alpha_2)} = \tfrac14 c(0,u,v)^2\cdot\Lambda^{2n-1}\cdot\frac{V(u,v)}{\sin(\alpha_1-\alpha_2)}.$$

Note that $\frac14 c(0,u,v)^2 = p_{\kappa_1}(0,u,v)$ by (0.15). Consequently, the inequality (1.17) and the fact that $|\Lambda|\leqslant 1$ yield

$$|\tau_2| = n\,\Lambda^{2(n-1)}p_{\kappa_1}\cdot\frac{|\Lambda|}{n}\cdot\frac{|V(u,v)|}{|\sin(\alpha_1-\alpha_2)|}\leqslant\frac{p_{\kappa_n}}{n}\cdot\frac{|V(u,v)|}{|\sin(\alpha_1-\alpha_2)|}. \tag{1.28}$$

Now we want to show that $|V(u,v)|\leqslant \mathrm{const}\cdot|\sin(\alpha_1-\alpha_2)|$. If we rewrite $A_1(u,v)$ and $A_2(u,v)$, defined in (1.24), using the formula

$$\frac{a^m-b^m}{a-b} = \sum_{v=0}^{m-1} a^{m-1-v}b^v, \qquad m\in\mathbb{N}^+,$$

for $m := N-2n \geqslant 1$, then (1.27) takes the form

$$V(u,v) = \sum_{v=0}^{m-1}\lambda_3^{2(m-1-v)}\left(\sin(\alpha_3+\alpha_1)\cdot\lambda_1^{2v+1} - \sin(\alpha_3+\alpha_2)\cdot\lambda_2^{2v+1}\right).$$

Now we substitute $\lambda_j = \sin\alpha_j$, $j=1,2$, by (1.23) and apply the well-known formula

$$(\sin\theta)^{2v+1} = \frac{1}{2^{2v}}\sum_{k=0}^{v}(-1)^{v-k}\binom{2v+1}{k}\sin(2v+1-2k)\theta.$$

This leads to the following representation:

$$V(u,v) = \sum_{v=0}^{m-1}\lambda_3^{2(m-1-v)}\frac{1}{2^{2v}}\sum_{k=0}^{v}(-1)^{v-k}\binom{2v+1}{k}B_{v,k}(\alpha_1,\alpha_2,\alpha_3), \tag{1.29}$$

where

$$B_{v,k}(\alpha_1,\alpha_2,\alpha_3)$$
$$:= \sin(\alpha_3+\alpha_1)\sin(2v+1-2k)\alpha_1 - \sin(\alpha_3+\alpha_2)\sin(2v+1-2k)\alpha_2.$$

By the formulas for the product of sines and the difference of cosines we obtain

$$\begin{aligned}
&B_{v,k}(\alpha_1,\alpha_2,\alpha_3)\\
&=\tfrac{1}{2}\left(\cos(\alpha_3-2(v-k)\alpha_1)-\cos(\alpha_3+2(v-k+1)\alpha_1)\right.\\
&\quad\left.-\cos(\alpha_3-2(v-k)\alpha_2)+\cos(\alpha_3+2(v-k+1)\alpha_2)\right)\\
&=\sin(\alpha_3-(v-k)(\alpha_1+\alpha_2))\sin((v-k)(\alpha_1-\alpha_2))\\
&\quad+\sin(\alpha_3+(v-k+1)(\alpha_1+\alpha_2))\sin((v-k+1)(\alpha_1-\alpha_2)).
\end{aligned}$$

Since $|\sin rx| \leqslant r|\sin x|$, $r \geqslant 0$, it follows that

$$|B_{v,k}(\alpha_1,\alpha_2,\alpha_3)| \leqslant (2v-2k+1)|\sin(\alpha_1-\alpha_2)|.$$

This and the obvious estimate of $|V(u,v)|$ from (1.29) yield

$$\frac{|V(u,v)|}{|\sin(\alpha_2-\alpha_1)|} \leqslant \sigma(m) := \sum_{v=0}^{m-1}\frac{1}{2^{2v}}\sum_{k=0}^{v}\binom{2v+1}{k}(2v-2k+1).$$

One can check by successive use of the formulas (4.2.1.6), (4.2.2.13) and (4.2.3.19) from [PBM, §4.2] that

$$\sigma(m) = \frac{4m^2-1}{3\cdot 4^{m-1}}\binom{2m-2}{m-1}.$$

Moreover, it can by easily proved by induction that

$$\sigma(m) \leqslant m^{3/2}, \qquad m \in \mathbb{N}.$$

Since $m = N-2n \geqslant 1$, (1.28) yields

$$|\tau_2| \leqslant \rho_{n,N}\cdot p_{\kappa_n}, \qquad \rho_{n,N} = \frac{(N-2n)^{3/2}}{n} = \left(\tfrac{N}{n}-2\right)\sqrt{N-2n}, \qquad N > 2n. \tag{1.30}$$

Now we come back to the representation (1.9) from Lemma 1.1 and estimation of its terms. By (1.14), (1.17) and (1.30), we deduce for $t \geqslant 0$ that

$$p_{K_t} \geqslant \left(\tfrac{N}{n}\,\Lambda^{2(N-n)} + (\tau_1-\rho_{n,N})\,t + t^2\right)\cdot p_{\kappa_n} = g(\xi_1,\xi_2,\xi_3)\cdot p_{\kappa_n}, \tag{1.31}$$

where $\xi_j = \lambda_j^{2(N-n)} \in [0,1]$ as in the proof of the previous lemma, and

$$g(\xi_1,\xi_2,\xi_3) := \tfrac{N}{n}\,\xi_1\xi_2\xi_3 + (\xi_1+\xi_2+\xi_3-\rho_{n,N})\,t + t^2. \tag{1.32}$$

The function g is non-decreasing for $t \geqslant 0$ with respect to each $\xi_j \in [0,1]$, hence for $t > \rho_{n,N} > 0$ we obtain the inequality

$$g(\xi_1,\xi_2,\xi_3) \geqslant g(0,0,0) = t(t-\rho_{n,N}) > 0.$$

For $t \leqslant 0$ we have

$$p_{K_t} \geqslant \left(\tfrac{N}{n}\,\Lambda^{2(N-n)} + (\tau_1+\rho_{n,N})\,t + t^2\right)\cdot p_{\kappa_n} = G(\xi_1,\xi_2,\xi_3)\cdot p_{\kappa_n}, \tag{1.33}$$

where the function

$$G(\xi_1, \xi_2, \xi_3) := \tfrac{N}{n}\, \xi_1\xi_2\xi_3 + (\xi_1 + \xi_2 + \xi_3 + \rho_{n,N})\, t + t^2 \tag{1.34}$$

is non-increasing for $t \leqslant -\frac{N}{n}$ with respect to each $\xi_j \in [0,1]$ and therefore

$$G(\xi_1, \xi_2, \xi_3) \geqslant G(1,1,1) = \tfrac{N}{n} + (3 + \rho_{n,N})\, t + t^2.$$

The roots of the latter quadratic polynomial are

$$-\tfrac{1}{2}\left(3 + \rho_{n,N} \pm \sqrt{(3+\rho_{n,N})^2 - 4\tfrac{N}{n}}\right),$$

so it has only positive values if $t < -\frac{1}{2}\left(3 + \rho_{n,N} + \sqrt{(3+\rho_{n,N})^2 - 4\frac{N}{n}}\right) \leqslant -\frac{N}{n}$. ∎

Note that Lemmas 1.5 and 1.6 give Theorem 1.1 by continuity. For the proof of Theorem 1.2, additionally to Theorem 1.1, we will also need lower estimates of p_{K_t} for t, which are the end points of the intervals excluded in (1.5) and (1.6) from the real line. In order to obtain these estimates, we first introduce additional notation.

Given two distinct points $z, w \in \mathbb{C}$, we denote by $L_{z,w}$ the line passing through z and w. Given three pairwise distinct points $z_1, z_2, z_3 \in \mathbb{C}$, we denote by $\angle(z_1, z_2, z_3)$ the smallest angle formed by the lines L_{z_1,z_2} and L_{z_1,z_3}. This angle belongs to $[0; \pi/2]$. If L and L' are lines, then $\angle(L, L')$ is the smallest angle between them. This angle belongs to $[0; \pi/2]$, too. Also, $\theta_V(L) := \angle(L, V)$ and $\theta_H(L) := \angle(L, H)$, where V and H are the vertical and horizontal lines, correspondingly. Furthermore, for a fixed constant $\tau \geqslant 1$, we set

$$\mathcal{O}_\tau = \left\{(z_1, z_2, z_3) : \frac{|z_i - z_j|}{|z_i - z_k|} \leqslant \tau \text{ for pairwise distinct } i, j, k \in \{1,2,3\}\right\}, \tag{1.35}$$

so that all the triangles with vertexes z_1, z_2 and z_3 in $\mathcal{O}_\tau$ have comparable sides.

Given $\alpha_0 \in (0, \pi/2)$ and (z_1, z_2, z_3), in what follows we will use the conditions

$$\theta_V(L_{z_1,z_2}) + \theta_V(L_{z_2,z_3}) + \theta_V(L_{z_1,z_3}) \geqslant \alpha_0 \tag{1.36}$$

and

$$\theta_H(L_{z_1,z_2}) + \theta_H(L_{z_2,z_3}) + \theta_H(L_{z_1,z_3}) \geqslant \alpha_0. \tag{1.37}$$

Note that (1.36) and (1.37) can be correspondingly replaced by the conditions

$$\theta_H(L_{z_1,z_2}) + \theta_H(L_{z_2,z_3}) + \theta_H(L_{z_1,z_3}) \leqslant \tfrac{3}{2}\pi - \alpha_0$$

and

$$\theta_V(L_{z_1,z_2}) + \theta_V(L_{z_2,z_3}) + \theta_V(L_{z_1,z_3}) \leqslant \tfrac{3}{2}\pi - \alpha_0.$$

To obtain the desired result, we first prove several geometrical lemmas.

Lemma 1.7. *Fix $\alpha_0 \in (0, \pi/2)$. Given $(0, u, v) \in \mathcal{O}_\tau$, if the condition* (1.36) *is satisfied, then*

$$\tau_1(0, u, v) = \lambda_1^{2(N-n)} + \lambda_2^{2(N-n)} + \lambda_3^{2(N-n)} \geqslant C_1(\alpha_0) > 0.$$

Proof. Clearly,

$$\lambda_1^2 = \sin^2\theta_V(L_{0,u}), \quad \lambda_2^2 = \sin^2\theta_V(L_{0,v}), \quad \lambda_3^2 = \sin^2\theta_V(L_{u,v}).$$

Moreover, from (1.36) it follows that at least one of the angles $\theta_V(L_{0,u})$, $\theta_V(L_{0,v})$, $\theta_V(L_{u,v})$ is not less than $\alpha_0/3$. Thus $\tau_1 \geqslant (\sin\frac{\alpha_0}{3})^{2(N-n)}$. ∎

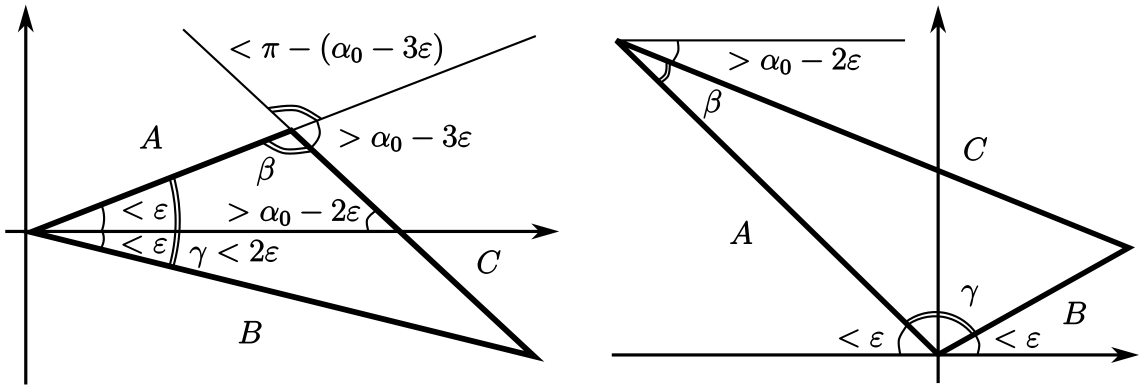

Figure 1.4: Triangles in the case 2 (the proof of Lemma 1.8).

Lemma 1.8. *Fix* $\alpha_0 \in (0, \pi/2)$. *Given* $(0, u, v) \in \mathcal{O}_\tau$, *if the condition* (1.37) *is satisfied, then*

$$\Upsilon(0, u, v) := 2 + (\lambda_1\lambda_2\lambda_3)^{2(N-n)} - (\lambda_1^{2(N-n)} + \lambda_2^{2(N-n)} + \lambda_3^{2(N-n)}) \geqslant C_2(\alpha_0, \tau) > 0.$$

Proof. First we note that

$$\Upsilon(0, u, v) \geqslant 2 + \lambda_1^2\lambda_2^2\lambda_3^2 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$$

as the function $2 + \xi_1\xi_2\xi_3 - (\xi_1 + \xi_2 + \xi_3)$ is non-increasing with respect to each $\xi_j \in [0, 1]$, $j = 1, 2, 3$, and $\lambda_j^2 \geqslant \lambda_j^{2(N-n)}$ for $N > n \geqslant 1$.

In addition, we have

$$\lambda_1^2 = 1 - \sin^2\theta_H(L_{0,u}), \quad \lambda_2^2 = 1 - \sin^2\theta_H(L_{0,v}), \quad \lambda_3^2 = 1 - \sin^2\theta_H(L_{u,v}),$$

and hence

$$\begin{aligned}\Upsilon(0, u, v) \geqslant{}& \sin^2\theta_H(L_{0,u})\sin^2\theta_H(L_{0,v}) + \sin^2\theta_H(L_{0,u})\sin^2\theta_H(L_{u,v}) \\ &+ \sin^2\theta_H(L_{0,v})\sin^2\theta_H(L_{u,v}) - \sin^2\theta_H(L_{0,u})\sin^2\theta_H(L_{0,v})\sin^2\theta_H(L_{u,v}) \\ \geqslant{}& \tfrac{2}{3}\left(\sin^2\theta_H(L_{0,v})\sin^2\theta_H(L_{0,u}) + \sin^2\theta_H(L_{0,u})\sin^2\theta_H(L_{u,v})\right. \\ &\left.+ \sin^2\theta_H(L_{0,v})\sin^2\theta_H(L_{u,v})\right).\end{aligned}$$

Consider a triangle $(0, u, v) \in \mathcal{O}_\tau$ such that (1.37) is satisfied. Fix some $\varepsilon \in (0; \alpha_0/3)$. Two cases are possible:

1. amongst $\theta_H(L_{0,u})$, $\theta_H(L_{0,v})$, $\theta_H(L_{u,v})$, there exists a pair of angles, each being greater than ε and then it is easily seen that $\Upsilon(0, u, v) \geqslant \frac{2}{3}\sin^4\varepsilon$;

2. amongst those, there exists no pair of angles, each being greater than ε.

Let us consider the second case in detail (see Figure 1.4). It is clear that at least two angles amongst $\theta_H(L_{0,u})$, $\theta_H(L_{0,v})$, $\theta_H(L_{u,v})$ are less than ε then. In other words, two sides of the triangle cut the horizontal line at angles less than ε. We call these sides A and B.

Furthermore, let the angle γ between A and B be acute; then obviously it is smaller than 2ε. Then the acute angle between the third side C and the horizontal line is greater than $\alpha_0 - 2\varepsilon$ and the acute angle between A and C is greater than $\alpha_0 - 3\varepsilon$. Consequently, the obtuse angle between A and C is smaller than $\pi - (\alpha_0 - 3\varepsilon)$. Thus we have for the angle β of the triangle:

$$\alpha_0 - 3\varepsilon < \beta < \pi - (\alpha_0 - 3\varepsilon).$$

Therefore by the law of sines, the inequalities $(2/\pi)\, x \leqslant \sin x \leqslant x$ for $x \in [0, \pi/2]$, and (1.35), we get

$$\frac{1}{\tau^2} \leqslant \frac{\text{length}(C)}{\text{length}(B)} = \frac{\sin\gamma}{\sin\beta} < \frac{\sin 2\varepsilon}{\sin(\alpha_0 - 3\varepsilon)} \leqslant \frac{\pi\varepsilon}{\alpha_0 - 3\varepsilon} \;\Rightarrow\; \varepsilon > \varepsilon_0(\alpha_0, \tau) := \frac{\alpha_0}{3 + \pi\tau^2}.$$

Now let the angle γ between A and B be not acute (it is greater than $\pi - 2\varepsilon$). Then for one of acute angles of the triangle, say β, we have

$$\beta < \varepsilon - (\alpha_0 - 2\varepsilon) = 3\varepsilon - \alpha_0 < 0, \qquad \varepsilon \in (0; \alpha_0/3),$$

which is impossible.

It follows from the aforesaid that there is a contradiction for $\varepsilon = \varepsilon_0(\alpha_0, \tau)$ in the second case and thus $\Upsilon(0, u, v) \geqslant \frac{2}{3} \sin^4 \varepsilon_0(\alpha_0, \tau)$. ∎

We will also need the following result.

Lemma 1.9 (Lemma 2.3 in [CMPT1]). *Fix $\alpha_0 \in (0, \pi/2)$. Given κ_n of the form* (0.16) *and $(z_1, z_2, z_3) \in \mathcal{O}_\tau$, if the condition* (1.36) *is satisfied, then*

$$p_{\kappa_n}(z_1, z_2, z_3) \geqslant C_3(\alpha_0, \tau) \cdot c(z_1, z_2, z_3)^2$$

for some $C_3(\alpha_0, \tau) > 0$.

Now we are able to obtain necessary lower pointwise estimates for p_{K_t} if t are the end points of the intervals excluded in (1.5) and (1.6) from the real line. Recall that $\rho_{n,N} = (\frac{N}{n} - 2)\sqrt{N - 2n}$.

Lemma 1.10. *Fix $\alpha_0 \in (0, \pi/2)$. Given K_t of the form* (1.1) *and $(z_1, z_2, z_3) \in \mathcal{O}_\tau$,*

(*i*) *if* (1.36) *is satisfied and $t = 2 - \frac{N}{n}$ for $n < N \leqslant 2n$ or $t = \rho_{n,N}$ for $N \geqslant 2n$, or*

(*ii*) *if* (1.36) *and* (1.37) *are satisfied and $t = -\frac{1}{2}\left(3 + \sqrt{9 - 4\frac{N}{n}}\right)$ for $n < N \leqslant 2n$ or $t = -\frac{1}{2}\left(3 + \rho_{n,N} + \sqrt{(3 + \rho_{n,N})^2 - 4\frac{N}{n}}\right)$ for $N \geqslant 2n$,*

then

$$p_{K_t}(z_1, z_2, z_3) \geqslant C(\alpha_0, \tau) \cdot c(z_1, z_2, z_3)^2$$

for some $C(\alpha_0, \tau) > 0$.

Proof. It is enough to prove it for triples $(0, u, v)$. What is more, the statement for $t = 0$ in (*i*), i.e. when $N = 2n$, is just Lemma 1.9 and therefore we may exclude it. We also recall the notation $\xi_j := \lambda_j^{2(N-n)} \in [0, 1]$, $j = 1, 2, 3$.

Now let $t = 2 - \frac{N}{n}$ and $n < N < 2n$. Then for the function given in (1.20) we have

$$f(\xi_1, \xi_2, \xi_3) = \tfrac{N}{n}\, \xi_1\xi_2\xi_3 + (2 - \tfrac{N}{n})(\xi_1 + \xi_2 + \xi_3) \geqslant (2 - \tfrac{N}{n})\tau_1,$$

where τ_1 is as in (1.15). If $t = \rho_{n,N}$ and $N > 2n$, then from (1.32) it follows that

$$g(\xi_1, \xi_2, \xi_3) = \tfrac{N}{n}\, \xi_1\xi_2\xi_3 + \rho_{n,N}(\xi_1 + \xi_2 + \xi_3) \geqslant \tfrac{N}{n}\tau_1.$$

Thus from the inequalities (1.19) for $n < N < 2n$ and (1.31) for $N > 2n$, both being valid for $t \geqslant 0$, and Lemmas 1.7 and 1.9 (with the assumption (1.36)) we get

$$p_{K_t}(0,u,v) \geqslant \tfrac{N}{n}\tau_1(0,u,v) \cdot p_{\kappa_n}(0,u,v) \geqslant \tfrac{N}{n}\, C_1(\alpha_0)\, C_3(\alpha_0,\tau) \cdot c(0,u,v)^2,$$

which is the required result in the case (i).

Now consider (ii). Set $t = -\frac{1}{2}\left(3 + \sqrt{9 - 4\frac{N}{n}}\right)$ and $n < N \leqslant 2n$. Then, by (1.22),

$$F(\xi_1,\xi_2,\xi_3) = \tfrac{N}{n}(\xi_1\xi_2\xi_3 - 1) + \tfrac{1}{2}(3 - (\xi_1+\xi_2+\xi_3))(3 + \sqrt{9 - 4\tfrac{N}{n}}).$$

Since $-\frac{1}{2}\left(3 + \sqrt{9 - 4\frac{N}{n}}\right) \leqslant -\frac{N}{n}$,

$$F(\xi_1,\xi_2,\xi_3) \geqslant \tfrac{N}{n}(2 + \xi_1\xi_2\xi_3 - (\xi_1+\xi_2+\xi_3)) \geqslant \tfrac{N}{n}\Upsilon.$$

The function (1.34) for

$$t = t_0 := -\tfrac{1}{2}\left(3 + \rho_{n,N} + \sqrt{(3+\rho_{n,N})^2 - 4\tfrac{N}{n}}\right)$$

and $N \geqslant 2n$ after some simplifications takes the form

$$G(\xi_1,\xi_2,\xi_3) = \tfrac{N}{n}(\xi_1\xi_2\xi_3 - 1) + (\xi_1+\xi_2+\xi_3 - 3)t_0,$$

and hence $G(\xi_1,\xi_2,\xi_3) \geqslant \frac{N}{n}\Upsilon$ since $t_0 \leqslant -\frac{N}{n}$ for $N \geqslant 2n$.

Thus for the last two values of t, by the inequalities (1.21) for $n < N \leqslant 2n$ and (1.33) for $N \geqslant 2n$, both being valid for $t \leqslant -\frac{N}{n}$, and Lemma 1.9 (with the assumption (1.36)), we get

$$p_{K_t}(0,u,v) \geqslant \tfrac{N}{n}\,\Upsilon(0,u,v) \cdot p_{\kappa_n}(0,u,v) \geqslant \tfrac{N}{n}\,\Upsilon(0,u,v)\, C_3(\alpha_0,\tau) \cdot c(0,u,v)^2.$$

If (1.37) is also satisfied, then by Lemma 1.8 we obtain the desired inequality

$$p_{K_t}(0,u,v) \geqslant \tfrac{N}{n}\, C_2(\alpha_0,\tau)\, C_3(\alpha_0,\tau) \cdot c(0,u,v)^2,$$

and we are done. ■

1.4 Examples

In this section we present triples $(0,u,v)$ such that the permutations $p_{K_t}(0,u,v)$ change sign for t mentioned in (1.7), namely, $t \in (-N/n, 0)$ for all n and N and $t \in \left(0, \frac{2}{e^{3/2}}\frac{N}{n}\right)$ for $N \gg n$. We use the notations of Lemma 1.1 below. Note that by this lemma, p_{K_t} can be calculated via p_{κ_m} for m, equal to n and N, and $\varphi_{n,N}$. To obtain p_{κ_m} we substitute (0.16) into (1.8); $\varphi_{n,N}$ is calculated by (1.10).

We first show that $p_{K_t}(0,u,v)$ is positive for any t if $u = a + i$ and $v = a - i$, where $a \in \mathbb{R} \setminus \{0\}$ is suitably chosen. By (1.8) and taking into account that $K_t(u) = K_t(v)$ and $K_t(u - v) \equiv 0$,

$$p_{K_t} = \left(\frac{a^{2N-1}}{(1+a^2)^N} + t \cdot \frac{a^{2n-1}}{(1+a^2)^n}\right)^2 = \frac{a^{4n-2}}{(1+a^2)^{2n}}\left(\frac{a^{2(N-n)}}{(1+a^2)^{N-n}} + t\right)^2,$$

which is positive for any real t if a is chosen so that the expression in the latter brackets does not vanish.

Now the aim is to show that for any fixed t from (1.7) there exist triples $(0,u,v)$ such that $p_{K_t}(0,u,v)$ is negative. To do so, we find families of $(0,u,v)$ such that the quadratic polynomial

$$\frac{p_{K_t}}{p_{\kappa_n}} = \frac{p_{\kappa_N}}{p_{\kappa_n}} + \frac{\varphi_{n,N}}{p_{\kappa_n}} t + t^2, \qquad p_{\kappa_n} > 0,$$

(with respect to t) has two different roots t_1 and t_2, depending on u and v, and thus p_{K_t} (of the form (1.9)) is negative for $t \in (t_1;t_2)$. In addition, we prove that the union of the intervals $(t_1;t_2)$ when $(0,u,v)$ runs the whole above-mentioned family is either the interval $(-N/n;0)$ or $\left(0;\frac{2}{e^{3/2}}\frac{N}{n}\right)$, indicated in (1.7).

Let us consider the case $t<0$.

Example 1.1. Set $u=-a+i$, $v=a+i$, where $a \in \mathbb{R}\setminus\{0\}$. Then

$$p_{\kappa_m} = -\frac{a^{2(2m-1)}}{(a^2+1)^{2m}} + \frac{a^{2m-2}}{2(a^2+1)^m} + \frac{a^{2m-2}}{2(a^2+1)^m} = \frac{a^{2m-2}\left((a^2+1)^m - a^{2m}\right)}{(a^2+1)^{2m}},$$

where m equals n or N, and

$$\begin{aligned}\varphi_{n,N} = &-\frac{a^{2N-1}}{(a^2+1)^N}\left(\frac{a^{2n-1}}{(a^2+1)^n} - \frac{1}{2a}\right)\\ &+\frac{a^{2N-1}}{(a^2+1)^N}\left(-\frac{a^{2n-1}}{(a^2+1)^n} + \frac{1}{2a}\right) - \frac{1}{2a}\left(-\frac{a^{2n-1}}{(a^2+1)^n} - \frac{a^{2n-1}}{(a^2+1)^n}\right)\\ =&\frac{a^{2n-2}\left((a^2+1)^N + (a^2+1)^n a^{2(N-n)} - 2a^{2N}\right)}{(a^2+1)^{N+n}}.\end{aligned}$$

From this by (1.9) we deduce that

$$\frac{p_{K_t}}{p_{\kappa_n}} = \frac{d_1(a)d_2(a)}{d_3(a)^2} + \frac{d_1(a)+d_2(a)}{d_3(a)} t + t^2, \qquad a \neq 0, \tag{1.38}$$

where

$$d_1(a) := (a^2+1)^n\left(1 - \frac{a^{2N}}{(a^2+1)^N}\right), \quad d_2(a) := (a^2+1)^n\left(\frac{a^{2(N-n)}}{(a^2+1)^{N-n}} - \frac{a^{2N}}{(a^2+1)^N}\right)$$

and $d_3(a) := (a^2+1)^n - a^{2n}$. The polynomial (1.38) has two different negative roots $t_1(a) = -d_1(a)/d_3(a)$ and $t_2(a) = -d_2(a)/d_3(a)$, where $d_1(a) > d_2(a) > 0$ and $d_3(a) > 0$. It is easy to check that the roots $t_1(a)$ and $t_2(a)$ run the intervals $(-N/n;-1)$ and $(-1;0)$, correspondingly, when a runs $(0;\infty)$. Furthermore, we see by continuity that

$$\bigcup_{a\in(0;\infty)} (t_1(a);t_2(a)) = (-N/n;0).$$

This means that p_{K_t} is negative for any t in $(-N/n;0)$ for a suitably chosen.

As we have already mentioned above, this example shows that (1.6) is sharp for $N = 2n$ in the sense of Remark 1.1. In addition, the left hand side of (1.6) is also sharp for $N = 2n+1$.

Now we give the example for $t>0$.

Example 1.2. Let n be fixed. Consider the triples $(0,u,v)$ such that

$$u=-r(1+\delta_N(q)i),\qquad v=r(1-\tfrac{1}{r}+\delta_N(q)i),\qquad \delta_N(q):=\sqrt{\frac{\ln q}{N-n}},\tag{1.39}$$

where $r>0$ and $q\geqslant e$. We can also calculate p_{κ_m} and $\varphi_{n,N}$ for these $(0,u,v)$ using (0.16), (1.8) and (1.10). However, the expression of p_{K_t}/p_{κ_n} obtained is too big and therefore we do not place it here. Instead, we give the following identity (the permutations are calculated for $(0,u,v)$ as in (1.39)):

$$P(t):=\lim_{r\to\infty}\frac{p_{K_t}}{p_{\kappa_n}}=c_N(q)+b_N(q)t+t^2,$$

where

$$c_N(q)=\frac{N}{n\,(1+\delta_N^2(q))^{2(N-n)}}\cdot\frac{(2N-1)\delta_N^2(q)+1}{(2n-1)\delta_N^2(q)+1}$$

and

$$b_N(q)=-\frac{\big(2(N-n)^2+N-4nN+n\big)\,\delta_N^2(q)-(n+N)}{n\,(1+\delta_N^2(q))^{N-n}((2n-1)\delta_N^2(q)+1)}.$$

Note that

$$c_N(q)\sim\frac{(2\ln q+1)N}{q^2n},\qquad b_N(q)\sim-\frac{(2\ln q-1)N}{qn},\qquad N\to\infty.$$

The quadratic polynomial P (with respect to t) has two different positive roots $t_1(N,q)$ and $t_2(N,q)$ if N is large enough (as the discriminant is positive). Additionally, one can check that

$$t_1(N,q)\sim\tilde t_1(q):=\frac{2\ln q+1}{q\,(2\ln q-1)},\quad t_2(N,q)\sim\tilde t_2(N,q):=\frac{(2\ln q-1)N}{qn},\quad N\to\infty.$$

Taking into account the properties

$$\tilde t_1(q)\to 0\text{ as }q\to\infty\qquad\text{and}\qquad\max_{q\in[e;\,\infty)}\frac{2\ln q-1}{q}=\frac{2}{e^{3/2}},$$

we deduce by continuity that

$$\bigcup_{q\in[e;\,\infty)}(\tilde t_1(q);\tilde t_2(q))=\left(0;\frac{2}{e^{3/2}}\frac{N}{n}\right).$$

Thus, $p_{K_t}(0,u,v)$ with u and v as in (1.39) are negative for any t in $\left(0;\frac{2}{e^{3/2}}\frac{N}{n}\right)$, if N (with respect to n) and r are large enough and q is suitably chosen.

1.5 Proof of Theorems 1.1 and 1.2

Recall that Lemmas 1.5 and 1.6 state that if K_t is of the form (1.1) and

$$t\in\mathbb{R}\setminus\left[-\tfrac{1}{2}\left(3+\sqrt{9-4\tfrac{N}{n}}\right),2-\tfrac{N}{n}\right],\qquad n<N\leqslant 2n,$$
$$t\in\mathbb{R}\setminus\left[-\tfrac{1}{2}\left(3+\rho_{n,N}+\sqrt{(3+\rho_{n,N})^2-4\tfrac{N}{n}}\right),\rho_{n,N}\right],\qquad N\geqslant 2n,$$

where $\rho_{n,N} = \left(\frac{N}{n} - 2\right)\sqrt{N-2n}$, then

$$p_{K_t} \geqslant C(t) \cdot p_{\kappa_n}, \qquad C(t) > 0.$$

These lemmas immediately give Theorem 1.1 by continuity if we take into account the fact that $p_{\kappa_n}(z_1, z_2, z_2) \geqslant 0$ for all $(z_1, z_2, z_3) \in \mathbb{C}^3$ (see (0.17)).

What is said from now on is related to Theorem 1.2.

First of all we note that the latter statement of Theorem 1.2, i.e. the one asserting that if the operator T_{K_t} is $L^2(\mathcal{H}^1\lfloor E)$-bounded, then E is rectifiable, is a corollary of the fact that the $L^2(\mathcal{H}^1\lfloor E)$-boundedness of T_{K_t} implies that $p_{K_t}(\mathcal{H}^1\lfloor E) < \infty$. This follows from the generalised Melnikov-Verdera identity (0.13) for the kernels K_t and permutations p_{K_t}.

Now we come to the proof of the former statement in Theorem 1.2.

The proof for t, mentioned in Lemmas 1.5 and 1.6 (see also the beginning of the current section), is direct via Theorem D, which states that if $p_{\kappa_n}(\mathcal{H}^1\lfloor E) < \infty$, then E is rectifiable. Indeed, if $p_{K_t}(\mathcal{H}^1\lfloor E) < \infty$ for such t, then $p_{\kappa_n}(\mathcal{H}^1\lfloor E) < \infty$ by the inequality $p_{K_t} \geqslant C(t) \cdot p_{\kappa_n}$, $C(t) > 0$, and thus the set E is rectifiable.

What is left is to prove the former statement in Theorem 1.2 for

$$t = -\tfrac{1}{2}\left(3 + \sqrt{9 - 4\tfrac{N}{n}}\right), \qquad t = 2 - \tfrac{N}{n}, \qquad n < N \leqslant 2n,$$
$$t = -\tfrac{1}{2}\left(3 + \rho_{n,N} + \sqrt{(3+\rho_{n,N})^2 - 4\tfrac{N}{n}}\right), \qquad t = \rho_{n,N}, \qquad N \geqslant 2n.$$

It requires some additional work and therefore for the reader's convenience we first make several observations, which could help to clarify the forthcoming proof.

We start with a very brief exposition of the proof of Theorem B given in [Leg] (note that one can find a modified version of the proof from [Leg] in [Tol5, Chapter 7] and follow that instead). Recall that Theorem B states that for a 1-set E, if $c^2(\mathcal{H}^1\lfloor E) < \infty$, then E is rectifiable. We emphasize again that it is essential in the proof that the curvature is non-negative.

The first step is to show that there exists a compact subset F of the given set E such that, among other things, $c^2(\mathcal{H}^1\lfloor F)$ is well-controlled and can be made very small (this is done in [Leg] by a quite standard uniformization procedure). Then the second and most important step follows: to prove that if μ is a measure satisfying a few special conditions (see Proposition 1.2), then there exists a Lipschitz graph Γ such that $\mu(\Gamma) \geqslant C \cdot \mu(\mathbb{C})$, where C is an absolute constant.

The problem is to choose an adequate coordinate system of $\mathbb{C}$ and construct a Lipschitz function A whose graph will be the one needed. For this purpose, the author of [Leg] first defines some functions used to measure how well the $\operatorname{spt}\mu$ is approximated by straight lines at a given location and a given scale. It is shown that these functions are related to the $c^2(\mu)$ in the case when the measure μ does not degenerate too much. These preliminary results are then used to construct the function A by stopping time arguments, which demand fine adjustments to many parameters and thresholds. Starting with choosing a point $x_0 \in \operatorname{spt}\mu$ and fixing an approximating line D_0 (which will be the domain of the function A) such that the mean distance from $\operatorname{spt}\mu$ to the line D_0 is suitably small, the author of [Leg] comes to cutting $\operatorname{spt}\mu$ in four disjoint pieces Z, F_1, F_2 and F_3 such that

$$\operatorname{spt}\mu = Z \cup F_1 \cup F_2 \cup F_3.$$

It is shown that Z is very nice for constructing the graph but the three others admit "bad events". Then the goal is to prove that these bad pieces carry only a small part of the measure μ, namely, $\mu(F_j) \leqslant 10^{-6}\mu(\mathbb{C})$ for each j and thus $\mu(Z) \geqslant \frac{99}{100}\mu(\mathbb{C})$. This allows to construct the required Lipschitz function $A : D_0 \to D_0^{\perp}$ such that the set Z is contained in the graph of A.

Coming back to the initial settings, if $\mu = \mathcal{H}^1\lfloor F$, where F is the above-mentioned subset of E, then there exists a Lipschitz graph Γ such that $\mathcal{H}^1(\Gamma \cap F) \geqslant C \cdot \mathcal{H}^1(F)$. This fact is used in the last step of the proof from [Leg], which is as follows. Since $\mathcal{H}^1(E) < \infty$ by the assumptions, the set E can be decomposed into a rectifiable and purely unrectifiable part, i.e. $E = E_{\text{rect}} + E_{\text{unrect}}$. Suppose that

$$\mathcal{H}^1(E_{\text{unrect}}) > 0. \tag{1.40}$$

Then there exists a compact set $F \subset E_{\text{unrect}}$ and Lipschitz graph Γ such that $\mathcal{H}^1(\Gamma \cap F) \geqslant C \cdot \mathcal{H}^1(F)$ that contradicts the fact that F is purely unrectifiable.

Let us now say a few words about the proof of Theorem D given in [CMPT1]. Recall that this theorem is an analogue of Theorem B, where the kernel $1/z$ and curvature squared $c^2(\mathcal{H}^1\lfloor E)$ are replaced by the kernels $\kappa_n(z) = (\operatorname{Re} z)^{2n-1}/|z|^{2n}$, $n \in \mathbb{N}$, and corresponding permutations p_{κ_n}. We will use the definitions given near the formula (1.35) and in the discussion of Theorem B above. First we mention that it is proved in [CMPT1] that the permutations $p_{\kappa_n}(z_1, z_2, z_3)$ behave similarly to $c^2(z_1, z_2, z_3)$ for all triangles with comparable sides, whose one side makes a big angle with the vertical line. More precisely (see Lemma 1.9), it is shown there that for a fixed $\alpha_0 \in (0, \pi/2)$ and given $(z_1, z_2, z_3) \in \mathcal{O}_\tau$, if the condition (1.36), i.e.

$$\theta_V(L_{z_1,z_2}) + \theta_V(L_{z_2,z_3}) + \theta_V(L_{z_1,z_3}) \geqslant \alpha_0,$$

or

$$\theta_H(L_{z_1,z_2}) + \theta_H(L_{z_2,z_3}) + \theta_H(L_{z_1,z_3}) \leqslant \tfrac{3}{2}\pi - \alpha_0,$$

is satisfied, then

$$p_{\kappa_n}(z_1, z_2, z_3) \geqslant C(\alpha_0, \tau) \cdot c(z_1, z_2, z_3)^2, \qquad C(\alpha_0, \tau) > 0.$$

This enables the authors of [CMPT1] to use the above-described scheme from [Leg] in order to construct the required Lipschitz graph Γ in the case when the first approximating line D_0 for $\operatorname{spt}\mu$ is far from the vertical line. Note that exchanging the curvature for the permutations p_{κ_n} still requires new arguments in several key points of the proof. Otherwise, when D_0 is close to the vertical line and the scheme from [Leg] does not work (as $\mu(F_3)$ may be too big), they tune thresholds and apply some coverings so that they can use the result for D_0, being far from the vertical line, to construct countably many Lipschitz graphs, which give Γ after appropriate joining.

We are now at the position to finish the proof of our Theorem 1.2. This will be an adaptation of the arguments from [CMPT1].

On the one hand, by the clause (i) of Lemma 1.10, for a fixed $\alpha_0 \in (0, \pi/2)$ and given $(z_1, z_2, z_3) \in \mathcal{O}_\tau$, if the condition (1.36), i.e. the same as in the result for $t = 0$ from [CMPT1] mentioned above, is satisfied and $t = 2 - \frac{N}{n}$ for $n < N \leqslant 2n$ or $t = \rho_{n,N}$ for $N \geqslant 2n$, then we also have

$$p_{K_t}(z_1, z_2, z_3) \geqslant C(\alpha_0, \tau) \cdot c(z_1, z_2, z_3)^2, \qquad C(\alpha_0, \tau) > 0. \tag{1.41}$$

It means that we can undeviatingly follow the scheme from [CMPT1] (exchanging p_{κ_n} for p_{K_t}) in order to get our result for $t = 2 - \frac{N}{n}$, $n < N \leqslant 2n$, and $t = \rho_{n,N}$, $N \geqslant 2n$.

On the other hand, by the clause (*ii*) of Lemma 1.10, the inequality (1.41) is true for

$$\begin{aligned} t &= -\tfrac{1}{2}\left(3+\sqrt{9-4\tfrac{N}{n}}\right), && n < N \leqslant 2n, \\ t &= -\tfrac{1}{2}\left(3+\rho_{n,N}+\sqrt{(3+\rho_{n,N})^2-4\tfrac{N}{n}}\right), && N \geqslant 2n, \end{aligned} \tag{1.42}$$

only if both the conditions (1.36) and (1.37) are satisfied, i.e.

$$\alpha_0 \leqslant \theta_V(L_{z_1,z_2}) + \theta_V(L_{z_2,z_3}) + \theta_V(L_{z_1,z_3}) \leqslant \tfrac{3}{2}\pi - \alpha_0,$$

or

$$\alpha_0 \leqslant \theta_H(L_{z_1,z_2}) + \theta_H(L_{z_2,z_3}) + \theta_H(L_{z_1,z_3}) \leqslant \tfrac{3}{2}\pi - \alpha_0,$$

and thus the triangles (z_1, z_2, z_3) are far from both the vertical and horizontal line.

Consequently, the scheme from [CMPT1] cannot be applied directly for t from (1.42). However, as we will see, it works after a few modifications (besides the exchange of p_{κ_n} for p_{K_t}) connected basically with adapting geometrical arguments to both the conditions (1.36) and (1.37). Since the cases where we are close to either the vertical or horizontal line are well-separated and similar geometrically, the arguments for the first approximating line D_0, being close (far) to (from) the vertical line, can be easily transferred into the ones for D_0, being close (far) to (from) the horizontal line.

We now reproduce the main steps of the proof, stemming from [CMPT1], with necessary changes when our permutations and the conditions (1.36) and (1.37) are involved. Several statements are formulated without proofs because they are the same as in [CMPT1] (or [Leg]) modulus the permutations involved.

Below we consider only t from (1.42). The following two propositions will then imply Theorem 1.2 by the same contradiction arguments as in the proof from [Leg] (see the arguments around (1.40) above). Note that one has to take $\mu = 40\mathcal{H}^1\lfloor F$ in Proposition 1.2, where the set F is from Proposition 1.1 (it may be suitably rescaled if necessary).

Proposition 1.1 (An analogue of Lemma 3.4 in [CMPT1] and Proposition 1.1 in [Leg])**.** *Let E be a 1-set and $p_{K_t}(\mathcal{H}^1\lfloor E) < \infty$. Then for all $\eta > 0$ there is a set $F \subset E$ such that*

- *F is compact,*
- *$p_{K_t}(\mathcal{H}^1\lfloor F) \leqslant \eta \operatorname{diam} F$,*
- *$\mathcal{H}^1(F) > \frac{1}{40} \operatorname{diam} F$,*
- *for all $z \in F$, for all $r > 0$, $\mathcal{H}^1(F \cap B(z,r)) \leqslant 3r$.*

Proposition 1.2. *For any $C_0 \geqslant 10$, there is $\eta > 0$ such that if a measure μ satisfies*

- *$\mu(B(0,1)) \geqslant 1$, $\mu(\mathbb{C} \setminus B(0,2)) = 0$,*
- *for any ball B, $\mu(B) \leqslant C_0 \operatorname{diam} B$,*
- *$p_{K_t}(\mu) \leqslant \eta$,*

then there exists a Lipschitz graph Γ such that $\mu(\Gamma) \geqslant 10^{-5}\mu(\mathbb{C})$.

The rest of the section is devoted to the proof of Proposition 1.2 which is an analogue of [CMPT1, Proposition 3.1]. First, we give several definitions that will be needed below, see [CMPT1, Leg] for further ones. Given a measure μ, set

$$p_{K_t,\tau}(\mu) := \iiint_{\mathcal{O}_\tau} p_{K_t}(z_1, z_2, z_3) d\mu(z_1) d\mu(z_2) d\mu(z_3),$$

see (0.6), (1.2) and (1.35). For a ball $B = B(x, r)$ set

$$\delta_\mu(x, r) := \frac{\mu(B(x, r))}{r}.$$

We will use a small density threshold $\delta > 0$ for this quantity.

Given a fixed $k > 1$, for any ball $B = B(x, r) \subset \mathbb{C}$ and D a line in $\mathbb{C}$, set

$$\beta^D_{1,\mu}(x, r) := \frac{1}{r} \int_{B(x,kr)} \frac{\operatorname{dist}(y, D)}{r}\, d\mu(y),$$

$$\beta^D_{2,\mu}(x, r) := \left(\frac{1}{r} \int_{B(x,kr)} \left(\frac{\operatorname{dist}(y, D)}{r} \right)^2 d\mu(y) \right)^{1/2}.$$

Geometrical notation connected with lines and angles is given near the formula (1.35) at the end of Section 1.3.

Lemma 1.11. *Let μ be a measure with C_0-linear growth and $B(x, r) \subset \mathbb{C}$ a ball with $\delta_\mu(x, r) \geqslant \delta$. Suppose that τ is big enough, then for any $\varepsilon > 0$, there exists some $\delta_1 = \delta_1(\delta, \varepsilon) > 0$ such that*

$$\frac{p_{K_t,\tau}(\mu\lfloor kB)}{\mu(B)} \leqslant \delta_1 \quad \Longrightarrow \quad \inf_D \beta^D_{2,\mu}(x, r) \leqslant \varepsilon.$$

Proof. The proof is the same as for Lemma 4.4 in [CMPT1]. We just have to use our Lemma 1.10 instead of Lemma 2.3 there for the case when both the conditions (1.37) and (1.36) are satisfied, and say that in the case

$$\theta_H(L_{z_1,z_2}) + \theta_H(L_{z_1,w}) + \theta_H(L_{z_2,w}) \leqslant \alpha_0$$

we obtain the same estimate for $\operatorname{dist}(w, L_{z_1,z_2})$ as in the case

$$\theta_V(L_{z_1,z_2}) + \theta_V(L_{z_1,w}) + \theta_V(L_{z_2,w}) \leqslant \alpha_0.$$

■

By Lemma 1.11, chosen a point $x_0 \in \operatorname{spt} \mu$, there exists an approximating line D_0 such that $\beta^{D_0}_{1,\mu}(x_0, 1) \leqslant \varepsilon$. The next step is to construct a first Lipschitz graph in the case when D_0 is far from both the horizontal and vertical lines.

To do so, one first has to introduce a family of stopping time regions and obtain the partition $\operatorname{spt} \mu = Z \cup F_1 \cup F_2 \cup F_3$ (see the exposition of the proof from [Leg] above). As this entirely repeats the corresponding part of [CMPT1, Section 5] (cf. [Leg, Subsection 3.1]), we omit it. We just have to mention that the thresholds θ_0 and α, arising there, have to be adapted to that D_0 is far from both the horizontal and vertical line. Namely, θ_0 is now a threshold for both $\theta_V(D_0)$ and $\theta_H(D_0)$. It means that one has to distinguish not only the cases $\theta_V(D_0) \geqslant \theta_0$ and $\theta_V(D_0) < \theta_0$ but also $\theta_H(D_0) \geqslant \theta_0$ and $\theta_H(D_0) < \theta_0$. Moreover, α is tuned as follows: if $\theta_V(D_0)$ or $\theta_H(D_0)$ are greater than θ_0, then $\alpha \leqslant \theta_0/10$; if $\theta_V(D_0)$ or $\theta_H(D_0)$ are not greater than θ_0, then $\alpha = 10\theta_0$.

Furthermore, see [Leg, CMPT1] for the way how one can define the Lipschitz function A on the line D_0, using Z, F_1, F_2, F_3, and appropriate thresholds.

Now we come to the main step of the proof of Proposition 1.2. The following lemma is an analogue of Lemma 6.1 from [CMPT1].

Lemma 1.12. *Under the assumptions of Proposition 1.2, if furthermore*

$$\theta_0 < \theta_V(D_0) < \tfrac{\pi}{2} - \theta_0,$$

then there exists a Lipschitz graph Γ *such that* $\mu(\Gamma) \geqslant \frac{99}{100}\mu(\mathbb{C})$.

For the proof one uses the above-mentioned function A to obtain the graph Γ, $Z \subset \Gamma$, and show that

$$\mu(F_1) + \mu(F_2) + \mu(F_3) \leqslant \tfrac{1}{100}\mu(\mathbb{C}).$$

Indeed, the following lemmas are valid (recall that $\mu(\mathbb{C}) \geqslant 1$ by the assumptions).

Lemma 1.13. *Under the assumptions of Proposition 1.2,*

$$\mu(F_1) \leqslant 10^{-6}.$$

Proof. This is an analogue of [CMPT1, Proposition 6.3], whose proof includes consideration of the two cases: 1) $\theta_V(D_0) > \theta_0$ and 2) $\theta_V(D_0) \leqslant \theta_0$ (see the proof of [CMPT1, Lemma 6.4]).

Under our settings, we have to consider three cases. Namely, the case 1) has to be exchanged for $\theta_0 < \theta_V(D_0) < \frac{\pi}{2} - \theta_0$, although the proof remains the same. The case 2) splits up into the following two: $\theta_V(D_0) \leqslant \theta_0$ and $\theta_V(D_0) \geqslant \frac{\pi}{2} - \theta_0$ (i.e. $\theta_H(D_0) \leqslant \theta_0$). Arguments in the latter case are the same as in the former one. ∎

Lemma 1.14 (An analogue of Proposition 6.2 in [CMPT1])**.** *Under the assumptions of Proposition 1.2,*

$$\mu(F_2) \leqslant 10^{-6}.$$

Lemma 1.15. *Under the assumptions of Lemma 1.12,*

$$\mu(F_3) \leqslant 10^{-6}.$$

Proof. The proof stems from the one of [CMPT1, Proposition 6.5], but with exchange of $\theta_V(D_0) > \theta_0$ for $\theta_0 < \theta_V(D_0) < \frac{\pi}{2} - \theta_0$ as in Lemma 1.12. ∎

Thus Proposition 1.2 is proved under the assumptions of Lemma 1.12. What is left is to consider the other case.

Lemma 1.16. *Under the assumptions of Proposition 1.2, if furthermore*

$$\theta_V(D_0) \leqslant \theta_0 \qquad \text{or} \qquad \theta_V(D_0) \geqslant \tfrac{\pi}{2} - \theta_0 \quad (\text{i.e.}\quad \theta_H(D_0) \leqslant \theta_0),$$

then there exists a Lipschitz graph Γ *such that* $\mu(\Gamma) \geqslant 10^{-5}\mu(\mathbb{C})$.

Proof. To prove this, we repeat arguments from the proof of [CMPT1, Lemma 7.1], given for $\theta_V(D_0) \leqslant \theta_0$, for the case $\theta_H(D_0) \leqslant \theta_0$. ∎

1.6 Additional remarks

In this section we generalise Theorem 1.2 to higher dimensions. Let us introduce necessary notation first. For $d \in \mathbb{N}^+$ and $E \subset \mathbb{R}^d$ with finite length we consider a SIO $\mathbf{T}_{K_t} = (T^j_{K_t})^d_{j=1}$ such that formally

$$T^j_{K_t} f(x) := \int_E f(y) K^j_t(x-y)\, d\mathcal{H}^1(y), \qquad K^j_t(x) := \kappa^j_N(x) + t \cdot \kappa^j_n(x),$$

where $\kappa^j_n(x) := x_j^{2n-1}/|x|^{2n}$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d \setminus \{0\}$. As before, we suppose that $N > n$, where $n, N \in \mathbb{N}^+$, and $t \in \mathbb{R}$. We also need the permutations

$$\mathbf{P}_{K_t}(x,y,z) := \sum_{j=1}^d P_{K^j_t}(x,y,z) \qquad \text{for distinct points } x,y,z \in \mathbb{R}^d,$$

where $P_{K^j_t}(x,y,z)$ are the same as in (0.5) with K^j_t instead of K. We also define $\mathbf{P}_{K_t}(\mu)$ analogously to (0.6).

Theorem 1.3. *Let t be as mentioned in Theorem 1.1. Given a Borel set $E \subset \mathbb{R}^d$ such that $0 < \mathcal{H}^1(E) < \infty$, if $\mathbf{P}_{K_t}(\mathcal{H}^1\lfloor E) < \infty$, then E is rectifiable. Moreover, if the operator $\mathbf{T}_{K_t}$ is $L^2(\mathcal{H}^1\lfloor E)$-bounded, then E is rectifiable.*

This result for $t = 0$ was recently proved in [CP, Theorem 1.2(1) and Theorem 6.2]. To prove Theorem 1.3 for all required t we only need to use our Lemmas 1.5, 1.6 and 1.10 in order to show that for all $x, y, z \in \mathbb{R}^d$ such that $(x,y,z) \in \mathcal{O}_\tau$ and the assumptions of Lemma 1.10 are satisfied,

$$P_{K^j_t}(x,y,z) \geqslant C(t,\alpha_0,\tau)\, P_{K^j_0}(x,y,z), \qquad C(t,\alpha_0,\tau) > 0. \tag{1.43}$$

See the definitions of α_0, τ and $\mathcal{O}_\tau$ before Lemma 1.7. Then by [CP, Proposition 3.3], adapted to the conditions (1.36) and (1.37), and the arguments similar to those in [CP, Section 6] and our Section 1.5 we immediately get the result. Note that [CP, Proposition 3.3] slightly simplifies the approach from [CMPT1] (and improves Lemma 1.9) in the case $t = 0$ as the parameter τ is not needed anymore. In our case this parameter is still necessary because of the inequality (1.43).

To finish, it is also worth mentioning here that under Ahlfors-David regularity assumption one can expect that for t as in Theorem 1.1 the L^2-boundedness of the operator associated with K_t implies uniform rectifiability. This is indeed true and will be proved in Chapter 2.

Chapter 2

Singular integral operators unsuitable for the curvature method whose L^2-boundedness still implies rectifiability

2.1 Introduction

The exposition in this chapter is based on [CMT1]. Below we continue working with the family of kernels defined in (0.24),

$$K_t(z) = \frac{(\operatorname{Re} z)^{2N-1}}{|z|^{2N}} + t \cdot \frac{(\operatorname{Re} z)^{2n-1}}{|z|^{2n}}, \quad t \in \mathbb{R}, \qquad K_\infty(z) = \frac{(\operatorname{Re} z)^{2n-1}}{|z|^{2n}},$$

and its subfamily with $(n, N) = (1, 2)$ defined in (0.21),

$$k_t(z) = \frac{(\operatorname{Re} z)^3}{|z|^4} + t \cdot \frac{\operatorname{Re} z}{|z|^2}, \quad t \in \mathbb{R}, \qquad k_\infty(z) = \frac{\operatorname{Re} z}{|z|^2}.$$

We are going to use the same definitions related to curvature and permutations as given in Introduction and Chapter 1 so it is recommended to recall them. We however emphasize that as in Chapter 1 it is more convenient for us to deal with the following simplified version of the permutations (0.5) introduced in (1.2) in Chapter 1:

$$\begin{aligned} &p_K(z_1, z_2, z_3) \\ &= K(z_1 - z_2)K(z_1 - z_3) + K(z_2 - z_1)K(z_2 - z_3) + K(z_3 - z_1)K(z_3 - z_2) \\ &= \tfrac{1}{2} P_K(z_1, z_2, z_3), \end{aligned}$$

where K is supposed to be an odd and real-valued kernel. Note that the kernels that we consider are odd and real-valued.

Take into account that, under the same assumption on K, by (1.3) and (1.4),

$$p_K(\mu_1, \mu_2, \mu_3) = \tfrac{1}{2} P_K(\mu_1, \mu_2, \mu_3), \qquad p_{K,\varepsilon}(\mu_1, \mu_2, \mu_3) = \tfrac{1}{2} P_{K,\varepsilon}(\mu_1, \mu_2, \mu_3)$$

and

$$p_K(\mu) = \tfrac{1}{2} p_K(\mu, \mu, \mu), \qquad p_{K,\varepsilon}(\mu) = \tfrac{1}{2} P_{K,\varepsilon}(\mu, \mu, \mu),$$

where μ, μ_1, μ_2 and μ_3 are measures.

Recall that by a *measure* we mean a *positive locally finite Borel measure on* $\mathbb{C}$.

We will need several results proved in Chapter 1. Namely, Theorem 1.1 states that if t belongs to the set

$$\Omega(n,N) := \begin{cases} \{0\} \cup \mathbb{R} \setminus \left(-\frac{1}{2}\left(3+\sqrt{9-4\frac{N}{n}}\right), 2-\frac{N}{n}\right) & \text{if } n < N \leqslant 2n, \\ \{0\} \cup \mathbb{R} \setminus \left(-\frac{1}{2}\left(\sigma_{n,M}+\sqrt{\sigma_{n,M}^2-4\frac{N}{n}}\right), \sigma_{n,M}-3\right) & \text{if } N \geqslant 2n, \end{cases} \tag{2.1}$$

where $\sigma_{n,M} := 3 + (\frac{N}{n}-2)\sqrt{N-2n}$, then

$$p_{K_t}(z_1,z_2,z_3) \geqslant 0 \quad \text{for all} \quad (z_1,z_2,z_3) \in \mathbb{C}^3. \tag{2.2}$$

Moreover, by Theorem 1.2, if the SIO T_{K_t}, where $t \in \Omega(n,N)$, is $L^2(\mathcal{H}^1\lfloor E)$-bounded, then E is rectifiable. From the other side, it follows from Remark 1.1 that there exist triples (z_1,z_2,z_3) such that $p_{K_t}(z_1,z_2,z_3)$ change sign if t belongs to the interval

$$\omega(n,N) := (-N/n, 0) \tag{2.3}$$

Obviously, $\omega(n,N) \subseteq \mathbb{R} \setminus \Omega(n,N)$. Note also that $\omega(n,2n) = (-2,0) = \mathbb{R} \setminus \Omega(n,2n)$. For this reason, a curvature-like method cannot be applied directly for $t \in \omega(n,N)$ (see also Remark 1.2 and Figure 0.2).

Thus we come to the question of what happens when $t \in \omega(n,N)$, i.e. the permutations $p_{K_t}(z_1,z_2,z_3)$ change sign and curvature-like methods as in [MMV, Leg, CMPT1, Chu] do not work. In this chapter a partial answer is given in the case of kernels (0.21). Namely, we show that for any fixed $t \in (-2,-\sqrt{2}) \subset \omega(1,2)$ the analogues of Theorems A and B are still valid (a plausible conjecture for the kernels (0.24) with $t \in \omega(n,N)$ is also stated). To the best of our knowledge, this is the first example of kernels with this property in the plane. We also establish an analogue of Theorem A for the kernels (0.24) with $t \in \Omega(n,N)$. The corresponding results are given in the next section.

2.2 Main results

The following two theorems are analogues of Theorems A and B for the kernels (0.21) with $t \in (-2,-\sqrt{2})$, whose corresponding permutations change sign and a curvature-like method cannot be applied directly. We will prove them in Section 2.3 by exploiting sharp estimates for permutations related to the kernels (0.16) but not to the ones in (0.21). This will form the perturbation method that we mentioned above. Recall that $\omega(1,2) = (-2,0)$, see (2.3).

Theorem 2.1. *Let μ be an AD-regular measure and k_t a kernel of the form* (0.21)*, where $t \in (-2,-\sqrt{2}) \subset \omega(1,2)$. The measure μ is uniformly rectifiable if and only if the SIO T_{k_t} is $L^2(\mu)$-bounded.*

Note that this theorem fails if $t = -1 \in \omega(1,2)$ as follows from Theorem E (take into account that the purely unrectifiable set there is AD-regular).

Theorem 2.2. *Let E be a 1-set and k_t a kernel of the form* (0.21)*, where $t \in (-2,-\sqrt{2}) \subset \omega(1,2)$. If the SIO T_{k_t} is $L^2(\mathcal{H}^1\lfloor E)$-bounded, then E is rectifiable.*

This theorem supplements the results about SIO T_{k_t}, see Figure 0.2.

Remark 2.1. As we will see at the end of Section 2.3, it is plausible that analogues of Theorems 2.1 and 2.2 are valid for the kernels (0.24) with $|t| > \sqrt{N/n}$. Note that in particular $(-N/n, -\sqrt{N/n}) \subset \omega(n,N)$, i.e. for t from this interval the corresponding permutations change sign.

We now formulate a Theorem A type result for the kernels (0.24), where $t \in \Omega(n,N)$ and thus the corresponding permutations are non-negative (see (2.1) and (2.2)).

Theorem 2.3. *Let μ be an AD-regular measure and K_t a kernel of the form* (0.24)*, where $t \in \Omega(n,N)$. The measure μ is uniformly rectifiable if and only if the SIO T_{K_t} is $L^2(\mu)$-bounded.*

Since the permutations are non-negative here, we can use a curvature-like method. The proof that will be given in Section 2.4 is more or less analogous to the one used for the kernels (0.16) in [CMPT1, Section 8].

2.3 Proof of Theorems 2.1 and 2.2

Recall that

$$k_\infty(z) = \kappa_1(z) = \frac{\operatorname{Re} z}{|z|^2}, \quad k_0(z) = \kappa_2(z) = \frac{(\operatorname{Re} z)^3}{|z|^4} \quad \text{and} \quad k_t(z) = \kappa_2(z) + t \cdot \kappa_1(z).$$

The following result from [CMPT1] will be necessary below.

Lemma 2.1 (Proof of Proposition 2.1 in [CMPT1])**.** *For any $u = (x,y)$ and $v = (a,b)$ in the complex plane,*

$$p_{\kappa_m}(0,u,v) = \sum_{k=1}^{m} \binom{m}{k} \frac{(ax(x-a))^{2(m-k)}}{|u|^{2m}|v|^{2m}|u-v|^{2m}}\, h_k(u,v), \tag{2.4}$$

where $h_k(u,v) := (ax)^{2k-1}(y-b)^{2k} + (x(x-a))^{2k-1}b^{2k} + (a(a-x))^{2k-1}y^{2k} \geqslant 0$.

To prove Theorems 2.1 and 2.2 we first obtain sharp pointwise estimates for the permutations related to the kernels (0.16).

Lemma 2.2. *It holds that*

$$p_{\kappa_2}(z_1,z_2,z_3) \leqslant 2p_{\kappa_1}(z_1,z_2,z_3) \qquad \textit{for all} \qquad (z_1,z_2,z_3) \in \mathbb{C}^3. \tag{2.5}$$

Proof. It is enough to prove (2.5) for $(z_1,z_2,z_3) = (0,u,v)$ as the permutations of the form (1.2) are invariant under translations (see Section 1.3). Given $u = (x,y)$ and $v = (a,b)$, by (2.4) we get

$$\begin{aligned}
2p_{\kappa_1}&(0,u,v) - p_{\kappa_2}(0,u,v) \\
&= \frac{2h_1(u,v)}{|u|^2|v|^2|u-v|^2} - \frac{2\,x^2a^2(x-a)^2\,h_1(u,v) + h_2(u,v)}{|u|^4|v|^4|u-v|^4} \\
&= \frac{2\left[|u|^2|v|^2|u-v|^2 - \, x^2a^2(x-a)^2\right]\, h_1(u,v) - h_2(u,v)}{|u|^4|v|^4|u-v|^4}.
\end{aligned}$$

Now we obtain a lower estimate of the expression in the square brackets before $h_1(u,v)$. Expanding $|u|^2|v|^2|u-v|^2$ gives

$$\begin{aligned}
&(x^2+y^2)(a^2+b^2)\left((x-a)^2+(y-b)^2\right) - x^2a^2(x-a)^2 \\
&= x^2a^2(y-b)^2 + (x^2b^2+a^2y^2+b^2y^2)\left((x-a)^2+(y-b)^2\right) \\
&\geqslant x^2a^2(y-b)^2 + (x^2b^2+a^2y^2)(x-a)^2.
\end{aligned}$$

Thus,

$$2p_{\kappa_1}(0,u,v) - p_{\kappa_2}(0,u,v) \geqslant \frac{G(x,y,a,b)}{|u|^4|v|^4|u-v|^4},$$

where

$$G(x,y,a,b) := 2(x^2a^2(y-b)^2 + (x^2b^2+a^2y^2)(x-a)^2)\, h_1(u,v) - h_2(u,v).$$

Notice that by Lemma 2.1,

$$\begin{aligned} h_1(u,v) &= ax(y-b)^2 + x(x-a)b^2 + a(a-x)y^2 \\ h_2(u,v) &= (ax)^3(y-b)^4 + (x(x-a))^3b^4 + (a(a-x))^3y^4. \end{aligned}$$

Consequently, to prove the required inequality it is enough to show that $G(x,y,a,b) \geqslant 0$. We separate the discussion into three cases.

1) Let $a=0$. Then

$$G(x,y,0,b) = 2x^4b^2 \cdot x^2b^2 - x^6b^4 = x^6b^4 \geqslant 0.$$

2) Let $b=0$. Then

$$\begin{aligned} &G(x,y,a,0) \\ &\quad = 2\left(a^2x^2y^2 + a^2y^2(x-a)^2\right)\left(axy^2 + a(a-x)y^2\right) - \left(a^3x^3y^4 + a^3(a-x)^3y^4\right) \\ &\quad = 2a^3y^4\left(x^2+(x-a)^2\right)\left(x+(a-x)\right) - a^3y^4\left(x^3-(x-a)^3\right) \\ &\quad = a^4y^4\left(2(x^2+(x-a)^2) - (x^2+x(x-a)+(x-a)^2)\right) \\ &\quad = a^4y^4\left(x^2 - x(x-a) + (x-a)^2\right) \\ &\quad = a^4y^4\left(x^2 - ax + a^2\right) \\ &\quad = a^4y^4\left((x-\tfrac{1}{2}a)^2 + \tfrac{3}{4}a^2\right) \geqslant 0. \end{aligned}$$

3) Let $a \neq 0$ and $b \neq 0$. We divide $G(x,y,a,b)$ by a^6b^4, put $\alpha = x/a$ and $\beta = y/b$ and take into account that by Lemma 2.1 in these settings one has

$$\frac{h_k(u,v)}{a^{4k-2}b^{2k}} = \alpha^{2k-1}(\beta-1)^{2k} + \alpha^{2k-1}(\alpha-1)^{2k-1} - (\alpha-1)^{2k-1}\beta^{2k}, \qquad k=1,2.$$

Therefore

$$\begin{aligned} &\frac{G(x,y,a,b)}{a^6b^4} \\ &\quad = 2\left(\alpha^2(\beta-1)^2 + (\alpha^2+\beta^2)(\alpha-1)^2\right)\left(\alpha(\beta-1)^2 + \alpha(\alpha-1) - (\alpha-1)\beta^2\right) \\ &\qquad - \left(\alpha^3(\beta-1)^4 + \alpha^3(\alpha-1)^3 - (\alpha-1)^3\beta^4\right). \end{aligned}$$

Removing brackets and further collecting terms give

$$\begin{aligned} \frac{G(x,y,a,b)}{a^6b^4} &= \left(\alpha^2-\alpha+1\right)\left(\beta^4 - 4\alpha\beta^3 + 6\alpha^2\beta^2 - 4\alpha^3\beta + \alpha^4\right) \\ &= \left((\alpha-\tfrac{1}{2})^2 + \tfrac{3}{4}\right)(\alpha-\beta)^4 \geqslant 0. \end{aligned}$$

Thus $G(x,y,a,b)$ is non-negative in all the cases and so we are done. ∎

Remark 2.2. The inequality (2.5) is sharp as it is known from Lemma 1.3 proved in Chapter 1 that

$$2\left[\tfrac{\mathrm{Re}\,(z_1-z_2)}{|z_1-z_2|}\tfrac{\mathrm{Re}\,(z_1-z_3)}{|z_1-z_3|}\tfrac{\mathrm{Re}\,(z_2-z_3)}{|z_2-z_3|}\right]^2 p_{K_1}(z_1,z_2,z_3) \leqslant p_{K_2}(z_1,z_2,z_3).$$

Indeed, when all sides of the triangle (z_1, z_2, z_3) make a small angle with the horizontal, the multiplier in the square brackets is close to 1 in modulus.

The estimate (2.5) allows us to obtain an inequality for L^2-norms.

Lemma 2.3. *Let μ be a finite measure with linear growth. Then for any $\varepsilon > 0$ we have*

$$\|T_{K_2,\varepsilon}1\|_{L^2(\mu)} \leqslant \sqrt{2}\,\|T_{K_1,\varepsilon}1\|_{L^2(\mu)} + C\sqrt{\mu(\mathbb{C})}, \qquad C > 0. \tag{2.6}$$

Proof. From Lemma 2.2 we immediately get that

$$p_{K_2,\varepsilon}(\mu) \leqslant 2p_{K_1,\varepsilon}(\mu). \tag{2.7}$$

Now we use the identity (0.13) for the permutations (1.2), i.e.

$$\|T_{K,\varepsilon}1\|^2_{L^2(\mu)} = \tfrac{1}{3}p_{K,\varepsilon}(\mu) + \mathcal{R}_{K,\varepsilon}(\mu), \qquad |\mathcal{R}_{K,\varepsilon}(\mu)| \leqslant C_K\mu(\mathbb{C}), \qquad C_K > 0,$$

where K is a real and odd CZ kernel with non-negative permutations. In these terms the inequality (2.7) gives

$$\tfrac{1}{3}p_{K_2,\varepsilon}(\mu) + \mathcal{R}_{K_2,\varepsilon}(\mu) \leqslant 2\left(\tfrac{1}{3}p_{K_1,\varepsilon}(\mu) + \mathcal{R}_{K_1,\varepsilon}(\mu)\right) + \mathcal{R}_{K_2,\varepsilon}(\mu) - 2\mathcal{R}_{K_1,\varepsilon}(\mu),$$

and, consequently,

$$\|T_{K_2,\varepsilon}1\|^2_{L^2(\mu)} \leqslant 2\|T_{K_1,\varepsilon}1\|^2_{L^2(\mu)} + C\mu(\mathbb{C}), \qquad C > 0.$$

Applying the inequality $\sqrt{ax^2+b} \leqslant \sqrt{a}x + \sqrt{b}$ valid for $a, b, x \geqslant 0$, we get (3.3). ■

Remark 2.3. Note that Lemma 2.2 is a particular case of [CMPT2, Lemma 7] but with an explicit constant. Nevertheless, the explicitness of the constant is essential here and actually enables us to obtain the result.

We are now ready to prove Theorems 2.1 and 2.2. By (2.6) and the triangle inequality,

$$\begin{aligned}\|T_{k_t,\varepsilon}1\|_{L^2(\mu)} &= \|(T_{K_2,\varepsilon} + t\cdot T_{K_1,\varepsilon})1\|_{L^2(\mu)}\\ &\geqslant |t|\|T_{K_1,\varepsilon}1\|_{L^2(\mu)} - \|T_{K_2,\varepsilon}1\|_{L^2(\mu)}\\ &\geqslant (|t|-\sqrt{2})\|T_{K_1,\varepsilon}1\|_{L^2(\mu)} - C\sqrt{\mu(\mathbb{C})}.\end{aligned}$$

Consequently,

$$\|T_{K_1,\varepsilon}1\|_{L^2(\mu)} \leqslant \frac{\|T_{k_t,\varepsilon}1\|_{L^2(\mu)} + C\sqrt{\mu(\mathbb{C})}}{|t|-\sqrt{2}}, \qquad |t| > \sqrt{2}, \tag{2.8}$$

and therefore for any cube $Q \subset \mathbb{C}$,

$$\|T_{K_1,\varepsilon}\chi_Q\|_{L^2(\mu\lfloor Q)} \leqslant \frac{\|T_{k_t,\varepsilon}\chi_Q\|_{L^2(\mu\lfloor Q)} + C\sqrt{\mu(Q)}}{|t|-\sqrt{2}}, \qquad |t| > \sqrt{2}.$$

Applying a variant of the $T1$ Theorem of Nazarov, Treil and Volberg from [Tol5, Theorem 9.40], we infer that the $L^2(\mu)$-boundedness of the SIO T_{k_t}, where t is fixed and such that $|t| > \sqrt{2}$, implies that T_{K_1} (or equivalently the Cauchy transform by (0.13) and (0.15)) is $L^2(\mu)$-bounded. Therefore, Theorems A and B give the desired result. Note that the "only if" part of Theorem 2.1 follows from [Dav1].

Remark 2.4. Computer experiments suggest that the following inequality holds:

$$p_{\kappa_N}(z_1, z_2, z_3) \leqslant \tfrac{N}{n} p_{\kappa_n}(z_1, z_2, z_3). \tag{2.9}$$

(Lemma 2.2 corresponds to the case $(n, N) = (1, 2)$.) Moreover, if $u = -\gamma + i$, $v = \gamma + i$ and $\gamma > 0$, then (see Example 1.1 in Chapter 1)

$$p_{\kappa_m}(0, u, v) = \frac{\gamma^{2m-2}\left((\gamma^2+1)^m - \gamma^{2m}\right)}{(\gamma^2+1)^{2m}}, \qquad m \in \mathbb{N},$$

and therefore

$$\lim_{\gamma\to\infty} \frac{p_{\kappa_N}(0, u, v)}{p_{\kappa_n}(0, u, v)} = \lim_{\gamma\to\infty} \frac{1 - \left(\gamma^2/(\gamma^2+1)\right)^N}{1 - \left(\gamma^2/(\gamma^2+1)\right)^n} = \frac{N}{n}.$$

It means that the constant N/n is sharp if (2.9) is true.

It would follow from (2.9) in the same manner as above that the $L^2(\mu)$-boundedness of T_{K_t}, where t is fixed and such that $|t| > \sqrt{N/n}$, implies that T_{κ_n} is $L^2(\mu)$-bounded. This would give the analogues of Theorems 2.1 and 2.2 for the more general class of kernels (0.24) via Theorems C and D. However, we are not able to prove (2.9) yet.

2.4 Proof of Theorem 2.3

We now consider the kernels (0.24) with $t \in \Omega(n, N)$ (see (2.1)). As mentioned above, the corresponding permutations are non-negative and hence a curvature-like method can be used directly. Namely, we will adapt the arguments from [CMPT1, Section 8], which in turn stem from [DS1], to our settings. Note that the "only if" part of Theorem 2.3 follows from [Dav1]. Thus we only need to prove the "if" part.

Suppose that μ is an AD-regular measure and T_{K_t} the SIO associated with the kernels (0.24), $t \in \Omega(n, N)$. It is proved in Lemmas 1.5 and 1.6 in Chapter 1 that if

$$t \in \mathbb{R} \setminus \left[-\tfrac{1}{2}\left(3 + \sqrt{9 - 4\tfrac{N}{n}}\right), 2 - \tfrac{N}{n}\right], \qquad n < N \leqslant 2n, \tag{2.10}$$

$$t \in \mathbb{R} \setminus \left[-\tfrac{1}{2}\left(\sigma_{n,M} + \sqrt{\sigma_{n,M}^2 - 4\tfrac{N}{n}}\right), \sigma_{n,M} - 3\right], \qquad N \geqslant 2n, \tag{2.11}$$

where $\sigma_{n,M} = 3 + \left(\frac{N}{n} - 2\right)\sqrt{N - 2n}$ as above, then

$$p_{K_t}(z_1, z_2, z_3) \geqslant C(t) \cdot p_{\kappa_n}(z_1, z_2, z_3), \qquad C(t) > 0, \qquad (z_1, z_2, z_3) \in \mathbb{C}^3.$$

Consequently, $p_{K_t,\varepsilon}(\mu) \geqslant C(t) \cdot p_{\kappa_n,\varepsilon}(\mu)$ and hence from (0.13) we conclude that for t as in (2.10) and (2.11) and any cube $Q \subset \mathbb{C}$,

$$\|T_{\kappa_n,\varepsilon}\chi_Q\|_{L^2(\mu\lfloor Q)} \leqslant C(t)\left(\|T_{K_t,\varepsilon}\chi_Q\|_{L^2(\mu\lfloor Q)} + C\sqrt{\mu(Q)}\right). \tag{2.12}$$

By a variant of the $T1$ Theorem from [Tol5, Theorem 9.40] and Theorem C (see also Lemma 3.50 that will be proved in Chapter 3), the measure μ is uniformly rectifiable.

What is left, according to (2.1), is to prove Theorem 2.3 for

$$t = 2 - \tfrac{N}{n}, \qquad n < N \leqslant 2n, \tag{2.13}$$

$$t = \sigma_{n,M} - 3, \qquad N \geqslant 2n, \tag{2.14}$$

$$t = -\tfrac{1}{2}\left(3 + \sqrt{9 - 4\tfrac{N}{n}}\right), \qquad n < N \leqslant 2n, \tag{2.15}$$

$$t = -\tfrac{1}{2}\left(\sigma_{n,M} + \sqrt{\sigma_{n,M}^2 - 4\tfrac{N}{n}}\right), \qquad N \geqslant 2n. \tag{2.16}$$

To manage these cases, we introduce additional notation. Given two distinct points $z, w \in \mathbb{C}$, we denote by $L_{z,w}$ the line passing through z and w. Given three pairwise distinct points $z_1, z_2, z_3 \in \mathbb{C}$, we denote by $\measuredangle(z_1, z_2, z_3)$ the smallest angle (belonging to $[0; \pi/2]$) formed by the lines L_{z_1,z_2} and L_{z_1,z_3}. If L and L' are lines, then $\measuredangle(L, L')$ is the smallest angle (belonging to $[0; \pi/2]$) between them. Also, $\theta_V(L) := \measuredangle(L, V)$, where V is the vertical. Furthermore, for a fixed constant $\tau \geqslant 1$ and complex numbers z_1, z_2 and z_2, set

$$\mathcal{O}_\tau := \left\{ (z_1, z_2, z_3) : \frac{|z_i - z_j|}{|z_i - z_k|} \leqslant \tau \text{ for pairwise distinct } i, j, k \in \{1, 2, 3\} \right\}, \tag{2.17}$$

so that all triangles with vertexes z_1, z_2 and z_3 in $\mathcal{O}_\tau$ have comparable sides.

Given $\alpha_0 \in (0, \pi/2)$ and $(z_1, z_2, z_3) \in \mathbb{C}^3$, in what follows we will sometimes need the conditions

$$\theta_V(L_{z_1,z_2}) + \theta_V(L_{z_2,z_3}) + \theta_V(L_{z_1,z_3}) \geqslant \alpha_0 \tag{2.18}$$

and

$$\theta_V(L_{z_1,z_2}) + \theta_V(L_{z_2,z_3}) + \theta_V(L_{z_1,z_3}) \leqslant \tfrac{3}{2}\pi - \alpha_0. \tag{2.19}$$

We will also use the following result which is Lemma 1.10 in Chapter 1.

Lemma 2.4. *Fix $\alpha_0 \in (0, \pi/2)$. Given K_t and $(z_1, z_2, z_3) \in \mathcal{O}_\tau$,*

- *(i) if* (2.18) *is satisfied and t is as in* (2.13) *or* (2.14)*,*

  or

- *(ii) if* (2.18) *and* (2.19) *are satisfied and t is as in* (2.15) *or* (2.16)*,*

then the following inequality holds:

$$p_{K_t}(z_1, z_2, z_3) \geqslant C(\alpha_0, \tau) \cdot c(z_1, z_2, z_3)^2, \qquad C(\alpha_0, \tau) > 0. \tag{2.20}$$

On the one hand, if we are in the clause (i) of Lemma 2.4, i.e. in the same settings as in [CMPT1], then we can undeviatingly follow the scheme from [CMPT1, Section 8] (exchanging p_{κ_n} for p_{K_t}) in order to get our result for t as in (2.13) or (2.14).

On the other hand, by the clause (ii) of Lemma 2.4, we can ensure that the inequality (2.20) is true for t as in (2.15) or (2.16) if the sides of the triangles (z_1, z_2, z_3) are far from both the vertical and horizontal. Consequently, the scheme from [CMPT1, Section 8] cannot be applied directly for such t. Nevertheless, as we show below, it works after several modifications (besides the exchange of p_{κ_n} for p_{K_t}) connected basically with adapting geometrical arguments to both the conditions (2.18) and (2.19). Note that some of the arguments in [CMPT1, Section 8] are very sketchy and so, for the sake of completeness, we give a proof that is more detailed than the corresponding one in [CMPT1, Section 8].

The fact that the $L^2(\mu)$-boundedness of T_{K_t} implies that μ is uniformly rectifiable will be proved by means of a corona type decomposition. We now recall how such a decomposition is defined in [DS1, Chapter 2] for a given AD-regular measure μ. The elements Q playing the role of dyadic cubes are usually called μ-cubes.

Given an AD regular measure μ on $\mathbb{C}$, for each $j \in \mathbb{Z}$ (or $j \geqslant j_0$ if $\mu(\mathbb{C}) < \infty$) there exists a family $\mathcal{D}_j$ of Borel subsets of $\operatorname{spt} \mu$, i.e. μ-cubes Q of the jth generation, such that

- each $\mathcal{D}_j$ is a disjoint partition of spt μ, i.e. if $Q, Q' \in \mathcal{D}_j$ and $Q \neq Q'$, then

$$\operatorname{spt} \mu = \bigcup\nolimits_{Q \in \mathcal{D}_j} Q \quad \text{and} \quad Q \cap Q' = \varnothing;$$

- if $Q \in \mathcal{D}_j$ and $Q' \in \mathcal{D}_k$ with $k \leqslant j$, then either $Q \subseteq Q'$ or $Q \cap Q' = \varnothing$;
- for all $j \in \mathbb{Z}$ and $Q \in D_j$, we have

$$2^{-j} \lesssim \operatorname{diam}(Q) \lesssim 2^{-j} \quad \text{and} \quad \mu(Q) \approx 2^{-j}.$$

In what follows, $\mathcal{D} := \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j$. Moreover, given $Q \in \mathcal{D}_j$, we define the *side length* of Q as $\ell(Q) = 2^{-j}$, which actually indicates the generation of Q. Obviously, $\ell(Q) \approx \operatorname{diam}(Q)$. The value of $\ell(Q)$ is not well defined if the μ-cube Q belongs to $\mathcal{D}_j \cap \mathcal{D}_k$ with $j \neq k$. To avoid this, one may consider a $Q \in \mathcal{D}_j$ as a couple (Q, j).

Given $\lambda > 1$ and $Q \in \mathcal{D}$, set

$$\lambda Q := \{x \in \operatorname{spt}\mu : \operatorname{dist}(x, Q) \leqslant (\lambda - 1)\ell(Q)\}.$$

We will also need the following version of P. Jones' β-numbers for μ-cubes (see [DS2]):

$$\beta_q(Q) := \inf_L \left(\frac{1}{\ell(Q)} \int_{\eta_1 Q} \left(\frac{\operatorname{dist}(x, L)}{\ell(Q)} \right)^q d\mu(x) \right)^{1/q}, \qquad 1 \leqslant q \leqslant \infty,$$

where $\eta_1 > 4$ is some constant to be fixed later and the infimum is taken over all affine lines L. We will mostly use $\beta_1(Q)$ and denote by L_Q the best approximating line for $\beta_1(Q)$.

Given $Q \in \mathcal{D}_j$, the *sons* of Q, forming the collection $\mathsf{Sons}(Q)$, are the μ-cubes $Q' \in \mathcal{D}_{j+1}$ such that $Q' \subseteq Q$.

By [DS1, Chapter 2], one says that μ admits a *corona decomposition* if there are parameters $\eta, \delta > 0$ and a triple $(\mathcal{B}, \mathcal{G}, \mathsf{Tree})$, where $\mathcal{B}$ and $\mathcal{G}$ are subsets of $\mathcal{D}$ and Tree is a family of subsets S of $\mathcal{G}$, such that the following conditions are satisfied:

1. $\mathcal{D} = \mathcal{B} \cup \mathcal{G}$ and $\mathcal{B} \cap \mathcal{G} = \varnothing$.
2. $\mathcal{B}$ satisfies a Carleson packing condition, i.e.

$$\sum_{Q\in\mathcal{B}: Q\subseteq R} \mu(Q) \lesssim_\eta \mu(R) \quad \text{for all} \quad R \in \mathcal{D}. \tag{2.21}$$

3. $\mathcal{G} = \bigcup_{S\in\mathsf{Tree}} S$ and the union is disjoint;
4. Each $S \in \mathsf{Tree}$ is called a *tree* and is *coherent*: each S has a unique maximal element Q_S, which contains all other elements of S as subsets, i.e.
   - a μ-cube $Q' \in \mathcal{D}$ belongs to S if $Q \subseteq Q' \subseteq Q_S$ for some $Q \in S$;
   - if $Q \in S$ then either all elements of $\mathsf{Sons}(Q)$ lie in S or none of them do.
5. For each $S \in \mathsf{Tree}$, there exists a (possibly rotated) Lipschitz graph Γ_S with constant smaller than η such that $\operatorname{dist}(x, \Gamma_S) \leqslant \delta \operatorname{diam}(Q)$ whenever $x \in 2Q$ and $Q \in S$.
6. The maximal μ-cubes Q_S, for $S \in \mathsf{Tree}$, satisfy the Carleson packing condition

$$\sum_{S\in\mathsf{Tree}:\, Q_S\subseteq R} \mu(Q_S) \lesssim \mu(R) \quad \text{for all} \quad R \in \mathcal{D}.$$

According to [DS1, Section 1, (C4) and (C6)], if μ is uniformly rectifiable, then it admits a corona decomposition for all $\eta, \delta > 0$. Conversely, the existence of a corona decomposition for a single set of η and δ implies that μ is uniformly rectifiable.

We now turn to constructing our corona decomposition. Let $\varepsilon > 0$ be some small constant to be chosen at the very end of the construction. From now on, $\mathcal{B}_0(\varepsilon)$ stands for the family of cubes $Q \in \mathcal{D}$ such that $\beta_1(Q) \geqslant \varepsilon$. Furthermore, $\mathcal{G}_0(\varepsilon) := \mathcal{D} \setminus \mathcal{B}_0(\varepsilon)$. The aim is to show that $\mathcal{B}_0(\varepsilon)$ satisfies a Carleson packing condition.

Note that constants in some inequalities below depend also on the AD-regularity constant and the constants η_1, η_2 and η_3 related to β-numbers. Since these constants are fixed at some point, we do not indicate this dependence explicitly. On the contrary, we emphasize the dependence of some forthcoming inequalities on the essential parameter ε.

Recall that $\ell(Q) \approx \mu(Q)$. We start with observing that for any $y, z \in \eta_1 Q$ we have

$$\begin{aligned}
\beta_2(Q)^2 &\leqslant \frac{1}{\ell(Q)} \int_{\eta_1 Q} \left(\frac{\operatorname{dist}(x, L_{y,z})}{\ell(Q)} \right)^2 d\mu(x) \\
&= \frac{1}{\ell(Q)} \Bigg(\int_{\substack{x \in \eta_1 Q: \\ \frac{\operatorname{dist}(x, L_{y,z})}{\ell(Q)} < \varepsilon^2}} \left(\frac{\operatorname{dist}(x, L_{y,z})}{\ell(Q)} \right)^2 d\mu(x) \\
&\qquad + \int_{\substack{x \in \eta_1 Q: \\ \frac{\operatorname{dist}(x, L_{y,z})}{\ell(Q)} \geqslant \varepsilon^2}} \left(\frac{\operatorname{dist}(x, L_{y,z})}{\ell(Q)} \right)^2 d\mu(x) \Bigg) \\
&\lesssim \varepsilon^4 + \frac{1}{\ell(Q)} \int_{\substack{x \in \eta_1 Q: \\ \frac{\operatorname{dist}(x, L_{y,z})}{\ell(Q)} \geqslant \varepsilon^2}} \left(\frac{\operatorname{dist}(x, L_{y,z})}{\ell(Q)} \right)^2 d\mu(x).
\end{aligned}$$

Lemma 2.5. *Let $B_1 = B(\zeta_1, r_1)$ and $B_2 = B(\zeta_2, r_2)$ be two balls such that $B_1 \cap \operatorname{spt} \mu \subset \eta_1 Q$, $B_2 \cap \operatorname{spt} \mu \subset \eta_1 Q$, $\operatorname{dist}(B_1, B_2) \approx \ell(Q)$ and $r_1 \approx r_2 \approx \ell(Q)$. If $y \in B_1$ and $z \in B_2$, then for ε small enough,*

$$\int_{\substack{x \in \eta_1 Q: \\ \frac{\operatorname{dist}(x, L_{y,z})}{\ell(Q)} \geqslant \varepsilon^2}} \operatorname{dist}(x, L_{y,z})^2 d\mu(x) \lesssim_\varepsilon \ell(Q)^2 p_{K_t}^{(\varepsilon;\, Q)}(\mu),$$

where

$$p_{K_t}^{(\varepsilon;\, Q)}(\mu) := \iiint_{\substack{(x,y,z) \in (\eta_1 Q)^3 \\ |x-y| \geqslant \varepsilon^2 \ell(Q), \\ |x-z| \geqslant \varepsilon^2 \ell(Q)}} p_{K_t}(x, y, z)\, d\mu(x) d\mu(y) d\mu(z).$$

Note that the existence of the above-mentioned balls B_1 and B_2 is guaranteed in the AD-regular case.

Proof. Note that the condition $\operatorname{dist}(x, L_{y,z}) \geqslant \varepsilon^2 \ell(Q)$ implies that $|x - y| \geqslant \varepsilon^2 \ell(Q)$ and $|x - z| \geqslant \varepsilon^2 \ell(Q)$. Consequently, since $x \in \eta_1 Q$, $y \in B_1$ and $z \in B_2$,

$$|x - z| \approx_\varepsilon |x - y| \approx_\varepsilon |y - z|. \tag{2.22}$$

We now separate two cases.

(1) Suppose that

$$\varepsilon^{10} \leqslant \theta_V(L_{x,y}) + \theta_V(L_{y,z}) + \theta_V(L_{x,z}) \leqslant \tfrac{3}{2}\pi - \varepsilon^{10}.$$

Then by the clause (*ii*) of Lemma 2.4, where we put $\alpha_0 = \varepsilon^{10}$ and $\tau = \tau(\varepsilon, \eta_1)$ chosen with respect to the constants in (2.22), we have $c(x, y, z)^2 \lesssim_\varepsilon p_{K_t}(x, y, z)$.

(2) Now let

$$\theta_V(L_{x,y}) + \theta_V(L_{y,z}) + \theta_V(L_{x,z}) < \varepsilon^{10}$$

or

$$\theta_V(L_{x,y}) + \theta_V(L_{y,z}) + \theta_V(L_{x,z}) > \tfrac{3}{2}\pi - \varepsilon^{10}.$$

In this case $\operatorname{dist}(x, L_{y,z}) \lesssim \varepsilon^{10}\ell(Q)$. Thus for ε small enough we get a contradiction with the assumption $\operatorname{dist}(x, L_{y,z}) \geqslant \varepsilon^2\ell(Q)$.

Summarizing,

$$\begin{aligned}
&\int_{\substack{x\in\eta_1 Q:\\ \frac{\operatorname{dist}(x,L_{y,z})}{\ell(Q)}\geqslant\varepsilon^2}} \operatorname{dist}(x, L_{y,z})^2 d\mu(x)\\
&\quad\lesssim_\varepsilon \frac{\ell(Q)^4}{\mu(B_1)\mu(B_2)}\int_{B_2}\int_{B_1}\int_{\substack{\eta_1 Q:\\ \frac{\operatorname{dist}(x,L_{y,z})}{\ell(Q)}\geqslant\varepsilon^2}}\left(\frac{\operatorname{dist}(x,L_{y,z})}{|x-y||x-z|}\right)^2 d\mu(x)d\mu(y)d\mu(z)\\
&\quad\lesssim_\varepsilon \ell(Q)^2\int_{\eta_1 Q}\int_{\eta_1 Q}\int_{\substack{\eta_1 Q:\\ \frac{\operatorname{dist}(x,L_{y,z})}{\ell(Q)}\geqslant\varepsilon^2}} c(x,y,z)^2 d\mu(x)d\mu(y)d\mu(z)\\
&\quad\lesssim_\varepsilon \ell(Q)^2 p_{K_t}^{(\varepsilon;\,Q)}(\mu).
\end{aligned}$$

We used the well-known identity $c(x, y, z) = \operatorname{dist}(x, L_{y,z})/(|x-y||x-z|)$. ■

The estimate for $\beta_2(Q)^2$ that we obtained above and Lemma 2.5 give

$$\beta_2(Q)^2 \lesssim \varepsilon^4 + \frac{C(\varepsilon)}{\ell(Q)}\, p_{K_t}^{(\varepsilon;\,Q)}(\mu), \qquad Q\in\mathcal{D}.$$

We now take into account that $\beta_1(Q) \lesssim \beta_2(Q)$ by Hölder's inequality and, consequently, if $Q \in \mathcal{B}_0(\varepsilon)$, i.e. $\beta_1(Q) \geqslant \varepsilon$, then $\beta_2(Q) \gtrsim \varepsilon$. From this we deduce for sufficiently small ε that

$$\mu(Q) \lesssim_\varepsilon p_{K_t}^{(\varepsilon;\,Q)}(\mu) \qquad \text{for any} \qquad Q\in\mathcal{B}_0(\varepsilon).$$

From this we immediately get that

$$\sum_{Q\in\mathcal{B}_0(\varepsilon)\colon Q\subseteq R}\mu(Q) \lesssim_\varepsilon \sum_{Q\in\mathcal{B}_0(\varepsilon)\colon Q\subseteq R} p_{K_t}^{(\varepsilon;Q)}(\mu).$$

To estimate the latter sum, we will use the notation

$$A_j(\varepsilon) := \{x : \varepsilon^2 2^{-j} \leqslant |x-y| \leqslant c\,2^{-j}\} \qquad \text{for some siutable } c>0. \tag{2.23}$$

Recall that $\ell(Q) = 2^{-j}$ if $Q \in \mathcal{D}_j$. Clearly, (2.23) are the concentric annuli $B(y, c\ell(Q)) \setminus B(y, \varepsilon^2\ell(Q))$ contained in the ball $B(y, c\ell(R))$ and having bounded overlap depending on ε and c. These observations lead to the following:

$$\begin{aligned}
&\sum_{Q\in\mathcal{B}_0(\varepsilon)\colon Q\subseteq R} p_{K_t}^{(\varepsilon;Q)}(\mu)\\
&\quad\leqslant \int_{\eta_1 R}\int_{\eta_1 R}\left(\sum_{j\geqslant j_0}\sum_{Q\in\mathcal{B}_0(\varepsilon)\cap\mathcal{D}_j(R)}\int_{\eta_1 Q\cap A_j(\varepsilon)} p_{K_t}(x,y,z)\,d\mu(x)\right)d\mu(y)d\mu(z)\\
&\quad\lesssim \int_{\eta_1 R}\int_{\eta_1 R}\left(\sum_{j\geqslant j_0}\int_{\eta_1 R\cap A_j(\varepsilon)} p_{K_t}(x,y,z)\,d\mu(x)\right)d\mu(y)d\mu(z)\\
&\quad\lesssim_\varepsilon p_{K_t}(\mu\lfloor(\eta_1 R)).
\end{aligned}$$

Since T_{K_t} is $L^2(\mu)$-bounded, we get $p_{K_t}(\mu\lfloor F) \lesssim \mu(F)$ for any $F\subset\mathbb{C}$. Consequently,

$$p_{K_t}(\mu\lfloor(\eta_1 R)) \lesssim \mu(R) \qquad \text{for all} \qquad R\in\mathcal{D},$$

and therefore we reach the desired inequality

$$\sum_{Q\in\mathcal{B}_0(\varepsilon):\,Q\subseteq R}\mu(Q)\lesssim_\varepsilon \mu(R)\qquad\text{for all}\qquad R\in\mathcal{D}.$$

Thus, for any $\varepsilon>0$, there exists the decomposition

$$\mathcal{D}=\mathcal{B}_0(\varepsilon)\cup\mathcal{G}_0(\varepsilon),\tag{2.24}$$

where $\mathcal{B}_0(\varepsilon)$ satisfies a Carleson packing condition and for any cube $Q\in\mathcal{G}_0(\varepsilon)$ there exists a line L_Q such that $\operatorname{dist}(x,L_Q)\lesssim\sqrt{\varepsilon}\,\ell(Q)$ for all $x\in\frac{1}{2}\eta_1 Q$ (since $\beta_1(Q)<\varepsilon$ for such cubes and $\beta_\infty(Q)\lesssim\sqrt{\beta_1(2Q)}$). More details can be found in [DS1, Ch. 6].

Using the decomposition (2.24), we now can apply [DS1, Lemma 7.1] in order to obtain a new decomposition (still depending on ε) but already with a family of stopping time regions. Suppose that θ_0 is small enough and $0<\varepsilon\ll\theta_0$ to prove the following assertion.

Lemma 2.6. *For all sufficiently small $\varepsilon>0$, there exists a decomposition $\mathcal{D}=\mathcal{B}\cup\mathcal{G}$, where $\mathcal{B}=\mathcal{B}(\varepsilon)$ satisfies a Carleson packing condition (with a constant depending on ε) and $\mathcal{G}=\mathcal{G}(\varepsilon)$ can be partitioned into a family* Tree *of coherent regions S, satisfying the following. For each $S\in$* Tree *denote*

$$\alpha(S):=\tfrac{1}{10}\theta_0\quad\textit{if}\quad\theta_0\leqslant\theta_V(L_{Q_S})\leqslant\pi/2-\theta_0$$

and

$$\alpha(S):=10\theta_0\quad\textit{if}\quad\theta_V(L_{Q_S})<\theta_0\quad\textit{or}\qquad\theta_V(L_{Q_S})>\pi/2-\theta_0.$$

Then

- *if $Q\in S$, then $\measuredangle(L_Q,L_{Q_S})\leqslant\alpha(S)$;*
- *if Q is a minimal cube of S, then either at least one element of* $\mathsf{Sons}(Q)$ *lies in $\mathcal{B}$ or else $\measuredangle(L_Q,L_{Q_S})\geqslant\frac{1}{2}\alpha(S)$.*

Here $\mathcal{G}\subseteq\mathcal{G}_0(\varepsilon)$ and therefore for any $Q\in\mathcal{G}$ one has $\beta_1(Q)<\varepsilon$.

Lemma 2.6 is an analogue of [CMPT1, Lemma 8.1] which comes from [DS1, Lemma 7.1]. The main difference between [CMPT1, Lemma 8.1] and [DS1, Lemma 7.1] is that two different values of the parameter $\alpha(S)$ have to be chosen, according to the angle $\theta_V(L_{Q_S})$. In our case the situations where the angle $\theta_V(L_{Q_S})$ is close to zero and $\pi/2$ have to be also distinguished.

To obtain the required Lipschitz function, one can follow the proof of [DS1, Proposition 8.2] to deduce the following lemma.

Lemma 2.7. *For each $S\in$* Tree *from Lemma 2.6, there exists a Lipschitz function $A_S: L_{Q_S}\to L_{Q_S}^{\perp}$ with norm $\lesssim\alpha(S)$ such that, denoting by Γ_S the graph of A_S,*

$$\operatorname{dist}(x,\Gamma_S)\lesssim\sqrt{\varepsilon}\,\ell(Q)$$

for all $x\in 2Q$ with $Q\in S$.

The proof will be completed if we show that the maximal μ-cubes Q_S, $S\in\mathsf{Tree}$, satisfy the Carleson packing condition

$$\sum_{S\in\mathsf{Tree}:\,Q_S\subseteq R}\mu(Q_S)\lesssim_\varepsilon\mu(R)\qquad\text{for all}\qquad R\in\mathcal{D}.$$

To do so, we will distinguish several types of trees.

Here and subsequently, $\mathsf{Stop}(S)$ denotes the family of the *minimal* μ-cubes of $S \in \mathsf{Tree}$, which may be empty. By Lemma 2.6, we can split $\mathsf{Stop}(S)$ as follows:

$$\mathsf{Stop}(S) = \mathsf{Stop}_\alpha(S) \cup \mathsf{Stop}_\beta(S), \qquad \mathsf{Stop}_\alpha(S) \cap \mathsf{Stop}_\beta(S) = \varnothing, \tag{2.25}$$

where $\mathsf{Stop}_\beta(S)$ contains all minimal μ-cubes Q such that at least one element of $\mathsf{Sons}(Q)$ belongs to $\mathcal{B}$ and $\mathsf{Stop}_\alpha(S)$ contains all minimal Q such that $\measuredangle(L_Q, L_{Q_S}) \geqslant \frac{1}{2}\alpha(S)$.

The first set that we will consider is

$$\Delta_1 := \left\{ S \in \mathsf{Tree} : \mu\left(Q_S \setminus \bigcup\nolimits_{Q\in\mathsf{Stop}(S)} Q\right) \geqslant \tfrac{1}{2}\,\mu(Q_S)\right\}.$$

Clearly, if $S \in \mathsf{Tree} \setminus \Delta_1$, then by (2.25),

$$\tfrac{1}{2}\,\mu(Q_S) < \mu\left(\bigcup\nolimits_{Q\in\mathsf{Stop}(S)} Q\right) = \mu\left(\bigcup\nolimits_{Q\in\mathsf{Stop}_\alpha(S)} Q\right) + \mu\left(\bigcup\nolimits_{Q\in\mathsf{Stop}_\beta(S)} Q\right). \tag{2.26}$$

Now let

$$\Delta_2 := \left\{ S \in \mathsf{Tree} \setminus \Delta_1 : \mu\left(\bigcup\nolimits_{Q\in\mathsf{Stop}_\beta(S)} Q\right) \geqslant \tfrac{1}{4}\,\mu(Q_S)\right\}.$$

The trees remained are in

$$\Delta_3 := \left\{ S \in \mathsf{Tree} \setminus (\Delta_1 \cup \Delta_2) : \mu\left(\bigcup\nolimits_{Q\in\mathsf{Stop}_\alpha(S)} Q\right) \geqslant \tfrac{1}{4}\,\mu(Q_S)\right\}.$$

Indeed, if $S \in \mathsf{Tree} \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3)$, then (2.26) is valid and moreover

$$\mu\left(\bigcup\nolimits_{Q\in\mathsf{Stop}_\alpha(S)} Q\right) < \tfrac{1}{4}\,\mu(Q_S) \quad \text{and} \quad \mu\left(\bigcup\nolimits_{Q\in\mathsf{Stop}_\beta(S)} Q\right) < \tfrac{1}{4}\,\mu(Q_S).$$

This means that $\mathsf{Tree} \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3) = \varnothing$.

We also split Δ_3 in the three disjoint sets:

$$\begin{aligned}
\Delta_3' &:= \{S \in \Delta_3 : \theta_0 \leqslant \theta_V(L_{Q_S}) \leqslant \pi/2 - \theta_0\}, \\
\Delta_3'' &:= \{S \in \Delta_3 : \theta_V(L_{Q_S}) < \theta_0\}, \\
\Delta_3''' &:= \{S \in \Delta_3 : \theta_V(L_{Q_S}) > \pi/2 - \theta_0\}.
\end{aligned}$$

So we have the disjoint union

$$\mathsf{Tree} = \Delta_1 \cup \Delta_2 \cup \Delta_3' \cup \Delta_3'' \cup \Delta_3'''.$$

The procedure now is to check the required Carleson packing condition for all components of this union.

For all $S \in \mathsf{Tree}$ the sets $Q_S \setminus \bigcup_{Q\in\mathsf{Stop}(S)} Q$ are pairwise disjoint and hence for $S \in \Delta_1$ we get

$$\sum_{S\in\Delta_1 \colon Q_S \subseteq R} \mu(Q_S) \leqslant 2 \sum_{S\in\mathsf{Tree}\colon Q_S\subseteq R} \mu\left(Q_S \setminus \bigcup\nolimits_{Q\in\mathsf{Stop}(S)} Q\right) \leqslant 2\mu(R).$$

If $S \in \Delta_2$, then by definition and the fact that $\mu(Q) \approx \mu(Q')$ for $Q' \in \mathsf{Sons}(Q)$,

$$\mu(Q_S) \leqslant 4\mu\left(\bigcup\nolimits_{Q\in\mathsf{Stop}_\beta(S)} Q\right) \lesssim \sum_{Q\in\mathsf{Stop}(S)} \sum_{Q'\in\mathcal{B}\cap\mathsf{Sons}(Q)} \mu(Q')$$

and consequently by Lemma 2.6,

$$\begin{aligned}\sum_{S\in\Delta_2\colon Q_S\subseteq R}\mu(Q_S) &\lesssim \sum_{S\in\Delta_2\colon Q_S\subseteq R}\ \sum_{Q\in\mathsf{Stop}(S)}\ \sum_{Q'\in\mathcal{B}\cap\mathsf{Sons}(Q)}\mu(Q')\\ &\leqslant \sum_{Q\in\mathcal{B}\colon Q_S\subseteq R}\mu(Q)\\ &\lesssim_\varepsilon \mu(R).\end{aligned}$$

Let us consider the case $S\in\Delta_3'$. We will need β-numbers defined for balls $B(x,r)$:

$$\beta_q(x,r):=\inf_L\left(\frac{1}{r}\int_{B(x,2r)}\left(\frac{\operatorname{dist}(x,L)}{r}\right)^q d\mu(x)\right)^{1/q},\qquad 1\leqslant q\leqslant\infty,$$

where the infimum is taken over all affine lines L.

It is claimed in [DS1, Section 12, Inequality (12.2)] that for all $S\in\Delta_3$ there exists $\eta_2>1$ such that

$$\mu(Q_S)\lesssim\iint_{X_S}\beta_1(x,\eta_2 r)^2\frac{d\mu(x)dr}{r},$$

where

$$X_S:=\{(x,r)\in\operatorname{spt}\mu\times\mathbb{R}^+ : x\in\eta_2 Q_S,\ \tfrac{1}{\eta_2}d(x)\leqslant r\leqslant\eta_2\operatorname{diam}(Q_S)\}\tag{2.27}$$

and

$$d(x):=\inf_{Q\in S}\{\operatorname{dist}(x,Q)+\operatorname{diam}(Q)\}.$$

By Holder's inequality, $\beta_1(x,\eta_2 r)\lesssim\beta_2(x,\eta_2 r)$. Moreover, it follows from [Leg, Lemma 2.5 and Proof of Proposition 2.4] (or by the arguments analogous to the ones in the proof of Lemma 2.5) that if μ is AD-regular, then there exists $\eta_3\geqslant 2$ such that for any $x\in\operatorname{spt}\mu$,

$$\beta_2(x,\eta_2 r)^2\lesssim\frac{1}{\eta_2 r}\iiint_{\mathcal{O}_{\eta_3}(x,\eta_2 r)}c(u,v,w)^2 d\mu(u)d\mu(v)d\mu(w),$$

where

$$\mathcal{O}_{\eta_3}(x,\rho):=\left\{(u,v,w)\in(B(x,\eta_3\rho))^3 : |u-v|\geqslant\frac{\rho}{\eta_3},|v-w|\geqslant\frac{\rho}{\eta_3},|u-w|\geqslant\frac{\rho}{\eta_3}\right\}.$$

Note also that for any $(u,v,w)\in\mathcal{O}_{\eta_3}(x,\rho)$ we have $|u-v|\leqslant 2\eta_3\rho$, $|v-w|\leqslant 2\eta_3\rho$ and $|u-w|\leqslant 2\eta_3\rho$, and thus for a fixed η_3,

$$|u-v|\approx|v-w|\approx|u-w|\approx\rho.$$

Therefore if a triple $(u,v,w)\in\mathcal{O}_{\eta_3}(x,\eta_2 r)$ with $(x,r)\in X_S$, then at least one side of the triangle (u,v,w) makes a big angle with the vertical and horizontal. Indeed, by construction, if η_1 is chosen much bigger than η_2, then $\beta_\infty(x,\eta_2 r)\lesssim\sqrt{\varepsilon}$, and consequently the angle between one side of (u,v,w) and the best approximating line L_{Q_S} is less than $C(\eta_3)\sqrt{\varepsilon}$ with some $C(\eta_3)>0$. Furthermore, $\theta_0\leqslant\theta_V(L_{Q_S})\leqslant\pi/2-\theta_0$ and thus the angle that one side of (u,v,w) makes with the vertical and horizontal belongs to the interval

$$\left(\tfrac{9}{10}\theta_0-C(\eta_3)\sqrt{\varepsilon},\pi/2-\tfrac{9}{10}\theta_0+C(\eta_3)\sqrt{\varepsilon}\right)\supseteq\left(\tfrac12\theta_0,\pi/2-\tfrac12\theta_0\right),$$

where ε is chosen sufficiently small. This fact enables us to use the clause (ii) of Lemma 2.4 and exchange the curvature for our permutation p_{K_t}:

$$\beta_2(x,\eta_2 r)^2 \lesssim_{\theta_0} \frac{1}{\eta_2 r}\iiint_{\mathcal{O}_{\eta_3}(x,\eta_2 r)} p_{K_t}(u,v,w)\; d\mu(u)d\mu(v)d\mu(w), \qquad (x,r)\in X_S.$$

Summarizing and taking into account that θ_0 is fixed, we get

$$\mu(Q_S) \lesssim \iint_{X_S}\iiint_{\mathcal{O}_{\eta_3}(x,\eta_2 r)} p_{K_t}(u,v,w)\; d\mu(u)d\mu(v)d\mu(w)\; \frac{d\mu(x)dr}{(\eta_2 r)^2}.$$

What is more, it is shown after [DS1, Lemma 7.9] that the regions X_S (see (2.27)) with $S\in\Delta_3^{'}$ have bounded overlap. By this reason,

$$\begin{aligned}
&\sum_{S\in\Delta_3^{'}\colon Q_S\subseteq R}\mu(Q_S)\\
&\lesssim \sum_{S\in\Delta_3^{'}\colon Q_S\subseteq R}\iint_{X_S}\iiint_{\mathcal{O}_{\eta_3}(x,\eta_2 r)} p_{K_t}(u,v,w)\; d\mu(u)d\mu(v)d\mu(w)\; \frac{d\mu(x)dr}{(\eta_2 r)^2}\\
&\lesssim \int_0^{2\eta_2\ell(R)}\int_{2\eta_2 R}\iiint_{\mathcal{O}_{\eta_3}(x,\eta_2 r)} p_{K_t}(u,v,w)\; d\mu(u)d\mu(v)d\mu(w)\; \frac{d\mu(x)dr}{(\eta_2 r)^2}\\
&\lesssim p_{K_t}(\mu\lfloor(2\eta_2 R)).
\end{aligned}$$

The third inequality is by Fubini's theorem. See the definition of $\mathcal{O}_\tau$ in (2.17). Finally, by the L^2-boundedness of T_{K_t}, we get

$$\sum_{S\in\Delta_3^{'}\colon Q_S\subseteq R}\mu(Q_S)\lesssim\mu(R).$$

Suppose now that $S\in\Delta_3^{''}$. If $Q\in\mathsf{Stop}_\alpha(S)$, then $\mathsf{Sons}(Q)\cap\mathcal{B}=\varnothing$ and by Lemma 2.6,

$$\measuredangle(L_Q,L_{Q_S})\leqslant\alpha(S), \qquad \measuredangle(L_Q,L_{Q_S})\geqslant\tfrac12\alpha(S), \qquad \alpha(S)=10\theta_0,$$

and thus

$$\begin{aligned}
\theta_V(L_Q)&\leqslant\measuredangle(L_Q,L_{Q_S})+\theta_V(L_{Q_S})<10\theta_0+\theta_0=11\theta_0,\\
\theta_V(L_Q)&\geqslant\measuredangle(L_Q,L_{Q_S})-\theta_V(L_{Q_S})>5\theta_0-\theta_0=4\theta_0.
\end{aligned}$$

Since $\beta_1(Q)<\varepsilon$, we can choose ε small enough so that $\measuredangle(L_Q,L_{Q'})\leqslant\theta_0$, $Q'\in\mathsf{Sons}(Q)$, and hence

$$3\theta_0<\theta_V(L_{Q'})<12\theta_0, \qquad Q'\in\mathsf{Sons}(Q).$$

Consequently, any element of $\mathsf{Sons}(Q)$ is the maximal μ-cube of a tree belonging either to Δ_1, Δ_2 or $\Delta_3^{'}$. Additionally, from the definition of Δ_3 and the fact that minimal cubes for a single tree are pairwise disjoint it follows that

$$\mu(Q_S)\leqslant 4\mu\Big(\bigcup\nolimits_{Q\in\mathsf{Stop}_\alpha(S)}Q\Big)=4\sum_{Q\in\mathsf{Stop}_\alpha(S)}\mu(Q)=4\sum_{Q\in\mathsf{Stop}_\alpha(S)}\sum_{Q'\in\mathsf{Sons}(Q)}\mu(Q').$$

From the above-mentioned we deduce that

$$\begin{aligned}
\sum_{S\in\Delta_3^{''}\colon Q_S\subseteq R}\mu(Q_S)&\leqslant 4\sum_{S\in\Delta_3^{''}\colon Q_S\subseteq R}\;\sum_{Q\in\mathsf{Stop}_\alpha(S)}\;\sum_{Q'\in\mathsf{Sons}(Q)}\mu(Q')\\
&\leqslant 4\sum_{S\in\Delta_1\cup\Delta_2\cup\Delta_3^{'}\colon Q_S\subseteq R}\mu(Q_S).
\end{aligned}$$

Take into account that the maximal cubes of all trees from $\Delta_1 \cup \Delta_2 \cup \Delta_3^{'}$ satisfy a Carleson packing condition (with a constant depending on ε). By this reason,

$$\sum_{S\in\Delta_3^{''}:\, Q_S\subseteq R} \mu(Q_S) \lesssim_\varepsilon \mu(R).$$

One can argue for $S \in \Delta_3^{'''}$ in the same manner as for $S \in \Delta_3^{''}$. Indeed, if ε is appropriately chosen and $Q \in \mathsf{Stop}_\alpha(S)$, then

$$\pi/2 - 12\theta_0 < \theta_V(L_{Q'}) < \pi/2 - 3\theta_0, \qquad Q' \in \mathsf{Sons}(Q),$$

and hence any element of $\mathsf{Sons}(Q)$ is the maximal μ-cube of a tree belonging either to Δ_1, Δ_2 or $\Delta_3^{'}$.

Summarizing, we proved that maximal cubes of all types of trees satisfy a Carleson packing condition and so the triple $(\mathcal{B}, \mathcal{G}, \mathsf{Tree})$ is a corona decomposition as required.

2.5 Additional remarks

To finish, we would like to mention a corollary of the results from the previous sections.

Let μ be a measure with linear growth and

$$t \in (-\infty, -\sqrt{2}) \cup (0, \infty). \tag{2.28}$$

If the SIO T_{k_t} is $L^2(\mu)$-bounded, then all 1-dimensional SIOs associated with a wide class of kernels defined around (0.23) are also $L^2(\mu)$-bounded. Indeed, it follows from (2.8) and (2.12) with $(n, N) = (1, 2)$ that for any t as in (2.28) and any cube $Q \subset \mathbb{C}$, one has

$$\|T_{\kappa_1,\varepsilon}\chi_Q\|_{L^2(\mu\lfloor Q)} \leqslant C(t)\left(\|T_{k_t,\varepsilon}\chi_Q\|_{L^2(\mu\lfloor Q)} + \sqrt{\mu(Q)}\right), \qquad C(t) > 0,$$

where T_{κ_1}, as we have already mentioned before, is the SIO associated with the real part of the Cauchy kernel, i.e. with the Cauchy kernel, up to a constant. Using the $T1$ Theorem from [Tol5, Theorem 9.40], we conclude that the $L^2(\mu)$-boundedness of T_{k_t} with t as in (2.28) implies that the Cauchy transform is $L^2(\mu)$-bounded. It is left to apply the results from [Tol2, Tol4].

Chapter 3

A family of singular integral operators which control the Cauchy transform

3.1 Introduction and Theorems

The exposition in this chapter is based on [CMT2]. We aim to study the behaviour of SIOs associated with the kernels (0.21), i.e. with

$$k_t(z) = \frac{(\operatorname{Re} z)^3}{|z|^4} + t \cdot \frac{\operatorname{Re} z}{|z|^2}, \quad t \in \mathbb{R}, \qquad k_\infty(z) = \frac{\operatorname{Re} z}{|z|^2},$$

in the case of small negative t. We will first show that for a measure with linear growth the corresponding L^2-norm of T_{k_0} controls the L^2-norm of T_{k_∞} and thus of the Cauchy transform. As a corollary, we prove that the $L^2(\mathcal{H}^1\lfloor E)$-boundedness of T_{k_t} with some $t \in (-t_0, 0)$, where $t_0 > 0$ is an absolute constant, implies that E is rectifiable (as above, E is a 1-set). This is so in spite of the fact that the usual curvature method fails to be directly applicable for $t \in (-2, 0)$ as shown in Chapter 1. As we will see, the study of kernels (0.21) for small negative t is more difficult than in the case of $t \in (-2, -\sqrt{2})$ considered in Chapter 2.

We will use the same definitions related to curvature and permutations as given in Introduction and Chapters 1 and 2. Now we will recall some of them. Again we will work with the following simplified version of the permutations P_K given in (0.5),

$$\begin{aligned} &p_K(z_1, z_2, z_3) \\ &= K(z_1 - z_2)K(z_1 - z_3) + K(z_2 - z_1)K(z_2 - z_3) + K(z_3 - z_1)K(z_3 - z_2) \\ &= \tfrac{1}{2} P_K(z_1, z_2, z_3), \end{aligned}$$

introduced in (1.2) in Chapter 1. The kernel K here is supposed to be odd and real-valued. Then by (1.2), (1.3) and (1.4), we also have

$$p_K(\mu_1, \mu_2, \mu_3) = \tfrac{1}{2} P_K(\mu_1, \mu_2, \mu_3), \qquad p_{K,\varepsilon}(\mu_1, \mu_2, \mu_3) = \tfrac{1}{2} P_{K,\varepsilon}(\mu_1, \mu_2, \mu_3)$$

and

$$p_K(\mu) = \tfrac{1}{2} p_K(\mu, \mu, \mu), \qquad p_{K,\varepsilon}(\mu) = \tfrac{1}{2} P_{K,\varepsilon}(\mu, \mu, \mu),$$

where μ, μ_1, μ_2 and μ_3 are measures.

Recall that by a *measure* we mean a *positive locally finite Borel measure on* $\mathbb{C}$.

Some of the identities from Introduction have a slightly different form in these terms. In particular, let K be an odd and real-valued CZ kernel with the permutations (1.2), being non-negative for any $(z_1, z_2, z_3) \in \mathbb{C}^3$. If μ is a finite measure with *C_*-linear growth*, i.e. there exists a constant $C_* > 0$ such that

$$\mu(B(z,r)) \leqslant C_* r \qquad \text{for all } z \in \operatorname{spt}\mu,$$

then the generalised Melnikov-Verdera identity (0.13) takes the form:

$$\|T_{K,\varepsilon}1\|^2_{L^2(\mu)} = \tfrac{1}{3} p_{K,\varepsilon}(\mu) + \mathcal{R}_{K,\varepsilon}(\mu), \qquad |\mathcal{R}_{K,\varepsilon}(\mu)| \leqslant c C_*^2 \mu(\mathbb{C}), \tag{3.1}$$

where $\varepsilon > 0$ and $c > 0$ is independent of ε. What is more, (0.15) becomes

$$p_{k_\infty}(z_1, z_2, z_3) = \tfrac{1}{4} c(z_1, z_2, z_3)^2, \qquad (z_1, z_2, z_3) \in \mathbb{C}^3.$$

Clearly, this implies that $p_{k_\infty}(\mu) = \frac{1}{4} c^2(\mu)$ for any measure μ.

Recall also that it is shown in Theorem 1.1 and Remark 1.1 in Chapter 1 that

$$\begin{cases} p_{k_t}(z_1, z_2, z_3) \geqslant 0 \text{ for any } (z_1, z_2, z_3) \in \mathbb{C}^3, \text{ if } t \notin (-2, 0), \\ p_{k_t}(z_1, z_2, z_3) \text{ may change sign for some } (z_1, z_2, z_3) \in \mathbb{C}^3, \text{ if } t \in (-2, 0), \end{cases}$$

see Figure 0.2. Moreover, by (2.5) from Lemma 2.2 proved in Chapter 2,

$$p_{k_0}(z_1, z_2, z_3) \leqslant 2 p_{k_\infty}(z_1, z_2, z_3) \text{ for any } (z_1, z_2, z_3) \in \mathbb{C}^3.$$

Now we give several definitions related to β-numbers and densities. For any closed ball $B = B(x, r)$ with center $x \in \mathbb{C}$ and radius $r > 0$ and $1 \leqslant p < \infty$, let

$$\beta_{\mu,p}(B) = \inf_L \left(\frac{1}{r} \int_B \left(\frac{\operatorname{dist}(y, L)}{r} \right)^p d\mu(y) \right)^{1/p}, \tag{3.2}$$

where the infimum is taken over all affine lines $L \subset \mathbb{C}$. The $\beta_{\mu,p}$ coefficients were introduced by David and Semmes [DS1] and are the generalization of the well-known Jones β-numbers [Jon2].

We will mostly deal with $\beta_{\mu,2}(2B_Q)$ and so by L_Q we denote a corresponding best approximating line, i.e. a line where the infimum is reached in (3.2) for $B = 2B_Q$ (see the definition of B_Q below) and $p = 2$.

Throughout the chapter we also use the following densities:

$$\Theta_\mu(B) := \Theta_\mu(x, r) = \frac{\mu(B(x, r))}{r}, \quad \text{where} \quad B = B(x, r), \qquad x \in \mathbb{C}, \qquad r > 0.$$

First we prove the following.

Theorem 3.1. *There exist absolute constants $t_0 > 0$ and $c > 0$ such that for any finite measure μ with C_*-linear growth it holds that*

$$\sup_{\varepsilon > 0} \|T_{k_\infty,\varepsilon}1\|_{L^2(\mu)} \leqslant t_0^{-1} \sup_{\varepsilon > 0} \|T_{k_0,\varepsilon}1\|_{L^2(\mu)} + c C_* \sqrt{\mu(\mathbb{C})}. \tag{3.3}$$

Roughtly speaking, this means that the L^2-norm of the Cauchy transform of measure is controlled by the L^2-norm of the operator T_{k_0}.

Note that by Lemma 2.3 in Chapter 2 under the same assumptions on μ,

$$\|T_{k_0,\varepsilon}1\|_{L^2(\mu)} \leqslant \sqrt{2} \|T_{k_\infty,\varepsilon}1\|_{L^2(\mu)} + c C_* \sqrt{\mu(\mathbb{C})},$$

where $\varepsilon > 0$ and $c > 0$ is independent of ε. With respect to the proof of Lemma 2.3, the proof of (3.3) is more difficult as we will see in this chapter.

As a corollary of Theorem 3.1, using the same t_0, we obtain the following result.

Theorem 3.2. *Let k_t be a kernel of the form* (0.21)*, where $t \in (-t_0, 0)$, and E a 1-set. If the SIO T_{k_t} is $L^2(\mathcal{H}^1\lfloor E)$-bounded, then E is rectifiable.*

This theorem complements Theorems B, D, E, F and Theorems 1.2 and 2.2 so that we have the overall picture as in Figure 0.2. It is clear from Theorem F that necessarily $t_0 \in (0, 3/4)$. What is more, it is very important here that the pointwise permutations, corresponding to $t \in (-t_0, 0)$, also change sign as in Theorem 2.2 so that the curvature method cannot be applied directly but L^2-boundedness still implies rectifiability.

Remark 3.1. By simple analysis one can show that the kernel k_t has

$$\begin{cases} \text{one zero line if } t \in (-\infty, -1) \cup [0, \infty], \\ \text{two zero lines if } t = -1, \\ \text{three zero lines if } t \in (-1, 0). \end{cases}$$

By a zero line we mean a straight line $L \subset \mathbb{C}$ such that $k_t(z) = 0$ for $z \in L$.

In this sense, it is interesting to compare Theorem 3.2 with Theorem F. Observing Figure 0.2, one can see that the number of zero lines along is not determinant.

Remark 3.2. Let t_1 and t_2 be such that $-\sqrt{2} \leqslant t_1 < t_2 \leqslant -t_0$. If there exist finite *purely unrectifiable* (i.e. concentrated on purely unrectifiable sets) measures μ_1 and μ_2 with linear growth such that $T_{k_{t_1}}$ is $L^2(\mu_1)$-bounded and $T_{k_{t_2}}$ is $L^2(\mu_2)$-bounded, then μ_1 is different from μ_2.

Indeed, let μ be a finite purely unrectifiable measure with linear growth such that $T_{k_{\tilde t}}$ is $L^2(\mu)$-bounded for a fixed $\tilde t \in [-\sqrt{2}, -t_0]$. By the triangle inequality for any real t,

$$\|T_{k_t}1\|_{L^2(\mu)} = \|(T_{k_0} + (t - \tilde t) \cdot T_{k_\infty} + \tilde t \cdot T_{k_\infty})1\|_{L^2(\mu)} \geqslant |t - \tilde t| \|T_{k_\infty}1\|_{L^2(\mu)} - \|T_{k_{\tilde t}}1\|_{L^2(\mu)}.$$

Consequently, $\|T_{k_t}1\|_{L^2(\mu)} = \infty$ for all $t \neq \tilde t$ as $\|T_{k_\infty}1\|_{L^2(\mu)} = \infty$ since μ is purely unrectifiable. Thus an example of a purely unrectifiable measure μ such that $T_{k_{\tilde t}}$ is $L^2(\mu)$-bounded for a fixed $\tilde t \in [-\sqrt{2}, -t_0]$ does not work for $t \neq \tilde t$.

3.2 Main Lemma and proofs of Theorems

Theorem 3.1 is implied by the following lemma.

Main Lemma. *There exist absolute constants $t_0 > 0$ and $c > 0$ such that for any finite measure μ with C_*-linear growth it holds that*

$$p_{k_\infty}(\mu) \leqslant t_0^{-2} p_{k_0}(\mu) + cC_*^2 \mu(\mathbb{C}). \tag{3.4}$$

The proof of this result is long and technical and actually takes the biggest part of this chapter. Note that (3.4) is a counterpart to the inequality $p_{k_0}(\mu) \leqslant 2p_{k_\infty}(\mu)$ that follows from (2.5).

3.2.1 Proof of Theorem 3.1

Suppose that Main Lemma holds. Then the identity (3.1) and inequality (3.4) yield

$$\begin{aligned} \sup_{\varepsilon>0} \|T_{k_\infty,\varepsilon}1\|^2_{L^2(\mu)} &\leqslant \tfrac{1}{3} p_{k_\infty}(\mu) + cC_*^2 \mu(\mathbb{C}) \\ &\leqslant \tfrac{1}{3}\, t_0^{-2}\, p_{k_0}(\mu) + cC_*^2 \mu(\mathbb{C}) \\ &\leqslant t_0^{-2} \sup_{\varepsilon>0} \|T_{k_0,\varepsilon}1\|^2_{L^2(\mu)} + cC_*^2 \mu(\mathbb{C}), \end{aligned}$$

where $c > 0$ is an absolute constant. Applying the inequality $\sqrt{ax^2 + b} \leqslant \sqrt{a}x + \sqrt{b}$ that is valid for $a, b, x \geqslant 0$, gives Theorem 3.1.

3.2.2 Proof of Theorem 3.2

We now apply the perturbation method from Chapter 2. By the triangle inequality and Theorem 3.1,

$$\begin{aligned}\sup_{\varepsilon>0}\|T_{k_t,\varepsilon}1\|_{L^2(\mu)} &= \sup_{\varepsilon>0}\|(T_{k_0,\varepsilon}+t\cdot T_{k_\infty,\varepsilon})1\|_{L^2(\mu)}\\ &\geqslant \sup_{\varepsilon>0}\|T_{k_0,\varepsilon}1\|_{L^2(\mu)}-|t|\sup_{\varepsilon>0}\|T_{k_\infty,\varepsilon}1\|_{L^2(\mu)}\\ &\geqslant (t_0-|t|)\sup_{\varepsilon>0}\|T_{k_\infty,\varepsilon}1\|_{L^2(\mu)}-cC_*\sqrt{\mu(\mathbb{C})}.\end{aligned}$$

Consequently,

$$\sup_{\varepsilon>0}\|T_{k_\infty,\varepsilon}1\|_{L^2(\mu)}\leqslant\frac{\sup_{\varepsilon>0}\|T_{k_t,\varepsilon}1\|_{L^2(\mu)}+cC_*\sqrt{\mu(\mathbb{C})}}{t_0-|t|},\qquad |t|<t_0,$$

and therefore for any cube $Q\subset\mathbb{C}$ we have

$$\sup_{\varepsilon>0}\|T_{k_\infty,\varepsilon}\chi_Q\|_{L^2(\mu\lfloor Q)}\leqslant\frac{\sup_{\varepsilon>0}\|T_{k_t,\varepsilon}\chi_Q\|_{L^2(\mu\lfloor Q)}+cC_*\sqrt{\mu(Q)}}{t_0-|t|},\qquad |t|<t_0. \tag{3.5}$$

Applying a variant of the $T1$ Theorem of Nazarov, Treil and Volberg from [Tol5, Theorem 9.40], we infer from (3.5) with $\mu=\mathcal{H}^1\lfloor E$ and 1-set E that the $L^2(\mathcal{H}^1\lfloor E)$-boundedness of T_{k_t} with a fixed t such that $|t|<t_0$ implies that T_{k_∞} and thus the Cauchy transform is $L^2(\mathcal{H}^1\lfloor E)$-bounded. Finally, Theorem B gives the desired result.

3.3 Several corollaries

Recall that a measure μ is *Ahlfors-David regular* (*AD-regular*) if it satisfies (0.4), i.e.

$$C^{-1}r\leqslant\mu(B(z,r))\leqslant Cr,\qquad\text{where}\qquad z\in\operatorname{spt}\mu,\qquad 0<r<\operatorname{diam}(\operatorname{spt}\mu),$$

and $C>1$ is some fixed constant. A measure μ is called *uniformly rectifiable* if it is AD-regular and $\operatorname{spt}\mu$ is contained in an AD-regular curve. One can summarise all up-to-date results characterising uniformly rectifiable measures via $L^2(\mu)$-bounded SIOs T_{k_t} as follows.

Corollary 3.1. *Let μ be an AD-regular measure and k_t a kernel of the form* (0.21) *with $t\in(-\infty,-\sqrt{2})\cup(-t_0,\infty]$. The measure μ is uniformly rectifiable if and only if the SIO T_{k_t} is $L^2(\mu)$-bounded.*

The part of Corollary 3.1 for $t=\infty$, i.e. for the Cauchy transform, was proved in [MMV]; for $t=0$ in [CMPT1]; and for $t\in(-\infty,-\sqrt{2})\cup(0,\infty)$ in Chapters 1 and 2.

Furthermore, one can formulate the following general result.

Corollary 3.2. *Let μ be a measure with linear growth and k_t a kernel of the form* (0.21) *with $t\in(-\infty,-\sqrt{2})\cup(-t_0,\infty]$. If the SIO T_{k_t} is $L^2(\mu)$-bounded, then so are all 1-dimensional SIOs associated with a wide class of kernels.*

The above-mentioned "wide class of kernels" was defined around (0.23) in Introduction. The part of Corollary 3.2 for $t=\infty$, i.e. for the Cauchy transform, was proved in [Tol2,Tol4] (see also [Gir]) and for $t\in(-\infty,-\sqrt{2})\cup(0,\infty)$ in Chapters 1 and 2.

3.4 Plan of the proof of Main Lemma

To prove Main Lemma, we will use a corona decomposition that is similar, for example, to the ones from [Tol6] and [AT]: it splits the David-Mattila dyadic lattice into some collections of cubes, which we will call "trees", where the density of μ does not oscillate too much and most of the measure is concentrated close to a graph of a Lipschitz function. To construct this function we will use a variant of the Whitney extension theorem adapted to the David-Mattila dyadic lattice. Further, we will show that the family of trees of the corona decomposition satisfies a packing condition by arguments inspired by some of the techniques used in [AT] and earlier in [Tol3] to prove the bilipschitz "invariance" of analytic capacity. More precisely, we will deduce Main Lemma from the two-sided estimate

$$p_{k_\infty}(\mu) \lesssim \sum_{R\in\mathsf{Top}} \Theta_\mu(2B_R)^2\mu(R) \lesssim p_{k_0}(\mu) + C_*^2\mu(\mathbb{C}), \tag{3.6}$$

where Top is the family of top cubes for the above-mentioned trees. Note that the left hand side inequality in (3.6) in essentially contained in [Tol6] and verifying the right hand side inequality is actually the main objective in the proof.

It is worth mentioning that the structure of our trees is more complicated than in [AT]. This is because we deal with permutations which are not comparable to curvature in some cases and this leads to additional technical difficulties. What is more, we are not able to use a nice theorem by David and Toro [DT] which shortens the proof in [AT] considerably. Indeed, this theorem would be useful to construct a chordal curve such that most of the measure μ is concentrated close to it. However, in our situation we need to control slope and therefore we have to deal with and to construct a graph of a Lipschitz function with well-controlled Lipschitz constant instead.

The plan of the proof of Main Lemma is the following. In Section 3.5 we recall the properties of the David-Mattila dyadic lattice. We construct the trees and establish their properties in Sections 3.6–3.12. The main properties are summarized in Section 3.13, where they are further used for constructing the corona type decomposition. The end of the proof of Main Lemma is given in Section 3.13.6.

Finally, in Sections 3.14 and 3.15 we show how one can slightly change the proof of Main Lemma in order to give another proof of a certain result from [AT] and also to extend the results in this chapter for a more general class of kernels.

Remark 3.3. In the forthcoming proof, we will usually write p_t instead of p_{k_t} in order to simplify notation.

Remark 3.4. The measure μ considered below is under assumptions of Main Lemma, i.e. μ is a finite measure with C_*-linear growth. Moreover, without loss of generality we additionally suppose that μ has *compact support.*

3.5 The David-Mattila lattice

We use the dyadic lattice of cubes with small boundaries constructed by David and Mattila [DM]. The properties of this lattice are summarized in the next lemma (for the case of $\mathbb{C}$).

Lemma 3.1 (Theorem 3.2 in [DM])**.** *Let μ be a measure, $E = \operatorname{spt}\mu$, and consider two constants $C_0 > 1$ and $A_0 > 5000\,C_0$. Then there exists a sequence of partitions of E into Borel subsets Q, $Q \in \mathcal{D}_k$, with the following properties:*

- *For each integer $k \geqslant 0$, E is the disjoint union of the "cubes" Q, $Q \in \mathcal{D}_k$, and if $k < l$, $Q \in \mathcal{D}_l$, and $R \in \mathcal{D}_k$, then either $Q \cap R = \varnothing$ or else $Q \subset R$.*
- *The general position of the cubes Q can be described as follows. For each $k \geqslant 0$ and each cube $Q \in \mathcal{D}_k$, there is a ball $B(Q) = B(z_Q, r(Q))$ such that*
$$z_Q \in Q, \qquad A_0^{-k} \leqslant r(Q) \leqslant C_0\, A_0^{-k},$$
$$E \cap B(Q) \subset Q \subset E \cap 28\, B(Q) = E \cap B(z_Q, 28r(Q)),$$
and
$$\text{the balls } 5B(Q),\ Q \in \mathcal{D}_k, \text{ are disjoint.}$$
- *The cubes $Q \in \mathcal{D}_k$ have small boundaries. That is, for each $Q \in \mathcal{D}_k$ and each integer $l \geqslant 0$, set*
$$N_l^{ext}(Q) = \{x \in E \setminus Q :\ \operatorname{dist}(x, Q) < A_0^{-k-l}\},$$
$$N_l^{int}(Q) = \{x \in Q :\ \operatorname{dist}(x, E \setminus Q) < A_0^{-k-l}\},$$
and
$$N_l(Q) = N_l^{ext}(Q) \cup N_l^{int}(Q).$$
Then
$$\mu(N_l(Q)) \leqslant (C^{-1} C_0^{-7} A_0)^{-l}\, \mu(90B(Q)).$$
- *Denote by $\mathcal{D}_k^{db}$ the family of cubes $Q \in \mathcal{D}_k$ for which*
$$\mu(100B(Q)) \leqslant C_0\, \mu(B(Q)). \tag{3.7}$$
If $Q \in \mathcal{D}_k \setminus \mathcal{D}_k^{db}$, then $r(Q) = A_0^{-k}$ and
$$\mu(100B(Q)) \leqslant C_0^{-l}\, \mu(100^{l+1}B(Q)) \quad \textit{for all } l \geqslant 1 \textit{ such that } 100^l \leqslant C_0.$$

We use the notation $\mathcal{D} = \bigcup_{k \geqslant 0} \mathcal{D}_k$. For $Q \in \mathcal{D}$, we set $\mathcal{D}(Q) = \{P \in \mathcal{D} : P \subset Q\}$. Observe that
$$r(Q) \approx \operatorname{diam}(Q).$$
Also we call z_Q the center of Q. We set $B_Q = 28\, B(Q) = B(z_Q, 28\, r(Q))$, so that
$$E \cap \tfrac{1}{28} B_Q \subset Q \subset B_Q.$$

We denote $\mathcal{D}^{db} = \bigcup_{k \geqslant 0} \mathcal{D}_k^{db}$ and $\mathcal{D}^{db}(Q) = \mathcal{D}^{db} \cap \mathcal{D}(Q)$. Note that, in particular, from (3.7) it follows that
$$\mu(100B(Q)) \leqslant C_0\, \mu(2B_Q) \qquad \text{if } Q \in \mathcal{D}^{db}. \tag{3.8}$$
For this reason we will call the cubes from $\mathcal{D}^{db}$ *doubling.*

As shown in [DM], any cube $Q \in \mathcal{D}$ can be covered μ-a.e. by doubling cubes.

Lemma 3.2 (Lemma 5.28 in [DM])**.** *Let $Q \in \mathcal{D}$. Suppose that the constants A_0 and C_0 in Lemma 3.1 are chosen suitably. Then there exists a family of doubling cubes $\{Q_i\}_{i \in I} \subset \mathcal{D}^{db}$, with $Q_i \subset Q$ for all i, such that their union covers μ-almost all Q.*

We denote by $J(Q)$ the number k such that $Q \in \mathcal{D}_k$.

Lemma 3.3 (Lemma 5.31 in [DM])**.** *Let $P \in \mathcal{D}$ and let $Q \subsetneq P$ be a cube such that all the intermediate cubes S, $Q \subsetneq S \subsetneq P$, are non-doubling (i.e. not in $\mathcal{D}^{db}$). Then*
$$\mu(100B(Q)) \leqslant A_0^{-20(J(Q)-J(P)-1)} \mu(100B(P)).$$

Recall that $\Theta_\mu(B) = \mu(B(x,r))/r$. From Lemma 3.3 one can easily deduce[1]

Lemma 3.4 (Lemma 2.4 in [AT]). *Let $Q, P \in \mathcal{D}$ be as in Lemma 3.3. Then*

$$\Theta_\mu(100B(Q)) \leqslant C_0 A_0^{-19(J(Q)-J(P)-1)+1}\, \Theta_\mu(100B(P)) \leqslant C_0 A_0\, \Theta_\mu(100B(P))$$

and

$$\sum_{S\in\mathcal{D}:Q\subset S\subset P} \Theta_\mu(100B(S)) \leqslant c\, \Theta_\mu(100B(P)), \qquad c = c(C_0, A_0).$$

We will assume that all implicit constants in the inequalities that follow may depend on C_0 and A_0. Moreover, we will assume that C_0 and A_0 are some big *fixed* constants so that the results stated in the lemmas below hold.

3.6 Balanced cubes and control on beta numbers through permutations

We first recall the properties of the so called balanced balls introduced in [AT].

Lemma 3.5 (Lemma 3.3 and Remark 3.2 in [AT]). *Let μ be a measure and consider the dyadic lattice $\mathcal{D}$ associated with μ from Lemma 3.1. Let $0 < \gamma < 1$ be small enough (with respect to some absolute constant), then there exist $\rho' = \rho'(\gamma) > 0$ and $\rho'' = \rho''(\gamma) > 0$ such that one of the following alternatives holds for every $Q \in \mathcal{D}^{db}$:*

(*a*) *There are balls $B_k = B(\xi_k, \rho'\, r(Q))$, $k = 1, 2$, where $\xi_1, \xi_2 \in B(Q)$, such that*

$$\mu\left(B_k \cap B(Q)\right) \geqslant \rho''\, \mu(Q), \qquad k = 1, 2,$$

and for any $y_k \in B_k \cap Q$, $k = 1, 2$,

$$\operatorname{dist}(y_1, y_2) \geqslant \gamma\, r(B_Q).$$

(*b*) *There exists a family of pairwise disjoint cubes $\{P\}_{P\in I_Q} \subset \mathcal{D}^{db}(Q)$ so that $\operatorname{diam}(P) \gtrsim \gamma \operatorname{diam}(Q)$ and $\Theta_\mu(2B_P) \gtrsim \gamma^{-1}\, \Theta_\mu(2B_Q)$ for each $P \in I_Q$, and*

$$\sum_{P\in I_Q} \Theta_\mu(2B_P)^2\, \mu(P) \gtrsim \gamma^{-2}\, \Theta_\mu(2B_Q)^2\, \mu(Q). \tag{3.9}$$

Let us mention that the densities in the latter inequality in the original Lemma 3.3 in [AT] are not squared. However, a slight variation of the proof of [AT, Lemma 3.3] gives (3.9) as stated.

Moreover, notice that in Lemma 3.5 the cubes Q and P, with $P \in I_Q$, are doubling. If the alternative (*a*) holds for a doubling cube Q with some γ, $\rho'(\gamma)$ and $\rho''(\gamma)$, then the corresponding ball $B(Q)$ is called γ-*balanced*. Otherwise, it is called γ-*unbalanced*. If $B(Q)$ is γ-balanced, then the cube Q is also called γ-*balanced*.

We are going to show now that the beta numbers $\beta_{\mu,2}(2B_Q)$ (see (3.2)) for γ-balanced cubes Q are controlled by a truncated version of the permutations $p_0(\mu\lfloor 2B_Q)$. To do so, we introduce some additional notation.

Given two distinct points $z, w \in \mathbb{C}$, we denote by $L_{z,w}$ the line passing through z and w. Given three pairwise distinct points $z_1, z_2, z_3 \in \mathbb{C}$, we denote by $\measuredangle(z_1, z_2, z_3)$ the smallest

[1]Note that there is an inaccuracy with constants in the original Lemma 2.4 in [AT].

angle formed by the lines L_{z_1,z_2} and L_{z_1,z_3} and belonging to $[0,\pi/2]$. If L and L' are lines, let $\measuredangle(L,L')$ be the smallest angle between them. This angle belongs to $[0,\pi/2]$, too. Also, we set $\theta_V(L)=\measuredangle(L,V)$, where V is the vertical line.

First we recall the following result of Chousionis and Prat [CP]. We say that a triple $(z_1,z_2,z_3)\in\mathbb{C}^3$ is *in the class* $\mathrm{V}_{\mathsf{Far}}(\theta)$ if it satisfies

$$\theta_V(L_{z_1,z_2})+\theta_V(L_{z_1,z_3})+\theta_V(L_{z_2,z_3})\geqslant\theta>0. \tag{3.10}$$

Lemma 3.6 (Proposition 3.3 in [CP]). *If* $(z_1,z_2,z_3)\in\mathrm{V}_{\mathsf{Far}}(\theta)$, *then*

$$p_0(z_1,z_2,z_3)\geqslant\mathsf{c}_1(\theta)\cdot p_\infty(z_1,z_2,z_3),\quad \textit{where}\quad 0<\mathsf{c}_1(\theta)\leqslant 2. \tag{3.11}$$

Note that the inequality $\mathsf{c}_1(\theta)\leqslant 2$ follows from (2.5) that was proved in [CMT1].

For measures μ_1, μ_2 and μ_3 and a cube Q we set

$$p_0^{[\delta,Q]}(\mu_1,\mu_2,\mu_3):=\iiint\limits_{\delta r(Q)\leqslant|z_1-z_2|\leqslant\delta^{-1}r(Q)}p_0(z_1,z_2,z_3)\,d\mu_1(z_1)d\mu_2(z_2)d\mu_3(z_3).$$

The parameter $\delta>0$ will be chosen later to be small enough for our purposes. If $\mu_1=\mu_2=\mu_3=\mu$, then we write $p_0^{[\delta,Q]}(\mu)$ instead of $p_0^{[\delta,Q]}(\mu,\mu,\mu)$, for short.

Now we are ready to state the above mentioned estimate of $\beta_{\mu,2}(2B_Q)$ for γ-balanced cubes Q via the truncated version of $p_0(\mu\lfloor 2B_Q)$. Pay attention that the first term in the estimate is a “non-summable” part which makes a big difference with the case of curvature or p_∞ (see Section 3.14).

Lemma 3.7. *If Q is γ-balanced, then for any $\varepsilon\in(0,1)$,*

$$\beta_{\mu,2}(2B_Q)^2\Theta_\mu(2B_Q)\leqslant 4\varepsilon^2\Theta_\mu(2B_Q)^2+C(\varepsilon,\gamma)\frac{p_0^{[\delta,Q]}(\mu\lfloor 2B_Q)}{\mu(Q)},\qquad 0<\delta\leqslant\gamma. \tag{3.12}$$

Moreover, for any $\varepsilon_0>0$, there exist $\varepsilon=\varepsilon(\varepsilon_0)>0$ and $\tilde\varepsilon=\tilde\varepsilon(\varepsilon_0,\gamma)>0$ such that if

$$\frac{p_0^{[\delta,Q]}(\mu\lfloor 2B_Q)}{\Theta_\mu(2B_Q)^2\mu(Q)}\leqslant\tilde\varepsilon, \tag{3.13}$$

then

$$\beta_{\mu,2}(2B_Q)^2\leqslant\varepsilon_0^2\Theta_\mu(2B_Q).$$

Proof. By Lemma 3.5, there exist balls $B_k=B(\xi_k,\rho' r(Q))$, $k=1,2$, where $\xi_k\in B(Q)$, such that $\mu(B_k\cap B(Q))\geqslant\rho''\mu(Q)$ and $\operatorname{dist}(y_1,y_2)\geqslant\gamma r(B_Q)$ for any $y_k\in B_k\cap Q$, $k=1,2$. From (3.2) it follows that

$$\beta_{\mu,2}(2B_Q)^2\leqslant\frac{1}{2r(B_Q)}\int_{2B_Q}\left(\frac{\operatorname{dist}(w,L_{y_1,y_2})}{2r(B_Q)}\right)^2d\mu(w).$$

We separate triples (w,y_1,y_2) that are in and not in $\mathrm{V}_{\mathsf{Far}}(\varepsilon)$. Clearly,

$$\operatorname{dist}(w,L_{y_1,y_2})\leqslant\operatorname{diam}(2B_Q)\sin\varepsilon\leqslant 4\,\varepsilon\, r(B_Q)\qquad\text{if }(w,y_1,y_2)\notin\mathrm{V}_{\mathsf{Far}}(\varepsilon).$$

Thus

$$
\begin{aligned}
&\beta_{\mu,2}(2B_Q)^2 \\
&\quad\leqslant \frac{4\varepsilon^2}{2r(B_Q)}\int_{2B_Q} d\mu(w) + \frac{1}{2r(B_Q)}\int_{2B_Q,\,(w,y_1,y_2)\in\mathsf{V}_{\mathsf{Far}}(\varepsilon)} \left(\frac{\operatorname{dist}(w, L_{y_1,y_2})}{2r(B_Q)}\right)^2 d\mu(w) \\
&\quad\leqslant 4\varepsilon^2\Theta_\mu(2B_Q) + 8r(B_Q)\int_{2B_Q,\,(w,y_1,y_2)\in\mathsf{V}_{\mathsf{Far}}(\varepsilon)} \left(\frac{2\operatorname{dist}(w, L_{y_1,y_2})}{|w-y_1||w-y_2|}\right)^2 d\mu(w) \\
&\quad= 4\varepsilon^2\Theta_\mu(2B_Q) + 8r(B_Q)\int_{2B_Q,\,(w,y_1,y_2)\in\mathsf{V}_{\mathsf{Far}}(\varepsilon)} c(w,y_1,y_2)^2 d\mu(w).
\end{aligned}
$$

We used that $|w-y_k| \leqslant \operatorname{diam}(2B_Q) = 4r(B_Q)$ as $w, y_1, y_2 \in 2B_Q$ and that

$$c(w,y_1,y_2) = \frac{2\operatorname{dist}(w, L_{y_1,y_2})}{|w-y_1||w-y_2|}.$$

Recall that $r(B_Q) = 28r(Q)$ by definition. By (0.15) and (3.11),

$$
\begin{aligned}
&\int_{2B_Q,\,(w,y_1,y_2)\in\mathsf{V}_{\mathsf{Far}}(\varepsilon)} c(w,y_1,y_2)^2 d\mu(w) \\
&\qquad\qquad \leqslant \frac{2}{\mathsf{c}_1(\varepsilon)}\int_{2B_Q,\,(w,y_1,y_2)\in\mathsf{V}_{\mathsf{Far}}(\varepsilon)} p_0(w,y_1,y_2) d\mu(w).
\end{aligned}
$$

Recall that $|y_1 - y_2| \geqslant \gamma r(Q)$ for any $y_k \in B_k \cap Q$, $k = 1, 2$. Furthermore, for any δ such that $0 < \delta \leqslant \gamma$ we can find $y_1 \in B_1$ and $y_2 \in B_2$ so that

$$\int_{2B_Q} p_0(w,y_1,y_2) d\mu(w) \leqslant \frac{p_0^{[\delta,Q]}(\mu\lfloor 2B_Q)}{\mu(B_1)\mu(B_2)} \leqslant \frac{p_0^{[\delta,Q]}(\mu\lfloor 2B_Q)}{(\rho'')^2\mu(Q)^2}.$$

By (3.7) and the fact that $E \cap B(Q) \subset Q$, we deduce that

$$\mu(Q) \geqslant C_0^{-1}\mu(100B(Q)) \geqslant C_0^{-1}\mu(56B(Q)) = C_0^{-1}\mu(2B_Q).$$

Consequently,

$$
\begin{aligned}
\beta_{\mu,2}(2B_Q)^2 &\leqslant 4\varepsilon^2\Theta_\mu(2B_Q) + \frac{16r(B_Q)p_0^{[\delta,Q]}(\mu\lfloor 2B_Q)}{\mathsf{c}_1(\varepsilon)(\rho'')^2\mu(Q)C_0^{-1}\mu(2B_Q)} \\
&= 4\varepsilon^2\Theta_\mu(2B_Q) + C(\varepsilon,\gamma)\frac{p_0^{[\delta,Q]}(\mu\lfloor 2B_Q)}{\Theta_\mu(2B_Q)\mu(Q)}.
\end{aligned}
$$

Multiplying both sides by $\Theta_\mu(2B_Q)$ finishes the proof of (3.12). Note that $\rho'' = \rho''(\gamma)$.

Let us prove the second statement. By the assumption (3.13),

$$\beta_{\mu,2}(2B_Q)^2 \leqslant (4\varepsilon^2 + C(\varepsilon,\gamma)\tilde{\varepsilon})\Theta_\mu(2B_Q).$$

For any $\varepsilon_0 > 0$, we put $\varepsilon = \frac{\sqrt{2}}{4}\varepsilon_0$ and choose $\tilde{\varepsilon}$ so that $\tilde{\varepsilon} \leqslant \frac{1}{2}\varepsilon_0^2/C(\varepsilon,\gamma)$. □

3.7 Parameters and thresholds

Recall that we work everywhere with the David-Mattila dyadic lattice $\mathcal{D}$ associated with the measure μ.

In what follows we will use many parameters and thresholds. Some of them depend on each other, some are independent. Let us give a list of the parameters:

- τ is the threshold for cubes with low density:

$$0 < \tau \ll 1.$$

- A is the threshold for cubes with high density:

$$0 < A^{-1} \leqslant \tau^2 \ll 1, \quad \text{i.e.} \quad A \gg 1.$$

- θ_0 is the threshold for the angle between best approximating lines associated to some cubes:

$$0 < \theta_0 \ll 1.$$

- γ is the parameter controlling unbalanced cubes:

$$0 < \gamma \leqslant \tau^3 \ll 1.$$

- ε_0 is the threshold controlling the $\beta_{2,\mu}$-numbers:

$$0 < \varepsilon_0 = \varepsilon_0(\gamma, \tau, A, \theta_0) \ll 1.$$

- α is the threshold controlling permutations of intermediate cubes:

$$0 < \alpha = \alpha(\tau, A, \varepsilon_0, \gamma, \theta_0) \ll 1.$$

- δ is the parameter controlling the truncation of permutations:

$$0 < \delta = \delta(\gamma, \varepsilon_0, \tau, A) \ll 1.$$

All the parameters and thresholds are supposed to be chosen (and fixed at the very end) so that the forthcoming results hold true. In what follows, we will again indicate step by step how the choice should be made.

3.8 Stopping cubes and trees

3.8.1 Stopping cubes

Let $R \in \mathcal{D}^{db}$. We use the parameters and thresholds given in Section 3.7. We denote by $\mathsf{Stop}(R)$ the family of the maximal cubes $Q \subset R$ for which one of the following holds:

(S1) $Q \in \mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R)$, where

- $\mathsf{HD}(R)$ is the family of *high density* doubling cubes $Q \in \mathcal{D}^{db}$ satisfying

$$\Theta_\mu(2B_Q) > A\,\Theta_\mu(2B_R);$$

- $\mathsf{LD}(R)$ is the family of *low density* cubes Q satisfying

$$\Theta_\mu(2B_Q) < \tau\,\Theta_\mu(2B_R);$$

- $\mathsf{UB}(R)$ is the family of *unbalanced* cubes $Q \in \mathcal{D}^{db} \setminus (\mathsf{HD}(R) \cup \mathsf{LD}(R))$ such that Q is γ-unbalanced;

(**S2**) $Q \in \mathsf{BP}(R)$ ("*big permutations*"), meaning $Q \notin \mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R)$ and

$$\sum_{Q \subset \tilde{Q} \subset R} \mathsf{perm}(\tilde{Q})^2 > \alpha^2, \qquad \mathsf{perm}(\tilde{Q})^2 := \frac{p_0^{[\delta,\tilde{Q}]}(\mu\lfloor 2B_{\tilde{Q}}, \mu\lfloor 2B_R, \mu\lfloor 2B_R)}{\Theta_\mu(2B_R)^2 \mu(\tilde{Q})}.$$

(**S3**) $Q \in \mathsf{BS}(R)$ ("*big slope*"), meaning $Q \notin \mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R) \cup \mathsf{BP}(R)$ and $Q \in \mathcal{D}^{db}$ so that

$$\measuredangle(L_Q, L_R) > \theta(R),$$

where $\theta(R)$ depends on some geometric properties of R and is comparable with the parameter $\theta_0 > 0$ mentioned in Section 3.7. The more precise description will be given in Section 3.12.

(**S4**) $Q \in \mathsf{F}(R)$ ("*big part of Q is far from best approximating lines for the doubling ancestors of Q*"), meaning $Q \notin \mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R) \cup \mathsf{BP}(R) \cup \mathsf{BS}(R)$ and

$$\mu(Q \setminus 2B_Q^{\mathsf{Cl}}) > \sqrt{\alpha}\, \mu(Q),$$

where

$$\begin{aligned} 2B_Q^{\mathsf{Cl}} := \{x \in R \cap 2B_Q : \ & \operatorname{dist}(x, L_{\tilde{Q}}) \leqslant 5\sqrt{\varepsilon_0}\, r(B_{\tilde{Q}}) \quad \forall \tilde{Q} \in \mathcal{D}^{db}(R) : \\ & 2B_Q \subset 2B_{\tilde{Q}} \text{ and } \tilde{Q} \text{ is not contained in any cube from} \\ & \mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R) \cup \mathsf{BP}(R) \cup \mathsf{BS}(R)\}. \end{aligned} \tag{3.14}$$

Let $\mathsf{Tree}(R)$ be the subfamily of the cubes from $\mathcal{D}(R)$ which are not *strictly* contained in any cube from $\mathsf{Stop}(R)$. We also set

$$\mathsf{DbTree}(R) := \mathcal{D}^{db} \cap (\mathsf{Tree}(R) \setminus \mathsf{Stop}(R)).$$

Note that all cubes in $\mathsf{Stop}(R)$ are disjoint.

Remark 3.5. It may happen that $\mathsf{Stop}(R)$ is empty. In this case there is no need to estimate the measure of stopping cubes and we may immediately go to Section 3.10. In the lemmas below related to estimating the measure of stopping cubes we naturally suppose that $\mathsf{Stop}(R)$ is not empty.

Generally speaking it is possible that $R \in \mathsf{Stop}(R)$ (and then $\mathsf{DbTree}(R)$ is empty). Clearly, $R \notin \mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{BS}(R)$ by definition but it may occur that $R \in \mathsf{UB}(R) \cup \mathsf{BP}(R) \cup \mathsf{F}(R)$. Firstly, we will not work with the family $\mathsf{UB}(R)$ before Section 3.13 so we may assume before that section that $R \notin \mathsf{UB}(R)$. Secondly, if $R \in \mathsf{BP}(R)$, then we may directly go to Lemma 3.14 and use the same estimate for the measure of stopping cubes from $\mathsf{BP}(R)$. Thirdly, it will follow from Lemmas 3.12 and 3.13 (see Remark 3.6) that if $R \notin \mathsf{UB}(R) \cup \mathsf{BP}(R)$, then $R \notin \mathsf{F}(R)$, i.e. the case $R \in \mathsf{F}(R)$ may be skipped.

It is also worth mentioning that if $R \in \mathsf{Stop}(R)$, then the Lipschitz function mentioned in Section 3.4 may be chosen identically zero and its graph is just L_R.

3.8.2 Properties of cubes in trees

Below, we will collect main properties of cubes from $\mathsf{Tree}(R)$ that readily follow from the stopping conditions. Before it we prove an additional result.

Lemma 3.8. *For any $Q \in \mathsf{Tree}(R)$, we have*

$$\Theta_\mu(2B_Q) \lesssim A\,\Theta_\mu(2B_R).$$

The implicit constant depends only on C_0 and A_0.

Proof. Let $Q \in \mathsf{Tree}(R)$. If $Q \in \mathcal{D}^{db}$, then there is nothing to prove. If not, then denote by $\tilde{Q} \in \mathcal{D}^{db}$ the first doubling ancestor of Q. Such a cube $\tilde{Q}$ exists and $\tilde{Q} \subset R$ because $R \in \mathcal{D}^{db}$ by construction. Since the intermediate cubes P, $Q \subsetneq P \subsetneq \tilde{Q}$, do not belong to $\mathcal{D}^{db}$, by Lemma 3.4 we have

$$\Theta_\mu(2B_Q) \lesssim \Theta_\mu(100B(Q)) \lesssim C_0 A_0 \Theta_\mu(100B(\tilde{Q})).$$

Using that $\tilde{Q} \in \mathcal{D}^{db}$, namely, the inequality (3.8), we get

$$\Theta_\mu(2B_Q) \lesssim C_0^2 A_0\, \Theta_\mu(2B_{\tilde{Q}}) \lesssim C_0^2 A_0\, A\, \Theta_\mu(2B_R),$$

and we are done. □

Lemma 3.9. *The following properties hold:*

$$\tau\,\Theta_\mu(2B_R) \leqslant \Theta_\mu(2B_Q) \lesssim A\,\Theta_\mu(2B_R), \quad \forall Q \in \mathsf{Tree}(R) \setminus (\mathsf{LD}(R) \cup \mathsf{HD}(R)). \tag{3.15}$$

$$Q \in \mathcal{D}^{db} \cap (\mathsf{Tree}(R) \setminus (\mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R))) \quad \Longrightarrow \quad Q \textit{ is } \gamma\textit{-balanced.} \tag{3.16}$$

$$\sum_{Q \subset \tilde{Q} \subset R} \mathsf{perm}(\tilde{Q})^2 < \alpha^2 \quad \forall Q \in \mathsf{Tree}(R) \setminus (\mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R) \cup \mathsf{BP}(R)). \tag{3.17}$$

$$\begin{gathered} \beta_{\mu,2}(2B_Q)^2 \leqslant \varepsilon_0^2 \Theta_\mu(2B_Q) \quad \textit{if } \alpha = \alpha(\gamma, \tau, \varepsilon_0) \textit{ is small enough and} \\ Q \in \mathcal{D}^{db} \cap (\mathsf{Tree}(R) \setminus (\mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R) \cup \mathsf{BP}(R))). \end{gathered} \tag{3.18}$$

$$\measuredangle(L_Q, L_R) \leqslant \theta(R) \qquad \forall Q \in \mathsf{DbTree}(R). \tag{3.19}$$

$$\mu(Q \setminus 2B_Q^{\mathsf{Cl}}) \leqslant \sqrt{\alpha}\,\mu(Q) \qquad \forall Q \in \mathsf{Tree}(R) \setminus \mathsf{Stop}(R). \tag{3.20}$$

Proof. The statement (3.15) follows from Lemma 3.8 and the stopping condition **(S1)**. The statements (3.16), (3.17), (3.19) and (3.20) immediately follow from the construction of $\mathsf{Stop}(R)$ and $\mathsf{Tree}(R)$, while (3.18) is implied by Lemma 3.7 and the stopping conditions **(S1)** and **(S2)**. □

The following property of γ-balanced cubes will be used many times below.

Lemma 3.10. *Let $\varepsilon_0 = \varepsilon_0(\gamma)$ be chosen small enough. Then for any $Q \in \mathcal{D}^{db} \cap (\mathsf{Tree}(R) \setminus (\mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R) \cup \mathsf{BP}(R)))$ there exist two sets $\mathcal{Z}_k \subset Q$, $k = 1, 2$, such that*

$$\mu(Q) \lesssim_\gamma \mu(\mathcal{Z}_k) \leqslant \mu(Q) \quad \textit{and} \quad \operatorname{dist}(\mathcal{Z}_1, \mathcal{Z}_2) \geqslant \gamma r(B_Q),$$

and moreover for any $z_1 \in \mathcal{Z}_1$ and $z_2 \in \mathcal{Z}_2$ we have

$$\operatorname{dist}_H(L_{z_1,z_2} \cap 2B_Q, L_Q \cap 2B_Q) \leqslant \sqrt{\varepsilon_0}\, r(B_Q).$$

Proof. Since $Q \in \mathcal{D}^{db} \cap (\mathsf{Tree}(R) \setminus (\mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R)))$, Q is γ-balanced by (3.16). Furthermore, by Lemma 3.5 there exist balls $B_k = B(\xi_k, \rho' r(Q))$, $k = 1, 2$, where $\xi_k \in B(Q)$, such that

$$\mu(B_k \cap B(Q)) \geqslant \rho'' \mu(Q) \quad \text{and} \quad \operatorname{dist}(y_1, y_2) \geqslant \gamma r(B_Q) \quad \text{for any } y_k \in B_k,\ k = 1, 2,$$

where ρ' and ρ'' depend on γ. Due to the estimate $\beta_{\mu,2}(2B_Q)^2 \leqslant \varepsilon_0^2 \Theta_\mu(2B_Q)$ (see (3.18)), by Chebyshev's inequality there exist $\mathcal{Z}_k \subset B_k \cap Q$ such that

$$\mu(Q) \lesssim_\gamma \mu(B_k) \lesssim \mu(\mathcal{Z}_k) \leqslant \mu(Q) \quad \text{and} \quad \sup_{z \in \mathcal{Z}_k} \operatorname{dist}(z, L_Q) \lesssim_\gamma \varepsilon_0\, r(B_Q), \quad k = 1, 2.$$

Thus for any $z_1 \in \mathcal{Z}_1$ and $z_2 \in \mathcal{Z}_2$ we have

$$\operatorname{dist}(z_k, L_Q) \lesssim_\gamma \varepsilon_0\, r(B_Q), \quad k = 1, 2, \qquad \operatorname{dist}(z_1, z_2) \gtrsim_\gamma r(B_Q).$$

This implies that $\measuredangle(L_{z_1,z_2}, L_Q) \lesssim_\gamma \varepsilon_0$ and therefore the following estimate for the Hausdorff distance holds:

$$\operatorname{dist}_H(L_{z_1,z_2} \cap 2B_Q, L_Q \cap 2B_Q) \lesssim_\gamma \varepsilon_0\, r(B_Q).$$

Choosing ε_0 small enough with respect to the implicit constant depending on γ, we obtain the required result. □

Clearly, it may happen that not all cubes in $\mathsf{Tree}(R)$ are γ-balanced as there may be undoubling cubes. However, for any cube in $\mathsf{Tree}(R)$, there is always an ancestor in $\mathsf{DbTree}(R)$ close by. Namely, the following result holds.

Lemma 3.11 (Lemma 6.3 in [AT]). *For any cube $Q \in \mathsf{Tree}(R)$ there exists a cube $\tilde{Q} \supset Q$ such that $\tilde{Q} \in \mathsf{DbTree}(R)$ and $\operatorname{diam}(\tilde{Q}) \leqslant \lambda \operatorname{diam}(Q)$ with some $\lambda = \lambda(A, \tau)$.*

Now we want to show that the measure of the set of points from R which are far from the best approximation lines for cubes in $\{R\} \cup (\mathsf{Tree}(R) \setminus \mathsf{Stop}(R))$ is small. Set

$$p_0^{[\delta,Q]}(x, \mu, \mu) := \iint_{\delta r(Q) \leqslant |x-y| \leqslant \delta^{-1} r(Q)} p_0(x, y, z)\, d\mu(y) d\mu(z)$$

and consider

$$R_{\mathsf{Far}} := \{x \in R : \quad \frac{p_0^{[\delta,Q]}(x, \mu\lfloor 2B_R, \mu\lfloor 2B_R)}{\Theta_\mu(2B_R)^2} \geqslant \mathsf{c}_2(\gamma, \tau, \varepsilon_0)$$
$$\text{for some } Q \in \{R\} \cup (\mathsf{Tree}(R) \setminus \mathsf{Stop}(R)) \text{ such that } x \in 2B_Q\},$$

where $\mathsf{c}_2(\gamma, \tau, \varepsilon_0) > 0$ will be defined precisely in the proof of Lemma 3.13.

Lemma 3.12. *If $R \notin \mathsf{UB}(R) \cup \mathsf{BP}(R)$ and $\alpha = \alpha(\gamma, \tau, \varepsilon_0)$ is chosen small enough, then*

$$\mu(R_{\mathsf{Far}}) \leqslant \alpha \mu(R).$$

Proof. By Chebyshev's inequality,

$$\begin{aligned}
&\mathsf{c}_2(\gamma, \tau, \varepsilon_0)\, \mu(R_{\mathsf{Far}}) \\
&\qquad \leqslant \int_R \sum_{Q \in \{R\} \cup (\mathsf{Tree}(R) \setminus \mathsf{Stop}(R)) :\, x \in 2B_Q} \frac{p_0^{[\delta,Q]}(x, \mu\lfloor 2B_R, \mu\lfloor 2B_R)}{\Theta_\mu(2B_R)^2}\, d\mu(x) \\
&\qquad \leqslant \sum_{Q \in \{R\} \cup (\mathsf{Tree}(R) \setminus \mathsf{Stop}(R))} \frac{p_0^{[\delta,Q]}(\mu\lfloor 2B_Q, \mu\lfloor 2B_R, \mu\lfloor 2B_R)}{\Theta_\mu(2B_R)^2} \\
&\qquad = \sum_{Q \in \{R\} \cup (\mathsf{Tree}(R) \setminus \mathsf{Stop}(R))} \frac{p_0^{[\delta,Q]}(\mu\lfloor 2B_Q, \mu\lfloor 2B_R, \mu\lfloor 2B_R)}{\Theta_\mu(2B_R)^2 \mu(Q)} \int \chi_Q(x)\, d\mu(x).
\end{aligned}$$

Changing the order of summation yields

$$
\begin{aligned}
&\mathsf{c}_2(\gamma,\tau,\varepsilon_0)\,\mu(R_{\mathsf{Far}})\\
&\leqslant \int_R \left(\frac{p_0^{[\delta,R]}(\mu\lfloor 2B_R)}{\Theta_\mu(2B_R)^2\mu(R)} + \sum_{Q\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R)\colon x\in Q} \frac{p_0^{[\delta,Q]}(\mu\lfloor 2B_Q,\mu\lfloor 2B_R,\mu\lfloor 2B_R)}{\Theta_\mu(2B_R)^2\mu(Q)} \right) d\mu(x)\\
&\leqslant \int_R \left(\mathsf{perm}(R)^2 + \sum_{Q\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R)\colon x\in Q} \mathsf{perm}(Q)^2 \right) d\mu(x)\\
&\leqslant 2\alpha^2\,\mu(R).
\end{aligned}
$$

Supposing that $2\alpha \leqslant \mathsf{c}_2(\gamma,\tau,\varepsilon_0)$ gives the required result. □

Recall the definition (3.14).

Lemma 3.13. *Let $\delta = \delta(\varepsilon_0)$ be chosen small enough. If $x \in (R \cap 2B_{\tilde{Q}}) \setminus 2B_{\tilde{Q}}^{\mathsf{Cl}}$ for some $\tilde{Q} \in \mathsf{Tree}(R)$, i.e. in particular there exists $Q \in \mathcal{D}^{db}(R)$ such that $2B_Q \supset 2B_{\tilde{Q}}$ and Q is not contained in any cube from $\mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R) \cup \mathsf{BP}(R) \cup \mathsf{BS}(R)$, then $x \in R_{\mathsf{Far}}$.*

Proof. Clearly, $x \in 2B_Q$ and $Q \in \mathcal{D}^{db} \cap (\mathsf{Tree}(R) \setminus (\mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{UB}(R) \cup \mathsf{BP}(R)))$. Therefore, by Lemma 3.10 we can find $\mathcal{Z}_k \subset Q$, $k = 1, 2$, such that for any $z_1 \in \mathcal{Z}_1$ and $z_2 \in \mathcal{Z}_2$ we have

$$\operatorname{dist}_H(L_{z_1,z_2} \cap 2B_Q, L_Q \cap 2B_Q) \leqslant \sqrt{\varepsilon_0}\, r(B_Q).$$

Consider triangle (x, z_1, z_2) which is wholly contained in $2B_Q$. It is easily seen that

$$\operatorname{dist}(x, L_{z_1,z_2}) \geqslant \operatorname{dist}(x, L_Q) - \operatorname{dist}_H(L_{z_1,z_2} \cap 2B_Q, L_Q \cap 2B_Q) \geqslant 4\sqrt{\varepsilon_0}\, r(B_Q). \tag{3.21}$$

This implies that one of the angle of the triangle (x, z_1, z_2) is at least

$$\frac{4\sqrt{\varepsilon_0}\, r(B_Q)}{\operatorname{diam}(2B_Q)} = \sqrt{\varepsilon_0},$$

and thus $(x, z_1, z_2) \in \mathrm{V}_{\mathsf{Far}}(\sqrt{\varepsilon_0})$ for any $z_1 \in \mathcal{Z}_1$ and $z_2 \in \mathcal{Z}_2$. Note also that (3.21) implies that $|x - z_1| \geqslant \delta r(Q)$ if $\delta = \delta(\varepsilon_0)$ is chosen small enough. Consequently, by the identity (0.15) and Lemma 3.6,

$$
\begin{aligned}
p_0^{[\delta,Q]}(x,\mu\lfloor 2B_R,\mu\lfloor 2B_R) &\geqslant \iint_{z_1\in\mathcal{Z}_1,\, z_2\in\mathcal{Z}_2} p_0(x,z_1,z_2)\, d\mu(z_1)d\mu(z_2)\\
&\geqslant \tfrac{1}{2}\mathsf{c}_1(\sqrt{\varepsilon_0}) \iint_{z_1\in\mathcal{Z}_1,\, z_2\in\mathcal{Z}_2} c(x,z_1,z_2)^2\, d\mu(z_1)d\mu(z_2)\\
&= \tfrac{1}{2}\mathsf{c}_1(\sqrt{\varepsilon_0}) \iint_{z_1\in\mathcal{Z}_1,\, z_2\in\mathcal{Z}_2} \left(\frac{2\operatorname{dist}(x,L_{z_1,z_2})}{|x-z_1||x-z_2|}\right)^2 d\mu(z_1)d\mu(z_2),
\end{aligned}
$$

where the constant c_1 is from Lemma 3.6. Furthermore, we apply (3.21) and the fact that $|x - z_k| \leqslant \operatorname{diam}(2B_Q) = 4r(B_Q)$ for $k = 1, 2$ to obtain the following:

$$p_0^{[\delta,Q]}(x,\mu\lfloor 2B_R,\mu\lfloor 2B_R) \geqslant \frac{\varepsilon_0\,\mathsf{c}_1(\sqrt{\varepsilon_0})}{8r(B_Q)^2}\,\mu(\mathcal{Z}_1)\mu(\mathcal{Z}_2).$$

Since $\mu(\mathcal{Z}_k) \gtrsim_\gamma \mu(Q)$ by Lemma 3.10, $\mu(Q) \gtrsim \mu(2B_Q)$ as $Q \in \mathcal{D}^{db}$ and $\Theta_\mu(2B_Q) \geqslant \tau\Theta_\mu(2B_R)$ by (3.15), we finally get

$$p_0^{[\delta,Q]}(x,\mu\lfloor 2B_R,\mu\lfloor 2B_R) \gtrsim_\gamma \varepsilon_0\,\mathsf{c}_1(\sqrt{\varepsilon_0})\,\tau^2\Theta_\mu(2B_R)^2 = \mathsf{c}_2(\gamma,\tau,\varepsilon_0)\,\Theta_\mu(2B_R)^2.$$

Consequently, $x \in R_{\mathsf{Far}}$ by definition. □

Remark 3.6. Suppose that $R \in \mathsf{F}(R)$ and thus $\mu(R \setminus 2B_R^{\mathsf{Cl}}) > \sqrt{\alpha}\,\mu(R)$ by definition. Then it is clear that $R \notin \mathsf{UB}(R) \cup \mathsf{BP}(R)$ (and furthermore $R \notin \mathsf{HD}(R) \cup \mathsf{LD}(R) \cup \mathsf{BS}(R)$, see Remark 3.5) and so $\mu(R_{\mathsf{Far}}) \leqslant \alpha\,\mu(R)$ by Lemma 3.12. Furthermore, $R \setminus 2B_R^{\mathsf{Cl}} \subset R_{\mathsf{Far}}$ by Lemma 3.13 (where one takes R for both Q and $\tilde{Q}$) and thus $\mu(R \setminus 2B_R^{\mathsf{Cl}}) \leqslant \alpha\mu(R)$ which contradicts the fact that $R \in \mathsf{F}(R)$ as $\alpha \ll 1$.

3.9 Measure of stopping cubes from $\mathsf{BP}(R)$ and $\mathsf{F}(R)$

Lemma 3.14. *It holds that*

$$\sum_{Q \in \mathsf{BP}(R)} \mu(Q) \leqslant \frac{1}{\alpha^2\,\Theta_\mu(2B_R)^2} \sum_{\tilde{Q} \in \mathsf{Tree}(R)} p_0^{[\delta,\tilde{Q}]}(\mu\lfloor 2B_{\tilde{Q}}, \mu\lfloor 2B_R, \mu\lfloor 2B_R).$$

What is more, if $\alpha = \alpha(\tau)$ is small enough, then

$$\sum_{Q \in \mathsf{F}(R)} \mu(Q) \leqslant \sqrt{\alpha}\,\mu(R) \leqslant \tfrac{1}{3}\sqrt{\tau}\,\mu(R).$$

Proof. All the cubes in $\mathsf{Stop}(R)$ are disjoint and so are the cubes in $\mathsf{BP}(R)$ and $\mathsf{F}(R)$. From **(S3)** we get

$$\begin{aligned}
\sum_{Q \in \mathsf{BP}(R)} \mu(Q) &\leqslant \frac{1}{\alpha^2} \sum_{Q \in \mathsf{BP}(R)} \sum_{Q \subset \tilde{Q} \subset R} \frac{p_0^{[\delta,\tilde{Q}]}(\mu\lfloor 2B_{\tilde{Q}}, \mu\lfloor 2B_R, \mu\lfloor 2B_R)}{\Theta_\mu(2B_R)^2 \mu(\tilde{Q})}\,\mu(Q) \\
&= \frac{1}{\alpha^2\,\Theta_\mu(2B_R)^2} \sum_{\tilde{Q} \in \mathsf{Tree}(R)} p_0^{[\delta,\tilde{Q}]}(\mu\lfloor 2B_{\tilde{Q}}, \mu\lfloor 2B_R, \mu\lfloor 2B_R) \sum_{Q \in \mathsf{BP}(R) \colon Q \subset \tilde{Q}} \frac{\mu(Q)}{\mu(\tilde{Q})} \\
&\leqslant \frac{1}{\alpha^2\,\Theta_\mu(2B_R)^2} \sum_{\tilde{Q} \in \mathsf{Tree}(R)} p_0^{[\delta,\tilde{Q}]}(\mu\lfloor 2B_{\tilde{Q}}, \mu\lfloor 2B_R, \mu\lfloor 2B_R).
\end{aligned}$$

By **(S4)** and Lemmas 3.12 and 3.13, we obtain

$$\sum_{Q \in \mathsf{F}(R)} \mu(Q) \leqslant \frac{1}{\sqrt{\alpha}} \sum_{Q \in \mathsf{F}(R)} \mu(Q \setminus 2B_Q^{\mathsf{Cl}}) \leqslant \frac{1}{\sqrt{\alpha}}\,\mu(R_{\mathsf{Far}}) \leqslant \sqrt{\alpha}\,\mu(R),$$

which finishes the proof. □

3.10 Construction of a Lipschitz function

We aim to construct a Lipschitz function $F : L_R \to L_R^\perp$ whose graph Γ_R is close to R up to the scale of cubes from $\mathsf{Stop}(R)$. We will mostly use the properties mentioned in Lemma 3.9. This task is quite technical and so we start with a bunch of auxiliary results. Note that, although we follow some of the methods from [Leg] and [Tol5, Chapter 7] quite closely, we need to adapt the whole construction to the David-Mattila lattice used in the current chapter (instead of the balls with controlled density used in [Leg] and [Tol5]).

Let us mention again that we may suppose that $R \notin \mathsf{Stop}(R)$ as otherwise we choose $F \equiv 0$ and the graph Γ_R of F is just L_R.

3.10.1 Auxiliary results

As before, we denote by L_Q a best approximating line for the ball $2B_Q$ in the sense of the beta numbers (3.2). We need now to estimate the angles between the best approximating lines corresponding to cubes that are near each other. This task is carried out in the next two lemmas. The first one is a well known result from [DS1, Section 5]. We formulate it for lines in the complex plane.

Lemma 3.15 ([DS1]). *Let $L_1, L_2 \subset \mathbb{C}$ be lines and $z_1, z_2 \in Z \subset \mathbb{C}$ be points so that*

(a) $d_1 = \operatorname{dist}(z_1, z_2)/\operatorname{diam}(Z) \in (0,1)$,

(b) $\operatorname{dist}(z_i, L_j) < d_2 \operatorname{diam}(Z)$ *for* $i = 1,2$ *and* $j = 1,2$, *where* $d_2 < d_1/4$.

Then for any $z \in L_2$,

$$\operatorname{dist}(z, L_1) \leqslant d_2 \left(\tfrac{4}{d_1} \operatorname{dist}(z, Z) + \operatorname{diam}(Z) \right). \tag{3.22}$$

We will use the preceding lemma to prove the following result.

Lemma 3.16. *Let $\varepsilon_0 = \varepsilon_0(\gamma)$ be chosen small enough. If $Q_1, Q_2 \in \mathsf{DbTree}(R)$ are such that $r(Q_1) \approx r(Q_2)$ and $\operatorname{dist}(Q_1, Q_2) \lesssim r(Q_j)$ for $j = 1, 2$, then*

$$\operatorname{dist}(w, L_{Q_2}) \lesssim \sqrt{\varepsilon_0}\,(\operatorname{dist}(w, Q_1) + r(Q_1)), \qquad w \in L_{Q_1}, \tag{3.23}$$

$$\operatorname{dist}(w, L_{Q_1}) \lesssim \sqrt{\varepsilon_0}\,(\operatorname{dist}(w, Q_2) + r(Q_2)), \qquad w \in L_{Q_2}, \tag{3.24}$$

$$\measuredangle(L_{Q_1}, L_{Q_2}) \lesssim \sqrt{\varepsilon_0}. \tag{3.25}$$

Proof. Let $Q \in \mathsf{DbTree}(R)$ be the smallest cube such that $2B_Q \supset B_{Q_1} \cap B_{Q_2}$. Clearly, $r(Q) \gtrsim r(Q_j)$, $j = 1, 2$. Moreover, we can also guarantee that

$$r(Q) \lesssim \operatorname{dist}(Q_1, Q_2) + \sum_{j=1}^{2} r(Q_j) \lesssim r(Q_j).$$

Now we use arguments similar to those in Lemma 3.10. Since $Q_j \in \mathsf{DbTree}(R)$ for $j = 1, 2$, by (3.16) and Lemma 3.5 there are balls $B_{k,j} = B(\xi_{k,j}, \rho'\, r(Q_j))$, $k = 1, 2$, where $\xi_{k,j} \in B(Q_j)$, such that $\mu(B_{k,j} \cap B(Q_j)) \geqslant \rho'' \mu(Q_j)$ and $\operatorname{dist}(y_{1,j}, y_{2,j}) \geqslant \gamma r(B_{Q_j}) \geqslant \gamma r(Q_j)$ for all $y_{k,j} \in B_{k,j} \cap Q_j$, where ρ' and ρ'' depend on γ. Consequently, by (3.18) and the fact that $r(B_{k,j}) \approx_\gamma r(Q_j)$ we get

$$\frac{1}{r(B_{k,j})} \int_{B_{k,j}} \left(\frac{\operatorname{dist}(w, L_{Q_j})}{r(B_{k,j})} \right)^2 d\mu(w) \lesssim_\gamma \beta_{\mu,2}(2B_{Q_j})^2 \lesssim_\gamma \varepsilon_0^2\, \Theta_\mu(2B_{Q_j}).$$

Since $r(Q) \approx r(Q_j)$, we analogously obtain

$$\frac{1}{r(B_{k,j})} \int_{B_{k,j}} \left(\frac{\operatorname{dist}(w, L_Q)}{r(B_{k,j})} \right)^2 d\mu(w) \lesssim_\gamma \beta_{\mu,2}(2B_Q)^2 \lesssim_\gamma \varepsilon_0^2\, \Theta_\mu(2B_Q).$$

Therefore, using Chebyshev's inequality and again the relation $r(Q) \approx r(Q_j)$, we can find $z_{k,j} \in B_{k,j} \cap Q_j$ such that

$$\max\{\operatorname{dist}(z_{k,j}, L_{Q_j}), \operatorname{dist}(z_{k,j}, L_Q)\} \lesssim_\gamma \varepsilon_0\, r(Q).$$

Since $\operatorname{dist}(z_{1,j}, z_{2,j}) \geqslant \gamma r(Q_j) \gtrsim \gamma r(Q)$, it follows by Lemma 3.15 that

$$\operatorname{dist}(w, L_Q) \lesssim_\gamma \varepsilon_0(\operatorname{dist}(w, Q_j) + r(Q_j)) \text{ for all } w \in L_{Q_j}, \qquad j = 1, 2,$$

and

$$\operatorname{dist}(w, L_{Q_j}) \lesssim_\gamma \varepsilon_0(\operatorname{dist}(w, Q_j) + r(Q_j)) \text{ for all } w \in L_Q, \qquad j = 1, 2.$$

From this, by the triangle inequality, choosing ε_0 small enough with respect to the implicit constant depending on γ, we obtain (3.23) and (3.24).

The inequality (3.25) follows from (3.23) and (3.24) by elementary geometry. □

Lemma 3.17. *Let $\alpha = \alpha(\gamma)$ and $\varepsilon_0 = \varepsilon_0(\gamma)$ be chosen small enough. If $Q_1, Q_2 \in \mathsf{DbTree}(R)$ are such that $2B_{Q_1} \subset 2B_{Q_2}$ and $x \in L_{Q_1} \cap 2B_{Q_1}$, then*

$$\operatorname{dist}(x, L_{Q_2}) \lesssim \varepsilon_0^{1/3}\, r(Q_2).$$

Proof. By Lemma 3.5 there exists a family of balls $B_k = B(\xi_k, \rho'\, r(Q_1))$, where $\xi_k \in B(Q_1)$, such that $\mu(B_k \cap B(Q_1)) \geqslant \rho''\mu(Q_1)$ and $\operatorname{dist}(y_1, y_2) \geqslant \gamma r(B_{Q_1}) \geqslant \gamma r(Q_1)$ for any $y_k \in B_k \cap Q_1$, $k = 1, 2$. Recall that ρ' and ρ'' depend on γ. Furthermore, we can choose $\alpha = \alpha(\gamma)$ in (3.20) small enough to guarantee that $B_k \cap B(Q_1) \cap 2B_{Q_1}^{\mathsf{Cl}} \neq \varnothing$. This and the definition of $2B_{Q_1}^{\mathsf{Cl}}$ imply that there exist $z_k \in B_k \cap B(Q_1) \cap 2B_{Q_1}^{\mathsf{Cl}}$, $k = 1, 2$, such that

$$\operatorname{dist}(z_k, L_{Q_j}) \lesssim \sqrt{\varepsilon_0}\, r(Q_j), \quad k = 1, 2, \quad j = 1, 2.$$

Let z_k' be the orthogonal projection of z_k onto L_{Q_1}. We easily get from the previous inequality that

$$\operatorname{dist}(z_k', L_{Q_2}) \lesssim \sqrt{\varepsilon_0}\, r(Q_2), \quad k = 1, 2. \tag{3.26}$$

Moreover, $\operatorname{dist}(z_1, z_2) \gtrsim_\gamma r(Q_1)$ implies that $\operatorname{dist}(z_1', z_2') \gtrsim_\gamma r(Q_1)$ and $z_k' \in 2B_{Q_1}$, if $\varepsilon_0 = \varepsilon_0(\gamma)$ is small enough. Having this and (3.26) in mind and taking into account that $x \in L_{Q_1} \cap 2B_{Q_1}$, by elementary geometry we get the required estimate for $\operatorname{dist}(x, L_{Q_2})$, assuming again that $\varepsilon_0 = \varepsilon_0(\gamma)$ is small enough. □

3.10.2 Lipschitz function F for the good part of R

For each given $R \in \mathcal{D}^{db}$, we first construct the required function F on the projection of the "good part" of R onto L_R and then extend it onto the whole L_R. In what follows, we will work a lot with the function

$$d(z) := \inf_{Q \in \mathsf{DbTree}(R)} \{\operatorname{dist}(z, Q) + \operatorname{diam}(Q)\}, \qquad z \in \mathbb{C}. \tag{3.27}$$

Let us mention that $\theta(R)$ is supposed to be comparable with the parameter θ_0, i.e. $\theta(R) \approx \theta_0$, where the implicit constants will be defined in Section 3.12.

Lemma 3.18. *Let $\varepsilon_0 = \varepsilon_0(\tau, A, \theta_0)$ and θ_0 be small enough. For any $z_1, z_2 \in cB_R$ we have*

$$|\Pi^\perp(z_1) - \Pi^\perp(z_2)| \lesssim \theta(R)|\Pi(z_1) - \Pi(z_2)| + c(\tau, A)\,(d(z_1) + d(z_2)),$$

where $\Pi(z)$ and $\Pi^\perp(z)$ are the projections of z onto L_R and $L_R^\perp$, correspondingly, and $c(\tau, A) > 0$.

Proof. Everywhere in the proof $k=1,2$. For a fixed $h>0$ and any $z_k \in cB_R$ one can always find $Q_k \in \mathsf{DbTree}(R)$ such that

$$\operatorname{dist}(z_k, Q_k) + \operatorname{diam}(Q_k) \leqslant d(z_k) + h, \qquad k=1,2.$$

Choose $z_k' \in Q_k$. Clearly, $|z_k - z_k'| \leqslant d(z_k)+h$.

Let $\tilde{Q}_k \in \mathsf{DbTree}(R)$ be the smallest cube such that $2B_{\tilde{Q}_k} \supset B_{Q_k}$ and

$$r(\tilde{Q}_k) \approx_{\tau,A} \varepsilon_0 |z_1 - z_2| + \sum\nolimits_k \operatorname{diam}(Q_k).$$

Now let $\tilde{Q} \in \mathsf{DbTree}(R)$ be the smallest cube such that $2B_{\tilde{Q}} \supset B_{\tilde{Q}_1} \cup B_{\tilde{Q}_2}$ and

$$r(\tilde{Q}) \approx_{\tau,A} |z_1 - z_2| + \sum\nolimits_k \operatorname{diam}(Q_k).$$

Note that $|z_1 - z_2| \lesssim r(R)$ as $z_k \in cB_R$ and thus the cubes $\tilde{Q}_k$ and $\tilde{Q}$ are well defined. Furthermore, we easily get that $\varepsilon_0\, r(\tilde{Q}) \lesssim_{\tau,A} r(\tilde{Q}_k)$. Consequently, the way how $\tilde{Q}_k$ and $\tilde{Q}$ are chosen and the inequalities (3.15) and (3.18) in Lemma 3.9 imply that

$$\begin{aligned}\frac{1}{\mu(B_{\tilde{Q}_k})} \int_{B_{\tilde{Q}_k}} \left(\frac{\operatorname{dist}(w, L_{\tilde{Q}})}{r(\tilde{Q})}\right)^2 d\mu(w) &\lesssim_{\tau,A} \frac{r(\tilde{Q})\, \beta_{\mu,2}(2B_{\tilde{Q}})^2}{\mu(2B_{\tilde{Q}_k})} \lesssim_{\tau,A} \varepsilon_0^2\, \frac{r(\tilde{Q})\Theta_\mu(2B_{\tilde{Q}})}{\mu(2B_{\tilde{Q}_k})} \\ &\lesssim_{\tau,A} \varepsilon_0\, \frac{\Theta_\mu(2B_{\tilde{Q}})}{\Theta_\mu(2B_{\tilde{Q}_k})} \lesssim_{\tau,A} \varepsilon_0 \lesssim \varepsilon_0^{3/4},\end{aligned}$$

if $\varepsilon_0 = \varepsilon_0(\tau, A)$ is chosen properly. Recall again that $r(B_Q) = 28 r(Q)$ by definition.

From the inequality just obtained we deduce by Chebyshev's inequality that there exist $z_k'' \in R \cap B_{\tilde{Q}_k}$, $k=1,2$, such that

$$|z_k'' - \pi(z_k'')| \lesssim \varepsilon_0^{3/8}\, r(\tilde{Q}) \lesssim \sqrt[4]{\varepsilon_0} \left(|z_1 - z_2| + \sum\nolimits_k \operatorname{diam}(Q_k)\right),$$

where $\pi(z_k'')$ stands for the orthogonal projection of z_k'' onto $L_{\tilde{Q}}$ and $\varepsilon_0 = \varepsilon_0(\tau, A)$ is small enough. Note also that

$$|z_k' - z_k''| \lesssim r(\tilde{Q}_k) \lesssim \sqrt[4]{\varepsilon_0} |z_1 - z_2| + c(\tau, A) \sum\nolimits_k \operatorname{diam}(Q_k),$$

if $\varepsilon_0 = \varepsilon_0(\tau, A)$ is small enough. Summarizing, we obtain the inequality

$$|z_k' - \pi(z_k'')| \leqslant |z_k' - z_k''| + |z_k'' - \pi(z_k'')| \lesssim \sqrt[4]{\varepsilon_0} |z_1 - z_2| + c(\tau, A) \sum\nolimits_k \operatorname{diam}(Q_k).$$

Furthermore, the triangle inequality yields

$$\begin{aligned}|\Pi^\perp(z_1') - \Pi^\perp(z_2')| &\leqslant |\Pi^\perp(\pi(z_1'')) - \Pi^\perp(\pi(z_2''))| + \sum\nolimits_k |\Pi^\perp(z_k') - \Pi^\perp(\pi(z_k''))| \\ &\leqslant |\Pi^\perp(\pi(z_1'')) - \Pi^\perp(\pi(z_2''))| + \sum\nolimits_k |z_k' - \pi(z_k'')|,\end{aligned}$$

and therefore we immediately obtain

$$|\Pi^\perp(z_1') - \Pi^\perp(z_2')| \lesssim |\Pi^\perp(\pi(z_1'')) - \Pi^\perp(\pi(z_2''))| + \sqrt[4]{\varepsilon_0} |z_1 - z_2| + c(\tau, A) \sum\nolimits_k \operatorname{diam}(Q_k).$$

From (3.19) in Lemma 3.9 applied to $\tilde{Q}$ and the triangle inequality we deduce that

$$\begin{aligned}
&|\Pi^{\perp}(\pi(z_1'')) - \Pi^{\perp}(\pi(z_2''))| \\
&\quad\lesssim \theta(R)|\Pi(\pi(z_1'')) - \Pi(\pi(z_2''))| \\
&\quad\lesssim \theta(R)\left(|\Pi(z_1) - \Pi(z_2)| + \sum\nolimits_k |\Pi(z_k) - \Pi(\pi(z_k''))|\right) \\
&\quad\lesssim \theta(R)\left(|\Pi(z_1) - \Pi(z_2)| + \sum\nolimits_k |z_k - \pi(z_k'')|\right) \\
&\quad\lesssim \theta(R)\left(|\Pi(z_1) - \Pi(z_2)| + \sum\nolimits_k \left(|z_k - z_k'| + |z_k' - \pi(z_k'')|\right)\right).
\end{aligned}$$

Recall the estimates for $|z_k - z_k'|$ and $|z_k' - \pi(z_k'')|$ and take into account that $\operatorname{diam}(Q_k) \leqslant d(z_k) + h$ and that ε_0 and θ_0 (and thus $\theta(R)$) are small. Consequently,

$$|\Pi^{\perp}(z_1') - \Pi^{\perp}(z_2')| \lesssim \theta(R)|\Pi(z_1) - \Pi(z_2)| + \sqrt[4]{\varepsilon_0}|z_1 - z_2| + c(\tau, A)\sum\nolimits_k (d(z_k) + h).$$

Additionally, the triangle inequality and the estimate for $|z_k - z_k'|$ lead to

$$|\Pi^{\perp}(z_1) - \Pi^{\perp}(z_2)| \leqslant |\Pi^{\perp}(z_1') - \Pi^{\perp}(z_2')| + \sum\nolimits_k (d(z_k) + h),$$

and thus

$$|\Pi^{\perp}(z_1) - \Pi^{\perp}(z_2)| \lesssim \theta(R)|\Pi(z_1) - \Pi(z_2)| + \sqrt[4]{\varepsilon_0}|z_1 - z_2| + c(\tau, A)\sum\nolimits_k (d(z_k) + h).$$

Take into account that $|z_1 - z_2| \leqslant |\Pi(z_1) - \Pi(z_2)| + |\Pi^{\perp}(z_1) - \Pi^{\perp}(z_2)|$ and choose ε_0 small enough with respect to θ_0 (and thus to $\theta(R)$) and to the implicit absolute constant in the latter inequality. Finally,

$$|\Pi^{\perp}(z_1) - \Pi^{\perp}(z_2)| \lesssim \theta(R)|\Pi(z_1) - \Pi(z_2)| + c(\tau, A)\sum\nolimits_k (d(z_k) + h).$$

Letting $h \to 0$ finishes the proof. □

We will also use the following notation:

$$G_R = \{x \in \mathbb{C} : d(x) = 0\}. \tag{3.28}$$

Lemma 3.18 implies that the map $\Pi : G_R \to L_R$ is injective and we can define the function F on $\Pi(G_R)$ by setting

$$F(\Pi(x)) = \Pi^{\perp}(x), \qquad x \in G_R. \tag{3.29}$$

Moreover, this F is Lipschitz with constant $\lesssim \theta(R)$.

We are now aimed to extend F onto the whole line L_R using a variant of the Whitney extension theorem. This approach is quite standard and is used, for example, in [DS1, Section 8], [Leg, Section 3.2] and [Tol5, Section 7.5]. Therefore we will skip some details and mostly give the results related to the adaptation of the scheme to the David-Mattila lattice that we use. These results will then imply the extension of F onto the whole L_R by repeating the "partition of unity" arguments presented in [Tol5, Section 7.5].

Let us define the function

$$D(z) := \inf_{x\in\Pi^{-1}(z)} d(x) = \inf_{Q\in\mathsf{DbTree}(R)} \{\operatorname{dist}(z, \Pi(Q)) + \operatorname{diam}(Q)\}, \qquad z \in L_R. \tag{3.30}$$

For each $z \in L_R$ such that $D(z) > 0$, i.e. $z \in L_R \setminus \Pi(G_R)$, we call J_z the largest dyadic interval from L_R containing z such that

$$\ell(J_z) \leqslant \tfrac{1}{20} \inf_{u\in J_z} D(u).$$

Let J_i, $i \in I$, be a relabelling of the set of all these intervals J_z, $z \in L_R \setminus \Pi(G_R)$, without repetition. Some properties of $\{J_i\}$ are summarized in the following lemma.

Lemma 3.19 (Analogue of Lemma 7.20 in [Tol5]). *The intervals in $\{J_i\}_{i\in I}$ have disjoint interiors in L_R and satisfy the properties:*

(a) *If $z \in 15J_i$, then $5\ell(J_i) \leqslant D(z) \leqslant 50\ell(J_i)$.*

(b) *There exists an absolute constant $c > 1$ such that if $15J_i \cap 15J_{i'} \neq \varnothing$, then*

$$c^{-1}\ell(J_i) \leqslant \ell(J_{i'}) \leqslant c\,\ell(J_i).$$

(c) *For each $i \in I$, there are at most N intervals $J_{i'}$ such that $15J_i \cap 15J_{i'} \neq \varnothing$, where N is some absolute constant.*

(d) *$L_R \setminus \Pi(G_R) = \bigcup_{i\in I} J_i = \bigcup_{i\in I} 15J_i$.*

Now we construct the function F on

$$U_0 = L_R \cap B_0, \qquad B_0 = B(\Pi(x_0), 10\,\mathrm{diam}\,(R)),$$

where $x_0 \in R$ is such that

$$\mathrm{dist}\,(x_0, \Pi(x_0)) = \mathrm{dist}\,(x_0, L_R) \leqslant \mathrm{diam}\,(R).$$

This x_0 exists due to the inequality (3.18) in Lemma 3.9. Note that by construction

$$R \subset B(\Pi(x_0), 2\,\mathrm{diam}\,(R)) \quad \text{and} \quad \Pi(R) \subset L_R \cap B(\Pi(x_0), 2\,\mathrm{diam}\,(R)). \tag{3.31}$$

We also define the following set of indexes:

$$I_0 = \{i \in I : J_i \cap U_0 \neq \varnothing\}.$$

Lemma 3.20. *The following holds.*
(a) *If $i \in I_0$, then $\ell(J_i) \leqslant \mathrm{diam}\,(R)$ and $3J_i \subset L_R \cap B(\Pi(x_0), 12\,\mathrm{diam}\,(R))$.*
(b) *If $J_i \cap B(\Pi(x_0), 3\,\mathrm{diam}\,(R)) = \varnothing$ (in particular if $i \notin I_0$), then*

$$\ell(J_i) \approx \mathrm{dist}\,(\Pi(x_0), J_i) \approx |\Pi(x_0) - z| \quad \textit{for all} \quad z \in J_i.$$

Proof. For (a), take J_i with $i \in I_0$ so that $J_i \cap U_0 \neq \varnothing$. Then we have

$$3J_i \subset L_R \cap B(\Pi(x_0), 10\,\mathrm{diam}\,(R) + 2\ell(J_i)).$$

It is necessary to estimate $\ell(J_i)$. Recall that

$$\ell(J_i) \leqslant \tfrac{1}{20} \inf_{u\in J_i} D(u).$$

Definitely, $\inf_{u\in J_i} D(u) \leqslant \max_{u\in U_0} D(u)$ in our case so we will estimate this maximum instead. To do so, we first notice that the definition (3.27) of d and the inequality (3.31) give

$$d(x) \leqslant \mathrm{dist}\,(x, R) + \mathrm{diam}\,(R) \leqslant 13\,\mathrm{diam}\,(R), \qquad x \in B_0.$$

This yields

$$\max_{u\in U_0} D(u) \leqslant \max_{x\in B_0} d(x) \leqslant 13\,\mathrm{diam}\,(R),$$

if we take into account the connection between d and D in (3.30). Thus

$$\ell(J_i) \leqslant \tfrac{13}{20}\,\mathrm{diam}\,(R)$$

and therefore

$$3J_i \subset L_R \cap B(\Pi(x_0), (10 + \tfrac{13}{10}) \operatorname{diam}(R)).$$

Now let us prove (b). Let $z \in J_i$. Clearly, $\operatorname{diam}(R) \leqslant \frac{1}{3}|\Pi(x_0) - z|$. Furthermore, we infer from this and the definition (3.30) that

$$D(z) \leqslant (|\Pi(x_0) - z| + 2\operatorname{diam}(R)) + \operatorname{diam}(R) \leqslant 2|\Pi(x_0) - z|.$$

From another side, by (3.30) and (3.31),

$$D(z) \geqslant \operatorname{dist}(z, \Pi(R)) \geqslant |\Pi(x_0) - z| - 2\operatorname{diam}(R) \geqslant \tfrac{1}{3}|\Pi(x_0) - z|.$$

Thus

$$\tfrac{1}{3}|\Pi(x_0) - z| \leqslant D(z) \leqslant 2|\Pi(x_0) - z|, \qquad z \in J_i.$$

Together with Lemma 3.19(a) this gives

$$\tfrac{5}{2}\ell(J_i) \leqslant |\Pi(x_0) - z| \leqslant 150\ell(J_i).$$

Moreover, since

$$|\Pi(x_0) - z| - \ell(J_i) \leqslant \operatorname{dist}(\Pi(x_0), J_i) \leqslant |\Pi(x_0) - z|, \qquad z \in J_i,$$

we get

$$\tfrac{3}{2}\ell(J_i) \leqslant \operatorname{dist}(\Pi(x_0), J_i) \leqslant 150\ell(J_i),$$

which finishes the proof. $\square$

Lemma 3.21. *Given $i \in I_0$, there exists a cube $Q_i \in \mathsf{DbTree}(R)$ such that*

(a) $\ell(J_i) \lesssim \operatorname{diam}(Q_i) \lesssim_{\tau,A} \ell(J_i)$*;*

(b) $\operatorname{dist}(J_i, \Pi(Q_i)) \lesssim \ell(J_i)$*.*

Proof. From the definition (3.30) of D it follows that there exists a cube $Q \in \mathsf{DbTree}(R)$ such that

$$\operatorname{dist}(z, \Pi(Q)) + \operatorname{diam}(Q) \leqslant 2D(z) \approx \ell(J_i), \qquad z \in J_i,$$

where the comparability is due to Lemma 3.19(a). This immediately gives (b) and the right hand side inequality in (a) for $Q_i = Q$. If the left hand side inequality in (a) does not hold, we can replace Q by its smallest doubling ancestor Q' satisfying $\operatorname{diam}(Q') \gtrsim \ell(J_i)$ so that all other inequalities are valid (recall Lemma 3.11). We rename Q' by Q_i then. $\square$

For $i \in I_0$, let F_i be the affine function $L_R \to L_R^{\perp}$ whose graph is the line L_{Q_i}. Moreover, F_i are Lipschitz functions with constant $\leqslant \theta(R)$ as $\measuredangle(L_{Q_i}, L_R) \leqslant \theta(R)$ by (3.19) in Lemma 3.9 taking into account that all $Q_i \in \mathsf{DbTree}(R)$. On the other hand, for $i \notin I_0$, we set $F_i \equiv 0$, i.e. the graph of F_i is just L_R in this case.

Lemma 3.22. *If $10J_i \cap 10J_{i'} \neq \varnothing$ for some $i, i' \in I$, then*

(a) $\operatorname{dist}(Q_i, Q_{i'}) \lesssim_{\tau,A} \ell(J_i)$ *if moreover $i, i' \in I_0$;*

(b) $|F_i(z) - F_{i'}(z)| \lesssim \varepsilon_0^{1/3}\, \ell(J_i)$ *for $z \in 100J_i$;*

(c) $|F_i' - F_{i'}'| \lesssim \varepsilon_0^{1/3}$.

Proof. For $i, i' \in I_0$, Lemmas 3.19(b) and 3.21(b) ensure that $\ell(Q_i) \approx \ell(Q_{i'})$ and

$$\operatorname{dist}(\Pi(Q_i), \Pi(Q_{i'})) \leqslant \operatorname{dist}(\Pi(Q_i), J_i) + \ell(J_i) + \operatorname{dist}(J_i, J_{i'}) + \ell(J_{i'}) + \operatorname{dist}(J_{i'}, \Pi(Q_{i'})) \lesssim \ell(J_i).$$

Keeping this in mind, we continue. For any $z_1 \in Q_i$ and $z_2 \in Q_{i'}$ by the triangle inequality and Lemma 3.18 we have

$$\begin{aligned} \operatorname{dist}(Q_i, Q_{i'}) \leqslant \operatorname{dist}(z_1, z_2) &\leqslant |\Pi^\perp(z_1) - \Pi^\perp(z_2)| + |\Pi(z_1) - \Pi(z_2)| \\ &\lesssim |\Pi(z_1) - \Pi(z_2)| + c(\tau, A)(d(z_1) + d(z_2)). \end{aligned}$$

Since $z_1 \in Q_i$ and $z_2 \in Q_{i'}$, we have $d(z_1) \leqslant \operatorname{diam}(Q_i)$ and $d(z_2) \leqslant \operatorname{diam}(Q_{i'})$. Moreover, if z_1 and z_2 are chosen so that

$$|\Pi(z_1) - \Pi(z_2)| \leqslant 2 \operatorname{dist}(\Pi(Q_i), \Pi(Q_{i'})),$$

then $\operatorname{dist}(Q_i, Q_{i'}) \lesssim \operatorname{dist}(\Pi(Q_i), \Pi(Q_{i'})) + \operatorname{diam}(Q_i) + \operatorname{diam}(Q_{i'}) \lesssim_{\tau,A} \ell(J_i)$ as in (a).

For $i, i' \in I_0$ the properties (b) and (c) follow from (a) and Lemma 3.16. Indeed, in this case

$$\operatorname{diam}(Q_i) \approx \operatorname{diam}(Q_{i'}) \approx_{\tau,A} \ell(J_i) \approx \ell(J_{i'}) \quad \text{and} \quad \operatorname{dist}(Q_i, Q_{i'}) \lesssim_{\tau,A} \ell(J_i).$$

Taking into account that L_{Q_i} and $L_{Q_{i'}}$ are the graphs of F_i and $F_{i'}$, correspondingly, by Lemma 3.16 we have

$$|F_i(z) - F_{i'}(z)| \lesssim_{\tau,A} \sqrt{\varepsilon_0}\, \ell(J_i) \lesssim \varepsilon_0^{1/3}\, \ell(J_i), \qquad z \in 100J_i,$$

if $\varepsilon_0 = \varepsilon_0(\tau, A)$ is chosen small enough. Moreover, by the same lemma $\measuredangle(L_{Q_i}, L_{Q_{i'}}) \lesssim_{\tau,A} \sqrt{\varepsilon_0}$ and thus

$$\begin{aligned} |F_i' - F_{i'}'| &= |\arctan \measuredangle(L_{Q_i}, L_R) - \arctan \measuredangle(L_{Q_{i'}}, L_R)| \\ &= |\arctan \measuredangle(L_{Q_i}, L_R) - \arctan(\measuredangle(L_{Q_i}, L_R) \pm \measuredangle(L_{Q_i}, L_{Q_{i'}}))| \\ &\lesssim |\arctan \measuredangle(L_{Q_i}, L_{Q_{i'}})| \\ &\lesssim_{\tau,A} \sqrt{\varepsilon_0} \\ &\lesssim \varepsilon_0^{1/3}, \end{aligned}$$

if $\varepsilon_0 = \varepsilon_0(\tau, A)$ is small enough.

For $i, i' \notin I_0$, $F_i \equiv F_{i'} \equiv 0$, and so ($b$) and ($c$) are trivial.

Finally, let $i \in I_0$ and $i' \notin I_0$. From the assumption $10J_i \cap 10J_{i'} \neq \varnothing$ and Lemma 3.19(b) we know that $\ell(J_i) \approx \ell(J_{i'})$. Moreover, by Lemma 3.20(a) we have $\ell(J_i) \leqslant \operatorname{diam}(R)$ as $i \in I_0$. From another side, by Lemma 3.20(b)

$$\ell(J_{i'}) \approx \operatorname{dist}(\Pi(x_0), J_{i'})$$

and additionally $\operatorname{dist}(\Pi(x_0), J_{i'}) \geqslant 10 \operatorname{diam}(R)$ as $i' \notin I_0$, i.e. $J_{i'} \cap U_0 = \varnothing$. From all these facts we conclude that

$$\ell(J_i) \approx \ell(J_{i'}) \approx_{\tau,A} \operatorname{diam}(R) \quad \text{and} \quad \operatorname{dist}(J_i, J_{i'}) \lesssim_{\tau,A} \operatorname{diam}(R).$$

Recall that $F_{i'} \equiv 0$ and $J_{i'} \subset L_R$. Then, using Lemma 3.21 and arguments close to those in the proof of Lemmas 3.10 and 3.16, one can show that L_{Q_i} is very close to L_R in cB_0, which yields (b) and (c) in this case if $\varepsilon_0 = \varepsilon_0(\tau, A)$ is chosen small enough. $\square$

3.10.3 Extension of F to the whole L_R

We are now ready to finish the definition of F on the whole L_R. Recall that F has already been defined on $\Pi(G_R)$ (see (3.29)). So it remains to define it only on $L_R \setminus \Pi(G_R)$. To this end, we first introduce a partition of unity on $L_R \setminus \Pi(G_R)$. For each $i \in I$, we can find a function $\tilde{\varphi}_i \in C^\infty(L_R)$ such that $\chi_{2J_i} \leqslant \tilde{\varphi}_i \leqslant \chi_{3J_i}$, with

$$|\tilde{\varphi}_i'| \leqslant \frac{c}{\ell(J_i)} \qquad \text{and} \qquad |\tilde{\varphi}_i''| \leqslant \frac{c}{\ell(J_i)^2}.$$

Then, for each $i \in I$, we set

$$\varphi_i = \frac{\tilde{\varphi}_i}{\sum_{j \in I} \tilde{\varphi}_j}. \tag{3.32}$$

It is clear that the family $\{\varphi_i\}_{i \in I}$ is a partition of unity subordinated to the sets $\{3J_i\}_{i \in I}$, and each function φ_i satisfies

$$|\varphi_i'| \leqslant \frac{c}{\ell(J_i)} \qquad \text{and} \qquad |\varphi_i''| \leqslant \frac{c}{\ell(J_i)^2},$$

taking into account Lemma 3.19.

Recall that $L_R \setminus \Pi(G_R) = \bigcup_{i \in I} J_i = \bigcup_{i \in I} 3J_i$. For $z \in L_R \setminus \Pi(G_R)$, we define

$$F(z) := \sum_{i \in I_0} \varphi_i(z) F_i(z).$$

Observe that in the preceding sum we can replace I_0 by I as $F_i \equiv 0$ for $i \in I \setminus I_0$.

We denote by Γ_R the graph $\{(z, F(z)) : z \in L_R\}$.

Using the lemmas proved above, one can undeviatingly follow the "partition of unity" arguments in [Tol5, Section 7.5] to prove the following.

Lemma 3.23. *The function $F : L_R \to L_R^\perp$ is supported on $L_R \cap B(\Pi(x_0), 12\,\mathrm{diam}\,(R))$ and is $C_F\,\theta(R)$-Lipschitz, where $C_F > 0$ is absolute. Also, if $z \in 15J_i$, $i \in I$, then*

$$|F''(z)| \lesssim \frac{\sqrt[4]{\varepsilon_0}}{\ell(J_i)}.$$

Recall that we suppose of course that the parameters and thresholds mentioned in Section 3.7 are chosen properly.

3.10.4 Γ_R and R are close to each other

Lemma 3.24. *There exists a constant $\mathsf{c}_3(\tau, A) > 0$ such that*

$$\mathrm{dist}\,(x, \Gamma_R) \leqslant \mathsf{c}_3(\tau, A) \cdot d(x) \quad \textit{for any } x \in B_0. \tag{3.33}$$

Proof. Let $y = (\Pi(x), F(\Pi(x)))$. By Lemma 3.18,

$$\mathrm{dist}\,(x, \Gamma_R) \leqslant |x - y| = |\Pi^\perp(x) - \Pi^\perp(y)| \lesssim_{\tau,A} d(x) + d(y). \tag{3.34}$$

If $\Pi(x) \in \Pi(G_R)$, then $y \in G_R$ and thus $d(y) \equiv 0$, which proves the lemma.

If $\Pi(x) \notin \Pi(G_R)$, let J_i, $i \in I$, be such that $\Pi(x) \in J_i$. Since $\Pi(x) \in J_i \cap B_0 \neq \varnothing$, $i \in I_0$ and therefore there exists a cube $Q_i \in \mathsf{DbTree}(R)$ described in Lemma 3.21. This gives

$$d(y) \leqslant \mathrm{dist}\,(y, Q_i) + \mathrm{diam}\,(Q_i) \lesssim_{\tau,A} \mathrm{dist}\,(y, Q_i) + \ell(J_i).$$

Let us estimate $\operatorname{dist}(y, Q_i)$. One can deduce from the definition of F that there exist $y' \in L_{Q_i}$ such that $\Pi(y') = \Pi(y)$ and $\operatorname{dist}(y, y') \lesssim \ell(J_i)$ (recall that L_{Q_i} is the graph of F_i and $\Pi(y) \in J_i$, see some details in [Tol5, Proof of Lemma 7.24]). Moreover, it follows in a similar way as in the proof of Lemmas 3.10 and 3.16 that there exist $\zeta \in Q_i$ and $\zeta' \in L_{Q_i}$ such that $\operatorname{dist}(\zeta, \zeta') \lesssim \sqrt{\varepsilon_0}\operatorname{diam}(Q_i)$. We know from Lemma 3.21 that $\operatorname{dist}(\Pi(y'), \Pi(\zeta)) \lesssim \ell(J_i)$. Furthermore, it holds that $\measuredangle(L_{Q_i}, L_R) \leqslant \theta(R)$ by (3.19) in Lemma 3.9 taking into account that all $Q_i \in \mathsf{DbTree}(R)$. These facts imply that $\operatorname{dist}(y', \zeta') \lesssim \ell(J_i)$. Summarizing, we obtain

$$\operatorname{dist}(y, Q_i) \leqslant \operatorname{dist}(y, y') + \operatorname{dist}(y', \zeta') + \operatorname{dist}(\zeta', \zeta) \lesssim \ell(J_i).$$

From this by Lemma 3.19(a) and the definition of D (see (3.30)), we conclude that

$$d(y) \lesssim_{\tau,A} \ell(J_i) \lesssim_{\tau,A} D(\Pi(x)) \lesssim_{\tau,A} d(x).$$

This fact together with (3.34) proves the lemma. □

Lemma 3.25. *Let $\varepsilon_0 = \varepsilon_0(A, \tau)$ be small enough. If $Q \in \mathsf{DbTree}(R)$ and $z \in \Gamma_R \cap 2B_Q$, then*

$$\operatorname{dist}(z, L_Q) \lesssim \sqrt[4]{\varepsilon_0}\, r(Q). \tag{3.35}$$

Proof. Let $z \in G_R$. Then there exists $Q' \in \mathsf{DbTree}(R)$ such that $z \in Q'$, $Q' \subset Q$ and $r(Q') \leqslant \varepsilon_0^{1/3} r(Q)$. By Lemma 3.10 there is $z' \in Q'$ such that $\operatorname{dist}(z', z'') \lesssim \sqrt{\varepsilon_0}\, r(Q')$, where $z'' \in L_{Q'} \cap 2B_{Q'}$. Furthermore, it is clear that $\operatorname{dist}(z, z') \lesssim r(Q') \lesssim \varepsilon_0^{1/3} r(Q)$. Using that $Q' \subset Q$, by Lemma 3.17 we get $\operatorname{dist}(z'', L_Q) \lesssim \varepsilon_0^{1/3} r(Q)$. Consequently,

$$\operatorname{dist}(z, L_Q) \leqslant \operatorname{dist}(z, z') + \operatorname{dist}(z', z'') + \operatorname{dist}(z'', L_Q) \lesssim \varepsilon_0^{1/3} r(Q).$$

Now let $z \notin G_R$ and $\zeta = \Pi(z)$. In this case

$$F(\zeta) = \sum_{i \in I_0} \varphi_i(\zeta) F_i(\zeta).$$

Now take into account (3.32) and distinguish two cases. Suppose first that

$$\sum_{i \in I_0} \varphi_i(\zeta) = 1.$$

In this case $(\zeta, F(\zeta))$ is a convex combination of the points $(\zeta, F_i(\zeta))$ for i such that $\varphi_i(\zeta) \neq 0$ (we will write $i \in \tilde{I}_0$ for these i s, $\tilde{I}_0 \subset I_0$). Therefore (3.35) follows if

$$\operatorname{dist}((\zeta, F_i(\zeta)), L_Q) \lesssim \varepsilon_0^{1/3} r(Q) \qquad \text{for all } i \in \tilde{I}_0. \tag{3.36}$$

To prove this estimate, notice that since $z \in 2B_Q$,

$$D(\zeta) \leqslant d(z) \lesssim r(Q).$$

Let $J_{i'}$, where $i' \in I_0$, be the interval that contains ζ. Then

$$\ell(J_{i'}) \leqslant \tfrac{1}{20} D(\zeta) \lesssim r(Q). \tag{3.37}$$

Recall that φ_i is supported on $3J_i$. Consequently, we necessarily have $3J_i \cap J_{i'} \neq \varnothing$ if $i \in \tilde{I}_0$. Therefore by Lemma 3.19(b) and 3.21(a),

$$\ell(J_i) \approx_{\tau,A} \operatorname{diam}(Q_i) \approx_{\tau,A} \operatorname{diam}(Q_{i'}) \approx_{\tau,A} \ell(J_{i'}) \lesssim_{\tau,A} r(Q).$$

Moreover, by Lemma 3.22(a),

$$\operatorname{dist}(\Pi(Q_i),\Pi(Q_{i'})) \leqslant \operatorname{dist}(Q_i,Q_{i'}) \lesssim_{\tau,A} \ell(J_i).$$

Taking into account that

$$\operatorname{dist}(\Pi(Q_{i'}),\Pi(Q)) \leqslant \operatorname{dist}(\Pi(Q_{i'}),J_{i'}) + \operatorname{diam}(J_{i'}) + \operatorname{dist}(J_{i'},\Pi(Q)) \lesssim \ell(J_{i'}) \lesssim_{\tau,A} r(Q),$$

we get

$$\operatorname{dist}(\Pi(Q_i),\Pi(Q)) \leqslant \operatorname{dist}(\Pi(Q_i),\Pi(Q_{i'}))+\operatorname{diam}(\Pi(Q_{i'}))+\operatorname{dist}(\Pi(Q_{i'}),\Pi(Q)) \lesssim_{\tau,A} r(Q).$$

From Lemma 3.18, applied for $z_1 \in Q_i$ and $z_2 \in Q$, we deduce that

$$\operatorname{dist}(Q_i,Q) \lesssim \operatorname{dist}(\Pi(Q_i),\Pi(Q)) + \operatorname{diam}(Q_i) + \operatorname{diam}(Q) \lesssim_{\tau,A} r(Q).$$

This means that $2B_{Q_i} \subset cB_Q$ with some $c = c(\tau, A) > 1$. Consequently, by Lemmas 3.11 and 3.16, we can find $Q' \in \mathsf{DbTree}(R)$ such that $cB_Q \subset 2B_{Q'}$, $\operatorname{diam}(Q') \approx_{A,\tau} \operatorname{diam}(Q)$ and

$$\operatorname{dist}(w,L_Q) \lesssim_{A,\tau} \sqrt{\varepsilon_0}(\operatorname{dist}(w,Q') + \operatorname{diam}(Q')), \qquad w \in L_{Q'}.$$

Choosing $\varepsilon_0 = \varepsilon_0(A,\tau)$ small enough, we get

$$\operatorname{dist}(w,L_Q) \lesssim \varepsilon_0^{1/3}(\operatorname{dist}(w,Q') + \operatorname{diam}(Q')), \qquad w \in L_{Q'}. \tag{3.38}$$

Recall that $(\zeta, F_i(\zeta)) \in L_{Q_i} \cap cB_{Q_i}$ and $2B_{Q_i} \subset 2B_{Q'}$ so Lemma 3.17 gives

$$\operatorname{dist}((\zeta,F_i(\zeta)),L_{Q'}) \lesssim \varepsilon_0^{1/3}\, r(Q').$$

Note that the parameters and thresholds in Lemma 3.17 are also supposed to be properly chosen. Together with (3.38) applied to $w = \mathsf{proj}_{L_{Q'}}(\zeta, F_i(\zeta))$, this yields (3.36) as required.

Suppose now that

$$\sum_{i\in I_0} \varphi_i(\zeta) < 1.$$

In this case, there exists some $J_{i'}$ with $i' \notin I_0$ such that $\zeta \in 3J_{i'}$ (as from (3.32) it follows that $\sum_{i\in I\setminus I_0}\varphi_i(\zeta) > 0$) and by Lemma 3.20($b$),

$$\operatorname{diam}(R) \lesssim \ell(J_{i'}) \approx \operatorname{dist}(\Pi(x_0),J_{i'}).$$

Moreover, if J_i is the interval that contains $\zeta = \Pi(z)$, $z \in 2B_Q$, then

$$\ell(J_i) \lesssim D(\Pi(z)) \lesssim d(z) \lesssim \operatorname{dist}(z,Q) + \operatorname{diam}(Q) \lesssim \operatorname{diam}(R),$$

where we used the definition of D, see (3.30).

By Lemma 3.19(b), $\ell(J_i) \approx \ell(J_{i'})$ as $J_i \cap 3J_{i'} \neq \varnothing$. That is why $\ell(J_{i'}) \approx \operatorname{diam}(R)$. This also implies that $\ell(J_m) \approx \operatorname{diam}(R)$ for any $m \in I_0$ such that $\zeta \in 3J_m$. By Lemma 3.21(a), it means that $\operatorname{diam}(Q_m) \approx_{\tau,A} \operatorname{diam}(R)$. Furthermore, it is clear that $\operatorname{dist}(Q_m,R) \equiv 0$ and so the assumptions of Lemma 3.16 are satisfied for Q_m and R. Consequently, L_{Q_m} and L_R are very close in cB_R for some $c > 1$ if the corresponding parameters are chosen properly, namely,

$$\operatorname{dist}_H(L_{Q_m} \cap cB_R, L_R \cap cB_R) \lesssim_{\tau,A} \sqrt{\varepsilon_0}\operatorname{diam}(R). \tag{3.39}$$

On the other hand, arguing as in (3.37), one deduces that $\ell(J_m) \lesssim_{\tau,A} r(Q)$, and from this we conclude that $r(Q) \approx_{\tau,A} \operatorname{diam}(R)$. By (3.39) then we get

$$|F_m(\zeta)| = \operatorname{dist}((\zeta, F_m(\zeta)), L_R) \lesssim_{\tau,A} \sqrt{\varepsilon_0}\operatorname{diam}(R) \lesssim_{\tau,A} \sqrt{\varepsilon_0}\, r(Q) \lesssim \varepsilon_0^{1/3}\, r(Q)$$

for all above-mentioned ms is $\varepsilon_0 = \varepsilon_0(\tau, A)$ is chosen small enough. Recall that we only need to sum up $i \in I_0$ such that $\zeta \in 3J_i$ and these are our $m \in I_0$. Thus

$$\begin{aligned}\operatorname{dist}((\zeta, F(\zeta)), L_R) &\leqslant \sum_{i\in I_0} \varphi_i(\zeta)|F_i(\zeta)| = \sum_{m\in I_0} \varphi_m(\zeta)|F_m(\zeta)| \\ &\leqslant \max_{m\in I_0} |F_m(\zeta)| \sum_{m\in I_0} \varphi_m(\zeta) \lesssim \varepsilon_0^{1/3}\, r(Q).\end{aligned}$$

Due to the fact that $r(Q) \approx \operatorname{diam}(R)$, by Lemma 3.16 lines L_R and L_Q are very close to each other in $2B_Q$, and thus

$$\operatorname{dist}((\zeta, F(\zeta)), L_Q) \lesssim \varepsilon_0^{1/3}\, r(Q)$$

as desired. □

Lemma 3.26. *For all $x \in R \setminus R_{\mathsf{Far}}$,*

$$\operatorname{dist}(x, \Gamma_R) \lesssim \sqrt[4]{\varepsilon_0}\; d(x). \tag{3.40}$$

Proof. Recall that if $d(x) = 0$, then $x \in \Gamma_R$ and we are done.

By Lemmas 3.13 and 3.25 any point $x \in R \setminus R_{\mathsf{Far}}$ is very close to L_R and (3.40) clearly holds if $d(x) \approx \operatorname{diam}(R)$. Hence, we may suppose below that $d(x)$ is small with respect to $\operatorname{diam}(R)$, say, $d(x) \ll (\mathsf{c}_3(\tau, A) + 2)\operatorname{diam}(R)$, where $\mathsf{c}_3(\tau, A) > 0$ is from Lemma 3.24.

Given $x \in R \setminus R_{\mathsf{Far}}$ with $d(x) > 0$, take a cube $Q \in \mathsf{DbTree}(R)$ such that

$$\operatorname{dist}(x, Q) + \operatorname{diam}(Q) \leqslant 2d(x).$$

Take any $z \in Q$ (note that $\operatorname{dist}(z, x) \leqslant 2d(x)$) and find $Q' \in \mathsf{DbTree}(R)$ such that

$$B(z, 2(\mathsf{c}_3(\tau, A) + 2)d(x)) \subset \tfrac{3}{2}B_{Q'}.$$

Recall that $d(x)$ is small with respect to $\operatorname{diam}(R)$ and thus Q' can be found. We can also guarantee that $r(Q') \approx_{\tau,A} d(x)$. Furthermore, it is clear that $x \in B(z, 2(\mathsf{c}_3(\tau, A) + 2)d(x))$ and thus $x \in \frac{3}{2}B_{Q'}$. Moreover, Lemma 3.24 gives

$$\operatorname{dist}(z, \Gamma_R) \leqslant \operatorname{dist}(z, x) + \operatorname{dist}(x, \Gamma_R) \leqslant (2 + \mathsf{c}_3(\tau, A))d(x),$$

which yields that $B(z, 2(\mathsf{c}_3(\tau, A) + 2)d(x)) \cap \Gamma_R \neq \varnothing$ and therefore

$$\tfrac{3}{2}B_{Q'} \cap \Gamma_R \neq \varnothing.$$

Take into account that $x \in \frac{3}{2}B_{Q'} \cap R \setminus R_{\mathsf{Far}} \subset 2B_{Q'}^{\mathsf{Cl}}$, i.e. $\operatorname{dist}(x, L_{Q'}) \lesssim \sqrt{\varepsilon_0}\, r(Q')$ and thus there is $x' \in L_{Q'} \cap 2B_{Q'}$ such that $\operatorname{dist}(x, x') \lesssim \sqrt{\varepsilon_0}\; r(Q')$. Furthermore, Lemma 3.25 says that $\operatorname{dist}(y, L_{Q'}) \leqslant c{\varepsilon_0}^{1/3}\, r(Q')$ for any $y \in \Gamma_R \cap 2B_{Q'}$ and some $c > 0$. In other words,

$$\Gamma_R \cap 2B_{Q'} \subset \mathcal{U}_{c{\varepsilon_0}^{1/3}\, r(Q')}(L_{Q'}),$$

and thus $\operatorname{dist}(x', \Gamma_R) \lesssim {\varepsilon_0}^{1/3}\; r(Q')$. Summarising, we get

$$\operatorname{dist}(x, \Gamma_R) \leqslant \operatorname{dist}(x, x') + \operatorname{dist}(x', \Gamma_R) \lesssim {\varepsilon_0}^{1/3}\; r(Q').$$

It is left to remember that $r(Q') \approx_{\tau,A} d(x)$ by construction and to choose $\varepsilon_0 = \varepsilon_0(\tau, A)$ small enough. □

Lemma 3.27. *For each $i \in I_0$,*

$$\operatorname{dist}(Q_i, \Gamma_R \cap \Pi^{-1}(J_i)) \lesssim_{\tau,A} \ell(J_i).$$

Proof. Let $x \in Q_i \subset B_0$. Then by Lemmas 3.24 and 3.21(a) we have

$$\operatorname{dist}(Q_i, \Gamma_R) \leqslant \operatorname{dist}(x, \Gamma_R) \lesssim_{\tau,A} d(x) \lesssim_{\tau,A} \operatorname{diam}(Q_i) \approx_{\tau,A} \ell(J_i).$$

Moreover, $\operatorname{dist}(J_i, \Pi(Q_i)) \lesssim \ell(J_i)$ by Lemma 3.21(b). From these two inequalities and Lemma 3.23, the required result follows. □

We finish this section with one more result which can be easily deduced from Lemmas 3.23 (look at $\operatorname{spt} F$) and 3.25.

Lemma 3.28. *For any $z \in \Gamma_R$, it holds that*

$$\operatorname{dist}(z, L_R) \lesssim \sqrt[4]{\varepsilon_0}\, r(R).$$

3.11 Small measure of the cubes from $\mathsf{LD}(R)$

In what follows we show that the measure of the low-density cubes is small.

Lemma 3.29. *If $\varepsilon_0 = \varepsilon_0(\tau, A)$ and τ are small enough, then*

$$\sum_{Q \in \mathsf{LD}(R)} \mu(Q) \leqslant \tfrac{1}{3}\sqrt{\tau}\, \mu(R). \tag{3.41}$$

Proof. Recall that the the parameters and thresholds from Section 3.7 are supposed to be chosen so that all above-stated results hold true. Taking this into account, note that by Lemma 3.12 with $\alpha = \alpha(\tau)$, being small enough, we have

$$\mu(R_{\mathsf{Far}}) \leqslant \tfrac{1}{6}\sqrt{\tau}\mu(R),$$

thus for obtaining (3.41) it suffices to show that

$$\mu(\mathcal{S}_{\mathsf{LD}}) \leqslant \tfrac{1}{6}\sqrt{\tau}\mu(R), \quad \text{where } \mathcal{S}_{\mathsf{LD}} := \bigcup_{Q \in \mathsf{LD}(R)} Q \setminus R_{\mathsf{Far}}. \tag{3.42}$$

By the Besicovitch covering theorem, there exist a countable collection of points $x_i \in \mathcal{S}_{\mathsf{LD}}$ such that

$$\mathcal{S}_{\mathsf{LD}} \subset \bigcup\nolimits_i B(x_i, r(Q_i)) \quad \text{and} \quad \sum\nolimits_i \chi_{B(x_i, r(Q_i))} \leqslant N,$$

where $Q_i \in \mathsf{LD}(R)$ is such that $x_i \in Q_i$, and N is some fixed constant. Note that $B(x_i, r(Q_i)) \subset 2B_{Q_i}$. From this it follows that

$$\mu(\mathcal{S}_{\mathsf{LD}}) \leqslant \sum\nolimits_i \mu(B(x_i, r(Q_i))) \leqslant \sum\nolimits_i \mu(2B_{Q_i}) \lesssim \sum\nolimits_i \Theta_\mu(2B_{Q_i}) r(Q_i).$$

Since $Q_i \in \mathsf{LD}(R)$, we have $\Theta_\mu(2B_{Q_i}) < \tau\Theta_\mu(2B_R)$ by definition. Furthermore, each $x_i \in \mathcal{S}_{\mathsf{LD}}$ satisfies Lemma 3.26 and moreover $d(x_i) \lesssim_{\tau,A} \operatorname{diam}(Q_i)$ (as x_i also belongs to the first doubling ancestor of Q_i with a comparable diameter with comparability constant $\lambda = \lambda(\tau, A)$, see Lemma 3.11) so that

$$\operatorname{dist}(x_i, \Gamma_R) \lesssim_{\tau,A} \sqrt[4]{\varepsilon_0}\, r(Q_i) \lesssim \sqrt[8]{\varepsilon_0}\, r(Q_i),$$

if $\varepsilon_0 = \varepsilon_0(\tau, A)$ is small enough. This means that Γ_R passes very close to the center of $B(x_i, r(Q_i))$ in terms of $r(Q_i)$. Consequently,

$$r(Q_i) \lesssim \mathcal{H}^1(\Gamma_R \cap B(x_i, r(Q_i)))$$

as Γ_R is a connected graph of a Lipschitz function. Thus we get

$$\mu(\mathcal{S}_{\mathsf{LD}}) \lesssim \tau\Theta_\mu(2B_R)\sum\nolimits_i \mathcal{H}^1(\Gamma_R \cap B(x_i, r(Q_i))).$$

Since $\sum_i \chi_{B(x_i, r(Q_i))} \leqslant N$ with an absolute constant N, we get by Lemma 3.23 that

$$\sum\nolimits_i \mathcal{H}^1(\Gamma_R \cap B(x_i, r(Q_i))) \lesssim \mathcal{H}^1\left(\Gamma_R \cap \bigcup\nolimits_i B(x_i, r(Q_i))\right) \lesssim \mathcal{H}^1(\Gamma_R \cap 2B_R) \lesssim r(B_R).$$

From this we deduce that

$$\mu(\mathcal{S}_{\mathsf{LD}}) \lesssim \tau\Theta_\mu(2B_R) r(B_R) \lesssim \tau\mu(2B_R) \lesssim \tau\mu(R),$$

where the latter inequality is due to the fact that $R \in \mathcal{D}^{db}$ by construction. Finally, we obtain (3.42) if τ is chosen small enough. $\square$

3.12 Small measure of the cubes from $\mathsf{BS}(R)$ for R whose best approximation line is far from the vertical

3.12.1 Auxiliaries and the key estimate for the measure of cubes from $\mathsf{BS}(R)$

Given some $\theta_0 > 0$, we say that

$$\begin{aligned} R \in \mathsf{T}_{VF}(\theta_0) \text{ and } \theta(R) = \theta_0, &\qquad \text{if} \quad \theta_V(L_R) \geqslant (1 + C_F)\,\theta_0; \\ R \notin \mathsf{T}_{VF}(\theta_0) \text{ and } \theta(R) = 2(1 + C_F)\,\theta_0, &\qquad \text{if} \quad \theta_V(L_R) < (1 + C_F)\,\theta_0. \end{aligned}$$

Note that $C_F > 0$ is an absolute constant from Lemma 3.23 where it is stated that the function F is $C_F\theta(R)$-Lipschitz. Recall that θ_0 and $\theta(R)$ were first introduced and used in Sections 3.7 and 3.8.1.

Let $R \in \mathsf{T}_{VF}(\theta_0)$. From the definition of the family $\mathsf{BS}(R)$ it follows that in this case we have

$$\measuredangle(L_Q, L_R) > \theta_0 \qquad \forall Q \in \mathsf{BS}(R). \tag{3.43}$$

On the other hand, if $Q \in \mathsf{DbTree}(R)$, then $\measuredangle(L_Q, L_R) \leqslant \theta_0$ and thus

$$\theta_V(L_Q) \geqslant (1 + C_F)\,\theta_0 - \measuredangle(L_Q, L_R) \geqslant C_F\,\theta_0 \qquad \forall Q \in \mathsf{DbTree}(R).$$

In this section we are going to deal with $R \in \mathsf{T}_{VF}(\theta_0)$ only. Our aim is to prove the following assertion.

Lemma 3.30. *For any $R \in \mathsf{T}_{VF}(\theta_0)$, if $\varepsilon_0 = \varepsilon_0(\tau)$ is chosen small enough, then*

$$\sum_{Q \in \mathsf{BS}(R)} \mu(Q) \leqslant \tfrac{1}{3}\sqrt{\tau}\mu(R).$$

The rest of this section is devoted to the proof of this lemma.

Remark 3.7. It is natural to suppose in this section that $\mathsf{BS}(R)$ is not empty. This and Remark 3.5 imply that $R \notin \mathsf{Stop}(R)$ and thus $\mathsf{Tree}(R) \setminus \mathsf{Stop}(R)$ is not empty.

3.12.2 The measure of cubes from $\mathsf{BS}(R)$ is controlled by the permutations of the Hausdorff measure restricted to Γ_R

Recall that the the parameters and thresholds from Section 3.7 are supposed to be chosen so that all above-stated results hold. Taking this into account, note that by Lemma 3.12 with $\alpha = \alpha(\tau)$, being small enough, we have

$$\mu(R_{\mathsf{Far}}) \leqslant \tfrac{1}{6}\sqrt{\tau}\mu(R),$$

thus to prove Lemma 3.30, it suffices to show that

$$\mu(\mathcal{S}_{\mathsf{BS}}) \leqslant \tfrac{1}{6}\sqrt{\tau}\mu(R), \qquad \text{where } \mathcal{S}_{\mathsf{BS}} := \bigcup_{Q\in\mathsf{BS}(R)} Q \setminus R_{\mathsf{Far}}. \tag{3.44}$$

The following results is the first step in proving (3.44). (Recall the identity (0.15).)

Lemma 3.31. *If θ_0 and $\varepsilon_0 = \varepsilon_0(\theta_0, \tau, A)$ are chosen small enough, then*

$$\mu(\mathcal{S}_{\mathsf{BS}}) \lesssim_A \frac{p_\infty(\Theta_\mu(2B_R)\,\mathcal{H}^1_{\Gamma_R})}{\theta_0^2\,\Theta_\mu(2B_R)^2}.$$

Proof. For every $x \in \mathcal{S}_{\mathsf{BS}}$ take the ball $B(x, r(Q_x))$, where $Q_x \in \mathsf{BS}(R)$ and is such that $x \in Q_x$. By the $5r$-covering theorem there exists a subfamily of pairwise disjoint balls $\{B(x_i, r(Q_i))\}_{i\in\hat{I}}$, where $Q_i = Q_{x_i}$, such that

$$\mathcal{S}_{\mathsf{BS}} \subset R \cap \bigcup\nolimits_{i\in\hat{I}} B(x_i, 5r(Q_i)).$$

Let $B_i = B(x_i, \frac{1}{2}r(Q_i))$, $i \in \hat{I}$. Clearly, $B_i \subset B_{Q_i}$. Moreover, take into account that $Q_i \in \mathcal{D}^{db}$ by the stopping condition **(S3)** and that $\mathcal{S}_{\mathsf{BS}} \cap R_{\mathsf{Far}} = \varnothing$ by definition. Therefore, by Lemma 3.26,

$$\operatorname{dist}(x_i, \Gamma_R) \lesssim \sqrt[4]{\varepsilon_0}\, d(x_i) \lesssim \sqrt[4]{\varepsilon_0}\, r(Q_i) < \tfrac{1}{4}r(Q_i),$$

if ε_0 is small enough. Thus $\Gamma_R \cap \frac{1}{2}B_i \neq \varnothing$ and therefore there exist $y_1, y_2 \in \Gamma_R \cap B_i$ such that

$$cr(Q_i) \leqslant |y_1 - y_2| \lesssim |\Pi(y_1) - \Pi(y_2)|$$

with some small fixed constant $c > 0$, where in the latter inequality we took into account that Γ_R is a graph of a Lipschitz function F (see Lemma 3.23).

Now, by Lemma 3.11, there exists $\tilde{Q}_i \in \mathsf{DbTree}(R)$ such that $Q_i \subset \tilde{Q}_i$ and moreover $\operatorname{diam}(Q_i) \approx_{\tau,A} \operatorname{diam}(\tilde{Q}_i)$. By Lemma 3.25,

$$\operatorname{dist}(y_k, L_{\tilde{Q}_i}) \lesssim {\varepsilon_0}^{1/3}\, r(\tilde{Q}_i), \qquad y_k \in \Gamma_R \cap B_i, \quad k = 1, 2.$$

At the same time, $\angle(L_{Q_i}, L_{\tilde{Q}_i}) \lesssim_{\tau,A} \sqrt[4]{\varepsilon_0}$ by arguments similar to those in the proof of Lemma 3.16 (this lemma cannot be applied directly as $Q_i \notin \mathsf{DbTree}(R)$ but the arguments can still be adapted if one of the cubes is in $\mathsf{BS}(R)$). Therefore, if $\varepsilon_0 = \varepsilon_0(\tau, A)$ is small enough, then one can show that

$$\operatorname{dist}(y_k, L_{Q_i}) \lesssim \sqrt[8]{\varepsilon_0}\, r(Q_i), \qquad y_k \in \Gamma_R \cap B_i \subset \Gamma_R \cap 2B_{Q_i}, \quad k = 1, 2.$$

Consequently, denoting by y_k' the orthogonal projections of y_k onto L_{Q_i}, we get

$$|y_k - y_k'| \lesssim \sqrt[8]{\varepsilon_0}\, r(Q_i), \qquad k = 1, 2.$$

Since $\measuredangle(L_{Q_i}, L_R) > \theta_0$ by (3.43) and $\varepsilon_0 = \varepsilon_0(\theta_0)$ is small enough, it holds that

$$
\begin{aligned}
&|F(\Pi(y_1)) - F(\Pi(y_2))| \\
&\quad = |\Pi^\perp(y_1) - \Pi^\perp(y_2)| \geqslant |\Pi^\perp(y_1') - \Pi^\perp(y_2')| - \sum\nolimits_k |y_k - y_k'| \\
&\quad \gtrsim \theta_0 |\Pi(y_1') - \Pi(y_2')| - \sum\nolimits_k |y_k - y_k'| \gtrsim \theta_0 |\Pi(y_1) - \Pi(y_2)| - 2\sum\nolimits_k |y_k - y_k'| \\
&\quad \gtrsim \theta_0 r(Q_i) - \sqrt[8]{\varepsilon_0} r(Q_i) \gtrsim \theta_0 r(Q_i),
\end{aligned}
$$

where $k = 1, 2$. Thus,

$$
\int_{\Pi(B_i)} |F'(z)| dz \geqslant \left| \int_{\Pi(y_1)}^{\Pi(y_2)} F'(z) dz \right| = |F(\Pi(y_1)) - F(\Pi(y_2))| \gtrsim \theta_0 r(Q_i).
$$

This and Hölder's inequality yield

$$
\theta_0 r(Q_i) \lesssim \sqrt{r(B_i)} \|F'\|_{2,\Pi(B_i)} \approx \sqrt{r(Q_i)} \|F'\|_{2,\Pi(B_i)},
$$

and finally

$$
r(Q_i) \lesssim \theta_0^{-2} \|F'\|_{2,\Pi(B_i)}^2.
$$

Since the balls $2B_i$, $i \in \hat{I}$, are pairwise disjoint by construction, so are the intervals $\Pi(B_i) \subset L_R$, $i \in \hat{I}$, if θ_0 is chosen small enough. This is a consequence of the fact that x_i, the centres of B_i, lie very close to Γ_R, namely, $\operatorname{dist}(x_i, \Gamma_R) \lesssim \sqrt[4]{\varepsilon_0}\, r(B_i)$, and moreover $\measuredangle(L_{x_i, x_j}, L_R) \lesssim \theta_0$ for all $i, j \in \hat{I}$ as Γ_R is Lipschits with constant $\lesssim \theta_0$, see Lemma 3.23. By this reason we have

$$
\begin{aligned}
\mu(\mathcal{S}_{\mathsf{BS}}) &\leqslant \sum\nolimits_{i \in \hat{I}} \mu(B(x_i, 5r(Q_i))) \lesssim \sum\nolimits_{i \in \hat{I}} \Theta_\mu(2B_{Q_i}) r(Q_i) \\
&\lesssim_A \theta_0^{-2} \Theta_\mu(2B_R) \sum\nolimits_{i \in \hat{I}} \|F'\|_{2,\Pi(B_i)}^2 \lesssim_A \theta_0^{-2} \Theta_\mu(2B_R) \|F'\|_2^2.
\end{aligned}
$$

Now take into account that under the assumption that $\|F'\|_\infty \leqslant 1/10$ (which is satisfied if θ_0 is sufficiently small) by [Tol5, Lemma 3.9] we have

$$
\|F'\|_2^2 \approx p_\infty(\mathcal{H}^1_{\Gamma_R}) \approx \Theta_\mu(2B_R)^{-3}\, p_\infty(\Theta_\mu(2B_R)\, \mathcal{H}^1_{\Gamma_R})
$$

with some absolute constants. □

We claim that $p_\infty(x, y, z)$ is well controlled by $p_0(x, y, z)$ for any $x, y \in \Gamma_R$ if $R \in \mathsf{T}_{VF}(\theta_0)$.

Lemma 3.32. *If $R \in \mathsf{T}_{VF}(\theta_0)$, then*

$$
p_\infty(x, y, z) \lesssim_{\theta_0} p_0(x, y, z) \qquad \textit{for any} \quad x, y \in \Gamma_R.
$$

Proof. The fact that the function F (whose graph is Γ_R) is $C_F \theta(R)$-Lipschitz by Lemma 3.23 and the definitions at the beginning of Subsection 3.12.1 yield

$$
\measuredangle(L_{xy}, L_R) \leqslant C_F\, \theta_0 \qquad \text{and} \qquad \theta_V(L_R) \geqslant (1 + C_F)\, \theta_0.
$$

Consequently,

$$
\theta_V(L_{xy}) \geqslant \theta_V(L_R) - \measuredangle(L_{xy}, L_R) \geqslant (1 + C_F)\, \theta_0 - C_F\, \theta_0 = \theta_0.
$$

Therefore $(x, y, z) \in \mathrm{V}_{\mathsf{Far}}(\theta_0)$ and it is left to use Lemma 3.6. □

For $x \in \mathbb{C}$ such that $\Pi(x) \notin \Pi(G_R)$, set

$$J_x = J_i, \quad i \in I, \quad \text{such that } \Pi(x) \in J_i,$$

and

$$\ell_x = \ell(J_x).$$

If $\Pi(x) \in \Pi(G_R)$, we write

$$J_x = \Pi(x) \qquad \text{and} \qquad \ell_x = 0,$$

i.e. one should think that in this case the point $\Pi(x)$ is a degenerate interval J_x with zero side length. To simplify notation, throughout this section we also write

$$x_1 = \Pi(x) \qquad \text{and} \qquad x_2 = \Pi^{\perp}(x).$$

Recall that the intervals $\{J_i\}$, $i \in I_0$, are the ones from $\{J_i\}$, $i \in I$, that intersect the ball $B_0 = B(\Pi(x_0), 10\,\mathrm{diam}\,(R))$, where $x_0 \in R$ is such that $\mathrm{dist}\,(x_0, L_R) \lesssim r(R)$ (see (3.31)). Observe that if $z \in U_0 = L_R \cap B_0$, then $D(z) \lesssim r(R)$. Thus $\ell(J_i) \lesssim r(R)$ for all $i \in I_0$. Thus, setting

$$\Gamma_{B_0} = G_R \cup \bigcup_{i \in I_0} \Gamma_R \cap \Pi^{-1}(J_i),$$

we deduce that $\Gamma_{B_0} \subset c' B_0$ with some fixed $c' > 0$. It is also true that $B_0 \subset c'' B_R$ with some $c'' > 0$ and thus

$$\Gamma_{B_0} \subset c B_R \quad \text{with some} \quad c > 0.$$

One can actually tune constants to guarantee that

$$\Gamma_{B_0} \subset \bigcup_{Q \in \mathsf{Tree}(R)} 2B_Q \subset 2B_R,$$

so we will suppose this in what follows.

Clearly, $\Pi(\Gamma_{B_0})$ is an interval on L_R and therefore Γ_{B_0} is a connected subset of Γ_R. We also set

$$\Gamma_{\mathbf{Ext}(B_0)} = \Gamma_R \setminus \Gamma_{B_0}.$$

First we will show that the part of the permutations of $\mathcal{H}^1_{\Gamma_R}$ that involves $\Gamma_{\mathbf{Ext}(B_0)}$ is very small.

Lemma 3.33. *We have*

$$p_\infty(\Theta_\mu(2B_R)\,\mathcal{H}^1_{\Gamma_{\mathbf{Ext}(B_0)}}, \Theta_\mu(2B_R)\,\mathcal{H}^1_{\Gamma_R}, \Theta_\mu(2B_R)\,\mathcal{H}^1_{\Gamma_R}) \lesssim \sqrt[8]{\varepsilon_0}\,\Theta_\mu(2B_R)^2\,\mu(R).$$

Proof. The proof is analogous (up to constants) to the proof of [Tol5, Lemma 7.36], where we should use our Lemmas 3.23 and 3.28 instead of [Tol5, Lemma 7.27 and Lemma 7.32]. □

What is more, it can be easily seen that

$$\begin{aligned} p_\infty(\Theta_\mu(2B_R)\,\mathcal{H}^1_{\Gamma_R}) \leqslant & p_\infty(\Theta_\mu(2B_R)\,\mathcal{H}^1_{\Gamma_{B_0}}) \\ & + 3p_\infty(\Theta_\mu(2B_R)\,\mathcal{H}^1_{\Gamma_{\mathbf{Ext}(B_0)}}, \Theta_\mu(2B_R)\,\mathcal{H}^1_{\Gamma_R}, \Theta_\mu(2B_R)\,\mathcal{H}^1_{\Gamma_R}). \end{aligned} \tag{3.45}$$

Consequently, taking into account Lemmas 3.31 and 3.33, we are now able to reduce the proof of Lemma 3.30 to the proof of a proper estimate for $p_\infty(\Theta_\mu(2B_R)\,\mathcal{H}^1_{\Gamma_{B_0}})$, where $\Gamma_{B_0}\subset cB_R$ with some $c>0$. For short, we will write

$$\sigma := \Theta_\mu(2B_R)\,\mathcal{H}^1_{\Gamma_{B_0}}.$$

Thus, using this notation, we are aimed to prove the following lemma in the forthcoming subsections.

Lemma 3.34. *It holds that*

$$p_\infty(\sigma) \lesssim \varepsilon_0^{1/40}\,\Theta_\mu(2B_R)^2\,\mu(R).$$

3.12.3 Estimates for the permutations of the Hausdorff measure restricted to Γ_R

Recall that, for $x\in\mathbb{C}$, we set $\ell_x=\ell(J_x)$. Let $x,y\in\Gamma_R$. We say that x and y are

- **very close** and write
$$(x,y)\in\mathbf{VC},\qquad \text{if } |x_1-y_1|\leqslant \ell_x+\ell_y;$$
- **close** and write
$$(x,y)\in\mathbf{C},\qquad \text{if } |x_1-y_1|\leqslant \varepsilon_0^{-1/20}(\ell_x+\ell_y);$$
- **far** and write
$$(x,y)\in\mathbf{F},\qquad \text{if } |x_1-y_1|> \varepsilon_0^{-1/20}(\ell_x+\ell_y).$$

Notice that the relations are symmetric with respect to x and y.

Given $(x,y,z)\in\Gamma^3_{B_0}$, there are three possibilities: either two of the points in the triple are very close, or no pair of points is very close but there is at least one pair that is close, or all the pairs of points are far. So we can split $p_\infty(\sigma)$ as follows:

$$\begin{aligned}
p_\infty(\sigma) &\leqslant 3\iiint_{(x,y)\in\mathbf{VC}} p_\infty(x,y,z)\,d\sigma(x)\,d\sigma(y)\,d\sigma(z)\\
&\quad+3\iiint_{\substack{(x,y)\in\mathbf{C}\setminus\mathbf{VC}\\(x,z)\notin\mathbf{VC}\\(y,z)\notin\mathbf{VC}}} p_\infty(x,y,z)\,d\sigma(x)\,d\sigma(y)\,d\sigma(z)\\
&\quad+\iiint_{\substack{(x,y)\in\mathbf{F}\\(x,z)\in\mathbf{F}\\(y,z)\in\mathbf{F}}} p_\infty(x,y,z)\,d\sigma(x)\,d\sigma(y)\,d\sigma(z)\\
&=: p_{\infty,\mathbf{VC}}(\sigma)+p_{\infty,\mathbf{C}\setminus\mathbf{VC}}(\sigma)+p_{\infty,\mathbf{F}}(\sigma).
\end{aligned}\tag{3.46}$$

A straightforward adaptation of the arguments from [Tol5, Section 7.8.2, Lemmas 7.38 and 7.39] to our settings gives the following.

Lemma 3.35. *If $\varepsilon_0=\varepsilon_0(\tau,A)$ and $\alpha=\alpha(\theta_0,\varepsilon_0,\tau,A)$ are chosen small enough, then*

$$p_{\infty,\mathbf{VC}}(\sigma)+p_{\infty,\mathbf{C}\setminus\mathbf{VC}}(\sigma)\lesssim \varepsilon_0^{1/40}\,\Theta_\mu(2B_R)^2\,\mu(R).$$

Now we are going to prove the following result that actually finishes the proof of Lemma 3.34 and therefore Lemma 3.30 taking into account (3.46) and Lemma 3.35.

Lemma 3.36. *If $\varepsilon_0 = \varepsilon_0(\tau, A)$, $\alpha = \alpha(\theta_0, \varepsilon_0, \tau, A)$ and $\delta = \delta(\varepsilon_0)$ are small enough, then*

$$p_{\infty,\mathbf{F}}(\sigma) \lesssim \varepsilon_0^{1/40}\, \Theta_\mu(2B_R)^2\, \mu(R). \tag{3.47}$$

The proof of Lemma 3.36 is similar to the one of [Tol5, Lemma 7.40] but necessary changes are not straightforward so we give details. First we need to approximate the measure σ by another measure absolutely continuous with respect to μ, of the form $g\mu$, with some $g \in L^\infty(\mu)$. This is carried out by the next lemma, where we say that

$$i \in I_0' \quad \text{if} \quad i \in I_0 \quad \text{and} \quad \mu(Q_i \setminus R_{\mathsf{Far}}) \geqslant \tfrac{3}{4}\mu(Q_i), \tag{3.48}$$

for the cubes $Q_i \in \mathsf{DbTree}(R)$ from Lemma 3.21 associated with the intervals J_i, $i \in I_0$. Recall the definition of R_{Far} in Section 3.8.2 and Lemmas 3.12 and 3.13. In what follows we will also write

$$\widehat{J}_i := \Gamma_R \cap \Pi^{-1}(J_i).$$

Lemma 3.37. *For each $i \in I_0'$ there exists a non-negative function $g_i \in L^\infty(\mu)$, supported on $A_i \subset Q_i \setminus R_{\mathsf{Far}}$, where $Q_i \in \mathsf{DbTree}(R)$ are associated with the intervals J_i by Lemma 3.21, and such that*

$$\int g_i\, d\mu = \Theta_\mu(2B_R)\mathcal{H}^1(\widehat{J}_i) = \sigma(\widehat{J}_i), \tag{3.49}$$

and

$$\sum\nolimits_{i\in I_0'} g_i \lesssim_{\tau,A} 1. \tag{3.50}$$

Proof. Assume first that the family $\{J_i\}_{i\in I_0'}$ is finite. Suppose also that $\ell(J_i) \leqslant \ell(J_{i+1})$ for all $i \in I_0'$. We will construct

$$g_i = \alpha_i \chi_{A_i}, \qquad \text{where} \quad \alpha_i \geqslant 0 \quad \text{and} \quad A_i \subset Q_i \setminus R_{\mathsf{Far}}.$$

We set

$$\alpha_1 = \frac{\sigma(\widehat{J}_1)}{\mu(A_1)} \qquad \text{and} \qquad A_1 = Q_1 \setminus R_{\mathsf{Far}},$$

so that $\int g_1\, d\mu = \sigma(\widehat{J}_1)$. Furthermore, by (3.15) in Lemma 3.9, Lemmas 3.21 and 3.23 and the condition (3.48) we get

$$\|g_1\|_\infty = \alpha_1 \lesssim_{\tau,A} \frac{\Theta_\mu(2B_R)\ell(J_1)}{\mu(Q_1)} \approx_{\tau,A} \frac{\Theta_\mu(2B_R)\operatorname{diam}(Q_1)}{\mu(2B_{Q_1})} \leqslant b' \quad \text{with } b' = b'(\tau, A) > 0.$$

Furthermore, we define g_k, $k \geqslant 2$, by induction. Suppose that $g_1, \dots, g_{k-1}$ have been constructed, satisfy (3.49) and the inequality $\sum_{i=1}^{k-1} g_i \leqslant b$ with some $b = b(\tau, A) > 0$ to be chosen later.

If Q_k is such that $Q_k \cap \bigcup_{i=1}^{k-1} Q_i = \varnothing$, then we set

$$\alpha_k = \frac{\sigma(\widehat{J}_k)}{\mu(A_k)} \qquad \text{and} \qquad A_k = Q_k \setminus R_{\mathsf{Far}},$$

so that $\int g_k\, d\mu = \sigma(\widehat{J}_k)$. Moreover, similarly to the case of α_1, we have

$$\|g_k\|_\infty = \alpha_k \leqslant b',$$

where $b' = b'(\tau, A)$ is obviously independent of k. Since $A_k \cap \bigcup_{i=1}^{k-1} A_i = \varnothing$, we have

$$g_k + \sum_{i=1}^{k-1} g_i \leqslant \max\{b, b'\}.$$

We choose $b = b'(\tau, A)$ in order to have (3.50).

Now suppose that $Q_k \cap \bigcup_{i=1}^{k-1} Q_i \neq \varnothing$ and let $Q_{s_1}, \dots, Q_{s_m}$ be the subfamily of $Q_1, \dots, Q_{k-1}$ such that $Q_{s_j} \cap Q_k \neq \varnothing$. Since $\ell(J_{s_j}) \leqslant \ell(J_k)$ (because of the non-decreasing sizes of $\ell(J_i)$, $i \in I_0'$), we deduce that $\operatorname{dist}(J_{s_j}, J_k) \lesssim \ell(J_k)$, and thus $J_{s_j} \subset c' J_k$, for some constant $c' > 0$. Using (3.49) for $i = s_j$, we get by (3.15) in Lemma 3.9, Lemmas 3.21 and 3.23 that

$$\begin{aligned}\sum_j \int g_{s_j}\, d\mu = \sum_j \sigma(\widehat{J}_{s_j}) &\leqslant \sigma(\Pi^{-1}(c' J_k)) \\ &\lesssim \Theta_\mu(2B_R)\ell(J_k) \lesssim \Theta_\mu(2B_R)\operatorname{diam}(Q_k) \leqslant c''\,\mu(Q_k)\end{aligned}$$

with some $c'' = c''(\tau, A) > 0$. Therefore, by Chebyshev's inequality,

$$\mu\left(\left\{\sum_j g_{s_j} > 2c''\right\}\right) \leqslant \frac{1}{2}\,\mu(Q_k).$$

So we set

$$A_k = \left(Q_k \cap \left\{\sum_j g_{s_j} \leqslant 2c''\right\}\right) \setminus R_{\mathsf{Far}},$$

and then $\mu(A_k) \geqslant \frac{1}{4}\mu(Q_k)$. As above, we put $\alpha_k = \sigma(\widehat{J}_k)/\mu(A_k)$ so that $g_k = \alpha_k \chi_{A_k}$ satisfies $\int g_k\, d\mu = \sigma(\widehat{J}_k)$. Consequently,

$$\alpha_k \leqslant \frac{\sigma(\widehat{J}_k)}{\frac{1}{4}\mu(Q_k)} \leqslant b'' \qquad \text{with some } b'' = b''(\tau, A) > 0,$$

which yields

$$g_k + \sum_j g_{s_j} \leqslant b'' + 2c''.$$

Recall that s_j are such that $Q_{s_j} \cap Q_k \neq \varnothing$. The latter inequality implies that

$$g_k + \sum_{i=1}^{k-1} g_i \leqslant \max\{b, b'' + 2c''\}.$$

In this case, we choose $b = b'' + 2c''$ and (3.50) follows. Clearly, this bound is independent of the number of functions.

Suppose now that $\{J_i\}_{i \in I_0'}$ is not finite. For each fixed M we consider a family of intervals $\{J_i\}_{1 \leqslant i \leqslant M}$. As above, we construct functions $g_1^M, \dots, g_M^M$ with $\operatorname{spt}(g_i^M) \subset Q_i \setminus R_{\mathsf{Far}}$ satisfying

$$\int g_i^M d\mu = \sigma(\widehat{J}_i) \quad \text{and} \quad \sum_{i=1}^M g_i^M \leqslant b = b(\tau, A).$$

Then there is a subsequence $\{g_1^k\}_{k \in I_1}$ which is convergent in the weak $*$ topology of $L^\infty(\mu)$ to some function $g_1 \in L^\infty(\mu)$. Now we take another convergent subsequence $\{g_2^k\}_{k \in I_2}$, $I_2 \subset I_1$, in the weak $*$ topology of $L^\infty(\mu)$ to another function $g_2 \in L^\infty(\mu)$, etc. We have $\operatorname{spt}(g_i) \subset Q_i \setminus R_{\mathsf{Far}}$. Furthermore, (3.49) and (3.50) also hold due to the weak $*$ convergence. $\square$

Recall that $G_R = \{z \in \mathbb{C} : d(z) = 0\}$ (see (3.28)) and clearly $G_R \subset R$. We will need the following result which can be proved analogously to [Tol5, Lemma 7.18] taking into account that the density $\Theta_\mu(2B_R)$ is involved in our case.

Lemma 3.38. *We have*

$$\mu\lfloor G_R = \rho_{G_R}\Theta_\mu(2B_R)\,\mathcal{H}^1_{G_R} = \rho_{G_R}\sigma\lfloor G_R,$$

where ρ_{G_R} is a function such that $c \leqslant \rho_{G_R} \leqslant c^{-1}$ with some constant $c = c(\tau, A) > 0$.

Let us mention now the following technical result proved in [Tol5, Subsection 4.6.1].

Lemma 3.39. *Let $x, y, z \in \mathbb{C}$ be pairwise distinct points, and let $x' \in \mathbb{C}$ be such that*

$$a^{-1}|x-y| \leqslant |x'-y| \leqslant a|x-y|,$$

where $a > 0$ is some constant. Then

$$|c(x,y,z) - c(x',y,z)| \leqslant (4+2a)\frac{|x-x'|}{|x-y||x-z|}.$$

Take into account that $p_\infty(x,y,z) = \frac{1}{2}c(x,y,z)^2$ by (0.15).
Recall that

$$\Gamma_{B_0} = G_R \cup \bigcup\nolimits_{i\in I_0} \widehat{J}_i \qquad \text{and} \qquad \widehat{J}_i = \Gamma_R \cap \Pi^{-1}(J_i).$$

In Lemma 3.37 we showed how $\sigma\lfloor\widehat{J}_i$ can be approximated by a measure supported on $Q_i \setminus R_{\mathsf{Far}}$, for each $i \in I_0'$, where I_0' is defined in (3.48). Notice that, by Lemma 3.27,

$$\operatorname{dist}(Q_i, \widehat{J}_i) \lesssim_{\tau,A} \ell(J_i), \qquad i \in I_0. \tag{3.51}$$

Now we consider the measures

$$\nu_i := g_i\,\mu, \qquad i \in I_0',$$

with g_i as in Lemma 3.37, and set

$$\nu := \sigma\lfloor G_R + \sum\nolimits_{i\in I_0'} \nu_i = \rho_{G_R}^{-1}\,\mu\lfloor G_R + \sum\nolimits_{i\in I_0'} g_i\,\mu. \tag{3.52}$$

This measure should be understood as an approximation of $\sigma = \Theta_\mu(2B_R)\mathcal{H}^1_{\Gamma_{B_0}}$, which coincides with σ on G_R due to Lemma 3.38 ($g_i \equiv 0$ in this case).

Using the measure ν, we will actually prove the inequality (3.47) in Lemma 3.36. This will be done in the forthcoming subsection.

3.12.4 Estimates for the permutations of the Hausdorff measure restricted to Γ_R in the case when points are far from each other

To proceed, we need to introduce some additional notation. Given measures τ_1, τ_2, τ_3, set

$$p_t(\tau_1,\tau_2,\tau_3) := \iiint p_t(x,y,z)\,d\tau_1(x)\,d\tau_2(y)\,d\tau_3(z), \qquad \text{where } t = 0 \text{ or } t = \infty.$$

We denote by $p_{t,\mathbf{F}}(\tau_1,\tau_2,\tau_3)$ the triple integral above restricted to (x,y,z) such that

$$\begin{aligned} |x_1 - y_1| &\geqslant \varepsilon_0^{-1/20}(\ell_x + \ell_y), \\ |x_1 - z_1| &\geqslant \varepsilon_0^{-1/20}(\ell_x + \ell_z), \\ |y_1 - z_1| &\geqslant \varepsilon_0^{-1/20}(\ell_y + \ell_z). \end{aligned} \tag{3.53}$$

So we have

$$
\begin{aligned}
p_{\infty,\mathbf{F}}(\sigma) = \quad & p_{\infty,\mathbf{F}}(\sigma\lfloor G_R) \\
& + p_{\infty,\mathbf{F}}(\sigma\lfloor\Gamma_{B_0} \setminus G_R) \\
& +3\, p_{\infty,\mathbf{F}}(\sigma\lfloor G_R, \sigma\lfloor\Gamma_{B_0} \setminus G_R, \sigma\lfloor\Gamma_{B_0} \setminus G_R) \\
& +3\, p_{\infty,\mathbf{F}}(\sigma\lfloor G_R, \sigma\lfloor G_R, \sigma\lfloor\Gamma_{B_0} \setminus G_R).
\end{aligned} \tag{3.54}
$$

1. Consider the term $p_{\infty,\mathbf{F}}(\sigma\lfloor G_R)$. In this case $\ell_x = \ell_y = \ell_z \equiv 0$ and the subscript $\mathbf{F}$ may be skipped. Moreover, using Lemmas 3.32 and 3.38, we get

$$
p_{\infty,\mathbf{F}}(\sigma\lfloor G_R) \lesssim_{\theta_0} p_0(\sigma\lfloor G_R) \approx_{\theta_0,\tau,A} p_0(\mu\lfloor G_R).
$$

Now we proceed very similarly to the proof of Lemma 3.12. For $\delta > 0$ from Lemma 3.7 (see also Section 3.7), taking into account Remark 3.7, we get

$$
\begin{aligned}
p_0(\mu\lfloor G_R) &\leqslant \int_{G_R} \sum_{Q\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R)\colon x\in 2B_Q} p_0^{[\delta,Q]}(x, \mu\lfloor G_R, \mu\lfloor G_R)\, d\mu(x) \\
&\leqslant \sum_{Q\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R)} p_0^{[\delta,Q]}(\mu\lfloor 2B_Q, \mu\lfloor 2B_R, \mu\lfloor 2B_R) \\
&= \sum_{Q\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R)} \frac{p_0^{[\delta,Q]}(\mu\lfloor 2B_Q, \mu\lfloor 2B_R, \mu\lfloor 2B_R)}{\mu(Q)} \int \chi_Q(x)\, d\mu(x).
\end{aligned}
$$

Changing the order of summation and the inequality (3.17) yield

$$
\begin{aligned}
\frac{p_0(\mu\lfloor G_R)}{\Theta_\mu(2B_R)^2} &\leqslant \int_R \sum_{Q\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R)\colon x\in Q} \frac{p_0^{[\delta,Q]}(\mu\lfloor 2B_Q, \mu\lfloor 2B_R, \mu\lfloor 2B_R)}{\Theta_\mu(2B_R)^2\mu(Q)}\, d\mu(x) \\
&= \int_R \sum_{Q\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R)\colon x\in Q} \mathsf{perm}(Q)^2\, d\mu(x).
\end{aligned}
$$

From this and the inequality (3.17) in Lemma 3.9 we deduce that

$$
p_0(\mu\lfloor G_R) \leqslant \alpha^2\, \Theta_\mu(2B_R)^2\mu(R).
$$

Finally, if $\alpha = \alpha(\theta_0, \varepsilon_0, \tau, A)$ is chosen small enough, then

$$
p_{\infty,\mathbf{F}}(\sigma\lfloor G_R) \lesssim \varepsilon_0^{1/40}\, \Theta_\mu(2B_R)^2\, \mu(R).
$$

2. Let us study $p_{\infty,\mathbf{F}}(\sigma\lfloor\Gamma_{B_0} \setminus G_R)$. In this case ℓ_x, ℓ_y and ℓ_z are strictly positive and so are the lengths of the associated doubling cubes from Lemma 3.21. We set

$$
p_{\infty,\mathbf{F}}(\sigma\lfloor\Gamma_{B_0} \setminus G_R) = \sum_{i,j,k\in I_0} p_{\infty,\mathbf{F}}\left(\sigma\lfloor\widehat{J}_i, \sigma\lfloor\widehat{J}_j, \sigma\lfloor\widehat{J}_k\right).
$$

First let us consider the case when at least one of the indices i, j or k is in $I_0 \setminus I_0'$, i.e. $\mu(Q_h \cap R_{\mathsf{Far}}) > \frac{3}{4}\mu(Q_h)$ for h being i, j or k, according to (3.48). By symmetry, we may consider just the case $i \in I_0 \setminus I_0'$. Moreover, then the required estimate will follow from a proper one for

$$
p_\infty\left(\sigma\lfloor\widehat{J}', \sigma\lfloor\Gamma_{B_0}, \sigma\lfloor\Gamma_{B_0}\right), \qquad \text{where } \widehat{J}' := \bigcup_{i\in I_0\setminus I_0'} \widehat{J}_i.
$$

Recall that

$$
\sigma = \Theta_\mu(2B_R)\mathcal{H}^1_{\Gamma_{B_0}} \quad \text{and} \quad \Gamma_{B_0} = G_R \cup \bigcup\nolimits_{i\in I_0} \widehat{J}_i.
$$

Lemma 3.40. *We have*

$$\mathcal{H}^1(\widehat{J'}) \leqslant \sqrt{\alpha}\, \operatorname{diam}(R).$$

Proof. Notice that for $i \in I_0 \setminus I_0'$ we have

$$\Theta_\mu(2B_R)\mathcal{H}^1(\widehat{J_i}) \lesssim \Theta_\mu(2B_R)\ell(J_i) \lesssim_{\tau,A} \Theta_\mu(2B_R) \operatorname{diam}(Q_i) \approx_{\tau,A} \mu(Q_i) \lesssim_{\tau,A} \mu(Q_i \cap R_{\mathsf{Far}}),$$

where $Q_i \in \mathsf{DbTree}(R)$ is the cube associated to the interval J_i by Lemma 3.21. By Vitali's covering lemma, there exists a subfamily of balls $2B_{Q_i}$, $i \in J \subset I_0 \setminus I_0'$, such that

- the balls $2B_{Q_i}$, $i \in J$, are disjoint,
- $\bigcup_{i\in I_0\setminus I_0'} 2B_{Q_i} \subset \bigcup_{i\in J} 10B_{Q_i}$.

Then, taking into account that $\mu(10B_{Q_i} \cap R) \approx_{\tau,A} \mu(2B_{Q_i}) \approx \mu(Q_i)$, we get

$$\begin{aligned}\Theta_\mu(2B_R)\mathcal{H}^1(\widehat{J'}) &\lesssim \sum_{i\in I_0\setminus I_0'} \mu(Q_i) \lesssim \sum_{i\in J} \mu(10B_{Q_i} \cap R)\\ &\lesssim_{\tau,A} \sum_{i\in J} \mu(Q_i) \lesssim_{\tau,A} \sum_{i\in J} \mu(Q_i \cap R_{\mathsf{Far}}) \lesssim_{\tau,A} \mu(R_{\mathsf{Far}}),\end{aligned}$$

because the cubes Q_i from the family J are disjoint. Since $\mu(R_{\mathsf{Far}}) \leqslant \alpha\mu(R)$ by Lemma 3.12, the lemma follows if $\alpha = \alpha(\tau, A)$ is chosen small enough. □

To continue, we need the following result from [Tol5].

Lemma 3.41 (Lemma 3.4 in [Tol5])**.** *Let μ_1, μ_2 and μ_3 be finite measures. Then*

$$\sum_{s\in\mathfrak{S}_3} \int \mathcal{C}_\varepsilon(\mu_{s_2})\overline{\mathcal{C}_\varepsilon(\mu_{s_3})}\, d\mu_{s_1} = c_\varepsilon^2(\mu_1,\mu_2,\mu_3) + \mathcal{R}, \qquad \mathcal{R} \leqslant C \sum_{s\in\mathfrak{S}_3} \int M_{\mathsf{R}}\mu_{s_2} M_{\mathsf{R}}\mu_{s_3}\, d\mu_{s_1},$$

where $\mathfrak{S}_3$ is the group of permutations of the three elements $\{1, 2, 3\}$, $\mathcal{C}_\varepsilon$ the truncated Cauchy integral, c_ε^2 the truncated curvature of measure (see (0.6) *and below) and M_{R} the* 1*-dimensional radial maximal operator.*

Lemma 3.42. *For $E \subset \Gamma_{B_0}$, we have*

$$c^2(\mathcal{H}^1_E, \mathcal{H}^1_{\Gamma_{B_0}}, \mathcal{H}^1_{\Gamma_{B_0}}) \lesssim \mathcal{H}^1(E)^{1/2} \operatorname{diam}(R)^{1/2}.$$

Proof. By Lemma 3.41, we have

$$\begin{aligned}c^2(\mathcal{H}^1_E, \mathcal{H}^1_{\Gamma_{B_0}}, \mathcal{H}^1_{\Gamma_{B_0}}) \lesssim{}& \limsup_{\varepsilon\to 0} \int_{\Gamma_{B_0}} |\mathcal{C}_\varepsilon(\mathcal{H}^1_E)\, \mathcal{C}_\varepsilon(\mathcal{H}^1_{\Gamma_{B_0}})|\, d\mathcal{H}^1\\ &+ \limsup_{\varepsilon\to 0} \int_E |\mathcal{C}_\varepsilon(\mathcal{H}^1_{\Gamma_{B_0}})|^2\, d\mathcal{H}^1\\ &+ \int_{\Gamma_{B_0}} |M_{\mathsf{R}}(\mathcal{H}^1_E)\, M_{\mathsf{R}}(\mathcal{H}^1_{\Gamma_{B_0}})|^2\, d\mathcal{H}^1\\ &+ \int_E |M_{\mathsf{R}}(\mathcal{H}^1_{\Gamma_{B_0}})|^2\, d\mathcal{H}^1\\ :={}& I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Regarding I_1, by the L^2-boundedness of the Cauchy transform on Lipschitz graphs (with respect to $\mathcal{H}^1_{\Gamma_R}$) we have

$$\begin{aligned} I_1 &\leqslant \limsup_{\varepsilon\to 0} \|\mathcal{C}_\varepsilon(\mathcal{H}^1_E)\|_{L^2(\mathcal{H}^1_{\Gamma_R})} \|\mathcal{C}_\varepsilon(\mathcal{H}^1_{\Gamma_{B_0}})\|_{L^2(\mathcal{H}^1_{\Gamma_R})} \\ &\lesssim \mathcal{H}^1(E)^{1/2}\, \mathcal{H}^1(\Gamma_{B_0})^{1/2} \\ &\lesssim \mathcal{H}^1(E)^{1/2} \operatorname{diam}(R)^{1/2}. \end{aligned}$$

For I_2 we use the L^4-boundedness of the Cauchy transform:

$$I_2 \leqslant \limsup_{\varepsilon\to 0} \mathcal{H}^1(E)^{1/2} \|\mathcal{C}_\varepsilon(\mathcal{H}^1_{\Gamma_{B_0}})\|^2_{L^4(\mathcal{H}^1_{\Gamma_R})} \lesssim \mathcal{H}^1(E)^{1/2} \operatorname{diam}(R)^{1/2}.$$

Using the fact that $M_{\mathsf{R}}(\mathcal{H}^1_{\Gamma_{B_0}}) \lesssim 1$, we derive

$$I_4 \leqslant \mathcal{H}^1(E) \lesssim \mathcal{H}^1(E)^{1/2} \operatorname{diam}(R)^{1/2},$$

and also

$$I_3 \lesssim \int_{\Gamma_{B_0}} |M_{\mathsf{R}}(\mathcal{H}^1_E)|\, d\mathcal{H}^1.$$

Since the operator $M_{\mathsf{R}}(\mathcal{H}^1_{\Gamma_R})$ is bounded in $L^2(\mathcal{H}^1_{\Gamma_R})$ (as it is comparable to the Hardy-Littlewood operator with respect to the measure $\mathcal{H}^1_{\Gamma_R}$), we deduce

$$I_3 \lesssim \|M_{\mathsf{R}}(\chi_E \mathcal{H}^1_{\Gamma_R})\|_{L^2(\mathcal{H}^1_{\Gamma_R})} \mathcal{H}^1(\Gamma_{B_0})^{1/2} \lesssim \mathcal{H}^1(E)^{1/2} \operatorname{diam}(R)^{1/2}.$$

So the lemma follows. $\square$

By Lemma 3.42 for $E = \widehat{J}'$ and Lemma 3.40 we derive that

$$c^2(\mathcal{H}^1_{\widehat{J}'}, \mathcal{H}^1_{\Gamma_{B_0}}, \mathcal{H}^1_{\Gamma_{B_0}}) \lesssim \mathcal{H}^1(\widehat{J}')^{1/2} \operatorname{diam}(R)^{1/2} \lesssim \alpha^{1/4} \operatorname{diam}(R).$$

Therefore, recalling that $p_\infty(x,y,z) = \frac{1}{2} c(x,y,z)^2$ (see (0.15)),

$$p_\infty\left(\sigma\lfloor \widehat{J}', \sigma\lfloor\Gamma_{B_0}, \sigma\lfloor\Gamma_{B_0}\right) \lesssim \alpha^{1/4}\Theta_\mu(2B_R)^3 \operatorname{diam}(R) \approx \alpha^{1/4}\, \Theta_\mu(2B_R)^2\, \mu(R).$$

Furthermore, choosing $\alpha = \alpha(\varepsilon_0)$ small enough, we get from the latter estimate that

$$\sum_{i\in I_0\setminus I_0',\, j,k\in I_0} p_{\infty,\mathbf{F}}\left(\sigma\lfloor \widehat{J}_i, \sigma\lfloor \widehat{J}_j, \sigma\lfloor \widehat{J}_k\right) \lesssim \varepsilon_0^{1/40}\, \Theta_\mu(2B_R)^2\, \mu(R), \tag{3.55}$$

and we are done with the case when at least one of the indices i, j or k is in $I_0 \setminus I_0'$.

Now let $(i,j,k) \in (I_0')^3$. By definition, if $p_{\infty,\mathbf{F}}\left(\sigma\lfloor \widehat{J}_i, \sigma\lfloor \widehat{J}_j, \sigma\lfloor \widehat{J}_k\right) \neq 0$, then there exist $x \in \widehat{J}_i$, $y \in \widehat{J}_j$ and $z \in \widehat{J}_k$ satisfying (3.53). Then it follows easily that

$$\begin{aligned} \operatorname{dist}(\widehat{J}_i, \widehat{J}_j) &\geqslant \tfrac{1}{2}\varepsilon_0^{-1/20}\left(\ell(J_i) + \ell(J_j)\right), \\ \operatorname{dist}(\widehat{J}_i, \widehat{J}_k) &\geqslant \tfrac{1}{2}\varepsilon_0^{-1/20}\left(\ell(J_i) + \ell(J_k)\right), \\ \operatorname{dist}(\widehat{J}_j, \widehat{J}_k) &\geqslant \tfrac{1}{2}\varepsilon_0^{-1/20}\left(\ell(J_j) + \ell(J_k)\right). \end{aligned} \tag{3.56}$$

We denote by $J_{\mathbf{F}}$ the set of those indices $(i,j,k) \in (I_0')^3$ such that the inequalities (3.56) hold, so that

$$p_{\infty,\mathbf{F}}(\sigma\lfloor\Gamma_{B_0} \setminus G_R) \leqslant \sum_{(i,j,k)\in J_{\mathbf{F}}} p_\infty\left(\sigma\lfloor \widehat{J}_i, \sigma\lfloor \widehat{J}_j, \sigma\lfloor \widehat{J}_k\right).$$

Consider $(i,j,k) \in J_{\mathbf{F}}$ and

$$x, x' \in \widehat{J}_i \cup Q_i, \qquad y, y' \in \widehat{J}_j \cup Q_j \qquad \text{and} \qquad z, z' \in \widehat{J}_k \cup Q_k.$$

Due to (3.56) and (3.51), taking into account that $\ell(J_h) \approx_{\tau,A} \operatorname{diam}(\widehat{J}_h) \approx_{\tau,A} \operatorname{diam}(Q_h)$ for each $h \in I$ by Lemma 3.21, the sets $\widehat{J}_i \cup Q_i$, $\widehat{J}_j \cup Q_j$ and $\widehat{J}_k \cup Q_k$ are far to each other in the sense that

$$\begin{aligned} &\operatorname{dist}(\widehat{J}_i \cup Q_i, \widehat{J}_j \cup Q_j) \gtrsim \varepsilon_0^{-1/20} \left(\ell(J_i) + \ell(J_j)\right), \\ &\operatorname{dist}(\widehat{J}_i \cup Q_i, \widehat{J}_k \cup Q_k) \gtrsim \varepsilon_0^{-1/20} \left(\ell(J_i) + \ell(J_k)\right), \\ &\operatorname{dist}(\widehat{J}_j \cup Q_j, \widehat{J}_k \cup Q_k) \gtrsim \varepsilon_0^{-1/20} \left(\ell(J_j) + \ell(J_k)\right), \end{aligned} \tag{3.57}$$

where ε_0 is chosen small enough. Furthermore, applying Lemma 3.39 three times gives

$$p_\infty(x,y,z) \leqslant 2\, p_\infty(x',y',z') + c\left(T_x(y,z) + T_y(x,z) + T_z(x,y)\right),$$

where

$$T_{z_1}(z_2,z_3) := \frac{\ell_{z_1}^2}{|z_1 - z_2|^2\,|z_1 - z_3|^2} \qquad \text{for} \quad z_1, z_2, z_3 \in \mathbb{C}.$$

Then, integrating on $x \in \widehat{J}_i$, $y \in \widehat{J}_j$, and $z \in \widehat{J}_k$ with respect to σ, we get

$$\begin{aligned} p_\infty\left(\sigma\lfloor\widehat{J}_i, \sigma\lfloor\widehat{J}_j, \sigma\lfloor\widehat{J}_k\right) &\leqslant 2\, p_\infty(x',y',z')\,\sigma(\widehat{J}_i)\,\sigma(\widehat{J}_j)\,\sigma(\widehat{J}_k) \\ &\quad + c \iiint_{\substack{x\in\widehat{J}_i\\ y\in\widehat{J}_j\\ z\in\widehat{J}_k}} \left[T_x(y,z) + T_y(x,z) + T_z(x,y)\right]\, d\sigma(x)\, d\sigma(y)\, d\sigma(z). \end{aligned}$$

On the other hand, by analogous arguments, we have

$$\begin{aligned} p_\infty(x',y',z')\,\|\nu_i\|\,\|\nu_j\|\,\|\nu_k\| &\leqslant 2\, p_\infty(\nu_i,\nu_j,\nu_k) \\ &\quad + c \iiint \left[T_x(y,z) + T_y(x,z) + T_z(x,y)\right]\, d\nu_i(x)\, d\nu_j(y)\, d\nu_k(z). \end{aligned}$$

Thus, recalling that $\|\nu_h\| = \sigma(\widehat{J}_h)$ for any $h \in I_0'$, from the preceding inequalities we get

$$\begin{aligned} p_\infty\left(\sigma\lfloor\widehat{J}_i, \sigma\lfloor\widehat{J}_j, \sigma\lfloor\widehat{J}_k\right) &\lesssim p_\infty(\nu_i,\nu_j,\nu_k) \\ &+ \iiint \left[T_x(y,z) + T_y(x,z) + T_z(x,y)\right]\, d\nu_i(x)\, d\nu_j(y)\, d\nu_k(z) \\ &\quad + \iiint_{\substack{x\in\widehat{J}_i\\ y\in\widehat{J}_j\\ z\in\widehat{J}_k}} \left[T_x(y,z) + T_y(x,z) + T_z(x,y)\right]\, d\sigma(x)\, d\sigma(y)\, d\sigma(z). \end{aligned} \tag{3.58}$$

Now recall that $A_h = \operatorname{spt}\nu_h \subset Q_h$ for any $h \in I_0'$. This and Lemma 3.24 imply that for each $x \in Q_i$ and $y \in Q_j$ there exist $\tilde{x} \in \Gamma_R$ and $\tilde{y} \in \Gamma_R$, correspondingly, such that $\operatorname{dist}(x,\tilde{x}) \lesssim_{\tau,A} \ell(J_i)$ and $\operatorname{dist}(y,\tilde{y}) \lesssim_{\tau,A} \ell(J_j)$. Due to this fact and (3.57), it holds that

$$\measuredangle(L_{x\tilde{y}}, L_{xy}) \lesssim \frac{|y-\tilde{y}|}{|x-y|} \lesssim_{\tau,A} \frac{\ell(J_j)}{\varepsilon_0^{-1/20}\left(\ell(J_i) + \ell(J_j)\right)} \lesssim_{\tau,A} \varepsilon_0^{1/20}$$

and

$$\measuredangle(L_{\tilde{x}y}, L_{xy}) \lesssim \frac{|x-\tilde{x}|}{|x-y|} \lesssim_{\tau,A} \frac{\ell(J_i)}{\varepsilon_0^{-1/20}\left(\ell(J_i) + \ell(J_j)\right)} \lesssim_{\tau,A} \varepsilon_0^{1/20}.$$

So it follows that $\measuredangle(L_{\tilde{x}\tilde{y}}, L_{xy}) \lesssim_{\tau,A} \varepsilon_0^{1/20}$. By Lemma 3.23 and the definitions at the beginning of Subsection 3.12.1,

$$\measuredangle(L_{xy}, L_R) \leqslant C_F\,\theta_0 \qquad \text{and} \qquad \theta_V(L_R) \geqslant (1+C_F)\,\theta_0.$$

Consequently,

$$\theta_V(L_{\tilde{x}\tilde{y}}) \geqslant \theta_V(L_R) - \measuredangle(L_{xy}, L_R) - \measuredangle(L_{\tilde{x}\tilde{y}}, L_{xy}) \geqslant \tfrac{1}{2}\theta_0,$$

if $\varepsilon_0 = \varepsilon_0(\theta_0, \tau, A)$ is chosen small enough. Now use Lemma 3.6 to conclude that

$$p_\infty(\nu_i, \nu_j, \nu_k) \lesssim_{\theta_0} p_0(\nu_i, \nu_j, \nu_k). \tag{3.59}$$

Moreover, from (3.57) and the fact that $\ell(J_h) \approx_{\tau,A} \operatorname{diam}(Q_h)$ for any h we conclude that

$$\begin{aligned} p_0(\nu_i, \nu_j, \nu_k) &\lesssim \int \sum_{Q\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R):\, x\in 2B_Q} p_0^{[\delta,Q]}(x, \nu_j, \nu_k)\, d\nu_i(x) \\ &\lesssim \sum_{Q\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R)} p_0^{[\delta,Q]}(\nu_i\lfloor 2B_Q, \nu_j, \nu_k), \end{aligned}$$

where $\delta = \delta(\varepsilon_0, \tau, A)$ is chosen small enough. Furthermore, using that $\nu = g\mu$ and arguing as in the case of $p_{\infty,\mathbf{F}}(\sigma\lfloor G_R)$ we get

$$\begin{aligned} \sum_{(i,j,k)\in J_{\mathbf{F}}} p_0(\nu_i, \nu_j, \nu_k) &\lesssim \sum_{Q\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R)} p_0^{[\delta,Q]}(\nu\lfloor 2B_Q, \nu, \nu) \\ &\lesssim_{\tau,A} \sum_{Q\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R)} p_0^{[\delta,Q]}(\mu\lfloor 2B_Q, \mu\lfloor 2B_R, \mu\lfloor 2B_R) \\ &\lesssim_{\tau,A} \alpha^2 \Theta_\mu(2B_R)^2 \mu(R) \\ &\lesssim \varepsilon_0^{1/20} \Theta_\mu(2B_R)^2 \mu(R), \end{aligned}$$

where $\alpha = \alpha(\theta_0, \varepsilon_0, \tau, A)$ is chosen small enough. From this, (3.58) and (3.59) by summing on $(i,j,k) \in J_{\mathbf{F}}$ we deduce that

$$\begin{aligned} &\sum_{(i,j,k)\in J_{\mathbf{F}}} p_\infty\left(\sigma\lfloor\widehat{J}_i, \sigma\lfloor\widehat{J}_j, \sigma\lfloor\widehat{J}_k\right) \\ &\quad\lesssim \varepsilon_0^{1/40}\Theta_\mu(2B_R)^2\mu(R) \\ &\qquad + \iiint_{\substack{|x-y|\geqslant \frac{1}{2}\varepsilon_0^{-1/20}(\ell_x+\ell_y)\\ |x-z|\geqslant \frac{1}{2}\varepsilon_0^{-1/20}(\ell_x+\ell_z)\\ |y-z|\geqslant \frac{1}{2}\varepsilon_0^{-1/20}(\ell_y+\ell_z)}} [T_x(y,z) + T_y(x,z) + T_z(x,y)]\, d\sigma(x)\, d\sigma(y)\, d\sigma(z) \\ &\qquad + \iiint_{\substack{|x-y|\geqslant \frac{1}{2}\varepsilon_0^{-1/20}(\ell_x+\ell_y)\\ |x-z|\geqslant \frac{1}{2}\varepsilon_0^{-1/20}(\ell_x+\ell_z)\\ |y-z|\geqslant \frac{1}{2}\varepsilon_0^{-1/20}(\ell_y+\ell_z)}} [T_x(y,z) + T_y(x,z) + T_z(x,y)]\, d\nu(x)\, d\nu(y)\, d\nu(z), \end{aligned} \tag{3.60}$$

where $\varepsilon_0 = \varepsilon_0(\theta_0)$ was chosen small enough. Recall the definition of ν in (3.52).

To estimate the first triple integral in the right side of (3.60), notice that

$$\begin{aligned}
&\iint_{\substack{|x-y|\geqslant\frac{1}{2}\varepsilon_0^{-1/20}(\ell_x+\ell_y)\\ |x-z|\geqslant\frac{1}{2}\varepsilon_0^{-1/20}(\ell_x+\ell_z)}} T_x(y,z)\,d\sigma(y)\,d\sigma(z)\\
&\quad\leqslant \left(\int_{|x-y|\geqslant\frac{1}{2}\varepsilon_0^{-1/20}\ell_x}\frac{\ell_x}{|x-y|^2}\,d\sigma(y)\right)\left(\int_{|x-z|\geqslant\frac{1}{2}\varepsilon_0^{-1/20}\ell_x}\frac{\ell_x}{|x-z|^2}\,d\sigma(z)\right) \qquad (3.61)\\
&\quad= \left(\int_{|x-y|\geqslant\frac{1}{2}\varepsilon_0^{-1/20}\ell_x}\frac{\ell_x}{|x-y|^2}\,d\sigma(y)\right)^2 \lesssim \varepsilon_0^{1/10}\,\Theta_\mu(2B_R)^2,
\end{aligned}$$

where the last inequality follows from splitting the domain $\{y : |x-y| \geqslant \frac{1}{2}\varepsilon_0^{-1/20}\ell_x\}$ into annuli and the linear growth of σ with constant $\lesssim \Theta_\mu(2B_R)$ (see (0.12)). Analogous estimates hold permuting x,y,z, and also interchanging σ by ν (the implicit constant in the analogue of (3.61) for ν depends on τ and A then). Indeed, this is a consequence of the following result.

Lemma 3.43. *It holds that*

$$\nu(B(x,r)) \lesssim_{\tau,A} \Theta_\mu(2B_R)\,r, \quad \textit{where } r\geqslant \ell_x>0 \textit{ and } x\in \operatorname{spt}\nu\subset R\setminus R_{\mathsf{Far}}.$$

Proof. Recall that $\nu = g\mu$ with g bounded by a constant depending in τ and A, see (3.52).

If $r >$ diam R, then $\operatorname{spt}\nu \subset B(x,r)$ and thus

$$\nu(B(x,r)) \lesssim_{\tau,A} \mu(2B_R) \approx_{\tau,A} \Theta_\mu(2B_R)\operatorname{diam}(R) \lesssim_{\tau,A} \Theta_\mu(2B_R)\,r.$$

Consequently, we may suppose below that $\ell_x \leqslant r \leqslant \operatorname{diam}(R)$.

First let $d(x) \leqslant C(\tau,A)\ell_x$, where $C(\tau,A) > 0$ will be chosen later. Then there should exist $P \in \mathsf{DbTree}(R)$ such that $B(x,r) \subset 2B_P$ and $\operatorname{diam}(P) \approx_{\tau,A} r$ so that

$$\nu(B(x,r)) \lesssim_{\tau,A} \mu(B(x,r)) \lesssim_{\tau,A} \mu(2B_P) \approx_{\tau,A} \Theta_\mu(2B_R)\operatorname{diam}(P) \approx_{\tau,A} \Theta_\mu(2B_R)r.$$

Now let $d(x) \geqslant C(\tau,A)\ell_x > 0$. Set $y = (\Pi(x), F(\Pi(x))) \in \Gamma_R$. As shown in the proof of Lemma 3.24, $d(y) \leqslant c(\tau,A)\ell_x$ with some $c(\tau,A) > 0$. Choose $Q' \in \mathsf{DbTree}(R)$ so that

$$\operatorname{dist}(y,Q') + \operatorname{diam}(Q') \leqslant 2d(y).$$

Taking into account that $x \in R\setminus R_{\mathsf{Far}}$, from Lemma 3.26 and the properties of Γ_R we deduce that $\operatorname{dist}(x,y) \lesssim \sqrt[4]{\varepsilon_0}\,d(x) \leqslant \sqrt[8]{\varepsilon_0}\,d(x)$ if ε_0 is chosen small enough. Thus

$$\begin{aligned}
d(x) &\leqslant \operatorname{dist}(x,Q') + \operatorname{diam}(Q') \leqslant \operatorname{dist}(x,y) + 2d(y) \leqslant \sqrt[8]{\varepsilon_0}\,d(x) + 2c(\tau,A)\ell_x\\
&\leqslant \sqrt[8]{\varepsilon_0}\,d(x) + \frac{2c(\tau,A)}{C(\tau,A)}d(x) \leqslant (\sqrt[8]{\varepsilon_0}+\tfrac{1}{2})d(x) < d(x),
\end{aligned}$$

if we choose $C(\tau,A) \geqslant 4c(\tau,A)$. Hence we get a contradiction if $d(x) \geqslant C(\tau,A)\ell_x > 0$. $\square$

By plugging the estimates obtained into (3.60), choosing $\varepsilon_0 = \varepsilon_0(\tau,A)$ small enough and recalling (3.55) we get

$$p_{\infty,\mathbf{F}}(\sigma\lfloor\Gamma_{B_0}\setminus G_R) \lesssim \varepsilon_0^{1/40}\,\Theta_\mu(2B_R)^2\,\mu(R).$$

Now it remains to estimate the last two terms of (3.54). The arguments are similar to the preceding ones.

3. Since $\sigma\lfloor G_R = \nu\lfloor G_R$, we have

$$p_{\infty,\mathbf{F}}(\sigma\lfloor G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R) = p_{\infty,\mathbf{F}}(\nu\lfloor G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R)$$

and

$$p_{\infty,\mathbf{F}}(\sigma\lfloor G_R, \sigma\lfloor G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R) = p_{\infty,\mathbf{F}}(\nu\lfloor G_R, \nu\lfloor G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R).$$

Concerning the term $p_{\infty,\mathbf{F}}(\sigma\lfloor G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R)$, the main difference with respect to the estimates above for $p_{\infty,\mathbf{F}}(\sigma\lfloor \Gamma_{B_0} \setminus G_R)$ is that $T_x(y,z)$ equals zero in this case, and instead of integrating over $\sigma\lfloor \widehat{J}_i$ and ν_i and then summing on i, one integrates over $\sigma\lfloor G_R$. Then one obtains

$$\begin{aligned} p_{\infty,\mathbf{F}}(\sigma\lfloor G_R, &\sigma\lfloor \Gamma_{B_0} \setminus G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R) \\ &\lesssim \varepsilon_0^{1/40} \Theta_\mu(2B_R)^2 \, \mu(R) \\ &\quad + \iiint_{\substack{|x-y| \geqslant \frac{1}{2}\varepsilon_0^{-1/20}\ell_y \\ |x-z| \geqslant \frac{1}{2}\varepsilon_0^{-1/20}\ell_z \\ |y-z| \geqslant \frac{1}{2}\varepsilon_0^{-1/20}(\ell_y+\ell_z)}} [T_y(x,z) + T_z(x,y)] \, d\sigma(x)\, d\sigma(y)\, d\sigma(z) \\ &\quad + \iiint_{\substack{|x-y| \geqslant \frac{1}{2}\varepsilon_0^{-1/20}\ell_y \\ |x-z| \geqslant \frac{1}{2}\varepsilon_0^{-1/20}\ell_z \\ |y-z| \geqslant \frac{1}{2}\varepsilon_0^{-1/20}(\ell_y+\ell_z)}} [T_y(x,z) + T_z(x,y)] \, d\nu(x)\, d\nu(y)\, d\nu(z). \end{aligned}$$

The last two triple integrals are estimated as in (3.61), and then it follows that

$$p_{\infty,\mathbf{F}}(\sigma\lfloor G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R) \lesssim \varepsilon_0^{1/40}\, \Theta_\mu(2B_R)^2 \, \mu(R).$$

4. Finally, the arguments for $p_{\infty,\mathbf{F}}(\sigma\lfloor G_R, \sigma\lfloor G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R)$ are very similar. In this case, both terms $T_x(y,z)$ and $T_y(x,z)$ vanish, and analogously we also get

$$p_{\infty,\mathbf{F}}(\sigma\lfloor G_R, \sigma\lfloor G_R, \sigma\lfloor \Gamma_{B_0} \setminus G_R) \lesssim \varepsilon_0^{1/40}\, \Theta_\mu(2B_R)^2 \, \mu(R).$$

This finishes the proof of Lemma 3.36.

3.13 The packing condition for Top cubes and the end of the proof of Main Lemma

3.13.1 Properties of the trees

In order to prove the packing condition for Top cubes we will first extract some necessary results from Lemmas 3.9, 3.14, 3.29, 3.30 and 3.38. We suppose that all the parameters and thresholds from Section 3.7 are chosen properly. Recall also the definition (3.28) of G_R.

Lemma 3.44. *Let μ be a finite measure with compact support such that*

$$p_0(\mu) < \infty.$$

Considering the David-Mattila dyadic lattice $\mathcal{D}$ associated with μ, let $R \in \mathcal{D}^{db}$. Then there exists a $C_F\theta_0$-Lipschitz function $F : L_R \to L_R^\perp$, where $C_F > 0$ is independent of R, a family of pairwise disjoint cubes $\mathsf{Stop}(R) \subset \mathcal{D}(R)$ and a set $G_R \subset R$ such that

(**a**) *G_R is contained in $\Gamma_R = F(L_R)$ and moreover $\mu\lfloor G_R$ is absolutely continuous with respect to $\Theta_\mu(2B_R)\mathcal{H}^1_{\Gamma_R}$;*

(**b**) *for any* $Q \in \mathsf{Tree}(R)$,

$$\Theta_\mu(2B_Q) \lesssim A\,\Theta_\mu(2B_R);$$

(**c**) *if* $R \in \mathsf{T}_{VF}(\theta_0)$, *then*

$$\sum_{\substack{Q\in \mathsf{Stop}(R)\\ Q\notin \mathsf{HD}(R)\cup\mathsf{UB}(R)}} \mu(Q) \leqslant \sqrt{\tau}\,\mu(R) + \frac{1}{\alpha^2\Theta_\mu(2B_R)^2}\sum_{\tilde{Q}\in\mathsf{Tree}(R)} p_0^{[\delta,\tilde{Q}]}(\mu\lfloor 2B_{\tilde{Q}}, \mu\lfloor 2B_R, \mu\lfloor 2B_R);$$

if $R \notin \mathsf{T}_{VF}(\theta_0)$, *then*

$$\sum_{\substack{Q\in \mathsf{Stop}(R)\\ Q\notin \mathsf{HD}(R)\cup\mathsf{UB}(R)\cup\mathsf{BS}(R)}} \mu(Q) \leqslant \sqrt{\tau}\,\mu(R) + \frac{1}{\alpha^2\Theta_\mu(2B_R)^2}\sum_{\tilde{Q}\in\mathsf{Tree}(R)} p_0^{[\delta,\tilde{Q}]}(\mu\lfloor 2B_{\tilde{Q}}, \mu\lfloor 2B_R, \mu\lfloor 2B_R).$$

3.13.2 New families of stopping cubes

According to Section 3.8 and Lemma 3.44, each $R \in \mathcal{D}^{db}$ generates several families of cubes fulfilling certain properties. In this subsection we will introduce some variants of these families. The idea is to have stopping cubes that are always different from R and are in $\mathcal{D}^{db}$, cf. Remark 3.5.

Recall that each cube in $\mathsf{HD}(R)$ is in $\mathcal{D}^{db}$ and is clearly different from R due to the fact that $Q \in \mathsf{HD}(R)$ satisfies $\Theta_\mu(2B_Q) > A\,\Theta_\mu(2B_R)$ with $A \gg 1$.

Now we turn our attention to the family $\mathsf{UB}(R)$. By Lemma 3.5, if $Q \in \mathsf{UB}(R)$, i.e. it is γ-unbalanced, there exists a family of pairwise disjoint cubes $\{P\}_{P\in I_Q} \subset \mathcal{D}^{db}(Q)$ such that $\operatorname{diam}(P) \gtrsim \gamma \operatorname{diam}(Q)$ and $\Theta_\mu(2B_P) \gtrsim \gamma^{-1}\,\Theta_\mu(2B_Q)$ for each $P \in I_Q$, and

$$\sum_{P\in I_Q} \Theta_\mu(2B_P)^2\,\mu(P) \gtrsim \gamma^{-2}\,\Theta_\mu(2B_Q)^2\,\mu(Q). \tag{3.62}$$

Let I'_Q be a family of (not necessarily doubling) cubes contained in Q, with side length comparable to $a \operatorname{diam}(Q)$ with some $a > 0$, disjoint from the ones from I_Q, so that

$$Q = \bigcup_{P\in I_Q} P \cup \bigcup_{P\in I'_Q} P.$$

To continue, we introduce additional notation. Given a cube $Q \in \mathcal{D}$, we denote by $\mathcal{MD}(Q)$ the family of maximal cubes (with respect to inclusion) from $\mathcal{D}^{db}(Q)$. By Lemma 3.2, this family covers μ-almost all Q. Furthermore, using the definition just given, we denote by $\widetilde{I}_Q$ the family $\bigcup_{P\in I'_Q}\mathcal{MD}(P)$. Moreover, we set

$$\widetilde{\mathsf{UB}}(R) = \bigcup_{Q\in\mathsf{UB}(R)} (I_Q \cup \widetilde{I}_Q).$$

One can deduce from (3.62) that $R \notin \widetilde{\mathsf{UB}}(R)$ for a and γ small enough.

Now consider $\mathsf{BS}(R)$. Each cube in this family is in $\mathcal{D}^{db}$ by construction. Moreover, $R \notin \mathsf{BS}(R)$ due to the condition $\measuredangle(L_Q, L_R) > \theta(R) > 0$ for each $Q \in \mathsf{BS}(R)$.

To continue, we set

$$\mathsf{O}(R) = \mathsf{Stop}(R) \setminus (\mathsf{HD}(R) \cup \mathsf{UB}(R) \cup \mathsf{BS}(R)) = \mathsf{LD}(R) \cup \mathsf{BP}(R) \cup \mathsf{F}(R)$$

and

$$\widetilde{\mathsf{O}}(R) = \left\{ \bigcup_{Q\in\mathcal{D}} \mathcal{MD}(Q) : Q \text{ is a son of some cube from } \mathsf{O}(R) \right\}.$$

This guarantees that $R \notin \tilde{\mathsf{O}}(R)$ as cubes in $\tilde{\mathsf{O}}(R)$ are descendants of cubes in $\mathsf{Tree}(R)$.

Finally, let

$$\mathsf{Next}(R) = \mathsf{HD}(R) \cup \widetilde{\mathsf{UB}}(R) \cup \widetilde{\mathsf{O}}(R) \cup \mathsf{BS}(R).$$

By construction, all cubes in $\mathsf{Next}(R)$ are disjoint, doubling and different from R. Moreover,

$$R \setminus \bigcup_{Q\in\mathsf{Next}(R)} Q = R \setminus \bigcup_{Q\in\mathsf{Stop}(R)} Q. \tag{3.63}$$

Using the small boundaries property of the David-Mattila lattice and the definition (3.28), one can also show that

$$\mu\left(R \setminus \bigcup_{Q\in\mathsf{Stop}(R)} Q\right) = \mu(G_R). \tag{3.64}$$

For the record, notice also that, by construction, if $P \in \mathsf{Next}(R)$, then

$$\Theta_\mu(2B_S) \lesssim_{\tau,A} \Theta_\mu(2B_R) \quad \text{for all } S \in \mathcal{D} \text{ such that } P \subset S \subset R. \tag{3.65}$$

3.13.3 The corona decomposition

Recall that we assumed that μ has compact support. Let

$$R_0 := \operatorname{spt}\mu.$$

Obviously we may suppose that $R_0 \in \mathcal{D}^{db}$. We will construct the family Top contained in R_0 inductively applying Lemma 3.44 so that $\mathsf{Top} = \bigcup_{k\geqslant 0} \mathsf{Top}_k$. Let

$$\mathsf{Top}_0 = \{R_0\}.$$

Assuming Top_k to be defined, we set

$$\mathsf{Top}_{k+1} = \bigcup_{R\in\mathsf{Top}_k} \mathsf{Next}(R).$$

Note that cubes in $\mathsf{Next}(R)$, with $R \in \mathsf{Top}_k$, are pairwise disjoint.

3.13.4 The families of cubes ID_H, ID_U and ID

We distinguish two types of cubes $R \in \mathsf{Top}$. We write $R \in ID_H$ (increasing density because of high density cubes) if

$$\mu\Big(\bigcup_{Q\in\mathsf{HD}(R)} Q\Big) \geqslant \frac{1}{4}\,\mu(R).$$

Also, we write $R \in ID_U$ (increasing density because of unbalanced cubes) if

$$\mu\Big(\bigcup_{Q\in\widetilde{\mathsf{UB}}(R)} Q\Big) \geqslant \frac{1}{4}\,\mu(R).$$

Additionally, let

$$ID = ID_H \cup ID_U.$$

Lemma 3.45 (Lemma 5.4 and its proof in [AT]). *If $R \in ID$, then*

$$\Theta_\mu(2B_R)^2\,\mu(R) \lesssim \frac{1}{A^2} \sum_{Q\in \mathsf{HD}(R)} \Theta_\mu(2B_Q)^2\,\mu(Q)$$

$$\Theta_\mu(2B_R)^2\,\mu(R) \lesssim \frac{\gamma^2}{\tau^2} \sum_{Q\in \widetilde{\mathsf{UB}}(R)} \Theta_\mu(2B_Q)^2\,\mu(Q).$$

Moreover, if A is such that $A^{-1} \leqslant \tau^2$ and $\gamma \leqslant \tau^3$, then

$$\Theta_\mu(2B_R)^2\,\mu(R) \leqslant c\tau^4 \sum_{Q\in \mathsf{Next}(R)} \Theta_\mu(2B_Q)^2\,\mu(Q),$$

where $c > 0$ is some absolute constant.

3.13.5 The packing condition

Recall that we assume linear growth of μ, i.e.

$$\mu(B(x,r)) \leqslant C_* r \qquad \forall x \in \operatorname{spt}\mu, \quad r > 0, \tag{3.66}$$

for some constant $C_* > 0$ (see (0.12)). Using this assumption, we will prove the following.

Lemma 3.46. *If the parameters and thresholds in Section 3.7 are chosen properly, then*

$$\sum_{R\in \mathsf{Top}} \Theta_\mu(2B_R)^2\,\mu(R) \leqslant \mathsf{c}_5\, p_0(\mu) + c\, C_*^2\, \mu(\mathbb{C}), \tag{3.67}$$

where $\mathsf{c}_5 = \mathsf{c}_5(\tau, A, \theta_0, \gamma, \varepsilon_0, \alpha, \delta) > 0$ and $c > 0$.

Proof. For a given $k \geqslant 0$, we set $\mathsf{Top}_0^k = \bigcup_{0\leqslant j\leqslant k} \mathsf{Top}_j$ and $ID_0^k = ID \cap \mathsf{Top}_0^k$.

To prove (3.67), first we deal with the cubes from the ID family. By Lemma 3.45,

$$\begin{aligned}\sum_{R\in ID_0^k} \Theta_\mu(2B_R)^2\,\mu(R) &\leqslant c\tau^2 \sum_{R\in ID_0^k}\ \sum_{Q\in \mathsf{Next}(R)} \Theta_\mu(2B_Q)^2\,\mu(Q)\\ &\leqslant c\tau^2 \sum_{R\in \mathsf{Top}_0^{k+1}} \Theta_\mu(2B_R)^2\,\mu(R),\end{aligned}$$

because the cubes from $\mathsf{Next}(R)$ with $R \in \mathsf{Top}_0^k$ belong to Top_0^{k+1}. So we have

$$\begin{aligned}&\sum_{R\in \mathsf{Top}_0^k} \Theta_\mu(2B_R)^2\,\mu(R)\\ &= \sum_{R\in \mathsf{Top}_0^k\setminus ID_0^k} \Theta_\mu(2B_R)^2\,\mu(R) + \sum_{R\in ID_0^k} \Theta_\mu(2B_R)^2\,\mu(R)\\ &\leqslant \sum_{R\in \mathsf{Top}_0^k\setminus ID_0^k} \Theta_\mu(2B_R)^2\,\mu(R) + c\tau^2 \sum_{R\in \mathsf{Top}_0^k} \Theta_\mu(2B_R)^2\,\mu(R) + c\tau^2 \sum_{R\in \mathsf{Top}_{k+1}} \Theta_\mu(2B_R)^2\,\mu(R)\\ &\leqslant \sum_{R\in \mathsf{Top}_0^k\setminus ID_0^k} \Theta_\mu(2B_R)^2\,\mu(R) + c\tau^2 \sum_{R\in \mathsf{Top}_0^k} \Theta_\mu(2B_R)^2\,\mu(R) + c\,\tau^2\, C_*^2 \mu(R_0),\end{aligned}$$

where we took into account that $\Theta_\mu(2B_R) \lesssim C_*$ for every $R \in \mathsf{Top}$ (and in particular for all $R \in \mathsf{Top}_{k+1}$). So, having τ small enough, we deduce that

$$\sum_{R\in\mathsf{Top}_0^k} \Theta_\mu(2B_R)^2\,\mu(R) \leqslant 1.1 \sum_{R\in\mathsf{Top}_0^k\setminus ID_0^k} \Theta_\mu(2B_R)^2\,\mu(R) + c\tau^2\,C_*^2\mu(R_0). \tag{3.68}$$

Let us estimate the first term in the right hand side of (3.68). First note that

$$\mu\Bigg(R\setminus \bigcup_{Q\in\mathsf{HD}(R)\cup\widetilde{\mathsf{UB}}(R)} Q\Bigg) \geqslant \frac{1}{2}\,\mu(R) \qquad \text{for any} R\in\mathsf{Top}_0^k\setminus ID_0^k.$$

Next, by applying the inequalities (c) in Lemma 3.44 and recalling (3.63) and (3.64) we get

$$\begin{aligned}
\mu(R) &\leqslant 2\,\mu\Bigg(R\setminus \bigcup_{Q\in\mathsf{Next}(R)} Q\Bigg) + 2\,\mu\Bigg(\bigcup_{Q\in\widetilde{\mathsf{O}}(R)\cup\mathsf{BS}(R)} Q\Bigg)\\
&\leqslant 2\,\mu(G_R) + 2\mu\Bigg(\bigcup_{Q\in\widetilde{\mathsf{O}}(R)} Q\Bigg) + 2\sum_{\substack{Q\in\mathsf{BS}(R)\\ (\text{if } R\in\mathsf{T}_{VF}(\theta_0))}} \mu(Q) + 2\sum_{\substack{Q\in\mathsf{BS}(R)\\ (\text{if } R\notin\mathsf{T}_{VF}(\theta_0))}} \mu(Q)\\
&\leqslant 2\,\mu(G_R) + 2\sqrt{\tau}\,\mu(R) + 2\sum_{\substack{Q\in\mathsf{BS}(R)\\ (\text{if } R\notin\mathsf{T}_{VF}(\theta_0))}} \mu(Q)\\
&\quad + \frac{2\alpha^{-2}}{\Theta_\mu(2B_R)^2}\sum_{Q\in\mathsf{Tree}(R)} p_0^{[\delta,Q]}(\mu\lfloor 2B_Q, \mu\lfloor 2B_R, \mu\lfloor 2B_R).
\end{aligned}$$

Suppose that τ is small enough to get

$$\begin{aligned}
\mu(R) \leqslant 2.1\,\mu(G_R) &+ 2.1\sum_{\substack{Q\in\mathsf{BS}(R)\\ (\text{if } R\notin\mathsf{T}_{VF}(\theta_0))}} \mu(Q)\\
&+ \frac{2.1\alpha^{-2}}{\Theta_\mu(2B_R)^2}\sum_{Q\in\mathsf{Tree}(R)} p_0^{[\delta,Q]}(\mu\lfloor 2B_Q, \mu\lfloor 2B_R, \mu\lfloor 2B_R).
\end{aligned}$$

So we deduce from (3.68) that

$$\begin{aligned}
&\sum_{R\in\mathsf{Top}_0^k} \Theta_\mu(2B_R)^2\,\mu(R)\\
&\qquad\leqslant 3\sum_{R\in\mathsf{Top}_0^k\setminus ID_0^k} \Theta_\mu(2B_R)^2\,\mu(G_R) \qquad\qquad (3.69)\\
&\qquad\quad + \frac{3}{\alpha^2}\sum_{R\in\mathsf{Top}_0^k}\sum_{Q\in\mathsf{Tree}(R)} p_0^{[\delta,Q]}(\mu\lfloor 2B_Q, \mu\lfloor 2B_R, \mu\lfloor 2B_R)\\
&\qquad\quad + 3\sum_{R\in\mathsf{Top}_0^k\setminus(ID_0^k\cup\mathsf{T}_{VF}(\theta_0))} \Theta_\mu(2B_R)^2\sum_{Q\in\mathsf{BS}(R)} \mu(Q)\\
&\qquad\quad + c\tau^2\,C_*^2\mu(R_0).
\end{aligned}$$

In order to deal with the first sum on the right hand side we take into account that $\Theta_\mu(2B_R) \lesssim C_*$ for all $R \in \mathsf{Top}$ by (3.66) and that the sets G_R with $R \in \mathsf{Top}$ are pairwise disjoint. Then we get

$$\sum_{R\in\mathsf{Top}_0^k\setminus ID_0^k} \Theta_\mu(2B_R)^2\,\mu(G_R) \leqslant c\,C_*^2\,\mu(R_0).$$

On the other hand, the double sum in (3.69) does not exceed

$$2\sum_{Q\in\mathcal{D}} p_0^{[\delta,Q]}(\mu\lfloor 2B_Q,\mu\lfloor 2B_R,\mu\lfloor 2B_R) \leqslant c(\delta)\,p_0(\mu),$$

by the finite superposition of the domains of integration. Recall that $\delta=\delta(\gamma,\varepsilon_0)$. So we obtain

$$\begin{aligned}\sum_{R\in\mathsf{Top}_0^k} \Theta_\mu(2B_R)^2\,\mu(R) \leqslant & c\,C_*^2\,\mu(R_0)+c(\tau,A,\gamma,\varepsilon_0,\alpha)\,p_0(\mu) \\ &+c\sum_{R\in\mathsf{Top}_0^k\setminus(ID_0^k\cup\mathsf{T}_{VF}(\theta_0))}\Theta_\mu(2B_R)^2\sum_{Q\in\mathsf{BS}(R)}\mu(Q).\end{aligned}\tag{3.70}$$

The third term in (3.70) without the constant may be written as the sum

$$\sum_{R\in\mathsf{Top}_0^k\setminus(ID_0^k\cup\mathsf{T}_{VF}(\theta_0))} \Theta_\mu(2B_R)^2(S_1(R)+S_2(R)),$$

where

$$S_1(R)=\sum_{Q\in\mathsf{BS}(R)\cap\mathsf{T}_{VF}(\theta_0)\setminus ID_0^{k+1}}\mu(Q)\quad\text{and}\quad S_2(R)=\sum_{Q\in\mathsf{BS}(R)\cap\mathsf{T}_{VF}(\theta_0)\cap ID_0^{k+1}}\mu(Q).$$

Note that we have the intersection with $\mathsf{T}_{VF}(\theta_0)$ in these sums. This is so because for any $Q\in\mathsf{BS}(R)$, where $R\in\mathsf{Top}\setminus\mathsf{T}_{VF}(\theta_0)$, it holds that

$$\theta_V(L_Q)\geqslant\measuredangle(L_Q,L_R)-\theta_V(L_R)\geqslant 2(1+C_F)\theta_0-(1+C_F)\theta_0=(1+C_F)\theta_0,$$

and thus $Q\in\mathsf{T}_{VF}(\theta_0)$.

Let us estimate $S_1(R)$. Since $Q\in\mathsf{T}_{VF}(\theta_0)\setminus ID_0^{k+1}$, we deduce from (c) in Lemma 3.44 that

$$\begin{aligned}\mu(Q)&\leqslant 2\,\mu\Big(Q\setminus\bigcup_{P\in\mathsf{Next}(Q)}P\Big)+2\,\mu\Big(\bigcup_{P\in\widetilde{\mathsf{O}}(Q)\cup\mathsf{BS}(Q)}P\Big)\\ &\leqslant 2\,\mu(G_Q)+2\mu\Big(\bigcup_{Q\in\widetilde{\mathsf{O}}(R)}Q\Big)+2\sum_{\substack{P\in\mathsf{BS}(Q)\\ (\text{if } Q\in\mathsf{T}_{VF}(\theta_0))}}\mu(P)\\ &\leqslant 2\,\mu(G_Q)+2\sqrt{\tau}\,\mu(Q)+\frac{2\alpha^{-2}}{\Theta_\mu(2B_Q)^2}\sum_{P\in\mathsf{Tree}(Q)}p_0^{[\delta,P]}(\mu\lfloor 2B_P,\mu\lfloor 2B_Q,\mu\lfloor 2B_Q).\end{aligned}$$

If τ is small enough, then

$$\mu(Q)\leqslant 2.1\,\mu(G_Q)+\frac{2.1\alpha^{-2}}{\Theta_\mu(2B_Q)^2}\sum_{P\in\mathsf{Tree}(Q)}p_0^{[\delta,P]}(\mu\lfloor 2B_P,\mu\lfloor 2B_Q,\mu\lfloor 2B_Q).$$

Recall that $\mathsf{BS}(R) \cap \mathsf{T}_{VF}(\theta_0) \setminus ID_0^{k+1} \subset \mathsf{Next}(R)$. So we deduce that

$$S_1(R) \leqslant 2.1 \sum_{Q \in \mathsf{Next}(R)} \Bigg(\mu(G_R) + \frac{\alpha^{-2}}{\Theta_\mu(2B_R)^2} \sum_{P \in \mathsf{Tree}(Q)} p_0^{[\delta,P]}(\mu\lfloor 2B_P, \mu\lfloor 2B_Q, \mu\lfloor 2B_Q) \Bigg).$$

Consequently, using that $\Theta_\mu(2B_R) \lesssim C_*$, we obtain

$$\begin{aligned} \sum_{R \in \mathsf{Top}_0^k \setminus (ID_0^k \cup \mathsf{T}_{VF}(\theta_0))} & \Theta_\mu(2B_R)^2 S_1(R) \\ & \leqslant c\, C_*^2 \sum_{R \in \mathsf{Top}_0^k} \sum_{Q \in \mathsf{Next}(R)} \mu(G_Q) \\ & \quad + \frac{c}{\alpha^2} \sum_{R \in \mathsf{Top}_0^k \setminus (ID_0^k \cup \mathsf{T}_{VF}(\theta_0))} \sum_{Q \in \mathsf{Next}(R)} \sum_{P \in \mathsf{Tree}(Q)} p_0^{[\delta,P]}(\mu\lfloor 2B_P, \mu\lfloor 2B_Q, \mu\lfloor 2B_Q) \\ & \leqslant c\, C_*^2 \sum_{R \in \mathsf{Top}_0^{k+1}} \mu(G_R) + \frac{c}{\alpha^2} \sum_{R \in \mathsf{Top}_0^{k+1}} \sum_{P \in \mathsf{Tree}(R)} p_0^{[\delta,P]}(\mu\lfloor 2B_P, \mu\lfloor 2B_R, \mu\lfloor 2B_R). \end{aligned}$$

Take into account that the sets G_R with $R \in \mathsf{Top}$ are disjoint and that the last (double) sum is controlled by $c(\delta)\, p_0(\mu)$ by the finite superposition of the domains of integration. So we have

$$\sum_{R \in \mathsf{Top}_0^k \setminus (ID_0^k \cup \mathsf{T}_{VF}(\theta_0))} \Theta_\mu(2B_R)^2 S_1(R) \leqslant c\, C_*^2 \mu(R_0) + c(\tau, A, \delta, \alpha)\, p_0(\mu).$$

Now we estimate $S_2(R)$. Since $\mathsf{BS}(R) \cap \mathsf{LD}(R) = \varnothing$, for each $Q \in \mathsf{BS}(R)$ we have $\Theta_\mu(2B_Q) \geqslant \tau \Theta_\mu(2B_R)$ and thus

$$S_2(R) \leqslant \frac{1}{\tau^2 \Theta_\mu(2B_R)^2} \sum_{Q \in \mathsf{BS}(R) \cap \mathsf{T}_{VF}(\theta_0) \cap ID_0^{k+1}} \Theta_\mu(2B_Q)^2 \mu(Q).$$

Since $Q \in ID_0^{k+1}$, by Lemma 3.45,

$$\begin{aligned} S_2(R) & \leqslant \frac{1}{\tau^2 \Theta_\mu(2B_R)^2} \sum_{Q \in \mathsf{BS}(R) \cap \mathsf{T}_{VF}(\theta_0) \cap ID_0^{k+1}} c\tau^4 \sum_{P \in \mathsf{Next}(Q)} \Theta_\mu(2B_P)^2 \mu(P) \\ & \leqslant \frac{c\tau^2}{\Theta_\mu(2B_R)^2} \sum_{Q \in \mathsf{BS}(R) \cap \mathsf{T}_{VF}(\theta_0) \cap ID_0^{k+1}} \sum_{P \in \mathsf{Next}(Q)} \Theta_\mu(2B_P)^2 \mu(P). \end{aligned}$$

Consequently, taking into account that $\mathsf{BS}(R) \cap \mathsf{T}_{VF}(\theta_0) \cap ID_0^{k+1} \subset \mathsf{Next}(R)$ and $\mathsf{Top}_0^k \setminus (ID_0^k \cup \mathsf{T}_{VF}(\theta_0)) \subset \mathsf{Top}_0^k$, we obtain

$$\begin{aligned} \sum_{R \in \mathsf{Top}_0^k \setminus (ID_0^k \cup \mathsf{T}_{VF}(\theta_0))} \Theta_\mu(2B_R)^2 S_2(R) & \leqslant c\tau^2 \sum_{R \in \mathsf{Top}_0^k} \sum_{Q \in \mathsf{Next}(R)} \sum_{P \in \mathsf{Next}(Q)} \Theta_\mu(2B_P)^2 \mu(P) \\ & \leqslant c\tau^2 \sum_{R \in \mathsf{Top}_0^{k+2}} \Theta_\mu(2B_R)^2 \mu(R) \\ & \leqslant c\tau^2 \sum_{R \in \mathsf{Top}_0^k} \Theta_\mu(2B_R)^2 \mu(R) + c\tau^2 C_*^2 \mu(R_0). \end{aligned}$$

Coming back to (3.70), we deduce that

$$\sum_{R\in\mathsf{Top}_0^k}\Theta_\mu(2B_R)^2\,\mu(R)$$
$$\leqslant c\,C_*^2\,\mu(R_0)+c(\tau,A,\delta,\alpha)\,p_0(\mu)+c\tau^2\sum_{R\in\mathsf{Top}_0^k}\Theta_\mu(2B_R)^2\mu(R).$$

Choosing τ small enough and recalling the information in Section 3.7 yield

$$\sum_{R\in\mathsf{Top}_0^k}\Theta_\mu(2B_R)^2\,\mu(R)\leqslant \mathsf{c}_5 p_0(\mu)+c\,C_*^2\,\mu(R_0),$$

where c_5 actually depends on all the parameters and thresholds mentioned in Section 3.7.

Letting $k\to\infty$ finishes the proof of Lemma 3.46. □

3.13.6 The end of the proof of Main Lemma

We first prove an additional property. For $Q,\tilde{Q}\in\mathcal{D}$ with $Q\subset\tilde{Q}$, define

$$\delta_\mu(Q,\tilde{Q})=\int_{2B_{\tilde{Q}}\setminus 2B_Q}\frac{1}{|y-z_Q|}d\mu(y),$$

where z_Q is the center of B_Q, see Lemma 3.1. Then the following statement holds.

Lemma 3.47. *For all $Q\in\mathsf{Next}(R)$ there exists a cube $\tilde{Q}\in\mathsf{DbTree}(R)$ such that $\delta_\mu(Q,\tilde{Q})\lesssim_{\tau,A}\Theta_\mu(2B_R)$ and $2B_{\tilde{Q}}\cap\Gamma_R\neq\varnothing$.*

Proof. Take $Q'\supset Q$ such that $Q'\in\mathsf{Stop}(R)$. By Lemma 3.11, there exists $\tilde{Q}\in\mathsf{DbTree}(R)$ such that $Q'\subset\tilde{Q}$ and $r(Q')\approx_{\tau,A} r(\tilde{Q})$. Moreover, one can easily deduce from Lemma 3.26 that $2B_{\tilde{Q}}\cap\Gamma_R\neq\varnothing$ if ε_0 is small enough (since $\tilde{Q}\in\mathsf{DbTree}(R)$, there is $x\in\tilde{Q}\setminus R_{\mathsf{Far}}$).

Furthermore, split

$$\delta_\mu(Q,\tilde{Q})=\int_{2B_{\tilde{Q}}\setminus 2B_{Q'}}\frac{1}{|y-z_Q|}d\mu(y)+\int_{2B_{Q'}\setminus 2B_Q}\frac{1}{|y-z_Q|}d\mu(y).$$

In the first integral we have $|y-z_Q|\gtrsim r(Q')\approx_{\tau,A} r(\tilde{Q})$ as $y\notin 2B_{Q'}$ and therefore

$$\int_{2B_{\tilde{Q}}\setminus 2B_{Q'}}\frac{1}{|y-z_Q|}d\mu(y)\lesssim_{\tau,A}\Theta_\mu(2B_{\tilde{Q}})\lesssim_{\tau,A}\Theta_\mu(2B_R),$$

where we also used the right hand side inequality in (3.15) in Lemma 3.9. To estimate the second integral we take into account that by construction there are no doubling cubes strictly between Q and Q'. This together with Lemma 3.4 and properties of Q' and $\tilde{Q}$ imply by standard estimates (in particular, splitting the domain of integration into annuli with respect to the intermediate cubes between Q and Q') that

$$\int\limits_{2B_{Q'}\setminus 2B_Q}\frac{1}{|y-z_Q|}d\mu(y)\lesssim\Theta_\mu(100B(Q'))\lesssim_{\tau,A}\Theta_\mu(2B_{\tilde{Q}})\lesssim_{\tau,A}\Theta_\mu(2B_R).$$

Thus we are done. □

Lemma 3.44, Lemma 3.47 and the property (3.65) allow us to use arguments as in [Tol6, Lemma 17.6] in order to show that if $\mu(B(x,r)) \leqslant C_* r$ for all $x \in \mathbb{C}$, then

$$c^2(\mu) \lesssim \sum_{R \in \mathsf{Top}} \Theta_\mu(2B_R)^2\, \mu(R)$$

for our family Top. By combining this estimate and the identity (0.15) with Lemma 3.46 for fixed suitable parameters from Section 3.7, we obtain

$$p_\infty(\mu) \lesssim p_0(\mu) + C_*^2 \mu(\mathbb{C})$$

as wished.

3.14 The case of curvature

Here we come back to the notion of curvature $c^2(\mu)$. Recall that $p_\infty(\mu) = \frac{1}{4} c^2(\mu)$ as has been already mentioned.

It is easy to see that one can exchange p_0 for c^2 in the stopping conditions. Then we can prove the following analogue of Lemma 3.46 by the arguments used above.

Lemma 3.48. *If the parameters and thresholds in Section 3.7 are chosen properly, then*

$$\sum_{R \in \mathsf{Top}} \Theta_\mu(2B_R)^2\, \mu(R) \leqslant \mathsf{c}_6\, c^2(\mu) + c\, C_*^2\, \mu(\mathbb{C}), \tag{3.71}$$

where $\mathsf{c}_6 = \mathsf{c}_6(\tau, A, \theta_0, \gamma, \varepsilon_0, \alpha, \delta) > 0$ *and* $c > 0$.

A more direct way to prove it is to use Lemma 3.46 and the inequality (2.5).

Now recall the following theorem from [AT]:

If μ is a finite compactly supported measure such that $\mu(B(x,r)) \leqslant r$ for all $x \in \mathbb{C}$ and $r > 0$, then

$$c^2(\mu) + \mu(\mathbb{C}) \approx \iint_0^\infty \beta_{\mu,2}(x,r)^2\, \Theta_\mu(x,r)\, \frac{dr}{r}\, d\mu(x) + \mu(\mathbb{C}), \tag{3.72}$$

where the implicit constants are absolute.

Note that the part $\lesssim$ of (3.72) was proved in [AT] by means of the David-Mattila lattice and a corona type construction similar to the one we considered in this chapter. However, the part $\gtrsim$ was proved in [AT] by the corona decomposition of [Tol3] that involved the usual dyadic lattice $\mathcal{D}(\mathbb{C})$, instead of the David-Mattila lattice $\mathcal{D}$.

Using Lemma 3.48 we can also prove the part $\gtrsim$ of (3.72) using only the David-Mattila lattice and an associated corona type construction and thus unify the approach with the proof of the part $\lesssim$ in [AT]. As predicted in [Tol6, Section 19], this indeed simplifies some of the technical difficulties arising from the lack of a well adapted dyadic lattice to the measure μ in [Tol3].

Clearly, we need to show that

$$\iint_0^\infty \beta_{\mu,2}(x,r)^2\, \Theta_\mu(x,r)\, \frac{dr}{r}\, d\mu(x) \lesssim c^2(\mu) + \mu(\mathbb{C})$$

or, equivalently, in a discrete form that

$$\sum_{Q \in \mathcal{D}} \beta_{\mu,2}(2B_Q)^2\, \Theta_\mu(2B_Q)\, \mu(Q) \lesssim c^2(\mu) + \mu(\mathbb{C}). \tag{3.73}$$

By the packing condition (3.71) for $C_* = 1$, to prove (3.73) it suffices to show that for every $R \in \mathsf{Top}$ the following estimate holds true:

$$S(R) = \sum_{Q \in \widetilde{\mathsf{Tree}}(R)} \beta_{\mu,2}(2B_Q)^2 \, \Theta_\mu(2B_Q) \, \mu(Q) \lesssim \Theta_\mu(2B_R)^2 \mu(R),$$

where $\widetilde{\mathsf{Tree}}(R)$ contains cubes in R not strictly contained in $\widetilde{\mathsf{Stop}}(R)$. By $\mathsf{St}(R)$ we denote cubes in $\mathsf{Stop}(R)$ not strictly contained in $\widetilde{\mathsf{Stop}}(R)$. Obviously, $\beta_{\mu,2}(2B_Q)^2 \leqslant 4\Theta_\mu(2B_Q)$ for any $Q \in \widetilde{\mathsf{Tree}}(R)$ and therefore

$$S(R) \leqslant \sum_{Q \in \mathsf{Tree}(R) \setminus \mathsf{Stop}(R)} \beta_{\mu,2}(2B_Q)^2 \, \Theta_\mu(2B_Q) \, \mu(Q) + \sum_{Q \in \mathsf{St}(R)} \Theta_\mu(2B_Q)^2 \, \mu(Q).$$

By Lemma 3.4, the density of all intermediate cubes between $\widetilde{\mathsf{Stop}}(R)$ and $\mathsf{Stop}(R)$, i.e. cubes in $\mathsf{St}(R)$, is controlled by the density of cubes from $\mathsf{Stop}(R)$ so it can be shown that

$$\sum_{P \in \mathsf{St}(R)} \Theta_\mu(2B_P)^2 \, \mu(P) \lesssim \sum_{Q \in \mathsf{Stop}(R)} \Theta_\mu(2B_Q)^2 \, \mu(Q).$$

Moreover,

$$\sum_{Q \in \mathsf{Stop}(Q)} \Theta_\mu(2B_Q)^2 \, \mu(Q) \lesssim_A \Theta_\mu(2B_R)^2 \sum_{Q \in \mathsf{Stop}(R)} \mu(Q) \lesssim_A \Theta_\mu(2B_R)^2 \mu(R),$$

as cubes in $\mathsf{Stop}(R)$ are disjoint subsets of R.

What is more, arguments similar to those in Lemmas 3.7 and 3.12 imply that

$$\begin{aligned}
&\sum_{Q \in \mathsf{Tree}(R) \setminus \mathsf{Stop}(R)} \beta_{\mu,2}(2B_Q)^2 \, \Theta_\mu(2B_Q) \, \mu(Q) \\
&\qquad \lesssim_\gamma \Theta_\mu(2B_R)^2 \sum_{Q \in \mathsf{Tree}(R) \setminus \mathsf{Stop}(R)} \frac{c^2_{[\delta,Q]}(\mu\lfloor 2B_Q)}{\Theta_\mu(2B_R)^2} \\
&\qquad \lesssim_{\alpha,\gamma} \Theta_\mu(2B_R)^2 \mu(R).
\end{aligned}$$

Thus, $S_R \lesssim_{\gamma,\alpha,A} \Theta_\mu(2B_R)^2 \mu(R)$, where γ, α and A depend on other parameters and thresholds and are suitably chosen and fixed at the end.

3.15 Further generalisations

Now consider the kernels defined in (0.24), i.e.

$$K_t(z) = \kappa_N(z) + t \cdot \kappa_n(z), \quad t \in \mathbb{R}, \qquad K_\infty(z) = \kappa_n(z), \quad n \leqslant N, \; n, N \in \mathbb{N},$$

where, according to (0.16),

$$\kappa_m(z) = \frac{(\operatorname{Re} z)^{2m-1}}{|z|^{2m}}, \qquad m \in \mathbb{N}.$$

Let us show that results similar to Theorems 3.1 and 3.2 also hold for them.

First of all, we recall that

$$p_{\kappa_n}(z_1, z_2, z_3) \leqslant C(n) \cdot c(z_1, z_2, z_3)^2 \quad \text{for all } (z_1, z_2, z_3) \in \mathbb{C}^2 \tag{3.74}$$

with some absolute constant $C(n) > 0$ as proved in [CMPT2, Lemma 7] (see also Remark 2.3 in Chapter 2). The inequality (3.74) and identity (0.15) readily imply that

$$p_{\kappa_n}(\mu) \leqslant C(n) \cdot p_{k_\infty}(\mu) \tag{3.75}$$

for any measure μ. Recall that $p_{k_\infty} \equiv p_{\kappa_1}$ by definition.

Furthermore, Lemma 3.6 that we used in the proof (see Section 3.6) may be exchanged for the following more general result.

Lemma 3.49 (Proposition 3.3 in [CP])**.** *If* $(z_1, z_2, z_3) \in \mathrm{V}_{\mathsf{Far}}(\theta)$*, then*

$$p_{\kappa_N}(z_1, z_2, z_3) \geqslant C(N, \theta) \cdot p_{k_\infty}(z_1, z_2, z_3) \quad \textit{for some } C(N, \theta) > 0. \tag{3.76}$$

Using this lemma, one can easily replace the permutations $p_{k_0} \equiv p_{\kappa_2}$ in the stopping conditions in Sections 3.8.1 with the permutations p_{κ_N}. Then the same arguments lead to the following analogue of Main Lemma for the kernels κ_N.

Lemma 3.50. *There exist absolute constants $C(N) > 0$ and $c > 0$ such that for any finite measure μ with C_*-linear growth it holds that*

$$p_{k_\infty}(\mu) \leqslant C(N) \cdot p_{\kappa_N}(\mu) + cC_*^2 \mu(\mathbb{C}). \tag{3.77}$$

Under the assumptions of Lemma 3.50, we infer from (3.75) and (3.77) that

$$p_{\kappa_n}(\mu) \leqslant C(n, N) \cdot p_{\kappa_N}(\mu) + cC_*^2 \mu(\mathbb{C}), \qquad n \leqslant N,$$

with some $C(n, N) > 0$. This and the perturbation method yield the following result.

Theorem 3.3. *There exist constants $\tilde{t}_0 = \tilde{t}_0(n, N) > 0$ and $c > 0$ such that for any finite measure μ with C_*-linear growth it holds that*

$$\sup_{\varepsilon > 0} \|T_{\kappa_n, \varepsilon} 1\|_{L^2(\mu)} \leqslant \tilde{t}_0^{-1} \sup_{\varepsilon > 0} \|T_{\kappa_N, \varepsilon} 1\|_{L^2(\mu)} + cC_* \sqrt{\mu(\mathbb{C})}. \tag{3.78}$$

As a corollary, we get the following theorems with the same $\tilde{t}_0$ as above.

Theorem 3.4. *Let K_t be a kernel of the form* (0.24)*, where $t \in (-\tilde{t}_0, \tilde{t}_0)$, and E a 1-set. If the SIO T_{K_t} is $L^2(\mathcal{H}^1 \lfloor E)$-bounded, then E is rectifiable.*

Theorem 3.5. *Let μ be an AD-regular measure and K_t a kernel of the form* (0.24)*, where $t \in (-\tilde{t}_0, \tilde{t}_0)$. The measure μ is uniformly rectifiable if and only if the SIO T_{K_t} is $L^2(\mu)$-bounded.*

Note that $\tilde{t}_0 \in (0, 1)$ as follows from Remark 1.2, see also Figure 1.1.

Theorems 3.4 and 3.5 complement Theorems 1.2, 2.2 and 2.3 dealing with the kernels K_t. It is important that the pointwise permutations p_{K_t} corresponding to $t \in (-\tilde{t}_0, \tilde{t}_0)$ change sign for n and N from Remark 1.1 so that the direct curvature method fails.

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