Universitat de Lleida

Generalizations of assignment games and information market games

Saadia El Obadi

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Universitat de Lleida

TESI DOCTORAL

Generalizations of assignment games and information market games

Saadia El Obadi

Memòria presentada per optar al grau de Doctor per la Universitat de Lleida
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I dedicate this thesis with love and gratitude to Aicha and Montse.

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Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

1.1.1 Cooperative game theory

Game theory is the field that studies situations of strategic interaction between rational decision makers. Von Neumann and Morgenstern (1944) distinguish between non-cooperative games and cooperative games.

The objective in non-cooperative games is to predict the best strategy for each agent, whereas in cooperative games, also known as coalitional games, the agents are allowed to write binding agreements with each other. The sinificant issue here is to predict which coalitions will form and the sharing of the value they obtain when they cooperate.

A classical model studied from the cooperative game theory is the bankruptcy situation (Auman and Mashler, 1985). Following is an example of a bankruptcy problem.

Example 1.1.1. *A company got bankrupt and left an estate* $E =$ 10000 *euro. There are 4 creditors who claim, respectively,* $d_1 = 3000$ *euro,* $d_2 = 2000$ *euro,* $d_3 = 5000$ *euro and* $d_4 = 8000$ *euro. We define the cooperative game* (N, v) *where the players are the creditors* $N = \{1, 2, 3, 4\}$ *and the worth of the grand coalition, N, is* $v(N) = E = 10000$.

Every coalition S in N can receive the estate if it pays the debt claimed by the creditors out of the coalition. The value of a coalition S in N is what remains from the estate E after paying the debt to the creditors in $N \setminus S$ *.*

For instance, if $S = \{1, 4\}$*, then,* $v(\{1, 4\}) = max\{0, E - d_2 - d_3\} = max\{0, 10000 - 2000 - 5000\} = 3000$*. Further, if* $S = \{2, 3\}$*, then,* $v(\{2, 3\}) = max\{0, E - d_1 - d_4\} = max\{0, 10000 - 3000 - 8000\} = 0$.

This way we define the game (N, v) *as follows:*

$$
\begin{array}{lll}
v(\{1\}) = 0, & v(\{1,2\}) = 0, & v(\{1,2,3\}) = 2000 \\
v(\{2\}) = 0, & v(\{1,3\}) = 0, & v(\{1,2,4\}) = 5000 \\
v(\{3\}) = 0, & v(\{2,3\}) = 0, & v(\{1,3,4\}) = 8000 \\
v(\{4\}) = 0, & v(\{1,4\}) = 3000, & v(\{2,3,4\}) = 7000 \\
 & v(\{2,4\}) = 2000 & \\
 & v(\{3,4\}) = 5000, & v(N) = 10000
\end{array}
$$

We also distinguish between two classes of cooperative games. They are transferable utility games with side payments and non-transferable utility games.

A cooperative game is said to be a transferable utility game if there is some medium, for instance money, of exchange between the agents, and the agents utilities are linear in money. In non-transferable utility games, such medium is not present or, if it is, the agents utilities are not linear in it.

In this dissertation, we focus on cooperative games with transferable utility.

In this games, the grand coalition N will form and the challenge will be to suggest a way of allocating the worth of the grand coaltion among agents.

Different solution concepts have been defined in the literature.

Gillies (1959) defines the core of coalitional games which is a set-solution concept. According to the definition, the player of any coalition receives at least as much as the joint revenues that they obtain by the cooperation.

Further, Shapley (1953) introduces the Shapley value which is a point-solution concept and assigns to each cooperative game, an efficient weighted average of all possible marginal contribution vectors. Furthermore, other point solutions, the $\tau-$value introduced by Tijs (1981) and the Nucleolus introduced by Shmeidler (1969), are discussed. In the next sections, all definitions of these solution concepts are introduced.

In this dissertation we focus on some models of cooperative games. Throughout the next four chapters, the different models are studied.

1.1.2 Information market games

In the second chapter we focus on some cooperative games defined from information markets. They are a generalization of information market games. Many papers appeared dealing with the issue of intentional sharing of information assuming perfect patent protection, we can find Gallini (1984) and Kamien et al. (1985), they examined licensing in oligopolistic markets. Muto et al.(1989) define the information market game. They analyse the cooperative behaviour of economic agents (firms), faced with the introduction of new technology, indispensable for the manufacturing of a new product. They consider that the market is divided into submarkets. For each submarket one knows the maximum profit achieved by producing and selling the new product in that submarket. They study the core and some point-solutions as the Shapley value and the τ-value. In this thesis we wonder what happens when we have more than one informed player (we can say they form a clan of patent holders).

Similar to Muto et al. (1989), the aim of this chapter is to analyze the cooperative behaviour of firms, faced again with the introduction of some new technologies owned by different patent holders, which are essential for manufacturing a new product. This situation, where more than one informed player is needed to produce the good, is considered. We name it clan information market.

In this new situation, the information is divided into several parts (or technologies) and each patent holder initially possess just one of the parts. As we assume perfect patent protection and also that the production of the new commodity needs all those patents, no single informed player has the technology required to produce the new commodity.

Each firm or group of firms has the possibility to manufacture the new product if they have all the technological information. They can obtain the maximum profit in the submarket they have access to only if they know the new technologies. By sharing these technologies with other firms (licensing), the clan may access and make a profit indirectly in other submarkets where the clan has no access by itself. Therefore, faced with this situation, cooperation is beneficial.

The corresponding cooperative game with side payments is defined in Chapter 2. It is named *clan information market game* (CIG). Muto et al. (1989) showed that the class of information market games is generalized by the class of big boss games, where the presence of the big boss is necessary for a coalition to attain any profit. When more than one agent is indispensable, Potters et al. (1989) introduced the class of clan games. In a short note, Tijs (1990) introduced clan information market games as an example of clan games with no further development of the model. Clearly, every clan information market game is a clan game, although the opposite is no longer true. The existence of population monotonic allocation schemes (PMAS) is studied. We show that the τ-value of a clan information market game also has some population monotonicity property: it yields a bi-mas. Finally, the final section characterizes the class of clan information market games and provides conditions on the market, under which the Shapley value belongs to the core.

1.1.3 Information interval games

In the third chapter we focus on other generalization of information market games. This time we consider that the profit attainable in each submarket is uncertain, it is given by an interval of real numbers. This way, we get a class of interval games. In interval games, the characteristic function assings a closed and bounded interval to each coalition.

Cooperative interval games are introduced by Branzei et al. (2003) to handle banckruptcy situations where the estate is known with certainty while the claims are given by bounded intervals of real numbers.

Carpente et al. (2005) propose a method to associate a coalitional interval game with each strategic game. Later, Alparslan Gök et al. (2009) consider selections of cooperative interval games which are classical cooperative games.

If we consider an OR problem with interval data, the corresponding cooperative game can be an interval game. For instance, this is the case of some connection problems (Moretti et al., 2008), lot sizing problems with uncertain demands (Drechsel and Kimms, 2008) or sequencing problems (Alparslan-Gök et al., 2008). A survey on cooperative interval games can be found in Branzei et al. (2010).

In Chapter 3 we study information interval games corresponding to information markets where the profit attainable in each submarket is uncertain. The aim of this chapter consists of providing interval solutions. Such a solution yields an interval of individual payoffs to each player such that the interval profit the gran coalition can obtain by cooperation, is allocated among the players.

1.1.4 Assignment games

In the second part of the thesis, Chapters 4 and 5, we consider another kind of markets, the assignment markets.

Shapley and Shubik (1972) defined assignment games as a model for two-sided markets where an indivisible product (houses, cars, etc.) is exchanged for money, and where sellers and buyers supply or demand exactly one unit respectively. The unit needs to be similar, and the same unit may have different values for differents participants. Each buyer has a value for every house and each seller has a reservation value. A valuation matrix represents the joint profit obtained by each mixed pair.

In Chapter 4 we study a class of assignment game where there is an agent who has a double role as a seller and as a buyer and who is needed for any exchange in the market, it is named the central player. We define a market where there is also a set of buyers N^1 and a set of sellers N^2, however these two sets are not disjoint but have one agent, the central player, in common, who can act both as a buyer and as a seller. As in Shapley and Shubik assignment game, each buyer wants to buy at most one unit and each seller has one unit on sale. A matrix $A = (a_{ij})_{\substack{i\in N^1 \\ j\in N^2}}$ summarizes the profit that players i and j get when the transaction between both players takes place.

However, an additional feature of our model is that no trade is possible if the central player does not participate in it. We can assume that this player, denoted by h, allows the trade between buyers and sellers. Therefore, the profit matrix is such that $a_{ij} = 0$ if $h \notin \{i, j\}$.

We can think of a social bank of flats where customers can sell and buy houses with reasonable prices under the supervision of the bank that can also buy and sell.

In Chapter 4 we define assignment games with a central player. We study its core and we analyze if it is a stable set. We give two point solutions, the τ-value and the nucleolus and we study when the first one lies in the core. Finally, we generalize the model allowing the central player to sell and buy more than one item and we study the competitive equilibria.

A different generalization of assignment games is introduced in Chapter 5.

We consider there that there is more than one player with a double role as a seller and as a buyer. In fact, these are middelmen who neither produce any good nor consume it. They buy the goods to the sellers and sell them to the buyers. Without any middelman, the trade is not possible. We assume that buyers and sellers cannot meet on their own. Further, each middelman may trade more than one unit. So,we assume that there is a third side in the market formed by a finite set of middlemen (disjoint with the set of buyers and sellers). This situation may represent a real estate market in which value is generated by the matching of a buyer and a seller but typically real estate agencies act as intermediaries. Moreover, the same house can be advertised in the website of several agencies, and each buyer can also search in several of these sites.

This situation resembles the firm-supplier-buyer in Stuart (1997), but there the value of a triplet is the sum of the value generated by firm and supplier and the value generated by supplier and buyer. In our case the middleman does not modify the value of the buyer-seller pair, that is, the profit generated by the trade of a buyer and a seller does not depend on who is the intermediary that connects them. Stuart's model is a particular instance of three-sided assignment game. Because the value of a firm-supplier-buyer triplet is defined additively, it can be guaranteed that the core of the associated three-sided assignment game is non-empty. It is also known, see Kaneko and Wooders (1982), that three-sided assignment games where values of triplets are defined arbitrarily may have an empty core.

In contrast to our model, each supplier in Stuart's model has unitary capacity, that is, each supplier can only connect one firm-buyer pair. The assignment markets with middlemen that we consider, are three-sided assignment markets with multiple partnership (on the side of middlemen). Two-sided assignment markets with multiple partnership have been studied for instance in Kaneko (1983), Thompson (1980), Sotomayor (1992, 2002) and Sánchez-Soriano (2001). When both sides of the market allow for multiple-partnership these games are sometimes called *transportation games*.

Transportation games have a non-empty core and this core contains the set of competitive equilibrium payoff vectors but, different to the one-to-one assignment game, this inclusion may be strict, that is, not all core allocations are supported by competitive prices. It is shown in Sotomayor (2002) that most of the properties of the core of the one-to-one assignment game, such as the lattice structure and the opposition of interests between the two sides of the market, are lost when we allow for multiple partnership.

In Chapter 5, after defining assignment games with middlemen, we provide some sufficient conditions for the non-emptyness of the core. The set of competitive equilibrium payoff vectors coincides with the solutions of the dual linear program corresponding to the linear program that is solved to obtain the worth of the grand coalition.

1.2 Preliminaries and solution concepts

This section deals with the preliminaires we need in order to explain the findings and contributions of this dissertation. In this section we recall some concepts related to cooperative games and assignment games. Further, some necessary notation about interval games is also included.

Besides these preliminaries, since each chapter in this thesis corresponds to a potential paper, each of them is self-contained. This means that the preliminaries needed in each chapter are also kept in the corresponding chapter, as well as the references corresponding to each chapter.

1.2.1 Cooperative games

This section is devoted to introduce some basic definitions from classical cooperative game theory. We will use these concepts throughout the thesis.

A *cooperative game (with side payments)* with player set N is a map $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Here, $v(S)$ is the maximal profit obtainable by the coalition $S \subseteq N$, without the help of any player outside coalition S.

A cooperative game v is called *monotonic* if $v(S) \leq v(T)$ for any two coalitions $S, T \subseteq N$ with $S \subseteq T$. It is called *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for any disjoint pair of coalitions S and T.

Shapley (1971) introduced the notion of convexity. A game (N, v) is *convex* if its characteristic function, $v : 2^N \to \mathbb{R}$, satisfies any of the following two equivalent conditions:

(i) $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$.

(ii) $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $i \in N$ and all $S, T \subseteq N$ such that $S \subseteq T \subseteq N \backslash \{i\}$.

So, from condition (ii), we can say that a game is convex if it satisfies the property of non-decreasing marginal returns when coalitions grow larger. A game (N, v) is *concave* if the oppossite equation holds, i.e. $v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$ for all $i \in N$ and all $S, T \subseteq N$ such that $S \subseteq T \subseteq N \backslash \{i\}$.

A game (N, v) is *simple* if $v(S) \in \{0, 1\}$ for all $S \subseteq N$ and $v(N) = 1$.

We denote $v = u_{S_1, S_2, \ldots, S_k}$ the monotonic simple game with minimal winning coalitions $S_1, S_2, \ldots, S_k$, i.e. $u_{S_1, S_2, \ldots, S_k}(S) = 1$ if $S_i \subseteq S$ for some $S_i \in \{S_1, S_2, \ldots, S_k\}$, and $u_{S_1, S_2, \ldots, S_k}(S) = 0$ otherwise. If the nonempty coalition $T \subseteq N$ is the only minimal winning coalition, the game (N, u_T) is called the *unanimity game*. The *dual* of a game (N, v) is denoted by (N, v^*) and is defined by $v^*(S) = v(N) - v(N \backslash S)$ for all $S \subseteq N$.

The theory of cooperative games offers some solution concepts. The *set of imputations* is the set of efficient and individually rational allocations, i.e. $I(v) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N\}$. Further, Gillies (1959) added the coalitional rationality and defined the *core* of a game (N, v) as the set $C(v) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N)$ and $\sum_{i \in S} x_i \geq v(S)$ for all $S \subseteq N\}$. Note that if a core allocation x is proposed, then no coalition S has an incentive to split off from the grand coalition N.

The core is said to be *stable* if for any imputation $y \notin C(v)$, there is a core allocation $x \in C(v)$ and a nonempty coalition $S \subseteq N$ such that $x_i > y_i$ for all $i \in S$ and $x(S) \leq v(S)$, where $x(S) = \sum_{i \in S} x_i$.

A game (N, v) is said to be *balanced* if it has a nonempty core, while it is said to be *totally balanced* if the core of every subgame is nonempty, where the subgame corresponding to some coalition $T \subseteq N$, $T \neq \emptyset$, is the game (T, v_T) with $v_T(S) = v(S)$ for all $S \subseteq T$.

Sprumont (1990) introduced population monotonic allocation schemes. Such schemes provide an allocation vector for any coalition. Given a game (N, v), the table $X = (x_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ is said to be a *population monotonic allocation scheme* (PMAS for short) if the following two conditions hold:

(i) efficiency: for all $S \in 2^N \setminus \{\emptyset\}$, we have $\sum_{i \in S} x_{S,i} = v(S)$;

(ii) monotonicity: for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$, and for all $i \in S$, we have $x_{S,i} \leq x_{T,i}$.

Note that a PMAS provides a core element for every coalition in the corresponding subgame in a monotonic way.

Finally, we recall three point solution concepts: the τ-value, the nucleolus and the Shapley value.

The *τ-value* (Tijs, 1981) of a game is essentially a compromise value between an upper bound payoff vector and a lower bound payoff vector. Let (N, v) be a game and let the vector $M(v) \in \mathbb{R}^N$ be such that its coordinates are the marginal contribution of each player to the grand coalition, i.e. $M_i(v) = v(N) - v(N \backslash \{i\})$, for all $i \in N$. The vector $M(v)$ is called the *utopia vector* and each $M_i(v)$ can be regarded as a maximal payoff (not always attainable) that player i can expect to obtain in the core of the cooperative game.

By using the utopia vector, we can now compute what remains for player $i \in N$ when coalition S forms, $i \in S$, and all players in $S \backslash \{i\}$ are paid their utopia payoff. The remainder for player i, $R^v(S, i)$ is defined by $R^v(S, i) = v(S) - \sum_{j \in S \backslash \{i\}} M_j(v)$.

Now, the vector $m(v) \in \mathbb{R}^N$ is defined by $m_i(v) = \max_{S: i \in S}\{R^v(S, i)\}$ for all $i \in N$. The vector $m(v)$ is called the *minimal rights vector*. Note that player $i \in N$ can be guaranteed the payoff $m_i(v)$ by offering the members of a suitable coalition (the one where the maximum is achieved) their utopia payoffs.

Let us now consider a coalitional game (N, v) such that $m(v) \neq M(v)$, $\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)$ and $m_i(v) \leq M_i(v)$ for all $i \in N$. Then, the *τ-value* of that game is the unique efficient payoff vector on the line segment between $m(v)$ and $M(v)$. Formally, $\tau(v) = \lambda m(v) + (1 - \lambda)M(v)$, where $\lambda \in [0, 1]$ is unique satisfying $\sum_{i \in N} \tau_i(v) = v(N)$.

The nucleolus was introduced by Schmeidler (1969). This solution concept is based on the excess of a coalition S at $x \in \mathbb{R}^N$ in the game v. For any $x \in \mathbb{R}^N$ and $S \subseteq N$ the *excess* of the coalition S is $e(x, S) = v(S) - x(S)$. It measures the gain or loss of the coalition S if its members depart from the allocation x in order to form their own coalition. Further, for any imputation x, let us define the vector $\theta(x) \in \mathbb{R}^{2^n-2}$ of excesses of all non-trivial coalitions at x, arranged in non-increasing order. That is to say, for all $k \in \{1, 2, \ldots, 2^n - 2\}$, $\theta(x)_k = e(S_k, x)$, where $\{S_1, S_2, \ldots, S_k, \ldots, S_{2^n-2}\}$ is the set of all nonempty coalitions in N different from N, and $e(S_k, x) \geq e(S_{k+1}, x)$.

Then the *nucleolus* of the game (N, v) is the imputation $\mu(v)$ which minimizes $\theta(x)$ with regard to the lexicographic order over the set of imputations: $\theta(\mu(v)) \leq_L \theta(x)$, for all $x \in I(v)$.

The last point solution considered in this section is the *Shapley value* (Shapley, 1953). According to the Shapley value $\phi(v)$ of the game (N, v), the value of player $i \in N$ is

$$\Phi_i(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} (w(P^\sigma(i) \cup i) - w(P^\sigma(i)))$$

where $n = |N|$, $\pi(N)$ is the set of all permutations $\sigma : N \to N$ and $P^\sigma(i)$ is the set of players that precede player i, i.e. $P^\sigma(i) = \{r \in N \text{ such that} \sigma^{-1}(r) < \sigma^{-1}(i)\}$ where σ^{-1} denote the entrance number of player i. In other words, the Shapley value of player $i \in N$ is $\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \alpha(S)(v(S \cup \{i\}) - v(S))$ where $\alpha(S) = \frac{s!(n-s-1)!}{n!}$,

$s = |S|$ and $n = |N|$. Some properties of the Shapley value are efficiency, i.e. $\sum_{i \in N} \phi_i(v) = v(N)$, and linearity, i.e. $\phi(v + w) = \phi(v) + \phi(w)$ and $\phi(av) = a\phi(v)$. The Shapley value may lie outside the core of the game, but if the game is convex, then the Shapley value is a core allocation.

If we consider an unanimity game u_T and its dual u_T^*, the Shapley value can be easily calculated as follows

$$\phi(u_T) = \phi(u_T^*) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$

1.2.2 Assigment games

Since Chapters 4 and 5 deal with assignment markets, we include here some notation related to these markets and the corresponding games.

Given a two-sided market, the assignment problem is defined by the triple (M_1, M_2, A) where $A = (a_{ij})_{\substack{i \in M_1 \\ j \in M_2}}$ is a non-negative real matrix. To solve the problem we must look for an optimal matching in A. A matching between M_1 and M_2 is a subset μ of $M_1 \times M_2$ such that each $k \in M_1 \cup M_2$ belongs at most to one pair in μ. When $(i, j) \in \mu$ we also denote with some abuse of notation $j = \mu(i)$ and $i = \mu(j)$. We will denote by $\mathcal{M}(M_1, M_2)$ the set of matchings. We say a matching μ is optimal for the problem (M_1, M_2, A), in short μ is optimal for A, if for all $\mu' \in \mathcal{M}(M_1, M_2)$, $\sum_{(i,j) \in \mu} a_{ij} \geq \sum_{(i,j) \in \mu'} a_{ij}$.

The set of optimal matchings of the problem (M_1, M_2, A) is denoted by $\mathcal{M}_A(M_1, M_2)$. Given $S \subseteq M_1$ and $T \subseteq M_2$, we denote by $\mathcal{M}(S,T)$ and $\mathcal{M}_A(S,T)$ the set of matchings and optimal matchings of the submarket $(S, T, A_{S\times T})$ defined by the subset S of buyers, the subset T of sellers and the restriction of A to $S \times T$. If $S = \emptyset$ or $T = \emptyset$, then the only possible matching is $\mu = \emptyset$ and, by convention, $\sum_{(i,j)\in\emptyset} a_{ij} = 0$.

Given an assignment problem (M_1, M_2, A), Shapley Shubik (1972) define a related coalitional game with transferable utility, the assignment game $(M_1 \cup M_2, w_A)$, as follows. The profits of the mixed-pair coalitions, $\{i, j\}$ where $i \in M_1$ and $j \in M_2$, are given by the non-negative matrix A,

$$w_A(\{i, j\}) = a_{ij} \geq 0,$$

and this matrix also determines the worth of any other coalition $S \cup T$, where $S \subseteq M_1$ and $T \subseteq M_2$, $w_A(S \cup T)$, in the following way $w_A(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \,|\, \mu \in \mathcal{M}(S, T)\}$.

1.2.3 Interval games

Let us name $I(\mathbb{R}_+)$ the set of all closed intervals in $\mathbb{R}_+$. If we consider the interval $R = [\underline{r}, \overline{r}]$, we say that $\underline{r}$ is the lower bound and $\overline{r}$ is the upper bound. In order to operate with cooperative interval games we first need to recall basic interval notation and calculus. The lenght of an interval $R = [\underline{r}, \overline{r}]$ is defined by $|R| = \overline{r} - \underline{r}$. Let $R_1 = [\underline{r}_1, \overline{r}_1]$ and $R_2 = [\underline{r}_2, \overline{r}_2]$ be two intervals, $R_1, R_2 \in I(\mathbb{R}_+)$, and let be $\alpha \in \mathbb{R}_+$. Then, $R_1 + R_2 = [\underline{r}_1 + \underline{r}_2, \overline{r}_1 + \overline{r}_2]$ and $\alpha R_1 = R_1 \alpha = [\alpha \underline{r}_1, \alpha \overline{r}_1]$. Let $R_1, R_2, \dots R_n$ be a finite set of intervals, we denote by $\sum_{i=1}^{n} R_i$ the sum of those n intervals, i.e. $\sum_{i=1}^{n} R_i = [\sum_{i=1}^{n} \underline{r}_i, \sum_{i=1}^{n} \overline{r}_i]$. Before defining the substraction of intervals, we need a preference relation. We say that R_1 is weakly preferred to R_2 ($R_1 \succcurlyeq R_2$) if and only if $\underline{r}_1 \geq \underline{r}_2$ and $\overline{r}_1 \geq \overline{r}_2$. Notice that it is only defined for not nested intervals. The substraction of two intervals R_1 and R_2, with $R_1 \succcurlyeq R_2$, is defined by $R_1 - R_2 = [\underline{r}_1 - \underline{r}_2, \overline{r}_1 - \overline{r}_2]$ only if $|R_1| \geq |R_2|$ (Alparslan et al., 2009). This last condition guarantees that the lower bound of the substraction interval is smaller or equal than its upper bound. Let $R_1, R_2, \dots R_n$ be a finite set of not nested intervals, the maximum interval is defined by $\max_{i \in \{1,2,\dots,n\}}\{R_i\} = \{R_{i^*} | R_{i^*} \succcurlyeq R_i$ for all $i \in \{1, 2, \dots, n\}\}$.

A *cooperative interval game* is an ordered pair (N, w) where $N = \{1, \dots, n\}$ is the set of players and $w : 2^N \to I(\mathbb{R})$ is the characteristic function such that $w(\emptyset) = [0, 0]$. Then, $w(S) = [\underline{w(S)}, \overline{w(S)}]$ where $\underline{w(S)}$ is the lower bound and $\overline{w(S)}$ is the upper bound of $w(S)$. We can associate three coalitional games to an interval game. Two of them are the border games $(N, \underline{w})$ and $(N, \overline{w})$ and the third one is the lenght game $(N, |w|)$ where $|w|(S) = |w(S)|$ for each $S \subseteq N$. Note that if all the worth intervals are degenerate intervals, $\underline{w(S)} = \overline{w(S)}$

for all $S \subseteq N$, then the interval game (N, w) corresponds to the coalitional game (N, v) with $v(S) = \underline{w(S)}$ for all $S \subseteq N$.

The *interval imputation set* $\mathcal{I}(w)$ of the interval game (N, w) is defined by

$$\mathcal{I}(w) = \{(I_1, ..., I_n) \in I(\mathbb{R}) | \sum_{i \in N} I_i = w(N) \text{ and } I_i \succcurlyeq w(\{i\}) \text{ for all } i \in N\}$$

and its *interval core* $\mathcal{C}(w)$ is defined by

$$\mathcal{C}(w) = \{(I_1, ..., I_n) \in \mathcal{I}(w) | \sum_{i \in S} I_i \succcurlyeq w(S) \text{ for all } S \subseteq N\}.$$

An interval game (N, w) is called *size-monotonic* if $(N, |w|)$ is monotonic, i.e. $|w|(S) \leq |w|(T)$ for all $S \subset T \subseteq N$. Let us name $SMIG^N$ the class of size monotonic interval games with player set N. For each $w \in SMIG^N$ and each $i \in N$, the interval marginal contribution of player i to the grand coalition in the game w is defined by $M_i(N, w) = w(N) - w(N \setminus \{i\})$.

The interval Shapley value $\Phi : SMIG^N \to I(\mathbb{R}^N)$ is defined by Alparslan-Gök et al. (2009) as follows,

$$\Phi(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w), \text{ for each } w \in SMIG^N$$

where $\Pi(N)$ is the set of all permutations $\sigma : N \to N$. The interval marginal operator $m^\sigma : SMIG^N \to I(\mathbb{R})$ corresponding to σ, associates with each $w \in SMIG^N$ the interval marginal vector $m^\sigma(w)$

defined by $w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i))$ for each $i \in N$. So, in other words, the interval Shapley value is given by

$$\Phi_i(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} (w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i)))$$

for any player $i \in N$.

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Chapter 2

Clan information market games

2.1 Introduction

This Chapter is based on El Obadi and Miquel (2017).

Muto et al. (1989) introduced information market games. They modelled the trading of information between one informed firm and some other initially non-informed firms. Later, Potters and Tijs (1990) allowed more than one initially informed firm to exist.

In both cases, in information market games with one informed player (Muto et al., 1989) and with more than one informed player (Potters and Tijs, 1990), the information is unique. In the first case, only one firm has the information, and in the second, more than one player has that same information.

Like Potters and Tijs (1990), we consider that more than one player has information. However, the information that each player has is not the same, but complementary.

Similarly to Muto et al. (1989), the aim of this chapter is to analyze the cooperative behaviour of firms, faced with the introduction of some new technology, which is essential for manufacturing a new product. However, now we ask what happens if the new product needs more than one new technology to manufacture it. In this chapter, this situation, which implies that more than one informed player is needed to produce the good, is considered.

Let us consider a simple example [1] : the production of waterproof books. Two technologies are needed to produce these: one for the paper and another for the suitable ink. Thus, we need two players, two patent holders, in order to be informed and be able to develop the new product.

Furthermore, we assume that the market for waterproof books is the whole of Europe, which is divided into submarkets (each corresponding, for instance, to a set of European regions). The firms or

[1] A recent example (Source: South China Morning Post, 2014/05/17): "Apple and Google have declared a ceasefire in their intellectual property wars. The two Silicon Valley technology giants said they are dropping lawsuits against one another and will work together to reform patent law."

group of firms have the right and possibility to enter one or another submarket. For each submarket, the maximal profit which can be achieved by producing and selling waterproof books is known.

In this new situation, the information is divided into several parts (or technologies) and each patent holder initially possesses just one of the parts. Let us call *clan* the finite set of patent holders. As we assume perfect patent protection and also that the production of the new commodity needs all those patents, all the members in the clan together may monopolize every submarket the clan has access to. Nevertheless, no single informed player, has the technology required to produce the new commodity.

Each firm or group of firms has the possibility to manufacture the new product if they have all the technological information. They can obtain maximum benefit in the submarket they have access to only if they know the new technologies. By sharing these technologies with other firms (licensing), the clan may access and make a profit indirectly in other submarkets where the clan has no access by itself. Therefore, faced with this situation, cooperation is beneficial.

Formally, such an information market where complementary pieces of information are distributed among more than one player, is defined by

$$\gamma = (N, C, (r_T)_{T \subseteq N, T \neq \emptyset}),$$

where $N = \{1, 2, \ldots, n\}$ is the set of firms, and $C \subseteq N$ is the clan, i.e.

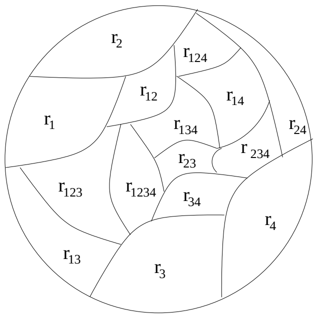

Figure 2.1: *Market partition (example with* $N = \{1, 2, 3, 4\}$*). For each submarket in the partition, only agents of a given coalition* $T \subseteq N$ *have access to and* $r_T \in \mathbb{R}_+$ *stands for the maximal attainable profit in that submarket.*

the set of patent holders. Further, Figure 2.1[2] shows the partition of the consumers market into submarkets. For each submarket, there is a set of firms $T \subseteq N$ who are the only ones able to access this submarket. Muto et al. (1989) named M_T the submarket where only firms in $T \subseteq N$ have access to it. Since this notation becomes superfluous here, we don't use it. The maximal profit obtainable in a submarket controlled by $T \subseteq N$ (whenever the information is available) is $r_T \in \mathbb{R}_+$.

It seems clear the convenience of cooperation among firms. The question is to which firms the informed players will sell their license

[2]For simplicity of notation, we write $r_{i_1 i_2 \ldots i_k}$ instead of $r_{\{i_1, i_2, \ldots, i_k\}}$.

rights and at what price.

Cooperation among players in information markets was also studied by Slikker et al. (2003) and Brânzei et al. (2001). They followed Aumann (1999) in assuming that players do not have perfect information on the true state of the world. They consider that the outcome of the decision that any player makes depends on the true state of the world. In the first case, different players have to make decisions and sharing their information might increase joint profits. In the second case, only one action taker can improve its action choices by gathering information from some players who are more informed about the situation.

This chapter is organized as follows. The next section presents concepts on cooperative game theory that will be referred to throughout the chapter. In section 2.3, the cooperation in information markets with more than one player owning part of the information is considered. The corresponding cooperative game with side payments is defined. It is named *clan information market game* (CIG). Muto et al. (1988) showed that the class of information market games is generalized by the class of big boss games, where the presence of the big boss is necessary for a coalition to attain any profit. When more than one agent is indispensable, Potters et al. (1989) introduced the class of clan games. In a short note, Tijs (1990) introduced clan information market games as an example of clan games with no further development of the model. Clearly, every clan information market game is a clan game, although the opposite is no longer true. The existence of population monotonic allocation schemes (PMAS) is studied in section 2.4. Section 2.5 shows that the τ-value of a clan information market game also has some population monotonicity property: it yields a bi-mas. Finally, section 2.6 characterizes the class of clan information market games and provides conditions on the market, under which the Shapley value belongs to the core.

2.2 Preliminaries

This section is devoted to introducing some basic definitions from classical cooperative game theory. We will use these conceps throughout the chapter.

A *cooperative game (with side payments)* with player set N is a map $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Here, $v(S)$ is the maximal profit obtainable by the coalition $S \subseteq N$, without the help of any player outside coalition S.

A cooperative game v is called *monotonic* if $v(S) \leq v(T)$ for any two coalitions $S, T \subseteq N$ with $S \subseteq T$. And v is called *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for any disjoint pair of coalitions S and T.

Shapley (1971) introduced the notion of convexity. A game (N, v) is *convex* if its characteristic function, $v : 2^N \to \mathbb{R}$, satisfies any of the following two equivalent conditions:

(i) $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$.

(ii) $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $i \in N$ and all $S, T \subseteq N$ such that $S \subseteq T \subseteq N \backslash \{i\}$.

So, from condition (ii), we can say that a game is convex if it satisfies the property of non-decreasing marginal returns when coalitions grow larger. A game (N, v) is *concave* if the oppossite equation holds, i.e. $v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T)$ for all $i \in N$ and all $S, T \subseteq N$ such that $S \subseteq T \subseteq N \backslash \{i\}$.

A game (N, v) is *simple* if $v(S) \in \{0, 1\}$ for all $S \subseteq N$ and $v(N) = 1$. A simple game (N, v) is *proper* if, and only if, $v(S) + v(N \backslash S) \leq 1$ for all $S \subseteq N$. We denote $v = u_{S_1, S_2, \ldots, S_k}$ the simple game with minimal winning coalitions $S_1, S_2, \ldots, S_k$, i.e. $u_{S_1, S_2, \ldots, S_k}(S) = 1$ if $S_i \subseteq S$ for some $S_i \in \{S_1, S_2, \ldots, S_k\}$, and $u_{S_1, S_2, \ldots, S_k}(S) = 0$ otherwise. If the nonempty coalition $T \subseteq N$ is the only minimal winning coalition, the game (N, u_T) is called the *unanimity game*. The *dual* of a game (N, v) is denoted by (N, v^*) and is defined by $v^*(S) = v(N) - v(N \backslash S)$ for all $S \subseteq N$.

The theory of cooperative games offers some solution concepts. The *set of imputations* is the set of efficient and individually rational allocations, i.e. $I(v) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N\}$. Further, Gillies (1959) added the coalitional rationality and defined the *core* of a game (N, v) as the set $C(v) = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N\}$. Note that if a core allocation x is proposed, then no coalition S has an incentive to split off from the grand coalition N.

The core is said to be *stable* if for any imputation $y \notin C(v)$, there is a core allocation $x \in C(v)$ and a nonempty coalition $S \subseteq N$ such that $x_i > y_i$ for all $i \in S$ and $x(S) \leq v(S)$, where $x(S) = \sum_{i \in S} x_i$.

A game (N, v) is said to be *balanced* if it has a nonempty core, while it is said to be *totally balanced* if the core of every subgame is nonempty, where the subgame corresponding to some coalition $T \subseteq N$, $T \neq \emptyset$, is the game (T, v_T) with $v_T(S) = v(S)$ for all $S \subseteq T$.

Sprumont (1990) introduced population monotonic allocation schemes. Such schemes provide an allocation vector for any coalition. Given a game (N, v), the table $X = (x_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ is said to be a *population monotonic allocation scheme* (PMAS for short) if the following two conditions hold:

(i) efficiency: for all $S \in 2^N \setminus \{\emptyset\}$, we have $\sum_{i \in S} x_{S,i} = v(S)$;

(ii) monotonicity: for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subseteq T$, and for all $i \in S$, we have $x_{S,i} \leq x_{T,i}$.

Note that a PMAS provides a core element for every coalition in the corresponding subgame in a monotonic way.

Finally, we recall three point solution concepts: the τ-value, the nucleolus and the Shapley value.

The *τ-value* (Tijs, 1981) of a game is essentially a compromise value between an upper bound payoff vector and a lower bound payoff vector. Let (N, v) be a game and let the vector $M(v) \in \mathbb{R}^N$ be such that its coordinates are the marginal contribution of each player to the grand coalition, i.e. $M_i(v) = v(N) - v(N \backslash \{i\})$, for all $i \in N$. The vector $M(v)$ is called the *utopia vector* and each $M_i(v)$ can be regarded as a maximal payoff (not always attainable) that player i can expect to obtain in the core of the cooperative game.

By using the utopia vector, we can now compute what remains for player $i \in N$ when coalition S forms, $i \in S$, and all players in $S \backslash \{i\}$ are paid their utopia payoff. The remainder for player i, $R^v(S, i)$ is defined by $R^v(S, i) = v(S) - \sum_{j \in S \backslash \{i\}} M_j(v)$.

Now, the vector $m(v) \in \mathbb{R}^N$ is defined by $m_i(v) = \max_{S: i \in S} \{R^v(S, i)\}$ for all $i \in N$. The vector $m(v)$ is called the *minimal rights vector*. Note that player $i \in N$ can be guaranteed the payoff $m_i(v)$ by offering the members of a suitable coalition (the one where the maximum is achieved) their utopia payoffs.

Let us now consider a coalitional game (N, v) such that $m(v) \neq M(v)$, $\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)$ and $m_i(v) \leq M_i(v)$ for all $i \in N$. Then, the *τ-value* of that game is the unique efficient payoff vector on the line segment between $m(v)$ and $M(v)$. Formally,

$\tau(v) = \lambda m(v) + (1-\lambda)M(v)$, where $\lambda \in [0,1]$ is unique satisfying $\sum_{i\in N} \tau_i(v) = v(N)$.

The nucleolus was introduced by Schmeidler (1969). This solution concept is based on the excess of a coalition S at $x \in \mathbb{R}^N$ in the game v. For any $x \in \mathbb{R}^N$ and $S \subseteq N$ the *excess* of the coalition S is $e(x,S) = v(S) - x(S)$. It measures the gain or loss of the coalition S if its members depart from the allocation x in order to form their own coalition. Further, for any imputation x, let us define the vector $\theta(x) \in \mathbb{R}^{2^n-2}$ of excesses of all non-trivial coalitions at x, arranged in non-increasing order. That is to say, for all $k \in \{1, 2, \ldots, 2^n - 2\}$, $\theta(x)_k = e(S_k, x)$, where $\{S_1, S_2, \ldots, S_k, \ldots, S_{2^n-2}\}$ is the set of all nonempty coalitions in N different from N, and $e(S_k, x) \geq e(S_{k+1}, x)$.

Then the *nucleolus* of the game (N, v) is the imputation $\mu(v)$ which minimizes $\theta(x)$ with regard to the lexicographic order over the set of imputations: $\theta(\mu(v)) \leq_L \theta(x)$, for all $x \in I(v)$.

The last point solution considered in this section is the *Shapley value* (Shapley, 1953). According to the Shapley value $\phi(v)$ of the game (N, v), the value of player $i \in N$ is $\phi_i(v) = \sum_{S\subseteq N\setminus\{i\}} \alpha(S)(v(S\cup \{i\}) - v(S))$ where $\alpha(S) = \frac{s!(n-s-1)!}{n!}$, $s = |S|$ and $n = |N|$. Some properties of the Shapley value are efficiency, i.e. $\sum_{i\in N} \phi_i(v) = v(N)$, and linearity, i.e. $\phi(v+w) = \phi(v) + \phi(w)$ and $\phi(av) = a\phi(v)$. The Shapley value may lie outside the core of the game, but if the game is convex, then the Shapley value is a core allocation.

If we consider a unanimity game u_T and its dual u_T^*, the Shapley value can be easily calculated as follows

$$\phi(u_T) = \phi(u_T^*) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$

2.3 Clan information market games

A *clan information market*, that is an information market with more than one player owning a part of the information, consists of

$$\gamma = (N, C, (r_T)_{T \subseteq N, T \neq \emptyset})$$

where $N = \{1, 2, ..., n\}$ is the set of players and $C = \{1, 2, ..., c\} \subseteq N$ is the set of players in the clan, with $0 < c \leq n$.

For each $T \in 2^N \backslash \{\emptyset\}$, $r_T \geq 0$ (see Figure 2.1) is the maximal profit obtainable from the submarket, which only the firms of T have access to.

From such an information market, we define a cooperative game with side payments in the same way as Muto et al. (1989) did for information market games.

Now, when considering our information market, for each coalition S not containing the fully informed clan C, i.e. $C \nsubseteq S$, the value is $v(S) = 0$.

If the clan C, with the whole information, belongs to the coalition S, then the firms in S can produce and sell to each submarket to which at least one of the firms of S has access. Then, they can earn the profit from all these submarkets. That is to say, $v(S) = \sum_{T\cap S\neq\emptyset} r_T$ if $C \subseteq S$.

So, we can define the corresponding cooperative game associated with a given clan information market as follows.

Definition 2.3.1. *Let $\gamma = (N, C, (r_T)_{T\subseteq N, T\neq\emptyset})$ be a clan information market. Then the corresponding clan information market game (N, v^γ), $v^\gamma : 2^N \to \mathbb{R}$, is given by*

$$v^\gamma(S) = \begin{cases} \displaystyle\sum_{T\cap S\neq\emptyset} r_T & \text{if} \quad C \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

Firstly, we show below that the above-defined game is monotonic and superadditive.

Proposition 2.3.2. *Let $\gamma = (N, C, (r_T)_{T\subseteq N, T\neq\emptyset})$ be a clan information market. Then, v^γ is monotonic and superadditive.*

Proof. First we show v^γ is monotonic. Let $R \subseteq S \subseteq N$ with $C \subseteq S$. Then,

$$v^\gamma(S) = \sum_{T\cap S\neq\emptyset} r_T = \sum_{T\cap R\neq\emptyset} r_T + \sum_{\substack{T\cap S\neq\emptyset \\ T\cap R=\emptyset}} r_T \geq v^\gamma(R),$$

since $r_T \geq 0$ for all $T \subseteq N$. Note that $v^\gamma(S) = v^\gamma(R) = 0$ if $C \nsubseteq S$.

Now we show v^γ is superadditive. Let $S, R \subseteq N$ with $S \cap R = \emptyset$ and $C \subseteq S \cup R$. Then,

$$v^\gamma(S \cup R) = \sum_{T\cap(S\cup R)\neq\emptyset} r_T = \sum_{(T\cap S)\cup(T\cap R)\neq\emptyset} r_T \geq$$
$$\max\{0, \sum_{T\cap S\neq\emptyset} r_T, \sum_{T\cap R\neq\emptyset} r_T\}$$

On the other hand, since $S \cap R = \emptyset$ and $C \subseteq S \cup R$, we just consider the following three cases:

i) if $C \subseteq S$, then $C \nsubseteq R$ and we have $v^\gamma(S) + v^\gamma(R) = \sum_{T\cap S\neq\emptyset} r_T$

ii) if $C \subseteq R$, then $C \nsubseteq S$ and we have $v^\gamma(S) + v^\gamma(R) = \sum_{T\cap R\neq\emptyset} r_T$

iii) if $C \nsubseteq S$ and $C \nsubseteq R$, then $v^\gamma(S) + v^\gamma(R) = 0$.

In all cases we obtain $v^\gamma(S \cup R) \geq v^\gamma(S) + v^\gamma(R)$ □

A cooperative game $v : 2^N \to \mathbb{R}$ is called a clan game (Potters et al., 1989) if the following holds:

(1) $v \geq 0$ and $M_i(v) = v(N) - v(N\backslash\{i\}) \geq 0$ for all $i \in N$, and

(2) there is a nonempty coalition $C \subseteq N$ such that

  (a) $v(S) = 0$ if $C \nsubseteq S$ (clan property) and

  (b) $v(N) - v(S) \geq \sum_{i\in N\backslash S} M_i(v)$ if $C \subseteq S$ (union property)

Next, we show that every clan information market game is a clan game, although the opposite inclusion does not hold.

Proposition 2.3.3. *Every clan information market game is a clan game.*

Proof. Let v^γ be a clan information market game. Now we check that v^γ fits the clan game definition.

(1) $v^\gamma \geq 0$ since $r_T \geq 0$ for all $T \subseteq N$. Moreover, $M_i(v^\gamma) = v^\gamma(N) - v^\gamma(N \backslash \{i\}) = r_i \geq 0$ for all $i \in N \backslash C$, and $M_i(v^\gamma) = v^\gamma(N) - v^\gamma(N \backslash \{i\}) = v^\gamma(N) \geq 0$ for all $i \in C$.

(2) There is a nonempty coalition $C \subseteq N$, the one formed by the informed players, such that

   (a) $v^\gamma(S) = 0$ if $C \nsubseteq S$ (by definition).

   (b) Let $S \subseteq N$ such that $C \subseteq S$. Then,

$$v^\gamma(N) - v^\gamma(S) = \sum_{T \subseteq N \backslash S} r_T \geq \sum_{i \in N \backslash S} r_i = \sum_{i \in N \backslash S} M_i(v^\gamma)$$

□

Remark 2.3.4. *A clan game is not necessarily a clan information market game since it is not necessarily monotonic.*

After this remark, a question arises. Does the class of clan information market games coincide with the class of monotonic clan games? The answer is negative as the following counterexample shows.

Example 2.3.5. *Let* (N, v) *be the following monotonic game with* $N = \{1, 2, 3, 4\}$*,*

$$\begin{array}{cccc} & v(\{1,2\}) = 4 & v(\{1,2,3\}) = 8 & \\ v(\{1\}) = 1 & v(\{1,3\}) = 4 & v(\{1,2,4\}) = 7 & v(N) = 10 \\ & v(\{1,4\}) = 3 & v(\{1,3,4\}) = 7 & \end{array}$$

$$\text{and} \quad v(S) = 0 \text{ for all } S \subseteq N \backslash \{1\}.$$

It can be checked that v *is a clan game with clan* $C = \{1\} \subseteq N = \{1, 2, 3, 4\}$*.*

However, it is not a clan information market game. If v *were a clan information market game, then* $r_2 = M_2(v) = 3$*,* $r_3 = M_3(v) = 3$ *and* $r_4 = M_4(v) = 2$*. Furthermore,*

(i) *From* $v(N) - v(\{1, 4\}) = 7$ *and* $v(N) - v(\{1, 4\}) = r_2 + r_3 + r_{23}$*, we have* $r_{23} = 1$*.*

(ii) *From* $v(N) - v(\{1, 3\}) = 6$ *and* $v(N) - v(\{1, 3\}) = r_2 + r_4 + r_{24}$*, we have* $r_{24} = 1$*.*

(iii) *From* $v(N) - v(\{1, 2\}) = 6$ *and* $v(N) - v(\{1, 2\}) = r_3 + r_4 + r_{34}$*, we have* $r_{34} = 1$*.*

Finally, since $9 = v(N) - v(\{1\}) = r_2 + r_3 + r_4 + r_{23} + r_{24} + r_{34} + r_{234}$ *and* r_{234} *should be non-negative, we conclude that* v *is not a clan information market game.*

As a consequence of Proposition 2.3.3, many of the results given by Potters et al. (1989) allow us to provide results in terms of the market data for clan information market games. Specifically, it allows us to describe the core of the game, and determine under which conditions it is stable and the game is convex. It also allows us to determine the nucleolus from the market data.

Certainly, the core of a clan information market game is nonempty. Moreover, it can be easily described as the following corollary of Proposition 2.3.3 shows.

Corollary 2.3.6. *Let $\gamma = (N, C, (r_T)_{T \subseteq N, T \neq \emptyset})$ be a clan information market. The core of v^γ is*

$$C(v^\gamma) = \left\{ x \in \mathbb{R}^n_+ : x(N) = v^\gamma(N) \ \textit{ and } \ 0 \leq x_i \leq r_i \ \textit{ for all } \ i \in N \backslash C \right\}$$

Proof. With Proposition 2.3.3 and Potters et al. (1989), taking into account that $M_i(v^\gamma) = r_i$ for all $i \in N \backslash C$. □ □

Note that the set $\{x \in \mathbb{R}^n_+ : x(N) = v^\gamma(N)$ and $0 \leq x_i \leq r_i$ for all $i \in N \backslash C\}$ is not empty since $r_i \geq 0$ for all $i \in N$ and $v^\gamma(N) \geq \sum_{i \in N \backslash C} r_i$.

Corollary 2.3.7. *Clan information market games are balanced games.*

Corollary 5.3.1 provides a nice expression of the core of a clan information market game. The bargaining set is another well-known set solution concept for cooperative games (see Aumann and Maschler, 1964).

Potters et al. (1988) proved that any cooperative game (N, v) with the properties: (i) there is a player $i_0 \in N$ such that $v(S) = 0$ if $i_0 \notin S$, and (ii) for all coalitions $S \subseteq N$ we have $0 \leq v(S) \leq v(N)$; is such that the bargaining set $\mathcal{M}(v)$ equals the core $C(v)$. Since clan information market games satisfy both conditions, the bargaining set and the core coincide for this class of games. So, the bargaining set of a clan information market game is

$$\begin{aligned}\mathcal{M}(v) &= C(v) \\ &= \left\{x \in \mathbb{R}^n_+ \,|\, x(N) = v(N) \text{ and } 0 \leq x_i \leq r_i \text{ for all } i \in N \backslash C\right\}.\end{aligned}$$

The core of any convex game is nonempty. However, the opposite does not hold. In general, a clan information market game is not convex as the following example shows.

Example 2.3.8. *Let $\gamma = (N, C, (r_T)_{T \subseteq N, T \neq \emptyset})$ be a clan information market with $N = \{1, 2, 3, 4\}$, clan $C = \{1, 2\}$ and $r_1 = r_2 = r_3 = r_4 = r_{34} = 1$ and otherwise $r_T = 0$. Let (N, v^γ) be the corresponding game.*

Since $v^\gamma(\{1, 2, 4\}) - v^\gamma(\{1, 2\}) = r_4 + r_{34} = 2$ and $v^\gamma(N) - v^\gamma(\{1, 2, 3\}) = r_4 = 1$, the marginal return of player $i = 4$ decreases when coalition grows larger. Therefore, v^γ is not convex.

The next proposition characterizes convexity for clan information market games in terms of the data of the market.

Proposition 2.3.9. *Let $\gamma = (N, C, (r_T)_{T \subseteq N, T \neq \emptyset})$ be a clan information market.*

Then, the following statements are equivalent:

(i) v^γ is convex

(ii) $r_T = 0$ for all coalitions $T \subseteq N \backslash C$ with $|T| \geq 2$

(iii) the core $C(v^\gamma)$ is stable

The reader will easily check that it follows from Theorem 5.1 in Potters et al. (1989) once its statement (2) is proved to be equivalent to our statement (ii) for clan information market games.

Note that, from Proposition 2.3.9, the condition under which clan information market games are convex, is quite restrictive. However, clan information market games always satisfy some concavity conditions.

Remark 2.3.10. *Notice that the concavity condition $v(P \cup \{i\}) - v(P) \geq v(Q \cup \{i\}) - v(Q)$ for $P \subseteq Q$ is satisfied whenever $C \subseteq P$.*

In fact, $v(P \cup \{i\}) - v(P) = \sum_{\substack{T: i \in T \\ T \backslash \{i\} \subseteq N \backslash P}} r_T \geq \sum_{\substack{T: i \in T \\ T \backslash \{i\} \subseteq N \backslash Q}} r_T = v(Q \cup \{i\}) - v(Q)$ *since* $N \backslash Q \subseteq N \backslash P$*.*

Until now, we have analized several properties of the class of clan information market games and obtained a simple description of the most relevant set solution concepts for these games: the core and the bargaining set. What remains of the chapter deals with the study of point solution concepts and their monotonicity properties when coalitions grow larger.

2.4 A population monotonic allocation scheme

The previous section shows that the core of a clan information market game is not empty. In other words, clan information market games are balanced games. Now we ask whether balancedness also holds for all the subgames.

Proposition 2.4.1. *Any subgame of a clan information market game is again a clan information market game.*

Proof. Let $\gamma = (N, C, (r_T)_{T \subseteq N, T \neq \emptyset})$ be a clan information market and (N, v^γ) its associated clan information market game. Take $R \subseteq N$ and v_R^γ the corresponding subgame. That is to say, (R, v_R^γ) is defined by $v_R^\gamma(S) = v^\gamma(S)$ for all $S \subseteq R$.

(i) If $C \nsubseteq R$, then $v_R^\gamma(S) = v^\gamma(S) = 0$ for all $S \subseteq R$, which means that v_R^γ is the zero-game. It is the clan information market game corresponding to the market $\gamma' = (R, C', (r'_T)_{T \subseteq R, T \neq \emptyset})$ with $C' = R$ and $r'_T = 0$ for all $T \subseteq R$.

(ii) If $C \subseteq R$, we will prove that (R, v_R^γ) is the clan information market game corresponding to the market $\gamma' = (R, C', (r'_T)_{T\subseteq R, T\neq\emptyset})$ with $C' = C$ and, for all $T \subseteq R$, $T \neq \emptyset$,

$$r'_T = \begin{cases} \sum\limits_{A\subseteq N\setminus C} r_{T\cup A} & \text{if } \; T \subseteq C \\ \sum\limits_{B\subseteq N\setminus R} r_{T\cup B} & \text{if } \; T \cap C = \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Take any $S \subseteq R$. If $C \nsubseteq S$, $v_R^\gamma(S) = v^{\gamma'}(S) = 0$. If $C \subseteq S$,

$$\begin{aligned} v_R^\gamma(S) = v^\gamma(S) &= \sum_{\substack{Q\subseteq N\\ Q\cap S\neq\emptyset}} r_Q = \sum_{\substack{Q\subseteq N\\ Q\cap S\neq\emptyset\\ Q\cap C\neq\emptyset}} r_Q + \sum_{\substack{Q\subseteq N\\ Q\cap S\neq\emptyset\\ Q\cap C=\emptyset}} r_Q \\ &= \sum_{\substack{T\subseteq C\subseteq R\\ T\cap S\neq\emptyset\\ A\subseteq N\setminus C}} r_{T\cup A} + \sum_{\substack{T\subseteq R\\ T\cap S\neq\emptyset\\ T\cap C=\emptyset\\ B\subseteq N\setminus R}} r_{T\cup B} \\ &= \sum_{\substack{T\subseteq C\subseteq R\\ T\cap S\neq\emptyset}} \sum_{A\subseteq N\setminus C} r_{T\cup A} + \sum_{\substack{T\subseteq R\\ T\cap S\neq\emptyset\\ T\cap C=\emptyset}} \sum_{B\subseteq N\setminus R} r_{T\cup B} + 0 \\ &= \sum_{\substack{T\cap S\neq\emptyset\\ T\subseteq C}} r'_T + \sum_{\substack{T\cap S\neq\emptyset\\ T\cap C=\emptyset}} r'_T + \sum_{\substack{T\cap S\neq\emptyset\\ T\cap C\neq\emptyset\\ T\nsubseteq C}} r'_T \\ &= \sum_{T\cap S\neq\emptyset} r'_T = v^{\gamma'}(S) \end{aligned}$$

where the fourth equality follows by taking $T = Q\cap C$ and $A = Q\setminus C$ in the first summation, which implies that $T \cap S \neq \emptyset$ since $C \subseteq S$ and that $A \subseteq N\setminus C$ since $Q \subseteq N$. On the other hand, the second summation follows by taking $T = Q \cap R$ and $B = Q \backslash R$, which implies that $T \cap S \neq \emptyset$ since $S \subseteq R$ and $S \cap Q \neq \emptyset$, and that $T \cap C = \emptyset$ since $Q \cap C = \emptyset$, and also that $B \subseteq N \backslash R$ since $Q \subseteq N$. $\square$ $\square$

Since any subgame of a clan information market game is again a clan information market game, every such subgame is balanced and the next corollary follows.

Corollary 2.4.2. *Clan information market games are totally balanced games.*

We already know that the core of a clan information market game is nonempty. Moreover, each subgame has a nonempty core. The question now is whether there is a PMAS for any clan information market game.

Voorneveld et al. (2002) proved the existence of a PMAS for *total clan games*. A game (N, v) is a *total clan game*, with clan $C \in 2^N \backslash \{\emptyset, N\}$, if (S, v_S) is a clan game (with clan C) for every coalition $S \subseteq N$ such that $C \subseteq S$.

Notice that a clan information market game is a total clan game. Indeed, by the proof of Proposition 2.4.1, part (ii), if $C \subseteq S$, any subgame v_S^γ is again a clan information market game and it is a clan game by Proposition 2.3.3. Thus, clan information market games with $C \neq N$ are total clan games and, by Voorneveld et al. (2002), the existence of a PMAS is guaranteed. Moreover, a clan information market game with $C = N$ trivially has PMAS since all subgames are zero games.

Furthermore, our aim is to show a particular PMAS that is described in terms of the data of the market. In this particular PMAS, the non-clan members get nothing while the clan members share equally the profit corresponding to the submarket they have access to.

Proposition 2.4.3. *Let $\gamma = (N, C, (r_T)_{T\subseteq N, T\neq\emptyset})$ be a clan information market and let $X = (x_{R,i})_{R\in 2^N, i\in R}$ be a table such that*

(i) if $C \nsubseteq R$, $x_{R,i} = 0$ for all $i \in R$,

(ii) if $C \subseteq R$,

$$x_{R,i} = \begin{cases} \displaystyle\sum_{\substack{T\subseteq C\\ i\in T}} \sum_{A\subseteq N\setminus C} \frac{r_{T\cup A}}{|T|} + \sum_{\substack{T\subseteq R\setminus C\\ T\neq\emptyset}} \sum_{B\subseteq N\setminus R} \frac{r_{T\cup B}}{|C|} & \text{if } i \in C \\ 0 & \text{if } i \in R\setminus C \end{cases}$$

Then, X is a PMAS of v^γ.

Proof. First we show the efficiency for each $R \subseteq N$. When $C \nsubseteq R$, easily we have that $x_{R,i} = 0$ for all $i \in R$ is an efficient allocation.

Let us now consider $R \subseteq N$ such that $C \subseteq R$, and show the efficiency of $(x_{R,i})_{i\in R}$. Cleary, if $C \subseteq R$, we have

$$\begin{aligned}
x(R) &= \sum_{i\in R} x_{R,i} = \sum_{i\in C} x_{R,i} \\
&= \sum_{i\in C} \sum_{\substack{T\subseteq C\\ i\in T}} \sum_{A\subseteq N\setminus C} \frac{r_{T\cup A}}{|T|} + \sum_{i\in C} \sum_{\substack{T\subseteq R\setminus C\\ T\neq\emptyset}} \sum_{B\subseteq N\setminus R} \frac{r_{T\cup B}}{|C|} \\
&= \sum_{\substack{T\subseteq C\subseteq R\\ T\neq\emptyset}} \sum_{A\subseteq N\setminus C} |T|\frac{r_{T\cup A}}{|T|} + \sum_{\substack{T\subseteq R\setminus C\\ T\neq\emptyset}} \sum_{B\subseteq N\setminus R} |C|\frac{r_{T\cup B}}{|C|} \\
&= \sum_{\substack{Q\cap C\neq\emptyset\\ Q\subseteq N}} r_Q + \sum_{\substack{Q\cap C=\emptyset\\ Q\cap R\neq\emptyset\\ Q\subseteq N}} r_Q \\
&= \sum_{\substack{Q\cap R\neq\emptyset\\ Q\subseteq N}} r_Q = v^\gamma(R) = v^\gamma_R(R)
\end{aligned}$$

Secondly we show the monotonicity of the scheme. Note that $x_{R,i} \geq 0$ for all $R \in 2^N$ and for all $i \in R$, since $r_T \geq 0$ for all $T \subseteq N$. Let $P, Q \in 2^N$ with $P \subseteq Q$. We show that $x_{P,i} \leq x_{Q,i}$ for all $i \in P$. Let us consider three cases:

(i) $C \not\subseteq Q$. In this case we have that $C \not\subseteq P$ and $x_{P,i} = x_{Q,i} = 0$ for all $i \in P$.

(ii) $C \subseteq Q$ and $C \not\subseteq P$. In this case we have that $x_{P,i} = 0 \leq x_{Q,i}$ for all $i \in P$.

(iii) $C \subseteq P$. If $i \in P \backslash C$, then $x_{P,i} = 0 \leq x_{Q,i}$. If $i \in C$, then

$$x_{Q,i} = \sum_{\substack{T \subseteq C \subseteq Q \\ i \in T}} \sum_{A \subseteq N \backslash C} \frac{r_{T \cup A}}{|T|} + \sum_{\substack{T \subseteq Q \backslash C \\ T \neq \emptyset}} \sum_{B \subseteq N \backslash Q} \frac{r_{T \cup B}}{|C|}$$

$$= \sum_{\substack{T \subseteq C \subseteq P \\ i \in T}} \sum_{A \subseteq N \backslash C} \frac{r_{T \cup A}}{|T|} + \sum_{\substack{T \subseteq P \backslash C \\ T \neq \emptyset}} \sum_{B \subseteq N \backslash P} \frac{r_{T \cup B}}{|C|} + \sum_{\substack{T \subseteq Q \backslash P \\ T \neq \emptyset}} \sum_{B \subseteq N \backslash Q} \frac{r_{T \cup B}}{|C|}$$

$$= x_{P,i} + \sum_{\substack{T \subseteq Q \backslash P \\ T \neq \emptyset}} \sum_{B \subseteq N \backslash Q} \frac{r_{T \cup B}}{|C|} \geq x_{P,i}$$

□

Note that the PMAS given above is such that the payoff of all non-clan agents is zero. Below, we will consider another notion of population monotonicity, the bi-monotonic allocation scheme, which allows for positive payoffs to non-clan agents.

2.5 A bi-monotonic allocation scheme

In this section we focus on the τ-value and we will see it always belongs to the core of the clan information market game.

The expression of the τ-value of a clan information market game with $|C| = 1$ follows from Muto et al. (1989). Let us see what happens when more than one player belongs to the clan.

Proposition 2.5.1. *Let $\gamma = (N, C, (r_T)_{T \subseteq N, T \neq \emptyset})$ be a clan information market with $|C| \geq 2$. Then, the τ-value of the associated game*

v^γ is

$$\tau_i(v^\gamma) = \begin{cases} \frac{v^\gamma(N)}{|C|v^\gamma(N)+\sum_{j\notin C} r_j}\, v^\gamma(N) & \text{if } i \in C \\ \\ \frac{v^\gamma(N)}{|C|v^\gamma(N)+\sum_{j\notin C} r_j}\, r_i & \text{if } i \notin C \end{cases}$$

Moreover, $\tau(v^\gamma) \in C(v^\gamma)$.

Proof. If (N, v^γ) is a clan information market game with clan $C \subseteq N$, then $M_i(v^\gamma) = v^\gamma(N)$ if $i \in C$ and $M_i(v^\gamma) = r_i$ if $i \notin C$.

Secondly, we compute $m_i(v^\gamma)$. Let us first consider a player $i \notin C$. Then, if $C \cap S \neq \emptyset$, since $v^\gamma(S) \leq v^\gamma(N)$ and $M_k(v^\gamma) = v^\gamma(N)$ for each $k \in C$, $v^\gamma(S) - \sum_{j\in S\setminus\{i\}} M_j(v^\gamma) \leq 0$. If $C \cap S = \emptyset$, since $v^\gamma(S) = 0$, $v^\gamma(S) - \sum_{j\in S\setminus\{i\}} M_j(v^\gamma) \leq 0$. This implies $m_i(v^\gamma) = 0$ for all $i \notin C$.

Next we consider a player $i \in C$. If $C \subseteq S$, since $M_k(v^\gamma) = v^\gamma(N)$ for each $k \in C$ and $|C| \geq 2$, $v^\gamma(S) - \sum_{j\in S\setminus\{i\}} M_j(v^\gamma) \leq 0$. If $C \nsubseteq S$, since $v^\gamma(S) = 0$, $v^\gamma(S) - \sum_{j\in S\setminus\{i\}} M_j(v^\gamma) \leq 0$.

Finally, if S consists of only one player, then $v^\gamma(S) - \sum_{j\in S\setminus\{i\}} M_j(v^\gamma) = v^\gamma(\{i\}) = 0$. Thus, $m_i(v^\gamma) = 0$ for any $i \in N$.

Therefore, the τ-value is proportional to the marginal contributions. Since the sum of all marginal contributions is $\sum_{i\in N} M_i(v^\gamma) = |C|v^\gamma(N) + \sum_{j\notin C} r_j$ and the τ-value is efficient,

$$\tau_i(v^\gamma) = \begin{cases} \frac{v^\gamma(N)}{|C|v^\gamma(N)+\sum_{j\notin C} r_j}\, v^\gamma(N) & \text{if } i \in C \\ \\ \frac{v^\gamma(N)}{|C|v^\gamma(N)+\sum_{j\notin C} r_j}\, r_i & \text{if } i \notin C \end{cases}$$

Moreover, with Corollary 5.3.1, $\tau(v^\gamma) \in C(v^\gamma)$. Clearly, $\tau(v^\gamma)$ is efficient and $\tau_i(v^\gamma) \leq r_i$ for all $i \notin C$ since $0 \leq \frac{v^\gamma(N)}{|C|v^\gamma(N)+\sum_{j\notin C} r_j} \leq 1$.

□

This proposition can be extended to arbitrary clan games (Potters et al., 1989) simply by replacing r_i with the marginal contribution $M_i(v)$ for all $i \notin C$.

There is another point solution concept, the nucleolus, which always exists and it belongs to the core whenever the core is nonempty. Potters et al. (1989) provided a way for finding the nucleolus of a clan game. Their expression in terms of the market data of a clan information market game, which is a subclass of the class of clan games, would be the following:

$$\mu_i = \begin{cases} t & \text{if } i \in C \\ \min\{t, \frac{1}{2}r_i\} & \text{if } i \notin C \end{cases}$$

where $t \geq 0$ is the only real number that guarantees efficiency.

Voorneveld et al. (2002) introduced the notion of *bi-monotonic allocation scheme*. Let (N, v) be a monotonic clan game with clan $C \in 2^N \backslash \{\emptyset, N\}$. A *bi-monotonic allocation scheme* (bi-mas for short) for the game is a table $X = (x_{S,i})_{S\in 2^N\backslash\{\emptyset\}, i\in S}$ of real numbers such that

(i) $\sum_{i\in S} x_{S,i} = v(S)$ for each $S \in 2^N \backslash \{\emptyset\}$,

(ii) $x_{S,i} \leq x_{T,i}$ if $S, T \in 2^N \backslash \{\emptyset\}$, $S \subseteq T$ and $i \in S \cap C$,

(iii) $x_{S,i} \geq x_{T,i}$ if $S, T \in 2^N \backslash \{\emptyset\}$, $S \subseteq T$ and $i \in S \backslash C$,

(iv) $(x_{S,i})_{i \in S}$ is a core element of the subgame (S, v) for each coalition $S \in 2^N \backslash \{\emptyset\}$.

Furthermore, they showed that the nucleolus applied to a monotonic clan game and its subgames does not necessarily yield a bi-mas. The following example shows that this is also the case for the τ-value applied to a CIG and its subgames.

Example 2.5.2. *Let (N, v^γ) be a clan information market game with set of players $N = \{1, 2, 3, 4, 5\}$ and clan $C = \{1, 2\}$ such that $r_3 = r_{45} = 1$ and $r_T = 0$ otherwise. Then $v^\gamma(N) = 2$ and the τ-value is $\tau(v^\gamma) = (\frac{2}{5}2, \frac{2}{5}2, \frac{2}{5}1, 0, 0)$. Now, for the subgame v^γ_S with $S = \{1, 2, 3\}$, $v^\gamma_S(S) = 1$ and $\tau(v^\gamma_S) = (\frac{1}{3}1, \frac{1}{3}1, \frac{1}{3}1)$. We can easily check that $\tau_3(v^\gamma) > \tau_3(v^\gamma_S)$ and hence the τ-value applied to all subgames is not a bi-mas.*

However, if we just consider convex CIGs, then the τ-value applied to the game and its subgames yields a bi-mas.

Proposition 2.5.3. *Let $\gamma = (N, C, (r_T)_{T \subseteq N, T \neq \emptyset})$ be a clan information market with $|C| \geq 2$ and $r_T = 0$ for all coalitions $T \subseteq N \backslash C$ with $|T| \geq 2$. Let (N, v^γ) be the corresponding convex CIG. Then the τ-value applied to the game v^γ and its subgames yields a bi-mas.*

Proof. For all $S \subseteq N$ with $C \subseteq S$ and for all $j \in S \backslash C$, by Proposition 2.5.1,

$$\tau_j(v^\gamma) = \frac{v^\gamma(N)}{|C|v^\gamma(N) + \sum_{i \in N \backslash C} r_i} r_j \quad \text{and} \quad \tau_j(v^\gamma_S) = \frac{v^\gamma(S)}{|C|v^\gamma(S) + \sum_{i \in S \backslash C} r'_i} r'_j,$$

where v^γ_S is the subgame with player set S and corresponding clan information market $\gamma' = (S, C, (r'_T)_{T \subseteq S, T \neq \emptyset})$. Next we show that $\tau_j(v^\gamma) \leq \tau_j(v^\gamma_S)$. Since $r_T = 0$ for all coalitions $T \subseteq N \backslash C$ with $|T| \geq 2$, from (2.1), $r'_i = r_i$ for all $i \in N \backslash C$. Thus, we show that $\dfrac{v^\gamma(N)}{|C|v^\gamma(N) + \sum_{i \in N \backslash C} r_i} \leq \dfrac{v^\gamma(S)}{|C|v^\gamma(S) + \sum_{i \in S \backslash C} r_i}$, which is equivalent to proving that $v^\gamma(N) \sum_{i \in S \backslash C} r_i \leq v^\gamma(S) \sum_{i \in N \backslash C} r_i$.

Indeed, as a consequence of Proposition 2.3.9, if the CIG is convex, then $v^\gamma(N) = v^\gamma(S) + \sum_{i \in N \backslash S} r_i$ for all $S \subseteq N$ with $C \subseteq S$, so

$$\begin{aligned} v^\gamma(N) \sum_{i \in S \backslash C} r_i \leq v^\gamma(S) \sum_{i \in N \backslash C} r_i & \\ \iff \sum_{i \in N \backslash S} r_i \cdot \sum_{i \in S \backslash C} r_i &\leq v^\gamma(S) \Big(\sum_{i \in N \backslash C} r_i - \sum_{i \in S \backslash C} r_i \Big) \\ \iff \sum_{i \in N \backslash S} r_i \cdot \sum_{i \in S \backslash C} r_i &\leq v^\gamma(S) \sum_{i \in N \backslash S} r_i \end{aligned}$$

Either $\sum_{i \in N \backslash S} r_i = 0$ and then the last inequality holds, or $\sum_{i \in N \backslash S} r_i > 0$ and we obtain $\sum_{i \in S \backslash C} r_i \leq v^\gamma(S)$ which also always holds. □

Voorneveld et al. (2002), in their Example 1, provided a convex game whose nucleolus applied to the game and its subgames does not yield a bi-mas. This game can be seen as a convex CIG (N, v^γ) with $N = \{1, 2, 3, 4\}$, clan $C = \{1, 2\}$ and where the maximal profit r_T is $r_{123} = 6$, $r_{124} = 99$, $r_{1234} = 5$ and $r_T = 0$ otherwise. Thus, even in a convex CIG, the nucleolus applied to the game and its subgames may not yield a bi-mas.

2.6 The Shapley value

This section is devoted to the Shapley value of a clan information market game. In general, it is not a core allocation. Therefore, it does not yield a PMAS neither a bi-mas. However, after a characterization of the class of clan information market games, we obtain an expression of the Shapley value in terms of the market data.

The class of clan information market games with player set N and clan $C = \{1, \ldots, c\} \subseteq N$ $(CIG^{N,C})$ can be characterized in terms of some simple games and their dual games, similarly to what Muto et al. (1989) did for information market games.

In particular, we consider the simple game $(N, u_{T,\{i\}_{i\in C}})$ where, for all $S \subseteq N$,

$$u_{T,\{i\}_{i\in C}}(S) = \begin{cases} 1 & \text{if } T \subseteq S \text{ or } i \in S \text{ for some } i \in C \\ 0 & \text{otherwise} \end{cases}$$

and its dual $(N, u^*_{T,\{i\}_{i\in C}})$, with

$$u^*_{T,\{i\}_{i\in C}}(S) = u_{T,\{i\}_{i\in C}}(N) - u_{T,\{i\}_{i\in C}}(N \backslash S).$$

Proposition 2.6.1. *$CIG^{N,C}$ is the cone generated over $\mathbb{R}_+$ by u_C and $\{u^*_{T,\{i\}_{i\in C}} \,|\, T \subseteq N\backslash C, T \neq \emptyset\}$. Specifically, for any $\gamma \in CIG^{N,C}$: $v^\gamma = r\, u_C + \sum_{T\subseteq N\backslash C} r_T\, u^*_{T,\{i\}_{i\in C}}$ where $r = \sum_{T\cap C\neq\emptyset} r_T$.*

Proof. Let be $\gamma \in CIG^{N,C}$ and $r = \sum_{T\cap C\neq\emptyset} r_T$. We show that $v^\gamma = r\, u_C + \sum_{T\subseteq N\backslash C} r_T\, u^*_{T,\{i\}_{i\in C}}$.

(i) If $C \nsubseteq S$, then $u_C(S) = 0$ and $C \cap (N\backslash S) \neq \emptyset$ which implies that $u_{T,\{i\}_{i\in C}}(N\backslash S) = 1$ for every $T \subseteq N\backslash C$, $T \neq \emptyset$. Therefore, for every $T \subseteq N\backslash C$, $T \neq \emptyset$, $u^*_{T,\{i\}_{i\in C}}(S) = u_{T,\{i\}_{i\in C}}(N) - u_{T,\{i\}_{i\in C}}(N\backslash S) = 0$. Thus, $(r\, u_C + \sum_{T\subseteq N\backslash C} r_T\, u^*_{T,\{i\}_{i\in C}})(S) = 0 = v^\gamma(S)$.

(ii) If $C \subseteq S$, then $u_C(S) = 1$ and $C \cap (N\backslash S) = \emptyset$. Let us consider $T \subseteq N\backslash C, T \neq \emptyset$. Then, $u_{T,\{i\}_{i\in C}}(N\backslash S) = 0$ if, and only if, $T \nsubseteq N\backslash S$, which is equivalent to $T \cap S \neq \emptyset$. So, $u^*_{T,\{i\}_{i\in C}}(S) = 1$ if, and only if, $T\cap S \neq \emptyset$. Thus, $(r\, u_C + \sum_{T\subseteq N\backslash C} r_T\, u^*_{T,\{i\}_{i\in C}})(S) = r + \sum_{\substack{T\cap S\neq\emptyset \\ T\subseteq N\backslash C}} r_T = \sum_{T\cap S\neq\emptyset} r_T = v^\gamma(S)$.

Secondly, we show that $v = r\, u_C + \sum_{T\subseteq N\backslash C} r_T\, u^*_{T,\{i\}_{i\in C}}$ is a clan information market game. Let $v = \lambda\, u_C + \sum_{T\subseteq N\backslash C} \lambda_T\, u^*_{T,\{i\}_{i\in C}}$ with $\lambda, \lambda_T \geq 0$ for all $T \subseteq N\backslash C$, $T \neq \emptyset$. Then v is the clan information market game with the following market data $r_T = \lambda_T$ if $T \subseteq N\backslash C$ and $\sum_{T:C\cap T\neq\emptyset} r_T = \lambda$.

□

Next, we provide the Shapley value of a clan information market game in terms of the market data.

With Proposition 2.6.1, clan information market games can be expressed as $v^\gamma = r\, u_C + \sum_{T \subseteq N \setminus C} r_T\, u^*_{T,\{i\}_{i\in C}}$. On the other hand, like any game, the monotonic simple game $u_{T,\{i\}_{i\in C}}$ may be linearly decomposed into unanimity games in a unique way. Lange and Kóczy (2013) were interested in voting games which are monotonic and proper simple games. They proved that the decomposition of a voting game (N, v) with set of minimal winning coalitions $\{S_1, \dots, S_m\}$ is given by $v = \sum_{A \subseteq \{S_1,\dots,S_m\}: A \neq \emptyset} (-1)^{|A|-1} u_{\hat{A}}$, where $\hat{A} = \bigcup_{i=1}^{m} A_i$ for any family of coalitions $A = \{A_1, \dots, A_m\}$. It is not difficult to check that their proof also holds for any monotonic simple game, with a nonempty set of minimal winning coalitions, with no need to require properness. Therefore, for any $T \subseteq N \setminus C$,

$$u_{T,\{i\}_{i\in C}} = \sum_{\substack{A \subseteq M^{T,C} \\ A \neq \emptyset}} (-1)^{|A|-1} u_{\hat{A}}, \qquad (2.2)$$

where $M^{T,C} = \{T, \{i\}_{i\in C}\}$ and $\hat{A} = \bigcup_{i=1}^{m} A_i$ for any $A = \{A_1, \dots, A_m\} \subseteq M^{T,C}$.

Then,

$$\begin{aligned}
\phi(v^\gamma) &= r\,\phi(u_C) + \sum_{T\subseteq N\setminus C} r_T\,\phi(u^*_{T,\{i\}_{i\in C}}) \\
&= r\,\phi(u_C) + \sum_{T\subseteq N\setminus C} r_T\,\phi(u_{T,\{i\}_{i\in C}}) \\
&= r\,\phi(u_C) + \sum_{T\subseteq N\setminus C} r_T\,\phi\Big(\sum_{\substack{A\subseteq M^{T,C}\\ A\neq\emptyset}} (-1)^{|A|-1} u_{\hat{A}}\Big) \\
&= r\,\phi(u_C) + \sum_{T\subseteq N\setminus C} r_T\,\Big(\sum_{\substack{A\subseteq M^{T,C}\\ A\neq\emptyset}} (-1)^{|A|-1} \phi(u_{\hat{A}})\Big) \\
&= \tfrac{r}{|C|}\mathbf{1}_{\mathrm{C}} + \sum_{\mathrm{T}\subseteq \mathrm{N}\setminus \mathrm{C}} \mathrm{r_T}\,\Big(\sum_{\substack{A\subseteq M^{T,C}\\ A\neq\emptyset}} (-1)^{|\mathrm{A}|-1} \frac{1}{|\hat{\mathrm{A}}|}\mathbf{1}_{\hat{\mathrm{A}}})\Big)
\end{aligned}$$

Note that the first and the fourth equalities follow from the linearity of the Shapley value, the second one because a game and its dual game have the same Shapley value (Funaki, 1995), the third one by (2.2) and the last one since $\phi(u_T) = \frac{1}{|T|}\mathbf{1}_{\mathrm{T}}$ (Shapley, 1953).

Example 2.6.2. *Let (N, v^γ) be a clan information market game with set of players $N = \{1,2,3,4\}$ and clan $C = \{1,2\}$. Let $r = \sum_{T\cap\{1,2\}\neq\emptyset} r_T$.*

Then, the Shapley value is

$$\begin{aligned}
\phi(v^\gamma) = \;& \tfrac{r}{2}\mathbf{1}_{\{1,2\}} + \\
& r_3(\mathbf{1}_{\{3\}} + \mathbf{1}_{\{1\}} + \mathbf{1}_{\{2\}} - \tfrac{1}{2}\mathbf{1}_{\{1,3\}} - \tfrac{1}{2}\mathbf{1}_{\{2,3\}} - \tfrac{1}{2}\mathbf{1}_{\{1,2\}} + \tfrac{1}{3}\mathbf{1}_{\{1,2,3\}}) + \\
& r_4(\mathbf{1}_{\{4\}} + \mathbf{1}_{\{1\}} + \mathbf{1}_{\{2\}} - \tfrac{1}{2}\mathbf{1}_{\{1,4\}} - \tfrac{1}{2}\mathbf{1}_{\{2,4\}} - \tfrac{1}{2}\mathbf{1}_{\{1,2\}} + \tfrac{1}{3}\mathbf{1}_{\{1,2,4\}}) + \\
& r_{34}(\tfrac{1}{2}\mathbf{1}_{\{3,4\}} + \mathbf{1}_{\{1\}} + \mathbf{1}_{\{2\}} - \tfrac{1}{3}\mathbf{1}_{\{1,3,4\}} - \tfrac{1}{3}\mathbf{1}_{\{2,3,4\}} - \tfrac{1}{2}\mathbf{1}_{\{1,2\}} + \tfrac{1}{4}\mathbf{1}_{\{1,2,3,4\}})
\end{aligned}$$

Furthermore, after the above vectorial operations,
$\phi(v^\gamma) = (\frac{1}{2}r + \frac{1}{3}r_3 + \frac{1}{3}r_4 + \frac{5}{12}r_{34}, \frac{1}{2}r + \frac{1}{3}r_3 + \frac{1}{3}r_4 + \frac{5}{12}r_{34}, \frac{1}{3}r_3 + \frac{1}{12}r_{34}, \frac{1}{3}r_4 + \frac{1}{12}r_{34})$.

Remark 2.6.3. *When a clan information market γ is such that $r_T = 0$ for all $T \subseteq N \backslash C$ with $|T| \geq 2$, the Shapley value is*

$$\phi_i(v^\gamma) = \begin{cases} \frac{1}{|C|}r + \sum_{j \in N \backslash C} \frac{1}{|C|+1} r_j & \text{if } i \in C \\ \\ \frac{1}{|C|+1} r_i & \text{if } i \notin C \end{cases}$$

Thus, since $\frac{1}{|C|+1} r_i < r_i$, it belongs to the core, which we already knew because of the convexity of this game.

Remark 2.6.4. *In case $r_T = 0$ for all T with $|T| \geq 2$ (the game is convex) and $N = C \cup \{i\}$ such that $v^\gamma(N) = r_i$, we have $\phi(v^\gamma) = \tau(v^\gamma) = \mu(v^\gamma)$.*

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Chapter 3

Information interval games

3.1 Introduction

Muto et al. (1989) define a class of cooperative games intended to model the trading of information between one informed firm and other initially not informed firms. The information needed to produce a new commodity is initially only possessed by one firm, the patent holder. The market is divided into several submarkets according to the group of firms which have the possibility to enter this market. The maximum profit attainable in each submarket by producing and selling the new commodity is obtained by the set of firms that control that submarket provided the informed firm has entrance to the submarket. By sharing the information with other firms, the patent holder can also gain profit from submarkets he has no access to. The corresponding cooperative game to this situation is named information market game.

Coalitional games have been extended to coalitional interval games. Branzei et at.(2003) introduce coalitional interval games to handle bankruptcy situations with uncertainty on the claims. These claims are, then, depicted by known bounded intervals of real numbers. They study these situations by using tools of interval analysis (Moore, 1979). Alparslan-Gök et.al (2009) consider selections of cooperative interval games which are classical cooperative games.

The model of information market games was extended by El Obadi and Miquel (2017) by considering that the initial owner of the information is not only one agent but it is a set of agents. In this chapter we consider a different extension of information market games. It is natural to think that the profit attainable in each submarket of an information market may not be known with certainty and this is what we assume in this chapter. So, the class of information market games is extended to information market interval games.

The aim of this chapter consists on providing solutions for sharing the profits obtained in the market where a patent holder is needed for the production of a new commodity with the particularity that the profit attainable in each submarket is uncertain. Such solutions provide an interval of individual payoffs to each player such that the interval profit that the grand coalition could obtain by cooperation, is distributed among the players.

The chapter is organized as follows. After the premilinaries section, in Section 3.3 we define information interval games and we give some properties. Next sections are devoted to different solutions. In Section 3.4 the interval core of the game is considered. Section 3.5 provides the interval Shapley value in terms of the market data and Section 3.6 shows how to find the interval $\mathcal{T}$-value of an information interval game.

3.2 Preliminaries

We start this section with definitions related to the model of information market games of Muto et al. (1989). After that, we add some basic concepts of coalitional games and, in particular, simple games. Later, we recall basic interval calculus (Moore, 1979) and define cooperative interval games. Some solutions for cooperative game theory under interval uncertainty are also considered in this section.

In fact, our model generalizes the one introduced by Muto et al. (1989). They defined an *information market* as the tuple $(N, \{1\}, (r_T)_{T\subset N, T\neq\emptyset})$, where N is the set of players and $r_T \in \mathbb{R}_+$ is the maximum profit obtainable from the submarket to which the firms of T have access to and no other firms.

Given such a market, the corresponding *information market game*

(N, v) is defined by $v(S) = \sum_{T \cap S \neq \emptyset} r_T$ if $\{1\} \subseteq S$ and $v(S) = 0$ if $\{1\} \nsubseteq S$. This is a model of cooperative games.

A *cooperative (coalitional) game* with player set N, (N, v), is a map $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The *subgame* of (N, v) restricted to a coalition $T \subseteq N$, (T, v^T), is the map $v^T : 2^T \to \mathbb{R}$ defined by $v^T(S) = v(S)$ for all $S \subseteq T$. A game (N, v) is *monotonic* if $S \subseteq T$ implies $v(S) \leq v(T)$.

The class of big boss games is another model of cooperative games. The cooperative game (N, v) is a *big boss game* if there is one player, denoted by i^*, satisfying the following conditions

(i) $v(S) = 0$ if $i^* \notin S$

(ii) $v(N) - v(S) \geq \sum_{i \in N \setminus S} (v(N) - v(N \setminus \{i\}))$ if $i^* \in S$

A big boss game is a *total big boss game* if it is monotonic and any subgame containing player i^* is a big boss game again.

Indeed, any information market game is a big boss game (Muto et al., 1988).

A game (N, v) is *simple* if $v(S) \in \{0, 1\}$ for all $S \subseteq N$ and $v(N) = 1$. Given a monotonic simple game (N, v), the coalition S is a *winning coalition* if $v(S) = 1$. We denote $v = u_{S_1, S_2, \ldots, S_k}$ the simple game with minimal winning coalitions $S_1, S_2, \ldots, S_k$, i.e. $u_{S_1, S_2, \ldots, S_k}(S) = 1$ if $S_i \subseteq S$ for some $S_i \in \{S_1, S_2, \ldots, S_k\}$, and $u_{S_1, S_2, \ldots, S_k}(S) = 0$ otherwise. If the nonempty coalition $T \subseteq N$ is the only minimal winning coalition, the game (N, u_T) is called the *unanimity game*. The *dual* of a game (N, v) is denoted by (N, v^*) and is defined by $v^*(S) = v(N) - v(N \backslash S)$ for all $S \subseteq N$.

According to the Shapley value $\phi(v)$ of the cooperative game (N, v), the value of player $i \in N$ is $\phi_i(v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} (v(P^\sigma(i) \cup \{i\}) - v(P^\sigma(i))$ where $n = |N|$ and $\Pi(N)$ is the set of all permutations $\sigma : N \to N$. Given a permutation σ, the set of players that precede player i is the set $P^\sigma(i) = \{r \in N$ such that $\sigma^{-1}(r) < \sigma^{-1}(i)\}$ where $\sigma^{-1}(i)$ denotes the entrance number of player i. Some properties of the Shapley value are efficiency, i.e. $\sum_{i \in N} \phi_i(v) = v(N)$, and linearity, i.e. $\phi(v + w) = \phi(v) + \phi(w)$ and $\phi(av) = a\phi(v)$. The Shapley value of the unanimity game u_T and its dual game u_T^* is

$$\phi(u_T) = \phi(u_T^*) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$

The Shapley value of $u_{T,1}$ and its dual $u_{T,1}^*$ are equal and can be found by linearity of ϕ. Since $u_{T,1} = u_T + u_1 - u_{T \cup \{1\}}$, $\phi(u_{T,1}^*) = \phi(u_{T,1}) = \phi(u_T) + \phi(u_1) - \phi(u_{T \cup \{1\}}) = \frac{1}{t} \mathbf{1_T} + \mathbf{1_1} - \frac{\mathbf{1}}{\mathbf{t+1}} \mathbf{1_{T \cup \{1\}}}$ [1] , where $t = |T|$.

Let us name $I(\mathbb{R}_+)$ the set of all closed intervals in $\mathbb{R}$. In order to operate with cooperative interval games we first need to recall basic interval notation and calculus. The lenght of an interval $R = [\underline{r}, \overline{r}]$ is defined by $|R| = \overline{r} - \underline{r}$. Let $R_1 = [\underline{r}_1, \overline{r}_1]$ and $R_2 = [\underline{r}_2, \overline{r}_2]$ be two intervals, $R_1, R_2 \in I(\mathbb{R}_+)$, and let be $\alpha \in \mathbb{R}_+$. Then, $R_1 + R_2 =$

[1] $(\mathbf{1}_\mathrm{T})_\mathrm{i} = 1$ if $i \in T$ and $(\mathbf{1}_\mathrm{T})_\mathrm{i} = 0$ if $i \notin T$

$[\underline{r}_1 + \underline{r}_2, \overline{r}_1 + \overline{r}_2]$ and $\alpha R_1 = R_1 \alpha = [\alpha \underline{r}_1, \alpha \overline{r}_1]$. Let $R_1, R_2, \ldots R_n$ be a finite set of intervals, we denote by $\sum_{i=1}^{n} R_i$ the sum of those n intervals, i.e. $\sum_{i=1}^{n} R_i = [\sum_{i=1}^{n} \underline{r}_i, \sum_{i=1}^{n} \overline{r}_i]$. Before defining the substraction of intervals, we need a preference relation. We say that R_1 is weakly preferred to R_2 ($R_1 \succcurlyeq R_2$) if and only if $\underline{r}_1 \geq \underline{r}_2$ and $\overline{r}_1 \geq \overline{r}_2$. Notice that it is only defined for not nested intervals. The substraction of two intervals R_1 and R_2, with $R_1 \succcurlyeq R_2$, is defined by $R_1 - R_2 = [\underline{r}_1 - \underline{r}_2, \overline{r}_1 - \overline{r}_2]$ only if $|R_1| \geq |R_2|$ (Alparslan et al., 2009). This last condition guarantees that the lower bound of the substraction interval is smaller or equal than its upper bound. Let $R_1, R_2, \ldots R_n$ be a finite set of not nested intervals, the maximum interval is defined by $\max_{i \in \{1,2,\ldots,n\}}\{R_i\} = \{R_{i^*} \in \{R_i\}$ such that $R_{i^*} \succcurlyeq R_i$ for all $i \in \{1, 2, \ldots, n\}\}$.

A *cooperative interval game* is an ordered pair (N, w) where $N = \{1, \ldots, n\}$ is the set of players and $w : 2^N \to I(\mathbb{R})$ is the characteristic function such that $w(\emptyset) = [0, 0]$. Then, $w(S) = [\underline{w(S)}, \overline{w(S)}]$ where $\underline{w(S)}$ is the lower bound and $\overline{w(S)}$ is the upper bound of $w(S)$. Note that if all the interval worths are degenerate intervals, i.e. $\underline{w(S)} = \overline{w(S)}$ for all $S \subseteq N$, then the interval game (N, w) corresponds to the coalitional game (N, v) with $v(S) = \underline{w(S)}$ for all $S \subseteq N$. We can associate three coalitional games to a given interval game (N, w). Two of them are the border games $(N, \underline{w})$ and $(N, \overline{w})$, where $\underline{w}(S) = \underline{w(S)}$ and $\overline{w}(S) = \overline{w(S)}$ respectively, and the third one is the lenght game $(N, |w|)$ where $|w|(S) = |w(S)|$ for all $S \subseteq N$.

The *interval imputation set* $\mathcal{I}(w)$ of the interval game (N, w) is defined by

$$\mathcal{I}(w) = \{(X_1, ..., X_n) \in I^N(\mathbb{R})| \\ \textstyle\sum_{i\in N} X_i = w(N) \text{ and } X_i \succcurlyeq w(\{i\}) \text{ for all } i \in N\}$$

and its *interval core* $\mathcal{C}(w)$ is defined by

$$\mathcal{C}(w) = \{(X_1, ..., X_n) \in \mathcal{I}(w)| \sum_{i\in S} X_i \succcurlyeq w(S) \text{ for all } S \subseteq N\}.$$

An interval game (N, w) is called *size-monotonic* if $(N, |w|)$ is monotonic, i.e. $|w|(S) \leq |w|(T)$ for all $S \subset T \subseteq N$. Let us name $SMIG^N$ the class of size monotonic interval games with player set N. For each $w \in SMIG^N$ and each $i \in N$, the interval marginal contribution of player i to the grand coalition in the game w is defined by $M_i(w) = w(N) - w(N \setminus \{i\})$.

The *interval Shapley value* $\Phi : SMIG^N \to I^N(\mathbb{R}^N)$ is defined by Alparslan-Gök et al. (2010) as follows,

$$\Phi(w) = \frac{1}{n!} \sum_{\sigma\in\Pi(N)} m^\sigma(w), \text{ for each } w \in SMIG^N$$

where $\Pi(N)$ is the set of all permutations $\sigma : N \to N$. The interval marginal operator $m^\sigma : SMIG^N \to I^N(\mathbb{R})$ corresponding to σ, associates with each $w \in SMIG^N$ the interval marginal vector $m^\sigma(w)$ defined by $w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i))$ for each $i \in N$. So, in other words, the interval Shapley value is given by

$$\Phi_i(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} (w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i)))$$

for any player $i \in N$.

An interval game (N, w) is said to be a big boss interval game if its border game $(N, \underline{w})$ and its lenght game $(N, |w|)$ are total big boss games.

3.3 Information market interval games

In this chapter we study information markets where the maximum profit obtainable in each submarket is uncertain. We address the uncertainty by considering the profit attainable in a submarket controlled by $T \subseteq N$ is given by the interval $R_T = [\underline{r}_T, \overline{r}_T]$. In this section, first of all, we introduce information markets with interval uncertainty and, later, the corresponding coalitional interval games. We also show some properties that these games satisfy.

Definition 3.3.1. *An information market with interval uncertainty is defined by the tuple* $(N, \{1\}, (R_T)_{T \subseteq N, T \neq \emptyset})$ *where* $N = \{1, \dots, n\}$ *is the set of firms and* $1 \in N$ *is the patent holder who owns the information.* $R_T = [\underline{r}_T, \overline{r}_T]$ *is the interval of profits potentially attainable in the submarket to which the firms of* T *have access to.*

For instance, an information market under uncertainty is the following one.

Example 3.3.2. *We can consider* $(N, \{1\}, (R_T)_{T\subseteq N, T\neq\emptyset})$*, an information market under uncertainty, with three firms* $N = \{1, 2, 3\}$*, where firm* 1 *is the patent holder, and with profits attainable in each submarket given as follows.*

$$R_{\{1\}} = [3,5] \quad R_{\{2\}} = [5,6] \quad R_{\{3\}} = [2,3]$$
$$R_{\{1,2\}} = [2,4] \quad R_{\{1,3\}} = [0,0] \quad R_{\{2,3\}} = [2,6]$$
$$R_{\{1,2,3\}} = [1,6]$$

Definition 3.3.3. *Let* $(N, \{1\}, (R_T)_{T\subseteq N, T\neq\emptyset})$ *be an information market with interval uncertainty. The corresponding information (market) interval game is defined by* (N, w) *where*

$$w(S) = \sum_{T\cap S\neq\emptyset} R_T$$

if $1 \in S$*, and* $w(S) = [0, 0]$ *otherwise.*

We name IIG^N the set of all information interval games with player set N.

Notice that the class of the classical information market games is a particular case of the class of information interval games. When an information interval game is such that, for each submarket controlled by $T \subseteq N$, the lower bound and the upper bound of the interval profit

$R_T = [\underline{r}_T, \overline{r}_T]$ coincide, $\underline{r}_T = \overline{r}_T$, then we have a classical information market game with same player set and the profit attainable in the submarket controlled by $T \subseteq N$ given by $r_T = \underline{r}_T = \overline{r}_T$.

Next, we show the information interval game corresponding to the market given in Example 3.3.2

Example 3.3.4. *Let us consider the information market under uncertainty in Example 3.3.2. From this market, we can define the information interval game* (N, w), *with characteristic funcion* $w(S) : 2^N \to I(\mathbb{R})$, *as follows.*

$$w(\{1\}) = R_{\{1\}} + R_{\{1,2\}} + R_{\{1,3\}} + R_{\{1,4\}} + R_{\{1,2,3\}} = [6, 15]$$
$$w(\{1,2\}) = w(1) + R_{\{2\}} + R_{\{2,3\}} = [13, 27]$$
$$w(\{1,3\}) = w(1) + R_{\{3\}} + R_{\{2,3\}} = [10, 24]$$
$$w(\{1,2,3\}) = w(1,2) + R_{\{3\}} = [15, 30]$$

and $w(S) = [0, 0]$ *otherwise.*

Notice that

$$\begin{aligned} w(S) &= \textstyle\sum_{T\cap S\neq\emptyset} R_T = \sum_{T\cap S\neq\emptyset} [\underline{r}_T, \overline{r}_T] \\ &= [\textstyle\sum_{T\cap S\neq\emptyset} \underline{r}_T, \sum_{T\cap S\neq\emptyset} \overline{r}_T] = [\underline{w}(S), \overline{w}(S)]. \end{aligned}$$

In fact, the border games $(N, \underline{w})$ and $(N, \overline{w})$ are the classical information market games obtained from the information markets $(N, \{1\}, (\underline{r}_T)_{T\subseteq N, T\neq\emptyset})$ and $(N, \{1\}, (\overline{r}_T)_{T\subseteq N, T\neq\emptyset})$, respectively.

Proposition 3.3.5. *Let us consider an information market with interval uncertainty* $(N, \{1\}, (R_T)_{T\subseteq N, T\neq\emptyset})$.

Then, the length game $(N, |w|)$ *is the classical information market game corresponding to the information market* $(N, \{1\}, (|R_T|)_{T\subseteq N, T\neq\emptyset})$

Proof. The length game is defined by $|w| = \overline{w(S)} - \underline{w(S)}$. So, for any coalition $S \subseteq N$ with $1 \in S$, $|w|(S) = \sum_{T\cap S\neq\emptyset} \overline{r}_T - \sum_{T\cap S\neq\emptyset} \underline{r}_T = \sum_{T\cap S\neq\emptyset}(\overline{r}_T - \underline{r}_T) = \sum_{T\cap S\neq\emptyset} |R_T|$. While for any coalition $S \subseteq N \setminus \{1\}$, $|w|(S) = 0 - 0 = 0$

Thus, the corresponding information market is $(N, \{1\}, (|R_T|)_{T\subseteq N, T\neq\emptyset})$. □

As a consequence of the above proposition, information interval games are size monotonic interval games (SMIG) since the corresponding length game is an information market game and, therefore, a monotonic game.

Further, by Muto et al. (1988), any information market game is a big boss game. Besides, they are total big boss games because they are also monotonic games. If we take into account that the border games, $(N, \overline{w})$ and $(N, \underline{w})$, and the lenght game $(N, |w|)$ are information market games and, therefore, they are total big boss games, by Alparslan et al. (2011), information interval games are big boss interval games [2].

In fact, the border game $(N, \overline{w})$ can be obtained as the sum of

[2] An interval game (N, w) is a big boss interval game if its border game $(N, \underline{w})$ and lenght game $(N, |w|)$ are total big boss games.

the border game $(N, \underline{w})$ and the lenght game $(N, |w|)$. That is to say, $\overline{w} = \underline{w} + |w|$.

A nice property, which is satisfied by all big boss interval games in general, is satisfied by information interval games in particular. This property is called 1-concavity and states that the interval marginal contribution of a player $i \in N \setminus \{1\}$ to a coalition containing the informed player does not increase with the size of that coalition. That is, $w(S \cup \{i\}) - w(S) \succcurlyeq w(T \cup \{i\}) - w(T)$ for all $S, T \in 2^N$ with $1 \in S \subseteq T \subseteq N \setminus \{i\}$. It can easily be proved for information interval games.

Proposition 3.3.6. *Let* $(N, \{1\}, (R_T)_{T \subseteq N, T \neq \emptyset})$ *be an information market with interval uncertainty and let* (N, w) *be the corresponding information interval game. Then,* (N, w) *satisfies the* 1*-concavity property.*

Proof. Let be $i \in N \setminus \{1\}$ and let $S, T \in 2^N$ be two coalitions with $1 \in S \subseteq T \subseteq N \setminus \{i\}$. Then,

$$\begin{aligned} w(S \cup \{i\}) - w(S) &= \sum_{Q \cap (S \cup \{i\}) \neq \emptyset} R_Q - \sum_{Q \cap S \neq \emptyset} R_Q \\ &= \sum_{\substack{i \in Q \\ Q \setminus \{i\} \subseteq N \setminus S}} R_Q \\ &\succcurlyeq \sum_{\substack{i \in Q \\ Q \setminus \{i\} \subseteq N \setminus T}} R_Q \\ &= w(T \cup \{i\}) - w(T) \end{aligned}$$

where the third step follows since $N \setminus T \subseteq N \setminus S$. $\square$

3.4 The core of information interval games

Alparslan et al. (2011) show that the interval core of a big boss interval game (N, u) is equal to the set $\mathcal{C}(u) = \{(X_1, \ldots, X_n) \in I^N(\mathbb{R}) : \sum_{i \in N} X_i = w(N), [0,0] \leq X_i \leq M_i(u)$ for all $i \in N \setminus \{1\}\}$.

Let us recall that any information interval game is a big boss interval game. Then, we can easily determine the interval core of an information interval game in terms of the market data. We just need to find the marginal contribution of any player $i \in N \setminus \{1\}$ to the grand coalition.

Corollary 3.4.1. *Let (N, w) be an information interval game associated to the market $(N, \{1\}, (R_T)_{T \subseteq N, T \neq \emptyset})$. Then,*

$$\begin{aligned} \mathcal{C}(w) = \; & \{(X_1, \ldots, X_n) \in I^N(\mathbb{R}) : \textstyle\sum_{i \in N} X_i = w(N), \\ & [0,0] \preccurlyeq X_i \preccurlyeq [\underline{r}_i, \overline{r}_i] \text{ for all } i \in N \setminus \{1\}\}. \end{aligned}$$

Proof. Let (N, w) be an information market interval game. Since it is also a big boss interval game, by Alparslan et al. (2011), the interval core payoff of a player $i \in N \setminus \{1\}$ is in between $[0,0]$ and its interval marginal contribution to the grand coalition $M_i(w)$. Further, this interval marginal contribution $M_i(w)$ is

$$M_i(w) = w(N) - w(N \setminus \{i\}) = \sum_{T \cap N \neq \emptyset} R_T - \sum_{T \cap (N \setminus \{i\}) \neq \emptyset} R_T = R_{\{i\}} = [\underline{r}_i, \overline{r}_i].$$

Since the class of information interval games is a subclass of the class of big boss interval games, the condition $[0,0] \preccurlyeq X_i \preccurlyeq [\underline{r}_i, \overline{r}_i]$ $\forall i \in N \setminus \{1\}$ together with the efficiency contition, $\sum_{i\in N} X_i = w(N)$, define the interval core of the game w. □

Let us recall the information interval game (N, w) in Example 3.3.4, where $N = \{1, 2, 3\}$. By Corollary 3.4.1, its core is the interval set $\mathcal{C}(w) = \{(X_1, X_2, X_3) \in I^3(\mathbb{R}) : X_1 + X_2 + X_3 = [15, 30], [0, 0] \preccurlyeq X_2 \preccurlyeq [5, 6]$ *and* $[0, 0] \preccurlyeq X_3 \preccurlyeq [2, 3]\}$. Notice that the interval allocations $([15, 30], [0, 0], [0, 0])$ and $([8, 21], [5, 6], [2, 3])$ belong to the interval core of this example. In the first one, the informed player benefits the most while, in the second one, the other players are the biggest winners.

Indeed, an immediate consequence of the interval core structure is that it always contain these two extreme interval allocations,

(i) $X_1 = w(N)$ and $X_i = [0, 0]$ for all $i \in N \setminus \{1\}$, and

(ii) $X_1 = w(N) - \sum_{i\in N\setminus\{1\}} [\underline{r}_i, \overline{r}_i]$ and $X_i = [\underline{r}_i, \overline{r}_i]$ for all $i \in N\setminus\{1\}$.

Following the notation of Alparslan et al. (2011), we can name the first one as the big boss interval point, $\mathcal{B}(w)$, and the second one as the union interval point, $\mathcal{U}(w)$.

Once we know what is the interval core of the game, we wonder how stable is this solution. The definition of a stable set for an interval game is given by Alparslan et al. (2008). They, first, define the domination between two interval allocations. Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be two interval imputations and let be $S \in 2^N \setminus \{\emptyset\}$. It is said that X dominates Y via coalition S, which is denoted by $X\, dom_S\, Y$, if

(i) $X_i \succcurlyeq Y_i$ for all $i \in S$,

(ii) $\sum_{i \in S} X_i \preccurlyeq w(S)$.

For an interval game (N, w) a subset A of the interval imputation set is a stable set if the following conditions hold:

(i) (Internal stability) There does not exist $X, Y \in A$ such that $X\, dom\, Y$.

(ii) (External stability) For each $X \notin A$ there exists $Y \in A$ such that $Y\, dom\, X$.

We show that the interval core of an interval information market game is not always a stable set. However, certain conditions on the market prove to be enough to guarantee the stability of the core.

Proposition 3.4.2. *Let $(N, \{1\}, (R_T)_{T \subseteq N, T \neq \emptyset})$ be an information market with interval uncertainty, let (N, w) be the corresponding game and let $\mathcal{C}(w)$ be its interval core. Then, $\mathcal{C}(w)$ is a stable set if and only if $R_T = [0, 0]$ for all $T \subseteq N \setminus \{1\}$ with $|T| \geq 2$.*

Proof. In order to show that the interval core is a stable set, we should check that it holds both, the internal stability and the external stability. However, the internal stability is always satisfied by an interval core of an interval game (Alparslan et al., 2008). So, we only need to show that the external stability is satisfied.

First, we prove the "if part". Let us consider an interval information market with $R_T = [0,0]$ for all $T \subseteq N \setminus \{1\}$ with $|T| \geq 2$. Then, the corresponding interval information market game is such that $w(S) = w(\{1\}) + \sum_{i \in S \setminus \{1\}} R_{\{i\}}$ for all $S \subseteq N$ with $1 \in S$ and $w(S) = 0$ otherwise. In particular, for $S = N$, $w(N) = w(\{1\}) + \sum_{i \in N \setminus \{1\}} R_{\{i\}}$.

Hence, any interval imputation $X \in \mathcal{I}(w)$ is such that $X_i \succcurlyeq [0,0]$ and $\sum_{i \in N} X_i = w(\{1\}) + \sum_{i \in N \setminus \{1\}} R_{\{i\}}$. Now, by Corollary 3.4.1, if $X \in \mathcal{I}(w) \setminus \mathcal{C}(w)$, there exists a player $i \in N \setminus \{1\}$ such that $X_i \succ R_{\{i\}}$. Let be $A = \{i \in N \setminus \{1\}$ such that $X_i \succ R_{\{i\}}\}$. Further, take $Y \in \mathcal{I}(w)$ such that $Y_i = R_{\{i\}}$ for all $i \in A$, $Y_1 = X_1 + \sum_{i \in A}(Y_i - R_{\{i\}})$, and $Y_j = X_j$ for all $j \in (N \setminus \{1\}) \setminus A$. Notice that, by Corollary 3.4.1, $Y \in \mathcal{C}(w)$. Moreover, notice that $Y_k \succcurlyeq X_k$ for all $k \in N \setminus A$ and $\sum_{k \in N \setminus A} Y_k = (w(\{1\}) + \sum_{i \in N \setminus \{1\}} R_{\{i\}}) - \sum_{i \in A} R_{\{i\}} \preccurlyeq w(N \setminus A)$, where the first equality holds by efficiency. So, if we take $S = N \setminus A$, we have $Y \, dom_S \, X$.

Secondly we prove the "only if part". Actually, we show that if there exists $T \subseteq N \setminus \{ 1\}$ with $|T| \geq 2$ and $R_T \neq [0,0]$, then $\mathcal{C}(w)$

is not a stable set. Let be $T \subseteq N \setminus \{1\}$ with $|T| \geq 2$ such that $R_T \succ [0,0]$. Then, take the imputation $X \in \mathcal{I}(w) \setminus \mathcal{C}(w)$ such that $X_i = R_{\{i\}} + \frac{1}{n}R_T$ for all $i \in N \setminus \{1\}$ and $X_1 = w(\{1\}) + \frac{1}{n}R_T$.

Next we look for an interval $Y \in \mathcal{C}(w)$ that dominates X. Since for all $i \in N \setminus \{1\}$, $Y_i \preccurlyeq R_{\{i\}}$ by Proposition 3.4.1 and, further, $R_{\{i\}} \prec X_i$, no $Y \in \mathcal{C}(w)$ can dominate X via a coalition containing players in $N \setminus \{1\}$. So, X can only be dominated by Y via coalition $\{1\}$. However, any $Y \in \mathcal{C}(w)$ with $Y_1 \succcurlyeq X_1 = w(\{1\}) + \frac{1}{n}R_T$ does not satisfy $Y_1 \preccurlyeq w(\{1\})$. Hence, no $Y \in \mathcal{C}(w)$ dominates $X \in \mathcal{I}(w) \setminus \mathcal{C}(w)$. So, $\mathcal{C}(w)$ is not a stable set. □

Together with Muto et al. (1989), a consequence of Proposition 3.4.2 is that the core of an information interval game is a stable set if and only if the core is stable for its border games $\underline{w}$ and $\overline{w}$.

3.5 Shapley value

Some characterizations of the interval Shapley value are given by Alparslan et al. (2010) and Palanci et al. (2015) for interval cooperative games. In the first reference, the characterization is given on those games generated by the additive cone $K = \{I_S u_S | S \in 2^N \setminus \{\emptyset\}, I_S \in I(\mathbb{R})\}$ (KIG^N) and, in the second one, on size monotonic interval games ($SMIG^N$). Next, we recall some properties of the interval Shapley value on size monotonic interval games, $\Phi : SMIG^N \to$

$I^N(\mathbb{R})$.

Efficiency: For every $w \in SMIG^N$, it holds that $\sum_{i \in N} \Phi_i(w) = w(N)$.

Null Player Property: If $i \in N$ is a null player in an interval game $w \in SMIG^N$, i.e. $w(S \cup \{i\}) = w(S)$ for each $S \in 2^{N \setminus \{i\}}$, then $\Phi_i(w) = [0, 0]$.

Fairness: If $i, j \in N$ are symmetric players in $w \in SMIG^N$, i.e. $w(S \cup \{j\}) - w(S) = w(S \cup \{i\}) - w(S)$ for each S with $i, j \notin S$, then $\Phi_i(w' + w) - \Phi_i(w') = \Phi_j(w' + w) - \Phi_j(w')$ for all $w' \in SMIG^N$.

Since information interval games are size monotonic interval games, by Palanci et al. (2015), its interval Shapley value is $\Phi : SMIG^N \to I^N(\mathbb{R})$ defined by $\Phi_i(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} (w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i)))$ and it is characterized by efficiency, the null player property and fairness.

In this section we find an expression for the interval Shapley value in terms of the data of the information market under uncertainty. To this end, we first provide a definition of information interval games equivalent to the one given in Definition 3.3.3.

Proposition 3.5.1. *Let $(N, \{1\}, (R_T)_{T \subseteq N, T \neq \emptyset})$ be an information market with interval uncertainty. Then (N, w) is the corresponding* *information interval game if and only if* $w = Ru_1 + \sum_{T \subseteq N \setminus \{1\}} R_T u^*_{T,1}$, *where* $R = \sum_{T:1 \in T} R_T$.

Proof. Firstly we show the "only if" part. If $1 \notin S$, then $w(S) = [0,0]$, $u_1(S) = 0$ and $u^*_{T,1}(S) = u_{T,1}(N) - u_{T,1}(N \setminus S) = 1 - 1 = 0$. Further, if $1 \in S$, then $u_1(S) = 1$ and $u^*_{T,1}(S) = 1$ if $T \cap S \neq \emptyset$ while $u^*_{T,1}(S) = 0$ otherwise. So, $(Ru_1 + \sum_{T \subseteq N \setminus \{1\}} R_T u^*_{T,1})(S) = R + \sum_{\substack{T: T\cap S \neq \emptyset \\ T \subseteq N \setminus \{1\}}} R_T = \sum_{T: T \cap S \neq \emptyset} R_T = w(S)$.

Next, we show the "if" part. Let $w = Au_1 + \sum_{T \subseteq N \setminus \{1\}} A_T u^*_{T,1}$ with $A, A_T \in I(\mathbb{R}_+)$ for all $T \subseteq N \setminus \{1\}, T \neq \emptyset$.

Then, this game is generated by the information market under uncertainty with $R_T = A_T$ if $T \subseteq N \setminus \{1\}$ and $\sum_{T:1 \in T} R_T = A$. □

After defining information interval games in terms of some simple games, we focus now on some properties of the interval Shapley value on the set of information interval games.

Lemma 3.5.2. *Let be* $\alpha, \beta \in \mathbb{R}_+$ *and let* (N, w_1) *and* (N, w_2) *be the two size-monotonic information interval games. Then,*

$$\Phi(\alpha\, w_1 + \beta\, w_2) = \alpha\, \Phi(w_1) + \beta\, \Phi(w_2).$$

Proof. Notice that for all $S \subseteq N$ with $1 \in S$, $w(S) - w(S \setminus \{1\}) = w(S)$. Meanwhile, $w(S) - w(S \setminus \{i\}) = \sum_{\substack{i \in T \\ T \setminus \{i\} \subseteq N \setminus S}} R_T$ for any $i \in S \subseteq$

$N \setminus \{1\}$. On the other hand, if $1 \notin S \subseteq N$, then $w(S) - w(S \setminus \{i\}) = [0, 0]$ for any $i \in N$. Then,

$$\begin{aligned}
&\Phi_i(\alpha\, w_1 + \beta\, w_2) \\
&\quad = \tfrac{1}{n!} \sum_{\sigma \in \Pi(N)} \sum_{\substack{i \in T \\ T\setminus\{i\} \subseteq N \setminus P^\sigma(i)}} \alpha\, R_{1T} + \beta\, R_{2T} \\
&\quad = \tfrac{1}{n!} \sum_{\sigma \in \Pi(N)} \left(\sum_{\substack{i \in T \\ T\setminus\{i\} \subseteq N \setminus P^\sigma(i)}} \alpha\, R_{1T} + \sum_{\substack{i \in T \\ T\setminus\{i\} \subseteq N \setminus P^\sigma(i)}} \beta\, R_{2T} \right) \\
&\quad = \tfrac{1}{n!} \left(\alpha \sum_{\sigma \in \Pi(N)} \sum_{\substack{i \in T \\ T\setminus\{i\} \subseteq N \setminus P^\sigma(i)}} R_{1T} + \beta \sum_{\sigma \in \Pi(N)} \sum_{\substack{i \in T \\ T\setminus\{i\} \subseteq N \setminus P^\sigma(i)}} R_{2T} \right) \\
&\quad = \alpha\, \tfrac{1}{n!} \sum_{\sigma \in \Pi(N)} \sum_{\substack{i \in T \\ T\setminus\{i\} \subseteq N \setminus P^\sigma(i)}} R_{1T} + \beta\, \frac{1}{n!} \sum_{\sigma \in \Pi(N)} \sum_{\substack{i \in T \\ T\setminus\{i\} \subseteq N \setminus P^\sigma(i)}} R_{2T} \\
&\quad = \alpha\, \Phi_i(w_1) + \beta\, \Phi_i(w_2)
\end{aligned}$$

□

If we consider an interval, say $A = [\underline{a}, \overline{a}]$, multiplied by a cooperative game (N, v), then we get an interval game, $w = Av$. Its interval Shapley value can be found by means of the Shapley value of the cooperative game as the following lemma shows.

Lemma 3.5.3. *Let (N, w) be an interval cooperative game with $w = Av$ where $A = [\underline{a}, \overline{a}] \in I(\mathbb{R})$ and (N, v) is a cooperative game. Then, $\Phi(w) = A\phi(v)$.*

Proof. Notice that w is size-monotinic since $|w(S)| = 0$ for all $S \subseteq N$. So we can calculate the interval Shapley value of w. For each player

$i \in N$,

$$\begin{aligned}\Phi_i(w) &= \tfrac{1}{n!} \sum_{\sigma\in\Pi(N)} (w(P^\sigma(i)\cup\{i\}) - w(P^\sigma(i))) \\ &= \tfrac{1}{n!} \sum_{\sigma\in\Pi(N)} (Av(P^\sigma(i)\cup\{i\}) - Av(P^\sigma(i))) \\ &= A\,\tfrac{1}{n!} \sum_{\sigma\in\Pi(N)} (v(P^\sigma(i)\cup\{i\}) - v(P^\sigma(i))) \\ &= A\phi_i(v)\end{aligned}$$

□

Next, we discuss one more property of the interval Shapley value. The interval Shapley value of an information interval game and the interval Shapley value of its dual coincide. Actually, as can be seen above, this property holds for the Shapley value of cooperative games.

Lemma 3.5.4. *Let $(N, \{1\}, (R_T)_{T\subseteq N, T\neq\emptyset})$ be an information market under uncertainty and let (N, w) be the corresponding information interval game. Then, $\Phi(w) = \Phi(w^*)$.*

Proof. The dual game w^* of the information interval game w is $w^*(S) = w(N) - w(N\setminus S) = \sum_{T\subseteq N} R_T - \sum_{T\cap(N\setminus S)\neq\emptyset} R_T = \sum_{T\subseteq S} R_T$ for all $S \subseteq N$.

Then, since w and w^* are size-monotonic interval games,

$$\begin{aligned}\Phi_i(w^*) &= \tfrac{1}{n!}\sum_{\sigma\in\Pi(N)}(w^*(P^\sigma(i)\cup\{i\})-(w^*(P^\sigma(i)))\\ &= \tfrac{1}{n!}\sum_{\sigma\in\Pi(N)}\left(\sum_{T\subseteq P^\sigma(i)\cup\{i\}}R_T-\sum_{T\subseteq P^\sigma(i)}R_T\right)\\ &= \tfrac{1}{n!}\sum_{\sigma\in\Pi(N)}\sum_{\substack{i\in T\\ T\setminus\{i\}\subseteq N\setminus P^\sigma(i)}}R_T\\ &= \Phi_i(w)\end{aligned}$$

□

The two Lemmas 3.5.2 and 3.5.3 allow us to find the interval Shapley value of an information interval game following the steps given in Muto et al. (1989) for information market games. Once information interval games have been defined in terms of some simple games (Proposition 3.5.1), we find their interval Shapley value taking into account some properties of simple games and their corresponding Shapley value.

Proposition 3.5.5. *Let $(N,\{1\},(R_T)_{T\subseteq N,T\neq\emptyset})$ be an information market under uncertainty and let (N,w) be the corresponding information interval game. Then, its interval Shapley value $\Phi(w)$: $IIG^N\to I(\mathbb{R})^N$ is*

$$\Phi(w)=R\cdot\mathbf{1}_1+\sum_{\substack{\mathrm{T}:\mathrm{T}\subseteq\mathrm{N}\setminus\{1\}\\ \mathrm{T}\neq\emptyset}}\mathrm{R_T}\left(\frac{|\mathrm{T}|}{|\mathrm{T}|+1}\mathbf{1}_1+\frac{1}{|\mathrm{T}|(|\mathrm{T}|+1)}\mathbf{1}_\mathrm{T}\right).$$

Proof. Since $w = Ru_1 + \sum_{T \subseteq N \setminus \{1\}} R_T u^*_{T,1}$ by Proposition 3.5.1 its Shapley value can be obtained as follows.

$$\begin{aligned} \Phi(w) &= R\phi(u_1) + \sum_{T \subseteq N \setminus \{1\}} R_T \phi(u^*_{T,1}) \\ &= R \cdot \mathbf{1}_1 + \sum_{\substack{\mathrm{T}:\mathrm{T} \subseteq \mathrm{N} \setminus \{1\} \\ \mathrm{T} \neq \emptyset}} \mathrm{R_T} \left(\frac{1}{|\mathrm{T}|} \mathbf{1}_{\mathrm{T}} + \mathbf{1}_1 + \frac{1}{|\mathrm{T}|+1} \mathbf{1}_{\mathrm{T} \cup \{1\}} \right) \\ &= R \cdot \mathbf{1}_1 + \sum_{\substack{\mathrm{T}:\mathrm{T} \subseteq \mathrm{N} \setminus \{1\} \\ \mathrm{T} \neq \emptyset}} \mathrm{R_T} \left(\frac{|\mathrm{T}|}{|\mathrm{T}|+1} \mathbf{1}_1 + \frac{1}{|\mathrm{T}|(|\mathrm{T}|+1)} \mathbf{1}_{\mathrm{T}} \right) \end{aligned}$$

where the first equality holds by Lemma 3.5.2 and Lemma 3.5.3. □

Then, the interval Shapley value of the information interval game of Example 3.3.4 can be obtained as follows: first we find $R = [3, 5] + [2, 4] + [0, 0] + [1, 6] = [6, 15]$ and then, $\Phi_1(w) = [6, 15] + \frac{1}{1+1}[5, 6] + \frac{1}{1+1}[2, 3] + \frac{2}{2+1}[2, 6] = [\frac{65}{6}, \frac{47}{2}]$, $\Phi_2(w) = \frac{1}{1 \cdot 2}[5, 6] + \frac{1}{2 \cdot 3}[2, 6] = [\frac{17}{6}, 4]$ and $\Phi_3 = \frac{1}{1 \cdot 2}[2, 3] + \frac{1}{2 \cdot 3}[2, 6] = [\frac{4}{3}, \frac{5}{2}]$. So,

$$\Phi(w) = \left(\left[\frac{65}{6}, \frac{47}{2} \right], \left[\frac{17}{6}, 4 \right], \left[\frac{4}{3}, \frac{5}{2} \right] \right)$$

which belongs to the interval core. This can be easily checked by Corollary 3.4.1.

Corollary 3.5.6. *Let $(N, \{1\}, (R_T)_{T \subseteq N, T \neq \emptyset})$ be an information market under uncertainty and let (N, w) be the corresponding information interval game. Let $(N, \underline{w})$ and $(N, \overline{w})$ be the corresponding border games. Then, $\Phi(w) = [\phi(\underline{w}), \phi(\overline{w})]$.*

Although the interval Shapley value of a size-monotonic interval game satisfies good properties as efficiency, null player property and fairness, it may lie outside the interval core. We just need to consider the information market with interval uncertainty in Example 3.3.2 with $R_{\{2,3\}} = [2, 10]$ instead of $R_{\{2,3\}} = [2, 6]$ and the corresponding information interval game $(\{1, 2, 3\}, w')$. Then, $\Phi_3(w') = [\frac{4}{3}, \frac{19}{6}] \not\preceq [2, 3] = R_{\{3\}}$ and therefore Φ does not belong to the interval core of w' (by Corollary 3.4.1).

Actually, Corollary 3.5.6 also shows that the interval Shapley value may not belong to the interval core since the upper bound of $\Phi(w)$ is the Shapley value of the information market game $\overline{w}$, that is $\phi(\overline{w})$, and it may not belong to the core of $\overline{w}$ beacuse there may exist a player $i \in N \setminus \{1\}$ with $\phi_i(\overline{w}) > \overline{r}_i$ (cf. Muto et al., 1989).

3.6 Interval core allocations with monotonicity

The interval Shapley value studied in the previous section is an interval solution for information interval games which can easily be found from the market data. However it may lie outside the interval core. This section is devoted to an interval solution that belongs to the interval core.

Before defining this interval solution, we need to introduce a property that interval games migth meet. Further, we will show that it is satisfied by any information interval game.

Definition 3.6.1. *Let (N, w) be an interval game. We say that w satisfies the size $i\tau$-condition if for any coalition $S \subseteq N$ and any $T \subset S$ with $|T| = |S| - 1$,*

$$|w(S)| + \sum_{i \in T} |w(N \setminus \{i\})| \geq |T| \cdot |w(N)| \tag{3.1}$$

Notice that this property is neither implied by size-monotonicity nor implies it, as next examples show.

First, we consider an interval game which is size-monotonic but does not satisfy the size iτ-condition.

Example 3.6.2. *Let us consider the interval game (N, w) with $N = \{1, 2, 3\}$ and*

$$w(\{1\}) = w(\{2\}) = w(\{3\}) = [0, 0]$$
$$w(\{1, 2\}) = w(\{1, 3\}) = w(\{2, 3\}) = [3, 3]$$
$$\textit{and } w(\{1, 2, 3\}) = [5, 6]$$

It is size-monotonic since $|w|(S) = |w(S)| = 0$ for any $S \subseteq N$ with $|S| < 3$ and $|w|(N) = |w(N)| = 1$. So, $|w|$ is a monotonic game. On the other hand, if we consider $S = \{1, 2\}$ and $T = \{1\}$ in expression (3.1), we can check that size $i\tau$-condition is not satisfied. Indeed, $|w(\{1, 2\})| + |w(N \setminus \{1\})| = 0 \ngeq (2 - 1)|w(N)| = 1$.

Now, we show an interval game which satisfies equation (3.1) but is not size-monotonic.

Example 3.6.3. *Let us consider the interval game* (N, w) *with* $N = \{1, 2, 3\}$ *and*

$$w(\{1\}) = w(\{2\}) = w(\{3\}) = [0, 0]$$
$$w(\{1, 2\}) = w(\{1, 3\}) = w(\{2, 3\}) = [1, 2]$$
$$\text{and } w(\{1, 2, 3\}) = [5, 5]$$

We can check condition (3.1) for $S = \{1, 2, 3\}$ *and any* $T = \{i, j\} \subset N$, $|w(\{1, 2, 3\})| + |w(N \setminus \{i\})| + |w(N \setminus \{j\})| = 2 \geq (3 - 1)|w(N)| = 0$, *and for any* S *with* $|S| = 2$ *and* $i \in S$, $|w(S)| + |w(N \setminus \{i\})| = 1 \geq (2 - 1)|w(N)| = 0$.

However w *is not size-monotonic since, for instance* $|w|(\{1, 2\}) = 1 \not\leq 0 = |w|(N)$.

If we consider the class of information interval games, both properties are satified. Any information interval game is size-monotonic by Proposition 3.3.5. Further, next proposition shows that also the size iτ-condition is satisfied.

Proposition 3.6.4. *Let* (N, w) *be an information interval game. Then* w *satisfies the size i*τ*-condition.*

Proof. An interval game satisfies the size iτ-condition if for any coalition $S \subseteq N$ and any $T \subset S$ with $|T| = |S| - 1$, $|w(S)| + \sum_{i \in T} |w(N \setminus \{i\})| \geq |T| \cdot |w(N)|$.

If we consider information interval games, for any coalition $S \subseteq N$ and any $T \subset S$ with $|T| = |S| - 1$,

$$\begin{aligned} |w(S)| + \sum_{i \in T} |w(N \setminus \{i\})| &= |w(S) + \sum_{i \in T} w(N \setminus \{i\})| \\ &= |w(S) + \sum_{i \in T} (w(N) - R_{\{i\}})| \\ &= |T| \cdot |w(N)| + |w(S) - \sum_{i \in T} R_{\{i\}}| \\ &\geq |T| \cdot |w(N)| \end{aligned}$$

Indeed, $|w(S) - \sum_{i \in T} R_{\{i\}}|$ is non-negative since $w(S) - \sum_{i \in T} R_{\{i\}} = R_{\{j\}} + \sum_{\substack{Q \cap S \\ |T| \geq 2}} R_Q$ where $\{j\} = S \setminus T$ and the sum of intervals is an interval. Therefore, its lower bound is smaller or equal than its upper bound and its length is not negative. □

The interval $\mathcal{T}$-value

We generalize here the τ-value defined for coalitional games by Tijs (1981) to the class of interval cooperative games.

Definition 3.6.5. *Let (N, w) be an interval game that satisfies size-monotonicity and size $i\tau$-condition. The interval $\mathcal{T}$-value of w is the interval allocation $\lambda M(w) + (1 - \lambda) m(w)$, with $0 \leq \lambda \leq 1$, that satisfies interval efficiency, i.e. $\sum_{i \in N} \lambda M_i(w) + \sum_{i \in N} (1 - \lambda) m_i(w) = w(N)$, where $M_i(w) = w(N) - w(N \setminus \{i\})$ and $m_i(w) = \max_{S : i \in S} \{w(S) - \sum_{j \in S \setminus \{i\}} M_j(w)\}$ for each $i \in N$.*

Notice $M_i(w)$ is well defined by the size monotonicity of w. Further, $m_i(w)$ is well defined by the size i$\mathcal{T}$-condition. Both properties are satisfied by information interval games.

Furthermore, it is not difficult to find the interval $\mathcal{T}$-value of an information interval game in terms of the market data. Indeed, $M_i(w) = w(N) - w(N \setminus \{i\}) = R_{\{i\}}$ for all $i \in N \setminus \{1\}$. Further, $M_1(w) = w(N) - w(N \setminus \{1\}) = w(N)$.

With respect to $m(w)$, notice that for all $S \subseteq N$ with $1 \in S$, $w(S) - \sum_{j\in S\setminus\{1\}} R_{\{j\}} = \sum_{T\cap S\neq\emptyset} R_T - \sum_{j\in S\setminus\{1\}} R_{\{j\}} = \sum_{\substack{T\cap S\neq\emptyset\\|T|\geq 2}} R_T + R_{\{1\}}$. So, $m_1(w) = w(N) - \sum_{j\in N\setminus\{1\}} R_{\{j\}}$ since $w(N) - \sum_{j\in N\setminus\{1\}} M_j(w) = \sum_{\substack{T\subseteq N\\|T|\geq 2}} R_T + R_{\{1\}} \succcurlyeq \sum_{\substack{T\cap S\neq\emptyset\\|T|\geq 2}} R_T + R_{\{1\}} = R^w(S,1)$ for any $S \subseteq N$ with $1 \in S$. Next, for any $i \in N \setminus \{1\}$,

(i) If $1 \in S$, $w(S) - \sum_{j\in S\setminus\{i\}} M_j(w) = w(S) - \sum_{\substack{j\in S\setminus\{i\}\\j\neq 1}} R_{\{j\}} - w(N) \preccurlyeq [0,0]$.

(ii) If $1 \notin S$ and $|S| \geq 2$, $w(S) - \sum_{j\in S\setminus\{i\}} M_j(w) = [0,0] - \sum_{j\in S\setminus\{i\}} M_j(w) \preccurlyeq [0,0]$.

(iii) If $1 \notin S$ and $|S| = 1$, $w(S) - \sum_{j\in S\setminus\{i\}} M_j(w) = w(\{i\}) = [0,0]$.

So, $m_i(w) = [0,0]$ for all $i \in N \setminus \{1\}$. Thus,

$$M(w) = (w(N), R_{\{2\}}, R_{\{3\}}, \ldots, R_{\{n\}}), \text{ and}$$

$$m(w) = (w(N) - \textstyle\sum_{j\in N\setminus\{1\}} R_{\{j\}}, [0,0], [0,0], \ldots, [0,0]).$$

Now, in order to find the interval $\mathcal{T}$-value, we first calculate λ in the equation $\lambda w(N)+\lambda \sum_{j\in N\setminus\{1\}} R_{\{j\}}+(1-\lambda)(w(N)-\sum_{j\in N\setminus\{1\}} R_{\{j\}})) = w(N)$. From this equation, we get $(2\lambda-1)\sum_{j\in N\setminus\{1\}} R_{\{j\}} = [0,0]$. So, either $\lambda = \frac{1}{2}$ or $R_{\{j\}} = [0,0]$ for all $j \in N \setminus \{1\}$.

If $R_{\{j\}} = [0,0]$ for all $j \in N \setminus \{1\}$, then

$$\mathcal{T} = M(w) = m(w) = (w(N), [0,0], [0,0], \ldots, [0,0]).$$

On the other hand, if $\lambda = \frac{1}{2}$, then

$$\mathcal{T} = (w(N) - \frac{1}{2}\sum_{j\in N\setminus\{1\}} R_{\{j\}}, \frac{1}{2}R_{\{2\}}, \frac{1}{2}R_{\{3\}}, \ldots, \frac{1}{2}R_{\{n\}}).$$

This corresponds to the definition given by Alparslan et al. (2011) for the interval $\mathcal{T}$-value of a big boss interval game, where the interval $\mathcal{T}$-value is defined by $\mathcal{T}(w) = \frac{1}{2}\mathcal{U}(w) + \frac{1}{2}\mathcal{B}(w)$.

For any interval game w, we can consider the set of coalitions containing the informed player $P_1 = \{S \subseteq N \mid 1 \in S\}$. The scheme $I = (I_{iS})_{\substack{i\in S\\ S\subseteq P_1}}$ is an interval allocation scheme for w if $(I_{iS})_{i\in S}$ belongs to the interval core of the subgame (S, w^S) for each coalition $S \in P_1$. Such allocation scheme I is called a bi-monotonic interval allocation scheme (bi-mas) for w if, for all $S, T \in P_1$ with $S \subseteq T$, we have $I_{is} \preccurlyeq I_{iT}$ for all $i \in S \setminus \{1\}$, and $I_{1s} \succcurlyeq I_{1T}$. That is to say, the informed player is weakly better off in large coalitions while the other players are weakly worse off.

Since information interval games are big boss interval games, by Alparslan-Gök et al. (2008), any allocation in the interval core of an information interval game is bi-mas extendable. In particular, the interval $\mathcal{T}$-value of an information interval game yields a bi-mas.

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Chapter 4

Assignment game with a central player

4.1 Introduction

A class of assignment markets is analyzed in Shapley Shubik (1972) by using the theory of cooperative games. In these markets, the set of agents is divided into two sectors (let us say sellers and buyers), and the model assumes that the objects of trade are indivisible, in such a way that each seller has a supply of exactly one item and each buyer desires exactly one item. In our market there is also a set of buyers N^1 and a set of sellers N^2, however the two sets are not disjoint, but have one agent in common, who can act both as a buyer and as a seller. We call it the central player, and it is denoted by

h. As in Shapley and Shubik assignment market, each buyer wants to buy at most one unit and each seller has one unit for sale. A matrix $A = (a_{ij})_{\substack{i \in N^1 \\ j \in N^2}}$ summarizes the profits from trades, where a_{ij} represents the amount that player i and j jointly obtain when the transaction between them takes place.

An additional feature of our model is that a trade is possible only if the central player participates in that trade. Therefore, the profit matrix is such that $a_{ij} = 0$ if $h \notin \{i, j\}$.

We may extend this model allowing the central player to trade several units. We can think then of a social bank of flats which acts as a referee in the housing market to ensure reasonable prices. The aim of establishing a social bank is to avoid a real estate bubble. Thus, the owners of flats in this market can only sell them to the social bank and buyers can only buy a flat from the social bank. However, the social bank may keep its flats as its own property and rent them (hence the bank may obtain some profit, $a_{hh} \geq 0$, by renting them), or it may sell them to the buyers.

Another model of a rental housing market can be found in Kaneko (1983). The market in our model is different due to the reservation value of the central player, and also different from other markets with middlemen, Oishi Sakaue (2014), Stuart (1997), Kaneko and Wooders (1982) and Johri and Leach (2002), where their competitive equilibria are studied.

The total value created in the market should be divided among the agents and this is the aim of this chapter. First we consider that the central player can sell and buy just one unit and we show how to find the core of the game. Any core allocation ensures that no coalition of players has an incentive to split off from the market. The nucleolus and the τ-value are single-point solutions, and can be found from the market data. Later, we consider that the central player may trade any finite number of units and show that any competitive equilibrium provides a core allocation.

We organize the chapter as follows. After Section 4.2, where notation and game theory definitions are provided, in Section 4.3, we define assignment games with a central agent. In Section 4.4 we analyze what the core looks like and some related set-solutions. Section 4.5 is devoted to point-solution concepts: the nucleolus and the τ-value. Finally, we generalize the model allowing the central player to buy and sell more than one item in Section 4.6.

4.2 Preliminaries

Given a two-sided market, the assignment problem is defined by the triple (M_1, M_2, A) where M_1 is the set of buyers, M_2 is the set of sellers, and $A = (a_{ij})_{\substack{i\in M_1\\ j\in M_2}}$ is a non-negative real matrix. To solve the problem we must look for an optimal matching in A. A matching between M_1 and M_2 is a subset μ of $M_1 \times M_2$ such that each $k \in M_1 \cup M_2$ belongs at most to one pair in μ. When $(i,j) \in \mu$ we also denote with some abuse of notation $j = \mu(i)$ and $i = \mu(j)$. We will denote by $\mathcal{M}(M_1, M_2)$ the set of matchings. We say a matching μ is optimal for the problem (M_1, M_2, A); in short μ is optimal for A and it is denoted by μ_A, if for all $\mu' \in \mathcal{M}(M_1, M_2)$, $\sum_{(i,j)\in\mu_A} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$. The set of optimal matchings of the problem (M_1, M_2, A) is denoted by $\mathcal{M}_A(M_1, M_2)$. Given $S \subseteq M_1$ and $T \subseteq M_2$, we denote by $\mathcal{M}(S,T)$ and $\mathcal{M}_A(S,T)$ the set of matchings and optimal matchings of the submarket $(S, T, A_{S\times T})$ defined by the subset S of buyers, the subset T of sellers and the restriction of A to $S \times T$. If $S = \emptyset$ or $T = \emptyset$, then the only possible matching is $\mu = \emptyset$ and, by convention, $\sum_{(i,j)\in\emptyset} a_{ij} = 0$.

A *transferable utility coalitional game* (a game) is an ordered pair (N, v) where $N = \{1, 2, \ldots, n\}$ is the set of players, and $v : 2^N \to \mathbf{R}$ is a real-valued function on the set 2^N of all subsets of N, with $v(\emptyset) = 0$. If no confusion regarding the set of players arises, we denote the game (N, v) only by v. The size of the set of players N is denoted by n.

Given an assignment problem (M_1, M_2, A), Shapley and Shubik (1972) defines a related coalitional game with transferable utility, the assignment game $(M_1 \cup M_2, w_A)$, as follows. The profits of the mixed-pair coalitions, $\{i, j\}$ where $i \in M_1$ and $j \in M_2$, are given by the non-negative matrix A, $w_A(\{i, j\}) = a_{ij} \geq 0$, and this matrix also determines the value of any other coalition $S \cup T$, $w_A(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\}$, where $S \subseteq M_1$ and $T \subseteq M_2$.

A game is *monotonic* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$. It is *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$, or equivalently, $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S \subseteq T \subseteq N \setminus \{i\}$ and for all $i \in N$.

A *payoff vector* is $x \in \mathbf{R}^n$, where, for all $i \in N$, x_i represents the payoff to player i. We denote $x(S) = \sum_{i\in S} x_i$. The set of payoff vectors that are efficient, $x(N) = v(N)$, and individually rational, $x_i \geq v(i)$ for all $i \in N$, is the *set of imputations* $I(v)$.

The *core* of a game, $C(v)$, is the set of payoff vectors $x \in \mathbf{R}^n$ which, besides being efficient, meet the coalitional rationality principle, that is $x(S) \geq v(S)$ for all $S \subset N$. A game (N, v) is said to be *balanced* if it has a non-empty core. It is said to be *totally balanced* if the core of every subgame is non-empty, where the subgame corresponding to some coalition $T \subseteq N$, $T \neq \emptyset$, is the game (T, v_T) with $v_T(S) = v(S)$ for all $S \subseteq T$.

The *marginal contribution* of a player $i \in N$ to the grand coalition, N, is defined by $b_i = v(N) - v(N\setminus\{i\})$. A *marginal worth vector* $m^\sigma(v)$ is a payoff vector defined for each ordering of the set of players $\sigma : N \to \{1, \ldots, n\}$ as $m_i^\sigma(v) = v(\{j \in N|\sigma(j) \leq \sigma(i)\}) - v(\{j \in N|\sigma(j) < \sigma(i)\})$ for all $i \in N$.

A balanced game (N, v) satisfies the *CoMa-property* if and only if $C(v) = conv\{m^\sigma(v)|m^\sigma(v) \in C(v)\}$, that is to say, the extreme points of its core are marginal worth vectors.

Let be $x \in I(v)$. The excess of a coalition $S \in 2^N \setminus \{\emptyset, N\}$ at x is the real number $\varepsilon(S,x) = v(S) - x(S)$. At any x, denote by $\theta(x) \in \mathbf{R}^{2^n-2}$ the respective excesses arranged in non-increasing order, i.e. $\theta_l(x) \geq \theta_{l'}(x)$ whenever $l < l'$. The *nucleolus* $\nu(v)$ is the unique imputation that lexicographically minimizes the vector $\theta(x)$. The nucleolus belongs to the core whenever it is non-empty.

The *utopia vector* $M(v) \in \mathbb{R}^N$ is defined by $M_i(v) = b_i$ for all $i \in N$. $M_i(v)$ can be regarded as a maximum payoff (not always attainable) that player i can expect to obtain in the core of the game. The *minimal rights vector* $m(v) \in \mathbb{R}^N$ is computed by using $M(v)$. For all $i \in N$, $m_i(v) = \max_{S:i\in S}\{R^v(i,S)\}$, where $R^v(i,S)$ is what remains for player $i \in N$ when coalition S forms, $i \in S$, and all players in $S \setminus \{i\}$ are paid their utopia payoff, $R^v(i,S) = v(S) - \sum_{j\in S\setminus\{i\}} M_j(v)$. Then, the τ-*value* of that game is the unique efficient payoff vector on the line segment between $m(v)$ and $M(v)$ whenever $\sum_{i\in N} m_i(v) \leq v(N) \leq \sum_{i\in N} M_i(v)$ and $m_i(v) \leq M_i(v)$ for all $i \in N$.

4.3 Assignment games with a central player

Let us consider an assignment market where there is one player, say player h, who is necessary for any trading among agents. So, h has veto power. Moreover, this player has a double role as a seller and as a buyer. Thus, player h has a central position in the market structure. Such a market is an assignment market with a central player, $(\{h\}, N_1, N_2, A)$, where $N = \{h\} \cup N_1 \cup N_2$ is the set of players: $N_1 \neq \emptyset$ is the set of sellers, $N_2 \neq \emptyset$ is the set of buyers and h is the central player. Since h is necessary for any trade among agents, matrix A with $a_{ij} \in \mathbf{R}_+$ is such that $a_{ij} = 0$ if $h \notin \{i, j\}$. The profits of the mixed-pair coalitions $\{h, i\}$ where $i \in N_1$ and $\{h, j\}$ where $j \in N_2$, can be obtained from the non-negative matrix A. They are a_{ih} and a_{hj} respectively. Finally, the matrix entry a_{hh} is a reservation value and may represent the profit obtained by the central player while keeping its ownership. For instance, this ownership can be rented to provide some profit.

Definition 4.3.1. *Let $(\{h\}, N_1, N_2, A)$ be an assignment market with a central player. Then, the corresponding assignment game with a central player (N, v_A) is defined by $v_A(S) = \max_{\substack{i\in S\cap N_1 \\ j\in S\cap N_2}} \{a_{hh}, a_{hj}+a_{ih}\}$ if $h \in S$, and $v_A(S) = 0$ otherwise.*

The deal between players in a coalition S is only possible if $h \in S$. Each coalition $S \subseteq N$, with $h \in S$ and at least one seller $i \in N_1$ and one buyer $j \in N_2$, maximizes among a_{hh} and $a_{hj}+a_{ih}$ with $h, i, j \in S$. In case $S \cap N_1 = \emptyset$ (or $S \cap N_2 = \emptyset$), coalition S maximizes between a_{hh} and a_{hj} (or a_{ih}).

Example 4.3.2. *Let $(\{h\}, \{1, 2\}, \{1', 2'\}, A)$ be an assignment market with a central player, where*

$$A = \begin{array}{c} \\ h \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} h & 1' & 2' \end{array} \\ \begin{pmatrix} 2 & 3 & 4 \\ 5 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix} \end{array},$$

where the rows correspond to sellers and the columns to buyers. Notice that the first column and the first row correspond to player h who has a double role as a seller and a buyer. Since the transaction among players in a coalition $S \subseteq N$ is only possible if player $h \in S$, the characteristic function is given by the following table. [1]

S	$\{h\}$	$\{h1\}$	$\{h2\}$	$\{h1'\}$	$\{h2'\}$
$v_A(S)$	2	5	5	3	4

S	$\{h12\}$	$\{h11'\}$	$\{h12'\}$	$\{h21'\}$	$\{h22'\}$	$\{h1'2'\}$
$v_A(S)$	5	8	9	8	9	4

S	$\{h121'\}$	$\{h122'\}$	$\{h11'2'\}$	$\{h21'2'\}$	N
$v_A(S)$	8	9	9	9	9

and $v_A(S) = 0$ for any other $S \subseteq N$.

Notice that assignment games with a central player are not assignment games since $v_A(\{h\})$ may be greater than 0.

[1] We write $\{i \dots j\}$ instead of $\{i, \dots, j\}$ for simplicity throughout the text. It is unlikely to lead to confusion.

Although assignment games are hardly ever convex games, we wonder if assignment games with a central player are convex. The answer is negative as shown by Example 4.3.2. Since $v_A(\{h121'2'\}) - v_A(\{h121'\}) = 9-8 = 1$ and $v_A(\{h122'\}) - v_A(\{h12\}) = 9-5 = 4$, we have that $v_A(\{h121'2'\}) - v_A(\{h121'\}) \not\geq v_A(\{h122'\}) - v_A(\{h12\})$. This means that the game is not convex.

Without loss of generality, henceforth, we will assume that buyers and sellers are respectively ordered by their non-increasing gains. So, matrix A is such that $a_{ih} \geq a_{i'h}$ for all $i, i' \in N_1$ with $i < i'$ and $a_{hj} \geq a_{hj'}$ for all $j, j' \in N_2$ with $j < j'$. Hence, $1 = argmax_{i \in N_1}\{a_{ih}\}$ and $1' = argmax_{j \in N_2}\{a_{hj}\}$.

Observe that $v_A(N) = a_{hh}$ means that $a_{hh} \geq a_{1h} + a_{h1}$ and, by the assumption above, $a_{hh} \geq a_{ih} + a_{hj}$ for all $i \in N_1$ and $j \in N_2$. In this case we say that the assignment game with a central player, v_A, is trivial. The central player would rather keep its ownership than be involved in any trade. The opposite situation is named intrinsic.

Lemma 4.3.3. *Let (N, v_A) be a trivial assignment game with a central player. Then, $v_A = a_{hh} \cdot u_{\{h\}}$, where $u_{\{h\}}$ denotes the unanimity game* [2] *for $\{h\}$.*

Proof. Notice that if $a_{hh} \geq a_{1h} + a_{h1}$, then $v_A(S) = a_{hh}$ for all $S \subseteq N$

[2]The unanimity game u_T, where $T \subseteq N$, is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise.

with $h \in S$, and $v_A(S) = 0$ otherwise. On the other hand, $u_{\{h\}}(S) = 1$ for all $S \subseteq N$ with $h \in S$, and $v_{\{h\}}(S) = 0$ otherwise. □

When v_A is a trivial assignment game with a central player, it is a convex game and there is only one core allocation. Indeed, $C(v_A) = \{(a_{hh}, 0, \ldots, 0)\}$[3]. Actually, since $v_A(N) = a_{hh}$ and $v_A(N \setminus \{i\}) = a_{hh}$ for all $i \in N \setminus \{h\}$, $b_i = 0$. Being trivial is sufficient to guarantee the convexity of assignment games with a central agent but it is not necessary, as we will see later on in the chapter.

4.4 The core

We have seen above what the core of a trivial assignment game with a central player is like. This section is devoted to studying the core of any assignment game with a central player.

On first consideration, the balancedness of any assignment game with a central player is guaranteed by Bahel (2016) where veto games are introduced. Actually, (N, v) is a veto game if there exists a coalition $T \in 2^N \setminus \{N, \emptyset\}$ satisfying $v(S) = 0$ for any $S \in 2^N$ such that $S \cap T \neq \emptyset$. Therefore, any assignment game with a central player is a veto game with $T = \{h\}$. An algorithm for finding extreme core

[3]The order of the components of an allocation $x \in \mathbf{R}^n$ is $(x_h, x_1, x_{1'}, x_2, x_{2'}, \ldots)$.

allocations of any veto game is provided by Bahel (2016). Moreover, by Bahel (2016) the core and the bargaining set[4] coincide.

Furthermore, assignment games with a central player are matrix-based pairing situations as defined by Tejada, Borm and Lohman (2014). Actually, we only need to consider a matrix-based pairing situation (N^1, N^2, A) with $N^1 = \{h\} \cup N_1$, $N^2 = \{h\} \cup N_2$ and notice that matrix A is such that $a_{ij} = 0$ if $h \notin \{i, j\}$.

Following Tejada, Borm and Lohman (2014), we consider the assignment game $(M^1 \cup M^2, w_A)$ corresponding to an assignment game with a central player $(\{h\} \cup N_1 \cup N_2, v_A)$ just by splitting the central player h into two players: player h^1 acting as seller and player h^2 acting as buyer, such that $M^1 = h^1 \cup N_1$ and $M^2 = h^2 \cup N_2$. Then, for any core allocation of the assignment game, $(u, v) \in C(w_A)$[5] the merger $m((u, v)) = (u_{h^1} + v_{h^2}, u_1, u_2, \ldots, u_{|N_1|}, v_1, v_2, \ldots, v_{|N_2|})$ is a core allocation of the assignment game with a central player $(\{h\} \cup N_1 \cup N_2, v_A)$

[4] The bargaining set of a game is the set of all imputations at which no player i has an objection (against some j) that is not met by a counter-objection (of j). Let $x \in I(v)$. An objection of i against j to x is a pair (S, y), $i \in S \subseteq N \setminus \{j\}$ and $y \in \mathbf{R}^s$, with $y(S) = v(S)$ such that $y_k > x_k$ for all $k \in S$. A counterobjection of j to (S, y) is a pair (T, z), $j \in T \subseteq N \setminus \{i\}$ and $z \in \mathbf{R}^t$, with $z(T) = v(T)$ such that $z_k \geq y_k$ for all $k \in T \cap S$ and $z_k \geq x_k$ for all $k \in T \setminus S$.

[5] A non-negative payoff vector $(u, v) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}$ belongs to the core of an assignment game $(M_1 \cup M_2, w_A)$ if and only if for any optimal matching μ_A, $u_i + v_j = a_{ij}$ for all $(i, j) \in \mu_A$, $u_i + v_j \geq a_{ij}$ for all $(i, j) \notin \mu_A$, and $u_i = 0$ and $v_j = 0$ if $i \in M_1$ and $j \in M_2$ are unmatched by μ_A.

since

$$C(v_A) = m(C(w_A)). \tag{4.1}$$

The next proposition shows that the set of extreme core allocations of the assignment game with a central player v_A coincides with the merger of the extreme core allocations of the associated assignment game w_A. This result does not hold in the more general matrix-based pairing situations where several agents have the double role of buyers and sellers.

Lemma 4.4.1. *Let* (N, v_A) *be an assignment game with a central player and let be* $x \in C(v_A)$*. Then* $x_i = 0$ *for all* $i \in N \setminus \{h, 1, 1'\}$*.*

Proof. Let us take $i \in N \setminus \{h, 1, 1'\}$. Then, $v_A(N) = v_A(N \setminus \{i\})$. So, $b_i = 0$. Therefore, $x_i \leq 0$ in any core allocation. Moreover, $x_i \geq v_A(\{i\}) = 0$ for these players in any core allocation. Consequently, for any $x \in C(v_A)$, $x_i = 0$ for all $i \in N \setminus \{h, 1, 1'\}$.

□

As a consequence of Lemma 4.4.1 the core of an assignment game with a central player can always be described only by the payoffs of the three players h, 1 and $1'$.

Proposition 4.4.2. *Let* $(\{h\} \cup N_1 \cup N_2, v_A)$ *be an assignment game with a central player and let* $(M^1 \cup M^2, w_A)$ *be the corresponding assignment game. Then,* $ext(C(v_A)) = m(ext(C(w_A)))$*.*

Proof. Let us first consider a trivial v_A, where $v_A(\{h\} \cup N_1 \cup N_2) = a_{hh}$. So, $w_A(M^1 \cup M^2) = a_{h^1h^2}$. Notice that $a_{hh} = a_{h^1h^2}$.

In this case, by Shapley and Shubik (1972), the core is a line segment with two extreme core allocations. When $u_{h^1} = a_{h^1h^2}$ we get one of them and when $v_{h^2} = a_{h^1h^2}$ we get the other one. By merging, both of them collapse to the only core allocation of the assignment game with a central player v_A. Thus, in this case, $(a_{hh}, 0, \ldots, 0) = ext(C(v_A)) = m(ext(C(w_A)))$.

Next, we consider an intrinsic v_A, where $v_A(\{h\} \cup N_1 \cup N_2) = w_A(M^1 \cup M^2) > a_{hh}$. In this case, $\mu_A(h^1) = 1' \in N_2$ and $\mu_A(h^2) = 1 \in N_1$. Hence, by Shapley and Shubik (1972), $u_{h^1} + v_1 = a_{h1}$ and $u_1 + v_{h^2} = a_{1h}$ for any $(u,v) \in C(w_A)$. For this case, first we show that for any two allocations $(u,v),(u',v') \in C(w_A)$, if $u_1 = u'_1$ and $v_1 = v'_1$, then $(u,v) = (u',v')$. Indeed, $v_1 = v'_1$ implies that $u_{h^1} = a_{h1} - v_1 = a_{h1} - v'_1 = u'_{h^1}$, and $u_1 = u'_1$ implies that $v_{h^2} = a_{1h} - u_1 = a_{1h} - u'_1 = v'_{h^2}$. Therefore, if two allocations in $C(w_A)$ are not equal, then the corresponding mergers are not either. Finally, we show that if $(u,v) \in ext(C(w_A))$, then $x = m((u,v)) \in ext(C(v_A))$. Let us consider an allocation $x \notin ext(C(v_A))$. Then, there are two allocations $x', x'' \in C(v_A)$ such that $x = \frac{1}{2}x' + \frac{1}{2}x''$. Since $x', x'' \in C(v_A)$, by Tejada Borm Lohman (2014), there exist $(u',v'),(u'',v'') \in C(w_A)$ such that $x' = m((u',v'))$ and $x'' = m((u'',v''))$. Now, $u_1 = x_1 = \frac{1}{2}x'_1 + \frac{1}{2}x''_1 = \frac{1}{2}u'_1 + \frac{1}{2}u''_1$ and similarly $v_1 = \frac{1}{2}v'_1 + \frac{1}{2}v''_1$. Further,

$u_{h^1} = a_{h1} - v_1 = a_{h1} - (\frac{1}{2}v'_1 + \frac{1}{2}v''_1) = \frac{1}{2}(a_{h1} - v'_1) + \frac{1}{2}(a_{h1} - v''_1) = \frac{1}{2}u'_{h^1} + \frac{1}{2}u''_{h^1}$, and similarly $v_{h^2} = \frac{1}{2}v'_{h^2} + \frac{1}{2}v''_{h^2}$. Therefore, $(u, v) \notin ext(C(w_A))$.

Notice that when $v_A(N)$ is intrinsic, there are as many extreme core allocations in v_A as extreme core allocations in w_A. Both sets are homeomorphic. □

We must remark at this point that there are simple algorithms to compute the extreme core points of an assignment game Izquierdo Núñez and Rafels (2007). Hence, the above proposition guarantees that these algorithms also provide all the extreme core points of our assignment game with a central player.

In fact, we can easily find a core element. It constists in giving the value of the grand coalition, $v_A(N)$, to player h.

Proposition 4.4.3. *Let* (N, v_A) *be an assignment game with a central player where* $N = \{h\} \cup N_1 \cup N_2$*. Let* $x \in \mathbf{R}^n$ *be an allocation with* $x_h = v_A(N)$ *and* $x_i = 0$ *for all* $i \in N \setminus \{h\}$*. Then,* $x \in C(v_A)$

Proof. Notice that $x(S) = 0 \geq v_A(S)$ for all $S \subseteq N \setminus \{h\}$, and $x(S) = v_A(N) \geq v_A(S)$ for all $S \subseteq N$ with $h \in S$. Besides, $v_A(N) = x_h = x(N)$.

□

Since the game is balanced, one might wonder whether its subgames are also balanced.

Proposition 4.4.4. *Every subgame of an assignment game with a central player is again an assignment game with a central player.*

Proof. Let us consider $T \subseteq N$ with $h \in T$. The subgame v_{A_T}, where $v_{A_T}(S) = v_A(S)$ for all $S \subseteq T$, is an assignment game with a central player and corresponding matrix A_T such that the rows correspond to players in $T \cap N_1$ and columns to players in $T \cap N_2$. If $T \subseteq N \setminus \{h\}$, then we have the zero-game which is an assignment game with a central player, say player 1, with $A = (0)$.

□

As an immediate consequence of the two results above, assignment games with a central player are totally balanced.

Corollary 4.4.5. *The assignment game with a central player is totally balanced.*

Next, we show which is the structure of the core of an assignment game with a central player.

Proposition 4.4.6. *let (N, v_A) be an assignment game with a central* *player with* $N = \{h\} \cup N_1 \cup N_2$. *Then,*

$$C(v_A) = \{x \in \mathbf{R}^n \text{ such that } \begin{aligned} & x(\{h,1,1'\}) = v_A(N) \\ & x_i = 0 \text{ for all } i \in N \setminus \{h,1,1'\} \\ & 0 \leq x_1 \leq b_1 \\ & 0 \leq x_{1'} \leq b_{1'} \\ & \text{and } x_h \geq a_{hh}\} \end{aligned} \tag{4.2}$$

Proof. If the game is trivial, then by Lemma 4.3.3, the only core allocation is $x \in C_A(v)$ with $x_h = a_{hh}$ and $x_i = 0$ for all $i \in N \setminus \{h\}$. Further, $b_1 = b_{1'} = 0$.

Let us now consider an intrinsic v_A. First, $x_i = 0$ for all $i \in N \setminus \{h,1,1'\}$ by Lemma 4.4.1. Further, we can consider a classical assignment game w_A with set of sellers $M_1 = \{h^1\} \cup N_1$ and set of buyers $M_2 = \{h^2\} \cup N_2$ as considered in (4.1). Let be $(u,v) \in C(w_A)$. Then, by Shapley and Shubik (1972),

(i)	$u_1 + v_{h^2} = a_{1h}$,	*(iv)*	$u_{h^1} \geq a_{h2}$,
(ii)	$u_{h^1} + v_1 = a_{h1}$,	*(v)*	$v_{h^2} \geq a_{2h}$,
(iii)	$u_{h^1} + v_{h^2} \geq a_{hh}$,	*(vi)*	$u_i \geq 0$ for all $i \in M_1$ and
		(vii)	$v_j \geq 0$ for all $j \in M_2$

Next, the core of v_A can be obtained by (4.1). Actually, all $x \in C(v_A)$ are obtained by merging the allocations $(u,v) \in C(w_A)$. That is to say, $x_h = u_{h^1} + v_{h^2}$, $x_i = u_i$ for all $i \in N_1$ and $x_{i'} = v_i$ for all $i \in N_2$. Therefore, by (iii), we get $x_h \geq a_{hh}$; by (i) and (ii), we get $x_h+x_1+x_{1'} = a_{1h}+a_{h1} = v_A(N)$; by (vi), (i) and (v), we get $0 \leq x_1 \leq a_{1h}-a_{2h}$; and by (vii), (ii) and (iv), we get $0 \leq x_{1'} \leq a_{h1}-a_{h2}$. Finally, since $x_1 + x_{1'} = a_{1h} + a_{h1} - x_h \leq a_{1h} + a_{h1} - a_{hh}$ and $x_{1'} \geq 0$, we have that $x_1 \leq a_{1h} + a_{h1} - a_{hh}$. Further, since the marginal contribution of the first seller to N is $b_1 = \min\{a_{1h} + a_{h1} - a_{hh}, a_{1h} - a_{2h}\}$, we can write $0 \leq x_1 \leq b_1$. Similarly, we get $0 \leq x_{1'} \leq b_{1'}$.

□

In fact, if we consider a three-player (N, v_A), all extreme core allocations can be simply obtained as the following corollary shows.

Corollary 4.4.7. *let (N, v_A) be an assignment game with a central player with $N = \{h, 1, 1'\}$. Then, $C(v_A) = conv\{(v_A(N) - b_1, b_1, 0), (v_A(N) - b_{1'}, 0, b_{1'}), (a_{hh}, b_1, v_A(N) - a_{hh} - b_1), (a_{hh}, v_A(N) - a_{hh} - b_{1'}, b_{1'}), (v_A(N), 0, 0)\}$.*

Proof. When the game is trivial, all five vectors coincide. The core is a unique point, $C(v_A) = \{(a_{hh}, 0, 0)\}$. On the other hand, when the game is intrinsic, by (4.2), the core is the set of efficient vectors such that $0 \leq x_1 \leq b_1$, $0 \leq x_{1'} \leq b_{1'}$ and $a_{hh} \leq x_h \leq v_A(N)$.

Since $b_1 + a_{hh} = a_{1h} + a_{h1} - \max\{a_{hh}, a_{h1}\} + a_{hh} \leq v_A(N)$, all lower and upper bounds are attainable in the core. Hence, we can obtain all five extreme core allocations by just combining the lower bound of a player with the upper bound of another player and determining the payoff of the third player by efficiency. Notice that some of them may coincide.

□

Since all extreme core points are marginal worth vectors, the three-player game $(\{h, 1, 1'\}, v_A)$ has the CoMa-property. Moreover it is easy to check that each marginal worth vector coincides with one of these extreme points and hence the game is convex.

Remark 4.4.8. *If we consider more than three players, the core given by those five vectors in Corollary 4.4.7 no longer determines the core. Example 4.3.2 is a counterexample:* $v_A(N) = 9$, $b_1 = 0$ *and, therefore, the vector* $x \in \mathbb{R}^5$ *where* $x_h = a_{hh} = 2$ *and* $x_1 = b_1 = 0$ *does not belong to the core of the game since* $x(\{h, 1\}) = 2 \ngeq v_A(\{h, 1\}) = 5$.

Notice that from any matrix A, we may consider another matrix A' where $a'_{hh} = v_A(N \backslash \{1, 1'\})$ and $a'_{ij} = a_{ij}$ otherwise. Then, the assignment games with a central player v_A and $v_{A'}$ have the same core, $C(v_A) = C(v_{A'})$, and the set of vectors $(x_h, x_1, x_{1'})$ such that $x \in C(v_A)$ coincide with the core of the subgame $(T, v_{A'_T})$ with $T = \{h, 1, 1'\}$ whose extreme points are given in Corollary 4.4.7.

When matrices A and A' are equal, i.e. $a_{hh} \geq a_{2h} + a_{h2}$, the extreme core points are marginal worth vectors where the order θ is such that the first three positions are occupied by players in $\{h, 1, 1'\}$. So, when $a_{hh} \geq a_{2h} + a_{h2}$, v_A satisfies the CoMa-property.

Once the structure of the core of the game is analyzed, let us study whether this core has additional stability properties.

Definition 4.4.9. *Let (N, v) be a coalitional game and let $C(v)$ be its core. The core $C(v)$ is a stable set if for each $y \in I(v) \setminus C(v)$ there exist $x \in C(v)$ and a nonempty coalition S such that (i) $x_i > y_i$ for all $i \in S$ and (ii) $x(S) \leq v(S)$. Then, we say that x dominates y.*

By definition, core elements are not dominated by any imputation, while an allocation outside the core is always dominated by some imputation. However this may not be a strong enough argument to discard an out-of-the-core allocation since the imputation that dominates it may also be outside the core. On the contrary, when the core is a stable set, each imputation outside the core is dominated by some core imputation.

First we consider assignment games with a central player with only three players, i.e $N = \{h, 1, 1'\}$. From the proof of Corollary 4.4.7, the core of this three-player game is the convex hull of the marginal worth vectors. Thus, the game is convex and its core is a stable set Shapley (1971). However, the question still remains open for more than three players. Let us regard Example 4.3.2 where we exchange the two last columns so that $a_{h1} = 4 > a_{h2} = 3$. It allows us to show that a non stable core of an assignment game with a central player can be found. Indeed, any $x \in C(v_A)$ is such that $x_h + x_1 + x_{1'} = 9$, $x_2 = x_{2'} = x_1 = 0$ and $0 \leq x_{1'} \leq 1$. Let us take $y = (7.5, 0, 1.5, 0, 0) \in I(v) \setminus C(v_A)$. Next we show that there is no core allocation $x \in C(v_A)$ that dominates y. If $x\, dom_S\, y$, then $h \in S$ and $\{2, 2', 1\} \notin S$. So, it must be $S = \{h, 1'\}$, but then, $x_{1'} > 1.5$ which contradicts the core condition $0 \leq x_{1'} \leq 1$.

Although, in general, the core is not a stable set, we look for matrices such that the corresponding game v_A has a stable core.

Proposition 4.4.10. *Let (N, v_A) be an assignment game with a central player.*

(i) In case $a_{h2} = a_{2h} = 0$, then $C(v_A)$ is a stable set.

(ii) In case $a_{2h} > 0$ ($a_{h2} > 0$), then $C(v_A)$ is a stable set if and only if $a_{hh} \geq a_{2h} + a_{h1}$ ($a_{hh} \geq a_{1h} + a_{h2}$).

Proof. (i) Notice that whenever $a_{h2} = a_{2h} = 0$, the game is equivalent to an assignment game with a central player with $N = \{h, 1, 1'\}$ and we have already seen that its core is a stable set.

(ii) In this case we first prove the "if" part. Let us consider the trivial v_A. The core then becomes a unique point $\{(v_A(N), 0, \ldots, 0)\}$ which dominates any other imputation via coalition $S = \{h\}$.

Next, we consider the intrinsic v_A. Since $a_{hh} \geq a_{1h} + a_{h2}$ and $a_{hh} \geq a_{2h} + a_{h1}$, then $b_1 = b_{1'} = v_A(N) - a_{hh}$. Further, since any imputation $y \in I(v_A)$ is such that $y_h \geq a_{hh}$ and $y(N) = v_A(N)$, then $y_1 \leq v_A(N) - a_{hh} = b_1$ and, similarly $y_{1'} \leq b_{1'}$. So, y satisfies these core conditions in (4.2). The only core condition in (4.2) that y may break is $y_i = 0$ for all $i \in N \setminus \{h, 1, 1'\}$. So, we can consider without loss of generality that $y_2 > 0$. In this case, y is dominated via coalition $\{h, 1, 1'\}$. Indeed, since $y_2 > 0$, $y_h < v_A(N)$ and $y_1 < b_1 = v_A(N) - a_{hh}$ (similarly $y_{1'} < b_{1'}$). So, we can define the allocation $x \in C(v_A)$ such that $x_h = y_h + \alpha \leq v_A(N)$, $x_1 = y_1 + \beta \leq b_1$, $x_{1'} = y_{1'} + \gamma \leq b_{1'}$ and $\alpha, \beta, \gamma \in \mathbb{R}_+$, with $\alpha + \beta + \gamma = y_2$. It holds that $x(\{h, 1, 1'\}) \leq v_A(\{h, 1, 1'\}) = v_A(N)$ and $x_h > y_h$, $x_1 > y_1$ and $x_{1'} > y_{1'}$. So, $x \, dom_{\{h,1,1'\}} \, y$. Thus, the core is stable.

Secondly, we prove the "only if" part. Let us suppose that $a_{2h} > 0$ and $a_{hh} < a_{2h} + a_{h1}$. We show that the core is not stable in this case by providing an imputation $y \in I(v_A) \setminus C(v_A)$ such that there is no core allocation $x \in C(v_A)$ which dominates y. Take the imputation $y = (a_{2h} + a_{h1} - \epsilon, a_{1h} - a_{2h} + \epsilon, 0, \ldots, 0)$. Since $a_{hh} < a_{2h} + a_{h1}$, $v_A(N \setminus \{1\}) = a_{2h} + a_{h1}$. So, $b_1 = a_{1h} - a_{2h}$ and $y_1 = a_{1h} - a_{2h} + \epsilon > b_1$. Therefore, $y \notin C(v_A)$. If there exists a coalition $S \subseteq N$ and a core allocation $x \in C(v_A)$ such that $x \, dom_S y$, then $1 \notin S$, since $x_1 > b_1$, and any $i \in N \setminus \{h, 1, 1'\}$ cannot belong to S either, since $x_i = 0$ for these players in the core. So, it must be $S = \{h, 1'\}$. However, $x(\{h, 1'\}) > y(\{h, 1'\}) = a_{2h} + a_{h1} - \epsilon \geq \max\{a_{hh}, a_{h1}\} = v_A(\{h, 1'\})$. Notice that the last inequality follows since $a_{2h} + a_{h1} - \epsilon \geq a_{hh}$ because

$a_{hh} < a_{2h} + a_{h1}$, and $a_{2h} + a_{h1} - \epsilon \geq a_{h1}$ because $a_{2h} > 0$. So, there is no S, such that a core allocation $x \in C(v_A)$ can dominate the above imputation y.

We would proceed in a similar way when $a_{h2} > 0$ to prove that $a_{hh} \geq a_{1h} + a_{h2}$ is necessary for the core to be a stable set.

□

Thus, core stability is achieved when the reservation value of the central player is above certain bounds. When the core is not stable it remains open whether stable sets exist for this model as they exist for the classical Shapley and Shubik assignment game Núñez and Rafels(2013).

4.5 The nucleolus and the τ-value

We already know how to find the core of an assignment game with a central player which, moreover, coincides with the bargaining set. However, it is a set-solution. In this section we present how to select just one allocation of the profit obtained by N. Well known point-solution concepts are the nucleolus, $\nu(v)$, and the τ-value.

Note that for trivial assignment games with a central player, the core consists of a unique allocation. This coincides with the nucleolus since it always belongs to the core whenever it is nonempty, it also coincides with the τ-value. Indeed, since $b_h = v_A(N)$ and $b_i = 0$

for all $i \in N \setminus \{h\}$, $M_h = m_h = v_A(N)$ and $M_i = m_i = 0$ for all $i \in N \setminus \{h\}$. So, the τ-value is $\tau(v) = (v_A(N), 0, \dots, 0)$.

The remainder of the section is devoted to intrinsic assignment games with a central player.

Since any assignment game with a central player is a veto game, its nucleolus can be calculated as shown by Bahel (2018). Then, $\nu(v_A) = x^*(v_A)$ [6] if and only if

$$\varepsilon(S, x^*(v_A)) \leq -\max\left\{\frac{b_1}{2}, \frac{b_{1'}}{2}\right\} \text{ for all } S \in N^*(v_A), \qquad (4.3)$$

where $N^*(v_A) = \{S \ni h \text{ such that } \varepsilon(S, x^*(v)) > \varepsilon(N \setminus S, x^*(v))\}$.

Lemma 4.5.1. *Let v_A be an intrinsic assignment game with a central player. Then, $x^*(v_A) = (v_A(N) - \frac{b_1}{2} - \frac{b_{1'}}{2}, \frac{b_1}{2}, \frac{b_{1'}}{2}, 0 \dots, 0)$.*

Proof. We show that the upper bounds $x_1^*(v_A) = \frac{b_1}{2}$, $x_{1'}^*(v_A) = \frac{b_{1'}}{2}$ and $x_i^*(v_A) = 0$ for all $i \in N \setminus \{h, 1, 1'\}$ are attainable and then $x_h^*(v_A) = v_A(N) - \frac{b_1}{2} - \frac{b_{1'}}{2}$ since $x_h^*(v_A) \geq \frac{b_1}{2}$ and $x_h^*(v_A) \geq \frac{b_{1'}}{2}$.

The first of these two inequalities is equivalent to $2v_A(N) - b_{1'} \geq 2b_1$, which in turn is equivalent to $v_A(N \setminus \{1'\}) + 2v_A(N \setminus \{1\}) \geq v_A(N)$. Likewise, the second inequality is equivalent to $v_A(N \setminus \{1\}) + 2v_A(N \setminus \{1'\}) \geq v_A(N)$. Both inequalities are always satisfied since $v_A(N \setminus \{1'\}) \geq a_{1h} + a_{h2}$ and $v_A(N \setminus \{1\}) \geq a_{2h} + a_{h1}$.

□

[6] When there is only one veto player h, $x^*(v) = \sum_{i \in N \setminus \{h\}} \min\{t^v, \frac{b_i}{2}\} e^i + t^v e^h$ where t^v is the unique $t > 0$ such that $v(N) = t + \sum_{i \in N \setminus \{h\}} \min\{t, \frac{b_i}{2}\}$

Notice that $x^*(v_A)$ is a core allocation.

Lemma 4.5.2. *Let v_A be an intrinsic assignment game with a central player. Then, $N^*(v_A) = \{h\} \cup \{T\}_{T \subseteq N \setminus \{h,1,1'\}}$ if $a_{1h} > a_{2h}$, $a_{h1} > a_{h2}$ and $a_{hh} > a_{2h} + a_{h2}$, and $N^*(v_A) = \emptyset$ otherwise.*

Proof. First we show that players 1 and $1'$ do not belong to any coalition in $N^*(v_A)$. Let us consider $S \supseteq \{h, 1, 1'\}$, then $\varepsilon(S, x^*(v_A)) \leq \varepsilon(N \setminus S, x^*(v_A))\} = 0$ since $x^*(v_A) \in C(v_A)$. So, $S \notin N^*(v_A)$. Let us now consider $\{h, 1\} \subseteq S \subseteq N \setminus \{1'\}$, then $\varepsilon(S, x^*(v_A)) = v_A(S) - v_A(N) + \frac{b_{1'}}{2} \leq -\frac{b_{1'}}{2} = \varepsilon(N \setminus S, x^*(v_A))\}$ since $v_A(N \setminus \{1'\}) \geq v_A(S)$. Again, $S \notin N^*(v_A)$. Similarly, it is shown for a coalition S containing $1'$ but not 1.

Next we show that, for all $S = \{h\} \cup \{T\}_{T \subseteq N \setminus \{h,1,1'\}}$, $\varepsilon(S, x^*(v_A)) > \varepsilon(N \setminus S, x^*(v_A))$ if $a_{1h} > a_{2h}$, $a_{h1} > a_{h2}$ and $a_{hh} > a_{2h} + a_{h2}$. Actually, for all $S = \{h\} \cup \{T\}_{T \subseteq N \setminus \{h,1,1'\}}$, $\varepsilon(S, x^*(v_A)) = a_{hh} - x^*_h = a_{hh} - v_A(N) + \frac{b_1}{2} + \frac{b_{1'}}{2} > -\frac{b_1}{2} - \frac{b_{1'}}{2} = \varepsilon(N \setminus S, x^*(v_A))$. Indeed, the inequality $b_1 + b_{1'} = 2v_A(N) - \max\{2a_{hh}, a_{1h} + a_{h1} + a_{2h} + a_{h2}, a_{hh} + a_{1h} + a_{h2}, a_{hh} + a_{h1} + a_{2h}\} > v_A(N) - a_{hh}$ holds since $v_A(N) > a_{hh} > a_{2h} + a_{h2}$ implies $v_A(N) - a_{hh} > 0$ and $v_A(N) - (a_{2h} + a_{h2}) > v_A(N) - a_{hh}$. Furthermore, since the game is intrinsic and $a_{1h} > a_{2h}$ and $a_{h1} > a_{h2}$, we obtain $v_A(N) - (a_{1h} + a_{h2}) > 0$ and $v_A(N) - (a_{2h} + a_{h1}) > 0$.

On the other hand, if $a_{1h} = a_{2h}$, then $b_1 = 0$. Therefore, for all $S = \{h\} \cup \{T\}_{T \subseteq N \setminus \{h,1,1'\}}$, $\varepsilon(S, x^*(v_A)) \leq a_{hh} - x^*_h = a_{hh} - v_A(N) +$

$\frac{b_{1'}}{2} \leq -\frac{b_{1'}}{2} = \varepsilon(N \setminus S, x^*(v_A))$, where the last equality holds since $b_{1'} = \min\{v_A(N) - a_{hh}, v_A(N) - a_{1h} - a_{h2}\}$ which implies that $b_{1'} \leq v_A(N) - a_{hh}$. So, $N^*(v_A) = \emptyset$. Similarly, it can be shown for $a_{h1} = a_{h2}$.

Finally, if $a_{hh} \leq a_{2h} + a_{h2}$, then $a_{hh} \leq a_{1h} + a_{h2}$ and $a_{hh} \leq a_{2h} + a_{h1}$. Therefore, $b_1 = a_{1h} - a_{2h}$ and $b_{1'} = a_{h1} - a_{h2}$. So, for any $S = \{h\} \cup \{T\}_{T \subseteq N \setminus \{h,1,1'\}}$, we have $\varepsilon(S, x^*(v_A)) \leq a_{2h} + a_{h2} - x^*_h = a_{2h} + a_{h2} - (a_{1h} + a_{h1}) + \frac{b_1}{2} + \frac{b_{1'}}{2} = -\frac{b_{1'}}{2} - \frac{b_1}{2} = \varepsilon(N \setminus S, x^*(v_A))$. In this case, $N^*(v_A)$ is also an empty set.

□

Next, we find out what intrinsic v_A with $\nu(v_A) = x^*(v_A)$ look like.

Lemma 4.5.3. *Let v_A be an intrinsic assignment game with a central player. Then v_A is a big boss game[7] if and only if $N^*(v_A) = \emptyset$.*

Proof. By Lemma 4.5.2, $N^*(v_A) = \emptyset$ if and only if $a_{1h} = a_{2h}$ or $a_{h1} = a_{h2}$ or $a_{hh} \leq a_{2h} + a_{h2}$. Let us consider a coalition $S \ni h$. If $1, 1' \in S$, then $v_A(N) - v_A(S) = 0 = \sum_{i \in N \setminus S} b_i$. If $1 \in S$ but $1' \notin S$, the required condition for big boss games also holds since $v_A(N) - v_A(S) \geq v_A(N) - v_A(N \setminus \{1'\}) = \sum_{i \in N \setminus S} b_i$ by monotonicity of v_A. Likewise, the condition holds if $1' \in S$ but $1 \notin S$.

[7]The monotonic game (N, v) is called a big boss game if there is one player $i^* \in N$ satisfying the two conditions: (i) $v(S) = 0$ if $i^* \notin S$, and (ii) $v(N) - v(S) \geq \sum_{i \in N \setminus S} v(N) - v(N \setminus \{i\})$ if $i^* \in S$

However, when $1, 1' \notin S$, then $v_A(N) - v_A(S) \geq b_1 + b_{1'} = \sum_{i \in N\setminus S} b_i$ if and only if $a_{1h} = a_{2h}$ or $a_{h1} = a_{h2}$ or $a_{hh} \leq a_{2h} + a_{h2}$. Let us start with the "if" part. Indeed, $v_A(N) - v_A(S) = v_A(N) - \max_{\substack{i \in S\cap N_1\setminus\{1\} \\ j \in S \cap N_2 \setminus\{1'\}}} \{a_{hh}, a_{ih} + a_{hj}\}$. If $a_{1h} = a_{2h}$, since $b_1 = 0$, we obtain $v_A(N) - v_A(S) \geq v_A(N) - \max_{j \in S\cap N_2\setminus\{1'\}}\{a_{hh}, a_{1h} + a_{hj}\} = b_{1'} = \sum_{i\in N\setminus S} b_i$. Similarly, it can be shown for $a_{h1} = a_{h2}$. Further, if $a_{hh} \leq a_{2h} + a_{h2}$, then $a_{hh} \leq a_{1h} + a_{h2}$ and $a_{hh} \leq a_{2h} + a_{h1}$. So, we get $b_1 = a_{1h} - a_{2h}$ and $b_{1'} = a_{h1} - a_{h2}$. Thus, $v_A(N) - v_A(S) \geq a_{1h} + a_{h1} - \max\{a_{hh}, a_{2h} + a_{h2}\} = b_1 + b_{1'} = \sum_{i\in N\setminus S} b_i$.

Next, we show the "only if" part. If $a_{1h} > a_{2h}$, $a_{h1} > a_{h2}$ and $a_{hh} > a_{2h} + a_{h2}$, then $v_A(N) - v_A(\{h\}) = v_A(N) - a_{hh} < v_A(N) - v_A(N\setminus\{1\}) + v_A(N) - v_A(N\setminus\{1'\}) < b_1 + b_{1'}$. Actually, $v_A(N\setminus\{1\}) + v_A(N\setminus\{1'\}) - a_{hh} = \max\{a_{hh}, a_{1h} + a_{h2}, a_{2h} + a_{h1}, v_A(N) + a_{2h} + a_{h2} - a_{hh}\} < a_{1h} + a_{h1} = v_A(N)$, where the last inequality holds because the game is intrinsic, $a_{h1} > a_{h2}$, $a_{1h} > a_{2h}$ and $a_{hh} > a_{2h} + a_{h2}$, respectively. Therefore, v_A is not a big boss game.

□

Proposition 4.5.4. *Let v_A be an intrinsic assignment game with a central player.*

(i) If v_A is a big boss game, then $\nu(v_A) = x^*(v_A)$.

(ii) If $\nu(v_A) = x^*(v_A)$, then either v_A is a big boss game or A is such that $a_{hh} \leq a_{1h} + a_{h2}$ and $a_{hh} \leq a_{2h} + a_{h1}$ when v_A is not a

big boss game.

Proof. (i) Let us recall that $\nu(v_A) = x^*(v_A)$ if and only if (4.3) holds. Obviously, if v_A is a big boss game, since by Lemma 4.5.3, $N^*(v_A) = \emptyset$, then, (4.3) is satisfied and, therefore, $\nu(v_A) = x^*(v_A)$.

(ii) We now show that if v_A is not a big boss game and $a_{hh} > a_{1h} + a_{h2}$ or $a_{hh} > a_{2h} + a_{h1}$, then condition (4.3) does not hold.

Let us first consider that $a_{hh} > a_{1h} + a_{h2}$ while $a_{hh} \leq a_{2h} + a_{h1}$, so $b_1 = v_A(N) - (a_{2h} + a_{h1})$ and $b_{1'} = v_A(N) - a_{hh}$. Take $S = \{h\}$.

$$\begin{aligned}\varepsilon(\{h\}, x^*(v_A)) &= a_{hh} - (v_A(N) - \tfrac{v_A(N)-a_{2h}-a_{h1}}{2} - \tfrac{v_A(N)-a_{hh}}{2}) \\ &= \tfrac{a_{hh}}{2} - \tfrac{a_{2h}+a_{h1}}{2} \\ &> -\tfrac{v_A(N)-a_{hh}}{2} = -\tfrac{b_{1'}}{2} \\ &\geq -\max_{i \in N\setminus\{h\}} \tfrac{b_i}{2}\end{aligned}$$

where the first inequality holds since, by Lemma 4.5.3, $a_{1h} > a_{2h}$ which implies $v_A(N) > a_{2h} + a_{h1}$. Likewise it can be proved if we consider $a_{hh} > a_{2h} + a_{h1}$ while $a_{hh} \leq a_{1h} + a_{h2}$.

On the other hand, if $a_{hh} > a_{1h} + a_{h2}$ and $a_{hh} > a_{2h} + a_{h1}$, then $b_1 = b_{1'} = v_A(N) - a_{hh} > 0$ since the game is intrinsic. So, $\varepsilon(\{h\}, x^*(v_A)) = a_{hh} - (v_A(N) - 2\frac{v_A(N)-a_{hh}}{2}) = 0 \geq -\max_{i \in N\setminus\{h\}} \frac{b_i}{2}$.

□

The opposite implication of statement (ii) in Proposition 4.5.4 does not hold true as shown by the next counterexample.

Example 4.5.5. *Let v_A be an intrinsic assignment game with a central player where $N = \{h\} \cup \{1,2\} \cup \{1',2'\}$ and let A be such that $a_{hh} = 7$, $a_{h1} = 6$, $a_{h2} = 3$, $a_{1h} = 5$ and $a_{2h} = 2$. Notice it is not a big boss game and $a_{hh} \leq a_{1h} + a_{h2}$ and $a_{hh} \leq a_{2h} + a_{h1}$. However, condition (4.3) does not hold since $b_1 = b_{1'} = 3$, so $x^*_h(v_A) = 8$ and therefore $\varepsilon(\{h\}, x^*(v_A)) = -1 > -\frac{3}{2} = -x^*_1(v_A) = -x^*_{1'}(v_A)$.*

Although the nucleolus in Example 4.5.5 is not $x^*(v_A)$, it can be obtained by taking into account the set of players that maximize $x^*_i(v_A)$ with $i \in N \setminus \{h\}$ is $U^*(v_A) = \{1, 1'\}$, and it does not contain players in any set of $N^*(v_A)$. Its size is $u^* = 2$. Now, following Bahel (2018), since $\frac{\varepsilon(\{h\},x^*(v_A))+\max_{i\in N\setminus\{h\}} x^*_i(v_A)}{1+u^*} = \frac{1}{6}$, the nucleolus is $\nu(v_A) = (y_h, y_1, y_{1'}, 0, \ldots, 0)$ with $y_1(v_A) = x^*_1(v_A) - \frac{1}{6} = \frac{3}{2} - \frac{1}{6}$, $y_{1'}(v_A) = x^*_{1'}(v_A) - \frac{1}{6} = \frac{3}{2} - \frac{1}{6}$ and $y_h(v_A) = x^*_h + 2\frac{1}{6} = 8 + \frac{2}{6}$.

In general, for any intrinsic v_A with matrix A such that $a_{hh} > a_{1h} + a_{h2}$ or $a_{hh} > a_{2h} + a_{h1}$ while $a_{1h} > a_{2h}$ and $a_{h1} > a_{h2}$, the nucleolus can be found in a maximum of two steps following Bahel (2018). Another algorithm is provided by Arin and Feltkamp (1997).

Until now we have found out how to obtain the nucleolus. A different point-solution is the τ-value wich can be easily found for the classical assignment game. Indeed, the τ-value of an assignment game w_A is the middle point between the two bounds $M(w_A)$ and $m(w_A)$ (see Núñez and Rafels (2003). We have shown above that this is also the case of trivial assignment games with a central player, where the core is a unique point which coincides with $M(w_A)$ and $m(w_A)$. However, when the game is not trivial but intrinsic, this only happens if the matrix entry for the central player is low enough as the next proposition shows.

Proposition 4.5.6. *Let (N, v_A) be an intrinsic assignment game with a central player, where $N = \{h\} \cup N_1 \cup N_2$. The τ-value is*

$$\tau(v_A) = \begin{cases} \frac{1}{2}M(v_A) + \frac{1}{2}m(v_A) & \textit{if } a_{hh} < a_{2h} + a_{h2} \\ \frac{1}{3}M(v_A) + \frac{2}{3}m(v_A) & \textit{otherwise} \end{cases}$$

Proof. Let us firstly find the upper bound $M(v_A)$. For all $i \in N\backslash\{h, 1, 1'\}$, $b_i = 0$ since $v_A(N\backslash\{i\}) = v_A(N)$. Further, $b_h = v_A(N)$. Finally, $b_1 = v_A(N) - a_{hh}$ if $a_{hh} > a_{2h} + a_{h1}$ and $b_1 = a_{1h} - a_{2h}$ otherwise. Similarly, $b_{1'} = v_A(N) - a_{hh}$ if $a_{hh} > a_{1h} + a_{h2}$ and $b_{1'} = a_{h1} - a_{h2}$ otherwise.

Next, we find the lower bound for each player. Clearly, for all $i \in N\backslash\{h\}$, $m_i = 0$ since $v_A(S) = 0$ if $h \notin S$ and $v_A(S) - b_h = v_A(S) - v_A(N) \leq 0$ by monotonicity. Further, if $a_{hh} < a_{2h} + a_{h2}$, then $R(h, N) = R(h, N\backslash\{1\}) = R(h, N\backslash\{1'\}) = v_A(N\backslash\{1\}) + v_A(N\backslash\{1'\}) - v_A(N) = a_{2h} + a_{h2} = v_A(N\backslash\{1, 1'\}) = R(h, N\backslash\{1, 1'\})$. Therefore, $m_h = a_{2h} + a_{h2}$ if $a_{hh} < a_{2h} + a_{h2}$. Otherwise, $m_h = a_{hh}$ since $R(h, N) = R(h, N\backslash\{1\}) = R(h, N\backslash\{1'\}) = v_A(N\backslash\{1\}) + v_A(N\backslash\{1'\}) - v_A(N) = \max\{2a_{hh} - v_A(N), a_{hh} - (a_{h1} - a_{h2}), a_{hh} - (a_{1h} - a_{2h}), a_{2h} +$

$a_{h2}\} \leq a_{hh} = v_A(N\backslash\{1,1'\}) = R(h, N\backslash\{1,1'\})$. So, $m_h = a_{hh}$ if $a_{hh} \geq a_{2h} + a_{h2}$.

Let us consider the following two situations.

i) If $a_{hh} < a_{2h} + a_{h2}$, then $a_{hh} < a_{1h} + a_{h2}$ and $a_{hh} < a_{2h} + a_{h1}$. Thus, $M = (M_h, M_1, M_{1'}, M_2, M_{2'}, \dots) = (v_A(N), a_{1h} - a_{2h}, a_{h1} - a_{h2}, 0, \dots, 0)$, and $m = (m_h, m_1, m_{1'}, m_2, m_{2'}, \dots) = (a_{2h} + a_{h2}, 0, \dots, 0))$. Therefore, $\tau(v_A) = \frac{1}{2}M + \frac{1}{2}m$.

ii) If $a_{hh} \geq a_{2h} + a_{h2}$, then $M = (M_h, M_1, M_{1'}, M_2, M_{2'}, \dots) = (v_A(N), v_A(N) - a_{hh}, v_A(N) - a_{hh}, 0, \dots, 0)$, and $m = (m_h, m_1, m_{1'}, m_2, m_{2'}, \dots) = (a_{hh}, 0, \dots, 0))$. Thus, $\tau(v_A) = \frac{1}{3}M + \frac{2}{3}m$.

□

In fact, we can provide the τ-value in terms of the matrix entries from the above proposition.

$$\tau(v_A)_{h,1,1'} = \begin{cases} (\frac{a_{1h}+a_{h1}+a_{2h}+a_{h2}}{2}, \frac{a_{1h}-a_{2h}}{2}, \frac{a_{h1}-a_{h2}}{2}) & \text{if } a_{hh} < a_{h2} + a_{2h} \\ (\frac{a_{1h}+a_{h1}}{3} + \frac{2a_{hh}}{3}, \frac{a_{1h}+a_{h1}-a_{hh}}{3}, \frac{a_{1h}+a_{h1}-a_{hh}}{3}) & \text{otherwise} \end{cases}$$

and $\tau_i(v_A) = 0$ for all $i \in N\backslash\{h, 1, 1'\}$.

Although the nucleolus always lies in the core since it is nonempty, this is no longer true for the τ-value. We just need to consider the game in Example 4.5.5 with $a_{h1} = 20$. Then, $\tau(v_A) = (13, 6, 6, 0, 0) \notin C(v_A)$. However, $\tau(v_A) \in C(v_A)$ when the τ-value is the middle point between the two bounds, i.e. $\tau(v_A) = \frac{1}{2}M + \frac{1}{2}m$, as the following proposition shows.

Proposition 4.5.7. *Let v_A be an intrinsic assignment game with a central player where matrix A is such that $a_{hh} < a_{2h} + a_{h2}$. Then, $\tau(v_A) \in C(v_A)$.*

Proof. If the τ-value is a core allocation, then $\tau_h(v_A) \geq a_{hh}$, $\tau_1(v_A) < b_1$, $\tau_{1'}(v_A) < b_{1'}$. We already have $\tau_i(v_A) = 0$ for all $i \in N \setminus \{h, 1, 1'\}$ and the efficiency is already guaranteed by the definition of the τ-value.

Let us consider that $a_{hh} < a_{h2} + a_{2h}$. We first check that $\tau_h(v_A) \geq a_{hh}$. Indeed, if $a_{hh} < a_{h2} + a_{2h}$, $\tau_h(v_A) = \frac{1}{2}(a_{1h} + a_{h1} + a_{2h} + a_{h2}) \geq a_{2h} + a_{h2} > a_{hh}$.

Finally, since $a_{hh} < a_{h2} + a_{2h}$, $\tau_1(v_A) = \frac{1}{2}(a_{1h} - a_{2h}) = \frac{1}{2}b_1 \leq b_1$ and $\tau_{1'}(v_A) = \frac{1}{2}(a_{h1} - a_{h2}) = \frac{1}{2}b_{1'} \leq b_{1'}$. Thus, it is a core allocation. □

4.6 A central agent with multiple partnership

In this section we allow the central player to sell and buy more than one item. Let us think that the social bank has capacity to manage r flats. Thus, player h may sell and buy r items. This situation is given by the tuple (h, N_1, N_2, r, A) where h is the central player, the sellers are denoted by $i \in N_1$, and the buyers by $j \in N_2$. So, the set of agents is $N = \{h\} \cup N_1 \cup N_2$. Matrix A with $a_{ij} \in \mathbb{R}_+$ is such that $a_{ij} = 0$ if $h \notin \{i, j\}$.

This situation is similar to the transportation situation defined by Sànchez Soriano (2001). Nevertheless, since there is one player that has a double role as a seller and a buyer, we will refer to this situation as an assignment market with a central player with multiple partnership.

Given an assignment market with a central player with multiple partnership, (h, N_1, N_2, r, A) and a coalition $S \subseteq N$ with $h \in S$, we can consider the optimization problem $T(S)$,

$$
\begin{aligned}
T(S): \quad max \quad & \sum_{i \in S \cap N^1} \sum_{j \in S \cap N^2} a_{ij} x_{ij} \\
s.t. \quad & \sum_{j \in S \cap N^2} x_{hj} \leq r \\
& \sum_{i \in S \cap N^1} x_{ih} \leq r \\
& \sum_{j \in S \cap N^2} x_{ij} \leq 1, \quad i \in S \cap N_1 \\
& \sum_{i \in S \cap N^1} x_{ij} \leq 1, \quad j \in S \cap N_2 \\
& x_{ij} \geq 0, \quad (i, j) \in S \cap N^1 \times S \cap N^2,
\end{aligned}
$$

where $N^1 = N_1 \cup \{h\}$ and $N^2 = N_2 \cup \{h\}$.

Notice that we can work with the LP relaxation of the integer problem because of the total unimodularity of the coeficients of the matrix defining the feasible set. In this way, the solutions will be integer numbers. On the other hand, since the feasible set is bounded, we can guarantee the existence of an optimal solution.

Definition 4.6.1. *Let (h, N_1, N_2, r, A) be an assignment market with a central player with multiple partnership, where the set of agents is $N = \{h\} \cup N_1 \cup N_2$. Then, the corresponding assignment game with a central player with multiple partnership (N, v_A) is defined by*

(i) $v_A(S)$ equals the optimal solution of $T(S)$ if $h \in S$, and

(ii) $v_A(S) = 0$ otherwise.

Since buying and selling flats is only possible if $h \in S$, assignment games with a central player with multiple partnership are veto games and, therefore, its core is nonempty. In order to study this core we may consider the dual problem associated to the optimization problem that defines the characteristic function of the game (as is done in Sànchez Soriano et.al (2001) for transportation games).

If we consider the optimization problem that defines the assignment game with a central player with multiple partnership $T(S)$, the associated dual minimization problem is given by

$$\begin{aligned} T^D(S):\quad min \quad & r(u_h+v_h)+\sum_{i\in S\cap N_1} u_i+\sum_{j\in S\cap N_2} v_j \\ s.t. \quad & u_h+v_h\geq a_{hh} \\ & u_h+v_j\geq a_{hj} \quad j\in S\cap N_2 \\ & u_i+v_h\geq a_{ih} \quad i\in S\cap N_1 \\ & u_i,v_j\geq 0. \end{aligned}$$

Notice that we have removed the constraints $u_i + v_j \geq a_{ij}$ for all $i \in N_1$ and $j \in N_2$ from the set of constraints that determine the feasible set of the dual problem because they are superfluous since $a_{ij} = 0$ when $h \notin \{i, j\}$. Actually, we already have $u_i, v_j \geq 0$ which implies $u_i + v_j \geq 0$.

Proposition 4.6.2. *Let (h, N_1, N_2, r, A) be an assignment market with a central player with multiple partnership and let (N, v_A) be the corresponding game. Let $(u^*, v^*) \in \mathbb{R}^{2+|N_1|+|N_2|}$ be an optimal solution of $T^D(N)$. Then*

$$(r(u_h^*+v_h^*);u_1^*,\ldots,u_{|N_1|}^*;v_1^*,\ldots,v_{|N_2|}^*)\in C(v_A).$$

Proof. Take an optimal solution of $T^D(N)$, (u^*, v^*). By the duality theorem, $r(u_h^* + v_h^*) + \sum_{i\in N_1} u_i^* + \sum_{j\in N_2} v_j^*$ coincides with the maximum $T(N)$ and, by definition, it is $v_A(N)$. Therefore, the payoff vector $(r(u_h^* + v_h^*); u_1^*, \ldots, u_{|N_1|}^*; v_1^*, \ldots, v_{|N_2|}^*)$ is efficient.

For any coalition such that $h \in S$, $v_A(S)$ is equal to the optimal value of $T^D(S)$ by the duality theorem. For all $S \subseteq N$, let us name $S \cap N_1 = \{i_1, \ldots, i_t, \ldots, i_{n_S}\}$ and $S \cap N_2 = \{j_1, \ldots, j_t, \ldots, j_{m_S}\}$ where we assume without loss of generality that $n_S < m_S$ and $1 \leq t \leq n_S$ is such that $a_{i_t h} + a_{h j_t} > a_{hh} > a_{t+1 h} + a_{h t+1}$. Then, for all $S \ni h$, $v_A(S) = \sum_{l=1}^{t} a_{i_l h} + \sum_{l=1}^{t} a_{h j_l} + (r-t) a_{hh}$. Further,

$$\begin{aligned}
&r(u_h^* + v_h^*) + \sum_{l=1}^{n_S} u_{i_l}^* + \sum_{l=1}^{m_S} v_{j_l}^* \\
&= \sum_{l=1}^{t} (u_{i_l}^* + v_{j_l}^*) + \sum_{l=t+1}^{n_S} u_{i_l}^* + \sum_{l=t+1}^{m_S} v_{j_l}^* + r(u_h^* + v_h^*) \\
&= \sum_{l=1}^{t} (u_{i_l}^* + v_h^*) + \sum_{l=1}^{t} (u_h^* + v_{j_l}^*) + \sum_{l=t+1}^{n_S} u_{i_l}^* + \sum_{l=t+1}^{m_S} v_{j_l}^* + (r-t)(u_h^* + v_h^*) \\
&\geq \sum_{l=1}^{t} a_{i_l h} + \sum_{l=1}^{t} a_{h j_l} + (r-t) a_{hh} = v_A(S)
\end{aligned}$$

where the last inequality follows because $u_{i_l}^* + v_h^* \geq a_{i_l h}$, $u_h^* + v_{j_l}^* \geq a_{h j_l}$ and $u_h^* + v_h^* \geq a_{hh}$ since $(u_h^*, v_h^*; u_1^*, \ldots, u_{|N_1|}^*; v_1^*, \ldots, v_{|N_2|}^*)$ is a solution of $T^D(N)$.

So, $(r(u_h^* + v_h^*); u_1^*, \ldots, u_{|N_1|}; v_1^*, \ldots, v_{|N_2|}^*)$ satisfies coalitional rationality whenever $h \in S$ and individual rationality for h.

Finally, the coalitional rationality when $h \notin S$ and the individual rationality for any player in $N_1 \cup N_2$ also hold by the constraints $u_i, v_j \geq 0$ and the fact that $v_A(S) = 0$ if $h \notin S$.

□

The opposite inclusion no longer holds as the following counterexample shows.

Example 4.6.3. *Let us consider an assignment market with a central player and multiple partnership with set of sellers* $N_1 = \{1\}$*, set of buyers* $N_2 = \{1'\}$ *and* $r = 2$ *items that player* h *may sell and buy. So,* $N = \{h, 1, 1'\}$*. The unitary profit is given by the following matrix* A*,*

$$\begin{array}{c} \\ h \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} h & 1' \end{array} \\ \begin{pmatrix} 4 & 7 \\ 2 & 0 \end{pmatrix} \end{array}.$$

The core of v_A *is the convex hull of the following four vectors,* $(13, 0, 0)$*,* $(8, 0, 5)$*,* $(8, 2, 3)$ *and* $(11, 2, 0)$*. Let us take the extreme core allocation* $(13, 0, 0)$*, the corresponding solution of the dual problem* $T^D(N)$ *should have* $u_1 = v_1 = 0$*. Then,* $u_h + v_h = \frac{13}{2}$*. Therefore,* $u_h + v_1 \leq \frac{13}{2}$*, which contradicts that any feasible solution of the dual problem should hold* $u_h + v_1 \geq 7$*.*

Besides that, not all core allocations of our assignment games with a central player and multiple partnership are supported by competitive prices. To show that, let us split h into h^1 and h^2 following Tejada, Borm and Lohman (2014) and, further, split h^1 and h^2 each of them into r copies of itself, $\{h_1^1, h_2^1, \ldots, h_{n_1}^1\} = H^1$ and $\{h_1^2, h_2^2, \ldots, h_{n_2}^2\} = H^2$ respectively. We then have a one-to-one assignment market where each h_i^1 has one object on sale and each h_j^2

wants to buy one object, such that $a_{h_i^1 h_j^2} = a_{hh}$ for all $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, r\}$, $a_{h_i^1 k} = a_{hk}$ for all $k \in N_2$ and $i \in \{1, \ldots, r\}$, and $a_{kh_j^2} = a_{kh}$ for all $k \in N_1$ and $j \in \{1, \ldots, r\}$.

Let $(p_{11}, p_{12}, \ldots, p_{1r}, p_1, \ldots, p_{n_1})$ be a vector of non-negative prices, one for each object on sale. We add a null object, 0, the price of which is always $p_0 = 0$ and is valued at 0 by any buyer, $a_{0h_j^2} = a_{0k} = 0$ for all $j \in \{1, \ldots, r\}$ and $k \in \{1, \ldots, n_2\}$. Then, a pair (p, μ) where p is a vector of prices and μ a matching between $H^1 \cup N_1$ and $H^2 \cup N_2$ is a competitive equilibrium if (i) $p_l = 0$ for all $l \in H^1 \cup N_1$ unassigned by μ, and (ii) for all $k \in H^2 \cup N_2$, $\mu(k) \in D_k(p) = \{l \in H^1 \cup N_1$ such that $a_{lk} - p_l = \max_{t \in H^1 \cup N_1 \cup \{0\}} \{a_{tk} - p_t\}\}$.

It is well known Gale (1960) that in one-to-one assignment markets, all core allocations come from competitive equilibria. This is not necessary the case in many-to-many markets like transportation games. Also, in our particular case, not all core allocations are supported by competitive prices. Take Example 4.6.3 where we know $(13, 0, 0) \in C(v_A)$. However, this allocation does not come from any competitive equilibrium of the related one-to-one market with corresponding matrix

$$\begin{array}{c} \\ h_1^1 \\ h_2^1 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} h_1^2 & h_2^2 & 1' \end{array} \\ \left(\begin{array}{ccc} 4 & 4 & 7 \\ 4 & 4 & 7 \\ 2 & 2 & 0 \end{array} \right) \end{array}.$$

Assume (p_{11}, p_{12}, p_1) are competitive prices and take any optimal matching, for instance $\mu = \{(h_1^1, 1'), (h_2^1, h_2^2), (1, h_1^2)\}$. Moreover, $(13, 0, 0) = (u_{h_1^1} + u_{h_2^1} + v_{h_1^2} + v_{h_2^2}, u_1, v_1)$. Then, $u_1 = 0$ implies $p_1 = 0$, and $v_1 = 0$ implies $7 - p_{11} = 0$, so $p_{11} = 7 = u_{h_1^1}$. For the vector $(u_{h_1^1}, u_{h_2^1}, u_1; v_{h_1^2}, v_{h_2^2}, v_1) = (7, u_{h_2^1}, 0; v_{h_1^2}, v_{h_2^2}, 0)$ to be in the core of the related one-to-one assignment game, $u_{h_2^1} + v_1 \geq 7$. So, $u_{h_2^1} \geq 7$, which contradicts $u_{h_2^1} + v_{h_2^2} = 4$. Then, the core element $(13, 0, 0)$ does not come from any core element (or any competitive equilibrium) of the related one-to-one assignment game.

Notice that in the case where $r = 1$, which indeed is the class of assignment games with a central player, by equality (4.1) the core is the solution of the dual problem $T^D(N)$. Moreover, when it is intrinsic, the correspondence is one to one, as is shown in the the following proposition.

Proposition 4.6.4. *Let (h, N_1, N_2, A) be an assignment situation with a central player and let be $N = \{h\} \cup N_1 \cup N_2$. Let (N, v_A) be the corresponding game with $v_A(N) = a_{1h} + a_{h1}$. Then, there exists a unique solution to the dual problem $T^D(N)$ corresponding to each core allocation.*

Proof. Let (N, v_A) be an intrinsic assignment game with a central player. Let be $x \in C(v_A)$. We may define a solution of the dual problem $T^D(N)$, $(u, v) \in \mathbb{R}^{2+|N_1|+|N_2|}$, such that $u_i = x_i$ for all $i \in N_1$

and $v_{i'} = x_{i'}$ for all $i' \in N_2$. Then $u_h + v_h = v_A(N) - x_1 - x_{1'} = x_h$ since $x_i = x_{i'} = 0$ for all $i \in N_1 \setminus \{1\}$ and $i' \in N_2 \setminus \{1'\}$ by Proposition 5.3.1.

Further, since (u, v) is a solution of the dual problem $T^D(N)$, $u_h + v_{1'} \geq a_{h1}$ and $u_1 + v_h \geq a_{1h}$. So, $u_h + x_{1'} \geq a_{h1}$ and $x_1 + v_h \geq a_{1h}$. In other words, $u_h \geq a_{h1} - x_{1'}$ and $v_h \geq a_{1h} - x_1$. Let us sum the two inequalities, $u_h + v_h \geq a_{h1} - x_{1'} + a_{1h} - x_1$. Next, since $v_A(N) = a_{1h} + a_{h1}$, $x_h = u_h + v_h \geq v_A(N) - x_{1'} - x_1 = x_h$. Therefore, the two inequalities are in fact equalities. So, $u_h = a_{h1} - x_{1'}$ and $v_h = a_{1h} - x_1$ are uniquely determined.

Finally we check that the inequality of the dual problem $u_h + v_h \geq a_{hh}$ is satisfied. Indeed, since $x \in C(v_A)$, $u_h + v_h = x_h \geq v_A(\{h\}) = a_{hh}$.

On the other hand, inequalities $u_h + v_{i'} \geq a_{hi}$ for all $i' \in N_2 \setminus \{1'\}$ and $u_i + v_h \geq a_{ih}$ for all $i \in N_1 \setminus \{1\}$ are also satisfied. Let us take the first ones, $u_h + v_{i'} = a_{h1} - x_{1'} + x_{i'} = a_{h1} - x_{1'} \geq a_{hi}$ for all $i' \in N_2 \setminus \{1'\}$. Since $x \in C(v_A)$, we know that $x_{1'} \leq b_{1'} = \min\{a_{1h} + a_{h1} - a_{hh}, a_{h1} - a_{h2}\} \leq a_{h1} - a_{h2} \leq a_{h1} - a_{hi}$. So, $a_{hi} \leq a_{h1} - x_{1'}$. In a similar way we obtain that $u_i + v_h \geq a_{ih}$ for all $i \in N_1 \setminus \{1\}$.

□

As a consequence, when $r = 1$, by Shapley and Shubik (1972) and Gale (1960), all core allocations are suppoted by competitve equilibria.

4.7 Concluding comments

The main feature of assignment games with a central player with multiple partnership is the possibility for a player to buy objects and simultaneously be the owner of objects to be sold. This double role and the existence of a reservation value for the central player make it difficult to consider competitive prices unless we consider his seller and buyer dimension separately. Further, we split the central player in as many sellers as objects on sale he has and as many buyers as objects he wants to buy. Hence we allow different prices for different objects owned by the central player to be closer to reality where a first object may be sold at a higher price than the next one. In this way we get the related one-to-one assignment game. Although in general not all core allocations are supported by competitive prices, when the central player has a capacity of just one unit they are.

Actually, in this case, the total profit should be allocated such that the central player obtained more than his reservation value and the other players less than their marginal contribution to the whole set of agents. Otherwise, some players may have incentives to leave the market. Further, when the allocation is such that $x_i = \frac{b_i}{2}$ for all $i \in N \setminus \{h\}$ and the central player receives the remainder, it coincides with the nucleolus if the reservation value of h is greater than $a_{2h}+a_{h2}$ and it is the τ-value if it is smaller.

We may say that assignment markets with a central player with multiple ownership are three sided markets with the singularity that one side has just one player and besides this player has a double role as a seller and as a buyer. It is left for further research to consider more than one player in the third side of the market allowing them to play as sellers and buyers of more than one unit. This would generalize assignment games with a central player with multiple ownersip besides the markets in Oishi and Sakaue (2014), Stuart (1997) and Kaneko and Wooders (1982).

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Chapter 5

Assignment games with middlemen

5.1 Introduction

In a two-sided market there are two disjoint finite sets of buyers and sellers that want to trade indivisible units of some good. Each buyer wants to buy one unit and each seller has only one unit on sale. Each buyer may place different values on the units of different sellers and a valuation matrix gathers the values of all possible buyer-seller pairs. This is the setting of the assignment game introduced by Shapley and Shubik (1972). The worth of a coalition of agents is the maximum value that can be obtained by matching buyers and sellers in the coalitions. Utiliy is fully transferable by means of the price a buyer pays for the assigned object and then the core consists of the set of payoff vectors (one payoff to each agent) such that no coalition blocks since it obtains at least its worth. Shapley and Shubik prove that the core of the assignment game is non-empty and each core allocation is supported by competitive prices. The core not only coincides with the set of competitive equilibrium payoff vectors but also with the set of solutions of the dual assignment problem.

In the present chapter we assume buyers and sellers cannot meet on their own. Some middleman is needed to connect them, and this middleman may connect several buyer-seller pairs. We assume then that there is a third side in the market formed by a finite set of middlemen (disjoint with the set of buyers and sellers). This situation may represent a real state market in which value is generated by the matching of a buyer and a seller but typically real state agencies act as intermediaries. Moreover, a same house can be advertised in the website of several agencies, and each buyer also searches in several of these sites.

This situation resembles the firm-supplier-buyer in Stuart (1997), but there the value of a triplet is the sum of the value generated by firm and supplier and the value generated by supplier and buyer. In our case the middleman does not modify the value of the buyer-seller pair, that is, the profit generated by the trade of a buyer and a seller does not depend on who is the intermediary that connects them. Stuart's model is a particular instance of three-sided assignment game. Because the value of a firm-supplier-buyer triplet is defined additively, it can be guaranteed that the core of the associated three-sided assignment game is non-empty. It is also known, see Kaneko and Wooders (1982), that three-sided assignment games where values of triplets are defined arbitrarily may have an empty core.

Different to our model, each supplier in Stuart's model has unitary capacity, that is, each supplier can only connect one firm-buyer pair. Hence, the assignment markets with middlemen we consider here are three-sided assignment markets with multiple partnership (on the side of middlemen). Two-sided assignment markets with multiple partnership have been studied for instance in Kaneko (1976), Thompson (1980), Sotomayor (1992, 2002) and Sánchez-Soriano (2001). When both sides of the market allow for multiple-partnership these games are sometimes called *transportation games*.

Transportation games have a non-empty core and this core contains the set of competitive equilibrium payoff vectors but, different to the one-to-one assignment game, this inclusion may be strict, that is, not all core allocations are supported by competitive prices. Also, the set of competitive payoff vectors (strictly) includes the set of solutions of the dual assignment problem. It is shown in Sotomayor (2002) that most of the properties of the core of the one-to-one assignment game, such as the lattice structure and the opposition of interest between the two sides of the market, are lost when we allow for multiple partnership.

Middlemen are introduced in the Shapley and Shubik assignment game in Oishi and Sakaue (2014). There, each middleman can take part in at most one partnership and buyers and sellers can decide whether to trade directly or indirectly through a middleman. Their analysis focuses in competitive equilibria.

The chapter is organized as follows. After defining our model of assignment games with middlemen in Section 5.2, in Section 5.3 we show that assignment games with middlemen, where the profit of a buyer-seller pair does not depend on who is the middleman that connects them, have a non-empty core when the number of middlemen in the market is enough to connect the maximum possible number of buyer-seller pairs. Further, we provide a sufficient condition to guarantee the non-emptiness of the core when this is not the case.

Obviously, if the profit of a buyer-seller pair does depend on who is the middleman that connects them, then the assignment game with middlemen contain the three-sided assignment game of Kaneko and Wooders (1982) and hence the core may be empty.

After the analysis of the core, in Section 5.4 we define competitive prices and competitive equilibrium payoff vectors and we study the relationship between this set and the core. We show that whenever the core is non-empty, the set of competitive equilibrium payoff vectors coincides with the set of solutions of the dual assignment problem.

5.2 Assignment games with middlemen

Let us consider a finite set of buyers B, a finite set of sellers S and a finite set of middlemen M. Each seller $k \in S$ has one object to sell and a reservation value $c_k \geq 0$. Each buyer $i \in B$ wants to buy one object and values in $h_{ik} \geq 0$ the object owned by seller $k \in S$. But the transaction can only be made by means of some middleman $j \in M$. We assume that the profit derived from a transaction does not depend on who is the middleman that takes part. Hence, for each triplet $(i, j, k) \in B \times M \times S$, the profit that can be attained is $a_{ik} = max\{h_{ik} - c_k, 0\}$. An assignment situation with middlemen is defined by the tuple (B, M, S, A) where $A = (a_{ik})_{\substack{i \in B \\ k \in S}}$ is a nonegative real matrix. It happens that each middleman $j \in M$ can take part in at most r_j partnerships.

Given a subset B' of buyers, a subset S' of sellers and a subset M' of middlemen, a matching μ is a set of triplets $(i, j, k) \in B' \times M' \times S'$ such that each buyer and seller can appear in only one triplet while each middleman $j \in M$ can appear in at most r_j. The set of matchings is $\mathcal{M}(B', M', S')$. A matching μ is optimal for the market

(B, M, S, A) if and only if $\sum_{(i,j,k)\in\mu} a_{ik} \geq \sum_{(i,j,k)\in\mu'} a_{ik}$ for all $\mu' \in \mathcal{M}(B, M, S)$.

An assignment matrix for that market situation is a 3-dimensional matrix $X = (x_{ijk})_{\substack{i\in B\\ j\in M\\ k\in S}}$ with $x_{ijk} \in \{0, 1\}$ for all $i \in B, j \in M, k \in S$.

Since buyers and sellers can be assigned at most once, while middlemen may take part in several transactions, we define a feasible assignment matrix as follows.

Definition 5.2.1. *We say that an assignment matrix X is feasible for the market $\gamma = (B, M, S, A)$ if it satisfies*

i) Demand feasibility: for all $i \in B$, $\sum_{j,k} x_{ijk} \leq 1$

ii) Supply feasibility: for all $k \in S$, $\sum_{i,j} x_{ijk} \leq 1$

iii) Mediation feasibility: for all $j \in M$, $\sum_{i,k} x_{ijk} \leq r_j$

Notice that there is a one-to-one correspondence between matchings in $\mathcal{M}(B, M, S)$ and feasible assignment matrices. Each matching $\mu \in \mathcal{M}(B, M, S)$ has one corresponding feasible assignment matrix X defined by $x_{ijk} = 1$ if and only if $(i, j, k) \in \mu$ and $x_{ijk} = 0$ otherwise. We denote by X^μ the assignment matrix corresponding to the matching μ. Likewise, we denote by μ^X the matching corresponding to the assignment matrix X.

Definition 5.2.2. *A feasible assignment matrix X^* is optimal for* *the market* $\gamma = (B, M, S, A)$ *if*

$$\sum_{i\in B}\sum_{j\in M}\sum_{k\in S} a_{ik}x^*_{ijk} \geq \sum_{i\in B}\sum_{j\in M}\sum_{k\in S} a_{ik}x_{ijk}$$

for every other feasible asignment matrix X*.*

Given a market (B, M, S, A), an optimal assignment matrix X is a solution of the following integer linear program, that we call the *assignment problem*,

$$\begin{aligned} P(N) = \max \quad & \sum_{i\in B}\sum_{j\in M}\sum_{k\in S} a_{ik}x_{ijk} \\ \text{such that} \quad & \sum_{k\in S}\sum_{j\in M} x_{ijk} \leq 1 \qquad \forall i \in B \\ & \sum_{i\in B}\sum_{j\in M} x_{ijk} \leq 1 \qquad \forall k \in S \\ & \sum_{i\in B}\sum_{k\in S} x_{ijk} \leq r_j \qquad \forall j \in M \\ & x_{ijk} \in \{0, 1\}. \end{aligned} \tag{5.1}$$

We then say that the total profit from the market γ is $P(N)$ and it is reached at X.

Given an assignment market with middlemen, we can define a coalitional game with set of players $N = B \cup M \cup S$ as follows.

Definition 5.2.3. *Let us consider an assignment market with middlemen* $\gamma = (B, M, S, A)$*. We define the assignment game with middlemen as* $(B\cup M\cup S, w_A)$*, where, for all* $T \subseteq B\cup M\cup S$*, if* $B\cap T \neq \emptyset$*,*

$M \cap T \neq \emptyset$ and $S \cap T \neq \emptyset$, then

$$w_A(T) = \max_{\mu \in \mathcal{M}(B\cap T, M\cap T, S\cap T)} \sum_{(i,j,k)\in\mu} a_{ik};$$

and $w_A(T) = 0$ otherwise.

Notice that, if $B \cap T \neq \emptyset$, $M \cap T \neq \emptyset$ and $S \cap T \neq \emptyset$, then $w_A(T) = P(T)$, where $P(T)$ is the optimal value of the integer linear problem:

$$\begin{aligned} P(T) = \max \quad & \sum_{i\in B\cap T} \sum_{j\in M\cap T} \sum_{k\in T\cap S} a_{ik} x_{ijk} & & \\ \text{such that} \quad & \sum_{k\in S\cap T} \sum_{j\in M\cap T} x_{ijk} \leq 1 & \forall i \in B \cap T & \\ & \sum_{i\in B\cap T} \sum_{j\in M\cap T} x_{ijk} \leq 1 & \forall k \in S \cap T & \qquad (5.2) \\ & \sum_{i\in B\cap T} \sum_{k\in S\cap T} x_{ijk} \leq r_j & \forall j \in M \cap T & \\ & x_{ijk} \in \{0,1\} & & \end{aligned}$$

Given a matching $\mu \in \mathcal{M}(B, M, S)$ and all $j \in M$, we denote

$$\mu_B(j) = \{i \in B \text{ such that there exists } k \in S \text{ and } (i,j,k) \in \mu\},$$

$$\mu_S(j) = \{k \in S \text{ such that there exists } i \in B \text{ and } (i,j,k) \in \mu\}.$$

We say that $i \in B$ is unassigned by μ if $i \notin \bigcup_{j\in M} \mu_B(j)$, $k \in S$ is unassigned by μ if $k \notin \bigcup_{j\in M} \mu_S(j)$ and $j \in M$ is unassigned by μ if $\mu_B(j) = \emptyset$ (or equivalently $\mu_S(j) = \emptyset$).

We will say that the set of assigned middlemen is $M_\mu = \{j \in M \text{ such that } \mu_B(j) \neq \emptyset\}$, the set of assigned buyers by μ is $B_\mu = \bigcup_{j \in M} \mu_B(j)$ and the set of assigned sellers is $S_\mu = \bigcup_{j \in M} \mu_S(j)$.

Definition 5.2.4. *The vector $(u, v, w) \in \mathbb{R}^B_+ \times \mathbb{R}^M_+ \times \mathbb{R}^S_+$ is called a feasible payoff vector for (B, M, S, A) if there is a feasible assignment matrix X such that*

$$\sum_{i \in B} u_i + \sum_{j \in M} v_j + \sum_{k \in S} w_k = \sum_{\substack{i \in B \\ j \in M \\ k \in S}} a_{ik} x_{ijk}.$$

Definition 5.2.5. *A feasible outcome $((u, v, w); X)$ is stable if*

(i) $u_i \geq 0$, $v_j \geq 0$ *and* $w_k \geq 0$

(ii) $\sum_{i \in B'} u_i + v_j + \sum_{k \in S'} w_k \geq w_A(B' \cup \{j\} \cup S')$ *for all* $j \in M$, $B' \subseteq B$ *and* $S' \subseteq S$ *with* $|B'| = |S'| \leq r_j$

Condition (i) reflects that a player always has the option of remaining unmatched and condition (ii) requires that the outcome is not blocked by any coalition $B' \cup \{j\} \cup S'$ formed by a single middleman and same number of sellers than buyers.

Proposition 5.2.6. *Let $((u, v, w); X)$ be a stable outcome for (B, M, S, A) and let μ^X be the matching corresponding to the assignment matrix X. Then*

(i) $\sum_{i \in \mu^X_B(j)} u_i + v_j + \sum_{k \in \mu^X_S(j)} w_k = \sum_{(i,j,k) \in \mu^X} a_{ik}$ *for all* j *assigned by* μ^X

(ii) $u_i = 0$ *if* $i \notin B_{\mu^X}$, $v_j = 0$ *if* $j \notin M_{\mu^X}$ *and* $w_k = 0$ *if* $k \notin S_{\mu^X}$

Proof. From feasibility and stability of $((u, v, w); x)$, we have

$$\begin{aligned}
\sum_{\substack{i\in B\\ j\in M\\ k\in S}} a_{ik}x_{ijk} &= \sum_{i\in B} u_i + \sum_{j\in M} v_j + \sum_{k\in S} w_k \\
&= \sum_{j\in M_{\mu^X}} \left(\sum_{i\in\mu_B^X(j)} u_i + v_j + \sum_{k\in\mu_S^X(j)} w_k \right) + \sum_{i\notin B_{\mu^X}} u_i + \sum_{j\notin M_{\mu^X}} v_j + \sum_{k\notin S_{\mu^X}} w_k \\
&\geq \sum_{j\in M_{\mu^X}} w_A(\mu_B^X(j) \cup \{j\} \cup \mu_S^X(j)) \\
&\geq \sum_{j\in M_{\mu^X}} \sum_{(i,j,k)\in\mu^X} a_{ik} \quad = \sum_{\substack{i\in B\\ j\in M\\ k\in S}} a_{ik}x_{ijk}
\end{aligned}$$

Hence, none of the inequalities above can be strict. In fact, they must be equalities. Therefore, since for all stable outcome, the payoff of each player is nonnegative, we get that $u_i = v_j = w_k = 0$ when $i \notin B_{\mu^X}$, $j \notin M_{\mu^X}$ and $k \notin S_{\mu^X}$ respectively. Moreover, $\sum_{i\in\mu_B^X(j)} u_i + v_j + \sum_{k\in\mu_S^X(j)} w_k = \sum_{(i,j,k)\in\mu^X} a_{ik}$ for all $j \in M_{\mu^X}$. □

5.3 The core

Our first aim is to determine whether the core of the assignment game with middlemen is always non-empty, as it is the case of two-sided markets with multiple partnership or may be empty in some cases, as it is the case of three-sided assignment games. To this end, we first explore if we need to consider all core constrains or if there is a smaller subset of constraints that are enough to characterize the core.

Proposition 5.3.1. *Let $\gamma = (B, M, S, A)$ be an assignment market with middlemen and let $(B \cup M \cup S, w_A)$ be the corresponding assignment game with middlemen. Let μ be an optimal matching for γ.*

The core of this game coincides with the set of stable payoff vectors that are compatible with μ.

Proof. Let μ be an optimal matching for γ and let us name G the set of stable payoff vectors compatible with μ. Then, by Definitions 5.2.4 and 5.2.5 and Proposition 5.2.6, G is the following set.

$$G = \left\{ \begin{array}{ll} \multicolumn{2}{l}{(u,v,w) \in \mathbb{R}^B_+ \times \mathbb{R}^M_+ \times \mathbb{R}^S_+ \text{ such that}} \\ (1) & u_i = 0 \text{ if } i \text{ unassigned by } \mu \\ & v_j = 0 \text{ if } j \text{ unassigned by } \mu \\ & w_k = 0 \text{ if } k \text{ unassigned by } \mu \\ (2) & \text{for all } j \in M, \\ & \displaystyle\sum_{i \in \mu_B(j)} u_i + v_j + \sum_{k \in \mu_S(j)} w_k = \sum_{(i,j,k) \in \mu} a_{ik} \\ (3) & \text{for all } B' \subseteq B, S' \subseteq S \text{ with } |B'| = |S'| \leq r_j, \\ & \displaystyle\sum_{i \in B'} u_i + v_j + \sum_{k \in S'} w_k \geq w_A(B' \cup \{j\} \cup S') \end{array} \right\}$$

So, we want to prove that $C(w_A) = G$.

First we show that $C(w_A) \subseteq G$. Let $(u, v, w) \in C(w_A)$. Then, for $B' \subseteq B$, $S' \subseteq S$ and $j \in M$ with $|B'| = |S'| \le r_j$, we have that $\sum_{i \in B'} u_i + v_j + \sum_{k \in S'} w_k \ge w_A(B' \cup \{j\} \cup S')$ by coalitional rationality.

Furthermore, by efficiency of core allocations, $\sum_{i \in B} u_i + \sum_{j \in M} v_j + \sum_{k \in S} w_k = w_A(B \cup M \cup S)$. Then, Proposition 5.2.6 guarantees $u_i = v_j = w_k = 0$ when $i \notin B_\mu$, $j \notin M_\mu$ and $k \notin S_\mu$; and $\sum_{i \in \mu_B(j)} u_i + v_j + \sum_{k \in \mu_S(j)} w_k = \sum_{(i,j,k) \in \mu} a_{ik} = w_A(\mu_B(j) \cup \{j\} \cup \mu_S(j))$ for all $j \in M$.

Next we show that $G \subseteq C(w_A)$.

Let be $(u, v, w) \in G$. First, we show its efficiency. We just need to sum, for all $j \in M$, the equalities $\displaystyle\sum_{i \in \mu_B(j)} u_i + v_j + \sum_{k \in \mu_S(j)} w_k = \sum_{(i,j,k) \in \mu} a_{ik}$ with μ optimal for the market $\gamma = (B, M, S, A)$. Indeed,

$$\begin{aligned}
&\sum_{i \in B} u_i + \sum_{j \in M} v_j + \sum_{k \in S} w_k = \\
&\quad = \sum_{j \in M_\mu} \Big(\sum_{i \in \mu_B(j)} u_i + v_j + \sum_{k \in \mu_S(j)} w_k \Big) + \sum_{i \notin B_\mu} u_i + \sum_{j \notin M_\mu} v_j + \sum_{k \notin S_\mu} w_k \\
&\quad = \sum_{j \in M_\mu} \sum_{(i,j,k) \in \mu} a_{ik} \\
&\quad = w_A(B \cup M \cup S).
\end{aligned}$$

where the second equality follows from (1) and (2) in G.

Next we show that all $(u, v, w) \in \mathbb{R}^B_+ \times \mathbb{R}^M_+ \times \mathbb{R}^S_+$ in G are also coalitionally rational. Let us consider a coalition $B' \cup M' \cup S'$ with $B' \subseteq B$, $M' \subseteq M$ and $S' \subseteq S$, and let μ' be an optimal matching for the submarket $(B', M', S', A_{B' \times S'})$ where $A_{B' \times S'}$ is the submatrix of A formed by rows corresponding to buyers in B' and columns to sellers in S'. Then,

$$\begin{aligned}
&\sum_{i\in B'} u_i + \sum_{j\in M'} v_j + \sum_{k\in S'} w_k = \\
&\quad = \sum_{j\in M'_{\mu'}} \Big(\sum_{i\in \mu'_{B'}(j)} u_i + v_j + \sum_{k\in \mu'_{S'}(j)} w_k \Big) + \sum_{i\notin B'_{\mu'}} u_i + \sum_{j\notin M'_{\mu'}} v_j + \sum_{k\notin S'_{\mu'}} w_k \\
&\quad \geq \sum_{j\in M'_{\mu'}} w_A(\mu'_{B'}(j) \cup \{j\} \cup \mu'_{S'}(j)) \\
&\quad = \sum_{j\in M'_{\mu'}} \sum_{(i,j,k)\in \mu'} a_{ik} \\
&\quad = w_A(B'_{\mu'} \cup M'_{\mu'} \cup S'_{\mu'}) = w_A(B' \cup M' \cup S').
\end{aligned}$$

where the inequality follows from (3) in G and the fact that all payoffs in G are nonnegative. □

The above proposition gives a characterization of the core of the assignment game with middlemen, but we still do not know if this core is always non-empty. With this aim, we first consider a particular market $\gamma = (B, M, S, A)$ where the set of middlemen is enough to satisfy all demands of buyers and supplies of sellers. That is to say, $\sum_{j\in M} r_j \geq \min\{b, s\}$.

Proposition 5.3.2. *Let $\gamma = (B, M, S, A)$ be an assignment market with middlemen where $\sum_{j\in M} r_j \geq \min\{b, s\}$ and let $(B\cup M\cup S, w_A)$ be the corresponding assignment game with middlemen. Then, $C(w_A) \neq \emptyset$.*

Proof. In case that $\sum_{j\in M} r_j \geq \min\{b,s\}$, if $(x_{ijk})_{(i,j,k)\in B\times M\times S}$ is optimal for the market (B,M,S,A), then $(x_{ik})_{(i,k)\in B\times S}$ is also optimal for the two-sided assignment market (B,S,A).

Also, if $\tilde{\mu}$ is an optimal matching for the two-sided assignment market (B,S,A), then we can always define an optimal matching μ for the market (B,M,S,A) in such a way that for all $(i,k)\in\tilde{\mu}$ and some $j\in M$, with j matched in at most r_j pairs (i,k), then $(i,j,k)\in\mu$.

This implies that the ILP in (5.2) has the same solution as its LP-relaxation. Then the dual of the LP relaxation of (5.2), when $T=N$, is

$$
\begin{array}{llll}
P^D(N) = & \min & \sum_{i\in B} u_i + \sum_{j\in M} r_j v_j + \sum_{k\in S} w_k & \\
 & \text{such that} & u_i + v_j + w_k \geq a_{ik} & \text{for all } (i,j,k)\in B\times M\times S \\
 & & u_i\geq 0,\ v_j\geq 0,\ w_k\geq 0, &
\end{array}
\tag{5.3}
$$

where $(u,v,w)\in\mathbb{R}^b\times\mathbb{R}^m\times\mathbb{R}^s$.

Take (u,v,w) a solution of $P^D(N)$. Let us see that $(u',v',w')\in C(w_A)$, where $u'_i=u_i$, for all $i\in B$, $v'_j=r_jv_j$ for all $j\in M$ and $w'_k=w_k$ for all $k\in S$. By the duality theorem, $\displaystyle\sum_{i\in B} u_i + \sum_{j\in M} r_j v_j + \sum_{k\in S} w_k = w_A(B\cup M\cup S)$.

Let $T=B_j\cup\{j\}\cup S_j$ where $j\in M$, $B_j\subseteq B$ and $S_j\subseteq S$

with $|B_j| = |S_j| \leq r_j$. Let μ_T be an optimal matching for T, then
$$\sum_{i \in B_j} u_i + r_j v_j + \sum_{k \in S} w_k \geq \sum_{(i,j,k) \in \mu_T} (u_i + v_j + w_k) \geq \sum_{(i,j,k) \in \mu_T} a_{ik} = w_A(T).$$
So, all core conditions are satisfied. Therefore, that allocation (u', v', w') belongs to the core and hence $C(w_A)$ is nonempty. □

Although under the assumptions of the above proposition each solution of the dual LP provides a core point of the game, not all core allocations can be obtained from solutions of the dual LP as the following example shows.

Example 5.3.3. *Let us consider an assignment game with middlemen where* $B = \{1, 2\}$*,* $M = \{1'\}$ *with* $r = 2$*, and* $S = \{1'', 2''\}$*. The profit matrix is the following one*

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix}$$

Notice that an optimal matching is $\mu = \{(1, 1', 1''), (2, 1', 2'')\}$ *and* $(u_1, u_2; v_1; w_1, w_2) = (3, 3; 1; 0, 0)$ *is a core element. If this allocation came from a solution* $(u_1, u_2; v_1'; w_1, w_2) = (3, 3; v_1'; 0, 0)$ *of the dual LP, then* $u_1 + u_2 + 2v_1' + w_1 + w_2 = 7$*, which means* $v_1' = \frac{1}{2}$*. But then* $u_2 + v_1' + w_2 = 3.5 < 4$ *and hence it does not satisfy the constraint* $u_2 + v_1' + w_2 \geq 4$*.*

We have proved that when no more middlemen are needed to increase the number of trades, the core of the corresponding game is nonempty. Nevertheless, when the set of middlemen is not enough, we don't know if the core is non-empty. We haven't found any counter-example yet.

Although under the scarcity of middlemen, we cannot guarantee the core is nonempty. However, when there is only one middleman, the core is always non-empty since we can define an allocation which always belongs to the core. This allocation consists of $v_j = w_A(N)$ for the only middleman $j \in M$ and $u_i = w_k = 0$ for all $i \in B$ and $k \in S$. Next proposition provides a further suficient condition for the nonemptyness of the core of an assignment game with middlemen when $\sum_{j\in M} r_j < \min\{b, s\}$.

To this end, we need to introduce some additional notation. Given a real number $d \in \mathbb{R}$ and a matrix $A = (a_{ik})_{\substack{i\in B\\k\in S}}$, we define $A_d = (a^d_{ik})_{\substack{i\in B\\k\in S}}$, where $a^d_{ik} = \max\{0, a_{ik} - d\}$. Hence, we have a two-sided market (with no middlemen) (B, S, A^d) and its corresponding assignment game $(B\cup S, v_{A^d})$, where as usual $v_d(R) = \max_{\mu\in\mathcal{M}(B,S)} \sum_{(i,k)\in\mu} a^d_{ik}$, for any $R \subseteq B \cup S$ containing at least a buyer and a seller, and $v_d(R) = 0$ otherwise.

Proposition 5.3.4. *Let (B, M, S, A) be an assignment market with middlemen and let $(B \cup M \cup S, w_A)$ be the corresponding assignment game with middlemen.*

If there exists $d > 0$ such that

$$w_A(B \cup M \cup S) = v_{A^d}(B \cup S) + \left(\sum_{j \in M} r_j \right) d, \tag{5.4}$$

then $C(w_A) \neq \emptyset$.

Proof. Take $(u', w') \in C(v_{A^d})$. Notice that such a core element exists since two-sided assignment games have a non-empty core. Define the payoff vector $(u, v, w) \in \mathbb{R}^B_+ \times \mathbb{R}^M_+ \times \mathbb{R}^S_+$ by

$$\begin{aligned} u_i &= u'_i && \text{for all } i \in B, \\ w_k &= w'_k && \text{for all } k \in S, \\ v_j &= r_j d && \text{for all } j \in M. \end{aligned}$$

Notice first that (u, v, w) is a feasible payoff vector,

$$\begin{aligned} &\textstyle\sum_{i \in B} u_i + \sum_{j \in M} v_j + \sum_{k \in S} w_k \\ &\quad = \textstyle\sum_{i \in B} u'_i + \sum_{k \in S} w'_k + \left(\sum_{j \in M} r_j \right) d \\ &\quad = v_{A^d}(B \cup S) + \left(\sum_{j \in M} r_j \right) d \\ &\quad = w_A(B \cup M \cup S), \end{aligned}$$

where the last equality follows from (5.4). Moreover (u, v, w) is a stable payoff vector. Indeed, take $j \in M$ and $B' \subseteq B$ and $S' \subseteq S$ such that $|B'| = |S'| \leq r_j$. Let μ' be an optimal matching for the submarket $(B', \{j\}, S', A_{|B' \times S'})$ and $\mu'_{-j} = \{(i, k) \in B' \times S' \mid (i, j, k) \in$

$\mu'\}$. Then,

$$\begin{aligned}
\textstyle\sum_{i\in B'} u_i + \sum_{k\in S'} w_k &\geq v_{A^d}(B'\cup S') \geq \textstyle\sum_{(i,k)\in\mu'_{-j}} a^d_{ik} \\
&= \textstyle\sum_{(i,k)\in\mu'_{-j}} \max\{0, a_{ik}-d\} \\
&\geq \textstyle\sum_{(i,k)\in\mu'_{-j}} (a_{ik}-d) \\
&= \textstyle\sum_{(i,k)\in\mu'_{-j}} a_{ik} - |B'|d \\
&\geq \textstyle\sum_{(i,k)\in\mu'_{-j}} a_{ik} - r_j d \\
&= w_A(B'\cup\{j\}\cup S') - r_j d,
\end{aligned}$$

and hence

$$\sum_{i\in B'} u_i + v_j + \sum_{k\in S'} w_k \geq w_A(B'\cup\{j\}\cup S').$$

By Proposition 5.3.1, feasibility and stability guarantee that $(u,v,w)\in C(w_A)$. □

Notice that when $\sum_{j\in M} r_j \geq \min\{b,s\}$, equality (5.4) holds with $d=0$. Hence, as we already know, the core is non-empty in this case. Then, the minimum core payoff of any middlemen is zero.

Given a matching for the assignment game with middlemen, $\mu\in\mathcal{M}(B,M,S)$, we define as $\mu_{-M}=\{(i,k)\in B\times S$ such that $(i,j,k)\in\mu\}$ the corresponding matching restricted to the sets of buyers and sellers.

Notice that, under the assumption (5.4) of Proposition 5.3.4, μ is optimal for (B,M,S,A) if and only if μ_{-M} is optimal for (B,S,A^d).

Indeed, if $\mu' \in \mathcal{M}(B,S)$ satisfies $\sum_{(i,k)\in\mu'} a^d_{ik} > \sum_{(i,k)\in\mu_{-M}} a^d_{ik}$, then

$$\sum_{(i,k)\in\mu'} a^d_{ik} > \sum_{(i,k)\in\mu_{-M}} a^d_{ik} \geq \sum_{(i,k)\in\mu_{-M}} a_{ik} - \left(\sum_{j\in M} r_j\right) d,$$

which implies $v_{A^d}(B\cup S)+\left(\sum_{j\in M} r_j\right) d > w_A(B\cup M\cup S)$, in contradiction with (5.4). The converse implication is also straightforward.

In the next example, we show that there exists $d > 0$ such that condition (5.4) is satisfied. Moreover, for different values of d, different core allocations can be obtained.

Example 5.3.5. *Let $\gamma = (B, M, S, A)$ be an assignment market with middlemen where $B = \{1, 2, 3\}$, $M = \{1', 2'\}$ with $r_j = 1$ for all $j \in M$, $S = \{1'', 2'', 3''\}$ and*

$$A = \begin{pmatrix} 0 & 7 & 4 \\ 7 & 10 & \mathbf{14} \\ 4 & \mathbf{8} & 10 \end{pmatrix}.$$

This is a situation where $\sum_{j\in M} r_j = 2$ while $\min\{b, s\} = 3$.

Let (N, w_A) with $N = B\cup M\cup S$ be the corresponding assignment game with middlemen. Notice that $w_A(N) = 22$ since the sum of the capacities of the middlemen is 2 and, therefore, each optimal matching is formed by only two triplets. An optimal matching is $\mu = \{(2, 1', 3''), (3, 2', 2'')\}$.

Instead, if we consider matrix A and the corresponding classical assignment game $(B \cup S, v_A)$, the worth of the grand coalition is

$v_A(N) = 25$ which can be obtained by considering the optimal matching $\mu' = \{(1, 2''), (2, 3''), (3, 1'')\}$.

Notice that $\mu_{-M} = \{(2, 3''), (3, 2'')\}$ is not included in μ'.

However, if we consider $d = 3$, then

$$A^d = \begin{pmatrix} 0 & 4 & 1 \\ 4 & 7 & \mathbf{11} \\ 1 & \mathbf{5} & 7 \end{pmatrix}.$$

If we consider the corresponding assignment game with middlemen $(B \cup M \cup S, w_{A^d})$ and the corresponding classical assigment game $(B \cup S, v_{A^d})$, the worth of the grand coalition coincides, $w_{A^d}(N) = v_{A^d}(N) = 16$. Actually, the optimal matching in the market (B, M, S, A^d), $\mu = \{(2, 1', 3''), (3, 2', 2'')\}$, restricted to the set of buyers and sellers, i.e. $\mu_{-M} = \{(2, 3''), (3, 2'')\}$, is now also an optimal matching in the market (B, S, A^d). When we consider the market (B, S, A^d), the two following matchings are optimal: the previous one $\mu' = \{(1, 2''), (2, 3''), (3, 1'')\}$ and a new one, μ_{-M}.

Then, if we consider a core allocation of the classical assignment game v_{A^d}, $(u_1, u_2, u_3; w_1, w_2, w_3) \in C(v_{A^d})$, then we can build a core allocation of the original assignment game with middlemen by adding a payoff of d for each of the two middlemen, $(u_1, u_2, u_3; d, d; w_1, w_2, w_3) \in C(w_A)$. For instance, with $d = 3$, $(0, 4, 1; 0, 4, 7) \in C(v_{A^d})$ and, therefore, $(0, 4, 1; 3, 3; 0, 4, 7) \in C(w_A)$.

Furthermore, let us consider d equal to the marginal contribution *of a middleman to the grand coalition, i.e.* $d = w_A(N) - w_A(N \setminus \{j\}) = 22 - 14 = 8$ *with* $j \in M$*. Then,*

$$A^d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & \mathbf{6} \\ 0 & \mathbf{0} & 2 \end{pmatrix}.$$

Again, $\mu_{-M} = \{(2, 3''), (3, 2'')\}$ *is an optimal matching in the market* (B, S, A^d) *and condition (5.4) is satisfied. So, for instance,* $(0, 2, 0; 8, 8; 0, 0, 4) \in C(w_A)$ *since* $(0, 2, 0; 0, 0, 4) \in C(v_{A^d})$*.*

When we look for a value of d such that equality (5.4) holds, some information about lower bounds of the middlemen payoffs in the core may be useful.

Proposition 5.3.6. *Let* $\gamma = (B, M, S, A)$ *be an assignment market with middlemen and let* $(B \cup M \cup S, w_A)$ *be the corresponding game. Let* μ *be an optimal matching for* γ*. If* $(u, v, w) \in C(w_A)$*, then for all* $j \in M$*,*

$$v_j \geq \max_{\substack{i \in B \setminus B_\mu \\ k \in S \setminus S_\mu}} \{a_{ik}\}. \tag{5.5}$$

Proof. Take any $j \in M$ and $(i', k') \in (B \setminus B_\mu) \times (S \setminus S_\mu)$ such that $a_{i'k'} = \max_{\substack{i \in B \setminus B_\mu \\ k \in S \setminus S_\mu}} \{a_{ik}\}$. Since $u_{i'} = w_{k'} = 0$, the core constraint $u_{i'} + v_j + w_{k'} \geq a_{i'k'}$ implies $v_j \geq a_{i'k'}$. □

The next result follows from Proposition 5.3.4.

Corollary 5.3.7. *Let $\gamma = (B, M, S, A)$ be an assignment market with middlemen such that $\sum_{j \in M} r_j < \min\{b, s\}$. Let $\mu \in \mathcal{M}(B, M, S)$ be an optimal matching for γ and $d = \max_{\substack{i \in B \setminus B_\mu \\ k \in S \setminus S_\mu}} \{a_{ik}\}$.*

If μ_{-M} is optimal for (B, S, A^d), then $C(w_A) \neq \emptyset$.

Proof. Since $\sum_{j \in M} r_j < \min\{b, s\}$, and given an optimal matching $\mu \in \mathcal{M}(B, M, S)$, each middlemen $j \in M$ is matched by μ to exactly r_j buyers and r_j sellers. Moreover, $a^d_{ik} = a_{ik} - d$ for all $(i, j, k) \in \mu$. Indeed, if there exists $(i_0, k_0) \in B_\mu \times S_\mu$ such that $a_{i_0 k_0} < d$, because of the definition of d, there exists $(i_1, k_1) \in (B \setminus B_\mu) \times (S \setminus S_\mu)$ such that $a_{i_1 k_1} = d > a_{i_0 k_0}$ and then $\mu_1 = (\mu \setminus \{(i_0, k_0)\}) \cup \{(i_1, k_1)\}$ contradicts the optimality of μ.

Then,

$$w_A(B \cup M \cup S) - \left(\sum_{j \in M} r_j\right) d = \sum_{(i,k) \in \mu_{-M}} a^d_{ik} \leq v_{A^d}(B \cup S)$$

always holds and the equality follows from the assumption that μ_{-M} is optimal for (B, S, A^d). Now, Proposition 5.3.4 guarantees that $C(w_A) \neq \emptyset$. □

The core allocations built in the proof of proposition 5.3.4 have the particularity that each middlemen receives the same payoff d from each buyer-seller pair this middlemen connects. However, this needs not be the case in each core allocation.

Nevertheless, when the number of transactions where a middleman can be involved is equal for all $j \in M$, all middlemen get the same payoff in each core allocation

Corollary 5.3.8. *Let $(B \cup M \cup S, w_A)$ be an assignment game with middlemen such that $r_j = r$ for all $j \in M$ and let $C(w_A)$ be its core. Then, in a core allocation (u, v, w), all middlemen get the same payoff: $v_j = v_{j'}$ for all $j, j' \in M$.*

Proof. We prove that for each $(u, v, w) \in C(w_A)$, $v_j = v_{j'}$ for all $j, j' \in M$.

Indeed, $w_A(B' \cup \{j\} \cup S') = w_A(B' \cup \{j'\} \cup S')$ for any j, j' in M. Therefore, if there is an optimal assignment for the grand coalition, μ, where $j \in M$ is assigned to the set of buyers $\mu_B(j) = B^*$ and the set of sellers $\mu_S(j) = S^*$, then there is also an optimal assignment μ' where $j' \in M$, $j' \neq j$, is assigned to such set of buyers $\mu'_B(j') = B^*$ and sellers $\mu'_S(j') = S^*$. So, the only difference between μ and μ' is the middlemen who is assigned to each subset of buyers and sellers.

Thus, since for all $j \in M$ we have that $\sum_{i \in \mu_B(j)} u_i + v_j + \sum_{k \in \mu_S(j)} w_k = \sum_{(i,j,k) \in \mu} a_{ik}$ in each core allocation, then, for that core allocation, it also holds that $\sum_{i \in \mu_B(j)} u_i + v_{j'} + \sum_{k \in \mu_S(j)} w_k = \sum_{(i,j,k) \in \mu} a_{ik}$ for any other $j' \in M$. So, $v_j = v_{j'}$. □

5.4 Competitive equilibrium

In this section we investigate the existence of competitive equilibria for assignment markets with middlemen and its relation with the core. In two-sided assignment games, competitive equilibria exist and the set of competitive equilibrium is equivalent to the core and to the set of solutions of the dual of the linear assignment problem (Shapley and Shubik, 1972). When each buyer has capacity one but each seller can take part in several partnerships, then it is proved in Kaneko (1976) that competitive equilibria exist, they coincide with the set of solutions of the dual linear assignment problem, but not all core element is supported by competitive prices.

For three-sided assignment markets, as introduced in Kaneko and Wooders (1982), the set of competitive equilibria is equivalent to the core (see Tejada, 2010), and hence it may be empty.

Our market has three sides, one of them with multiple partnership, but also has some reminiscence of a two-sided market, since the valuation of a triplet only depends on the buyer and the seller, not on the middleman that connects them. Let us introduce competitive prices for this assignment market with middlemen.

Define a price vector as $p \in \mathbb{R}_+^M \times \mathbb{R}_+^S$ where p_j is the price of the service provided by middleman $j \in M$ and p_k is the price of the object of seller $k \in S$. Now, agents are price takers. Given a price vector $p \in \mathbb{R}_+^M \times \mathbb{R}_+^S$, middlemen and sellers supply units of services or goods (up to their capacity) to maximize revenues at p, and buyers demand any combination of middleman and seller that maximizes his/her valuation at p.

Hence, the supply of middleman $j \in M$ at price $p_j \geq 0$ is

$$s_j(p_j) = \begin{cases} r_j & \text{if } p_j > 0, \\ \{0, 1, 2, \ldots, r_j\} & \text{if } p_j = 0. \end{cases}$$

The supply of seller $k \in S$ at price $p_k \geq 0$ is

$$s_k(p_k) = \begin{cases} 1 & \text{if } p_k > 0, \\ \{0, 1\} & \text{if } p_k = 0. \end{cases}$$

The demand of buyer $i \in B$ at price vector $p \in \mathbb{R}_+^M \times \mathbb{R}_+^S$ is

$$D_i(p) = \{(j,k) \in M \times S \mid a_{ik} - p_j - p_k \geq a_{ik'} - p_{j'} - p_{k'} \text{ for all } (j', k') \in M \times S\}.$$

At this point it is convenient to introduce a dummy middleman j_0 and a dummy seller k_0 such that $a_{ik_0} = 0$ for all $i \in B$, and $p_{j_0} = p_{k_0} = 0$ at any price vector. Notice that, if the prices are high enough, a buyer $i \in B$ will demand the pair (j_0, k_0) and in this way we can guarantee that his/her demand is always non-empty.

Given any matching $\mu \in \mathcal{M}(B, M, S)$, if $(i, j, k) \in \mu$ we will write $(j, k) = \mu(i)$. Whenever buyer $i \in B$ is not matched by μ, we understand that $\mu(i) = (j_0, k_0)$.

We can now introduce the notion of competitive equilibrium.

Definition 5.4.1. *A competitive equilibrium for a market* (B, M, S, A) *is a pair* (p, μ)*, where* $p \in \mathbb{R}_+^M \times \mathbb{R}_+^S$ *is a price vector and* $\mu \in \mathcal{M}(B, M, S)$ *a matching, such that:*

1. *For each* $i \in B$*,* $\mu(i) \in D_i(p)$*;*
2. *For each* $j \in M$*, if* $|\mu_B(j)| = |\mu_S(j)| < r_j$*, then* $p_j = 0$*;*
3. *For each* $k \in S$*, if* k *is unassigned by* μ*, then* $p_k = 0$*.*

If (p, μ) is a competitive equilibrium of a market (B, M, S, A), then we say p is a vector of competitive prices and μ is a compatible matching.

The first question is whether each core allocation (u, v, w) of an assignment market with middlemen (B, M, S, A) comes from a competitive equilibrium (p, μ). This would imply that $w_k = p_k$ for all $k \in S$; $v_j = r_j p_j$ for all $j \in M$ and $u_i = a_{ik} - p_j - p_k$ for all $i \in B$ such that $(i, j, k) \in \mu$. Example 5.3.3 shows that the answer is in the negative. Notice that if the core element $(3, 3; 1; 0, 0)$ were supported by a competitive equilibrium (p, μ), then $p_{1''} = p_{2''} = 0$ and $p_{1'} = \frac{1}{2}$. But then, $u_1 = 3 - \frac{1}{2} - 0 = 2.5 \neq 3$.

Hence, whenever the core is non-empty, core and competitive equilibrium payoff vectors may not coincide. In the above example, the core allocation $(3, 3; 1; 0, 0)$ would be supported by different prices of the middleman for the different connections he/she guarentees. That is price 0 to connect buyer 1 and seller $1''$ and price 1 to connect buyer 2 and seller $2''$.

Like in two-sided assignment markets, the matching in a competitive equilibrium of an assignment market with middlemen is an optimal matching. And moreover, any optimal matching is compatible with a vector of competitive prices.

Lemma 5.4.2. *Let (B, M, S, A) be an assignment market with middlemen and (p, μ) a competitive equilibrium for this matching. Then,*

1. *μ is an optimal matching for this market and*

2. *given any other optimal matching μ', the pair (p, μ') is also a competitive equilibrium.*

Proof. Assume that (p, μ) is a competitive equilibrium but μ is not optimal. Then, there exists $\mu' \in \mathcal{M}(B, M, S)$ such that

$$\sum_{(i,j,k)\in\mu'} a_{ik} > \sum_{(i,j,k)\in\mu} a_{ik}$$

and

$$\begin{aligned}\sum_{(i,j,k)\in\mu'}(a_{ik}-p_j-p_k) &\geq \sum_{(i,j,k)\in\mu'} a_{ik} - \sum_{j\in M} r_j p_j - \sum_{k\in S} p_k \\ &> \sum_{(i,j,k)\in\mu} a_{ik} - \sum_{j\in M} r_j p_j - \sum_{k\in S} p_k \\ &= \sum_{(i,j,k)\in\mu}(a_{ik}-p_j-p_k) \\ &\quad - \sum_{j\in M}(r_j - r_j(\mu))p_j - \sum_{k\in S\setminus S_\mu} p_k \\ &= \sum_{(i,j,k)\in\mu}(a_{ik}-p_j-p_k)\end{aligned}$$

where the first inequality follows from the fact that prices are non-negative and the last equality follows from the fact that (p,μ) is a competitive equilibrium. Then, $\sum_{(i,j,k)\in\mu'}(a_{ik}-p_j-p_k) > \sum_{(i,j,k)\in\mu}(a_{ik}-p_j-p_k)$ implies that there exists $i \in B$ such that $(i,j_1,k_1)\in\mu'$, $(i,j_2,k_2)\in\mu$ and

$$a_{ik_1} - p_{j_1} - p_{k_1} > a_{ik_2} - p_{j_2} - p_{k_2}$$

which contradicts that (p,μ) is a competitive equilibrium.

As for the second statement in the lemma, since both μ and μ' are optimal matchings, $\sum_{(i,j,k)\in\mu} a_{ik} = \sum_{(i,j,k)\in\mu'} a_{ik}$, we have

$$\sum_{(i,j,k)\in\mu} a_{ik} - \sum_{j\in M} r_j p_j - \sum_{k\in S} p_k = \sum_{(i,j,k)\in\mu'} a_{ik} - \sum_{j\in M} r_j p_j - \sum_{k\in S} p_k. \quad (5.6)$$

On the other hand,

$$\begin{aligned}
&\sum_{(i,j,k)\in\mu} a_{ik} - \sum_{j\in M} r_j p_j - \sum_{k\in S} p_k \\
&\quad = \sum_{(i,j,k)\in\mu} (a_{ik} - p_j - p_k) - \sum_{j\in M} (r_j - r_j(\mu)) p_j - \sum_{k\in S\setminus S_\mu} p_k \\
&\quad \geq \sum_{(i,j,k)\in\mu'} (a_{ik} - p_j - p_k) \\
&\quad \geq \sum_{(i,j,k)\in\mu'} (a_{ik} - p_j - p_k) - \sum_{j\in M} (r_j - r_j(\mu')) p_j - \sum_{k\in S\setminus S_{\mu'}} p_k \\
&\quad = \sum_{(i,j,k)\in\mu'} a_{ik} - \sum_{j\in M} r_j p_j - \sum_{k\in S} p_k,
\end{aligned}$$

where the first inequality follows because (p, μ) is a competitive equilibrium and the second one from the fact that prices are non-negative. Now, equality (5.6) implies that the inequalities above cannot be strict and as a consequence

(i) if there exists $j \in M$ such that $r_j - r_j(\mu') > 0$, then $p_j = 0$,

(ii) $p_k = 0$ for all $k \in S \setminus S_{\mu'}$ and

(iii) for all $i \in B$, if $(i, j_1, k_1) \in \mu$ and $(i, j_2, k_2) \in \mu'$, then $a_{ik_1} - p_{j_1} - p_{k_1} = a_{ik_2} - p_{j_2} - p_{k_2}$ and hence $\mu'(i) \in D_i(p)$.

This guarantees that (p, μ') is also a competitive equilibrium. □

If we go now to the second question, that is, whether competitive equilibria always exist for assignment markets with middlemen. We can only give a partial answer.

We show that whenever an assignment market with middlemen has a non-empty core, then the set of competitive equilibria coincides with the set of solutions of the dual assignment problem. We know from Quint (1991), that an assignment market with middlemen has a non-empty core if and only if the solution of the assignment problem (5.1) coincides with that of its relaxation where the variables x_{ijk}, for $(i, j, k) \in B \times M \times S$, are not constrained to have values in $\{0, 1\}$ but just $x_{ijk} \geq 0$. In this case, the optimal value of the assignment problem coincides with the optimal value of its dual linear program.

Proposition 5.4.3. *Let (B, M, S, A) be an assignment market with middlemen such that $C(w_A) \neq \emptyset$. Then, the set of competitive equilibrium payoff vectors coincides with the solutions of the dual linear program (5.3).*

Proof. Let (u, v, w) be a solution of the dual LP in (5.3). Define the price vector $p \in \mathbb{R}^M_+ \times \mathbb{R}^S_+$ such that $p_j = v_j$ for all $j \in M$ and $p_k = w_k$ for all $k \in S$; and take $\mu \in \mathcal{M}(B, M, S)$ any optimal matching in the market (B, M, S, A). Let us see that (p, μ) is a competitive equilibrium.

Since $C(w_A) \neq \emptyset$, from the duality theory and the fact that (u, v, w) is a solution of the dual LP, $\sum_{i\in B} u_i + \sum_{j\in M} r_j v_j + \sum_{k\in S} w_k =$

$\sum_{(i,j,k)\in\mu} a_{ik}$. Then,

$$\begin{aligned}\sum_{(i,j,k)\in\mu} a_{ik} &= \sum_{i\in B} u_i + \sum_{j\in M} r_j v_j + \sum_{k\in S} w_k \\ &= \sum_{(i,j,k)\in\mu} (u_i + v_j + w_k) + \sum_{j\in M}(r_j - r_j(\mu))v_j - \sum_{k\in S\setminus S_\mu} w_k \\ &\geq \sum_{(i,j,k)\in\mu} a_{ik},\end{aligned}$$

where the inequality follows since (u, v, w) satisfies all the constraints of the dual LP. As a consequence, if $r_j - r_j(\mu) > 0$, then $p_j = v_j = 0$ and for all $k \in S \setminus S_\mu$, $p_k = w_k = 0$. Moreover, if $(i, j, k) \in \mu$, then

$$a_{ik} - p_j - p_k = a_{ik} - v_j - w_k = u_i \geq a_{il} - v_l - w_t = a_{il} - p_l - p_t,$$

for all $l \in M$ and $t \in S$, and this completes the proof that (p, μ) is a competitive equilibrium and its payoff vector is (u, v, w).

Conversely, if (p, μ) is a competitive equilibrium, consider its related payoff vector $(u, v, w) \in \mathbb{R}^B_+ \times \mathbb{R}^M_+ \times \mathbb{R}^S_+$, that is,

$$\begin{aligned}&v'_j = p_j \geq 0 && \text{for all } j \in M, \\ &w'_k = p_k \geq 0 && \text{for all } k \in S, \\ &u'_i = a_{ik} - p_j - p_k && \text{for all } i \in B \text{ such that } (i, j, k) \in \mu.\end{aligned}$$

Recall that when some $i \in B$ is unassigned we assume he/she is assigned to the dummy middleman j_0 and seller k_0 and by the convention $p_{j_0} = p_{k_0} = a_{ik_0} = 0$ the above equality also holds. Moreover, $u'_i \geq 0$ for all $i \in B$.

Now, since (p,μ) is a competitive equilibrium,

$$\sum_{i\in B} u_i' + \sum_{j\in M} r_j v_j' + \sum_{k\in S} w_k' = \sum_{(i,j,k)\in\mu} (u_i' + v_j' + w_k') + \sum_{j\in M}(r_j - r_j(\mu))p_j + \sum_{k\in S\setminus S_\mu} p_k = \sum_{(i,j,k)\in\mu} (u_i' + v_j' + w_k') = \sum_{(i,j,k)\in\mu} a_{ik},$$

where, by duality theory, $\sum_{(i,j,k)\in\mu} a_{ik}$ is the optimal value of the dual linear program, whenever $C(w_A) \neq \emptyset$.

Moreover, for all $(i,l,t) \in B \times M \times S$, if $(i,j,k) \in \mu$,

$$u_i' = a_{ik} - p_j - p_k \geq a_{it} - p_l - p_t = a_{it} - v_l' - w_t',$$

which concludes that (u', v', w') is a solution of the dual linear program (5.3). □

The above result resembles that in Kaneko (1976) for two-sided markets where buyers have unitary capacity while sellers can establish multiple partnerships. In our three-sided assignment market with middlemen, where also buyers have capacity one, the set of competitive equilibria also coincides with the set of solutions of the dual assignment problem, provided the core of the assignment market game is non-empty.

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Conclusions

In this dissertation we focus on the study of some models of coalitional games with the particularity of the existence of some special players who possess some essential information and therefore they accumulate power.

The power agregation is currently a widespread phenomenon. Company mergers, takeovers of companies or agreements between companies in order to avoid competition are more and more frequent. We study how to allocate a joint profit among all the agents involved in an economic activity taking into account that only some of these agents accumulate most of the power.

Indeed, we consider two kinds of markets, information markets and assignment markets. In the first case, the special players are patent holders and their knowledge is necessary to produce a new commodity which will be sold on a market. This market is divided into submarkets and each of them might be controlled by agents different from those patent holders. On the other hand, we consider assignment markets where, besides buyers and sellers, some other players are necessary to make any trade possible. They are the middlemen who have a double role as buyers and as sellers.

Chapters 2 and 3 are devoted to information markets while Chapters 4 and 5 are devoted to assignment markets.

Actually, in Chapters 2 and 3, two different generalizations of information market games (Muto et al., 1989) are studied. Firstly (in Chapter 2), the necessary information to produce a new commodity is divided into several parts (different technologies) and each patent holder possesses just one part. Therefore, instead of the existence of one informed player (as it is in Muto et al., 1989), there might be more than one informed player. The clan is the set of all informed players. Thus, the corresponding market is named clan information market. In the corresponding game, the clan information market game, all the players in the clan should belong to a coalition to enable the coalition to produce the new commodity and sell it on those markets controlled by the players in the coalition. This means that no single informed player has the technology required to produce the new commodity. However, all clan members together may share their technologies, even with other firms (licencing), and they all may access and make profit in submarkets where the clan formerly had no access.

The second generalization of information market games is related to the market data in the problem, that is to say, the profit attainable in each submarket. Unlike information market games (Muto et al., 1989), the profit attainable in each submarket is no longer known with certainty. Actually, we represent the uncertainty by intervals, so that the profit attainable in each submarket is given by an interval of real numbers. Thus, we know a lower bound and an upper bound of profits for each submarket.

A well known solution concept for coalitional games is the core. A generalized solution to interval games is the interval core. We can observe in both generalizations of information market games that the most a non-patent-holder can reach in the core is the profit attainable in the submarket where this agent is the only one who has access to. This means that in clan information market games, the core payoff of each player not belonging to the clan, $i \in N \setminus C$, is no greater than $r_{\{i\}}$, while in information interval games, the interval core payoff of any player but the informed one is an interval $[\underline{x}_i, \overline{x}_i] \preccurlyeq [\underline{r}_{\{i\}}, \overline{r}_{\{i\}}]$. Further, the core payoff for an informed player might be the worth of the grand coalition. Moreover, in both generalizations we can impose conditions on the market data to the core (and the interval core) to be a stable set.

The point solution concept firstly studied for these games is the Shapley value. We show how to obtain it in terms of the market data for clan information market games and, in a similar way, we find the interval Shapley value of information interval games in terms of the interval market data. Although the Shapley value satisfies desirable properties as efficiency, additivity or null player property among others, this solution does not always belong to the core of the clan information market game. Further we have the same problem with the interval Shapley value of an information interval game.

Therefore, we study another point solution. This is the τ-value, which, in clan information market games, is always a core allocation. Moreover, it yields a bi-mas. For information interval games, it was not so simple. Indeed, we couldn't find the definition of the interval τ-value in the literature. Hence, we first define the interval τ-value of an interval game and, afterwards, we study how to obtain this interval τ-value of an information interval game in terms of the market data. Although the interval τ-value can only be well defined for a subclass of interval games, the point is that information interval games belong to this subclass and therefore we are able to find the interval τ-value of information interval games in terms of the market data and even show that it is bi-mas extendable.

Chapters 4 and 5 are devoted to assignment markets. Indeed, the model of assignment games with middlemen studied in Chapter 5 considers the existence of one or more middlemen while the model of assignment games with a central player studied in Chapter 4 considers the existence of only one middleman. A particularity of this model is that the middleman may have a resarvation value. this is not the case of the middlemen in Chapter 5.

The assumptions in the classical assignment game of Shapley and Shubik (1972) are slightly modified in Chapter 4. We consider that there is a central player who has a double role as a buyer and as a seller. Moreover, the central player is necessary for any trade. This means that if a coalition does not contain the central player, then the worth of this coalition is equal to zero. We can think that this player is a kind of middleman in this market.

The other players are either buyers or sellers and, as in the classical assignment game, each buyer desires exactly one item and each seller has a supply of exactly one item. Further, in the first part of the chapter, we consider that the central player can only buy and sell one item, whereas after, this condition is relaxed. Then, we allow the central player to buy and sell more than one item. Furthermore, in Chapter 5, we consider the possibility that there were more than one middlemen who have the double role as buyers and as sellers and, in addition, they may trade more than one item. This assumptions bring us closer and closer to real situations.

Some results about assignment games with a central player are discussed in Chapter 4. Actually, its core is always non-empty. We first show how to obtain the extreme core allocations of an assignment game with a central player by means of a related assignment game. Later, we provide the core as a set of allocations that meet certain conditions. We also show what is the market data like in order to guarantee the core to be a stable set. Point solution concepts as the nucleolus and the τ-value have been provided in terms of the market data.

When we allow the central player to sell and buy more than one item, we say the central player has multiple partnership. The core of the corresponding game is also non-empty and strictly contains the optimal solution of the dual problem to the optimization problem solved to find the worth of the grand coalition.

Further, in Chapter 5, we still assume that buyers and sellers connot meet on their own. Now, however, there is more than one middleman and each of them can connect several buyer-seller pairs. With this generalization we are able to show that the corresponding assignment game with middlemen has a non-empty core if the number of middlemen is enough to connect the maximum possible number of buyer-seller pairs. Moreover, some sufficient conditions guarentee the non-emptyness of the core although the capacity of all middlemen is smaller than the maximum possible number of buyer-seller pairs. When the core is non-empty, it is characterized and the set of competitive equilibrium payoff vectors coincides with the set of solutions of the dual assignment problem.

With these concluding remarks, we have reached the end of the PhD Thesis and new questions arise. It would be interesting to study clan information games under uncertainty and obtain a new class of interval games. Moreover, we could check if the interval τ-value is well defined in this new class of interval games. Define and discuss about an interval nucleolus would also be very interesting. Finally, deeper discussions within assignment games with middlemen, would allow us to know the problem much better and to be able to prove if its core is always non-empty or it is not. Adressing these further problems might keep us, and possible readers, busy some more months.

Resum

Aquesta tesi consta de quatre capítols principals. El primer capítol analitza el comportament cooperatiu de les empreses i considera la introducció de noves tecnologies essencials per fabricar un nou producte. Proporcionem algunes solucions puntuals en termes de les dades del mercat per al joc cooperatiu que es deriva d'aquest mercat d'informació en el que interves més d'un titular de patent.

En el segon capítol, considerem que el benefici que es pot obtenir en cada submercat en que es divideix el mercat d'informació és incert. Només es coneixen les fites superior i inferior. L'objectiu del capítol consisteix a proporcionar solucions d'interval per al joc definit per intervals que es deriva d'aquell mercat.

En el tercer capítol, estudiem una clase de jocs d'assignació on hi ha un jugador central amb doble paper com a comprador i venedor, necessari per a qualsevol intercavi comercial. Estudiem el nucli d'aquest joc i analitzem si és un conjunt estable. Proporcionem algunes solucions puntuals en termes de les dades del mercat. Final-

ment, generalitzem el model permetent al jugador central vendre i comprar més dun article.

En el darrer capítol, considerem el cas on hi ha més d'un jugador amb un doble paper com vendedor i comprador. De fet, aquests són intermediaris que compren els productes dels vendedors i els venen als compradors. De forma que, sense intermediaris, l'activitat comercial no és possible. A més, cada intermediari pot intercanviar més d'una unitat. Proporcionem condicions suficients perquè el nucli del joc cooperatiu corresponent no sigui buit i estudiem els equilibris competitius.

Resumen

Esta tesis consta de cuatro capítulos principales. El primer capítulo analiza el comportamiento cooperativo de les empresas cuando se desea introducir cierta tecnología que es esencial para la fabricación de un nuevo producto. En el modelo consideramos más de un jugador con parte de la tecnología necesaria para producir el bien. Estudiamos la existencia de PMAS y mostramos que el el valor τ de este juego tiene propiedades de monotonicidad. También proporcionamos condiciones sobre el mercado, bajo las cuales el valor de Shapley pertenece a núcleo.

En el segundo capítulo, consideramos incertidumbre sobre el beneficio que se puede obtener en cada submercado. Éste no se conoce con certeza aunque si se conoce entre qué valores puede estar. De esta forma consideramos un modelo de juegos definidos por intervalos. El objetivo del capítulo consiste en proporcionar soluciones dadas por intervalos para el modelo bajo incertidumbre.

En el tercero capítulo, estudiamos una clase de juegos de asig-

nación donde hay un jugador central, necesario para cualquier intercambio que tenga lugar. Estudiamos el núcleo de este juego, analizamos si es un conjunto estable. Proporcionamos algunas soluciones puntuales en términos de los datos del mercado. También generalizamos el modelo permitiendo al jugador especial de vender y comprar más de un artículo.

Finalmente, consideramos el caso donde hay más de un intermediario que no produce bienes, ni tampoco consume. Compra los productos de los vendedores y los vende a los compradores de forma que sin estos intermediarios, el comercio no es posible. Además, cada intermediario puede intercambiar más de una unidad. Proporcionamos condiciones suficientes para que el núcleo del juego cooperativo correspondiente no sea vacío y estudiamos los equilibrios competitivos.

Summary

This tesis consists of four main chapters, the first chapter, entitled Clan information market games, analyzes the cooperative behaviour of firms, faced with the introduction of some technology with its ownership divided among several patent holders. All of them are necessary for manufacturing a new product. We study the existence of PMAS of the corresponding cooperative game and show that the τ value of this game has some monotonicity properties. We also provide conditions on the market under which the Shapley value belongs to the core.

In the second chapter, we consider uncertainty on the profit obtainable in each submarket into which the information market is divided. We only know the lower and the upper bounds of that profits. The aim of the chapter consists in providing interval solutions for the associated interval game.

In the third chapter, we study a class of assignment games where there is a central player, with a double role as buyer and as a seller,

such that the trade is not possible without him. We provide some point solutions in term of the merket data. Further, we generalize the model allowing the special player to sell and buy more than one item.

Finally, we consider the case where there is more than one player with a double role as a seller and as a buyer. They buy the goods from the sellers and they sell them to the buyers. So, without any middleman, the trade is not possible. However, they don't add value to the trade. We provide sufficient conditions for the non-emptyness of the core of the corresponding cooperative game and we study the competitive equilibrium.