Non-Relativistic Bound States in QCD: Towards the N^3LL heavy quarkonium spectrum

Daniel Moreno Torres

Non-Relativistic Bound States in QCD: Towards the N^{3}LL heavy quarkonium spectrum

A dissertation by

Daniel Moreno Torres

aimed to the achievement of

Ph.D. in physics

granted by
Departament de Física
Universitat Autònoma de Barcelona

May 3rd, 2019

Supervised by
Dr. Antonio Miguel Pineda Ruiz

Acknowledgements

Firstly, I would like to thank all the persons in the University of Barcelona (UB) who wrote me a recommendation letter to apply for ph.D. positions. In particular, I would like to stress my gratitude to Dr. F. Mescia and Prof. Dr. J. Soto whose letters happened to be crucial to get a ph.D. position at the institute of high energy physics (IFAE) at the Autonomous University of Barcelona (UAB).

I would like to express my enormous gratitude to my supervisor, Dr. A. Pineda, who gave me the opportunity to take this project and who has guided me during these four years and a half. During the ph.D. I have been involved in very demanding calculations which I could never have done without his guidance. Giving me that project has allowed me to develop myself as a researcher and become a competitive one for future positions. I would like to thank him for sharing his vast knowledge on effective field theories and heavy quarkonium physics. I enjoyed quite a lot working in this topic and I hope I will continue contributing to that field of research, as well as, collaborating with him. I also thank him for encouraging me to attend to schools and conferences. I not only thank him for giving me the project summarized in the present dissertation, but also for giving me an outstanding project which has allowed me to be offered four postdoctoral positions around the world. To that respect, I also would like to express my gratitude to Dr. A. A. Penin and Prof. Dr. J. Soto who very kindly wrote me recommendation letters to apply for these positions, and of course, to Dr. A. Pineda who wrote the most important one.

I would like to thank Prof. Dr. M. Steinhauser for welcoming me at the Karlsruhe Institute of Technology (KIT) in Germany, as well as, for dedicating me time for fruitful discussions. I also would like to thank Dr. A. A. Penin for the discussions about the computation of the α^2/m^3 potential, as well as, for the checks.

Thank you to the organizers of the "XIIIth Quark confinement and the hadron spectrum" conference and the "13th International Workshop on Heavy Quarkonium" for giving me the opportunity to contribute with a talk.

Thank you to the people I have met in the schools I have attended: The "QCD Master Class" in Saint-Jacut-de-la-Mer (France) in 2016, the 55th international Schladming winter school on theoretical physics "Bound states and resonances" in Admont (Austria) in 2017 and the summer school "Effective Field Theory in Particle Physics and Cosmology" in Les Houches (France) in 2017, for the kindness and the exciting discussions about different topics of particle physics. In particular, thank you to Dra. Marcela Peláez for sharing her Mathematica knowledge and time when I was still unexperienced since, thanks to her help, I could write the Mathematica program which helped me in the computation of the LL running of the $1/m^3$ HQET Lagrangian operators.

Concerning the paper "Chromopolarizabilities of a heavy quark at weak coupling", the authors would like to thank Prof. Dr. M. Steinhauser for the reading of the manuscript. That work was supported in part by the Spanish grants FPA2014-55613-P, FPA2013-43425-P and SEV-2016-0588, which we also thank.

Concerning the paper "HQET renormalization group improved Lagrangian at $\mathcal{O}(1/m^3)$ with leading logarithmic accuracy: Spin-dependent case", the authors thank Dr. A. Pineda for useful discussions and for reading over the manuscript. That work was supported in part by the Spanish grants FPA2014-55613-P, SEV-2016-0588 and the Barcelona Institute of Science and Technology (BIST). It was also supported by a Clarendon Scholarship from the University of Oxford. We thank all these grants. I also would like to thank my collaborators X. Lobregat and R. Petrossian-Byrne for taking part in this project.

Concerning the paper "Spectator effects in the HQET renormalization group improved Lagrangian at $\mathcal{O}(1/m^3)$ with leading logarithmic accuracy: Spin-dependent case", I thank Dr. A. Pineda for reading over the manuscript. That work was supported by the Spanish grants FPA2014-55613-P, FPA2017-86989-P and SEV-2016-0588, which we also thank.

Finally, concerning the paper "S-wave heavy quarkonium spectrum with next-to-next-to-next-to-leading logarithmic accuracy" the authors would like to thank discussions with Dr. A. A. Penin, Dr. C. Peset, Dr. M. Stahlhofen and Prof. Dr. M. Steinhauser. We also thank Dr. A. A. Penin for partial checks of the computations of this paper. This work was supported in part by the Spanish grants FPA2014-55613-P, FPA2017-86989-P and SEV-2016-0588, which we also thank.

I would like to thank Dr. M. Pavón and Dr. L. Geng for inviting me to give a talk at the Beihang University in Beijing. I am specially grateful to Manolo for giving me a taste of China, both literally and figuratively, and for his kindness. It has been an enriching experience and I felt as being at home. I thank Dr. F. Guo for inviting me to give a talk to the theoretical particle physics group at the Chinese Academy of Sciences, also in Beijing.

Thank you to all the IFAE people. In particular, I would like to thank Dr. J. Matias for telling me the need to cover an A3.6 associate professor position, and for encouraging me to apply. Getting this position has been very important for me to be able to finish the present dissertation with a much better quality.

I would like to thank Dr. A. A. Penin, Prof. Dr. J. Soto and Prof. Dr. M. Steinhauser for accepting my invitation to be part of the evaluation committee of this dissertation.

On the personal side, I would like to thank my wife Nadia. You always have encouraged me to pursue my dreams and to work in what I like. You have been every single day of this ph.D. thesis on my side and it would not have been possible without you, I can take it for granted. Finally, thank you to all my family and friends for being there.

Abstract

In this dissertation, we obtain the renormalization group improved expressions of the Wilson coefficients associated to the $\mathcal{O}(1/m^3)$ HQET Lagrangian operators with leading-logarithmic (LL) approximation, in the Coulomb gauge. The Wilson coefficients include the heavy quark chromopolarizabilities. The analysis incorporates the effects induced by spectator quarks, which are considered to be massless. Special attention is paid to gauge independent combinations of Wilson coefficients.

The previous results are applied to pNRQCD at the weak coupling regime obtaining the renormalization group improved $\mathcal{O}(\alpha^2/m^3)$ and $\mathcal{O}(\alpha/m^4)$ spin-independent potentials in the off-shell Coulomb gauge matching scheme. The $\mathcal{O}(\alpha^3/m^2)$ potential is also computed, but in the off-shell Feynman gauge matching scheme, up to missing contributions proportional to c_k^2, d_{ss} and d_{vs}.

Such potentials are necessary to obtain the next-to-leading-logarithmic (NLL) potential and soft running of the Wilson coefficient $\tilde{D}_d^{(2)}$ associated to the spin-independent delta-like potential, which we also compute. The obtained result is complete up to missing contributions proportional to c_k^2, d_{ss}, d_{vs} and $\bar{c}_1^{hl\,(i)\,\mathrm{NLL}}$ of the NLL soft running. The first three are expected to be of the order of the computed contribution to the soft running, whereas the latter is of $\mathcal{O}(T_f n_f m\alpha^6 \ln\alpha)$, which is expected to be numerically subleading compared to the other contributions. The NLL ultrasoft running is also incorporated, as well as a contribution to the potential running of ultrasoft origin. The scheme independence of the potential renormalization group equation is explored via field redefinitions.

Presently, obtaining the NLL running of $\tilde{D}_d^{(2)}$ is the missing link to obtain the complete next-to-next-to-next-to-leading-logarithmic (N^{3}LL) pNRQCD Lagrangian. That is the necessary precision to obtain the spin-average (spin-independent part) heavy quarkonium spectrum relevant for S-wave (zero angular momentum) states with N^{3}LL ($\mathcal{O}(m\alpha^{5+n}\ln^n\alpha)$ with $n \in \mathbb{Z}$ and $n \geq 0$) accuracy. We carry out this computation up to a missing contribution coming from the missing contribution of $\tilde{D}_d^{(2)}$.

List of Publications

1. **Chromopolarizabilities of a heavy quark at weak coupling**

   D. Moreno, and A. Pineda, Phys. Rev. D **97** 016012 (2018); **98** 059902(E) (2018).

   arXiv:1710.07647 [hep-ph].

2. **HQET renormalization group improved Lagrangian at $\mathcal{O}(1/m^3)$ with leading logarithmic accuracy: Spin-dependent case**

   X. Lobregat, D. Moreno, and R. Petrossian-Byrne Phys. Rev. D **97**, 054018 (2018).

   arXiv:1802.07767 [hep-ph].

3. **Spectator effects in the HQET renormalization group improved Lagrangian at $\mathcal{O}(1/m^3)$ with leading logarithmic accuracy: Spin-dependent case**

   D. Moreno, Phys. Rev. D **98**, 034016 (2018).

   arXiv:1806.09323 [hep-ph].

4. **S-wave heavy quarkonium spectrum with next-to-next-to-next-to-leading logarithmic accuracy**

   C. Anzai, D. Moreno, and A. Pineda, Phys. Rev. D **98**, 114034 (2018).

   arXiv:1810.11031 [hep-ph].

Abbreviations

b bottom quark

c charm quark

CG Coulomb gauge

d down quark

dof degrees of freedom

EFT effective field theory

EFTs effective field theories

EOM equation of motion

EOMs equations of motion

FG Feynman gauge

HQET heavy quark effective theory

IR infrared

LL leading-logarithmic

LO leading order

MS minimal subtraction

$\overline{\textbf{MS}}$ modified MS

NLL next-to-leading-logarithmic

NLO next-to-leading order

N^2LL next-to-next-to-leading-logarithmic

N^2LO next-to-next-to-leading order

N^3LL next-to-next-to-next-to-leading-logarithmic

N^3LO next-to-next-to-next-to-leading order

N^4LL next-to-next-to-next-to-next-to-leading-logarithmic

N^4LO next-to-next-to-next-to-next-to-leading order

NRQCD non-relativistic QCD

NRQED non-relativistic QED

pNRQCD potential NRQCD

pNRQED potential NRQED

QCD quantum chromodynamics

QED quantum electrodynamics

QFT quantum field theory

RG renormalization group

RGE RG equation

RGEs RG equations

s strange quark

SD spin-dependent

SI spin-independent

SL single leading-logarithmic

SM Standard Model

t top quark

u up quark

UV ultraviolet

W W boson

Z Z boson

Contents

Introduction

According to Big Bang Cosmology, the first stages of the Universe's existence date from 13.8 billion years ago approximately. Between 10^{-12}-10^{-6} seconds after the Big bang, the Universe, with a temperature of approximately 10^{12}-10^{22} K (100 MeV-10^{9} GeV), was basically a "soup" of quarks, gluons (quark-gluon plasma) and leptons in what is called the quark epoch of the Universe. The Universe's temperature was then too high for hadrons, bound states of quarks due to the strong interaction, to exist. These are the energies we are able to reproduce presently in large accelerator facilities[1].

While the Universe was cooling due to its expansion, between 10^{-6}-1 second Universe's life, the temperature, of about 10^{10}-10^{12} K (1-100 MeV) started to be low enough to allow the first bound states to be formed, the hadrons. For this reason, this period of time is known as the hadron epoch. By the end of it, only stable baryons, like the proton and the neutron remained. Since then it is impossible to understand the Universe, as it is now, without understanding the physics of bound states, because since then, as a consequence of the different interactions, ordinary matter started regrouping in bound states consisting in larger and larger number of particles.

We have to wait until the Universe was about 10-10^{3} seconds old (with a temperature of around 10^{7}-10^{9} K i.e. between 100 keV-1 keV) to find the first primordial nuclei, bound states of protons and neutrons due to residual strong interactions, to be synthesized. Among them, hydrogen (^{1}H), helium-4 (^{4}He), small amounts of deuterium (^{2}H), helium-3 (^{3}He), and lithium-7 (^{7}Li) appeared. That is called the Big Bang nucleosynthesis epoch.

Much more time had to be passed, of around $380 \cdot 10^{3}$ years, for the Universe to be cold enough to find electrons bounded with atomic nuclei, due to the electromagnetic interaction, to form light neutral atoms like hydrogen or helium. This is known as the recombination epoch, and universe's temperature was around 4000 K (0.4 eV).

As the Universe was cooling, large amounts of molecular hydrogen (bound states of two hydrogen atoms due to residual electromagnetic interactions) and helium, scattered in the space, bounded again due to residual electromagnetic forces, forming regions of space filled in with a medium of macroscopic gas. In some cases, this macroscopic gas started to collapse due to the gravitational force giving rise to the formation of stars. The period which comprises the formation of the first stars goes from $380 \cdot 10^{3}$ years to $150 \cdot 10^{6}$ years. The Universe temperature was around 4000-60 K by then.

With the existence of these massive objects, the gravitational interaction, irrelevant since the Planck epoch[2], started to play a role again. Then, several star-kind objects

[1] Indeed, the highest total energy reached in the LHC has been 13 TeV.

[2] Period of time smaller than 10^{-43} s, where current physical theories lose their predictability. Physics

started to form bound states, giving rise to bigger astronomical objects like supermassive black holes, stars cumulus, planetary systems, and finally, the first galaxies. The period of formation of the first galaxies comprises the period of times from $150 \cdot 10^6$ to $1 \cdot 10^9$ years. The Universe's temperature lowered to between 60-19 K. Later, also due to the gravitational interaction, galaxies started regrouping making more complex bound states called galaxy clusters and superclusters. This started occurring in the period of time in which the Universe was between $1 \cdot 10^9$ and $10 \cdot 10^9$ years. The Universe's temperature was between 19-4 K.

In the meanwhile, since the formation of the first stars, the strong and electromagnetic interactions followed its course as a cause of the formation of more and more particle composite bound states. High temperature inside stars formed heavier and heavier nuclei, depending on the mass of the star, and then on its capacity to fuse them. At the end of the stars life, an envelope of gas is expelled from it. The expelled gas, which is relatively rich in heavy nuclei created within the star, cools as it moves away from the star, allowing molecules (bound states of atoms) and dust particles (bound states of molecules or atoms) to form. This kind of processes enriched the medium with a variety of different mass atoms.

In particular, 4.6 billion years ago approximately, our Solar system (also a bound state of an star, the Sun, planets and other less important in mass objects), started its formation and evolution in a medium already enriched with a wide range of atoms and molecules. In a particular planet, the Earth, the conditions were such that these molecules bounded in more and more complex molecules which gave rise to the cell, the smallest unit of life. Any form of known life, including ourselves, is a bound state of cells.

As we have seen, bound states are present in a vast range of scales, from the tiniest length scales presently accessible in accelerators, the hadron's scale length (10^{-15} m), to the biggest superclusters of galaxies (10^{22}-10^{23} m) that have been observed. Bound states are important, not only to understand the origin and evolution of the Universe, but also to understand the origin and evolution of life. Thus, understanding bound states is crucial to understand how the ordinary matter[3] behaves. In particular, simple bound states (formed by two or three particles) are of vital importance to understand the interactions that produce them. The Earth orbiting around the Sun was crucial to the understanding of the classical gravitational interaction, the hydrogen atom (a bound state of a proton and an electron) to understand the electromagnetic interactions at the quantum level and heavy quarkonium (a bound state of a heavy quark and a heavy antiquark) to improve our comprehension of the strong interactions.

The research carried out in this dissertation is focused on the study of bound states due to the strong interactions, so on the tiniest ones that can be presently measured. In particular, we focus our attention to the most compact ones of them, heavy quarkonia $Q\bar{Q}$, bound states made out of a heavy quark and a heavy antiquark[4]. These states, conceived

in this epoch is though to be dominated by quantum effects of gravity.

[3]Ordinary matter accounts for a 5% of the total Universe's energy. The rest is accounted for the dark matter (23%) and dark energy (72%) whose origin is the most puzzling problem that theoretical physics faces nowadays.

[4]If $m \gg \Lambda_{\rm QCD}$, where m is the quark mass and $\Lambda_{\rm QCD} \sim$ 200-300 MeV is the scale of confinement, i.e. the typical energy scale that separates the regions of large and small strong coupling, then Q is called a heavy quark. There are three light quarks: the u, d and s, and three heavy quarks: the c, b and t, in the

as the simplest bound states one can study in the framework of the strong interactions, not only help us to understand how the universe looked like at the first stages of its existence (since these particles only existed "naturally" in the hadron epoch), but also to improve our understanding of the strong interaction. The description of the experiments presently performed in large accelerator facilities indeed requires a solid knowledge of this interaction. In fact, this is unavoidable in order to solve the most important problems that high energy physics faces nowadays:

- To discern the differences between the SM predictions and experimental data i.e. to detect new physics.
- To quantitatively understand the strong interactions.

Both items will be addressed by carrying out perturbative high precision computations of heavy quarkonium related observables. The reason why we focus on heavy quarkonium is because, for large enough masses, perturbation theory can be used and the heavy quark dynamics simplifies, making it much simpler to study than other strongly interacting systems. Therefore, in the present dissertation, we will only study the weak coupling regime of the strong interaction.

The theory which describes the strong interaction is QCD. However, using full QCD to compute high order perturbative computations in heavy quarkonium can be very complicated, to say the least. Instead, these kind of computations requires the use of EFTs, which take advantage of the hierarchy of scales of the problem. The resulting EFTs make the computations much simpler than in the underlying theory. In this context, the motivation of the present dissertation is to use EFTs of QCD in strong interaction processes, in particular, to perform perturbative high precision calculations of heavy quark and heavy quarkonium properties.

The non-relativistic nature of heavy quarkonium characterizes its dynamics by three widely separated scales, the hard scale m, which is of the order of the heavy quark mass, the soft scale mv, which is of the order of the relative momentum between the two heavy quarks in the center of mass frame, or likewise, of the order of the inverse Bohr radius of the system, and the ultrasoft scale mv^2, which is of the order of the binding energy of the system, where v is the relative velocity between the two heavy quarks in the center of mass frame. Since the system is non-relativistic ($v \ll 1$), the hierarchy of scales is such that $m \gg mv \gg mv^2$ and one can take advantage of it by constructing subsequent EFTs by integrating out the hard and soft modes ending up with the EFTs called NRQCD and pNRQCD, respectively. In Fig. 1 we show schematically how the construction of these EFTs is carried out.

Still we can find two different situations depending on the relative size of $\Lambda_{\rm QCD}$ with respect to the soft scale:

- The weak coupling regime ($mv \gg mv^2 \gtrsim \Lambda_{\rm QCD}$): The matching between NRQCD and pNRQCD can be carried out perturbatively. This situation is satisfied for very large masses.

SM of particle physics. However, only the b and/or the c form heavy quarkonium bound states. The t is too massive and it decays before any bound state can be formed.

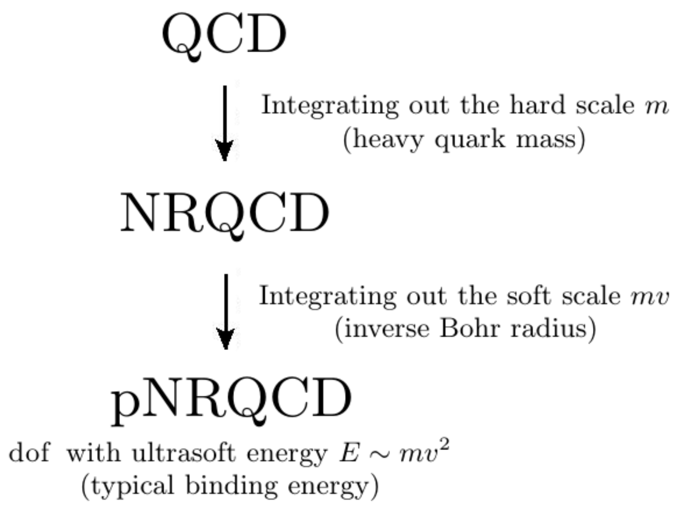


Figure 1: Schematic construction of pNRQCD. The hard scale is firstly integrated out to obtain the EFT called NRQCD and, subsequently, the soft scale is also integrated out obtaining the EFT called pNRQCD.

- The strong coupling regime ($mv \gtrsim \Lambda_{\rm QCD}$): The matching between NRQCD and pNRQCD can not be carried out perturbatively.

On the one hand, the quantity $R_{\rm had} \sim 1/\Lambda_{\rm QCD}$ determines the typical size of the light hadrons. On the other hand, at energy scales close to the confinement scale, the strong coupling constant becomes large[5], so perturbation theory breaks down. Light hadrons are non-perturbative systems. Contrarily, at the energy scale of the heavy quark mass the coupling constant $\alpha(m)$ is small, implying that on length scales comparable to the Compton wavelength $\lambda \sim 1/m$ of the heavy quark the strong interactions are perturbative and very much like the electromagnetic interactions. In particular, the system under study, namely heavy quarkonium, have a size of the order of $R_{Q\bar{Q}} \sim 1/(m\alpha(m)) \ll R_{had}$, being perturbative and very much hydrogen-like. Thus, in a first approximation, the heavy quarkonium spectrum can be described by solving the Schrödinger equation with the singlet Coulomb potential, obtaining the energy levels

$$E(n) = -C_F^2 \frac{m_r \alpha^2}{2n^2} \,, \tag{1}$$

where n is the principal quantum number, m_r is the reduced mass of the system and the color factor $C_F = 4/3$ in QCD. Comparing the binding energy with the ultrasoft scale we identify that $v \sim \alpha/n$, so the identification $v \sim \alpha$ is a good approximation for the lower levels of heavy quarkonium.

Therefore, in the present dissertation we will assume the strict weak coupling regime $mv \gg mv^2 \gg \Lambda_{\rm QCD}$. In that case, pNRQCD [1, 2] provides a well founded connection

[5]Particularly, $\alpha(\Lambda_{\rm QCD})$ becomes infinite in MS-like renormalization schemes.

between QCD and descriptions of the heavy quarkonium dynamics in terms of Schrödinger-like equations. This connection is obtained using standard QFT techniques such as dimensional regularization and renormalization. Schematically, pNRQCD is described by a Schrödinger-like equation where relativistic corrections to the potential and interactions with other low energy degrees of freedom can be incorporated systematically. The weak coupling regime is a good starting point for some particular systems. The top-antitop production near threshold can be obtained in a first approximation using a weak coupling analysis. This also applies, to a large extent, to the lower energy levels of bottomonium ($b\bar{b}$), charmonium ($c\bar{c}$) and the B_c ($b\bar{c}$ or $\bar{b}c$).

Computations in the previously mentioned systems become feasible in the framework of pNRQCD with otherwise unreachable very high accuracy. Nowadays, the achieved precision for the heavy quarkonium mass is N^{2}LL [3] and N^{3}LO [4], and the precision battle ground is N^{3}LL [5, 6] and N^{4}LO. For inclusive radiative decays, non-relativistic sum rules, or the top-antitop production near threshold the N^{3}LO [7] expressions are known, whereas the N^{2}LL [8] expressions are sought. For M1 radiative transitions the achieved precision is N^{2}LO [9].

Such computations and their comparison with the experiment will give us valuable information of the QCD dynamics. They are also of vital importance for the most accurate determination of some of the SM parameters like the masses of the heavy quarks: top, bottom and charm, or the strong coupling constant [10]. Moreover, several of these results can be easily applied to atomic physics opening the possibility of accurately determining, lepton masses, the electromagnetic coupling constant, as well low energy hadronic constants.

The present dissertation is devoted to the complete determination of the heavy quarkonium spectrum with N^{3}LL accuracy, leaving for future projects the computation of other observables. That is a very demanding task which has been already considered in previous works [4, 6, 11, 12, 13], where several contributions were computed. At present, the missing link to obtain the complete N^{3}LL heavy quarkonium spectrum is the spin-average (spin-independent) contribution to the S-wave (zero angular momentum) states. The spectrum for S-wave states can be splitted as

$$E_{\text{S-wave}} = E_{\text{aver}} + \mathbf{S}_1 \cdot \mathbf{S}_2 E_{\text{hfs}} \,, \tag{2}$$

where E_{aver} is the spin-average contribution and E_{hfs} is the hyperfine splitting contribution. Particularly, E_{aver} comes from the expectation value of the spin-independent delta-like interaction potential between the two heavy quarks, whose Wilson coefficient is needed with NLL accuracy. The determination of that Wilson coefficient with the necessary accuracy is the computation we address in this thesis. Its running is divided in three different parts: the soft running, the potential running and the ultrasoft running. Partial results will be obtained for the soft running, whereas the potential running will be completely determined. Finally, we will include the ultrasoft running which was already computed in Ref. [14].

The dissertation is divided as follows. In Ch. 1 we introduce some EFT concepts qualitatively. In Ch. 2 we obtain the LL running of the Wilson coefficients associated to the $1/m^3$ operators of the HQET Lagrangian. Of those, the ones associated to the spin-independent operators are relevant to obtain the potential running of the spin-independent delta-like potential with NLL precision. In Ch. 3 we introduce the NRQCD Lagrangian and compute the already known [15] NLO matching of the spin-independent four fermion operators in the different mass case. This section is devoted to a reader interested in how the matching between QCD and NRQCD is performed. It also set the details for a future evaluation of the N^2LO matching of the four fermion operators, which could be useful to determine the missing contribution of the NLL soft running of the spin-independent delta-like potential. In Ch. 4 we compute the matching between NRQCD and pNRQCD to obtain the spin-independent α^2/m^3 and α/m^4 interaction potentials, which are necessary to obtain the NLL potential running of the spin-independent delta-like potential. We partially compute the spin-independent part of the α^3/m^2 potential which allows us to partially obtain the NLL soft running of the spin-independent delta-like potential. The already known ultrasoft running is also included. Therefore, in this chapter we obtain the pNRQCD Lagrangian relevant for S-wave states with N^3LL accuracy. Finally, we compute the heavy quarkonium mass of spin-average states with N^3LL accuracy. These results are complete up to a missing contribution coming from the NLL soft running. Conclusions and prospects of future research are also discussed. At the end the thesis we include an appendix in order to fix notation, summarize useful results and mathematical identities, display master integrals, and perform examples of computations we have been involved in.

Chapter 1

Effective Field Theories in Particle Physics

1.1 Introduction

An EFT of a physical system is a low-energy (long-distance) theory specially designed to the description of that system such that it is computationally simpler than its underlying high-energy theory. This is due to the so called high energy degrees of freedom decouple and disappear completely from the dynamics of the low energy system (see from Ref. [16] to Ref. [23] for reviews). The details of the high-energy (short distance) theory become irrelevant to the description of the low energy system and their effects are encoded into the parameters of the EFT Lagrangian, called Wilson coefficients. Its construction is characterized by a scale Λ, acting as a cut-off, which splits the high and low energy degrees of freedom and under which the EFT is expected to describe the physical system as good as the underlying high energy theory does. Indeed, both theories are imposed to be equivalent at energies smaller than the cut-off i.e in the IR. It is in the UV, where both theories differ. The EFT is mathematically described by an effective Lagrangian $\mathcal{L}_{\rm EFT}$ which can be systematically improved by introducing operators of higher dimensionality, obtaining a power series of corrections in $1/\Lambda$. Therefore, given an operator it can be determined to what order in $1/\Lambda$ will contribute, in other words, power counting rules arise.

Typically, there are two situations in which EFTs are used depending on if the underlying high energy theory is known or not. In the case it is, the EFT is used in a "top-down" approach. This is done when one wants a simpler low energy theory to describe a system. One can construct the effective Lagrangian from an explicit calculation in the high energy theory, i.e. from first principles, finding the operators that enter in the EFT. Examples are when we integrate out heavy particles like the t, the W and the Z or when we use HQET to describe the b and the c. In the case the underlying high energy theory is not known, the EFT is used in a "bottom-up" approach. One must construct the underlying theory with constraints based on the symmetries the high energy theory is supposed to have (like Lorentz and gauge invariance), which are the relevant degrees of freedom and "naturalness" considerations (like no fine tunning), i.e. it can not be constructed from first principles. An example of that case is the extension of the SM introducing operators of higher dimensionality to describe physics at energies closer to some heavy particle that has been integrated out and whose effects do not show up in the SM. A "bottom-up" approach is also taken when the underlying theory is known, but the matching is too difficult to be performed. An example of that case is the construction of an EFT to describe non-perturbative dynamics in QCD.

The EFTs considered in this thesis follow a "top-down" approach since the full theory QCD is well-known and the regime where we work is perturbative, so there are no difficulties when performing the matching between the full and the effective theory. In this context, the way of disentangling the low and high energy degrees of freedom follows two steps. First, one must identify the high energy (heavy) degrees of freedom in the full theory and integrate them out of the action, ending up with an effective action with non-local interactions between the low energy (light) degrees of freedom. The second step is the expansion of this non-local effective action in local operators which describe local interactions between the low energy degrees of freedom. This way of understanding the construction of EFTs is known as the Wilsonian approach to EFTs. The effective Lagrangian resulting from this process can be written as

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_{<D} + \mathcal{L}_D + \sum_{n=1}^{\infty} \mathcal{L}_{D+n} = \mathcal{L}_{<D} + \mathcal{L}_D + \sum_{n=1}^{\infty} c_n \frac{\mathcal{O}_{D+n}}{\Lambda^n} \,, \tag{1.1}$$

where $\mathcal{L}_i$ means that it contains operators of mass dimension i, $\mathcal{O}_i$ is an operator of mass dimension i, Λ is the scale of new physics where the EFT loses its validity, and c_n are dimensionless coefficients called Wilson coefficients which encode the physics at short distances. Three different kind of operators can be identified:

1. Relevant: $[\mathcal{O}] < M^D$, are dominant.
2. Marginal: $[\mathcal{O}] = M^D$, remain constant.
3. Irrelevant: $[\mathcal{O}] > M^D$, are subdominant.

Remember that traditional textbooks argue that any QFT must contain only relevant and marginal operators or otherwise it loses its predictive power because an infinite number of counterterms are needed to renormalize an infinite number of parameters. However, this statement is in controversy with EFTs since, in addition to relevant and marginal operators, they are constructed with an infinite number of irrelevant operators with any possible mass dimension. The key point is that the predictive power is not lost as long as our goal is to make a calculation with a given precision. Once the aimed accuracy is fixed, the series is truncated and the infinities arising from loop integrals at a given order can be absorbed by a finite number of counterterms. Another crucial property demanded to any EFT is a well-defined power counting. Let us look at the irrelevant operators of Eq. (1.1). Since the scale of new physics Λ is much larger that any typical momentum and energy of the system, the series define a power counting in $1/\Lambda$. Note that if we want an EFT to be renormalizable at $\mathcal{O}(1/\Lambda^r)$, the effective Lagrangian must be truncated to incorporate operators of dimension up to $[\mathcal{O}] = M^{D+r}$. As it can be seen, the dimensionality of the operators is intimately related with the power counting.

1.2 Matching

In the construction of an EFT we need to map the underlying theory to the Wilson coefficients of the effective theory. That map is called matching and consists on imposing both theories to be equal (up to corrections in $1/\Lambda$) at a given energy scale Λ, called the matching scale, which is the cut-off of the effective theory. In practice, this is done by imposing S-matrix elements or Green functions to be equal in the underlying and the effective theory after expanding the former in powers of $1/\Lambda$. In the cases at hand, such a computation can be carried out in perturbation theory. Typically, the matching is a computation in the underlying theory since loop integrals in the EFT appear to be scaleless and are set to zero in dimensional regularization. By construction, both theories have the same IR behaviour, so IR divergences cancel in the matching process[1]. However, UV divergences appear and they are canceled by renormalization counterterms determining part of the running of the Wilson coefficients of the EFT. The matching determines the Wilson coefficients evaluated at the matching scale[2] $c(\Lambda)$, which can be expressed as a power series in the underlying theory expansion parameter, the coupling constant α

$$c(\Lambda) = \sum_m a_m \alpha^m(\Lambda)\,, \tag{1.2}$$

where a_m are constants and the index m runs till the aimed accuracy.

After the construction of the effective theory and the subsequent matching, an interaction mediated by heavy fields in the underlying theory is described by higher-dimensional operators in the EFT. It means that, non-local interactions in the underlying theory, are approximated in the effective theory by local contact interactions with coupling $c(\Lambda)/\Lambda^{n-D}$, where n is the mass dimension of the operator we are considering.

Therefore, there are two expansion parameters in the effective theory which must be taken into account in order to determine an observable with a certain precision, the coupling constant α and the inverse of the cut-off $1/\Lambda$. The size of a certain contribution is then measured in powers of α^r/Λ^s.

1.3 Renormalization Group

Once the matching calculation is done, we obtain the Wilson coefficients at fixed order $c(\nu = \Lambda)$, where ν is the cut-off of the EFT. However, the expressions of these Wilson coefficients still need to be improved via RG in what is called the resummation of large logarithms. The RG improvement is specially important when the $\ln(\nu/\Lambda)$ is large, which happens when ν gets closer to the next physical scale of the problem, and it allows to find $\mathcal{O}(\ln^n(\nu/\Lambda))$ contributions that can not be found by just working at fixed order.

The way to determine the RGE of a Wilson coefficient is by computing S-matrix elements or Green functions. For instance, in the one-loop approximation, we have that

[1] A kind of IR divergence called pinch singularity can occur. These IR divergences are eliminated by subtracting to the diagrams of the underlying theory the ones of the effective theory [8].

[2] Note that, for simplicity in the discussion, we have chosen the matching scale to be equal to the expansion parameter, but they do not have to be necessarily the same.

$$\mathcal{A} = \mathcal{A}^{\text{tree}} + \mathcal{A}^{\text{1-loop}}\,, \tag{1.3}$$

where $\mathcal{A}$ is the S-matrix element, which is a physical observable, so it can not depend on the renormalization scale ($\nu d\mathcal{A}/d\nu = 0$) and it must be a finite quantity. On the contrary, the one-loop contribution to the S-matrix $\mathcal{A}^{\text{1-loop}}$ is UV divergent. Such a divergence gets absorbed in the tree level contribution to the S-matrix $\mathcal{A}^{\text{tree}}$, which at the Wilson coefficient level allows to determine the counterterms of the Wilson coefficients in $\mathcal{A}^{\text{tree}}$. Likewise, this allows to determine their RGE. Let us see how it works from a schematic example with a single Wilson coefficient. Thus

$$\mathcal{A}^{\text{tree}} = c_B A\,, \tag{1.4}$$

$$\mathcal{A}^{\text{1-loop}} = ag^2\frac{1}{\epsilon}M^{2\epsilon}A\,, \tag{1.5}$$

where A stands for some momentum structure, c_B is the bare Wilson coefficient, M is some energy or momentum scale of the problem and a is a constant. The quantity $\mathcal{A}^{\text{tree}} + \mathcal{A}^{\text{1-loop}}$ must be finite and ν-independent. Since the second term is UV divergent, such divergence must be canceled by the coefficient in $\mathcal{A}^{\text{tree}}$, so

$$c_B A + ag^2\frac{1}{\epsilon}M^{2\epsilon}A = \text{finite and } \nu\text{-independent}\,. \tag{1.6}$$

At the level of the Wilson coefficients and rewriting $c_B = c(\nu) + \delta c(\nu)$, being $c(\nu)$ the renormalized Wilson coefficient and $\delta c(\nu)$ the counterterm, we have that

$$c(\nu) + \delta c(\nu) + ag^2\frac{1}{\epsilon}M^{2\epsilon} = \text{finite and } \nu\text{-independent}\,. \tag{1.7}$$

Since the left hand side of Eq. (1.7) must be finite, we can fix the counterterm

$$\delta c(\nu) = -ag^2\frac{1}{\epsilon}\nu^{2\epsilon}\,. \tag{1.8}$$

Finally, since the left hand side of Eq. (1.7) must be ν-independent

$$\nu\frac{d}{d\nu}\left(c(\nu) + \delta c(\nu) + ag^2\frac{1}{\epsilon}M^{2\epsilon}\right) = 0\,, \tag{1.9}$$

which allows us to find the RGE of the Wilson coefficient c

$$\nu\frac{d}{d\nu}c(\nu) = -\nu\frac{d}{d\nu}\delta c(\nu) = 2ag^2\,. \tag{1.10}$$

After solving the RGEs, the RG improved Wilson coefficients are obtained. They can be expressed as $c(\nu) = c(\Lambda) + \Delta c(\nu)$, with $\Delta c(\nu = \Lambda) = 0$. If we expand $c(\nu)$ in powers of the coupling constant $\alpha(\Lambda)$, the resulting expression is a tower of logarithms (from here it comes the term resummation of logarithms) of the form

$$c(\nu) = \sum_m\left(a_m\alpha^m(\Lambda) + \sum_{n=1}^{\infty} b_{mn}\alpha^{m+n}(\Lambda)\ln^n\left(\frac{\nu}{\Lambda}\right)\right), \tag{1.11}$$

where a_m and b_{mn} are constants and the index m runs till the aimed accuracy of a particular calculation. If the logarithms $\ln^n(\nu/\Lambda)$ are large, then the resummation of them become numerically important compared to the fixed order computation, and they must be computed. We will refer to these expressions where the RG evolution has been taken into account as RG improved expressions and the procedure to compute them as the resummation of large logarithms. Such a resummation, which is the sum over the index n, is achieved by solving the RGEs.

In general, we will use the terms LO, NLO, $\mathrm{N^2LO}$ and so on to refer to the first, second, third and so on non-vanishing fixed order contribution to the Wilson coefficients. Once the Wilson coefficients are RG improved we will refer to their accuracy as being LL, NLL, $\mathrm{N^2LL}$ and so on.

Chapter 2

Heavy Quark Effective Theory (HQET)

2.1 Introduction

The expansion in inverse powers of the heavy quark mass is a powerful tool for the study of hadrons containing one or more heavy quarks. This expansion is formulated more systematically in terms of an effective theory and of its associated effective Lagrangian. For the one-heavy quark sector, this effective theory is the HQET [24]. Once obtained, its Lagrangian can be applied, for instance, to physical observables associated to the B or D mesons such as their spectrum or decays. The HQET Lagrangian is also instrumental in the description of systems with more than one heavy quark, in particular if we fix our attention to the heavy quark-antiquark sector, i.e. heavy quarkonium, as the HQET Lagrangian corresponds to one of the building blocks of the NRQCD Lagrangian [25, 26]. The Wilson coefficients of the HQET Lagrangian operators also enter into the Wilson coefficients of the operators (i.e. the interaction potentials) of the pNRQCD Lagrangian [1, 27], an effective field theory optimized for the description of heavy quark-antiquark systems near threshold (for reviews, see Refs. [2, 8]). The dependence of the potentials with the Wilson coefficients of the HQET Lagrangian is a consequence of the matching between NRQCD and pNRQCD.

In this chapter we address the computation of the LL running of the Wilson coefficients associated to the $1/m^3$ operators of the HQET Lagrangian. The Wilson coefficients we compute have applications in heavy quark and heavy quarkonium physics. On the one hand, what is more important for the present work is that the computed Wilson coefficients that are associated to spin-independent operators are necessary ingredients to obtain the pNRQCD Lagrangian with N^{3}LL accuracy, which in turn is the necessary precision to obtain the complete heavy quarkonium spectrum with N^{3}LL accuracy, i.e. with $\mathcal{O}(m\alpha^5 + m\alpha^6 \ln\alpha + m\alpha^7 \ln^2\alpha + \ldots)$ accuracy. They are also necessary for the computation of the production and annihilation of heavy quarkonium with N^{2}LL precision. Actually, this is one of the main motivations we undertake this work. More precisely, they play a role in the computation of the NLL potential running of the Wilson coefficient $\tilde{D}_d^{(2)}$ associated to the spin-independent delta-like potential, which is of $\mathcal{O}(\alpha/m^2)$ at its LO. Let us see the reason why we need this Wilson coefficient to this order. In momentum space, the spin-independent delta-like potential can be written as $\tilde{V}_d \sim \tilde{D}_d^{(2)}/m^2$. Then, the potential V_d in position space contributes to the spectrum in the following way

$$\begin{aligned} V_d &\sim d^3\mathbf{k}\tilde{V}_d \sim \frac{d^3\mathbf{k}}{m^2}\tilde{D}_d^{(2)} \sim m\alpha^3(\alpha+\alpha^2\ln\alpha+\ldots+\alpha^2+\alpha^3\ln\alpha+\ldots) \\ &= \underbrace{\underbrace{m\alpha^4}_{\mathrm{N^2LO}} + m\alpha^5\ln\alpha+\ldots}_{\mathrm{N^2LL}} + \underbrace{\underbrace{m\alpha^5}_{\mathrm{N^3LO}} + m\alpha^6\ln\alpha+\ldots}_{\mathrm{N^3LL}}, \end{aligned} \tag{2.1}$$

where we have used the power counting rules given in Ref. [8]. The $\mathcal{O}(\alpha,\alpha^2)$ terms come from the initial matching condition of $\tilde{D}_d^{(2)}$ at tree level and at one loop, respectively. The $\mathcal{O}(\alpha^2\ln\alpha+\ldots,\alpha^3\ln\alpha+\ldots)$ terms come from the RG evolution of $\tilde{D}_d^{(2)}$ at LL and at NLL, respectively. As it can be seen, in order to determine the contribution of the spin-independent delta-like potential to the N^{3}LL heavy quarkonium spectrum the running of $\tilde{D}_d^{(2)}$ must be determined with NLL accuracy. That means

$$\delta\tilde{D}_d^{(2)\,\mathrm{NLL}} \sim \alpha^2+\alpha^3\ln\alpha+\alpha^4\ln^2\alpha+\ldots \Rightarrow \nu\frac{d}{d\nu}\delta\tilde{D}_d^{(2)\,\mathrm{NLL}} \sim \alpha^3+\alpha^4\ln\alpha+\ldots\,, \tag{2.2}$$

where $\delta\tilde{D}_d^{(2)\,\mathrm{NLL}}$ stands for the purely NLL contribution to $\tilde{D}_d^{(2)}$, more precisely, $\tilde{D}_d^{(2)} = \tilde{D}_d^{(2)\,\mathrm{LL}}+\delta\tilde{D}_d^{(2)\,\mathrm{NLL}}$. In particular, we will see in Ch. 4 that the potential RGE[1] of $\delta\tilde{D}_d^{(2)\,\mathrm{NLL}}$ has the form

$$\nu\frac{d}{d\nu}\delta\tilde{D}_{d,p}^{(2)\,\mathrm{NLL}} \sim \alpha^3\big(A_{12}(2c_{A_1}+\bar{c}_{A_2})+A_{34}(2c_{A_3}+c_{A_4})\big)+\ldots\,, \tag{2.3}$$

where A_{12} and A_{34} are constants, c_{A_i} are Wilson coefficients associated to the $\mathcal{O}(1/m^3)$ spin-independent operators of the HQET Lagrangian, and the dots stand for irrelevant terms for the present discussion. Comparing Eq. (2.2) with Eq. (2.3) it can be seen that the c_{A_i} must be determined with LL precision, namely with the following accuracy

$$c_{A_i} \sim 1+\alpha\ln\alpha+\alpha^2\ln^2\alpha+\ldots\,. \tag{2.4}$$

On the other hand, the computed Wilson coefficients associated to spin-dependent operators start to be relevant when considering the pNRQCD Lagrangian with N^{4}LO and N^{4}LL accuracy, which is the necessary precision to determine the N^{4}LO, i.e the $\mathcal{O}(m\alpha^6)$, and the N^{4}LL, i.e. the $\mathcal{O}(m\alpha^6+m\alpha^7\ln\alpha+m\alpha^8\ln^2\alpha+\ldots)$, heavy quarkonium spectrum. This computation is beyond the scope of this work, though. These results are also instrumental in the determination of higher order logarithms for NRQED bound states, like in hydrogen and muonic hydrogen-like atoms.

[1]The NLL contribution to $\tilde{D}_d^{(2)}$ is conveniently divided in four pieces, $\delta\tilde{D}_d^{(2)\,\mathrm{NLL}} = \tilde{D}_d^{(2)\,\mathrm{NLO}} + \delta\tilde{D}_{d,us}^{(2)\,\mathrm{NLL}}+\delta\tilde{D}_{d,s}^{(2)\,\mathrm{NLL}}+\delta\tilde{D}_{d,p}^{(2)\,\mathrm{NLL}}$. In Ch. 4 we will define what each contribution is. At the moment, we only need to know that $\delta\tilde{D}_d^{(2)\,\mathrm{NLL}}$ is splitted in different contributions.

The computation presented here also provides a cross-check of some of the reparametrization invariance relations given in Ref. [28] (except for c_M and d_1^{hl}, for which we introduce some controversy) and gives a solution in the standard basis settled on by the same reference.

At present, the operator structure of the HQET Lagrangian, and the tree-level values of their Wilson coefficients, is known to $\mathcal{O}(1/m^3)$ in the case with no light quarks [28]. The inclusion of light quarks has been considered in Ref. [29]. The Wilson coefficients were computed with LL accuracy in Refs. [30, 31, 32] to $\mathcal{O}(1/m^2)$ and with NLO accuracy in Ref. [28] also to $\mathcal{O}(1/m^2)$ but without considering dimension 6 heavy-light operators. The LL running to $\mathcal{O}(1/m^3)$ without considering spectator effects was considered in Refs. [33, 34], which turned out to have internal discrepancies between their explicit SL results and their own anomalous dimension matrix. The inclusion of heavy-light operators to $\mathcal{O}(1/m^3)$ was considered in Ref. [29], but only SL results for the Wilson coefficients were provided. The resummed expressions of the Wilson coefficients with LL approximation, to $\mathcal{O}(1/m^3)$, and including spectator quarks were obtained in Refs. [35, 36, 37], which corrected the inconsistencies found in Refs. [33, 34] and found disagreement with some of the SL results presented in Ref. [29]. The present chapter basically summarizes and details the work done in Refs. [35, 36, 37].

In summary, in this chapter we obtain the RG improved expressions of the Wilson coefficients of the HQET Lagrangian operators with LL approximation to $\mathcal{O}(1/m^3)$, which includes the heavy quark chromopolarizabilites. The analysis includes the effects induced by spectator quarks, which are considered to be massless. The Coulomb gauge will be used throughout.

The chapter is organized as follows. In Sec. 2.2 we give a brief introduction to HQET and talk about heavy quark symmetry. In Sec. 2.3 we introduce the HQET Lagrangian up to $\mathcal{O}(1/m^3)$ including relevant heavy-light operators (see Ref. [29] for a complete basis of heavy-light operators). In Sec. 2.4 we compute the Compton scattering, which allows us to identify gauge-independent combinations of Wilson coefficients. The Sec. 2.5 is dedicated to the computation of the anomalous dimensions and it is divided in three subsections: In Sec. 2.5.1 and Sec. 2.5.2 the RGEs for the Wilson coefficients associated to the $1/m^3$ heavy-light operators and to the $1/m^3$ heavy-gluon operators are presented, respectively. In Sec. 2.5.3 physical combinations are sought, and their associated RGEs are presented. The solution of the RGEs for the physical quantities we have found is displayed in Sec. 2.5.5. A numerical analysis of the solution is also performed. We compare with the earlier work done in Refs. [29, 33, 34] in Sec. 2.6. We dedicate Sec. 2.7 to show how the UV pole of the loop integrals is obtained. Finally, in Sec. B we display the necessary Feynman rules, master integrals and some useful color algebra and gamma matrices relations. The problem with c_M and d_1^{hl} not satisfying reparametrization invariance in the Coulomb gauge is also discussed.

2.2 Heavy Quark Symmetry

As we already discussed in Ch. 1, in particle physics, it is very common that the effect that a given heavy particle has in the dynamics of a certain process becomes irrelevant. In that case, it is useful to construct and EFT where such a particle no longer appears. The resulting theory happens to be mathematically more tractable than the full theory.

The case in which one wants to describe the properties of hadrons which contain a heavy quark [38] is quite similar to the case described above, where the heavy particle role is played by the heavy quark, but with some subtleties. The difference is that, in this case, it is not possible to completely remove the heavy quark from the effective theory. Instead, what it is possible is to integrate out the "small components" in the full heavy quark spinor, which describe fluctuations around the mass shell.

The starting point for the construction of HQET is the use of the phenomenological fact that a heavy quark Q of mass $m \gg \Lambda_{\rm QCD}$ inside an hadron is almost on-shell and moves with very good approximation with the hadron's velocity v. Thus, its four momentum can be written as

$$P_Q = mv + k\,, \tag{2.5}$$

where k is the residual momentum of the heavy quark which describes small fluctuations with respect to the on shell condition $P_Q = mv$. Since the heavy quark is almost on shell, $k \ll mv$. The first step is to introduce large and small component fields

$$Q_v(x) = e^{imv\cdot x} P_+ Q(x)\,, \tag{2.6}$$

$$q_v(x) = e^{imv\cdot x} P_- Q(x)\,, \tag{2.7}$$

respectively, so that

$$Q(x) = e^{-imv\cdot x}\big(Q_v(x) + q_v(x)\big)\,, \tag{2.8}$$

where $Q(x)$ is the full heavy quark spinor appearing in the Dirac Lagrangian for the heavy quark

$$\mathcal{L}_Q = \bar{Q}(i\not{D} - m)Q\,, \tag{2.9}$$

and $P_\pm = \frac{1}{2}(1 \pm \not{v})$ are projection operators. In the rest frame, where $v = (1, \mathbf{0})$, Q_v corresponds to the upper two components of Q and q_v to the lower two ones. These new fields satisfy $\not{v} Q_v = Q_v$ and $\not{v} q_v = -q_v$. If the heavy quark was perfectly on shell (this happens in the infinite heavy quark mass limit), the field q_v would be absent. For describing an hadron containing a heavy antiquark we just must replace $v \to -v$ in Eqs. (2.6-2.8).

At the classical level, the heavy degrees of freedom (small component) q_v can be eliminated by just using Eq. (2.8) to replace the expression of Q in terms of the new fields into the equation of motion of the heavy quark, $(i\not{D} - m)Q = 0$, obtaining

$$i\not{D} Q_v + (i\not{D} - 2m)q_v = 0\,. \tag{2.10}$$

From Eq. (2.10) it is apparent that Q_v describes massless degrees of freedom, whereas q_v corresponds to degrees of freedom with twice the heavy quark mass. For this reason, we called q_v heavy degrees of freedom above. The small component field q_v is the degree of freedom which is integrated out of the theory deriving a nonlocal effective action for the large component Q_v. Such effective action can then be expanded in local operators. Multiplying Eq. (2.10) by $P_\pm$ we obtain

$$-\, iv\cdot DQ_v = i\not{D}_\perp q_v\,, \tag{2.11}$$

$$(iv\cdot D + 2m)q_v = i\not{D}_\perp Q_v\,, \tag{2.12}$$

where $D_\perp^\mu = D^\mu - v^\mu v\cdot D$. Eq. (2.12) can be solved and gives

$$q_v = \frac{1}{iv\cdot D + 2m - i\eta} i\not{D}_\perp Q_v\,, \tag{2.13}$$

showing that the small component field q_v is suppressed by the heavy quark mass. Inserting it into Eq. 2.11 we obtain

$$-\, iv\cdot DQ_v = i\not{D}_\perp \frac{1}{iv\cdot D + 2m - i\eta} i\not{D}_\perp Q_v\,, \tag{2.14}$$

which is the equation of motion for Q_v. From the EOM it is easy to guess the effective Lagrangian:

$$\mathcal{L}_{\rm eff} = \bar{Q}_v(iv\cdot D)Q_v + \bar{Q}_v i\not{D}_\perp \frac{1}{iv\cdot D + 2m - i\eta} i\not{D}_\perp Q_v\,, \tag{2.15}$$

which is, in the absence of radiative corrections, the HQET Lagrangian. This Lagrangian was derived more elegantly from the generating functional of QCD in Ref. [39].

Derivatives acting on Q_v produces powers of the residual momenta $k \ll m$. Hence, the nonlocal effective Lagrangian Eq. (2.15) can be expanded in powers of $iv\cdot D/m$, defining the operator product expansion of the HQET Lagrangian as a series of local operators. It can be seen that, to first order in $1/m$

$$\mathcal{L}_{\rm eff} = \bar{Q}_v(iv\cdot D)Q_v - \frac{1}{2m}\bar{Q}_v D_\perp^2 Q_v - \frac{g}{4m}\bar{Q}_v\sigma_{\alpha\beta}G^{\alpha\beta}Q_v + \mathcal{O}(1/m^2)\,. \tag{2.16}$$

In the present discussion we stop the derivation of the HQET Lagrangian to $\mathcal{O}(1/m)$ because we only want to be illustrative. However, this thesis is concerned with the HQET Lagrangian up to $\mathcal{O}(1/m^3)$. In the rest frame the above Lagrangian is written as

$$\mathcal{L}_{\rm eff}^{\rm rest\ frame} = Q^\dagger\left(iD_0 + \frac{1}{2m}\mathbf{D}^2 + \frac{g}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}\right)Q + \mathcal{O}(1/m^2)\,. \tag{2.17}$$

The term $\mathbf{D}^2/m$ can be identified as the kinetic energy associated to the off-shell residual motion of the heavy quark, whereas the term $\boldsymbol{\sigma}\cdot\mathbf{B}/m$ is the Pauli term, which describes the interaction of the heavy quark spin with the gluon field. In the $m\to\infty$ limit, these two operators do not appear. On the one hand, it implies that hadronic states with different heavy quark spin have the same properties. This is the origin of the heavy quark spin symmetry. On the other hand, it implies that hadron properties are independent of the flavor of the heavy quark it contains. This is the origin of the heavy quark flavor symmetry. Together, they are referred as the heavy quark symmetry.

These approximate symmetries allow the determination of relations between hadrons containing a heavy quark, like the B, D, B^* and D^* heavy mesons or the Λ_b and Λ_c heavy baryons. These symmetries are broken by relativistic effects suppressed by powers of the heavy quark mass, which are of the order of powers of $\Lambda_{\rm QCD}/m$. As far as $m \gg \Lambda_{\rm QCD}$, heavy quark symmetry is a good approximation to the description of the hadrons containing a quark of mass m.

2.3 The HQET Lagrangian

2.3.1 The HQET Lagrangian without light fermions

The HQET Lagrangian is defined uniquely up to field redefinitions. In this thesis we use the HQET Lagrangian density for a quark of mass $m \gg \Lambda_{\rm QCD}$ in the special frame $v = (1,0,0,0)$ given in Ref. [28]:

$$\mathcal{L}_{\rm HQET} = \mathcal{L}_g + \mathcal{L}_Q\,, \tag{2.18}$$

$$\mathcal{L}_g = -\frac{1}{4}G^{\mu\nu\,a}G^a_{\mu\nu} + c_1^g\frac{g}{4m^2}f^{abc}G^a_{\mu\nu}G^{\mu\,b}{}_{\alpha}G^{\nu\alpha\,c} + \mathcal{O}\left(\frac{1}{m^4}\right), \tag{2.19}$$

$$\begin{aligned}
\mathcal{L}_Q = Q^\dagger \Bigg\{ & iD_0 + \frac{c_k}{2m}\mathbf{D}^2 + \frac{c_F}{2m}\boldsymbol{\sigma}\cdot g\mathbf{B} \\
& + \frac{c_D}{8m^2}\left(\mathbf{D}\cdot g\mathbf{E} - g\mathbf{E}\cdot\mathbf{D}\right) + i\,\frac{c_S}{8m^2}\boldsymbol{\sigma}\cdot\left(\mathbf{D}\times g\mathbf{E} - g\mathbf{E}\times\mathbf{D}\right) \\
& + \frac{c_4}{8m^3}\mathbf{D}^4 + ic_M\,g\frac{\mathbf{D}\cdot[\mathbf{D}\times\mathbf{B}] + [\mathbf{D}\times\mathbf{B}]\cdot\mathbf{D}}{8m^3} + c_{A_1}\,g^2\,\frac{\mathbf{B}^2-\mathbf{E}^2}{8m^3} - c_{A_2}\frac{g^2\mathbf{E}^2}{16m^3} \\
& + c_{W_1}\,g\frac{\{\mathbf{D^2},\boldsymbol{\sigma}\cdot\mathbf{B}\}}{8m^3} - c_{W_2}\,g\frac{\mathbf{D}^i\,\boldsymbol{\sigma}\cdot\mathbf{B}\,\mathbf{D}^i}{4m^3} + c_{p'p}\,g\frac{\boldsymbol{\sigma}\cdot\mathbf{D}\,\mathbf{B}\cdot\mathbf{D} + \mathbf{D}\cdot\mathbf{B}\,\boldsymbol{\sigma}\cdot\mathbf{D}}{8m^3} \\
& + c_{A_3}\,g^2\,\frac{1}{N_c}{\rm Tr}\left(\frac{\mathbf{B}^2-\mathbf{E}^2}{8m^3}\right) - c_{A_4}\,g^2\,\frac{1}{N_c}{\rm Tr}\left(\frac{\mathbf{E}^2}{16m^3}\right) \\
& + ic_{B_1}\,g^2\,\frac{\boldsymbol{\sigma}\cdot(\mathbf{B}\times\mathbf{B} - \mathbf{E}\times\mathbf{E})}{8m^3} - ic_{B_2}\,g^2\,\frac{\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{E})}{8m^3}\Bigg\}Q + \mathcal{O}\left(\frac{1}{m^4}\right),
\end{aligned} \tag{2.20}$$

where Q is a non-relativistic fermion field represented by a Pauli spinor. The components of the vector $\boldsymbol{\sigma}$ are the Pauli matrices. We define $iD^0 = i\partial^0 - gA^0$, $i\mathbf{D} = i\boldsymbol{\nabla} + g\mathbf{A}$, $\mathbf{E}^i = G^{i0}$ and $\mathbf{B}^i = -\epsilon^{ijk}G^{jk}/2$, where ϵ^{ijk} is the three-dimensional totally antisymmetric tensor[2] with $\epsilon^{123} = 1$ and $(\mathbf{a}\times\mathbf{b})^i \equiv \epsilon^{ijk}\mathbf{a}^j\mathbf{b}^k$. Note also that we have rescaled by a factor $1/N_c$ the coefficients $c_{A_{3,4}}$, as compared to the definitions given in Ref. [28]. In general, we will refer to the c_{A_i} as the chromopolarizabilities.

[2] In dimensional regularization several prescriptions are possible for the ϵ^{ijk} tensors and $\boldsymbol{\sigma}$, and the same prescription as for the calculation of the Wilson coefficients must be used.

2.3.2 The HQET Lagrangian with massless fermions

By including n_f massless fermions to the HQET Lagrangian, it has the following structure:

$$\mathcal{L}_{\rm HQET} = \mathcal{L}_g + \mathcal{L}_Q + \mathcal{L}_l, \tag{2.21}$$

$$\mathcal{L}_l = \sum_{i=1}^{n_f} \bar{q}_i i\not{D} q_i + \frac{\delta\mathcal{L}_q^{(2)}}{m^2} + \frac{\delta\mathcal{L}_{Qq}^{(2)}}{m^2} + \frac{\delta\mathcal{L}_q^{(3)}}{m^3} + \frac{\delta\mathcal{L}_{Qq}^{(3)}}{m^3} + \mathcal{O}\left(\frac{1}{m^4}\right), \tag{2.22}$$

where q_i is a relativistic fermion field represented by a Dirac spinor. The complete set of operators to $\mathcal{O}(1/m^2)$ can be found in Ref. [32]. They read

$$\begin{aligned}\delta\mathcal{L}_{Qq}^{(2)} =& \frac{c_1^{hl}}{8} g^2 \sum_{i=1}^{n_f} Q^\dagger T^a Q \bar{q}_i \gamma^0 T^a q_i - \frac{c_2^{hl}}{8} g^2 \sum_{i=1}^{n_f} Q^\dagger \boldsymbol{\sigma}^j T^a Q \bar{q}_i \gamma^j \gamma_5 T^a q_i \\ &+ \frac{c_3^{hl}}{8} g^2 \sum_{i=1}^{n_f} Q^\dagger Q \bar{q}_i \gamma^0 q_i - \frac{c_4^{hl}}{8} g^2 \sum_{i=1}^{n_f} Q^\dagger \boldsymbol{\sigma}^j Q \bar{q}_i \gamma^j \gamma_5 q_i, \end{aligned} \tag{2.23}$$

$$\begin{aligned}\delta\mathcal{L}_q^{(2)} =& \frac{c_D^l}{4} \bar{q}_i \gamma_\nu D_\mu G^{\mu\nu} q_i \\ &+ \frac{c_1^{ll}}{8} g^2 \sum_{i,j=1}^{n_f} \bar{q}_i T^a \gamma^\mu q_i \; \bar{q}_j T^a \gamma_\mu q_j + \frac{c_2^{ll}}{8} g^2 \sum_{i,j=1}^{n_f} \bar{q}_i T^a \gamma^\mu \gamma_5 q_i \; \bar{q}_j T^a \gamma_\mu \gamma_5 q_j \\ &+ \frac{c_3^{ll}}{8} g^2 \sum_{i,j=1}^{n_f} \bar{q}_i \gamma^\mu q_i \; \bar{q}_j \gamma_\mu q_j + \frac{c_4^{ll}}{8} g^2 \sum_{i,j=1}^{n_f} \bar{q}_i \gamma^\mu \gamma_5 q_i \; \bar{q}_j \gamma_\mu \gamma_5 q_j. \end{aligned} \tag{2.24}$$

However, the light-light operators $\delta\mathcal{L}_q^{(2)}$ and $\delta\mathcal{L}_q^{(3)}$, as well as the $1/m^2$ gluonic operator with associated Wilson coefficient c_1^g, contribute at NLL or beyond, so we will not consider them any further.

The $\mathcal{O}(1/m^3)$ (dimension 7) heavy-light operators were considered in detail in Ref. [29] and they can be found in the Eq. (10) of that reference. However, we will not consider all of them, but only those that get LL running and that could affect the LL running of the Wilson coefficients of the heavy-gluon operators: c_4, c_M, c_{A_1}, c_{A_2}, c_{A_3}, c_{A_4}, $c_{p'p}$, c_{W_1}, c_{W_2}, c_{B_1} and c_{B_2}. After disregarding some of them because they are proportional to the energy of the heavy quark (so they become subleading after using the heavy quark EOM), we find that the only relevant operators are

$$\mathcal{M}_{4\pm}^{(3h)s/o} = \pm g^2 [\bar{q}_l \gamma^\mu \mathcal{C}_{s/o}^a q_l][\bar{h}_v \mathcal{C}_{s/o}^a i D_\mu^\pm h_v], \tag{2.25}$$

$$\mathcal{M}_{6\pm}^{(3h)s/o} = \pm g^2 [\bar{q}_l i\sigma^{\mu\lambda} v_\lambda \mathcal{C}_{s/o}^a q_l][\bar{h}_v \mathcal{C}_{s/o}^a i D_\mu^\pm h_v], \tag{2.26}$$

$$\mathcal{M}_{2\pm}^{(3l)s/o} = \pm g^2 [\bar{q}_l \mathcal{C}_{s/o}^a (ivD^\pm) q_l][\bar{h}_v \mathcal{C}_{s/o}^a h_v], \tag{2.27}$$

$$\mathcal{M}_{3\pm}^{(3l)s/o} = \pm g^2 [\bar{q}_l \not{v} \mathcal{C}_{s/o}^a (ivD^\pm) q_l][\bar{h}_v \mathcal{C}_{s/o}^a h_v], \tag{2.28}$$

$$\mathcal{M}_{4\pm}^{(3l)s/o} = \pm g^2 [\bar{q}_l i\sigma^{\lambda\nu} v_\lambda \mathcal{C}_{s/o}^a i D_\nu^\pm q_l][\bar{h}_v \mathcal{C}_{s/o}^a h_v], \tag{2.29}$$

$$\mathcal{M}_{3\pm}^{(3h)s/o} = \pm g_s^2 [\bar{q}_l \gamma_\mu \mathcal{C}_{s/o}^a q_l][\bar{h}_v i\sigma^{\mu\nu} \mathcal{C}_{s/o}^a i D_\nu^\pm h_v], \tag{2.30}$$

$$\mathcal{M}_{5\pm}^{(3h)s/o} = \pm g_s^2 [\bar{q}_l i\sigma_{\mu\lambda} v^\lambda \mathcal{C}^a_{s/o} q_l][\bar{h}_v i\sigma^{\mu\nu} \mathcal{C}^a_{s/o} iD^\pm_\nu h_v] \,, \tag{2.31}$$

$$\mathcal{M}_{7\pm}^{(3h)s/o} = \pm g_s^2 [\bar{q}_l \gamma_5 \not{v} \mathcal{C}^a_{s/o} q_l][\bar{h}_v \gamma_5 \mathcal{C}^a_{s/o} i\not{D}^\pm h_v] \,, \tag{2.32}$$

$$\mathcal{M}_{9\pm}^{(3h)s/o} = \pm g_s^2 [\bar{q}_l \gamma_5 \mathcal{C}^a_{s/o} q_l][\bar{h}_v \gamma_5 \mathcal{C}^a_{s/o} i\not{D}^\pm h_v] \,, \tag{2.33}$$

$$\mathcal{M}_{5\pm}^{(3l)s/o} = \pm g_s^2 [\bar{q}_l i\sigma^{\mu\nu} \mathcal{C}^a_{s/o} (ivD^\pm) q_l][\bar{h}_v i\sigma_{\mu\nu} \mathcal{C}^a_{s/o} h_v] \,, \tag{2.34}$$

$$\mathcal{M}_{6\pm}^{(3l)s/o} = \pm g_s^2 [\bar{q}_l \gamma_5 \not{v} \mathcal{C}^a_{s/o} iD^\pm_\mu q_l][\bar{h}_v \gamma_5 \gamma^\mu \mathcal{C}^a_{s/o} h_v] \,, \tag{2.35}$$

$$\mathcal{M}_{7\pm}^{(3l)s/o} = \pm g_s^2 [\bar{q}_l \gamma_5 \gamma^\mu \mathcal{C}^a_{s/o} (ivD^\pm) q_l][\bar{h}_v \gamma_5 \gamma_\mu \mathcal{C}^a_{s/o} h_v] \,, \tag{2.36}$$

$$\mathcal{M}_{8\pm}^{(3l)s/o} = \pm g_s^2 [\bar{q}_l \gamma_5 \mathcal{C}^a_{s/o} iD^\pm_\mu q_l][\bar{h}_v \gamma_5 \gamma^\mu \mathcal{C}^a_{s/o} h_v] \,, \tag{2.37}$$

$$\mathcal{M}_{10\pm}^{(3l)s/o} = \pm g_s^2 [\bar{q}_l \gamma_\nu \mathcal{C}^a_{s/o} iD^\pm_\mu q_l][\bar{h}_v i\sigma^{\mu\nu} \mathcal{C}^a_{s/o} h_v] \,, \tag{2.38}$$

where $iD^+_\mu = i \overrightarrow{\partial}_\mu - gA^a_\mu T^a$ and $iD^-_\mu = i \overleftarrow{\partial}_\mu + gA^a_\mu T^a$, meaning the arrows over the derivatives that they act over the fields in the left/right hand depending on the direction of the arrow (they only act over heavy quark fields or over light quark fields), $\mathcal{C}^a_s = 1$, $\mathcal{C}^a_o = T^a$, and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. It is also understood that in the octet case the covariant derivative stands left/right of the color matrix when acting to the left/right. In our case, we work in the rest frame, so that $v^\mu = (1, \mathbf{0})$ and $h_v \equiv Q$. Moreover, we are in the heavy-quark sector, and not in the antiquark one, so we can project to this sector. Note that we have not displayed the operator $\mathcal{M}_{9\pm}^{(3l)s/o}$ because it is wrong (there are typographic mistakes and even free indices) and should be corrected. Fortunately, as we will see later on, this operator is not relevant for the computation of the LL running of the Wilson coefficients, since the operators that are left are enough to absorb all the divergences coming from one-loop diagrams. After all these simplifications, the previous operators can be written as

$$\mathcal{M}_{4\pm}^{(3h)s/o} = \pm g^2 [\bar{q}_l \gamma^i \mathcal{C}^a_{s/o} q_l][Q^\dagger \mathcal{C}^a_{s/o} i\mathbf{D}^{i\,\pm} Q] \,, \tag{2.39}$$

$$\mathcal{M}_{6\pm}^{(3h)s/o} = \mp g^2 [\bar{q}_l \gamma^i \gamma^0 \mathcal{C}^a_{s/o} q_l][Q^\dagger \mathcal{C}^a_{s/o} i\mathbf{D}^{i\,\pm} Q] \,, \tag{2.40}$$

$$\mathcal{M}_{2\pm}^{(3l)s/o} = \pm g^2 [\bar{q}_l \mathcal{C}^a_{s/o} (iD^\pm_0) q_l][Q^\dagger \mathcal{C}^a_{s/o} Q] \,, \tag{2.41}$$

$$\mathcal{M}_{3\pm}^{(3l)s/o} = \pm g^2 [\bar{q}_l \gamma^0 \mathcal{C}^a_{s/o} (iD^\pm_0) q_l][Q^\dagger \mathcal{C}^a_{s/o} Q] \,, \tag{2.42}$$

$$\mathcal{M}_{4\pm}^{(3l)s/o} = \pm g^2 [\bar{q}_l \gamma^i \gamma^0 \mathcal{C}^a_{s/o} i\mathbf{D}^{i\,\pm} q_l][Q^\dagger \mathcal{C}^a_{s/o} Q] \,, \tag{2.43}$$

$$\mathcal{M}_{3\pm}^{(3h)s/o} = \mp g_s^2 [\bar{q}_l \gamma^i \mathcal{C}^a_{s/o} q_l][Q^\dagger i\epsilon^{ijk} \boldsymbol{\sigma}^k \mathcal{C}^a_{s/o} i\mathbf{D}^{j\,\pm} Q] \,, \tag{2.44}$$

$$\mathcal{M}_{5\pm}^{(3h)s/o} = \pm g_s^2 [\bar{q}_l \gamma^i \gamma^0 \mathcal{C}^a_{s/o} q_l][Q^\dagger i\epsilon^{ijk} \boldsymbol{\sigma}^k \mathcal{C}^a_{s/o} i\mathbf{D}^{j\,\pm} Q] \,, \tag{2.45}$$

$$\mathcal{M}_{7\pm}^{(3h)s/o} = \mp g_s^2 [\bar{q}_l \gamma_5 \gamma^0 \mathcal{C}^a_{s/o} q_l][Q^\dagger \boldsymbol{\sigma}^i \mathcal{C}^a_{s/o} i\mathbf{D}^{i\,\pm} Q] \,, \tag{2.46}$$

$$\mathcal{M}_{9\pm}^{(3h)s/o} = \mp g_s^2 [\bar{q}_l \gamma_5 \mathcal{C}^a_{s/o} q_l][Q^\dagger \boldsymbol{\sigma}^i \mathcal{C}^a_{s/o} i\mathbf{D}^{i\,\pm} Q] \,, \tag{2.47}$$

$$\mathcal{M}_{5\pm}^{(3l)s/o} = \mp g_s^2 [\bar{q}_l \gamma^i \gamma^j \mathcal{C}^a_{s/o} iD^\pm_0 q_l][Q^\dagger i\epsilon^{ijk} \boldsymbol{\sigma}^k \mathcal{C}^a_{s/o} Q] \,, \tag{2.48}$$

$$\mathcal{M}_{6\pm}^{(3l)s/o} = \mp g_s^2[\bar{q}_l\gamma_5\gamma^0\mathcal{C}^a_{s/o}i\mathbf{D}^{i\,\pm}q_l][Q^\dagger\boldsymbol{\sigma}^i\mathcal{C}^a_{s/o}Q]\,, \tag{2.49}$$

$$\mathcal{M}_{7\pm}^{(3l)s/o} = \pm g_s^2[\bar{q}_l\gamma_5\gamma^i\mathcal{C}^a_{s/o}iD_0^{\pm}q_l][Q^\dagger\boldsymbol{\sigma}^i\mathcal{C}^a_{s/o}Q]\,, \tag{2.50}$$

$$\mathcal{M}_{8\pm}^{(3l)s/o} = \mp g_s^2[\bar{q}_l\gamma_5\mathcal{C}^a_{s/o}i\mathbf{D}^{i\,\pm}q_l][Q^\dagger\boldsymbol{\sigma}^i\mathcal{C}^a_{s/o}Q]\,, \tag{2.51}$$

$$\mathcal{M}_{10\pm}^{(3l)s/o} = \mp g_s^2[\bar{q}_l\gamma^j\mathcal{C}^a_{s/o}i\mathbf{D}^{i\,\pm}q_l][Q^\dagger i\epsilon^{ijk}\boldsymbol{\sigma}^k\mathcal{C}^a_{s/o}Q]\,. \tag{2.52}$$

We then have

$$\delta\mathcal{L}^{(3)}_{Qq} = \sum_{l=1}^{n_f}\sum_m d_m^{hl}\mathcal{O}_m\,, \tag{2.53}$$

where the $\mathcal{O}_m$ operators are all the possible linear independent combinations of the $\mathcal{M}$ operators given in Eqs. (2.39-2.52). In the present work, only those linear combinations whose associated Wilson coefficient gets LL running will be defined. The discussion is reserved to Sec. 2.5.1.

2.4 Compton scattering

In order to explore the existence of physical combinations involving the Wilson coefficients that we aim to calculate, i.e. c_4, c_M, c_{A_1}, c_{A_2}, c_{A_3}, c_{A_4}, c_{W_1}, c_{W_2}, $c_{p'p}$, c_{B_1} and c_{B_2}, we compute the Compton scattering amplitude, which is the amplitude of the scattering of a heavy quark with a gluon[3] $Qg \to Qg$. We compute it at tree level up to $\mathcal{O}(1/m^3)$ in the mass expansion and in the Coulomb gauge, though obviously the Compton scattering amplitude is a gauge-independent quantity. It is precisely this property the one that will allow us to identify gauge-independent combinations of Wilson coefficients.

We take the external incoming and outgoing quarks to have four-momentum $p = (E_1, \mathbf{p})$ and $p' = (E_1', \mathbf{p}\,')$, respectively. We take the gluon four-momenta as outgoing and label them by k_1, i, a and k_2, j, b with respect to Lorentz and color indices. This also implies the on-shell condition $k_1^0 = -|\mathbf{k_1}|$ and $k_2^0 = |\mathbf{k_2}|$. We work in the incoming quark rest frame, i.e $E_1 = 0$ and $\mathbf{p} = \mathbf{0}$, so $\mathbf{p}\,' = -(\mathbf{k_1} + \mathbf{k_2})$ and $E_1' = -(k_1^0 + k_2^0)$. In addition, we define the unit vectors $\mathbf{n_1} = \mathbf{k_1}/|\mathbf{k_1}|$ and $\mathbf{n_2} = \mathbf{k_2}/|\mathbf{k_2}|$. The relation

$$|\mathbf{k}_2| = \frac{|\mathbf{k}_1|}{1 + \frac{|\mathbf{k}_1|}{m}(1 + \mathbf{n}_1\cdot\mathbf{n}_2)} \tag{2.54}$$

holds from four-momenta conservation.

By inserting the appropriate Wilson coefficients up to $\mathcal{O}(1/m^3)$, the diagrams we have to consider for the computation are listed in Fig. 2.1. We split the amplitude in its spin-dependent $\mathcal{A}^{ij\,ab}_{\rm SD}$ and spin-independent $\mathcal{A}^{ij\,ab}_{\rm SI}$ parts

$$\mathcal{A}^{ij\,ab} = \mathcal{A}^{ij\,ab}_{\rm SD} + \mathcal{A}^{ij\,ab}_{\rm SI}\,, \tag{2.55}$$

where

[3]More precisely, we will compute the scattering of a heavy quark with a transverse gluon.

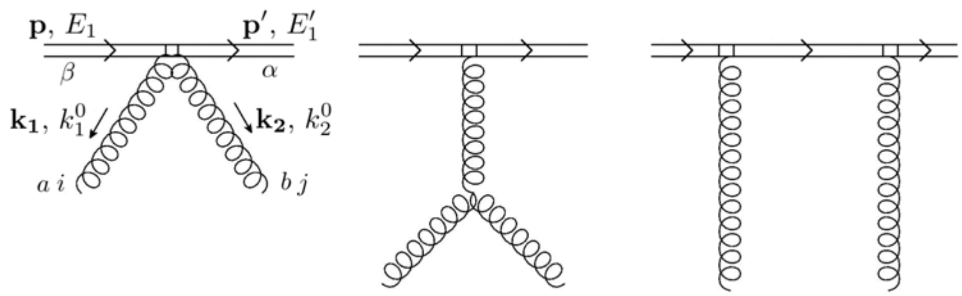


Figure 2.1: Topologies contributing to Compton scattering at tree level up to $\mathcal{O}(1/m^3)$. Diagrams are generated from these topologies by considering all possible vertices and kinetic insertions contributing up to $\mathcal{O}(1/m^3)$.

$$
\begin{aligned}
\mathcal{A}^{ij\,ab}_{\rm SD} \;=\;& c_F\frac{g^2}{2m}\boldsymbol{\sigma}^k\epsilon^{ijk}[T^a,T^b]_{\alpha\beta}\\
&+c_F\frac{g^2}{4m}\frac{1}{1+\mathbf{n}_1\cdot\mathbf{n}_2}\Big((\boldsymbol{\sigma}\times\mathbf{n}_2)^i\mathbf{n}_1^j-(\boldsymbol{\sigma}\times\mathbf{n}_1)^j\mathbf{n}_2^i+\boldsymbol{\sigma}^i(\mathbf{n}_1\times\mathbf{n}_2)^j+\boldsymbol{\sigma}^j(\mathbf{n}_1\times\mathbf{n}_2)^i\\
&+2(\boldsymbol{\sigma}\times\mathbf{n}_1)^i\mathbf{n}_1^j-2(\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i\Big)[T^a,T^b]_{\alpha\beta}\\
&+c_Fc_k\frac{g^2}{4m^2}|\mathbf{k}_1|\big((\boldsymbol{\sigma}\times\mathbf{n}_1)^i\mathbf{n}_1^j+(\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i\big)[T^a,T^b]_{\alpha\beta}\\
&+\frac{g^2}{8m^2}|\mathbf{k}_1|\Big[(2c_Fc_k-c_Sc_k)\big((\boldsymbol{\sigma}\times\mathbf{n}_1)^i\mathbf{n}_1^j-(\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i\big)\\
&+c_Sc_k\big((\boldsymbol{\sigma}\cdot\mathbf{n}_1)\mathbf{n}_1^k\epsilon^{ijk}+(\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_2^k\epsilon^{ijk}\big)\\
&+c_F^2\big((\boldsymbol{\sigma}\cdot\mathbf{n}_1)\mathbf{n}_2^k\epsilon^{ijk}+(\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_1^k\epsilon^{ijk}+\boldsymbol{\sigma}^i(\mathbf{n}_1\times\mathbf{n}_2)^j+\boldsymbol{\sigma}^j(\mathbf{n}_1\times\mathbf{n}_2)^i\big)\Big]\{T^a,T^b\}_{\alpha\beta}\\
&+(c_{B_1}-2c_{W_1}-c_F^2c_k-c_Sc_k)\frac{g^2}{16m^3}|\mathbf{k}_1|^2\Big((\boldsymbol{\sigma}\cdot\mathbf{n}_1)\mathbf{n}_2^k\epsilon^{ijk}+(\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_1^k\epsilon^{ijk}\\
&+\boldsymbol{\sigma}^i(\mathbf{n}_1\times\mathbf{n}_2)^j+\boldsymbol{\sigma}^j(\mathbf{n}_1\times\mathbf{n}_2)^i\Big)[T^a,T^b]_{\alpha\beta}\\
&-(2c_{W_1}-2c_{W_2}+2c_Fc_k^2+c_Sc_k+c_Sc_F)\\
&\times\frac{g^2}{16m^3}|\mathbf{k}_1|^2\big((\boldsymbol{\sigma}\times\mathbf{n}_1)^i\mathbf{n}_1^j-(\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i\big)[T^a,T^b]_{\alpha\beta}\\
&-(2c_{W_1}-2c_{W_2}-c_Sc_F+c_Sc_k-2c_Fc_k^2)\\
&\times\frac{g^2}{16m^3}|\mathbf{k}_1|^2\big((\boldsymbol{\sigma}\times\mathbf{n}_1)^i\mathbf{n}_1^j+(\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i\big)\{T^a,T^b\}_{\alpha\beta}\\
&+(c_{B_2}+c_{B_1}-2c_{W_1}-c_Sc_F-c_Sc_k)\frac{g^2}{8m^3}|\mathbf{k}_1|^2\boldsymbol{\sigma}^k\epsilon^{ijk}[T^a,T^b]_{\alpha\beta}\\
&-c_{p'p}\frac{g^2}{16m^3}|\mathbf{k}_1|^2\Big[\big((\mathbf{n}_1\times\mathbf{n}_2)^j\boldsymbol{\sigma}^i-(\mathbf{n}_1\times\mathbf{n}_2)^i\boldsymbol{\sigma}^j\\
&+\epsilon^{ijk}(\mathbf{n}_1-\mathbf{n}_2)^k(\boldsymbol{\sigma}\cdot(\mathbf{n}_1+\mathbf{n}_2))\big)\{T^a,T^b\}_{\alpha\beta}\\
&-\big((\mathbf{n}_1\times\mathbf{n}_2)^j\boldsymbol{\sigma}^i+(\mathbf{n}_1\times\mathbf{n}_2)^i\boldsymbol{\sigma}^j-\epsilon^{ijk}(\mathbf{n}_1+\mathbf{n}_2)^k(\boldsymbol{\sigma}\cdot(\mathbf{n}_1+\mathbf{n}_2))\big)[T^a,T^b]_{\alpha\beta}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+c_F^2\frac{g^2}{8m^3}|\mathbf{k}_1|^2(1+\mathbf{n}_1\cdot\mathbf{n}_2)\big((\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_1^k\epsilon^{ijk}+\boldsymbol{\sigma}^j(\mathbf{n}_1\times\mathbf{n}_2)^i\big)[T^a,T^b]_{\alpha\beta}\\
&-c_Fc_k\frac{g^2}{4m^3}|\mathbf{k}_1|^2(1+\mathbf{n}_1\cdot\mathbf{n}_2)(\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i[T^a,T^b]_{\alpha\beta}\\
&-\frac{g^2}{8m^3}|\mathbf{k}_1|^2(1+\mathbf{n}_1\cdot\mathbf{n}_2)\Big[c_S\big((\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i+(\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_2^k\epsilon^{ijk}\big)\\
&+c_F^2\big((\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_1^k\epsilon^{ijk}+\boldsymbol{\sigma}^j(\mathbf{n}_1\times\mathbf{n}_2)^i\big)-2c_Fc_k(\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i\Big]\{T^a,T^b\}_{\alpha\beta}\,, \qquad (2.56)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_{\rm SI}^{ij\,ab} \;=\;& -ig^2\delta^{ij}\frac{1}{|\mathbf{k}_1|}\frac{1}{1+\mathbf{n}_1\cdot\mathbf{n}_2}[T^a,T^b]_{\alpha\beta}-c_k\frac{ig^2}{2m}\delta^{ij}\{T^a,T^b\}_{\alpha\beta}-\frac{ig^2}{2m}\delta^{ij}[T^a,T^b]_{\alpha\beta}\\
&+\frac{ig^2}{4m^2}|\mathbf{k}_1|\big(\delta^{ij}(2-c_F^2\mathbf{n}_1\cdot\mathbf{n}_2)+c_F^2\mathbf{n}_2^i\mathbf{n}_1^j\big)[T^a,T^b]_{\alpha\beta}\\
&+\frac{ig^2}{16m^3}|\mathbf{k}_1|^2\Big[\delta^{ij}\big((4c_4+4c_M-2c_{A_1}-c_{A2}+2c_Sc_F)\\
&+(4c_4-2c_{A_1})\mathbf{n}_1\cdot\mathbf{n}_2\big)+2c_{A_1}\mathbf{n}_2^i\mathbf{n}_1^j\Big]\{T^a,T^b\}_{\alpha\beta}\\
&-c_F^2\frac{ig^2}{8m^3}|\mathbf{k}_1|^2(\delta^{ij}\mathbf{n}_1\cdot\mathbf{n}_2-\mathbf{n}_2^i\mathbf{n}_1^j)(\mathbf{n}_1\cdot\mathbf{n}_2)\{T^a,T^b\}_{\alpha\beta}\\
&-\frac{ig^2}{8m^3}|\mathbf{k}_1|^2\big(\delta^{ij}(2-c_F^2\mathbf{n}_1\cdot\mathbf{n}_2)+c_F^2\mathbf{n}_2^i\mathbf{n}_1^j\big)(1+\mathbf{n}_1\cdot\mathbf{n}_2)[T^a,T^b]_{\alpha\beta}\,. \qquad (2.57)
\end{aligned}
$$

Note that c_D, which is gauge dependent, does not appear explicitly. It only appears implicitly through c_M (as c_M is related with c_D by reparametrization invariance[4] [28]). One can also observe that three combinations of Wilson coefficients always appear in the observable: $\bar{c}_{A_2}\equiv c_{A_2}-4c_M$, $\bar{c}_W\equiv c_{W_1}-c_{W_2}$ and $\bar{c}_{B_1}\equiv c_{B_1}-2c_{W_1}$. These combinations, together with the Wilson coefficients c_4, c_{A_1}, c_{A_3}, c_{A_4}, c_{B_2} and $c_{p'p}$, are physical quantities, i.e. they are gauge-independent. This implies that the RGEs of these physical quantities can only depend on physical quantities, in particular, on physical combinations of Wilson coefficients. Later on, we will see that this is indeed the case. We suspect that individually c_M, c_{W_1}, c_{W_2} and c_{B_1} are gauge dependent quantities, since our results are in agreement with Ref. [33] (where the calculation was done in the Feynman gauge) at the level of the SL precision for the physical quantities defined earlier, but we disagree for each of these four individually.

It is worth mentioning that, for the observables we have studied (the Compton scattering here and the heavy quarkonium spectrum in Ch. 4), we see that c_{A_2} always appear in the combination $\tilde{c}_{A_2}\equiv 2c_{A_1}+\bar{c}_{A_2}$. A similar analysis for the elastic scattering of a heavy quark with a massless quark shows that $\bar{c}_1^{hl}\equiv c_D+c_1^{hl}$ is physical. Note, however, that c_D and c_1^{hl} individually are not; indeed they are gauge dependent (for instance, see Ref. [3] for a discussion on this issue).

[4]Even though we will introduce some controversy about this relation in the coming sections (the relation found in Ref. [28] is satisfied in the Feynman gauge, but it is not in the Coulomb gauge), c_M will be still related with c_D and with other gauge dependent quantities like c_1^{hl} and d_1^{hl}.

For QED, we obtain

$$\mathcal{A}^{ij} = \mathcal{A}^{ij}_{\rm SD} + \mathcal{A}^{ij}_{\rm SI}\,, \tag{2.58}$$

where

$$\begin{aligned}
\mathcal{A}^{ij}_{\rm SD} \;=\;& \frac{g^2}{4m^2}|\mathbf{k}_1|\Big[(2c_Fc_k - c_S)\big((\boldsymbol{\sigma}\times\mathbf{n}_1)^i\mathbf{n}_1^j - (\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i\big)\\
&+c_S\big((\boldsymbol{\sigma}\cdot\mathbf{n}_1)\mathbf{n}_1^k\epsilon^{ijk} + (\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_2^k\epsilon^{ijk}\big)\\
&+c_F^2\big((\boldsymbol{\sigma}\cdot\mathbf{n}_1)\mathbf{n}_2^k\epsilon^{ijk} + (\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_1^k\epsilon^{ijk} + \boldsymbol{\sigma}^i(\mathbf{n}_1\times\mathbf{n}_2)^j + \boldsymbol{\sigma}^j(\mathbf{n}_1\times\mathbf{n}_2)^i\big)\Big]\\
&-(2c_{W_1} - 2c_{W_2} - c_Sc_F + c_Sc_k - 2c_Fc_k^2)\frac{g^2}{8m^3}|\mathbf{k}_1|^2\big((\boldsymbol{\sigma}\times\mathbf{n}_1)^i\mathbf{n}_1^j + (\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i\big)\\
&-c_{p'p}\frac{g^2}{8m^3}|\mathbf{k}_1|^2\big((\mathbf{n}_1\times\mathbf{n}_2)^j\boldsymbol{\sigma}^i - (\mathbf{n}_1\times\mathbf{n}_2)^i\boldsymbol{\sigma}^j + \epsilon^{ijk}(\mathbf{n}_1-\mathbf{n}_2)^k(\boldsymbol{\sigma}\cdot(\mathbf{n}_1+\mathbf{n}_2))\big)\\
&-\frac{g^2}{4m^3}|\mathbf{k}_1|^2(1+\mathbf{n}_1\cdot\mathbf{n}_2)\Big[c_S\big((\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i + (\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_2^k\epsilon^{ijk}\big)\\
&+c_F^2\big((\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_1^k\epsilon^{ijk} + \boldsymbol{\sigma}^j(\mathbf{n}_1\times\mathbf{n}_2)^i\big) - 2c_Fc_k(\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i\Big]\,,
\end{aligned} \tag{2.59}$$

$$\begin{aligned}
\mathcal{A}^{ij}_{\rm SI} \;=\;& -c_k\frac{ig^2}{m}\delta^{ij}\\
&+\frac{ig^2}{8m^3}|\mathbf{k}_1|^2\delta^{ij}\big((2 + 4c_4 - 2c_{A_1} + 2c_Sc_F + 4c_M - c_{A_2} - 2c_F^2)\\
&+(2 + 4c_4^{(1)} - 2c_{A_1} + 2c_F^2c_k - 2c_F^2)\mathbf{n}_1\cdot\mathbf{n}_2\big)\\
&+\frac{ig^2}{4m^3}|\mathbf{k}_1|^2\delta^{ij}\big(c_F^2(1-\mathbf{n}_1\cdot\mathbf{n}_2) - 1\big)(1+\mathbf{n}_1\cdot\mathbf{n}_2)\\
&+\frac{ig^2}{4m^3}|\mathbf{k}_1|^2\big((c_F^2 + c_{A_1} - c_F^2c_k) + c_F^2\mathbf{n}_1\cdot\mathbf{n}_2\big)\mathbf{n}_2^i\mathbf{n}_1^j\,.
\end{aligned} \tag{2.60}$$

Note that there is no $\mathcal{O}(1/m^0)$ contribution. Note also that, in the spin-dependent amplitude, there is no $\mathcal{O}(1/m)$ contribution and that, in the spin-independent one, there is no $\mathcal{O}(1/m^2)$ contribution. Setting the Wilson coefficients to their tree level values we obtain

$$\begin{aligned}
\mathcal{A}^{ij}_{\rm SD} \;=\;& \frac{g^2}{4m^2}|\mathbf{k}_1|\Big[(\boldsymbol{\sigma}\times\mathbf{n}_1)^i\mathbf{n}_1^j - (\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i + (\boldsymbol{\sigma}\cdot\mathbf{n}_1)\mathbf{n}_1^k\epsilon^{ijk} + (\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_2^k\epsilon^{ijk}\\
&+(\boldsymbol{\sigma}\cdot\mathbf{n}_1)\mathbf{n}_2^k\epsilon^{ijk} + (\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_1^k\epsilon^{ijk} + \boldsymbol{\sigma}^i(\mathbf{n}_1\times\mathbf{n}_2)^j + \boldsymbol{\sigma}^j(\mathbf{n}_1\times\mathbf{n}_2)^i\Big]\\
&+\frac{g^2}{4m^3}|\mathbf{k}_1|^2(1+\mathbf{n}_1\cdot\mathbf{n}_2)(-\boldsymbol{\sigma}^j(\mathbf{n}_1\times\mathbf{n}_2)^i - (\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_1^k\epsilon^{ijk}\\
&+(\boldsymbol{\sigma}\times\mathbf{n}_2)^j\mathbf{n}_2^i - (\boldsymbol{\sigma}\cdot\mathbf{n}_2)\mathbf{n}_2^k\epsilon^{ijk})\,,
\end{aligned} \tag{2.61}$$

$$
\begin{aligned}
\mathcal{A}^{ij}_{\rm SI} &= -\frac{ig^2}{m}\delta^{ij} + \frac{ig^2}{2m^3}|\mathbf{k}_1|^2\delta^{ij}(1+\mathbf{n}_1\cdot\mathbf{n}_2) \\
&\quad -\frac{ig^2}{4m^3}|\mathbf{k}_1|^2(\delta^{ij}\mathbf{n}_1\cdot\mathbf{n}_2 - \mathbf{n}_2^i\mathbf{n}_1^j)(1+\mathbf{n}_1\cdot\mathbf{n}_2)\,.
\end{aligned}
\tag{2.62}
$$

These expressions agree with Eq. (19) in Ref. [40].

The above analysis gives us the set of Wilson coefficients and their combinations that appear in physical observables:

$$
\{c_4, c_{A_1}, \bar{c}_{A_2}, c_{A_3}, c_{A_4}, \bar{c}_W, \bar{c}_{B_1}, c_{B_2}, c_{p'p}\}\,. \tag{2.63}
$$

We compute the anomalous dimensions for these, but also for the unphysical set:

$$
\{c_4, c_M, c_{A_1}, c_{A_2}, c_{A_3}, c_{A_4}, c_{W_1}, c_{W_2}, c_{B_1}, c_{B_2}, c_{p'p}\}\,, \tag{2.64}
$$

in the Coulomb gauge, as it can be important for future research about the possible gauge dependence of these Wilson coefficients, and also it is an intermediate step we can not skip.

For completeness, we also define polarizabilities. The concept of polarizability is potentially ambiguous, as it is defined after subtracting what are called Born terms (which indeed can be defined in several ways) from the Compton scattering computation. Indeed, this discussion already appears in the context of QED in the elastic scattering of photons with protons (see, for instance, Refs. [41, 42, 43]). One possible definition is the one used in Ref. [44], which adapted to the notation of the present work reads

$$
\frac{4m^3}{\alpha}\alpha_{E1} \equiv -c_{A_1} - \frac{\bar{c}_{A_2}}{2} + c_F^2 - c_F + 1\,, \tag{2.65}
$$

$$
\frac{4m^3}{\alpha}\beta_{M1} \equiv -1 + c_{A_1}\,. \tag{2.66}
$$

Note that, even in QED, the polarizabilities are not low energy constants, as they depend on the renormalization scale.

2.5 RG improved Wilson coefficients of the $1/m^3$ operators with LL accuracy

In this section, we obtain the renormalization group improved expressions of the Wilson coefficients associated to the $1/m^3$ operators of the HQET Lagrangian with leading logarithmic approximation, which includes the heavy quark chromopolarizabilities. In order to do so, we need to determine their anomalous dimension with $\mathcal{O}(\alpha)$ accuracy. Our analysis includes the effects induced by spectator quarks.

In principle, one would like to only compute irreducible diagrams. However, as indicated in Ref. [35], this would involve considering a more extensive basis of operators, including those that vanish on shell. Instead, since we want to work in a minimal basis of operators, we will also need to consider reducible diagrams in a computation that resembles that of an S-matrix element.

In particular, we will compute the divergent part of the amplitude for the elastic scattering of a heavy quark with a transverse gluon, which allows to determine the anomalous dimensions of the heavy-gluon operators, and of a heavy quark with a massless quark, which allows to determine the anomalous dimensions of the heavy-light operators. Both computations are carried at one-loop order. In general, external particles will be considered to be on shell i.e. free asymptotic states, so the free EOMs[5] will be used throughout. The computation is done in the Coulomb gauge and in dimensional regularization. The divergences coming from Feynman diagrams cancel with the divergences of the Wilson coefficients determining their anomalous dimension. The computation is organized in powers of $1/m$, up to $\mathcal{O}(1/m^3)$, by considering all possible insertions of the HQET Lagrangian operators. This statement requires some qualifications once $1/m^3$ operators with massless fermions fields are involved. For those, we do not seek the anomalous dimension of all their Wilson coefficients and we are not exhaustive in the search of a complete basis of operators. The reason is that most of them start to contribute at NLL, playing a subleading role in heavy quarkonium physics[6]. We will only consider those that get LL running paying special attention to those that contribute to the LL running of the chromopolarizabilities.

As a cross-check, we will also compute the elastic scattering of a heavy quark with a longitudinal gluon, which allows us to determine the anomalous dimension of the combinations $2c_{A_1}+c_{A_2}$, $2c_{A_3}+c_{A_4}$ and $c_{B_1}+c_{B_2}$. Furthermore, we compute the one transverse gluon-matrix element of the heavy quark, which allows us to cross-check the anomalous dimension of c_4, c_M, c_{W_1}, c_{W_2} and $c_{p'p}$. In the latter, only irreducible diagrams enter the calculation.

It is important to recall that the Compton scattering analysis at $\mathcal{O}(1/m^3)$ showed that c_4, c_{A_1}, $\bar{c}_{A_2}=c_{A_2}-4c_M$, c_{A_3}, c_{A_4}, $\bar{c}_W=c_{W_1}-c_{W_2}$, $\bar{c}_{B_1}=c_{B_1}-2c_{W_1}$, c_{B_2} and $c_{p'p}$ are physical combinations i.e. they are gauge-independent. This observation will be crucial in order to determine which combinations of Wilson coefficients associated to heavy-light operators are gauge-independent.

The Wilson coefficients of the kinetic term will be kept explicit for tracking purposes even though they are protected by reparametrization invariance ($c_k=c_4=1$ to any order in perturbation theory) [45]. We will compute the LL running of c_4 as a check. The Wilson coefficients $c_{p'p}=c_F-1$ and the physical combination $\bar{c}_W=c_{W_1}-c_{W_2}=1$ are fixed by reparametrization invariance [28], as well. We will check by explicit calculation that all these relations are satisfied at LL. In principle, c_M is also fixed by reparametrization invariance. It was originally determined in Ref. [28]. Recently, a new result, $2c_M = c_D-c_F$, was obtained [44], which differs by a sign of the old one. Working off shell, our computation in QED in Sec. 2.5.4 indeed confirms (at one loop and in the Coulomb gauge)

---

[5]The heavy quark EOM is $E=c_k\frac{\mathbf{p}^2}{2m}$, the massless quark EOM is $\not{k}_i=0$, $i=1,2$ and the gluon EOM is $\mathbf{k}_i^2=(k_i^0)^2$, $i=1,2$.

[6]They play a subleading role, for instance, in the determination of the N^{3}LL heavy quarkonium spectrum. Indeed, only the ones associated to spin-independent operators are needed to this order. The Wilson coefficients of the spin-dependent operators start to be relevant at higher orders and they are computed just for completeness.

$2c_M = c_D - c_F$. Nevertheless, in QCD working on shell and in the Coulomb gauge, we observe that this relation is not satisfied. More precisely, the RGE we find for c_M at one-loop order is not equal to the one of $c_D - c_F$. The constraints imposed by reparametrization invariance once operators with light fermion fields are included have also been studied in Ref. [44] for the QED case, where the relations[7] $d_1^{hl} = c_1^{hl}/16$ and $d_4^{hl} = -c_2^{hl}/16$ were deduced. The results we have obtained for QED are in agreement with these relations. However, for QCD working on shell and in the Coulomb gauge the relation for d_1^{hl} is violated, as it happened with c_M. More precisely, the RGE we find for d_1^{hl} at one-loop order is not equal to the one of $c_1^{hl}/16$. Indeed, we can see in Sec. 2.5.3 and in App. B.3 that the violation of these two equations seems to be related. Nevertheless, the Wilson coefficient d_1^{hl} does not enter into the running of the chromopolarizabilities.

The Coulomb gauge will be used throughout this computation. On the one hand, this significantly reduces the number of diagrams but, on the other hand, the complexity of each one of them increases. It also makes it difficult to use standard routines designed for computations of Feynman diagrams in covariant gauges and relativistic setups. However, since we are only looking for the UV pole, the calculation is feasible. The renormalization of the heavy quark field, gluon fields, the strong coupling g and the Wilson coefficients c_F, c_D and c_S are needed. The relations between the bare and renormalized fields and couplings are

$$
\begin{aligned}
&g_B = Z_g g_R\,, \quad \mathbf{A}_B = Z_{\mathbf{A}}^{1/2}\mathbf{A}_R\,, \quad A^0_B = Z_{A^0}^{1/2}A^0_R\,, \quad \psi_B = Z_h^{1/2}\psi_R\,, \quad q_B = Z_l^{1/2}q_R\,,\\
&c_{k,B} = c_{k,R}\,, \quad c_{F,B} = c_{F,R} + \delta c_F\,, \quad c_{S,B} = c_{S,R} + \delta c_S\,, \quad c_{D,B} = c_{D,R} + \delta c_D\,, \qquad (2.67)
\end{aligned}
$$

where the subscript B stands for bare and R for renormalized quantities. Often the subscript R will be removed in the following when it is understood. In the Coulomb gauge, the renormalization constants read (we define $D = 4 + 2\epsilon$ and $d = 3 + 2\epsilon$):

$$
\begin{aligned}
&Z_{A^0}^{-1/2} = Z_g = 1 + \frac{11}{6}C_A\frac{\alpha}{4\pi}\frac{1}{\epsilon} - \frac{2}{3}T_F n_f\frac{\alpha}{4\pi}\frac{1}{\epsilon}\,, \qquad Z_{\mathbf{A}}^{1/2} = 1 - \frac{C_A}{2}\frac{\alpha}{4\pi}\frac{1}{\epsilon} - \frac{2}{3}T_F n_f\frac{\alpha}{4\pi}\frac{1}{\epsilon}\,,\\
&Z_g^2 Z_{\mathbf{A}} = 1 + \frac{8}{3}C_A\frac{\alpha}{4\pi}\frac{1}{\epsilon}\,, \quad Z_l = 1 + C_F\frac{\alpha}{4\pi}\frac{1}{\epsilon}\,, \quad Z_h = 1 + \frac{\mathbf{p}^2}{m^2}\frac{4}{3}C_F\frac{\alpha}{4\pi}\frac{1}{\epsilon}\,,\\
&\delta c_F = -c_{F,R}C_A\frac{\alpha}{4\pi}\frac{1}{\epsilon}\,, \quad \delta c_S = -2c_{F,R}C_A\frac{\alpha}{4\pi}\frac{1}{\epsilon}\,,\\
&\delta c_D = -\frac{1}{3}\left(11C_A c_{D,R} - 16(C_A + C_F)c_{k,R}^2 - 5C_A c_{F,R}^2\right)\frac{\alpha}{4\pi}\frac{1}{\epsilon}\,, \qquad (2.68)
\end{aligned}
$$

where

$$
C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}\,, \qquad C_A = N_c = 3\,. \qquad (2.69)
$$

Since for any Wilson coefficient $c_{i,B} = c_{i,R}(\nu) + \delta c_i(\nu)$, once the counterterm $\delta c_i(\nu)$ is determined, its RGE can be obtained from the expression

[7] Look at Eqs. (2.72) and (2.75) in order to see the operators with associated Wilson coefficients d_1^{hl} and d_4^{hl}.

$$\nu \frac{d}{d\nu} c_{i,R} = -\nu \frac{d}{d\nu} \delta c_i = \gamma_{c_i} \,, \tag{2.70}$$

where γ_{c_i} is the anomalous dimension of the Wilson coefficient c_i, which is a finite quantity. We will only present explicitly the RGEs, but a reader interested in obtaining the counterterms can do it straightforwardly from the anomalous dimension

$$c_{i,B} = c_{i,R}(\nu) - \frac{1}{2\epsilon} \gamma_{c_i} \,. \tag{2.71}$$

2.5.1 Anomalous dimensions of $1/m^3$ heavy-light operators

Let us consider firstly the inclusion of spectator quarks to the computation. The coefficients c_D^l, c^{ll} and the ones associated to $1/m^3$ light-light operators are NLL, and we will neglect them in the following. The c_i^{hl}, associated to $1/m^2$ heavy-light operators, were computed with LL accuracy in Ref. [32]. The Wilson coefficients c_i^{hl} and d_i^{hl}, the latter associated to $1/m^3$ heavy-light operators, evaluated at the hard scale are of $\mathcal{O}(\alpha)$, so at the order of interest, i.e. at leading order, their matching condition is zero. This is so because such operators can not be generated at tree level in the underlying theory, QCD. Given this condition, the only way they can get LL running is through mixing with other Wilson coefficients that get LL running.

In order to determine which operators of Eqs. (2.39-2.52) are relevant i.e. which operators get LL running, we compute the scattering of a heavy quark with a massless quark at one loop order. As previously mentioned, the Wilson coefficients associated to these operators will get LL running if there is a mixing with other Wilson coefficients that get LL running. That is, if there is a mixing with the Wilson coefficients of the heavy-gluon sector up to $\mathcal{O}(1/m^3)$ or with the Wilson coefficients associated to heavy-light operators up to $\mathcal{O}(1/m^2)$. Obviously, one also has to compute the self-running with the Wilson coefficients associated to the $\mathcal{O}(1/m^3)$ heavy-light operators which get LL running. Divergences coming from Feynman diagrams will be absorbed in the Wilson coefficients d_i^{hl} determining their running. What we find is what we already advanced in previous sections, not all the operators in Eqs. (2.39-2.52) get LL running, but only a combination of some of them. The other possible operators are irrelevant for our calculation because their associated Wilson coefficients do not mix with Wilson coefficients that get LL running and their matching condition, i.e. the Wilson coefficient evaluated at the hard scale, is zero at tree level. At least their matching condition is $\mathcal{O}(\alpha)$. These two properties together make the contribution of these operators subleading. In particular, there are eleven different operators relevant for our discussion, which read

$$
\begin{aligned}
\mathcal{O}_1 &= \mathcal{M}_{4+}^{(3h)o} + \mathcal{M}_{4-}^{(3h)o}\,, & (2.72)\\
\mathcal{O}_2 &= \mathcal{M}_{3+}^{(3l)o} + \mathcal{M}_{3-}^{(3l)o}\,, & (2.73)\\
\mathcal{O}_3 &= \mathcal{M}_{3+}^{(3l)s} + \mathcal{M}_{3-}^{(3l)s}\,, & (2.74)\\
\mathcal{O}_4 &= \mathcal{M}_{7+}^{(3h)o} + \mathcal{M}_{7-}^{(3h)o}\,, & (2.75)\\
\mathcal{O}_5 &= \mathcal{M}_{7+}^{(3h)s} + \mathcal{M}_{7-}^{(3h)s}\,, & (2.76)\\
\mathcal{O}_6 &= \mathcal{M}_{6+}^{(3l)o} + \mathcal{M}_{6-}^{(3l)o}\,, & (2.77)\\
\mathcal{O}_7 &= \mathcal{M}_{6+}^{(3l)s} + \mathcal{M}_{6-}^{(3l)s}\,, & (2.78)\\
\mathcal{O}_8 &= \mathcal{M}_{7+}^{(3l)o} + \mathcal{M}_{7-}^{(3l)o}\,, & (2.79)\\
\mathcal{O}_9 &= \mathcal{M}_{7+}^{(3l)s} + \mathcal{M}_{7-}^{(3l)s}\,, & (2.80)\\
\mathcal{O}_{10} &= \mathcal{M}_{10+}^{(3l)o} - \mathcal{M}_{10-}^{(3l)o}\,, & (2.81)\\
\mathcal{O}_{11} &= \mathcal{M}_{10+}^{(3l)s} - \mathcal{M}_{10-}^{(3l)s}\,, & (2.82)
\end{aligned}
$$

where Eqs. (2.72-2.74) are spin-independent operators, so they are relevant to obtain the $\mathrm{N^3LL}$ heavy quarkonium spectrum, whereas Eqs. (2.75-2.82) are spin-dependent ones, so they start to be relevant to compute the $\mathrm{N^4LO}$ heavy quarkonium spectrum. The Feynman rules associated to these operators are displayed in App. B.1. The running of these operators is obtained from the topologies drawn in Fig. 2.2. They produce around 124 diagrams to be computed without counting crossed and inverted ones. The RGEs we obtain are

$$
\nu\frac{d}{d\nu}d_1^{hl} = \left[\frac{1}{4}(2\beta_0 - 3C_A)d_1^{hl} - C_A\left(\frac{1}{96}c_M - \frac{1}{192}c_Sc_F - \frac{1}{16}c_k^3 - \frac{5}{64}c_kc_F^2\right)\right]\frac{\alpha}{\pi}\,, \tag{2.83}
$$

$$
\begin{aligned}
\nu\frac{d}{d\nu}d_2^{hl} = &\Bigg[d_2^{hl}\left(\frac{4}{3}C_F - \frac{17}{12}C_A + \frac{\beta_0}{2}\right) - \frac{1}{4}c_3^{hl}c_k - \frac{1}{16}c_2^{hl}c_FC_A\\
&- (8C_F - 3C_A)\bigg(\frac{1}{24}c_M - \frac{1}{96}c_{A_2} + \frac{1}{16}c_4 + \frac{1}{48}c_Sc_F\\
&- \frac{5}{48}c_k^3 - \frac{5}{24}c_kc_F^2 + \frac{1}{16}c_Dc_k + \frac{1}{16}c_1^{hl}c_k\bigg)\Bigg]\frac{\alpha}{\pi}\,,
\end{aligned} \tag{2.84}
$$

$$
\begin{aligned}
\nu\frac{d}{d\nu}d_3^{hl} = &\Bigg[\left(\frac{4}{3}C_F + \frac{\beta_0}{2}\right)d_3^{hl} - C_F(C_A - 2C_F)\bigg(\frac{1}{12}c_M - \frac{1}{48}c_{A_2} - \frac{1}{48}c_{A_4}\\
&+ \frac{1}{8}c_4 + \frac{1}{24}c_Sc_F - \frac{5}{24}c_k^3 - \frac{5}{12}c_kc_F^2 + \frac{1}{8}c_Dc_k + \frac{1}{8}c_1^{hl}c_k\bigg)\Bigg]\frac{\alpha}{\pi}\,,
\end{aligned} \tag{2.85}
$$

$$
\begin{aligned}
\nu\frac{d}{d\nu}d_4^{hl} = &\frac{\alpha}{\pi}\Bigg[(8C_F - 3C_A)\bigg(\frac{1}{32}c_{W_1} - \frac{1}{32}c_{W_2} + \frac{1}{16}c_{p'p} + \frac{1}{64}c_Sc_k\\
&- \frac{1}{32}c_Sc_F - \frac{5}{32}c_Fc_k^2 + \frac{5}{64}c_F^2c_k\bigg) - \frac{1}{4}d_4^{hl}(3C_A - 2\beta_0)\Bigg]\,,
\end{aligned} \tag{2.86}
$$

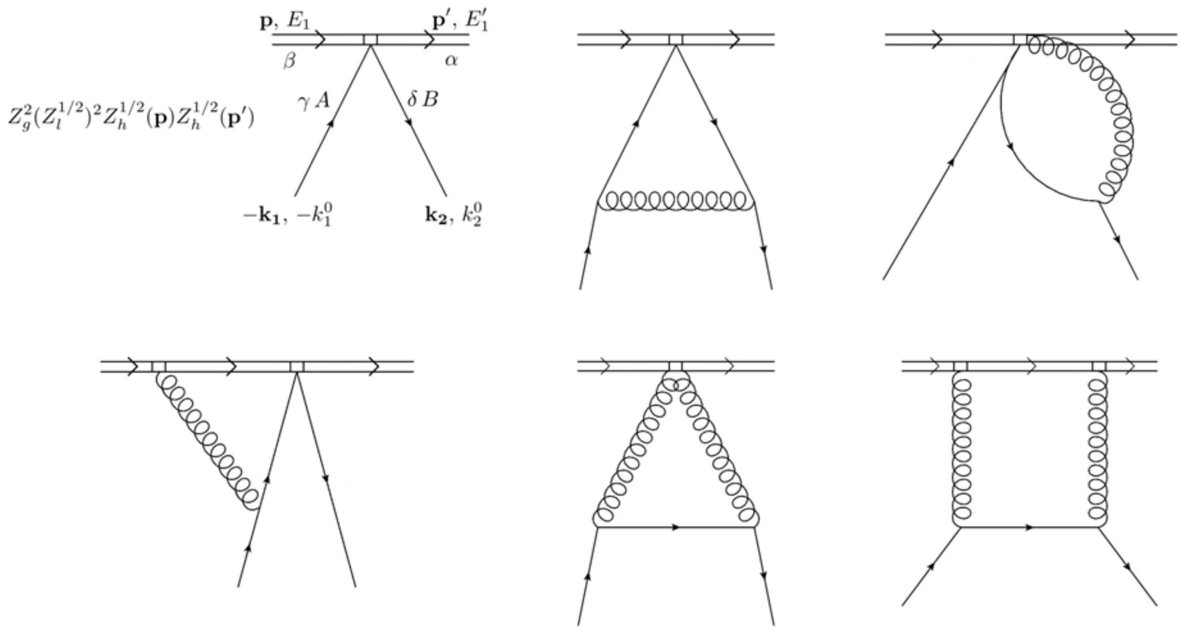


Figure 2.2: Topologies contributing to the LL running of the Wilson coefficients associated to the $1/m^3$ heavy-light operators. The first diagram is the tree level diagram multiplied by the renormalization of the external fields and coupling. The other diagrams are the one-loop topologies that also contribute. In general the depicted gluon can be either longitudinal or transverse. All possible vertices and insertions with the right counting in $1/m$ should be considered to generate the diagrams.

$$\begin{aligned}\nu\frac{d}{d\nu}d_5^{hl} &= \frac{\alpha}{\pi}\Bigg[C_F(2C_F-C_A)\bigg(-\frac{1}{16}c_{W_1}+\frac{1}{16}c_{W_2}-\frac{1}{8}c_{p'p}-\frac{1}{32}c_Sc_k \\ &\quad +\frac{1}{16}c_Sc_F+\frac{5}{16}c_Fc_k^2-\frac{5}{32}c_F^2c_k\bigg)+\frac{1}{2}d_5^{hl}\beta_0\Bigg], \end{aligned} \tag{2.87}$$

$$\begin{aligned}\nu\frac{d}{d\nu}d_6^{hl} &= \frac{\alpha}{\pi}\Bigg[\frac{1}{192}c_{B_1}C_A+\frac{1}{192}c_Sc_FC_A-\frac{5}{96}c_F^2c_kC_A+\frac{1}{64}c_2^{hl}c_F(8C_F-3C_A) \\ &\quad +\frac{1}{16}c_4^{hl}c_F+\frac{1}{3}d_4^{hl}(8C_F-3C_A)+\frac{4}{3}d_5^{hl}+\frac{1}{6}d_6^{hl}(5C_F-5C_A+3\beta_0) \\ &\quad +\frac{5}{12}d_8^{hl}(2C_F-C_A)+\frac{1}{12}d_{10}^{hl}C_A\Bigg], \end{aligned} \tag{2.88}$$

$$\begin{aligned}\nu\frac{d}{d\nu}d_7^{hl} &= \frac{\alpha}{\pi}\Bigg[-C_F(2C_F-C_A)\bigg(\frac{1}{32}c_2^{hl}c_F+\frac{2}{3}d_4^{hl}\bigg)+\frac{1}{6}d_7^{hl}(5C_F+3\beta_0) \\ &\quad +\frac{5}{6}d_9^{hl}C_F\Bigg], \end{aligned} \tag{2.89}$$

$$
\begin{aligned}
\nu\frac{d}{d\nu}d_8^{hl} &= \frac{\alpha}{\pi}\Bigg[-\frac{1}{32}c_{W_1}C_A-\frac{1}{192}c_{B_1}C_A-\frac{1}{96}c_{B_2}C_A-\frac{1}{64}c_Dc_FC_A-\frac{1}{64}c_Sc_kC_A \\
&+\frac{1}{192}c_Sc_FC_A+\frac{5}{32}c_Fc_k^2C_A-\frac{5}{96}c_F^2c_kC_A-\frac{1}{64}c_1^{hl}c_FC_A-\frac{1}{16}c_2^{hl}c_k(8C_F-3C_A) \\
&+\frac{1}{32}c_2^{hl}c_F(8C_F-3C_A)-\frac{1}{4}c_4^{hl}c_k+\frac{1}{8}c_4^{hl}c_F-\frac{1}{12}d_4^{hl}(8C_F-3C_A) \\
&-\frac{1}{3}d_5^{hl}+\frac{1}{6}d_6^{hl}(3C_F-2C_A)+\frac{1}{2}d_8^{hl}(C_F-2C_A+\beta_0)+\frac{1}{6}d_{10}^{hl}C_A\Bigg], \qquad (2.90)
\end{aligned}
$$

$$
\begin{aligned}
\nu\frac{d}{d\nu}d_9^{hl} &= \frac{\alpha}{\pi}\Bigg[C_F(2C_F-C_A)\left(\frac{1}{8}c_2^{hl}c_k-\frac{1}{16}c_2^{hl}c_F+\frac{1}{6}d_4^{hl}\right)+\frac{1}{2}d_7^{hl}C_F \\
&+\frac{1}{2}d_9^{hl}(C_F+\beta_0)\Bigg], \qquad (2.91)
\end{aligned}
$$

$$
\begin{aligned}
\nu\frac{d}{d\nu}d_{10}^{hl} &= \frac{\alpha}{\pi}\Bigg[\frac{1}{48}c_{W_1}C_A+\frac{1}{192}c_{W_2}C_A-\frac{7}{192}c_{B_1}C_A-\frac{1}{48}c_{B_2}C_A+\frac{1}{384}c_{p'p}C_A \\
&+\frac{1}{128}c_Dc_FC_A+\frac{1}{96}c_Sc_kC_A-\frac{7}{384}c_Sc_FC_A+\frac{5}{64}c_F^2c_kC_A \\
&+\frac{1}{128}c_1^{hl}c_FC_A-\frac{1}{128}c_2^{hl}c_F(8C_F-3C_A)-\frac{1}{32}c_4^{hl}c_F \\
&+\frac{1}{24}d_4^{hl}(8C_F-3C_A)+\frac{1}{6}d_5^{hl}-\frac{1}{24}d_6^{hl}(4C_F-3C_A) \\
&-\frac{1}{12}d_8^{hl}(2C_F-C_A)-\frac{1}{24}d_{10}^{hl}(11C_A-12\beta_0)\Bigg], \qquad (2.92)
\end{aligned}
$$

$$
\begin{aligned}
\nu\frac{d}{d\nu}d_{11}^{hl} &= \frac{\alpha}{\pi}\Bigg[C_F(2C_F-C_A)\left(\frac{1}{64}c_2^{hl}c_F-\frac{1}{12}d_4^{hl}\right)-\frac{1}{6}d_7^{hl}C_F \\
&-\frac{1}{6}d_9^{hl}C_F+\frac{1}{2}d_{11}^{hl}\beta_0\Bigg]. \qquad (2.93)
\end{aligned}
$$

Note that, as we previously advanced, d_1^{hl} does not satisfy reparametrization invariance[8]. However, we will see later on that it does not contribute to the running of the gauge-independent chromopolarizabilities. The RGEs of the remaining Wilson coefficients d_i^{hl}, with $i, j > 11$, have the structure

$$
\nu\frac{d}{d\nu}d_i^{hl} = \frac{\alpha}{\pi}A_{ij}d_j^{hl}\,. \qquad (2.94)
$$

and, for this reason, they are NLL.

[8]See Sec. 2.5.3 and App. B.3 for a more detailed discussion about this issue.

2.5.2 Anomalous dimensions of $1/m^3$ heavy-gluon operators

Let us consider the $1/m^3$ heavy-gluon operators. Firstly, for the pure gluonic sector, we have that c_1^g is NLL, so we will neglect it in the following.

The running of the unphysical set of Wilson coefficients, Eq. (2.64), is determined from the topologies drawn in Fig. 2.3. From these, we generate all possible diagrams up to order $1/m^3$ by considering all possible vertices to the appropriate order in $1/m$ and/or kinetic insertions, which correspond to the expansion of the non-static heavy quark propagator. Note that diagrams of lower order than $1/m^3$ also must be considered, at least those that depend on the energy, as the use of the heavy quark EOM, $E = c_k \frac{\mathbf{p}^2}{2m}$, adds extra powers of $1/m$. This generates around 420 diagrams (without taking into account permutations and crossing) in both cases: the elastic scattering with a transverse gluon and, similarly, with a longitudinal gluon.

In the case of scattering with a transverse gluon, for diagrams proportional to $1/m^3$ operators, only irreducible ones need to be considered. Note that this is not true for the case of scattering with a longitudinal gluon because the Coulomb vertex does not add extra powers of $1/m$. When one considers diagrams proportional to iterations of $1/m^2$ and/or $1/m$ operators one also has to consider reducible diagrams in both cases. One has to keep in mind that Taylor expanding reducible diagrams in the energy can produce non-local terms which cancel at the end of the calculation and all divergences can be absorbed by local counterterms that correspond to operators of the Lagrangian[9]. It is also worth mentioning that we find that the sum of all reducible diagrams whose sub-irreducible one-loop diagram is $1/m$ or below ($1/m^2$ or below in the case of the scattering with a longitudinal gluon) cancel with the renormalization of the tree level reducible diagrams. Therefore, non-local terms coming from expanding these diagrams in the energy vanish at all orders in the expansion.

Let us consider the calculation of the one transverse gluon exchange, which has a peculiarity which deserves a comment. This calculation allows us to determine the anomalous dimensions of c_4, c_M, c_{W_1}, c_{W_2}, $c_{p'p}$ and c_S. The necessary topologies to produce the diagrams are shown in Fig. 2.4. They produce around 100-120 diagrams without counting inverted ones. Note that there are only irreducible diagrams in this case. What is interesting in this calculation is that one obtains a spin-dependent structure which does not look like any structure of the spin-dependent $1/m^3$ operators, i.e. the $1/m^3$ vertices with a single transverse gluon. So at first sight, it would look like a problem, since the divergence proportional to this structure could not be absorbed by any operator in the theory, leading one to suspect that there might be operators missing. However, this is not the case. The explanation is the following: in principle, one would consider c_S as an $\mathcal{O}(1/m^2)$ operator. Nevertheless, the vertex with an external transverse gluon is proportional to k^0, so it becomes $\mathcal{O}(1/m^3)$ after using the heavy quark EOM. Once we use it, the resulting structure is the one that could not be absorbed by the $1/m^3$ operators. Therefore, in order properly renormalize the theory and to determine the running of c_S through the calculation of the

[9]If we only compute irreducible diagrams we would need a larger number of operators, in particular those that vanish on shell.

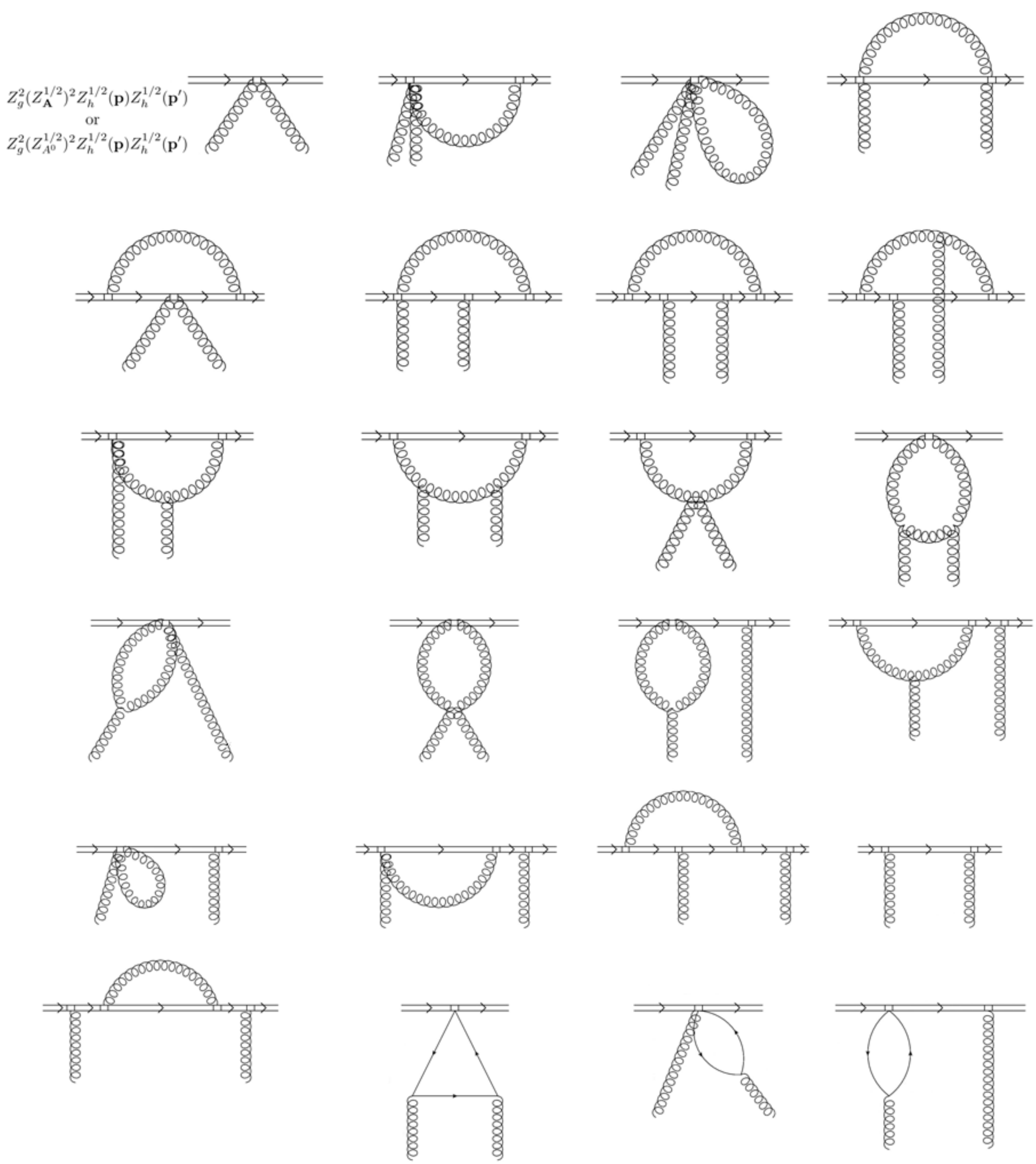


Figure 2.3: Topologies contributing to the anomalous dimensions of the Wilson coefficients associated to $1/m^3$ heavy-gluon operators in QCD. The double-line represents the heavy fermion, whereas the curly line represents either a transverse or a longitudinal gluon. Both external gluons are transverse or longitudinal depending on the kind of scattering we are considering. All diagrams are generated from these topologies by considering all possible vertices up to $\mathcal{O}(1/m^3)$. Tree level diagrams should be understood to be multiplied by Wilson coefficient, field and strong coupling counterterms.

one transverse gluon exchange, one must consider this operator[10] as an $\mathcal{O}(1/m^3)$ operator.

[10]The operator proportional to c_S responsible to the one transverse gluon exchange.

Only in this way is the correct running of c_S (expected from reparametrization invariance) obtained. So everything must be made physical, meaning put on shell, in order to arrive to proper results. Note that the running of c_S will appear also in the determination of the running of Wilson coefficients at higher orders in $1/m$ if it is done through the calculation of the one transverse gluon matrix element of a heavy quark, because the heavy quark EOM has corrections in $1/m$. In particular, it will appear at $\mathcal{O}(1/m^5)$. This is important to keep in mind in future calculations. Another important aspect of the calculation is that it allows us to determine the running of c_M independently of c_{A_2}. The result is in agreement with the one obtained from the two transverse gluon exchange, which in turn is in disagreement with reparametrization invariance.

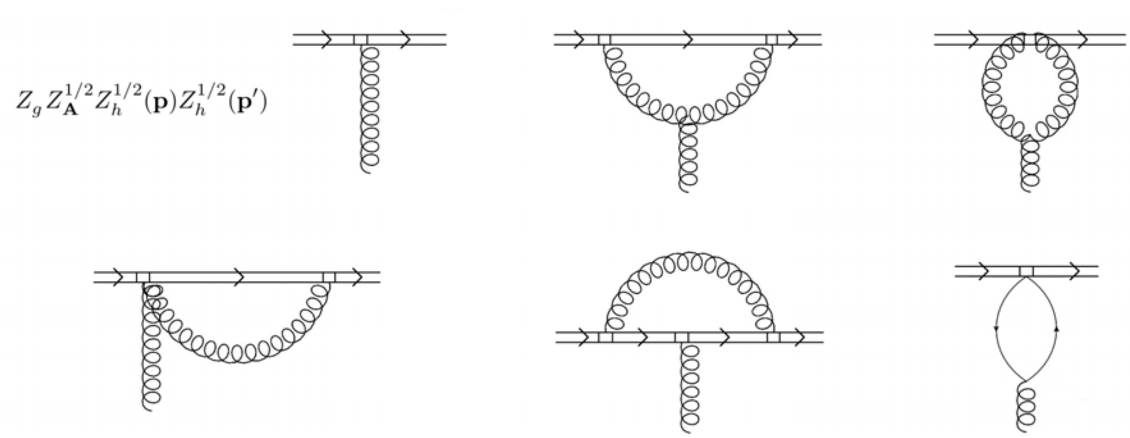


Figure 2.4: Topologies contributing to the one transverse gluon exchange. All diagrams are generated from these topologies by considering all possible vertices up to $\mathcal{O}(1/m^3)$. While the external gluon is transverse, internal gluons must be understood as either longitudinal or transverse.

Since we work in the massless limit, diagrams proportional to d_i^{hl} do not depend on the specific light fermion, so the result of each of them has to be multiplied by the number of light fermions n_f.

The RGEs for the unphysical set Eq.(2.64), in the Coulomb gauge, read

$$\nu \frac{d}{d\nu} c_4 = 0 \,, \tag{2.95}$$

$$\begin{aligned} \nu \frac{d}{d\nu} c_M \;=\; & \frac{\alpha}{\pi} \left[\frac{7}{6} c_M C_A + \frac{1}{2} c_4 C_A - \frac{1}{3} c_S c_F C_A - \frac{1}{6} c_k^3 (8C_F + 11C_A) \right. \\ & \left. - \frac{1}{12} c_F^2 c_k C_A + \frac{16}{3} d_1^{hl} T_F n_f \right] , \end{aligned} \tag{2.96}$$

$$\nu\frac{d}{d\nu}c_{A_1} = \frac{\alpha}{\pi}\left[\frac{1}{2}c_{A_1}C_A - \frac{11}{48}c_{A_2}C_A + \frac{11}{12}c_M C_A - c_4 C_A - \frac{1}{24}c_S c_F C_A \right.$$
$$\left. +\frac{1}{3}c_k^3(16C_F + 11C_A) + \frac{3}{2}c_F^3 C_A + \frac{73}{24}c_F^2 c_k C_A - \frac{16}{3}d_2^{hl}T_F n_f\right], \qquad (2.97)$$

$$\nu\frac{d}{d\nu}c_{A_2} = \frac{\alpha}{\pi}\left[-c_{A_1}C_A + \frac{11}{12}c_{A_2}C_A + 10c_4 C_A + c_M C_A - \frac{7}{6}c_S c_F C_A - \frac{2}{3}c_k^3(40C_F + 39C_A)\right.$$
$$\left. -3c_F^3 C_A - \frac{15}{2}c_F^2 c_k C_A + \frac{64}{3}T_F n_f d_1^{hl} + \frac{64}{3}T_F n_f d_2^{hl}\right], \qquad (2.98)$$

$$\nu\frac{d}{d\nu}c_{A_3} = \frac{\alpha}{\pi}\left[-\frac{1}{2}c_{A_1}C_A - \frac{11}{48}c_{A_2}C_A + \frac{11}{12}c_M C_A - \frac{11}{24}c_{A_4}C_A - \frac{1}{24}c_S c_F C_A\right.$$
$$\left. -\frac{3}{2}c_F^3 C_A + \frac{73}{24}c_F^2 c_k C_A + \frac{16}{3}d_2^{hl}T_F n_f - \frac{32}{3}d_3^{hl}C_A T_F n_f\right], \qquad (2.99)$$

$$\nu\frac{d}{d\nu}c_{A_4} = \frac{\alpha}{\pi}\left[c_{A_1}C_A + \frac{11}{12}c_{A_2}C_A - \frac{11}{3}c_M C_A + \frac{11}{6}c_{A_4}C_A + \frac{8}{3}c_4 C_A - \frac{17}{6}c_S c_F C_A\right.$$
$$\left. -\frac{8}{3}c_k^3 C_A + 3c_F^3 C_A - \frac{31}{6}c_F^2 c_k C_A - \frac{64}{3}d_2^{hl}T_F n_f + \frac{128}{3}d_3^{hl}C_A T_F n_f\right], \qquad (2.100)$$

$$\nu\frac{d}{d\nu}c_{W_1} = \frac{\alpha}{\pi}\left[\frac{13}{12}c_{W_1}C_A + \frac{7}{12}c_{W_2}C_A - \frac{1}{4}c_{B_1}C_A - \frac{1}{8}c_{B_2}C_A + \frac{1}{24}c_{p'p}C_A\right.$$
$$+\frac{7}{24}c_S c_k C_A - \frac{1}{6}c_S c_F C_A - \frac{1}{12}c_F c_k^2(16C_F + 15C_A)$$
$$\left. +\frac{7}{8}c_F^2 c_k C_A - T_F n_f\left(\frac{8}{3}d_6^{hl} - \frac{8}{3}d_8^{hl} + \frac{16}{3}d_{10}^{hl}\right)\right], \qquad (2.101)$$

$$\nu\frac{d}{d\nu}c_{W_2} = \frac{\alpha}{\pi}\left[\frac{7}{12}c_{W_1}C_A + \frac{13}{12}c_{W_2}C_A - \frac{1}{4}c_{B_1}C_A - \frac{1}{8}c_{B_2}C_A + \frac{1}{24}c_{p'p}C_A\right.$$
$$-\frac{5}{24}c_S c_k C_A - \frac{1}{6}c_S c_F C_A - \frac{1}{12}c_F c_k^2(16C_F + 3C_A)$$
$$\left. +\frac{7}{8}c_F^2 c_k C_A - T_F n_f\left(\frac{8}{3}d_6^{hl} - \frac{8}{3}d_8^{hl} + \frac{16}{3}d_{10}^{hl}\right)\right], \qquad (2.102)$$

$$\nu\frac{d}{d\nu}c_{B_1} = \frac{\alpha}{\pi}\left[\frac{1}{6}c_{W_1}C_A + \frac{1}{6}c_{W_2}C_A + c_{B_1}C_A - \frac{1}{3}c_{B_2}C_A + \frac{7}{12}c_{p'p}C_A\right.$$
$$+\frac{1}{12}c_S c_k C_A - \frac{1}{4}c_S c_F C_A + \frac{7}{6}c_F c_k^2 C_A + \frac{7}{6}c_F^2 c_k C_A$$
$$\left. +\frac{3}{2}c_F^3 C_A - T_F n_f\left(\frac{8}{3}d_6^{hl} - 8d_8^{hl} + \frac{32}{3}d_{10}^{hl}\right)\right], \qquad (2.103)$$

$$\nu \frac{d}{d\nu} c_{B_2} = \frac{\alpha}{\pi}\Bigg[c_{W_2} C_A - \frac{1}{2} c_{B_1} C_A + \frac{7}{6} c_{B_2} C_A - \frac{4}{3} c_S c_k (4C_F + C_A) - \frac{1}{6} c_S c_F C_A + \frac{4}{3} c_F c_k^2 (2C_F - C_A) + \frac{2}{3} c_F^2 c_k C_A - \frac{3}{2} c_F^3 C_A - T_F n_f \left(\frac{16}{3} d_6^{hl} + \frac{16}{3} d_8^{hl}\right)\Bigg], \tag{2.104}$$

$$\nu \frac{d}{d\nu} c_{p'p} = \frac{\alpha}{\pi}\left[\frac{1}{2} c_{p'p} C_A - \frac{1}{2} c_S c_k C_A + c_F c_k^2 C_A\right]. \tag{2.105}$$

Note that, as we previously advanced, c_M does not satisfy reparametrization invariance[11].

2.5.3 Anomalous dimensions of physical quantities

Previously, in Sec. 2.5.1 and Sec. 2.5.2, we found the running of the Wilson coefficients associated to the $1/m^3$ HQET Lagrangian operators including spectator quarks. However, it is well known from the analysis done in Sec. 2.4 that Eqs. (2.96, 2.98, 2.101-2.103) are not physical. The gauge dependence of these RGEs imply that Eqs. (2.83, 2.88, 2.90, 2.92) are also gauge dependent.

The aim of this section is to find the RGEs of the gauge-independent combinations of Wilson coefficients. For the heavy-gluon operators, these combinations were found in Sec. 2.4. So the next step, is to compute the RGEs for the the known physical quantities c_4, c_{A_1}, $\bar{c}_{A_2} = c_{A_2} - 4c_M$, c_{A_3}, c_{A_4} $\bar{c}_W = c_{W_1} - c_{W_2}$, $\bar{c}_{B_1} = c_{B_1} - 2c_{W_1}$, c_{B_2} and $c_{p'p}$. They read

$$\nu \frac{d}{d\nu} c_4 = 0, \tag{2.106}$$

$$\nu \frac{d}{d\nu} c_{A_1} = \frac{\alpha}{\pi}\Bigg[\frac{1}{2} c_{A_1} C_A - \frac{11}{48} \bar{c}_{A_2} C_A - c_4 C_A - \frac{1}{24} c_S c_F C_A + \frac{1}{3} c_k^3 (16C_F + 11C_A) + \frac{3}{2} c_F^3 C_A + \frac{73}{24} c_F^2 c_k C_A - \frac{16}{3} d_2^{hl} T_F n_f\Bigg], \tag{2.107}$$

$$\nu \frac{d}{d\nu} \bar{c}_{A_2} = \frac{\alpha}{\pi}\Bigg[- c_{A_1} C_A + \frac{11}{12} \bar{c}_{A_2} C_A + 8 c_4 C_A + \frac{1}{6} c_S c_F C_A - \frac{8}{3} c_k^3 (8C_F + 7C_A) - 3 c_F^3 C_A - \frac{43}{6} c_F^2 c_k C_A + \frac{64}{3} d_2^{hl} T_F n_f\Bigg], \tag{2.108}$$

[11]See Sec. 2.5.3 and App. B.3 for a more detailed discussion about this issue.

$$\nu\frac{d}{d\nu}c_{A_3} = \frac{\alpha}{\pi}\left[-\frac{1}{2}c_{A_1}C_A - \frac{11}{48}\bar{c}_{A_2}C_A - \frac{11}{24}c_{A_4}C_A - \frac{1}{24}c_Sc_FC_A - \frac{3}{2}c_F^3C_A \right.$$
$$\left. +\frac{73}{24}c_F^2c_kC_A + \frac{16}{3}d_2^{hl}T_Fn_f - \frac{32}{3}d_3^{hl}C_AT_Fn_f\right], \tag{2.109}$$

$$\nu\frac{d}{d\nu}c_{A_4} = \frac{\alpha}{\pi}\left[c_{A_1}C_A + \frac{11}{12}\bar{c}_{A_2}C_A + \frac{11}{6}c_{A_4}C_A + \frac{8}{3}c_4C_A - \frac{17}{6}c_Sc_FC_A - \frac{8}{3}c_k^3C_A \right.$$
$$\left. +3c_F^3C_A - \frac{31}{6}c_F^2c_kC_A - \frac{64}{3}d_2^{hl}T_Fn_f + \frac{128}{3}d_3^{hl}C_AT_Fn_f\right], \tag{2.110}$$

$$\nu\frac{d}{d\nu}\bar{c}_W = \frac{\alpha}{\pi}\left[\frac{1}{2}\bar{c}_WC_A + \frac{1}{2}c_Sc_kC_A - c_Fc_k^2C_A\right] = 0\,, \tag{2.111}$$

$$\nu\frac{d}{d\nu}\bar{c}_{B_1} = \frac{\alpha}{\pi}\left[\frac{3}{2}\bar{c}_{B_1}C_A + \bar{c}_WC_A - \frac{1}{12}c_{B_2}C_A + \frac{1}{2}c_{p'p}C_A - \frac{1}{2}c_Sc_kC_A + \frac{1}{12}c_Sc_FC_A \right.$$
$$\left. +\frac{1}{3}c_Fc_k^2(8C_F + 11C_A) - \frac{7}{12}c_F^2c_kC_A + \frac{3}{2}c_F^3C_A + \frac{8}{3}\bar{d}_8^{hl}T_Fn_f\right], \tag{2.112}$$

$$\nu\frac{d}{d\nu}c_{B_2} = \frac{\alpha}{\pi}\left[-\frac{1}{2}\bar{c}_{B_1}C_A - \bar{c}_WC_A + \frac{7}{6}c_{B_2}C_A - \frac{4}{3}c_Sc_k(4C_F + C_A) - \frac{1}{6}c_Sc_FC_A \right.$$
$$\left. +\frac{4}{3}c_Fc_k^2(2C_F - C_A) + \frac{2}{3}c_F^2c_kC_A - \frac{3}{2}c_F^3C_A - \frac{16}{3}\bar{d}_8^{hl}T_Fn_f\right], \tag{2.113}$$

$$\nu\frac{d}{d\nu}c_{p'p} = \frac{\alpha}{\pi}\left[\frac{1}{2}c_{p'p}C_A - \frac{1}{2}c_Sc_kC_A + c_Fc_k^2C_A\right] = \nu\frac{d}{d\nu}c_F\,. \tag{2.114}$$

The last equality in Eqs. (2.111,2.114) can be easily deduced by using relations between Wilson coefficients imposed by reparametrization invariance ($c_k = 1$, $c_S = 2c_F - 1$ and $c_{p'p} = c_F - 1$) [28]. In general, we find that reparametrization invariance is satisfied ($c_4 = 1$, $c_{W_1} - c_{W_2} = 1$, $c_{p'p} = c_F - 1$ and $d_4^{hl} = -c_2^{hl}/16$) even with the inclusion of spectator quarks, but this is not true for $2c_M = c_D - c_F$ and $d_1^{hl} = c_1^{hl}/16$ (See Sec. 2.5.1 and Sec. 2.5.2). This leads us to think that these two relations depend on the gauge, and that in particular, they are satisfied in covariant or Feynman-like gauges, but they are not in gauges that break Lorentz invariance, like the Coulomb gauge. Instead, we propose the following relation between these Wilson coefficients: $2c_M + 16d_1^{hl} = c_D + c_1^{hl} - c_F$, which is satisfied in both, the Feynman and the Coulomb gauges. If we define a new physical quantity $\bar{d}_1^{hl} \equiv c_M + 8d_1^{hl}$, the previous relation can be written as $2\bar{d}_1^{hl} = \bar{c}_1^{hl} - c_F$. Since $\bar{c}_1^{hl}$ and c_F are well known to be gauge-independent, also $\bar{d}_1^{hl}$ must be. We must understand the proposed relation as an ansatz, so it should be demonstrated more rigorously. It is quite remarkable that if we determine c_M in the Feynman gauge using $2c_M^{\rm FG} = c_D^{\rm FG} - c_F$ (the superscript refers to quantities in the Feynman gauge) and compute $d_1^{hl\,{\rm FG}}$ in the Feynman gauge through the relation $\bar{d}_1^{hl} = c_M^{\rm FG} + 8d_1^{hl\,{\rm FG}}$ (where $\bar{d}_1^{hl}$ has been computed in the Coulomb gauge, but it is supposed to be a gauge-independent quantity) we find[12] that $d_1^{hl\,{\rm FG}} = c_1^{hl\,{\rm FG}}/16$, which strengthens our argument that these two relations are gauge dependent, satisfied in the Feynman gauge, and that the truly gauge invariant relation that reparametrization invariance should obtain is $2\bar{d}_1^{hl} = \bar{c}_1^{hl} - c_F$.

Let us find the physical combinations involving the Wilson coefficients of the heavy-light operators. On the one hand, from Eqs. (2.106-2.110) we learn that d_2^{hl} and d_3^{hl} are physical quantities because they appear in the running of physical quantities and they do not combine with each other. Note that the RGE of $\bar{d}_1^{hl} = c_M + 8d_1^{hl}$ only depends on physical quantities and on $\bar{d}_1^{hl}$ itself (see Eq. (2.115)). This, together with the relation $2c_M+16d_1^{hl} = c_D+c_1^{hl}-c_F$ which is satisfied in both, the Coulomb and the Feynman gauges, makes us to think that this quantity is gauge-independent, since the right hand side of the equation is well-known to be gauge-independent. Also note that nor d_1^{hl}, neither $\bar{d}_1^{hl}$ appear in the running of the chromopolarizabilities, so they are not necessary to determine their running. On the other hand, from Eqs. (2.111-2.114), we learn that $\bar{d}_8^{hl} = d_6^{hl} + d_8^{hl}$ must be physical, as it appears in the running of physical combinations. Indeed, since the running of d_6^{hl} and d_8^{hl} can not be written in terms of gauge-independent quantities[13], $\bar{d}_8^{hl}$ must be a physical combination, whereas d_6^{hl} and d_8^{hl} alone are gauge dependent. The gauge independence of the RGE for $\bar{d}_8^{hl}$ also implies the existence of another physical combination, $\bar{d}_{10}^{hl} = 8d_6^{hl}+8d_{10}^{hl}-c_{W_1}$, whose running also depends only on physical quantities, as expected. The running of these two physical combinations also depends on d_4^{hl} and d_5^{hl}, which happen to be gauge-independent, as their running only depends on physical quantities and on themselves, and they do not combine with any gauge dependent quantity in the running of gauge-independent combinations. In particular, d_4^{hl} satisfies reparametrization invariance ($d_4^{hl} = -c_2^{hl}/16$) [44]. The Wilson coefficients d_7^{hl}, d_9^{hl} and d_{11}^{hl} do not mix with $c_{p'p}$, $\bar{c}_W$, $\bar{c}_{B_1}$, c_{B_2}, d_4^{hl}, d_5^{hl}, $\bar{d}_8^{hl}$ and $\bar{d}_{10}^{hl}$, so they are not necessary to determine their running. Since they do not appear in known physical quantities we do not dare to talk about their gauge dependence. The RGEs for the physical set of heavy-light Wilson coefficients read

$$\begin{aligned}\nu\frac{d}{d\nu}\bar{d}_1^{hl} &= \left[\frac{13}{12}\bar{d}_1^{hl}C_A + \frac{1}{2}c_4C_A - \frac{7}{24}c_Sc_FC_A - \frac{4}{3}c_k^3(C_F+C_A)\right.\\ &\quad \left.+\frac{13}{24}c_F^2c_kC_A\right]\frac{\alpha}{\pi}\,, \end{aligned} \tag{2.115}$$

[12]See App. B.3 for a detailed discussion.

[13]If one assumes that d_6^{hl} and d_8^{hl} are gauge-independent, their RGEs can be written only in terms of c_{W_1} and d_{10}^{hl}, which should combine in a gauge-independent way. However, the combination in the RGEs of d_6^{hl} and d_8^{hl} is different. Therefore, this possibility must be discarded.

$$\nu\frac{d}{d\nu}d_2^{hl} = \left[d_2^{hl}\left(\frac{4}{3}C_F-\frac{17}{12}C_A+\frac{\beta_0}{2}\right)-\frac{1}{4}c_3^{hl}c_k-\frac{1}{16}c_2^{hl}c_FC_A\right.$$
$$-(8C_F-3C_A)\left(-\frac{1}{96}\bar{c}_{A_2}+\frac{1}{16}c_4+\frac{1}{48}c_Sc_F-\frac{5}{48}c_k^3\right.$$
$$\left.\left.-\frac{5}{24}c_kc_F^2+\frac{1}{16}\bar{c}_1^{hl}c_k\right)\right]\frac{\alpha}{\pi}\,, \tag{2.116}$$

$$\nu\frac{d}{d\nu}d_3^{hl} = \left[\left(\frac{4}{3}C_F+\frac{\beta_0}{2}\right)d_3^{hl}-C_F(C_A-2C_F)\left(-\frac{1}{48}\bar{c}_{A_2}-\frac{1}{48}c_{A_4}\right.\right.$$
$$\left.\left.+\frac{1}{8}c_4+\frac{1}{24}c_Sc_F-\frac{5}{24}c_k^3-\frac{5}{12}c_kc_F^2+\frac{1}{8}\bar{c}_1^{hl}c_k\right)\right]\frac{\alpha}{\pi}\,, \tag{2.117}$$

$$\nu\frac{d}{d\nu}d_4^{hl} = \frac{\alpha}{\pi}\left[-\frac{1}{4}d_4^{hl}(3C_A-2\beta_0)+(8C_F-3C_A)\left(\frac{1}{32}\bar{c}_W+\frac{1}{16}c_{p'p}\right.\right.$$
$$\left.\left.+\frac{1}{64}c_Sc_k-\frac{1}{32}c_Sc_F-\frac{5}{32}c_Fc_k^2+\frac{5}{64}c_F^2c_k\right)\right]=-\nu\frac{d}{d\nu}\frac{c_2^{hl}}{16}\,, \tag{2.118}$$

$$\nu\frac{d}{d\nu}d_5^{hl} = \frac{\alpha}{\pi}\left[\frac{1}{2}d_5^{hl}\beta_0+C_F(2C_F-C_A)\left(-\frac{1}{16}\bar{c}_W-\frac{1}{8}c_{p'p}\right.\right.$$
$$\left.\left.-\frac{1}{32}c_Sc_k+\frac{1}{16}c_Sc_F+\frac{5}{16}c_Fc_k^2-\frac{5}{32}c_F^2c_k\right)\right]\,, \tag{2.119}$$

$$\nu\frac{d}{d\nu}\bar{d}_8^{hl} = \frac{\alpha}{\pi}\left[-\frac{1}{96}c_{B_2}C_A+\frac{1}{4}d_4^{hl}(8C_F-3C_A)+d_5^{hl}+\frac{1}{12}\bar{d}_8^{hl}(16C_F-17C_A+6\beta_0)\right.$$
$$+\frac{1}{32}\bar{d}_{10}^{hl}C_A-\frac{1}{64}c_Sc_kC_A+\frac{1}{96}c_Sc_FC_A+\frac{5}{32}c_Fc_k^2C_A-\frac{5}{48}c_F^2c_kC_A-\frac{1}{64}\bar{c}_1^{hl}c_FC_A$$
$$\left.-\frac{1}{16}c_2^{hl}c_k(8C_F-3C_A)+\frac{3}{64}c_2^{hl}c_F(8C_F-3C_A)-\frac{1}{4}c_4^{hl}c_k+\frac{3}{16}c_4^{hl}c_F\right]\,, \tag{2.120}$$

$$\nu\frac{d}{d\nu}\bar{d}_{10}^{hl} = \frac{\alpha}{\pi}\left[-\frac{1}{24}c_{B_2}C_A+3d_4^{hl}(8C_F-3C_A)+12d_5^{hl}+\frac{2}{3}\bar{d}_8^{hl}(8C_F-15C_A+3\beta_0)\right.$$
$$+\frac{35}{24}\bar{d}_{10}^{hl}C_A+\frac{13}{24}\bar{c}_WC_A-\frac{1}{48}c_{p'p}C_A-\frac{5}{24}c_Sc_kC_A+\frac{1}{16}c_Sc_FC_A$$
$$+\frac{1}{12}c_Fc_k^2(16C_F+15C_A)-\frac{2}{3}c_F^2c_kC_A+\frac{1}{16}\bar{c}_1^{hl}c_FC_A$$
$$\left.+\frac{1}{16}c_2^{hl}c_F(8C_F-3C_A)+\frac{1}{4}c_4^{hl}c_F\right]\,, \tag{2.121}$$

$$\nu\frac{d}{d\nu}d_7^{hl} = \frac{\alpha}{\pi}\bigg[-\frac{2}{3}d_4^{hl}C_F(2C_F-C_A)+\frac{1}{6}d_7^{hl}(5C_F+3\beta_0)+\frac{5}{6}d_9^{hl}C_F -\frac{1}{32}c_2^{hl}c_FC_F(2C_F-C_A)\bigg], \quad (2.122)$$

$$\nu\frac{d}{d\nu}d_9^{hl} = \frac{\alpha}{\pi}\bigg(\frac{1}{6}d_4^{hl}C_F(2C_F-C_A)+\frac{1}{2}d_7^{hl}C_F+\frac{1}{2}d_9^{hl}(C_F+\beta_0) +\frac{1}{8}c_2^{hl}c_kC_F(2C_F-C_A)-\frac{1}{16}c_2^{hl}c_FC_F(2C_F-C_A)\bigg), \quad (2.123)$$

$$\nu\frac{d}{d\nu}d_{11}^{hl} = \frac{\alpha}{\pi}\bigg(-\frac{1}{12}d_4^{hl}C_F(2C_F-C_A)-\frac{1}{6}d_7^{hl}C_F-\frac{1}{6}d_9^{hl}C_F+\frac{1}{2}d_{11}^{hl}\beta_0 +\frac{1}{64}c_2^{hl}c_FC_F(2C_F-C_A)\bigg). \quad (2.124)$$

As before, the last equality in Eq. (2.118) can be easily deduced by using relations between Wilson coefficients imposed by reparametrization invariance. It it quite remarkable that the RGEs depend only on gauge-independent combinations of Wilson coefficients: $\bar{c}_{A_2}$, $\bar{c}_W$, $\bar{c}_{B_1}$, $\bar{c}_1^{hl}$, $\bar{d}_1^{hl}$, $\bar{d}_8^{hl}$ and $\bar{d}_{10}^{hl}$. This is a very strong check, as at intermediate steps we get contributions from c_D, c_M, c_{A_2}, c_{W_1}, c_{W_2}, c_{B_1}, c_1^{hl}, d_1^{hl}, d_6^{hl}, d_8^{hl} and d_{10}^{hl} which only at the end of the computation arrange themselves in gauge-independent combinations. Note that we include the Wilson coefficients d_7^{hl}, d_9^{hl} and d_{11}^{hl} despite of we do not know if they are physical or not. We do so because we will solve also these RGEs in the next section, as it can be useful in the future.

2.5.4 Anomalous dimensions: The QED limit with $n_f = 0$

In this section we analyze the purely Abelian case of QED without spectator effects. To do this, we just need to take the appropriate limit of the results found in Sec. 2.5.2 i.e. to take $C_F = 1$, $C_A = 0$ and $n_f = 0$. Note that the operators proportional to c_{A_3}, c_{A_4}, c_{B_1} and c_{B_2} do not appear now. The diagrams that contribute are obtained from the topologies drawn in Fig. 2.3 and Fig. 2.4 disregarding the non-Abelian ones. The result is the following

$$c_{4,B}=c_4\,,\quad c_{M,B}=c_M+c_k^3\frac{2}{3}\frac{\alpha}{\pi}\frac{1}{\epsilon}\,,\quad c_{A_1,B}=c_{A_1}-c_k^3\frac{8}{3}\frac{\alpha}{\pi}\frac{1}{\epsilon}\,,\quad c_{A_2,B}=c_{A_2}+c_k^3\frac{40}{3}\frac{\alpha}{\pi}\frac{1}{\epsilon}\,,$$

$$c_{W_1,B}=c_{W_1}+\frac{4}{3}c_Fc_k^2C_F\frac{\alpha}{\pi}\frac{1}{2\epsilon}\,,\quad c_{W_2,B}=c_{W_2}+\frac{4}{3}c_Fc_k^2C_F\frac{\alpha}{\pi}\frac{1}{2\epsilon}\,,\quad c_{p'p,B}=c_{p'p}\,,\quad (2.125)$$

where the subscript B stands for the bare Wilson coefficients, whereas the renormalized Wilson coefficients do not have associated subscript. The first and second lines of Eq. (2.125) are Wilson coefficients associated to spin-independent and spin-dependent operators, respectively. We obtain the following RGEs

$$\nu\frac{d}{d\nu}c_4 = 0\,,\qquad \nu\frac{d}{d\nu}c_M = -c_k^3\frac{4}{3}\frac{\alpha}{\pi}\,,\qquad \nu\frac{d}{d\nu}c_{A_1} = c_k^3\frac{16}{3}\frac{\alpha}{\pi}\,,\qquad \nu\frac{d}{d\nu}c_{A_2} = -c_k^3\frac{80}{3}\frac{\alpha}{\pi}\,,$$

$$\nu\frac{d}{d\nu}c_{W_1} = -\frac{4}{3}c_F c_k^2 C_F\frac{\alpha}{\pi}\,,\qquad \nu\frac{d}{d\nu}c_{W_2} = -\frac{4}{3}c_F c_k^2 C_F\frac{\alpha}{\pi}\,,\qquad \nu\frac{d}{d\nu}c_{p'p} = 0\,. \tag{2.126}$$

This result is in agreement with the explicit SL results given in Refs. [33, 34], where the calculation was done in the Feynman gauge. This is not that strange; for instance, the running of c_D in QED happens to be equal in the Coulomb and Feynman gauges (see the discussion in Ref. [3]). The analysis done in Sec. 2.4 suggests that the physical objects are still $\bar{c}_{A_2}$ and $\bar{c}_W$, though. It is important to mention that, in order to determine the running of c_M and c_{A_2} separately from the scattering of a heavy fermion with a transverse photon, we had to consider the photons to be off shell; otherwise, we could not have distinguished the Feynman rule of c_M and c_{A_2} but only of the physical quantity $\bar{c}_{A_2}$. In QED, this seems to be fine. However, the running of c_M can be obtained from the one transverse photon exchange independently if we work on shell or off shell. For QCD, working off shell produces divergences that can not be absorbed by the operators of the Lagrangian, so a more extensive basis of operators would be needed. That is the main reason we have worked on shell.

In the QED case, the obtained results are in agreement with the reparametrization invariance relations given in Ref. [28], since c_4, $\bar{c}_W$ and $c_{p'p}$ do not renormalize (reparametrization invariance fixes $c_4 = 1$, $\bar{c}_W = 1$ and $c_{p'p} = c_F - 1$, and c_F does not run at one loop in QED), and the relation $2c_M = c_D - c_F$ is satisfied.

2.5.5 LL running: solution and numerical analysis

In this section we solve the RGEs for the physical quantities found in Sec. 2.5.3. We only solve these because they are the combinations that always will appear in observables, whereas the RGEs found in Sec. 2.5.1 or Sec. 2.5.2 may depend on the gauge, and as a consequence, will never appear alone in observables. These RGEs can be written compactly by defining a vector

$$\mathbf{A} = \{c_{A_1}, \bar{c}_{A_2}, c_{A_3}, c_{A_4}, d_2^{hl}, d_3^{hl}, \bar{c}_{B_1}, c_{B_2}, d_5^{hl}, \bar{d}_8^{hl}, \bar{d}_{10}^{hl}, d_7^{hl}, d_9^{hl}, d_{11}^{hl}\}\,. \tag{2.127}$$

Note that we do not include the RGEs of c_4, $c_{p'p}$, $\bar{c}_W$ and d_4^{hl} because their solution can be easily found using the reparametrization invariant relations given in Refs. [28, 44], for which their RGEs are in agreement. The Wilson coefficient $\bar{d}_1^{hl}$ is nor included because it can be obtained from the new relation $2\bar{d}_1^{hl} = \bar{c}_1^{hl} - c_F$. Note that $\bar{d}_1^{hl}$ and d_4^{hl} can be determined analytically using the analytic expressions of $\bar{c}_1^{hl}$, c_2^{hl} and c_F [32] (see App. B.2). Therefore, we will not give explicit expressions for any of them. As pointed out in Sec. 2.5.3, we do not know if the Wilson coefficients d_7^{hl}, d_9^{hl} and d_{11}^{hl} are physical or not, but we will solve their RGEs anyway. The RGEs are simplified to the single equation

$$\nu\frac{d\mathbf{A}}{d\nu} = \frac{\alpha}{\pi}(\mathbf{MA} + \mathbf{F}(\alpha))\,. \tag{2.128}$$

The matrix $\mathbf{M}$ and the vector $\mathbf{F}$ follow from the RGEs given in Sec. 2.5.3. As it can be seen, the matrix $\mathbf{M}$ is not diagonal, so there is a mixing between the different Wilson coefficients which lead to a coupled system of differential equations. The only dependence of Eq. (2.128) with the energy scale is through the strong coupling α. Therefore, in order to solve the system, it is more convenient to write Eq. (2.128) in terms of a derivative with respect to the strong coupling instead of with respect to the soft scale ν. Since we are only interested in the LL running, it is enough to take the one-loop β-function i.e. the LL running of the strong coupling α,

$$\nu\frac{d\alpha}{d\nu} \equiv \beta(\alpha_s) = -2\alpha\left\{\beta_0\frac{\alpha}{4\pi} + \cdots\right\}, \tag{2.129}$$

which leads to the well-known solution

$$\alpha(\nu) = \frac{\alpha(m)}{1+\alpha(m)\frac{\beta_0}{2\pi}\ln\left(\frac{\nu}{m}\right)}, \tag{2.130}$$

where

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f\,, \tag{2.131}$$

and n_f is the number of dynamical (active) quarks i.e. the number of massless quarks. In this approximation, the Eq. (2.128) can be simplified to

$$\frac{d\mathbf{A}}{d\alpha} = -\frac{2}{\beta_0\alpha}(\mathbf{M}\mathbf{A} + \mathbf{F}(\alpha))\,. \tag{2.132}$$

It is more convenient to define a new variable $z \equiv \left(\frac{\alpha(\nu)}{\alpha(m)}\right)^{\frac{1}{\beta_0}} \simeq 1 - \frac{1}{2\pi}\alpha(m)\ln(\frac{\nu}{m})$ in order to obtain more compact results. In terms of z the equation above reads

$$\frac{d\mathbf{A}}{dz} = -\frac{2}{z}(\mathbf{M}\mathbf{A} + \mathbf{F}(z))\,. \tag{2.133}$$

In order to solve Eq. 2.133, we need the tree level initial matching conditions at the hard scale ν_h, which is taken to be equal to the heavy quark mass m. On the one hand, for the heavy-gluon operators, they have been determined in Ref. [28] and read $c_k = c_F = c_D = c_S = c_4 = c_{A_1} = c_{W_1} = c_{B_1} = 1$ and $c_M = c_{A_2} = c_{A_3} = c_{A_4} = c_{W_2} = c_{p'p} = c_{B_2} = 0$. Note that the matching coefficients of the kinetic term are protected by reparametrization invariance ($c_k = c_4 = 1$ to any order in perturbation theory) [45]. Nevertheless, even if we set them to one when solving the RGEs, we have kept them explicit in the RGEs for tracking purposes. On the other hand, there is no tree level contribution to the Wilson coefficients associated to heavy-light operators[14], so their initial matching conditions are $c_i^{hl} = 0$, $i = 1, \ldots 4$ and $d_i^{hl} = 0$, $i = 1, \ldots, 11$. The Wilson coefficients c_k, c_F, $c_S = 2c_F - 1$, $c_4 = 1$, $c_{p'p} = c_F - 1$, $\bar{c}_W = 1$, $\bar{c}_1^{hl}$, c_2^{hl}, c_3^{hl}, c_4^{hl} and $d_4^{hl} = -c_2^{hl}/16$ are needed with LL

[14] The initial matching conditions of the c_i^{hl} and d_i^{hl} are $\mathcal{O}(\alpha)$ unless the operators can be generated at tree level. This is not the case with the basis we consider, but it is if one eliminates the Darwin operator c_D i.e. if we work in terms of $\bar{c}_1^{hl}$.

accuracy. They can be found in Refs. [8, 28, 32]. We summarize their expressions in Sec. B.2.

After solving the RGEs we obtain the LL running of the physical combinations of Wilson coefficients associated to the $1/m^3$ operators of the HQET Lagrangian. For the case without massless fermions ($n_f = 0$), we obtain analytic results. They read

$$
\begin{aligned}
c_{A_1} &= \frac{75}{17} z^{-3C_A} - \frac{29}{3} z^{-2C_A} - \frac{z^{-C_A}}{11} \\
&+ \left(\frac{64C_F}{\sqrt{157}C_A} + \frac{42184}{561\sqrt{157}} + \frac{1780}{561} \right) z^{-\frac{1}{12}\left(17+\sqrt{157}\right)C_A} \\
&+ \left(-\frac{64C_F}{\sqrt{157}C_A} - \frac{42184}{561\sqrt{157}} + \frac{1780}{561} \right) z^{\frac{1}{12}\left(\sqrt{157}-17\right)C_A} ,
\end{aligned} \tag{2.134}
$$

$$
\begin{aligned}
\bar{c}_{A_2} &= -\frac{216}{17} z^{-3C_A} + 34 z^{-2C_A} + \frac{2z^{-C_A}}{11} + \frac{128}{11} + \frac{256C_F}{11C_A} \\
&- \left(\frac{640C_F}{11\sqrt{157}C_A} + \frac{128C_F}{11C_A} + \frac{29720}{187\sqrt{157}} + \frac{3096}{187} \right) z^{-\frac{1}{12}\left(17+\sqrt{157}\right)C_A} \\
&+ \left(\frac{640C_F}{11\sqrt{157}C_A} - \frac{128C_F}{11C_A} + \frac{29720}{187\sqrt{157}} - \frac{3096}{187} \right) z^{\frac{1}{12}\left(\sqrt{157}-17\right)C_A} ,
\end{aligned} \tag{2.135}
$$

$$
\begin{aligned}
c_{A_3} &= \left(\frac{32C_F}{11C_A} + \frac{344}{55} \right) z^{-\frac{11C_A}{3}} - \frac{75}{17} z^{-3C_A} + \frac{88}{15} z^{-2C_A} + \frac{23z^{-C_A}}{11} - \frac{38}{11} - \frac{32C_F}{11C_A} \\
&- \left(\frac{64C_F}{\sqrt{157}C_A} + \frac{42184}{561\sqrt{157}} + \frac{1780}{561} \right) z^{-\frac{1}{12}\left(17+\sqrt{157}\right)C_A} \\
&+ \left(\frac{64C_F}{\sqrt{157}C_A} + \frac{42184}{561\sqrt{157}} - \frac{1780}{561} \right) z^{\frac{1}{12}\left(\sqrt{157}-17\right)C_A} ,
\end{aligned} \tag{2.136}
$$

$$
\begin{aligned}
c_{A_4} &= -\left(\frac{128C_F}{11C_A} + \frac{1376}{55} \right) z^{-\frac{11C_A}{3}} + \frac{216}{17} z^{-3C_A} - \frac{64}{5} z^{-2C_A} - \frac{24z^{-C_A}}{11} - \frac{64}{11} - \frac{128C_F}{11C_A} \\
&+ \left(\frac{640C_F}{11\sqrt{157}C_A} + \frac{128C_F}{11C_A} + \frac{29720}{187\sqrt{157}} + \frac{3096}{187} \right) z^{-\frac{1}{12}\left(17+\sqrt{157}\right)C_A} \\
&+ \left(-\frac{640C_F}{11\sqrt{157}C_A} + \frac{128C_F}{11C_A} - \frac{29720}{187\sqrt{157}} + \frac{3096}{187} \right) z^{\frac{1}{12}\left(\sqrt{157}-17\right)C_A} ,
\end{aligned} \tag{2.137}
$$

$$
\begin{aligned}
\bar{c}_{B_1} &= -\frac{86}{123} - \frac{32C_F}{123C_A} - \left(\frac{25}{9} + \frac{16C_F}{9C_A} \right) z^{-C_A} + z^{-2C_A} - 15 z^{-3C_A} \\
&+ \left(\frac{3040}{369} - \frac{5077}{369}\sqrt{\frac{2}{5}} + \frac{376C_F}{369C_A} - \frac{928C_F}{369C_A}\sqrt{\frac{2}{5}} \right) z^{\frac{1}{6}(-16+\sqrt{10})C_A} \\
&+ \left(\frac{3040}{369} + \frac{5077}{369}\sqrt{\frac{2}{5}} + \frac{376C_F}{369C_A} + \frac{928C_F}{369C_A}\sqrt{\frac{2}{5}} \right) z^{-\frac{1}{6}(16+\sqrt{10})C_A} ,
\end{aligned} \tag{2.138}
$$

$$c_{B_2} = -\frac{24}{41} - \frac{192C_F}{41C_A} + \left(\frac{11}{3} + \frac{32C_F}{3C_A}\right) z^{-C_A} + z^{-2C_A} + 18z^{-3C_A}$$
$$+ \left(-\frac{1358}{123} + \frac{64}{41}\sqrt{\frac{2}{5}} + \frac{298}{123}\sqrt{10} - \frac{368C_F}{123C_A} + \frac{8C_F}{123C_A}\sqrt{\frac{2}{5}}\right) z^{\frac{1}{6}(-16+\sqrt{10})C_A}$$
$$+ \left(-\frac{1358}{123} - \frac{64}{41}\sqrt{\frac{2}{5}} - \frac{298}{123}\sqrt{10} - \frac{368C_F}{123C_A} - \frac{8C_F}{123C_A}\sqrt{\frac{2}{5}}\right) z^{-\frac{1}{6}(16+\sqrt{10})C_A}. \quad (2.139)$$

After the inclusion of light fermions, the solution of the RGEs is numerical. We show the result for $n_f = 4$ light fermions where $\alpha(m)$ has n_f active light flavours

$$c_{A_1} = 1.08839 \times 10^{-9} + \frac{5.9421}{z^{9.}} - \frac{3.66729 \times 10^{-19}}{z^{8.33333}} + \frac{3.44328}{z^{7.99055}} - \frac{2.07923}{z^{6.83333}} + \frac{2.56988}{z^{6.5}}$$
$$-\frac{2.31629}{z^{6.}} + \frac{8.05227}{z^{3.}} - \frac{10.0312}{z^{2.87467}} - \frac{4.58085}{z^{1.02367}}, \quad (2.140)$$

$$\bar{c}_{A_2} = 23.9049 - \frac{3.83509 \times 10^{-20}}{z^{14.5556}} - \frac{17.807}{z^{9.}} + \frac{4.00069 \times 10^{-19}}{z^{8.33333}} - \frac{11.1876}{z^{7.99055}} + \frac{6.49126}{z^{6.83333}}$$
$$-\frac{7.90732}{z^{6.5}} + \frac{11.9489}{z^{6.}} - \frac{16.1045}{z^{3.}} + \frac{19.1876}{z^{2.87467}} - \frac{8.5262}{z^{1.02367}}, \quad (2.141)$$

$$c_{A_3} = -4.73174 + \frac{1.11951}{z^{14.5556}} - \frac{5.9421}{z^{9.}} + \frac{12.652}{z^{8.33333}} - \frac{3.44328}{z^{7.99055}} + \frac{1.45822}{z^{6.83333}}$$
$$+\frac{1.5325608 \times 10^{-21}}{z^{41/6}} + \frac{0.621007}{z^{6.83333}} - \frac{5.10498}{z^{6.5}} + \frac{5.522246 \times 10^{-21}}{z^{13/2}} + \frac{1.500000}{z^{6}}$$
$$-\frac{6.68835}{z^{6.}} + \frac{1.500000}{z^{3}} - \frac{7.55227}{z^{3.}} + \frac{10.0312}{z^{2.87467}} + \frac{4.58085}{z^{1.02367}}, \quad (2.142)$$

$$c_{A_4} = -12.9779 - \frac{4.47804}{z^{14.5556}} + \frac{17.807}{z^{9.}} - \frac{50.6079}{z^{8.33333}} + \frac{11.1876}{z^{7.99055}} - \frac{6.49126}{z^{6.83333}} + \frac{18.0477}{z^{6.5}}$$
$$+\frac{24.0697}{z^{6.}} + \frac{14.1045}{z^{3.}} - \frac{19.1876}{z^{2.87467}} + \frac{8.5262}{z^{1.02367}}, \quad (2.143)$$

$$d_2^{hl} = -0.124078 - \frac{0.101624}{z^{9.}} - \frac{0.084417}{z^{7.99055}} + \frac{0.0354398}{z^{6.83333}} - \frac{0.0802099}{z^{6.83333}} + \frac{0.0880307}{z^{6.5}}$$
$$+\frac{0.387618}{z^{6.}} + \frac{1.04971}{z^{3.}} - \frac{1.29563}{z^{2.87467}} + \frac{0.125167}{z^{1.02367}}, \quad (2.144)$$

$$
\begin{aligned}
d_3^{hl} &= 0.00135186 - \frac{0.0310975}{z^{14.5556}} + \frac{1.63677\times 10^{-19}}{z^{9.}} + \frac{0.263583}{z^{8.33333}} + \frac{1.0633\times 10^{-19}}{z^{7.99055}} \\
&\quad + \frac{1.56003\times 10^{-20}}{z^{6.83333}} - \frac{0.0891247}{z^{6.5}} - \frac{0.144712}{z^{6.}} + \frac{4.54348\times 10^{-12}}{z^{3.}} \\
&\quad + \frac{1.68781\times 10^{-19}}{z^{2.87467}} - \frac{1.60402\times 10^{-20}}{z^{1.02367}}\,, \qquad (2.145)
\end{aligned}
$$

$$
\begin{aligned}
\bar{c}_{B_1} &= -0.695 + \frac{0.045788}{z^{11.333333}} + \frac{13.6766}{z^{9.762121}} - \frac{10.4869}{z^{9}} + \frac{0.0690}{z^{8.333333}} - \frac{0.3586}{z^{8.24892}} - \frac{1.83813}{z^{6.833333}} \\
&\quad + \frac{1.34179}{z^{6.549986}} + \frac{1}{z^{6}} + \frac{4.49328}{z^{3.833333}} - \frac{4.95805}{z^{3.577865}} - \frac{3.290}{z^{3}}\,, \qquad (2.146)
\end{aligned}
$$

$$
\begin{aligned}
c_{B_2} &= -2.966 - \frac{0.065821}{z^{11.333333}} - \frac{16.4471}{z^{9.762121}} + \frac{13.4869}{z^{9}} - \frac{0.0493}{z^{8.333333}} + \frac{0.2388}{z^{8.24892}} - \frac{2.94100}{z^{6.833333}} \\
&\quad + \frac{4.6355}{z^{6.549986}} - \frac{4.3137}{z^{6}} - \frac{15.2080}{z^{3.833333}} + \frac{16.0570}{z^{3.577865}} + \frac{7.572}{z^{3}}\,, \qquad (2.147)
\end{aligned}
$$

$$
\begin{aligned}
d_5^{hl} &= 0.01 - \frac{2.22704\cdot 10^{-22}}{z^{11.333333}} - \frac{5.46119\cdot 10^{-20}}{z^{9.762121}} + \frac{1.24130\cdot 10^{-21}}{z^{9.5}} + \frac{4.30351\cdot 10^{-20}}{z^{9}} \\
&\quad - \frac{0.00851190}{z^{8.333333}} + \frac{3.836\cdot 10^{-23}}{z^{8.24892}} + \frac{1.10836\cdot 10^{-21}}{z^{6.833333}} - \frac{9.1837\cdot 10^{-23}}{z^{6.549986}} \\
&\quad - \frac{0.01190476}{z^{6}} + \frac{5.1030\cdot 10^{-22}}{z^{3.833333}} - \frac{1.84872\cdot 10^{-21}}{z^{3.577865}} + \frac{0.0104167}{z^{3}}\,, \qquad (2.148)
\end{aligned}
$$

$$
\begin{aligned}
\bar{d}_8^{hl} &= -0.1149 + \frac{0.0069309}{z^{11.333333}} + \frac{0.206216}{z^{9.762121}} - \frac{0.211554}{z^{9}} - \frac{0.00663}{z^{8.333333}} + \frac{0.03645}{z^{8.24892}} + \frac{0.235510}{z^{6.833333}} \\
&\quad - \frac{0.090907}{z^{6.549986}} - \frac{0.24908}{z^{6}} - \frac{2.88931}{z^{3.833333}} + \frac{3.27297}{z^{3.577865}} - \frac{0.1957}{z^{3}}\,, \qquad (2.149)
\end{aligned}
$$

$$
\begin{aligned}
\bar{d}_{10}^{hl} &= -0.851 - \frac{0.000658}{z^{11.333333}} + \frac{1.52703}{z^{9.762121}} - \frac{1.3162}{z^{9.5}} - \frac{1.39575}{z^{9}} + \frac{0.2628}{z^{8.333333}} + \frac{1.024}{z^{8.24892}} \\
&\quad + \frac{0.81541}{z^{6.833333}} + \frac{0.0125353}{z^{6.549986}} - \frac{0.1392}{z^{6}} - \frac{7.7003}{z^{3.833333}} + \frac{8.65108}{z^{3.577865}} - \frac{1.890}{z^{3}}\,, \qquad (2.150)
\end{aligned}
$$

$$
\begin{aligned}
d_7^{hl} &= -0.000867 + \frac{0.0000799}{z^{11.888889}} + \frac{3.346\cdot 10^{-24}}{z^{11.333333}} + \frac{4.0769\cdot 10^{-22}}{z^{9.5}} + \frac{0.020957}{z^{9}} \\
&\quad - \frac{0.036600}{z^{8.333333}} - \frac{3.188\cdot 10^{-22}}{z^{8.24892}} + \frac{0.01792}{z^{6.833333}} - \frac{4.0091\cdot 10^{-24}}{z^{6.549986}} \\
&\quad - \frac{0.000611}{z^{6}} - \frac{0.00086}{z^{3.833333}} - \frac{2.06422\cdot 10^{-21}}{z^{3.577865}} - \frac{0.00002}{z^{3}}\,, \qquad (2.151)
\end{aligned}
$$

$$
\begin{aligned}
d_9^{hl} &= -0.006752 + \frac{0.0000479}{z^{11.888889}} - \frac{1.5783 \cdot 10^{-24}}{z^{11.333333}} - \frac{1.9232 \cdot 10^{-22}}{z^{9.5}} - \frac{0.009862}{z^{9}} \\
&+ \frac{0.036600}{z^{8.333333}} + \frac{1.504 \cdot 10^{-22}}{z^{8.24892}} - \frac{0.067979}{z^{6.833333}} + \frac{1.89125 \cdot 10^{-24}}{z^{6.549986}} \\
&+ \frac{0.019841}{z^{6}} + \frac{0.05322}{z^{3.833333}} + \frac{9.7377 \cdot 10^{-22}}{z^{3.577865}} - \frac{0.025117}{z^{3}}\,, \qquad (2.152)
\end{aligned}
$$

$$
\begin{aligned}
d_{11}^{hl} &= -0.000769 - \frac{0.00001598}{z^{11.888889}} + \frac{7.019 \cdot 10^{-25}}{z^{11.333333}} + \frac{8.553 \cdot 10^{-23}}{z^{9.5}} + \frac{0.0006164}{z^{9}} \\
&- \frac{0.006880}{z^{8.333333}} - \frac{6.69 \cdot 10^{-23}}{z^{8.24892}} + \frac{0.013287}{z^{6.833333}} - \frac{8.4104 \cdot 10^{-25}}{z^{6.549986}} - \frac{0.009005}{z^{6}} \\
&+ \frac{0.008295}{z^{3.833333}} - \frac{4.33038 \cdot 10^{-22}}{z^{3.577865}} - \frac{0.005529}{z^{3}}\,. \qquad (2.153)
\end{aligned}
$$

Note that, if we are interested in the solution of the gauge dependent quantities in the Coulomb gauge, we just must solve the RGEs given in Sec. 2.5.1 and Sec. 2.5.2. Notice that the running of c_{A_2} and d_1^{hl} can be obtained in the Feynman gauge by using the expression of c_M determined from the reparametrization invariance relation given in Ref. [28].

The SL results can be found analytically by solving the RGEs of Sec. 2.5.3 taking the tree level values of the Wilson coefficients that appear and considering α as a constant, $\alpha(m)$. They can also be obtained by expanding the above solutions in powers of $\alpha(m)$. We obtain

$$
\begin{aligned}
c_{A_1} &= 1+\left(\frac{16}{3}C_F+\frac{23}{3}C_A\right)\frac{\alpha(m)}{\pi}\ln\left(\frac{\nu}{m}\right)+\mathcal{O}(\alpha^2)\,, && (2.154)\\
\bar{c}_{A_2} &= 0-\left(\frac{64}{3}C_F+\frac{65}{3}C_A\right)\frac{\alpha(m)}{\pi}\ln\left(\frac{\nu}{m}\right)+\mathcal{O}(\alpha^2)\,, && (2.155)\\
c_{A_3} &= 0+C_A\frac{\alpha(m)}{\pi}\ln\left(\frac{\nu}{m}\right)+\mathcal{O}(\alpha^2)\,, && (2.156)\\
c_{A_4} &= 0-4C_A\frac{\alpha(m)}{\pi}\ln\left(\frac{\nu}{m}\right)+\mathcal{O}(\alpha^2)\,, && (2.157)\\
d_2^{hl} &= 0+\frac{1}{6}\left(8C_F-3C_A\right)\frac{\alpha(m)}{\pi}\ln\left(\frac{\nu}{m}\right)+\mathcal{O}(\alpha^2)\,, && (2.158)\\
d_3^{hl} &= 0+\frac{1}{3}C_F\left(C_A-2C_F\right)\frac{\alpha(m)}{\pi}\ln\left(\frac{\nu}{m}\right)+\mathcal{O}(\alpha^2)\,, && (2.159)\\
\bar{c}_{B_1} &= -1+\left(\frac{8}{3}C_F+\frac{11}{3}C_A\right)\frac{\alpha(m)}{\pi}\ln\left(\frac{\nu}{m}\right)+\mathcal{O}(\alpha^2)\,, && (2.160)\\
c_{B_2} &= -\left(\frac{8}{3}C_F+\frac{25}{6}C_A\right)\frac{\alpha(m)}{\pi}\ln\left(\frac{\nu}{m}\right)+\mathcal{O}(\alpha^2)\,, && (2.161)\\
d_5^{hl} &= \frac{1}{8}C_F(2C_F-C_A)\frac{\alpha(m)}{\pi}\ln\left(\frac{\nu}{m}\right)+\mathcal{O}(\alpha^2)\,, && (2.162)\\
\bar{d}_8^{hl} &= \mathcal{O}(\alpha^2)\,, && (2.163)\\
\bar{d}_{10}^{hl} &= -1+\left(\frac{4}{3}C_F-\frac{5}{12}C_A\right)\frac{\alpha(m)}{\pi}\ln\left(\frac{\nu}{m}\right)+\mathcal{O}(\alpha^2)\,, && (2.164)\\
d_7^{hl} &= \mathcal{O}(\alpha^2)\,, && (2.165)\\
d_9^{hl} &= \mathcal{O}(\alpha^2)\,, && (2.166)\\
d_{11}^{hl} &= \mathcal{O}(\alpha^2)\,. && (2.167)
\end{aligned}
$$

It is important to realise that the SL does not depend on n_f. Note that $\bar{d}_8^{hl}$ and d_i^{hl}, for $i=7,9,11$, are zero at the level of the SL. This means that the first contribution will be of $\mathcal{O}(\alpha^2\ln^2(\nu/m))$ and, as a consequence, their running is expected to be small compared to the other Wilson coefficients because the SL dominates the expansion in the strong coupling, $\alpha(m)$.

In Figs. 2.5, 2.6 and 2.7 we plot the results obtained in this section applied to the bottom heavy quark case, illustrating the importance of incorporating large logarithms in heavy quark physics. Only physical combinations and specific combinations that appear in physical observables are represented. For instance, in heavy quarkonium physics applications like the N^{3}LL spectrum and the N^{2}LL running of the Wilson coefficient of the electromagnetic current we observe that only the combinations $\tilde{c}_{A_2}\equiv 2c_{A_1}+\bar{c}_{A_2}$, $\tilde{c}_{A_4}\equiv 2c_{A_3}+c_{A_4}$ appear [5]. For the Compton scattering discussed in Sec. 2.4, we observe that $c_{A_{1,3}}$, and again $\tilde{c}_{A_{2,4}}$, appear. We remind that those can be understood as linear combinations of the chromopolarizabilities of the heavy quark. The combination $\tilde{c}_{B_1}=\bar{c}_{B_1}+c_{B_2}$ also appears in the Compton scattering. We run the Wilson coefficients from the heavy quark mass to 1 GeV for zero and four massless fermions. For illustrative purposes, we take $m_b=4.73$

GeV and $\alpha(m_b) = 0.215943$.

Concerning the numerical analysis of the spin-independent operators, we observe that the effect due to the logarithms is very large in most cases. This is due to very large coefficients multiplying the logarithms (even in the Abelian limit the coefficient is quite large). We also observe that the LL resummation is basically saturated by the SL in all cases except for c_{A_4} and $\tilde{c}_{A_4}$. For most cases, the incorporation of light fermions plays a minor role. Let us discuss in more detail every Wilson coefficient. We observe that c_{A_1} changes from 1 to -2 after running. The resummation of logarithms introduces a change of approximately 0.2, with respect to the SL result, after running. The case of $\bar{c}_{A_2}$ is even more dramatic, it goes from 0 to 10, and the resummation of logarithms introduces a change after running of 0.5 with respect to the SL result. The change after running of c_{A_3} is more moderate, even though certainly sizable, and so is for c_{A_4}. They change from 0 to -0.2 and -0.05 (even though there is a maximum at 0.35), respectively. For c_{A_3}, the resummation of large logarithms introduces a change after running of 0.12 with respect to the SL result. For c_{A_4} the resummation of logarithms happens to be very important since its behaviour is not saturated by the SL. It is interesting to note that the running of $\tilde{c}_{A_2}$ is smaller than the running of $\bar{c}_{A_2}$, but still rather large; it changes by nearly a factor of 3. The qualitative behaviour of $\tilde{c}_{A_4}$ is similar to c_{A_4}, and it changes from 0 to roughly -0.3 after running. Comparatively, the running of d_2^{hl} and d_3^{hl} is much smaller (they go from 0 to -0.036 and -0.015, respectively) confirming that spectator quarks associated corrections are subleading numerically. In these cases, the resummation of logarithms introduces a change after running of around 0.006 and 0.001 with respect to the SL result, respectively. Spectator quark effects in heavy-gluon operators introduce a change after running of 0.2, 1, 0.03, 0.35, 0.4, 0.16 to c_{A_1}, $\bar{c}_{A_2}$, c_{A_3}, c_{A_4}, $\tilde{c}_{A_2}$ and $\tilde{c}_{A_4}$, respectively, with respect to the LL result with $n_f = 0$.

Concerning the numerical analysis of spin-dependent operators, we observe that the effect due to the logarithms is large in general (not for QED though, where the only physical combination that appears, $\bar{c}_W$, does not run). This is because the coefficients multiplying the logarithms are large, in particular, those that multiply the non-Abelian color factor C_A. We also observe that the LL result is saturated by the SL in all cases except in the combination $\bar{c}_{B_1} + c_{B_2}$. In general, the incorporation of spectator effects plays a minor role. Let us discuss in more detail every Wilson coefficient. We observe that $\bar{c}_{B_1}$ changes from -1 to -2.25 after running. The case of c_{B_2} is rather similar, it goes from 0 to 1.4 after running. In general, the effect due to the resummation of logarithms is not quite large, but certainly sizable. It introduces a change of approximately 0.3-0.4 with respect to the SL result after running. For the combination $\bar{c}_{B_1} + c_{B_2}$ the effect is very small. It goes from -1 to -0.99 even though it has a maximum of -0.95. In this case, the resummation of logarithms is important because the behaviour is not saturated by the SL. The inclusion of spectator quarks slightly changes the running of $\bar{c}_{B_1}$ and c_{B_2}, but that change is small (of approximately 0.1 after running, with respect to the LL result with $n_f = 0$), so the effect induced by them is numerically subleading. However, the effect induced by the spectators tends to get away the curve from the SL one, so it makes the resummation of large logarithms more important. The change in combinations that appear in Compton scattering, like $\bar{c}_{B_1} + c_{B_2}$ is sizable, but even smaller than before. It changes by 0.02 after

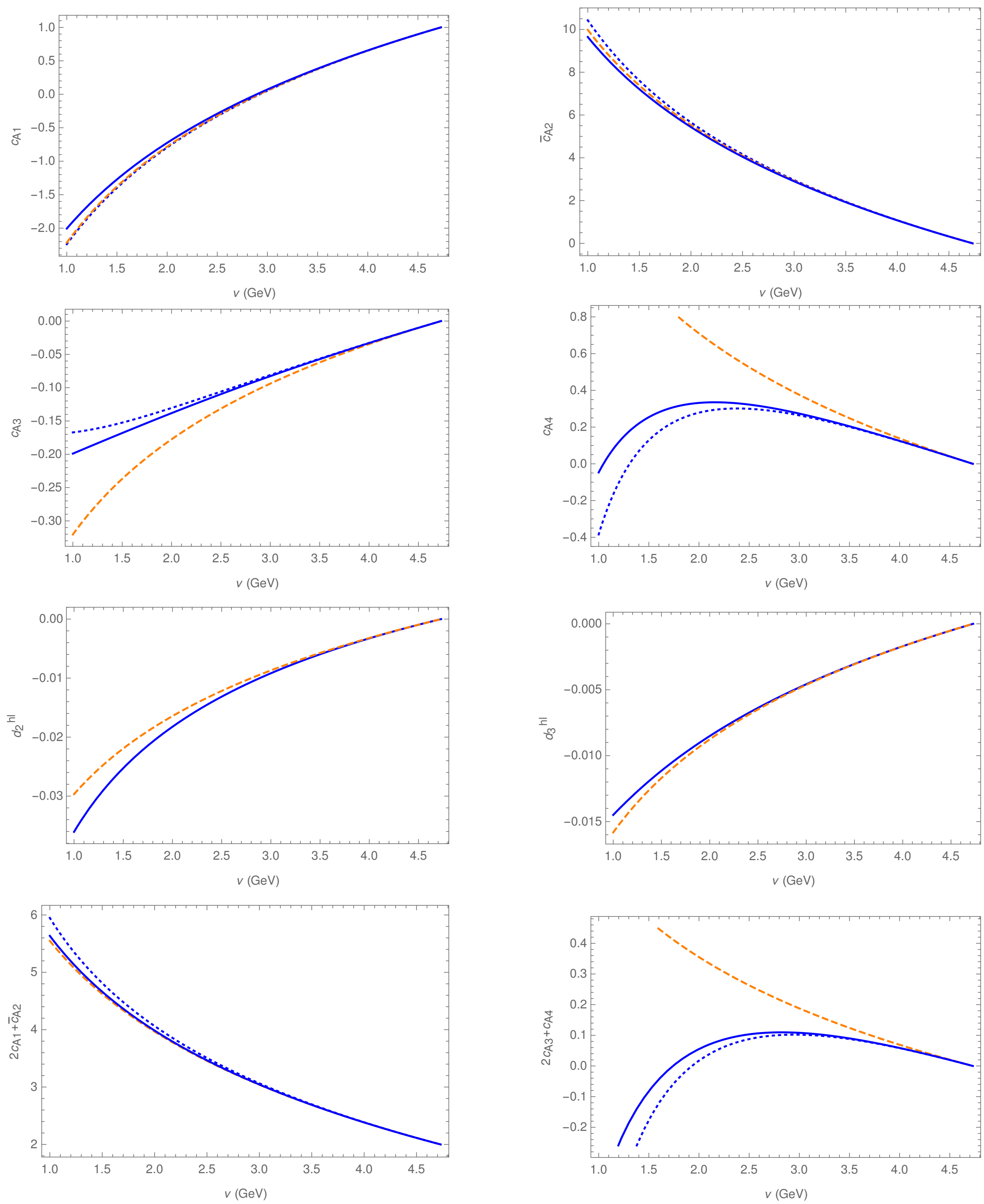


Figure 2.5: Running of the Wilson coefficients associated to the $1/m^3$ spin-independent operators: c_{A_1}, $\bar{c}_{A_2}$, c_{A_3}, c_{A_4}, d_2^{hl}, d_3^{hl}, $2c_{A_1} + \bar{c}_{A_2}$ and $2c_{A_3} + c_{A_4}$, applied to the bottom heavy quark case. The continuous line is the LL result with $n_f = 4$, the dotted line is the LL result with $n_f = 0$, and the dashed line is the SL result (it does not depend on n_f).

running with respect to the LL result with $n_f = 0$. Concerning the Wilson coefficients associated to heavy-light operators, we find that their running is small but sizable in some cases. The running is saturated by the SL in d_5^{hl} and $\bar{d}_{10}^{hl}$. In particular, d_5^{hl} changes from 0 to 0.006. In that case, the resumation of logarithms happens to be unimportant. In the case of $\bar{d}_{10}^{hl}$, which runs from -1 to -1.042, the resummation of logarithms roughly introduces a difference of 0.015 at 1 GeV, with respect to the SL value. The resummation of logarithms happens to be qualitatively very important for $\bar{d}_8^{hl}$, d_7^{hl}, d_9^{hl} and d_{11}^{hl}, even though their running is small, because their behaviour is not saturated by the SL result. They go from 0 to $8.2\cdot 10^{-4}$, $-3\cdot 10^{-5}$, $-1.5\cdot 10^{-4}$ and $-5\cdot 10^{-5}$, respectively, after running at approximately 1.5 GeV.

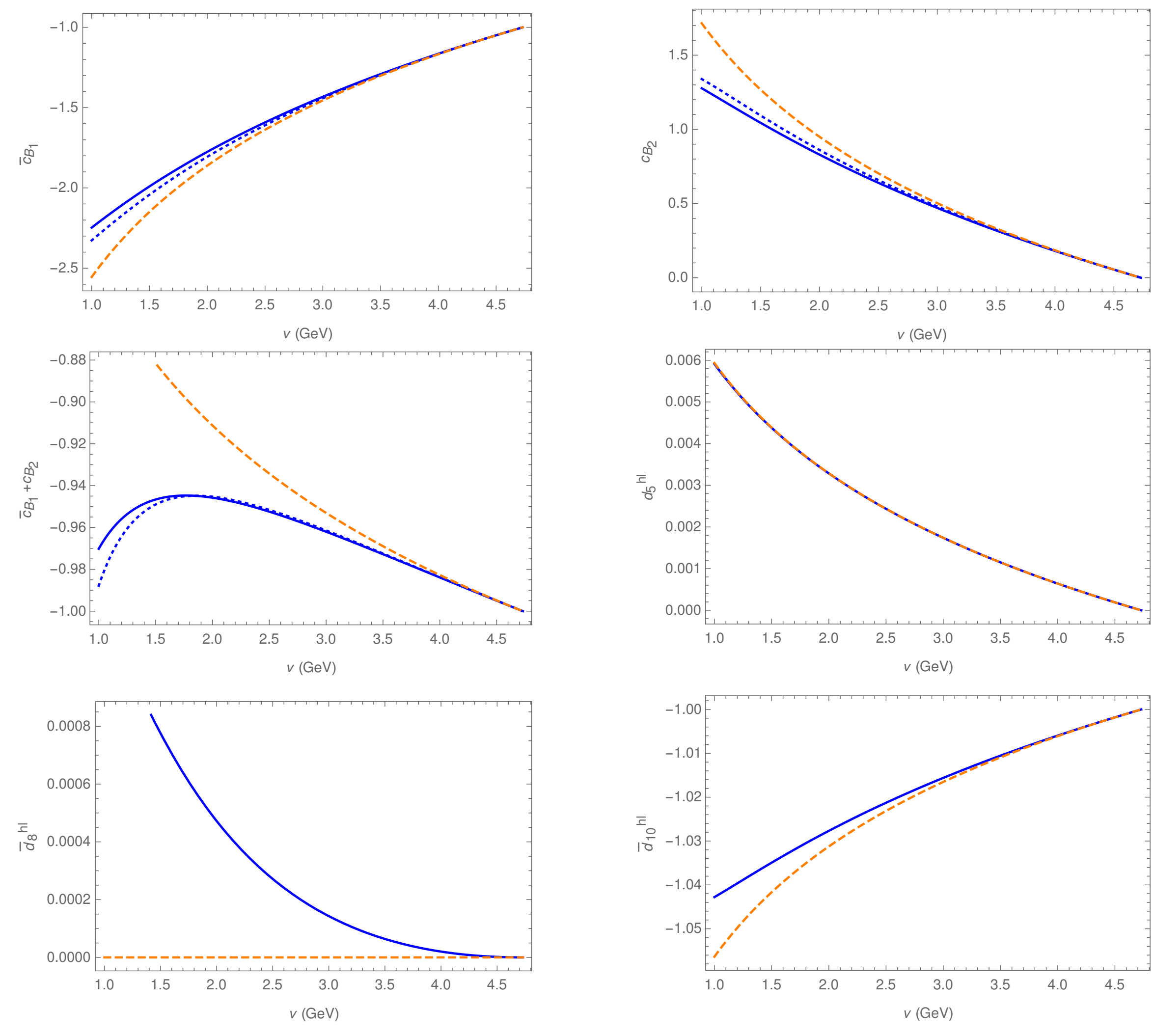


Figure 2.6: Running of the Wilson coefficients associated to the $1/m^3$ spin-dependent operators: $\bar{c}_{B_1}$, c_{B_2}, $\bar{c}_{B_1} + c_{B_2}$, d_5^{hl}, $\bar{d}_8^{hl}$ and $\bar{d}_{10}^{hl}$, applied to the bottom heavy quark case. The continuous line is the LL result with $n_f = 4$, the dotted line is the LL result with $n_f = 0$ and the dashed line is the SL result (it does not depend on n_f).

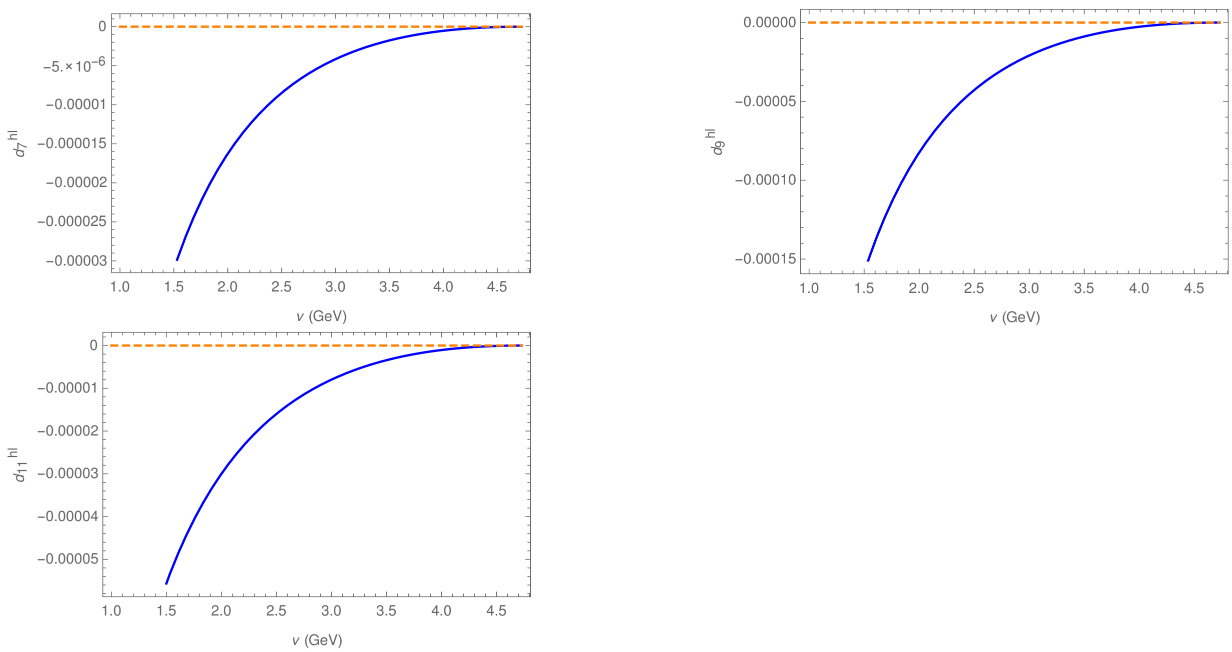


Figure 2.7: Running of the Wilson coefficients associated to the $1/m^3$ spin-dependent operators: d_7^{hl}, d_9^{hl} and d_{11}^{hl}, applied to the bottom heavy quark case. The continuous line is the LL result with $n_f = 4$ and the dashed line is the SL result (it does not depend on n_f).

2.6 Comparison with earlier work

The LL running of the Wilson coefficients associated to the $1/m^3$ operators of the HQET Lagrangian was first addressed in Refs. [29, 33, 34]. For the case without light fermions, expressions for the anomalous dimension matrix and explicit expressions for the Wilson coefficients with SL accuracy are given. We find that these results are mutually inconsistent, as the anomalous dimension matrix produces different expressions for the SL results, compared to the explicit SL expressions written in these references (except for $c_{12}^{(3)}$).

The basis of operators these results were obtained from is different from the basis used in this thesis, so in order to compare our results, we have to change the operator basis. This is done via field redefinitions, which at the order we are working in, is equivalent to using the full EOMs to order $1/m$. To this purpose, we start with the HQET Lagrangian in a general frame[15], Eq. (8) in Ref. [28]

[15] We take this opportunity to correct a misprint in the term proportional to $c_{p'p}$, where the minus sign appearing there should be a plus sign in order to reproduce the Lagrangian Eq. (6) and Eq. (7) in Ref. [28].

$$\begin{aligned}
\mathcal{L}_v &= \bar{Q}_v\Big\{ iD\cdot v - c_k\frac{D_\perp^2}{2m} + c_4\frac{D_\perp^4}{8m^3} - c_F g\frac{\sigma_{\alpha\beta}G^{\alpha\beta}}{4m} - c_D g\frac{v^\alpha[D_\perp^\beta, G_{\alpha\beta}]}{8m^2} \\
&+ ic_S g\frac{v_\lambda\sigma_{\alpha\beta}\{D_\perp^\alpha, G^{\lambda\beta}\}}{8m^2} + c_{W_1}g\frac{\{D_\perp^2, \sigma_{\alpha\beta}G^{\alpha\beta}\}}{16m^3} - c_{W_2}g\frac{D_\perp^\lambda\sigma_{\alpha\beta}G^{\alpha\beta}D_{\perp\,\lambda}}{8m^3} \\
&+ c_{p'p}g\frac{\sigma^{\alpha\beta}(D_\perp^\lambda G_{\lambda\alpha}D_{\perp\,\beta} + D_{\perp\,\beta}G_{\lambda\alpha}D_\perp^\lambda + D_\perp^\lambda G_{\alpha\beta}D_{\perp\,\lambda})}{8m^3} \\
&- ic_M g\frac{D_{\perp\,\alpha}[D_{\perp\,\beta}G^{\alpha\beta}] + [D_{\perp\beta}G^{\alpha\beta}]D_{\perp\,\alpha}}{8m^3} + c_{A_1}g^2\frac{G_{\alpha\beta}G^{\alpha\beta}}{16m^3} \\
&+ c_{A_2}g^2\frac{G_{\mu\alpha}G^{\mu\beta}v_\alpha v_\beta}{16m^3} + \frac{c_{A_3}}{N_c}g^2\,\mathrm{Tr}\left(\frac{G_{\alpha\beta}G^{\alpha\beta}}{16m^3}\right) + \frac{c_{A_4}}{N_c}g^2\,\mathrm{Tr}\left(\frac{G_{\mu\alpha}G^{\mu\beta}v_\alpha v_\beta}{16m^3}\right) \\
&- ic_{B_1}g^2\frac{\sigma_{\alpha\beta}[G^{\mu\alpha}, G_\mu{}^\beta]}{16m^3} - ic_{B_2}g^2\frac{\sigma_{\alpha\beta}[G^{\mu\alpha}, G^{\nu\beta}]v_\mu v_\nu}{16m^3}\Big\}Q_v\,, \qquad (2.168)
\end{aligned}$$

where $D_\perp^\mu = D^\mu - v^\mu v\cdot D$ and $\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^\alpha, \gamma^\beta]$. We use the EOMs of this Lagrangian at $\mathcal{O}(1/m)$ to move terms of the $1/m$ and $1/m^2$ operators to $1/m^3$ operators, as well as, to remove terms of the $1/m^3$ operators, which are sent to higher orders in $1/m$. The EOMs read

$$\bar{Q}_v(iv\cdot D) = c_k\frac{1}{2m}\bar{Q}_v D_\perp^2 + c_F\frac{g}{4m}\bar{Q}_v\sigma_{\alpha\beta}G^{\alpha\beta}\,, \qquad (2.169)$$

$$(iv\cdot D)Q_v = c_k\frac{1}{2m}D_\perp^2 Q_v + c_F\frac{g}{4m}\sigma_{\alpha\beta}G^{\alpha\beta}Q_v\,. \qquad (2.170)$$

Once done, we write every operator of the physical operator basis[16] given in Refs. [33, 34] as a combination of the operators of Eq. 2.168. To this purpose it is necessary to write the field strength tensor in terms of covariant derivatives through the relation $G_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$. We obtain the following relations between the Wilson coefficients of the $1/m^3$ operators in the two basis

[16]In Refs. [33, 34] the HQET Lagrangian is defined as

$$\mathcal{L}_{\mathrm{HQET}} = \bar{Q}_v(iv\cdot D)Q_v + \sum_{n=1}^{\infty}\frac{1}{(2m)^n}\sum_i c_i^{(n)}\mathcal{O}_i^{(n)}\,, \qquad (2.171)$$

where $c_i^{(n)}$ are the Wilson coefficients associated to the dimension $D+n$ operators, $\mathcal{O}_i^{(n)}$, which can be found in the same references.

$$
\begin{aligned}
c_1^{(3)} &= -2c_M + \frac{1}{2}c_{A_2} + c_k c_F^2 + c_S c_F \,, && (2.172)\\
c_2^{(3)} &= c_4 + 2c_M - c_k^3 - c_D c_k \,, && (2.173)\\
c_3^{(3)} &= 2c_M + c_{A_1} - c_k c_F^2 - c_S c_F \,, && (2.174)\\
c_4^{(3)} &= -4c_M - c_{A_1} + c_k c_F^2 + c_S c_F \,, && (2.175)\\
c_{12}^{(3)} &= \frac{1}{12}c_{A_3} \,, && (2.176)\\
c_{13}^{(3)} &= \frac{1}{12}c_{A_4} \,, && (2.177)\\
c_5^{(3)} &= -c_{B_2} - c_k c_F^2 - c_S c_F \,, && (2.178)\\
c_6^{(3)} &= -c_{W_1} - c_{p'p} + c_k^2 c_F + \frac{1}{2}c_D c_F + \frac{1}{2}c_S c_k \,, && (2.179)\\
c_7^{(3)} &= 2c_{W_2} - 2c_{p'p} - c_{B_1} + c_k c_F^2 + c_S c_F \,, && (2.180)\\
c_8^{(3)} &= -c_{W_1} - c_{p'p} + c_k^2 c_F + \frac{1}{2}c_D c_F + \frac{1}{2}c_S c_k \,, && (2.181)\\
c_9^{(3)} &= -c_{B_1} + c_k c_F^2 + c_S c_F \,, && (2.182)\\
c_{10}^{(3)} &= c_{p'p} + c_{B_1} - c_k c_F^2 - c_S c_F \,, && (2.183)\\
c_{11}^{(3)} &= c_{p'p} + c_{B_1} - c_k c_F^2 - c_S c_F \,. && (2.184)
\end{aligned}
$$

Where Eqs. (2.172-2.177) are Wilson coefficients associated to spin-independent operators and Eqs. (2.178-2.184) to spin-dependent ones. Note that $c_6^{(3)} = c_8^{(3)}$ and $c_{10}^{(3)} = c_{11}^{(3)}$. This is to be expected since it is well-known from Ref. [28] that there are five spin-dependent operators and five different Wilson coefficients, whereas in Refs. [33, 34] there are seven operators and seven different Wilson coefficients. Thereby, one must find that two Wilson coefficients have to be equal to another two. We also find the following relations between the Wilson coefficients of $1/m$ and $1/m^2$ operators

$$
c_1^{(1)} = c_k \,, \qquad c_2^{(1)} = c_F \,, \qquad c_1^{(2)} = -c_D \,, \qquad c_2^{(2)} = c_S \,. \qquad (2.185)
$$

With these relations, we can think about reproducing the anomalous dimension matrix given in Ref. [33] from our results. However, this is not possible for all the entries, as some of the Wilson coefficients are gauge dependent and the computation in Ref. [33] was done in the Feynman gauge, whereas we performed it in the Coulomb gauge. For the spin-independent case, this is not a problem, since all gauge dependence comes from c_D and c_M, which are known at LL in the Feynman gauge (c_D is known and c_M is determined from the reparametrization invariance relation $2c_M = c_D - c_F$, which is satisfied in the Feynman gauge). The spin-dependent case is different, though, because the gauge dependent Wilson coefficients in the Ref. [28] basis are not known in the Feynman gauge with LL accuracy. For the spin-dependent ones, we could only determine the anomalous dimensions for $c_5^{(3)}$ and $c_7^{(3)}$, which are gauge-independent. However, it is not worthwhile because it is a bit cumbersome due to $c_6^{(3)} = c_8^{(3)}$ and $c_{10}^{(3)} = c_{11}^{(3)}$, which makes ambiguous some of the entries of the anomalous dimension matrix. Instead, we compute it only for the spin-independent operators: $\mathcal{O}_{1-4,12,13}$. We obtain (we use the same ordering and notation that in Ref. [33])

$$\hat{\gamma}_l^{(3)A} = \begin{pmatrix} -11/12 & 0 & 11/24 & -11/24 & 11/288 & -11/72 \\ -4 & 0 & 1 & -1 & 0 & -2/9 \\ -3 & 5/6 & 5/6 & -5/3 & 1/12 & -7/18 \\ -7/2 & 5/6 & 4/3 & -13/6 & 1/24 & -11/36 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 11/24 & -11/6 \end{pmatrix}, \tag{2.186}$$

$$\hat{\gamma}_l^{(111)A} = \begin{pmatrix} 16/3 & 0 & -5/2 & 7/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -2/3 & -11/6 & 5/3 & -1/4 & 1/2 \\ 3/2 & 0 & -3/2 & 3/2 & 1/8 & -1/4 \end{pmatrix}, \tag{2.187}$$

$$\hat{\gamma}_l^{(111)F} = \begin{pmatrix} 32/3 & 0 & -8/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.188}$$

$$\hat{\gamma}_l^{(12)A} = \begin{pmatrix} 4 & -5/6 & -3/2 & 2 & 0 & 2/9 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 5/6 & 1/3 & -1/12 & -1/4 & 1/144 & 11/36 \end{pmatrix}, \tag{2.189}$$

$$\hat{\gamma}_l^{(12)F} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.190}$$

On the one hand, in all matrices except the last one, we find discrepancies with the entries given in the Appendix of Ref. [33]. The differences do not follow a clear pattern. On the other hand, remarkably enough, our anomalous dimension matrix produces the same SL expressions as those given in Table II of Ref. [34] (note that the expression for $c_1^{(3)p}$ is different from the one it can be found in Table I of Ref. [33]).

It is also interesting (and the only possibility to compare the spin-dependent operators) to make the comparison backward and compute the RGEs for physical quantities in our basis from the anomalous dimension matrix given in Refs. [33, 34]. To do this, one needs the inverse relations between the Wilson coefficients in the two bases

$$c_M = -\frac{1}{2}(c_3^{(3)} + c_4^{(3)})\,, \tag{2.191}$$

$$c_4 = c_2^{(3)} + c_3^{(3)} + c_4^{(3)} + c_1^{(1)\,3} - c_1^{(1)} c_1^{(2)}\,, \tag{2.192}$$

$$c_{A_1} = 2c_3^{(3)} + c_4^{(3)} + c_1^{(1)} c_2^{(1)\,2} + c_2^{(1)} c_2^{(2)}\,, \tag{2.193}$$

$$\bar{c}_{A_2} = 2c_1^{(3)} - 2c_1^{(1)} c_2^{(1)\,2} - 2c_2^{(2)} c_2^{(1)}\,, \tag{2.194}$$

$$c_{A_3} = 12c_{12}^{(3)}\,, \tag{2.195}$$

$$c_{A_4} = 12c_{13}^{(3)}\,, \tag{2.196}$$

$$c_{p'p} = c_9^{(3)} + c_{10}^{(3)}\,, \tag{2.197}$$

$$c_{W_1} = -c_6^{(3)} - c_9^{(3)} - c_{10}^{(3)} + c_1^{(1)\,2} c_2^{(1)} - \frac{1}{2} c_1^{(2)} c_2^{(1)} + \frac{1}{2} c_2^{(2)} c_1^{(1)}\,, \tag{2.198}$$

$$c_{W_2} = \frac{1}{2} c_7^{(3)} + \frac{1}{2} c_9^{(3)} + c_{10}^{(3)}\,, \tag{2.199}$$

$$c_{B_1} = -c_9^{(3)} + c_1^{(1)} c_2^{(1)\,2} + c_2^{(2)} c_2^{(1)}\,, \tag{2.200}$$

$$c_{B_2} = -c_5^{(3)} - c_1^{(1)} c_2^{(1)\,2} - c_2^{(2)} c_2^{(1)}\,. \tag{2.201}$$

Concerning the Wilson coefficients of the spin-independent operators, and for the gauge invariant combinations we have computed, the following RGEs are obtained

$$\nu \frac{d}{d\nu} c_4 = \frac{\alpha}{\pi}\left[C_A\left(\mathbf{\frac{1}{3}} c_D c_k - \mathbf{\frac{5}{6}} c_F^2 c_k - \mathbf{\frac{1}{6}} c_k^3\right) - \mathbf{\frac{8}{3}} c_k^3 C_F\right], \tag{2.202}$$

$$\begin{aligned}\nu \frac{d}{d\nu} c_{A_1} = {} & \frac{\alpha}{\pi}\left[C_A\left(-c_4 + \frac{1}{2} c_{A_1} - \frac{11}{48} \bar{c}_{A_2} + \mathbf{\frac{121}{24}} c_F^2 c_k - \mathbf{\frac{37}{24}} c_F c_S + \mathbf{0} c_F^3\right)\right. \\ & \left. + \left(\frac{11}{3} C_A + \frac{16}{3} C_F\right) c_k^3\right], \end{aligned} \tag{2.203}$$

$$\begin{aligned}\nu \frac{d}{d\nu} \bar{c}_{A_2} = {} & \frac{\alpha}{\pi}\left[C_A\left(8c_4 - c_{A_1} + \frac{11}{12} \bar{c}_{A_2} - \mathbf{\frac{67}{6}} c_F^2 c_k + \mathbf{\frac{19}{6}} c_S c_F + \mathbf{0} c_F^3\right)\right. \\ & \left. + \left(-\frac{56}{3} C_A - \frac{64}{3} C_F\right) c_k^3\right], \end{aligned} \tag{2.204}$$

$$\nu\frac{d}{d\nu}c_{A_3} = \frac{\alpha}{\pi}C_A\left[-\frac{1}{2}c_{A_1}-\frac{11}{48}\bar{c}_{A_2}-\frac{11}{24}c_{A_4}-\frac{3}{2}c_F^3+\frac{73}{24}c_F^2c_k-\frac{1}{24}c_Fc_S\right], \quad (2.205)$$

$$\nu\frac{d}{d\nu}c_{A_4} = \frac{\alpha}{\pi}C_A\left[\frac{8}{3}c_4+c_{A_1}+\frac{11}{12}\bar{c}_{A_2}+\frac{\mathbf{1}}{\mathbf{6}}c_{A_4}-\mathbf{3}c_F^3-\frac{31}{6}c_F^2c_k-\frac{8}{3}c_k^3-\frac{17}{6}c_Fc_S\right]. \quad (2.206)$$

where numbers in bold indicate a discrepancy with respect to our results. Note that the anomalous dimension matrix given in Ref. [33] yields different RGEs as those we found in Sec. 2.5.3 (nor even the running of c_4 is zero, neither gauge-independent) and also different SL expressions as those we found in Eqs. (2.154-2.157), except for Eq.(2.156). Contrarily, the SL results given in Table II of Ref. [34] yield results in agreement with our SL results in Eqs. (2.154-2.157). We also observe that the running of c_M obtained from Refs. [33, 34] (which we do not display explicitly) agrees with the result predicted by reparametrization invariance (in the Feynman gauge).

Concerning the Wilson coefficients of the spin-dependent operators, firstly note that the anomalous dimension matrix displayed in Ref. [33] gives $c_6^{(3)} \neq c_8^{(3)}$. This already disagrees with our results and with the explicit SL results given in Table II of that reference. We continue with the comparison nonetheless. We take the expression for $c_6^{(3)}$, which is the one which minimizes the discrepancies. The RGEs read

$$\nu\frac{d}{d\nu}c_{p'p} = \frac{\alpha}{\pi}\left[C_Ac_Fc_k^2+\frac{1}{2}C_Ac_{p'p}-\frac{1}{2}C_Ac_Sc_k\right], \quad (2.207)$$

$$\begin{aligned}\nu\frac{d}{d\nu}\bar{c}_W = \frac{\alpha}{\pi}\Big[&\frac{\mathbf{1}}{\mathbf{6}}C_Ac_Dc_F-\frac{\mathbf{5}}{\mathbf{12}}C_Ac_F^3-\frac{\mathbf{3}}{\mathbf{4}}C_Ac_F^2c_k-\frac{\mathbf{1}}{\mathbf{12}}C_Ac_Fc_k^2-\frac{\mathbf{4}}{\mathbf{3}}C_Fc_Fc_k^2\\&-\frac{\mathbf{1}}{\mathbf{4}}C_Ac_Sc_k+\frac{1}{2}C_A\bar{c}_W\Big], \quad (2.208)\end{aligned}$$

$$\begin{aligned}\nu\frac{d}{d\nu}\bar{c}_{B_1} = \frac{\alpha}{\pi}\Big[&\frac{3}{2}C_A\bar{c}_{B_1}-\frac{1}{12}C_Ac_{B_2}-\frac{\mathbf{1}}{\mathbf{3}}C_Ac_Dc_F+\frac{\mathbf{5}}{\mathbf{6}}C_Ac_F^3+\frac{\mathbf{35}}{\mathbf{12}}C_Ac_F^2c_k+\frac{\mathbf{11}}{\mathbf{6}}C_Ac_Fc_k^2\\&+\frac{\mathbf{16}}{\mathbf{3}}C_Fc_Fc_k^2+\frac{1}{2}C_Ac_{p'p}-\frac{\mathbf{17}}{\mathbf{12}}C_Ac_Fc_S+\mathbf{1}C_Ac_Sc_k+C_A\bar{c}_W\Big], \quad (2.209)\end{aligned}$$

$$\begin{aligned}\nu\frac{d}{d\nu}c_{B_2} = \frac{\alpha}{\pi}\Big[&-\frac{1}{2}C_A\bar{c}_{B_1}+\frac{7}{6}C_Ac_{B_2}-\frac{\mathbf{4}}{\mathbf{3}}C_Ac_F^2c_k-\frac{4}{3}C_Ac_Fc_k^2+\frac{8}{3}C_Fc_Fc_k^2+\frac{\mathbf{4}}{\mathbf{3}}C_Ac_Fc_S\\&-\frac{4}{3}C_Ac_Sc_k-\frac{16}{3}C_Fc_Sc_k-C_A\bar{c}_W+\mathbf{0}c_F^3\Big], \quad (2.210)\end{aligned}$$

where, as before, numbers in bold indicate a discrepancy with respect to our results. In general, we find disagreement for all RGEs (even in QED) except for Eq. (2.207), which satisfies reparametrization invariance. Conceptually, the disagreement with Eqs. (2.208-2.209) is important, since they do not depend only on physical combinations of Wilson coefficients due to the explicit appearance of c_D, which is gauge dependent. In addition, Eq. (2.208) does not satisfy reparametrization invariance.

On the contrary, it is remarkable that using the SL results given in Table II of Ref. [34] one finds agreement with the SL results for the physical quantities we have computed, i.e with Eqs. (2.160,2.161). Nevertheless, we find disagreement for the unphysical quantities c_M, c_{A_2}, c_{W_1}, c_{W_2} and c_{B_1} (whose SL results are not presented explicitly, but they can be easily obtained from the RGEs of Sec. 2.5.2). If we trust the explicit SL results presented in this reference, this is a clear indication these Wilson coefficients are gauge dependent.

Spectator effects in HQET up to $\mathcal{O}(1/m^3)$ were already studied in Ref. [29]. However, no anomalous dimension matrix was given, but only SL results. At this level, we can compare our results with the ones given in that reference after changing the operator basis. For the spin-independent heavy-light operators we find that $c_{3-}^{(3l)o} = 8d_2^{hl}$ and $c_{3-}^{(3l)s} = 8d_3^{hl}$. Such results disagree with Eqs. (2.158-2.159) by a factor of two. The first thing we observe concerning the spin-dependent heavy-light operators is that, in Ref. [29], it is stated that they change the SL results of the spin-dependent heavy-gluon sector found previously in Ref. [33]. That is strange, because the initial matching condition of the heavy-light operators is zero at tree level, and therefore, they should not change the SL expressions. After a more detailed comparison, taking the SL results given in Ref. [29] and using Eqs. (2.197-2.201) to change the operator basis, we find that, for physical combinations, the results remain unchanged and are still in agreement with ours and with what we find in this thesis (that the SL remain unchanged after including spectators). Concerning the running of the spin-dependent heavy-light operators, we find that $c_{7+}^{(3h)o} = c_{7-}^{(3h)o} = 8d_4^{hl}$. The first equality is already in disagreement with Ref. [29], and for the explicit SL results given in it, only the term proportional to C_F agrees with ours. We also find that $c_{7+}^{(3h)s} = c_{7-}^{(3h)s} = 8d_5^{hl}$, which leads to agreement between the SL expression presented in Ref. [29] and ours. Also the given results for d_7^{hl}, d_9^{hl} and d_{11}^{hl}, whose SL result is zero, are in agreement with ours. We find that $c_{6-}^{(3l)o} + c_{7-}^{(3l)o} = 8\bar{d}_8^{hl}$, which also agrees. Finally, we find that $\bar{d}_{10}^{hl} = c_6^{(3l)o} - (c_{7+}^{(3h)o} - c_{7-}^{(3h)o})/2 - c_{W_1}^{\rm FG}$ (where $c_{W_1}^{\rm FG}$ is the Wilson coefficient c_{W_1} evaluated in the Feynman gauge, whose SL expression was found in Ref. [33]), for which we find disagreement. It is worth mentioning that a change of sign in the SL of $c_6^{(3l)o}$ plus the condition $c_{7+}^{(3h)o} = c_{7-}^{(3h)o}$, expected to reproduce d_4^{hl} correctly, would lead to agreement. This also would imply a change of sign in the SL expression of $c_{7-}^{(3l)o}$.

2.7 Loop integrals UV pole

Loop integrals in the Coulomb gauge are presumably more difficult than in any other covariant gauge where several computational tools have been developed. In particular, the integrals we face have the difficulty that they are explicitly non-relativistic i.e. they can not be written in a covariant way. As a consequence, an splitting between the energy and d-momentum integrals occurs. Note that there is no problem in using dimensional regularization (indeed it is what we formally use throughout the thesis) to regulate the integrals, but the standard techniques for solving them, like the introduction of Feynman parametrization, are not feasible. This makes the exact solution of the integrals very difficult to achieve. However, since we are only interested in the UV pole or the logarithm (both are connected) of the integrals, we can introduce a hard cutoff and expand the integrals for high loop momentum compared to external momenta and energies, while keeping only logarithmically divergent terms (terms with superficial degree of divergence $\mathcal{D}=0$). Once we solve the integrals for these terms, we find a logarithm which is the same one would obtain in dimensional regularization. The construction of the pole one would obtain in dimensional regularization is then straightforward. Let us make these statements quantitative by computing an example which can be solved exactly, too. The integral we aim to compute is

$$I = g^4 \int \frac{d^d\mathbf{q}}{(2\pi)^d}|\mathbf{q}|\left(\delta^{kl} - \frac{\mathbf{q}^k\mathbf{q}^l}{\mathbf{q}^2}\right)\int \frac{dq^0}{2\pi}\frac{1}{(q^0)^2-\mathbf{q}^2+i\eta_g}\frac{1}{q^0-E_1-i\eta_q}\frac{1}{q^0-E_1'-i\eta_q}\,. \tag{2.211}$$

Firstly, let us compute the exact solution. Integrating over q^0, we obtain

$$I = -\frac{ig^4}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\left(\delta^{kl} - \frac{\mathbf{q}^k\mathbf{q}^l}{\mathbf{q}^2}\right)\frac{1}{|\mathbf{q}|-E_1-i\eta_q}\frac{1}{|\mathbf{q}|-E_1'-i\eta_q}\,. \tag{2.212}$$

The remaining integral can be solved by changing to d-dimensional spherical coordinates and integrating over the solid angle and over $|\mathbf{q}|$. However, in order to do that, we must have an scalar integral. To this purpose, it is enough to realise that the tensor structure of the solution is $I = A\delta^{kl}$. Therefore, the only thing that remains to be determined is the global factor A given by the following scalar integral

$$A = -\frac{ig^4}{2}\frac{d-1}{d}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{|\mathbf{q}|-E_1-i\eta_q}\frac{1}{|\mathbf{q}|-E_1'-i\eta_q}\,. \tag{2.213}$$

By changing to spherical coordinates and integrating over the solid angle, we find that

$$A = -\frac{ig^4}{2}\frac{d-1}{d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty \frac{d|\mathbf{q}|}{(2\pi)^d}\frac{|\mathbf{q}|^{d-1}}{(|\mathbf{q}|-E_1-i\eta_q)(|\mathbf{q}|-E_1'-i\eta_q)}\,, \tag{2.214}$$

or, written in terms of ϵ ($d=3+2\epsilon$),

$$A = -\frac{ig^4}{2}\frac{2+2\epsilon}{3+2\epsilon}\frac{2\pi^{3/2+\epsilon}}{\Gamma(3/2+\epsilon)}\frac{1}{(2\pi)^{3+2\epsilon}}\int_0^\infty d|\mathbf{q}|\frac{|\mathbf{q}|^{2+2\epsilon}}{(|\mathbf{q}|-E_1-i\eta_q)(|\mathbf{q}|-E_1'-i\eta_q)}\,. \tag{2.215}$$

The integral over $|\mathbf{q}|$ can be computed with Mathematica. The solution reads

$$A = -\frac{ig^4}{2}\frac{2+2\epsilon}{3+2\epsilon}\frac{2\pi^{3/2+\epsilon}}{\Gamma(3/2+\epsilon)}\frac{1}{(2\pi)^{3+2\epsilon}}\frac{\pi((-E_1-i\eta_q)^{2+2\epsilon}-(-E_1'-i\eta_q)^{2+2\epsilon})}{(-E_1+E_1')\sin(2\pi\epsilon)}\,. \tag{2.216}$$

Adding and subtracting $\nu^{2\epsilon}$ and expanding around $\epsilon=0$, it gives

$$A = \frac{ig^4}{2}\nu^{2\epsilon}\left(\frac{E_1+E_1'}{6\pi^2\epsilon} + \frac{E_1^2\ln\left(\frac{-E_1-i\eta_q}{\nu}\right) - E_1'^2\ln\left(\frac{-E_1'-i\eta_q}{\nu}\right)}{3\pi^2(E_1-E_1')} + \dots\right) + \mathcal{O}(\epsilon)\,. \tag{2.217}$$

The counterterm should be chosen in such a way that it cancels the UV divergence, so

$$\delta A = -\frac{ig^4}{12\pi^2\epsilon}\nu^{2\epsilon}(E_1+E_1')\,. \tag{2.218}$$

However, as we commented previously, the exact solution is not always available and we must find an alternative way to extract the UV pole. To do so, we focus in the high energy behaviour of the integral, and expand around zero external momenta and energies, while keeping the loop momentum and energy of the same order $q^0 \sim |\mathbf{q}|$, and much larger than the external ones. In order to make the expansion valid, we will have to integrate in a region where such expansion holds. In other words, we will have to introduce two hard cutoffs to fix the integration regime: an upper or UV cutoff Λ_{UV}, and a lower or IR cutoff Λ_{IR}. We will only pay attention to the UV, and the IR is only introduced to make sense of the integral. The cutoff is introduced in the integral over the modulus of the loop momentum, which is where the divergence in four dimensions comes from. The expansion is done with the aim of finding all terms which are logarithmically divergent in the UV i.e. those terms with zero superficial degree of divergence. For the q^0 integral summation and integration commute, so we expand at the very beginning. Since the superficial degree of divergence of the integral is $\mathcal{D} = 1$, we only need to expand to first order (note that, as higher order terms are considered in the expansion, more IR the integral becomes). Thus

$$I_{\text{UV}} = g^4(E_1+E_1')\int\frac{d^3\mathbf{q}}{(2\pi)^3}|\mathbf{q}|\left(\delta^{kl} - \frac{\mathbf{q}^k\mathbf{q}^l}{\mathbf{q}^2}\right)\int\frac{dq^0}{2\pi}\frac{1}{(q^0)^2-\mathbf{q}^2+i\eta_g}\frac{1}{(q^0-i\eta_q)^3}\,. \tag{2.219}$$

Integrating over q^0, we obtain

$$I_{\text{UV}} = -\frac{ig^4}{2}(E_1+E_1')\int\frac{d^3\mathbf{q}}{(2\pi)^3}\left(\delta^{kl} - \frac{\mathbf{q}^k\mathbf{q}^l}{\mathbf{q}^2}\right)\frac{1}{|\mathbf{q}|^3}\,. \tag{2.220}$$

Like before, we realise that the integral has the tensor structure $I = A_{\text{UV}}\delta^{kl}$, and the problem reduces to the computation of the following scalar integral

$$A_{\text{UV}} = -\frac{ig^4}{3}(E_1+E_1')\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{|\mathbf{q}|^3}\,, \tag{2.221}$$

which changing to spherical coordinates and integrating over the solid angle, gives

$$A_{\text{UV}} = -\frac{ig^4}{6\pi^2}(E_1+E_1')\int_{\Lambda_{IR}}^{\Lambda_{\text{UV}}} d|\mathbf{q}|\frac{1}{|\mathbf{q}|}\,, \tag{2.222}$$

and finally,

$$A_{\text{UV}} = -\frac{ig^4}{6\pi^2}(E_1+E_1')\ln\left(\frac{\Lambda_{\text{UV}}}{\Lambda_{IR}}\right)\,, \tag{2.223}$$

from where it can be identified that the logarithm of $\Lambda_{\rm UV}$ is the logarithm of ν appearing in Eq. (2.217) or, likewise, it is the logarithm that appears if we expand Eq. (2.218) around $\epsilon = 0$. Since we know the exact solution, the counterterm can be determined by just identifying $\nu \leftrightarrow \Lambda_{\rm UV}$ in Eq. (2.218), obtaining

$$\delta A = -\frac{ig^4}{12\pi^2\epsilon}(E_1 + E_1')\Lambda_{\rm UV}^{2\epsilon}\,. \tag{2.224}$$

Note that the scaling with the UV cutoff is $\Lambda_{\rm UV}^{2\epsilon}$ because the integral has non-integer mass dimension $M^{2\epsilon}$. This is a general fact which will allow us to reconstruct the counterterm from the logarithm of $\Lambda_{\rm UV}$. Therefore, provided the solution (the logarithmic term) of a Feynman integral where hard cut-offs have been introduced

$$I \equiv -\gamma \ln\left(\frac{\Lambda_{\rm UV}}{\Lambda_{IR}}\right)\,, \tag{2.225}$$

the counterterm will always be reconstructed as follows

$$\delta I = -\frac{1}{2\epsilon}\gamma\Lambda_{\rm UV}^{2\epsilon}\,, \tag{2.226}$$

where γ is defined just as the term multiplying the logarithm. Since we can identify $\nu \leftrightarrow \Lambda_{\rm UV}$, the counterterm we would obtain from a calculation in dimensional regularization would be

$$\delta I = -\frac{1}{2\epsilon}\gamma\nu^{2\epsilon}\,. \tag{2.227}$$

Chapter 3

Non-Relativistic Quantum Chromodynamics (NRQCD)

3.1 Heavy Quarkonium, $Q\bar{Q}$

Since the simultaneous discovery of the J/ψ particle (charmonium $c\bar{c}$) in 1975 [46, 47] at SLAC and the Brookhaven National Laboratory, and three years later of the Υ particle (bottomonium $b\bar{b}$) [48] in Fermilab, the study of non-relativistic bound states made of two heavy quarks, heavy quarkonium for short, has proven to be very important for our understanding of the strong interactions, as its nature falls in the interplay between the perturbative and non-perturbative regime of QCD. Moreover it opened the possibility to study a non-relativistic picture in the framework of QCD. Such a picture became even more relevant with the discovery of the B_c ($\bar{b}c$) and $\bar{B}_c$ ($b\bar{c}$) bound states in 1998 [49].

The interaction between the two heavy quarks in the bound state is understood in two assymptotic regions: the very short distance region, described by a Coulomb-like $1/r$ (plus relativistic corrections) interaction potential, and the very long distance region, described by a linear in r interaction potential, being r the separation between the two heavy quarks. Whereas perturbation theory is successful to the description of the potential at very short distances, it completely fails to describe the potential as soon as we move away from the very short distance regime, since QCD becomes non-perturbative.

A crucial aspect of heavy quarkonium is that its physics is characterized by three widely separated scales: the hard scale m, which is the heavy quark mass, the soft scale $|\mathbf{p}| \sim mv$, which is the relative momentum between the two heavy quarks (or the inverse Bohr radius), and the ultrasoft scale $E \sim mv^2$, which is the typical binding energy of the system, being v the heavy quark velocity. Since the system is non-relativistic ($v \ll 1$), the following hierarchy between the different scales is expected: $m \gg mv \gg mv^2$. Moreover, by definition of heavy quark, its mass $m \gg \Lambda_{\rm QCD}$, the hadronization scale. Therefore, processes happening at the scale m can be described using perturbation theory due to the asymptotic freedom of QCD. However, the scales $|\mathbf{p}|$ and E may or may not be accessible to perturbation theory. The appearance of all these scales makes the quantitative study of heavy quarkonium extremely difficult.

In the present work we will focus our attention in the perturbative part of the heavy quarkonium interaction. We have to say that, even for the perturbative contributions, it took some time to find a way to incorporate radiative corrections systematically due to UV divergences in the non-relativistic approach, based on the use of the Schrödinger equation, and due to technical difficulties in the relativistic one, based on the Bethe-Salpeter formalism [50]. A new approach was proposed by Caswell and Lepage [25] and Bodwin, Braaten and Lepage [26]. It consisted in a reformulation of QCD in terms of an EFT made of quarks and gluons describing fluctuations below the hard scale m, and where fluctuations above the scale m are encoded in the Wilson coefficients of the theory. That effective theory was called NRQCD. However, NRQCD only takes advantage of the fact that $m \gg |\mathbf{p}|$, E, $\Lambda_{\rm QCD}$, but it forgets about all other scale separation, and this generates important problems. Despite of the fact that this new theory provided a framework to treat UV divergences, it presents several complications: NRQCD is inconsistent in dimensional regularization, there are not well-defined counting rules and it is not possible to sum large logarithms of v. Consistent calculations are only possible in cut-off schemes which are well-known to be troublesome specially if one wants to compute radiative corrections beyond one loop. As a conclusion, the theory was found to be not completely optimised to systematically incorporate radiative corrections in bound state calculations.

The solution came by taking advantage of the full scale separation characterizing heavy quarkonium. The NRQCD framework was rewritten in terms of a Schrödinger theory, where the dominant binding of the $Q\bar{Q}$ pair is described by the well-known non-relativistic Schrödinger equation with the non-Abelian version of the Coulomb potential, and where higher order corrections such as v-suppressed potentials and retardation effects can be computed separately as perturbations. That was the born of a new EFT called pNRQCD, devised by Pineda and Soto [1]. The new theory turns up to have well-defined counting rules in v and it is formulated in dimensional regularization.

3.2 NRQCD vs HQET

HQET and NRQCD are EFTs devised to describe the interaction of one heavy quark of mass m and of two heavy quarks of mass m_1 and m_2, respectively, which are almost on shell. If we compare two HQET Lagrangians (one for each heavy quark appearing in the NRQCD Lagrangian) with the NRQCD Lagrangian, we find that both EFTs look identical as far as the form of the Lagrangian operators is concerned, except for the fact that in NRQCD there are four fermion operators. However, these two EFTs are not equivalent, and not only for this reason. On the one hand, HQET is thought to describe hadronic properties as far as the momentum transferred is $p^\mu \sim \Lambda_{\rm QCD} \ll m$. Therefore, the expansion in HQET is an expansion in powers of $\Lambda_{\rm QCD}/m$. On the other hand, NRQCD is applied to bound state systems formed by two non-relativistic heavy quarks, a $Q\bar{Q}$ bound state. The NRQCD Lagrangian also has an expansion in $1/m$, but in this case, the momentum transferred is of order mv, and the energy transferred of order mv^2, where v is the relative velocity between the two heavy quarks in the center of mass frame. Therefore, the small expansion parameter in NRQCD is v. The basic difference between HQET and NRQCD can be seen from the first two terms in the effective Lagrangian,

$$\mathcal{L} = Q^\dagger (iD^0) Q + Q^\dagger \frac{\mathbf{D}^2}{2m} Q \,. \tag{3.1}$$

In HQET, the first term is of order $\Lambda_{\rm QCD}$ and the second term is of order $\Lambda_{\rm QCD}^2/m$, so the second term is subleading and it can be eliminated in a first approximation. In NRQCD both terms are of order mv^2 instead and, therefore, the second term can not be eliminated. As a consequence, the propagators of both EFTs are different. In HQET the propagator is $i/(k^0+i\eta)$, whereas in NRQCD the it is $i/(k^0-\mathbf{k}^2/2m+i\eta)$.

Since the HQET propagator is m independent, in the matching calculation of QCD with HQET, the contribution of a graph to a given order is completely determined by the power of the mass of every operator. The power counting is manifest in the Lagrangian. Differently, the matching calculation in NRQCD is more subtle. Due to the form of the NRQCD propagator, we cannot compute matching corrections using it, since the v power counting breaks down (see Refs. [28, 51]). The counting in v is not manifest in the Lagrangian anymore. Instead, the matching conditions for NRQCD should be computed using the HQET power counting, by expanding in powers of p^μ/m. The same applies to the computation of the renormalization group improved Wilson coefficients of the Lagrangian. After the HQET Lagrangian has been computed, it can be used for computing bound state properties using the NRQCD velocity power counting rules. That is the reason why we will use the results obtained in Ch. 2 to Ch. 4.

3.3 NRQCD Lagrangian

In the previous chapter we studied aspects of the dynamics of a heavy quark interacting with gluons and light quarks. Such dynamics is described by the HQET Lagrangian. That Lagrangian is indeed one of the main ingredients of the NRQCD Lagrangian, aimed to describe bound states made of two non-relativistic heavy quarks.

Important to present work is that the Wilson coefficients of the NRQCD Lagrangian [25, 26] are instrumental in the determination of the Wilson coefficients of the pNRQCD Lagrangian. Since the aim of this thesis is to compute the N^{3}LL heavy quarkonium spectrum relevant to S-wave spin-independent states, we only include those NRQCD Lagrangian operators relevant to that analysis.

The HQET Lagrangian up to $\mathcal{O}(1/m^3)$ can be found in Ref. [28], and including light fermions, though in a different basis, in Ref. [29]. In the present dissertation, we use the basis and notation from Ref. [35], which also includes massless fermions. That is precisely the basis given is Ch. 2. In Ref. [35] one can find the resummed expressions of the Wilson coefficients with LL accuracy for the $1/m^3$ spin-independent operators. For the spin-dependent $1/m^3$ operators, not relevant to this program, the LL running can be found in Refs. [36, 37]. How these results were obtained is described in Ch. 2. Note that there are not purely gluonic operators of dimension seven, neither four heavy-fermion operators of dimension seven [52].

Concerning the $\mathcal{O}(1/m^4)$ NRQCD Lagrangian operators, not all of them are needed for the purposes of this thesis, but only some few operators. For the heavy-gluon sector, the complete set of operators was written for QED in Ref. [44] and for QCD in Ref. [53] (in the last case without light fermions). Of those we can neglect most. We do not need the spin-dependent $1/m^4$ operators, nor terms proportional to a single $\mathbf{B}$, neither terms with two (either $\mathbf{B}$ or $\mathbf{E}$) terms. The reason is that we only need $\mathcal{O}(1/m^4)$ tree level potentials. Therefore, we can take all relevant operators from the QED case generalized to QCD. Following the notation of Ref. [44], the relevant $\mathcal{O}(1/m^4)$ HQET Lagrangian operators are

$$\begin{aligned}\delta\mathcal{L}_Q^{(4)} &= c_{X1}^{(1)} g \frac{Q^\dagger[\mathbf{D}^2, \mathbf{D}\cdot\mathbf{E}+\mathbf{E}\cdot\mathbf{D}]Q}{m^4} + c_{X2}^{(1)} g \frac{Q^\dagger\{\mathbf{D}^2, [\boldsymbol{\nabla}\cdot\mathbf{E}]\}Q}{m^4} \\ &\quad + c_{X3}^{(1)} g \frac{Q^\dagger[\boldsymbol{\nabla}^2, \boldsymbol{\nabla}\cdot\mathbf{E}]Q}{m^4} + \cdots\,, \end{aligned} \tag{3.2}$$

and similarly for the antiquark. The dots stand for terms that one can trivially see do not contribute to the S-wave spin-independent spectrum at N^{3}LL, either because they involved the emission of two gluons or because they are spin-dependent, so in principle, we need three new coefficients. Nevertheless, we will see in the next chapter that only c_{X1} contributes to the running of the spin-independent delta-like potential. Still, we will compute any tree level potential proportional to c_{X1}, c_{X2} and c_{X3}.

The fact that we need c_{X1}, one of the Wilson coefficients of the $1/m^4$ heavy-gluon operators of the HQET Lagrangian, could make it necessary to consider the Wilson coefficients of the $1/m^4$ heavy-light operators as well (light-light operators are subleading for the same reason they are at $\mathcal{O}(1/m^3)$), as they may enter through RG mixing. Fortunately, this is not the case, since c_{X1} can be determined by reparametrization invariance [44]

$$32 c_{X1}^{(i)} = \frac{5Z}{4} - c_F^{(i)} + c_D^{(i)}\,, \tag{3.3}$$

where one should take $Z = 1$ for QCD. Note that it depends on c_D, so indeed $c_{X1}^{(i)}$ is gauge dependent. Nevertheless, we will see in Sec. 4.5.3 of Ch. 4 that it always combines with c_M to produce gauge invariant combinations. This indeed is a nontrivial check of the computation. Note also that the above coefficient has an Abelian term, so it can be checked with QED computations.

Finally, we consider the heavy four-fermion sector of the $\mathcal{O}(1/m^4)$ Lagrangian. They generate local or quasilocal potentials, which do not produce divergent potential loops. The same happens for the potentials generated by c_{X2} and c_{X3}. Therefore, in both cases, such potentials do not generate contributions to the heavy quarkonium mass at N^{3}LL, and we can neglect them. Consequently, we have the LL running of all the necessary Wilson coefficients of the $\mathcal{O}(1/m^4)$ NRQCD Lagrangian operators.

Summarizing, the NRQCD Lagrangian we need is the following

$$\begin{aligned}\mathcal{L}_{\text{NRQCD}} &= \mathcal{L}_{Q_1\chi_{c,2}} + \mathcal{L}_{\text{HQET}}(Q \to Q_1, m \to m_1) \qquad (3.4)\\ &\quad + \mathcal{L}_{\text{HQET}}(Q \to \chi_{c,2}, g \to -g, T^a \to (T^a)^T, m \to m_2)\,, \qquad (3.5)\end{aligned}$$

where $\mathcal{L}_{\text{HQET}}$ is given by the sum of Eq. (2.21) and Eq. (3.2), Q_1 is a non-relativistic heavy quark field of mass m_1 represented by a Pauli spinor, $\chi_{c,2} = -i\sigma^2\chi_2^*$ is a non-relativistic heavy anti-quark field of mass m_2 also represented by a Pauli spinor, $(T^a)^T$ stands for the transposed matrix of T^a, and the change $T^a \to (T^a)^T$ only applies to the matrices contracted to the heavy quark color indices. The rest of the definitions are as in Sec. 2.3. Finally, The four heavy fermion operators to $\mathcal{O}(1/m^2)$ are given by

$$\mathcal{L}_{Q_1\chi_{c,2}} = -\frac{d_{ss}}{m_1 m_2} Q_1^\dagger Q_1 \chi_{c,2}^\dagger \chi_{c,2} + \frac{d_{sv}}{m_1 m_2} Q_1^\dagger \boldsymbol{\sigma}_1 Q_1 \chi_{c,2}^\dagger \boldsymbol{\sigma}_2 \chi_{c,2} \tag{3.6}$$

$$-\frac{d_{vs}}{m_1 m_2} Q_1^\dagger T^a Q_1 \chi_{c,2}^\dagger (T^a)^T \chi_{c,2} + \frac{d_{vv}}{m_1 m_2} Q_1^\dagger T^a \boldsymbol{\sigma}_1 Q_1 \chi_{c,2}^\dagger (T^a)^T \boldsymbol{\sigma}_2 \chi_{c,2} \,. \tag{3.7}$$

3.4 Matching QCD with NRQCD

In this section we compute the matching of the $1/m^2$ spin-independent four fermion operators of the NRQCD Lagrangian with the QCD Lagrangian at one loop order and for two quarks of different masses. The reason we do not consider the equal mass case is for simplicity, since for different masses annihilation type diagrams are not possible. This problem was already addressed in Ref. [15] for both, equal and different masses, but we compute it here as an illustrative example. As mentioned in Sec. 3.2, the matching must be done as in HQET. The section has two purposes, to give a pedagogical and detailed explanation of how the matching between QCD and NRQCD is performed, and also to set up the problem for a future two loop computation which could help to find the missing contribution of the NLL soft running of the Wilson coefficient $\tilde{D}_d^{(2)}$ associated to the spin-independent delta-like potential (see Sec. 4.4.2 and Sec. 4.5.1). To this last purpose, it should be enough to compute the divergent part of the two loop diagrams appearing in the matching calculation.

In order to carry out the matching, we compute the scattering of a heavy quark of mass m_1 with a heavy anti-quark of mass m_2 in QCD and in the Feynman gauge. The necessary QCD Feynman rules are displayed in Sec. C.1. Since there are no derivative terms in the $\mathcal{O}(1/m^2)$ four fermion operators, we can expand the matrix element around zero residual momentum to the zeroth order. In other words, it is enough to compute the matrix element for the four quarks at rest. This also means that the amputated legs in a digram only have to be multiplied either by $p_+ = (1+\gamma^0)/2$ (projector on the particle at rest subspace) or $p_- = (1-\gamma^0)/2$ (projector on the antiparticle at rest subspace), and the kinematic factor $\sqrt{m/E}$ relating relativistic and non-relativistic normalizations can be put to one.

It can be seen from the NRQCD Feynman rules of the four fermion operators displayed in Sec. C.1 that the structure of the spin-independent vertices is proportional to $\delta_{\mathcal{AB}}\delta_{\mathcal{B}'\mathcal{A}'}$, whereas the structure of the spin dependent ones is proportional to $\boldsymbol{\sigma}_{\mathcal{AB}}\boldsymbol{\sigma}_{\mathcal{B}'\mathcal{A}'}$. Therefore, once the external legs are projected to the particle or the antiparticle sector, $I_{\mathcal{AB}\,\mathcal{B}'\mathcal{A}'} = I_{AB\,B'A'}(p_+)_{\mathcal{A}A}(p_+)_{B\mathcal{B}}(p_-)_{A'\mathcal{A}'}(p_-)_{\mathcal{B}'B'}$, the result of any Feynman integral will be always proportional to these two structures. In other words, the result of any Feynman integral (to the order of interest) can be written as

$$I_{\mathcal{AB}\,\mathcal{B}'\mathcal{A}'} = I_{\rm SI}\delta_{\mathcal{AB}}\delta_{\mathcal{B}'\mathcal{A}'} + I_{\rm SD}\boldsymbol{\sigma}_{\mathcal{AB}}\boldsymbol{\sigma}_{\mathcal{B}'\mathcal{A}'} \,, \tag{3.8}$$

where $I_{\rm SI}$ and $I_{\rm SD}$ still have color indices. Since we are only interested in the spin-independent part, we can multiply by the projectors to the particle and the antiparticle sectors $(p_+)_{\mathcal{BA}}(p_-)_{\mathcal{A}'\mathcal{B}'} = [(1+\gamma^0)_{\mathcal{BA}}/2][(1-\gamma^0)_{\mathcal{A}'\mathcal{B}'}/2]$. Thus

$$I_{\mathcal{AB}\,\mathcal{B}'\mathcal{A}'}\frac{(1+\gamma^0)_{\mathcal{BA}}}{2}\frac{(1-\gamma^0)_{\mathcal{A}'\mathcal{B}'}}{2} = I_{\rm SI}\,{\rm Tr}\left(\frac{1+\gamma^0}{2}\right){\rm Tr}\left(\frac{1-\gamma^0}{2}\right). \tag{3.9}$$

Therefore, $I_{\rm SI}$ can be obtained from the expression

$$I_{\rm SI} = \frac{1}{D^2}I_{\mathcal{AB}\,\mathcal{B}'\mathcal{A}'}(1+\gamma^0)_{\mathcal{BA}}(1-\gamma^0)_{\mathcal{A}'\mathcal{B}'}\,, \tag{3.10}$$

where we have considered D-dimensional gamma matrices with $D = 4+2\epsilon$.

Let us go to the computation. For the case of different masses, only two diagrams contribute. They are shown in Fig. [3.1]. The amplitude for the diagram (1) of Fig. [3.1] reads

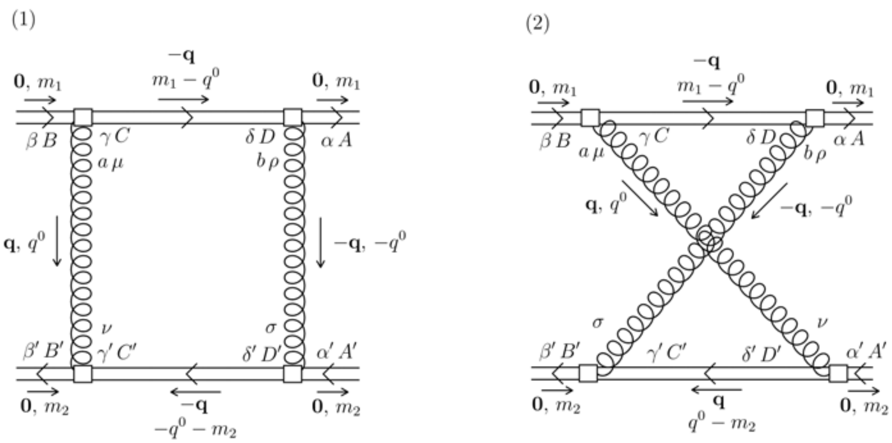


Figure 3.1: QCD diagrams contributing to the Wilson coefficients associated to the $1/m^2$ four fermion operators of the NRQCD Lagrangian in the case of two different masses.

$$\begin{aligned}(I_1)_{AB\,B'A'} \;=\; & \int\frac{d^Dq}{(2\pi)^D}(-ig)(\gamma^\mu)_{CB}(T^a)_{\gamma\beta}\frac{i}{q^2+i\eta_g}g_{\mu\nu}(-ig)(\gamma^\nu)_{B'C'}(T^a)_{\beta'\gamma'}\\ & \times\frac{i(\gamma^0(-q^0+m_1)-\gamma^i(-q)^i+m_1)_{DC}}{(-q^0+m_1)^2-\vec{q}^{\,2}-m_1^2+i\eta_q}\delta_{\delta\gamma}\\ & \times\frac{i(\gamma^0(-q^0-m_2)-\gamma^i(-q)^i+m_2)_{C'D'}}{(-q^0-m_2)^2-\vec{q}^{\,2}-m_2^2+i\eta_q}\delta_{\gamma'\delta'}\\ & \times(-ig)(\gamma^\rho)_{AD}(T^b)_{\alpha\delta}\frac{i}{q^2+i\eta_g}g_{\rho\sigma}(-ig)(\gamma^\sigma)_{D'A'}(T^b)_{\delta'\alpha'}\,.\end{aligned} \tag{3.11}$$

Projecting to the particle and the antiparticle sector correspondingly, we find that

$$
\begin{aligned}
(I_1)_{\mathcal{A}\mathcal{B}\mathcal{B}'\mathcal{A}'} &= g^4(T^bT^a)_{\alpha\beta}(T^aT^b)_{\beta'\alpha'}g_{\mu\nu}g_{\rho\sigma}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2+i\eta_g)^2}\\
&\times\frac{(p_+\gamma^\rho(-\gamma^\chi q_\chi+m_1(1+\gamma^0))\gamma^\mu p_+)_{\mathcal{A}\mathcal{B}}}{q^2-2m_1q^0+i\eta_q}\\
&\times\frac{(p_-\gamma^\nu(-\gamma^\xi q_\xi+m_2(1-\gamma^0))\gamma^\sigma p_-)_{\mathcal{B}'\mathcal{A}'}}{q^2+2m_2q^0+i\eta_q}\,. \qquad (3.12)
\end{aligned}
$$

Using Eq. (3.10) we find that

$$
\begin{aligned}
I_{1,\mathrm{SI}} &= g^4(T^bT^a)_{\alpha\beta}(T^aT^b)_{\beta'\alpha'}\frac{1}{D^2}g_{\mu\nu}g_{\rho\sigma}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2+i\eta_g)^2}\\
&\times\frac{\mathrm{Tr}(\gamma^\rho(-\gamma^\chi q_\chi+m_1(1+\gamma^0))\gamma^\mu(1+\gamma^0))}{q^2-2m_1q^0+i\eta_q}\\
&\times\frac{\mathrm{Tr}(\gamma^\nu(-\gamma^\xi q_\xi+m_2(1-\gamma^0))\gamma^\sigma(1-\gamma^0))}{q^2+2m_2q^0+i\eta_q}\,, \qquad (3.13)
\end{aligned}
$$

and computing the traces, we obtain

$$
\begin{aligned}
I_{1,\mathrm{SI}} &= g^4(T^bT^a)_{\alpha\beta}(T^aT^b)_{\beta'\alpha'}\int\frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2+i\eta_g)^2}\frac{1}{q^2-2m_1q^0+i\eta_q}\frac{1}{q^2+2m_2q^0+i\eta_q}\\
&\times[-2q^2+(2-D)(q^0)^2+2(m_1-m_2)q^0+4m_1m_2]\,. \qquad (3.14)
\end{aligned}
$$

At this point, it is better to write the integral in a covariant form. In order to do that, we introduce an auxiliary vector $v^\mu=(1,\mathbf{0})$, such that $q\cdot v=q^0$. Then, $I_{1,\mathrm{SI}}$ can be expressed as

$$
\begin{aligned}
I_{1,\mathrm{SI}} &= \frac{1}{4}g^4(\{T^a,T^b\}_{\alpha\beta}\{T^a,T^b\}_{\beta'\alpha'}-[T^a,T^b]_{\alpha\beta}[T^a,T^b]_{\beta'\alpha'})\\
&\times\int\frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2+i\eta_g)^2}\frac{1}{q^2-2m_1q\cdot v+i\eta_q}\frac{1}{q^2+2m_2q\cdot v+i\eta_q}\\
&\times[-(2g_{\mu\nu}+(D-2)v_\mu v_\nu)q^\mu q^\nu+2(m_1-m_2)v_\mu q^\mu+4m_1m_2]\,. \qquad (3.15)
\end{aligned}
$$

Similarly, for the diagram (2) of Fig.[3.1] we obtain

$$
\begin{aligned}
I_{2,\mathrm{SI}} &= \frac{1}{4}g^4(\{T^a,T^b\}_{\alpha\beta}\{T^a,T^b\}_{\beta'\alpha'}+[T^a,T^b]_{\alpha\beta}[T^a,T^b]_{\beta'\alpha'})\\
&\times\int\frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2+i\eta_g)^2}\frac{1}{q^2-2m_1q\cdot v+i\eta_q}\frac{1}{q^2-2m_2q\cdot v+i\eta_q}\\
&\times[(2g_{\mu\nu}+(D-2)v_\mu v_\nu)q^\mu q^\nu-2(m_1+m_2)v_\mu q^\mu+4m_1m_2]\,. \qquad (3.16)
\end{aligned}
$$

Using the master integrals of Sec. C.2, the sum of both diagrams yields

$$I_{1,\mathrm{SI}}+I_{2,\mathrm{SI}}=\Bigg[m_1^2(m_2^2)^\epsilon\Bigg(8C_Fm_1m_2(-1+\epsilon-2\epsilon^2)+C_A\Bigg(m_2^2\epsilon(-3+2\epsilon)+m_1^2(1+2\epsilon)$$
$$+3m_1m_2(1+\epsilon(-1+2\epsilon))\Bigg)\Bigg)-m_2^2(m_1^2)^\epsilon\Bigg(8C_Fm_1m_2(-1+\epsilon-2\epsilon^2)+C_A\Bigg(m_1^2\epsilon(-3+2\epsilon)+m_2^2(1+2\epsilon)$$
$$+3m_1m_2(1+\epsilon(-1+2\epsilon))\Bigg)\Bigg)\Bigg]\frac{i4^{-3-\epsilon}g^4\pi^{-2-\epsilon}(3+2\epsilon)\Gamma(-\epsilon)(T^a)_{\alpha\beta}(T^a)_{\beta'\alpha'}}{m_1^2(m_1-m_2)m_2^2(m_1+m_2)(-3+2\epsilon)(-1+2\epsilon)(1+2\epsilon)}$$
$$+\frac{i2^{-5-2\epsilon}(C_A-2C_F)C_Fg^4\Big((m_1^2)^\epsilon m_2^2-m_1^2(m_2^2)^\epsilon\Big)\pi^{-2-\epsilon}\Big(3-\epsilon(1-4\epsilon(1+\epsilon))\Big)\Gamma(-\epsilon)\delta_{\alpha\beta}\delta_{\beta'\alpha'}}{m_1(m_1-m_2)m_2(m_1+m_2)(-3+2\epsilon)(-1+2\epsilon)(1+2\epsilon)}. \tag{3.17}$$

The amplitude Eq. (3.17) is divergent and needs to be renormalized, i.e. we need to subtract the divergent part. We do it in the $\overline{\mathrm{MS}}$ renormalization scheme, characterized by the renormalization scale

$$\bar{\nu}^{2\epsilon}=\nu^{2\epsilon}\left(\frac{e^{\gamma_E}}{4\pi}\right)^\epsilon. \tag{3.18}$$

Once the amplitude is renormalized, we can compute $d_{ss}^{\overline{\mathrm{MS}}}$ and $d_{vs}^{\overline{\mathrm{MS}}}$ by comparing Eq. (3.17) with the amplitude in NRQCD

$$I_{1,\mathrm{SI}}^{\overline{\mathrm{MS}}}+I_{2,\mathrm{SI}}^{\overline{\mathrm{MS}}}=-i\frac{d_{ss}^{\overline{\mathrm{MS}}}}{m_1m_2}\delta_{\alpha\beta}\delta_{\beta'\alpha'}-i\frac{d_{vs}^{\overline{\mathrm{MS}}}}{m_1m_2}(T^a)_{\alpha\beta}(T^a)_{\beta'\alpha'}\,, \tag{3.19}$$

from where we can identify

$$d_{ss}^{\overline{\mathrm{MS}}}=-C_F\left(\frac{C_A}{2}-C_F\right)\frac{\alpha^2}{m_1^2-m_2^2}\Bigg\{m_1^2\Bigg(\frac{1}{3}+\ln\left(\frac{m_2^2}{\nu^2}\right)\Bigg)-m_2^2\Bigg(\frac{1}{3}+\ln\left(\frac{m_1^2}{\nu^2}\right)\Bigg)\Bigg\}, \tag{3.20}$$

$$\begin{aligned}d_{vs}^{\overline{\mathrm{MS}}} &= -\frac{2C_F\alpha^2}{m_1^2-m_2^2}\Bigg\{m_1^2\Bigg(\frac{1}{3}+\ln\left(\frac{m_2^2}{\nu^2}\right)\Bigg)-m_2^2\Bigg(\frac{1}{3}+\ln\left(\frac{m_1^2}{\nu^2}\right)\Bigg)\Bigg\}\\ &\quad+\frac{C_A\alpha^2}{4(m_1^2-m_2^2)}\Bigg\{3\Bigg[m_1^2\Bigg(\frac{1}{3}+\ln\left(\frac{m_2^2}{\nu^2}\right)\Bigg)-m_2^2\Bigg(\frac{1}{3}+\ln\left(\frac{m_1^2}{\nu^2}\right)\Bigg)\Bigg]\\ &\quad+\frac{1}{m_1m_2}\Bigg[m_1^4\Bigg(\frac{10}{3}+\ln\left(\frac{m_2^2}{\nu^2}\right)\Bigg)-m_2^4\Bigg(\frac{10}{3}+\ln\left(\frac{m_1^2}{\nu^2}\right)\Bigg)\Bigg]\Bigg\},\end{aligned} \tag{3.21}$$

which is in agreement with Ref. [15].

Chapter 4

Potential Non-Relativistic Quantum Chromodynamics (pNRQCD)

4.1 Preliminaries

As it was mentioned in the previous chapter, from all the hierarchy of scales which characterizes heavy quarkonium, NRQCD only takes advantage of the fact that $m \gg mv, mv^2, \Lambda_{\rm QCD}$. However, if one is interested in describing physics at the scale of the binding energy $E \sim mv^2$, then NRQCD still contains degrees of freedom that never will show up as physical states. These degrees of freedom are soft heavy quarks, light quarks and gluons. A solution is to integrate out these unphysical degrees of freedom in an EFT approach. The implementation of this idea was devised by Pineda and Soto in Ref. [1]. It was the born of a new EFT called pNRQCD.

Two different situations arise depending on the relative size of $\Lambda_{\rm QCD}$ with respect to $|\mathbf{p}| \sim mv$ and E:

- The weak coupling regime: if $|\mathbf{p}| \gg E \gtrsim \Lambda_{\rm QCD}$ (satisfied for very large masses).
- The strong coupling regime: if $|\mathbf{p}| \gtrsim \Lambda_{\rm QCD}$.

The present dissertation is only involved with pNRQCD in the weak coupling regime, that is the approximation that the b and c quarks are heavy enough. Therefore, the strong coupling regime (see Ref. [27] for a detailed review) is beyond the scope of this work.

In the weak coupling regime we can integrate out the unphysical degrees of freedom, with energy of the order of $|\mathbf{p}|$, using perturbation theory. Once done, the Wilson coefficients of pNRQCD depend on the momentum of the heavy quark $\mathbf{p}$ and the heavy antiquark $\mathbf{p}'$ very often in the combination $\mathbf{k} = \mathbf{p} - \mathbf{p}'$, the momentum transferred. This produces non-local terms which can be identified as different contributions to the interaction potential between the two heavy quarks. These contributions to the potential are relativistic corrections to the leading Coulomb potential. This feature of pNRQCD is very interesting, as it provides an interpretation of the usual interaction potentials in non-relativistic quantum mechanics in an EFT framework, as well as it provides a link between QFT and non-relativistic quantum mechanics, and a Schrödinger-like formulation of QCD. Therefore, in this regime, heavy quarkonium can be described with a Schrödinger-like equation with heavy quarks interacting via the leading interaction potential, the non-Abelian version of the Coulomb potential, and relativistic corrections to this potential can be implemented as perturbations as one would do in non-relativistic quantum mechanics. Ultrasoft radiation is also implemented in the theory. As a consequence, in this regime heavy quarkonium is very much like positronium.

The particle content of pNRQCD will be quark-antiquark pairs, gluons, and light quarks with momentum $|\mathbf{p}|$ and energy E. For the description of the heavy quark-antiquark pairs there are two possibilities. We can use a field for the quark and another one for the antiquark, or we can use a single field representing the quark-antiquark pair. This can be rigorously achieved in a non-relativistic system because the particle and the antiparticle numbers are separately conserved. This single field can be decomposed to a color singlet and color octet components. The representation we use will hardly depend on the kind of the calculation we aim to do. For instance, in the matching between NRQCD and pNRQCD we will use the first representation, since it is more smoothly connected to NRQCD, whereas in the computation of potential loops we will use the latter. Finally, the computations in pNRQCD are easily implemented in the Coulomb gauge, so we will use it throughout this chapter.

4.2 Introduction

High order perturbative computations in heavy quarkonium require the use of EFTs, as they efficiently deal with the different scales of the system. One such EFT is pNRQCD [1, 27] (for reviews see Refs. [2, 8]). The key ingredient of the EFT is, obviously, its Lagrangian. At present, the pNRQCD Lagrangian is known with N^{3}LO accuracy [4, 11].

One of the major advantages of using EFTs is that it facilitates the systematic resummation of the large logarithms generated by the ratios of the different scales of the problem. For the case at hand, we are talking about

- the hard scale (m, the heavy quark mass),
- the soft scale (mv, the inverse Bohr radius of the problem),
- the ultrasoft scale (mv^2, the typical binding energy of the system).

At present, the pNRQCD Lagrangian is known with N^{3}LL precision as far as P-wave states are concerned [6]. For S-wave observables the present precision is N^{2}LL [3]. The missing link to obtain the complete N^{3}LL pNRQCD Lagrangian is the N^{3}LL running of the delta-like potentials[1]. For the spin-dependent case, such precision for the running has already been achieved in Refs. [12, 13]. Therefore, what is left is to obtain the N^{3}LL result for the spin-independent delta potential. This is an extremely challenging computation which we undertake in this thesis [5].

[1]We use the term "delta-like potentials" for the delta potential and the potentials generated by the Fourier transform of $\ln^n k$ (in practice only $\ln k$).

It is convenient to describe the energy levels of an nS ($l = 0$) state by dividing it up into a spin-independent part called the spin-averaged part and a part dependent on the spin of the heavy fermions, called the hyperfine splitting part[2]

$$E(J, n) = E_{\rm aver}(n) + \mathbf{S}_1 \cdot \mathbf{S}_2 E_{\rm hfs}(n)\,, \tag{4.4}$$

where J is the total spin value and $\mathbf{S}_1$ and $\mathbf{S}_2$ are the spins of the fermion and the anti-fermion, respectively. The aim of this thesis is to compute $E_{\rm aver}(n)$ with N^{3}LL precision. Partial results are obtained, whereas the missing contributions are expected to be small.

The new results we obtain are the following:

- In Secs. 4.4.1 and 4.4.2 we compute the α/m^4 and the α^2/m^3 spin-independent potentials. These potentials are finite. The expectation value of them produces energy shifts of $\mathcal{O}(m\alpha^6)$, which contribute to the heavy quarkonium mass at N^{4}LO. Nevertheless, since some expectation values are divergent, some of energy shifts are logarithmic enhanced, i.e. of order $\mathcal{O}(m\alpha^6 \ln(\frac{\nu}{m\alpha}))$. Such corrections contribute to the heavy quarkonium mass at N^{3}LL. These divergences, and the associated factorization scale ν, get canceled by the corresponding divergence in the spin-independent delta potential. By incorporating the HQET Wilson coefficients with LL accuracy[3] in the α/m^4 and the α^2/m^3 spin-independent potentials, the divergent structure of the potential loops determines the piece associated to these potentials to the RGE of the spin-independent delta potential with N^{3}LL precision.

- In Sec. 4.4.3 we compute the soft α^3/m^2 contribution to the spin-independent delta-like potential proportional to $c_F^{(1)\,2}$, $c_F^{(2)\,2}$, $\bar{c}_1^{(1)hl}$ and $\bar{c}_1^{(2)hl}$. Unlike before, this potential is divergent. Therefore, for future use, we also give the renormalized expression. The divergent part produces corrections of $\mathcal{O}(m\alpha^6 \ln(\frac{\nu}{m\alpha}))$ i.e. of N^{3}LL precision. From these divergences, we generate in Sec. 4.5.1 the soft contribution to the RGE of the spin-independent delta potential and resum logarithms with N^{3}LL precision. In order to reach this accuracy, we need the NLL running of the $1/m$ and $1/m^2$ HQET Wilson coefficients. For c_F this is known [55, 56] but not for $\bar{c}_1^{hl}$. The associated missing term is of $\mathcal{O}(T_f n_f m\alpha^6 \ln(1/\alpha))$ and is expected to be quite small. Its computation will be carried out in future publications. The possible mixing between the soft α^3/m^2 and the α^2/m^3 spin-independent potentials is also quantified in Sec. 4.4.4.

---

[2]Another possible definition is to divide it up into a spin-dependent part, called likewise spin-averaged part, and a part dependent on the total spin, called likewise hyperfine splitting part

$$E(J, n) = E'_{\rm aver}(n) + \mathbf{S}^2 E'_{\rm hfs}(n)\,, \tag{4.1}$$

This definition is adopted, for instance, in Refs. [12, 13, 14]. Using Eq. (A.14) we can easily change from one definition to the other

$$E'_{\rm aver}(n) = E_{\rm aver}(n) - \frac{3}{4} E_{\rm hfs}(n)\,, \tag{4.2}$$

$$E'_{\rm hfs}(n) = \frac{1}{2} E_{\rm hfs}(n)\,. \tag{4.3}$$

[3]These are known at $\mathcal{O}(1/m)$ [54], $\mathcal{O}(1/m^2)$ [31, 32] and $\mathcal{O}(1/m^3)$ [35].

The computation of the soft α^3/m^2 contribution to the spin-independent delta-like potential proportional to the other NRQCD Wilson coefficients: $c_k^{(1)\,2}$, $c_k^{(2)\,2}$, $c_k^{(1)}c_k^{(2)}$, d_{ss} and d_{vs} will be performed in future publications. The associated contribution to the running is expected to be small compared to the total running of the heavy quarkonium potential. We will estimate its size using the result of the running of the already computed soft contribution.

- The N^{3}LL ultrasoft running of the static, $1/m$ and $1/m^2$ potentials was originally computed in Refs. [57, 58, 59] (see also Refs. [60, 61]). This is enough for P-wave analyses [6], where such corrections produce a N^{3}LL shift to the energy. Nevertheless, it is not so for S-wave states, as already noted in Refs. [12, 13] for the case of the hyperfine splitting. The reason is the generation of singular potentials through divergent ultrasoft loops. We revisit it in Sec. 4.5.2 and incorporate in Sec. 4.5.3 the missing contributions needed to have the complete ultrasoft-potential running that produces N^{3}LL shifts to the energy.

- Finally, in Sec. 4.5.3, we compute the complete potential contribution to the RGE of the spin-independent delta-like potential with N^{3}LL accuracy, which is the first nonzero contribution. Solving this equation, we obtain the complete N^{3}LL potential running of the spin-independent delta potential. By adding everything together, we obtain the spin-average S-wave mass (spin-independent contribution proportional to δ_{l0}) with N^{3}LL accuracy, except to a missing and already mentioned contribution which is expected to be small. The N^{3}LL running of the delta potential is also one of the missing blocks to obtain the complete N^{2}LL RG improved expression of the Wilson coefficient of the electromagnetic current. This is indeed what is needed to achieve N^{2}LL precision for non-relativistic sum rules and the t-$\bar{t}$ production near threshold. Since the spin-dependent contribution for $l \neq 0$ (P-wave) and for $l = 0$ (hyperfine splitting) has already been computed in earlier works [6, 12, 13], we only consider here terms relevant for the N^{3}LL S-wave spin-average energy. By adding those results to the S-wave spin-average, the complete N^{3}LL heavy quarkonium spectrum is obtained (P-wave and S-wave).

Throughout this thesis we work in the $\overline{\text{MS}}$ renormalization scheme, where the bare and renormalized couplings are related by ($D = 4 + 2\epsilon$)

$$g_B^2 = g^2\left[1 + \frac{g^2\bar{\nu}^{2\epsilon}}{(4\pi)^2}\beta_0\frac{1}{\epsilon} + \left(\frac{g^2\bar{\nu}^{2\epsilon}}{(4\pi)^2}\right)^2\left[\beta_0^2\frac{1}{\epsilon^2} + \beta_1\frac{1}{\epsilon}\right] + \mathcal{O}(g^6)\right], \qquad \bar{\nu}^{2\epsilon} = \nu^{2\epsilon}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{\epsilon}, \tag{4.5}$$

where

$$\begin{aligned} \beta_0 &= \frac{11}{3}C_A - \frac{4}{3}T_F n_f\,, \\ \beta_1 &= \frac{34}{3}C_A^2 - \frac{20}{3}C_A T n_f - 4C_F T n_f\,. \end{aligned} \tag{4.6}$$

being n_f the number of dynamical (active) quarks and $\alpha = g^2\nu^{2\epsilon}/(4\pi)$. This definition is slightly different from the one used, for instance, in Ref. [62].

In the following we will only distinguish between the bare coupling g_B and the $\overline{\rm MS}$ renormalized coupling g when necessary. The running of α is governed by the β function defined through

$$\frac{1}{2}\nu\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\alpha}{\pi} = \nu^2\frac{\mathrm{d}}{\mathrm{d}\nu^2}\frac{\alpha}{\pi} = \beta(\alpha) = -\frac{\alpha}{\pi}\left\{\beta_0\frac{\alpha}{4\pi} + \beta_1\left(\frac{\alpha}{4\pi}\right)^2 + \cdots\right\}. \tag{4.7}$$

The coupling $\alpha(\nu)$ has n_f active light flavours. Note that, with the precision achieved in this work, we need in some cases the two-loop running of the coupling when solving the RGEs. Finally, like in Ch. 2, we define

$$z \equiv \left(\frac{\alpha(\nu)}{\alpha(\nu_h)}\right)^{\frac{1}{\beta_0}} \simeq 1 - \frac{1}{2\pi}\alpha(\nu_h)\ln\left(\frac{\nu}{\nu_h}\right). \tag{4.8}$$

4.3 pNRQCD Lagrangian

Integrating out the soft modes in NRQCD we end up with the EFT named pNRQCD. The most general pNRQCD Lagrangian compatible with the symmetries of QCD that can be constructed with a singlet and an octet quarkonium field, as well as an ultrasoft gluon field to NLO in the multipole expansion has the form [1, 27]

$$\begin{aligned}\mathcal{L}_{\rm pNRQCD} =& \int d^3\mathbf{r}\ \mathrm{Tr}\left\{\mathrm{S}^\dagger\left(i\partial_0 - h_s(\mathbf{r},\mathbf{p},\mathbf{P_R},\mathbf{S}_1,\mathbf{S}_2)\right)\mathrm{S} + \mathrm{O}^\dagger\left(iD_0 - h_o(\mathbf{r},\mathbf{p},\mathbf{P_R},\mathbf{S}_1,\mathbf{S}_2)\right)\mathrm{O}\right\}\\ &+V_A(r)\mathrm{Tr}\left\{\mathrm{O}^\dagger\mathbf{r}\cdot g\mathbf{E}\,\mathrm{S} + \mathrm{S}^\dagger\mathbf{r}\cdot g\mathbf{E}\,\mathrm{O}\right\} + \frac{V_B(r)}{2}\mathrm{Tr}\left\{\mathrm{O}^\dagger\mathbf{r}\cdot g\mathbf{E}\,\mathrm{O} + \mathrm{O}^\dagger\mathrm{O}\mathbf{r}\cdot g\mathbf{E}\right\}\\ &-\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu\,a} + \sum_{i=1}^{n_f}\bar{q}_i\,i\not{D}\,q_i\,,\end{aligned} \tag{4.9}$$

$$h_s(\mathbf{r},\mathbf{p},\mathbf{P_R},\mathbf{S}_1,\mathbf{S}_2) = \frac{\mathbf{p}^2}{2m_r} + \frac{\mathbf{P}^2_\mathbf{R}}{2M} + V_s(\mathbf{r},\mathbf{p},\mathbf{P_R},\mathbf{S}_1,\mathbf{S}_2), \tag{4.10}$$

$$h_o(\mathbf{r},\mathbf{p},\mathbf{P_R},\mathbf{S}_1,\mathbf{S}_2) = \frac{\mathbf{p}^2}{2m_r} + \frac{\mathbf{P}^2_\mathbf{R}}{2M} + V_o(\mathbf{r},\mathbf{p},\mathbf{P_R},\mathbf{S}_1,\mathbf{S}_2), \tag{4.11}$$

where $iD_0\mathrm{O} \equiv i\partial_0\mathrm{O} - g[A_0(\mathbf{R},t),\mathrm{O}]$, $\mathbf{P_R} = -i\boldsymbol{\nabla}_\mathbf{R}$ for the singlet, $\mathbf{P_R} = -i\mathbf{D_R}$ for the octet (where the covariant derivative is in the adjoint representation), $\mathbf{p} = -i\boldsymbol{\nabla}_\mathbf{r}$,

$$m_r = \frac{m_1 m_2}{m_1+m_2}\,, \tag{4.12}$$

and $M = m_1 + m_2$. We adopt the color normalization

$$\mathrm{S} = S\,\mathbf{1}_c/\sqrt{N_c}\,, \qquad \mathrm{O} = O^a\mathrm{T}^a/\sqrt{T_F}\,, \tag{4.13}$$

for the singlet field $S(\mathbf{r},\mathbf{R},t)$ and the octet field $O^a(\mathbf{r},\mathbf{R},t)$. Here and throughout this thesis we denote the quark-antiquark distance vector by $\mathbf{r}$, the center-of-mass position of the quark-antiquark system by $\mathbf{R}$, and the time by t.

Both h_s and the potential V_s are operators acting on the Hilbert space of a heavy quark-antiquark system in the singlet configuration[4]. Likewise, h_o and the potential V_o are operators acting on the Hilbert space of a heavy quark-antiquark system in the octet configuration. Both, V_s and V_o can be Taylor expanded in powers of $1/m$ (up to logarithms). For the purposes of this thesis we only need to care about h_s, so we will not pay too much attention to h_o. Therefore, at low orders we have

$$\begin{aligned} V_s &= V^{(0)} + \frac{V^{(1)}}{m_r} + \frac{V^{(2)}_{\mathbf{L}^2}}{m_1 m_2}\frac{\mathbf{L}^2}{r^2} + \frac{1}{2m_1m_2}\left\{\mathbf{p}^2, V^{(2)}_{\mathbf{p}^2}(r)\right\} + \frac{V_r^{(2)}}{m_1m_2} \\ &+\frac{1}{m_1m_2}V^{(1,1)}_{S^2}(r)\mathbf{S}_1\cdot\mathbf{S}_2 + \frac{1}{m_1m_2}V^{(1,1)}_{\mathbf{S}_{12}}(r)\mathbf{S}_{12}(\mathbf{r}) \\ &+\frac{1}{m_1m_2}V^{(2)}_{LS_1}(r)\mathbf{L}\cdot\mathbf{S}_1 + \frac{1}{m_1m_2}V^{(2)}_{LS_2}(r)\mathbf{L}\cdot\mathbf{S}_2 + \mathcal{O}(1/m^3), \end{aligned} \tag{4.14}$$

where $\mathbf{S}_1 = \boldsymbol{\sigma}_1/2$, $\mathbf{S}_2 = \boldsymbol{\sigma}_2/2$, $\mathbf{L} \equiv \mathbf{r}\times\mathbf{p}$, and $\mathbf{S}_{12}(\mathbf{r}) \equiv \dfrac{3\mathbf{r}\cdot\boldsymbol{\sigma}_1\,\mathbf{r}\cdot\boldsymbol{\sigma}_2}{r^2} - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$.

The static potential $V^{(0)}$ is known with N^{3}LL accuracy [57, 58]. The N^{3}LL result for the spin-independent and momentum dependent $1/m$ and $1/m^2$ potentials is also known in different matching schemes [6, 59, 63]: on-shell, off-shell (Coulomb, Feynman) and Wilson. In terms of the original definitions used in these papers, and in four dimensions, they read

$$V^{(1)} = V^{(1,0)}(r) = V^{(0,1)} \equiv -\frac{C_F C_A D^{(1)}}{4r^2}\,, \tag{4.15}$$

$$\frac{V^{(2)}_{\mathbf{L}^2}}{m_1m_2} \equiv \frac{V^{(2,0)}_{\mathbf{L}^2}(r)}{m_1^2} + \frac{V^{(0,2)}_{\mathbf{L}^2}(r)}{m_2^2} + \frac{V^{(1,1)}_{\mathbf{L}^2}(r)}{m_1m_2} \equiv \frac{C_F D_2^{(2)}}{2m_1m_2 r}\,, \tag{4.16}$$

$$\frac{V^{(2)}_{\mathbf{p}^2}}{m_1m_2} \equiv \frac{V^{(2,0)}_{\mathbf{p}^2}(r)}{m_1^2} + \frac{V^{(0,2)}_{\mathbf{p}^2}(r)}{m_2^2} + \frac{V^{(1,1)}_{\mathbf{p}^2}(r)}{m_1m_2} \equiv -\frac{C_F D_1^{(2)}}{m_1m_2 r}\,. \tag{4.17}$$

The spin-dependent and momentum-dependent potentials are also known with N^{3}LL precision [6]. We use the following definitions in this thesis (again we refer to Ref. [6]):

$$\frac{1}{m_1m_2}V^{(2)}_{LS_1}(r) \equiv \left(\frac{1}{m_1^2}V^{(2,0)}_{LS}(r) + \frac{1}{m_1m_2}V^{(1,1)}_{L_2S_1}(r)\right) \equiv \frac{3C_F D^{(2)}_{LS_1}}{2m_1m_2}\,, \tag{4.18}$$

$$\frac{1}{m_1m_2}V^{(2)}_{LS_2}(r) \equiv \left(\frac{1}{m_2^2}V^{(0,2)}_{LS}(r) + \frac{1}{m_1m_2}V^{(1,1)}_{L_1S_2}(r)\right) \equiv \frac{3C_F D^{(2)}_{LS_2}}{2m_1m_2}\,. \tag{4.19}$$

More delicate are $V^{(1,1)}_{S^2}$ and $V^{(2)}_r$, as their running is sensitive to potential loops, which are more efficiently computed in momentum space. Therefore, it is more convenient to work with the potential in momentum space, which is defined as follows:

$$\tilde{V}_s \equiv \langle \mathbf{p}'|V_s|\mathbf{p}\rangle\,. \tag{4.20}$$

[4]Therefore, in a more mathematical notation, $h \to \hat{h}$, $V_s(\mathbf{r},\mathbf{p}) \to \hat{V}_s(\hat{\mathbf{r}},\hat{\mathbf{p}})$. We will however avoid this notation in order to facilitate the reading.

Then, the potential reads

$$
\begin{aligned}
\tilde{V}_s \;=\;& -4\pi C_F \frac{\alpha_{\tilde{V}}}{\mathbf{q}^2} - \mathbf{p}^4 \left(\frac{c_4^{(1)}}{8m_1^3} + \frac{c_4^{(2)}}{8m_2^3} \right) (2\pi)^d \delta^{(d)}(\mathbf{q}) \\
& - C_F C_A \tilde{D}^{(1)} \frac{\pi^2}{2m_r |\mathbf{q}|^{1-2\epsilon}} \\
& - \frac{2\pi C_F \tilde{D}_1^{(2)}}{m_1 m_2} \frac{\mathbf{p}^2 + \mathbf{p}'^2}{\mathbf{q}^2} + \frac{\pi C_F \tilde{D}_2^{(2)}}{m_1 m_2} \left(\left(\frac{\mathbf{p}^2 - \mathbf{p}'^2}{\mathbf{q}^2} \right)^2 - 1 \right) \\
& + \frac{\pi C_F \tilde{D}_d^{(2)}}{m_1 m_2} - \frac{4\pi C_F \tilde{D}_{S^2}^{(2)}}{d\, m_1 m_2} [\mathbf{S}_1^i, \mathbf{S}_1^j][\mathbf{S}_2^i, \mathbf{S}_2^j] \\
& + \frac{4\pi C_F \tilde{D}_{S_{12}}^{(2)}}{d\, m_1 m_2} [\mathbf{S}_1^i, \mathbf{S}_1^r][\mathbf{S}_2^i, \mathbf{S}_2^j] \left(\delta^{rj} - d\, \frac{\mathbf{q}^r \mathbf{q}^j}{\mathbf{q}^2} \right) \\
& - \frac{6\pi C_F}{m_1 m_2} \frac{\mathbf{p}^i \mathbf{q}^j}{\mathbf{q}^2} \left(\tilde{D}_{LS_1}^{(2)} [\mathbf{S}_1^i, \mathbf{S}_1^j] + \tilde{D}_{LS_2}^{(2)} [\mathbf{S}_2^i, \mathbf{S}_2^j] \right) ,
\end{aligned}
\tag{4.21}
$$

where the Wilson coefficients $\tilde{D}$ generically stand for the Fourier transform of the original Wilson coefficients in position space D. For them, and also for $\alpha_{\tilde{V}}$, we use the power counting LL/LO for the first nonvanishing correction, and so on.

The potential $V_{S^2}^{(1,1)}$ is indeed known with the required N^{3}LL accuracy [12, 13] (one should be careful when comparing though, as there is a change in the basis of potentials used there, compared with the one we use here). In terms of $\tilde{D}_{S^2}^{(2)}$, it reads

$$
\begin{aligned}
\frac{V_{S^2}^{(1,1)}}{m_1 m_2} \;\equiv\;& \delta^{(3)}(\mathbf{r}) \frac{8\pi C_F \tilde{D}_{S^2}^{(2)}}{3 m_1 m_2} \\
& + \frac{8\pi C_F \tilde{D}_{S^2}^{(2)}}{3 m_1 m_2} \left[-\frac{1}{4\pi} \mathrm{reg} \frac{1}{r^3} - \ln \nu\, \delta^{(3)}(\mathbf{r}) \right] \left(k \frac{d}{dk} \tilde{D}_{S^2}^{(2)} \right) \Bigg|_{k=\nu}^{\mathrm{LL}} ,
\end{aligned}
\tag{4.22}
$$

where

$$
-\frac{1}{4\pi} \mathrm{reg} \frac{1}{r^3} \equiv \int \frac{d^3 k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{r}} \ln k \,, \tag{4.23}
$$

and we neglect higher order logarithms, as they are subleading.

Finally we consider V_r, which in terms of $\tilde{D}_d^{(2)}$ reads

$$
\begin{aligned}
\frac{V_r^{(2)}}{m_1 m_2} &\equiv \frac{V_r^{(2,0)}(r)}{m_1^2} + \frac{V_r^{(0,2)}(r)}{m_2^2} + \frac{V_r^{(1,1)}(r)}{m_1 m_2} \\
&\equiv \delta^{(3)}(\mathbf{r}) \frac{\pi C_F \tilde{D}_d^{(2)}}{m_1 m_2} + \frac{\pi C_F}{m_1 m_2} \left[-\frac{1}{4\pi} \mathrm{reg} \frac{1}{r^3} - \ln \nu\, \delta^{(3)}(\mathbf{r}) \right] \left(k \frac{d}{dk} \tilde{D}_d^{(2)} \right) \Bigg|_{k=\nu}^{\mathrm{LL}} .
\end{aligned}
\tag{4.24}
$$

Unlike all the other potentials, we do not know $V_r^{(2)}$ with N^{3}LL precision. Presently, only the N^{2}LL expression is known [3]. This leads us to the main purpose of this chapter, the computation of V_r with N^{3}LL accuracy. This is equivalent to obtaining the NLL expression of $\tilde{D}_d^{(2)}$. This will require the use of the other Wilson coefficients of the potentials to one less order, namely LL. Indeed in Eq. (4.21) we have already approximated the Fourier transform of $V_{\mathbf{L}^2}^{(2)}$ by its N^{2}LL expression. Otherwise the momentum dependence is more complicated.

At LL the Wilson coefficients are equal in position and momentum space. We only explicitly display those that we will need later. For the static potential we have that $\alpha_V = \alpha_{\tilde{V}} = \alpha$ at LL. For the rest, we show the results in the off-shell Coulomb (which are equal to the Feynman at this order) and the on-shell matching schemes, except for $D_{LS_i}^{(2)}$, which we do not need for the S-wave:

$$D_{\rm CG}^{(1)\,\rm LL} = \tilde{D}_{\rm CG}^{(1)\,\rm LL} = \alpha^2(\nu) + \frac{16}{3\beta_0}\left(\frac{C_A}{2}+C_F\right)\alpha^2(\nu)\ln\left(\frac{\alpha(\nu)}{\alpha(\nu^2/\nu_h)}\right), \tag{4.25}$$

$$D_{\rm ON}^{(1)\,\rm LL} = \tilde{D}_{\rm ON}^{(1)\,\rm LL} = \alpha^2(\nu)\left[1-\frac{2C_F}{C_A}\frac{m_r^2}{m_1m_2}\right] + \frac{16}{3\beta_0}\left(\frac{C_A}{2}+C_F\right)\alpha^2(\nu)\ln\left(\frac{\alpha(\nu)}{\alpha(\nu^2/\nu_h)}\right), \tag{4.26}$$

$$D_1^{(2)\,\rm LL} = \tilde{D}_1^{(2)\,\rm LL} = \alpha(\nu) + \frac{(m_1+m_2)^2}{m_1m_2}\frac{2C_A}{3\beta_0}\alpha(\nu)\ln\left(\frac{\alpha(\nu)}{\alpha(\nu^2/\nu_h)}\right), \tag{4.27}$$

$$D_{S_{12}}^{(2)\,\rm LL} = \tilde{D}_{S_{12}}^{(2)\,\rm LL} = \alpha(\nu)c_F^2(\nu)\,, \tag{4.28}$$

$$D_{S^2}^{(2)\,\rm LL} = \tilde{D}_{S^2}^{(2)\,\rm LL} = \alpha(\nu)c_F^2(\nu) - \frac{3}{2\pi C_F}(d_{sv}(\nu)+C_Fd_{vv}(\nu))\,. \tag{4.29}$$

The Eqs. (4.25-4.29) can be trivially written in terms of z defined in Eq. (4.8), and in the one-loop coupling approximation, using the relation

$$\frac{\alpha(\nu)}{\alpha(\nu^2/\nu_h)} = 2 - \frac{\alpha(\nu)}{\alpha(\nu_h)} = 2 - z^{\beta_0}\,. \tag{4.30}$$

The above precision is enough for the purposes of this thesis. We now turn to $\tilde{D}_d^{(2)}$. Expanding $\tilde{D}_d^{(2)}(k,\nu)$ in powers of $\ln k$, we obtain

$$\tilde{D}_d^{(2)}(k,\nu) = \tilde{D}_d^{(2)}(\nu_s,\nu_p,\nu_p^2/\nu_h)\Big|_{\nu_s=\nu_p=\nu} + k\frac{d}{dk}\tilde{D}_d^{(2)}(k,\nu)\Big|_{k=\nu}\ln\left(\frac{k}{\nu}\right) + \dots\,, \tag{4.31}$$

where we have made explicit the dependence on the different factorization scales.

So far we have not made explicit the dependence on $\nu_h \sim m$. Nevertheless, it will play an important role later, when solving the RGEs. Therefore, in the following, we use the notation $\tilde{D}_d^{(2)}(\nu_s,\nu_p,\nu_p^2/\nu_h)\Big|_{\nu_s=\nu_p=\nu} \equiv \tilde{D}_d^{(2)}(\nu_h;\nu)$.

The Wilson coefficient $\tilde{D}_d^{(2)}(\nu_h;\nu)$ can be written in several ways: as a sum of the LL term $\tilde{D}_d^{(2)\,\rm LL}(\nu_h;\nu)$ and the NLL correction $\delta\tilde{D}_d^{(2)\,\rm NLL}(\nu_h;\nu)$, or as the sum of the initial condition at the hard scale $\tilde{D}_d^{(2)}(\nu_h;\nu_h) \equiv \tilde{D}_d^{(2)}(\nu_h)$ up to NLO and the pure running contribution up to NLL $\delta\tilde{D}_d^{(2)}(\nu_h;\nu)$, where $\delta\tilde{D}_d^{(2)}(\nu_h;\nu_h)=0$:

$$\tilde{D}_d^{(2)}(\nu_h;\nu) = \tilde{D}_d^{(2)}(\nu_h) + \delta\tilde{D}_d^{(2)}(\nu_h;\nu) = \tilde{D}_d^{(2)\,\rm LL}(\nu_h;\nu) + \delta\tilde{D}_d^{(2)\,\rm NLL}(\nu_h;\nu)\,. \tag{4.32}$$

This Wilson coefficient may depend on the matching scheme. Here we mainly consider the off-shell Coulomb gauge matching scheme. Still, for later discussion, we also give expressions in the on-shell matching scheme (see Ref. [11] for more details). The LL running is known [3]:

$$D^{(2)\,\mathrm{LL}}_{d,\mathrm{CG}}(\nu)=\tilde{D}^{(2)\,\mathrm{LL}}_{d,\mathrm{CG}}(\nu) = 2\alpha(\nu)+\frac{1}{\pi C_F}[d_{ss}(\nu)+C_F\bar{d}_{vs}(\nu)] +\frac{(m_1+m_2)^2}{m_1m_2}\frac{8}{3\beta_0}\left(\frac{C_A}{2}-C_F\right)\alpha(\nu)\ln\left(\frac{\alpha(\nu)}{\alpha(\nu^2/\nu_h)}\right), \tag{4.33}$$

$$D^{(2)\,\mathrm{LL}}_{d,\mathrm{ON}}(\nu)=\tilde{D}^{(2)\,\mathrm{LL}}_{d,\mathrm{ON}}(\nu) = \alpha(\nu)+\frac{1}{\pi C_F}[d_{ss}(\nu)+C_F\bar{d}_{vs}(\nu)] +\frac{(m_1+m_2)^2}{m_1m_2}\frac{8}{3\beta_0}\left(\frac{C_A}{2}-C_F\right)\alpha(\nu)\ln\left(\frac{\alpha(\nu)}{\alpha(\nu^2/\nu_h)}\right), \tag{4.34}$$

where

$$\bar{d}_{vs}(\nu)=\frac{1}{2}\pi\alpha(\nu)m_1m_2\left(\frac{c_D^{(1)}}{m_1^2}+\frac{c_D^{(2)}}{m_2^2}\right)+d_{vs}(\nu) \tag{4.35}$$

is a gauge invariant combination of NRQCD Wilson coefficients whose LL running can be found in Refs. [3, 14]. We summarize them in Sec. E.4.1. Again, Eqs. (4.33-4.35) can be trivially written in terms of z and in the one-loop coupling approximation using Eq. (4.30). In order to visualize the relative importance of the NLL corrections compared with the LL term, we plot the latter in Fig. 4.1 in the Coulomb gauge[5]. For reference, in these and later figures, we use the following numerical values for the heavy quark masses and $\alpha(\nu_h)$: $m_b=4.73$ GeV, $\alpha(m_b)=0.216547$, $m_c=1.5$ GeV, $\alpha(m_c)=0.348536$ and $\alpha(2m_bm_c/(m_b+m_c))=0.290758$. So, $\nu_h=m_b$ for bottomonium, $\nu_h=m_c$ for charmonium, and $\nu_h=2m_r=2m_bm_c/(m_b+m_c)$ for the B_c system.

From the LL result, using the ν_s independence of the potential at LO, one obtains

$$\begin{aligned} k\frac{d}{dk}\tilde{D}^{(2)}_{d,\mathrm{CG}}\Big|^{\mathrm{LL}}_{k=\nu}(\nu_h;\nu) &= -\beta_0\frac{\alpha^2}{\pi}+\frac{\alpha^2}{\pi}\left(2C_F-\frac{C_A}{2}\right)c_k^{(1)}c_k^{(2)} \\ &+\frac{\alpha^2}{\pi}\left[\frac{m_1}{m_2}\left(\frac{1}{3}T_fn_f\bar{c}_1^{hl(2)}-\frac{4}{3}(C_A+C_F)c_k^{(2)\,2}-\frac{5}{12}C_Ac_F^{(2)\,2}\right)\right. \\ &\left.+\frac{m_2}{m_1}\left(\frac{1}{3}T_fn_f\bar{c}_1^{hl(1)}-\frac{4}{3}(C_A+C_F)c_k^{(1)\,2}-\frac{5}{12}C_Ac_F^{(1)\,2}\right)\right] \\ &-\frac{(m_1+m_2)^2}{m_1m_2}\frac{4}{3}\left(\frac{C_A}{2}-C_F\right)\frac{\alpha^2}{\pi}\left[\ln\left(\frac{\alpha(\nu)}{\alpha(\nu^2/\nu_h)}\right)+1\right]. \end{aligned} \tag{4.36}$$

This term contributes to the N^{3}LL energy shift of the spectrum.

Since we know the NLO expression of $\tilde{D}^{(2)}_d$, we can determine the initial matching condition. It reads

[5] Unlike in the other plots, we use here the two-loop running for α, which we compute using the Mathematica package RunDec. The effect is small.

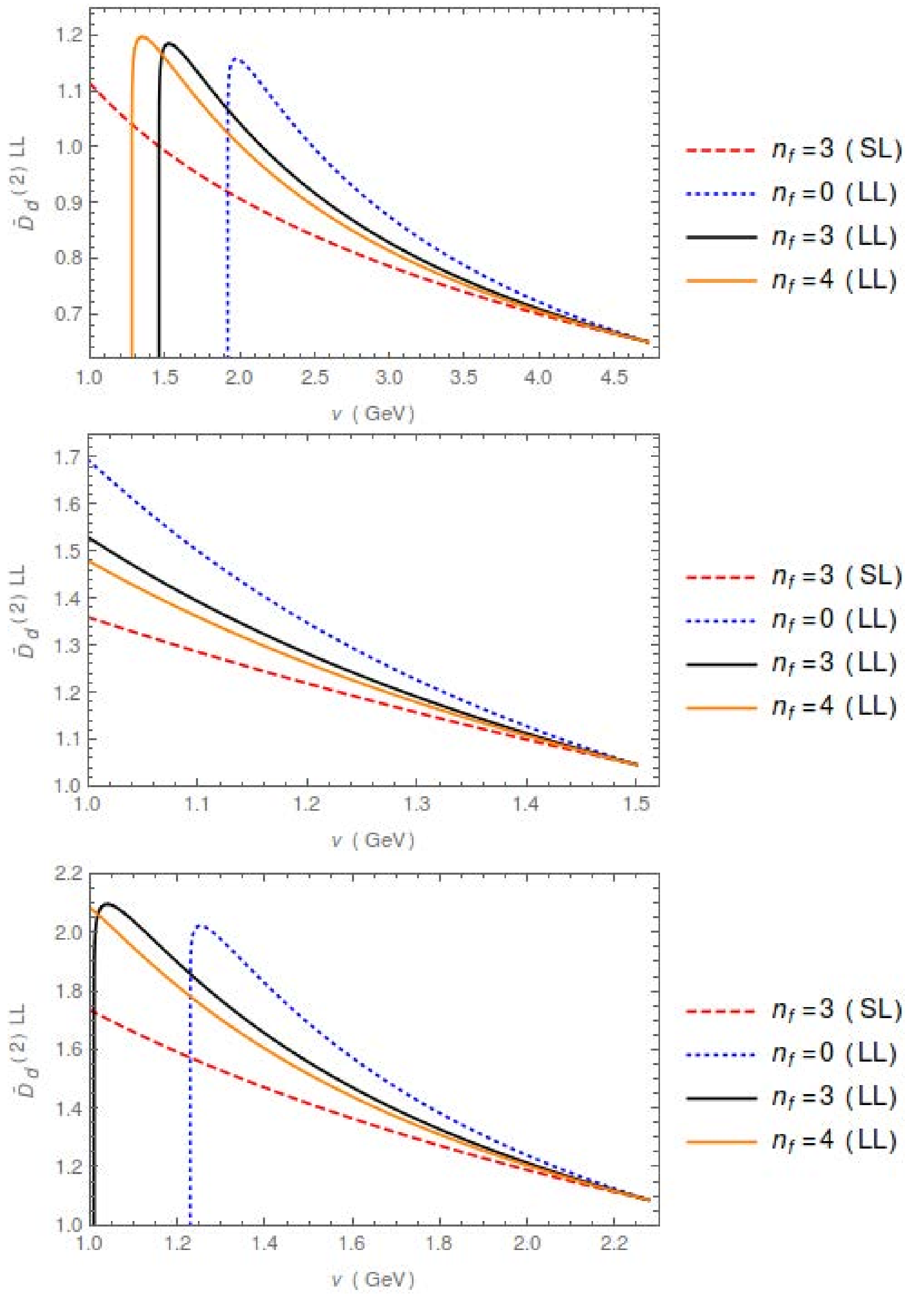


Figure 4.1: Plot of Eq. (4.33), the LL running in the off-shell (Coulomb/Feynman) matching scheme of $\tilde{D}_d^{(2)}$ for different values of n_f (0,3,4) and in the SL approximation (in this case only with $n_f = 3$). **Upper panel:** Plot for bottomonium with $\nu_h = m_b$. **Middle panel:** Plot for charmonium with $\nu_h = m_c$. **Lower panel:** Plot for B_c with $\nu_h = 2m_b m_c/(m_b + m_c)$.

$$\begin{aligned}
\tilde{D}^{(2)}_{d,\mathrm{ON}}(\nu_h) &= \alpha(\nu_h) + \frac{\alpha^2(\nu_h)}{4\pi}\Bigg(\frac{28}{9}C_A + \frac{4}{3}C_F - \frac{20}{9}T_F n_f \\
&\quad + \left(\frac{m_1}{m_2} + \frac{m_2}{m_1}\right)\left[\frac{25}{18}C_A - \frac{10}{9}T_F n_f\right]\Bigg) + \frac{1}{\pi C_F}\left(d_{ss}(\nu_h) + C_F \bar{d}_{vs}(\nu_h)\right) ,
\end{aligned} \tag{4.37}$$

$$
\begin{aligned}
\tilde{D}^{(2)}_{d,\mathrm{CG}}(\nu_h) &= 2\alpha(\nu_h) + \frac{\alpha^2(\nu_h)}{4\pi}\left(\frac{62}{9}C_A + \frac{4}{3}C_F - \frac{32}{3}C_A\ln 2 - \frac{28}{9}T_F n_f \right.\\
&\left. + \left(\frac{m_1}{m_2}+\frac{m_2}{m_1}\right)\left[-\frac{10}{9}T_F n_f + \left(\frac{61}{18}-\frac{16}{3}\ln 2\right)C_A\right]\right)\\
&+\frac{1}{\pi C_F}\left(d_{ss}(\nu_h) + C_F \bar{d}_{vs}(\nu_h)\right)\,. \qquad (4.38)
\end{aligned}
$$

The heavy-gluon Wilson coefficient c_D and the four-fermion Wilson coefficients d_{ss} and d_{vs} were computed at one-loop order in Ref. [28] and Ref. [15] respectively, where one can find the explicit expressions. For practical reasons, we summarize them in Sec. E.4.2.

The NLL correction $\delta\tilde{D}^{(2)\,\mathrm{NLL}}_d(\nu_h;\nu)$ can be conveniently splitted into the following pieces

$$
\begin{aligned}
\delta\tilde{D}^{(2)\,\mathrm{NLL}}_d(\nu_h;\nu) &= \tilde{D}^{(2)\,\mathrm{NLO}}_d(\nu_h) \qquad (4.39)\\
&+\delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,us}(\nu_h;\nu) + \delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,s}(\nu_h;\nu) + \delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,p}(\nu_h;\nu)\,,
\end{aligned}
$$

where $\delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,us}(\nu_h;\nu)$, $\delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,s}(\nu_h;\nu)$ and $\delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,p}(\nu_h;\nu)$ stand for the ultrasoft, soft and potential NLL running of $\tilde{D}^{(2)}_d$. The second line is zero when $\nu = \nu_h$. The matching condition $\tilde{D}^{(2)\,\mathrm{NLO}}_d(\nu_h)$ is the $\mathcal{O}(\alpha^2)$ term of Eq. (4.37) or Eq. (4.38), depending on the matching scheme. Their numerical values in the Coulomb gauge matching scheme are; for bottomonium, 0.042, 0.052 and 0.081 for n_f=4, 3, and 0 respectively; for charmonium, 0.108, 0.134 and 0.211 for n_f=4, 3, and 0 respectively; and for B_c, 0.048, 0.072 and 0.142 for n_f=4, 3, and 0 respectively. We nicely observe that these numbers generate small corrections to the leading order results.

At present, the NLL running is only known for the ultrasoft term [59]:

$$
\begin{aligned}
\delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,us}(\nu_h;\nu) &= \frac{(m_1+m_2)^2}{m_1 m_2}\frac{4\pi}{\beta_0}\left(\frac{C_A}{2}-C_F\right)\alpha(\nu)\left\{\frac{2}{3\pi}\ln\left(\frac{\alpha(\nu)}{\alpha(\nu^2/\nu_h)}\right)a_1\frac{\alpha(\nu)}{4\pi}\right.\\
&\left.+(\alpha(\nu^2/\nu_h)-\alpha(\nu))\left(\frac{8}{3}\frac{\beta_1}{\beta_0}\frac{1}{(4\pi)^2} - \frac{1}{27\pi^2}\left(C_A\left(47+6\pi^2\right) - 10T_F n_f\right)\right)\right\}\,, \qquad (4.40)
\end{aligned}
$$

where $a_1 = 31C_A/9 - 20T_F n_f/9$. This equation can also be conveniently written in terms of z and in the one-loop coupling approximation using Eq. (4.30). We show the size of this correction in Fig. 4.2. Note that the ultrasoft contribution to the delta potential vanishes in the large N_c limit (it is $1/N_c^2$ suppressed). Nevertheless, it quickly becomes big at relatively small scales because the overall coefficient is large and the ultrasoft scale quickly becomes small. Finally, note also that part of the ultrasoft correction (proportional to $\ln k$) is included in Eq. (4.36).

The missing terms to obtain the complete NLL running of $\tilde{D}^{(2)}_d$ are then $\delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,s}(\nu_h;\nu)$ and $\delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,p}(\nu_h;\nu)$. For $\delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,s}(\nu_h;\nu)$ we need the two-loop soft computation[6] of $\tilde{D}^{(2)}_d$

[6] That is the UV divergent part of the two loop matching between NRQCD and pNRQCD at $\mathcal{O}(1/m^2)$. More precisely, the one that gets absorbed in the delta-like potential.

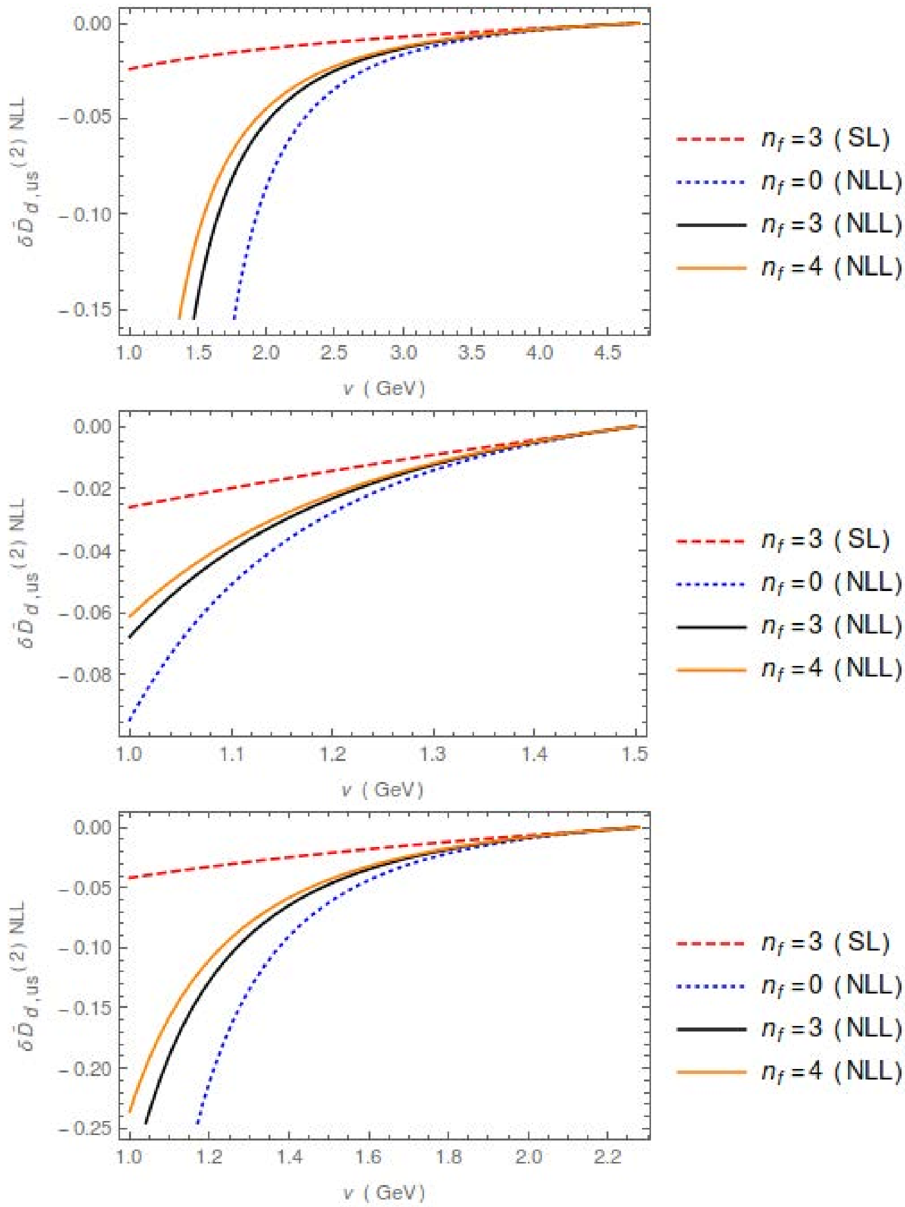


Figure 4.2: Plot of Eq. (4.40), the NLL ultrasoft running in the off-shell (Coulomb/Feynman) matching scheme of $\tilde{D}_d^{(2)}$ for different values of n_f (0,3,4) and in the SL approximation (in this case only with $n_f = 3$). **Upper panel:** Plot for bottomonium with $\nu_h = m_b$. **Middle panel:** Plot for charmonium with $\nu_h = m_c$. **Lower panel:** Plot for B_c with $\nu_h = 2m_b m_c/(m_b + m_c)$.

and the associated soft RGE, which we partially obtain in Sec. 4.4.3 and Sec. 4.5.1, respectively. In Sec. 4.4.4 we discuss the mixing between the α^3/m^2 and α^2/m^3 potentials coming from the use of the EOM in energy-dependent potentials. For $\delta\tilde{D}_{d,p}^{(2)\,\mathrm{NLL}}(\nu_h;\nu)$ we need to determine and solve the potential RGE. This requires first the matching between NRQCD and pNRQCD to higher orders in $1/m$, which we do in Sec. 4.4.1 and Sec. 4.4.2, an extra ultrasoft associated running, which we obtain in Sec. 4.5.2, and obtaining the potential RGE, which we do in Sec. 4.5.3.

4.4 Matching NRQCD with pNRQCD: The SI potential

In this section we compute the potentials whose expectation values produce corrections to the $\mathcal{O}(m\alpha^6)$ S-wave spectrum[7]. This means the $\mathcal{O}(\alpha/m^4)$, $\mathcal{O}(\alpha^2/m^3)$ and $\mathcal{O}(\alpha^3/m^2)$ potentials[8]. Of them, we mostly care about those that produce logarithmic enhanced contributions to the spectrum. On the one hand, the $\mathcal{O}(\alpha/m^4)$ and $\mathcal{O}(\alpha^2/m^3)$ potentials are finite (the soft, potential and ultrasoft contributions to their running are zero), so their expectation value does not produce logarithmically enhanced corrections. Some of them can be traced back from the QED computation. We mainly compare with Ref. [64], but one could also look into Ref. [65] for the equal mass case. Since they are finite, the only way they can produce logarithmically enhanced corrections is through the divergences generated when inserting these potentials into potential loops. These divergences get then absorbed by the $\mathcal{O}(\alpha^3/m^2)$ potential producing logarithmically enhanced corrections. On the other hand, there is another source of logarithmically enhanced contributions to the spectrum which is not generated by potential loops, but by the divergent structure of the $\mathcal{O}(\alpha^3/m^2)$ potential itself i.e. by the divergences appearing in the matching calculation between NRQCD and pNRQCD. We refer to the running of the $\mathcal{O}(\alpha^3/m^2)$ potential produced by potential loops as potential running and the one produced by the matching calculation as soft running. The former case will be discussed in Sec. 4.5.3, whereas the latter in Sec. 4.5.1.

The spin-dependent case was computed in Refs. [12, 13]. Explicit expressions for the potentials can be found in the Appendix of Ref. [14]. They produced corrections to the hyperfine splitting, but not to the fine splittings, as shown in Ref. [6].

[7]An example of how these computations are caried out is given in Sec. D.1

[8]The expectation value of the $\mathcal{O}(\alpha^5)$ Coulomb potential and the $\mathcal{O}(\alpha^4/m)$ potential also contribute to the spectrum at $\mathcal{O}(m\alpha^6)$, and their divergences produce logarithmically enhanced corrections. However, these are only relevant for P-wave states i.e. their contributions are not proportional to δ_{l0}, and for this reason they are not considered. The p^6/m^5 correction to the kinetic term does not has to be considered neither, since it does not give ultraviolet divergent corrections

4.4.1 $\mathcal{O}(\alpha/m^4)$ potential

From a tree level computation (see topology (a) in Fig. 4.3) we obtain the complete spin-independent $\mathcal{O}(\alpha/m^4)$ potential in momentum space:

$$
\begin{aligned}
\tilde{V}_{\rm tree} = & -c_D^{(1)}c_D^{(2)}C_F\frac{g^2}{64m_1^2m_2^2}\mathbf{k}^2 \\
& -C_Fg^2\left(\frac{c_{X1}^{(1)}}{m_1^4}+\frac{c_{X1}^{(2)}}{m_2^4}\right)\frac{(\mathbf{p}^2-\mathbf{p}'^{\,2})^2}{\mathbf{k}^2} \\
& -C_Fg^2\left(\frac{c_{X2}^{(1)}}{m_1^4}+\frac{c_{X2}^{(2)}}{m_2^4}\right)(\mathbf{p}^2+\mathbf{p}'^2) \\
& -C_Fg^2\left(\frac{c_{X3}^{(1)}}{m_1^4}+\frac{c_{X3}^{(2)}}{m_2^4}\right)\mathbf{k}^2 \\
& +C_F\frac{g^2c_k^{(1)2}c_k^{(2)2}}{16m_1^2m_2^2}\frac{1}{\mathbf{k}^4}(\mathbf{p}^2-\mathbf{p}'^2)^2\left(2(\mathbf{p}^2+\mathbf{p}'^2)-\mathbf{k}^2-\frac{(\mathbf{p}^2-\mathbf{p}'^2)^2}{\mathbf{k}^2}\right) \\
& +C_F\frac{g^2}{16m_1m_2}\left(\frac{c_4^{(1)}c_k^{(2)}}{m_1^2}+\frac{c_4^{(2)}c_k^{(1)}}{m_2^2}\right)\frac{\mathbf{p}^2+\mathbf{p}'^2}{\mathbf{k}^2}\left(2(\mathbf{p}^2+\mathbf{p}'^2)-\mathbf{k}^2-\frac{(\mathbf{p}^2-\mathbf{p}'^2)^2}{\mathbf{k}^2}\right) \\
& -C_F\frac{g^2}{16m_1m_2}\left(\frac{c_M^{(1)}c_k^{(2)}}{m_1^2}+\frac{c_M^{(2)}c_k^{(1)}}{m_2^2}\right)\left(\frac{(\mathbf{p}^2-\mathbf{p}'^2)^2}{\mathbf{k}^2}-(\mathbf{p}+\mathbf{p}')^2\right)\,.
\end{aligned}
\tag{4.41}
$$

In this result we have already used the full EOM replacing [66]

$$
(k^0)^2\to -c_k^{(1)}c_k^{(2)}\frac{(\mathbf{p}^2-\mathbf{p}'^2)^2}{4m_1m_2}\,. \tag{4.42}
$$

Such $(k^0)^2$ terms are generated by Taylor expanding in powers of the transferred energy k^0 the denominator of the transverse gluon propagator.

Not all terms in Eq. (4.41) contribute to the NLL running of the delta potential. The ones that are local (or pseudolocal) do not contribute, as they do not produce potential loop divergences, since the expectation values of these potentials are proportional to $|\psi(0)|^2$ and/or analytic derivatives of it (kind of $\boldsymbol{\nabla}^2|\psi(0)|^2$), which are finite. This happens for instance for the potentials proportional to c_D^2, c_{X2} and c_{X3}. It is also this fact that allows us to neglect $1/m^4$ potentials generated by dimension eight four-heavy fermion operators of the NRQCD Lagrangian.

As we have incorporated the LL running of the HQET Wilson coefficients, these potentials are already RG improved.

Note that with trivial modifications these potentials are also valid for QED.

4.4.2 $\mathcal{O}(\alpha^2/m^3)$ potential

We now compute the complete set of the $\mathcal{O}(\alpha^2/m^3)$ spin-independent potentials. We show the relevant topologies that contribute to the α^2/m^3 potential in Fig. 4.3. By properly changing the vertices all potentials are generated.

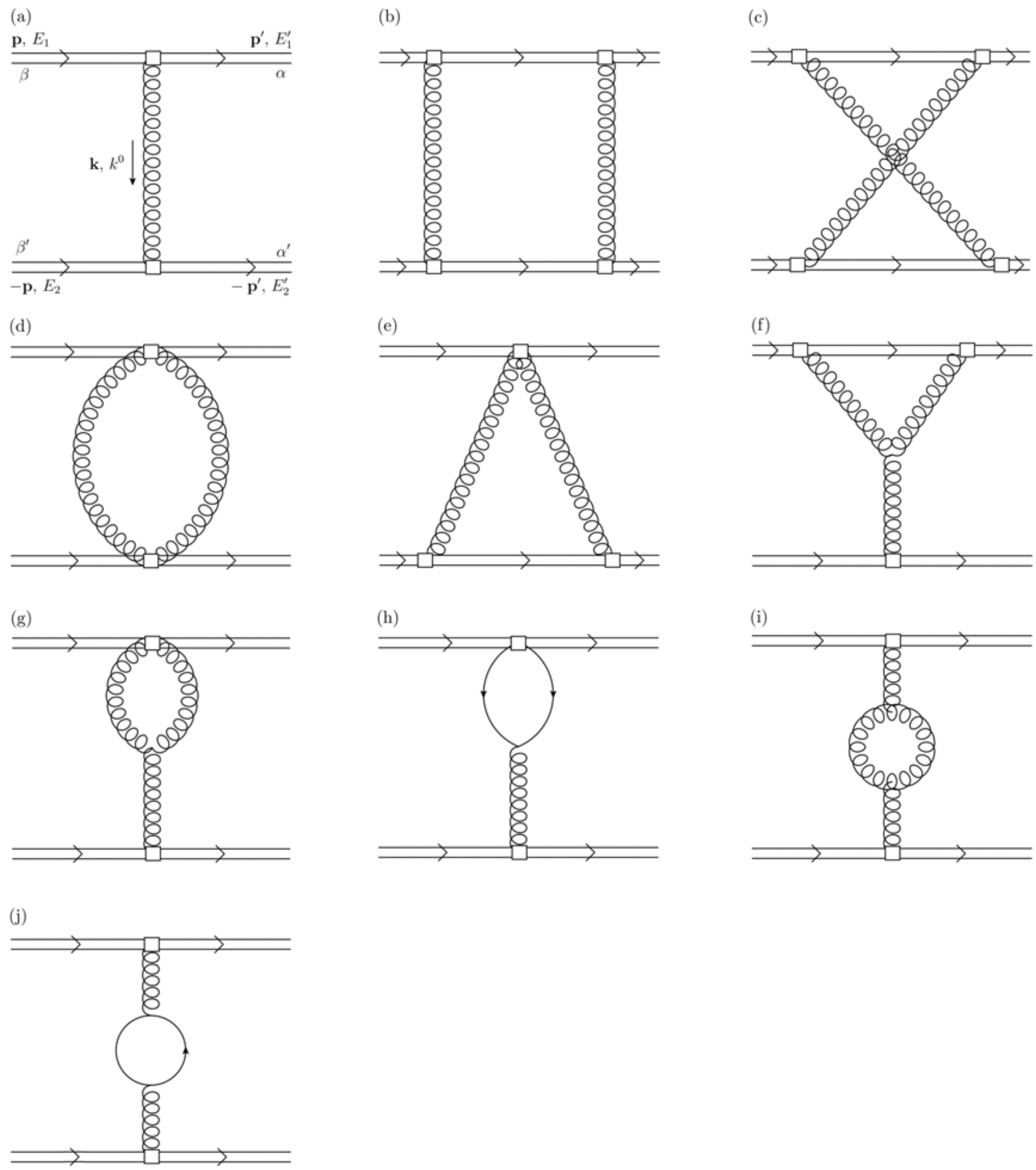


Figure 4.3: The first diagram is the only topology that contributes to the tree level potential. Properly changing the vertex and/or Taylor expanding the denominator of the propagators all potentials are generated. The other diagrams are the general topologies that contribute to the α^2/m^3 potential. Again, properly changing the vertices and/or Taylor expanding the denominator of the propagators, all potentials are generated.

The (b)-type diagrams in Fig. 4.3 do not generate $\mathcal{O}(\alpha^2/m^3)$ potentials in the Coulomb gauge.

The (c)-type diagrams in Fig. 4.3 do generate $\mathcal{O}(\alpha^2/m^3)$ potentials. They read

$$\begin{aligned}\tilde{V}^{(c,1)}_{\rm 1loop} &= -C_F\left(C_F-\frac{C_A}{2}\right)c_k^{(1)}c_k^{(2)}\frac{g^4}{512m_1m_2}\frac{E_1+E_2}{|\mathbf{k}|^{3-2\epsilon}} \\ &\quad\times\left(2(\mathbf{p}^2+\mathbf{p}'^2)-\mathbf{k}^2-\frac{(\mathbf{p}^2-\mathbf{p}'^2)^2}{\mathbf{k}^2}-\frac{8(\mathbf{p}\cdot\mathbf{k})(\mathbf{p}'\cdot\mathbf{k})}{\mathbf{k}^2}\right), \end{aligned} \tag{4.43}$$

$$\begin{aligned}\tilde{V}^{(c,2)}_{\rm 1loop} &= -C_F\left(C_F-\frac{C_A}{2}\right)\frac{g^4}{256m_1m_2}\left(\frac{c_k^{(1)\,2}c_k^{(2)}}{m_1}+\frac{c_k^{(1)}c_k^{(2)\,2}}{m_2}\right)|\mathbf{k}|^{1+2\epsilon} \\ &\quad\times\left(3(\mathbf{p}^2+\mathbf{p}'^2)\frac{(\mathbf{p}\cdot\mathbf{k})(\mathbf{p}'\cdot\mathbf{k})}{\mathbf{k}^6}-\frac{2(\mathbf{p}^2+\mathbf{p}'^2)}{\mathbf{k}^2}+\frac{11}{4}\right. \\ &\quad\left.-\frac{1}{4}\frac{(\mathbf{p}^2-\mathbf{p}'^2)^2}{\mathbf{k}^4}-\frac{1}{2}\frac{(\mathbf{p}^2+\mathbf{p}'^2)^2}{\mathbf{k}^4}\right). \end{aligned} \tag{4.44}$$

The (d)-type diagrams in Fig. 4.3 do not generate $\mathcal{O}(\alpha^2/m^3)$ potentials.

The (e)-type diagrams in Fig. 4.3 do generate $\mathcal{O}(\alpha^2/m^3)$ potentials. They read

$$\begin{aligned}\tilde{V}^{(e,1)}_{\rm 1loop} &= C_F\left(2C_F-\frac{C_A}{2}\right)\frac{g^4}{512m_1m_2}\left(\frac{c_k^{(1)\,2}c_k^{(2)}}{m_1}+\frac{c_k^{(1)}c_k^{(2)\,2}}{m_2}\right)|\mathbf{k}|^{1+2\epsilon} \\ &\quad\times\left(\frac{5(\mathbf{p}^2+\mathbf{p}'^2)}{\mathbf{k}^2}-\frac{7}{2}-\frac{3}{2}\frac{(\mathbf{p}^2-\mathbf{p}'^2)^2}{\mathbf{k}^4}\right), \end{aligned} \tag{4.45}$$

$$\begin{aligned}\tilde{V}^{(e,2)}_{\rm 1loop} &= -C_F\left(2C_F-\frac{C_A}{2}\right)\frac{g^4}{256}\left(\frac{c_{A_1}^{(1)}}{m_1^3}+\frac{c_{A_1}^{(2)}}{m_2^3}\right)|\mathbf{k}|^{1+2\epsilon} \\ &\quad -C_F\left(2C_F-\frac{C_A}{2}\right)\frac{g^4}{512}\left(\frac{c_{A_2}^{(1)}}{m_1^3}+\frac{c_{A_2}^{(2)}}{m_2^3}\right)|\mathbf{k}|^{1+2\epsilon} \\ &\quad -C_F\left(2C_F-\frac{C_A}{2}\right)\frac{g^4}{128m_1m_2}\left(\frac{c_F^{(1)\,2}c_k^{(2)}}{m_1}+\frac{c_F^{(2)\,2}c_k^{(1)}}{m_2}\right)|\mathbf{k}|^{1+2\epsilon} \\ &\quad -C_FC_A\frac{g^4}{256m_1m_2}\left(\frac{c_D^{(1)}c_k^{(2)}}{m_1}+\frac{c_D^{(2)}c_k^{(1)}}{m_2}\right)|\mathbf{k}|^{1+2\epsilon} \\ &\quad -\frac{T_F}{N_c}C_F\frac{g^4}{128}\left(\frac{c_{A_3}^{(1)}}{m_1^3}+\frac{c_{A_3}^{(2)}}{m_2^3}\right)|\mathbf{k}|^{1+2\epsilon} \\ &\quad -\frac{T_F}{N_c}C_F\frac{g^4}{256}\left(\frac{c_{A_4}^{(1)}}{m_1^3}+\frac{c_{A_4}^{(2)}}{m_2^3}\right)|\mathbf{k}|^{1+2\epsilon}. \end{aligned} \tag{4.46}$$

The (f)-type diagrams in Fig. 4.3 do generate $\mathcal{O}(\alpha^2/m^3)$ potentials. They read

$$\tilde{V}^{(f,1)}_{\rm 1loop} = -C_FC_A\frac{g^4}{128}\left(\frac{c_F^{(1)\,2}c_k^{(1)}}{m_1^3}+\frac{c_F^{(2)\,2}c_k^{(2)}}{m_2^3}\right)|\mathbf{k}|^{1+2\epsilon}\frac{\mathbf{p}\cdot\mathbf{p}'}{\mathbf{k}^2}, \tag{4.47}$$

$$\begin{aligned}\tilde{V}^{(f,2)}_{\rm 1loop} &= -C_F C_A \frac{g^4}{512 m_1 m_2}\left(\frac{c_k^{(2)} c_k^{(1)\,2}}{m_1} + \frac{c_k^{(1)} c_k^{(2)\,2}}{m_2}\right)|\mathbf{k}|^{1+2\epsilon}\left(1 - \frac{3}{2}\frac{\mathbf{p}^2+\mathbf{p}'^2}{\mathbf{k}^2}\right)\\ &\times\left(\frac{2(\mathbf{p}^2+\mathbf{p}'^2)}{\mathbf{k}^2} - 1 - \frac{(\mathbf{p}^2-\mathbf{p}'^2)^2}{\mathbf{k}^4}\right)\\ &-C_F C_A \frac{g^4}{512}\left(\frac{c_k^{(1)\,3}}{m_1^3} + \frac{c_k^{(2)\,3}}{m_2^3}\right)|\mathbf{k}|^{1+2\epsilon}\left(\frac{\mathbf{p}\cdot\mathbf{p}'}{\mathbf{k}^2} + \frac{5(\mathbf{p}\cdot\mathbf{k})(\mathbf{p}'\cdot\mathbf{k})}{\mathbf{k}^4} - \frac{12(\mathbf{p}\cdot\mathbf{p}')^2}{\mathbf{k}^4}\right.\\ &\left.+\frac{2\mathbf{p}^2\mathbf{p}'^2}{\mathbf{k}^4} + \frac{6(\mathbf{p}\cdot\mathbf{p}')(\mathbf{p}\cdot\mathbf{k})(\mathbf{p}'\cdot\mathbf{k})}{\mathbf{k}^6}\right), \end{aligned} \tag{4.48}$$

$$\begin{aligned}\tilde{V}^{(f,3)}_{\rm 1loop} &= -c_k^{(1)} c_k^{(2)} C_F C_A \frac{3g^4}{128 m_1 m_2}|\mathbf{k}|^{-1+2\epsilon}\left(\mathbf{p}\cdot\mathbf{p}' - \frac{(\mathbf{p}\cdot\mathbf{k})(\mathbf{p}'\cdot\mathbf{k})}{\mathbf{k}^2}\right)\\ &\times\frac{(E_1+E_1')+(E_2+E_2')}{\mathbf{k}^2}\\ &-C_F C_A\frac{g^4}{256}\left(\frac{c_k^{(1)\,2}}{m_1^2}(E_1+E_1') + \frac{c_k^{(2)\,2}}{m_2^2}(E_2+E_2')\right)|\mathbf{k}|^{-1+2\epsilon}\\ &\times\left(\frac{5\mathbf{p}\cdot\mathbf{p}'}{\mathbf{k}^2} - \frac{3(\mathbf{p}\cdot\mathbf{k})(\mathbf{p}'\cdot\mathbf{k})}{\mathbf{k}^4}\right), \end{aligned} \tag{4.49}$$

$$\begin{aligned}\tilde{V}^{(f,4)}_{\rm 1loop} &= -C_F C_A\frac{g^4}{256}\left(\frac{c_k^{(1)\,3}}{m_1^3} + \frac{c_k^{(2)\,3}}{m_2^3}\right)|\mathbf{k}|^{1+2\epsilon}\left(-1 + \frac{\mathbf{p}^2+\mathbf{p}'^2}{\mathbf{k}^2}\right.\\ &+\frac{3(\mathbf{p}^4+\mathbf{p}'^4) + (\mathbf{p}^2+\mathbf{p}'^2)(\mathbf{p}\cdot\mathbf{p}') - 6(\mathbf{p}\cdot\mathbf{p}')^2}{\mathbf{k}^4}\\ &\left.+\frac{-3(\mathbf{p}^6+\mathbf{p}'^6) + 4(\mathbf{p}^4+\mathbf{p}'^4)(\mathbf{p}\cdot\mathbf{p}') - 2(\mathbf{p}\cdot\mathbf{p}')^3}{\mathbf{k}^6}\right), \end{aligned} \tag{4.50}$$

$$\begin{aligned}\tilde{V}^{(f,5)}_{\rm 1loop} &= -C_F C_A\frac{g^4}{128}|\mathbf{k}|^{1+2\epsilon}\left[\frac{c_k^{(2)\,2}}{m_2^2}\left(\frac{3(E_1+E_1')(\mathbf{p}\cdot\mathbf{k})(\mathbf{p}'\cdot\mathbf{k})}{\mathbf{k}^6} + \frac{(E_1+E_1')(\mathbf{p}\cdot\mathbf{p}')}{\mathbf{k}^4}\right.\right.\\ &\left.+\frac{2(E_1\mathbf{p}^4+E_1'\mathbf{p}'^4)}{\mathbf{k}^6} - \frac{2(E_1\mathbf{p}^2+E_1'\mathbf{p}'^2)(\mathbf{p}\cdot\mathbf{p}'+\mathbf{k}^2)}{\mathbf{k}^6}\right)\\ &+\frac{c_k^{(1)\,2}}{m_1^2}\left(\frac{3(E_2+E_2')(\mathbf{p}\cdot\mathbf{k})(\mathbf{p}'\cdot\mathbf{k})}{\mathbf{k}^6} + \frac{(E_2+E_2')(\mathbf{p}\cdot\mathbf{p}')}{\mathbf{k}^4} + \frac{2(E_2\mathbf{p}^4+E_2'\mathbf{p}'^4)}{\mathbf{k}^6}\right.\\ &\left.\left.-\frac{2(E_2\mathbf{p}^2+E_2'\mathbf{p}'^2)(\mathbf{p}\cdot\mathbf{p}'+\mathbf{k}^2)}{\mathbf{k}^6}\right)\right], \end{aligned} \tag{4.51}$$

$$\tilde{V}^{(f,6)}_{\rm 1loop} = -C_F C_A\frac{7g^4}{256 m_1 m_2}\left(\frac{c_k^{(1)\,2} c_k^{(2)}}{m_1} + \frac{c_k^{(1)} c_k^{(2)\,2}}{m_2}\right)|\mathbf{k}|^{1+2\epsilon}\left(\frac{2(\mathbf{p}^2+\mathbf{p}'^2)}{\mathbf{k}^2} - 1 - \frac{(\mathbf{p}^2-\mathbf{p}'^2)^2}{\mathbf{k}^4}\right), \tag{4.52}$$

$$
\begin{aligned}
\tilde{V}^{(f,7)}_{\rm 1loop} &= C_F C_A \frac{g^4}{256 m_1 m_2}\left(\frac{c_D^{(1)} c_k^{(2)}}{m_1} + \frac{c_D^{(2)} c_k^{(1)}}{m_2}\right) |\mathbf{k}|^{1+2\epsilon} \\
&+ C_F C_A \frac{g^4}{1024}\left(\frac{c_4^{(1)}}{m_1^3} + \frac{c_4^{(2)}}{m_2^3}\right) |\mathbf{k}|^{1+2\epsilon}\left(\frac{10(\mathbf{p}^2+\mathbf{p}'^2)}{\mathbf{k}^2} - 7 + \frac{5(\mathbf{p}^2-\mathbf{p}'^2)^2}{\mathbf{k}^4}\right) \\
&- C_F C_A \frac{g^4}{256}\left(\frac{c_M^{(1)}}{m_1^3} + \frac{c_M^{(2)}}{m_2^3}\right) |\mathbf{k}|^{1+2\epsilon} \\
&- C_F C_A \frac{g^4}{512}\left(\frac{c_F^{(1)} c_S^{(1)}}{m_1^3} + \frac{c_F^{(2)} c_S^{(2)}}{m_2^3}\right) |\mathbf{k}|^{1+2\epsilon}, \qquad (4.53)
\end{aligned}
$$

$$
\tilde{V}^{(f,8)}_{\rm 1loop} = C_F C_A \frac{g^4}{64} |\mathbf{k}|^{-5+2\epsilon}\left[\left(c_k^{(1)}\frac{E_1^2}{m_1} + c_k^{(2)}\frac{E_2^2}{m_2}\right)(\mathbf{p}\cdot\mathbf{k}) - \left(c_k^{(1)}\frac{E_1'^2}{m_1} + c_k^{(2)}\frac{E_2'^2}{m_2}\right)(\mathbf{p}'\cdot\mathbf{k})\right], \qquad (4.54)
$$

$$
\tilde{V}^{(f,9)}_{\rm 1loop} = C_F C_A \frac{g^4}{128}\left(\frac{c_F^{(1)\,2}}{m_1^2}(E_1+E_1') + \frac{c_F^{(2)\,2}}{m_2^2}(E_2+E_2')\right) |\mathbf{k}|^{-1+2\epsilon}. \qquad (4.55)
$$

The rest of the topologies: (g), (h), (i) and (j), do not contribute. Note that those topologies include, in particular, the one-loop diagrams proportional to c_i^{hl} or d_i^{hl}, as they may produce $\mathcal{O}(\alpha^2/m^3)$ potentials. We find that such contributions vanish.

As we have incorporated the LL running of the HQET Wilson coefficients, these potentials are already RG improved.

Note that with trivial modifications these potentials are also valid for QED.

4.4.3 $\mathcal{O}(\alpha^3/m^2)$ V_r potential

In this section we perform a partial computation of the $\mathcal{O}(\alpha^3/m^2)$ soft contribution to the V_r potential. The contributions we compute here are those proportional to the HQET Wilson coefficients $\bar{c}_1^{(i)hl}$ and $c_F^{(i)}$. We define

$$
\frac{\tilde{D}_d^{(2)}}{m_1 m_2} = \frac{\tilde{D}_d^{(2,0)}}{m_1^2} + \frac{\tilde{D}_d^{(0,2)}}{m_2^2} + \frac{\tilde{D}_d^{(1,1)}}{m_1 m_2}. \qquad (4.56)
$$

Using the notation of Ref. [11],

$$
\pi C_F \tilde{D}_{d,B}^{(2,0)} = \tilde{D}_r^{(2,0)} = g_B^2 C_F\left\{D_{r,1}^{(2,0)} + \frac{g_B^2 k^{2\epsilon}}{(4\pi)^2} D_{r,2}^{(2,0)} + \frac{g_B^4 k^{4\epsilon}}{(4\pi)^4} D_{r,3}^{(2,0)} + \cdots\right\}, \qquad (4.57)
$$

the bare new result reads

$$
\begin{aligned}
\tilde{D}^{(2,0)}_{r,3} &= \bar{c}_1^{hl\,(1)}\Bigg[T_F n_f\Bigg(C_A\Bigg(-\frac{2^{-8\epsilon-4}\pi^{\frac{5}{2}-2\epsilon}3(2\epsilon^2+7\epsilon+4)\csc(2\pi\epsilon)\csc(\pi\epsilon)}{\epsilon(2\epsilon+3)\Gamma(2\epsilon+5/2)} \\
&-\frac{2^{-6\epsilon-3}\pi^{\frac{3}{2}-2\epsilon}(40\epsilon^4+160\epsilon^3+240\epsilon^2+167\epsilon+44)\csc(2\pi\epsilon)\Gamma^2(\epsilon+1)}{\epsilon(2\epsilon+3)\Gamma(\epsilon+5/2)\Gamma(3\epsilon+3)} \\
&+\frac{2^{-6\epsilon-3}\pi^{\frac{3}{2}-2\epsilon}(4\epsilon^4+12\epsilon^3+12\epsilon^2+13\epsilon+6)\sin(2\pi\epsilon)\csc^2(\pi\epsilon)\Gamma(-2\epsilon-3)\Gamma(\epsilon+2)}{\epsilon\Gamma(\epsilon+5/2)}\Bigg) \\
&+C_F\Bigg(\frac{2^{-8\epsilon-4}\pi^{2-2\epsilon}(2\epsilon+1)(2\epsilon+3)(\epsilon^2+2\epsilon+2)\csc(\pi\epsilon)\sec(\pi\epsilon)\Gamma(\epsilon+2)\Gamma(2\epsilon+2)}{\epsilon^2\Gamma^2(\epsilon+5/2)\Gamma(3\epsilon+3)} \\
&-\frac{2^{-8\epsilon-5}\pi^{3-2\epsilon}(\epsilon+1)(2\epsilon+3)(2\epsilon^2+\epsilon+2)\csc^2(\pi\epsilon)}{\epsilon\Gamma^2(\epsilon+5/2)}\Bigg)\Bigg) \\
&+\frac{(T_F n_f)^2 2^{-8\epsilon-3}\pi^{3-2\epsilon}(\epsilon+1)^2\csc^2(\pi\epsilon)}{\Gamma^2(\epsilon+5/2)}\Bigg] \\
&+c_F^{(1)\,2}\frac{1}{3}C_A 2^{-8\epsilon-7}\pi^{-2\epsilon}\Bigg[C_A\Bigg(\frac{2^{4\epsilon+5}3\Gamma(1-2\epsilon)\Gamma^3(\epsilon)}{(4\epsilon(\epsilon+2)+3)^2\Gamma(3\epsilon+3)}(\epsilon(\epsilon(\epsilon(\epsilon(2\epsilon(18\epsilon(2\epsilon+11)+401) \\
&+661)+33)-283)-165)-30)+\frac{\pi 2^{4\epsilon+5}\Gamma(1-2\epsilon)\Gamma^3(\epsilon+1/2)}{\Gamma(3\epsilon+3/2)} \\
&+\frac{3\pi^3(\epsilon(\epsilon(22-\epsilon(12\epsilon+17))+45)+15)\csc^2(\pi\epsilon)}{\epsilon\Gamma^2(\epsilon+5/2)}+\frac{12\pi^4(2\epsilon-1)\sec^2(\pi\epsilon)}{\Gamma^2(\epsilon+1)} \\
&+\frac{24\pi^{5/2}(\epsilon(\epsilon(\epsilon(4\epsilon(\epsilon+12)+127)+130)+65)+15)\csc(\pi\epsilon)\csc(2\pi\epsilon)}{\epsilon(4\epsilon(\epsilon+2)+3)\Gamma(2\epsilon+5/2)}\Bigg) \\
&+\frac{24\pi^{3/2}n_f T_F}{(2\epsilon+3)^2}\Bigg(\frac{4^{\epsilon+1}\Gamma(\epsilon+1)\epsilon(4\epsilon+3)\cot(\pi\epsilon)\Gamma(-2\epsilon-1)}{\Gamma(\epsilon+3/2)} \\
&-\frac{4^{\epsilon+1}\Gamma(\epsilon+1)(6\epsilon^2+9\epsilon+4)(2\epsilon(2\epsilon+5)+5)\csc(2\pi\epsilon)\Gamma(\epsilon)}{\Gamma(\epsilon+3/2)\Gamma(3\epsilon+3)} \\
&-\frac{\pi(2\epsilon+1)^2(2\epsilon+3)\csc(\pi\epsilon)\csc(2\pi\epsilon)}{\Gamma(2\epsilon+5/2)}\Bigg)\Bigg]. \qquad (4.58)
\end{aligned}
$$

With obvious changes the same result is obtained for $\tilde{D}^{(0,2)}_{r,3}$. It is worth emphasizing that this expression vanishes in pure QED. A nontrivial check of this result is that c_D and c_1^{hl} appear in the gauge invariant combination $\bar{c}_1^{hl} = c_D + c_1^{hl}$. Another nontrivial check is that the counterterm is independent of k and that the $1/\epsilon^2$ terms comply with the constraints from RG. This computation has been done in the Feynman gauge (with a general gauge parameter ξ) in the kinematic configuration $\mathbf{p} = \mathbf{k}$ and $\mathbf{p}' = 0$. We also set the external energy to zero. Not setting it to zero produces subleading corrections (we recall that the one-loop computation of this contribution has no energy dependence [11]). The result is shown to be independent of the gauge fixing parameter ξ.

For future computations, it is useful to explain the convention we have taken for the D-dimensional spin matrices. For the $c_F^{(i)}$ vertex we typically take a covariant notation $\sigma^{\mu\nu}$ (see for instance Ref. [30]) and project to the particle or the antiparticle sector, depending if the external legs are quarks or antiquarks, in order to single out the spin-independent part, as we did when we computed the one-loop matching of four fermion operators in Ch. 3. In particular, we can use Eq.(3.10) because the structure of the delta-like potential is the same that the one of the four fermion vertices. Indeed, in the matching between NRQCD and pNRQCD, the contribution coming from these operators gets absorbed into the delta-like potential Wilson coefficient. Basically, the numerator in the loop integrals becomes:

$$\sim \frac{\mathrm{Tr}[(\ldots)(1+\gamma_0)/2]\,\mathrm{Tr}[(\ldots)(1-\gamma_0)/2]}{\mathrm{Tr}[(1+\gamma^0)/2]\,\mathrm{Tr}[(1-\gamma^0)/2]} \tag{4.59}$$

Where the dots stand for the numerators of the Feynman integrals coming from the particle and the antiparticle, respectively, before any projection is carried out. At one-loop, this procedure gives the same result as using Pauli matrices with the conventions used in Ref. [11].

Though not directly relevant for this work, we also give the $\overline{\mathrm{MS}}$ renormalized expression of the bare potential computed above. It will be of relevance for future computations of the spectrum at N^{4}LO and for decays. The result reads ($\alpha=\alpha(\nu)$)

$$\begin{aligned}
\tilde{D}^{(2,0)}_{r,\overline{\mathrm{MS}}}(\mathbf{k}) \;=\;& \frac{C_F\alpha^2}{2}\left[\frac{13}{36}c_F^{(1)\,2}C_A-\frac{5}{9}\bar{c}_1^{hl\,(1)}T_Fn_f+\left(-C_A\frac{5}{6}c_F^{(1)\,2}+\frac{2}{3}\bar{c}_1^{hl\,(1)}T_Fn_f\right)\ln\left(\frac{k}{\nu}\right)\right]\\
&+c_F^{(1)\,2}C_FC_A^2\frac{\alpha^3}{2\pi}\left[\frac{1080\zeta(3)+706-900\gamma+432\pi^2-81\pi^4+900\ln(4\pi)}{5184}\right.\\
&\quad\left.-\frac{179}{108}\ln\left(\frac{k}{\nu}\right)+\frac{10}{9}\ln^2\left(\frac{k}{\nu}\right)\right]\\
&+c_F^{(1)\,2}C_FC_An_fT_F\frac{\alpha^3}{2\pi}\left[\frac{-3581+750\gamma-750\ln(4\pi)}{2592}+\frac{91}{54}\ln\left(\frac{k}{\nu}\right)\right.\\
&\left.-\frac{7}{12}\ln^2\left(\frac{k}{\nu}\right)\right]\\
&+\bar{c}_1^{hl}C_Fn_fT_FC_A\frac{\alpha^3}{2\pi}\left[\frac{-1008\zeta(3)+627-130\gamma+130\ln(4\pi)}{864}\right.\\
&\quad\left.+\frac{5}{6}\ln\left(\frac{k}{\nu}\right)-\frac{31}{36}\ln^2\left(\frac{k}{\nu}\right)\right]\\
&+\bar{c}_1^{hl}C_F^2n_fT_F\frac{\alpha^3}{2\pi}\left[\frac{48\zeta(3)-55+6\gamma-6\ln(4\pi)}{48}+\frac{1}{2}\ln\left(\frac{k}{\nu}\right)\right]\\
&+\bar{c}_1^{hl}C_Fn_f^2T_F^2\frac{\alpha^3}{2\pi}\left[\frac{25}{81}-\frac{20}{27}\ln\left(\frac{k}{\nu}\right)+\frac{4}{9}\ln^2\left(\frac{k}{\nu}\right)\right],
\end{aligned} \tag{4.60}$$

where we have also included the $\mathcal{O}(\alpha^2)$ term. Note that this contribution does not mix with $V_{\mathbf{L}}^{(2)}$. Therefore, it really corresponds to the contributions proportional to $c_F^{(1)}$ and $\bar{c}_1^{hl(1)}$ of $D_r^{(2,0)}$, as defined in Ref. [11]. With obvious changes a similar expression is obtained for $\tilde{D}^{(0,2)}_{r,\overline{\mathrm{MS}}}(\mathbf{k})$.

Note that there is a missing term proportional to $c_k^{(1)\,2}$ ($c_k^{(2)\,2}$) in $\tilde{D}_d^{(2,0)}$ ($\tilde{D}_d^{(0,2)}$) and other missing terms proportional to $c_k^{(1)}c_k^{(2)}$, d_{ss} and d_{vs} in $\tilde{D}_d^{(1,1)}$, which are the only contributions it has. Finally, note that the missing part of the soft term should carefully be computed in a way consistent with the scheme we have used for the rest of the computation, in particular for the α^2/m^3 potential, as a strong mixing (if using the EOMs/field redefinitions) of the terms proportional to c_k^2 is expected. In the next section we will see how this mixing takes place for the terms proportional to c_F^2.

4.4.4 Equations of motion

Some of the potentials we have obtained in Sec. 4.4.2 are energy-dependent. If we want to eliminate such energy dependence, and write an energy independent potential, it can be achieved by using field redefinitions. At the order we are working, it is enough to use the full EOM at the leading order, which includes the Coulomb potential. We first consider Eq. (4.43), which depends on the total energy of the heavy quarkonium and, for this reason, does not contribute to the running of the delta potential. We next consider Eq. (4.55), which is the only energy-dependent potential proportional to $c_F^{(i)\,2}$. Such a potential is generated by the following interaction Lagrangian:

$$\begin{aligned} L_{\tilde{V}_{\rm 1loop}^{(f,9)}} = & -C_F C_A \frac{g^4}{128}\frac{c_F^{(1)\,2}}{m_1^2}\int d^3x_1 d^3x_2 (\psi^\dagger(i\partial_0\psi(t,\mathbf{x}_1)) - (i\partial_0\psi^\dagger)\psi(t,\mathbf{x}_1)) \\ & \times \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|^{1-2\epsilon}}\chi_c^\dagger\chi_c(t,\mathbf{x}_2) \\ & -C_F C_A \frac{g^4}{128}\frac{c_F^{(2)\,2}}{m_2^2}\int d^3x_1 d^3x_2 \psi^\dagger\psi(t,\mathbf{x}_1) \\ & \times \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|^{1-2\epsilon}}(\chi_c^\dagger i\partial_0\chi_c(t,\mathbf{x}_2) - (i\partial_0\chi_c^\dagger)\chi_c(t,\mathbf{x}_2))\,. \end{aligned} \tag{4.61}$$

For this Lagrangian one can use the EOMs

$$\left(i\partial_0 + \frac{\boldsymbol{\nabla}^2}{2m_1}\right)\psi(t,\mathbf{x}) - \int d^3x_2 \psi(t,\mathbf{x})V_C(\mathbf{x}-\mathbf{x}_2)\chi_c^\dagger\chi_c(t,\mathbf{x}_2) = 0\,, \tag{4.62}$$

where $V_C(\mathbf{x}) = -C_F\alpha/|\mathbf{x}|$, and similarly for the other fields. We then obtain

$$\begin{aligned} L_{\tilde{V}_{\rm 1loop}^{(f,9)}} = & -C_F C_A \frac{g^4}{128}\frac{c_F^{(1)\,2}}{m_1^2}\int d^3x_1 d^3x_2 \left[\psi^\dagger\left(-\frac{\boldsymbol{\nabla}^2}{2m_1}\psi(t,\mathbf{x}_1)\right) + \left(-\frac{\boldsymbol{\nabla}^2}{2m_1}\psi^\dagger\right)\psi(t,\mathbf{x}_1)\right] \\ & \times \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|^{1-2\epsilon}}\chi_c^\dagger\chi_c(t,\mathbf{x}_2) \\ & -C_F C_A \frac{g^4}{64}\frac{c_F^{(1)\,2}}{m_1^2}\int d^3x_1 d^3x_2 d^3x_3 \psi^\dagger\psi(t,\mathbf{x}_1)V_C(\mathbf{x}_1-\mathbf{x}_3) \\ & \times \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{|\mathbf{k}|^{1-2\epsilon}}\chi_c^\dagger\chi_c(t,\mathbf{x}_2)\chi_c^\dagger\chi_c(t,\mathbf{x}_3) + \cdots\,, \end{aligned} \tag{4.63}$$

where the dots stand for the analogous contribution for the antiparticle.

The first term in Eq. (4.63) yields the potential we obtain after using the free on-shell EOMs in Eq. (4.55). It reads

$$\tilde{V}^{(f,9)}_{\rm 1loop} = C_F C_A \frac{g^4}{256}\left(\frac{c_F^{(1)\,2} c_k^{(1)}}{m_1^3} + \frac{c_F^{(2)\,2} c_k^{(2)}}{m_2^3}\right)|\mathbf{k}|^{1+2\epsilon}\frac{\mathbf{p}^2+\mathbf{p}'^2}{\mathbf{k}^2}\,. \tag{4.64}$$

The second term is a six-fermion field term. After contracting two of them[9], a new α^3/m^2 potential is generated. Here we only care about the divergent part because we are only interested in logarithmically enhanced contributions to the spectrum. It reads

$$\delta\tilde{V}^{(f,9)}_{\rm 1loop} = \frac{1}{32} C_F^2 C_A \frac{g^6 k^{4\epsilon}}{(4\pi)^2}\left(\frac{c_F^{(1)2}}{m_1^2} + \frac{c_F^{(2)2}}{m_2^2}\right)\frac{1}{\epsilon}\,. \tag{4.65}$$

It is worth mentioning that this contribution has a different color structure compared to the one of the purely soft contributions computed in Sec. 4.4.3. It is also π^2 enhanced compared to them. Therefore, one could expect it to be more important than the strictly purely soft contribution.

Remarkably enough, we will see later that the contributions from Eqs. (4.64) and (4.65) to the running of the delta-like potential cancel each other in the equal mass case, but not for different masses. This was to be expected, since in the equal mass case, the potential can be written in terms of the total energy of the heavy quarkonium, which does not produce divergences that should be absorbed in the delta-like potential.

It is worth mentioning that this exhausts all possible $c_F^{(i)2}$ structures that can be generated. To be sure of this statement, we have to check that the result does not depend on the gauge. Therefore, we have redone the diagrams proportional to $c_F^{(i)2}$ (i.e. the associated contributions to $\tilde{V}^{(e,2)}_{1loop}$, $\tilde{V}^{(f,1)}_{1loop}$ and $\tilde{V}^{(f,9)}_{1loop}$) in the Feynman gauge and have found the same result.

The other potentials that are dependent on the energy are proportional to c_k^2. As before, these contributions will mix with the α^3/m^2 purely soft contribution proportional to c_k^2, which we have not computed anyhow. Therefore, in this thesis, we only include the explicit contribution generated using the free EOMs and postpone the incorporation of the other contribution to have the full result. The contributions we explicitly include in this thesis read

[9] We use $\{\psi(\mathbf{x}),\psi^\dagger(\mathbf{y})\} = \delta^{(3)}(\mathbf{x}-\mathbf{y})$ and $\{\chi_c(\mathbf{x}),\chi_c^\dagger(\mathbf{y})\} = \delta^{(3)}(\mathbf{x}-\mathbf{y})$.

$$
\begin{aligned}
\tilde{V}^{(f,3)}_{\rm 1loop} &= -C_F C_A \frac{3g^4}{1024 m_1 m_2}\left(\frac{c_k^{(1)\,2}c_k^{(2)}}{m_1}+\frac{c_k^{(1)}c_k^{(2)\,2}}{m_2}\right)|\mathbf{k}|^{1+2\epsilon}\frac{\mathbf{p}^2+\mathbf{p}'^2}{\mathbf{k}^2} \\
&\times\left(\frac{2(\mathbf{p}^2+\mathbf{p}'^2)}{\mathbf{k}^2}-1-\frac{(\mathbf{p}^2-\mathbf{p}'^2)^2}{\mathbf{k}^4}\right) \\
&-C_F C_A \frac{g^4}{512}\left(\frac{c_k^{(1)\,3}}{m_1^3}+\frac{c_k^{(2)\,3}}{m_2^3}\right)|\mathbf{k}|^{1+2\epsilon}\frac{\mathbf{p}^2+\mathbf{p}'^2}{\mathbf{k}^2} \\
&\times\left(\frac{5\mathbf{p}\cdot\mathbf{p}'}{\mathbf{k}^2}-\frac{3(\mathbf{p}\cdot\mathbf{k})(\mathbf{p}'\cdot\mathbf{k})}{\mathbf{k}^4}\right)\,, \qquad (4.66)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{V}^{(f,5)}_{\rm 1loop} &= -C_F C_A \frac{g^4}{256 m_1 m_2}|\mathbf{k}|^{1+2\epsilon}\left(\frac{c_k^{(1)\,2}c_k^{(2)}}{m_1}+\frac{c_k^{(1)}c_k^{(2)\,2}}{m_2}\right)\left(\frac{3(\mathbf{p}^2+\mathbf{p}'^2)(\mathbf{p}\cdot\mathbf{k})(\mathbf{p}'\cdot\mathbf{k})}{\mathbf{k}^6}\right. \\
&-\frac{2(\mathbf{p}^4+\mathbf{p}'^4)}{\mathbf{k}^4}+\frac{(\mathbf{p}^2+\mathbf{p}'^2)(\mathbf{p}\cdot\mathbf{p}')}{\mathbf{k}^4}+\frac{2(\mathbf{p}^6+\mathbf{p}'^6)}{\mathbf{k}^6} \\
&\left.-\frac{2(\mathbf{p}^4+\mathbf{p}'^4)(\mathbf{p}\cdot\mathbf{p}')}{\mathbf{k}^6}\right)\,, \qquad (4.67)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{V}^{(f,8)}_{\rm 1loop} &= C_F C_A \frac{g^4}{512}\left(\frac{c_k^{(1)\,3}}{m_1^3}+\frac{c_k^{(2)\,3}}{m_2^3}\right)|\mathbf{k}|^{1+2\epsilon} \\
&\times\left(\frac{2(\mathbf{p}^6+\mathbf{p}'^6)}{\mathbf{k}^6}-\frac{(\mathbf{p}^4+\mathbf{p}'^4)(\mathbf{p}^2+\mathbf{p}'^2)}{\mathbf{k}^6}+\frac{\mathbf{p}^4+\mathbf{p}'^4}{\mathbf{k}^4}\right)\,. \qquad (4.68)
\end{aligned}
$$

4.5 NLL running of $\tilde{D}_d^{(2)}$

In this section we compute the NLL soft and potential running of $\tilde{D}_d^{(2)}$.

4.5.1 Soft running

From the results obtained in Sec. 4.4.3 we can obtain the $\mathcal{O}(\alpha^3)$ soft RGE of $\tilde{D}_d^{(2)}$ (the $\mathcal{O}(\alpha^2)$ soft RGE can be found in Ref. [3]) proportional to $c_F^{(i)\,2}$ and $c_1^{hl(i)}$. In practice, such a computation can be understood as getting the NLL soft running of $d_{ss}+C_F\bar{d}_{vs}$ (see

Eq. (4.33) or Eq. (4.34)). It reads

$$\nu_s \frac{d}{d\nu_s}(d_{ss}+C_F\bar{d}_{vs})\Big|_{\rm soft} = C_F\alpha^2\left(2C_F-\frac{C_A}{2}\right)c_k^{(1)}c_k^{(2)} \tag{4.69}$$
$$+C_F\alpha^2\left[\frac{m_1}{m_2}\left(\frac{1}{3}T_fn_f\bar{c}_1^{hl(2)}-\frac{4}{3}(C_A+C_F)[c_k^{(2)}]^2-\frac{5}{12}C_A[c_F^{(2)}]^2\right)\right.$$
$$\left.+\frac{m_2}{m_1}\left(\frac{1}{3}T_fn_f\bar{c}_1^{hl(1)}-\frac{4}{3}(C_A+C_F)[c_k^{(1)}]^2-\frac{5}{12}C_A[c_F^{(1)}]^2\right)\right]$$
$$+C_F\frac{\alpha^3}{4\pi}\left[\frac{m_1}{m_2}\left(-\frac{T_Fn_f}{54}(65C_A-54C_F)\bar{c}_1^{hl(2)}-\frac{C_A}{18}(25C_A-\frac{125}{3}T_Fn_f)[c_F^{(2)}]^2\right)\right.$$
$$\left.+\frac{m_2}{m_1}\left(-\frac{T_Fn_f}{54}(65C_A-54C_F)\bar{c}_1^{hl(1)}-\frac{C_A}{18}(25C_A-\frac{125}{3}T_Fn_f)[c_F^{(1)}]^2\right)\right]$$
$$+\mathcal{O}(\alpha^3)\,.$$

The $\mathcal{O}(\alpha^3)$ stands for terms proportional to NRQCD Wilson coefficients different from $c_F^{(i)\,2}$ and $c_1^{hl(i)}$. This equation is meant to represent the pure-soft running of the NRQCD Wilson coefficients. It does not give the full running of $\tilde{D}_d^{(2)}$, as one should also include the potential and ultrasoft running. We fix the initial matching condition to zero, since we only need the initial matching condition of the total potential, which can be added in the final step, when combining the different contributions.

The strict NLL contribution to the solution of this equation[10] reads (the LL is already

[10]See App. E.5 in order to see a more detailed discussion of how this contribution is computed, as well as, to find some necessary Wilson coefficients.

included in Eq. (4.33))

$$
\begin{aligned}
&\pi C_F\delta\tilde{D}^{(2)\,\rm NLL}_{d,s} = [d_{ss}+C_F\bar{d}_{vs}]^{\rm NLL} = -\alpha^2(\nu_h)C_F\Bigg[\Bigg(465C_A^6(757m_1^2-306m_1m_2+757m_2^2)\\
&-13824C_F^2(2m_1^2-3m_1m_2+2m_2^2)n_f^4T_F^4+C_A^5\Big(5580C_F(53m_1^2+102m_1m_2+53m_2^2)\\
&+(-590218m_1^2+342117m_1m_2-590218m_2^2)n_fT_F\Big)-C_A^4n_fT_F\Big(34C_F(8347m_1^2\\
&+38772m_1m_2+8347m_2^2)-3(115117m_1^2-101466m_1m_2+115117m_2^2)n_fT_F\Big)\\
&+32C_An_f^3T_F^3\Big(81C_F^2(70m_1^2-83m_1m_2+70m_2^2)-4C_F(5m_1^2-459m_1m_2+5m_2^2)n_fT_F\\
&+120(m_1^2+m_2^2)n_f^2T_F^2\Big)-8C_A^2n_f^2T_F^2\Big(81C_F^2(566m_1^2-563m_1m_2+566m_2^2)-3C_F(193m_1^2\\
&-17595m_1m_2+193m_2^2)n_fT_F+2(739m_1^2+1080m_1m_2+739m_2^2)n_f^2T_F^2\Big)\\
&+6C_A^3n_fT_F\Big(360C_F^2(106m_1^2-93m_1m_2+106m_2^2)+C_F(10129m_1^2+187731m_1m_2\\
&+10129m_2^2)n_fT_F-4(2536m_1^2-4959m_1m_2+2536m_2^2)n_f^2T_F^2\Big)\Bigg)\\
&\times\frac{1}{36m_1m_2(31C_A-16n_fT_F)(5C_A-4n_fT_F)(11C_A-4n_fT_F)^2(2C_A-n_fT_F)}\\
&+5C_A(m_1^2+m_2^2)\Big(397C_A^3+48C_Fn_f^2T_F^2+11C_A^2(33C_F-35n_fT_F)+10C_An_fT_F(-21C_F\\
&+10n_fT_F)\Big)z^{\frac{1}{3}(5C_A-4n_fT_F)}\frac{1}{6m_1m_2(5C_A-4n_fT_F)(11C_A-4n_fT_F)^2}\\
&-\frac{1}{468m_1m_2(11C_A-4n_fT_F)^2}\Big(1989C_A^3(8m_1^2+3m_1m_2+8m_2^2)+8C_Fn_fT_F\Big(81C_F(6m_1^2\\
&+13m_1m_2+6m_2^2)+1240(m_1^2+m_2^2)n_fT_F\Big)+2C_An_fT_F\Big(C_F(-15134m_1^2+5967m_1m_2\\
&-15134m_2^2)+3100(m_1^2+m_2^2)n_fT_F\Big)+2C_A^2\Big(3978C_F(2m_1^2-3m_1m_2+2m_2^2)\\
&-5(2263m_1^2+351m_1m_2+2263m_2^2)n_fT_F\Big)\Big)z^{\frac{2}{3}(11C_A-4n_fT_F)}\\
&+\frac{2(5C_A+8C_F)(m_1^2+m_2^2)n_fT_F(-1327C_A+594C_F+620n_fT_F)z^{\frac{31}{6}C_A-\frac{8}{3}n_fT_F}}{117m_1m_2(31C_A-16n_fT_F)(11C_A-4n_fT_F)}\\
&-C_A(m_1^2+m_2^2)\Big(15C_A^3-188C_A^2n_fT_F-2n_f^2T_F^2(27C_F+10n_fT_F)+C_An_fT_F(216C_F\\
&+137n_fT_F)\Big)z^{\frac{8}{3}(2C_A-n_fT_F)}\frac{1}{12m_1m_2(11C_A-4n_fT_F)^2(2C_A-n_fT_F)}\\
&-\frac{5C_A^2\Big(1-z^{\frac{1}{3}(5C_A-4n_fT_F)}\Big)\Big(m_2^2\ln\left(\frac{\nu_h}{m_1}\right)+m_1^2\ln\left(\frac{\nu_h}{m_2}\right)\Big)}{2m_1m_2(5C_A-4n_fT_F)}\Bigg]. \qquad (4.70)
\end{aligned}
$$

We do not aim in this thesis to give a full-fledged phenomenological analysis. Still, we

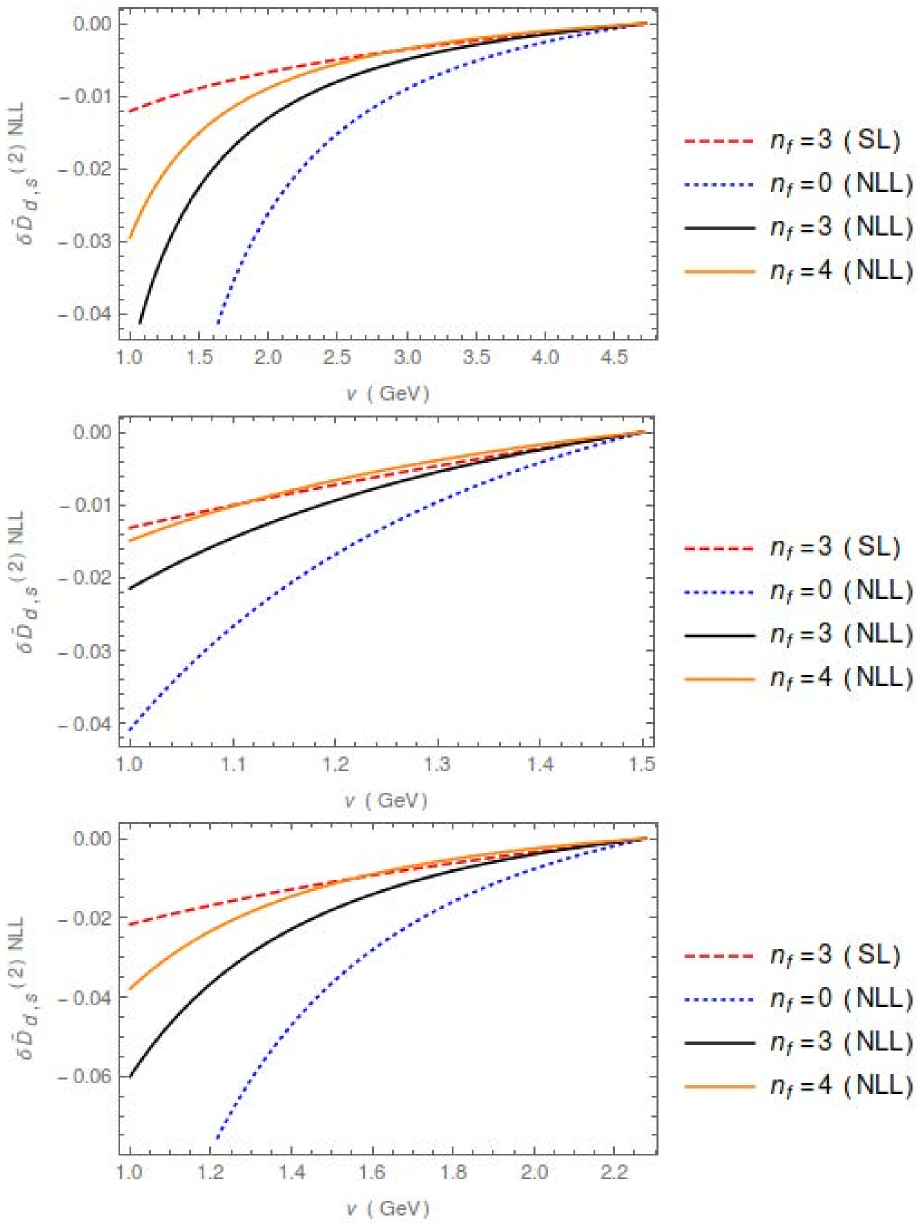


Figure 4.4: Plot of the NLL soft running due to Eq. (4.70) to $\delta \tilde{D}_{d,s}^{(2)\,\mathrm{NLL}}$ for different values of n_f (0,3,4) and in the SL approximation (in this case only with $n_f = 3$). **Upper panel:** Plot for bottomonium with $\nu_h = m_b$. **Middle panel:** Plot for charmonium with $\nu_h = m_c$. **Lower panel:** Plot for B_c with $\nu_h = 2m_b m_c/(m_b + m_c)$.

compute numerically the running of $\delta \tilde{D}_{d,s}^{(2)\,\mathrm{NLL}}$ to see its size. We show the result in Fig. 4.4. The contribution is small compared to the LL, as expected, and it is comparable to the NLL ultrasoft running.

To this contribution one should also add the contributions generated by the new α^3/m^2 potentials that appear after using the full EOM. Of those, we only computed the potentials

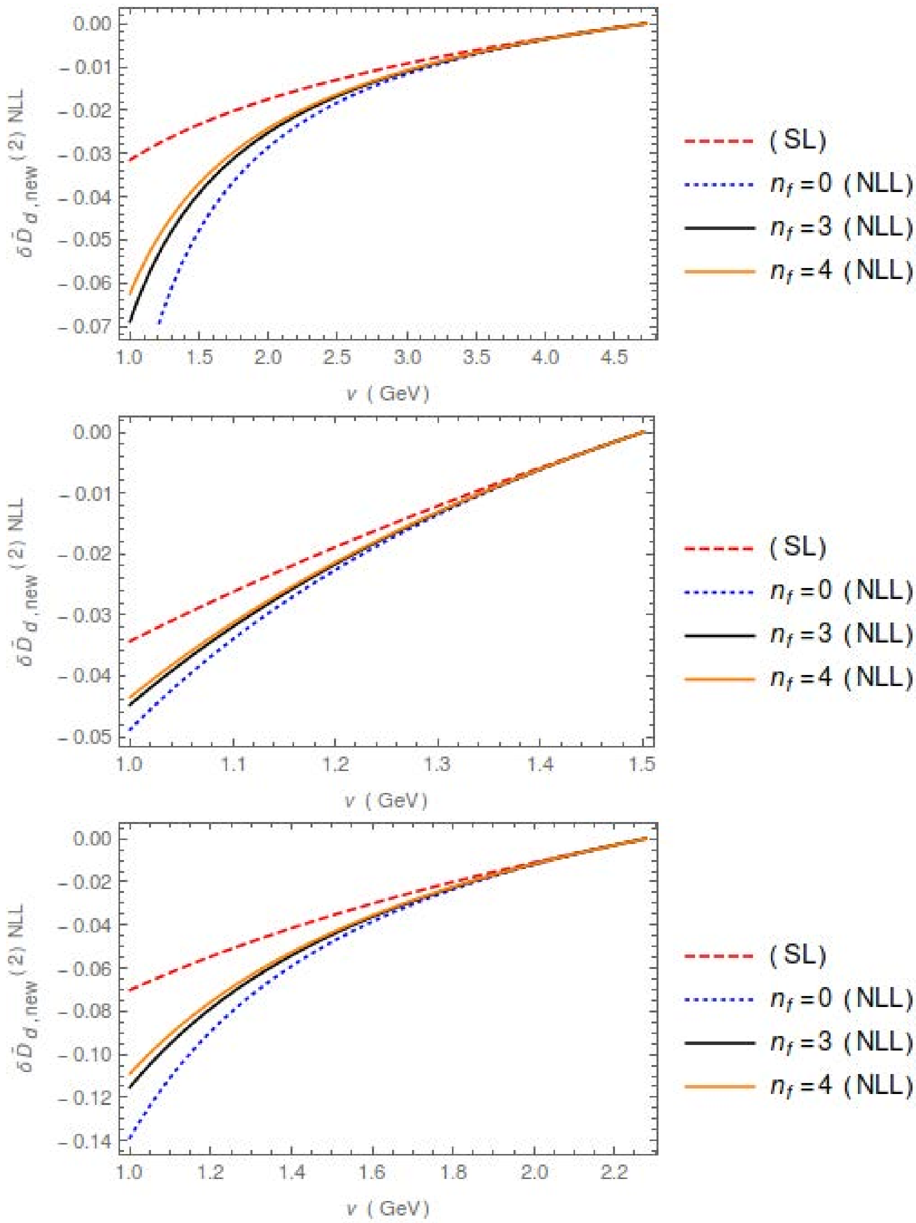


Figure 4.5: Plot of the extra contribution to the NLL soft running, $\delta\tilde{D}_{d,s}^{(2)\,\mathrm{NLL}}$, due to Eq. (4.72), for different values of n_f (0,3,4) and in the SL approximation (in this case only with $n_f = 3$). **Upper panel:** Plot for bottomonium with $\nu_h = m_b$. **Middle panel:** Plot for charmonium with $\nu_h = m_c$. **Lower panel:** Plot for B_c with $\nu_h = 2m_b m_c/(m_b + m_c)$.

proportional to $c_F^{(i)2}$ and c_1^{hl} (the latter happened to be zero). In other words, we must add the contribution coming from the potential given in Eq. (4.65). This generates a new contribution to the soft RGE:

$$\nu_s \frac{d}{d\nu_s}(d_{ss} + C_F \bar{d}_{vs})\bigg|_{\mathrm{soft}} = \cdots + \frac{1}{4}\pi C_F^2 C_A \alpha^3 \left(\frac{m_2}{m_1} c_F^{(1)\,2} + \frac{m_1}{m_2} c_F^{(2)\,2}\right), \qquad (4.71)$$

where the dots stand for the already computed soft contribution. Its solution reads

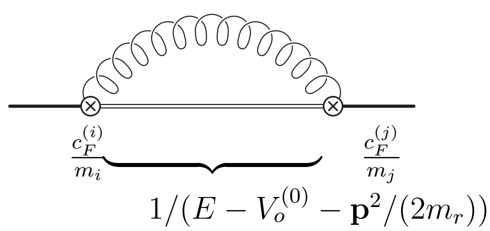
$$\delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,new} = \frac{1}{\pi C_F}(d_{ss} + C_F\bar{d}_{vs}) = -\frac{\pi C_A C_F (m_1^2+m_2^2)(1-z^{-2(C_A-\beta_0)})\alpha^2(\nu_h)}{4m_1m_2(C_A-\beta_0)}\,. \qquad (4.72)$$

We show the size of this new contribution in Fig. 4.5. The soft running will change to $\delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,s} \to \delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,s} + \delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,new}$.

Let us note that the c_k^2 terms can also mix with the α^2/m^3 potentials through field redefinitions. This contribution could be different for other matching schemes (see the discussion in Sec. 4.7).

4.5.2 Ultrasoft running

To obtain the complete potential RGE, we also need an extra potential divergence that is generated by ultrasoft divergences. This term was already computed in Ref. [14] and applied to the spin-dependent case. Here, we give the full term, which contributes to both, the spin-dependent and spin-independent terms. It is generated by the following diagram



which produces the following ultrasoft RGE

$$\nu_{us}\frac{\mathrm{d}V_{\mathbf{S}^2,1/r^3}}{\mathrm{d}\nu_{us}} = \frac{4C_F}{3}\left[\frac{2\mathbf{S_1}\cdot\mathbf{S}_2 c_F^{(1)}(\nu_{us})c_F^{(2)}(\nu_{us})}{m_1m_2} - \frac{3}{4}\left(\frac{c_F^{(1)2}(\nu_{us})}{m_1^2}+\frac{c_F^{(2)2}(\nu_{us})}{m_2^2}\right)\right] \times\left((V_o-V_s)^3 + \left(\frac{1}{m_1}+\frac{1}{m_2}\right)\frac{(V_o-V_s)^2}{2r^2}\right)\left[\frac{\alpha(\nu_{us})}{2\pi}\right]\,, \qquad (4.73)$$

or alternatively (but equivalent at this order)

$$\nu_{us}\frac{\mathrm{d}V_{\mathbf{S}^2,1/r^3}}{\mathrm{d}\nu_{us}} = \frac{4C_F}{3}\left[\frac{2\mathbf{S_1}\cdot\mathbf{S}_2 c_F^{(1)}(\nu_{us})c_F^{(2)}(\nu_{us})}{m_1m_2} - \frac{3}{4}\left(\frac{c_F^{(1)2}(\nu_{us})}{m_1^2}+\frac{c_F^{(2)2}(\nu_{us})}{m_2^2}\right)\right] \times V_o(V_o-V_s)^2\left[\frac{\alpha(\nu_{us})}{2\pi}\right]\,. \qquad (4.74)$$

Using that the LL running of c_F is independent of the masses (we take the initial matching condition to be ν_h for both heavy quarks), its solution reads

$$V_{\mathbf{S}^2,1/r^3} = \frac{4C_F}{3}\left[\frac{2\mathbf{S}_1\cdot\mathbf{S}_2}{m_1m_2} - \frac{3}{4}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\right]V_o(V_o-V_s)^2 D_{1/r^3,\mathbf{S}^2}\,, \qquad (4.75)$$

or

$$V_{\mathbf{S}^2,1/r^3} = \frac{4C_F}{3}\left[\frac{2\mathbf{S}_1\cdot\mathbf{S}_2}{m_1m_2} - \frac{3}{4}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)\right] D_{\mathbf{S}^2,1/r^3} \times\left((V_o-V_s)^3 + \left(\frac{1}{m_1}+\frac{1}{m_2}\right)\frac{(V_o-V_s)^2}{2r^2}\right), \qquad (4.76)$$

where (we use the same notation that in Ref. [14])

$$D_{\mathbf{S}^2,1/r^3} = \frac{1}{2C_A}\left[\left(\frac{\alpha(\nu_h)}{\alpha(\nu_{us})}\right)^{2C_A/\beta_0} - \left(\frac{\alpha(\nu_h)}{\alpha(1/r)}\right)^{2C_A/\beta_0}\right]. \qquad (4.77)$$

The potential $V_{\mathbf{S}^2,1/r^3}$ is singular and will contribute to the potential running of $\tilde{D}_d^{(2)}$. When solving the potential RGE we write Eq. (4.77) in terms of z, and in the one-loop approximation, which can be done trivially.

4.5.3 Potential running

We now have all the necessary preliminary ingredients to obtain the complete potential RGE. The next step is to compute all potential loops[11] that produce UV divergences that get absorbed in $\tilde{D}_d^{(2)}$ and are at most of $\mathcal{O}(\alpha^3)$. Since the delta-like potential is of $\mathcal{O}(1/m^2)$, we must construct potential loop diagrams of $\mathcal{O}(\alpha^n/m^2)$ with $n \leq 3$, describing the interaction between the two heavy quarks in the bound state through several potentials. The first nonvanishing contribution to the potential running is indeed of $\mathcal{O}(\alpha^3/m^2)$. To construct such potential loop diagrams, we must consider the power of α and m of each potential and take into account that each propagator adds an extra power of the mass in the numerator. We summarize all kind of diagrams that contribute to the NLL potential running of $\tilde{D}_d^{(2)}$ in Figs. [4.6-4.9], where V_C is the tree level, $\mathcal{O}(\alpha)$, Coulomb potential, V_{α^r/m^s} is the $\mathcal{O}(\alpha^r/m^s)$ potential and V_{1/m^3} corresponds to the first relativistic correction to the kinetic energy, and it is proportional to c_4. The UV divergences arising in such diagrams must be absorbed in the $1/m^2$ potentials. However, after the computation, we observe that all divergences are only absorbed by the delta-like potential. It is important to mention that the iteration of two or more spin-dependent potentials can give a contribution to $\tilde{D}_d^{(2)}$, associated to a spin-independent potential.

It is interesting to discuss in more detail which of the novel α^2/m^3 potentials computed in Sec. 4.4.2 and Sec. 4.4.4 (we remind that here we use the potentials after using the free equations of motion, i.e. the expressions in Sec. 4.4.4 for the energy-dependent potentials) contribute to the running of $\tilde{D}_d^{(2)}$. The potentials in Eqs. (4.43-4.44) do not contribute to the running of $\tilde{D}_d^{(2)}$: Eq. (4.43) does not because it is proportional to a total derivative, whereas Eq. (4.44) does not because of the following argument: the only possible potential loop that can be constructed with an $\mathcal{O}(\alpha^2/m^3)$ potential is the iteration of it with a Coulomb potential. As a consequence, the α^2/m^3 potential is always applied to an external momentum. When the high loop momentum limit is taken in the integral

[11]A detailed example of the computation of a potential loop is given in Sec. E.2.

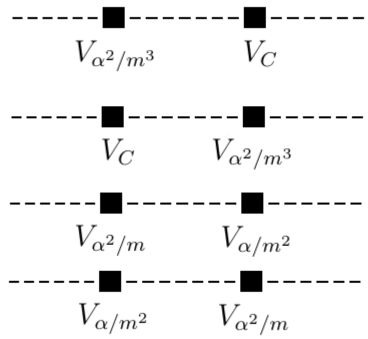


Figure 4.6: Divergent diagrams with one potential loop that contribute to the running of $\tilde{D}_d^{(2)}$ at $\mathcal{O}(\alpha^3)$.

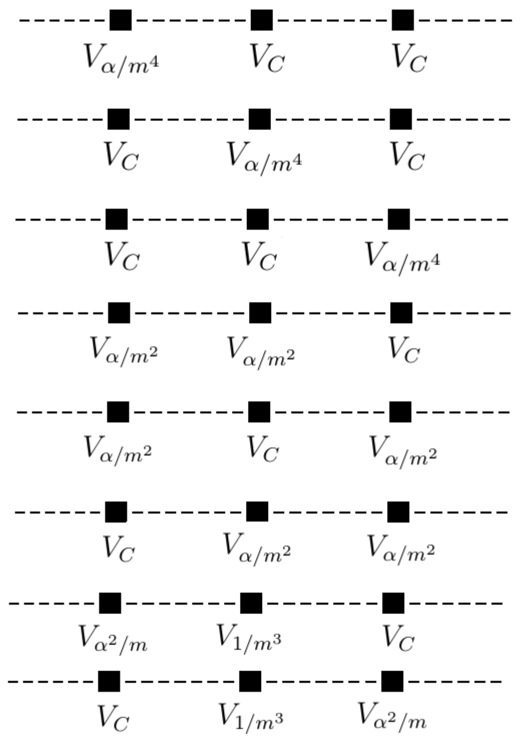


Figure 4.7: Divergent diagrams with two potential loops that contribute to the running of $\tilde{D}_d^{(2)}$ at $\mathcal{O}(\alpha^3)$.

in order to find the UV pole, all these external momenta vanish and all the terms become proportional to $|\mathbf{k}|^{1+2\epsilon}$. After doing so, and summing all the terms, the overall coefficient is zero, explaining the fact that they do not contribute. This argument also applies to $\tilde{V}^{(e,1)}$ and $\tilde{V}^{(f,i)}$ (with $i = 1$ to 6). Contrarily, $\tilde{V}^{(e,2)}$ and $\tilde{V}^{(f,7/8/9)}$ do contribute to the running. Note that $\tilde{V}^{(f,8)}$ and $\tilde{V}^{(f,9)}$ were originally dependent on the energy.

Diagrams with V_{1/m^3} in the extremes of a potential loop, i.e. acting over an external momentum, have not been drawn because they do not produce UV divergences. Similarly, diagrams with V_{1/m^5} do not produce UV divergences. Then, one can easily convince oneself

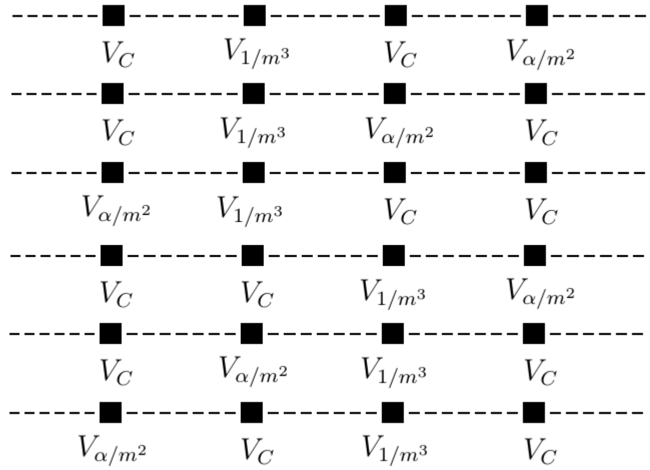


Figure 4.8: Divergent diagrams with three potential loops that contribute to the running of $\tilde{D}_d^{(2)}$ at $\mathcal{O}(\alpha^3)$.

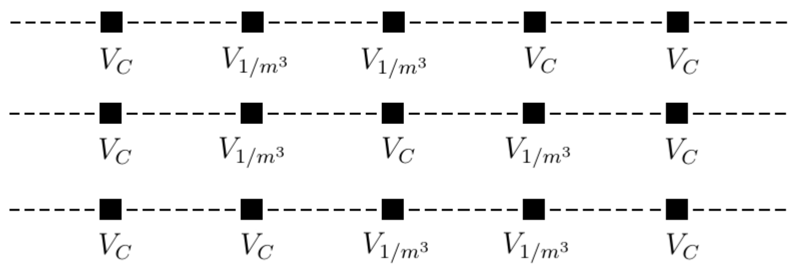


Figure 4.9: Divergent diagrams with four potential loops that contribute to the running of $\tilde{D}_d^{(2)}$ at $\mathcal{O}(\alpha^3)$.

that there are no diagrams with five or more potential loops contributing to the $\mathcal{O}(\alpha^3)$ anomalous dimension of $\tilde{D}_d$. The above discussion exhausts all possible contributions.

The potential RGE, reads

$$
\begin{aligned}
&\nu\frac{d}{d\nu}\delta\tilde{D}^{(2)\,\mathrm{NLL}}_{d,p} = -2C_F^2\alpha_V^2 m_r^3\left(\frac{c_4^{(1)}}{m_1^3}+\frac{c_4^{(2)}}{m_2^3}\right)\tilde{D}^{(2)}_d \\
&+C_F^2\alpha_V\frac{m_r^2}{m_1m_2}\left(\tilde{D}^{(2)\,2}_d-8\tilde{D}^{(2)}_d\tilde{D}^{(2)}_1+12\tilde{D}^{(2)\,2}_1-\frac{5}{6}\tilde{D}^{(2)\,2}_{S_{12}}+\frac{4}{3}\tilde{D}^{(2)\,2}_{S^2}\right) \\
&+2C_F^2\alpha_V^2 m_r^3\left(\frac{c_4^{(1)}}{m_1^3}+\frac{c_4^{(2)}}{m_2^3}\right)\left(4\tilde{D}^{(2)}_1\right) \\
&+C_FC_A\left[2\tilde{D}^{(2)}_1\tilde{D}^{(1)}-\tilde{D}^{(2)}_d\tilde{D}^{(1)}+\tilde{D}^{(1)}\alpha_V m_r m_1 m_2\left(\frac{c_4^{(1)}}{m_1^3}+\frac{c_4^{(2)}}{m_2^3}\right)\right] \\
&+C_F^2\alpha_V^3 m_r^4 m_1 m_2\left(\frac{c_4^{(1)}}{m_1^3}+\frac{c_4^{(2)}}{m_2^3}\right)^2 \\
&+\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}\right)\frac{C_A^2(C_A-2C_F)\alpha_s^3}{2}D^{(2)}_{\mathbf{S}^2,1/r^3} \\
&-2C_F^2\alpha_V^2\alpha m_r^2\left[16m_1m_2\left(\frac{c_{X1}^{(1)}}{m_1^4}+\frac{c_{X1}^{(2)}}{m_2^4}\right)+2\left(\frac{c_4^{(1)}c_k^{(2)}}{m_1^2}+\frac{c_4^{(2)}c_k^{(1)}}{m_2^2}\right)+\frac{c_k^{(1)\,2}c_k^{(2)\,2}}{m_1m_2}\right. \\
&\left.+\left(\frac{c_M^{(1)}c_k^{(2)}}{m_1^2}+\frac{c_M^{(2)}c_k^{(1)}}{m_2^2}\right)\right]+C_F\left(2C_F-\frac{C_A}{2}\right)\alpha^2\alpha_V m_r m_1 m_2\left[\frac{1}{2}\left(\frac{c_{A_1}^{(1)}}{m_1^3}+\frac{c_{A_1}^{(2)}}{m_2^3}\right)\right. \\
&\left.+\frac{1}{4}\left(\frac{c_{A_2}^{(1)}}{m_1^3}+\frac{c_{A_2}^{(2)}}{m_2^3}\right)+\frac{1}{m_1m_2}\left(\frac{c_F^{(1)\,2}c_k^{(2)}}{m_1}+\frac{c_F^{(2)\,2}c_k^{(1)}}{m_2}\right)\right] \\
&-\frac{1}{2}C_FC_A\alpha^2\alpha_V m_r m_1 m_2\left[2\left(\frac{c_4^{(1)}}{m_1^3}+\frac{c_4^{(2)}}{m_2^3}\right)+\left(\frac{c_k^{(1)\,3}}{m_1^3}+\frac{c_k^{(2)\,3}}{m_2^3}\right)-\left(\frac{c_M^{(1)}}{m_1^3}+\frac{c_M^{(2)}}{m_2^3}\right)\right. \\
&\left.+\left(\frac{c_F^{(1)\,2}c_k^{(1)}}{m_1^3}+\frac{c_F^{(2)\,2}c_k^{(2)}}{m_2^3}\right)-\left(\frac{c_F^{(1)}c_S^{(1)}}{m_1^3}+\frac{c_F^{(2)}c_S^{(2)}}{m_2^3}\right)\right] \\
&+\frac{T_F}{N_c}C_F\alpha^2\alpha_V m_r m_1 m_2\left[\left(\frac{c_{A_3}^{(1)}}{m_1^3}+\frac{c_{A_3}^{(2)}}{m_2^3}\right)+\frac{1}{2}\left(\frac{c_{A_4}^{(1)}}{m_1^3}+\frac{c_{A_4}^{(2)}}{m_2^3}\right)\right]\,.
\end{aligned}
\tag{4.78}
$$

The first five lines are generated by potential loops with the α^2/m and α/m^2 potentials, and the $\mathbf{p}^4/m^3$ correction to the kinetic energy (besides the iteration of the Coulomb potential, accounted for by α_V). The sixth line is the term generated by the potential computed in Sec. 4.5.2. The last six lines are generated by potential loops with the α^2/m^3 and α/m^4 potentials (besides the iteration of the Coulomb potential, accounted for again by α_V). The number of potential loops involved in each term is accounted by m_r, since the heavy quarkonium propagator is the only source of explicit powers of m_r and the number of propagators is equal to the number of potential loops. The Wilson coefficients on the right hand side must be understood as having an accuracy such that $\tilde{D}^{(2)\,\mathrm{NLL}}_{d,p}$ is NLL. Note that, for simplicity, we have already used $c_k^{(i)}=1$ [45] in the heavy quarkonium propagator.

In this way we can write it in terms of the reduced mass. If we want to recover all the c_k factors we just must replace $m_r \to m_1 m_2/(c_k^{(1)} m_2 + c_k^{(2)} m_1)$. A part of this equation was already computed in Ref. [67]. The QED limit of several of these terms can also be checked with the computations done in Ref. [64].

If one is interested in writing the counterterm it is enough to know that it scales as $\nu^{4\epsilon}$. Then, since $\delta \tilde{D}_{d,p\,B}^{(2)\,\rm NLL} = \delta \tilde{D}_{d,p\,R}^{(2)\,\rm NLL} + \delta_p \tilde{D}_{d,p}^{(2)\,\rm NLL}$, the anomalous dimension, which we show in Eq. (4.78) is given by

$$\nu \frac{d}{d\nu} \delta \tilde{D}_{d,p\,R}^{(2)\,\rm NLL} = -\nu \frac{d}{d\nu} \delta_p \tilde{D}_{d,p}^{(2)\,\rm NLL} \equiv \gamma_{\delta \tilde{D}_{d,p}^{(2)\,\rm NLL}} \,, \tag{4.79}$$

and the counterterm by

$$\delta_p \tilde{D}_{d,p}^{(2)\,\rm NLL} = -\frac{1}{4} \gamma_{\delta \tilde{D}_{d,p}^{(2)\,\rm NLL}} \frac{1}{\epsilon} \nu^{4\epsilon} \,. \tag{4.80}$$

It is interesting to see that there is a matching scheme dependence of the individual α^2/m^3 and α/m^4 potentials that cancels out in the sum. In Sec. 4.7 we perform a detailed proof of this statement. In the above expression the coefficients c_{A_2}, c_D, c_M and c_{X1} appear. Note that the last two coefficients are dependent on c_D due to reparametrization invariance, so they are gauge-dependent quantities. From Ch. 2 we know that c_{A_2} is also gauge dependent. Such gauge dependence should vanish in the final result, since the Wilson coefficient $\tilde{D}_d^{(2)}$ is directly related with the S-wave heavy quarkonium spectrum, and indeed, it does. This is actually a strong check of the computation. In Eq. (4.78) we can approximate $\alpha_V = \alpha$ since everything is needed with LL accuracy. Then we can show that everything can be written in terms of $\bar{c}_{A_2}$, which is gauge independent (it is an observable in the low energy limit of the Compton scattering; see the discussion in Sec. 2.4 or in Ref. [35]), and the explicit dependence on c_D, c_M, c_{X1} and c_{A_2} disappears[12]. The

[12]More precisely, everything can be written in terms of $\bar{c}_{A_2}$ and a new physical quantity $\bar{c}_{X1} \equiv 64 c_{X1} - c_{A_2} = 5/2 - \bar{c}_{A_2}$. The expression of $\bar{c}_{X1}$ in terms of $\bar{c}_{A_2}$ have been obtained using relations between Wilson coefficients imposed by reparametrization invariance, in particular, the relation $2c_M^{\rm FG} = c_D^{\rm FG} - c_F$, which we found to be not satisfied in the Coulomb gauge. However, since $\bar{c}_{X1}$ is gauge-independent, it can be obtained in the Feynman gauge using $2c_M^{\rm FG} = c_D^{\rm FG} - c_F$, which is satisfied in that gauge. The fact that $2c_M = c_D - c_F$ is not satisfied in the Coulomb gauge implies that the relation $32 c_{X1} = 5/4 - c_F + c_D$ is neither satisfied in that gauge. Indeed, we find that the true gauge independent relation that reparametrization invariance should obtain is $32 c_{X1} = 5/4 - c_F + c_D + c_1^{hl} - 16 d_1^{hl}$. Since in the Feynman gauge $c_1^{hl\,\rm FG} - 16 d_1^{hl\,\rm FG} = 0$, then $32 c_{X1}^{\rm FG} = 5/4 - c_F + c_D^{\rm FG}$ is satisfied. A proof of this statement by explicit calculation would reinforce our argument. However such a proof is left for future research.

resulting expression reads

$$
\begin{aligned}
&\nu\frac{d}{d\nu}\delta\tilde{D}^{(2)\,\rm NLL}_{d,p} = -2c_4C_F^2\alpha_V^2m_r^3\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)\tilde{D}^{(2)}_d\\
&+C_F^2\alpha_{V_s}\frac{m_r^2}{m_1m_2}\left(\tilde{D}^{(2)\,2}_d-8\tilde{D}^{(2)}_d\tilde{D}^{(2)}_1+12\tilde{D}^{(2)\,2}_1-\frac{5}{6}\tilde{D}^{(2)\,2}_{S_{12}}+\frac{4}{3}\tilde{D}^{(2)\,2}_{S^2}\right)\\
&+2c_4C_F^2\alpha_V^2m_r^3\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)\left(4\tilde{D}^{(2)}_1\right)\\
&+C_FC_A\left[2\tilde{D}^{(2)}_1\tilde{D}^{(1)}-\tilde{D}^{(2)}_d\tilde{D}^{(1)}+c_4\tilde{D}^{(1)}\alpha_Vm_rm_1m_2\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)\right]\\
&+c_4^2C_F^2\alpha_V^3m_r^4m_1m_2\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)^2\\
&+\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}\right)\frac{C_A^2(C_A-2C_F)\alpha^3}{2}D^{(2)}_{\mathbf{S}^2,1/r^3}\\
&-2C_F^2\alpha^3m_r^2\left[\frac{5}{8}m_1m_2\left(\frac{1}{m_1^4}+\frac{1}{m_2^4}\right)+2\left(\frac{c_4^{(1)}}{m_1^2}+\frac{c_4^{(2)}}{m_2^2}\right)+\frac{1}{m_1m_2}\right]\\
&+C_F\left(2C_F-\frac{C_A}{2}\right)\alpha^3m_rm_1m_2\left[\frac{1}{2}\left(\frac{c^{(1)}_{A_1}}{m_1^3}+\frac{c^{(2)}_{A_1}}{m_2^3}\right)+\frac{1}{4}\left(\frac{\bar{c}^{(1)}_{A_2}}{m_1^3}+\frac{\bar{c}^{(2)}_{A_2}}{m_2^3}\right)\right.\\
&\left.+\frac{1}{m_1m_2}\left(\frac{c_F^{(1)\,2}}{m_1}+\frac{c_F^{(2)\,2}}{m_2}\right)\right]-\frac{1}{2}C_FC_A\alpha^3m_rm_1m_2\left[2\left(\frac{c_4^{(1)}}{m_1^3}+\frac{c_4^{(2)}}{m_2^3}\right)+\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)\right.\\
&\left.+\left(\frac{c_F^{(1)\,2}}{m_1^3}+\frac{c_F^{(2)\,2}}{m_2^3}\right)-\left(\frac{c_F^{(1)}c_S^{(1)}}{m_1^3}+\frac{c_F^{(2)}c_S^{(2)}}{m_2^3}\right)\right]\\
&-\left(C_F-\frac{C_A}{2}\right)C_F\alpha^3m_rm_1m_2\left[\left(\frac{c^{(1)}_{A_3}}{m_1^3}+\frac{c^{(2)}_{A_3}}{m_2^3}\right)+\frac{1}{2}\left(\frac{c^{(1)}_{A_4}}{m_1^3}+\frac{c^{(2)}_{A_4}}{m_2^3}\right)\right], \qquad (4.81)
\end{aligned}
$$

where we have used Eq. (A.23) in order to write the equation above in a explicitly gauge-independent way. Note that, for simplicity, we have used $c_k^{(i)}=1$ [45].

From this result one may think that c_{A_3} and c_{A_4} contribute to the Abelian case. Nevertheless, the LO matching condition is zero for these Wilson coefficients, and all the running vanishes in the Abelian limit. Therefore, there is no contradiction with the pure QED case.

In order to solve Eq. (4.81), we need to introduce the D's, the Wilson coefficients of the potentials. The necessary expressions can be found in Sec. 4.3. Note that in those expressions we have already correlated the ultrasoft factorization scale ν_{us} with ν and ν_h using $\nu_{us}=\nu^2/\nu_h$. We also do so in Eq. (4.77), where we also set $1/r=\nu$, consistent with the precision of our computation. This correlation of scales was first introduced and motivated in Ref. [68]. We also write the RGE in terms of z. Schematically, if we write Eq. (4.81) as

$$
\nu\frac{d}{d\nu}\delta\tilde{D}^{(2)\,\rm NLL}_{d,p}=F(\alpha(\nu))\,, \qquad (4.82)
$$

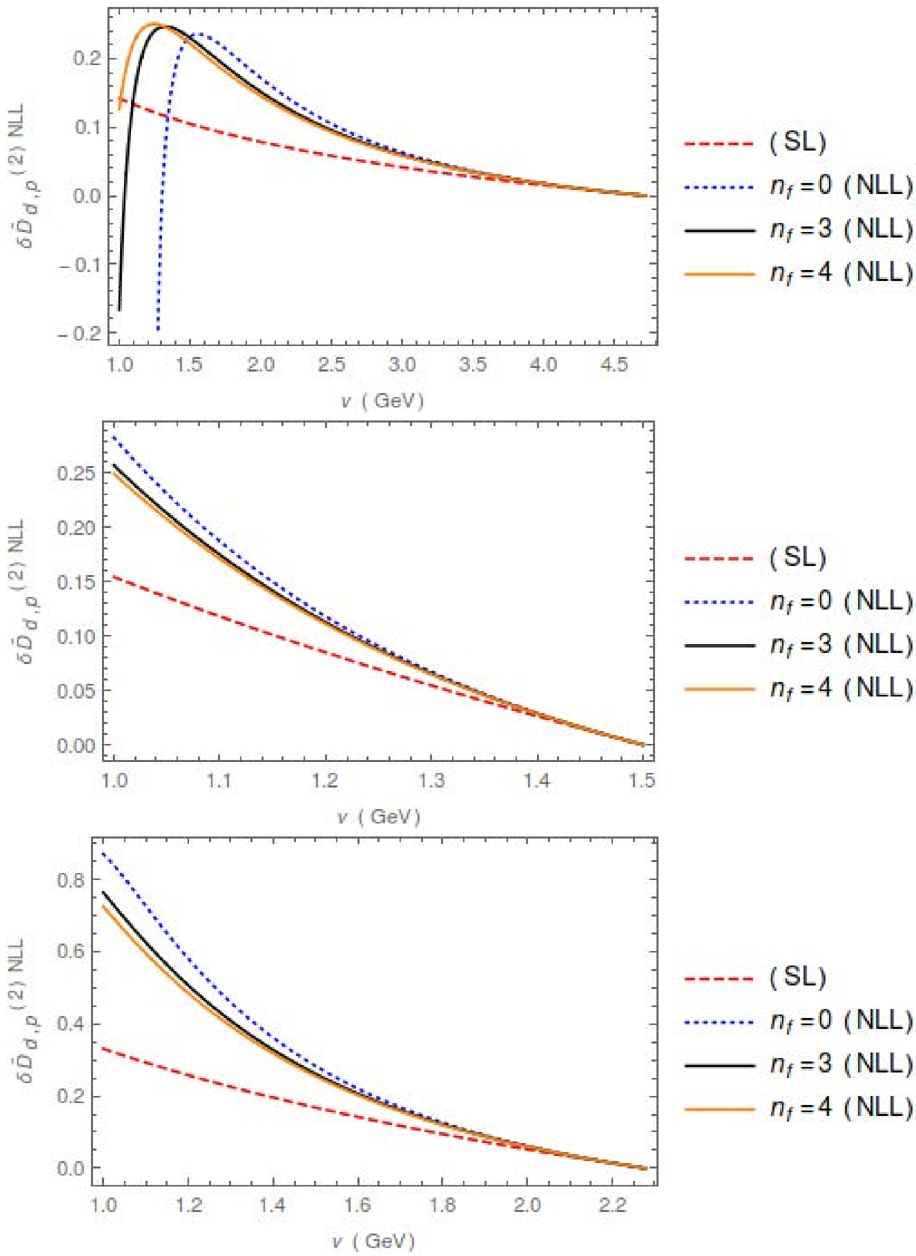


Figure 4.10: Plot of $\delta\tilde{D}_{d,p}^{(2)NLL}$ for different values of n_f (0,3,4) and in the single log (SL) approximation (in this case only with $n_f = 3$). **Upper panel:** Plot for bottomonium with $\nu_h = m_b$. **Middle panel:** Plot for charmonium with $\nu_h = m_c$. **Lower panel:** Plot for B_c with $\nu_h = 2m_b m_c/(m_b + m_c)$.

then, in the one-loop running coupling approximation, and in terms of a derivative with respect to α, it can be written as

$$\frac{d}{d\alpha}\delta\tilde{D}_{d,p}^{(2)\,\mathrm{NLL}} = -\frac{2\pi}{\beta_0\alpha^2}F(\alpha)\,, \tag{4.83}$$

and finally, in terms of z as

$$\frac{d}{dz}\delta\tilde{D}_{d,p}^{(2)\,\mathrm{NLL}} = -\frac{2\pi}{\alpha(\nu_h)}\frac{F(z)}{z^{1+\beta_0}}\,. \tag{4.84}$$

Which is the equation we solve. For $n_f = 3$ or 4 it is not possible to get an analytic solution of the RGE, more specifically, for the coefficients multiplying the different z functions. Note that this comes back to the fact that the polarizability Wilson coefficients c_{A_1}, c_{A_2}, c_{A_3} and c_{A_4} cannot be computed analytically. On top of that the resulting expressions are too long. Therefore, we only explicitly show the analytic result for $n_f = 0$. It reads

$$\delta\tilde{D}_{d,p}^{(2)\,\mathrm{NLL}} = \pi\alpha(\nu_h)^2\Bigg[-\frac{1}{48656036000C_A^3m_1^5m_2^5}\Bigg(4561503375C_A^4m_1^4m_2^4(m_1^2+m_2^2)$$
$$-1754688000C_F^4m_1^2m_2^2(18m_1^4+142m_1^3m_2+101m_1^2m_2^2+142m_1m_2^3+18m_2^4)m_r^2$$
$$-14250C_A^3C_Fm_1^3m_2^3\Big(480m_1^3(1924m_2-5885m_r)+481m_1^2m_2(720m_2-1331m_r)$$
$$+481m_1m_2^2(1920m_2-1331m_r)-2824800m_2^3m_r\Big)-98800C_AC_F^3m_1m_2$$
$$\times\Big(-293040m_2^5m_r^3-222m_1m_2^4m_r^2(691m_2+5720m_r)+m_1^2m_2^3m_r(-144320m_2^2$$
$$+2370183m_2m_r-293040m_r^2)+2m_1^5(497280m_2^3-72160m_2^2m_r-76701m_2m_r^2$$
$$-146520m_r^3)+111m_1^4m_2(15680m_2^3+21353m_2m_r^2-11440m_r^3)+444m_1^3m_2^2(2240m_2^3$$
$$-1831m_2m_r^2-660m_r^3)\Big)-4C_A^2C_F^2\Big(-2741700m_1^2m_2^5(5109m_2-1100m_r)m_r^2$$
$$-5767851375m_1m_2^6m_r^3+3317457000m_2^6m_r^4-25m_1^3m_2^3m_r(298447930m_2^3$$
$$+693732351m_2^2m_r+230714055m_2m_r^2-265396560m_r^3)+27417m_1^4m_2^3(380000m_2^3$$
$$+332750m_2^2m_r-922808m_2m_r^2-210375m_r^3)+685425m_1^5m_2^2(9440m_2^3+13310m_2^2m_r$$
$$-25303m_2m_r^2+4400m_r^3)+25m_1^6(416738400m_2^4-298447930m_2^3m_r-560293812m_2^2m_r^2$$
$$-230714055m_2m_r^3+132698280m_r^4)\Big)\Bigg)+\frac{1}{1375C_A^2m_1^3m_2^3}\Bigg(6C_F^2\Big(100C_F(2m_1^4-3m_1^3m_2$$
$$+4m_1^2m_2^2-3m_1m_2^3+2m_2^4)+C_A(200m_1^4+75m_1^3m_2+334m_1^2m_2^2+75m_1m_2^3$$
$$+200m_2^4)\Big)m_r^2z^{5C_A/3}\Bigg)-\frac{3C_F^2(25m_1^4-32m_1^2m_2^2+25m_2^4)m_r^2z^{10C_A/3}}{500C_Am_1^3m_2^3}$$
$$+\frac{24C_F(43C_A^2-66C_AC_F-40C_F^2)(m_1^3+m_2^3)m_rz^{11C_A/3}}{605C_A^2m_1^2m_2^2}-\frac{1}{1331C_A^3m_1^4m_2^4}6C_F$$
$$\times\Bigg(32C_F^3m_1m_2(2m_1^4+36m_1^3m_2+19m_1^2m_2^2+36m_1m_2^3+2m_2^4)m_r^2+11C_A^3m_1^2m_2^2\Big(3m_1^2m_2^2$$
$$+8m_1m_2^3+8m_1^3(m_2-m_r)-8m_2^3m_r\Big)+8C_AC_F^2m_1m_2\Big(132m_1m_2^3m_r^2-m_2^3m_r^2(8m_2$$
$$+77m_r)+8m_1^4(6m_2^2-m_r^2)+8m_1^2(6m_2^4-7m_2^2m_r^2)+m_1^3(96m_2^3+132m_2m_r^2-77m_r^3)\Big)$$
$$+2C_A^2C_F\Big(44m_2^5m_r^3-m_1m_2^4m_r^2(64m_2+55m_r)+4m_1^2m_2^3m_r(-22m_2^2+32m_2m_r+11m_r^2)$$
$$+m_1^4m_2(38m_2^3+128m_2m_r^2-55m_r^3)+m_1^3m_2^2(96m_2^3-71m_2m_r^2+44m_r^3)+m_1^5(96m_2^3$$
$$-88m_2^2m_r-64m_2m_r^2+44m_r^3)\Big)\Bigg)z^{11C_A/3}-\frac{99(3C_A-8C_F)C_F(m_1^3+m_2^3)m_rz^{13C_A/3}}{442C_Am_1^2m_2^2}$$
$$+\frac{(C_A-2C_F)C_F(m_1^3+m_2^3)m_rz^{16C_A/3}}{10C_Am_1^2m_2^2}$$
$$+\frac{(C_A-4C_F)C_F(11m_1^3+3m_1^2m_2+3m_1m_2^2+11m_2^3)m_rz^{16C_A/3}}{16C_Am_1^2m_2^2}+\frac{3}{352C_A^2m_1^4m_2^4}$$
$$\times\Big(11C_A^3m_1^3m_2^3(m_1^2+m_2^2)-22C_A^2C_Fm_1^2m_2^2(m_1^3+m_2^3)m_r-16C_F^3m_1m_2(2m_1^4$$
$$-3m_1^3m_2+4m_1^2m_2^2-3m_1m_2^3+2m_2^4)m_r^2-4C_AC_F^2(m_1^2+m_2^2)m_r^2\Big(-19m_1^2m_2^2+8m_1m_2^3$$

$$+m_1^3(8m_2-11m_r)-11m_2^3m_r\Big)\Big)z^{16C_A/3}+\frac{6C_F^2(m_1^3+m_2^3)m_rz^{19C_A/3}}{19C_Am_1^2m_2^2}$$
$$+\frac{3C_F(51C_A^2-54C_AC_F-224C_F^2)(m_1^3+m_2^3)m_rz^{22C_A/3}}{121C_A^2m_1^2m_2^2}$$
$$+\frac{1}{5324C_A^3m_1^5m_2^5}3C_F\Big(-64C_F^3m_1^2m_2^2(10m_1^4-2m_1^3m_2+25m_1^2m_2^2-2m_1m_2^3+10m_2^4)m_r^2$$
$$+2C_A^3m_1^3m_2^3\Big(42m_1^2m_2^2+112m_1m_2^3+209m_2^3m_r+m_1^3(112m_2-209m_r)\Big)$$
$$-32C_AC_F^2m_1m_2\Big(-33m_2^5m_r^3+m_1m_2^4m_r^2(32m_2+11m_r)-m_1^2m_2^3m_r^2(47m_2+33m_r)$$
$$+m_1^5(16m_2^3+32m_2m_r^2-33m_r^3)+m_1^3m_2^2(16m_2^3+87m_2m_r^2-33m_r^3)+m_1^4m_2(4m_2^3$$
$$-47m_2m_r^2+11m_r^3)\Big)+C_A^2C_F\Big(m_1^2m_2^5(221m_2-1320m_r)m_r^2+880m_1m_2^6m_r^3-484m_2^6m_r^4$$
$$+8m_1^3m_2^3m_r(44m_2^3+356m_2^2m_r+110m_2m_r^2-121m_r^3)-8m_1^5m_2^2(42m_2^3-356m_2m_r^2$$
$$+165m_r^3)+4m_1^4m_2^3(4m_2^3+149m_2m_r^2+220m_r^3)+m_1^6(16m_2^4+352m_2^3m_r$$
$$+221m_2^2m_r^2+880m_2m_r^3-484m_r^4)\Big)\Big)z^{22C_A/3}-\frac{1}{11949113C_A^2m_1^2m_2^2}\Big(4(3C_A-10C_F)C_F$$
$$\times\Big((2691765+51647\sqrt{157})C_A+272(10205-269\sqrt{157})C_F\Big)(m_1^3+m_2^3)m_r$$
$$\times z^{-\frac{1}{12}(-71+\sqrt{157})C_A}\Big)+\frac{1}{11949113C_A^2m_1^2m_2^2}\Big(4(3C_A-10C_F)C_F\Big((-2691765$$
$$+51647\sqrt{157})C_A-272(10205+269\sqrt{157})C_F\Big)(m_1^3+m_2^3)m_rz^{\frac{1}{12}(71+\sqrt{157})C_A}\Big)$$
$$+\frac{1}{32m_1m_2}\Big(3(C_A-2C_F)(m_1^2+m_2^2)\,{}_2F_1(-6/11,1;27/11;-1)\Big)$$
$$-\frac{1}{2^{49/11}m_1m_2(2-z^{11C_A/3})^{6/11}}\Big(3(C_A-2C_F)(m_1^2+m_2^2)z^{22C_A/3}(-1+2z^{-11C_A/3})^{6/11}$$
$$\times\,{}_2F_1(-6/11,16/11;27/11;z^{11C_A/3}/2)\Big)-\frac{1}{18C_A^2m_1^3m_2^3}\Big(C_F(m_1^2+m_2^2)$$
$$\times\Big(C_A^2m_1^2m_2^2+2C_F^2(m_1+m_2)^2m_r^2+C_AC_F\big(2m_1m_2m_r^2+m_2^2m_r^2+m_1^2(2m_2^2+m_r^2)\big)\Big)$$
$$\times\,{}_2F_1(1,1;38/11;-1)\Big)+\frac{1}{36C_A^2m_1^3m_2^3}\Big(C_F(C_A+2C_F)(m_1^2+m_2^2)$$
$$\times\Big(C_Am_1^2m_2^2+C_F(m_1+m_2)^2m_r^2\Big)z^{9C_A}\,{}_2F_1(1,27/11;38/11;z^{11C_A/3}/2)\Big)$$
$$-\frac{1}{3025C_A^2m_1^3m_2^3}8C_F^2\Big(100C_F^2(2m_1^2-3m_1m_2+2m_2^2)^2+50C_AC_F(16m_1^4-18m_1^3m_2$$
$$+23m_1^2m_2^2-18m_1m_2^3+16m_2^4)+C_A^2(400m_1^4+300m_1^3m_2+1673m_1^2m_2^2+300m_1m_2^3$$
$$+400m_2^4)\Big)m_r^2\ln(z)-\frac{96C_F(m_1+m_2)}{1331C_A^3m_1^4m_2^4}\Big(-11C_A^3m_1^2m_2^2(m_1^2-m_1m_2+m_2^2)m_r$$

$$
\begin{aligned}
&+24C_F^3 m_1 m_2 (m_1+m_2)^3 m_r^2 + 2C_A C_F^2 (m_1+m_2)\Big(2m_1^2 m_2^2 m_r^2 + 12 m_1 m_2^3 m_r^2 - 11 m_2^3 m_r^3 \\
&+m_1^3(24m_2^3 + 12 m_2 m_r^2 - 11 m_r^3)\Big) + C_A^2 C_F\Big(- 22 m_1^2 m_2^4 m_r - 11 m_1 m_2^3 m_r^3 - 11 m_2^4 m_r^3 \\
&+m_1^4(13m_2^3 - 22 m_2^2 m_r - 11 m_r^3) + m_1^3(13 m_2^4 + 22 m_2^3 m_r - 11 m_2 m_r^3)\Big)\Big) \ln(2 - z^{11C_A/3}) \\
&-\frac{1}{1331 C_A^3 m_1^3 m_2^3} 48 C_F \Big(C_A^3 m_1^2 m_2^2 (8m_1^2 + 3 m_1 m_2 + 8 m_2^2) - 56 C_F^3 m_1 m_2 (m_1+m_2)^2 m_r^2 \\
&+C_A^2 C_F \Big(19 m_1 m_2^3 m_r^2 + 8 m_2^4 m_r^2 + 19 m_1^3 m_2(-2m_2^2 + m_r^2) + 8 m_1^4 (m_2^2 + m_r^2) + m_1^2 (8 m_2^4 \\
&+22 m_2^2 m_r^2)\Big) - 2 C_A C_F^2 \Big(11 m_1 m_2^3 m_r^2 - 4 m_2^4 m_r^2 + m_1^4 (8 m_2^2 - 4 m_r^2) + 11 m_1^3 m_2 (4 m_2^2 + m_r^2) \\
&+m_1^2(8 m_2^4 + 30 m_2^2 m_r^2)\Big)\Big) z^{11C_A/3} \ln(2 - z^{11C_A/3}) + \frac{1}{22 C_A^2 m_1^3 m_2^3}\Big(3 C_F (C_A + 2C_F) \\
&\times (m_1^2 + m_2^2)\Big(C_A m_1^2 m_2^2 + C_F (m_1+m_2)^2 m_r^2\Big) z^{16C_A/3} \ln(2 - z^{11C_A/3})\Big) \\
&+\frac{24 C_F}{1331 C_A^3 m_1^4 m_2^4}\Big(8 C_F^3 m_1 m_2 (m_1+m_2)^2 (3m_1^2 - m_1 m_2 + 3 m_2^2) m_r^2 + C_A^3 m_1^2 m_2^2 \Big(3 m_1^2 m_2^2 \\
&+8 m_1 m_2^3 + m_1^3 (8 m_2 - 11 m_r) - 11 m_2^3 m_r\Big) + 2 C_A C_F^2 \Big(2 m_1 m_2^4 (8 m_2 - 11 m_r) m_r^2 \\
&+m_1^2 m_2^3 (15 m_2 - 11 m_r) m_r^2 - 11 m_2^5 m_r^3 + m_1^4 m_2 (4 m_2^3 + 15 m_2 m_r^2 - 22 m_r^3) \\
&+m_1^3 m_2^2 (16 m_2^3 - 2 m_2 m_r^2 - 11 m_r^3) + m_1^5 (16 m_2^3 + 16 m_2 m_r^2 - 11 m_r^3)\Big) \\
&+C_A^2 C_F \Big(2 m_1 m_2^4 (4 m_2 - 11 m_r) m_r^2 - 11 m_2^5 m_r^3 + m_1^2 m_2^3 m_r (-22 m_2^2 + 19 m_2 m_r - 11 m_r^2) \\
&+m_1^4 m_2 (-12 m_2^3 + 19 m_2 m_r^2 - 22 m_r^3) + m_1^5 (21 m_2^3 - 22 m_2^2 m_r + 8 m_2 m_r^2 - 11 m_r^3) \\
&+m_1^3 m_2^2 (21 m_2^3 + 22 m_2 m_r^2 - 11 m_r^3)\Big)\Big) z^{22C_A/3} \ln(2 - z^{11C_A/3}) \\
&+\frac{1}{1331 C_A^3 m_1^3 m_2^3}\Big(768 C_F^2 (m_1+m_2)^2 \Big(C_A^2 m_1^2 m_2^2 + C_F^2 (m_1+m_2)^2 m_r^2 + C_A C_F \big(2 m_1 m_2 m_r^2 \\
&+m_2^2 m_r^2 + m_1^2 (2 m_2^2 + m_r^2)\big)\Big) \ln^2(2 - z^{11C_A/3})\Big) - \frac{1}{1331 C_A^3 m_1^3 m_2^3}\Big(192 C_F^2 (m_1+m_2)^2 \\
&\times \Big(C_A^2 m_1^2 m_2^2 + C_F^2 (m_1+m_2)^2 m_r^2 + C_A C_F \big(2 m_1 m_2 m_r^2 + m_2^2 m_r^2 + m_1^2 (2 m_2^2 + m_r^2)\big)\Big) \\
&\times z^{22C_A/3} \ln^2(2 - z^{11C_A/3})\Big)\Big] \,. \qquad (4.85)
\end{aligned}
$$

Finally, in Fig. 4.10, we give the numerical evaluation $\delta \tilde{D}_{d,p}^{(2)\,\mathrm{NLL}}$ for different values of n_f. The contribution is sizable, and definitely more important than the soft and ultrasoft running.

4.5.4 Potential running, spin-dependent delta potential

Even though it is not relevant for this thesis, we profit to present the potential RGE of $\tilde{D}^{(2)}_{S^2}$ in the basis we use in the present work, which is different from the basis used in Ref. [14]. The final solution is nevertheless the same:

$$\begin{aligned}
\frac{d\tilde{D}^{(2)}_{S^2}}{d\ln\nu} &= -2m_r^3\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)C_F^2\alpha_V^2\tilde{D}^{(2)}_{S^2}+\frac{m_r^2}{m_1m_2}C_F^2\alpha_V\left(2\tilde{D}^{(2)}_d\tilde{D}^{(2)}_{S^2}\right.\\
&\left.-\frac{4}{3}\left(\tilde{D}^{(2)}_{S^2}\right)^2-8\tilde{D}^{(2)}_{S^2}\tilde{D}^{(2)}_1-\frac{5}{12}\left(\tilde{D}^{(2)}_{S_{12}}\right)^2\right)-C_AC_F\tilde{D}^{(1)}\tilde{D}^{(2)}_{S^2}\\
&+\frac{1}{2}\alpha^3C_F^2m_r^2\left(\frac{c^{(1)}_{p'p}}{m_1^2}c_F^{(2)}+c_F^{(1)}\frac{c^{(2)}_{p'p}}{m_2^2}\right)-2\alpha^3C_F^2\frac{m_r^2}{m_1m_2}c_F^{(1)}c_F^{(2)}\\
&-\frac{1}{4}\alpha^3C_F^2\frac{m_r^2}{m_1m_2}c_S^{(1)}c_S^{(2)}-\alpha^3C_F^2m_r^2\left(c_F^{(1)}\frac{c_S^{(2)}}{m_2^2}+\frac{c_S^{(1)}}{m_1^2}c_F^{(2)}\right)\\
&+\frac{1}{2}\alpha^3C_F(4C_F-C_A)m_r\left(c_F^{(1)}\frac{c_S^{(2)}}{m_2}+\frac{c_S^{(1)}}{m_1}c_F^{(2)}\right)\\
&+\frac{1}{8}\alpha^3C_FC_Am_r\left(c_F^{(1)}\frac{c_F^{(2)\,2}}{m_2}+\frac{c_F^{(1)\,2}}{m_1}c_F^{(2)}\right)-\frac{C_A^2(C_A-2C_F)\alpha^3}{2}D^{(2)}_{\mathbf{S}^2,1/r^3}\,. \quad (4.86)
\end{aligned}$$

This equation has slightly changed with respect to Eq. (36) in Ref. [14] because of the change in the basis of potentials[13]. In particular, the term proportional to $\tilde{D}^{(2)}_{S^2}$ changes to compensate for the fact that $\tilde{D}^{(2)}_d$ is also different, so that the result is the same.

4.6 N^{3}LL heavy quarkonium mass

For the organization of the computation and the presentation of the results we closely follow the notation of Ref. [6]. In particular, we split the total RG improved potential in the following way:

$$V_s^{\mathrm{N}^i\mathrm{LL}}(\nu_h,\nu)=V_s^{\mathrm{N}^i\mathrm{LO}}(\nu)+\delta V_s^{\mathrm{N}^i\mathrm{LL}}(\nu_h,\nu)\,, \quad (4.89)$$

where $V_s^{\mathrm{N}^i\mathrm{LO}}(\nu)\equiv V_s^{\mathrm{N}^i\mathrm{LL}}(\nu_h=\nu,\nu)$. We then split the total energy into the N^{3}LO result and the new contribution associated to the resummation of logarithms. The spectrum at N^{3}LO was obtained in Ref. [69] for the ground state, in Refs. [70, 71] for S-wave states,

[13] The change in the basis of potentials implies the following relations between the Wilson coefficients in the two basis

$$\tilde{D}'^{(2)}_{S^2}=\tilde{D}^{(2)}_{S^2}\,, \quad (4.87)$$

$$\tilde{D}'^{(2)}_d=\tilde{D}^{(2)}_d-2\tilde{D}^{(2)}_{S^2}\,, \quad (4.88)$$

where the Wilson coefficients with the primes refer to the basis of potentials given in Ref. [14].

and in Refs. [72, 73] for general quantum numbers but for the equal mass case. The result for the unequal mass case was obtained in Ref. [11].

From the RG improved potential one obtains the $\mathrm{N}^i\mathrm{LL}$ shift to the energy levels

$$E_{\mathrm{N}^i\mathrm{LL}}(\nu_h,\nu) = E_{\mathrm{N}^i\mathrm{LO}}(\nu) + \delta E_{\mathrm{RG}}(\nu_h,\nu)\Big|_{\mathrm{N}^i\mathrm{LL}}\,, \tag{4.90}$$

where the explicit expression for $E_{\mathrm{N}^i\mathrm{LO}}(\nu)$ can be found in Ref. [11], and in a different spin basis in the Appendix B of Ref. [6].

The LO and NLO energy levels are unaffected by the RG improvement, i.e.

$$\delta E_{\mathrm{RG}}\Big|_{\mathrm{LL}} = \delta E_{\mathrm{RG}}\Big|_{\mathrm{NLL}} = 0. \tag{4.91}$$

We now determine the variations with respect to the $\mathrm{N}^2\mathrm{LO}$ and $\mathrm{N}^3\mathrm{LO}$ results. We are here interested in the corrections associated to the resummation of logarithms. In order to obtain the spectrum at $\mathrm{N}^2\mathrm{LL}$ and $\mathrm{N}^3\mathrm{LL}$ we need to add the following energy shift to the $\mathrm{N}^2\mathrm{LO}$ and $\mathrm{N}^3\mathrm{LO}$ spectrum:

$$\delta E_{\mathrm{RG}}\Big|_{\mathrm{N}^2\mathrm{LL}} = \langle nl|\delta V_s^{\mathrm{N}^2\mathrm{LL}}(\nu_h,\nu)|nl\rangle\,, \tag{4.92}$$

which was computed in Ref. [3], and

$$\begin{aligned}\delta E_{nl,\mathrm{RG}}\Big|_{\mathrm{N}^3\mathrm{LL}} &= \langle nl|\delta V_s^{\mathrm{N}^3\mathrm{LL}}(\nu_h,\nu)|nl\rangle + 2\langle nl|V_1\frac{1}{(E_n^C-h)'}\delta V_s^{\mathrm{N}^2\mathrm{LL}}(\nu_h,\nu)|nl\rangle \\ &\quad + [\delta E_{\mathrm{US}}(\nu,\nu_{us}) - \delta E_{\mathrm{US}}(\nu,\nu)]\end{aligned} \tag{4.93}$$

Note that $\langle nl|\delta V_s^{\mathrm{N}^3\mathrm{LL}}(\nu_h,\nu)|nl\rangle$ includes $\langle nl|\delta V_s^{\mathrm{N}^2\mathrm{LL}}(\nu_h,\nu)|nl\rangle$.

$\delta E_{nl,\mathrm{RG}}\Big|_{\mathrm{N}^3\mathrm{LL}}$ was computed for $l\neq 0$ in Ref. [6], and for $l=0$, $s=1$ in Refs. [12, 13]. To have the complete result for S-wave states, one needs to compute and add the new term for $l=0$

$$\delta E^{\mathrm{new}}_{n0,\mathrm{RG}}\Big|_{\mathrm{N}^3\mathrm{LL}} = \langle n0|[\delta V_r^{\mathrm{N}^3\mathrm{LL}} - \delta V_r^{\mathrm{N}^2\mathrm{LL}}](\nu_h,\nu)|n0\rangle + 2\langle n0|V_1\frac{1}{(E_n^C-h)'}\delta V_r^{\mathrm{N}^2\mathrm{LL}}(\nu_h,\nu)|n0\rangle\,, \tag{4.94}$$

where

$$V_1 = -\frac{C_F\alpha}{r}\frac{\alpha}{4\pi}(2\beta_0\ln(\nu r e^{\gamma_E}) + a_1)\,, \tag{4.95}$$

and $\delta V_r^{\mathrm{N}^i\mathrm{LL}}$ is the delta-related potential contribution to $\delta V_s^{\mathrm{N}^i\mathrm{LL}}$. The new term generated by $\tilde{D}_d^{(2)}$ reads

$$
\begin{aligned}
\delta E^{\rm new}_{n0,{\rm RG}}\Big|_{\rm N^3LL} &= \frac{1}{m_1 m_2}\pi C_F[\delta\tilde{D}_d^{(2)\,{\rm NLL}}(\nu_h,\nu)-\delta\tilde{D}_d^{(2)\,{\rm NLL}}(\nu,\nu)]\frac{(m_r C_F\alpha)^3}{\pi n^3}\delta_{l0}\\
&+2\frac{1}{m_1 m_2}\pi C_F[\tilde{D}_d^{(2)\,{\rm LL}}(\nu_h,\nu)-\tilde{D}_d^{(2)\,{\rm LL}}(\nu,\nu)]\left[-\frac{\alpha}{4\pi}\right]\frac{(m_r C_F\alpha)^3}{\pi n^3}\delta_{l0}\\
&\times\left\{2\beta_0\left[\frac{1}{2}+\frac{\pi^2 n}{6}-n\Sigma_2^{(k)}(n,0)-\frac{3}{2}\ln\left(\frac{na\nu}{2}\right)-\frac{3}{2}S_1(n)\right]-\frac{3}{2}a_1\right\}\\
&+\frac{\pi C_F}{m_1 m_2}\left[-\frac{1}{4\pi}\right]\frac{2(m_r C_F\alpha)^3}{n^3}\left(\ln\frac{na\nu}{2}-S_1(n)-\frac{n-1}{2n}\right)2\delta_{l0}\\
&\times\left[\left(k\frac{d}{dk}\tilde{D}_d^{(2)}\right)\Bigg|^{\rm LL}_{k=\nu}(\nu_h;\nu)-\left(k\frac{d}{dk}\tilde{D}_d^{(2)}\right)\Bigg|^{\rm LL}_{k=\nu}(\nu;\nu)\right], \qquad (4.96)
\end{aligned}
$$

where $\delta\tilde{D}_d^{(2)\,{\rm NLL}}$ is defined in Eq. (4.39). The first three lines are generated by the term proportional to $\delta^{(3)}(\mathbf{r})$. The last two lines are the contribution to the S-wave ($l=0$) energy from the last term of Eq. (4.24). The contribution to the P-wave energy, proportional to the $1-\delta_{l0}$ term, is already included in Ref. [6]. Therefore, we do not include it in the expression above. To this contribution we have explicitly subtracted the fixed order contribution already included in the N^{3}LO result.

By adding $\delta E^{\rm new}_{n0,{\rm RG}}\Big|_{\rm N^3LL}$ to the results computed in these references[14] one obtains the complete result.

4.7 Matching scheme independence

In this section, we study the matching scheme dependence of Eq. (4.81). On the one hand, the potentials obtained in Sec. 4.4 are computed in the Coulomb gauge. On the other hand, the potential RGE obtained in Sec. 4.5 is generated by potential loops, which are independent of the gauge/matching scheme. The matching scheme dependence of Eq. (4.81) is generated implicitly by the Wilson coefficients of the potentials, such as $\tilde{D}_d^{(2)}$ or $\tilde{D}^{(1)}$, and also explicitly, since we put the explicit expressions for the $1/m^3$ and $1/m^4$ potentials obtained in the Coulomb gauge. This last point makes that Eq. (4.81) can only be used in the Coulomb gauge matching scheme, though with not much effort it could be written in terms of general structures of the $1/m^3$ and $1/m^4$ potentials that would make it also useful for a computation in a general matching scheme. Nevertheless, since we do not know the $1/m^3$ and $1/m^4$ potentials in other matching schemes, we refrain from doing so in this thesis. Still, it is worth studying how the differences between different matching schemes show up in the terms where all the matching scheme dependence is encoded in the D's (the first four lines in Eq. (4.81)).

[14]Note though that one should change $2=S(S+1)$ by $S(S+1)-3/2$ in the result obtained in Refs. [12, 13] to account for the change of operator basis to the one we use here. One should also change from the on-shell to the Coulomb basis of potentials in Ref. [6] (this is very easy to do, as the ultrasoft running is not affected by this transformation).

At $\mathcal{O}(m\alpha^4)$ the Coulomb and Feynman matching schemes produce the same potential but the on-shell scheme does not. At this order, the relation between the Wilson coefficients of the delta-like and the $1/m$ potentials in the off-shell Coulomb gauge (equal to the Feynman gauge at this order) and the on-shell schemes is given by

$$\tilde{D}^{(2)}_{d,\text{CG}} = \tilde{D}^{(2)}_{d,\text{ON}} + \alpha(\nu)\,, \tag{4.97}$$

$$\tilde{D}^{(1)}_{\text{CG}} = \tilde{D}^{(1)}_{\text{ON}} + \alpha^2(\nu)\frac{2C_F}{C_A}\frac{m_r^2}{m_1 m_2}\,. \tag{4.98}$$

At the order we are working in, the Eqs. (4.97,4.98) produce the following difference between the potential RGEs of $\tilde{D}^{(2)}_d$ in the two matching schemes (for the first four lines in Eq. (4.81)):

$$\nu\frac{d}{d\nu}(\tilde{D}^{(2)}_{d,\text{CG}} - \tilde{D}^{(2)}_{d,\text{ON}}) = C_F^2\frac{m_r^2}{m_1 m_2}\left(-4\alpha^2\tilde{D}^{(2)}_1 + \alpha^3 - \alpha\frac{C_A}{C_F}\frac{m_1 m_2}{m_r^2}\tilde{D}^{(1)}_{\text{CG}}\right)\,, \tag{4.99}$$

which does not vanish. This difference can be understood through field redefinitions. After applying a field redefinition to move from the off-shell Coulomb scheme to the on-shell scheme, new potentials arise in the latter, which produce a contribution to the potential RGE such that it cancels the difference given by Eq. (4.99). The field redefinition that moves from the off-shell Coulomb to the on-shell schemes was already discussed in Refs. [11, 74]. In the second reference, the discussion was focused on effects to the spectrum up to $\mathcal{O}(m\alpha^5)$. We now need to see the logarithmically enhanced differences of $\mathcal{O}(m\alpha^6)$. They can be traced back by using the following Hamiltonian in the Coulomb (Feynman) gauge:

$$h_{\text{CG}} = h^{(0)} + h^{(2)}_{\text{CG}} + h^{(4)}_{\text{CG}}\,, \tag{4.100}$$

where $h^{(0)} \sim mv^2$ is the leading order Hamiltonian:

$$h^{(0)} = \frac{\mathbf{p}^2}{2m_r} + V^{(0)}(r)\,, \tag{4.101}$$

the term $h^{(2)}_{\text{CG}} \sim mv^4$ is the first relativistic correction to the Hamiltonian, with the explicit potentials:

$$\begin{aligned} h^{(2)}_{\text{CG}} &= -c_4\frac{\mathbf{p}^4}{8m_1^3} - c_4\frac{\mathbf{p}^4}{8m_2^3} - \frac{C_F C_A D^{(1)}}{4m_r r^2} \\ &\quad -\frac{C_F D^{(2)}_1}{2m_1 m_2}\left\{\frac{1}{r},\mathbf{p}^2\right\} + \frac{C_F D^{(2)}_2}{2m_1 m_2}\frac{1}{r^3}\mathbf{L}^2 + \frac{\pi C_F D^{(2)}_d}{m_1 m_2}\delta^{(3)}(\mathbf{r}) \\ &\quad +\frac{8\pi C_F D^{(2)}_{S^2}}{3m_1 m_2}\mathbf{S}_1\cdot\mathbf{S}_2\delta^{(3)}(\mathbf{r}) + \frac{3C_F}{2m_1 m_2}\frac{1}{r^3}\mathbf{L}\cdot(D^{(2)}_{LS_1}\mathbf{S}_1 + D^{(2)}_{LS_2}\mathbf{S}_2) + \frac{C_F D^{(2)}_{S_{12}}}{4m_1 m_2}\frac{1}{r^3}S_{12}(\hat{\mathbf{r}})\,, \end{aligned} \tag{4.102}$$

and $h^{(4)}_{\text{CG}} \sim mv^6$ is the next relativistic correction to the Hamiltonian. Important to the present discussion are only those potentials which produce logarithmically enhanced contributions to the spin-average S-wave spectrum, which were computed in momentum space in Sec. 4.4. For this reason, we do not display them explicitly. The Hamiltonian $h_{\rm CG}$ correctly reproduces the $\mathcal{O}(m\alpha^6)$ spectrum once the Wilson coefficients of the potentials are known with the necessary precision. For the purpose of the comparison, we can take the $\mathcal{O}(\alpha)$ static potential, so

$$V^{(0)} = -C_F \frac{\alpha_{V_s}}{r^{1+2\epsilon}} \simeq -C_F \frac{\Gamma(1/2+\epsilon)}{\pi^{1/2+\epsilon}} \frac{\alpha}{r^{1+2\epsilon}} \,. \tag{4.103}$$

We now consider the field redefinition that transforms $h_{\rm CG}$ into the on-shell Hamiltonian $h_{\rm ON}$:

$$U = \exp\left(-\frac{i}{m_r}\{\mathbf{W}(r), \mathbf{p}\}\right) . \tag{4.104}$$

$\mathbf{W}$ can be determined from the equation:

$$V^{(1)}_{\rm ON} - V^{(1)}_{\rm CG} = \frac{2}{m_r} \mathbf{W} \cdot (\boldsymbol{\nabla} V^{(0)}) \,. \tag{4.105}$$

Since the only possible tensor structure of $\mathbf{W}$ is $\mathbf{W} = W(r^2)\mathbf{r}^i$, the above equation can be written as

$$V^{(1)}_{\rm ON} - V^{(1)}_{\rm CG} = \frac{2}{m_r} W(r^2)\mathbf{r}^i \cdot (\boldsymbol{\nabla}^i V^{(0)}) \,. \tag{4.106}$$

We then obtain

$$\mathbf{W}^i = \frac{\pi}{2g_B^2} C_A (D^{(1)}_{\rm CG} - D^{(1)}_{\rm ON}) \frac{\mathbf{r}^i}{r^{1+2\epsilon}} \,, \tag{4.107}$$

and

$$h_{\rm ON} = U^\dagger h_{\rm CG} U = h_{\rm CG} + \delta h = h^{(0)} + h^{(2)}_{\rm CG} + \delta h^{(2)} + h^{(4)}_{\rm CG} + \delta h^{(4)} = h^{(0)} + h^{(2)}_{\rm ON} + h^{(4)}_{\rm ON} + \cdots \,. \tag{4.108}$$

Both Hamiltonians, $h_{\rm CG}$ and $h_{\rm ON}$, must produce the same spectrum because it is an observable and its value can not depend on how the computation is carried out, in particular, on the matching scheme. Therefore, δh cannot produce energy shifts, and any change in the potential RGE when changing from one scheme to another through transformations like Eqs. (4.97,4.98) has to be compensated among different terms in such a way that the potential RGE has exactly the same form written in one scheme or another, i.e. that it is scheme independent. In other words, the change in the spectrum (the potential RGE) caused by $\delta h^{(2)}$ must be compensated by the change caused by $\delta h^{(4)}$. Note that $h^{(4)}_{\rm CG}$ does not contribute to δh, so this piece of the Hamiltonian is not relevant to compute the differences between both Hamiltonians to this order. The piece $h^{(2)}_{\rm ON} = h^{(2)}_{\rm CG} + \delta h^{(2)}$ produces the differences reported in Eq. (4.99). Such differences should be eliminated by the piece $h^{(4)}_{\rm ON} = h^{(4)}_{\rm CG} + \delta h^{(4)}$ and indeed, they are. In momentum space $\delta \tilde{h}^{(4)}$ reads

$$\begin{aligned} \delta \tilde{h}^{(4)} \;=\;& C_F^2 g_B^2 m_r \frac{\pi D_1^{(2)}}{8 m_1^2 m_2^2} \left(|\mathbf{k}|^{1+2\epsilon} + 4(\mathbf{p}\cdot\mathbf{p}')|\mathbf{k}|^{-1+2\epsilon} + \frac{2(\mathbf{p}\cdot\mathbf{k})(\mathbf{p}'\cdot\mathbf{k})}{|\mathbf{k}|^{3-2\epsilon}} \right) \\ & + C_F \frac{\pi}{m_1 m_2} \left(-\frac{1}{4} C_F C_A D^{(1)}_{\rm CG} + \frac{1}{4} \frac{m_r^2}{m_1 m_2} C_F^2 \frac{g_B^4}{16\pi^2} \right) \frac{g_B^2}{4\pi} \frac{1}{\epsilon} \frac{1}{|\mathbf{k}|^{-4\epsilon}} \\ & + \cdots \,. \end{aligned} \tag{4.109}$$

Note that the term proportional to $|\mathbf{k}|^{1+2\epsilon}$ in the first line gives a contribution to the potential RGE through potential loops. It is equivalent to generating a new α^2/m^3 potential. The other two terms in the first line do not contribute to the potential RGE. Looking at the second line, it is also interesting to see that there is a kind of soft contribution (divergent α^3/m^2 potential). The second term in the second line can also be interpreted as a pure-soft contribution. This brings the interesting observation that even if the potential RGE can be written in a matching-scheme-independent way, the implicit scheme dependence of the potentials allows for a mixing with the soft computation (at least in the on-shell scheme). Finally, for the dots in the third line we refer to extra contributions to $\delta\tilde{h}^{(4)}$, generated by the field redefinitions, which do not contribute to the running.

Conclusions

The research carried out in this dissertation has focused on the use EFTs of QCD to perform high precision calculations of heavy quark and heavy quarkonium properties. The major aim has been to make a step towards the complete determination of the heavy quarkonium spectrum with N^{3}LL accuracy. The obtained results will help us to improve our understanding of the strong interaction, necessary to successfully describe the experiments presently performed in large accelerator facilities.

In Ch. 2 we have obtained the LL running of the Wilson coefficients associated to the $1/m^3$ HQET Lagrangian operators, in the Coulomb gauge, including n_f massless spectator quarks. The LL running is also obtained for combinations of Wilson coefficients that are gauge-independent. These combinations are determined by computing the tree level Compton scattering at $\mathcal{O}(1/m^3)$ plus inspecting the RGEs. We find that the RGEs of physical quantities depend only on gauge-independent quantities, as expected. This is a very strong check of the computation. For the gauge-independent combinations, the RGEs have been solved. The solution is presented for the particular case of $n_f = 4$, since the anomalous dimension matrix can not be diagonalized for a general n_f. For this reason, in the obtained expressions, the coefficients that multiply the different functions of z are numerical. The Wilson coefficients evaluated at $n_f = 3$ are also used throughout this thesis despite of they are not presented explicitly. The reason is that they can be computed easily once the RGEs are known. We also have computed the case of $n_f = 0$ for which we obtain analytic results. The Wilson coefficients have been computed with SL approximation, too. These Wilson coefficients include the heavy quark chromopolarizabilities induced by the strong interactions.

We have performed a numerical analysis of these results. Concerning the SI heavy-gluon operators, we observe that the running produces a very large effect for the coefficients c_{A_1} and c_{A_2}. For c_{A_3} and c_{A_4} the running is smaller but certainly sizeable. For the combinations $2c_{A_1}+\bar{c}_{A_2}$ and $2c_{A_3}+c_{A_4}$, that appear in physical observables such as the heavy quarkonium spectrum or the Compton scattering, the running is more moderate but still quite large. For the spin-dependent heavy-gluon operators, we observe that the running produces a large effect except for the combination $\bar{c}_{B_1} + c_{B_2}$, which appears in the Compton scattering, and whose running is smaller. In general, we conclude that the running produces a very important effect over the Wilson coefficients, specially over the spin-independent ones. The resummation of large logarithms introduces relevant corrections with respect to the SL results, and it is particularly important is those cases where the behaviour is not saturated by the SL expression. Spectator quark effects are observed to be numerically subleading with respect to the ones coming from the heavy-gluon sector. However, they produce corrections that have to be included formally.

We observe that the relations between Wilson coefficients imposed by reparametrization invariance are satisfied when they involve gauge independent combinations of Wilson coefficients. We observe that the relations $2c_M = c_D - c_F$ [28] and $d_1^{hl} = c_1^{hl}/16$ [44], also imposed by reparametrization invariance, are not satisfied in the Coulomb gauge. Assuming that $\bar{d}_1^{hl} = c_M + 8d_1^{hl}$ is gauge independent and that in the Feynman gauge $2c_M = c_D - c_F$ is satisfied, we find that $d_1^{hl} = c_1^{hl}/16$ also must be satisfied in that gauge. Therefore, our results do not contradict reparametrization invariance as long as we work in the Feynman gauge (and possibly in any covariant gauge). Moreover, we propose a new gauge independent relation, $2c_M + 16d_1^{hl} = c_D + c_1^{hl} - c_F$, which we find to be satisfied in both, the Coulomb and the Feynman gauge.

We have compared our results with the previous work done in Refs. [33, 34], where the computation of the LL running of the $1/m^3$ heavy-gluon operators of the HQET Lagrangian was considered in the Feynman gauge. For the gauge invariant combinations we have computed, the anomalous dimension matrix given in Ref. [33] yields different RGEs that those we found in Sec. 2.5.2, and also yields different SL expressions compared to those explicitly given in these references. Nevertheless, it is remarkable that we find agreement with the explicit SL results given in Refs. [33, 34]. If we trust the SL results explicitly given there, we find a clear indication that the quantities we computed are gauge independent. Likewise, they are useful to test which Wilson coefficients are gauge dependent. The inclusion of spectator quarks was considered in Ref. [29] in the Feynman gauge, but only SL results were presented. For the gauge invariant combinations we have computed, the SL expressions presented in this reference are in agreement with ours, except for d_2^{hl}, d_3^{hl}, d_4^{hl} and $\bar{d}_{10}^{hl}$.

The Wilson coefficients we find correspond to a set of operators written in a more standard basis, set by Ref. [28], than the one used previously by Refs. [29, 33, 34], and they are connected more closely to observables, as the quantities we computed are gauge independent.

Among the Wilson coefficients we have computed, the ones associated to SI operators are necessary building blocks for the determination of the pNRQCD Lagrangian with N^{3}LL accuracy, which in turn is the necessary precision to completely determine the production and annihilation of heavy quarkonium near threshold with N^{2}LL accuracy and the heavy quarkonium spectrum with N^{3}LL precision. In the latter case, they enter into the N^{3}LL S-wave spin average contribution to the spectrum, which we partially compute in this thesis. The ones associated to spin-dependent operators start to be relevant at higher orders. In particular, they are necessary ingredients to obtain the pNRQCD Lagrangian with N^{4}LO and N^{4}LL accuracy, which is the necessary precision to obtain the N^{4}LO and the N^{4}LL heavy quarkonium spectrum. They also have applications in QED bound states like in muonic hydrogen.

In order to compute the N^{3}LL S-wave spin average contribution to the spectrum several pieces are needed. All of them appear in the same place: the NLL running of the spin-independent delta-like potential. These pieces are the soft, potential and ultrasoft contributions to its running. This computation has been addressed in Ch. 4.

Concerning the soft running, we have computed the bare and renormalized $\mathcal{O}(\alpha^3/m^2)$

spin-independent delta-like potential proportional to $c_F^{(1)\,2}$, $c_F^{(2)\,2}$, $\bar{c}_1^{(1)hl}$ and $\bar{c}_1^{(2)hl}$, and obtained and solved the soft RGE. The missing terms to obtain the full result are the NLL running of $\bar{c}_1^{hl}$, whose associated missing contribution is of $\mathcal{O}(T_f n_f m\alpha^6 \ln\alpha)$, which is expected to be quite small, and the piece of the soft running proportional to $c_k^{(1)\,2}$, $c_k^{(2)\,2}$, $c_k^{(1)}c_k^{(2)}$, d_{ss} and d_{vs}. The magnitude of this contribution is estimated to be smaller compared to the potential running, since the soft running is π^2 suppressed with respect to it. It is also expected to be smaller than the complete running of the heavy quarkonium potential. Even though the contribution from the missing terms is expected to be small, a detailed phenomenological analysis is postponed to future publications.

Concerning the potential running, we needed the HQET Wilson coefficients computed in Ch. 2, since they enter into the potentials of pNRQCD. Of all of them, we needed to compute some which were previously unknown: the spin-independent $\mathcal{O}(\alpha^2/m^3)$ and $\mathcal{O}(\alpha/m^4)$ potentials. The computation have been carried out in the Coulomb gauge. We have obtained the potential RGE of the spin-independent delta-like potential with NLL accuracy, which is the first nonzero contribution to the potential running. An extra contribution to the potential running with ultrasoft origin has also been included. We have solved the RGE and obtained, after including the already known ultrasoft running [59], the complete potential and ultrasoft running of the spin-independent delta-like potential with NLL accuracy.

We have quantified the mixing between the α^3/m^2 and the α^2/m^3 potentials proportional to c_F^2, which takes place when using the full EOMs in energy dependent potentials. More precisely, we have observed that using the full EOMs in energy dependent α^2/m^2 potentials produces a finite α^2/m^3 potential (coming from the free EOMs) and a divergent α^3/m^2 potential (coming from the Coulomb term in the full EOMs). We have not quantified the corresponding mixing proportional to c_k^2 because the soft contribution proportional to c_k^2 is missing anyway.

The solution of the RGE is analytic in general. However, for the case of $n_f = 3, 4$, the coefficients multiplying the different functions of z in the contribution coming from the potential running are numerical. That is a direct consequence of the fact that, for $n_f = 3, 4$, we could not find an analytical solution for the coefficients multiplying the different functions of z in the chromopolarizabilities.

Combining the results we have found in the present dissertation (plus the ones that are missing) with the results given in Refs. [11, 12, 13] we obtain the S-wave (spin average and hyperfine splitting) heavy quarkonium mass with N^{3}LL accuracy. After incorporating the results of Ref. [6], the complete N^{3}LL heavy quarkonium spectrum would be obtained.

The scheme independence of the potential RGE have been studied. By using field redefinitions we moved from the Coulomb gauge to the on-shell Hamiltonians and we computed the difference between the potential RGE of the spin-independent delta-like potential in both schemes.

Finally, we remark that significant parts of the computations we have carried out in this dissertation are necessary building blocks for a future evaluation of the N^{4}LO heavy quarkonium spectrum.

Future prospects

In the present dissertation, the NLL running of the Wilson coefficient associated to the spin-independent delta-like potential have been completely determined up to two missing soft contributions. The first one, which is expected to be more important, comes from the UV divergent part of the $\mathcal{O}(\alpha^3/m^2)$ potential proportional to $c_k^{(1)\,2}$, $c_k^{(2)\,2}$, $c_k^{(1)}c_k^{(2)}$, d_{ss} and d_{vs}, which is determined by matching NRQCD with pNRQCD. In principle, this computation is feasible for the contributions proportional to d_{ss} and d_{vs}. The second one, which is expected to be less important, comes from the need to have the NLL running of the HQET Wilson coefficient $\bar{c}_1^{hl}$, which enters into the LL soft running of the spin-independent delta-like potential. On the one hand, and once we will have determined the NLL soft RGE proportional to d_{ss} and d_{vs}, the c_k^2 missing term could be extracted from the two-loop matching of the NRQCD four fermion operators taking advantage of the fact that all the other terms in the soft RGE are known. Such a computation is starting to be considered. On the other hand, the NLL running of $\bar{c}_1^{hl}$ is not that important, since it is expected to be quite small because it is of $\mathcal{O}(T_f n_f m\alpha^6 \ln\alpha)$, and its contribution could even be negligible in practice. However, it has to be computed formally. In order to determine it, we need to compute in the framework of HQET to $\mathcal{O}(1/m^2)$, the two-loop correction to the one gluon vertex, which allows to determine c_D, and the scattering of a heavy quark with a massless quark, which allows to determine c_1^{hl}. Then, the coefficient we need is given by $\bar{c}_1^{hl} = c_D + c_1^{hl}$. In summary, the determination of such contributions is mandatory in order to perform a complete phenomenological analysis of the N^{3}LL S-wave spin-average heavy quarkonium spectrum. Once it is determined, the full spectrum will be obtained to this order, since the other contributions have already been computed in the past.

The complete determination of the NLL running of the spin-independent delta-like potential is necessary to obtain the Wilson coefficient of the electromagnetic current with N^{2}LL precision. That is indeed what is needed to achieve N^{2}LL precision for non-relativistic sum rules and the t-$\bar{t}$ production near threshold. These computations are also of vital importance for the most accurate determination of some of the Standard Model parameters, like the masses of the heavy quarks: top, bottom and charm, as well as the strong coupling constant [10]. Moreover, several of these results can be easily applied to atomic physics opening the possibility of accurately determining lepton masses, the electromagnetic coupling constant, as well as low energy hadronic constants.

Another interesting computation would be the determination of the $\mathcal{O}(\alpha^2/m^3)$ and the $\mathcal{O}(\alpha/m^4)$ potentials in the on-shell matching scheme, define a basis of potentials, and see how the pNRQCD Lagrangian in the on-shell and in the Coulomb gauge matching schemes relate with each other via field redefinitions. This would allow us to see how the Wilson coefficients of the $\mathcal{O}(\alpha^2/m^3)$ and $\mathcal{O}(\alpha/m^4)$ potentials transform.

Once the full N^{3}LL heavy quarkonium spectrum will have been obtained, the next goal will be to obtain N^{4}LO and the N^{4}LL heavy quarkonium spectrum, in other words, the spectrum at $\mathcal{O}(m\alpha^6)$ and at $\mathcal{O}(m\alpha^6 + m\alpha^7 \ln\alpha + \ldots)$. For instance, the expectation value of the α^2/m^3 and the α/m^4 potentials contribute to these orders, so we have already got some results towards this computation.

It also would be interesting to compute the LL running of the $1/m^3$ HQET Lagrangian operators in the Feynman gauge in order to strengthen the confidence in the results we have obtained, in particular, in order to check which Wilson coefficients are gauge dependent, and which Wilson coefficients and combinations of them are gauge independent. This also would help to understand why the relations $2c_M = c_D - c_F$ and $d_1^{hl} = c_1^{hl}/16$ are gauge dependent. A rigorous prove that the truly gauge independent combination is $2c_M + 16d_1^{hl} = c_D + c_1^{hl} - c_F$ is lacking and should be obtained.

During this thesis we wrote a software to assist the computation of the LL running of the Wilson coefficients associated to the $1/m^3$ HQET Lagrangian operators. It would be interesting to improve that software in order to be able to compute the LL running of the Wilson coefficients of the $1/m^4$ operators of the HQET Lagrangian, which was recently obtained in Ref. [53], since some of them are important to compute the N^{4}LL heavy quarkonium spectrum (see Eq. (4.41)).

Appendix A

Conventions and Identities

A.1 Fourier Tranforms

In order to compute the Feynman rules associated to the different Lagrangian operators we deal with in this thesis, it is useful to fix the convention we use for the Fourier transform of N variables in Minkowski space

$$f(x; x_1, \ldots, x_N) = \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_N}{(2\pi)^4} e^{-ik_1(x-x_1)} \cdots e^{-ik_N(x-x_N)} \tilde{f}(k_1, \ldots, k_N)\,, \tag{A.1}$$

$$\tilde{f}(k_1, \ldots, k_N) = \int d^4x_1 \ldots d^4x_N \, e^{ik_1(x-x_1)} \ldots e^{ik_N(x-x_N)} f(x; x_1, \ldots, x_N)\,. \tag{A.2}$$

For the d-dimensional potentials, the convention for the Fourier transform reads

$$V(\mathbf{r}) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} e^{-i\mathbf{k}\cdot\mathbf{r}} \tilde{V}(\mathbf{k})\,, \tag{A.3}$$

$$\tilde{V}(\mathbf{k}) = \int d^d\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} V(\mathbf{r})\,. \tag{A.4}$$

The following d-dimensional Fourier transform which can be found in Ref. [8] proves to be very useful in the computation carried out in Sec. 4.7

$$\mathcal{F}_n(r) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{|\mathbf{k}|^n} = \frac{2^{-n}\pi^{-d/2}}{|\mathbf{r}|^{d-n}} \frac{\Gamma(d/2 - n/2)}{\Gamma(n/2)}\,. \tag{A.5}$$

Sometimes we will use the notation $k \equiv |\mathbf{k}|$ and $r \equiv |\mathbf{r}|$, indistinctly.

A.2 Quantum Mechanics Notation

Let us fix some Quantum Mechanics conventions and notation, as well as, to summarize some useful identities. The four-momentum operator is given by

$$p^\mu = i\partial^\mu \,. \tag{A.6}$$

The identity operator expressed in terms of position and momentum eigenstates in d-dimensional space is given by

$$\mathbb{I} = \int \frac{d^d p}{(2\pi)^d} |\mathbf{p}\rangle\langle\mathbf{p}| \,, \tag{A.7}$$

$$\mathbb{I} = \int d^d r |\mathbf{r}\rangle\langle\mathbf{r}| \,. \tag{A.8}$$

Other useful identities are

$$\langle\mathbf{r}|\mathbf{p}\rangle = e^{i\mathbf{p}\cdot\mathbf{r}} \,, \tag{A.9}$$

$$\langle\mathbf{p}|\mathbf{r}\rangle = e^{-i\mathbf{p}\cdot\mathbf{r}} \,, \tag{A.10}$$

$$\langle\mathbf{r}|\mathbf{r}'\rangle = \delta^{(d)}(\mathbf{r}-\mathbf{r}') \,, \tag{A.11}$$

$$\langle\mathbf{p}|\mathbf{p}'\rangle = (2\pi)^d \delta^{(d)}(\mathbf{p}-\mathbf{p}') \,. \tag{A.12}$$

It is useful to give the Dirac delta operator expressed as a projector using the position eigenstates

$$\delta^{(d)}(\mathbf{r}) = |\mathbf{r}'=0\rangle\langle\mathbf{r}'=0| \,. \tag{A.13}$$

Finally, the following relation is used to change the spin basis in which the results are presented

$$\mathbf{S}_1 \otimes \mathbf{S}_2 = \frac{1}{2}\mathbf{S}^2 - \frac{3}{4} I_1 \otimes I_2 \,, \tag{A.14}$$

where $\mathbf{S}_1$ is the spin operator of the heavy fermion, $\mathbf{S}_2$ is the spin operator of the heavy antifermion and $\mathbf{S}$ is the total spin operator of the heavy fermion-antifermion system.

A.3 Quantum Chromodynamics Notation

Let us fix some definitions and conventions in QCD. The covariant derivative is given by

$$D_\mu = \partial_\mu + igA^a_\mu T^a \,. \tag{A.15}$$

The field strenght tensor is defined by

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \,. \tag{A.16}$$

In terms of the four-potential $A^{\mu\,a} = (A^{0\,a}, \mathbf{A}^a)$, the chromoelectric and chromomagnetic fields are given by

$$\mathbf{E}^i = \mathbf{E}^{i\,a}T^a = G^{i0\,a}T^a = -\partial_0\mathbf{A}^{i\,a}T^a - \partial_i A^{0\,a}T^a + ig\mathbf{A}^{i\,b}A^{0\,c}[T^b, T^c]\,, \tag{A.17}$$

$$\mathbf{B}^i = \mathbf{B}^{i\,a}T^a = -\frac{1}{2}\epsilon^{ijk}G^{jk\,a}T^a = \epsilon^{ijk}\partial_j\mathbf{A}^{k\,a}T^a - \frac{ig}{2}\epsilon^{ijk}\mathbf{A}^{j\,b}\mathbf{A}^{k\,c}[T^b, T^c]\,. \tag{A.18}$$

or, likewise

$$\mathbf{E}^i = -\partial_0\mathbf{A}^{i\,a}T^a - \boldsymbol{\nabla}^i A^{0\,a}T^a + ig\mathbf{A}^{i\,b}A^{0\,c}[T^b, T^c]\,, \tag{A.19}$$

$$\mathbf{B}^i = (\boldsymbol{\nabla}\times\mathbf{A}^a)^i T^a - \frac{ig}{2}(\mathbf{A}^b\times\mathbf{A}^c)^i[T^b, T^c]\,. \tag{A.20}$$

In QED, the non-Abelian terms do no appear and the generators must be replaced by one ($T^a \to 1$). After doing it, we obtain the electric and magnetic fields expressed in terms of the four-potential components

$$\mathbf{E}^i = -\partial_0\mathbf{A}^i - \boldsymbol{\nabla}^i A^0\,, \tag{A.21}$$

$$\mathbf{B}^i = (\boldsymbol{\nabla}\times\mathbf{A})^i\,. \tag{A.22}$$

A.4 Mass relations

The following identities are very useful for Sec. 4.5.3 in order to write the potential RGE in terms of gauge independent combinations of Wilson coefficients, as well as, to make comparisons with Ref. [64]

$$m_r\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right) = \frac{m_r^2}{m_1 m_2}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right) + m_r^2\left(\frac{1}{m_1^4}+\frac{1}{m_2^4}\right)\,, \tag{A.23}$$

$$m_r\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right) = \left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right) - \frac{1}{m_1 m_2}\,, \tag{A.24}$$

where m_1 is the mass of the heavy fermion, m_2 is the mass of the heavy antifermion and $m_r = m_1 m_2/(m_1+m_2)$ is the reduced mass of the heavy fermion-antifermion system.

A.5 Feynman parametrization and Feynman integrals

The following standard Feynman integrals in d-dimensional Euclidean space are very useful [75]

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2}\,, \tag{A.25}$$

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(n-d/2-1)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2-1}\,, \tag{A.26}$$

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^4}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{d(d+2)}{4}\frac{\Gamma(n-d/2-2)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2-2}\,, \tag{A.27}$$

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{(\ell_E^2)^\alpha}{(\ell_E^2 + \Delta)^\beta} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\alpha + d/2)\Gamma(\beta - \alpha - d/2)}{\Gamma(\beta)\Gamma(d/2)} \left(\frac{1}{\Delta}\right)^{\beta-\alpha-d/2} , \tag{A.28}$$

as well as, the introduction of Feynman parametrization [76]

$$\frac{1}{A_1 A_2 \dots A_n} = \int_0^1 dx_1 \int_0^1 dx_2 \dots \int_0^1 dx_{n-1}$$
$$\times \frac{(n-1)! x_1^{n-2} x_2^{n-3} \dots x_{n-2}}{[A_1 x_1 \dots x_{n-1} + A_2 x_1 \dots x_{n-2}(1 - x_{n-1}) + \dots + A_{n-1} x_1 (1 - x_2) + A_n (1 - x_1))]^n} . \tag{A.29}$$

The following particular cases appear quite often

$$\frac{1}{A_1^\alpha A_2^\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[A_1 x + A_2(1-x)]^{\alpha+\beta}} , \tag{A.30}$$

$$\frac{1}{A_1^\alpha A_2^\beta A_3^\gamma} = \frac{\Gamma(\alpha + \beta + \gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 dx\, x \int_0^1 dy \frac{(xy)^{\alpha-1}(x(1-y))^{\beta-1}(1-x)^{\gamma-1}}{[A_1 xy + A_2 x(1-y) + A_3(1-x)]^{\alpha+\beta+\gamma}} . \tag{A.31}$$

Appendix B

HQET RG

B.1 HQET Feynman rules

Here we collect the Feynman rules in the Coulomb gauge, coming from the HQET Lagrangian Eqs. (2.19,2.20,2.21), which are necessary for the computation of the LL running of the Wilson coefficients of the $1/m^3$ HQET Lagrangian operators. They complement those that can be found in Ref. [8], which we also include. We also take the occasion to correct a misprint of Eq. (205) of that reference. The correct Feynman rule is displayed in Eq. (B.21). The conventions are shown in Figs. B.1,B.2,B.3.

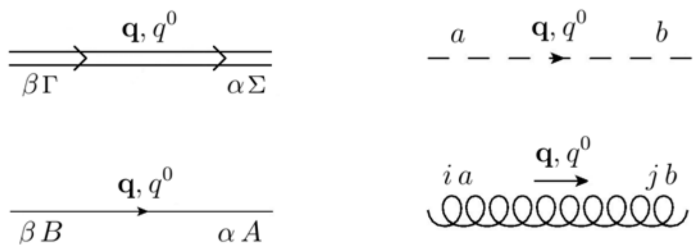


Figure B.1: Conventions for the propagators of the HQET Lagrangian. The double line represents a heavy quark, the single line a massless quark, and the curly and dashed lines a transverse and a longitudinal gluon, respectively.

B.1.1 Propagators and Coulomb vertex

Heavy quark propagator:

$$\frac{i}{q^0 - c_k \frac{\mathbf{q}^2}{2m} + c_4 \frac{\mathbf{q}^4}{8m^3} + i\eta_q} \delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.1}$$

Massless quark propagator:

$$\frac{i(\not{q})_{AB}}{q^2 + i\eta} \delta_{\alpha\beta} \tag{B.2}$$

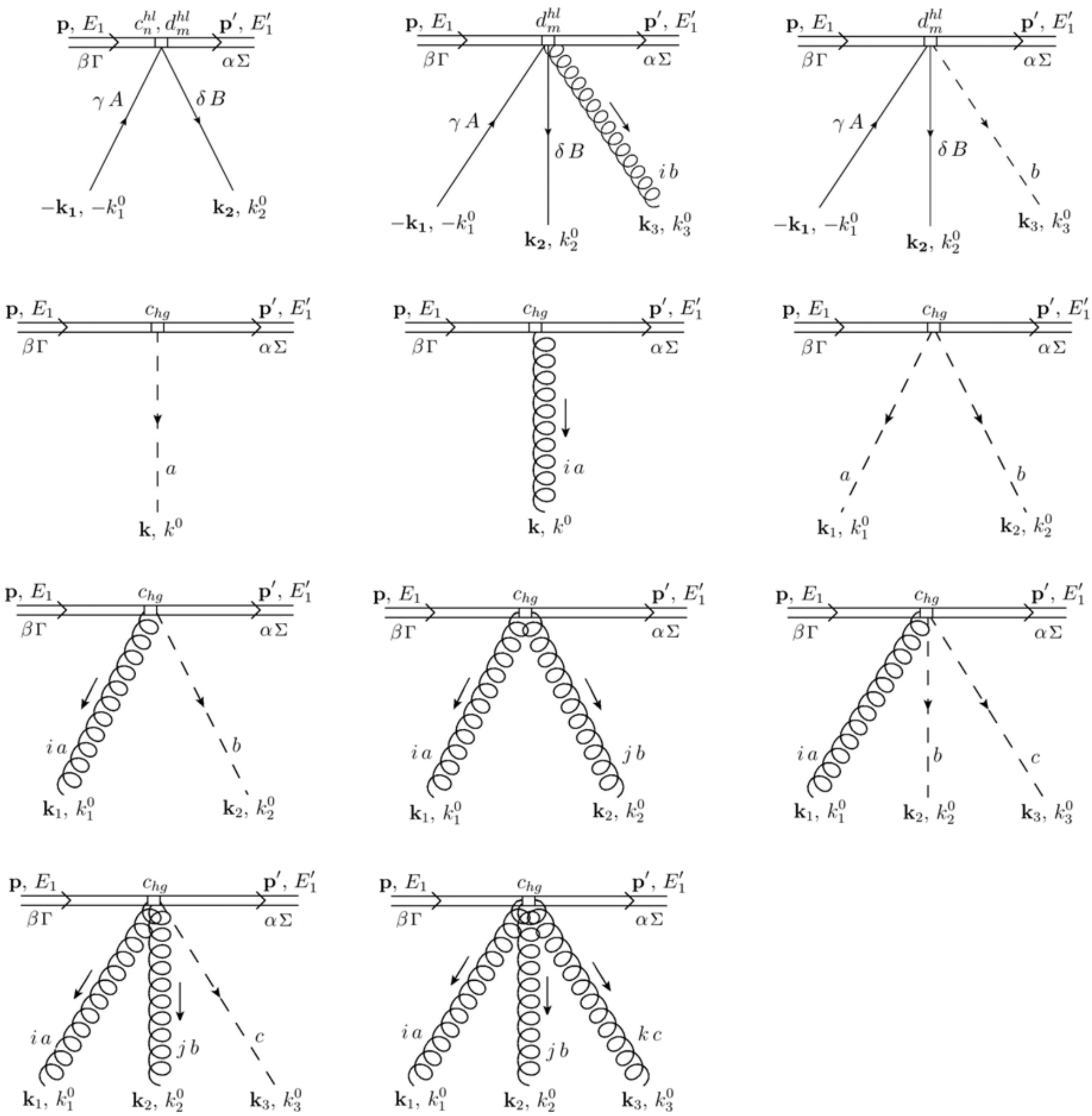


Figure B.2: Conventions for the Feynman rules of the HQET Lagrangian. The double line represents a heavy quark, the single line a massless quark, and the curly and dashed lines a transverse and a longitudinal gluon, respectively. The subindex m goes from 1 to 11, n goes from 1 to 4 and hg stands for all subindices of the heavy-gluon Wilson coefficients. Note that, by four-momentum conservation, $k \equiv \sum_{i=1}^{n} k_i = p - p'$.

Longitudinal gluon propagator:

$$\frac{i\delta^{ab}}{\mathbf{q}^2} \tag{B.3}$$

Transverse gluon propagator:

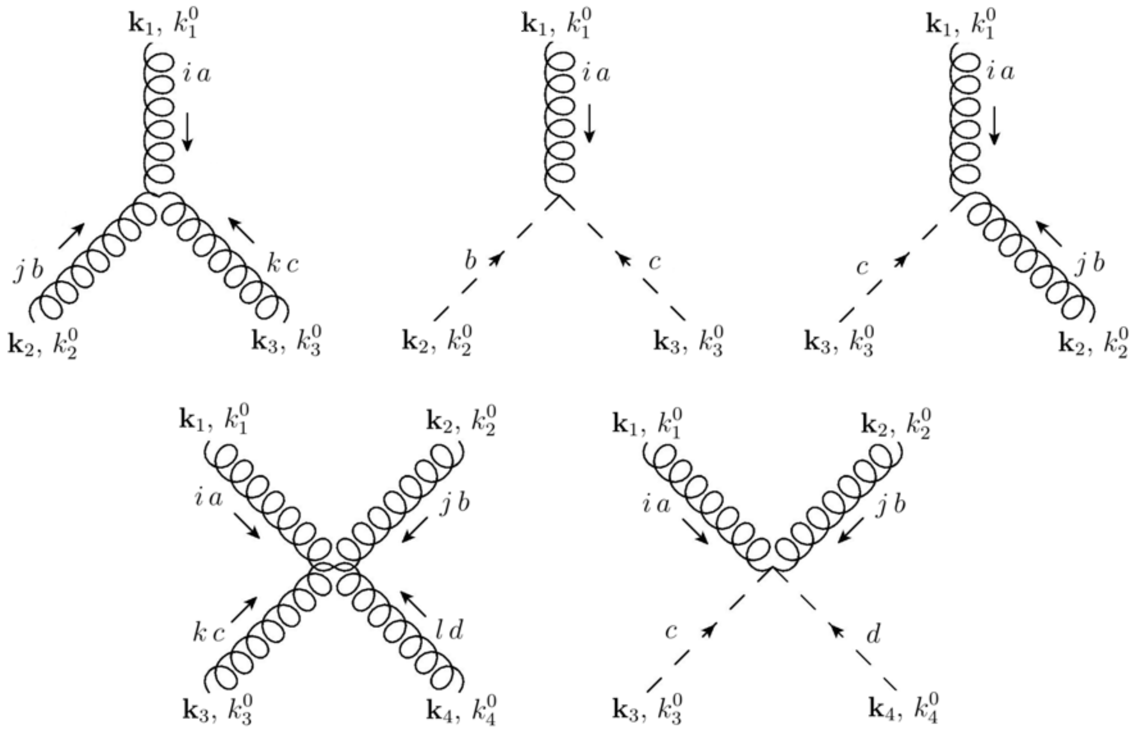


Figure B.3: Conventions for the Feynman rules of $\mathcal{L}_g$. The curly and dashed lines represent a transverse and a longitudinal gluon, respectively.

$$\frac{i\delta^{ab}}{q^2 + i\eta_g}\left(\delta^{ij} - \frac{\mathbf{q}^i\mathbf{q}^j}{\mathbf{q}^2}\right) \tag{B.4}$$

Coulomb vertex:

$$\mathcal{V}^a = -ig(T^a)_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.5}$$

B.1.2 Gluon self-interaction

$$\mathcal{V}^{i\,abc} = gf^{abc}(\mathbf{k}_2 - \mathbf{k}_3)^i \tag{B.6}$$

$$\mathcal{V}^{ij\,abc} = gf^{abc}\delta^{ij}(k_1^0 - k_2^0) \tag{B.7}$$

$$\mathcal{V}^{ijk\,abc} = gf^{abc}[\delta^{ik}(\mathbf{k}_1 - \mathbf{k}_3)^j + \delta^{ij}(\mathbf{k}_2 - \mathbf{k}_1)^k + \delta^{jk}(\mathbf{k}_3 - \mathbf{k}_2)^i] \tag{B.8}$$

$$\mathcal{V}^{ijkl\,abcd} = -ig^2[f^{abe}f^{cde}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}) + f^{ace}f^{bde}(\delta^{ij}\delta^{kl} - \delta^{il}\delta^{jk}) + f^{ade}f^{cbe}(\delta^{ik}\delta^{jl} - \delta^{ij}\delta^{kl})] \tag{B.9}$$

$$\mathcal{V}^{ijkl\,abcd} = ig^2\delta^{ij}(f^{ace}f^{bde} - f^{ade}f^{cbe}) \tag{B.10}$$

B.1.3 Light quark-gluon interactions

$$\mathcal{V} = -ig(\gamma^0)_{AB}(T^a)_{\alpha\beta} \tag{B.11}$$

$$\mathcal{V}^i = ig(\gamma^i)_{AB}(T^a)_{\alpha\beta} \tag{B.12}$$

B.1.4 Heavy quark-gluon interactions

Proportional to c_k

$$\mathcal{V}_{c_k}^{i\,a} = c_k \frac{ig}{2m}(\mathbf{p}+\mathbf{p}')^i (T^a)_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.13}$$

$$\mathcal{V}_{c_k}^{ij\,ab} = -c_k \frac{ig^2}{2m}\delta^{ij}\{T^a, T^b\}_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.14}$$

Proportional to c_F

$$\mathcal{V}_{c_F}^{i\,a} = c_F \frac{g}{2m}(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k})^i (T^a)_{\alpha\beta} \tag{B.15}$$

$$\mathcal{V}_{c_F}^{ij\,ab} = c_F \frac{g^2}{2m}\boldsymbol{\sigma}^k_{\Sigma\Gamma}\epsilon^{ijk}[T^a, T^b]_{\alpha\beta} \tag{B.16}$$

Proportional to c_D

$$\mathcal{V}_{c_D}^{a} = c_D \frac{ig}{8m^2}\mathbf{k}^2 (T^a)_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.17}$$

$$\mathcal{V}_{c_D}^{i\,ab} = c_D \frac{ig^2}{4m^2}\mathbf{k}_2^i [T^a, T^b]_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.18}$$

$$\mathcal{V}_{c_D}^{ij\,ab} = c_D \frac{ig^2}{8m^2}\delta^{ij}(k_1^0 - k_2^0)[T^a, T^b]_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.19}$$

$$\mathcal{V}_{c_D}^{ij\,abc} = c_D \frac{ig^3}{8m^2}\delta^{ij}([T^a,[T^b,T^c]]_{\alpha\beta} + [T^b,[T^a,T^c]]_{\alpha\beta})\delta_{\Sigma\Gamma} \tag{B.20}$$

Proportional to c_S

$$\mathcal{V}_{c_S}^{a} = c_S \frac{g}{4m^2}\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot(\mathbf{p}'\times\mathbf{p})(T^a)_{\alpha\beta} \tag{B.21}$$

$$\mathcal{V}_{c_S}^{i\,a} = -c_S \frac{g}{8m^2}k^0(\boldsymbol{\sigma}_{\Sigma\Gamma}\times(\mathbf{p}+\mathbf{p}'))^i (T^a)_{\alpha\beta} \tag{B.22}$$

$$\mathcal{V}_{c_S}^{i\,ab} = c_S \frac{g^2}{8m^2}[(\boldsymbol{\sigma}_{\Sigma\Gamma}\times(\mathbf{p}+\mathbf{p}'))^i [T^a, T^b]_{\alpha\beta} + (\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_2)^i\{T^a, T^b\}_{\alpha\beta}] \tag{B.23}$$

$$\mathcal{V}_{c_S}^{ij\,ab} = c_S \frac{g^2}{8m^2} \boldsymbol{\sigma}^k_{\Sigma\Gamma} \epsilon^{kij} (k_2^0 - k_1^0) \{T^a, T^b\}_{\alpha\beta} \tag{B.24}$$

$$\mathcal{V}_{c_S}^{ij\,abc} = -c_S \frac{g^3}{8m^2} \boldsymbol{\sigma}^k_{\Sigma\Gamma} \epsilon^{kij} (\{T^a, [T^b, T^c]\}_{\alpha\beta} - \{T^b, [T^a, T^c]\}_{\alpha\beta}) \tag{B.25}$$

Proportional to c_4

$$\mathcal{V}_{c_4}^{i\,a} = -c_4 \frac{ig}{8m^3} (\mathbf{p}^2 + \mathbf{p}'^2)(\mathbf{p} + \mathbf{p}')^i (T^a)_{\alpha\beta} \delta_{\Sigma\Gamma} \tag{B.26}$$

$$\mathcal{V}_{c_4}^{ij\,ab} = c_4 \frac{ig^2}{8m^3} \left(\delta^{ij} (\mathbf{p}^2 + \mathbf{p}'^2) \{T^a, T^b\}_{\alpha\beta} + 4\mathbf{p}'^i \mathbf{p}^j (T^a T^b)_{\alpha\beta} + 4\mathbf{p}^i \mathbf{p}'^j (T^b T^a)_{\alpha\beta} \right) \delta_{\Sigma\Gamma} \tag{B.27}$$

$$\begin{aligned}
\mathcal{V}_{c_4}^{ijk\,abc} = \; & -c_4 \frac{ig^3}{4m^3} \left[\delta^{ij} \left((T^c \{T^a, T^b\})_{\alpha\beta} \mathbf{p}'^k + (\{T^a, T^b\} T^c)_{\alpha\beta} \mathbf{p}^k \right) \right. \\
& + \delta^{jk} \left((T^a \{T^b, T^c\})_{\alpha\beta} \mathbf{p}'^i + (\{T^b, T^c\} T^a)_{\alpha\beta} \mathbf{p}^i \right) && \text{(B.28)} \\
& \left. + \delta^{ik} \left((T^b \{T^a, T^c\})_{\alpha\beta} \mathbf{p}'^j + (\{T^a, T^c\} T^b)_{\alpha\beta} \mathbf{p}^j \right) \right] \delta_{\Sigma\Gamma} && \text{(B.29)}
\end{aligned}$$

Proportional to c_M

$$\mathcal{V}_{c_M}^{i\,a} = -c_M \frac{ig}{8m^3} \mathbf{k}^2 (\mathbf{p} + \mathbf{p}')^i (T^a)_{\alpha\beta} \delta_{\Sigma\Gamma} \tag{B.30}$$

$$\begin{aligned}
\mathcal{V}_{c_M\,CG}^{ij\,ab} = \; & c_M \frac{ig^2}{8m^3} \left(\delta^{ij} \{T^a, T^b\}_{\alpha\beta} (\mathbf{k}_1^2 + \mathbf{k}_2^2) - \delta^{ij} [T^a, T^b]_{\alpha\beta} ((\mathbf{p} + \mathbf{p}') \cdot (\mathbf{k}_1 - \mathbf{k}_2)) \right) \\
& + 4 [T^a, T^b]_{\alpha\beta} (\mathbf{p}^i \mathbf{k}_1^j - \mathbf{k}_2^i \mathbf{p}^j)) \, \delta_{\Sigma\Gamma} && \text{(B.31)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}_{c_M\,CG}^{ijk\,abc} = \; & c_M \frac{ig^3}{8m^3} \left([T^a, [T^b, T^c]]_{\alpha\beta} (\delta^{ik} (\mathbf{p}' + \mathbf{p})^j - \delta^{ij} (\mathbf{p}' + \mathbf{p})^k) \right. \\
& + [T^b, [T^a, T^c]]_{\alpha\beta} (\delta^{jk} (\mathbf{p}' + \mathbf{p})^i - \delta^{ij} (\mathbf{p}' + \mathbf{p})^k) \\
& + [T^c, [T^a, T^b]]_{\alpha\beta} (\delta^{jk} (\mathbf{p}' + \mathbf{p})^i - \delta^{ik} (\mathbf{p}' + \mathbf{p})^j) \\
& + \{T^a, [T^b, T^c]\}_{\alpha\beta} (2\delta^{ik} \mathbf{k}_3^j - 2\delta^{ij} \mathbf{k}_2^k + \delta^{jk} (\mathbf{k}_2 - \mathbf{k}_3)^i) \\
& + \{T^b, [T^a, T^c]\}_{\alpha\beta} (2\delta^{jk} \mathbf{k}_3^i - 2\delta^{ij} \mathbf{k}_1^k + \delta^{ik} (\mathbf{k}_1 - \mathbf{k}_3)^j) \\
& \left. + \{T^c, [T^a, T^b]\}_{\alpha\beta} (2\delta^{jk} \mathbf{k}_2^i - 2\delta^{ik} \mathbf{k}_1^j + \delta^{ij} (\mathbf{k}_1 - \mathbf{k}_2)^k) \right) \delta_{\Sigma\Gamma} && \text{(B.32)}
\end{aligned}$$

Proportional to c_{A_1}

$$\mathcal{V}^{ij\,ab}_{c_{A_1}} = c_{A_1}\frac{ig^2}{8m^3}(\delta^{ij}(k_1^0k_2^0 - \mathbf{k}_1\cdot\mathbf{k}_2) + \mathbf{k}_2^i\mathbf{k}_1^j)\{T^a,T^b\}_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.33}$$

$$\begin{aligned}\mathcal{V}^{ijk\,abc}_{c_{A_1}} \;=\;& c_{A_1}\frac{ig^3}{8m^3}\left[(\delta^{ij}\mathbf{k}_1^k - \delta^{ik}\mathbf{k}_1^j)\{[T^b,T^c],T^a\}_{\alpha\beta} + (\delta^{ij}\mathbf{k}_2^k - \delta^{jk}\mathbf{k}_2^i)\{[T^a,T^c],T^b\}_{\alpha\beta}\right.\\ &\left.+(\delta^{ik}k_3^j - \delta^{jk}\mathbf{k}_3^i)\{[T^a,T^b],T^c\}_{\alpha\beta}\right]\delta_{\Sigma\Gamma}\end{aligned} \tag{B.34}$$

$$\mathcal{V}^{i\,ab}_{c_{A_1}} = -c_{A_1}\frac{ig^2}{8m^3}k_1^0\mathbf{k}_2^i\{T^a,T^b\}_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.35}$$

$$\mathcal{V}^{ij\,abc}_{c_{A_1}} = -c_{A_1}\frac{ig^3}{8m^3}\delta^{ij}(k_1^0\{T^a,[T^b,T^c]\}_{\alpha\beta} + k_2^0\{T^b,[T^a,T^c]\}_{\alpha\beta})\delta_{\Sigma\Gamma} \tag{B.36}$$

$$\mathcal{V}^{ab}_{c_{A_1}} = c_{A_1}\frac{ig^2}{8m^3}\mathbf{k}_1\cdot\mathbf{k}_2\{T^a,T^b\}_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.37}$$

$$\mathcal{V}^{i\,abc}_{c_{A_1}} = c_{A_1}\frac{ig^3}{8m^3}(\mathbf{k}_2^i\{T^b,[T^a,T^c]\}_{\alpha\beta} + \mathbf{k}_3^i\{T^c,[T^a,T^b]\}_{\alpha\beta})\delta_{\Sigma\Gamma} \tag{B.38}$$

Proportional to c_{A_2}

$$\mathcal{V}^{ij\,ab}_{c_{A_2}} = c_{A_2}\frac{ig^2}{16m^3}\delta^{ij}k_1^0k_2^0\{T^a,T^b\}_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.39}$$

$$\mathcal{V}^{i\,ab}_{c_{A_2}} = -c_{A_2}\frac{ig^2}{16m^3}k_1^0\mathbf{k}_2^i\{T^a,T^b\}_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.40}$$

$$\mathcal{V}^{ij\,abc}_{c_{A_2}} = -c_{A_2}\frac{ig^3}{16m^3}\delta^{ij}(k_1^0\{T^a,[T^b,T^c]\}_{\alpha\beta} + k_2^0\{T^b,[T^a,T^c]\}_{\alpha\beta}])\delta_{\Sigma\Gamma} \tag{B.41}$$

$$\mathcal{V}^{ab}_{c_{A_2}} = c_{A_2}\frac{ig^2}{16m^3}\mathbf{k}_1\cdot\mathbf{k}_2\{T^a,T^b\}_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.42}$$

$$\mathcal{V}^{i\,abc}_{c_{A_2}} = c_{A_2}\frac{ig^3}{16m^3}(\mathbf{k}_2^i\{T^b,[T^a,T^c]\}_{\alpha\beta} + \mathbf{k}_3^i\{T^c,[T^a,T^b]\}_{\alpha\beta})\delta_{\Sigma\Gamma} \tag{B.43}$$

Proportional to c_{A_3}

$$\mathcal{V}^{ij\,ab}_{c_{A_3}} = c_{A_3}\frac{ig^2}{4m^3}\frac{T_F}{N_c}(\delta^{ij}(k_1^0k_2^0 - \mathbf{k}_1\cdot\mathbf{k}_2) + \mathbf{k}_2^i\mathbf{k}_1^j)\delta^{ab}\delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.44}$$

$$\mathcal{V}^{ijk\,abc}_{c_{A_3}} = c_{A_3}\frac{g^3}{4m^3}\frac{T_F}{N_c}f^{abc}\left[\delta^{ik}(\mathbf{k}_1-\mathbf{k}_3)^j + \delta^{jk}(\mathbf{k}_3-\mathbf{k}_2)^i + \delta^{ij}(\mathbf{k}_2-\mathbf{k}_1)^k\right]\delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.45}$$

$$\mathcal{V}^{i\,ab}_{c_{A_3}} = -c_{A_3}\frac{ig^2}{4m^3}\frac{T_F}{N_c}k_1^0\mathbf{k}_2^i\delta^{ab}\delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.46}$$

$$\mathcal{V}^{ij\,abc}_{c_{A_3}} = c_{A_3}\frac{g^3}{4m^3}\frac{T_F}{N_c}\delta^{ij}f^{abc}(k_1^0 - k_2^0)\delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.47}$$

$$\mathcal{V}^{ab}_{c_{A_3}} = c_{A_3}\frac{ig^2}{4m^3}\frac{T_F}{N_c}\mathbf{k}_1\cdot\mathbf{k}_2\delta^{ab}\delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.48}$$

$$\mathcal{V}^{i\,abc}_{c_{A_3}} = -c_{A_3}\frac{g^3}{4m^3}\frac{T_F}{N_c}f^{abc}(\mathbf{k}_3 - \mathbf{k}_2)^i\delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.49}$$

Proportional to c_{A_4}

$$\mathcal{V}^{ij\,ab}_{c_{A_4}} = c_{A_4}\frac{ig^2}{8m^3}\frac{T_F}{N_c}\delta^{ij}k_1^0k_2^0\delta^{ab}\delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.50}$$

$$\mathcal{V}^{i\,ab}_{c_{A_4}} = -c_{A_4}\frac{ig^2}{8m^3}\frac{T_F}{N_c}k_1^0\mathbf{k}_2^i\delta^{ab}\delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.51}$$

$$\mathcal{V}^{ij\,abc}_{c_{A_4}} = c_{A_4}\frac{g^3}{8m^3}\frac{T_F}{N_c}\delta^{ij}f^{abc}(k_1^0 - k_2^0)\delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.52}$$

$$\mathcal{V}^{ab}_{c_{A_4}} = c^{(1)}_{A_4}\frac{ig^2}{8m^3}\frac{T_F}{N_c}\mathbf{k}_1\cdot\mathbf{k}_2\delta^{ab}\delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.53}$$

$$\mathcal{V}^{i\,abc}_{c_{A_4}} = -c^{(1)}_{A_4}\frac{g^3}{8m^3}\frac{T_F}{N_c}f^{abc}(\mathbf{k}_3 - \mathbf{k}_2)^i\delta_{\alpha\beta}\delta_{\Sigma\Gamma} \tag{B.54}$$

Proportional to c_{W_1}

$$\mathcal{V}^{i\,a}_{c_{W_1}} = -c_{W_1}\frac{g}{8m^3}(\mathbf{p}^2 + \mathbf{p}'^2)(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k})^i(T^a)_{\alpha\beta} \tag{B.55}$$

$$\begin{aligned}\mathcal{V}^{ij\,ab}_{c_{W_1}} \;=\; & -c_{W_1}\frac{g^2}{8m^3}[(\boldsymbol{\sigma}^k_{\Sigma\Gamma}\epsilon^{kij}(\mathbf{p}^2 + \mathbf{p}'^2) - (\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_1)^i\mathbf{k}_1^j + \mathbf{k}_2^i(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_2)^j)[T^a,T^b]_{\alpha\beta} \\ & -((\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_1)^i(\mathbf{p}+\mathbf{p}')^j + (\mathbf{p}+\mathbf{p}')^i(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_2)^j)\{T^a,T^b\}_{\alpha\beta}]\end{aligned} \tag{B.56}$$

$$\begin{aligned}\mathcal{V}^{ijk\,abc}_{c_{W_1}} \;=\; & c_{W_1}\frac{g^3}{8m^3}\boldsymbol{\sigma}^m_{\Sigma\Gamma}\Big(\epsilon^{mjk}\big(\{T^a,[T^b,T^c]\}_{\alpha\beta}(\mathbf{p}+\mathbf{p}')^i - [T^a,[T^b,T^c]]_{\alpha\beta}(\mathbf{k}_2+\mathbf{k}_3)^i\big) \\ & +\epsilon^{mki}\big(\{T^b,[T^c,T^a]\}_{\alpha\beta}(\mathbf{p}+\mathbf{p}')^j - [T^b,[T^c,T^a]]_{\alpha\beta}(\mathbf{k}_1+\mathbf{k}_3)^j\big) \\ & +\epsilon^{mij}\big(\{T^c,[T^a,T^b]\}_{\alpha\beta}(\mathbf{p}+\mathbf{p}')^k - [T^c,[T^a,T^b]]_{\alpha\beta}(\mathbf{k}_1+\mathbf{k}_2)^k\big) \\ & -\epsilon^{mri}\delta^{jk}\{T^a,\{T^b,T^c\}\}_{\alpha\beta}\mathbf{k}_1^r - \epsilon^{mrj}\delta^{ik}\{T^b,\{T^c,T^a\}\}_{\alpha\beta}\mathbf{k}_2^r \\ & -\epsilon^{mrk}\delta^{ij}\{T^c,\{T^a,T^b\}\}_{\alpha\beta}\mathbf{k}_3^r\Big)\end{aligned} \tag{B.57}$$

Proportional to c_{W_2}

$$\mathcal{V}^{i\,a}_{c_{W_2}} = c_{W_2}\frac{g}{4m^3}(\mathbf{p}\cdot\mathbf{p}')(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k})^i(T^a)_{\alpha\beta} \tag{B.58}$$

$$\begin{aligned}\mathcal{V}^{ij\,ab}_{c_{W_2}} \;=\;& -c_{W_2}\frac{g^2}{8m^3}[((\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_2)^j\mathbf{k}_2^i - (\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_1)^i\mathbf{k}_1^j - 2\boldsymbol{\sigma}^k_{\Sigma\Gamma}\epsilon^{kij}(\mathbf{p}\cdot\mathbf{p}'))[T^a,T^b]_{\alpha\beta}\\ &+((\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_1)^i(\mathbf{p}+\mathbf{p}')^j + (\mathbf{p}+\mathbf{p}')^i(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_2)^j)\{T^a,T^b\}_{\alpha\beta}]\end{aligned} \tag{B.59}$$

$$\begin{aligned}\mathcal{V}^{ijk\,abc}_{c_{W_2}} \;=\;& -c_{W_2}\frac{g^3}{4m^3}\boldsymbol{\sigma}^l_{\Sigma\Gamma}\Big(-\epsilon^{lij}([T^a,T^b]T^c)_{\alpha\beta}\mathbf{k}^k - \epsilon^{lki}([T^c,T^a]T^b)_{\alpha\beta}\mathbf{k}^j\\ &-\epsilon^{ljk}([T^b,T^c]T^a)_{\alpha\beta}\mathbf{k}^i + \epsilon^{lij}\{T^c,[T^a,T^b]\}_{\alpha\beta}\mathbf{p}^k + \epsilon^{lki}\{T^b,[T^c,T^a]\}_{\alpha\beta}\mathbf{p}^j\\ &+\epsilon^{ljk}\{T^a,[T^b,T^c]\}_{\alpha\beta}\mathbf{p}^i - \epsilon^{lrj}\delta^{ki}(T^aT^bT^c+T^cT^bT^a)_{\alpha\beta}\mathbf{k}_2^r\\ &-\epsilon^{lrk}\delta^{ij}(T^aT^cT^b+T^bT^cT^a)_{\alpha\beta}\mathbf{k}_3^r - \epsilon^{lri}\delta^{jk}(T^bT^aT^c+T^cT^aT^b)_{\alpha\beta}\mathbf{k}_1^r\Big)\end{aligned} \tag{B.60}$$

Proportional to $c_{p'p}$

$$\mathcal{V}^{i\,a}_{c_{p'p}} = c_{p'p}\frac{g}{8m^3}\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot(\mathbf{p}+\mathbf{p}')(\mathbf{p}\times\mathbf{p}')^i(T^a)_{\alpha\beta} \tag{B.61}$$

$$\begin{aligned}\mathcal{V}^{ij\,ab}_{c_{p'p}} \;=\;& c_{p'p}\frac{g^2}{16m^3}\Big\{\big[((\mathbf{p}+\mathbf{p}')\times\mathbf{k}_1)^i\boldsymbol{\sigma}^j_{\Sigma\Gamma} + \boldsymbol{\sigma}^i_{\Sigma\Gamma}((\mathbf{p}+\mathbf{p}')\times\mathbf{k}_2)^j\\ &+\epsilon^{ijk}(\mathbf{k}_1-\mathbf{k}_2)^k(\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot(\mathbf{p}+\mathbf{p}'))\big]\{T^a,T^b\}_{\alpha\beta} + \big[(\mathbf{k}_1\times\mathbf{k}_2)^i\boldsymbol{\sigma}^j_{\Sigma\Gamma} + \boldsymbol{\sigma}^i_{\Sigma\Gamma}(\mathbf{k}_1\times\mathbf{k}_2)^j\\ &-\epsilon^{ijk}(\mathbf{k}^k(\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{k}) + 2((\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{p})\mathbf{p}'^k + (\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{p}')\mathbf{p}^k))\big][T^a,T^b]_{\alpha\beta}\Big\}\end{aligned} \tag{B.62}$$

$$\begin{aligned}\mathcal{V}^{ijk\,abc}_{c_{p'p}} \;=\;& c_{p'p}\frac{g^3}{8m^3}\Big[(\epsilon^{ljk}\boldsymbol{\sigma}^i_{\Sigma\Gamma}\mathbf{p}^l + \epsilon^{ijk}(\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{p}))(T^a[T^b,T^c])_{\alpha\beta}\\ &+(\epsilon^{ljk}\boldsymbol{\sigma}^i_{\Sigma\Gamma}\mathbf{p}'^l + \epsilon^{ijk}(\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{p}'))([T^b,T^c]T^a)_{\alpha\beta}\\ &+(\epsilon^{lik}\boldsymbol{\sigma}^j_{\Sigma\Gamma}\mathbf{p}^l - \epsilon^{ijk}(\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{p}))(T^b[T^a,T^c])_{\alpha\beta}\\ &+(\epsilon^{lik}\boldsymbol{\sigma}^j_{\Sigma\Gamma}\mathbf{p}'^l - \epsilon^{ijk}(\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{p}'))([T^a,T^c]T^b)_{\alpha\beta}\\ &+(\epsilon^{lij}\boldsymbol{\sigma}^k_{\Sigma\Gamma}\mathbf{p}^l + \epsilon^{ijk}(\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{p}))(T^c[T^a,T^b])_{\alpha\beta}\\ &+(\epsilon^{lij}\boldsymbol{\sigma}^k_{\Sigma\Gamma}\mathbf{p}'^l + \epsilon^{ijk}(\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{p}'))([T^a,T^b]T^c)_{\alpha\beta}\\ &-(\epsilon^{ljk}\boldsymbol{\sigma}^i_{\Sigma\Gamma} - \epsilon^{lij}\boldsymbol{\sigma}^k_{\Sigma\Gamma})\mathbf{k}_2^l(T^aT^bT^c+T^cT^bT^a)_{\alpha\beta}\\ &+(\epsilon^{lik}\boldsymbol{\sigma}^j_{\Sigma\Gamma} + \epsilon^{ljk}\boldsymbol{\sigma}^i_{\Sigma\Gamma})\mathbf{k}_3^l(T^bT^cT^a+T^aT^cT^b)_{\alpha\beta}\\ &-(\epsilon^{lij}\boldsymbol{\sigma}^k_{\Sigma\Gamma} + \epsilon^{lik}\boldsymbol{\sigma}^j_{\Sigma\Gamma})\mathbf{k}_1^l(T^cT^aT^b+T^bT^aT^c)_{\alpha\beta}\Big]\end{aligned} \tag{B.63}$$

Proportional to c_{B_1}

$$\mathcal{V}^{ab}_{c_{B_1}} = -c_{B_1}\frac{g^2}{8m^3}\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot(\mathbf{k}_1\times\mathbf{k}_2)[T^a,T^b]_{\alpha\beta} \tag{B.64}$$

$$\mathcal{V}^{i\,ab}_{c_{B_1}} = -c_{B_1}\frac{g^2}{8m^3}k_1^0(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_2)^i[T^a,T^b]_{\alpha\beta} \tag{B.65}$$

$$\begin{aligned}\mathcal{V}^{ij\,ab}_{c_{B_1}} &= c_{B_1}\frac{g^2}{16m^3}\Big(\epsilon^{ijk}(\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{k}_1)\mathbf{k}_2^k+\epsilon^{ijk}(\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{k}_2)\mathbf{k}_1^k+\boldsymbol{\sigma}^i_{\Sigma\Gamma}(\mathbf{k}_1\times\mathbf{k}_2)^j\\ &+\boldsymbol{\sigma}^j_{\Sigma\Gamma}(\mathbf{k}_1\times\mathbf{k}_2)^i-2\boldsymbol{\sigma}^k_{\Sigma\Gamma}\epsilon^{kij}k_1^0k_2^0\Big)[T^a,T^b]_{\alpha\beta}\end{aligned} \tag{B.66}$$

$$\mathcal{V}^{i\,abc}_{c_{B_1}} = -c_{B_1}\frac{g^3}{8m^3}((\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_2)^i[T^b,[T^a,T^c]]_{\alpha\beta}+(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_3)^i[T^c,[T^a,T^b]]_{\alpha\beta}) \tag{B.67}$$

$$\mathcal{V}^{ij\,abc}_{c_{B_1}} = c_{B_1}\frac{g^3}{8m^3}\boldsymbol{\sigma}^k_{\Sigma\Gamma}\epsilon^{kij}(k_1^0[T^a,[T^b,T^c]]_{\alpha\beta}-k_2^0[T^b,[T^a,T^c]]_{\alpha\beta}) \tag{B.68}$$

$$\begin{aligned}\mathcal{V}^{ijk\,abc}_{c_{B_1}} =&- c_{B_1}\frac{g^3}{8m^3}\left\{\left(\epsilon^{ijk}\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{k}_1-\epsilon^{ljk}\boldsymbol{\sigma}^i_{\Sigma\Gamma}\mathbf{k}_1^l\right)[T^a,[T^b,T^c]]_{\alpha\beta}\right.\\ &-\left(\epsilon^{ijk}\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{k}_2-\epsilon^{ilk}\boldsymbol{\sigma}^j_{\Sigma\Gamma}\mathbf{k}_2^l\right)[T^b,[T^a,T^c]]_{\alpha\beta}\\ &\left.+\left(\epsilon^{ijk}\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot\mathbf{k}_3-\epsilon^{lij}\boldsymbol{\sigma}^k_{\Sigma\Gamma}\mathbf{k}_3^l\right)[T^c,[T^a,T^b]]_{\alpha\beta}\right\}\end{aligned} \tag{B.69}$$

Proportional to c_{B_2}

$$\mathcal{V}^{ab}_{c_{B_2}} = -c_{B_2}\frac{g^2}{8m^3}\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot(\mathbf{k}_1\times\mathbf{k}_2)[T^a,T^b]_{\alpha\beta} \tag{B.70}$$

$$\mathcal{V}^{i\,ab}_{c_{B_2}} = -c_{B_2}\frac{g^2}{8m^3}k_1^0(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_2)^i[T^a,T^b]_{\alpha\beta} \tag{B.71}$$

$$\mathcal{V}^{ij\,ab}_{c_{B_2}} = -c_{B_2}\frac{g^2}{8m^3}\boldsymbol{\sigma}^k_{\Sigma\Gamma}\epsilon^{kij}k_1^0k_2^0[T^a,T^b]_{\alpha\beta} \tag{B.72}$$

$$\mathcal{V}^{i\,abc}_{c_{B_2}} = -c_{B_2}\frac{g^3}{8m^3}((\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_2)^i[T^b,[T^a,T^c]]_{\alpha\beta}+(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k}_3)^i[T^c,[T^a,T^b]]_{\alpha\beta}) \tag{B.73}$$

$$\mathcal{V}^{ij\,abc}_{c_{B_2}} = c_{B_2}\frac{g^3}{8m^3}\boldsymbol{\sigma}^k_{\Sigma\Gamma}\epsilon^{kij}(k_1^0[T^a,[T^b,T^c]]_{\alpha\beta}-k_2^0[T^b,[T^a,T^c]]_{\alpha\beta}) \tag{B.74}$$

B.1.5 Heavy quark-light quark interactions

Proportional to c_i^{hl}

$$\mathcal{V} = c_1^{hl}\frac{ig^2}{8m^2}\delta_{\Sigma\Gamma}(T^a)_{\alpha\beta}(T^a)_{\delta\gamma}(\gamma^0)_{BA} \tag{B.75}$$

$$\mathcal{V} = -c_2^{hl}\frac{ig^2}{8m^2}\boldsymbol{\sigma}^i_{\Sigma\Gamma}(T^a)_{\alpha\beta}(T^a)_{\delta\gamma}(\gamma^i\gamma_5)_{BA} \tag{B.76}$$

$$\mathcal{V} = c_3^{hl}\frac{ig^2}{8m^2}\delta_{\Sigma\Gamma}\delta_{\alpha\beta}\delta_{\delta\gamma}(\gamma^0)_{BA} \tag{B.77}$$

$$\mathcal{V} = -c_4^{hl}\frac{ig^2}{8m^2}\boldsymbol{\sigma}^i_{\Sigma\Gamma}\delta_{\alpha\beta}\delta_{\delta\gamma}(\gamma^i\gamma_5)_{BA} \tag{B.78}$$

Proportional to d_1^{hl}

$$\mathcal{V} = -d_1^{hl}\frac{ig^2}{m^3}(T^a)_{\alpha\beta}(T^a)_{\delta\gamma}(\gamma^i)_{BA}(\mathbf{p}+\mathbf{p}')^i\delta_{\Sigma\Gamma} \tag{B.79}$$

$$\mathcal{V}^{i\,b} = d_1^{hl}\frac{ig^3}{m^3}\{T^a,T^b\}_{\alpha\beta}(T^a)_{\delta\gamma}(\gamma^i)_{BA}\delta_{\Sigma\Gamma} \tag{B.80}$$

Proportional to d_2^{hl}

$$\mathcal{V} = -d_2^{hl}\frac{ig^2}{m^3}(T^a)_{\alpha\beta}(T^a)_{\delta\gamma}(\gamma^0)_{BA}(k_1^0-k_2^0)\delta_{\Sigma\Gamma} \tag{B.81}$$

$$\mathcal{V}^b = -d_2^{hl}\frac{ig^3}{m^3}(T^a)_{\alpha\beta}\{T^a,T^b\}_{\delta\gamma}(\gamma^0)_{BA}\delta_{\Sigma\Gamma} \tag{B.82}$$

Proportional to d_3^{hl}

$$\mathcal{V} = -d_3^{hl}\frac{ig^2}{m^3}\delta_{\alpha\beta}\delta_{\delta\gamma}(\gamma^0)_{BA}(k_1^0-k_2^0)\delta_{\Sigma\Gamma} \tag{B.83}$$

$$\mathcal{V}^b = -d_3^{hl}\frac{2ig^3}{m^3}\delta_{\alpha\beta}(T^b)_{\delta\gamma}(\gamma^0)_{BA}\delta_{\Sigma\Gamma} \tag{B.84}$$

Proportional to d_4^{hl}

$$\mathcal{V} = -d_4^{hl}\frac{ig^2}{m^3}(\gamma^0\gamma_5)_{BA}(T^a)_{\alpha\beta}(T^a)_{\delta\gamma}\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot(\mathbf{p}+\mathbf{p}') \tag{B.85}$$

$$\mathcal{V}^{i\,b} = d_4^{hl}\frac{ig^3}{m^3}(\gamma^0\gamma_5)_{BA}(T^a)_{\delta\gamma}\{T^a,T^b\}_{\alpha\beta}\boldsymbol{\sigma}^i_{\Sigma\Gamma} \tag{B.86}$$

Proportional to d_5^{hl}

$$\mathcal{V} = -d_5^{hl}\frac{ig^2}{m^3}(\gamma^0\gamma_5)_{BA}\delta_{\alpha\beta}\delta_{\delta\gamma}\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot(\mathbf{p}+\mathbf{p}') \tag{B.87}$$

$$\mathcal{V}^{i\,b} = d_5^{hl}\frac{2ig^3}{m^3}(\gamma^0\gamma_5)_{BA}\delta_{\delta\gamma}(T^b)_{\alpha\beta}\boldsymbol{\sigma}^i_{\Sigma\Gamma} \tag{B.88}$$

Proportional to d_6^{hl}

$$\mathcal{V} = d_6^{hl}\frac{ig^2}{m^3}(\gamma^0\gamma_5)_{BA}(T^a)_{\alpha\beta}(T^a)_{\delta\gamma}\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot(\mathbf{k}_1-\mathbf{k}_2) \tag{B.89}$$

$$\mathcal{V}^{i\,b} = d_6^{hl}\frac{ig^3}{m^3}(\gamma^0\gamma_5)_{BA}\{T^a,T^b\}_{\delta\gamma}(T^a)_{\alpha\beta}\boldsymbol{\sigma}^i_{\Sigma\Gamma} \tag{B.90}$$

Proportional to d_7^{hl}

$$\mathcal{V} = d_7^{hl}\frac{ig^2}{m^3}(\gamma^0\gamma_5)_{BA}\delta_{\alpha\beta}\delta_{\delta\gamma}\boldsymbol{\sigma}_{\Sigma\Gamma}\cdot(\mathbf{k}_1-\mathbf{k}_2) \tag{B.91}$$

$$\mathcal{V}^{i\,b} = d_7^{hl}\frac{2ig^3}{m^3}(\gamma^0\gamma_5)_{BA}(T^b)_{\delta\gamma}\delta_{\alpha\beta}\boldsymbol{\sigma}^i_{\Sigma\Gamma} \tag{B.92}$$

Proportional to d_8^{hl}

$$\mathcal{V} = d_8^{hl}\frac{ig^2}{m^3}(\gamma^i\gamma_5)_{BA}(T^a)_{\alpha\beta}(T^a)_{\delta\gamma}\boldsymbol{\sigma}^i_{\Sigma\Gamma}(k_1^0-k_2^0) \tag{B.93}$$

$$\mathcal{V}^{b} = d_8^{hl}\frac{ig^3}{m^3}(\gamma^i\gamma_5)_{BA}\{T^a,T^b\}_{\delta\gamma}(T^a)_{\alpha\beta}\boldsymbol{\sigma}^i_{\Sigma\Gamma} \tag{B.94}$$

Proportional to d_9^{hl}

$$\mathcal{V} = d_9^{hl}\frac{ig^2}{m^3}(\gamma^i\gamma_5)_{BA}\delta_{\alpha\beta}\delta_{\delta\gamma}\boldsymbol{\sigma}^i_{\Sigma\Gamma}(k_1^0-k_2^0) \tag{B.95}$$

$$\mathcal{V}^{b} = d_9^{hl}\frac{2ig^3}{m^3}(\gamma^i\gamma_5)_{BA}(T^b)_{\delta\gamma}\delta_{\alpha\beta}\boldsymbol{\sigma}^i_{\Sigma\Gamma} \tag{B.96}$$

Proportional to d_{10}^{hl}

$$\mathcal{V} = d_{10}^{hl}\frac{g^2}{m^3}(T^a)_{\alpha\beta}(T^a)_{\delta\gamma}(\gamma^i)_{BA}(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k})^i \tag{B.97}$$

$$\mathcal{V}^{i\,b} = d_{10}^{hl}\frac{g^3}{m^3}(\gamma^j)_{BA}[T^a,T^b]_{\delta\gamma}(T^a)_{\alpha\beta}\epsilon^{ijk}\boldsymbol{\sigma}^k_{\Sigma\Gamma} \tag{B.98}$$

Proportional to d_{11}^{hl}

$$\mathcal{V} = d_{11}^{hl}\frac{g^2}{m^3}\delta_{\alpha\beta}\delta_{\delta\gamma}(\gamma^i)_{BA}(\boldsymbol{\sigma}_{\Sigma\Gamma}\times\mathbf{k})^i \tag{B.99}$$

B.2 Resummed Wilson coefficients of the $1/m$ and $1/m^2$ operators with LL accuracy

In order to solve the RGEs of Sec. 2.5, we need the resummed expressions of the Wilson coefficients associated to the $1/m$ and $1/m^2$ operators of the HQET Lagrangian with LL accuracy. They can be found in Refs. [8, 28, 32], but summarize them in the following for practical reasons

$$c_F(\nu) = z^{-C_A}\,, \tag{B.100}$$

$$\bar{c}_1^{hl}(\nu) = z^{-2C_A} + \left(\frac{20}{13}+\frac{32C_F}{13C_A}\right)(1-z^{-13C_A/6})\,, \tag{B.101}$$

$$\begin{aligned} c_2^{hl}(\nu) \;=\; & (8C_F-3C_A)\left(\frac{1}{2\beta_0-7C_A}\left(z^{-2C_A}-z^{-\beta_0+\frac{3C_A}{2}}\right)\right. \\ & \left. -\frac{2}{2\beta_0-5C_A}\left(z^{-C_A}-z^{-\beta_0+\frac{3C_A}{2}}\right)-\frac{3}{2\beta_0-3C_A}\left(1-z^{-\beta_0+\frac{3C_A}{2}}\right)\right)\,, \end{aligned} \tag{B.102}$$

$$c_3^{hl}(\nu) = 0\,, \tag{B.103}$$

$$c_4^{hl}(\nu) = \left(1-\frac{1}{N_c^2}\right)\left(\frac{1}{2\beta_0-4C_A}(z^{-2C_A}-z^{-\beta_0})-\frac{1}{\beta_0-C_A}(z^{-C_A}-z^{-\beta_0})-\frac{3}{2\beta_0}(1-z^{-\beta_0})\right)\,. \tag{B.104}$$

The other Wilson coefficients we need are known from reparametrization invariance [28] relations. They read $c_k = c_4 = 1$, $c_S = 2c_F - 1$, $c_{W_1} - c_{W_2} = 1$, $c_{p'p} = c_F - 1$ and $d_4^{hl} = -c_2^{hl}/16$.

B.3 About the problem with reparametrization invariance

Reparametrization invariance [28] obtains relations between Wilson coefficients by imposing Lorentz invariance as a symmetry of the HQET Lagrangian. Since this is done without assuming any particular gauge, the obtained relations should be, in principle, gauge independent. However, in Sec. 2.5 we found that the relations imposed by reparametrization invariance $2c_M = c_D - c_F$ and $d_1^{hl} = c_1^{hl}/16$ are violated at LL when the computation is done in the Coulomb gauge, proving that these two relations indeed depend on the gauge.

In order to solve this problem, a new gauge invariant relation was proposed in Sec. 2.5.3 by adding the two relations which are not satisfied in the Coulomb gauge but that are supposed to be in some particular gauge (we will see later on that in the Feynman gauge they are satisfied). The motivation to add these two equations is to make it to appear $\bar{c}_1^{hl}$, which is well-known to be gauge independent. The new relation $2c_M+16d_1^{hl}=c_D+c_1^{hl}-c_F$, or likewise, $2\bar{d}_1^{hl}=\bar{c}_1^{hl}-c_F$ implies that $\bar{d}_1^{hl}$ must be also gauge independent. It can be shown that the proposed relation is satisfied in both, the Coulomb gauge, and the gauge where the two "problematic" relations are satisfied.

In this section we want to show that, assuming that $\bar{d}_1^{hl}$ (obtained in a Coulomb gauge computation) is gauge independent, and that in the Feynman gauge $2c_M^{\rm FG}=c_D^{\rm FG}-c_F$ is satisfied[1], we can obtain $d_1^{hl\,\rm FG}$ and see that it satisfies reparametrization invariance, $d_1^{hl\,\rm FG}=c_1^{hl\,\rm FG}/16$.

Firstly, let us summarize the RGEs of the unphysical quantities c_D, c_1^{hl}, c_M and d_1^{hl} in the Coulomb gauge. They read

$$\nu\frac{d}{d\nu}c_D^{\rm CG}=\left[\frac{11}{6}c_1^{hl\,\rm CG}C_A+\frac{11}{6}c_D^{\rm CG}C_A-\frac{5}{6}c_F^2C_A-\frac{8}{3}c_k^2C_A-\frac{8}{3}c_k^2C_F-\frac{1}{2}c_1^{hl\,\rm CG}\beta_0\right]\frac{\alpha}{\pi}\,, \quad \text{(B.105)}$$

$$\nu\frac{d}{d\nu}c_1^{hl\,\rm CG}=\left[-\frac{3}{4}c_1^{hl\,\rm CG}C_A-\frac{3}{4}c_D^{\rm CG}C_A+\frac{3}{4}c_F^2C_A+c_k^2C_A+\frac{1}{2}c_1^{hl\,\rm CG}\beta_0\right]\frac{\alpha}{\pi}\,, \quad \text{(B.106)}$$

$$\nu\frac{d}{d\nu}c_M^{\rm CG}=\left[\frac{7}{6}c_M^{\rm CG}C_A+\frac{1}{2}c_4C_A-\frac{1}{3}c_Sc_FC_A-\frac{1}{6}c_k^3(8C_F+11C_A)-\frac{1}{12}c_F^2c_kC_A+\frac{16}{3}d_1^{hl\,\rm CG}T_Fn_f\right]\frac{\alpha}{\pi}\,, \quad \text{(B.107)}$$

$$\nu\frac{d}{d\nu}d_1^{hl\,\rm CG}=\left[\frac{1}{4}(2\beta_0-3C_A)d_1^{hl\,\rm CG}-C_A\left(\frac{1}{96}c_M^{\rm CG}-\frac{1}{192}c_Sc_F-\frac{1}{16}c_k^3-\frac{5}{64}c_kc_F^2\right)\right]\frac{\alpha}{\pi}\,. \quad \text{(B.108)}$$

From these equations it can be seen that $2c_M^{\rm CG}\neq c_D^{\rm CG}-c_F$ and $d_1^{hl\,\rm CG}\neq c_1^{hl\,\rm CG}/16$. We can also obtain the RGEs of the physical quantities $\bar{c}_1^{hl}$ and $\bar{d}_1^{hl}$, which read

$$\nu\frac{d}{d\nu}\bar{c}_1^{hl}=\left[\frac{13}{12}C_A\bar{c}_1^{hl}-\frac{20}{12}C_Ac_k^2-\frac{32}{12}C_Fc_k^2-\frac{1}{12}C_Ac_F^2\right]\frac{\alpha}{\pi}\,, \quad \text{(B.109)}$$

$$\nu\frac{d}{d\nu}\bar{d}_1^{hl}=\left[\frac{13}{12}\bar{d}_1^{hl}C_A+\frac{1}{2}c_4C_A-\frac{7}{24}c_Sc_FC_A-\frac{4}{3}c_k^3(C_F+C_A)+\frac{13}{24}c_F^2c_kC_A\right]\frac{\alpha}{\pi}\,. \quad \text{(B.110)}$$

We also need the RGEs of c_D and $\bar{c}_1^{hl}$ in the Feynman gauge. They were computed in Ref. [32], and read

[1]At the level of the SL, it has been proven in Refs. [28, 33, 34].

$$\nu\frac{d}{d\nu}c_D^{\rm FG} = \left[\frac{1}{3}C_A c_D^{\rm FG} - \left(\frac{1}{6}C_A + \frac{8}{3}C_F\right)c_k^2 - \frac{5}{6}C_A c_F^2 + \frac{2}{3}T_f n_f c_1^{hl\,{\rm FG}}\right]\frac{\alpha}{\pi}\,, \tag{B.111}$$

$$\nu\frac{d}{d\nu}c_1^{hl\,{\rm FG}} = \left[\frac{3}{4}C_A c_D^{\rm FG} - \frac{3}{2}C_A c_k^2 + \frac{3}{4}C_A c_F^2 - \left(\frac{3}{4}C_A - \frac{1}{2}\beta_0\right)c_1^{hl\,{\rm FG}}\right]\frac{\alpha}{\pi}\,. \tag{B.112}$$

What it remains to be computed are the RGEs of c_M and d_1^{hl} in the Feynman gauge. Firstly, we compute $c_M^{\rm FG}$ assuming that the relation $2c_M^{\rm FG} = c_D^{\rm FG} - c_F$ is satisfied. Thus, from the new relation $2\bar{d}_1^{hl} = 2c_M^{\rm FG} + 8d_1^{hl\,{\rm FG}} = \bar{c}_1^{hl} - c_F$ proposed in Sec. 2.5.3 we can compute $d_1^{hl\,{\rm FG}}$. The RGEs of these two Wilson coefficients in the Feynman gauge read

$$\nu\frac{d}{d\nu}c_M^{\rm FG} = \left[\frac{1}{6}C_A c_D^{\rm FG} - \frac{1}{12}(C_A + 16C_F)c_k^2 - \frac{5}{12}C_A c_F^2 - \frac{1}{4}c_F C_A + \frac{1}{3}T_f n_f c_1^{hl\,{\rm FG}}\right]\frac{\alpha}{\pi}\,, \tag{B.113}$$

$$\begin{aligned}\nu\frac{d}{d\nu}d_1^{hl\,{\rm FG}} &= \left[\frac{13}{96}\bar{d}_1^{hl}C_A + \frac{1}{16}c_4 C_A + \frac{13}{192}c_F C_A - \frac{5}{32}C_A c_k^3 + \frac{3}{64}c_F^2 c_k C_A\right.\\ &\left.\quad -\frac{1}{48}C_A c_D^{\rm FG} - \frac{1}{24}T_f n_f c_1^{hl\,{\rm FG}}\right]\frac{\alpha}{\pi}\,.\end{aligned} \tag{B.114}$$

If we compare the running of d_1^{hl} with the running of c_1^{hl} in the Feynman gauge (obviously assuming $2c_M^{\rm FG} = c_D^{\rm FG} - c_F$ and $d_1^{hl\,{\rm FG}} = c_1^{hl\,{\rm FG}}/16$) we find that

$$\nu\frac{d}{d\nu}(16d_1^{hl\,{\rm FG}} - c_1^{hl\,{\rm FG}}) = 0\,, \tag{B.115}$$

so everything is consistent, the relations $2c_M = c_D - c_F$ and $d_1^{hl} = c_1^{hl}/16$ are satisfied in the Feynman gauge, but not in the Coulomb gauge. The correct gauge-independent relation is $16d_1^{hl} + 2c_M = c_D + c_1^{hl} - c_F$, since it involves truly gauge independent quantities, $\bar{d}_1^{hl}$ and $\bar{c}_1^{hl}$.

Possibly, the source of the problem is that we have to include heavy-light operators when imposing reparametrization invariance in order to arrive to gauge-independent quantities. Otherwise, part of the Lagrangian is missing, and gauge dependencies arise. In particular, imposing Lorentz invariance only in the heavy-gluon sector does not guarantee that the obtained relations between Wilson coefficients are gauge-independent. Contrarily, it seems that the imposition of Lorentz invariance in the heavy-gluon and the heavy-light sectors separately, fixes somehow a particular gauge or class of gauges for which the relations $2c_M = c_D - c_F$ and $d_1^{hl} = c_1^{hl}/16$ are satisfied. A rigorous proof of this statement is beyond the scope of this work, and it is left for future investigation.

B.4 HQET Master Integrals

In this section we display the integrals needed for the calculation of the LL running of the HQET Wilson coefficients. We always integrate first over the energy. The following integrals appear

$$I(n,m) \equiv \int \frac{dq^0}{2\pi} \frac{1}{((q^0)^2 - \mathbf{q}^2 + i\eta_g)^n (q^0 - i\eta_q)^m} = -\frac{i}{\Gamma(n)} \frac{d^{(n-1)}}{dq^0} \left(\frac{1}{(q^0)^m (q^0 + |\mathbf{q}|)^n} \right)_{q^0 = |\mathbf{q}|}, \tag{B.116}$$

$$I(0,1) = \int \frac{dq^0}{2\pi} \frac{1}{q^0 - i\eta} = \frac{i}{2}, \tag{B.117}$$

where $n, m \in \mathbb{N}$. Later, we integrate over the d-momentum ($d = 3 + 2\epsilon$). The following logarithmically divergent integrals appear

$$I = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{|\mathbf{q}|^3} \to \frac{1}{4\pi^2} \frac{1}{\epsilon} \nu^{2\epsilon}, \tag{B.118}$$

$$I^{ij} = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{\mathbf{q}^i\mathbf{q}^j}{|\mathbf{q}|^5} \to \frac{1}{12\pi^2} \frac{1}{\epsilon} \delta^{ij} \nu^{2\epsilon}, \tag{B.119}$$

$$I^{ijkl} = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{\mathbf{q}^i\mathbf{q}^j\mathbf{q}^k\mathbf{q}^l}{|\mathbf{q}|^7} \to \frac{1}{60\pi^2} \frac{1}{\epsilon} (\delta^{ik}\delta^{jl} + \delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk}) \nu^{2\epsilon}, \tag{B.120}$$

$$\begin{aligned} I^{ijklmn} &= \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{\mathbf{q}^i\mathbf{q}^j\mathbf{q}^k\mathbf{q}^l\mathbf{q}^m\mathbf{q}^n}{|\mathbf{q}|^9} \to \frac{1}{420\pi^2} \frac{1}{\epsilon} \big(\delta^{ij}(\delta^{kl}\delta^{mn} + \delta^{km}\delta^{ln} + \delta^{kn}\delta^{lm}) \\ &+\delta^{ik}(\delta^{jl}\delta^{mn} + \delta^{jm}\delta^{ln} + \delta^{jn}\delta^{lm}) + \delta^{il}(\delta^{jk}\delta^{mn} + \delta^{jm}\delta^{kn} + \delta^{jn}\delta^{km}) \\ &+\delta^{im}(\delta^{jk}\delta^{ln} + \delta^{jl}\delta^{kn} + \delta^{jn}\delta^{kl}) + \delta^{in}(\delta^{jk}\delta^{lm} + \delta^{jl}\delta^{km} + \delta^{jm}\delta^{kl}) \big) \nu^{2\epsilon}, \end{aligned} \tag{B.121}$$

$$I^{ijklmnrs} = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{\mathbf{q}^i\mathbf{q}^j\mathbf{q}^k\mathbf{q}^l\mathbf{q}^m\mathbf{q}^n\mathbf{q}^r\mathbf{q}^s}{|\mathbf{q}|^{11}} \to \frac{1}{3780\pi^2} \frac{1}{\epsilon} \Big(\delta^{is}(\delta^{jr}(\delta^{kn}\delta^{lm} + \delta^{km}\delta^{ln} + \delta^{kl}\delta^{mn})$$

$$+\delta^{jn}(\delta^{kr}\delta^{lm} + \delta^{km}\delta^{lr} + \delta^{kl}\delta^{mr}) + \delta^{jm}(\delta^{kr}\delta^{ln} + \delta^{kn}\delta^{lr} + \delta^{kl}\delta^{nr}) + \delta^{jl}(\delta^{kr}\delta^{mn} + \delta^{kn}\delta^{mr} + \delta^{km}\delta^{nr})$$

$$+\delta^{jk}(\delta^{lr}\delta^{mn} + \delta^{ln}\delta^{mr} + \delta^{lm}\delta^{nr})) + \delta^{ir}(\delta^{js}(\delta^{kn}\delta^{lm} + \delta^{km}\delta^{ln} + \delta^{kl}\delta^{mn}) +$$

$$\delta^{jn}(\delta^{ks}\delta^{lm} + \delta^{km}\delta^{ls} + \delta^{kl}\delta^{ms}) + \delta^{jm}(\delta^{ks}\delta^{ln} + \delta^{kn}\delta^{ls} + \delta^{kl}\delta^{ns}) + \delta^{jl}(\delta^{ks}\delta^{mn} + \delta^{kn}\delta^{ms} + \delta^{km}\delta^{ns})$$

$$+\delta^{jk}(\delta^{ls}\delta^{mn} + \delta^{ln}\delta^{ms} + \delta^{lm}\delta^{ns})) + \delta^{in}(\delta^{js}(\delta^{kr}\delta^{lm} + \delta^{km}\delta^{lr} + \delta^{kl}\delta^{mr}) + \delta^{jr}(\delta^{ks}\delta^{lm} + \delta^{km}\delta^{ls} + \delta^{kl}\delta^{ms})$$

$$+\delta^{jm}(\delta^{ks}\delta^{lr} + \delta^{kr}\delta^{ls} + \delta^{kl}\delta^{rs}) + \delta^{jl}(\delta^{ks}\delta^{mr} + \delta^{kr}\delta^{ms} + \delta^{km}\delta^{rs}) + \delta^{jk}(\delta^{ls}\delta^{mr} + \delta^{lr}\delta^{ms} + \delta^{lm}\delta^{rs}))$$

$$+\delta^{im}(\delta^{js}(\delta^{kr}\delta^{ln} + \delta^{kn}\delta^{lr} + \delta^{kl}\delta^{nr}) + \delta^{jr}(\delta^{ks}\delta^{ln} + \delta^{kn}\delta^{ls} + \delta^{kl}\delta^{ns}) + \delta^{jn}(\delta^{ks}\delta^{lr} + \delta^{kr}\delta^{ls} + \delta^{kl}\delta^{rs})$$

$$+\delta^{jl}(\delta^{ks}\delta^{nr} + \delta^{kr}\delta^{ns} + \delta^{kn}\delta^{rs}) + \delta^{jk}(\delta^{ls}\delta^{nr} + \delta^{lr}\delta^{ns} + \delta^{ln}\delta^{rs})) + \delta^{il}(\delta^{js}(\delta^{kr}\delta^{mn} + \delta^{kn}\delta^{mr} + \delta^{km}\delta^{nr})$$

$$
\begin{aligned}
&+\delta^{jr}(\delta^{ks}\delta^{mn}+\delta^{kn}\delta^{ms}+\delta^{km}\delta^{ns})+\delta^{jn}(\delta^{ks}\delta^{mr}+\delta^{kr}\delta^{ms}+\delta^{km}\delta^{rs})+\delta^{jm}(\delta^{ks}\delta^{nr}+\delta^{kr}\delta^{ns}+\delta^{kn}\delta^{rs})\\
&+\delta^{jk}(\delta^{ms}\delta^{nr}+\delta^{mr}\delta^{ns}+\delta^{mn}\delta^{rs}))+\delta^{ik}(\delta^{js}(\delta^{lr}\delta^{mn}+\delta^{ln}\delta^{mr}+\delta^{lm}\delta^{nr})+\delta^{jr}(\delta^{ls}\delta^{mn}+\delta^{ln}\delta^{ms}+\delta^{lm}\delta^{ns})\\
&+\delta^{jn}(\delta^{ls}\delta^{mr}+\delta^{lr}\delta^{ms}+\delta^{lm}\delta^{rs})+\delta^{jm}(\delta^{ls}\delta^{nr}+\delta^{lr}\delta^{ns}+\delta^{ln}\delta^{rs})+\delta^{jl}(\delta^{ms}\delta^{nr}+\delta^{mr}\delta^{ns}+\delta^{mn}\delta^{rs}))\\
&+\delta^{ij}(\delta^{ks}(\delta^{lr}\delta^{mn}+\delta^{ln}\delta^{mr}+\delta^{lm}\delta^{nr})+\delta^{kr}(\delta^{ls]}\delta^{mn}+\delta^{ln}\delta^{ms}+\delta^{lm}\delta^{ns})+\delta^{kn}(\delta^{ls}\delta^{mr}+\delta^{lr}\delta^{ms}+\delta^{lm}\delta^{rs})\\
&\qquad+\delta^{km}(\delta^{ls}\delta^{nr}+\delta^{lr}\delta^{ns}+\delta^{ln}\delta^{rs})+\delta^{kl}(\delta^{ms}\delta^{nr}+\delta^{mr}\delta^{ns}+\delta^{mn}\delta^{rs}))\Bigg)\nu^{2\epsilon}\,, \qquad \text{(B.122)}
\end{aligned}
$$

$$
\begin{aligned}
I^{ijklmnrstu} &= \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{\mathbf{q}^i\mathbf{q}^j\mathbf{q}^k\mathbf{q}^l\mathbf{q}^m\mathbf{q}^n\mathbf{q}^r\mathbf{q}^s\mathbf{q}^t\mathbf{q}^u}{|\mathbf{q}|^{13}} \to \frac{1}{41580\pi^2}\frac{1}{\epsilon}\Bigg(\delta^{mn}\delta^{lr}\delta^{ks}\delta^{ju}\delta^{iv}+\delta^{ln}\delta^{mr}\delta^{ks}\delta^{ju}\delta^{iv}\\
&+\delta^{lm}\delta^{nr}\delta^{ks}\delta^{ju}\delta^{iv}+\delta^{mn}\delta^{kr}\delta^{ls}\delta^{ju}\delta^{iv}+\delta^{kn}\delta^{mr}\delta^{ls}\delta^{ju}\delta^{iv}+\delta^{km}\delta^{nr}\delta^{ls}\delta^{ju}\delta^{iv}+\delta^{ln}\delta^{kr}\delta^{ms}\delta^{ju}\delta^{iv}\\
&+\delta^{kn}\delta^{lr}\delta^{ms}\delta^{ju}\delta^{iv}+\delta^{kl}\delta^{nr}\delta^{ms}\delta^{ju}\delta^{iv}+\delta^{lm}\delta^{kr}\delta^{ns}\delta^{ju}\delta^{iv}+\delta^{km}\delta^{lr}\delta^{ns}\delta^{ju}\delta^{iv}+\delta^{kl}\delta^{mr}\delta^{ns}\delta^{ju}\delta^{iv}\\
&+\delta^{lm}\delta^{kn}\delta^{rs}\delta^{ju}\delta^{iv}+\delta^{km}\delta^{ln}\delta^{rs}\delta^{ju}\delta^{iv}+\delta^{kl}\delta^{mn}\delta^{rs}\delta^{ju}\delta^{iv}+\delta^{mn}\delta^{lr}\delta^{js}\delta^{ku}\delta^{iv}+\delta^{ln}\delta^{mr}\delta^{js}\delta^{ku}\delta^{iv}\\
&+\delta^{lm}\delta^{nr}\delta^{js}\delta^{ku}\delta^{iv}+\delta^{mn}\delta^{jr}\delta^{ls}\delta^{ku}\delta^{iv}+\delta^{jn}\delta^{mr}\delta^{ls}\delta^{ku}\delta^{iv}+\delta^{jm}\delta^{nr}\delta^{ls}\delta^{ku}\delta^{iv}+\delta^{ln}\delta^{jr}\delta^{ms}\delta^{ku}\delta^{iv}\\
&+\delta^{jn}\delta^{lr}\delta^{ms}\delta^{ku}\delta^{iv}+\delta^{jl}\delta^{nr}\delta^{ms}\delta^{ku}\delta^{iv}+\delta^{lm}\delta^{jr}\delta^{ns}\delta^{ku}\delta^{iv}+\delta^{jm}\delta^{lr}\delta^{ns}\delta^{ku}\delta^{iv}+\delta^{jl}\delta^{mr}\delta^{ns}\delta^{ku}\delta^{iv}\\
&+\delta^{lm}\delta^{jn}\delta^{rs}\delta^{ku}\delta^{iv}+\delta^{jm}\delta^{ln}\delta^{rs}\delta^{ku}\delta^{iv}+\delta^{jl}\delta^{mn}\delta^{rs}\delta^{ku}\delta^{iv}+\delta^{mn}\delta^{kr}\delta^{js}\delta^{lu}\delta^{iv}+\delta^{kn}\delta^{mr}\delta^{js}\delta^{lu}\delta^{iv}\\
&+\delta^{km}\delta^{nr}\delta^{js}\delta^{lu}\delta^{iv}+\delta^{mn}\delta^{jr}\delta^{ks}\delta^{lu}\delta^{iv}+\delta^{jn}\delta^{mr}\delta^{ks}\delta^{lu}\delta^{iv}+\delta^{jm}\delta^{nr}\delta^{ks}\delta^{lu}\delta^{iv}+\delta^{kn}\delta^{jr}\delta^{ms}\delta^{lu}\delta^{iv}\\
&+\delta^{jn}\delta^{kr}\delta^{ms}\delta^{lu}\delta^{iv}+\delta^{jk}\delta^{nr}\delta^{ms}\delta^{lu}\delta^{iv}+\delta^{km}\delta^{jr}\delta^{ns}\delta^{lu}\delta^{iv}+\delta^{jm}\delta^{kr}\delta^{ns}\delta^{lu}\delta^{iv}+\delta^{jk}\delta^{mr}\delta^{ns}\delta^{lu}\delta^{iv}\\
&+\delta^{km}\delta^{jn}\delta^{rs}\delta^{lu}\delta^{iv}+\delta^{jm}\delta^{kn}\delta^{rs}\delta^{lu}\delta^{iv}+\delta^{jk}\delta^{mn}\delta^{rs}\delta^{lu}\delta^{iv}+\delta^{ln}\delta^{kr}\delta^{js}\delta^{mu}\delta^{iv}+\delta^{kn}\delta^{lr}\delta^{js}\delta^{mu}\delta^{iv}\\
&+\delta^{kl}\delta^{nr}\delta^{js}\delta^{mu}\delta^{iv}+\delta^{ln}\delta^{jr}\delta^{ks}\delta^{mu}\delta^{iv}+\delta^{jn}\delta^{lr}\delta^{ks}\delta^{mu}\delta^{iv}+\delta^{jl}\delta^{nr}\delta^{ks}\delta^{mu}\delta^{iv}+\delta^{kn}\delta^{jr}\delta^{ls}\delta^{mu}\delta^{iv}\\
&+\delta^{jn}\delta^{kr}\delta^{ls}\delta^{mu}\delta^{iv}+\delta^{jk}\delta^{nr}\delta^{ls}\delta^{mu}\delta^{iv}+\delta^{kl}\delta^{jr}\delta^{ns}\delta^{mu}\delta^{iv}+\delta^{jl}\delta^{kr}\delta^{ns}\delta^{mu}\delta^{iv}+\delta^{jk}\delta^{lr}\delta^{ns}\delta^{mu}\delta^{iv}\\
&+\delta^{kl}\delta^{jn}\delta^{rs}\delta^{mu}\delta^{iv}+\delta^{jl}\delta^{kn}\delta^{rs}\delta^{mu}\delta^{iv}+\delta^{jk}\delta^{ln}\delta^{rs}\delta^{mu}\delta^{iv}+\delta^{lm}\delta^{kr}\delta^{js}\delta^{nu}\delta^{iv}+\delta^{km}\delta^{lr}\delta^{js}\delta^{nu}\delta^{iv}\\
&+\delta^{kl}\delta^{mr}\delta^{js}\delta^{nu}\delta^{iv}+\delta^{lm}\delta^{jr}\delta^{ks}\delta^{nu}\delta^{iv}+\delta^{jm}\delta^{lr}\delta^{ks}\delta^{nu}\delta^{iv}+\delta^{jl}\delta^{mr}\delta^{ks}\delta^{nu}\delta^{iv}+\delta^{km}\delta^{jr}\delta^{ls}\delta^{nu}\delta^{iv}\\
&+\delta^{jm}\delta^{kr}\delta^{ls}\delta^{nu}\delta^{iv}+\delta^{jk}\delta^{mr}\delta^{ls}\delta^{nu}\delta^{iv}+\delta^{kl}\delta^{jr}\delta^{ms}\delta^{nu}\delta^{iv}+\delta^{jl}\delta^{kr}\delta^{ms}\delta^{nu}\delta^{iv}+\delta^{jk}\delta^{lr}\delta^{ms}\delta^{nu}\delta^{iv}\\
&+\delta^{kl}\delta^{jm}\delta^{rs}\delta^{nu}\delta^{iv}+\delta^{jl}\delta^{km}\delta^{rs}\delta^{nu}\delta^{iv}+\delta^{jk}\delta^{lm}\delta^{rs}\delta^{nu}\delta^{iv}+\delta^{lm}\delta^{kn}\delta^{js}\delta^{ru}\delta^{iv}+\delta^{km}\delta^{ln}\delta^{js}\delta^{ru}\delta^{iv}\\
&+\delta^{kl}\delta^{mn}\delta^{js}\delta^{ru}\delta^{iv}+\delta^{lm}\delta^{jn}\delta^{ks}\delta^{ru}\delta^{iv}+\delta^{jm}\delta^{ln}\delta^{ks}\delta^{ru}\delta^{iv}+\delta^{jl}\delta^{mn}\delta^{ks}\delta^{ru}\delta^{iv}+\delta^{km}\delta^{jn}\delta^{ls}\delta^{ru}\delta^{iv}\\
&+\delta^{jm}\delta^{kn}\delta^{ls}\delta^{ru}\delta^{iv}+\delta^{jk}\delta^{mn}\delta^{ls}\delta^{ru}\delta^{iv}+\delta^{kl}\delta^{jn}\delta^{ms}\delta^{ru}\delta^{iv}+\delta^{jl}\delta^{kn}\delta^{ms}\delta^{ru}\delta^{iv}+\delta^{jk}\delta^{ln}\delta^{ms}\delta^{ru}\delta^{iv}\\
&+\delta^{kl}\delta^{jm}\delta^{ns}\delta^{ru}\delta^{iv}+\delta^{jl}\delta^{km}\delta^{ns}\delta^{ru}\delta^{iv}+\delta^{jk}\delta^{lm}\delta^{ns}\delta^{ru}\delta^{iv}+\delta^{lm}\delta^{kn}\delta^{jr}\delta^{su}\delta^{iv}+\delta^{km}\delta^{ln}\delta^{jr}\delta^{su}\delta^{iv}\\
&+\delta^{kl}\delta^{mn}\delta^{jr}\delta^{su}\delta^{iv}+\delta^{lm}\delta^{jn}\delta^{kr}\delta^{su}\delta^{iv}+\delta^{jm}\delta^{ln}\delta^{kr}\delta^{su}\delta^{iv}+\delta^{jl}\delta^{mn}\delta^{kr}\delta^{su}\delta^{iv}+\delta^{km}\delta^{jn}\delta^{lr}\delta^{su}\delta^{iv}\\
&+\delta^{jm}\delta^{kn}\delta^{lr}\delta^{su}\delta^{iv}+\delta^{jk}\delta^{mn}\delta^{lr}\delta^{su}\delta^{iv}+\delta^{kl}\delta^{jn}\delta^{mr}\delta^{su}\delta^{iv}+\delta^{jl}\delta^{kn}\delta^{mr}\delta^{su}\delta^{iv}+\delta^{jk}\delta^{ln}\delta^{mr}\delta^{su}\delta^{iv}\\
&+\delta^{kl}\delta^{jm}\delta^{nr}\delta^{su}\delta^{iv}+\delta^{jl}\delta^{km}\delta^{nr}\delta^{su}\delta^{iv}+\delta^{jk}\delta^{lm}\delta^{nr}\delta^{su}\delta^{iv}+\delta^{mn}\delta^{lr}\delta^{ks}\delta^{iu}\delta^{jv}+\delta^{ln}\delta^{mr}\delta^{ks}\delta^{iu}\delta^{jv}\\
&+\delta^{lm}\delta^{nr}\delta^{ks}\delta^{iu}\delta^{jv}+\delta^{mn}\delta^{kr}\delta^{ls}\delta^{iu}\delta^{jv}+\delta^{kn}\delta^{mr}\delta^{ls}\delta^{iu}\delta^{jv}+\delta^{km}\delta^{nr}\delta^{ls}\delta^{iu}\delta^{jv}+\delta^{ln}\delta^{kr}\delta^{ms}\delta^{iu}\delta^{jv}\\
&+\delta^{kn}\delta^{lr}\delta^{ms}\delta^{iu}\delta^{jv}+\delta^{kl}\delta^{nr}\delta^{ms}\delta^{iu}\delta^{jv}+\delta^{lm}\delta^{kr}\delta^{ns}\delta^{iu}\delta^{jv}+\delta^{km}\delta^{lr}\delta^{ns}\delta^{iu}\delta^{jv}+\delta^{kl}\delta^{mr}\delta^{ns}\delta^{iu}\delta^{jv}
\end{aligned}
$$

$$+\delta^{lm}\delta^{kn}\delta^{rs}\delta^{iu}\delta^{jv}+\delta^{km}\delta^{ln}\delta^{rs}\delta^{iu}\delta^{jv}+\delta^{kl}\delta^{mn}\delta^{rs}\delta^{iu}\delta^{jv}+\delta^{mn}\delta^{lr}\delta^{is}\delta^{ku}\delta^{jv}+\delta^{ln}\delta^{mr}\delta^{is}\delta^{ku}\delta^{jv}$$
$$+\delta^{lm}\delta^{nr}\delta^{is}\delta^{ku}\delta^{jv}+\delta^{mn}\delta^{ir}\delta^{ls}\delta^{ku}\delta^{jv}+\delta^{in}\delta^{mr}\delta^{ls}\delta^{ku}\delta^{jv}+\delta^{im}\delta^{nr}\delta^{ls}\delta^{ku}\delta^{jv}+\delta^{ln}\delta^{ir}\delta^{ms}\delta^{ku}\delta^{jv}$$
$$+\delta^{in}\delta^{lr}\delta^{ms}\delta^{ku}\delta^{jv}+\delta^{il}\delta^{nr}\delta^{ms}\delta^{ku}\delta^{jv}+\delta^{lm}\delta^{ir}\delta^{ns}\delta^{ku}\delta^{jv}+\delta^{im}\delta^{lr}\delta^{ns}\delta^{ku}\delta^{jv}+\delta^{il}\delta^{mr}\delta^{ns}\delta^{ku}\delta^{jv}$$
$$+\delta^{lm}\delta^{in}\delta^{rs}\delta^{ku}\delta^{jv}+\delta^{im}\delta^{ln}\delta^{rs}\delta^{ku}\delta^{jv}+\delta^{il}\delta^{mn}\delta^{rs}\delta^{ku}\delta^{jv}+\delta^{mn}\delta^{kr}\delta^{is}\delta^{lu}\delta^{jv}+\delta^{kn}\delta^{mr}\delta^{is}\delta^{lu}\delta^{jv}$$
$$+\delta^{km}\delta^{nr}\delta^{is}\delta^{lu}\delta^{jv}+\delta^{mn}\delta^{ir}\delta^{ks}\delta^{lu}\delta^{jv}+\delta^{in}\delta^{mr}\delta^{ks}\delta^{lu}\delta^{jv}+\delta^{im}\delta^{nr}\delta^{ks}\delta^{lu}\delta^{jv}+\delta^{kn}\delta^{ir}\delta^{ms}\delta^{lu}\delta^{jv}$$
$$+\delta^{in}\delta^{kr}\delta^{ms}\delta^{lu}\delta^{jv}+\delta^{ik}\delta^{nr}\delta^{ms}\delta^{lu}\delta^{jv}+\delta^{km}\delta^{ir}\delta^{ns}\delta^{lu}\delta^{jv}+\delta^{im}\delta^{kr}\delta^{ns}\delta^{lu}\delta^{jv}+\delta^{ik}\delta^{mr}\delta^{ns}\delta^{lu}\delta^{jv}$$
$$+\delta^{km}\delta^{in}\delta^{rs}\delta^{lu}\delta^{jv}+\delta^{im}\delta^{kn}\delta^{rs}\delta^{lu}\delta^{jv}+\delta^{ik}\delta^{mn}\delta^{rs}\delta^{lu}\delta^{jv}+\delta^{ln}\delta^{kr}\delta^{is}\delta^{mu}\delta^{jv}+\delta^{kn}\delta^{lr}\delta^{is}\delta^{mu}\delta^{jv}$$
$$+\delta^{kl}\delta^{nr}\delta^{is}\delta^{mu}\delta^{jv}+\delta^{ln}\delta^{ir}\delta^{ks}\delta^{mu}\delta^{jv}+\delta^{in}\delta^{lr}\delta^{ks}\delta^{mu}\delta^{jv}+\delta^{il}\delta^{nr}\delta^{ks}\delta^{mu}\delta^{jv}+\delta^{kn}\delta^{ir}\delta^{ls}\delta^{mu}\delta^{jv}$$
$$+\delta^{in}\delta^{kr}\delta^{ls}\delta^{mu}\delta^{jv}+\delta^{ik}\delta^{nr}\delta^{ls}\delta^{mu}\delta^{jv}+\delta^{kl}\delta^{ir}\delta^{ns}\delta^{mu}\delta^{jv}+\delta^{il}\delta^{kr}\delta^{ns}\delta^{mu}\delta^{jv}+\delta^{ik}\delta^{lr}\delta^{ns}\delta^{mu}\delta^{jv}$$
$$+\delta^{kl}\delta^{in}\delta^{rs}\delta^{mu}\delta^{jv}+\delta^{il}\delta^{kn}\delta^{rs}\delta^{mu}\delta^{jv}+\delta^{ik}\delta^{ln}\delta^{rs}\delta^{mu}\delta^{jv}+\delta^{lm}\delta^{kr}\delta^{is}\delta^{nu}\delta^{jv}+\delta^{km}\delta^{lr}\delta^{is}\delta^{nu}\delta^{jv}$$
$$+\delta^{kl}\delta^{mr}\delta^{is}\delta^{nu}\delta^{jv}+\delta^{lm}\delta^{ir}\delta^{ks}\delta^{nu}\delta^{jv}+\delta^{im}\delta^{lr}\delta^{ks}\delta^{nu}\delta^{jv}+\delta^{il}\delta^{mr}\delta^{ks}\delta^{nu}\delta^{jv}+\delta^{km}\delta^{ir}\delta^{ls}\delta^{nu}\delta^{jv}$$
$$+\delta^{im}\delta^{kr}\delta^{ls}\delta^{nu}\delta^{jv}+\delta^{ik}\delta^{mr}\delta^{ls}\delta^{nu}\delta^{jv}+\delta^{kl}\delta^{ir}\delta^{ms}\delta^{nu}\delta^{jv}+\delta^{il}\delta^{kr}\delta^{ms}\delta^{nu}\delta^{jv}+\delta^{ik}\delta^{lr}\delta^{ms}\delta^{nu}\delta^{jv}$$
$$+\delta^{kl}\delta^{im}\delta^{rs}\delta^{nu}\delta^{jv}+\delta^{il}\delta^{km}\delta^{rs}\delta^{nu}\delta^{jv}+\delta^{ik}\delta^{lm}\delta^{rs}\delta^{nu}\delta^{jv}+\delta^{lm}\delta^{kn}\delta^{is}\delta^{ru}\delta^{jv}+\delta^{km}\delta^{ln}\delta^{is}\delta^{ru}\delta^{jv}$$
$$+\delta^{kl}\delta^{mn}\delta^{is}\delta^{ru}\delta^{jv}+\delta^{lm}\delta^{in}\delta^{ks}\delta^{ru}\delta^{jv}+\delta^{im}\delta^{ln}\delta^{ks}\delta^{ru}\delta^{jv}+\delta^{il}\delta^{mn}\delta^{ks}\delta^{ru}\delta^{jv}+\delta^{km}\delta^{in}\delta^{ls}\delta^{ru}\delta^{jv}$$
$$+\delta^{im}\delta^{kn}\delta^{ls}\delta^{ru}\delta^{jv}+\delta^{ik}\delta^{mn}\delta^{ls}\delta^{ru}\delta^{jv}+\delta^{kl}\delta^{in}\delta^{ms}\delta^{ru}\delta^{jv}+\delta^{il}\delta^{kn}\delta^{ms}\delta^{ru}\delta^{jv}+\delta^{ik}\delta^{ln}\delta^{ms}\delta^{ru}\delta^{jv}$$
$$+\delta^{kl}\delta^{im}\delta^{ns}\delta^{ru}\delta^{jv}+\delta^{il}\delta^{km}\delta^{ns}\delta^{ru}\delta^{jv}+\delta^{ik}\delta^{lm}\delta^{ns}\delta^{ru}\delta^{jv}+\delta^{lm}\delta^{kn}\delta^{ir}\delta^{su}\delta^{jv}+\delta^{km}\delta^{ln}\delta^{ir}\delta^{su}\delta^{jv}$$
$$+\delta^{kl}\delta^{mn}\delta^{ir}\delta^{su}\delta^{jv}+\delta^{lm}\delta^{in}\delta^{kr}\delta^{su}\delta^{jv}+\delta^{im}\delta^{ln}\delta^{kr}\delta^{su}\delta^{jv}+\delta^{il}\delta^{mn}\delta^{kr}\delta^{su}\delta^{jv}+\delta^{km}\delta^{in}\delta^{lr}\delta^{su}\delta^{jv}$$
$$+\delta^{im}\delta^{kn}\delta^{lr}\delta^{su}\delta^{jv}+\delta^{ik}\delta^{mn}\delta^{lr}\delta^{su}\delta^{jv}+\delta^{kl}\delta^{in}\delta^{mr}\delta^{su}\delta^{jv}+\delta^{il}\delta^{kn}\delta^{mr}\delta^{su}\delta^{jv}+\delta^{ik}\delta^{ln}\delta^{mr}\delta^{su}\delta^{jv}$$
$$+\delta^{kl}\delta^{im}\delta^{nr}\delta^{su}\delta^{jv}+\delta^{il}\delta^{km}\delta^{nr}\delta^{su}\delta^{jv}+\delta^{ik}\delta^{lm}\delta^{nr}\delta^{su}\delta^{jv}+\delta^{mn}\delta^{lr}\delta^{js}\delta^{iu}\delta^{kv}+\delta^{ln}\delta^{mr}\delta^{js}\delta^{iu}\delta^{kv}$$
$$+\delta^{lm}\delta^{nr}\delta^{js}\delta^{iu}\delta^{kv}+\delta^{mn}\delta^{jr}\delta^{ls}\delta^{iu}\delta^{kv}+\delta^{jn}\delta^{mr}\delta^{ls}\delta^{iu}\delta^{kv}+\delta^{jm}\delta^{nr}\delta^{ls}\delta^{iu}\delta^{kv}+\delta^{ln}\delta^{jr}\delta^{ms}\delta^{iu}\delta^{kv}$$
$$+\delta^{jn}\delta^{lr}\delta^{ms}\delta^{iu}\delta^{kv}+\delta^{jl}\delta^{nr}\delta^{ms}\delta^{iu}\delta^{kv}+\delta^{lm}\delta^{jr}\delta^{ns}\delta^{iu}\delta^{kv}+\delta^{jm}\delta^{lr}\delta^{ns}\delta^{iu}\delta^{kv}+\delta^{jl}\delta^{mr}\delta^{ns}\delta^{iu}\delta^{kv}$$
$$+\delta^{lm}\delta^{jn}\delta^{rs}\delta^{iu}\delta^{kv}+\delta^{jm}\delta^{ln}\delta^{rs}\delta^{iu}\delta^{kv}+\delta^{jl}\delta^{mn}\delta^{rs}\delta^{iu}\delta^{kv}+\delta^{mn}\delta^{lr}\delta^{is}\delta^{ju}\delta^{kv}+\delta^{ln}\delta^{mr}\delta^{is}\delta^{ju}\delta^{kv}$$
$$+\delta^{lm}\delta^{nr}\delta^{is}\delta^{ju}\delta^{kv}+\delta^{mn}\delta^{ir}\delta^{ls}\delta^{ju}\delta^{kv}+\delta^{in}\delta^{mr}\delta^{ls}\delta^{ju}\delta^{kv}+\delta^{im}\delta^{nr}\delta^{ls}\delta^{ju}\delta^{kv}+\delta^{ln}\delta^{ir}\delta^{ms}\delta^{ju}\delta^{kv}$$
$$+\delta^{in}\delta^{lr}\delta^{ms}\delta^{ju}\delta^{kv}+\delta^{il}\delta^{nr}\delta^{ms}\delta^{ju}\delta^{kv}+\delta^{lm}\delta^{ir}\delta^{ns}\delta^{ju}\delta^{kv}+\delta^{im}\delta^{lr}\delta^{ns}\delta^{ju}\delta^{kv}+\delta^{il}\delta^{mr}\delta^{ns}\delta^{ju}\delta^{kv}$$
$$+\delta^{lm}\delta^{in}\delta^{rs}\delta^{ju}\delta^{kv}+\delta^{im}\delta^{ln}\delta^{rs}\delta^{ju}\delta^{kv}+\delta^{il}\delta^{mn}\delta^{rs}\delta^{ju}\delta^{kv}+\delta^{mn}\delta^{jr}\delta^{is}\delta^{lu}\delta^{kv}+\delta^{jn}\delta^{mr}\delta^{is}\delta^{lu}\delta^{kv}$$
$$+\delta^{jm}\delta^{nr}\delta^{is}\delta^{lu}\delta^{kv}+\delta^{mn}\delta^{ir}\delta^{js}\delta^{lu}\delta^{kv}+\delta^{in}\delta^{mr}\delta^{js}\delta^{lu}\delta^{kv}+\delta^{im}\delta^{nr}\delta^{js}\delta^{lu}\delta^{kv}+\delta^{jn}\delta^{ir}\delta^{ms}\delta^{lu}\delta^{kv}$$
$$+\delta^{in}\delta^{jr}\delta^{ms}\delta^{lu}\delta^{kv}+\delta^{ij}\delta^{nr}\delta^{ms}\delta^{lu}\delta^{kv}+\delta^{jm}\delta^{ir}\delta^{ns}\delta^{lu}\delta^{kv}+\delta^{im}\delta^{jr}\delta^{ns}\delta^{lu}\delta^{kv}+\delta^{ij}\delta^{mr}\delta^{ns}\delta^{lu}\delta^{kv}$$
$$+\delta^{jm}\delta^{in}\delta^{rs}\delta^{lu}\delta^{kv}+\delta^{im}\delta^{jn}\delta^{rs}\delta^{lu}\delta^{kv}+\delta^{ij}\delta^{mn}\delta^{rs}\delta^{lu}\delta^{kv}+\delta^{ln}\delta^{jr}\delta^{is}\delta^{mu}\delta^{kv}+\delta^{jn}\delta^{lr}\delta^{is}\delta^{mu}\delta^{kv}$$
$$+\delta^{jl}\delta^{nr}\delta^{is}\delta^{mu}\delta^{kv}+\delta^{ln}\delta^{ir}\delta^{js}\delta^{mu}\delta^{kv}+\delta^{in}\delta^{lr}\delta^{js}\delta^{mu}\delta^{kv}+\delta^{il}\delta^{nr}\delta^{js}\delta^{mu}\delta^{kv}+\delta^{jn}\delta^{ir}\delta^{ls}\delta^{mu}\delta^{kv}$$
$$+\delta^{in}\delta^{jr}\delta^{ls}\delta^{mu}\delta^{kv}+\delta^{ij}\delta^{nr}\delta^{ls}\delta^{mu}\delta^{kv}+\delta^{jl}\delta^{ir}\delta^{ns}\delta^{mu}\delta^{kv}+\delta^{il}\delta^{jr}\delta^{ns}\delta^{mu}\delta^{kv}+\delta^{ij}\delta^{lr}\delta^{ns}\delta^{mu}\delta^{kv}$$
$$+\delta^{jl}\delta^{in}\delta^{rs}\delta^{mu}\delta^{kv}+\delta^{il}\delta^{jn}\delta^{rs}\delta^{mu}\delta^{kv}+\delta^{ij}\delta^{ln}\delta^{rs}\delta^{mu}\delta^{kv}+\delta^{lm}\delta^{jr}\delta^{is}\delta^{nu}\delta^{kv}+\delta^{jm}\delta^{lr}\delta^{is}\delta^{nu}\delta^{kv}$$

$$+\delta^{jl}\delta^{mr}\delta^{is}\delta^{nu}\delta^{kv}+\delta^{lm}\delta^{ir}\delta^{js}\delta^{nu}\delta^{kv}+\delta^{im}\delta^{lr}\delta^{js}\delta^{nu}\delta^{kv}+\delta^{il}\delta^{mr}\delta^{js}\delta^{nu}\delta^{kv}+\delta^{jm}\delta^{ir}\delta^{ls}\delta^{nu}\delta^{kv}$$
$$+\delta^{im}\delta^{jr}\delta^{ls}\delta^{nu}\delta^{kv}+\delta^{ij}\delta^{mr}\delta^{ls}\delta^{nu}\delta^{kv}+\delta^{jl}\delta^{ir}\delta^{ms}\delta^{nu}\delta^{kv}+\delta^{il}\delta^{jr}\delta^{ms}\delta^{nu}\delta^{kv}+\delta^{ij}\delta^{lr}\delta^{ms}\delta^{nu}\delta^{kv}$$
$$+\delta^{jl}\delta^{im}\delta^{rs}\delta^{nu}\delta^{kv}+\delta^{il}\delta^{jm}\delta^{rs}\delta^{nu}\delta^{kv}+\delta^{ij}\delta^{lm}\delta^{rs}\delta^{nu}\delta^{kv}+\delta^{lm}\delta^{jn}\delta^{is}\delta^{ru}\delta^{kv}+\delta^{jm}\delta^{ln}\delta^{is}\delta^{ru}\delta^{kv}$$
$$+\delta^{jl}\delta^{mn}\delta^{is}\delta^{ru}\delta^{kv}+\delta^{lm}\delta^{in}\delta^{js}\delta^{ru}\delta^{kv}+\delta^{im}\delta^{ln}\delta^{js}\delta^{ru}\delta^{kv}+\delta^{il}\delta^{mn}\delta^{js}\delta^{ru}\delta^{kv}+\delta^{jm}\delta^{in}\delta^{ls}\delta^{ru}\delta^{kv}$$
$$+\delta^{im}\delta^{jn}\delta^{ls}\delta^{ru}\delta^{kv}+\delta^{ij}\delta^{mn}\delta^{ls}\delta^{ru}\delta^{kv}+\delta^{jl}\delta^{in}\delta^{ms}\delta^{ru}\delta^{kv}+\delta^{il}\delta^{jn}\delta^{ms}\delta^{ru}\delta^{kv}+\delta^{ij}\delta^{ln}\delta^{ms}\delta^{ru}\delta^{kv}$$
$$+\delta^{jl}\delta^{im}\delta^{ns}\delta^{ru}\delta^{kv}+\delta^{il}\delta^{jm}\delta^{ns}\delta^{ru}\delta^{kv}+\delta^{ij}\delta^{lm}\delta^{ns}\delta^{ru}\delta^{kv}+\delta^{lm}\delta^{jn}\delta^{ir}\delta^{su}\delta^{kv}+\delta^{jm}\delta^{ln}\delta^{ir}\delta^{su}\delta^{kv}$$
$$+\delta^{jl}\delta^{mn}\delta^{ir}\delta^{su}\delta^{kv}+\delta^{lm}\delta^{in}\delta^{jr}\delta^{su}\delta^{kv}+\delta^{im}\delta^{ln}\delta^{jr}\delta^{su}\delta^{kv}+\delta^{il}\delta^{mn}\delta^{jr}\delta^{su}\delta^{kv}+\delta^{jm}\delta^{in}\delta^{lr}\delta^{su}\delta^{kv}$$
$$+\delta^{im}\delta^{jn}\delta^{lr}\delta^{su}\delta^{kv}+\delta^{ij}\delta^{mn}\delta^{lr}\delta^{su}\delta^{kv}+\delta^{jl}\delta^{in}\delta^{mr}\delta^{su}\delta^{kv}+\delta^{il}\delta^{jn}\delta^{mr}\delta^{su}\delta^{kv}+\delta^{ij}\delta^{ln}\delta^{mr}\delta^{su}\delta^{kv}$$
$$+\delta^{jl}\delta^{im}\delta^{nr}\delta^{su}\delta^{kv}+\delta^{il}\delta^{jm}\delta^{nr}\delta^{su}\delta^{kv}+\delta^{ij}\delta^{lm}\delta^{nr}\delta^{su}\delta^{kv}+\delta^{mn}\delta^{kr}\delta^{js}\delta^{iu}\delta^{lv}+\delta^{kn}\delta^{mr}\delta^{js}\delta^{iu}\delta^{lv}$$
$$+\delta^{km}\delta^{nr}\delta^{js}\delta^{iu}\delta^{lv}+\delta^{mn}\delta^{jr}\delta^{ks}\delta^{iu}\delta^{lv}+\delta^{jn}\delta^{mr}\delta^{ks}\delta^{iu}\delta^{lv}+\delta^{jm}\delta^{nr}\delta^{ks}\delta^{iu}\delta^{lv}+\delta^{kn}\delta^{jr}\delta^{ms}\delta^{iu}\delta^{lv}$$
$$+\delta^{jn}\delta^{kr}\delta^{ms}\delta^{iu}\delta^{lv}+\delta^{jk}\delta^{nr}\delta^{ms}\delta^{iu}\delta^{lv}+\delta^{km}\delta^{jr}\delta^{ns}\delta^{iu}\delta^{lv}+\delta^{jm}\delta^{kr}\delta^{ns}\delta^{iu}\delta^{lv}+\delta^{jk}\delta^{mr}\delta^{ns}\delta^{iu}\delta^{lv}$$
$$+\delta^{km}\delta^{jn}\delta^{rs}\delta^{iu}\delta^{lv}+\delta^{jm}\delta^{kn}\delta^{rs}\delta^{iu}\delta^{lv}+\delta^{jk}\delta^{mn}\delta^{rs}\delta^{iu}\delta^{lv}+\delta^{mn}\delta^{kr}\delta^{is}\delta^{ju}\delta^{lv}+\delta^{kn}\delta^{mr}\delta^{is}\delta^{ju}\delta^{lv}$$
$$+\delta^{km}\delta^{nr}\delta^{is}\delta^{ju}\delta^{lv}+\delta^{mn}\delta^{ir}\delta^{ks}\delta^{ju}\delta^{lv}+\delta^{in}\delta^{mr}\delta^{ks}\delta^{ju}\delta^{lv}+\delta^{im}\delta^{nr}\delta^{ks}\delta^{ju}\delta^{lv}+\delta^{kn}\delta^{ir}\delta^{ms}\delta^{ju}\delta^{lv}$$
$$+\delta^{in}\delta^{kr}\delta^{ms}\delta^{ju}\delta^{lv}+\delta^{ik}\delta^{nr}\delta^{ms}\delta^{ju}\delta^{lv}+\delta^{km}\delta^{ir}\delta^{ns}\delta^{ju}\delta^{lv}+\delta^{im}\delta^{kr}\delta^{ns}\delta^{ju}\delta^{lv}+\delta^{ik}\delta^{mr}\delta^{ns}\delta^{ju}\delta^{lv}$$
$$+\delta^{km}\delta^{in}\delta^{rs}\delta^{ju}\delta^{lv}+\delta^{im}\delta^{kn}\delta^{rs}\delta^{ju}\delta^{lv}+\delta^{ik}\delta^{mn}\delta^{rs}\delta^{ju}\delta^{lv}+\delta^{mn}\delta^{jr}\delta^{is}\delta^{ku}\delta^{lv}+\delta^{jn}\delta^{mr}\delta^{is}\delta^{ku}\delta^{lv}$$
$$+\delta^{jm}\delta^{nr}\delta^{is}\delta^{ku}\delta^{lv}+\delta^{mn}\delta^{ir}\delta^{js}\delta^{ku}\delta^{lv}+\delta^{in}\delta^{mr}\delta^{js}\delta^{ku}\delta^{lv}+\delta^{im}\delta^{nr}\delta^{js}\delta^{ku}\delta^{lv}+\delta^{jn}\delta^{ir}\delta^{ms}\delta^{ku}\delta^{lv}$$
$$+\delta^{in}\delta^{jr}\delta^{ms}\delta^{ku}\delta^{lv}+\delta^{ij}\delta^{nr}\delta^{ms}\delta^{ku}\delta^{lv}+\delta^{jm}\delta^{ir}\delta^{ns}\delta^{ku}\delta^{lv}+\delta^{im}\delta^{jr}\delta^{ns}\delta^{ku}\delta^{lv}+\delta^{ij}\delta^{mr}\delta^{ns}\delta^{ku}\delta^{lv}$$
$$+\delta^{jm}\delta^{in}\delta^{rs}\delta^{ku}\delta^{lv}+\delta^{im}\delta^{jn}\delta^{rs}\delta^{ku}\delta^{lv}+\delta^{ij}\delta^{mn}\delta^{rs}\delta^{ku}\delta^{lv}+\delta^{kn}\delta^{jr}\delta^{is}\delta^{mu}\delta^{lv}+\delta^{jn}\delta^{kr}\delta^{is}\delta^{mu}\delta^{lv}$$
$$+\delta^{jk}\delta^{nr}\delta^{is}\delta^{mu}\delta^{lv}+\delta^{kn}\delta^{ir}\delta^{js}\delta^{mu}\delta^{lv}+\delta^{in}\delta^{kr}\delta^{js}\delta^{mu}\delta^{lv}+\delta^{ik}\delta^{nr}\delta^{js}\delta^{mu}\delta^{lv}+\delta^{jn}\delta^{ir}\delta^{ks}\delta^{mu}\delta^{lv}$$
$$+\delta^{in}\delta^{jr}\delta^{ks}\delta^{mu}\delta^{lv}+\delta^{ij}\delta^{nr}\delta^{ks}\delta^{mu}\delta^{lv}+\delta^{jk}\delta^{ir}\delta^{ns}\delta^{mu}\delta^{lv}+\delta^{ik}\delta^{jr}\delta^{ns}\delta^{mu}\delta^{lv}+\delta^{ij}\delta^{kr}\delta^{ns}\delta^{mu}\delta^{lv}$$
$$+\delta^{jk}\delta^{in}\delta^{rs}\delta^{mu}\delta^{lv}+\delta^{ik}\delta^{jn}\delta^{rs}\delta^{mu}\delta^{lv}+\delta^{ij}\delta^{kn}\delta^{rs}\delta^{mu}\delta^{lv}+\delta^{km}\delta^{jr}\delta^{is}\delta^{nu}\delta^{lv}+\delta^{jm}\delta^{kr}\delta^{is}\delta^{nu}\delta^{lv}$$
$$+\delta^{jk}\delta^{mr}\delta^{is}\delta^{nu}\delta^{lv}+\delta^{km}\delta^{ir}\delta^{js}\delta^{nu}\delta^{lv}+\delta^{im}\delta^{kr}\delta^{js}\delta^{nu}\delta^{lv}+\delta^{ik}\delta^{mr}\delta^{js}\delta^{nu}\delta^{lv}+\delta^{jm}\delta^{ir}\delta^{ks}\delta^{nu}\delta^{lv}$$
$$+\delta^{im}\delta^{jr}\delta^{ks}\delta^{nu}\delta^{lv}+\delta^{ij}\delta^{mr}\delta^{ks}\delta^{nu}\delta^{lv}+\delta^{jk}\delta^{ir}\delta^{ms}\delta^{nu}\delta^{lv}+\delta^{ik}\delta^{jr}\delta^{ms}\delta^{nu}\delta^{lv}+\delta^{ij}\delta^{kr}\delta^{ms}\delta^{nu}\delta^{lv}$$
$$+\delta^{jk}\delta^{im}\delta^{rs}\delta^{nu}\delta^{lv}+\delta^{ik}\delta^{jm}\delta^{rs}\delta^{nu}\delta^{lv}+\delta^{ij}\delta^{km}\delta^{rs}\delta^{nu}\delta^{lv}+\delta^{km}\delta^{jn}\delta^{is}\delta^{ru}\delta^{lv}+\delta^{jm}\delta^{kn}\delta^{is}\delta^{ru}\delta^{lv}$$
$$+\delta^{jk}\delta^{mn}\delta^{is}\delta^{ru}\delta^{lv}+\delta^{km}\delta^{in}\delta^{js}\delta^{ru}\delta^{lv}+\delta^{im}\delta^{kn}\delta^{js}\delta^{ru}\delta^{lv}+\delta^{ik}\delta^{mn}\delta^{js}\delta^{ru}\delta^{lv}+\delta^{jm}\delta^{in}\delta^{ks}\delta^{ru}\delta^{lv}$$
$$+\delta^{im}\delta^{jn}\delta^{ks}\delta^{ru}\delta^{lv}+\delta^{ij}\delta^{mn}\delta^{ks}\delta^{ru}\delta^{lv}+\delta^{jk}\delta^{in}\delta^{ms}\delta^{ru}\delta^{lv}+\delta^{ik}\delta^{jn}\delta^{ms}\delta^{ru}\delta^{lv}+\delta^{ij}\delta^{kn}\delta^{ms}\delta^{ru}\delta^{lv}$$
$$+\delta^{jk}\delta^{im}\delta^{ns}\delta^{ru}\delta^{lv}+\delta^{ik}\delta^{jm}\delta^{ns}\delta^{ru}\delta^{lv}+\delta^{ij}\delta^{km}\delta^{ns}\delta^{ru}\delta^{lv}+\delta^{km}\delta^{jn}\delta^{ir}\delta^{su}\delta^{lv}+\delta^{jm}\delta^{kn}\delta^{ir}\delta^{su}\delta^{lv}$$
$$+\delta^{jk}\delta^{mn}\delta^{ir}\delta^{su}\delta^{lv}+\delta^{km}\delta^{in}\delta^{jr}\delta^{su}\delta^{lv}+\delta^{im}\delta^{kn}\delta^{jr}\delta^{su}\delta^{lv}+\delta^{ik}\delta^{mn}\delta^{jr}\delta^{su}\delta^{lv}+\delta^{jm}\delta^{in}\delta^{kr}\delta^{su}\delta^{lv}$$
$$+\delta^{im}\delta^{jn}\delta^{kr}\delta^{su}\delta^{lv}+\delta^{ij}\delta^{mn}\delta^{kr}\delta^{su}\delta^{lv}+\delta^{jk}\delta^{in}\delta^{mr}\delta^{su}\delta^{lv}+\delta^{ik}\delta^{jn}\delta^{mr}\delta^{su}\delta^{lv}+\delta^{ij}\delta^{kn}\delta^{mr}\delta^{su}\delta^{lv}$$
$$+\delta^{jk}\delta^{im}\delta^{nr}\delta^{su}\delta^{lv}+\delta^{ik}\delta^{jm}\delta^{nr}\delta^{su}\delta^{lv}+\delta^{ij}\delta^{km}\delta^{nr}\delta^{su}\delta^{lv}+\delta^{ln}\delta^{kr}\delta^{js}\delta^{iu}\delta^{mv}+\delta^{kn}\delta^{lr}\delta^{js}\delta^{iu}\delta^{mv}$$
$$+\delta^{kl}\delta^{nr}\delta^{js}\delta^{iu}\delta^{mv}+\delta^{ln}\delta^{jr}\delta^{ks}\delta^{iu}\delta^{mv}+\delta^{jn}\delta^{lr}\delta^{ks}\delta^{iu}\delta^{mv}+\delta^{jl}\delta^{nr}\delta^{ks}\delta^{iu}\delta^{mv}+\delta^{kn}\delta^{jr}\delta^{ls}\delta^{iu}\delta^{mv}$$

$$+\delta^{jn}\delta^{kr}\delta^{ls}\delta^{iu}\delta^{mv}+\delta^{jk}\delta^{nr}\delta^{ls}\delta^{iu}\delta^{mv}+\delta^{kl}\delta^{jr}\delta^{ns}\delta^{iu}\delta^{mv}+\delta^{jl}\delta^{kr}\delta^{ns}\delta^{iu}\delta^{mv}+\delta^{jk}\delta^{lr}\delta^{ns}\delta^{iu}\delta^{mv}$$
$$+\delta^{kl}\delta^{jn}\delta^{rs}\delta^{iu}\delta^{mv}+\delta^{jl}\delta^{kn}\delta^{rs}\delta^{iu}\delta^{mv}+\delta^{jk}\delta^{ln}\delta^{rs}\delta^{iu}\delta^{mv}+\delta^{ln}\delta^{kr}\delta^{is}\delta^{ju}\delta^{mv}+\delta^{kn}\delta^{lr}\delta^{is}\delta^{ju}\delta^{mv}$$
$$+\delta^{kl}\delta^{nr}\delta^{is}\delta^{ju}\delta^{mv}+\delta^{ln}\delta^{ir}\delta^{ks}\delta^{ju}\delta^{mv}+\delta^{in}\delta^{lr}\delta^{ks}\delta^{ju}\delta^{mv}+\delta^{il}\delta^{nr}\delta^{ks}\delta^{ju}\delta^{mv}+\delta^{kn}\delta^{ir}\delta^{ls}\delta^{ju}\delta^{mv}$$
$$+\delta^{in}\delta^{kr}\delta^{ls}\delta^{ju}\delta^{mv}+\delta^{ik}\delta^{nr}\delta^{ls}\delta^{ju}\delta^{mv}+\delta^{kl}\delta^{ir}\delta^{ns}\delta^{ju}\delta^{mv}+\delta^{il}\delta^{kr}\delta^{ns}\delta^{ju}\delta^{mv}+\delta^{ik}\delta^{lr}\delta^{ns}\delta^{ju}\delta^{mv}$$
$$+\delta^{kl}\delta^{in}\delta^{rs}\delta^{ju}\delta^{mv}+\delta^{il}\delta^{kn}\delta^{rs}\delta^{ju}\delta^{mv}+\delta^{ik}\delta^{ln}\delta^{rs}\delta^{ju}\delta^{mv}+\delta^{ln}\delta^{jr}\delta^{is}\delta^{ku}\delta^{mv}+\delta^{jn}\delta^{lr}\delta^{is}\delta^{ku}\delta^{mv}$$
$$+\delta^{jl}\delta^{nr}\delta^{is}\delta^{ku}\delta^{mv}+\delta^{ln}\delta^{ir}\delta^{js}\delta^{ku}\delta^{mv}+\delta^{in}\delta^{lr}\delta^{js}\delta^{ku}\delta^{mv}+\delta^{il}\delta^{nr}\delta^{js}\delta^{ku}\delta^{mv}+\delta^{jn}\delta^{ir}\delta^{ls}\delta^{ku}\delta^{mv}$$
$$+\delta^{in}\delta^{jr}\delta^{ls}\delta^{ku}\delta^{mv}+\delta^{ij}\delta^{nr}\delta^{ls}\delta^{ku}\delta^{mv}+\delta^{jl}\delta^{ir}\delta^{ns}\delta^{ku}\delta^{mv}+\delta^{il}\delta^{jr}\delta^{ns}\delta^{ku}\delta^{mv}+\delta^{ij}\delta^{lr}\delta^{ns}\delta^{ku}\delta^{mv}$$
$$+\delta^{jl}\delta^{in}\delta^{rs}\delta^{ku}\delta^{mv}+\delta^{il}\delta^{jn}\delta^{rs}\delta^{ku}\delta^{mv}+\delta^{ij}\delta^{ln}\delta^{rs}\delta^{ku}\delta^{mv}+\delta^{kn}\delta^{jr}\delta^{is}\delta^{lu}\delta^{mv}+\delta^{jn}\delta^{kr}\delta^{is}\delta^{lu}\delta^{mv}$$
$$+\delta^{jk}\delta^{nr}\delta^{is}\delta^{lu}\delta^{mv}+\delta^{kn}\delta^{ir}\delta^{js}\delta^{lu}\delta^{mv}+\delta^{in}\delta^{kr}\delta^{js}\delta^{lu}\delta^{mv}+\delta^{ik}\delta^{nr}\delta^{js}\delta^{lu}\delta^{mv}+\delta^{jn}\delta^{ir}\delta^{ks}\delta^{lu}\delta^{mv}$$
$$+\delta^{in}\delta^{jr}\delta^{ks}\delta^{lu}\delta^{mv}+\delta^{ij}\delta^{nr}\delta^{ks}\delta^{lu}\delta^{mv}+\delta^{jk}\delta^{ir}\delta^{ns}\delta^{lu}\delta^{mv}+\delta^{ik}\delta^{jr}\delta^{ns}\delta^{lu}\delta^{mv}+\delta^{ij}\delta^{kr}\delta^{ns}\delta^{lu}\delta^{mv}$$
$$+\delta^{jk}\delta^{in}\delta^{rs}\delta^{lu}\delta^{mv}+\delta^{ik}\delta^{jn}\delta^{rs}\delta^{lu}\delta^{mv}+\delta^{ij}\delta^{kn}\delta^{rs}\delta^{lu}\delta^{mv}+\delta^{kl}\delta^{jr}\delta^{is}\delta^{nu}\delta^{mv}+\delta^{jl}\delta^{kr}\delta^{is}\delta^{nu}\delta^{mv}$$
$$+\delta^{jk}\delta^{lr}\delta^{is}\delta^{nu}\delta^{mv}+\delta^{kl}\delta^{ir}\delta^{js}\delta^{nu}\delta^{mv}+\delta^{il}\delta^{kr}\delta^{js}\delta^{nu}\delta^{mv}+\delta^{ik}\delta^{lr}\delta^{js}\delta^{nu}\delta^{mv}+\delta^{jl}\delta^{ir}\delta^{ks}\delta^{nu}\delta^{mv}$$
$$+\delta^{il}\delta^{jr}\delta^{ks}\delta^{nu}\delta^{mv}+\delta^{ij}\delta^{lr}\delta^{ks}\delta^{nu}\delta^{mv}+\delta^{jk}\delta^{ir}\delta^{ls}\delta^{nu}\delta^{mv}+\delta^{ik}\delta^{jr}\delta^{ls}\delta^{nu}\delta^{mv}+\delta^{ij}\delta^{kr}\delta^{ls}\delta^{nu}\delta^{mv}$$
$$+\delta^{jk}\delta^{il}\delta^{rs}\delta^{nu}\delta^{mv}+\delta^{ik}\delta^{jl}\delta^{rs}\delta^{nu}\delta^{mv}+\delta^{ij}\delta^{kl}\delta^{rs}\delta^{nu}\delta^{mv}+\delta^{kl}\delta^{jn}\delta^{is}\delta^{ru}\delta^{mv}+\delta^{jl}\delta^{kn}\delta^{is}\delta^{ru}\delta^{mv}$$
$$+\delta^{jk}\delta^{ln}\delta^{is}\delta^{ru}\delta^{mv}+\delta^{kl}\delta^{in}\delta^{js}\delta^{ru}\delta^{mv}+\delta^{il}\delta^{kn}\delta^{js}\delta^{ru}\delta^{mv}+\delta^{ik}\delta^{ln}\delta^{js}\delta^{ru}\delta^{mv}+\delta^{jl}\delta^{in}\delta^{ks}\delta^{ru}\delta^{mv}$$
$$+\delta^{il}\delta^{jn}\delta^{ks}\delta^{ru}\delta^{mv}+\delta^{ij}\delta^{ln}\delta^{ks}\delta^{ru}\delta^{mv}+\delta^{jk}\delta^{in}\delta^{ls}\delta^{ru}\delta^{mv}+\delta^{ik}\delta^{jn}\delta^{ls}\delta^{ru}\delta^{mv}+\delta^{ij}\delta^{kn}\delta^{ls}\delta^{ru}\delta^{mv}$$
$$+\delta^{jk}\delta^{il}\delta^{ns}\delta^{ru}\delta^{mv}+\delta^{ik}\delta^{jl}\delta^{ns}\delta^{ru}\delta^{mv}+\delta^{ij}\delta^{kl}\delta^{ns}\delta^{ru}\delta^{mv}+\delta^{kl}\delta^{jn}\delta^{ir}\delta^{su}\delta^{mv}+\delta^{jl}\delta^{kn}\delta^{ir}\delta^{su}\delta^{mv}$$
$$+\delta^{jk}\delta^{ln}\delta^{ir}\delta^{su}\delta^{mv}+\delta^{kl}\delta^{in}\delta^{jr}\delta^{su}\delta^{mv}+\delta^{il}\delta^{kn}\delta^{jr}\delta^{su}\delta^{mv}+\delta^{ik}\delta^{ln}\delta^{jr}\delta^{su}\delta^{mv}+\delta^{jl}\delta^{in}\delta^{kr}\delta^{su}\delta^{mv}$$
$$+\delta^{il}\delta^{jn}\delta^{kr}\delta^{su}\delta^{mv}+\delta^{ij}\delta^{ln}\delta^{kr}\delta^{su}\delta^{mv}+\delta^{jk}\delta^{in}\delta^{lr}\delta^{su}\delta^{mv}+\delta^{ik}\delta^{jn}\delta^{lr}\delta^{su}\delta^{mv}+\delta^{ij}\delta^{kn}\delta^{lr}\delta^{su}\delta^{mv}$$
$$+\delta^{jk}\delta^{il}\delta^{nr}\delta^{su}\delta^{mv}+\delta^{ik}\delta^{jl}\delta^{nr}\delta^{su}\delta^{mv}+\delta^{ij}\delta^{kl}\delta^{nr}\delta^{su}\delta^{mv}+\delta^{lm}\delta^{kr}\delta^{js}\delta^{iu}\delta^{nv}+\delta^{km}\delta^{lr}\delta^{js}\delta^{iu}\delta^{nv}$$
$$+\delta^{kl}\delta^{mr}\delta^{js}\delta^{iu}\delta^{nv}+\delta^{lm}\delta^{jr}\delta^{ks}\delta^{iu}\delta^{nv}+\delta^{jm}\delta^{lr}\delta^{ks}\delta^{iu}\delta^{nv}+\delta^{jl}\delta^{mr}\delta^{ks}\delta^{iu}\delta^{nv}+\delta^{km}\delta^{jr}\delta^{ls}\delta^{iu}\delta^{nv}$$
$$+\delta^{jm}\delta^{kr}\delta^{ls}\delta^{iu}\delta^{nv}+\delta^{jk}\delta^{mr}\delta^{ls}\delta^{iu}\delta^{nv}+\delta^{kl}\delta^{jr}\delta^{ms}\delta^{iu}\delta^{nv}+\delta^{jl}\delta^{kr}\delta^{ms}\delta^{iu}\delta^{nv}+\delta^{jk}\delta^{lr}\delta^{ms}\delta^{iu}\delta^{nv}$$
$$+\delta^{kl}\delta^{jm}\delta^{rs}\delta^{iu}\delta^{nv}+\delta^{jl}\delta^{km}\delta^{rs}\delta^{iu}\delta^{nv}+\delta^{jk}\delta^{lm}\delta^{rs}\delta^{iu}\delta^{nv}+\delta^{lm}\delta^{kr}\delta^{is}\delta^{ju}\delta^{nv}+\delta^{km}\delta^{lr}\delta^{is}\delta^{ju}\delta^{nv}$$
$$+\delta^{kl}\delta^{mr}\delta^{is}\delta^{ju}\delta^{nv}+\delta^{lm}\delta^{ir}\delta^{ks}\delta^{ju}\delta^{nv}+\delta^{im}\delta^{lr}\delta^{ks}\delta^{ju}\delta^{nv}+\delta^{il}\delta^{mr}\delta^{ks}\delta^{ju}\delta^{nv}+\delta^{km}\delta^{ir}\delta^{ls}\delta^{ju}\delta^{nv}$$
$$+\delta^{im}\delta^{kr}\delta^{ls}\delta^{ju}\delta^{nv}+\delta^{ik}\delta^{mr}\delta^{ls}\delta^{ju}\delta^{nv}+\delta^{kl}\delta^{ir}\delta^{ms}\delta^{ju}\delta^{nv}+\delta^{il}\delta^{kr}\delta^{ms}\delta^{ju}\delta^{nv}+\delta^{ik}\delta^{lr}\delta^{ms}\delta^{ju}\delta^{nv}$$
$$+\delta^{kl}\delta^{im}\delta^{rs}\delta^{ju}\delta^{nv}+\delta^{il}\delta^{km}\delta^{rs}\delta^{ju}\delta^{nv}+\delta^{ik}\delta^{lm}\delta^{rs}\delta^{ju}\delta^{nv}+\delta^{lm}\delta^{jr}\delta^{is}\delta^{ku}\delta^{nv}+\delta^{jm}\delta^{lr}\delta^{is}\delta^{ku}\delta^{nv}$$
$$+\delta^{jl}\delta^{mr}\delta^{is}\delta^{ku}\delta^{nv}+\delta^{lm}\delta^{ir}\delta^{js}\delta^{ku}\delta^{nv}+\delta^{im}\delta^{lr}\delta^{js}\delta^{ku}\delta^{nv}+\delta^{il}\delta^{mr}\delta^{js}\delta^{ku}\delta^{nv}+\delta^{jm}\delta^{ir}\delta^{ls}\delta^{ku}\delta^{nv}$$
$$+\delta^{im}\delta^{jr}\delta^{ls}\delta^{ku}\delta^{nv}+\delta^{ij}\delta^{mr}\delta^{ls}\delta^{ku}\delta^{nv}+\delta^{jl}\delta^{ir}\delta^{ms}\delta^{ku}\delta^{nv}+\delta^{il}\delta^{jr}\delta^{ms}\delta^{ku}\delta^{nv}+\delta^{ij}\delta^{lr}\delta^{ms}\delta^{ku}\delta^{nv}$$
$$+\delta^{jl}\delta^{im}\delta^{rs}\delta^{ku}\delta^{nv}+\delta^{il}\delta^{jm}\delta^{rs}\delta^{ku}\delta^{nv}+\delta^{ij}\delta^{lm}\delta^{rs}\delta^{ku}\delta^{nv}+\delta^{km}\delta^{jr}\delta^{is}\delta^{lu}\delta^{nv}+\delta^{jm}\delta^{kr}\delta^{is}\delta^{lu}\delta^{nv}$$
$$+\delta^{jk}\delta^{mr}\delta^{is}\delta^{lu}\delta^{nv}+\delta^{km}\delta^{ir}\delta^{js}\delta^{lu}\delta^{nv}+\delta^{im}\delta^{kr}\delta^{js}\delta^{lu}\delta^{nv}+\delta^{ik}\delta^{mr}\delta^{js}\delta^{lu}\delta^{nv}+\delta^{jm}\delta^{ir}\delta^{ks}\delta^{lu}\delta^{nv}$$
$$+\delta^{im}\delta^{jr}\delta^{ks}\delta^{lu}\delta^{nv}+\delta^{ij}\delta^{mr}\delta^{ks}\delta^{lu}\delta^{nv}+\delta^{jk}\delta^{ir}\delta^{ms}\delta^{lu}\delta^{nv}+\delta^{ik}\delta^{jr}\delta^{ms}\delta^{lu}\delta^{nv}+\delta^{ij}\delta^{kr}\delta^{ms}\delta^{lu}\delta^{nv}$$

$$+\delta^{jk}\delta^{im}\delta^{rs}\delta^{lu}\delta^{nv}+\delta^{ik}\delta^{jm}\delta^{rs}\delta^{lu}\delta^{nv}+\delta^{ij}\delta^{km}\delta^{rs}\delta^{lu}\delta^{nv}+\delta^{kl}\delta^{jr}\delta^{is}\delta^{mu}\delta^{nv}+\delta^{jl}\delta^{kr}\delta^{is}\delta^{mu}\delta^{nv}$$

$$+\delta^{jk}\delta^{lr}\delta^{is}\delta^{mu}\delta^{nv}+\delta^{kl}\delta^{ir}\delta^{js}\delta^{mu}\delta^{nv}+\delta^{il}\delta^{kr}\delta^{js}\delta^{mu}\delta^{nv}+\delta^{ik}\delta^{lr}\delta^{js}\delta^{mu}\delta^{nv}+\delta^{jl}\delta^{ir}\delta^{ks}\delta^{mu}\delta^{nv}$$

$$+\delta^{il}\delta^{jr}\delta^{ks}\delta^{mu}\delta^{nv}+\delta^{ij}\delta^{lr}\delta^{ks}\delta^{mu}\delta^{nv}+\delta^{jk}\delta^{ir}\delta^{ls}\delta^{mu}\delta^{nv}+\delta^{ik}\delta^{jr}\delta^{ls}\delta^{mu}\delta^{nv}+\delta^{ij}\delta^{kr}\delta^{ls}\delta^{mu}\delta^{nv}$$

$$+\delta^{jk}\delta^{il}\delta^{rs}\delta^{mu}\delta^{nv}+\delta^{ik}\delta^{jl}\delta^{rs}\delta^{mu}\delta^{nv}+\delta^{ij}\delta^{kl}\delta^{rs}\delta^{mu}\delta^{nv}+\delta^{kl}\delta^{jm}\delta^{is}\delta^{ru}\delta^{nv}+\delta^{jl}\delta^{km}\delta^{is}\delta^{ru}\delta^{nv}$$

$$+\delta^{jk}\delta^{lm}\delta^{is}\delta^{ru}\delta^{nv}+\delta^{kl}\delta^{im}\delta^{js}\delta^{ru}\delta^{nv}+\delta^{il}\delta^{km}\delta^{js}\delta^{ru}\delta^{nv}+\delta^{ik}\delta^{lm}\delta^{js}\delta^{ru}\delta^{nv}+\delta^{jl}\delta^{im}\delta^{ks}\delta^{ru}\delta^{nv}$$

$$+\delta^{il}\delta^{jm}\delta^{ks}\delta^{ru}\delta^{nv}+\delta^{ij}\delta^{lm}\delta^{ks}\delta^{ru}\delta^{nv}+\delta^{jk}\delta^{im}\delta^{ls}\delta^{ru}\delta^{nv}+\delta^{ik}\delta^{jm}\delta^{ls}\delta^{ru}\delta^{nv}+\delta^{ij}\delta^{km}\delta^{ls}\delta^{ru}\delta^{nv}$$

$$+\delta^{jk}\delta^{il}\delta^{ms}\delta^{ru}\delta^{nv}+\delta^{ik}\delta^{jl}\delta^{ms}\delta^{ru}\delta^{nv}+\delta^{ij}\delta^{kl}\delta^{ms}\delta^{ru}\delta^{nv}+\delta^{kl}\delta^{jm}\delta^{ir}\delta^{su}\delta^{nv}+\delta^{jl}\delta^{km}\delta^{ir}\delta^{su}\delta^{nv}$$

$$+\delta^{jk}\delta^{lm}\delta^{ir}\delta^{su}\delta^{nv}+\delta^{kl}\delta^{im}\delta^{jr}\delta^{su}\delta^{nv}+\delta^{il}\delta^{km}\delta^{jr}\delta^{su}\delta^{nv}+\delta^{ik}\delta^{lm}\delta^{jr}\delta^{su}\delta^{nv}+\delta^{jl}\delta^{im}\delta^{kr}\delta^{su}\delta^{nv}$$

$$+\delta^{il}\delta^{jm}\delta^{kr}\delta^{su}\delta^{nv}+\delta^{ij}\delta^{lm}\delta^{kr}\delta^{su}\delta^{nv}+\delta^{jk}\delta^{im}\delta^{lr}\delta^{su}\delta^{nv}+\delta^{ik}\delta^{jm}\delta^{lr}\delta^{su}\delta^{nv}+\delta^{ij}\delta^{km}\delta^{lr}\delta^{su}\delta^{nv}$$

$$+\delta^{jk}\delta^{il}\delta^{mr}\delta^{su}\delta^{nv}+\delta^{ik}\delta^{jl}\delta^{mr}\delta^{su}\delta^{nv}+\delta^{ij}\delta^{kl}\delta^{mr}\delta^{su}\delta^{nv}+\delta^{lm}\delta^{kn}\delta^{js}\delta^{iu}\delta^{rv}+\delta^{km}\delta^{ln}\delta^{js}\delta^{iu}\delta^{rv}$$

$$+\delta^{kl}\delta^{mn}\delta^{js}\delta^{iu}\delta^{rv}+\delta^{lm}\delta^{jn}\delta^{ks}\delta^{iu}\delta^{rv}+\delta^{jm}\delta^{ln}\delta^{ks}\delta^{iu}\delta^{rv}+\delta^{jl}\delta^{mn}\delta^{ks}\delta^{iu}\delta^{rv}+\delta^{km}\delta^{jn}\delta^{ls}\delta^{iu}\delta^{rv}$$

$$+\delta^{jm}\delta^{kn}\delta^{ls}\delta^{iu}\delta^{rv}+\delta^{jk}\delta^{mn}\delta^{ls}\delta^{iu}\delta^{rv}+\delta^{kl}\delta^{jn}\delta^{ms}\delta^{iu}\delta^{rv}+\delta^{jl}\delta^{kn}\delta^{ms}\delta^{iu}\delta^{rv}+\delta^{jk}\delta^{ln}\delta^{ms}\delta^{iu}\delta^{rv}$$

$$+\delta^{kl}\delta^{jm}\delta^{ns}\delta^{iu}\delta^{rv}+\delta^{jl}\delta^{km}\delta^{ns}\delta^{iu}\delta^{rv}+\delta^{jk}\delta^{lm}\delta^{ns}\delta^{iu}\delta^{rv}+\delta^{lm}\delta^{kn}\delta^{is}\delta^{ju}\delta^{rv}+\delta^{km}\delta^{ln}\delta^{is}\delta^{ju}\delta^{rv}$$

$$+\delta^{kl}\delta^{mn}\delta^{is}\delta^{ju}\delta^{rv}+\delta^{lm}\delta^{in}\delta^{ks}\delta^{ju}\delta^{rv}+\delta^{im}\delta^{ln}\delta^{ks}\delta^{ju}\delta^{rv}+\delta^{il}\delta^{mn}\delta^{ks}\delta^{ju}\delta^{rv}+\delta^{km}\delta^{in}\delta^{ls}\delta^{ju}\delta^{rv}$$

$$+\delta^{im}\delta^{kn}\delta^{ls}\delta^{ju}\delta^{rv}+\delta^{ik}\delta^{mn}\delta^{ls}\delta^{ju}\delta^{rv}+\delta^{kl}\delta^{in}\delta^{ms}\delta^{ju}\delta^{rv}+\delta^{il}\delta^{kn}\delta^{ms}\delta^{ju}\delta^{rv}+\delta^{ik}\delta^{ln}\delta^{ms}\delta^{ju}\delta^{rv}$$

$$+\delta^{kl}\delta^{im}\delta^{ns}\delta^{ju}\delta^{rv}+\delta^{il}\delta^{km}\delta^{ns}\delta^{ju}\delta^{rv}+\delta^{ik}\delta^{lm}\delta^{ns}\delta^{ju}\delta^{rv}+\delta^{lm}\delta^{jn}\delta^{is}\delta^{ku}\delta^{rv}+\delta^{jm}\delta^{ln}\delta^{is}\delta^{ku}\delta^{rv}$$

$$+\delta^{jl}\delta^{mn}\delta^{is}\delta^{ku}\delta^{rv}+\delta^{lm}\delta^{in}\delta^{js}\delta^{ku}\delta^{rv}+\delta^{im}\delta^{ln}\delta^{js}\delta^{ku}\delta^{rv}+\delta^{il}\delta^{mn}\delta^{js}\delta^{ku}\delta^{rv}+\delta^{jm}\delta^{in}\delta^{ls}\delta^{ku}\delta^{rv}$$

$$+\delta^{im}\delta^{jn}\delta^{ls}\delta^{ku}\delta^{rv}+\delta^{ij}\delta^{mn}\delta^{ls}\delta^{ku}\delta^{rv}+\delta^{jl}\delta^{in}\delta^{ms}\delta^{ku}\delta^{rv}+\delta^{il}\delta^{jn}\delta^{ms}\delta^{ku}\delta^{rv}+\delta^{ij}\delta^{ln}\delta^{ms}\delta^{ku}\delta^{rv}$$

$$+\delta^{jl}\delta^{im}\delta^{ns}\delta^{ku}\delta^{rv}+\delta^{il}\delta^{jm}\delta^{ns}\delta^{ku}\delta^{rv}+\delta^{ij}\delta^{lm}\delta^{ns}\delta^{ku}\delta^{rv}+\delta^{km}\delta^{jn}\delta^{is}\delta^{lu}\delta^{rv}+\delta^{jm}\delta^{kn}\delta^{is}\delta^{lu}\delta^{rv}$$

$$+\delta^{jk}\delta^{mn}\delta^{is}\delta^{lu}\delta^{rv}+\delta^{km}\delta^{in}\delta^{js}\delta^{lu}\delta^{rv}+\delta^{im}\delta^{kn}\delta^{js}\delta^{lu}\delta^{rv}+\delta^{ik}\delta^{mn}\delta^{js}\delta^{lu}\delta^{rv}+\delta^{jm}\delta^{in}\delta^{ks}\delta^{lu}\delta^{rv}$$

$$+\delta^{im}\delta^{jn}\delta^{ks}\delta^{lu}\delta^{rv}+\delta^{ij}\delta^{mn}\delta^{ks}\delta^{lu}\delta^{rv}+\delta^{jk}\delta^{in}\delta^{ms}\delta^{lu}\delta^{rv}+\delta^{ik}\delta^{jn}\delta^{ms}\delta^{lu}\delta^{rv}+\delta^{ij}\delta^{kn}\delta^{ms}\delta^{lu}\delta^{rv}$$

$$+\delta^{jk}\delta^{im}\delta^{ns}\delta^{lu}\delta^{rv}+\delta^{ik}\delta^{jm}\delta^{ns}\delta^{lu}\delta^{rv}+\delta^{ij}\delta^{km}\delta^{ns}\delta^{lu}\delta^{rv}+\delta^{kl}\delta^{jn}\delta^{is}\delta^{mu}\delta^{rv}+\delta^{jl}\delta^{kn}\delta^{is}\delta^{mu}\delta^{rv}$$

$$+\delta^{jk}\delta^{ln}\delta^{is}\delta^{mu}\delta^{rv}+\delta^{kl}\delta^{in}\delta^{js}\delta^{mu}\delta^{rv}+\delta^{il}\delta^{kn}\delta^{js}\delta^{mu}\delta^{rv}+\delta^{ik}\delta^{ln}\delta^{js}\delta^{mu}\delta^{rv}+\delta^{jl}\delta^{in}\delta^{ks}\delta^{mu}\delta^{rv}$$

$$+\delta^{il}\delta^{jn}\delta^{ks}\delta^{mu}\delta^{rv}+\delta^{ij}\delta^{ln}\delta^{ks}\delta^{mu}\delta^{rv}+\delta^{jk}\delta^{in}\delta^{ls}\delta^{mu}\delta^{rv}+\delta^{ik}\delta^{jn}\delta^{ls}\delta^{mu}\delta^{rv}+\delta^{ij}\delta^{kn}\delta^{ls}\delta^{mu}\delta^{rv}$$

$$+\delta^{jk}\delta^{il}\delta^{ns}\delta^{mu}\delta^{rv}+\delta^{ik}\delta^{jl}\delta^{ns}\delta^{mu}\delta^{rv}+\delta^{ij}\delta^{kl}\delta^{ns}\delta^{mu}\delta^{rv}+\delta^{kl}\delta^{jm}\delta^{is}\delta^{nu}\delta^{rv}+\delta^{jl}\delta^{km}\delta^{is}\delta^{nu}\delta^{rv}$$

$$+\delta^{jk}\delta^{lm}\delta^{is}\delta^{nu}\delta^{rv}+\delta^{kl}\delta^{im}\delta^{js}\delta^{nu}\delta^{rv}+\delta^{il}\delta^{km}\delta^{js}\delta^{nu}\delta^{rv}+\delta^{ik}\delta^{lm}\delta^{js}\delta^{nu}\delta^{rv}+\delta^{jl}\delta^{im}\delta^{ks}\delta^{nu}\delta^{rv}$$

$$+\delta^{il}\delta^{jm}\delta^{ks}\delta^{nu}\delta^{rv}+\delta^{ij}\delta^{lm}\delta^{ks}\delta^{nu}\delta^{rv}+\delta^{jk}\delta^{im}\delta^{ls}\delta^{nu}\delta^{rv}+\delta^{ik}\delta^{jm}\delta^{ls}\delta^{nu}\delta^{rv}+\delta^{ij}\delta^{km}\delta^{ls}\delta^{nu}\delta^{rv}$$

$$+\delta^{jk}\delta^{il}\delta^{ms}\delta^{nu}\delta^{rv}+\delta^{ik}\delta^{jl}\delta^{ms}\delta^{nu}\delta^{rv}+\delta^{ij}\delta^{kl}\delta^{ms}\delta^{nu}\delta^{rv}+\delta^{kl}\delta^{jm}\delta^{in}\delta^{su}\delta^{rv}+\delta^{jl}\delta^{km}\delta^{in}\delta^{su}\delta^{rv}$$

$$+\delta^{jk}\delta^{lm}\delta^{in}\delta^{su}\delta^{rv}+\delta^{kl}\delta^{im}\delta^{jn}\delta^{su}\delta^{rv}+\delta^{il}\delta^{km}\delta^{jn}\delta^{su}\delta^{rv}+\delta^{ik}\delta^{lm}\delta^{jn}\delta^{su}\delta^{rv}+\delta^{jl}\delta^{im}\delta^{kn}\delta^{su}\delta^{rv}$$

$$+\delta^{il}\delta^{jm}\delta^{kn}\delta^{su}\delta^{rv}+\delta^{ij}\delta^{lm}\delta^{kn}\delta^{su}\delta^{rv}+\delta^{jk}\delta^{im}\delta^{ln}\delta^{su}\delta^{rv}+\delta^{ik}\delta^{jm}\delta^{ln}\delta^{su}\delta^{rv}+\delta^{ij}\delta^{km}\delta^{ln}\delta^{su}\delta^{rv}$$

$$+\delta^{jk}\delta^{il}\delta^{mn}\delta^{su}\delta^{rv}+\delta^{ik}\delta^{jl}\delta^{mn}\delta^{su}\delta^{rv}+\delta^{ij}\delta^{kl}\delta^{mn}\delta^{su}\delta^{rv}+\delta^{lm}\delta^{kn}\delta^{jr}\delta^{iu}\delta^{sv}+\delta^{km}\delta^{ln}\delta^{jr}\delta^{iu}\delta^{sv}$$

$$+\delta^{kl}\delta^{mn}\delta^{jr}\delta^{iu}\delta^{sv} + \delta^{lm}\delta^{jn}\delta^{kr}\delta^{iu}\delta^{sv} + \delta^{jm}\delta^{ln}\delta^{kr}\delta^{iu}\delta^{sv} + \delta^{jl}\delta^{mn}\delta^{kr}\delta^{iu}\delta^{sv} + \delta^{km}\delta^{jn}\delta^{lr}\delta^{iu}\delta^{sv}$$

$$+\delta^{jm}\delta^{kn}\delta^{lr}\delta^{iu}\delta^{sv} + \delta^{jk}\delta^{mn}\delta^{lr}\delta^{iu}\delta^{sv} + \delta^{kl}\delta^{jn}\delta^{mr}\delta^{iu}\delta^{sv} + \delta^{jl}\delta^{kn}\delta^{mr}\delta^{iu}\delta^{sv} + \delta^{jk}\delta^{ln}\delta^{mr}\delta^{iu}\delta^{sv}$$

$$+\delta^{kl}\delta^{jm}\delta^{nr}\delta^{iu}\delta^{sv} + \delta^{jl}\delta^{km}\delta^{nr}\delta^{iu}\delta^{sv} + \delta^{jk}\delta^{lm}\delta^{nr}\delta^{iu}\delta^{sv} + \delta^{lm}\delta^{kn}\delta^{ir}\delta^{ju}\delta^{sv} + \delta^{km}\delta^{ln}\delta^{ir}\delta^{ju}\delta^{sv}$$

$$+\delta^{kl}\delta^{mn}\delta^{ir}\delta^{ju}\delta^{sv} + \delta^{lm}\delta^{in}\delta^{kr}\delta^{ju}\delta^{sv} + \delta^{im}\delta^{ln}\delta^{kr}\delta^{ju}\delta^{sv} + \delta^{il}\delta^{mn}\delta^{kr}\delta^{ju}\delta^{sv} + \delta^{km}\delta^{in}\delta^{lr}\delta^{ju}\delta^{sv}$$

$$+\delta^{im}\delta^{kn}\delta^{lr}\delta^{ju}\delta^{sv} + \delta^{ik}\delta^{mn}\delta^{lr}\delta^{ju}\delta^{sv} + \delta^{kl}\delta^{in}\delta^{mr}\delta^{ju}\delta^{sv} + \delta^{il}\delta^{kn}\delta^{mr}\delta^{ju}\delta^{sv} + \delta^{ik}\delta^{ln}\delta^{mr}\delta^{ju}\delta^{sv}$$

$$+\delta^{kl}\delta^{im}\delta^{nr}\delta^{ju}\delta^{sv} + \delta^{il}\delta^{km}\delta^{nr}\delta^{ju}\delta^{sv} + \delta^{ik}\delta^{lm}\delta^{nr}\delta^{ju}\delta^{sv} + \delta^{lm}\delta^{jn}\delta^{ir}\delta^{ku}\delta^{sv} + \delta^{jm}\delta^{ln}\delta^{ir}\delta^{ku}\delta^{sv}$$

$$+\delta^{jl}\delta^{mn}\delta^{ir}\delta^{ku}\delta^{sv} + \delta^{lm}\delta^{in}\delta^{jr}\delta^{ku}\delta^{sv} + \delta^{im}\delta^{ln}\delta^{jr}\delta^{ku}\delta^{sv} + \delta^{il}\delta^{mn}\delta^{jr}\delta^{ku}\delta^{sv} + \delta^{jm}\delta^{in}\delta^{lr}\delta^{ku}\delta^{sv}$$

$$+\delta^{im}\delta^{jn}\delta^{lr}\delta^{ku}\delta^{sv} + \delta^{ij}\delta^{mn}\delta^{lr}\delta^{ku}\delta^{sv} + \delta^{jl}\delta^{in}\delta^{mr}\delta^{ku}\delta^{sv} + \delta^{il}\delta^{jn}\delta^{mr}\delta^{ku}\delta^{sv} + \delta^{ij}\delta^{ln}\delta^{mr}\delta^{ku}\delta^{sv}$$

$$+\delta^{jl}\delta^{im}\delta^{nr}\delta^{ku}\delta^{sv} + \delta^{il}\delta^{jm}\delta^{nr}\delta^{ku}\delta^{sv} + \delta^{ij}\delta^{lm}\delta^{nr}\delta^{ku}\delta^{sv} + \delta^{km}\delta^{jn}\delta^{ir}\delta^{lu}\delta^{sv} + \delta^{jm}\delta^{kn}\delta^{ir}\delta^{lu}\delta^{sv}$$

$$+\delta^{jk}\delta^{mn}\delta^{ir}\delta^{lu}\delta^{sv} + \delta^{km}\delta^{in}\delta^{jr}\delta^{lu}\delta^{sv} + \delta^{im}\delta^{kn}\delta^{jr}\delta^{lu}\delta^{sv} + \delta^{ik}\delta^{mn}\delta^{jr}\delta^{lu}\delta^{sv} + \delta^{jm}\delta^{in}\delta^{kr}\delta^{lu}\delta^{sv}$$

$$+\delta^{im}\delta^{jn}\delta^{kr}\delta^{lu}\delta^{sv} + \delta^{ij}\delta^{mn}\delta^{kr}\delta^{lu}\delta^{sv} + \delta^{jk}\delta^{in}\delta^{mr}\delta^{lu}\delta^{sv} + \delta^{ik}\delta^{jn}\delta^{mr}\delta^{lu}\delta^{sv} + \delta^{ij}\delta^{kn}\delta^{mr}\delta^{lu}\delta^{sv}$$

$$+\delta^{jk}\delta^{im}\delta^{nr}\delta^{lu}\delta^{sv} + \delta^{ik}\delta^{jm}\delta^{nr}\delta^{lu}\delta^{sv} + \delta^{ij}\delta^{km}\delta^{nr}\delta^{lu}\delta^{sv} + \delta^{kl}\delta^{jn}\delta^{ir}\delta^{mu}\delta^{sv} + \delta^{jl}\delta^{kn}\delta^{ir}\delta^{mu}\delta^{sv}$$

$$+\delta^{jk}\delta^{ln}\delta^{ir}\delta^{mu}\delta^{sv} + \delta^{kl}\delta^{in}\delta^{jr}\delta^{mu}\delta^{sv} + \delta^{il}\delta^{kn}\delta^{jr}\delta^{mu}\delta^{sv} + \delta^{ik}\delta^{ln}\delta^{jr}\delta^{mu}\delta^{sv} + \delta^{jl}\delta^{in}\delta^{kr}\delta^{mu}\delta^{sv}$$

$$+\delta^{il}\delta^{jn}\delta^{kr}\delta^{mu}\delta^{sv} + \delta^{ij}\delta^{ln}\delta^{kr}\delta^{mu}\delta^{sv} + \delta^{jk}\delta^{in}\delta^{lr}\delta^{mu}\delta^{sv} + \delta^{ik}\delta^{jn}\delta^{lr}\delta^{mu}\delta^{sv} + \delta^{ij}\delta^{kn}\delta^{lr}\delta^{mu}\delta^{sv}$$

$$+\delta^{jk}\delta^{il}\delta^{nr}\delta^{mu}\delta^{sv} + \delta^{ik}\delta^{jl}\delta^{nr}\delta^{mu}\delta^{sv} + \delta^{ij}\delta^{kl}\delta^{nr}\delta^{mu}\delta^{sv} + \delta^{kl}\delta^{jm}\delta^{ir}\delta^{nu}\delta^{sv} + \delta^{jl}\delta^{km}\delta^{ir}\delta^{nu}\delta^{sv}$$

$$+\delta^{jk}\delta^{lm}\delta^{ir}\delta^{nu}\delta^{sv} + \delta^{kl}\delta^{im}\delta^{jr}\delta^{nu}\delta^{sv} + \delta^{il}\delta^{km}\delta^{jr}\delta^{nu}\delta^{sv} + \delta^{ik}\delta^{lm}\delta^{jr}\delta^{nu}\delta^{sv} + \delta^{jl}\delta^{im}\delta^{kr}\delta^{nu}\delta^{sv}$$

$$+\delta^{il}\delta^{jm}\delta^{kr}\delta^{nu}\delta^{sv} + \delta^{ij}\delta^{lm}\delta^{kr}\delta^{nu}\delta^{sv} + \delta^{jk}\delta^{im}\delta^{lr}\delta^{nu}\delta^{sv} + \delta^{ik}\delta^{jm}\delta^{lr}\delta^{nu}\delta^{sv} + \delta^{ij}\delta^{km}\delta^{lr}\delta^{nu}\delta^{sv}$$

$$+\delta^{jk}\delta^{il}\delta^{mr}\delta^{nu}\delta^{sv} + \delta^{ik}\delta^{jl}\delta^{mr}\delta^{nu}\delta^{sv} + \delta^{ij}\delta^{kl}\delta^{mr}\delta^{nu}\delta^{sv} + \delta^{kl}\delta^{jm}\delta^{in}\delta^{ru}\delta^{sv} + \delta^{jl}\delta^{km}\delta^{in}\delta^{ru}\delta^{sv}$$

$$+\delta^{jk}\delta^{lm}\delta^{in}\delta^{ru}\delta^{sv} + \delta^{kl}\delta^{im}\delta^{jn}\delta^{ru}\delta^{sv} + \delta^{il}\delta^{km}\delta^{jn}\delta^{ru}\delta^{sv} + \delta^{ik}\delta^{lm}\delta^{jn}\delta^{ru}\delta^{sv} + \delta^{jl}\delta^{im}\delta^{kn}\delta^{ru}\delta^{sv}$$

$$+\delta^{il}\delta^{jm}\delta^{kn}\delta^{ru}\delta^{sv} + \delta^{ij}\delta^{lm}\delta^{kn}\delta^{ru}\delta^{sv} + \delta^{jk}\delta^{im}\delta^{ln}\delta^{ru}\delta^{sv} + \delta^{ik}\delta^{jm}\delta^{ln}\delta^{ru}\delta^{sv} + \delta^{ij}\delta^{km}\delta^{ln}\delta^{ru}\delta^{sv}$$

$$+\delta^{jk}\delta^{il}\delta^{mn}\delta^{ru}\delta^{sv} + \delta^{ik}\delta^{jl}\delta^{mn}\delta^{ru}\delta^{sv} + \delta^{ij}\delta^{kl}\delta^{mn}\delta^{ru}\delta^{sv} + \delta^{lm}\delta^{kn}\delta^{jr}\delta^{is}\delta^{uv} + \delta^{km}\delta^{ln}\delta^{jr}\delta^{is}\delta^{uv}$$

$$+\delta^{kl}\delta^{mn}\delta^{jr}\delta^{is}\delta^{uv} + \delta^{lm}\delta^{jn}\delta^{kr}\delta^{is}\delta^{uv} + \delta^{jm}\delta^{ln}\delta^{kr}\delta^{is}\delta^{uv} + \delta^{jl}\delta^{mn}\delta^{kr}\delta^{is}\delta^{uv} + \delta^{km}\delta^{jn}\delta^{lr}\delta^{is}\delta^{uv}$$

$$+\delta^{jm}\delta^{kn}\delta^{lr}\delta^{is}\delta^{uv} + \delta^{jk}\delta^{mn}\delta^{lr}\delta^{is}\delta^{uv} + \delta^{kl}\delta^{jn}\delta^{mr}\delta^{is}\delta^{uv} + \delta^{jl}\delta^{kn}\delta^{mr}\delta^{is}\delta^{uv} + \delta^{jk}\delta^{ln}\delta^{mr}\delta^{is}\delta^{uv}$$

$$+\delta^{kl}\delta^{jm}\delta^{nr}\delta^{is}\delta^{uv} + \delta^{jl}\delta^{km}\delta^{nr}\delta^{is}\delta^{uv} + \delta^{jk}\delta^{lm}\delta^{nr}\delta^{is}\delta^{uv} + \delta^{lm}\delta^{kn}\delta^{ir}\delta^{js}\delta^{uv} + \delta^{km}\delta^{ln}\delta^{ir}\delta^{js}\delta^{uv}$$

$$+\delta^{kl}\delta^{mn}\delta^{ir}\delta^{js}\delta^{uv} + \delta^{lm}\delta^{in}\delta^{kr}\delta^{js}\delta^{uv} + \delta^{im}\delta^{ln}\delta^{kr}\delta^{js}\delta^{uv} + \delta^{il}\delta^{mn}\delta^{kr}\delta^{js}\delta^{uv} + \delta^{km}\delta^{in}\delta^{lr}\delta^{js}\delta^{uv}$$

$$+\delta^{im}\delta^{kn}\delta^{lr}\delta^{js}\delta^{uv} + \delta^{ik}\delta^{mn}\delta^{lr}\delta^{js}\delta^{uv} + \delta^{kl}\delta^{in}\delta^{mr}\delta^{js}\delta^{uv} + \delta^{il}\delta^{kn}\delta^{mr}\delta^{js}\delta^{uv} + \delta^{ik}\delta^{ln}\delta^{mr}\delta^{js}\delta^{uv}$$

$$+\delta^{kl}\delta^{im}\delta^{nr}\delta^{js}\delta^{uv} + \delta^{il}\delta^{km}\delta^{nr}\delta^{js}\delta^{uv} + \delta^{ik}\delta^{lm}\delta^{nr}\delta^{js}\delta^{uv} + \delta^{lm}\delta^{jn}\delta^{ir}\delta^{ks}\delta^{uv} + \delta^{jm}\delta^{ln}\delta^{ir}\delta^{ks}\delta^{uv}$$

$$+\delta^{jl}\delta^{mn}\delta^{ir}\delta^{ks}\delta^{uv} + \delta^{lm}\delta^{in}\delta^{jr}\delta^{ks}\delta^{uv} + \delta^{im}\delta^{ln}\delta^{jr}\delta^{ks}\delta^{uv} + \delta^{il}\delta^{mn}\delta^{jr}\delta^{ks}\delta^{uv} + \delta^{jm}\delta^{in}\delta^{lr}\delta^{ks}\delta^{uv}$$

$$+\delta^{im}\delta^{jn}\delta^{lr}\delta^{ks}\delta^{uv} + \delta^{ij}\delta^{mn}\delta^{lr}\delta^{ks}\delta^{uv} + \delta^{jl}\delta^{in}\delta^{mr}\delta^{ks}\delta^{uv} + \delta^{il}\delta^{jn}\delta^{mr}\delta^{ks}\delta^{uv} + \delta^{ij}\delta^{ln}\delta^{mr}\delta^{ks}\delta^{uv}$$

$$+\delta^{jl}\delta^{im}\delta^{nr}\delta^{ks}\delta^{uv} + \delta^{il}\delta^{jm}\delta^{nr}\delta^{ks}\delta^{uv} + \delta^{ij}\delta^{lm}\delta^{nr}\delta^{ks}\delta^{uv} + \delta^{km}\delta^{jn}\delta^{ir}\delta^{ls}\delta^{uv} + \delta^{jm}\delta^{kn}\delta^{ir}\delta^{ls}\delta^{uv}$$

$$+\delta^{jk}\delta^{mn}\delta^{ir}\delta^{ls}\delta^{uv} + \delta^{km}\delta^{in}\delta^{jr}\delta^{ls}\delta^{uv} + \delta^{im}\delta^{kn}\delta^{jr}\delta^{ls}\delta^{uv} + \delta^{ik}\delta^{mn}\delta^{jr}\delta^{ls}\delta^{uv} + \delta^{jm}\delta^{in}\delta^{kr}\delta^{ls}\delta^{uv}$$

$$+\delta^{im}\delta^{jn}\delta^{kr}\delta^{ls}\delta^{uv} + \delta^{ij}\delta^{mn}\delta^{kr}\delta^{ls}\delta^{uv} + \delta^{jk}\delta^{in}\delta^{mr}\delta^{ls}\delta^{uv} + \delta^{ik}\delta^{jn}\delta^{mr}\delta^{ls}\delta^{uv} + \delta^{ij}\delta^{kn}\delta^{mr}\delta^{ls}\delta^{uv}$$
$$+\delta^{jk}\delta^{im}\delta^{nr}\delta^{ls}\delta^{uv} + \delta^{ik}\delta^{jm}\delta^{nr}\delta^{ls}\delta^{uv} + \delta^{ij}\delta^{km}\delta^{nr}\delta^{ls}\delta^{uv} + \delta^{kl}\delta^{jn}\delta^{ir}\delta^{ms}\delta^{uv} + \delta^{jl}\delta^{kn}\delta^{ir}\delta^{ms}\delta^{uv}$$
$$+\delta^{jk}\delta^{ln}\delta^{ir}\delta^{ms}\delta^{uv} + \delta^{kl}\delta^{in}\delta^{jr}\delta^{ms}\delta^{uv} + \delta^{il}\delta^{kn}\delta^{jr}\delta^{ms}\delta^{uv} + \delta^{ik}\delta^{ln}\delta^{jr}\delta^{ms}\delta^{uv} + \delta^{jl}\delta^{in}\delta^{kr}\delta^{ms}\delta^{uv}$$
$$+\delta^{il}\delta^{jn}\delta^{kr}\delta^{ms}\delta^{uv} + \delta^{ij}\delta^{ln}\delta^{kr}\delta^{ms}\delta^{uv} + \delta^{jk}\delta^{in}\delta^{lr}\delta^{ms}\delta^{uv} + \delta^{ik}\delta^{jn}\delta^{lr}\delta^{ms}\delta^{uv} + \delta^{ij}\delta^{kn}\delta^{lr}\delta^{ms}\delta^{uv}$$
$$+\delta^{jk}\delta^{il}\delta^{nr}\delta^{ms}\delta^{uv} + \delta^{ik}\delta^{jl}\delta^{nr}\delta^{ms}\delta^{uv} + \delta^{ij}\delta^{kl}\delta^{nr}\delta^{ms}\delta^{uv} + \delta^{kl}\delta^{jm}\delta^{ir}\delta^{ns}\delta^{uv} + \delta^{jl}\delta^{km}\delta^{ir}\delta^{ns}\delta^{uv}$$
$$+\delta^{jk}\delta^{lm}\delta^{ir}\delta^{ns}\delta^{uv} + \delta^{kl}\delta^{im}\delta^{jr}\delta^{ns}\delta^{uv} + \delta^{il}\delta^{km}\delta^{jr}\delta^{ns}\delta^{uv} + \delta^{ik}\delta^{lm}\delta^{jr}\delta^{ns}\delta^{uv} + \delta^{jl}\delta^{im}\delta^{kr}\delta^{ns}\delta^{uv}$$
$$+\delta^{il}\delta^{jm}\delta^{kr}\delta^{ns}\delta^{uv} + \delta^{ij}\delta^{lm}\delta^{kr}\delta^{ns}\delta^{uv} + \delta^{jk}\delta^{im}\delta^{lr}\delta^{ns}\delta^{uv} + \delta^{ik}\delta^{jm}\delta^{lr}\delta^{ns}\delta^{uv} + \delta^{ij}\delta^{km}\delta^{lr}\delta^{ns}\delta^{uv}$$
$$+\delta^{jk}\delta^{il}\delta^{mr}\delta^{ns}\delta^{uv} + \delta^{ik}\delta^{jl}\delta^{mr}\delta^{ns}\delta^{uv} + \delta^{ij}\delta^{kl}\delta^{mr}\delta^{ns}\delta^{uv} + \delta^{kl}\delta^{jm}\delta^{in}\delta^{rs}\delta^{uv} + \delta^{jl}\delta^{km}\delta^{in}\delta^{rs}\delta^{uv}$$
$$+\delta^{jk}\delta^{lm}\delta^{in}\delta^{rs}\delta^{uv} + \delta^{kl}\delta^{im}\delta^{jn}\delta^{rs}\delta^{uv} + \delta^{il}\delta^{km}\delta^{jn}\delta^{rs}\delta^{uv} + \delta^{ik}\delta^{lm}\delta^{jn}\delta^{rs}\delta^{uv} + \delta^{jl}\delta^{im}\delta^{kn}\delta^{rs}\delta^{uv}$$
$$+\delta^{il}\delta^{jm}\delta^{kn}\delta^{rs}\delta^{uv} + \delta^{ij}\delta^{lm}\delta^{kn}\delta^{rs}\delta^{uv} + \delta^{jk}\delta^{im}\delta^{ln}\delta^{rs}\delta^{uv} + \delta^{ik}\delta^{jm}\delta^{ln}\delta^{rs}\delta^{uv} + \delta^{ij}\delta^{km}\delta^{ln}\delta^{rs}\delta^{uv}$$
$$+ \delta^{jk}\delta^{il}\delta^{mn}\delta^{rs}\delta^{uv} + \delta^{ik}\delta^{jl}\delta^{mn}\delta^{rs}\delta^{uv} + \delta^{ij}\delta^{kl}\delta^{mn}\delta^{rs}\delta^{uv}\Big)\nu^{2\epsilon}\,. \tag{B.123}$$

Note that all these integrals are zero in dimensional regularization because they are scaleless. Instead, they are evaluated by introducing a hard cutoff. After the logarithm is found, the UV pole is reconstructed. The replacement rule we indicate corresponds to the counterterm necessary to cancel the UV divergence that would appear in dimensional regularization if we would not have expanded over external momenta and energies.

B.5 Color Algebra and gamma matrices

B.5.1 Color Algebra

We will use the Gell-Mann matrices as a representation of the infinitesimal generators of the special unitary grup SU(3). The Lie algebra of this group has dimension eight and, therefore, it has a set of eight linearly independent generators, which we write as T^a, with $a = 1, \ldots, 8$. These Lie algebra elements satisfy the following commutation and anticommutation relations

$$[T^a, T^b]_{\alpha\beta} = i f^{abc} (T^c)_{\alpha\beta}\,, \tag{B.124}$$

$$\{T^a, T^b\}_{\alpha\beta} = \frac{1}{N_c}\delta^{ab}\delta_{\alpha\beta} + d^{abc}(T^c)_{\alpha\beta}\,, \tag{B.125}$$

where f^{abc} and d^{abc} are antisymmetric and symmetric structure constants, respectively. The normalization is the following

$$\mathrm{Tr}[T^a T^b] = T_F \delta^{ab}\,, \tag{B.126}$$

and the color factors are

$$C_A = N_c\,, \tag{B.127}$$

$$C_F = T_F \frac{N_c^2 - 1}{N_c}\,, \tag{B.128}$$

$$T_F = \frac{1}{2}\,. \tag{B.129}$$

For QCD, $C_A = N_c = 3$ and $C_F = 4/3$. The following relations, which can be found in Ref. [76], are necessary to rewrite the color structure appearing in the HQET diagrams we compute as the one of appearing in the Feynman rules given in Sec. B.1

$$(T^a T^a)_{\alpha\beta} = C_F \delta_{\alpha\beta}\,, \tag{B.130}$$

$$(T^b T^a T^b)_{\alpha\beta} = -\frac{T_F}{N_c}(T^a)_{\alpha\beta} = \left(C_F - \frac{C_A}{2}\right)(T^a)_{\alpha\beta}\,, \tag{B.131}$$

$$f^{abc} f^{dbc} = C_A \delta^{ad}\,, \tag{B.132}$$

$$d^{abb} = 0\,, \tag{B.133}$$

$$d^{abc} d^{dbc} = \left(N_c - \frac{4}{N_c}\right)\delta^{ad} = (8C_F - 3C_A)\delta^{ad}\,. \tag{B.134}$$

Particularly, we computed the following relations that are difficult to find in the literature and that turned out to be very important for the computation of diagrams appearing in the scattering of a heavy quark with a gluon

$$(T^c \{T^a, T^b\} T^c)_{\alpha\beta} = \left(C_F - \frac{C_A}{2}\right)\{T^a, T^b\}_{\alpha\beta} + T_F \delta^{ab}\delta_{\alpha\beta}\,, \tag{B.135}$$

$$(T^c [T^a, T^b] T^c)_{\alpha\beta} = \left(C_F - \frac{C_A}{2}\right)[T^a, T^b]_{\alpha\beta}\,, \tag{B.136}$$

$$(T^c T^a T^b T^c)_{\alpha\beta} = \left(C_F - \frac{C_A}{2}\right)(T^a T^b)_{\alpha\beta} + \frac{T_F}{2}\delta^{ab}\delta_{\alpha\beta}\,, \tag{B.137}$$

and the following ones which typically show up in the computation of diagrams appearing in the scattering of a heavy quark with a light quark

$$\{T^a, T^b\}_{\alpha\beta}\{T^a, T^b\}_{\delta\gamma} = -2C_F(2C_F - C_A)\delta_{\alpha\beta}\delta_{\delta\gamma} + (8C_F - 3C_A)(T^a)_{\alpha\beta}(T^a)_{\delta\gamma}\,, \tag{B.138}$$

$$\{\{T^a, T^b\}, T^b\}_{\alpha\beta} = (4C_F - C_A)(T^a)_{\alpha\beta}\,, \tag{B.139}$$

$$[T^a, T^b]_{\alpha\beta}[T^a, T^b]_{\delta\gamma} = -C_A (T^a)_{\alpha\beta}(T^a)_{\delta\gamma}\,. \tag{B.140}$$

B.5.2 Properties of the gamma matrices

The gamma matrices or Dirac matrices are a set of conventional matrices with certain anticommutation relations such that ensure they generate a matrix representation of the Clifford algebra. In general, there are several representations for the gamma matrices and in those cases we need to specify it, we will use the Dirac representation. These relations are standard, and read

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4 \,, \tag{B.141}$$

$$\{\gamma_5, \gamma^\mu\} = 0 \,, \tag{B.142}$$

with

$$(\gamma^0)^2 = I_4 \,, \quad (\gamma^i)^2 = -3I_4 \,. \tag{B.143}$$

The following standard traces of gamma matrices are needed for the computation of the scattering of a heavy and light quarks

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \,, \tag{B.144}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) = 4(g^{\mu\nu}g^{\sigma\rho} - g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma}) \,, \tag{B.145}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = -4i\epsilon^{\mu\nu\rho\sigma} \,, \tag{B.146}$$

$$\begin{aligned} \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\beta\gamma^\alpha\gamma^\gamma\gamma^\rho) \;=\; & 4g^{\gamma\rho}(g^{\mu\nu}g^{\beta\alpha} - g^{\mu\beta}g^{\nu\alpha} + g^{\mu\alpha}g^{\nu\beta}) - 4g^{\alpha\rho}(g^{\mu\nu}g^{\beta\gamma} - g^{\mu\beta}g^{\nu\gamma} + g^{\mu\gamma}g^{\nu\beta}) \\ & +4g^{\beta\rho}(g^{\mu\nu}g^{\alpha\gamma} - g^{\mu\alpha}g^{\nu\gamma} + g^{\mu\gamma}g^{\nu\alpha}) - 4g^{\nu\rho}(g^{\mu\beta}g^{\alpha\gamma} - g^{\mu\alpha}g^{\beta\gamma} + g^{\mu\gamma}g^{\beta\alpha}) \\ & +4g^{\mu\rho}(g^{\nu\beta}g^{\alpha\gamma} - g^{\nu\alpha}g^{\beta\gamma} + g^{\nu\gamma}g^{\beta\alpha}) \,, \end{aligned} \tag{B.147}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta\gamma_5) = -4i(g^{\mu\nu}\epsilon^{\rho\sigma\alpha\beta} - g^{\mu\rho}\epsilon^{\nu\sigma\alpha\beta} + g^{\rho\nu}\epsilon^{\mu\sigma\alpha\beta} - g^{\alpha\beta}\epsilon^{\sigma\mu\nu\rho} + g^{\sigma\beta}\epsilon^{\alpha\mu\nu\rho} - g^{\sigma\alpha}\epsilon^{\beta\mu\nu\rho}) \,. \tag{B.148}$$

We define $\epsilon^{123} = 1$ and $\epsilon^{0123} = -\epsilon_{0123} = 1$, so the relation between the three dimensional Levi-Civita symbol in euclidean space with its four dimensional counterpart in Minskowski space is $\epsilon^{0ijk} = \epsilon^{ijk}$. Also the following relations are useful to compute the three-fermion loop

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^T \,, \tag{B.149}$$

$$C^{-1}\gamma_5 C = (\gamma_5)^T \,, \tag{B.150}$$

where C is the charge conjugation operator.

B.5.3 Some algebraic relations

The following Levi-Civita symbol contraction identities have been very useful

$$\epsilon^{ijk}\epsilon^{imn} = \delta^{jm}\delta^{kn} - \delta^{jn}\delta^{km}\,, \tag{B.151}$$

$$\epsilon^{jmn}\epsilon^{imn} = 2\delta^{ij}\,, \tag{B.152}$$

$$\epsilon^{ijk}\epsilon^{ijk} = 6\,. \tag{B.153}$$

The contraction of the Levi-Civita symbol with three vectors and the contraction of Eq. (B.151) with four vectors give rise to the following useful properties

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})\,, \tag{B.154}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\,. \tag{B.155}$$

Other, less common in the literature, relations involving cross products can be obtained by contracting the following expressions containing three Levi-Civita symbols and three vectors

$$\epsilon^{ijk}(\mathbf{c}\times(\mathbf{a}\times\mathbf{b}))^k\,, \tag{B.156}$$

$$\mathbf{a}^k\mathbf{b}^l\mathbf{c}^m\epsilon^{rkl}(\epsilon^{nim}\epsilon^{njr} + \epsilon^{njm}\epsilon^{nir})\,, \tag{B.157}$$

$$\mathbf{a}^k\mathbf{b}^l\mathbf{c}^m\epsilon^{rkl}(\epsilon^{nim}\epsilon^{njr} - \epsilon^{njm}\epsilon^{nir})\,. \tag{B.158}$$

The obtained relations are the following

$$\mathbf{c}^i(\mathbf{a}\times\mathbf{b})^j - \mathbf{c}^j(\mathbf{a}\times\mathbf{b})^i = (\mathbf{b}\cdot\mathbf{c})\mathbf{a}^k\epsilon^{ijk} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b}^k\epsilon^{ijk}\,, \tag{B.159}$$

$$2\delta^{ij}\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{a}^i(\mathbf{b}\times\mathbf{c})^j + \mathbf{a}^j(\mathbf{b}\times\mathbf{c})^i - \mathbf{b}^i(\mathbf{a}\times\mathbf{c})^j - \mathbf{b}^j(\mathbf{a}\times\mathbf{c})^i + \mathbf{c}^i(\mathbf{a}\times\mathbf{b})^j + \mathbf{c}^j(\mathbf{a}\times\mathbf{b})^i\,, \tag{B.160}$$

$$\mathbf{c}^i(\mathbf{a}\times\mathbf{b})^j - \mathbf{c}^j(\mathbf{a}\times\mathbf{b})^i = -\mathbf{a}^i(\mathbf{b}\times\mathbf{c})^j + \mathbf{a}^j(\mathbf{b}\times\mathbf{c})^i + \mathbf{b}^i(\mathbf{a}\times\mathbf{c})^j - \mathbf{b}^j(\mathbf{a}\times\mathbf{c})^i\,. \tag{B.161}$$

These relations are crucial to write the spin-dependent part of the one-loop scattering of a heavy quark with a transverse gluon in terms of the structures of the spin-dependent vertices of the HQET Lagrangian.

Appendix C

Matching QCD with NRQCD

C.1 QCD and NRQCD Feynman rules

In this section we summarize the Feynman rules necessary for the computation carried out in Sec. 3.4. For QCD, we do not display all of them, but only the ones we need.

Heavy quark propagator:

$$\frac{i(\not{p}+m)_{AB}}{p^2-m^2+i\eta_q}\delta_{\alpha\beta} \tag{C.1}$$

Gluon propagator (Feynman gauge):

$$\frac{ig_{\mu\nu}\delta^{ab}}{q^2+i\eta_g} \tag{C.2}$$

Quark-gluon vertex:

$$-ig(\gamma^\mu)_{AB}(T^a)_{\alpha\beta} \tag{C.3}$$

The needed NRQCD Feynman rules are the ones associated to the $1/m^2$ four fermion operators. They are shown in Fig. [C.1]

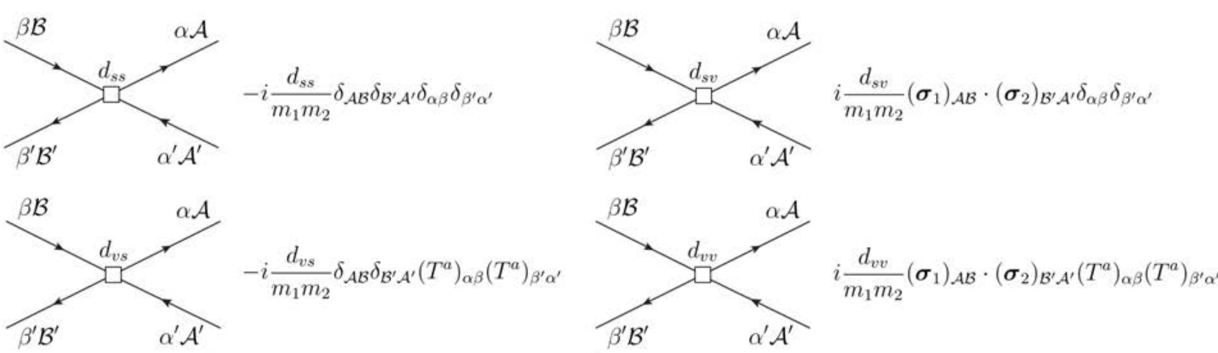


Figure C.1: NRQCD Feynman rules of the $1/m^2$ four fermion operators.

C.2 QCD Master Integrals

The following master integrals are necessary for the computation carried out in Sec. 3.4

$$\begin{aligned}
I(n;m_1,m_2) &= \int \frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2+i\eta_g)^n}\frac{1}{q^2-2m_1q\cdot v+i\eta_q}\frac{1}{q^2+2m_2q\cdot v+i\eta_q} \\
&= \frac{i}{(4\pi)^2}\left(\frac{-1}{m_1^2m_2^2}\right)^n\frac{1}{m_1+m_2}\left(m_1(m_2^2)^n\left(\frac{m_1^2}{4\pi}\right)^\epsilon+(m_1^2)^nm_2\left(\frac{m_2^2}{4\pi}\right)^\epsilon\right) \\
&\times\frac{\Gamma(n-\epsilon)\Gamma(1-2n+2\epsilon)}{\Gamma(2-n+2\epsilon)}\,, \qquad \text{(C.4)}
\end{aligned}$$

$$\begin{aligned}
I^\mu(n;m_1,m_2) &= \int \frac{d^Dq}{(2\pi)^D}\frac{q^\mu}{(q^2+i\eta_g)^n}\frac{1}{q^2-2m_1q\cdot v+i\eta_q}\frac{1}{q^2+2m_2q\cdot v+i\eta_q} \\
&= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{(-1)^n}{m_1+m_2}\left(-(m_1^2)^{1-n}\left(\frac{m_1^2}{4\pi}\right)^\epsilon+(m_2^2)^{1-n}\left(\frac{m_2^2}{4\pi}\right)^\epsilon\right) \\
&\times\frac{\Gamma(n-\epsilon)\Gamma(3-2n+2\epsilon)}{(-1+n-\epsilon)\Gamma(3-n+2\epsilon)}v^\mu\,, \qquad \text{(C.5)}
\end{aligned}$$

$$\begin{aligned}
I^{\mu\nu}(n;m_1,m_2) &= \int \frac{d^Dq}{(2\pi)^D}\frac{q^\mu q^\nu}{(q^2+i\eta_g)^n}\frac{1}{q^2-2m_1q\cdot v+i\eta_q}\frac{1}{q^2+2m_2q\cdot v+i\eta_q} \qquad \text{(C.6)} \\
&= \frac{1}{2}\frac{i}{(4\pi)^2}\frac{(-1)^n}{m_1+m_2}\left((m_1)^{3-2n}\left(\frac{m_1^2}{4\pi}\right)^\epsilon+(m_2)^{3-2n}\left(\frac{m_2^2}{4\pi}\right)^\epsilon\right) \\
&\times\frac{\Gamma(4-2n+2\epsilon)}{(-3+2n-2\epsilon)\Gamma(4-n+2\epsilon)}(g^{\mu\nu}\Gamma(-1+n-\epsilon)-2\Gamma(n-\epsilon)v^\mu v^\nu)\,.
\end{aligned}$$

C.3 Traces in D dimensions

In Sec. B.5.2 we gave some properties of the gamma matrices in four dimensions. However, in the computation of Sec. 3.4 we are interested in the full D-dimensional result, so these properties must be rewritten in $D = 4+2\epsilon$ dimensions. The necessary ones read

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = Dg^{\mu\nu}\,, \qquad \text{(C.7)}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) = D(g^{\mu\nu}g^{\sigma\rho}-g^{\mu\sigma}g^{\nu\rho}+g^{\mu\rho}g^{\nu\sigma})\,, \qquad \text{(C.8)}$$

$$\begin{aligned}
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\beta\gamma^\alpha\gamma^\gamma\gamma^\rho) &= D[g^{\gamma\rho}(g^{\mu\nu}g^{\beta\alpha}-g^{\mu\beta}g^{\nu\alpha}+g^{\mu\alpha}g^{\nu\beta})-g^{\alpha\rho}(g^{\mu\nu}g^{\beta\gamma}-g^{\mu\beta}g^{\nu\gamma}+g^{\mu\gamma}g^{\nu\beta}) \\
&+g^{\beta\rho}(g^{\mu\nu}g^{\alpha\gamma}-g^{\mu\alpha}g^{\nu\gamma}+g^{\mu\gamma}g^{\nu\alpha})-g^{\nu\rho}(g^{\mu\beta}g^{\alpha\gamma}-g^{\mu\alpha}g^{\beta\gamma}+g^{\mu\gamma}g^{\beta\alpha}) \\
&+g^{\mu\rho}(g^{\nu\beta}g^{\alpha\gamma}-g^{\nu\alpha}g^{\beta\gamma}+g^{\nu\gamma}g^{\beta\alpha})]\,. \qquad \text{(C.9)}
\end{aligned}$$

Appendix D

Matching NRQCD with pNRQCD

D.1 Matching NRQCD with pNRQCD: An example

In Sec. 4.4 the $\mathcal{O}(\alpha/m^4)$, $\mathcal{O}(\alpha^2/m^3)$ and $\mathcal{O}(\alpha^3/m^2)$ potentials are presented. These potentials happen to contribute to the $\mathcal{O}(m\alpha^6)$ heavy quarkonium spectrum. In this section, we give a particular example of the one-loop matching calculation to $\mathcal{O}(\alpha^2/m^3)$ between NRQCD and pNRQCD. In the matching procedure, NRQCD Green functions in the kinematical situation E, p^0, $p'^0 \sim mv^2$ and $|\mathbf{p}|$, q^0, $\mathbf{q}$, $\mathbf{p'} \sim mv$ are matched to interaction potentials in pNRQCD. These counting rules give a well defined and systematic way of computing the potentials that contribute, for instance, to the heavy quarkonium spectrum at a given order, and they allow us to determine to which order in v a given potential contributes. Moreover, the leading contribution to the binding energy of the bound state is given by

$$E(n) = -C_F^2 \frac{m_r \alpha^2}{2n^2} \,, \tag{D.1}$$

which is obtained by solving the Schrödinger equation with the leading Coulomb potential. In addition, the binding energy of a non-relativistic bound state is of the order of the ultrasoft scale $m_r v^2$, and we can identify that $v \sim \alpha/n$. For the lower levels of heavy quarkonium, the counting $v \sim \alpha$ is a good approximation. We will take this counting from now on and throughout this thesis. As an example, let us do the counting for the Coulomb potential i.e. let us see to which order it contributes to the spectrum

$$\tilde{V}_C \sim \frac{\alpha}{\mathbf{k}^2} \sim \frac{\alpha}{m^2\alpha^2} \Rightarrow V_C \sim d^3\mathbf{k}\tilde{V}_C \sim m^3\alpha^3 \frac{\alpha}{m^2\alpha^2} \sim m\alpha^2 \,, \tag{D.2}$$

which is in agreement with Eq. (D.1). The $\mathcal{O}(\alpha^2/m^3)$ interaction potential contributes to order

$$\tilde{V}_{\alpha^2/m^3} \sim \frac{\alpha^2}{m^3}|\mathbf{k}| \sim \frac{\alpha^2}{\mathbf{k}^2}\frac{|\mathbf{k}|^3}{m^3} \Rightarrow V_{\alpha^2/m^3} \sim m\alpha^6 \,, \tag{D.3}$$

where we have used the counting of the Coulomb potential to determine the counting of V_{α^2/m^3} quite fastly. The same applies for all the other potentials. The term $|\mathbf{k}|$ could be a more complicated function of the external and transfered momentum, but the power counting is the same. Likewise, it can be seen that the $\mathcal{O}(\alpha/m^4)$ and $\mathcal{O}(\alpha^2/m^3)$ potentials also contribute to the spectrum at $\mathcal{O}(m\alpha^6)$.

The diagram we have chosen as an example, which is shown in Fig.[D.3], happens to be irrelevant for the computation of the potential running of $\tilde{D}_d^{(2)}$ because the $\mathcal{O}(\alpha^2/m^3)$ contribution turns out to be zero, but it has all the ingredients that can appear in the matching calculation, so it is a good pedagogical diagram. In addition, it gives an $\mathcal{O}(\alpha^2/m^2)$ contribution which we also compute. During the calculation, we use the non-static heavy quark propagator and expand it to the necessary order[1]. Also energy dependent propagators are expanded to the necessary order. The computation is carried out in the center of mass frame of the heavy quark-antiquark pair.

As explained in Ref. [8], NRQCD Green functions are splitted in two contributions corresponding to the two kinematical situations where the loop energy is either soft ($q^0 \sim mv$) or ultrasoft ($q^0 \sim mv^2$). In the former case, green functions can be associated to a local in time, but not in space, interaction potential. The green functions are then matched to potential terms in pNRQCD. In the latter case, green functions can not be associated to a local in time potential, and that contribution corresponds to the usual iteration of potentials in non-relativistic quantum mechanics, which we refer in this thesis as potential loops.

Let us look closer to Fig. [D.3]. The loop diagrams on the left are NRQCD four-fermion diagrams, whereas the ones on the right are the same kind of diagrams in pNRQCD. Since the same diagram exists in both theories one has to subtract to the diagrams of the underlying theory, the ones of the EFT. The remainder, if there is any, can be absorbed in a new non-local in space term (interaction potential) in the EFT. This new interaction potential can be finite, but also IR or UV divergent. In the former case, the divergences are typically cancelled by the ultrasoft, and in the latter, they are absorbed by the counterterms of the potentials determining their anomalous dimensions. Both kind of diagrams must be considered to be in the kinematical situation E, p^0, $p'^0 \sim mv^2$ and $|\mathbf{p}|$, q^0, $\mathbf{q}$, $\mathbf{p}' \sim mv$, so note that the pNRQCD diagrams are not potential loops. The effect that subtracting these pNRQCD diagrams has is to regulate the NRQCD diagrams by eliminating the contribution of the heavy quark propagator poles that have already been included in potential loops, and that in NRQCD diagrams produce a kind of IR divergences called pinch singularities ($\pm i\eta$ poles).

Before going to the particular one-loop calculation, we need the tree level matching due to a longitudinal gluon exchange shown in Fig. [D.1], and due to a transverse gluon exchange with two c_k vertices shown in Fig. [D.2]. The matching will be carried out in the off-shell Coulomb gauge matching scheme. From Fig. [D.1] we have that

$$-ig(T^a)_{\alpha\beta}\frac{i}{\mathbf{k}^2}ig(T^a)_{\beta'\alpha'} = -i\tilde{V}_C\,. \tag{D.4}$$

Therefore

$$-i\tilde{V}_C = (T^a)_{\alpha\beta}(T^a)_{\beta'\alpha'}\frac{ig^2}{\mathbf{k}^2}\,. \tag{D.5}$$

[1] Another option is to use the static heavy quark propagator and consider corrections coming from the expansion of the non-static one as kinetic insertions. Both options are equivalent.

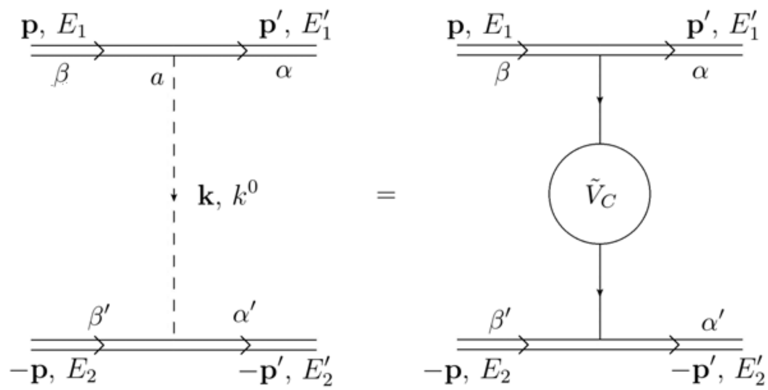


Figure D.1: Tree level $1/m^0$ matching between NRQCD and pNRQCD due to a longitudinal gluon exchange. The vertex comes from the $1/m^0$ heavy-gluon operator of the NRQCD Lagrangian. It gives the leading interaction potential between the two heavy quarks, the Coulomb potential.

It is worth noting that, since in the Coulomb gauge the longitudinal gluon (the time component of the vector potential) has an energy-independent propagator, the interaction associated with the exchange of it corresponds to an instantaneous potential. From Fig.[D.2] we have that

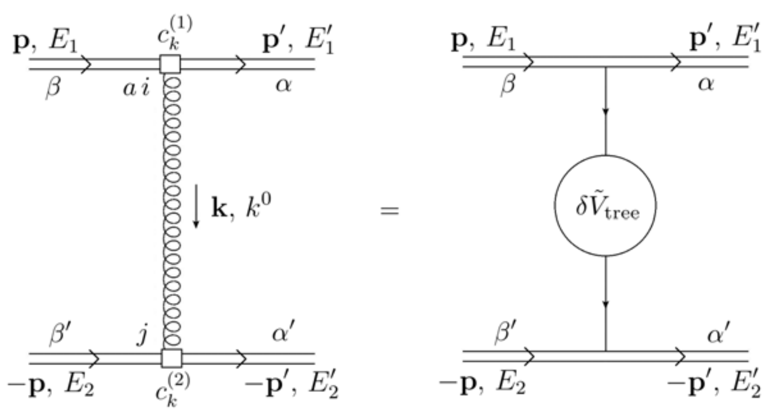


Figure D.2: Tree level $1/m^2$ spin-independent matching between NRQCD and pNRQCD due to a transverse gluon exchange. The vertex comes from the $1/m$ spin-independent heavy-gluon operator of the NRQCD Lagrangian, which is proportional to c_k.

$$c_k^{(1)} \frac{ig}{2m_1} (\mathbf{p} + \mathbf{p}')^i (T^a)_{\alpha\beta} \frac{i}{k^2} \left(\delta^{ij} - \frac{\mathbf{k}^i \mathbf{k}^j}{\mathbf{k}^2 + i\eta_g} \right) c_k^{(2)} \frac{-ig}{2m_2} (-\mathbf{p} - \mathbf{p}')^j (T^a)_{\beta'\alpha'} = -i\delta\tilde{V}_{\rm tree} \,, \quad \text{(D.6)}$$

$$-\, i\delta\tilde{V}_{\rm tree} = -c_k^{(1)} c_k^{(2)} \frac{ig^2}{m_1 m_2} (T^a)_{\alpha\beta} (T^a)_{\beta'\alpha'} \frac{1}{(k^0)^2 - \mathbf{k}^2 + i\eta_g} \mathbf{p}^i \mathbf{p}^j \left(\delta^{ij} - \frac{\mathbf{k}^i \mathbf{k}^j}{\mathbf{k}^2} \right) . \quad \text{(D.7)}$$

Note that, due to the energy dependence of the transverse gluon propagator, the above expression have to be expanded and it contributes to the spectrum at different orders of $m\alpha$. In particular, it contributes to all orders in $m\alpha^{4+2n}$, with $n \in \mathbb{Z}^+$. Expanding up to the order of interest ($m\alpha^6$)

$$-i\delta\tilde{V}_{\text{tree}} = c_k^{(1)}c_k^{(2)}\frac{ig^2}{m_1 m_2}(T^a)_{\alpha\beta}(T^a)_{\beta'\alpha'}\frac{1}{\mathbf{k}^2}\left(1+\frac{(k^0)^2}{\mathbf{k}^2}\right)\mathbf{p}^i\mathbf{p}^j\left(\delta^{ij}-\frac{\mathbf{k}^i\mathbf{k}^j}{\mathbf{k}^2}\right), \qquad \text{(D.8)}$$

where we removed the $i\eta$ prescription because it will not be necessary even when we will introduce this potential inside the pNRQCD loops, since there will not be energy dependence in the denominator, and therefore, no pole in the complex energy plane. At this point, we can start the computation of the one loop diagrams shown in Fig.[D.3]. Schematically we have

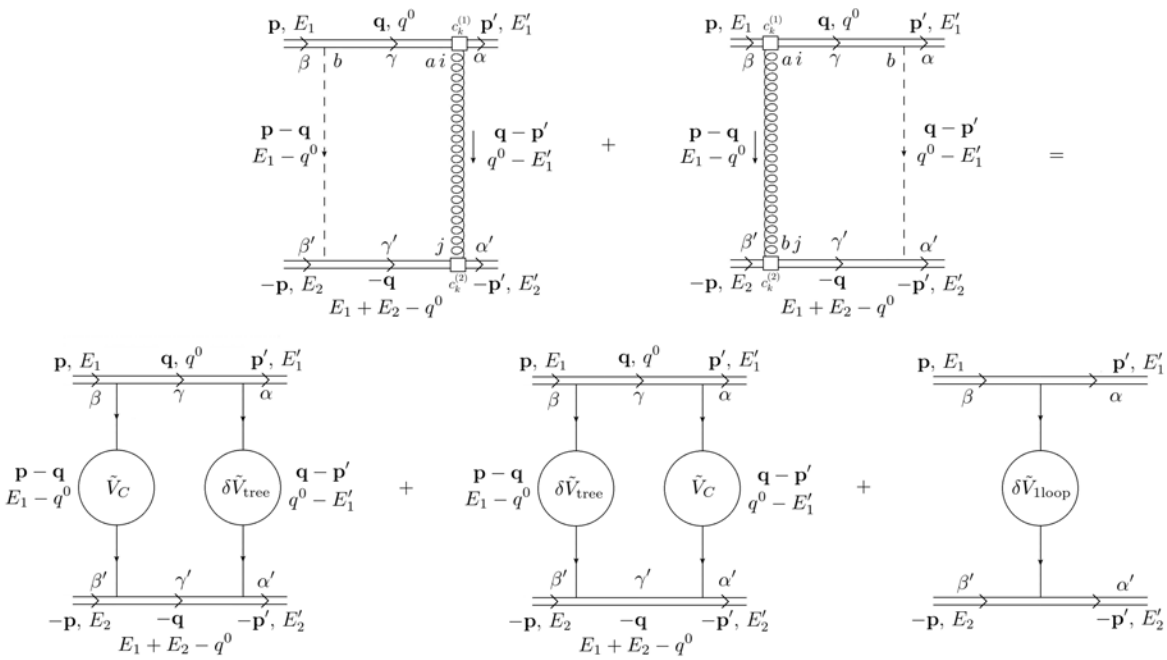


Figure D.3: One-loop contribution to the $\mathcal{O}(\alpha^3/m^2)$ matching between NRQCD and pNRQCD due to a longitudinal and a transverse gluon exchange. Diagrams above are NRQCD ones, whereas diagrams below are pNRQCD ones. The one-loop pNRQCD diagrams regulate the pinch singularities of the NRQCD diagrams. The transverse and longitudinal gluon exchange vertices come from the $1/m$ spin-independent and $1/m^0$ heavy-gluon operators of the NRQCD Lagrangian, respectively.

$$I_1 + I_2 = I_1^p + I_2^p - i\delta\tilde{V}_{\text{1loop}}\,. \qquad \text{(D.9)}$$

Let us begin with I_1

$$
\begin{aligned}
I_1 &= \int \frac{d^D q}{(2\pi)^D} c_k^{(1)} \frac{ig}{2m_1} (\mathbf{p}+\mathbf{q})^i (T^a)_{\gamma\beta} \frac{i}{(p-q)^2 + i\eta_g} \left(\delta^{ij} - \frac{(\mathbf{p}-\mathbf{q})^i (\mathbf{p}-\mathbf{q})^j}{(\mathbf{p}-\mathbf{q})^2} \right) \\
&\times c_k^{(2)} \frac{-ig}{2m_2} (-\mathbf{p}-\mathbf{q})^j (T^a)_{\beta'\gamma'} (-ig) (T^b)_{\alpha\gamma} \frac{i}{(\mathbf{q}-\mathbf{p}')^2} (ig) (T^b)_{\gamma'\alpha'} \frac{i}{q^0 + i\eta_q} \\
&\times \frac{i}{E_1 + E_2 - q^0 + i\eta_q} \, . \qquad \text{(D.10)}
\end{aligned}
$$

Introducing the change of variables $\mathbf{Q} = \mathbf{q} - \mathbf{p}$ the integral can be simplified to

$$
\begin{aligned}
I_1 &= c_k^{(1)} c_k^{(2)} \frac{g^4}{m_1 m_2} (T^b T^a)_{\alpha\beta} (T^a T^b)_{\beta'\alpha'} \mathbf{p}^i \mathbf{p}^j \int \frac{d^d \mathbf{Q}}{(2\pi)^d} \left(\delta^{ij} - \frac{\mathbf{Q}^i \mathbf{Q}^j}{\mathbf{Q}^2} \right) \frac{1}{(\mathbf{Q}+\mathbf{k})^2} \\
&\times \int \frac{dq^0}{2\pi} \frac{1}{(q^0 - E_1)^2 - \mathbf{Q}^2 + i\eta_g} \frac{1}{q^0 + i\eta_q} \frac{1}{q^0 - E_1 - E_2 - i\eta_q} \, . \qquad \text{(D.11)}
\end{aligned}
$$

By Taylor expanding following the hierarchies described at the beginning of the section and up to $\mathcal{O}(m\alpha^6)$, I_1 takes the form

$$
\begin{aligned}
I_1 &= c_k^{(1)} c_k^{(2)} \frac{g^4}{m_1 m_2} (T^b T^a)_{\alpha\beta} (T^a T^b)_{\beta'\alpha'} \mathbf{p}^i \mathbf{p}^j \int \frac{d^d \mathbf{Q}}{(2\pi)^d} \left(\delta^{ij} - \frac{\mathbf{Q}^i \mathbf{Q}^j}{\mathbf{Q}^2} \right) \frac{1}{(\mathbf{Q}+\mathbf{k})^2} \qquad \text{(D.12)} \\
&\times \int \frac{dq^0}{2\pi} \frac{1}{(q^0)^2 - \mathbf{Q}^2 + i\eta_g} \frac{1}{q^0 + i\eta_q} \frac{1}{q^0 - i\eta_q} \left(1 + \frac{2q^0 E_1}{(q^0)^2 - \mathbf{Q}^2 + i\eta_g} + \frac{E_1 + E_2}{q^0 - i\eta_q} \right) .
\end{aligned}
$$

Now let us compute the diagram in the EFT (pNRQCD), I_1^p. The amplitude can be written as

$$
I_1^p = \int \frac{d^D q}{(2\pi)^D} (-i\tilde{V}^{(o)}_{tree}) \frac{i}{q^0 + i\eta_q} \frac{i}{E_1 + E_2 - q^0 + i\eta_q} (-i\tilde{V}_C) \, , \qquad \text{(D.13)}
$$

or explicitly, as

$$
\begin{aligned}
I_1^p &= \int \frac{d^D q}{(2\pi)^D} c_k^{(1)} c_k^{(2)} \frac{ig^2}{m_1 m_2} (T^a)_{\gamma\beta} (T^a)_{\beta'\gamma'} \frac{1}{(\mathbf{q}-\mathbf{p})^2} \left(1 + \frac{(q^0 - E_1)^2}{(\mathbf{q}-\mathbf{p})^2} \right) \mathbf{p}^i \mathbf{p}^j \qquad \text{(D.14)} \\
&\times \left(\delta^{ij} - \frac{(\mathbf{q}-\mathbf{p})^i (\mathbf{q}-\mathbf{p})^j}{(\mathbf{q}-\mathbf{p})^2} \right) \frac{i}{q^0 + i\eta_q} \frac{i}{E_1 + E_2 - q^0 + i\eta_q} (T^b)_{\alpha\gamma} (T^b)_{\gamma'\alpha'} \frac{ig^2}{(\mathbf{q}-\mathbf{p}')^2} \, .
\end{aligned}
$$

Doing the change $\mathbf{Q} = \mathbf{q} - \mathbf{p}$, the integral can be written as

$$
\begin{aligned}
I_1^p &= -c_k^{(1)} c_k^{(2)} \frac{g^4}{m_1 m_2} (T^b T^a)_{\alpha\beta} (T^a T^b)_{\beta'\alpha'} \mathbf{p}^i \mathbf{p}^j \int \frac{d^d \mathbf{Q}}{(2\pi)^d} \left(\delta^{ij} - \frac{\mathbf{Q}^i \mathbf{Q}^j}{\mathbf{Q}^2} \right) \frac{1}{\mathbf{Q}^2} \frac{1}{(\mathbf{Q}+\mathbf{k})^2} \\
&\times \int \frac{dq^0}{2\pi} \left(1 + \frac{(q^0 - E_1)^2}{\mathbf{Q}^2} \right) \frac{1}{q^0 + i\eta_q} \frac{1}{q^0 - E_1 - E_2 - i\eta_q} \, . \qquad \text{(D.15)}
\end{aligned}
$$

Finally, expanding up to the desired order, we obtain

$$
\begin{aligned}
I_1^p &= -c_k^{(1)} c_k^{(2)} \frac{g^4}{m_1 m_2} (T^b T^a)_{\alpha\beta} (T^a T^b)_{\beta'\alpha'} \mathbf{p}^i \mathbf{p}^j \int \frac{d^d\mathbf{Q}}{(2\pi)^d} \left(\delta^{ij} - \frac{\mathbf{Q}^i \mathbf{Q}^j}{\mathbf{Q}^2} \right) \frac{1}{\mathbf{Q}^2} \frac{1}{(\mathbf{Q}+\mathbf{k})^2} \\
&\times \int \frac{dq^0}{2\pi} \frac{1}{q^0 + i\eta_q} \frac{1}{q^0 - i\eta_q} \left(1 + \frac{(q^0)^2 - 2q^0 E_1}{\mathbf{Q}^2} + \frac{E_1 + E_2}{q^0 - i\eta_q} \left(1 + \frac{(q^0)^2}{\mathbf{Q}^2} \right) \right). \quad \text{(D.16)}
\end{aligned}
$$

The quantity $I_1 - I_1^p$ must be pinch singularity free, and the integral over q^0 can be computed. Before the q^0 integral is performed $I_1 - I_1^p$ is given by

$$
\begin{aligned}
I_1 - I_1^p &= c_k^{(1)} c_k^{(2)} \frac{g^4}{m_1 m_2} (T^b T^a)_{\alpha\beta} (T^a T^b)_{\beta'\alpha'} \mathbf{p}^i \mathbf{p}^j \int \frac{d^d\mathbf{Q}}{(2\pi)^d} \left(\delta^{ij} - \frac{\mathbf{Q}^i \mathbf{Q}^j}{\mathbf{Q}^2} \right) \frac{1}{\mathbf{Q}^2} \frac{1}{(\mathbf{Q}+\mathbf{k})^2} \\
&\times \int \frac{dq^0}{2\pi} \frac{1}{q^0 + i\eta} \frac{1}{q^0 - i\eta} \frac{1}{(q^0)^2 - \mathbf{Q}^2 + i\eta} \left((q^0)^2 + \frac{E_1 + E_2}{q^0 - i\eta} \frac{(q^0)^4}{\mathbf{Q}^2} \right), \quad \text{(D.17)}
\end{aligned}
$$

where we have used that antisymmetric functions integrated over symmetric integration limits vanish. Note that we have also removed the scaleless integral ($\int dq^0/2\pi$) that appeared. A priori, such integral is ill-defined, so we have to regulate it by extrapolating the q^0 integral to $d' = 1 + 2\epsilon$ dimensions. The resulting integral is scaleless and vanishes in dimensional regularization. Integrating by residues we find that the term proportional to $E_1 + E_2$ vanishes, so there is no $\mathcal{O}(\alpha^2/m^3)$ contribution to the potential, but only an $\mathcal{O}(\alpha^2/m^2)$ contribution. After integrating over q^0 the following d-momentum integral is left

$$
I_1 - I_1^p = -c_k^{(1)} c_k^{(2)} \frac{ig^4}{2m_1 m_2} (T^b T^a)_{\alpha\beta} (T^a T^b)_{\beta'\alpha'} \mathbf{p}^i \mathbf{p}^j \int \frac{d^d\mathbf{Q}}{(2\pi)^d} \left(\delta^{ij} - \frac{\mathbf{Q}^i \mathbf{Q}^j}{\mathbf{Q}^2} \right) \frac{1}{|\mathbf{Q}|^3} \frac{1}{(\mathbf{Q}+\mathbf{k})^2}. \quad \text{(D.18)}
$$

Analogously, for $I_2 - I_2^p$, we find

$$
I_2 - I_2^p = -c_k^{(1)} c_k^{(2)} \frac{ig^4}{2m_1 m_2} (T^b T^a)_{\alpha\beta} (T^a T^b)_{\beta'\alpha'} \mathbf{p}'^i \mathbf{p}'^j \int \frac{d^d\mathbf{Q}}{(2\pi)^d} \left(\delta^{ij} - \frac{\mathbf{Q}^i \mathbf{Q}^j}{\mathbf{Q}^2} \right) \frac{1}{|\mathbf{Q}|^3} \frac{1}{(\mathbf{Q}+\mathbf{k})^2}. \quad \text{(D.19)}
$$

Therefore, the contribution to the $\mathcal{O}(\alpha^2/m^2)$ potential is given by

$$
\begin{aligned}
\delta\tilde{V}_{1\text{loop}} &= c_k^{(1)} c_k^{(2)} \frac{g^4}{2m_1 m_2} (T^b T^a)_{\alpha\beta} (T^a T^b)_{\beta'\alpha'} (\mathbf{p}^i \mathbf{p}^j + \mathbf{p}'^i \mathbf{p}'^j) \\
&\times \int \frac{d^d\mathbf{Q}}{(2\pi)^d} \left(\delta^{ij} - \frac{\mathbf{Q}^i \mathbf{Q}^j}{\mathbf{Q}^2} \right) \frac{1}{|\mathbf{Q}|^3} \frac{1}{(\mathbf{Q}+\mathbf{k})^2}, \quad \text{(D.20)}
\end{aligned}
$$

which an be expressed in terms of the master integrals of Sec. D.2 as

$$\delta\tilde{V}_{1\text{loop}} = c_k^{(1)} c_k^{(2)} \frac{g^4}{2m_1m_2}(T^bT^a)_{\alpha\beta}(T^aT^b)_{\beta'\alpha'}(\mathbf{p}^i\mathbf{p}^j + \mathbf{p}'^i\mathbf{p}'^j)\left(\delta^{ij}I_1(1,3/2) - I_1^{ij}(1,5/2)\right) . \tag{D.21}$$

In $d = 3+2\epsilon$ dimensions and projecting to the singlet sector using the relations of Sec. D.3, Eq. (D.21) gives

$$\begin{aligned}\delta\tilde{V}_{1\text{loop}} &= c_k^{(1)} c_k^{(2)} \frac{g^4}{2m_1m_2}\frac{1}{8\pi\sqrt{\pi}}\frac{1}{(4\pi)^\epsilon}C_F^2(\mathbf{p}^i\mathbf{p}^j + \mathbf{p}'^i\mathbf{p}'^j)|\mathbf{k}|^{-2+2\epsilon} \\ &\times\Bigg[-\frac{\Gamma(2-\epsilon)\Gamma(1/2+\epsilon)\Gamma(1+\epsilon)}{\Gamma(5/2)\Gamma(3/2+2\epsilon)}\frac{\mathbf{k}^i\mathbf{k}^j}{\mathbf{k}^2} \\ &+\Gamma(1-\epsilon)\Gamma(\epsilon)\left(\frac{\Gamma(1/2+\epsilon)}{\Gamma(3/2)\Gamma(1/2+2\epsilon)} - \frac{\Gamma(5/2+\epsilon)}{(3+2\epsilon)\Gamma(5/2)\Gamma(3/2+2\epsilon)}\right)\delta^{ij}\Bigg] .\end{aligned} \tag{D.22}$$

By Taylor expanding to $\mathcal{O}(\epsilon^0)$ we obtain

$$\begin{aligned}\delta\tilde{V}_{1\text{loop}} &= c_k^{(1)} c_k^{(2)} \frac{g^4}{6m_1m_2}\frac{1}{\pi^2}C_F^2|\mathbf{k}|^{-2+2\epsilon}\left[\frac{1}{2}\left(\frac{1}{\epsilon}+1+\gamma_E-\ln(\pi)\right)(\mathbf{p}^2+\mathbf{p}'^2)\right. \\ &\left.-\frac{(\mathbf{p}\cdot\mathbf{k})^2+(\mathbf{p}'\cdot\mathbf{k})^2}{\mathbf{k}^2}\right] + \mathcal{O}(\epsilon) .\end{aligned} \tag{D.23}$$

As it can be seen, this contribution to the potential is divergent. That divergence has IR origin, since the superficial degree of divergence of Eq. (D.20) is $\mathcal{D} = -2$, and it is not absorbed by any Wilson coefficient of the potential, but it is cancelled by the ultrasoft [66] instead. Afterwards, the resulting contribution to the $\mathcal{O}(\alpha^2/m^2)$ potential is finite.

D.2 NRQCD Master Integrals

The following d-momentum master integrals are needed to carry out the matching between NRQCD and pNRQCD, in particular, to compute the $\mathcal{O}(\alpha^2/m^3)$ potential presented in Sec. 4.4.2:

$$\begin{aligned}I_1(\alpha,\beta) &\equiv \int\frac{d^d\mathbf{Q}}{(2\pi)^d}\frac{1}{(\mathbf{Q}\pm\mathbf{k})^{2\alpha}\mathbf{Q}^{2\beta}} \\ &= \frac{\Gamma(\alpha+\beta-3/2-\epsilon)\Gamma(3/2-\alpha+\epsilon)\Gamma(3/2-\beta+\epsilon)}{(4\pi)^{3/2+\epsilon}\Gamma(\alpha)\Gamma(\beta)\Gamma(3-\alpha-\beta+2\epsilon)}|\mathbf{k}|^{-2\alpha-2\beta+3+2\epsilon} ,\end{aligned} \tag{D.24}$$

$$\begin{aligned}I_1^i(\alpha,\beta;\pm\mathbf{k}) &\equiv \int\frac{d^d\mathbf{Q}}{(2\pi)^d}\frac{\mathbf{Q}^i}{(\mathbf{Q}\pm\mathbf{k})^{2\alpha}\mathbf{Q}^{2\beta}} \\ &= \mp\frac{1}{8\pi\sqrt{\pi}}\frac{1}{(4\pi)^\epsilon}\frac{\Gamma(\alpha+\beta-3/2-\epsilon)\Gamma(3/2-\alpha+\epsilon)\Gamma(5/2-\beta+\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(4-\alpha-\beta+2\epsilon)} \\ &\times|\mathbf{k}|^{-2\alpha-2\beta+3+2\epsilon}\mathbf{k}^i ,\end{aligned} \tag{D.25}$$

$$\begin{aligned} I_1^{ij}(\alpha,\beta) &\equiv \int \frac{d^d\mathbf{Q}}{(2\pi)^d}\frac{\mathbf{Q}^i\mathbf{Q}^j}{(\mathbf{Q}\pm\mathbf{k})^{2\alpha}\mathbf{Q}^{2\beta}} \\ &= \frac{1}{8\pi\sqrt{\pi}}\frac{1}{(4\pi)^\epsilon}\frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(5-\alpha-\beta+2\epsilon)}|\mathbf{k}|^{-2\alpha-2\beta+5+2\epsilon} \\ &\quad\times\Bigg(\frac{\Gamma(5/2+\epsilon)\Gamma(\alpha+\beta-5/2-\epsilon)\Gamma(5/2-\alpha+\epsilon)\Gamma(5/2-\beta+\epsilon)}{(3+2\epsilon)\Gamma(3/2+\epsilon)}\delta^{ij} \\ &\quad+\Gamma(\alpha+\beta-3/2-\epsilon)\Gamma(3/2-\alpha+\epsilon)\Gamma(7/2-\beta+\epsilon)\frac{\mathbf{k}^i\mathbf{k}^j}{\mathbf{k}^2}\Bigg), \end{aligned} \tag{D.26}$$

$$\begin{aligned} I_1^{ijk}(\alpha,\beta;\pm\mathbf{k}) &\equiv \int \frac{d^d\mathbf{Q}}{(2\pi)^d}\frac{\mathbf{Q}^i\mathbf{Q}^j\mathbf{Q}^k}{(\mathbf{Q}\pm\mathbf{k})^{2\alpha}\mathbf{Q}^{2\beta}} \\ &= \mp\frac{1}{8\pi\sqrt{\pi}}\frac{1}{(4\pi)^\epsilon}\frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(6-\alpha-\beta+2\epsilon)}|\mathbf{k}|^{-2\alpha-2\beta+5+2\epsilon} \\ &\quad\times\Bigg(\frac{1}{2}\Gamma(\alpha+\beta-5/2-\epsilon)\Gamma(5/2-\alpha+\epsilon)\Gamma(7/2-\beta+\epsilon) \\ &\quad\times(\mathbf{k}^k\delta^{ij}+\mathbf{k}^j\delta^{ik}+\mathbf{k}^i\delta^{jk})+\Gamma(\alpha+\beta-3/2-\epsilon)\Gamma(3/2-\alpha+\epsilon) \\ &\quad\times\Gamma(9/2-\beta+\epsilon)\frac{\mathbf{k}^i\mathbf{k}^j\mathbf{k}^k}{\mathbf{k}^2}\Bigg), \end{aligned} \tag{D.27}$$

$$\begin{aligned} I_1^{ijkl}(\alpha,\beta) &\equiv \int \frac{d^d\mathbf{Q}}{(2\pi)^d}\frac{\mathbf{Q}^i\mathbf{Q}^j\mathbf{Q}^k\mathbf{Q}^l}{(\mathbf{Q}\pm\mathbf{k})^{2\alpha}\mathbf{Q}^{2\beta}} \\ &= \frac{1}{8\pi\sqrt{\pi}}\frac{1}{(4\pi)^\epsilon}\frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(7-\alpha-\beta+2\epsilon)}|\mathbf{k}|^{-2\alpha-2\beta+7+2\epsilon} \\ &\quad\times\Bigg(\frac{1}{4}\Gamma(\alpha+\beta-7/2-\epsilon)\Gamma(7/2-\alpha+\epsilon)\Gamma(7/2-\beta+\epsilon) \\ &\quad\times(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}) \\ &\quad+\frac{1}{2}\Gamma(\alpha+\beta-5/2-\epsilon)\Gamma(5/2-\alpha+\epsilon)\Gamma(9/2-\beta+\epsilon)\frac{1}{\mathbf{k}^2}(\delta^{ij}\mathbf{k}^k\mathbf{k}^l+\delta^{ik}\mathbf{k}^j\mathbf{k}^l \\ &\quad+\delta^{il}\mathbf{k}^j\mathbf{k}^k+\delta^{jk}\mathbf{k}^i\mathbf{k}^l+\delta^{jl}\mathbf{k}^i\mathbf{k}^k+\delta^{kl}\mathbf{k}^i\mathbf{k}^j) \\ &\quad+\Gamma(\alpha+\beta-3/2-\epsilon)\Gamma(3/2-\alpha+\epsilon)\Gamma(11/2-\beta+\epsilon)\frac{\mathbf{k}^i\mathbf{k}^j\mathbf{k}^k\mathbf{k}^l}{\mathbf{k}^4}\Bigg). \end{aligned} \tag{D.28}$$

Also the following q^0 integrals are very useful

$$I_2 = \int\frac{dq^0}{2\pi}\frac{1}{q^0+i\eta} = -\frac{i}{2}\,, \tag{D.29}$$

$$I_3 = \int\frac{dq^0}{2\pi}\frac{1}{(q^0)^2-\mathbf{Q}^2+i\eta_g}\frac{1}{q^0+i\eta_q} = \frac{i}{2}\frac{1}{\mathbf{Q}^2}\,, \tag{D.30}$$

$$I_4 = \int\frac{dq^0}{2\pi}\frac{1}{(q^0)^2-\mathbf{Q}^2+i\eta_g}\frac{1}{(q^0)^2-(\mathbf{Q}-\mathbf{k})^2+i\eta_g}\frac{1}{q^0+i\eta_q} = -\frac{i}{2}\frac{1}{(\mathbf{Q}-\mathbf{k})^2\mathbf{Q}^2}\,. \tag{D.31}$$

D.3 Color structure

Since we are only interested in observable heavy quarkonium, i.e. in heavy quarkonium in a color singlet state, we are only interested in the color singlet potential, which describes the interaction between the two heavy quarks inside the bound state without changing the color of the bounded system[2]. Therefore, we need to project the potential to the singlet sector $|s\rangle = \frac{1}{\sqrt{N_c}}\delta_{\alpha\alpha'}$ ($\langle s| = \frac{1}{\sqrt{N_c}}\delta_{\beta\beta'}$). The following projections are necessary to that purpose

$$\langle s|\delta_{\alpha\beta}\delta_{\beta'\alpha'}|s\rangle = 1\,, \tag{D.32}$$

$$\langle s|(T^a)_{\alpha\beta}(T^a)_{\beta'\alpha'}|s\rangle = C_F\,, \tag{D.33}$$

$$\langle s|(T^aT^b)_{\alpha\beta}(T^aT^b)_{\beta'\alpha'}|s\rangle = C_F\left(C_F - \frac{C_A}{2}\right)\,, \tag{D.34}$$

$$\langle s|(T^bT^a)_{\alpha\beta}(T^aT^b)_{\beta'\alpha'}|s\rangle = C_F^2\,, \tag{D.35}$$

$$\langle s|f^{abc}(T^a)_{\alpha\beta}(T^bT^c)_{\beta'\alpha'}|s\rangle = \frac{i}{2}C_FC_A\,, \tag{D.36}$$

$$\langle s|f^{abc}\{T^b, T^c\}_{\alpha\beta}(T^a)_{\beta'\alpha'}|s\rangle = 0\,. \tag{D.37}$$

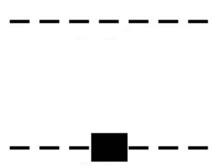
[2]Contrarily, heavy quarkonium in an octet state brings color charge, and therefore, it is not observable as an asymptotically free state. The interaction potential between two heavy quarks in a color octet state is the interaction potential projected to the octet state. Heavy quarkonium in a singlet state can also interact with ultrasoft gluons and with octet quarkonium fields as intermediate steps. However, this kind of interactions are accounted in the ultrasoft.

Appendix E

pNRQCD RG

E.1 Feynman Rules of pNRQCD

In Fig.[E.1], we display the pNRQCD Feynman rules needed for the calculation of the potential running of $\tilde{D}_d^{(2)}$. A more complete set of Feynman rules can be found in Refs. [2, 8].



$$= -iG_c^{(0)} = \frac{i}{E - \frac{\hat{\mathbf{p}}^2}{2m_r} + i\eta}$$

$$= -i\delta\hat{V}$$

Figure E.1: pNRQCD Feynman rules needed for the calculation of the potential running of $\tilde{D}_d^{(2)}$. Above the $Q\bar{Q}$ singlet propagator and below the iteration of a perturbative interaction potential. Hats over quantities mean we are dealing with operators.

E.2 NLL Potential Running of $\tilde{D}_d^{(2)}$: An Example

In this section, we give an example of the computation of a potential loop contributing to the NLL potential running of $\tilde{D}_d^{(2)}$, as well as, we determine its contribution to the potential RGE. In general, there are one-loop and two-loop contributions at $\mathcal{O}(\alpha^3)$. The master integrals relevant for the complete calculation are summarized in Sec. E.3.

The potential running is the contribution to the full running (soft, potential and ultrasoft) coming from potential loops, which is equivalent to non-relativistic quantum mechanics perturbation theory. Potential loops correspond to the region of NRQCD green functions where the loop energy is of the order of the ultrasoft scale. More precisely, when E, p^0, p'^0, $q^0 \sim mv^2$ and $|\mathbf{p}|$, $|\mathbf{p}|'$, $|\mathbf{q}| \sim mv$. The anomalous dimension is determined from the UV divergences arising in diagrams of heavy quark-antiquark pairs interacting through different order in α and m interaction potentials, that are treated as perturbations. Therefore, we are only interested in the UV pole, i.e. the $1/\epsilon$, of these diagrams. Unlike in Ch.

4, we will keep hats in the present section to symbolize we are dealing with operators in order to make it more clear for an unexperienced reader.

Let us go into the computation. Firstly, let us explicitly write the Coulomb and delta-like potentials in momentum space, as they are the ones we will use in this example. They are given by

$$\tilde{V}_C = -C_F \frac{g_{V_s}^2}{\mathbf{k}^2}\,, \tag{E.1}$$

$$\tilde{V}_d = C_F \frac{\pi \tilde{D}_d^{(2)}}{m_1 m_2}\,, \tag{E.2}$$

respectively. We also need to fix the normalization, which is given by the diagram shown in Fig.[E.2]. The amplitude reads

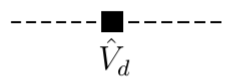


Figure E.2: Potential loop of a single $1/m^2$ potential. It fixes de normalization.

$$-\,i\mathcal{A}_0 = \langle \mathbf{p}'|(-i)\hat{V}_d|\mathbf{p}\rangle = -i\tilde{V}_d = -iC_F \frac{\pi \tilde{D}_d^{(2)}}{m_1 m_2}\,, \tag{E.3}$$

where we used that the object $\langle \mathbf{p}'|\hat{V}|\mathbf{p}\rangle = \tilde{V}$ is the Fourier transform of the d-dimensional potential in position space i.e. it is the d-dimensional potential in momentum space. As an example, we consider a two-loop calculation which consists in the iteration of the potentials shown in Fig.[E.3]. Let us compute first the potential loops with the Coulomb potential applied to external momenta. On the one hand, we have that $-i\mathcal{A}_3 = -i\mathcal{A}_1(\mathbf{p} \leftrightarrow \mathbf{p}')$, so we only need to compute one of them. On the other hand, $-i\mathcal{A}_1$ is given by

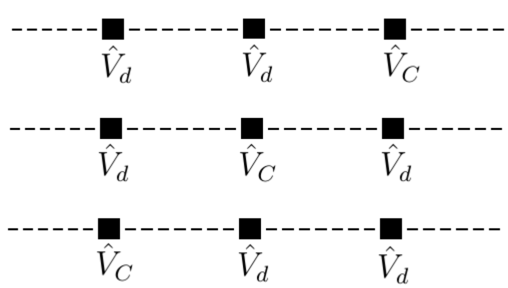


Figure E.3: Potential loop that contributes to the $\mathcal{O}(\alpha^3)$ potential anomalous dimension of $\tilde{D}_d^{(2)}$ consisting in the iteration of two $1/m^2$ potentials and one Coulomb potential.

$$
\begin{aligned}
-i\mathcal{A}_1 &= \langle \mathbf{p}'|(-i)\hat{V}_d \frac{i}{E-\frac{\hat{\mathbf{p}}^2}{2m_r}+i\eta}(-i)\hat{V}_d \frac{i}{E-\frac{\hat{\mathbf{p}}^2}{2m_r}+i\eta}(-i)\hat{V}_C|\mathbf{p}\rangle \\
&= -4im_r^2\langle \mathbf{p}'|\hat{V}_d \frac{1}{\hat{\mathbf{p}}^2-2m_r(E+i\eta)}\hat{V}_d \frac{1}{\hat{\mathbf{p}}^2-2m_r(E+i\eta)}\hat{V}_C|\mathbf{p}\rangle\,. \qquad \text{(E.4)}
\end{aligned}
$$

Note that every propagator adds an extra power of m_r. Introducing identities in terms of momentum eigenstates we can write the amplitude as the following two-loop integral

$$
\begin{aligned}
-i\mathcal{A}_1 &= -4im_r^2\int\frac{d^d\mathbf{q}}{(2\pi)^d}\int\frac{d^d\mathbf{q}'}{(2\pi)^d}\langle \mathbf{p}'|\hat{V}_d|\mathbf{q}\rangle\frac{1}{\mathbf{q}^2-2m_r(E+i\eta)}\langle \mathbf{q}|\hat{V}_d|\mathbf{q}'\rangle \\
&\quad\times\frac{1}{\mathbf{q}'^2-2m_r(E+i\eta)}\langle \mathbf{q}'|\hat{V}_C|\mathbf{p}\rangle\,. \qquad \text{(E.5)}
\end{aligned}
$$

Using that $\tilde{V}=\langle \mathbf{p}'|\hat{V}|\mathbf{p}\rangle$, we find

$$
\begin{aligned}
-i\mathcal{A}_1 &= 4i\pi^2C_F^3g_{V_s}^2\tilde{D}_d^{(2)\,2}\frac{m_r^2}{m_1^2m_2^2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\int\frac{d^d\mathbf{q}'}{(2\pi)^d}\frac{1}{\mathbf{q}^2-2m_r(E+i\eta)} \\
&\quad\times\frac{1}{\mathbf{q}'^2-2m_r(E+i\eta)}\frac{1}{(\mathbf{q}'-\mathbf{p})^2}\,. \qquad \text{(E.6)}
\end{aligned}
$$

Always that we arrive at this stage we make a delicate step. We set all the external momenta $\mathbf{p}$ and $\mathbf{p}'$ equal to zero. This step can only be justified if all the integrals are UV, and then killing the external momentum does no ruin the regulation of any IR divergence, and also if the two integrals are together logarithmically divergent. This is so because the appearance of an external momentum lowers the superficial degree of divergence, making it more and more IR. Therefore, if the integral is logarithmically divergent, the appearance of an external momentum would make it IR instead of UV. In that case, we can set all the external momenta to zero and all the divergences can be absorbed in the delta-like potential. Contrarily, if the superficial degree of divergence is such that the integrals are UV and power-like, then this step would not be justified, as the appearance of a factorized external momentum is necessary to make the integral logarithmically divergent. In that case, the integrals would contribute to the potential running of all other spin-independent $1/m^2$ potentials. Fortunately, by explicit calculation, we see that this is not the case and that all the integrals satisfy both previous conditions. Therefore only the delta-like potential gets potential running. Note that the regulator of the integrals is the term $-2m_r(E+i\eta)$ coming from the singlet propagator. Thus

$$
\begin{aligned}
-i\mathcal{A}_1 &= 4i\pi^2C_F^3g_{V_s}^2\tilde{D}_d^{(2)\,2}\frac{m_r^2}{m_1^2m_2^2}\int\frac{d^d\mathbf{q}'}{(2\pi)^d}\frac{1}{\mathbf{q}'^2(\mathbf{q}'^2-2m_r(E+i\eta))}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{\mathbf{q}^2-2m_r(E+i\eta)} \\
&= 4i\pi^2C_F^3g_{V_s}^2\tilde{D}_d^{(2)\,2}\frac{m_r^2}{m_1^2m_2^2}I_{\rm PL}(1,1)I_{\rm PL}(0,1)\,, \qquad \text{(E.7)}
\end{aligned}
$$

where $I_{\rm PL}(1,1)$ and $I_{\rm PL}(0,1)$ are master integrals defined in Sec. E.3.1. These two integrals are not divergent. Therefore, $-i\mathcal{A}_1$ does not contribute to the running. For the same reason $-i\mathcal{A}_3$ neither contributes. The other possible contribution is the one with the Coulomb insertion between the two delta-like potentials. For this case, the amplitude reads

$$-i\mathcal{A}_2 = \langle \mathbf{p}'|(-i)\hat{V}_d \frac{i}{E-\frac{\hat{\mathbf{p}}^2}{2m_r}+i\eta}(-i)\hat{V}_C \frac{i}{E-\frac{\hat{\mathbf{p}}^2}{2m_r}+i\eta}(-i)\hat{V}_d|\mathbf{p}\rangle \,. \tag{E.8}$$

By introducing identities in terms of momentum eigenstates and consequently writing the amplitude in terms of momentum space potentials, we find the following expression for the amplitude

$$\begin{aligned} -i\mathcal{A}_2 &= 4i\pi^2 C_F^3 g_{V_s}^2 \tilde{D}_d^{(2)\,2}\frac{m_r^2}{m_1^2m_2^2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{\mathbf{q}^2-2m_r(E+i\eta)} \\ &\quad \times \int \frac{d^d\mathbf{q}'}{(2\pi)^d}\frac{1}{(\mathbf{q}'-\mathbf{q})^2(\mathbf{q}'^2-2m_r(E+i\eta))} \\ &= 4i\pi^2 C_F^3 g_{V_s}^2 \tilde{D}_d^{(2)\,2}\frac{m_r^2}{m_1^2m_2^2} I_{\rm PL}(0,1,1,0,1), \end{aligned} \tag{E.9}$$

where the two-loop integral $I_{\rm PL}(0,1,1,0,1)$ is defined and evaluated in Sec. E.3.2. The amplitude is finally given by

$$-i\mathcal{A}_2 = -\frac{i}{16} C_F^3 g_{V_s}^2 \tilde{D}_d^{(2)\,2}\frac{m_r^2}{m_1^2m_2^2}\frac{1}{\epsilon}(-2m_r(E+i\eta))^{2\epsilon} + \mathcal{O}(\epsilon)\,, \tag{E.10}$$

which is divergent. This divergence gets absorbed in the spin-independent delta-like potential, more particularly into its Wilson coefficient $\tilde{D}_d^{(2)}$. The counterterm that cancels the UV divergence above is given by

$$\delta\tilde{D}_d^{(2)} = -\frac{1}{16\pi} C_F^2 g_{V_s}^2 \tilde{D}_d^{(2)\,2}\frac{m_r^2}{m_1m_2}\frac{1}{\epsilon}\nu^{4\epsilon}\,. \tag{E.11}$$

Its contribution to the potential RGE of $\tilde{D}_d^{(2)}$ can be obtained from the relation

$$\nu\frac{d}{d\nu}\tilde{D}_{d,R}^{(2)} = -\nu\frac{d}{d\nu}\delta\tilde{D}_d^{(2)}\,, \tag{E.12}$$

since $\tilde{D}_{d,B}^{(2)} = \tilde{D}_{d,R}^{(2)} + \delta\tilde{D}_d^{(2)}$. Therefore

$$\nu\frac{d}{d\nu}\tilde{D}_d^{(2)} = C_F^2\alpha_{V_s}\tilde{D}_d^{(2)\,2}\frac{m_r^2}{m_1m_2}\,. \tag{E.13}$$

Let us stress that Eqs. (E.11,E.13) are nor the complete counterterm neither the complete RGE despite of the notation of Ch. 4, and they must be understood in this section as the contributions coming from the potential loops we have computed.

E.3 Potential Loop Master Integrals

In this section, we summarize the integrals that appear in the potential loops we compute to determine the potential running of $\tilde{D}_d^{(2)}$ in Sec. 4.5.3. All the integrals can be computed using the standard technique of Feynman parametrization and the master integrals given in Sec. A.5. In general, we are only interested in the divergent part of the integrals, since we only want to compute the potential anomalous dimension of $\tilde{D}_d^{(2)}$.

E.3.1 One-loop Master Integrals

The general structure of the one-loop integrals that appear is the following

$$\begin{aligned} I_{\rm PL}(\alpha,\beta) &\equiv \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{(\mathbf{q}^2)^\alpha(\mathbf{q}^2-2m_r(E+i\eta))^\beta} \\ &= \frac{1}{(4\pi)^{3/2+\epsilon}} \frac{\Gamma(\alpha+\beta-3/2-\epsilon)\Gamma(3/2-\alpha+\epsilon)}{\Gamma(\beta)\Gamma(3/2+\epsilon)} (-2m_r(E+i\eta))^{-\alpha-\beta+3/2+\epsilon}. \end{aligned} \tag{E.14}$$

These particular cases in which the integral above is divergent appear in the calculation

$$I_{\rm PL}(1/2-\epsilon,1) = -\frac{1}{8\pi^2\epsilon}(-2m_r(E+i\eta))^{2\epsilon} + \mathcal{O}(\epsilon)\,, \tag{E.15}$$

$$I_{\rm PL}(-1/2-\epsilon,2) = -\frac{1}{8\pi^2\epsilon}(-2m_r(E+i\eta))^{2\epsilon} + \mathcal{O}(\epsilon)\,. \tag{E.16}$$

Also the particular cases $I_{\rm PL}(1,1)$, $I_{\rm PL}(0,1)$ and $I_{\rm PL}(-1,2)$ appear. All of them are finite and do not contribute to the anomalous dimension.

E.3.2 Two-loop Master Integrals

The general structure of the two-loop integrals that appear is the following

$$\begin{aligned} I_{\rm PL}(\alpha,\beta,\lambda,\sigma,\rho) &\equiv \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{1}{(\mathbf{q}^2)^\alpha(\mathbf{q}^2-2m_r(E+i\eta))^\beta} \\ &\quad \times \int \frac{d^d\mathbf{q}'}{(2\pi)^d} \frac{1}{[(\mathbf{q}'-\mathbf{q})^2]^\lambda(\mathbf{q}'^2)^\sigma(\mathbf{q}'^2-2m_r(E+i\eta))^\rho} \\ &= \frac{1}{(4\pi)^{3+2\epsilon}} \frac{\Gamma(\alpha+\beta+\lambda+\sigma+\rho-3-2\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\lambda)\Gamma(\sigma)\Gamma(\rho)} \\ &\quad \times(-2m_r(E+i\eta))^{-\alpha-\beta-\lambda-\sigma-\rho+3+2\epsilon} \int_0^1 dx\, x^{-\rho+1/2+\epsilon}(1-x)^{\rho-1} \\ &\quad \times \int_0^1 dy\, y^{-\sigma-\rho+1/2+\epsilon}(1-y)^{\sigma-1}(1-xy)^{-\lambda-\sigma-\rho+3/2+\epsilon} \\ &\quad \times \int_0^1 dz\, z^{\alpha+\beta-1}(1-z)^{\lambda+\sigma+\rho-5/2-\epsilon} \int_0^1 dw\, w^{\alpha-1}(1-w)^{\beta-1} \\ &\quad \times \left(z(1-w) + \frac{(1-x)(1-z)}{xy(1-xy)} \right)^{-\alpha-\beta-\lambda-\sigma-\rho+3+2\epsilon}. \end{aligned} \tag{E.17}$$

The expression above is rather general but not very powerful, since the integrals over Feynman parameters can not be computed analytically in the general case. It can not be computed analytically even in some particular cases. Instead, one can use this expression and try different techniques to extract the $1/\epsilon$, like expanding the term

$$\left(z(1-w)+\frac{(1-x)(1-z)}{xy(1-xy)}\right)^{2\epsilon}, \tag{E.18}$$

which appears when the integral is logarithmically divergent, to $\mathcal{O}(\epsilon^0)$. Note that, in those cases where some argument of the integral is zero or a negative integer, the integral must be regulated. That can be done by shifting these arguments by a small amount δ and then taking the limit $\delta \to 0$ after doing the integral and before expanding in ϵ. The following particular cases in which the integral above is divergent appear in the calculation

$$I_{\rm PL}(0,1,1,0,1) = -\frac{1}{64\pi^2\epsilon}(-2m_r(E+i\eta))^{2\epsilon} + \mathcal{O}(\epsilon)\,, \tag{E.19}$$

$$I_{\rm PL}(-1,2,1,-1,2) = -\frac{1}{64\pi^2\epsilon}(-2m_r(E+i\eta))^{2\epsilon} + \mathcal{O}(\epsilon)\,, \tag{E.20}$$

$$I_{\rm PL}(-1,2,1,0,1) = -\frac{1}{64\pi^2\epsilon}(-2m_r(E+i\eta))^{2\epsilon} + \mathcal{O}(\epsilon)\,. \tag{E.21}$$

Also the particular cases

$$\begin{aligned}
&I_{\rm PL}(-1,1,1,1,1)\,,\ I_{\rm PL}(-2,1,2,1,1)\,,\ I_{\rm PL}(0,1,2,-1,1)\,,\ I_{\rm PL}(-1,1,2,0,1)\,,\\
&I_{\rm PL}(1,1,1,-1,1)\,,\ I_{\rm PL}(-3,1,3,1,1)\,,\ I_{\rm PL}(1,1,3,-3,1)\,,\ I_{\rm PL}(-1,1,3,-1,1)\,,\\
&I_{\rm PL}(0,1,3,-2,1)\,,\ I_{\rm PL}(-2,1,3,0,1)\,,\ I_{\rm PL}(1,1,-1,1,1)\,,\ I_{\rm PL}(-3,3,1,1,1)\,,\\
&I_{\rm PL}(-2,2,1,1,1)\,,\ I_{\rm PL}(-3,2,2,1,1)\,,\ I_{\rm PL}(-1,2,2,-1,1)\,,\ I_{\rm PL}(-2,2,2,0,1)\,,
\end{aligned} \tag{E.22}$$

appear. All of them are finite and do not contribute to the anomalous dimension.

E.4 Necessary Wilson coefficients

Here we present some Wilson coefficients needed for the computation of the NLL running of the Wilson coefficient associated to the spin-independent delta-like potential.

E.4.1 For the plot of the LL running of $\tilde{D}_d^{(2)}$ with the two-loop running coupling

In order to visualize the relative importance of the NLL corrections compared with the LL term, we plotted the later in the Coulomb gauge in Sec. 4.3. To this purpose we needed the LL expressions of the the following NRQCD Wilson coefficients which can be found in Refs. [3, 14]

$$
\begin{aligned}
c_D(\nu)\Big|^{\rm LL} &= c_D(\nu_h)\Big|^{\rm LO} - 1 + \frac{9C_A}{9C_A+8T_Fn_f}\Bigg\{ -\frac{5C_A+4T_Fn_f}{4C_A+4T_Fn_f}z^{-2C_A} \\
&+\frac{C_A+16C_F-8T_Fn_f}{2(C_A-2T_Fn_f)} \\
&+\frac{-7C_A^2+32C_AC_F-4C_AT_Fn_f+32C_FT_Fn_f}{4(C_A+T_Fn_f)(2T_Fn_f-C_A)}z^{\frac{4}{3}T_Fn_f-\frac{2}{3}C_A} \\
&+\frac{8T_Fn_f}{9C_A}\left[z^{-2C_A}+\left(\frac{20}{13}+\frac{32C_F}{13C_A}\right)\left(1-z^{-\frac{13}{6}C_A}\right)\right]\Bigg\}\,,
\end{aligned} \tag{E.23}
$$

$$
d_{ss}(\nu)\Big|^{\rm LL} = d_{ss}(\nu_h)\Big|^{\rm LO} + 4C_F\left(C_F-\frac{C_A}{2}\right)\frac{\pi}{\beta_0}\alpha(\nu_h)(z^{\beta_0}-1)\,, \tag{E.24}
$$

$$
\begin{aligned}
d_{vs}(\nu)\Big|^{\rm LL} &= d_{vs}(\nu_h)\Big|^{\rm LO} - \alpha(\nu_h)\left[4C_F-\frac{3C_A}{2}-\frac{5}{4}C_A\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}\right)\right]\frac{2\pi}{\beta_0}(z^{\beta_0}-1) \\
&-\alpha(\nu_h)\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}\right)\frac{27C_A^2}{9C_A+8T_Fn_f}\frac{\pi}{2\beta_0}\Bigg\{-\frac{5C_A+4T_Fn_f}{4C_A+4T_Fn_f}\frac{\beta_0}{\beta_0-2C_A} \\
&\times(z^{\beta_0-2C_A}-1)+\frac{C_A+16C_F-8T_Fn_f}{2(C_A-2T_Fn_f)}(z^{\beta_0}-1) \\
&+\frac{-7C_A^2+32C_AC_F-4C_AT_Fn_f+32C_FT_Fn_f}{4(C_A+T_Fn_f)(2T_Fn_f-C_A)}\frac{3\beta_0}{3\beta_0+4T_Fn_f-2C_A} \\
&\times(z^{\beta_0+4T_Fn_f/3-2C_A/3}-1)+\frac{8T_Fn_f}{9C_A}\Bigg[\frac{\beta_0}{\beta_0-2C_A}(z^{\beta_0-2C_A}-1) \\
&+\left(\frac{20}{13}+\frac{32}{13}\frac{C_F}{C_A}\right)\left((z^{\beta_0}-1)-\frac{6\beta_0}{6\beta_0-13C_A}(z^{\beta_0-13C_A/6}-1)\right)\Bigg]\Bigg\},
\end{aligned} \tag{E.25}
$$

where the initial matching conditions are

$$
d_{ss}(\nu_h)+C_Fd_{vs}(\nu_h)\Big|^{\rm LO} = 0\,, \tag{E.26}
$$

$$
c_D(\nu_h)\Big|^{\rm LO} = 1\,. \tag{E.27}
$$

Note that, for the unequal mass case, $d_{ss}(\nu_h)\Big|^{\rm LO}=0$ and $d_{vs}(\nu_h)\Big|^{\rm LO}=0$ separately, at $\mathcal{O}(\alpha)$. This is not true for the equal mass case, since there is a non-vanishing contribution to the initial matching conditions at $\mathcal{O}(\alpha)$ coming from QCD annihilation diagrams. However, Eq. (E.26) is satisfied in both cases and it is enough for the evaluation of $\tilde{D}_d^{(2)\,\rm LL}$.

E.4.2 For the numerical evaluation of $\tilde{D}_d^{(2)}$ at NLO

In this section, we summarize the needed expressions of the NRQCD Wilson coefficients to compute the numerical value of the Wilson coefficient $\tilde{D}_d^{(2)}$ at NLO. They can be found in Refs. [15, 28]. The label NLO refers to the purely NLO contribution i.e. without the inclusion of the LO contribution.

Different masses: B_c

$$d_{ss}(\nu_h)\Big|^{\rm NLO} = -C_F\left(\frac{C_A}{2}-C_F\right)\frac{\alpha(\nu_h)^2}{m_1^2-m_2^2}\left[m_1^2\left(\ln\left(\frac{m_2^2}{\nu_h^2}\right)+\frac{1}{3}\right)-m_2^2\left(\ln\left(\frac{m_1^2}{\nu_h^2}\right)+\frac{1}{3}\right)\right], \tag{E.28}$$

$$\begin{aligned} d_{vs}(\nu_h)\Big|^{\rm NLO} = & -2C_F\frac{\alpha(\nu_h)^2}{m_1^2-m_2^2}\left[m_1^2\left(\ln\left(\frac{m_2^2}{\nu_h^2}\right)+\frac{1}{3}\right)-m_2^2\left(\ln\left(\frac{m_1^2}{\nu_h^2}\right)+\frac{1}{3}\right)\right] \\ & +\frac{C_A}{4}\frac{\alpha(\nu_h)^2}{m_1^2-m_2^2}\left[3\left(m_1^2\left(\ln\left(\frac{m_2^2}{\nu_h^2}\right)+\frac{1}{3}\right)-m_2^2\left(\ln\left(\frac{m_1^2}{\nu_h^2}\right)+\frac{1}{3}\right)\right)\right. \\ & \left.+\frac{1}{m_1m_2}\left(m_1^4\left(\ln\left(\frac{m_2^2}{\nu_h^2}\right)+\frac{10}{3}\right)-m_2^4\left(\ln\left(\frac{m_1^2}{\nu_h^2}\right)+\frac{10}{3}\right)\right)\right], \end{aligned} \tag{E.29}$$

$$c_D^{(1)}(\nu_h)\Big|^{\rm NLO} = \frac{\alpha(\nu_h)}{2\pi}C_A-\frac{4\alpha(\nu_h)}{15\pi}\left(1+\frac{m_1^2}{m_2^2}\right)T_F+\frac{\alpha(\nu_h)}{\pi}\left(\frac{8}{3}C_F+\frac{2}{3}C_A\right)\ln\left(\frac{m_1}{\nu_h}\right), \tag{E.30}$$

$$c_D^{(2)}(\nu_h)\Big|^{\rm NLO} = \frac{\alpha(\nu_h)}{2\pi}C_A-\frac{4\alpha(\nu_h)}{15\pi}\left(1+\frac{m_2^2}{m_1^2}\right)T_F+\frac{\alpha(\nu_h)}{\pi}\left(\frac{8}{3}C_F+\frac{2}{3}C_A\right)\ln\left(\frac{m_2}{\nu_h}\right). \tag{E.31}$$

Equal masses: bottomonium/charmonium

$$d_{ss}(\nu_h)\Big|^{\rm NLO} = -\frac{d_{ss}^a}{2N_c}-\frac{3d_{sv}^a}{2N_c}-\frac{N_c^2-1}{4N_c^2}d_{vs}^a-3\frac{N_c^2-1}{4N_c^2}d_{vv}^a+\frac{2}{3}C_F\left(\frac{C_A}{2}-C_F\right)\alpha(\nu_h)^2 \tag{E.32}$$

$$d_{vs}(\nu_h)\Big|^{\rm NLO} = -d_{ss}^a-3d_{sv}^a+\frac{d_{vs}^a}{2N_c}+\frac{3d_{vv}^a}{2N_c}+\left(\frac{4}{3}C_F+\frac{11}{12}C_A\right)\alpha(\nu_h)^2 \tag{E.33}$$

$$d_{ss}^a\Big|^{\rm NLO} = \alpha(\nu_h)^2C_F\left(\frac{C_A}{2}-C_F\right)(2-2\ln 2+i\pi) \tag{E.34}$$

$$d_{sv}^a\Big|^{\rm NLO} = 0 \tag{E.35}$$

$$d_{vs}^a\Big|^{\rm NLO} = \frac{\alpha(\nu_h)^2}{2}\left(-\frac{3}{2}C_A+4C_F\right)(2-2\ln 2+i\pi) \tag{E.36}$$

$$d^a_{vv}\Big|^{\rm NLO} = -\alpha(\nu_h)^2\left(T_F\left(\frac{1}{3}n_f\left(2\ln 2-\frac{5}{3}-i\pi\right)-\frac{8}{9}\right)+C_A\frac{109}{36}-4C_F\right) \tag{E.37}$$

$$c_D^{(i)}(\nu_h)\Big|^{\rm NLO} = \frac{\alpha(\nu_h)}{2\pi}C_A - 16 d_2(\nu_h)\Big|^{\rm LO} \tag{E.38}$$

$$d_2(\nu_h)\Big|^{\rm LO} = \frac{\alpha(\nu_h)}{60\pi}T_F \tag{E.39}$$

Let's recall that $d^a_{vv}\Big|^{\rm LO} = -\pi\alpha(\nu_h)$, whereas $d^a_{ss}\Big|^{\rm LO} = d^a_{sv}\Big|^{\rm LO} = d^a_{vs}\Big|^{\rm LO} = 0$.

E.5 Details of the soft RGE

The soft RGE of $\tilde{D}^{(2)}_d$ was displayed in Sec. 4.5.1. It reads

$$\begin{aligned}
&\nu_s\frac{d}{d\nu_s}(d_{ss}+C_F\bar{d}_{vs})\Big|_{\rm soft} = C_F\alpha^2\left(2C_F-\frac{C_A}{2}\right)c_k^{(1)}c_k^{(2)} \\
&+C_F\alpha^2\left[\frac{m_1}{m_2}\left(\frac{1}{3}T_f n_f \bar{c}_1^{hl(2)}-\frac{4}{3}(C_A+C_F)c_k^{(2)\,2}-\frac{5}{12}C_A c_F^{(2)\,2}\right)\right. \\
&\qquad\left.+\frac{m_2}{m_1}\left(\frac{1}{3}T_f n_f \bar{c}_1^{hl(1)}-\frac{4}{3}(C_A+C_F)c_k^{(1)\,2}-\frac{5}{12}C_A c_F^{(1)\,2}\right)\right] \\
&+C_F\frac{\alpha^3}{4\pi}\left[\frac{m_1}{m_2}\left(-\frac{T_F n_f}{54}(65C_A-54C_F)\bar{c}_1^{hl(2)}+\#c_k^{(2)\,2}-\frac{C_A}{18}(25C_A-\frac{125}{3}T_F n_f)c_F^{(2)\,2}\right)\right. \\
&\qquad\left.+\frac{m_2}{m_1}\left(-\frac{T_F n_f}{54}(65C_A-54C_F)\bar{c}_1^{hl(1)}+\#c_k^{(1)\,2}-\frac{C_A}{18}(25C_A-\frac{125}{3}T_F n_f)c_F^{(1)\,2}\right)\right] \\
&+C_F\frac{\alpha^3}{4\pi}\left[\#c_k^{(1)}c_k^{(2)}+\#d_{ss}+\#d_{vs}\right],
\end{aligned} \tag{E.40}$$

where the numbers "#" are presently unknown. Remember that our aim is to determine the purely NLL, i.e. the $\mathcal{O}(\alpha^2+\alpha^3\ln\alpha+\ldots)$, contribution to the soft running of $d_{ss}+C_F\bar{d}_{vs}$. In order to isolate this correction from the LL part, we start splitting the Wilson coefficients into their LL and NLL contributions, namely $c_i = [c_i]^{\rm LL}+[c_i]^{\rm NLL}$. We also set to zero the unknown terms (including the NLL running of $\bar{c}_1^{hl}$ which is also unknown at present, so we take it with LL precision). Thus

$$\nu_s \frac{d}{d\nu_s}(d_{ss} + C_F \bar{d}_{vs})\Big|_{\rm soft} = C_F \alpha^2 \left(2C_F - \frac{C_A}{2}\right) \tag{E.41}$$
$$+C_F\alpha^2 \left[\frac{m_1}{m_2}\left(\frac{1}{3}T_F n_f [\bar{c}_1^{hl(2)}]^{\rm LL} - \frac{4}{3}(C_A + C_F) - \frac{5}{12}C_A([c_F^{(2)}]^{\rm LL} + [c_F^{(2)}]^{\rm NLL})^2\right)\right.$$
$$\left.+\frac{m_2}{m_1}\left(\frac{1}{3}T_F n_f [\bar{c}_1^{hl(1)}]^{\rm LL} - \frac{4}{3}(C_A + C_F) - \frac{5}{12}C_A([c_F^{(1)}]^{\rm LL} + [c_F^{(1)}]^{\rm NLL})^2\right)\right]$$
$$+C_F\frac{\alpha^3}{4\pi}\left[\frac{m_1}{m_2}\left(- \frac{T_F n_f}{54}(65C_A - 54C_F)[\bar{c}_1^{hl(2)}]^{\rm LL} - \frac{C_A}{18}(25C_A - \frac{125}{3}T_F n_f)([c_F^{(2)}]^{\rm LL})^2\right)\right.$$
$$\left.+\frac{m_2}{m_1}\left(- \frac{T_F n_f}{54}(65C_A - 54C_F)[\bar{c}_1^{hl(1)}]^{\rm LL} - \frac{C_A}{18}(25C_A - \frac{125}{3}T_F n_f)([c_F^{(1)}]^{\rm LL})^2\right)\right].$$

The Wilson coefficients at LL do not depend on the mass, so

$$\nu_s \frac{d}{d\nu_s}(d_{ss} + C_F \bar{d}_{vs})\Big|_{\rm soft} = C_F \alpha^2 \left(2C_F - \frac{C_A}{2}\right)$$
$$+C_F\alpha^2 \left(\frac{m_1}{m_2} + \frac{m_2}{m_1}\right)\left(\frac{1}{3}T_F n_f [\bar{c}_1^{hl}]^{\rm LL} - \frac{4}{3}(C_A + C_F) - \frac{5}{12}C_A([c_F]^{\rm LL})^2\right)$$
$$-\frac{5}{6}C_A C_F \alpha^2 [c_F]^{\rm LL}\left(\frac{m_1}{m_2}[c_F^{(2)}]^{\rm NLL} + \frac{m_2}{m_1}[c_F^{(1)}]^{\rm NLL}\right)$$
$$+C_F\frac{\alpha^3}{4\pi}\left(\frac{m_1}{m_2} + \frac{m_2}{m_1}\right)\left(- \frac{T_F n_f}{54}(65C_A - 54C_F)[\bar{c}_1^{hl}]^{\rm LL}\right.$$
$$\left.-\frac{C_A}{18}(25C_A - \frac{125}{3}T_F n_f)([c_F]^{\rm LL})^2\right). \tag{E.42}$$

Let us writte the RGE above in terms of a derivative with respect to α instead of a derivative with respect to ν_s

$$\frac{d}{d\alpha}(d_{ss} + C_F \bar{d}_{vs})\Big|_{\rm soft} = -\frac{1}{\beta_0\frac{\alpha^2}{2\pi} + \beta_1\frac{\alpha^3}{8\pi^2}}\left[C_F \alpha^2 \left(2C_F - \frac{C_A}{2}\right)\right.$$
$$+C_F\alpha^2 \left(\frac{m_1}{m_2} + \frac{m_2}{m_1}\right)\left(\frac{1}{3}T_F n_f [\bar{c}_1^{hl}]^{\rm LL} - \frac{4}{3}(C_A + C_F) - \frac{5}{12}C_A([c_F]^{\rm LL})^2\right)$$
$$-\frac{5}{6}C_A C_F \alpha^2 [c_F]^{\rm LL}\left(\frac{m_1}{m_2}[c_F^{(2)}]^{\rm NLL} + \frac{m_2}{m_1}[c_F^{(1)}]^{\rm NLL}\right)$$
$$+C_F\frac{\alpha^3}{4\pi}\left(\frac{m_1}{m_2} + \frac{m_2}{m_1}\right)\left(- \frac{T_F n_f}{54}(65C_A - 54C_F)[\bar{c}_1^{hl}]^{\rm LL}\right.$$
$$\left.\left.-\frac{C_A}{18}(25C_A - \frac{125}{3}T_F n_f)([c_F]^{\rm LL})^2\right)\right]. \tag{E.43}$$

Expanding in powers of α and taking only the relevant terms to determine $[d_{ss}+C_F\bar{d}_{vs}]^{\rm NLL}$, where $d_{ss} + C_F\bar{d}_{vs} = [d_{ss} + C_F\bar{d}_{vs}]^{\rm LL} + [d_{ss} + C_F\bar{d}_{vs}]^{\rm NLL}$, we obtain the following RGE

$$
\begin{aligned}
&\frac{d}{d\alpha}(d_{ss}+C_F\bar{d}_{vs})\Big|_{\rm soft}^{\rm NLL} = \frac{2\pi}{\beta_0}\left[\frac{\beta_1}{\beta_0}\frac{\alpha}{4\pi}C_F\left(2C_F-\frac{C_A}{2}\right)\right.\\
&+\frac{\beta_1}{\beta_0}\frac{\alpha}{4\pi}C_F\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}\right)\left(\frac{1}{3}T_Fn_f[\bar{c}_1^{hl}]^{\rm LL}-\frac{4}{3}(C_A+C_F)-\frac{5}{12}C_A([c_F]^{\rm LL})^2\right)\\
&+\frac{5}{6}C_AC_F[c_F]^{\rm LL}\left(\frac{m_1}{m_2}[c_F^{(2)}]^{\rm NLL}+\frac{m_2}{m_1}[c_F^{(1)}]^{\rm NLL}\right)\\
&+C_F\frac{\alpha}{4\pi}\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}\right)\left(\frac{T_Fn_f}{54}(65C_A-54C_F)[\bar{c}_1^{hl}]^{\rm LL}\right.\\
&\left.\left.+\frac{C_A}{18}(25C_A-\frac{125}{3}T_Fn_f)([c_F]^{\rm LL})^2\right)\right], \qquad \text{(E.44)}
\end{aligned}
$$

which written in terms of z, reads

$$
\begin{aligned}
&\frac{d}{dz}(d_{ss}+C_F\bar{d}_{vs})\Big|_{\rm soft}^{\rm NLL} = 2\pi\alpha(\nu_h)z^{-1+\beta_0}\left[\frac{\beta_1}{\beta_0}\frac{\alpha(\nu_h)z^{\beta_0}}{4\pi}C_F\left(2C_F-\frac{C_A}{2}\right)\right.\\
&+\frac{\beta_1}{\beta_0}\frac{\alpha(\nu_h)z^{\beta_0}}{4\pi}C_F\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}\right)\left(\frac{1}{3}T_Fn_f[\bar{c}_1^{hl}]^{\rm LL}-\frac{4}{3}(C_A+C_F)-\frac{5}{12}C_A([c_F]^{\rm LL})^2\right)\\
&+\frac{5}{6}C_AC_F[c_F]^{\rm LL}\left(\frac{m_1}{m_2}[c_F^{(2)}]^{\rm NLL}+\frac{m_2}{m_1}[c_F^{(1)}]^{\rm NLL}\right)\\
&+C_F\frac{\alpha(\nu_h)z^{\beta_0}}{4\pi}\left(\frac{m_1}{m_2}+\frac{m_2}{m_1}\right)\left(\frac{T_Fn_f}{54}(65C_A-54C_F)[\bar{c}_1^{hl}]^{\rm LL}\right.\\
&\left.\left.+\frac{C_A}{18}(25C_A-\frac{125}{3}T_Fn_f)([c_F]^{\rm LL})^2\right)\right]. \qquad \text{(E.45)}
\end{aligned}
$$

This is precisely the equation we solve to obtain Eq. (4.70).

E.5.1 NLL running of c_F

Note that, in order to solve Eq. (E.45), the Wilson coefficients $\bar{c}_1^{hl}$ and c_F are needed with LL and NLL precision, respectively. The expressions with LL precision are summarized in Sec. B.2. The full NLL running of c_F was computed in Ref. [13] and it is given by

$$
c_F^{(i)}(m_i) = z^{-\gamma_0/2}\left[1+\frac{\alpha(\nu_h)}{4\pi}\left(c_1+\frac{\gamma_0}{2}\ln\left(\frac{\nu_h^2}{m_i^2}\right)\right)+\frac{\alpha(\nu_h)-\alpha(\nu_s)}{4\pi}\left(\frac{\gamma_1}{2\beta_0}-\frac{\gamma_0\beta_1}{2\beta_0^2}\right)\right], \qquad \text{(E.46)}
$$

where

$$
\gamma_0 = 2C_A\,, \quad \gamma_1 = \frac{68}{9}C_A^2-\frac{52}{9}C_AT_Fn_f\,, \quad c_1 = 2(C_A+C_F)\,. \qquad \text{(E.47)}
$$

This expression can be splitted into its purely LL and NLL contributions, $c_F^{(i)}(m_i) = [c_F^{(i)}(m_i)]^{\rm LL}+[c_F^{(i)}(m_i)]^{\rm NLL}$. The purely NLL contribution is given by

$$\begin{aligned}
[c_F^{(i)}(m_i)]^{\rm NLL} &= c_F^{(i)}(m_i) - [c_F^{(i)}(m_i)]^{\rm LL} \\
&= z^{-C_A}\,\frac{\alpha(\nu_h)}{4\pi}\left[\left(c_1 + 2C_A \ln\left(\frac{\nu_h}{m_i}\right)\right) + (1 - z^{\beta_0})\left(\frac{\gamma_1}{2\beta_0} - \frac{C_A\beta_1}{\beta_0^2}\right)\right].
\end{aligned} \tag{E.48}$$

Bibliography

[1] A. Pineda, and J. Soto, Nucl. Phys. Proc. Suppl. **64**, 428 (1998)

[2] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Rev. Mod. Phys. **77**, 1423 (2005).

[3] A. Pineda, Phys. Rev. D **65**, 074007 (2002).

[4] B. A. Kniehl, A. A. Penin, V. A. Smirnov, and M. Steinhauser, Nucl. Phys. **B635**, 357 (2002).

[5] C. Anzai, D. Moreno, and A. Pineda, Phys. Rev. D **98**, 114034 (2018).

[6] C. Peset, A. Pineda, and J. Segovia, Phys. Rev. D **98**, 094003 (2018).

[7] M. Beneke, Y. Kiyo, P. Marquard, A. Penin, J. Piclum, D. Seidel, and M. Steinhauser, Phys. Rev. Lett. **112**, 151801 (2014).

[8] A. Pineda, Prog. Part. Nucl. Phys. **67**, 735 (2012).

[9] N. Brambilla, Y. Jia, and A. Vairo Phys. Rev. D **73**, 054005 (2006).

[10] A. Bazavov, N. Brambilla, X. Garcia i Tormo, P. Petreczky, Joan Soto, and A. Vairo, Phys. Rev. D **86**, 114031 (2012).

[11] C. Peset, A. Pineda, and M. Stahlhofen, J. High Energy Phys. 05, (2016) 017.

[12] B. A. Kniehl, A. A. Penin, A. Pineda, V. A. Smirnov, and M. Steinhauser, Phys. Rev. Lett. **92**, 242001 (2004); **104**, 199901(E) (2010).

[13] A. A. Penin, A. Pineda, V. A. Smirnov, and M. Steinhauser, Phys. Lett. B **593**, 124 (2004); **677**, 343(E) (2009).

[14] A. A. Penin, A. Pineda, V. A. Smirnov, and M. Steinhauser, Nucl. Phys. **B699**, 183 (2004); **B829**, 398(E) (2010).

[15] A. Pineda, and J. Soto, Phys. Rev. D **58**, 114011 (1998).

[16] C. P. Burgess, *An Introduction to Effective Field Theory*, Ann. Rev. Nucl. Part. Sci. Vol. **57** (2007).

[17] H. Georgi, *Effective Field Theory*, Annu. Rev. Nucl. Part. Sci. Vol. **43** (1993).

[18] H. Georgi, Nucl. Phys. Proc. Suppl. **29BC** (1992), 1-10.

[19] A. V. Manohar Lect. Notes Phys. **479** (1997), 311-362.

[20] M. Neubert, *Effective field theory and heavy quark physics* (2005), arXiv: hep-ph/0512222.

[21] A. Pich *Effective field theory: course* (1998), arXiv: hep-ph/9806303.

[22] I. Rothsein, *TASI Lectures on Effective Field Theories* (2003), arXiv: hep-ph/0308266.

[23] Castellani, E. (2002) *Reductionism, Emergence, and Effective Field Theories*, Studies in History and Philosophy of Modern Physics **33**: 251-267.

[24] M.B. Voloshin, and M.A. Shifman, Sov. J. Nucl. Phys. **45**, 292 (1987); H.D. Politzer, and M.B. Wise, Phys. Lett. B **206**, 681 (1988); N. Isgur, and M.B. Wise, Phys. Lett. B **232**, 113 (1989); E. Eichten, and B. Hill, Phys. Lett. B **234**, 511 (1990); H. Georgi, Phys. Lett. B **240**, 447 (1990); B. Grinstein, Nucl. Phys. B **339**, 253 (1990).

[25] W. E. Caswell, and G. P. Lepage, Phys. Lett. B **167B**, 437 (1986).

[26] G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D **51**, 1125 (1995); **55**, 5853(E) (1997).

[27] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Nucl. Phys. **B566**, 275 (2000).

[28] A. V. Manohar, Phys. Rev. D **56**, 230 (1997).

[29] C. Balzereit, Phys. Rev. D **59**, 094015 (1999).

[30] M. Finkemeier, and M. McIrvin, Phys. Rev. D **55**, 377 (1997).

[31] B. Blok, J. G. Korner, D. Pirjol, and J. C. Rojas, Nucl. Phys. **B496**, 358 (1997).

[32] C. W. Bauer, and A. V. Manohar, Phys. Rev. D **57**, 337 (1998).

[33] C. Balzereit, Phys. Rev. D **59**, 034006 (1999).

[34] C. Balzereit, *Reparametrization invariance of heavy quark effective theory at* $\mathcal{O}(1/m_Q^3)$, arXiv: hep-ph/9809226.

[35] D. Moreno, and A. Pineda, Phys. Rev. D **97** 016012 (2018); **98** 059902(E) (2018).

[36] X. Lobregat, D. Moreno, and R. Petrossian-Byrne, Phys. Rev. D **97**, 054018 (2018).

[37] D. Moreno, Phys. Rev. D **98**, 034016 (2018).

[38] M. Neubert, Phys. Rept. **245**, 259 (1994).

[39] T. Mannel, W. Roberts, and Z. Ryzak, Nucl. Phys. **B368**, 204 (1992).

[40] S. Balk, J. G. Korner and D. Pirjol, Nucl. Phys. **B428**, 499 (1994).

[41] S. Scherer, A. Y. Korchin, and J. H. Koch, Phys. Rev. C **54**, 904 (1996).

[42] H. W. Fearing, and S. Scherer, Few Body Syst. **23**, 111 (1998).

[43] D. Nevado, and A. Pineda, Phys. Rev. C **77**, 035202 (2008).

[44] R. J. Hill, G. Lee, G. Paz, and M. P. Solon, Phys. Rev. D **87**, 053017 (2013).

[45] M. E. Luke, and A. V. Manohar, Phys. Lett. B **286**, 348 (1992).

[46] J. E. Augustin et al., Phys. Rev. Lett. **33**, 1406 (1974).

[47] J. J. Aubert, U. Becker, P. J. Biggs, J. Burger, M. Chen, G. Everhart, P. Goldhagen, J. Leong, T. McCorriston, T. G. Rhoades, M. Rohde, Samuel C. C. Ting, Sau Lan Wu, and Y. Y. Lee, Phys. Rev. Lett. **33**, 1404 (1974).

[48] S. W. Herb, D. C. Hom, L. M. Lederman, J. C. Sens, H. D. Snyder, J. K. Yoh, J. A. Appel, B. C. Brown, C. N. Brown, W. R. Innes, K. Ueno, T. Yamanouchi, A. S. Ito, H. Jöstlein, D. M. Kaplan, and R. D. Kephart, Phys. Rev. Lett. **39**, 252 (1977).

[49] F. Abe et al. (CDF Collaboration), Phys. Rev. Lett. **81**, 2432 (1998).

[50] H. A. Bethe, and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms*, Academic Press Inc., New York, 1957.

[51] M. Luke, and A. V. Manohar, Phys. Rev. D **55** 4129 (1997).

[52] N. Brambilla, E. Mereghetti, and A. Vairo, Phys. Rev. D **79**, 074002 (2009); **83**, 079904(E) (2011).

[53] A. Gunawardana, and G. Paz, J. High Energy Phys. 07, (2017) 137.

[54] E. Eichten, and B. Hill, Phys. Lett. B **243**, 427 (1990).

[55] G. Amorós, M. Beneke, and M. Neubert, Phys. Lett. B **401**, 81 (1997).

[56] A. Czarnecki, and A. G. Grozin, Phys. Lett. B **405**, 142 (1997); **650**, 447(E) (2007).

[57] N. Brambilla, A. Vairo, X. Garcia i Tormo, and J. Soto, Phys. Rev. D **80**, 034016 (2009).

[58] A. Pineda, and M. Stahlhofen, Phys. Rev. D **84**, 034016 (2011).

[59] A. Pineda, Phys. Rev. D **84**, 014012 (2011).

[60] A. H. Hoang, and M. Stahlhofen, Phys. Rev. D **75**, 054025 (2007).

[61] A. H. Hoang, and M. Stahlhofen, J. High Energy Phys. 06, (2011) 088.

[62] J. Collins, Cambridge Monogr. Part. Phys. Nucl. Phys. Cosmol. **32**, 1 (2011).

[63] C. Peset, A. Pineda, and M. Stahlhofen, Eur. Phys. J. C **77**, 681 (2017).

[64] I. B. Khriplovich, A. I. Milstein, and A. S. Yelkhovsky, Phys. Scr. **T46**, 252, (1993).

[65] A. Czarnecki, K. Melnikov, and A. Yelkhovsky, Phys. Rev. A **59**, 4316 (1999).

[66] A. Pineda, and J. Soto, Phys. Rev. **59**, 016005 (1999).

[67] A. V. Manohar, and I. W. Stewart, Phys. Rev. Lett. **85**, 2248 (2000).

[68] M. E. Luke, A. V. Manohar, and I. Z. Rothstein, Phys. Rev. D **61**, 074025 (2000).

[69] A. A. Penin, and M. Steinhauser, Phys. Lett. B **538**, 335 (2002).

[70] A. A. Penin, V. A. Smirnov, and M. Steinhauser, Nucl. Phys. **B716**, 303 (2005).

[71] M. Beneke, Y. Kiyo, and K. Schuller, Nucl. Phys. **B714**, 67 (2005).

[72] Y. Kiyo, and Y. Sumino, Phys. Lett. B **730**, 76 (2014).

[73] Y. Kiyo, and Y. Sumino, Nucl. Phys. **B889**, 156 (2014)

[74] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Phys. Rev. D **63**, 014023 (2001).

[75] M. E. Peskin, and D. V. Schroeder, *An Introduction to quantum field theory*. Perseus Books, Cambridge, 1995.

[76] P. Pascual, and R. Tarrach, *QCD: Renormalization for the Practitioner*, Lecture Notes in Physics 194, Springer-Verlag, 1984.