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Arithmetic applications of the Euler systems of Beilinson–Flach elements and diagonal cycles

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Aos meus pais, Alberto e Josefa, por quererme sempre de xeito incondicional.

"Tra bufalo e locomotiva la differenza salta agli occhi: la locomotiva ha la strada segnata, il bufalo può scartare di lato e cadere." Francesco De Gregori.

Contents

Abstract

In this thesis we study several applications of the theory of Euler systems and p-adic L-functions, with an emphasis on special value formulas, exceptional zeros, and congruence relations.

The first chapters deal with different kinds of exceptional zero phenomena. The main result we obtain is the proof of a conjecture of Darmon, Lauder and Rotger on special values of the Hida–Rankin p-adic L-function, which may be regarded both as the proof of a Gross–Stark type conjecture, or as the determination of the $\mathcal{L}$-invariant corresponding to the adjoint representation of a weight one modular form. The proof recasts to Hida theory and to the ideas developed by Greenberg–Stevens, and makes use of the Galois deformation techniques introduced by Bellaïche and Dimitrov. We further discuss a similar exceptional zero phenomenon from the Euler system side, leading us to the construction of *derived* Beilinson–Flach classes. This allows us to give a more conceptual proof of the previous result, using the underlying properties of this Euler system. We also discuss other instances of this formalism, studying exceptional zeros at the level of cohomology classes both in the scenario of elliptic units and diagonal cycles.

The last part of the thesis aims to start a systematic study of the Artin formalism for Euler systems. This relies on ideas regarding factorizations of p-adic L-functions, and also recasts to the theory of Perrin-Riou maps and the study of canonical periods attached to weight two modular forms. We hope that these results could be extended to different settings concerning the other Euler systems studied in this memoir.

Keywords: exceptional zeros, Euler systems, p-adic L-functions, Elliptic Stark Conjecture, Eisenstein congruences.

MSC2010: 11F67, 11F80, 11G40, 11F33.

Resum

En aquesta tesi estudiem diverses aplicacions de la teoria dels sistemes d'Euler i les funcions L p-àdiques, posant l'èmfasi en fórmules de valors especials, zeros excepcionals i relacions de congruències entre formes modulars.

Als primers capítols es consideren diferents fenòmens de zeros excepcionals. El primer resultat que obtenim és la prova d'una conjectura de Darmon, Lauder i Rotger al voltant dels valors especials de la funció L p-àdica de Hida–Rankin en el cas autoadjunt. Aquest teorema es pot veure, per una banda, com un cas més de les conjectures de Gross–Stark, i per l'altra, com la determinació de l'invariant $\mathcal{L}$ corresponent a la representació adjunta d'una forma modular de pes 1. La prova fa servir teoria de Hida, així com algunes de les idees desenvolupades per Greenberg–Stevens i també les tècniques de deformacions de Galois introduïdes per Bellaïche i Dimitrov. Discutim després un fenomen semblant de zeros excepcionals des del punt de vista dels sistemes d'Euler, la qual cosa ens porta a la construcció de classes de Beilinson–Flach *derivades*. Això ens permetrà donar una prova més conceptual del resultat precedent, fent servir per a això les propietats d'aquest sistema d'Euler derivat. Discutim per últim altres exemples on s'observen aquests fenòmens al nivell de sistemes d'Euler, centrant-nos en els casos d'unitats el·líptiques i cicles diagonals.

La darrera part de la tesi pretén començar un estudi sistemàtic del formalisme d'Artin pels sistemes d'Euler. Aquesta teoria fa servir factoritzacions de les funcions L p-àdiques, i també requereix un estudi de les aplicacions de Perrin-Riou i dels períodes canònics associats a les formes modulars de pes 2. Esperem que aquests resultats es puguin estendre a altres casos relatius als sistemes d'Euler estudiats al llarg d'aquesta memòria.

Paraules claus: zeros excepcionals, sistemes d'Euler, funcions L p-àdiques, conjectures de Stark, congruències d'Eisenstein.

MSC2010: 11F67, 11F80, 11G40, 11F33.

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Introduction

The Birch and Swinnerton-Dyer conjecture and beyond

The main objective of this dissertation is the exploration of certain arithmetic applications of the Euler systems of Beilinson–Flach elements and diagonal cycles. Euler systems have been proved to be a very powerful tool for the study of Iwasawa theory and Selmer groups. Roughly speaking, they are collections of Galois cohomology classes satisfying certain compatibility relations, and are typically constructed using the étale cohomology of algebraic varieties. The genesis of the concept comes from Kolyvagin, who used them to fully prove the Birch and Swinnerton–Dyer conjecture in analytic rank one, and also from Rubin, who proposed a systematic framework to understand this cohomological tool. In the last years, many new constructions and results around these Euler systems have been obtained, and the aim of this thesis is to look at some of their arithmetic applications towards exceptional zeros, special value formulas, and Eisenstein congruences.

Any historical motivation of the problems studied in this monograph should necessarily begin with the Birch and Swinnerton-Dyer conjecture. Let E be an elliptic curve defined over the field of rational numbers, and consider its Hasse–Weil L-function, $L(E,s)$, which is defined in terms of an Euler product of local factors which converges for $\Re(s) > 3/2$. It is known that E is modular over $\mathbb{Q}$, and hence the L-function admits an analytic continuation to the whole complex plane and a functional equation relating the values at s and $2-s$. Hence, it makes sense to consider its order of vanishing at $s=1$, $\mathrm{ord}_{s=1}L(E,s)$.

The Birch and Swinnerton-Dyer conjecture (BSD for short), as stated by Tate, admits the following formulation.

Conjecture 0.0.1 (Birch–Swinnerton-Dyer)**.** *Let E be an elliptic curve defined over $\mathbb{Q}$, and let r denote the rank of its rational points as a $\mathbb{Z}$-module. Then, the following are true:*

1. $r = \mathrm{ord}_{s=1}L(E,s)$.
2. *The r-th term of the Taylor expansion, $L^{(r)}(E,1)$, satisfies that*

$$\frac{L^{(r)}(E,1)}{r! \cdot \Omega_E \cdot \mathrm{Reg}_E} = \frac{|\mathrm{Sha}(E)| \cdot \prod_{p|N} c_p}{|E_{\mathrm{tors}}|^2}. \tag{1}$$

Here, Ω_E is the canonical period attached to the elliptic curve; Reg_E *is the regulator of the Néron–Tate height pairing on E;* $\mathrm{Sha}(E)$*, its Shafarevich group, which is conjectured to be finite; and c_p is the so-called local Tamagawa number at p, only depending on the behavior of E over $\mathbb{Q}_p$. The cardinality of a finite group G has been denoted as $|G|$.*

The Shafarevich group $\mathrm{Sha}(E)$ (sometimes called Tate–Shafarevich group) is, in rough terms, the set of cohomology classes which are trivial at every local place, so it can be thought as a measure of the failure of the Hasse–Minkowski principle for elliptic curves. Its finiteness is often stated as part of the BSD conjecture.

Recall that the L-function can be also understood in terms of compatible systems of Galois representations. More precisely, for any prime ℓ, we may introduce the Tate module $V_\ell(E)$ as

$$V_\ell(E) = \left(\lim_{\leftarrow} E[\ell^n] \right) \otimes \mathbb{Q}_\ell.$$

This gives rise to a Galois action of the absolute Galois group $G_{\mathbb{Q}}$, and yields a representation

$$\varrho_{E,\ell} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(V_\ell(E)) \simeq \mathrm{GL}_2(\mathbb{Q}_\ell).$$

The family $\{V_\ell(E)\}_\ell$ is a compatible system of Galois representations, in the sense that for any $p \neq \ell$, the characteristic polynomial of Fr_p, the Frobenius element at p, has coefficients over $\mathbb{Z}$ which are independent of ℓ. Any such system gives rise to an L-function $L(\{V_\ell(E)\}_\ell, s)$, defined as the product of its local factors.

More generally, let $H/\mathbb{Q}$ be a finite Galois extension and let $\varrho : \mathrm{Gal}\,(H/\mathbb{Q}) \to \mathrm{GL}_n(L)$ be an Artin representation of degree n (here, $L/\mathbb{Q}$ is a finite extension). The L-function of E twisted by ϱ is

$$L(E, \varrho, s) = L(\{V_\ell(E) \otimes \varrho\}_\ell, s).$$

Similarly, we may define the ϱ-isotypic component of the Mordell–Weil group $E(H)$ as

$$E(H)[\varrho] = \mathrm{Hom}_{G_{\mathbb{Q}}}(V_\varrho, E(H) \otimes L),$$

where V_ϱ is the underlying L-vector space associated to the representation ϱ. Hence, it makes sense to consider the following strengthened version of the BSD conjecture.

Conjecture 0.0.2 (Equivariant BSD)**.** *The L-function $L(E, \varrho, s)$ admits analytic continuation and satisfies a functional equation relating $L(E, \varrho, s)$ to $L(E, \varrho^\vee, 2 - s)$, and moreover*

$$\dim_L E(H)[\varrho] = \mathrm{ord}_{s=1} L(E, \varrho, s).$$

Here, $\varrho^\vee$ stands for the dual representation of ϱ (usually called contragradient representation).

As an extra piece of notation, we refer to the order of vanishing of $L(E, \varrho, s)$ at $s = 1$ as the *analytic rank*, and write $r_{\mathrm{an}}(E, \varrho)$. Similarly, the value of $\dim_L E(H)[\varrho]$ is the so-called algebraic rank, and we write $r_{\mathrm{alg}}(E, \varrho)$.

Of course these conjectures are instances of a more general program, generally due to Beilinson, Bloch, and Kato. The general philosophy behind these predictions is that the vanishing of the L-functions associated to certain algebraic motives provides us with a systematic supply of rational cycles over the associated algebraic varieties. In the case of the BSD conjecture, we expect that the larger the order of vanishing of the L-function, the more the number of rational points over the elliptic curve.

Not many results are known about this conjecture. Coates and Wiles [CW77], in 1977, were the first who derived some evidence in the case that E has complex multiplication by an imaginary quadratic field and $L(E, 1) \neq 0$. The key point of their proof was the use of the system of elliptic units, that we later subsume in the general theory of Euler systems. During the eighties, Gross and Zagier [GZ86] envisaged a path to prove the conjecture when the analytic rank is 1, by establishing a relation between the derivative $L'(E, 1)$ and the height pairing of a Heegner point, which can be understood in rough terms as the analogue of the elliptic unit when the elliptic curve does not have complex multiplication. This result was used by Kolyvagin in [Kol88a] and [Kol88b] to give a complete proof of the BSD conjecture when the analytic rank is at most one[1], by showing

[1] The method of Kolyvagin establishes the first part of Conjecture 0.0.1. In the last years, there has been substantial progress towards establishing the second part, by proving that the p-adic valuation of both sides in (1) agrees. We refer the reader to the work of Jetchev, Skinner, and Wan [JSW17].

that the existence of a systematic supply of cohomology classes (Heegner points varying over ring class extensions of an imaginary quadratic field) also yields an upper bound on the size of the Selmer group and consequently on the size of the Mordell–Weil group. The following result follows after combining the Gross–Zagier formula with Kolyvagin's result, stated here in the realm of the equivariant BSD conjecture.

Theorem 0.0.3 (Gross–Zagier, Kolyvagin)**.** *Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field, and consider a character $\psi : \mathrm{Gal}\,(H/K) \to L^\times$, where H/K is abelian, and $H/\mathbb{Q}$ is Galois and dihedral. Let*

$$\varrho_\psi = \mathrm{Ind}(\psi) : \mathrm{Gal}\,(H/\mathbb{Q}) \longrightarrow \mathrm{GL}(V_\psi) \simeq \mathrm{GL}_2(L).$$

Then, if $r_{\mathrm{an}}(E, \varrho_\psi) = r \in \{0, 1\}$, it holds that $r_{\mathrm{alg}}(E, \varrho_\psi) = r$.

It was essentially the aforementioned work of Kolyvagin that brought the interest to these families of cohomology classes, commonly known under the name of Euler systems. The main reason is that it suggests the possibility of extending those results to other Artin representations, and also looking for possible generalizations to extend the methods to higher rank situations, where Heegner points are futile.

One of the most important features of Euler systems is that they arise as a *geometric realization of a p-adic L-function.* Of course this sentence needs further clarification, that will be provided along the monograph. p-adic L-functions constitute one of the most important objects of this dissertation, and can be seen as p-adic analogues of the classical (complex) L-functions, arising via the p-adic interpolation of special values of these L-functions and constructed via very different approaches (either with automorphic methods, with coherent cohomology, and even with Emerton's completed cohomology). Alternatively, and following the more classical vision of Iwasawa and his school, they can arise from the arithmetic of cyclotomic fields, and this treatment is very convenient towards studying the so-called Iwasawa main conjectures, of an outstanding importance in number theory nowadays. These conjectures, that we later recall, also establish a link between an analytic object (the p-adic L-function) and an algebraic one (the Selmer group). The deep fact beyond all this comes from the interaction between Galois representations and geometry, which leads to a strong connection between p-adic L-functions, encoding the local behavior at p of a Galois representation, and Euler systems, which are some kind of geometric avatar of the analytic objects.

The easiest examples of Euler systems come with circular and elliptic units. The latter already arose when we recalled Coates–Wiles' breakthrough on the Birch and Swinnerton–Dyer conjecture, but their relevance goes beyond that fact. For example, circular units were the key tool in the proof of the classical Iwasawa main conjecture, while elliptic units were crucial to generalize this result to quadratic imaginary fields (this result is essentially due to Rubin [Rub92]). At this point of the discussion, it seems evident that the construction of Euler systems is important in order to study bounds on Selmer groups, applications to Iwasawa theory, BSD-type results and many other arithmetical phenomena.

The Euler systems that appear in this thesis arise when trying to look at new instances of the equivariant Birch and Swinnerton-Dyer conjecture. The genesis of some of the Euler systems that are extensively discussed along this memoir goes back to Kato. He proved the following.

Theorem 0.0.4 (Kato)**.** *Let $\varrho : \mathrm{Gal}\,(H/\mathbb{Q}) \to L^\times$ be a Dirichlet character. If $r_{\mathrm{an}}(E, \varrho) = 0$, then*

$$\mathrm{Hom}_{G_\mathbb{Q}}(V_\varrho, E(H) \otimes L) = 0.$$

The proof of this result relies on the construction of cohomology classes in the Galois cohomology of the elliptic curve, varying as predicted by the theory of Euler systems. This idea was later generalized to two other settings we would like to mention: the case where ϱ is an odd, irreducible, two-dimensional Artin representation satisfying some mild restrictions; and the case where $\varrho =$

$\varrho_1 \otimes \varrho_2$, being ϱ_1 and ϱ_2 two odd, irreducible, two-dimensional Galois representations, which is self-dual and satisfies some other restrictions.

We can work out these ideas on the realm of modular forms. For the former, let E be an elliptic curve defined over $\mathbb{Q}$, and let $f \in S_2(N)$ denote its associated newform. Let $g \in S_1(N_g, \chi)$ be a weight one modular form. The Rankin–Selberg L-function associated to the eigenforms (f, g) is

$$L(f, g, s) = L(\{V_\ell(f) \otimes V_\ell(g)\}_\ell, s).$$

The equivariant BSD conjecture is then expressed as follows.

Conjecture 0.0.5. *We have that*

$$\mathrm{ord}_{s=1} L(f, g, s) = \dim_L E(H)[\varrho_g],$$

where $\varrho_g : G_{\mathbb{Q}} \to \mathrm{GL}_2(L)$ is the Artin representations attached to g, and H is the field cut out by it.

Bertolini, Darmon and Rotger proved in [BDR15a] and [BDR15b] that when $L(f, g, 1) \neq 0$, we have $\dim_L E(H)[\varrho_g] = 0$. The proof is based on the use of p-adic families of global Galois cohomology classes arising from a new Euler system, that of Beilinson–Flach elements.

The second setting we want to explore is related with the so-called triple product L-functions. Again, let E be an elliptic curve, and let $f \in S_2(N_f)$ stand for its associated weight two modular form. Let $g \in S_1(N_g, \chi)$ and $h \in S_1(N_h, \bar{\chi})$ be weight one modular forms. The triple product L-function of eigenforms (f, g, h) of weights $(2, 1, 1)$ is

$$L(f, g, h, s) = L(\{V_\ell(f) \otimes V_\ell(g) \otimes V_\ell(h)\}_\ell, s).$$

Conjecture 0.0.6. *We have that*

$$\mathrm{ord}_{s=1} L(f, g, h, s) = \dim_L E(H)[\varrho_g \otimes \varrho_h],$$

where $\varrho_g, \varrho_h : G_{\mathbb{Q}} \to \mathrm{GL}_2(L)$ are the Artin representations attached to g and h, and H is the field cut out by their tensor product.

Darmon and Rotger showed in [DR14] and [DR17] that in the rank zero situation where $L(f, g, h, 1) \neq 0$, it happens that $\dim_L E(H)[\varrho] = 0$, where $\varrho := \varrho_g \otimes \varrho_h$. However, they went further by studying a higher rank situation, which sheds some light for the first time in the BSD conjecture beyond ranks 0 and 1. In particular, let $\mathrm{Sel}_p(E, \varrho)$ stand for the ϱ-isotypic component of the Bloch–Kato Selmer group of $E(H)$, as defined in [DR17, equation (154)]. Then, they prove that under the assumptions that $L(E, \varrho, 1) = 0$ and $\mathscr{L}_p{}^{g_\alpha}(f, g, h) \neq 0$, then $\dim_{L_p} \mathrm{Sel}_p(E, \varrho) \geq 2$. Here, $\mathscr{L}_p{}^{g_\alpha}(f, g, h)$ is a special value of a p-adic L-function which may be understood as a p-adic avatar of the second derivative of the classical L-function. Their strategy is based on the construction of two canonical classes in the corresponding Selmer group, which are proved to be linearly independent assuming the non-vanishing of the p-adic L-value. The key tool for deriving these results is the systematic study of p-adic families of global Galois cohomology classes arising from Gross–Kudla–Schoen diagonal cycles in a tower of triple products of modular curves.

Further, the applications of these methods go beyond these results, and the program pioneered by Darmon and Rotger has, as one of its ultimate goals, the proof of the rationality of Stark–Heegner points (also known as Darmon points). These points are conjectural substitutes for Heegner points when the imaginary quadratic field of the theory of complex multiplication is replaced by a real quadratic field K, which can be seen as a piece of evidence towards the understanding of explicit class field theory over real quadratic fields. Although they are constructed analytically as local points on elliptic curves, they are conjectured to be rational over ring class fields of K. In particular, Darmon conjectured in [Dar01] that any linear combination of Stark–Heegner points weighted by the values of a ring class character ψ of K belongs to the corresponding piece of the Mordell–Weil group over the corresponding ring class field, being non-trivial when $L'(E/K,\psi,1) \neq 0$. Darmon and Rotger [DR20b], and also Bertolini, Seveso and Venerucci [BSV20a] with slightly different methods, showed that these linear combinations arise from global classes in the appropriate pro-p Selmer group, and are non-trivial when $\mathscr{L}_p(\mathbf{f}/K,\psi)$ does not vanish at the point associated to $(E/K,\psi)$. Here, $\mathscr{L}_p(\mathbf{f}/K,\psi)$ refers to the so-called Mazur–Kitagawa p-adic L-function attached to the real quadratic field K, as introduced in [BD14]; we come back to this later on. This proof requires the construction of a three-variable family $\kappa(\mathbf{f},\mathbf{g},\mathbf{h})$ of cohomology classes associated to a triple of Hida families. In the same way that their work explores weight one specializations of the families $(\mathbf{g},\mathbf{h})$, the first chapters of the monograph will be concerned with the same phenomenon, but now in the setting of Beilinson–Flach elements.

These Hida families, that will appear all along this dissertation and whose properties are crucial to derive some of our main results, can be understood as families of modular forms continuously varying over a rigid analytic space (the weight space). Their arithmetic applications date back to the work of Greenberg–Stevens [GS94], who proved results around the p-adic BSD conjecture in rank 0. p-adic analogues of the BSD conjecture had been introduced in 1986 by Mazur, Tate and Teitelbaum [MTT86], who observed the presence of some phenomena that were absent in the archimedean case. Among them, maybe the most interesting one was the so-called exceptional zero phenomenon, suggesting an extra vanishing of the corresponding p-adic L-function associated to some additional zeros introduced by Euler factors at the prime p. Similar situations were also studied e.g. by Bertolini and Darmon in [BD07], with applications towards the arithmetic of elliptic curves, and more recently by Venerucci [Ven16], proving under some mild assumptions a remarkable conjecture formulated by Perrin-Riou.

We emphasize the point that the existence of Hida families is something intrinsically p-adic, and in the complex world only Eisenstein series admit a parametrization by a weight variable, which resembles the p-adic notion of *family*; this suggests that certain questions involving modular forms are easier to study from a p-adic perspective, and this approach will be present all along this memoir.

A taste of the main results: a p-adic Gross–Stark formula

We now give a brief summary of one of the main results discussed along this thesis. We do this without properly introducing all the notations and definitions that this task would require, so we refer to the introduction of Chapter 3 for a more detailed treatment.

The aforementioned article of Mazur, Tate and Teitelbaum [MTT86] can be seen as a foundational work in the study of p-adic L-functions and exceptional zeros. As before, let E be an elliptic curve defined over $\mathbb{Q}$, with stable reduction modulo a prime p. The works of Mazur–Swinnerton-Dyer [MSD74], Amice–Velu [AV75], Vishik [Vis14] and Mazur–Tate–Teitalbaum [MTT86] allow us to associate a p-adic L-function to the elliptic curve, that we denote $L_p(E,s)$. This function can be defined in terms of a certain interpolation property of the corresponding complex values, and is analytic for $s \in \mathbb{Z}_p$. In particular,

$$L_p(E,1) = (1-\alpha_p^{-1})(1-\beta_p p^{-1}) \cdot \frac{L(E,1)}{\Omega_E}, \tag{2}$$

where α_p is the unit root of the p-th Hecke polynomial; $\beta_p = p/\alpha_p$ if E is ordinary and 0 in the split multiplicative case; and Ω_E is the canonical period attached to E. It may be tempting to formulate a p-adic BSD conjecture claiming that the order of vanishing of the p-adic L-function at $s=1$ also agrees with the rank of E. Unfortunately, this is no longer true: when $L(E,1) \neq 0$ and $\alpha_p = 1$ (i.e., E has split multiplicative reduction), the above formula (2) shows that $L_p(E,1)=0$. In this case, by Tate's p-adic uniformization theory, there is a p-adic integer $q_E \in p\mathbb{Z}_p$ and a p-adic analytic isomorphism

$$E(\bar{\mathbb{Q}}_p) \simeq \bar{\mathbb{Q}}_p^\times / q_E^{\mathbb{Q}}$$

which is defined over $\mathbb{Q}_p$. Let $\log_p$ be the usual p-adic logarithm on $\mathbb{Z}_p^\times$, extended to $\mathbb{Q}_p^\times$ by setting $\log_p(p) = 0$; and let ord_p stand for the normalized valuation. Define the $\mathcal{L}$-invariant of E by

$$\mathcal{L}_p(E) := \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}.$$

Greenberg and Stevens [GS94] showed that for a prime $p \geq 5$ satisfying that E has split multiplicative reduction at p, then

$$L_p'(E,1) = \mathcal{L}_p(E) \cdot \frac{L(E,1)}{\Omega_E}.$$

In general, it has been conjectured in [MTT86] that under the split multiplicative condition,

$$\mathrm{ord}_{s=1} L_p(E,s) = 1 + \mathrm{ord}_{s=1} L(E,s).$$

There are other well-known instances where the vanishing of an Euler factor gives rise to interesting arithmetic phenomena. Let η be a primitive Dirichlet character modulo N, taking values in a number field L, and let $p \nmid N$ be a fixed prime. The p-adic L-function of Kubota–Leopoldt $L_p(\eta\omega, s)$ satisfies the interpolation property

$$L_p(\eta\omega, 1-j) = (1 - (\eta\omega^{1-j})(p)p^{j-1}) L(\eta\omega^{1-j}, 1-j), \quad j \geq 1. \tag{3}$$

Under the assumption that $\eta(p) = 1$ and η is odd, $L(\eta, 0) \neq 0$, but the factor $1 - (\eta\omega^{1-j})(p)p^{j-1}$ vanishes at $j = 1$ and therefore $L_p(\eta\omega, s)$ has a trivial zero at $s = 0$. Write H for the field cut out by η, seen as a Galois character. Fix a prime $\mathfrak{P}$ of H above p and write H_p for the completion of H at $\mathfrak{P}$; this determines two $\mathbb{Z}$-module homomorphisms

$$\mathrm{ord}_{\mathfrak{P}} : \mathcal{O}_H[1/p]^\times \longrightarrow \mathbb{Z}, \quad \mathrm{Log}_{\mathfrak{P}} : \mathcal{O}_H[1/p]^\times \longrightarrow \mathbb{Z}_p,$$

where the latter is defined by

$$\mathrm{Log}_{\mathfrak{P}}(u) = \log_p(\mathbb{N}_{H_p/\mathbb{Q}_p}(u)).$$

The presence of an exceptional zero is related with the fact that, under the assumption that $\eta(p) = 1$, the group of p-units $(\mathcal{O}_H[1/p]^\times \otimes L)^{\eta^{-1}}$ is one-dimensional and we may fix a basis of it, v_η. Then, we may define

$$\mathcal{L}(\eta) := -\frac{\mathrm{Log}_{\mathfrak{P}}(v_\eta)}{\mathrm{ord}_{\mathfrak{P}}(v_\eta)},$$

and it holds that

$$L_p'(\eta\omega, 0) = \mathcal{L}(\eta) \cdot L(\eta, 0).$$

It may be instructive to compare the previous result with the more familiar situation where one considers an even character χ. Then, one recovers the celebrated Leopoldt's formula, which asserts that for a primitive, non-trivial even Dirichlet character of conductor N,

$$L_p(\chi, 1) = -\frac{1 - \chi(p)p^{-1}}{\mathfrak{g}(\bar{\chi})} \log_p(c_\chi), \tag{4}$$

where $L_p(\chi, s)$ is the Kubota–Leopoldt p-adic L-function associated to the Dirichlet character χ; $\mathfrak{g}(\bar{\chi})$ is the Gauss sum attached to a choice of primitive N-th root of unity ζ_N, and c_χ is a certain circular unit obtained as a weighted combination of cyclotomic units

$$\mathfrak{g}(\bar{\chi}) = \sum_{a=1}^{N-1} \bar{\chi}(a)\zeta_N^a, \quad c_\chi = \prod_{a=1}^{N-1} (1 - \zeta_N^a)^{\bar{\chi}(a)}.$$

The situation of exceptional zeros we discuss in the first part of the thesis shares some common features with the previous ones, as it is concerned with the convolution of two weight one modular forms (g, h). The first main result we obtain, proved in Chapter 3, may be seen as a generalization of the previous formulas, and as another instance of the Gross–Stark philosophy, which aims to relate special values of the (p-adic) L-functions attached to Artin representations with the arithmetic of number fields. More precisely, let $g \in S_1(N, \chi)$ be a weight one modular form, and let $g^* \in S_1(N, \bar{\chi})$ stand for the twist by the inverse of its nebentype (in terms of Fourier expansions, this corresponds to the complex conjugation of the coefficients). Then, one may consider the Hida–Rankin p-adic L-function $L_p(g, g^*, s)$, whose construction is indeed rather indirect, since it is based on the p-adic interpolation of both g and g^* along a Hida family (see [Hi85] and [Hi88]). This interpolation, however, relies on the choice of a p-stabilization for g and g^*. The special value $L_p(g, g^*, 1)$ (or alternatively $L_p(g, g^*, 0)$ since both are linked via a functional equation) does depend on the p-stabilization of g (but does not depend on that of h). Hence, if we write

$$x^2 - a_p(g) + \chi(p) = (x - \alpha)(x - \beta),$$

we refer to the special value $L_p(g, g^*, 1)$ associated to the α p-stabilization of g as $\mathscr{L}_p{}^{g_\alpha}$. In order to describe it, we need to introduce certain units and p-units u and v. Let H stand for the field cut out by the Artin representation attached to the adjoint representation of g, and let L denote the field of coefficients of g; enlarging it if necessary, we assume that it contains both α and β. As an extra piece of notation, let $V_{gg^*} = V_g \otimes V_{g^*}$ denote the tensor product of the Galois representations attached to the weight one modular forms g and g^*; similarly, let $\mathrm{ad}^0(g)$ stand for the adjoint representation of g, that is, the quotient of V_{gg^*} by the trivial representation.

Under certain regularity assumptions that we later state in a precise way,

$$\dim_L(\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}}} = 1, \quad \dim_L(\mathcal{O}_H[1/p]^\times/p^{\mathbb{Z}} \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}}} = 2,$$

and we may fix a basis $\{u, v\}$ of the latter, such that $u \in (\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}}}$. As a $G_{\mathbb{Q}_p}$-module, $\mathrm{ad}^0(g)$ decomposes as $\mathrm{ad}^0(g) = L \oplus L^{\alpha\otimes\bar{\beta}} \oplus L^{\beta\otimes\bar{\alpha}}$, where each line is characterized by the property that the arithmetic Frobenius Fr_p acts on it with eigenvalue 1, α/β, β/α, respectively. Let H_p denote the completion of H in $\bar{\mathbb{Q}}_p$, and let

$$u_1,\, u_{\alpha\otimes\bar{\beta}},\, u_{\beta\otimes\bar{\alpha}},\, v_1,\, v_{\alpha\otimes\bar{\beta}},\, v_{\beta\otimes\bar{\alpha}} \,\in\, H_p^\times \otimes_{\mathbb{Q}} L$$

denote the projection of the elemets u and v in $(H_p^\times \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}_p}}$ to the above lines.

Then, we have the following result.

Theorem 0.0.7. *With the previous notations, let $\mathscr{L}_p{}^{g_\alpha}$ denote the special value $L_p(g, g^*, 1)$ associated to the α p-stabilization of g. Then, the following equality holds up to $L^\times$:*

$$\mathscr{L}_p{}^{g_\alpha} = \frac{\log_p(v_1) \cdot \log_p(u_{\alpha\otimes\bar{\beta}}) - \log_p(u_1) \cdot \log_p(v_{\alpha\otimes\bar{\beta}})}{\log_p(u_{\alpha\otimes\bar{\beta}})}.$$

Alternatively, this result may be understood in the framework of p-adic iterated integrals, which is more convenient towards computational purposes.

In this thesis, we provide two proofs of the aforementioned result, one based on the use of Galois deformations following the work of Bellaïche and Dimitrov [BeDi16], and the other using just the properties of Beilinson–Flach elements. For the former, the main ingredients are the following ones:

1. The modular forms g and g^* can be p-adically interpolated along Hida families $\mathbf{g}$ and $\mathbf{g}^*$. Further, Hida [Hi85], [Hi88] constructed a three-variable p-adic L-function $L_p(\mathbf{g}, \mathbf{g}^*)$ indexed by the variables (y, z, s), where (y, z) are the weights of $(\mathbf{g}, \mathbf{g}^*)$ and s is a cyclotomic variable.

2. Hida [Hi04] proved the existence of an improved p-adic L-function, with good interpolation properties which allows us to *delete* the Euler factor which vanishes in this self-dual case.

3. The previous results allow us to reduce the proof to a Galois deformation problem, which can be solved using the techniques developed mainly by Bellaïche and Dimitrov [BeDi16], in the setting of [DLR18].

Exceptional zeros of Euler systems

The first chapters of the thesis are based on the tantalizing interaction among three different mathematical concepts: the arithmetic of certain number fields (and in particular, the study of the groups of units and p-units); the Beilinson–Flach classes attached to two Hida families $(\mathbf{g}, \mathbf{h})$ interpolating weight one modular forms (see for example [BDR15b], [LLZ14], [KLZ20] and [KLZ17]); and the Hida–Rankin p-adic L-function, also attached to two Hida families. It had already been shown in [KLZ17, §10.2] that this p-adic L-function is the image under a certain big logarithm of the Euler system of Beilinson–Flach elements. But while in the applications towards the BSD conjecture it was convenient to specialize the Hida families $(\mathbf{g}, \mathbf{h})$ at weights $(2, 1)$, we focus now in the more intriguing setting of weights $(1, 1)$.

This big-logarithm map (or Perrin-Riou map) is a map associated to a family of p-adic Galois representations, interpolating either the dual exponential map or the Bloch–Kato logarithm. It is natural to ask ourselves how our previous result on the special value of a Hida–Rankin p-adic L-function fits with the general theory of Euler systems and Beilinson–Flach classes. This is not straightforward at all since some of the Euler factors which appear in the explicit reciprocity law connecting the classes with the p-adic L-function vanish under the self-duality assumption. This leads us to construct a *derived* cohomology class and establish a derived reciprocity law in this framework. These derived classes turn out to encode relevant arithmetic information, and indeed we are able to reprove Theorem 0.0.7 just by using their properties.

Let us be more precise. Given two modular forms (g, h) and an integer s satisfying certain weight relations, it is possible to construct the so-called Eisenstein class $\mathrm{Eis}^{[g,h,s]}$ in étale cohomology. However, this is no longer possible in weight 1, and one must proceed again in a rather indirect way: one takes Hida families $(\mathbf{g}, \mathbf{h})$ going through some p-stabilizations of the weight one modular forms, and that way obtains Λ-adic classes which when specialized at geometric points allow us to recover, up to appropriate Euler factors, the previous constructions of Eisenstein classes (in strike analogy with the interpolation of critical L-values). It is not surprising at all saying that at weight one, these classes reproduce similar phenomena than the corresponding p-adic L-functions. We write $\kappa(g_\alpha, h_\alpha)$ for the class associated to the choice of the p-stabilizations g_α and h_α. When $h = g^*$, the p-th Hecke eigenvalues for g^* are $\{1/\alpha, 1/\beta\}$, and one has that

$$\kappa(g_\alpha, g^*_{1/\beta}) = \kappa(g_\beta, g^*_{1/\alpha}) = 0.$$

In these cases, it is possible to construct *derived classes* and prove *derived reciprocity laws*. This concept is quite subtle and it is present in different parts of the monograph.

- In Section 3 of Chapter 3, we construct derived classes along a weight direction. Moreover, we prove a reciprocity law connecting this derived class with the p-adic L-function, and we recover the same $\mathcal{L}$-invariant which arises when working with p-adic L-functions.

- In Chapter 5, however, we take cyclotomic derivatives, and this gives us more flexibility and encodes more information. What happens here is that over the space of p-adic units, one direction will recover the p-adic logarithm, while the other will give the order map. This allows us to reprove Theorem 0.0.7.

As we later recall, we also explore the concept of derived classes in the setting of elliptic units (Chapter 6) and Gross–Kudla–Schoen cycles (Chapter 7). This shows another ostensible parallelism with the theory of p-adic L-functions, where this kind of results had already been studied, and nowadays there is a well understood framework and quite general conjectures which are widely believed to be true. We focus on the cases of elliptic units and diagonal cycles, which allow us to obtain results towards the Elliptic Stark Conjecture of [DLR15a]. This conjecture may be understood as an analogue of the more well-known Gross–Stark conjectures, but here an elliptic curve comes into play, showing once more this analogy between the arithmetic of number fields and elliptic curves. When these exceptional zero phenomena occur, one is led to study higher order derivatives of the Euler system in order to extract the arithmetic information which is usually encoded in the explicit reciprocity laws which make the link with p-adic L-functions.

Artin formalism and Euler systems

Let V_1 and V_2 denote two Artin representations of the absolute Galois group $G_{\mathbb{Q}}$, and write $L(V_1, s)$ and $L(V_2, s)$ for the corresponding complex L-functions. Unwinding the definitions, one observes that the L-function attached to the direct sum $V_1 \oplus V_2$, $L(V_1 \oplus V_2, s)$ factors as

$$L(V_1 \oplus V_2, s) = L(V_1, s) \cdot L(V_2, s).$$

This is usually called *Artin formalism*.

A recurrent problem when dealing with p-adic methods is obtaining similar formulas when complex L-functions are replaced by their p-adic counterparts. There are relatively few such results in the literature. One example is Gross' celebrated work [Gro80] for the Katz p-adic L-series associated to the restriction of a Dirichlet character (their method is indeed based on a comparison between circular units and elliptic units). A more recent advance has been made by Dasgupta [Das16], who proved a factorization formula exploding the decomposition $V_f \otimes V_f = \mathrm{Sym}^2(V_f) \oplus \chi\varepsilon_{\mathrm{cyc}}^{k-1}$, where $f \in S_k(N, \chi)$, V_f is its associated Galois representation, and $\varepsilon_{\mathrm{cyc}}$ stands for the cyclotomic character. The Artin formalism yields an equality of primitive L-series

$$L(f \otimes f, s) = L(\mathrm{Sym}^2 f, s) \cdot L(\chi, s - k + 1),$$

and Dasgupta succeeded on proving a p-adic counterpart, which in rough terms asserts that

$$L_p(f \otimes f, s) = L_p(\mathrm{Sym}^2 f, s) \cdot L_p(\chi, s - k + 1)^2.$$

A rather related issue arises when taking modulo p reductions, that is, if one considers a cuspidal modular form f whose mod p reduction is Eisenstein, and hence one may wonder if there is a mod p Artin formalism for this p-adic L-functions. Mazur [Maz79], and later Stevens [St82], Vatsal [Va99], or Greenberg [GV00] dealt with that question, which was also studied more recently in the anticyclotomic setting by Kriz [Kr16]. This crucially depends on the appropriate definition of certain canonical periods attached to the p-adic L-functions.

The Perrin-Riou formalism connecting p-adic L-functions and Euler systems suggests the existence of a similar Artin formalism, allowing us to decompose an Euler system attached to a p-adic representation $V = V_1 \oplus V_2$ as the sum of two other Euler systems attached to V_1 and V_2. The easiest instance appears in the case where one considers the Kato Euler system attached to a weight two modular form f which is congruent modulo p to an Eisenstein series. The Beilinson–Kato cohomology class κ_f associated to f can give rise to two different components modulo p, and we discuss congruence relations connecting those components to explicit expressions involving circular units. See the introduction of Chapter 8 for a precise formulation of the results.

[2] See [Das16, Theorem 1] for a precise formulation. The formula we have given here requires N to be coprime with p and must be restricted to half of the weight space.

Organization of the thesis

The monograph is organized in three parts. The first one corresponds to Chapters 1 and 2 and is devoted to review the state of the art and discusses certain preliminary results.

The second part is the core of the thesis: it corresponds to Chapters 3–7 and it is there where we develop the results concerning exceptional zeros, as well as the interaction between generalized cohomology classes and p-adic L-functions, dealing with several instances where the arithmetic phenomena are specially rich. In particular, Chapter 3 develops two of the main results we have mentioned in this Introduction: the special value formula for the Hida–Rankin p-adic L-function, and also the study of the exceptional specializations for the Beilinson–Flach classes. The following four chapters, 4, 5, 6 and 7, further develop ideas around those points: Chapter 4 studies the previous special value conjecture for arbitrary weight one modular forms (g,h); Chapter 5 offers a reinterpretation of the previous results on Beilinson–Flach elements and gives a new proof of Theorem 0.0.7; and finally, both Chapter 6 and Chapter 7 deal with other instances of the exceptional zero phenomenon which are particularly interesting due to its arithmetic relevance (elliptic units and diagonal cycles, respectively).

The third part of the thesis corresponds to Chapter 8, and there we study the interaction of different Euler systems in a situation where one or more cuspidal modular forms are congruent to Eisenstein series. This gives rise to an Artin formalism for Euler systems, and it is just the first insight into a topic that I hope to continue exploring in my future research.

Except for the first two chapters, each one corresponds to a different research article that has been produced during the realization of this thesis (or which is currently work in progress). They are intended to be read independently, and for that reason each one contains its own introduction and its own section of preliminary results.

- Chapter 3 corresponds to the article *Derived Beilinson–Flach elements and the arithmetic of the adjoint of a modular form*. This article, which is a joint work with Victor Rotger, has been accepted for publication in the *Journal of the European Mathematical Society*.

- Chapter 4 corresponds to the article *Beilinson–Flach elements, Stark units and p-adic iterated integrals*. It is also a joint work with Victor Rotger, that was published in *Forum Mathematicum* **31** (2019), no. 6, 1517–1532.

- Chapter 5 corresponds to the article *Cyclotomic derivatives of Beilinson–Flach classes and a new proof of a Gross–Stark formula*. This article will be soon posted at the Arxiv repository.

- Chapter 6 corresponds to the article *The exceptional zero phenomenon for elliptic units*. This article has been accepted for publication in *Revista Matemática Iberoamericana*.

- Chapter 7 corresponds to the article *Generalized Kato classes and exceptional zero conjectures*. This article has been accepted for publication in *Indiana University Mathematics Journal*.

- Chapter 8 corresponds to *Eisenstein congruences between circular units and Beilinson–Kato elements*. This article, which is a joint work with Victor Rotger, will be soon posted at the Arxiv repository.

Chapter 1

Background material I: Euler systems and L-functions

The first two chapters recall several concepts which play a prominent role along the dissertation. There is no claim of originality and all the results can be found in the references we provide. We hope that this short presentation, emphasizing the most important points and stressing the link among the different concepts, can help the reader of this memoir.

We begin by providing background on several topics from where our dissertation builds up. This includes a short summary of BSD-type conjectures, making precise the analogy with certain results about the arithmetic of number fields. We continue with the introduction of Euler systems, which are present all along this thesis and constitute one of the main tools to deal with BSD-type conjectures; in spite of being introduced via a cumbersome definition, they arise quite naturally in different frameworks, and behind the scenes one can already envisage a connection with the theory of (p-adic) L-functions. We recall the most well-known examples, namely circular units, elliptic units, Kato elements, and Beilinson–Flach classes. Moreover, we emphasize the fact that with a more relaxed view of this theory we can also consider other kinds of families of cohomology classes such as those coming from Heegner points or diagonal cycles. The latter should be understood as the bottom layer of an ubiquitous Euler system that has not been built up yet. We finally point out that new Euler systems are appearing in the literature, based on a better understanding of bigger arithmetic groups (symplectic or orthogonal groups). In the third section of the chapter, we review the theory of p-adic L-functions. We begin by presenting them as a natural p-adic avatar of the complex L-functions, since they interpolate classical L-values. However, we are mainly interested in the connection with Euler systems, and in this direction it is particularly relevant the so-called *Perrin-Riou theory*. The philosophy behind this, as we have already mentioned in the introduction, is that p-adic L-functions can be understood as the image under a *big-regulator map* of an Euler system.

1.1 BSD-type conjectures

The Birch and Swinnerton-Dyer conjecture, stated as Conjecture 0.0.1 in the introduction, is a milestone of number theory and of mathematics in general, not only for being one of the celebrated *millennium problems* posed by the Clay Mathematic Institute, but for the link it suggests among different branches of mathematics: the contraposition between an algebraic avatar (the Mordell–Weil rank) and an analytic one (the order of vanishing of an L-function); but also between a global object (the rank of the $\mathbb{Q}$-rational points of an elliptic curve) and a local one (the L-function is obtained as an Euler product of local factors counting the number of points of the elliptic curve over residue fields). One of the key ideas of this thesis is that there is a rather imperfect dictionary between the arithmetic of number fields and the arithmetic of elliptic curves. The analytic class number formula is a good instance of this phenomenon.

Moreover, some of the results and conjectures available for elliptic curves have been also formulated in other settings, and its comprehension shall help us to better understand elliptic curves (and in particular, the BSD conjecture, that can be thought as one of the leitmotifs of this thesis). We begin by reviewing the arithmetic of number fields and the analytic class number formula; we follow by explaining the Bloch–Kato conjecture and discussing some of its easiest cases; and we finish the section by recalling the Beilinson conjecture, which proposes a tantalising connection between L-series and cycles in algebraic varieties.

The arithmetic of number fields

Let F denote an algebraic number field. Its Dedekind zeta function is initially defined for complex numbers s with real part $\Re(s) > 1$ by the Dirichlet series

$$\zeta_F(s) = \sum_{I \subset \mathcal{O}_F} \frac{1}{(\mathbb{N}_{F/\mathbb{Q}}(I))^s},$$

where I ranges through the non-zero ideals of the ring of integers of $\mathcal{O}_F$ and $\mathbb{N}_{F/\mathbb{Q}}(I)$ denotes the norm of the ideal I. Alternatively,

$$\zeta_F(s) = \prod_{\mathfrak{p} \subset \mathcal{O}_F} \frac{1}{1 - (\mathbb{N}_{F/\mathbb{Q}}(\mathfrak{p}))^{-s}},$$

where the product runs over the set of non-zero prime ideals of F. It was Hecke who first proved that $\zeta_F(s)$ has an analytic continuation to the complex plane as a meromorphic function, having a simple pole at $s = 1$. With a slight abuse of notation, we keep the name of $\zeta_F(s)$ for its analytic extension.

Associated to the number field F, there is a positive real number known as its regulator, Reg_F, formed out of the determinant of a matrix with entries given by the logarithms of all but one of the real and complex absolute values of F applied to a set of generators of the unit group $\mathcal{O}_F^\times$, modulo torsion.

The following result can be seen as a quite good analogue of the BSD conjecture. The first part says that the order of vanishing of ζ_F at $s = 0$ agrees with the rank of the group of units $\mathcal{O}_F^\times$. Moreover, both the first and the second part provide expressions either for the leading term in the Taylor expansion at $s = 0$ or the residue at $s = 1$ in terms of the arithmetic of the number field F.

Theorem 1.1.1 (Analytic class number formula)**.** *Let F be a number field. Then, the following holds:*

(a) ζ_F is holomorphic at $s = 0$ and its order of vanishing is $r = r_1 + r_2 - 1$, where the quantities r_1 and r_2 are respectively the number of real and complex places of F. Moreover,

$$\lim_{s \to 0} s^{-r} \zeta_F(s) = -\frac{h_F \, \mathrm{Reg}_F}{w_F},$$

where h_F is the class number of F and w_F is the number of roots of unity in F.

(b) ζ_F has a simple pole at $s = 1$ and

$$\lim_{s \to 1} (s-1)\zeta_F(s) = \frac{2^{r_1} (2\pi)^{r_2} h_F \, \mathrm{Reg}_F}{w_F |d_F|^{1/2}},$$

where d_F stands for the discriminant of F.

Following this analogy, the natural replacement of the rank of the elliptic curve is the rank of the group of units, which agrees with $r_1 + r_2 - 1$ by Dirichlet's unit theorem. The analogue of the size of the Shafarevich group is the class number h_F, whose finiteness is proved using Minkowski's theorem. Finally, the obvious analogue of the size of the torsion group is w_F, the number of roots of unity. Of course the regulator of the elliptic curve is substituted by Reg_F.

The Bloch–Kato conjecture

We now present the Bloch–Kato conjecture, following the wonderful survey [Bel09]. The statement appeared in 1990 in The Grothendieck Festchrift [BK93], a collection of papers to commemorate Grothendieck's 60th birthday. It can be seen as a generalization of the BSD conjecture, but in some sense as a *second-order conjecture*, that is, it talks about objects whose basic properties (or even definitions!) depend on unproved conjectures. In its most basic formulation, the Bloch–Kato conjecture relates two objects attached to a geometric Galois representation.

As it is customary, a *geometric* Galois representation is a semisimple continuous representation of the absolute Galois group of a number field F on a finite dimensional vector space V over $\mathbb{Q}_p$ which satisfies certain properties fulfilled by the Galois representations that appear in the étale cohomology of proper and smooth varieties.

Let V be a representation of G_F unramified outside a finite set of places Σ. If there is a finite set of places $\Sigma' \supset \Sigma$ such that the characteristic polynomial of the Frobenius Fr_v on V has coefficients in $\bar{\mathbb{Q}}$ when $v \notin \Sigma'$, we say that V is *Σ'-algebraic*. Furthermore, we say that V is *pure of weight* w if there is a finite set of places $\Sigma' \supset \Sigma$ such that V is Σ'-algebraic and all the roots of the characteristic polynomial of Fr_v have complex absolute values (for all embeddings $\bar{\mathbb{Q}}$ to $\mathbb{C}$) $q_v^{-w/2}$, where q_v is the cardinality of the residue field k_v of F at v. In this case, w is called the *weight* or the *motivic weight*.

The Bloch–Kato conjecture involves different Galois cohomology groups. The most important ones are the different subgroups of $H^1(G_F, V)$, where V is a p-adic Galois representation of F. More precisely, there is a distinguished subgroup, denoted as $H^1_{\mathrm{f}}(G_F, V)$ and usually referred to as the *finite Bloch–Kato Selmer group*, which roughly speaking consists on *geometric* classes, that is, those which are unramified almost everywhere. See [Bel09, Section 2.2] for the definitions at the level of local fields and Section 2.3 for the global objects.

There are two morphisms induced by Kummer maps that have special relevance for us. At the level of units in number fields, one has a map

$$\mathcal{O}_F^\times \otimes_{\mathbb{Z}} \mathbb{Q}_p \longrightarrow H^1_{\mathrm{f}}(G_F, \mathbb{Q}_p(1)); \tag{1.1}$$

which happens to be an isomorphism, and at the level of points of elliptic curves, there is an injection

$$E(F) \otimes_{\mathbb{Z}} \mathbb{Q}_p \hookrightarrow H^1_{\mathrm{f}}(G_F, V_p(E)), \tag{1.2}$$

where $V_p(E)$ is the p-adic Tate module of E.

As anticipated in the introduction, given a p-adic geometric representation of G_F, V, we may consider its associated L-function $L(V, s)$ as a product of the local factors $L_v(V, s)$, where v varies over the set of finite places of F. More precisely,

$$L_v(V, s) = \det\left((\mathrm{Fr}_v^{-1} q_v^{-s} - \mathrm{Id})|_{V^{I_v}} \right),$$

where s is a complex argument, q_v is the cardinality of the residue field of F at v, and the matrix of Fr_v is seen as a complex matrix using a fixed embedding of $\mathbb{Q}_p$ into $\mathbb{C}$. More generally, we denote by $L_\Sigma(V, s)$ the L-function with the factors at Σ removed.

Proposition 1.1.2. *Let Σ be a finite set of finite places containing all places above p, and all places where V ramifies. If V is Σ-pure of weight w, then the Euler product defining $L_\Sigma(V, s)$ converges absolutely and uniformly on all compact sets on the domain $\Re(s) > w/2 + 1$.*

In general, it is not known if $L(V, s)$ can be analytically continued. However, it is conjectured that when V is a geometric p-adic representation of G_F, pure of weight w, the L-function $L(V, s)$ admits a meromorphic continuation on all the complex plane. Moreover, $L(V, s)$ has no zeros on $\Re(s) > w/2 + 1$. If V is irreducible, $L(V, s)$ has no poles, except if $V \simeq \mathbb{Q}_p(n)$, in which case

$L(V,s)$ has a unique pole at $s=n+1$, which is simple. Here, $\mathbb{Q}_p(n)$ denotes the twist of the trivial representation by the n-th power of the cyclotomic character.

We are now ready to state the Bloch–Kato conjecture. As before, let F denote a number field, and V a pure geometric representation of the absolute group G_F. We assume that the L-function $L(V,s)$ has a meromorphic continuation to the entire plane.

Conjecture 1.1.3 (Bloch–Kato)**.** *The following equality holds:*

$$\dim H^1_{\mathrm{f}}(G_F, V^*(1)) - \dim H^0(G_F, V^*(1)) = \mathrm{ord}_{s=0} L(V,s).$$

The H^0 terms in the left hand side is zero unless V contains $\mathbb{Q}_p(1)$ as a quotient; this can be usually ignored, although in our applications to exceptional zeros it is convenient to keep it in mind.

This conjecture relates two very different objects attached to V. Firstly, the Selmer group $H^1_{\mathrm{f}}(G_F,V)$, which is a global invariant of V, that contains deep number-theoretical information attached to the representation V, the motive M of which it is the p-adic realization, and also, the algebraic variety where it comes from. Next, the L-function is built on local information, and via a process of analytic continuation it gives rise to a mysterious integer, the order at $s=0$ of the L-function.

The easiest example arises when $V=\mathbb{Q}_p$. In this case, $L(V,s)$ is the Dedekind zeta function $\zeta_F(s)$, whose order of vanishing at $s=0$ is given by r_1+r_2-1. Similarly,

$$H^1_{\mathrm{f}}(G_F, V^*(1)) \simeq \mathcal{O}_F^\times \otimes_{\mathbb{Z}} \mathbb{Q}_p \simeq \mathbb{Q}_p^{r_1+r_2-1}, \tag{1.3}$$

where the first equality comes from Kummer theory (see equation (1.1)) and the second, from Dirichlet's unit theorem. In this case, the conjecture clearly holds.

The situation for elliptic curves is more interesting. Let $V=V_p(E)$, whence $V^*(1)\simeq V$ by the Weil's pairing, and so V is pure of weight one. The Bloch–Kato conjecture amounts to the prediction

$$\dim H^1_{\mathrm{f}}(G_F, V_p(E)) = \mathrm{ord}_{s=0} L(V_p(E),s) = \mathrm{ord}_{s=1} L(E,s). \tag{1.4}$$

We have seen that the dimension of $\dim H^1_{\mathrm{f}}(G_F,V)$ is at least the rank of $E(F)$; it turns out that equality holds if and only if $\mathrm{Sha}(E)[p^\infty]$. Hence, the BSD conjecture together with the fact that $\mathrm{Sha}(E)[p^\infty]$ is finite imply the Bloch–Kato conjecture for V. Similarly, assuming the finiteness of the p-part of the Shafarevich group, the *classical* BSD conjecture is equivalent to the Bloch–Kato conjecture for $V=V_p(E)$.

The Beilinson conjecture

We now recall another well-known conjecture linking L-functions with algebraic objects. Beilinson's conjecture, however, is a more geometric statement, and directly relates the vanishing of the L-function with the existence of a supply of algebraic cycles over an algebraic variety. To fix our framework, let X be a smooth and proper algebraic variety of dimension $n\geq 0$ over a number field F. It is known that the étale cohomology of the variety is trivial beyond ranks $0\leq i\leq 2n$. For this set of values, however, the étale cohomology gives rise to a compatible system of Galois representations of G_F, denoted as $\{H^i_{\mathrm{et}}(X_{\bar F},\mathbb{Q}_\ell)\}_\ell$.

The Beilinson conjecture relates two different objects. The first one is already familiar to us: the L-function $L(X,i,s) := L(\{H^i_{\mathrm{et}}(X_{\bar F},\mathbb{Q}_\ell)\}_\ell, s)$ associated to the compatible system of Galois representation. As before, $L(X,i,s)$ is a product over the local factors $L_v(X,i,s)$, with

$$L_v(X,i,s) = \det\Big((\mathrm{Fr}_v^{-1} q_v^{-s} - \mathrm{Id})|_{V^{I_v}} \Big), \tag{1.5}$$

being V the ℓ-adic Galois representation $H^i_{\mathrm{et}}(X_{\bar F},\mathbb{Q}_\ell)$, where Fr_v acts as an endomorphism in the part fixed by inertia.

The statement of the conjecture requires the introduction of the Chow group $\mathrm{CH}^c(X,n)$ of the variety, or more generally, its higher Chow groups. A good reference for the study of this theory, far beyond the scope of this thesis, can be found in [MVW06]. To introduce the higher Chow groups, let

$$\Delta_n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{A}^{n+1} \text{ such that } x_0 + x_1 + \ldots + x_n = 1\}$$

be the usual n-dimensional symplex.

Definition 1.1.4. Let

$$\widetilde{\mathrm{CH}}^c(X,n) = \Big\{ \sum n_i Z_i \mid n_i \in \mathbb{Z} \Big\},$$

where $Z_i \subset X \times \Delta_n$ is an irreducible subvariety of dimension $d+n-c$ properly intersecting all the subfaces of Δ_n (that is, all the subsets $X \times F$, where F is a subface of Δ_n).

We also define the boundary maps between the previous groups.

Definition 1.1.5. The boundary map δ_n is defined by

$$\delta_n : \widetilde{\mathrm{CH}}^c(X,n) \longrightarrow \widetilde{\mathrm{CH}}^c(X,n-1), \quad Z \mapsto \sum_{k=0}^{n} (-1)^k (Z \cap (X \times \Delta_n^k)),$$

where $\Delta_n^i = \{(x_0, \ldots, x_n) \in \Delta_n \mid x_i = 0\}$.

The map $\delta_0 : \widetilde{\mathrm{CH}}^c(X,0) \to H_{2d-2c}(X)$ is defined by sending $Z \mapsto [Z]$, since $Z = \sum n_i Z_i$, where Z_i is of complex codimension $d-c$ (and therefore of real dimension $2d-2c$).

We want to consider now a chain of groups of the form

$$\widetilde{\mathrm{CH}}^c(X,n+1) \xrightarrow{\partial_{n+1}} \widetilde{\mathrm{CH}}^c(X,n) \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_1} \widetilde{\mathrm{CH}}^c(X,0) \xrightarrow{\partial_0} H_{2d-2c}(X) \simeq H^{2c}(X),$$

where in the last step, $H^{2c}(X)$ denotes any of the standard cohomologies attached to X (either étale, de Rham,...).

It is not difficult to show that this defines a complex of abelian groups, and one may consider the associated cohomology groups, namely

$$\mathrm{CH}^c(X,n) := \frac{\ker \partial_n}{\mathrm{Im}\ \partial_{n+1}}.$$

The following conjecture is generally due to Beilinson. We have opted here for presenting its most basic formulation, without discussing certain subtleties around it and the precise hypothesis which are needed for the validity of the main statement.

Conjecture 1.1.6. *Let F be a number field and let X/F be an algebraic variety of dimension d satisfying appropriate mild assumptions. Fix integer numbers i, c such that $0 \leq i \leq 2d$, and $c < i/2 + 1$. Then, the order of vanishing*

$$\mathrm{ord}_{s=c} L(X,i,s)$$

agrees with the rank of the higher Chow group $\mathrm{CH}^{-c+i+1}(X, -2c+i+1)$.

Along this thesis we will implicitly use some concrete descriptions of these higher Chow groups.

- Let $X = \mathrm{Spec}(F)$, and fix $i = 0$. Then, $L(\mathrm{Spec}(F), 0, s) = \zeta_F(s)$. This function has a simple pole at $s = 1$ and for any integer $c < 1$, the order of vanishing of $\zeta_F(s)$ at $s = c$ is given by: (a) $r_1 + r_2 - 1$ when $c = 0$; (b) $r_1 + r_2$ when c is negative and even; (c) r_2 when c is negative and odd. Hence, the Beilinson conjecture predicts that those are precisely the dimensions of the Chow groups $\mathrm{CH}^{1-c}(X, 1-2c)$, that may be identified with the K-groups $K_{1-2c}(\mathcal{O}_F)$. A theorem of Borel [Bo08, Section 11] guarantees that these are precisely the dimensions of those groups (Borel's results assert that the rank of $K_j(\mathcal{O}_F)$ is 0 when j is even; $r_1 + r_2$ when j is 1 modulo 4; and r_2 when j is 3 modulo 4).

- For an algebraic variety X, $\mathrm{CH}^c(X,0)$ is the set of elements of the form $Z = \sum n_i Z_i$, where the sum is finite and Z_i are cycles of codimension c in X which are homologically trivial and are taken modulo the equivalence relation induced by the image of ∂_1. In the particular case that X is a curve, $\mathrm{CH}^1(X,0) = \mathrm{Jac}(X)$, where $\mathrm{Jac}(X)$ stands for the jacobian of X.

- For an algebraic variety X, $\mathrm{CH}^1(X,1)$ consists on elements $Z \subset X \times \mathbb{A}^1$ of codimension 1. There is a natural inclusion $\mathcal{O}_X^\times \hookrightarrow \mathrm{CH}^1(X,1)$ that under quite general assumptions is an isomorphism. It is given by sending a function u to its graph:
$$u \mapsto Z_u := \{(x, u(x)) \text{ where } x \in X\}.$$

- Let X stand for an algebraic surface. Then, $\mathrm{CH}^2(X,1)$ is given as the set of finite sums of the form $\sum n_i(C_i, u_i)$, where C_i is a curve in X, $u_i \in K_{C_i}$, and $\sum n_i \operatorname{Div}(u_i) = 0$. Here, K_{C_i} stands for the field of functions of the curve C_i.

We point out that these higher Chow groups are endowed with a rich algebraic and geometric structure (for instance, they come with a cup product and an excision sequence), that will play a relevant role in different parts of the exposition.

Another conjecture we would like to discuss, and which can be seen as another relation between algebraic and analytic objects, is the so-called Iwasawa main conjecture, as well as its variants for elliptic curves. However, we postpone the discussion until the introduction of p-adic L-functions in the third section of the chapter.

1.2 Euler systems

The term of *Euler system* was initially coined by Kolyvagin, and the origins of the concept can be traced to two independent developments. First, Thaine's cyclotomic method for bounding the exponents of the ideal class groups of cyclotomic fields. Then, Kolyvagin's fundamental articles, which replace the use of circular and elliptic units in the works around the Iwasawa main conjecture by certain norm-compatible points on a modular elliptic curve, the so-called Heegner points. These two methods, although applied to different situations, exhibit many formal similarities. Quoting H. Darmon [Dar02], *"Euler systems have cropped up in a rich variety of guises and played key roles in many of the important number theoretic breakthroughs of the last decades"*.

We begin this section by reviewing the basis of the theory and recalling the axiomatic view of the topic; continue with a study of Kato's Euler system and their natural generalization to the case of Beilinson–Flach elements; and finish with an overview of other relevant instances. The two first sections are based on the extensive surveys of Loeffler [Loe17b] and Loeffler–Zerbes [LZ18].

Definitions and first examples

Let K be a number field, $\bar{K}$ an algebraic closure, and $G_K = \mathrm{Gal}(\bar{K}/K)$ its absolute Galois group. Let p be a prime number. We are interested in representations of the group G_K on finite-dimensional $\mathbb{Q}_p$-vector spaces V. We always assume that the following two conditions hold:

1. $\varrho : G_K \to \mathrm{Aut}(V) \simeq \mathrm{GL}_d(\mathbb{Q}_p)$ is continuous (where $d = \dim(V)$) with respect to the profinite topology of G_K and the p-adic topology on $\mathrm{GL}_d(\mathbb{Q}_p)$.

2. V is *unramified almost everywhere*: for all but finitely many prime ideals v of K, we have $\varrho(I_v) = \{1\}$, where I_v is an inertia group at v (observe that this depends on the choice of a prime v of $\bar{K}$ above v, but since this is only up to conjugation in G_K, whether or not V is unramified at v is well-defined).

Let V be a $G_{\mathbb{Q}}$-representation; $T \subset V$ a $G_{\mathbb{Q}}$-stable $\mathbb{Z}_p$-lattice; and Σ a finite set of primes containing p and all ramified primes for V. Since V is a $G_{\mathbb{Q}}$-representation, we can consider it as a G_K-representation, for any number field K. In particular, there are corestriction or norm maps

$$\mathbb{N}_{L/K} : H^i(L, V) \to H^i(K, V) \quad \text{if } L \supset K.$$

When K is Galois, $H^i(K, V)$ is a module over $\mathbb{Q}_p[\mathrm{Gal}\,(K/\mathbb{Q})]$. This works in the same way for the cohomology of lattices $H^i(K, T)$.

Definition 1.2.1. An Euler system for (T, Σ) is a collection of classes $\mathbf{c} = (c_m)_{m\geq 1}$, with $c_m \in H^1(\mathbb{Q}(\mu_m), T)$ and satisfying the following compatibility relations for any $m \geq 1$ and ℓ prime:

$$\mathbb{N}_{\mathbb{Q}(\mu_{m\ell})/\mathbb{Q}(\mu_m)}(c_{m\ell}) = \begin{cases} c_m & \text{if } \ell \in \Sigma \text{ or } \ell \mid m \\ P_\ell(V^*(1), \sigma_\ell^{-1}) \cdot c_m & \text{otherwise,} \end{cases}$$

where σ_ℓ is the image of Fr_ℓ in $\mathrm{Gal}\,(\mathbb{Q}(\mu_m)/\mathbb{Q})$.

An *Euler system* for V is an Euler system for (T, Σ), for some $T \subset V$ and some Σ.

Roughly speaking, the element c_m is related with the L-function $L(V^*(1), s)$, with the Euler factors at primes dividing $m\Sigma$ removed. Then, when elements for different m are compared, the Euler factors appear. The first motivation for the study of Euler systems comes from the following theorem due to Rubin, building on earlier work of Kolyvagin. Here, $\mathrm{Sel}_{\mathrm{strict}}$ is the set of Selmer classes which are trivial at p.

Theorem 1.2.2. *Suppose* $\mathbf{c}$ *is an Euler system for* (T, Σ)*, with* c_1 *non-zero, and suppose* V *satisfies various technical conditions. Then,* $\mathrm{Sel}_{\mathrm{strict}}(\mathbb{Q}, V^*(1))$ *is zero.*

See [Rub00] and [MR04] for a more exhaustive treatment of the topic and a precise statement of the previous result.

For our purposes, we will be interested in dealing with a more flexible notion of Euler system, also encompassing the so-called *anticyclotomic Euler systems*. In this case, one has a representation V of G_K, a quadratic extension L/K, and cohomology classes for V over the anticyclotomic extension of L, which are the abelian extensions of L on which conjugation by $\mathrm{Gal}\,(L/K)$ acts on their Galois groups by -1. The most important example is given by Kolyvagin's Euler system of Heegner points, where $K = \mathbb{Q}$, $V = V_p(E)$ for E an elliptic curve, and L is an imaginary quadratic field. Other examples of anticyclotomic Euler systems have recently been found by Cornut, and by Jetchev and his coauthors.

The easiest instance of an Euler system is provided by *circular units*. First of all, and after fixing an embedding $\iota : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}^\times$, one considers the root of unity $\zeta_m = \iota^{-1}(e^{2\pi i/m})$. Then, for all $m \geq 1$, we set

$$u_m = 1 - \zeta_m \in \mathbb{Q}(\zeta_m)^\times.$$

It is easy to show that these elements *almost* give an Euler system:

$$\mathbb{N}_{\mathbb{Q}(\mu_m\ell)/\mathbb{Q}(\mu_m)} = \begin{cases} u_m & \text{if } \ell \mid m; \\ (1 - \sigma_\ell^{-1}) \cdot u_m & \text{if } \ell \nmid m \text{ and } m > 1; \\ \ell & \text{if } m = 1. \end{cases}$$

However, this precludes the possibility of defining u_1; and moreover, we are seeing Euler factors at all primes, and we only want them for primes outside a non-empty set Σ (that in particular, has to contain p). These difficulties can be circumvented by defining

$$v_m = \begin{cases} u_m & \text{if } p \mid m, \\ \mathbb{N}_{\mathbb{Q}(\mu_{pm})/\mathbb{Q}(\mu_m)}(u_{pm}) & \text{if } p \nmid m. \end{cases}$$

Let

$$\kappa_p : L^\times \longrightarrow H^1(L, \mathbb{Z}_p(1))$$

stand for the Kummer map associated to any finite extension L/K.

Proposition 1.2.3. *With the previous notations, the classes $c_m = \kappa_p(v_m)$ are an Euler system for $(\mathbb{Z}_p(1), \{p\})$.*

It is also interesting to mention the so-called Soulé twists, that arises when we take not just $\mathbb{Z}_p(1)$ but a twist by a finite order character χ, that we write $\mathbb{Z}_p(\chi)(1)$. The reason is that the cohomology classes that we can consider for this representation, and that are constructed after a suitable modification of c_m, are strongly linked with the Kubota–Leopoldt p-adic L-function. We come to this later on.

The first natural generalization of the previous construction is given by *elliptic units*, which constitute the natural replacement of circular units when the field of rational numbers is replaced by an imaginary quadratic field, that we denote by K. In this case, there are two disjoint $\mathbb{Z}_p$-extensions, which are usually labelled *cyclotomic* and *anticyclotomic*, according to the action of the complex conjugation of $\mathrm{Gal}\,(K/\mathbb{Q})$. Alternatively, elliptic units can be understood in terms of the values of modular units at the CM points of a modular curve.

For the sake of simplicity, we may assume that K has class number 1. Let E be an elliptic curve with complex multiplication by an order $\mathcal{O}$ contained in $\mathcal{O}_K$, the ring of integers of K. The construction of the Euler system of elliptic unit requires the choice of an ideal $\mathfrak{a}$ of $\mathcal{O}$ coprime to 6. If $\Delta(E)$ stands for the discriminant of E and $\gamma \in \mathcal{O}$ is a generator of the ideal $\mathfrak{a}$, we define

$$\Theta_{E,\mathfrak{a}} = \gamma^{-12} \Delta(E)^{\mathbb{N}\mathfrak{a}-1} \prod_{P \in E[\mathfrak{a}]-O} (x - x(P))^{-6}. \tag{1.6}$$

This function is independent of the choice of a model, and is defined over the field of definition of E. From that function, we may define $\Lambda_{E,\mathfrak{a}}$ as a product of some translates of $\Theta_{E,\mathfrak{a}}$. These functions, both $\Theta_{E,\mathfrak{a}}$ and $\Lambda_{E,\mathfrak{a}}$, satisfy nice distribution relations, which allow us to define elliptic units.

Let ψ be the Hecke character attached to E with conductor $\mathfrak{f}$. Choose a prime $\mathfrak{p}$ of K not dividing $6f$, and let p be the rational prime below it. As before, fix an ideal $\mathfrak{a}$ of $\mathcal{O}$ coprime to $6\mathfrak{p}\mathfrak{f}$, and let $\mathcal{R}$ stand for the set of square free ideals of $\mathcal{O}$ coprime to $6\mathfrak{f}\mathfrak{p}\mathfrak{a}$. Finally, fix an analytic isomorphism $\xi : \mathbb{C}/L \xrightarrow{\sim} E(\mathbb{C})$, where $L = \Omega\mathcal{O}$ and $\Omega \in \mathbb{C}$.

Definition 1.2.4. Given an integer $n \geq 0$ and an integral ideal $\tau \in \mathcal{R}$, define

$$\eta_n^{(\mathfrak{a})}(\tau) = \Lambda_{E,\mathfrak{a}}(\xi(\psi(\mathfrak{p}^n\tau)^{-1}\Omega)).$$

The pair given by $\{\eta_n^{(\mathfrak{a})}(\tau)\}_{n\geq 1}$ and $\tau \in \mathcal{R}$ is called the set of *elliptic units.*

It is a nice exercise to analyze the fields of definitions of these units, for which we refer the interested reader to [Rub92]. In the following chapters we will recover these elliptic units and give more useful descriptions for our purposes. See for instance the first chapters of [DD06].

Kato's Euler systems

A problem that has called the attraction of different number theorists is the construction of Euler systems. There is no a systematic procedure for that, and basically all known constructions come from geometry. This is because for an algebraic variety X/K, the étale cohomology groups $H^i_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_p)$ are an interesting source of Galois representations, as reflected in celebrated results like the Fontaine–Mazur conjecture.

Starting from étale cohomology, there are several ways to obtain Euler systems, making use of the rich structure of Galois cohomology groups. As anticipated before, one has cup products, Kummer maps, or pushforward maps; indeed, if $Z \subset X$ is a closed subvariety of codimension d, with X and Z smooth, there are maps

$$H^i(Z, \mathbb{Q}_p(n)) \longrightarrow H^{i+2d}(X, \mathbb{Q}_p(n+d)). \tag{1.7}$$

In particular the pushforward of the identity class $1_Z \in H^0(Z, \mathbb{Q}_p(0))$ is a class in $H^{2d}(X, \mathbb{Q}_p(d))$, the *cycle class* of Z (see [Mil] for a review of all these concepts on étale cohomology). Therefore, one of the easiest ways to get useful cohomology classes is using the source of units in the coordinate ring of our variety, or even a subvariety of it. In the case of modular curves, there are plenty of units at our disposal; among them, we are interested in the so called *modular units*. This is also the case for Shimura varieties. Recall that for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, a modular unit of level Γ is a nowhere-vanishing, Γ-invariant, holomorphic function $\mathbb{H} \to \mathbb{C}$ with poles of finite order at the cusps.

This fact is particularly relevant for constructing two of the Euler systems that will appear more frequently along this dissertation: Kato's Euler system, and the Euler system of Beilinson–Flach elements. In these two cases, the key input is provided by Siegel units.

Definition 1.2.5 (Siegel units)**.** Let $\alpha, \beta \in \mathbb{Q}/\mathbb{Z}$, not both zero. Define the function $g_{\alpha,\beta} : \mathbb{H} \to \mathbb{C}$ as follows: write $(\alpha, \beta) = (a/N, b/N)$ for some $N \geq 1$ and $a, b \in \mathbb{Z}$, with $0 \leq a < N$. Then,

$$g_{\alpha,\beta}(\tau) = q^w \prod_{n \geq 0} \left(1 - q^{n+a/N}\zeta_N^b\right) \prod_{n \geq 1} \left(1 - q^{n-a/N}\zeta_N^{-b}\right),$$

where $q = e^{2\pi i \tau}$ and $w = \frac{1}{12} - \frac{a}{N} + \frac{a^2}{2N^2}$.

For $c > 1$ coprime to 6 and to the order of α and β in $\mathbb{Q}/\mathbb{Z}$, let

$${}_cg_{\alpha,\beta} = \frac{(g_{\alpha,\beta})^{c^2}}{g_{c\alpha,c\beta}}.$$

The units ${}_cg_{\alpha,\beta}$, with $(\alpha, \beta) \in (1/N\mathbb{Z}/\mathbb{Z})^{\oplus 2} - \{(0,0)\}$ are all defined over $\mathbb{Q}(\mu_N)$. Moreover, there is an action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, under which they transform nicely. The interaction between the Hecke and the Galois action over these units, which play a key role in the last part of the thesis, is discussed in [St82]. See also [LLZ14, Sections 2 and 3] for details.

Let $N \geq 2$ be an integer coprime to c. We define modular units u_N and v_N by

$$u_N(\tau) \;=\; {}_cg_{1/N,0}(N\tau), \quad v_N(\tau) \;=\; {}_dg_{0,1/N}(\tau).$$

While u_N is defined over $\mathbb{Q}$, v_N is not, and it is in fact defined over $\mathbb{Q}(\mu_N)$. For $A \geq 1$ such that N and AN have the same prime factors, letting π denote the natural map $Y_1(AN) \to Y_1(N)$, we have that

$$\mathbb{N}_{\mathbb{Q}(\mu_{AN})/\mathbb{Q}(\mu_N)}(u_{AN}) = \pi^*(u_N),$$
$$\pi_*(v_{AN}) = v_N.$$

Here, we are using the same conventions regarding modular curves than in Kato's seminal paper [Ka04] (see also [LLZ14]).

Kato classes are then constructed by taking the cup product in cohomology of the two Siegel units, as illustrated in the following definition.

Definition 1.2.6. For integers m, N, with $m \geq 2$ and $m \mid N$, we define

$$z_{N,m} = \kappa_p(u_m) \cup \kappa_p(v_N) \in H^2_{\mathrm{et}}(Y_1(N)_{\mathbb{Q}(\mu_m)}, \mathbb{Z}_p(2)).$$

We first observe that when $m \mid N \mid N'$, and N and N' have the same prime factors, then

$$(\pi_{N'/N})_*(z_{N',m}) = z_{N,},$$

where $\pi_{N'/N}$ is the natural map $Y_1(N') \to Y_1(N)$.

Theorem 1.2.7 (Kato). *If ℓ is prime with $\ell \mid m$, then*

$$\mathbb{N}_{\mathbb{Q}(\mu_{m\ell})/\mathbb{Q}(\mu_m)}(z_{N,m\ell}) = z_{N,m}.$$

If $\ell \nmid mN$, then

$$\mathbb{N}_{\mathbb{Q}(\mu_{m\ell})/\mathbb{Q}(\mu_m)}(z_{N,m\ell}) = (1 - \langle \ell \rangle^{-1} T_\ell \sigma_\ell^{-1} + \ell \langle \ell \rangle^{-1} \sigma_\ell^{-2}) z_{N,m},$$

where $\langle \ell \rangle$ and T_ℓ are the usual Hecke operators.

When projecting to the quotient $H^1(\mathbb{Q}(\mu_m), V_p(f)(2))$ of $H^2_{\text{et}}(Y_1(N)_{\mathbb{Q}(\mu_m)}, \mathbb{Q}_p(2))$, the Hecke operators T_ℓ and $\langle \ell \rangle$ act as $a_\ell(f)$ and $\chi(\ell)$, respectively. Hence, the Euler factor appearing becomes $(1 - \chi(\ell)^{-1} a_\ell(f) X + \ell \chi(\ell)^{-1} X^2$ evaluated at $X = \sigma_\ell^{-1}$. Observe that if $V = V_p(f)(2)$, then $V^*(1) = V_p(f \otimes \chi_f^{-1})$, and so the Euler factor is exactly $P_\ell(V^*(1), X)$ evaluated at σ_ℓ^{-1}. However, observe that we have Euler factors in our norm relations for all primes, so also for p. This can be remedied by replacing $z_{N,m}$ with $z_{N,m}^{(p)} = \mathbb{N}_{\mathbb{Q}(\mu_{mp})/\mathbb{Q}(\mu_m)}(z_{N,mp})$.

Remark 1.2.8. In forthcoming chapters we will use slightly different notations regarding the Galois representations attached to modular forms; in particular, we closely follow the notations of the series of papers of Darmon–Rotger, which depart from the conventions of Kato and other recent works. See Chapter 3 for details.

We would like to point out that these are not the exact kind of Kato elements we will deal with in subsequent chapters. In [BD14] the authors consider the modular unit u_{χ_1,χ_2} associated to the choice of two Dirichlet characters, and characterized by the fact that its logarithmic derivative agrees with $E_2(\chi_1, \chi_2)$, the Eisenstein series of characters χ_1 and χ_2. Moreover, one has to impose an additional scaling property. This system can be seen as a *weighted* variant of Kato's original construction, which gives more flexibility towards our arithmetic applications. In a certain sense that we later formulate in a clearer way, this can be though as an Euler system not for V_f, but for $V_f \otimes V_E$, where V_E is the Galois representation attached to the aforementioned Eisenstein series.

Kato's strategy was later generalized to the Beilinson–Flach case, firstly for weight two forms by Lei, Loeffler and Zerbes [LLZ14] and then allowing higher weights and variation in families, as developed in the works of Bertolini–Darmon–Rotger [BDR15b] and Kings–Loeffler–Zerbes [KLZ20], [KLZ17].

We come back to this general construction in the following chapter, but let us mention the main ideas around the weight two case. The natural approach for obtaining an Euler system for the Rankin–Selberg convolution $V_p(f) \otimes V_p(g)$ is to find curves $C \subset Y \times Y$, where $Y = Y_1(N)$, and suitable units on C. The obvious guess is to take C to be the diagonally-embedded copy of Y in $Y \times Y$ and as units, the corresponding Siegel units. In this case, the curve C varies, and get some contribution to the norm-compatibility relations. Morally, the compatibility in the *level* direction comes from the unit, while the compatibility in the *field* direction comes from the choice of the curve C.

We give a rough sketch of the construction. For integers $M \mid N$, let

$$\Gamma(M,N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ such that } a \equiv d \equiv 1,\, c \equiv 0 \text{ modulo } N;\, b \equiv 0 \text{ modulo } M \right\}.$$

The curve associated to this group, $Y(M,N)$, has a $\mathbb{Q}$-model, but its group action is only defined over $\mathbb{Q}(\mu_M)$. To overcome this, let $C_{M,N} \subset Y(M,N)^2$ be the curve defined by

$$\left\{ (P,Q) \in Y(M,N)^2 \text{ with } Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot P \right\}.$$

This is defined over $\mathbb{Q}(\mu_M)$ and satisfies the key compatibility relations. In order to define the appropriate unit on it, let us first introduce $\phi_1^M : Y(M,N) \to Y_1(N)$ as the natural quotient map, and consider the twisted map $\hat{\phi}_1^M : Y(M,N) \to Y_1(N)$ corresponding to $z \mapsto z/M$ on $\mathbb{H}$.

Definition 1.2.9. For $M \mid N$ and $c > 1$ coprime to $6N$, we define

$$\xi_{M,N} = \Big((\hat{\phi}_1^M \times \hat{\phi}_1^M)_* \circ (\iota_{M,N}) \circ \kappa_p \Big) ({}_c g_{0,1/N}) \in H^3_{\text{et}}(Y_1(N)^2_{\mathbb{Q}(\mu_M)}, \mathbb{Z}_p(2)),$$

where $\iota_{M,N}$ denotes the inclusion $C_{M,N} \hookrightarrow Y(M,N)^2$ and κ_p is the Kummer map.

These elements satisfy norm-compatibility in N, and we can extend the definition of ${}_c\xi_{M,N}$ to all M. To achieve that, let ℓ be a prime with $\ell \mid M$ and $\ell \mid N$. Then,

$$\mathbb{N}_{\mathbb{Q}(\mu_{M\ell})/\mathbb{Q}(\mu_M)}({}_c\xi_{M\ell,N}) = (U'_\ell \times U'_\ell) \cdot {}_c\xi_{M,N}, \tag{1.8}$$

where U'_ℓ is the transpose Hecke operator, defined by the double coset of $\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}$. When there is no ℓ dividing both M and N, one obtains analogue results after performing a few intricate calculations of double cosets (see again [LLZ14, Section 3]).

As with Kato's elements, we finally want to project these classes in étale cohomology to the (f,g)-component, where f and g are weight two cuspidal modular forms. The point of [KLZ20] is that this construction can be extended to higher weight modular forms via the so-called *Rankin–Eisenstein classes*. Even more, in [KLZ17] they establish the existence of Λ-adic classes that fit into Hida families $(\mathbf{f}, \mathbf{g})$, via Kings' theory of Λ-sheaves.

Generalizations and further work

Rubin defined a so-called anticyclotomic Euler system as a compatible family of classes satisfying the same norm relations, but only over ring class extensions [Rub92, Section 9.3]; he moreover allows certain flexibility in the shape of the Euler factors. This allows us to work with Heegner points in the realm of this theory. Observe that this system is one of the starting points of this series of developments, and has deep implications towards BSD-type results.

Heegner points are constructed via the theory of complex multiplication. Let N be a fixed positive integer, and let $M_0(N)$ be the ring of 2×2 matrices with entries in $\mathbb{Z}$ which are upper-triangular modulo N. Given $\tau \in \mathbb{H}$, we may define its associated order $\mathcal{O}_\tau^{(N)}$ to be the set of $\gamma \in M_0(N)$ such that $\gamma\tau = \tau$, together with the zero matrix.

If E is an elliptic curve of conductor N, we fix a modular parametrization $\phi_N : X_0(N) \to E$ and for any $\tau \in \mathbb{H} \cap K$, we consider its order, $\mathcal{O}_\tau^{(N)}$, with associated ring class field H/K. Then, we have

$$\Phi_N(\tau) \in E(H). \tag{1.9}$$

This is the key result for constructing the easies instance of an anticyclotomic Euler system.

Indeed, let $\mathcal{O}$ be any order of discriminant prime to N. Its set of CM points (that is, those with $\mathcal{O}_\tau^{(N)} = \mathcal{O}$) is non-empty if and only if all the primes dividing N split in $K/\mathbb{Q}$, so we keep this assumption for the moment. More generally, if the cardinality of the set of inert primes is even, a parallel treatment is available replacing the modular curve by an indefinite Shimura curve. If n is any integer prime to N, and $\mathcal{O}_n$ is the order of K of conductor n, a point of the form $\Phi_N(\tau)$, where τ has CM by $\mathcal{O}_n$, is a Heegner point of conductor n. These points satisfy norm compatibility relations when n varies, as predicted by this *relaxed* theory of anticyclotomic Euler systems. See [Dar04, Sections 3 and 4] for a proof of the compatibility relations and a discussion on how to move to the setting of Shimura curves.

Like circular units and elliptic units, Heegner points arise as a universal norm of a compatible system of points defined over anti-cyclotomic extensions of K. After letting $\Lambda_K^- = \mathbb{Z}_p[[\mathrm{Gal}\,(K_\infty^-/K)]]$

be the Iwasawa algebra attached to the anticyclotomic $\mathbb{Z}_p$-extension, this norm compatible collection of Heegner points can be parlayed into the construction of a global cohomology class

$$\kappa_{E,K,\infty} \in H^1(K, V_p(E) \otimes \Lambda_K^-), \tag{1.10}$$

where $V_p(E)$ is the Galois representation attached to the p-adic Tate module of E. The module $\Lambda_K^- \otimes_{\mathbb{Z}_p} V_p(E)$ is a deformation of $V_p(E)$ which p-adically interpolates the twists of $V_p(E)$ by anticyclotomic Hecke characters. These results were further extended by Castella [Cas20] to allow variation along Hida families.

The second instance we would like to address is the following one. The Euler systems of Kato and Beilinson–Flach can be reinterpreted as follows: the former arises when we take a modular curve and two modular units over it; the latter is the result of considering the two-fold of the modular curve and a modular unit over the diagonal. This suggests a third option, just considering the three-fold of the modular curve and a codimension two cycle on it, and that way we get the so-called diagonal cycles, already studied in the seminal work of Gross and Kudla [GK93]. Let (f, g, h) a triple of normalized primitive cuspidal eigenforms of weights $k, \ell, m \geq 2$, levels $N_f, N_g, N_h \geq 1$ and nebentype characters χ_f, χ_g, χ_h, respectively. Let $N = \mathrm{lcm}(N_f, N_g, N_h)$ and assume that $\chi_f \chi_g \chi_h = 1$, so that in particular $k + \ell + m$ is even. We say that (k, ℓ, m) is balanced if the largest weight is strictly smaller than the sum of the other two. Let $c := \frac{k+\ell+m}{2} - 1$, and assume that $\gcd(N_f, N_g, N_h) = 1$. Under these circumstances, and following the notations and ideas of [Tale14], we may construct appropriate cycles in the Galois representation $V_{fgh} = V_f \otimes V_g \otimes V_h$, where as usual V_f, V_g and V_h arise for the Galois representations attached to the modular forms f, g and h, respectively.

Let us briefly recall the main ideas involved in the construction. Let $\mathcal{E}$ denote the universal generalised elliptic curve fibered over $X = X_1(N)$. For any $n \geq 0$, let $\mathcal{E}^n$ be the n-th Kuga–Sato variety over X, which is an $n+1$-dimensional variety. The p-adic Galois representation V_{fgh} occurs in the middle cohomology of the triple product $W := \mathcal{E}^{k-2} \times \mathcal{E}^{\ell-2} \times \mathcal{E}^{m-2}$. Under the previous assumptions, the conjectures of Bloch–Kato and Beilinson predict (because of the vanishing of $L(f, g, h, c)$ due to sign reasons) that there exists a non-trivial cycle in the Chow group $\mathbb{Q} \otimes \mathrm{CH}^c(W)_0$ of rational equivalence classes of null-homologous cycles of codimension c on the variety W. Darmon and Rotger, in [DR14, §3.1], introduce cycles $\Delta_{f,g,h} \in \mathbb{Q} \otimes \mathrm{CH}^c(W)_0$ which are natural candidates to fulfill these expectations. Setting $r = c - 2$, there is an essentially unique way to embed the Kuga–Sato variety $\mathcal{E}^r$ in W. Its image gives rise to an element in $\mathrm{CH}^{r+2}(W)$ which, suitably modified, becomes homologically trivial, giving rise to $\Delta_{k,\ell,m} \in \mathrm{CH}^{r+2}(W)_0$. When $k = \ell = m = 2$, this is just the modified cycle considered by Gross–Kudla and Gross–Schoen. The cycles $\Delta_{f,g,h}$ are just the (f, g, h)-isotypical component of the null-homologous cycles $\Delta_{k,\ell,m}$ with respect to the action of the Hecke operators. See also [YZZ15] for an account of the arithmetic properties of these cycles.

Again, subsequent work of [DR17] (varying only one weight variable at a time) and [DR20b] and [BSV20a] allow us to consider the variation of this cycle in Hida families. Unfortunately, these classes do not yield an Euler system; even for weights $(2, 2, 2)$, we obtain classes in $H^1(\mathbb{Q}, V_{fgh}(-1))$ and although we expect that they should be realized as the bottom layer of an Euler system, this has not been accomplished yet. A rather related setting is concerned with the theory of the so-called Hirzebruch–Zagier cycles and twisted diagonal cycles. See for instance the work of Liu [Liu17].

Finally, and since this chapter is merely expositive, we would like to list other instances of different Euler systems that have been constructed during the last years. This topic seems to be very active lastly, and this enumeration is likely to become obsolete in a few years:

- Loeffler, Skinner and Zerbes [LSZ20a] have constructed an Euler system for Galois representations associated to cohomological cuspidal automorphic representations of $\mathrm{GSp}(4)$, using the pushforwards of Eisenstein classes for $\mathrm{GL}_2 \times \mathrm{GL}_2$. This was further generalized in recent work of Hsu, Jin and Sakamoto to produce an Euler system for $\mathrm{GSp}(4) \times \mathrm{GL}_2$.

- Cauchi and Jacinto [CJ20] have constructed global cohomology classes in the middle degree cohomology of the Shimura variety of the sympletic group $\mathrm{GSp}(6)$ compatible when varying the level at p. These classes are expected to form an Euler system for the Galois representations appearing in these cohomology groups.

- Loeffler, Skinner and Zerbes [LSZ20b] have recently constructed an Euler system for the unitary group $\mathrm{GU}(2,1)$. In this case the base field is not $\mathbb{Q}$, but an imaginary quadratic field E, and therefore the Euler system consists of classes over all of the abelian extensions of E, most of which are not abelian over $\mathbb{Q}$. In this work, they introduce a new strategy for proving norm-compatibility relations, based on cyclicity results for local Hecke algebras.

- Lei, Loeffler and Zerbes [LLZ18] constructed an Euler system for the Galois representations appearing in the geometry of Hilbert modular surfaces. The non-triviality of the Euler system relies on a conjecture of Bloch and Kato about the injectivity of regulator maps.

1.3 p-adic L-functions

The theory of p-adic L-functions has experimented an enormous growth during the last years, due to the increasing interest on Iwasawa theory and the development of the theory of Euler systems, to which it is unavoidable linked (it is often said that Euler systems are the geometric or cohomological incarnation of the appropriate p-adic L-functions). Along this section, we review the first and easiest examples of p-adic L-functions, emphasizing the connection with the theory of Hida families; we then move to the study of the p-adic L-functions of Hida–Rankin–Selberg type, where the connection with Euler systems is crystal-clear; finally, we come back to the ideas presented in the first section and introduce the Iwasawa main conjecture.

p-adic L-functions: first examples

The complex ζ-function is a function $\zeta : \mathbb{C} \to \mathbb{C}$ with complex analytic properties, and which is rational at negative integers. Since $\mathbb{Z}$ is a subset of both $\mathbb{C}$ and $\mathbb{Z}_p$, we may ask for an analogous function $\zeta_p : \mathbb{Z}_p \to \mathbb{C}_p$ which is p-adic analytic. The same occurs when we look at the L-function of an elliptic curve.

The use of p-adic L-functions dates back to Serre [Ser72]. Since this circle of ideas is specially well-known and relevant for further constructions, we briefly recall it here. Observe that if k is even and greater or equal than 2, the classical L-function $L(\chi, 1-k)$ can be realised as the constant terms of an holomorphic Eisenstein series

$$E_{k,\chi}(q) = \frac{L(\chi, 1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n, \quad \sigma_{k-1,\chi}(n) = \sum_{d \mid n} \chi(d) d^{k-1}$$

of weight k, level N and character χ. If p is any prime, its ordinary p-stabilisation is

$$E_{k,\chi}^{(p)}(q) = E_{k,\chi}(q) - \chi(p) p^{k-1} E_{k,\chi}(q^p),$$

and its Fourier expansion is given by

$$E_{k,\chi}^{(p)}(q) = L_p(\chi, 1-k) + 2 \sum_{n=1}^{\infty} \sigma_{k-1,\chi}^{(p)}(n) q^n, \quad \sigma_{k-1,\chi}^{(p)}(n) = \sum_{p \nmid d \mid n} \chi(d) d^{k-1}.$$

For all $n \geq 1$, the function on $\mathbb{Z}^{\geq 1}$ sending k to the n-th Fourier coefficient $\sigma_{k-1,\chi}^{(p)}(n)$ extends to a p-adic analytic function of $k \in (\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{Z}_p$. The point here is that the constant term inherits that same property. More generally, we have the following theorem.

Theorem 1.3.1 (Kubota–Leopoldt). *Let χ be an even, primitive Dirichlet character of conductor Mp^r for some $r \geq 0$ and $M \geq 1$ prime to p. Given an embedding of $\bar{\mathbb{Q}}$ in $\mathbb{C}_p$, there exists a unique p-adic analytic (aside from $s = 1$ when $\chi = 1$, where it has a simple pole with residue $1 - p^{-1}$) function $L_p(\chi, s)$ on $\mathbb{Z}_p$ satisfying*

$$L_p(\chi, 1-i) = (1 - \chi\omega^{-i}(p)p^{i-1})L(\chi\omega^{-i}, 1-i)$$

for all $i \geq 1$. Here, ω stands for the Teichmüller character.

The function $L_p(\chi, s)$ is the Kubota–Leopoldt p-adic L-function of χ. Different normalizations are sometimes adopted, depending on the weight space we consider, either $\mathbb{Z}_p[[1+p\mathbb{Z}_p]]$ or $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$; note that the former requires the introduction of suitable powers of the Teichmüller character in the interpolation formula, as illustrated in the previous result. In any case, both conventions agree if we restrict ourselves to a choice of residue class modulo $p-1$. See for instance the discussion of [Das99, Section 3].

The importance of the Kubota–Leopoldt p-adic L-function also comes here from the connection with circular units. As anticipated in the introduction, one has the following result, due to Leopoldt: when χ is a non-trivial, even primitive Dirichlet character of conductor N,

$$L_p(\chi, 1) = -\frac{(1-\chi(p)p^{-1})}{\mathfrak{g}(\bar{\chi})} \sum_{a=1}^{N-1} \bar{\chi}(a) \log_p(1 - \zeta_N^a), \tag{1.11}$$

where ζ_N is a primitive N-th root of unity and $\mathfrak{g}(\bar{\chi})$ is the corresponding Gauss sum.

But a more general result holds, and this is the germ of the easiest *explicit reciprocity law*, connecting a p-adic L-function with (the bottom layer of) an Euler system. The values $L_p(\chi, k)$ can be also recovered from subsequent specializations of the Kummer image of the whole cyclotomic system of circular units. More precisely, writing $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, one can construct a Λ-adic cohomology class

$$\kappa_{\chi,\infty} \in H^1(\mathbb{Q}, \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p,\chi}(\bar{\chi})(1))$$

by gluing the image of different cyclotomic units under the Kummer map. Here, we are assuming that Λ is endowed with the tautological Galois action. After applying the $G_{\mathbb{Q}}$-equivariant specializations maps

$$\nu_{k,\xi} : \Lambda \longrightarrow \mathbb{Q}_{p,\xi}(\bar{\xi})(k-1),$$

where k is an integer and ξ a character of p-power conductor, we get a family of compatible classes

$$\kappa_{k,\chi\xi} := \nu_{k,\xi}(\kappa_{\chi,\infty}) \quad \in \quad H^1(\mathbb{Q}, \mathbb{Q}_{p,\chi\xi}(\bar{\chi}\bar{\xi})(k)).$$

See Chapters 6 and 8 for a more exhaustive discussion around this construction, emphasizing the connection with the exceptional zero phenomenon.

Furthermore, for all $k \geq 1$,

$$L_p(k, \chi) = \frac{(1-\chi(p)p^{-k})}{(1-\bar{\chi}(p)p^{k-1})} \times \frac{(-t)^k}{(k-1)!\mathfrak{g}(\bar{\chi})} \times \log_{k,\chi}(\kappa_{k,\chi}), \tag{1.12}$$

where t is Fontaine's p-adic analogue of $2\pi i$ and $\log_{k,\chi}$ is the Bloch–Kato logarithm associated to the p-adic representation $H^1(\mathbb{Q}_p, \mathbb{Q}_{p,\chi}(k)((\chi)^{-1}))$. For $k \leq 0$ a similar result holds, but replacing the Bloch–Kato logarithm by the dual exponential map. Along the different chapters of the thesis we will explore more deeply this idea, which can be summarized under the idea that there exists an ubiquitous map, known under the names *Perrin-Riou map, big regulator map, big logarithm map,...*, and which interpolates either the dual exponential map or the Bloch–Kato logarithm, according to the Hodge–Tate weights of the representation. This is specially useful when considering variation in families, and will play a prominent role in the discussion of the Euler systems of Beilinson–Flach and diagonal cycles. See e.g. [KLZ17, Section 8.2] for an axiomatic treatment of this map, and the different chapters of this memoir for the study of distinct instances of this application.

This suggests that there may be a plethora of p-adic L-functions, that we are going to consider along this thesis, at least one for each Euler system! Due to its special significance and since it is the first example in this framework, we are going to introduce now the so-called two-variable Mazur–Kitagawa p-adic L-function. This is a natural generalization of earlier results which establish the existence of p-adic L-functions attached to a single modular form. This was firstly studied in the early 1970s by Mazur and Swinnerton-Dyer [MSD74], who constructed a p-adic L-function attached to a modular elliptic curve for each prime of good, ordinary reduction. In [AV75] and [Vis14] the construction was generalized to higher weight modular forms, to supersingular primes, and to primes of bad reduction. See also [MTT86] for a very accurate description of the scenario regarding p-adic L-functions of modular forms.

The idea we want to develop is the following: once we have a notion of p-adic variation of modular forms provided by the theory of Hida families, it makes sense to formulate the question if whether or not we can relate their different p-adic L-functions, that is, if we can introduce a two-variable (analytic) p-adic L-function, with one of the variables afforded by the weight. It is important to keep in mind that the setting of Hida families is unavoidably linked to the ordinariness assumptions; we refer for instance to the work of Pollack [Pol03] for a discussion of the supersingular case, with very nice applications towards Iwasawa theory.

For our later convenience, let us fix the axiomatic framework to work with Hida families, with a discussion borrowed from [DR14, Section 2]. Let $\mathcal{W} = \mathrm{Spf}(\Lambda) = \mathrm{Hom}(\Lambda, \mathbb{Z}_p)$. The subset of classical characters of $\mathcal{W}$ is defined to be

$$\mathcal{W}_{\mathrm{cl}} = \{(\gamma \mapsto \gamma^k), \quad \text{with } k \in \mathbb{Z}^{\geq 2}\}.$$

Given a finite flat extension $\Lambda_{\mathbf{f}}$ of Λ, let $\mathcal{W}_{\mathbf{f}} = \mathrm{Spf}(\Lambda_{\mathbf{f}})$. This space is endowed with a natural p-adic topology and is equipped with a natural projection $\kappa : \mathcal{W}_{\mathbf{f}} \to \mathcal{W}$ induced by the inclusion $\Lambda \subset \Lambda_{\mathbf{f}}$. A point $x \in \mathcal{W}_{\mathbf{f}}$ for which $\kappa(x)$ belongs to $\mathcal{W}_{\mathrm{cl}}$ is called a classical point, and set of such points is written as $\mathcal{W}_{\mathbf{f},\mathrm{cl}}$.

Definition 1.3.2. A Hida family of tame level N is a quadruple $(\Lambda_{\mathbf{f}}, \mathcal{W}_{\mathbf{f}}, \mathcal{W}_{\mathbf{f},\mathrm{cl}}, \mathbf{f})$, where:

(a) $\Lambda_{\mathbf{f}}$ is a finite flat extension of $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$.

(b) $\mathcal{W}_{\mathbf{f}}$ is a non-empty subset of $\mathfrak{X}_{\mathbf{f}} = \mathrm{Hom}_{\mathrm{cont}}(\Lambda_{\mathbf{f}}, \mathbb{C}_p)$, and $\mathcal{W}_{\mathbf{f},\mathrm{cl}}$ is a p-adically dense subset of $\mathcal{W}_{\mathbf{f}}$ whose image under κ lies in $\mathcal{W}_{\mathrm{cl}}$.

(c) $\mathbf{f} = \sum a_n q^n \in \Lambda_{\mathbf{f}}[[q]]$ is a formal q-series such that, for all $x \in \mathcal{W}_{\mathbf{f},\mathrm{cl}}$, $f_x = \sum x(a_n) q^n$ is the q-series of the ordinary p-stabilization of a normalized eigenform of weight $\kappa(x)$ on $\Gamma_1(N)$. When there is no risk of confusion, we use $\kappa(x)$ for the point x itself; it is also customary to write $f_x = \sum a_n(x) q^n$.

Hida's theorem asserts that given a normalized eigenform of weight $k \geq 1$ on $\Gamma_1(N)$, where $p \nmid N$ and such that p is ordinary for f, there exists a Hida family $(\Lambda_{\mathbf{f}}, \mathcal{W}_{\mathbf{f}}, \Omega_{\mathbf{f},\mathrm{cl}}, \mathbf{f})$ and a classical point $x_0 \in \mathcal{W}_{\mathbf{f},\mathrm{cl}}$ such that $\kappa(x_0) = k$ and f_{x_0} is an ordinary p-stabilization of f, and therefore unique if $k > 1$. In this case, we write f_x° for the modular form whose ordinary p-stabilization is f_x. See Section 2 of Chapter 3 for more details on Hida families, with applications towards the results of this thesis. There are also more geometric approaches to the topic, relying on the geometric theory of modular curves and the so-called Igusa tower. Extending Hida's work, Coleman and Mazur [CoMa98] constructed a geometric object, the eigencurve, which parametrizes p-adic families of modular form. The study of its geometry plays a crucial role in several moments of this dissertation, whose main results rely on the results of Bellaïche and Dimitrov [BeDi16]. They show that the p-adic eigencurve is smooth at classical weight 1 points which are regular at p, and give a precise criterion for étaleness over the weight space at those points, using for that purpose the theory of Galois representations.

Let $\mathbf{f} = \sum_{n=1}^{\infty} a_n(k)q^n \in \Lambda_{\mathbf{f}}[[q]]$ be a Hida family of tame level N, which is defined over a finite flat extension of the usual Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^{\times}]]$; assume that the weight two specialization corresponds to the ordinary p-stabilization of the eigenform attached to an elliptic curve E. Fix a neighborhood U of $2 \in \mathcal{W}_{\mathbf{f}}$, and suppose for simplicity that U is contained in the residue class of 2 modulo $p-1$. The weight k specialization, denoted for simplicity as f_k, is a normalised eigenform of weight k on $\Gamma_0(Np)$, which is new at the primes dividing N. More precisely, if k belongs to $U \cap \mathbb{Z}^{>2}$, the modular form f_k is the p-stabilization of a normalised eigenform on $\Gamma_0(N)$, denoted now f_k°. If $(p,n) = 1$, then $a_n(f_k^{\circ}) = a_n(f_k)$. Letting

$$1 - a_p(f_k^{\circ}) + p^{k-1-2s} = (1 - \alpha_p(k)p^{-s})(1 - \beta_p(k)p^{-s})$$

denote the Euler factor at p that appears in the L-series of f_k°, we may order the roots in such a way that

$$\alpha_p(k) = a_p(f_k), \quad \beta_p(k) = p^{k-1}a_p(f_k)^{-1};$$

then, $f_k(z) = f_k^{\circ}(z) - \beta_p(k)f_k^{\circ}(pz)$. For each k, we choose the Shimura periods $\Omega_k^+ := \Omega_{f_k}^+$ and $\Omega_k^- := \Omega_{f_k}^-$, requiring that

$$\Omega_2^+\Omega_2^- = \langle f, f \rangle, \quad \Omega_k^+\Omega_k^- = \langle f_k^{\circ}, f_k^{\circ} \rangle \text{ for } k > 2[1].$$

Let χ denote a primitive Dirichlet character of conductor m satisfying $\chi(-1) = w_\infty$, where w_∞ is the Fricke eigenvalue of $\mathbf{f}$. For a fixed modular form g of weight k, an integer j with $1 \leq j \leq k-1$, and a Dirichlet character χ satisfying $\chi(-1) = (-1)^{j-1}w_\infty$, we define

$$L^{\mathrm{alg}}(g, \chi, j) = \frac{(j-1)!\mathfrak{g}(\chi)}{(-2\pi i)^{j-1}\Omega_g} L(g, \chi, j), \tag{1.13}$$

where $\Omega_g = \Omega_g^+$ if its Fricke eigenvalue $w_\infty = 1$, and Ω_g^- elsewhere. An algebraicity result usually attributed to Shimura asserts that this quantity belongs to K_g, the field of coefficients of g.

The Mazur–Kitagawa two-variable p-adic L-function attached to χ is a function of $(k, s) \in \mathcal{W}_{\mathbf{f}} \times \mathcal{W}$ denoted as $L_p(\mathbf{f}, \chi)$, and usually defined in terms of modular symbols. See [Ki94] for a detailed construction of that function. The main property we want to underline is summarized in the following proposition.

Proposition 1.3.3. *Suppose that k belongs to $U \cap \mathbb{Z}^{\geq 2}$, and that $1 \leq j \leq k-1$ satisfies $\chi(-1) = (-1)^{j-1}w_\infty$. Then,*

$$L_p(\mathbf{f}, \chi)(k, j) = \lambda(k)(1 - \chi(p)a_p(k)^{-1}p^{j-1})L^{\mathrm{alg}}(f_k, \chi, j).$$

Here, $\lambda(k) \in \mathbb{C}_p$ is a p-adic period arising from the p-adic interpolation of modular symbols and that can be made explicit; in particular, $\lambda(2) = 1$.

[1] In the last chapter we will need slightly different normalizations for these periods, but which of course do not affect to the algebraicity results.

Alternatively,

$$L_p(\mathbf{f},\chi)(k,j) = \lambda(k)(1-\chi(p)\alpha_p(k)^{-1}p^{j-1})(1-\chi(p)\alpha_p(k)^{-1}p^{k-j-1})^{\epsilon_k} L^{\mathrm{alg}}(f_k,\chi,j),$$

where $\epsilon_k = 1$ if $f_k \neq f_k^\circ$ and $\epsilon_k = 0$ otherwise.

This function is related with Kato's Euler system; more precisely, with Ochiai's variant allowing the variation of the modular form along a Hida family. See e.g. [Och03], [BD09] or [Ven16].

p-adic L-functions and Euler systems

As we have already mentioned, our main interest for the study of p-adic L-function is the connection with Euler systems. We have already seen this phenomenon when relating special values of the Kubota–Leopoldt p-adic L-function with circular units. The same occurs with elliptic units, whose values are encoded in Katz's two-variable p-adic L-function. This also suggests that when the Eisenstein series is replaced by a cuspidal form and elliptic units are substituted by Heegner points, there should be a cuspidal analogue to Katz's function. This is provided by Bertolini, Darmon and Prasanna. Let $\Sigma_{f,c}$ be the set of Hecke characters ψ of conductor c and trivial central character for which $L(f,\psi^{-1},s)$ is self-dual and has $s=0$ as its central critical point. Then, there is a unique p-adic analytic function

$$L_p(f,K) \,:\, \hat{\Sigma}_{f,c} \to \mathbb{C}_p \tag{1.14}$$

interpolating the critical values $L(f,\psi^{-1},0)$ for those characters of infinity type $(\kappa+2,-\kappa)$, with $\kappa \geq 0$ (the infinity type of a Hecke character of K is a pair of two integer numbers related with the archimedean behavior of it). One can prove an *explicit reciprocity law* connecting this function with the anticyclotomic Euler system of Heegner points. Both the setting of elliptic units and Heegner points require the choice of a pair (p,K), where p is a prime and K is an imaginary quadratic field in which p splits. Andreatta and Iovita [AI19] have recently extended those constructions, allowing the prime p to be inert or ramified in K.

There is a different scenario arising from the Rankin–Selberg product. Let (f,g,h) be a triple of eigenforms of weights (k,ℓ,m), with $k=\ell+m+2r$ and $r\geq 0$. Let

$$\delta_m = \frac{1}{2\pi i}\Big(\frac{d}{d\tau}+\frac{m}{\tau-\bar{\tau}}\Big)$$

stand for the Shimura–Maass derivative operator. The main point is that the Petersson scalar product

$$I(f,g,h) := \langle f, g\times \delta_m^r h\rangle \tag{1.15}$$

may be p-adically interpolated. For that purpose, one has to consider Hida families $(\mathbf{f},\mathbf{g},\mathbf{h})$ going through (f,g,h), as well as different operators. These are e_{ord}, which stands for Hida's ordinary projector; and $d=q\frac{d}{dq}$, which denotes the Atkin–Serre operator that raises the weight of a modular form by two. This operator admits the same algebraic description as δ in terms of the Gauss–Manin connection and can be seen as a p-adic replacement of it. Then, for any triple of points (x,y,z) over the weight space, of weights (k,ℓ,m), with $k=\ell+m+2r$, $r\geq 0$, define

$$\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z) = \frac{\langle (\mathbf{f}_x^*)^\circ, e_{\mathrm{ord}}(d^r\mathbf{g}_y^{[p]}\times \mathbf{h}_z)\rangle}{\langle (\mathbf{f}_x^*)^\circ, (\mathbf{f}_x^*)^\circ\rangle},$$

where f^* denotes the modular form whose q-expansion is given by the complex conjugation of the coefficients of f, and $[p]$ is the p-depletion operator acting on modular forms. See [DR14, Section 4] for the detailed construction.

When both $\mathbf{g}$ and $\mathbf{h}$ are Eisenstein series we obtain a p-adic L-function which is essentially the product of two Mazur–Kitagawa functions attached to $\mathbf{f}$ (but one needs to be careful with the choice of Shimura's periods!). See e.g. [BD14, Section 3]. If only $\mathbf{h}$ is Eisenstein but both $\mathbf{f}$ and $\mathbf{g}$ are cuspidal, we obtain the so-called three-variable Hida–Rankin p-adic L-function $L_p(\mathbf{f},\mathbf{g})$ attached to two Hida families, already constructed in [Hi85] and [Hi88]; and when all three Hida families are cuspidal, this is the triple product p-adic L-function constructed in different instances by Ichino, Harris–Tilouine, Darmon–Rotger or more recently by Hsieh [Hs20], whose construction also works for interpolating along the *balanced* region, where $k \leq \ell + m$; $\ell \leq m + k$; and $m \leq k + \ell$ (this generalizes earlier work of Greenberg and Seveso [GS20]). We extensively discuss these different instances and the main properties of each p-adic L-function along the text. More precisely, in Chapter 3 we revisit the Hida–Rankin p-adic L-function and in Chapter 7 we recall the main properties of the triple product p-adic L-function.

The following table shows different instances of p-adic L-functions and the connection with Euler systems. In some of the cases, we are not emphasizing the number of variables. For instance, Kato's elements were originally constructed for a fixed modular form, and in that case the connection is with the so-called Mazur–Swinnerton-Dyer p-adic L-function; later, Ochiai's work [Och03] allowed to extend the construction to Hida families and connected the two-variable Euler system with the Mazur–Kitagawa p-adic L-function. Something similar happens with Beilinson–Flach elements: one can consider up to 3 variables (one for each Hida family and a third one corresponding to cyclotomic twists), and the same occurs for the Hida–Rankin p-adic L-function, that we have denoted by $L_p(\mathbf{g},\mathbf{h})$ in the introduction. In the case of diagonal cycles the situation is not so well understood. There is always a three-variable Euler system which encodes information about an unbalanced p-adic L-function (there are three options depending on the region of interpolation); one would expect to construct as a putative refinement a four variable p-adic L-function, but there is no Ichino formula giving an algebraicity result for non-central values.

Euler system	p-adic L-function
Circular units	Kubota–Leopoldt p-adic L-function
Elliptic units	Katz's two-variable p-adic L-function
Heegner points	Bertolini–Darmon–Prasanna anticyclotomic p-adic L-function
Kato elements	Mazur–Swinnerton-Dyer p-adic L-function
Beilinson–Flach	Hida–Rankin p-adic L-function
Diagonal cycles	Harris–Tilouine triple product p-adic L-function

As we have mentioned in our discussion around Euler systems, one can find in the literature a plethora of new instances of this formalism. For instance, Loeffler and Zerbes have recently obtained a reciprocity law for the symplectic group GSp(4), which allows them to obtain positive results towards the study of the Bloch–Kato conjecture and the Iwasawa main conjecture [LZ20].

Iwasawa main conjectures

Iwasawa theory is by itself one of the most interesting topics in number theory nowadays. The goal of classical Iwasawa theory is to study the growth of the class group in towers of cyclotomic fields. Although the most relevant points for our work are those related with the arithmetic of elliptic curves, we begin by recalling the formulation of the classical Iwasawa main conjecture. For that purpose, we closely follow [Sh18].

Let $F_n = \mathbb{Q}(\mu_{p^n})$, and let A_n denote the p-part of the class group of F_n, which is a module over $\mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})^\times]$. For $n \geq m$, we may consider the maps $A_m \to A_n$ and $A_n \to A_m$ induced by the inclusion of ideal groups and the norm map on ideals, respectively. We let

$$A_\infty = \varinjlim A_n \quad \text{and} \quad X_\infty = \varprojlim A_n,$$

with the direct and inverse limits taken with respect to these maps. Endowing A_∞ with the discrete topology and X_∞ with the profinite topology, they become continuous modules over the completed group ring $\mathbb{Z}_p[[\mathbb{Z}_p^\times]] = \lim_{\leftarrow} \mathbb{Z}_p[(\mathbb{Z}/p^n\mathbb{Z})]$.

The group $\Gamma_0 = 1 + p\mathbb{Z}_p$ is procyclic, generated for example by $1+p$. The profinite $\mathbb{Z}_p$-algebra $\Lambda_0 = \mathbb{Z}_p[[\Gamma_0]]$ is isomorphic to the completed group ring $\mathbb{Z}_p[[T]]$. The structure of the modules over Λ_0 is quite well understood, and we refer to any standard text on Iwasawa theory for details on that point. The nice point is that every localization of Λ_0 at a height one prime is a principal ideal domain, and in this case the structure theory of finitely generated modules is standard.

A first remarkable fact, that follows by Nakayama's lemma, is that the group X_∞ is a finitely generated, torsion Λ_0-module. We want a finer decomposition of X_∞ as a $\mathbb{Z}_p[(\mathbb{Z}/p\mathbb{Z})^\times]$-module, by singling out one of the $p-1$ components with respect to the action of $(\mathbb{Z}/p\mathbb{Z})^\times$. Each of these eigenspaces is denoted as $X^{(i)}$, where i ranges over the set of residue classes modulo $p-1$, and is a Λ_0-module.

To state the Iwasawa main conjecture we need to introduce the p-adic L-function that best fits with our purposes, which is nothing but a suitable modification of Kubota–Leopoldt's one. Indeed, consider an element f_k of the total quotient ring of $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ such that when k is not congruent to 0 modulo $p-1$ satisfies that

$$f_k(v^s - 1) = L_p(\omega^k, s).$$

This element can be constructed from the theory of Stickelberger elements. When $k \equiv 0$ modulo $p-1$, we set $f_k = 1$.

Theorem 1.3.4 (Iwasawa main conjecture, Mazur–Wiles)**.** *Let k be an even integer. Then,*

$$\mathrm{Char}(X_\infty^{(1-k)}) = (f_k),$$

where Char *is the ideal of the* Λ_0*-module.*

There are different approaches to this result. Proving an equality of ideals like the previous one requires checking both divisibilities; however, in this particular case, the nice behavior of class groups makes that with one of them suffices. Mazur and Wiles proved it for an odd prime p, and Wiles treated the case $p = 2$ and a generalization of the conjecture to totally real number fields. Their proof uses Hida theory. Rubin gave another proof exhibiting the opposite divisibility, bounding the size of the unramified Iwasawa module by exploiting the method of Euler systems (in particular, as we already anticipated, the use of circular units).

Beginning with work of Mazur and Swinnerton–Dyer in the seventies and specially in subsequent papers of Greenberg, those ideas were extended to elliptic curves and other p-adic Galois representations. Each instance has its own main conjecture (at least conjecturally!) relating certain Galois cohomology groups (algebraic side) with a p-adic L-function (analytic side). And like as it happens with the original main conjecture, these facts have deep consequences for the related special value formulas.

Let us discuss the formulation of the Iwasawa main conjecture (IMC) for elliptic curves by recalling the main ingredients involved in its formulation. The main reference for this part is [Sk18]. We assume all the time that $p > 2$. Let F be a fixed number field, and let F_∞/F be a $\mathbb{Z}_p^d$-extension of F, with $d \geq 1$. This is a profinite abelian extension of F such that $\Gamma = \mathrm{Gal}\,(F_\infty/F)$ is isomorphic to $\mathbb{Z}_p^d$. Now, let $\Lambda = \mathbb{Z}_p[[\Gamma]]$ stand for the completed group ring of Γ over $\mathbb{Z}_p$. For simplicity, assume that F_∞ is ramified at each place $v \,|\, p$. Let $T = T_p(E)$ be the p-adic Tate module of E, which is a free $\mathbb{Z}_p$-module with a continuous $\mathbb{Z}_p$-linear action of G_F, that we denote as ϱ. Let $\Lambda^\vee = \mathrm{Hom}_{\mathrm{cont}}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p)$ be the Pontryagin dual of Λ, and let $\Psi : G_F \to \Gamma$ stand for the canonical projection. Put $M = T \otimes \Lambda^\vee$ and let G_F act via $\varrho \otimes \Psi^{-1}$.

We fix a finite set of places of F, Σ, containing all those dividing p or at which E has bad reduction. We are going to define $S(E/F_\infty)$ as a subgroup of $H^1(G_{F,\Sigma}, M)$, where $G_{F,\Sigma}$ is the Galois group of the maximal extension of F unramified outside Σ. Towards that purpose, and for the sake of simplicity, we assume that E has either good ordinary, multiplicative, or supersingular reduction at each $v \,|\, p$. Finally, let S_p^{ord} be set of $v \,|\, p$ at which E has good ordinary or multiplicative reduction, and therefore there exists a G_{F_v}-filtration $0 \subset T_v^+ \subset T$, with T_v^+ a rank one $\mathbb{Z}_p$-summand. It is characterized by the property that T/T_v^+ is unramified and Fr_v acts as multiplication by the unit root α_v of $x^2 - a_v(E) + p$ (if E has good reduction at v) or by $a_v(E)$ (if E has multiplicative reduction at v). Then, define

$$S(E/F_\infty) = \ker\Big(H^1(G_{F,\Sigma}, M) \xrightarrow{\text{res}} \prod_{v\in\Sigma, v\nmid p} H^1(F_v, M) \times \prod_{v\in S_p^{\text{ord}}} H^1(I_v, T/T_v^+ \otimes_{\mathbb{Z}_p} \Lambda^\vee)\Big).$$

To formulate the IMC, we content ourselves with the case where $F = \mathbb{Q}$. The p-adic L-function that appears in this setting is an element $\mathcal{L}(E/\mathbb{Q}_\infty) \in \Lambda = \mathbb{Z}_p[[\Gamma]]$, closely connected with the Mazur–Kitagawa p-adic L-function when the weight is kept fixed. To characterize it, we must fix some notations. Given a primitive p^t-th root of unity ζ, ψ_ζ is the finite order character of $G_\mathbb{Q}$ obtained by projecting to Γ and composing with the character of Γ and composing with the character of Γ that sends a topological generator γ to ζ. We also denote by ψ_ζ the Dirichlet character of $(\mathbb{Z}/p^{t+1}\mathbb{Z})^\times$ of Λ sending T to $\zeta - 1$. Similarly, $\phi_\zeta : \Lambda \to \mathbb{Z}_p[\zeta]$ is the homomorphism sending $\gamma \in \Gamma$ to ζ. Associated to the elliptic curve E there is a modular form f_E whose associated L-function is $L(f_E, s)$, that can be twisted by the Dirichlet character ψ_ζ, $L(f_E, \psi_\zeta, s)$ (we insist on the fact that this is essentially the Mazur–Swinnerton-Dyer p-adic L-function, which already appeared in previous sections).

Then, $\mathcal{L}(E/\mathbb{Q}_\infty)$ is characterized by the property that for any primitive p^t-th root of unity ζ,

$$\phi_\zeta(\mathcal{L}(E/\mathbb{Q}_\infty)) = e_p(\zeta)\frac{L(f_E, \psi_\zeta^{-1}, 1)}{\Omega_{f_E}}, \tag{1.16}$$

where Ω_{f_E} is a canonical period of f_E and

$$e_p(\zeta) = \begin{cases} \alpha_p^{-(t+1)} \frac{p^{t+1}}{\mathfrak{g}(\psi_\zeta^{-1})} & \text{if } \zeta \neq 1, \\ \alpha_p^{-1}(1-\alpha_p^{-1})^{m_p} & \text{if } \zeta = 1. \end{cases}$$

Here, $m_p = 2$ if E has good ordinary reduction and $m_p = 1$ if E has multiplicative reduction.

Conjecture 1.3.5 (Cyclotomic Iwasawa–Greenberg main conjecture for E)**.** *Assume that E has good ordinary or multiplicative reduction at p. The Pontryagin dual $X(E/\mathbb{Q}_\infty)$ of $S(E/\mathbb{Q}_\infty)$ is a torsion Λ-module and its characteristic ideal* $\mathrm{Char}(E/\mathbb{Q}_\infty) = \mathrm{Char}(X(E/\mathbb{Q}_\infty))$ *is generated by $\mathcal{L}(E/\mathbb{Q}_\infty)$ in $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and even in Λ if $E[p]$ is an irreducible $G_\mathbb{Q}$-representation.*

There is a formulation of the IMC which does not require p-adic L-functions. Let

$$S_{\text{str}}(E/\mathbb{Q}_\infty) = \ker\Big(S(E/\mathbb{Q}_\infty) \to H^1(\mathbb{Q}_p, M)\Big), \quad X_{\text{str}}(E/\mathbb{Q}_\infty) = S_{\text{str}}(E/\mathbb{Q}_\infty)^\vee.$$

Kato constructed a free Λ-module $Z_{\text{Kato}} \subset H^1(\mathbb{Z}[1/p], T \otimes_{\mathbb{Z}_p} \Lambda)$. The IMC without L-functions, in this case, asserts that $H^1(\mathbb{Z}[1/p], T \otimes_{\mathbb{Z}_p} \Lambda)$ is a torsion-free rank one Λ-module, that $Z_{\text{Kato}} \neq 0$, and that

$$\mathrm{Char}(H^1(\mathbb{Z}[1/p], T \otimes_{\mathbb{Z}_p} \Lambda)/Z_{\text{Kato}}) = \mathrm{Char}(X_{\text{str}}(E/\mathbb{Q}_\infty)). \tag{1.17}$$

It is quite a pleasant exercise to show that both formulations are equivalent; it basically relies on Poitou–Tate duality and the reciprocity law connecting Kato's class with the p-adic L-function.

The main conjecture for CM elliptic curves with ordinary reduction at p was established by Rubin. Kato establishes one of the divisibilities under certain mild assumptions, relying on the existence of an Euler system, which gives appropriate bounds for the Selmer group. In particular, he proved that $X(E/\mathbb{Q}_\infty)$ is a torsion Λ-module and that

$$(\mathcal{L}(E/\mathbb{Q}_\infty)) \subset \mathrm{Char}(E/\mathbb{Q}_\infty),$$

if E has good ordinary reduction at p. Greenberg and Vatsal [GV00] later explored this fact together with the classical IMC for Dirichlet characters so as to deduce the cyclotomic main conjecture for some elliptic curves E for which $E[p]$ is reducible as a $G_{\mathbb{Q}}$-representation. The opposite divisibility was established by Skinner and Urban [SU14], who constructed non-trivial elements of the p-adic Selmer group from congruences between Eisenstein series and cusp forms of $\mathrm{GU}(2,2)$ of an appropriate quadratic imaginary field (the Eisenstein series is properly selected so that its constant terms involves L-values of f). This is a generalization of the strategy initiated by Ribet and extended by Mazur–Wiles in the proof of the cyclotomic IMC. Finally, we point out that this result was extended to the case of split multiplicative reduction by Wan [Wan15], where he works with the unitary group $\mathrm{GU}(3,1)$.

The IMC has deep applications towards the Birch and Swinnerton–Dyer conjecture, for the moment restricted to ranks 0 and 1, although recent work of Castella and Hsieh [CH20] deals with an anticyclotomic version of it in a rank 2 situation. Under certain assumptions, Jetchev, Skinner and Wan [JSW17] have recently proved that the p-part of the BSD conjecture is true in rank 1.

Theorem 1.3.6. *Let E be a semistable elliptic curve of conductor N_E and p a prime of good reduction such that $a_p(E) = 0$ if E has supersingular reduction at p. Suppose that $E[p]$ is irreducible as a $G_{\mathbb{Q}}$-module. If E has analytic rank one, then*

$$\Big|\frac{L'(E,1)}{\Omega_E \cdot \mathrm{Reg}_E}\Big|_p^{-1} = \Big||\mathrm{Sha}(E)| \cdot \prod_{\ell \mid N_E} c_\ell\Big|_p^{-1}.$$

The arguments were extended to the case of multiplicative reduction by Castellà [Cas18b].

Some variants of the Iwasawa main conjecture for elliptic curves and modular forms are particulary interesting, specially those concerning quadratic imaginary fields, where also a lot of progress has been made (see for instance [How04], [BD05] and [ChHs05]). Another instance of the IMC is related with the tensor product of the Galois representations attached to two modular forms, where the results of Lei, Kings, Loeffler and Zerbes [LLZ14], [KLZ17] give the proof of one of the divisibilities, due to the existence of an Euler system in that case. See also the related works of Lei, Loeffler and Zerbes [LLZ14] and Büyükboduk and Lei [BL18], [BL20], where the authors discuss the construction of an Euler system attached to the twist of a modular form by a grössencharacter of an imaginary quadratic field, and apply this to bounding Selmer groups.

In an ongoing project with Raúl Alonso and Francesc Castellà [ACR21], we expect to partially adapt these results to the setting of diagonal cycles and triple products p-adic L-functions, combining several of the ideas discussed in this thesis with Castella's earlier work on anticyclotomic Iwasawa theory [Cas17], [CH18].

Chapter 2

Background material II: Gross–Stark units, exceptional zeros and congruences

In this chapter, we continue with a review of the state of the art, focusing now on some specific topics which play a key role in this dissertation. The first section serves as a quick survey of the different Stark and Gross–Stark conjectures that have been studied in the literature, with special emphasis again on the connection between the arithmetic of units in number fields and the arithmetic of elliptic curves, following the analogy of [DLR15a] and [DLR16]. The second part is a short section devoted to the theory of exceptional zeros and improved p-adic L-functions. We go back to the origins of the concept and see its evolution, finishing with the general conjectures posed by Greenberg, Benois, and others. Finally, we close the chapter by emphasizing some of the most relevant points of the theory of congruences between modular forms, beginning with the foundational works of Mazur and Ribet and going until the last studies of Sharifi and Fukaya–Kato.

2.1 Gross–Stark units

Stark's conjectures give complex analytic formulas for units in number fields (more precisely, for their logarithms) in terms of the leading terms of Artin L-functions at $s = 0$. This was later studied in the p-adic setting, where Gross was the first one who envisaged how the use of p-adic techniques could yield to new results. It is also natural to wonder if there are similar formulas for algebraic points on elliptic curves. As it has already been mentioned, Heegner points are the right analogue to circular or elliptic units (arising when the elliptic curve does not have CM), and it is known that their heights are related to L-series via the celebrated Gross–Zagier formula.

The aim of this section is threefold: we begin with a short overview of different Gross–Stark type conjectures available in the literature; we then move to the setting of points in elliptic curves; and we finish stressing the parallelism between the arithmetic of units over number fields and that of elliptic curves by comparing different constructions for real quadratic fields, appearing when the field of complex multiplication is no longer available. We had already anticipated in the introduction the importance of this topic when we discussed the interaction between the arithmetic of triple product p-adic L-functions and Darmon points, and the possibility of carrying out a similar study when one takes their natural replacements in the Eisenstein case (Hida–Rankin p-adic L-functions and Darmon–Dasgupta units).

Stark's conjectures

The easiest cases where a connection between special values of (p-adic) L-values and units arise are related with Dirichlet character. For simplicity, say that $\chi : (\mathbb{Z}/N\mathbb{Z})^\times$ is a non-trivial even Dirichlet character. Then, if $H = \mathbb{Q}(\zeta_N)$, the space $U_\chi = (\mathcal{O}_H^\times \otimes E)^{\bar{\chi}}$ is one-dimensional, and fixing a generator u_χ, we have that

$$L'(\chi, 0) = -\frac{1}{2}\log(u_\chi), \quad L(\bar{\chi}, 1) = -\frac{1}{\mathfrak{g}(\chi)}\log(u_\chi).$$

Here, log is the usual complex logarithm.

The counterpart to this result in the non-archimedean setting was already presented (this is Leopoldt's formula again). With the conventions of Theorem 1.3.1,

$$L_p(\bar{\chi}, 1) = -\frac{1 - \chi(p)p^{-1}}{\mathfrak{g}(\chi)}\log_p(u_\chi),$$

where now $\log_p$ is the p-adic logarithm. Similarly, and with the usual conventions adopted in the study of Stark's conjectures,

$$L_p(\chi, 0) = -(1 - \chi\omega^{-1}(p))B_{\chi\omega^{-1},1},$$

where $B_{\chi\omega^{-1},1}$ is a generalised Bernoulli number. We come back to this formula later on in Chapter 5, but let us just mention that this is particularly interesting due to the presence of a certain Euler factor which may vanish, and this happens exactly when $\chi\omega^{-1}(p) = 1$.

In 1980, Gross conjectured a formula for the expected leading term at $s = 0$ of the Deligne–Ribet p-adic L-function associated to a totally even character ψ of a totally real field F. It states that after scaling by $L(\psi\omega^{-1}, 0)$, this value is equal to a p-adic regulator of units in the abelian extension of F. To fix notations, let F be a totally real field and let

$$\chi : G_F \longrightarrow \bar{\mathbb{Q}}^\times$$

be a totally odd character of the absolute Galois group of F. Let H denote the CM, cyclic extension cut out by χ, and let H stand for the number field of definition of χ. Consider the L-function associated to χ with the Euler factors at primes above p removed:

$$L^*(\chi, s) = L(\chi, s) \cdot \prod_{\mathfrak{p}|p}(1 - \chi(\mathfrak{p})(\mathbb{N}\mathfrak{p})^{-s}).$$

There is a unique meromorphic p-adic L-function

$$L_p(\chi\omega, s) : \mathbb{Z}_p \longrightarrow \mathbb{C}_p \tag{2.1}$$

determined by the interpolation property

$$L_p(\chi\omega, n) = L^*(\chi\omega^n) \text{ for all } n \leq 0.$$

The set of primes above p in F may be split as $R \cup R'$, where R consists on those $\mathfrak{p}$ with $\chi(\mathfrak{p}) = 1$ and R' is its complementary. The cardinality of R is denoted as $r_p(\chi)$. Gross conjectured that

$$\mathrm{ord}_{s=0}L_p(\chi\omega, s) = r_p(\chi),$$

and proposed a formula for the leading term $L_p^{(r)}(\chi\omega, 0)$.

In the case where $r_p(\chi) = 1$ and R consists on a single prime $\mathfrak{p}$, this is just the following.

Theorem 2.1.1 (Darmon–Dasgupta–Pollack). *With the previous assumptions,*

$$\frac{L_p'(\chi\omega,0)}{L(\chi,0)} = \mathcal{L}(\chi)\cdot\prod_{\mathfrak{q}\in R'}(1-\chi(\mathfrak{q})),$$

where the $\mathcal{L}$-invariant $\mathcal{L}(\chi)$ is given by

$$\mathcal{L}(\chi) = -\frac{\log_{\mathfrak{p}}(v_\chi)}{\mathrm{ord}_{\mathfrak{p}}(v_\chi)}.$$

Here, v_χ is a generator of $U_\chi = (\mathcal{O}_H[1/p]^\times \otimes L)^{\bar\chi}$.

When $r_p(\chi) > 1$ there is an analogue expression in terms of a regulator matrix which also involves p-adic logarithm of units. In this setting, the conjecture was proved by Dasgupta, Kakde and Ventullo [DKV18]. Along different recent works, Dasgupta and Kakde have achieved a better comprehension of this question, studying the so-called Brumer–Stark conjecture and deriving evidence towards Hilbert's 12th Problem. More precisely, let H/F be a finite abelian extension of number fields with F totally real and H a CM field. Let S and T be disjoint sets of places of F satisfying appropriate conditions. The Brumer–Stark conjecture states that the Stickelberger element $\Theta_{S,T}^{H/F}$ annihilates the T-smoothed class group of H. We refer to [DK20] for more details and for a proof away from $p=2$ (that is, after tensoring with $\mathbb{Z}[1/2]$).

Let us discuss another instance, connected with weight one modular forms and of great importance in this monograph. Let

$$g = \sum_{n\geq 1} a_n q^n \in S_1(N,\chi)$$

be a cusp form of weight one, level N, and odd character χ. Deligne and Serre associated to it an odd, two-dimensional Artin representation

$$\varrho_g : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C}).$$

Conversely, work of Buzzard–Taylor and Khare–Winterberger asserts that when ϱ is an odd, irreducible, two-dimensional Artin representation, there is a weight one newform g satisfying $L(\varrho,s) = L(g,s)$.

As usual, let H stand for the field cut out by the Artin representation ϱ_g, and let L be a number field large enough to contain the Fourier coefficients of g.

Conjecture 2.1.2 (Stark). *Let g be a cuspidal newform of weight one, with Fourier coefficients in L. Then, there is a unit $u_g \in (\mathcal{O}_H^\times \otimes L)^+$ satisfying*

$$L'(g,0) = \log(u_g),$$

where $(\mathcal{O}_H^\times \otimes L)^+$ stands for the fixed part of the unit group under the action of complex conjugation.

This conjecture admits a more general formulation in the setting of arbitrary Artin representations, and moreover some of its generalizations and variants have raised a lot of interest. This includes for instance the so-called Brumer–Stark conjecture, and also its higher rank variants studied by Burns, Popescu and Rubin, among others.

The Elliptic Stark Conjecture

We want to present now an alternative Gross–Stark-type conjecture connecting p-adic L-functions (alternatively, a p-adic iterated integral) with p-adic logarithms of global points over elliptic curves. This is not new at all, since results of this kind had already appeared in other settings: the Katz p-adic L-function (Rubin, 1992); the Mazur–Swinnerton-Dyer p-adic L-function (Perrin-Riou, 1993);

various types of p-adic Rankin L-function attached to $f \otimes \theta_\psi$, where θ_ψ is the theta series attached to a character of a quadratic imaginary field ψ (Bertolini–Darmon, 1995; Bertolini–Darmon–Prasanna, 2008); p-adic Garett–Rankin L-functions attached to $f \otimes g \otimes h$ (Darmon–Rotger, 2012)... In this section we present the main conjecture of [DLR15a], which intends to shed some light on a rank two instance of the BSD conjecture. Moreover, we recover one of the triple product L-functions discussed along the previous part, interpolating central critical values corresponding to (f_k, g_ℓ, h_m), where $\ell \geq k + m$. Therefore, it is natural to ask ourselves about the value at a point lying outside the region of classical interpolation, in this case the point $(2, 1, 1)$ (for balanced points, the values of the function are related with the Bloch–Kato logarithm of certain diagonal cycles).

To fix notations, consider a triple of normalized weight two newforms

$$f \in S_2(N_f), \quad g \in M_1(N_g, \bar{\chi}), \quad h \in M_1(N_h, \chi),$$

and assume for simplicity that f is attached to a rational elliptic curve, and that L is a number field which is large enough to contain the Fourier coefficients of both g and h, and also the roots of their p-th Hecke polynomials. We further assume that $h = g^*$, that is, the twist by the inverse of its nebentype. Finally, V_g, V_h and V_{gh} stand for the Galois representations attached to g, h and $g \otimes h$, respectively.

Let us consider the functional equation for the triple product classical L-function $L(E, V_{gh}, s) = L(f \otimes g \otimes h, s)$, connecting s and $2 - s$. The sign of the functional equation, expressed in terms of the root number $\epsilon(E, V_{gh})$, is a product of local signs, as

$$\epsilon(E, V_{gh}) = \prod_{v | N\infty} \epsilon_v(E, V_{gh}),$$

where $N = \mathrm{lcm}(N_f, N_g, N_h)$. In this case, $\epsilon_\infty(E, V_{gh}) = 1$ since we are in the balanced region (it can be easily proved that the sign at infinity is -1 if and only if the triple is balanced; this is coherent with the expectation provided by the Beilinson conjecture and the fact that diagonal cycles are available precisely in that framework). We now make several assumptions, always following the setting of [DLR15a].

(A) Local sign hypothesis: for all finite places $v \mid N$, $\epsilon_v(E, V_{gh}) = 1$.

Under this assumption, $L(E, V_{gh}, s)$ vanishes to even order at $s = 1$, and we can relate its value at 1 with the value of the trilinear form

$$I : S_2(N)_{\mathbb{C}} \times M_1(N, \bar{\chi})_{\mathbb{C}} \times M_1(N, \chi)_{\mathbb{C}} \to \mathbb{C}$$

given by $I(\tilde{f}, \tilde{g}, \tilde{h}) = \langle \tilde{f}, \tilde{g}\tilde{h} \rangle$. Harris and Kudla proved that the restriction

$$I_{fgh} : S_2(N)[f] \times M_1(N, \bar{\chi})[g] \times M_1(N, \chi)[h] \to \mathbb{C}$$

is identically zero if and only if the central critical value of the L-function vanishes.

(B) Global vanishing hypothesis: the L-function $L(E, V_{gh}, s)$ vanishes at $s = 1$ (and therefore the trilinear form I_{fgh} is identically zero).

Let $d = q\frac{d}{dq}$ stand for Serre's differential operator, raising by 2 the weight in the space of p-adic modular forms. If $\tilde{f} \in S_2(N)$ is *overconvergent* (see [DR14] for a discussion on the importance of this fact), which is marked with the superscript "oc", then its primitive $d^{-1}\tilde{f}$ may be understood as a p-adic limit of d^t when t goes to -1 p-adically. Let $\tilde{F} = d^{-1}\tilde{f} \in S_0^{\mathrm{oc}}(N)$ and define the p-adic iterated integral attached to

$$(\tilde{f}, \tilde{\gamma}, \tilde{h}) \in S_2(Np)_L \times M_k^{\mathrm{ord}}(Np, \chi)_L^\vee \times M_k(Np, \chi)_L$$

as

$$I'_p(\tilde{f}, \tilde{\gamma}, \tilde{h}) = \tilde{\gamma}(e_{\mathrm{ord}}(\tilde{F} \cdot \tilde{h})) \in \mathbb{C}_p. \tag{2.2}$$

(C) Classicality property for the p-stabilization g_α: let $g_\alpha(q) = g(q) - \beta_g g(q^p)$ and write

$$S_1^{\mathrm{oc,ord}}(N,\chi)_{\mathbb{C}_p}[[g_\alpha^*]] = \cup_{n\geq 1}\ker(I_{g_\alpha}^n)$$

for the generalised eigenspace attached to g_α; here I_{g_α} is the ideal associated to the system of Hecke eigenvalues of g_α. Then, this space is non-trivial and consists only on classical forms.

This hypothesis is usually satisfied in practice. Indeed, Bellaiche and Dimitrov showed that when $g \in S_1(N,\bar{\chi})$, the natural inclusion

$$S_1(Np,\chi)_{\mathbb{C}_p}[g_\alpha^*] \hookrightarrow S_1^{\mathrm{oc,ord}}(N,\chi)[[g_\alpha^*]]$$

is an isomorphism of $\mathbb{C}_p$-vector spaces if and only if V_g is not induced from a character of a real quadratic field in which p splits.

(C') The modular form g satisfies one of the following properties: (i) it is a cusp form, regular at p, and it is not the theta series of a real quadratic field in which p splits; (ii) it is an Eisenstein series irregular at p, i.e., $V_g = \chi_1 \oplus \chi_2$, with $\chi_1(p) = \chi_2(p)$.

Hypothesis A and B together imply that $r_{\mathrm{an}}(E, V_{gh}) \geq 2$, and of course the same is expected for the algebraic rank. When this value is 2, we may fix an algebraic basis $\{P, Q\}$ for $\mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}, E(H)\otimes L)$. Choose a basis of V_{gh} compatible with the Frobenius decomposition

$$e_{\alpha\alpha} \in V_{gh}^{\alpha\alpha}, \quad e_{\alpha\beta} \in V_{gh}^{\alpha\beta}, \quad e_{\beta\alpha} \in V_{gh}^{\beta\alpha}, \quad e_{\beta\beta} \in V_{gh}^{\beta\beta},$$

and recall that one may identify $\mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}, E(H)\otimes L) \simeq (E(H)\otimes V_{gh}^\vee)^{G_\mathbb{Q}}$. Write

$$P = P_{\alpha\alpha}\otimes e_{\beta\beta}^\vee + P_{\alpha\beta}\otimes e_{\beta\alpha}^\vee + P_{\beta\alpha}\otimes e_{\alpha\beta}^\vee + P_{\beta\beta}\otimes e_{\alpha\alpha}^\vee,$$

and similarly for Q. Here, the arithmetic Frobenius Fr_p acts on $P_{\alpha\alpha}$ with eigenvalue $\beta_g\beta_h$, and similarly for the other components[1].

The regulator attached to E and V_{gh} is

$$\mathrm{Reg}_{g_\alpha}(E, V_{gh}) = \log_p(P_{\alpha\alpha})\cdot\log_p(Q_{\alpha\beta}) - \log_p(Q_{\alpha\alpha})\cdot\log_p(P_{\alpha\beta}).$$

Finally, let $\mathrm{ad}^0(g) := \mathrm{Hom}^0(V_g, V_g)$ denote the three-dimensional adjoint representation attached to V_g, on which the Frobenius acts with eigenvalues 1, α_g/β_g, and β_g/α_g. In these cases, we may attach to g_α a Stark unit $u_{g_\alpha} \in \mathcal{O}_{H_g}[1/p]^\times)_L^{\mathrm{ad}_g}$, where H_g is the number field cut out by ad_g^0.

Conjecture 2.1.3. *Assume that hypotheses A, B and C-C' are satisfied. If $r(E, V_{gh}) > 2$, the trilinear form I_p' of (2.2) is zero. Otherwise, there are test vectors*

$$(\tilde{f}, \tilde{\gamma}, \tilde{h}) \in S_2(Np)_L[f] \times M_1(Np,\chi)_L^\vee[g_\alpha] \times M_1(Np,\chi)_L[h]$$

for which

$$I_p(\tilde{f}, \tilde{\gamma}, \tilde{h}) = \frac{\mathrm{Reg}_{g_\alpha}(E, V_{gh})}{\log_p(u_{g_\alpha})}. \tag{2.3}$$

Remark 2.1.4. This conjecture is usually presented in terms of triple product p-adic L-functions. As shown in [DLR15a, Proposition 2.6], the trilinear form may be recast as the special value $\mathscr{L}_p{}^g(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at the point $(2,1,1)$. See Chapter 7 for further discussions on this.

[1] Along the different works around the Elliptic Stark Conjecture, one may find different terminologies regarding the local decomposition at p. We have tried to be consistent along the memoir, but keep in mind that in some articles this could be slightly different.

The case where f is Eisenstein and therefore is no longer attached to an elliptic curve is equally interesting. In that situation, the global points over elliptic curves are replaced by Gross–Stark units. This is extensively discussed in subsequent chapters.

A last comment is that there are several cases, namely those where V_{gh} decomposes as the direct sum of smaller representations, where the formulation of the conjecture is very explicit. This is the case where g and h are theta series of the same quadratic imaginary field where p splits, and the regulator becomes just the product of the logarithms of two Heegner points. There is another interesting instance: when g and h are theta series of a real quadratic field where p is inert, we expect to get logarithms of Stark–Heegner points (the natural replacement of Heegner points over real quadratic fields, we come back to them in the following section). And of course we also expect this at the level of units when f is Eisenstein: elliptic units in the quadratic imaginary case, and Darmon–Dasgupta units (sometimes called Gross–Stark units) in the real case. We explore this parallelism in the following section and prove results in this direction along the first part of the memoir.

An analogy between units and points

At this point of the discussion is already clear that there is a fruitful connection between the arithmetic of points in elliptic curves and that of units in number fields. One of the objectives of Chapter 3 is mimicking the approach of [DR20b] so as to study Stark–Heegner points, but in the case of units in number fields and with diagonal classes replaced by Beilinson–Flach elements. The conjectural p-adic replacement for Stark–Heegner points or Darmon points is provided by certain p-adic units, referred either as Gross–Stark units or Darmon–Dasgupta units; we adopt this name for the rest of the text. Let us discuss this parallelism and present a general view of Darmon–Dasgupta units, which are encoded at the weight one specializations of Beilinson–Flach elements, as shown in Section 6 of the following chapter.

To begin with this analogy, let us mention that in [BD09], the authors give evidence towards the rationality of Darmon points for genus characters by using the ideas that led to the proof of the celebrated Kronecker limit formula. The same ideas were also used for Park in its study of Darmon–Dasgupta units, the real quadratic replacement of elliptic units. The following table summarizes the different settings we find.

	Classical setting	p-adic setting
Units in number fields	Elliptic units	Darmon–Dasgupta units
Points in elliptic curves	Heegner points	Darmon points

In both p-adic settings, linear combinations of the logarithms of those objects (points or units), weighted by genus characters, are shown to be equal (up to some factors) to the logarithm of their classical counterparts, given evidence towards its rationality. These proofs, as it is underlined in the Introduction of [BD09], are inspired precisely by the Kronecker limit formula, that relates weighted combinations of logarithms of elliptic units with the logarithm of a fundamental unit of a real quadratic field. Let us sketch the treatment which is followed in each of the approaches.

Elliptic units: Kronecker's limit formula. Let K be an imaginary quadratic field and fix an order $\mathcal{O}_D$ in K of discriminant D; fix also $N > 0$ relatively prime with D. Let $\mathcal{H}^D$ be the set of $\tau \in \mathbb{H} \cap K$ such that $\mathcal{O}_\tau = \mathcal{O}_D$, where

$$\mathcal{O}_\tau = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R \text{ such that } a\tau + b = c\tau^2 + d\tau \right\} \subset K,$$

being R the subring of matrices which are upper-triangular modulo N. Let G_D be the class group of K. A non-trivial quadratic (genus) character χ of G_D corresponds to a pair of Dirichlet characters

χ_1 and χ_2 which are even and odd, respectively, cutting out quadratic extensions K_1 and K_2 of $\mathbb{Q}$. Let ϵ_1 be the fundamental unit of K_1 and denote by h_j the class number of K_j and by w_2 the number of roots of unity in K_2.

Let
$$\eta^*(\tau) := |D|^{-1/4}\sqrt{2y}|\eta(\tau)|^2;$$
Kronecker showed that for any $\tau \in \mathcal{H}^D$,
$$\sum_{\sigma \in G_D} \chi(\sigma) \log \eta^*(\tau^\sigma) = -\frac{2h_1h_2}{w_2}\log(\epsilon_1).$$

Observe that this gives a solution of the Pell equation $x^2 - D_1y^2 = 1$ in terms of the function η^* evaluated at suitable imaginary quadratic arguments (here, D_1 is the discriminant of K_1). The proof uses three main ingredients:

1. The Kronecker limit formula, which expresses the left hand side of the previous equation in terms of $L(K,\chi,s)$. Namely,
$$-\sum_{\sigma \in G_D} \chi(\sigma) \log \eta^*(\tau^\sigma) = \frac{d}{ds}L(K,\chi,s)|_{s=0}.$$

2. A factorisation of $L(K,\chi,s)$ as a product of the Dirichlet L-series attached to χ_1 and χ_2 (where χ_1 is odd and χ_2 even):
$$L(K,\chi,s) = L(\chi_1,s)L(\chi_2,s).$$

3. Dirichlet's class number formula, which affirms that
$$L'(\chi_1,0) = h_1\log(\epsilon_1), \qquad L(\chi_2,0) = \frac{2h_2}{w_2}.$$

Darmon–Dasgupta units. Darmon–Dasgupta units, as introduced in [DD06], are natural substitutes of elliptic units when the imaginary quadratic field of the theory of complex multiplication is replaced by a real quadratic field K. These units are defined over a local field, but it is conjectured that linear combinations of them, weighted by the values of ring class characters of K, belong to the group of global p-units of the Hilbert class field of K.

We begin by recalling the main ingredients that are needed for defining Darmon–Dasgupta units. Let p be a rational prime and K a real quadratic field of discriminant D in which p is inert. Fix N such that $(N,pD)=1$, and fix also a formal sum
$$\delta = \sum_{d|N, d>0} n_d[d],$$
such that $n_d \in \mathbb{Z}$, $\sum n_d = \sum n_d \cdot d = 0$.

Let $\mathbb{H}_p$ denote the p-adic upper half plane, and let $R \subset M_2(\mathbb{Z}[1/p])$ be the subring of matrices that are upper triangular modulo N. The $\mathbb{Z}[1/p]$-order associated to $\tau \in \mathbb{H}_p \cap K$, that we denote as $\mathcal{O}_\tau$, is defined to be the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in R such that $a\tau + b = c\tau^2 + d\tau$. Via the morphism that sends a matrix to $c\tau+d$, we can see this order inside K. We refer to the $\mathbb{Z}[1/p]$-order of K of discriminant D as $\mathcal{O}_D$, and let
$$\mathcal{H}_p^D := \{\tau \in \mathbb{H}_p \cap K \text{ such that } \mathcal{O}_\tau = \mathcal{O}_D\}.$$

The set $\mathcal{H}_p^D$ is non-empty (since p is inert in K), hence we may fix $\tau \in \mathcal{H}_p^D$. Moreover, $\mathcal{H}_p^D$ is endowed with an action of an arithmetic group, and the quotient space is finite and has a transitive action of $\mathrm{Pic}(\mathcal{O})$; we denote it as $\mathfrak{a} * \tau$, where $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$.

Using the theory of p-adic multiplicative integrals, Darmon and Dasgupta constructed an elliptic unit for K associated to δ and τ,

$$u(\delta, \tau) \in K_p^\times.$$

They conjectured that $u(\delta, \tau)$ is a global p-unit lying in $\mathcal{O}_H[1/p]^\times$, where H is the Hilbert class field of K. More precisely:

Conjecture 2.1.5. *Let $\tau \in \mathcal{H}_p^D$. Then, $u(\delta, \tau) \in \mathcal{O}_H[1/p]^\times$, where H is the Hilbert class field of K, and*

$$u(\delta, \mathfrak{a} * \tau) = \mathrm{rec}(\mathfrak{a})^{-1} u(\delta, \tau),$$

where $\mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O})$, rec is the usual Shimura reciprocity law map, and $\mathfrak{a} * \tau$ denotes the canonical action of the narrow Picard group in $\mathcal{H}_p^{\mathcal{O}}/\Gamma$.

Let us see the connection with p-adic zeta functions. Consider the $\Gamma_0(N)$-Eisenstein series of weight $2r$ (with $r \geq 1$) given by the test vector

$$F_{2r}(z) = -24 \cdot \sum_{d|N, d>0} n_d \cdot d \cdot E_{2r}(dz).$$

Darmon and Dasgupta established the existence of a family μ of $\mathbb{Z}$-valued p-adic measures on $\mathbb{Q}_p^2 - \{(0,0)\}$, indexed by pairs $(r, s) \in \Gamma \cdot \infty \times \Gamma \cdot \infty$ and denoted $\mu\{r \to s\}$, such that

$$\int_{(\mathbb{Z}_p^2)'} Q_\tau(z, 1)^{r-1}\, d\mu\{\infty \to \gamma_\tau \cdot \infty\}(x, y) = (1 - p^{2r-2}) \cdot \int_{\infty}^{\gamma_\tau \infty} Q_\tau(z, 1)^{r-1} F_{2r}(z)\, dz.$$

This allows us to define a p-adic zeta series

$$\zeta_p(\delta, \tau, s) = \frac{1}{12} \cdot \int_{(\mathbb{Z}_p^2)'} \langle Q_\tau(x, y) \rangle^{-s}\, d\mu\{\infty \to \gamma_\tau \cdot \infty\}(x, y),$$

where γ_τ is the generator of the stabilizer of τ in Γ (which is a rank one module), and the dependence on δ is encoded in the measure μ.

To any $\tau \in \mathcal{H}_p^D/\Gamma$ we can attach a primitive integral indefinite binary quadratic form Q_τ of discriminant D. Let

$$\zeta(\tau, s) := \zeta_{Q_\tau}(s), \qquad \zeta(\delta, \tau, s) := \sum_{d|N, d>0} n_d \cdot d^s \cdot \zeta(d\tau, s).$$

It holds that $\zeta(\tau, s) = \zeta(\mathfrak{a}, s) - \zeta(\mathfrak{a}^*, s)$, where $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_D)$ is the narrow ideal class associated to Q_τ, $\mathfrak{a}^*$ is the narrow ideal corresponding to $\alpha\mathfrak{a}$ (with $\alpha \in K^\times$ of negative norm) and

$$\zeta(\mathfrak{a}, s) = \sum_{I \in \mathfrak{a}} \frac{1}{\mathbb{N}(I)^s}.$$

Darmon and Dasgupta show in [DD06, Section 4] the following relation between the p-adic zeta function and the units they have previously constructed.

Theorem 2.1.6. *The p-adic zeta function $\zeta_p(\delta, \tau, s)$ vanishes at $s = 0$ and its derivative satisfies*

$$\zeta_p'(\delta, \tau, 0) = -\frac{1}{12} \log_p(\mathbb{N}_{K_p/\mathbb{Q}_p}(u(\delta, \tau))).$$

From now on, we focus on the ψ-version of the conjecture, where ψ is a finite order ring class character. We define the ψ-component of the Gross-Stark unit as

$$u(\delta,\tau)_\psi := \prod_{\mathfrak{a}\in \mathrm{Pic}(\mathcal{O})} u(\delta, \mathfrak{a} * \tau)^{\psi(\mathfrak{a})}.$$

The dependence of $u(\delta,\tau)$ on τ is not relevant for the study of the rationality of these units, since their logarithm only depends on τ up to scaling in $L^\times$: indeed, given any other $\tau' \in \mathcal{H}_p^D$, since the action of $\mathrm{Pic}(\mathcal{O})$ is transitive, there exists $\mathfrak{b} \in \mathrm{Pic}(\mathcal{O})$ such that $\tau' = \mathfrak{b} * \tau$ and then

$$\log(u(\delta,\tau')_\psi) = \sum_{\mathfrak{a}\in \mathrm{Pic}(\mathcal{O})} \psi(\mathfrak{a})\cdot u(\delta,(\mathfrak{a}\mathfrak{b})*\tau) = \psi(\mathfrak{b})^{-1}\cdot \log(u(\delta,\tau)_\psi).$$

Hence, we may forget about the variable τ and just write $u(\delta)_\psi$. The ψ-equivariant version of the Darmon–Dasgupta conjecture says the following. Here, $V_\psi = \mathrm{Ind}_K^{\mathbb{Q}}\,\psi$.

Conjecture 2.1.7. *Let ψ be an odd, finite order ring class character. Then,*

$$u(\delta)_\psi \in \mathcal{O}_H[1/p]^\times[V_\psi].$$

In the case of quadratic genus characters, the work of Park [Park10] provides theoretical evidence in favor of the conjecture. When $\psi^2 \neq 1$, the induced representation V_ψ is irreducible; the field it cuts out cannot be embedded in any compositum of ring class fields of imaginary quadratic fields, and the Darmon–Dasgupta conjecture cannot be studied via the theory of elliptic units in this case. Assume then that

$$V_\psi = \mathrm{Ind}_K^{\mathbb{Q}}\,\psi = \chi_1 \oplus \chi_2$$

decomposes as the sum of two one-dimensional Galois representations attached to quadratic Dirichlet characters of odd signature satisfying $\chi_1(p) = -\chi_2(p)$. In particular, the work of Park establishes that $\log(u(\delta)_\psi^+)$ is equal (up to some explicit factors) to $\log_p(\mathfrak{u})$, where $\mathfrak{u}$ is an elliptic unit. Here, $u(\delta)_\psi = \mathbb{N}_{K_p/\mathbb{Q}_p}(u(\delta)_\psi)$.

The different p-adic ζ functions, as τ varies along the class group, are encoded in terms of what we refer to as $L_p(K,\psi,s)$, which is defined through the equation

$$\sum \psi(\mathfrak{a})\cdot \zeta_p(\delta,\mathfrak{a}*\tau,s) = 2\Big(\sum_{d|N,d>0} \epsilon_d n_d \langle d\rangle^s\Big)\cdot L_p(K,\psi,s),$$

where $\epsilon_d := \psi(\mathfrak{a}\mathfrak{a}_d^{-1})$ (according to [Park10, Lemma 4.2], this quantity only depends on d and not on $\mathfrak{a}$). Theorem 2.1.6 can be adapted to the equivariant version of the conjecture and in this case says that

$$\log(u(\delta)_\psi^+) = 24\Big(\sum_{d|N,d>0} \epsilon_d n_d\Big)\cdot L_p'(K,\psi,0). \tag{2.4}$$

One of the main results of [Park10] gives a connection between this p-adic L-function and the classical elliptic unit of a quadratic imaginary field.

Proposition 2.1.8 (Park)**.** *Assume that $D = D_1\cdot D_2$, with $D_1, D_2 < 0$, and let ψ be the genus character associated to that decomposition (with $\psi_{D_1}(p) = -\psi_{D_2}(p) = 1$). There exists a unique $\mathfrak{u}_{D_1} \in \mathbb{Q}\otimes \mathcal{O}_{\mathbb{Q}(\sqrt{D_1})}[1/p]^\times$ such that*

$$\log(u(\delta)_\psi^+) = \frac{96h_2}{w_2}\Big(\sum_{d|N,d>0} \epsilon_d n_d\Big)\cdot \log_p(\mathfrak{u}_{D_1}),$$

where h_2 is the class number of $\mathbb{Q}(\sqrt{D_2})$, w_2 is the number of roots of unity and $\epsilon_d := \chi(\mathfrak{a}\mathfrak{a}_d^{-1})$, being $\mathfrak{a}_d \in \mathrm{Pic}^+(\mathcal{O}_K)$ such that $\zeta(d(\mathfrak{a}*\tau),s) = \zeta(\mathfrak{a}_d,s) - \zeta(\mathfrak{a}_d^*,s)$ (ϵ_d does not depend on $\mathfrak{a}$).

This result is only available for genus characters; however Theorem 2.1.6 always holds. The principle behind the proof is to compare the Darmon–Dasgupta units to suitable elliptic units, exploiting the fact that the field over which $u(\delta)_\psi^+$ is conjecturally defined is a biquadratic extension of $\mathbb{Q}$ and is also contained in ring class fields of imaginary quadratic fields. The argument of Park shares some ideas with the original method for proving Kronecker's limit formula. He uses the following tools.

1. The formula relating a Darmon–Dasgupta unit with the derivative of a zeta function, in particular with the value at 0 of $\zeta_p'(\delta,\tau,0)$:
$$-12\cdot\zeta_p'(\delta,\tau,0)=\log_p(\mathbb{N}_{K_p/\mathbb{Q}_p}(u(\delta,\tau))).$$
As we have seen, this implies that
$$\log(u(\delta)_\chi^+)=24\Big(\sum_{d|N,d>0}\epsilon_d n_d\Big)\cdot L_p'(K,\chi,0). \tag{2.5}$$

2. A p-adic analogue of the factorization formula of Kronecker, given by
$$L_p(K,\chi,s)=L_p(\chi_1\cdot\omega,s)\cdot L_p(\chi_2\cdot\omega,s),$$
where ω is the Teichmuller character.

3. Some known cases of Stark's conjectures, relating the logarithm of elliptic units with special values of the derivatives of p-adic L-functions. In particular,
$$L_p(\chi_{D_2}\cdot\omega,0)=\frac{4h_2}{w_2},\qquad L_p'(\chi_{D_1}\cdot\omega,0)=-\log_p(\mathfrak{u}_{D_1}).$$

(Classical) Heegner points. Bertolini and Darmon [BD07] have carried out the first two steps in the previous guide, showing in [BD07, Theorem 1] that when E has at least two primes of semistable reduction (this is a technical assumption they need since they work over certain Shimura curves) there is a global point $P\in E(\mathbb{Q})\otimes\mathbb{Q}$ and a scalar $\ell\in\mathbb{Q}^\times$ such that
$$\frac{d^2}{dk^2}L_p(\mathbf{f})(k,k/2)|_{k=2}=\ell\cdot\log_E(P)^2.$$
Some of the common features with the previous settings are the following:

1. A formula relating the logarithm of a Heegner point with the derivative of a p-adic L-functions. With their notations,
$$\frac{d}{dk}\mathcal{L}_p(\mathbf{f}/K,\chi,k)|_{k=2}=\frac{1}{2}(\log_E(P_\chi)+a_p\log_E(\mathrm{Fr}_p(P_\chi))),$$
or using the more familiar L-function $L_p(\mathbf{f}/K,\chi)(k)$,
$$\frac{d^2}{dk^2}L_p(\mathbf{f}/K,\chi)(k,k/2)|_{k=2}=2\log_E(P_\chi)^2,$$
provided that $\chi_1(p)=a_p$ (and it is zero elsewhere).

2. A factorization formula that asserts that for all genus characters χ,
$$L_p(\mathbf{f}/K,\chi)(k,k/2)=\eta(k)\cdot L_p(\mathbf{f},\chi_1)(k,k/2)\cdot L_p(\mathbf{f},\chi_2)(k,k/2),$$
being $\eta(k)\in\mathbb{C}_p$ an explicit factor such that $\eta(2)=1$. We recall that χ_1 and χ_2 are respectively the even and odd characters attached to χ

We come back to this setting in the following section, to analyze it in the realm of the exceptional zero phenomenon.

Darmon (Stark–Heegner) points. Bertolini and Darmon followed a similar strategy to prove the main conjecture about Stark–Heegner points in the case of a genus character χ. Each of the steps carried out in the case of Darmon–Dasgupta units admits a counterpart in this scenario:

1. A formula relating the Stark–Heegner point P_χ to the leading term of the Mazur–Kitagawa p-adic L-function over a quadratic imaginary field $L_p(\mathbf{f}/K,\chi)(k,k/2)$. For this, they derive a deep relation between periods of Hida families and Stark–Heegner points. With the notations introduced in [BD14] (observe that in the p-adic setting one has to single out a prime p and hence there is a factorization $N = pM$, where some additional requirements are imposed),
$$\frac{d}{dk}\mathcal{L}_p(\mathbf{f}/K,\chi,k)|_{k=2} = \frac{1}{2}(1-\chi(-M)w_M)\log_E(P_\chi),$$
or in terms of the more familiar p-adic L-function $L_p(\mathbf{f}/K,\chi(k,k/2)$,
$$\frac{d^2}{dk^2}L_p(\mathbf{f}/K,\chi)(k,k/2)|_{k=2} = 2\log_E^2(P_\chi),$$
provided that $\chi_1(-M) = -w_M$.

2. A factorisation of $L_p(\mathbf{f}/K,\chi)$ as a product of two Mazur–Kitagawa p-adic L-functions associated to Dirichlet characters χ_j, $L_p(\mathbf{f},\chi_j)(k,s)$.

3. The previous theorem of Bertolini and Darmon (proved in [BD07]) relating the second derivative of a Mazur–Kitagawa p-adic L-function with a classical Heegner point.

2.2 The exceptional zero phenomenon

The first references to the exceptional zero phenomenon date back to Mazur, Tate, and Teitelbaum [MTT86], who were some of the first number theorists that studied p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. In the introduction of their paper, they made the following comment: *"it seemed to us to be an appropriate time to embark on the project of formulating a p-adic analogue of the conjecture of Birch and Swinnerton-Dyer, and gathering numerical data in its support [...] The project has proved to be anything but routine".* The first main contributions to the conjecture were done by Greenberg and Stevens [GS94], who proved it when the analytic rank is 0; and by Bertolini and Darmon [BD07], who did it partially in analytic rank 1. However, their results are restricted to the central critical line of the Mazur–Kitagawa p-adic L-function, which is not completely satisfactory (a priori, we expect a formula for the derivative of the p-adic L-function when the weight is fixed). It was Venerucci [Ven16] who proved firstly a conjecture of Perrin-Riou, relating p-adic Beilinson–Kato elements to Heegner points, and then a large part of the rank one case of the Mazur–Tate–Teitelbaum exceptional zero conjecture for the cyclotomic p-adic L-function. We first review the general statement of the conjectures, and then focus on the work of Venerucci; we finish by presenting more general conjectures, following Benois [Ben14b].

A conjecture of Mazur, Tate and Teitelbaum

Mazur, Tate and Teitelbaum [MTT86] observed that certain *p-adic multipliers* involved in the definition of the p-adic L-function affected the formulation of the conjecture, measuring in a certain way the discrepancy between the p-adic and classical special values. Unless otherwise stated, assume that E is an elliptic curve defined over $\mathbb{Q}$ with split multiplicative reduction at p. In this case, it has been conjectured, also in [MTT86], that

$$\mathrm{ord}_{s=1}L_p(E,s) = 1 + \mathrm{ord}_{s=1}L(E,s).$$

We will see that this may be understood in terms of the vanishing of one of the Euler factors in the interpolation property.

The Euler factors at p play a prominent role in the proof given by Greenberg and Stevens, who covered the rank 0 case. The main tool they use is the p-adic variation of the modular form f_E attached to the elliptic curve E along a Hida family $\mathbf{f}$. For each $k \geq 2$ in the same congruence class than 2 modulo $p-1$, we denote by $f_k \in S_k(Np)$ the specialization of f_k at weight k. Let $a_p(k)$ stand for the trace of the Frobenius acting on the Tate module. Then, for each integer $k \geq 2$, the p-th Euler factor of the complex L-function $L(f_k,s)$ of f_k has the form

$$\frac{1}{(1-\alpha_p(k)p^{-s})(1-\beta_p(k)p^{-s})},$$

where $\alpha_p(k) = a_p(k)$ and

$$\beta_p(k) = \begin{cases} p^{k-1}/\alpha_p(k) & \text{if } k > 2 \text{ and } k \equiv 2 \pmod{p-1} \\ 0 & \text{otherwise .} \end{cases}$$

Then, for each integer $k \geq 2$ and each integer s_0 with $0 < s_0 < k$ and $s_0 \equiv 1$ modulo $p-1$, the p-adic L-function $L_p(f_k,s)$ satisfies the following interpolation property:

$$L_p(f_k,s_0) = (1-\beta_p(k)p^{-s_0})(1-\alpha_p(k)^{-1}p^{s_0-1}) \cdot \frac{L(f_k,s_0)}{\Omega_{f_k}}. \tag{2.6}$$

When $s_0 = 1$ the second Euler factor can be interpolated as an Iwasawa function, namely $(1-a_p(k)^{-1})$. It vanishes at $k=2$, and so it is a non-unit in the Iwasawa algebra. Further, $L_p(k,1)$ is also an Iwasawa function in k, which is shown to be divisible by $(1-a_p(k)^{-1})$. The quotient $L_p^*(k,1)$ is an Iwasawa function in k, that we call the *improved* p-adic L-function. It satisfies the interpolation property

$$L_p^*(k,1) = (1-\beta_p(k)p^{-1}) \cdot \frac{L(f_k,1)}{\Omega_{f_k}}.$$

This yields to an exceptional vanishing of the p-adic L-function attached to an elliptic curve, $L_p(E,s)$, at the central critical point $s=1$. Greenberg and Stevens introduced in this setting what they called an $\mathcal{L}$-invariant, characterized by the fact that, in a rank zero situation,

$$L_p'(E,s)|_{s=1} = \mathcal{L}_E \cdot \frac{L(E,1)}{\Omega_E}. \tag{2.7}$$

From the functional equation satisfied by the Mazur–Kitagawa p-adic L-function, one has that $L_p(f_k,k/2) = 0$ for all k. Then, the linear term in the Taylor expansion of $L_p(\mathbf{f})$ around $(k,s) = (2,1)$ is given by $c \cdot (-\frac{1}{2}(k-2)+(s-1))$ for some $c \in \mathbb{Z}_p$. It is clear that $c = L_p'(E,1)$, but at the same time the interpolation formula also gives a formula for c in terms of the derivative of a_p:

$$\mathcal{L}_p(E) = -2 \cdot \frac{da_p(k)}{dk}\Big|_{k=2}. \tag{2.8}$$

Here, $a_p(k)$ is seen as a p-adic analytic function on the p-adic variable $k \in \mathcal{W}_{\mathbf{f}}$, for a certain open disk $\mathcal{W}_{\mathbf{f}}$ of the weight space.

Then, we just have to connect the quantity $-2a_p'(2)$ with Tate's uniformizer. This is shown by using the Λ-adic Galois representation $\mathbb{V}_{\mathbf{f}}$ attached to the Hida family $\mathbf{f}$ so as to compute an annihilator under the local Tate pairing of a certain class

$$\kappa \in H^1(\mathbb{Q}_p, \mathbb{Q}_p) \simeq \mathrm{Hom}(G_{\mathbb{Q}_p}, \mathbb{Q}_p).$$

The construction of this element is rather indirect, and it uses Kummer theory and Galois deformation techniques. The space $H^1(\mathbb{Q}_p, \mathbb{Q}_p)$ has a canonical basis given by two distinguished classes κ_{nr} and κ_{cyc}. The former is defined as the unique unramified homomorphism sending the Frobenius element Fr_p to 1; the latter is given by the logarithm of the cyclotomic character. Therefore, writing

$$\kappa = x \cdot \kappa_{\mathrm{nr}} + y \cdot \kappa_{\mathrm{cyc}}, \qquad \text{with } x, y \in \mathbb{Q}_p,$$

the quotient $-x/y$ is univocally determined and is precisely the $\mathcal{L}$-invariant we are interested in. See [GS94] for the details of the proof; it may be illustrative to compare it with that of the Gross conjecture in [DDP11], where the authors follow a quite similar approach and also need to compute an annihilator under the local Tate pairing. This same scenario is studied, with much more generality, in [DKV18], where the authors are able to deal with a situation where the order of vanishing is greater than 1 (with more or less similar ideas, although there are different technical difficulties, partially studied in Ventullo's PhD thesis). In the framework of elliptic curves, we will see that the problem becomes much more complicated.

As we already discussed, Bertolini and Darmon [BD07] explored the counterpart of the previous phenomenon, but when the sign of the functional equation is odd. Indeed, among the general properties of $L_p(f_k, s)$, there is a functional equation relating $L_p(\mathbf{f})(k, s)$ to $L_p(\mathbf{f})(k, k-s)$. The sign is independent on k, and we can denote it as $\mathrm{sign}(\mathbf{f})$. If E has split multiplicative reduction at p, then

$$\mathrm{sign}(\mathbf{f}) = -\mathrm{sign}(E/\mathbb{Q}),$$

where the latter stands for the sign in the functional equation of the classical Hasse–Weil L-function $L(E, s)$. This discrepancy reflects the fact that $L_p(f, s)$ has an *exceptional zero* at the central critical point $s = 1$, arising from the fact that p is a prime of split multiplicative reduction for E. While Greenberg and Stevens considered the case where $\mathrm{sign}(E, \mathbb{Q}) = 1$, we are going to assume now that $\mathrm{sign}(E, \mathbb{Q}) = -1$. Since $L(E, 1) = 0$, the Birch and Swinnerton–Dyer conjecture predicts that $E(\mathbb{Q})$ is infinite. The results of Greenberg and Stevens show that

$$\frac{\partial}{\partial s} L_p(\mathbf{f})(k, s)|_{(2,1)} = \frac{\partial}{\partial k} L_p(\mathbf{f})(k, s)|_{(2,1)} = 0,$$

and in particular, $L_p(\mathbf{f})(k, s)$ vanishes to order at least 2 at $(k, s) = (2, 1)$. Since $\mathrm{sign}(f_\infty) = 1$, the restriction of $L_p(\mathbf{f})(k, s)$ to the central critical line $s = k/2$ need not vanish identically, and the main result of [BD07] is the following, that we have just discussed in the previous section as a part of the big analogy between points and units.

Theorem 2.2.1 (Bertolini–Darmon). *Suppose that E has at least two primes of semistable reduction.*

1. *There is a global point $P \in E(\mathbb{Q}) \otimes \mathbb{Q}$ and a scalar $\ell \in \mathbb{Q}^\times$ such that*

$$\frac{d^2}{dk^2} L_p(\mathbf{f})(k, k/2)|_{k=2} = \ell \cdot \log_E(P)^2.$$

2. *The point P is of infinite order if and only if $L'(E, 1) \neq 0$.*

It remains of course to study higher rank situations, and this study suggests that this phenomenon may be studied in many other settings. The next section is devoted to study other derivatives beyond the central critical line.

A conjecture of Perrin-Riou and Venerucci's theorem

As before, let $f \in S_2(\Gamma_0(Np), \mathbb{Z})$ be the weight two newform associated to E by the modularity theorem, and let $\mathbf{f} = \sum_{n=1}^{\infty} a_n(k)q^n \in \Lambda_{\mathbf{f}}[[q]]$ be the Hida family passing through it. For simplicity, we single out the congruence class of 2 modulo $p-1$. For every classical point $k \in \mathcal{W}_{\mathbf{f}}$, the q-expansion of $f_k \in S_k(\Gamma_1(Np), \mathbb{Z}_p)$ is an N-new p-ordinary Hecke eigenform of weight k, and moreover $f_2 = f$. We also have the Mazur–Kitagawa two-variable p-adic L-function $L_p(\mathbf{f})(k,s)$, in such a way that $L_p(\mathbf{f})(2,s) = L_p(E,s)$. One of the main results of [Ven16] is a formula for $\frac{d^2}{ds^2} L_p(E,s)$ at the point $s=1$, where it vanishes to order two.

However, and as a part of his program, he first establishes a tantalizing connection between two a priori different Euler systems. Perrin-Riou had already conjectured that, in a rank one situation, the logarithm of the p-adic Beilinson–Kato class equals the square of the logarithm of a Heegner point on the elliptic curve, up to a non-zero rational factor. In particular, she had predicted that the Beilinson–Kato class is non-zero precisely if the Hasse–Weil L-function has a simple zero at $s=1$. Along this section, we call κ_E the p-adic Beilinson–Kato class attached to E. According to Kato's reciprocity law

$$\exp^*_{\mathrm{BK}}(\mathrm{res}_p(\kappa_E)) = (1-p^{-1}) \frac{L(E,1)}{\Omega_E^+} \in \mathbb{Q}, \tag{2.9}$$

where Ω_E^+ is a Néron–Tate period attached to E.

Remark 2.2.2. The Euler system κ_E used by Venerucci is not exactly the one discussed in the previous chapter, but a rather connected one. As we will discuss in the last chapter, the Beilinson–Kato element is usually attached to the convolution $f \otimes E_2(\chi_1, \chi_2)$, where E_2 is the Eisenstein series attached to the auxiliary characters χ_1 and χ_2. This explains the discrepancy between the reciprocity laws presented by Venerucci and Bertolini–Darmon in [BD14].

Remark 2.2.3. The relations between different Euler systems appears once more. In this case, we are connecting a *cyclotomic* Euler system (that of Beilinson–Kato) and an *anticyclotomic* one (Heegner points).

Theorem 2.2.4 (Venerucci)**.** *Assume that $L(E,1)=0$, and that κ_E belongs to the Bloch–Kato Selmer group. Then, there exists a non-zero rational number $\ell_1 \in \mathbb{Q}^\times$ and a rational point $P \in E(\mathbb{Q}) \otimes \mathbb{Q}$ such that*

$$\log_E(\mathrm{res}_p(\kappa_E)) = \ell_1 \cdot \log_E^2(P).$$

Further, P is non zero if and only if $L(E,s)$ has a simple zero at $s=1$.

Later, he gets a refined version of the result of Bertolini–Darmon, being able to express the derivative of $L_p(E,s)$ in terms of a height pairing. This is a deep result that requires the introduction of the so-called Nekovar's extended Selmer group $\tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, V_p(E))$. It is a $\mathbb{Q}_p$-module, equipped with a natural inclusion $E^\dagger(\mathbb{Q}) \otimes \mathbb{Q}_p \hookrightarrow \tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, V_p(E))$, where $E^\dagger(\mathbb{Q})$ stands for the extended Mordell–Weil group, as introduced in [MTT86]. In general, $\tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, V_p(E))$ is canonically isomorphic to the direct sum of the Bloch–Kato Selmer group $H^1_{\mathrm{f}}(\mathbb{Q}, V_p(E))$ and the 1-dimensional vector space $\mathbb{Q}_p \cdot q_E$ generated by the Tate period of $E_{\mathbb{Q}_p}$. This can be thought as the *extra* element explaining the additional zero in the case of split multiplicative reduction, in the same way that the p-unit gives rise to an extra zero in the setting of units over number fields.

Let I stand for the ideal of functions vanishing at the point $(k,s)=(2,1)$. In [Ven16, §4], a canonical $\mathbb{Q}_p$-bilinear pairing

$$\langle\langle \cdot, \cdot \rangle\rangle_p \; : \; \tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, V_p(E)) \otimes_{\mathbb{Q}_p} \tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, V_p(E)) \to I/I^2,$$

is introduced, called the *cyclotomic height-weight pairing.* It can be decomposed as

$$\langle\langle \cdot, \cdot \rangle\rangle_p = \langle \cdot, \cdot \rangle_p^{\mathrm{cyc}} \cdot \{s-1\} + \langle \cdot, \cdot \rangle_p^{\mathrm{wt}} \cdot \{k-2\},$$

in terms of two different pairings, whose definition is also recalled in loc. cit.

Finally, for $x, y \in H^1_{\mathrm{f}}(\mathbb{Q}, V_p(E))$, let

$$\langle x, y\rangle_p^{\mathrm{Sch}} := \langle x, y\rangle_p^{\mathrm{cyc}} - \frac{\log_E(\mathrm{res}_p(x)) \cdot \log_E(\mathrm{res}_p(y))}{\log_p(q_E)},$$

where res_p is the map corresponding to localization at p.

Theorem 2.2.5 (Venerucci)**.** *Assume that $L(E, 1) = 0$ and that the restriction map*

$$\mathrm{res}_p : H^1_{\mathrm{f}}(\mathbb{Q}, V_pE) \to E(\mathbb{Q}_p)\hat{\otimes}\mathbb{Q}_p$$

is non-zero. Let $P \in E(\mathbb{Q}) \otimes \mathbb{Q}$ a Heegner point. Then, $L_p(E, s)$ vanishes to order at least 2 at $s = 1$, and there exists a non-zero rational number $\ell_2 \in \mathbb{Q}^\times$ such that

$$\frac{d^2}{ds^2} L_p(E, s)|_{s=1} = \ell_2 \cdot \mathcal{L}_p(E) \cdot \langle P, P\rangle_p^{\mathrm{Sch}}.$$

Towards a general theory of exceptional zeros

The exceptional zero phenomenon can be explored from a broader perspective. There is a particularly nice sentence in the Introduction to Chapter III of Perrin-Riou saying the following: *"Nous avons toutefois supposé pour simplifier que les opérateurs $1 - \phi$ et $1 - p^{-1}\phi^{-1}$ sont inversibles laissant les autres cas, pourtant extremement intéressant pour plus tard."* (Here, ϕ is the terminology she uses for the Frobenius element). Benois formulated a conjecture about extra zeros of p-adic L-functions at near central points, proving that it is compatible with Perrin-Riou's theory of p-adic L-functions. Let us briefly explain the philosophy behind this general framework, following mainly [Ben14b].

Let M be a pure motive over $\mathbb{Q}$, and assume that its complex L-function $L(M, s)$ extends to a meromorphic function on the whole complex plane $\mathbb{C}$. As we have already emphasized, Perrin-Riou formulated precise conjectures about the existence and arithmetic properties of p-adic L-functions when the p-adic realisation V of M is crystalline at p. Let $\mathbb{D}_{\mathrm{cris}}(V)$ denote the filtered Dieudonné module attached to V by the theory of Fontaine, and let D be a subspace of it of dimension $d_+(V) = \dim_{\mathbb{Q}_p} V^{c=1}$ and stable under the action of Frobenius (here, c is the complex conjugation). We say that D is regular if we can associate to D a p-adic analogue of the six-term exact sequence of Fontaine and Perrin-Riou (see [Ben14b] for the definition). Fix a lattice T of V stable under the action of the Galois group and a lattice N of a regular module D. Perrin-Riou conjectured that we can associate to this data a p-adic L-function $L_p(T, N, s)$ satisfying an explicit interpolation property. Observe that this same phenomenon occurs when V is semistable and non-crystaline at p; here, the functor $\mathbb{D}_{\mathrm{cris}}$ is replaced by $\mathbb{D}_{\mathrm{st}}$.

In particular, we can formulate the following expectation, concerning both the existence of the p-adic L-function and the exceptional zero phenomenon (we do it in the semistable case for its resemblance with the case of elliptic curves with split multiplicative reduction at p).

Conjecture 2.2.6 (Trivial zero conjecture)**.** *Let V be a geometric p-adic representation of $G_{\mathbb{Q}}$ which is semistable at p and critical in the sense of Deligne. Given a regular submodule $D \subset \mathbb{D}_{\mathrm{st}}(V_p)$, where $V_p = V|_{G_{\mathbb{Q}_p}}$, there exists a p-adic L-function $L_p(V, D, s)$ satisfying an interpolation formula of the form*

$$L_p(V, D, 0) = \mathcal{E}(V_p, D) \frac{L(V, 0)}{\Omega_V},$$

where Ω_V is a Deligne period, $L(V, s)$ is the complex L-function, and $\mathcal{E}(V_p, D)$ is a product of linear Euler factors at p. Moreover, if e is the number of vanishing Euler factors in $\mathcal{E}(V_p, D)$ and

$\mathcal{E}^+(V_p, D)$ is the product of the remaining non-vanishing ones, then $L_p(V, D, s)$ vanishes to order at least e at $s = 0$ and

$$L_p^{(e)}(V, D, 0) = e!\mathcal{L}(V, D) \cdot \mathcal{E}^+(V_p, D) \cdot \frac{L(V, 0)}{\Omega_V},$$

where $\mathcal{L}(V, D)$ is an arithmetic $\mathcal{L}$-invariant. The trivial zero conjecture for a geometric representation V of G_F agrees with that for $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} V$.

This conjecture is known in relatively few cases, but there is a general belief that it must be true. Its study over totally number fields have been recently carried out by Barrera, Dimitrov, and Jorza [BDJ18], and other generalizations have been appearing in the mathematical literature in the last years.

Therefore, one of the obstructions to have a better understanding of the theory is the good definition of $\mathcal{L}$-invariants. We expect to be able to define them from different point of views, and in fact, this is a question that has been studied in several contexts, beginning with the foundational work of Greenberg and Stevens [GS94], but also with the more geometric approaches of Coleman and Coleman–Iovita [Co89].

It may be instructive to look at the aforementioned works [DDP11] and [DKV18], where the authors relate in a very simple case the special value of a p-adic L-function with the logarithm of a suitable p-unit, using for that purpose tools from Galois cohomology and Galois deformations, but also an explicit computation of the Fourier coefficient of certain Eisenstein series. In that work, for instance, we see different ways of introducing the $\mathcal{L}$-invariant. Although this has been carried out from many different perspective, our interest will be mainly in these three approaches: (a) the **analytic** $\mathcal{L}$-invariant, given as a special value of a p-adic L-function or its derivative (alternatively, as the value of some rigid analytic function connected with the Fourier coefficients of a modular form varying over a Hida family); (b) the **algebraic** $\mathcal{L}$-invariant, given in terms of the slope of a certain cohomology class (alternatively, Greenberg recasts this formulation in terms of class field theory and universal norms, see for instance the work of Roset–Rotger–Vatsal [RRV19] for a discussion of this case in the setting of the adjoint); and (c) the $\mathcal{L}$-invariant à la **Greenberg–Stevens**, expressed in terms of the quotient of logarithm and order maps in suitable spaces of units (or more generally, regulators defined in terms of logarithm and order maps or several units). There are many other instances where the necessity of seeing the $\mathcal{L}$-invariant from different perspectives is crucial. This idea will be present in subsequent chapters and will be carefully developed in different scenarios.

2.3 Congruences between modular forms

The theory of congruences between modular forms has been widely studied in the last decades, beginning with the works of Swinnerton-Dyer [SD73], Ribet [Rib76], or the seminal papers of Mazur [Maz77], [Maz79] on the arithmetic of the Eisenstein ideal and the arithmetic of special values of L-functions. There are different kind of congruences one can be interested in. For example, the study of Fermat's last theorem and other modularity results put the emphasis on the so called *level raising* and *level lowering* results, which also play a role in problems related with the theory of Euler systems, like the study of the indivisibility of Heegner points [Zh14]. Other topics of interest include, for instance, the theory of congruences between classical holomorphic or meromorphic modular forms, and whose genesis is at various conjectures of Ramanujan concerning the partition function. In this thesis, our main focus is on the so-called Eisenstein congruences, that is, the case where a normalised cuspidal eigenform is congruent, modulo a prime ideal, to an Eisenstein series.

Let us begin by recalling one of the most prototypical examples. We refer the reader to the wonderful survey [Vonk20] for a more detailed treatment. The Ramanujan Δ-function is a cusp form of weight 12 whose q-expansion is given by the infinite product due to Jacobi

$$\Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Consider also the weight k normalised Eisenstein series

$$E_k(q) = \frac{-B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \qquad \text{where } \sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

When $k = 12$, the constant term is equal to $\frac{691}{65520}$, whereas for $k = 6$ the constant term is $\frac{-1}{504}$. Since the space of weight 12 modular forms of level 1 is two-dimensional and spanned by E_{12} and the cuspidal form Δ, the form E_6^2 must be a linear combination of the two. Computing the first two terms of all three q-expansions, we find that

$$\frac{691}{65520} \cdot 504^2 \cdot E_6(q)^2 = E_{12}(q) - \frac{756}{65} \Delta(q),$$

and since all three modular forms involve 691-integral q-expansions, we obtain as a consequence the congruence $E_{12}(q) \equiv \Delta(q)$ modulo 691. In particular, for any prime p, we recover the celebrated Ramanujan conjecture

$$\tau(p) \equiv 1 + p^{11} \pmod{691}.$$

There are other remarkable examples which shed light into some interesting phenomena, with applications in other areas of mathematics beyond number theory, like the theory of partitions of integers. We refer the reader to e.g. [EO96] for further references on this topic.

We fix now some notations and definitions. Let $f = q + \sum_{i=2}^{\infty} a_i q^i$ be a normalised cuspidal eigenform, and let $g = \sum_{i=0}^{\infty} b_i q^i$ be another normalised eigenform (either cuspidal or Eisenstein). We assume, for the sake of simplicity, that both are of weight 2, trivial nebentype and level N_f and N_g, respectively. We warn the reader that the assumption on the nebentype can be easily removed, as we will discuss later on. Let L_f and L_g be the fields generated by the coefficients of f and g, and let L be their composite field. We denote by $\mathcal{O}_f$, $\mathcal{O}_g$, and $\mathcal{O}$ their rings of integers. Let $p > 2$ be a prime and let ϱ_f (resp. ϱ_g) be the 2-dimensional p-adic representation associated to f (resp. g), with values in $\mathcal{O}_{f,p} := \mathcal{O}_f \otimes \mathbb{Z}_p$ (resp. $\mathcal{O}_{g,p}$). Fix a place $\mathfrak{p} \,|\, p$ in L, and let us denote also by $\mathfrak{p}$ its restrictions to L_f and to L_g.

Definition 2.3.1. We say that the modular forms f and g are congruent modulo $\mathfrak{p}^t$ (and write $f \equiv g \pmod{\mathfrak{p}^t}$) if $a_\ell \equiv b_\ell$ modulo $\mathfrak{p}^t$ for almost every prime ℓ.

One of the most important points towards our applications, and which has allowed an enormous development of this theory, is that congruences may be encoded in terms of Galois representations. Let $\bar{\varrho}_{f,\mathfrak{p}^t}$ (resp. $\bar{\varrho}_{g,\mathfrak{p}^t}$) stand for the reduction modulo $\mathfrak{p}^t$ of the p-adic Galois representation associated to f (resp. g). As a consequence of Chebotarev's density theorem, the congruence property is equivalent to saying that the semisimplifications of $\bar{\varrho}_{f,\mathfrak{p}^t}$ and $\bar{\varrho}_{g,\mathfrak{p}^t}$ are isomorphic (this further needs the Brauer–Nesbitt theorem, which guarantees that these elements completely determine the representation).

Proposition 2.3.2. *The congruence property $f \equiv g \pmod{\mathfrak{p}^t}$ is equivalent to*

$$(\bar{\varrho}_{f,\mathfrak{p}^t})^{\mathrm{ss}} \sim (\bar{\varrho}_{g,\mathfrak{p}^t})^{\mathrm{ss}},$$

where we have written $\sim$ to indicate that the representations are isomorphic, and ss *stands for the semisimplification.*

In the following sections, we recover some of the main properties of these Galois representations.

Ribet's converse to Herbrand theorem

In 1976, Ribet [Rib76] proved a refinement of Kummer's criterion for the regularity of an odd prime p. Recall that a prime p is said to be regular if the ideal class group of the cyclotomic field $\mathbb{Q}(\mu_p)$ has no p-torsion. Indeed, Kummer's criterion asserts that an odd prime p is irregular if and only if there exists an even integer $2 \leq k \leq p-3$ such that p divides the numerator of the k-th Bernoulli number B_k. In particular, an odd prime p is irregular if and only if there exists an even integer $2 \leq k \leq p-3$ such that p divides the numerator of $\zeta(1-k)$.

Herbrand had proved a refinement of Kummer's criterion, showing that the p-divisibility of a specific Bernoulli number could only occur if a corresponding character occurs in the action of $\mathrm{Gal}\,(\mathbb{Q}(\mu_p)/\mathbb{Q})$ on the p-part of the class group of $\mathbb{Q}(\mu_p)$. To fix notations, let A be the class group of $\mathbb{Q}(\mu_p)$ and let C be the $\mathbb{F}_p$-vector space A/A^p (the $\mathbb{F}_p$-vector structure is induced by the structure of A as a $\mathbb{Z}$-module). The absolute Galois group acts on C through its abelian quotient $\Delta = \mathrm{Gal}\,(\mathbb{Q}(\mu_p)/\mathbb{Q})$. This allows to give a decomposition of C according to the Δ-action as

$$C = \bigoplus_{i \bmod p-1} C^{(i)},$$

where $C^{(i)}$ is the ω^i-isotypical component of C as a Δ-module. Ribet proved the following.

Theorem 2.3.3 (Ribet)**.** *Let k be an even integer, $2 \leq k \leq p-3$. Then, p divides the numerator of B_k if and only if $C^{(1-k)} \neq 0$.*

Ribet's proof is based on the theory of congruences between modular forms. More precisely, he constructs a cuspidal eigenform which is congruent modulo p to the Eisenstein series E_k. Hence, one may deduce that the reduction modulo p of the Galois representation $\varrho : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_{\mathfrak{p}}) \simeq \mathrm{GL}(V_{\mathfrak{p}})$ attached to this modular form can be written as

$$\begin{pmatrix} 1 & * \\ 0 & \varepsilon_{\mathrm{cyc}}^{k-1} \end{pmatrix}.$$

Here, $K_{\mathfrak{p}}$ is a finite extension of $\mathbb{Q}_p$, $V_{\mathfrak{p}}$ a two-dimensional $K_{\mathfrak{p}}$-representation, and ϱ comes from considering the Galois action on the Tate module of the abelian variety attached to the modular curve. Then, he establishes that there exists a two-dimensional lattice $T \subset V_{\mathfrak{p}}$ such that the previous representation is not semi-simple. As this image is the Galois group of some normal extension of $\mathbb{Q}$, it turns out that it is precisely these elements of order p that correspond to p-extension of $\mathbb{Q}(\mu_p)$. The harder part in his argument is of course proving that the representation is unramified at p.

Regarding this Galois representation, the following result is essentially [Rib76, Theorem 1.3].

Theorem 2.3.4. *Suppose p divides B_k. Then, there exists a finite field $\mathbb{F} \supset \mathbb{F}_p$ and a continuous representation*

$$\bar{\varrho} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{F})$$

such that:

(a) $\bar{\varrho}$ is unramified at all primes $\ell \neq p$.

(b) The representation $\bar{\varrho}$ is reducible over $\mathbb{F}$, in the sense that $\bar{\varrho}$ is isomorphic to a representation of the form

$$\begin{pmatrix} 1 & * \\ 0 & \varepsilon_{\mathrm{cyc}}^{k-1} \end{pmatrix}.$$

This means that $\bar{\varrho}$ is an extension of the 1-dimensional representation with character $\varepsilon_{\mathrm{cyc}}^{k-1}$ by the trivial 1-dimensional representation.

(c) The image of $\bar{\varrho}$ has order divisible by p (this means that $\bar{\varrho}$ is not diagonalizable).

(d) Let D_p be a decomposition group for p. Then, $\bar{\varrho}(D_p)$ has order prime to p (that is, $\bar{\varrho}_{|D_p}$ is diagonalizable).

The coefficient $*$ in the previous matrix gives a cohomology class in $H^1_{\mathrm{f}}(\mathbb{Q}, \mathbb{Q}_p(k-1))$ (and this of course becomes more interesting when we consider non-trivial nebentype, with interesting applications towards the Bloch–Kato conjecture). Indeed, this idea is very powerful, as shown for instance in the recent preprints of Kakde and Dasgupta on the Brumer–Stark conjectures.

Congruences between L-values

One of the main sources of motivation for this dissertation has been the article of Vatsal [Va03], where he shows how congruences between the Fourier coefficients of Hecke eigenforms give rise to corresponding congruences between the algebraic parts of the critical values of the associated L-functions. As already mentioned in loc. cit., this topic had already been object of study in works of Mazur like [Maz79] (where it was made clear how congruences for analytic L-values were closely related to the integral structure of appropriate Hecke rings and cohomology groups) Stevens [St82], [St85], Rubin–Wiles [RW82] (who used congruences to study the behavior of elliptic curves in towers of cyclotomic fields), and the more recent works of Ono–Skinner [OS98] or James [Jam98]. The result of Mazur, which serves as one of the starting points in this theory, can be summarized as follows. Let N be a prime, and f be a weight two newform for $\Gamma_0(N)$. Write $L^{\mathrm{alg}}(f,\chi,1)$ for the algebraic part of $L(f,\chi,1)$, whose definition we later explore. Mazur showed that the residue class of $L^{\mathrm{alg}}(f,\chi,1)$ modulo the Eisenstein ideal gives information about the arithmetic of $X_0(N)$. As an application, in [Maz79], he also derives a congruence formula for the p-adic L-function, connecting special values of this function at $s=2$ with logarithms of circular units.

The starting point in the work of Vatsal is Shimura's algebraicity result, which asserts that, given a modular form f of weight k there exist complex periods $\Omega_f^{\pm}$ such that, for each integer m with $0 \leq m \leq k-2$, and every Dirichlet character χ, the quantity

$$\mathfrak{g}(\bar{\chi}) \cdot m! \frac{L(f,\chi,m+1)}{(-2\pi i)^{m+1}\Omega_f^{\pm}}$$

is algebraic. Here, the sign $\pm$ of $\Omega^{\pm}$ is determined by $\pm 1 = \chi(-1)$.

If we take another eigenform $g = \sum b_n q^n$, where the Fourier coefficients b_n are related to those of f by a congruence $a_n \equiv b_n \pmod{\mathfrak{p}}$, the general arguments from Iwasawa theory suggest that algebraic parts of special values should reflect algebraic properties, so that for critical m, there must be a congruence

$$\mathfrak{g}(\bar{\chi}) \cdot m! \frac{L(f,\chi,m+1)}{(-2\pi i)^{m+1}\Omega_f^{\pm}} \equiv \mathfrak{g}(\bar{\chi}) \cdot m! \frac{L(g,\chi,m+1)}{(-2\pi i)^{m+1}\Omega_g^{\pm}} \pmod{\mathfrak{p}}.$$

This is achieved by a careful determination of the periods $\Omega_*^{\pm}$, since Shimura's theorem only specifies them up to an algebraic constant. In the case of two cuspidal eigenforms, the main result is Theorem 1.10 and the Corollary 1.11. For our purposes, we are particularly interested in the results developed in the Eisenstein scenario, under certain assumptions on the freeness of the appropriate modules. In this case, the main result is Theorem 2.10, that we recover here for the sake of completeness.

Theorem 2.3.5. *Let f be a p-stabilized cuspidal newform of weight 2 and level Np. Assume that there exists a p-stabilized Eisenstein series $E(\psi_1,\psi_2)$ such that $f \equiv E \pmod{\mathfrak{p}^r}$. Under certain technical assumptions (see [Va03, Section 2]), for each non-trivial primitive Dirichlet character χ of conductor prime to Np there exists a period Ω_E which is a p-adic unit such that the following congruence holds:*

$$\mathfrak{g}(\bar{\chi}) \cdot \frac{L(f,\chi,1)}{(-2\pi i)\Omega_f^{\alpha}} \equiv \mathfrak{g}(\bar{\chi}) \cdot \frac{L(E,\chi,1)}{(-2\pi i)\Omega_E} \pmod{\mathfrak{p}^r},$$

where $\alpha = -\psi_1(-1)$.

The proof crucially relies on the aforementioned of Stevens [St82] around the study of periods of modular forms. The author also mentions that the requirement that f is p-stabilized is harmless, and that the same result holds at level N. There is an important remark we want to emphasize. In the proof of the result, it is clear that the period Ω_E does not depend on the Dirichlet character χ! Indeed, the main point here is that the period Ω_f itself is not well-defined: it depends on the choice of an isomorphism between two free rank-one modules over the ring of integers of a finite extension of $\mathbb{Q}_p$ (see [Va03, Sections 1.2, 1.3] for an exhaustive discussion).

Note that $L(E,\chi,1) = L(\psi_1\chi,1)\cdot L(\psi_2\chi,0)$, so the algebraic part of $L(f,\chi,1)$ may be written as the product of two special values of L-functions of Dirichlet characters for a particular choice of the period Ω_f such that the corresponding unit $\Omega_E = 1$.

Other works of Greenberg and Greenberg–Vatsal study similar issues, with applications towards the Iwasawa theory of elliptic curves (see [GV00]). The relation between certain periods and congruences has been also the topic of other influential works, like that of Prasanna [Pr06].

Goldfeld's conjecture and other applications

Another of the applications which motivated our study is related with the so-called Goldfeld's conjecture, which is related with the theory of families of twists by quadratic characters, where one aims to have quantitative estimates for the number of quadratic twists of a given modular form having non-vanishing L-function at $s = 1$. Following the presentation of Kriz–Li [KrLi19], the conjecture can be stated as follows. Let $E/\mathbb{Q}$ be an elliptic curve, and let $r_{\mathrm{an}}(E)$ stand for its analytic rank. This conjecture is concerned about the distribution of $r_{\mathrm{an}}(E)$ when E varies in families. The simplest 1-parameter family is given by the quadratic twists of E by $\mathbb{Q}(\sqrt{d})$, that we denote $E^{(d)}$. Goldfeld's conjecture asserts that $r_{\mathrm{an}}(E^{(d)})$ tends to be as low as possible (compatible with the sign of the equation). Namely, in the quadratic twists family $\{E^{(d)}\}$, r_{an} is 0 (resp. 1) for half of d's. More precisely, we have the following.

Conjecture 2.3.6 (Goldfeld)**.** *Let*

$$N_r(E,X) = \{|d| < X \textit{ with } r_{\mathrm{an}}(E^{(d)}) = r\}.$$

Then, for $r \in \{0,1\}$,

$$N_r(E,X) \sim \frac{1}{2}\sum_{|d|<X} 1, \quad X \to \infty.$$

Here, d runs over all fundamental discriminants.

A weaker conjecture asserts that a positive proportion of its quadratic twists have analytic rank 0 (resp. 1).

Kriz and Li derive theoretical evidence towards the weak Goldfeld conjecture by studying congruences between Heegner points, proving it when the elliptic curve has a rational 3-isogeny. They establish for that purpose a congruence formula between p-adic logarithms of Heegner points, and apply it in the special cases $p = 3$ and $p = 2$ to construct the desired twists explicitly. As a by-product, they prove the corresponding p-part of the BSD conjecture for these explicit twists. The proof of the congruence between the logarithms of Heegner points, expressed as Theorem 1.16 of loc. cit., relies on techniques of p-adic integration, and can be seen as an incarnation of a congruence formula for the Bertolini–Darmon–Prasanna p-adic L-function. This is further extended in Theorem 1.20 to the case of Eisenstein primes.

Kriz [Kr16] also considers the case of a normalized eigenform $f \in S_k(\Gamma_0(N),\chi)$, whose non-constant term Fourier coefficients are congruent to those of an Eisenstein series modulo some prime ideal above a rational prime p. In this situation, he establishes a congruence between the anticyclotomic p-adic L-function of Bertolini–Darmon–Prasanna and the Katz two-variable p-adic

L-function. This allows the author to derive congruences between images under the p-adic Abel Jacobi map of generalized Heegner cycles attached to f and special values of the Katz p-adic L-function. The main results are presented as Theorem 3 and Corollary 4 in the introduction of his paper.

All these evidences suggest that one should expect to prove Eisenstein congruences among different kinds of Euler systems. Recall that this analogy is quite natural: as it has already been pointed out, p-adic L-functions are realizations of Euler systems after applying a suitable Perrin-Riou map! The starting point in this study is that the Galois representation is no longer irreducible modulo p, and hence the corresponding cohomology classes must split as the sum of two classes for smaller groups, and we aim to relate them with appropriate *canonical* Euler systems attached to those groups. As a possible application, if we can prove the non-vanishing of the Euler system for the *smaller group* modulo p, we can get positive results towards the non-vanishing of the Euler system for the *bigger*.

Sharifi's conjectures

One of the key tools in our investigations around congruences between modular forms have been the tools developed around the study of Sharifi's conjectures [Sha11], most notably in the work of Fukaya and Kato [FK12]. Concepts like the Eisenstein quotient, the cohomology of the modular curves, or the p-adic L-function of Mazur and Kitagawa also appear, suggesting a strong connection with the previous notions we have discussed.

Sharifi proposes, for primes $p \geq 5$, a conjecture relating the values of cup products in the Galois cohomology of the maximal unramified outside p-extension of a cyclotomic field on cyclotomic p-units to the values of p-adic L-functions of cuspidal eigenforms that satisfy mod p congruences with Eisenstein series. Passing up to the cyclotomic and Hida towers, the author constructs an isomorphism that allows to compare the value of a reciprocity map on a particular norm compatible system of p-units to the two-variable p-adic L-function of Mazur and Kitagawa. We may think on his conjectures as a link between the geometric theory of GL_2 and the arithemtic theory of GL_1. An incarnation of this philosophy is given by the simple and explicit map

$$[u : v] \mapsto \{1 - \zeta_N^u, 1 - \zeta_N^v\},$$

where $[u : v]$ is a Manin symbol in the relative homology group of the modular curve $X_1(N)$, and $\{1 - \zeta_N^u, 1 - \zeta_N^v\}$ is a Steinberg symbol in the algebraic K-group $K_2(\mathbb{Z}[\zeta_N, 1/N])$.

Let us be more precise, assuming for simplicity that the tame level is 1. For $r \geq 1$, let ζ_{p^r} be a primitive p^r-th root of unity, and let $\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))\{p\}$ be the p-power part of the ideal class group $\mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))$ of the cyclotomic field $\mathbb{Q}(\zeta_{p^r})$. Let

$$X = \varprojlim \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))\{p\},$$

where the inverse limit is taken with respect to the norm maps of ideal class groups. Then, X is a finitely generated $\mathbb{Z}_p$-modules, which decomposes as $X = X^+ \oplus X^-$ according to the action of complex conjugation.

Consider now H_r, the ordinary part of $H^1(X_1(p^r)(\mathbb{C}), \mathbb{Z}_p) = H^1_{\mathrm{et}}(X_1(p^r) \otimes \bar{\mathbb{Q}}, \mathbb{Z}_p)$ with respect to the dual Hecke operator T_p^*. Let $\mathfrak{h}_r$ be the subring of the $\mathbb{Z}_p$-endomorphisms of H_r generated by the dual Hecke operators T_n^*. The Eisenstein ideal $I_r \subset \mathfrak{h}_r$ is the ideal of $\mathfrak{h}_r$ generated by $T_p^* - 1$ and $T_\ell^* - \ell\langle\ell\rangle^{-1} - 1$ for primes $\ell \neq p$. Let

$$H = \varprojlim H_r, \quad \mathfrak{h} = \varprojlim \mathfrak{h}_r, \quad I = \varprojlim I_r \subset \mathfrak{h}.$$

Then, H/IH and $\mathfrak{h}/I$ are finitely generated as $\mathbb{Z}_p$-modules. The module H admits, as an $\mathfrak{h}$-module, a decomposition $H = H^+ \oplus H^-$ induced by complex conjugation. Sharifi introduces maps

$$\varpi : H^-/IH^- \longrightarrow X^-, \quad \Upsilon : X^- \longrightarrow H^-/IH^-$$

relating modular curves and cyclotomic fields.

Conjecture 2.3.7 (Sharifi)**.** *Under certain technical assumptions, the maps ϖ and Υ are isomorphisms, and one is the inverse of the other.*

Fukaya and Kato obtain positive results towards the study of the conjecture, stated as Theorem 0.14 and Theorem 0.15 in the Introduction of his influential paper [FK12]. The crucial ingredient is the use of Beilinson elements (also called Beilinson–Kato elements). This allows them to construct a map $H^- \to K_2$, being K_2 a certain p-adic completion of the inverse system of K_2 of $X_1(p^r)$; in rough terms, it is the inverse limit of the maps which in finite levels sends $[u : v]_r^- \in H_r^-/IH_r^-$ to the Beilinson element $\{g_{0,u/p^r}, g_{0,v/p^r}\}$ in this K_2 (here, $g_{0,u/p^r}$ is a specific instance of Siegel unit). The evaluation of such Siegel units at infinity produces a map to X^-, and the composition of both maps is precisely ϖ. Another crucial ingredient is the study of Galois cohomology, carried out in Section 9 and whose role will be clear in subsequent chapters of the thesis.

Chapter 3

Derived Beilinson–Flach elements and the arithmetic of the adjoint of a modular form

Kings, Lei, Loeffler and Zerbes constructed in [LLZ14], [KLZ17] a three-variable Euler system $\kappa(\mathbf{g},\mathbf{h})$ of Beilinson-Flach elements associated to a pair of Hida families $(\mathbf{g},\mathbf{h})$. They exploited it to obtain applications to the arithmetic of elliptic curves by specializing the Euler system to points of weights $(2,1,1)$, extending the earlier work [BDR15b]. As anticipated before, this Euler system also encodes arithmetic information at points of weights $(1,1,0)$, concerning the group of units of the associated number fields. The setting becomes specially novel and intriguing when $\mathbf{g}$ and $\mathbf{h}$ specialize in weight 1 to p-stabilizations of eigenforms such that one is dual of another. We encounter an exceptional zero phenomenon which forces the specialization of $\kappa(\mathbf{g},\mathbf{h})$ at $(1,1,0)$ to vanish and we are led to study the *derivative* of this class. The main result we obtain is the proof of the main conjecture of [DLR16] on special values of the Hida–Rankin p-adic L-function and the main conjecture of [DR16] for Beilinson-Flach elements in the adjoint setting. Our main point is that the methods of [DLR15a], [DLR16] or [CH20], where the above conjectures are proved when the weight 1 eigenforms have CM, do not apply to our setting and new ideas are required. In loc. cit. a crucial ingredient is a factorization of p-adic L-functions, which in our scenario is not available due to the lack of critical points. Instead we resort to the principle of *improved* Euler systems and p-adic L-functions to reduce our problems to questions which can be resolved using Galois deformation theory.

The results presented at this chapter are the content of the research article [RR20a], which is a joint work with Victor Rotger. We hope that the background provided along the preceding chapter could help the reader to a better comprehension of the main theorems. Further, Chapters 4 and 5 can be read as a companion and sequel to this, providing theoretical evidence and partial proofs to the Elliptic Stark Conjecture of [DLR16].

3.1 Introduction

The purpose of this chapter is proving two results conjectured by H. Darmon, A. Lauder and V. Rotger in [DLR16] and [DR16] respectively, concerning the interplay between the arithmetic of units in number fields, the theory of Coleman iterated integrals, and the Rankin p-adic L-function and Euler system of Beilinson–Flach elements associated to a pair $(\mathbf{g},\mathbf{h})$ of Hida families of modular forms.

The first theorem of this chapter is Theorem A (together with its equivalent form given in Theorem A'), which yields a proof of the main conjecture of [DLR16] in the adjoint setting. Although Beilinson–Flach elements are certainly behind the scenes, we find interesting to remark that this

Euler system is not involved neither in the statement nor the proof we provide of Theorems A, A', and can be phrased in purely analytic terms by means of p-adic iterated integrals and p-adic L-functions. It is only later in the chapter that we explore the consequences that Theorem A has on the weight one specialisations of the Euler system of Beilinson-Flach elements, as described in Theorems B and C.

Let χ be a Dirichlet character of level $N \geq 1$ and let $M_k(N,\chi)$ (resp. $S_k(N,\chi)$) denote the space of (resp. cuspidal) modular forms of weight k, level N and nebentype χ. Let $g = \sum_{n\geq 1} a_n q^n \in S_1(N,\chi)$ be a normalized newform and let $g^* = g \otimes \chi^{-1}$ denote its twist by the inverse of its nebentype. Let

$$\varrho_g : \mathrm{Gal}\,(H_g/\mathbb{Q}) \hookrightarrow \mathrm{GL}(V_g) \simeq \mathrm{GL}_2(L), \quad \varrho_{\mathrm{ad}^0(g)} : \mathrm{Gal}\,(H/\mathbb{Q}) \hookrightarrow \mathrm{GL}(\mathrm{ad}^0(g)) \simeq \mathrm{GL}_3(L)$$

denote the Artin representations associated to g and its adjoint, respectively. Here $H_g \supseteq H$ denote the finite Galois extensions of $\mathbb{Q}$ cut out by these representations, and L is a sufficiently large finite extension of $\mathbb{Q}$ containing their traces. The three-dimensional representation $\mathrm{ad}^0(g)$ may be identified with the subspace of $\mathrm{End}(V_g)$ of null-trace endomorphisms, on which $G_{\mathbb{Q}}$ acts by conjugation. There is a natural decomposition of $L[G_{\mathbb{Q}}]$-modules $\mathrm{ad}(g) := \mathrm{End}(V_g) = L \oplus \mathrm{ad}^0(g)$, where L stands for the trivial representation.

Fix a prime $p \nmid N$ and let $S_k^{\mathrm{oc}}(N,\chi)$ denote the space of overconvergent p-adic modular forms of weight k, tame level N and character χ. Fix an embedding $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$ and let L_p denote the completion of L in $\bar{\mathbb{Q}}_p$.

Label and order the roots of the p-th Hecke polynomial of g as

$$X^2 - a_p(g)X + \chi(p) = (X-\alpha)(X-\beta).$$

We assume throughout that

(H1) The reduction of ϱ_g mod p is irreducible;

(H2) g is p-distinguished, i.e. $\alpha \not\equiv \beta \pmod{p}$, and

(H3) ϱ_g is not induced from a character of a real quadratic field in which p splits.

Let

$$g_\alpha(q) = g(q) - \beta g(q^p)$$

denote the p-stabilization of g on which the Hecke operator U_p acts with eigenvalue α. Enlarge L if necessary so that it contains all Fourier coefficients of g_α.

As shown in [BeDi16] and [DLR15a], the above hypotheses ensure that any generalized overconvergent modular form with the same generalized eigenvalues as g_α is classical, and hence simply a multiple of g_α. This allows to define a canonical projector

$$e_{g_\alpha} : S_1^{\mathrm{oc}}(N,\chi) \longrightarrow L_p$$

which extracts from an overconvergent modular form its coefficient at g_α with respect to an orthonormal basis: cf. [DLR15a] for more details.

One can attach a p-adic invariant $I_p(g) \in L_p$ to g_α as follows. Let

$$Y := Y_1(N) \quad \subset \quad X := X_1(N)$$

denote the models over $\mathbb{Q}$ of the (affine and projective, respectively) modular curves classifying pairs (A,P) where A is a (generalised) elliptic curve and P is a point of order N on A. We keep these notations for subsequent chapters.

For $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, let $\mathfrak{g}_{a;N}$ be Kato's *Siegel unit* whose q-expansion is given by

$$\mathfrak{g}_{a;N} := (1-\zeta_N^a)q^{1/12}\prod_{n=1}^{\infty}(1-\zeta_N^a q^n)(1-\zeta_N^{-a}q^n), \tag{3.1}$$

where ζ_N is a fixed primitive N-th root of unity. The unit $\mathfrak{g}_{a;N}$ can naturally be viewed as belonging to $\mathbb{Q}\otimes\mathcal{O}_Y^\times$, and its q-expansion is defined over $\mathbb{Q}(\mu_N)$.

Set

$$\mathfrak{g} = \frac{1}{2}\otimes\prod_{a=1}^{N}\mathfrak{g}_{a;N} \in \mathbb{Q}\otimes\mathcal{O}_Y^\times, \tag{3.2}$$

and define

$$E_0 = \log_p \mathfrak{g}.$$

This is a locally analytic modular form of weight 0 on Y. It follows from (3.1) and (3.2) that the logarithmic derivative of $\mathfrak{g}$ is

$$dE_0 = E_2\frac{dq}{q} \in \Omega_Y^1, \tag{3.3}$$

where

$$E_2 = \frac{1}{2}\cdot\zeta(-1) + \sum_{n=1}^{\infty}\left(\sum_{d|n} d\right)q^n$$

is the classical Eisenstein series of weight 2.

Let $E_0^{[p]} := \sum_{p\nmid n} a_n(E_0)q^n$ denote the p-depletion of E_0; this is an overconvergent modular form and we may define

$$I_p(g) := e_{g_\alpha}(E_0^{[p]}\cdot g_\alpha). \tag{3.4}$$

The invariant $I_p(g)$ can be recast as a p-adic iterated integral, denoted $I'_p(E_2, g, g^*)$ in [DLR16]. The first main result of this chapter, conjectured in [DLR16], asserts that $I_p(g)$ is equal to the following motivic expression. As shown in [DLR16, Lemma 1.1], we have

$$\dim_L(\mathcal{O}_H^\times\otimes\mathrm{ad}^0(g))^{G_\mathbb{Q}} = 1, \quad \dim_L(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z}\otimes\mathrm{ad}^0(g))^{G_\mathbb{Q}} = 2.$$

Fix a generator u of $(\mathcal{O}_H^\times\otimes\mathrm{ad}^0(g))^{G_\mathbb{Q}}$ and also an element v of $(\mathcal{O}_H^\times[1/p]^\times\otimes\mathrm{ad}^0(g))^{G_\mathbb{Q}}$ in such a way that $\{u,v\}$ is a basis of $(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z}\otimes\mathrm{ad}^0(g))^{G_\mathbb{Q}}$. The element v may be chosen to have p-adic valuation $\mathrm{ord}_p(v)=1$, and we do so.

Viewed as a $G_{\mathbb{Q}_p}$-module, $\mathrm{ad}^0(g)$ decomposes as $\mathrm{ad}^0(g) = L\oplus L^{\alpha\otimes\bar\beta}\oplus L^{\beta\otimes\bar\alpha}$, where each line is characterized by the property that the arithmetic Frobenius Fr_p acts on it with eigenvalue 1, α/β and β/α, respectively[1]. Let H_p denote the completion of H in $\bar{\mathbb{Q}}_p$ and let

$$u_1,\ u_{\alpha\otimes\bar\beta},\ u_{\beta\otimes\bar\alpha},\ v_1,\ v_{\alpha\otimes\bar\beta},\ v_{\beta\otimes\bar\alpha} \in H_p^\times\otimes_\mathbb{Q} L \pmod{L^\times}$$

denote the projection of the elements u and v in $(H_p^\times\otimes\mathrm{ad}^0(g))^{G_{\mathbb{Q}_p}}$ to the above lines: cf. (3.50) for more details. By construction we have $u_1, v_1\in\mathbb{Q}_p^\times$ and

$$\mathrm{Fr}_p(u_{\alpha\otimes\bar\beta}) = \frac{\beta}{\alpha}u_{\alpha\otimes\bar\beta},\ \mathrm{Fr}_p(v_{\alpha\otimes\bar\beta}) = \frac{\beta}{\alpha}v_{\alpha\otimes\bar\beta},\ \mathrm{Fr}_p(u_{\beta\otimes\bar\alpha}) = \frac{\alpha}{\beta}u_{\beta\otimes\bar\alpha},\ \mathrm{Fr}_p(v_{\beta\otimes\bar\alpha}) = \frac{\alpha}{\beta}v_{\beta\otimes\bar\alpha}.$$

Let

$$\log_p : H_p^\times\otimes L \longrightarrow H_p\otimes L$$

[1]The decomposition of $\mathrm{ad}^0(g)$ as the direct sum of three canonical lines is also available when $\alpha/\beta=\beta/\alpha=-1$, see (3.45) and (3.50) for details.

denote the usual p-adic logarithm.

Theorem A. *The following equality holds in* $L_p \subset H_p \otimes L \pmod{L^\times}$*:*

$$I_p(g) = \frac{1}{\log_p(u_{\alpha\otimes\bar\beta})} \times \det \begin{pmatrix} \log_p u_1 & \log_p u_{\alpha\otimes\bar\beta} \\ \log_p v_1 & \log_p v_{\alpha\otimes\bar\beta} \end{pmatrix}. \tag{3.5}$$

This is [DLR16, Conjecture 1.2] specialized to the pair (g, g^*): note that both statements are equivalent because $(\mathcal{O}_H^\times \otimes V_g^\alpha)^{G_{\mathbb{Q}_p}} \simeq \mathrm{Hom}_{G_{\mathbb{Q}_p}}(V_g^\beta, \mathcal{O}_H^\times)$.

Let $\mathcal{L}(\mathrm{ad}^0(g_\alpha))$ be Hida's analytic $\mathcal{L}$-invariant attached to the adjoint Galois representation of g_α; cf. (3.23). As we recall in Proposition 3.2.5, there are several equivalent definitions of this invariant.

These $\mathcal{L}$-invariants are in general difficult to compute, and explicit expressions for them are rather rare. The proof of the above result, which we describe further below in this introduction, shows that Theorem A may be recast as a formula for $\mathcal{L}(\mathrm{ad}^0(g_\alpha))$. Since this might be of independent interest, specially for the reader less acquainted with the Stark elliptic conjectures of [DLR15a], [DLR16] and more familiar with the theory of $\mathcal{L}$-invariants, we quote it below as a separate statement:

Theorem A'. *The analytic $\mathcal{L}$-invariant of* $\mathrm{ad}^0(g_\alpha)$ *is*

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) = \frac{\log_p(u_1)\log_p(v_{\alpha\otimes\bar\beta}) - \log_p(u_{\alpha\otimes\bar\beta})\log_p(v_1)}{\log_p(u_{\alpha\otimes\bar\beta})} \pmod{L^\times}. \tag{3.6}$$

Remark 3.1.1. The prototypical case where an explicit formula for the $\mathcal{L}$-invariant is known arises of course when $E/\mathbb{Q}$ is an elliptic curve of split multiplicative reduction at p (see the previous chapter). In that case $\mathcal{L}_p(E) = \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}$ where $q_E \in p\mathbb{Z}_p$ is a Tate period for E. Observe that this expression and the one in Theorem A' are in fact very similar, as both may be recast as

$$\mathcal{L} = \frac{\log_p(\kappa)}{\mathrm{ord}_p(\kappa)} \tag{3.7}$$

where κ is an element in $H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ arising from some global class in motivic cohomology. In the classical case one has $\kappa = q_E$, which might be regarded as an element in the *extended* Mordell-Weil group $\tilde{E}(\mathbb{Q})$, or Nekovář's extended Selmer group of E. In Theorem A' one has, up to scalar, $\kappa = \log_p(v_{\alpha\otimes\bar\beta})u_1 - \log_p(u_{\alpha\otimes\bar\beta})v_1$ in the unit group of H; note that recipe (3.7) indeed gives rise to (3.6) because $\mathrm{ord}_p(u_1) = 0$ and $\mathrm{ord}_p(v_1) = 1$.

The second main result of this chapter may be regarded as a conceptual explanation of Theorem A, as it establishes a connection between the iterated integral $I_p(g)$, a special value of Hida's p-adic Rankin L-function associated to the pair (g, g^*), and the generalized Kato classes arising from the three-variable Euler system of Beilinson–Flach elements constructed in [KLZ17].

In order to describe it more precisely, let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ denote the Iwasawa algebra and denote $\mathcal{W} = \mathrm{Spf}(\Lambda)$ the associated weight space. As shown by Wiles in [Wi88] (cf. also [KLZ17, Section 7.2] for the normalizations we adopt), associated to g_α there are:

1. a finite flat extension $\Lambda_{\mathbf{g}}$ of Λ, giving rise to a covering $\mathrm{w} : \mathcal{W}_{\mathbf{g}} = \mathrm{Spf}(\Lambda_{\mathbf{g}}) \longrightarrow \mathcal{W}$;

2. a family of overconvergent p-adic ordinary modular forms $\mathbf{g}$ with coefficients in $\Lambda_{\mathbf{g}}$ specializing to g_α at some point $y_0 \in \mathcal{W}_{\mathbf{g}}$ of weight $\mathrm{w}(y_0) = 1$; our running assumptions imply that the above Hida family passing through g_α is unique by [BeDi16].

3. a Galois representation $\varrho_{\mathbf{g}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\Lambda_{\mathbf{g}})$ characterized by the property that all its classical specializations coincide with Deligne's Galois representation associated to the corresponding specialization of the Hida family.

Let $\mathbb{V}_{\mathbf{g}}$ denote the rank two $\Lambda_{\mathbf{g}}$-module realizing the Galois representation $\varrho_{\mathbf{g}}$ as specified e.g. in [KLZ17, Section 7.2], where this is denoted $M(\mathbf{g})^*$. Let $\mathbf{g}^* := \mathbf{g} \otimes \chi^{-1}$ denote the twist of $\mathbf{g}$ by the inverse of its tame nebentype. Note that $\mathbf{g}^*$ specializes at y_0 to the eigenform $g_\alpha \otimes \chi^{-1} = g^*_{1/\beta}$, namely the p-stabilization of g^* on which U_p acts with eigenvalue $1/\beta$.

Let

$$\varepsilon_{\mathrm{cyc}} : G_{\mathbb{Q}} \to \mathrm{Gal}\,(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \to \mathbb{Z}_p^\times$$

denote the p-adic cyclotomic character. Let $\underline{\varepsilon}_{\mathrm{cyc}}$ be the composition of $\varepsilon_{\mathrm{cyc}}$ with the natural inclusion $\mathbb{Z}_p^\times \subset \Lambda^\times$ taking z to the group-like element $[z]$ in $\Lambda^\times$.

Define the three-variable Iwasawa algebra $\Lambda_{\mathbf{gg}^*} := \Lambda_{\mathbf{g}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda_{\mathbf{g}^*} \hat{\otimes}_{\mathbb{Z}_p} \Lambda$ and the $\Lambda_{\mathbf{gg}^*}[G_{\mathbb{Q}}]$-module

$$\mathbb{V}_{\mathbf{gg}^*} := \mathbb{V}_{\mathbf{g}} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{V}_{\mathbf{g}^*} \hat{\otimes}_{\mathbb{Z}_p} \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1}).$$

The article [KLZ17] attaches to $(\mathbf{g}, \mathbf{g}^*)$ a Λ-adic global cohomology class

$$\kappa(\mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gg}^*})$$

parametrized by the triple product of weight spaces $\mathcal{W}_{\mathbf{gg}^*} := \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}$, where the first two variables are afforded by the weight of the Hida families $\mathbf{g}$ and $\mathbf{g}^*$, and the third one is the cyclotomic variable.

The common key strategy in several recent works on the arithmetic of elliptic curves ([BDR15b], [DR17],[LLZ14],[KLZ17],[CH20]) consists in proving a *reciprocity law* relating a suitable specialization of a Λ-adic class as the one above to the critical value of the underlying classical L-function. The non-vanishing of the appropriate (classical or p-adic) L-value can then be invoked to show that the associated global cohomology class is not trivial, and one can derive striking arithmetic consequences from this (cf. e.g. [DR17, Theorems A and B], [KLZ17, Corollary C]).

Our setting differs in several key aspects from the previous ones. Namely, while it is also natural to consider the specialization

$$\kappa(g_\alpha, g^*_{1/\beta}) := \kappa(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 0) \in H^1(\mathbb{Q}, V_g \otimes V_{g^*}(1)) \tag{3.8}$$

of the Euler system of Beilinson–Flach elements at the point $(y_0, y_0, 0)$, one can show that this global cohomology class is trivial and thus no arithmetic information can be extracted directly from it: cf. Theorem 3.3.5.

The fact that $\kappa(g_\alpha, g^*_{1/\beta}) = 0$ might be regarded as an *exceptional zero phenomenon*, albeit quite different from the ones one typically encounters in e.g. the pioneering work [MTT86] of Mazur, Tate and Teitelbaum, because our case corresponds to a point lying *outside* the classical region of interpolation of the associated Hida–Rankin p-adic L-function. But it still keeps the same flavor, because the vanishing of the global cohomology class (3.8) is caused by the cancellation of the Euler-like factor at p arising in the interpolation process of the construction of the Euler system. To be more precise, the situation is more delicate than that: for classical points $y \in \mathcal{W}_{\mathbf{g}}$ of weights $\ell > 1$, the triples $(y, y, \ell - 1)$ do lie in the region of *geometric* interpolation of the Euler system of Beilinson–Flach elements, and the main theorem of [KLZ17] applies and implies that $\kappa(\mathbf{g}, \mathbf{g}^*)(y, y, \ell - 1) = 0$. The vanishing of $\kappa(g_\alpha, g^*_{1/\beta})$ then follows by a density argument exploiting various Λ-adic Perrin-Riou regulators; we refer to Section 3.3 for more details.

We are hence placed to work with the *derived class* $\kappa'(g_\alpha, g^*_{1/\beta}) \in H^1(\mathbb{Q}, \mathrm{ad}^0(g)(1))$, which is introduced in Section 3.3 as the derivative along the weight of $\mathbf{g}^*$ of the Λ-adic class $\kappa(\mathbf{g}, \mathbf{g}^*)$ at $(y_0, y_0, 0)$.

In [DR16] it was laid a conjecture proposing an explicit description of the generalized Kato classes arising from the Euler systems of diagonal cycles of [DR17]. This conjecture, together with the expected (but so far also unproved) behavior of certain periods arising from Hida theory, is shown to imply the main conjecture of [DLR15a] for twists of elliptic curves by Artin representations. This is seen to provide a conceptual interpretation of the numerical examples computed in [DLR15a].

In the case of arbitrary pairs (g,h) where $h \neq g^*$ and there are no exceptional zero phenomena, the constructions and conjectures of [DR16] can be adapted in an easier way. We refer to Chapter 4 for the details.

However, pairs (g,g^*) present a completely different scenario, as we already hinted at above. Not only $\kappa(g_\alpha, g^*_{1/\beta}) = 0$, but it also turns out that the derived class $\kappa'(g_\alpha, g^*_{1/\beta})$ is *not* crystalline at p, as opposed to the set-up of [DR16] and [RR19] (the latter contained as part of Chapter 4). Yet it is still possible to formulate a conjecture analogous to loc. cit., proposing an explicit description of $\kappa'(g_\alpha, g^*_{1/\beta})$ in terms of the $\mathrm{ad}^0(g)$-isotypical component of the group of p-units of H. We can in fact prove it, giving rise to the second main result of this chapter.

Let $H^1_{\mathrm{f},p}(\mathbb{Q}, \mathrm{ad}^0(g)(1))$ denote the subspace of $H^1(\mathbb{Q}, \mathrm{ad}^0(g)(1))$ consisting of classes that are de Rham at p and unramified at all remaining places. Kummer theory (cf. Proposition 3.3.11 below) gives rise to a canonical isomorphism

$$\mathcal{O}_H[1/p]^\times[\mathrm{ad}^0(g)] \otimes L_p \xrightarrow{\simeq} H^1_{\mathrm{f},p}(\mathbb{Q}, \mathrm{ad}^0(g)(1)) \tag{3.9}$$

and we shall use (3.9) throughout to identify these two spaces.

Theorem B. *Assume that $\mathcal{L}(\mathrm{ad}^0(g_\alpha)) \neq 0$. The equality*

$$\kappa'(g_\alpha, g^*_{1/\beta}) = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) \cdot \frac{\log_p(v_{\alpha\otimes\bar\beta})u - \log_p(u_{\alpha\otimes\bar\beta})v}{\log_p(u_{\alpha\otimes\bar\beta})} \tag{3.10}$$

holds in $\frac{\mathcal{O}_H[1/p]^\times}{p^{\mathbb{Z}}}[\mathrm{ad}^0(g)] \otimes L_p \pmod{L^\times}$.

Here it again becomes apparent the notable differences between the phenomena occurring in say [BDR15b], [DR17], [KLZ17] and this study. In loc. cit., the non-vanishing of the central critical L-value is shown to imply the triviality of the Mordell-Weil group; conversely, the vanishing of the central L-value should imply the non-triviality of the Mordell-Weil group.

In contrast, in our setting we have $L(g,g^*,1) \neq 0$ and nonetheless the associated motivic groups $\mathcal{O}_H^\times[\mathrm{ad}^0(g)]$ and $\mathcal{O}_H[1/p]^\times[\mathrm{ad}^0(g)]$ have positive rank. This is ultimately explained by the fact that $L(g,g^*,s) = L(\mathrm{ad}^0(g),s)\cdot\zeta(s)$ factors as the product of two zeta functions having a simple zero and a simple pole at $s=1$, respectively.

Remark 3.1.2. As we show in Section 3.5, it is actually possible to prove that $\mathcal{L}(\mathrm{ad}^0(g_\alpha)) \neq 0$ in many dihedral cases, where ϱ_g is induced from a finite order character of the Galois group of a (real or imaginary) quadratic field. We refer to loc. cit. for more details. On the other hand, when g is exotic, we still expect the $\mathcal{L}$-invariant not to vanish systematically, but proving this properly appears to be a less accessible question in the theory of transcendental p-adic numbers.

The main ingredients in the proof of Theorems A, A' and B are the following:

(I) Let $L_p(\mathbf{g},\mathbf{g}^*)$ be the three-variable Hida–Rankin p-adic L-function as introduced in [Hi85] or [Das16, Section 3.6], and let $L_p(g_\alpha, g^*_{1/\beta}, s) := L_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,s)$ denote the restriction to $\{y_0,y_0\}\times\mathcal{W}$. This function satisfies a functional equation relating the values at s and $1-s$.

As explained in [DLR16], the p-adic iterated integral introduced in (3.4) may be recast as a special value of a p-adic L-function, namely [DLR16, Lemma 4.2] asserts that under our running assumptions H1-H2-H3 that are in place throughout the chapter, we have

$$I_p(g) = L_p(g_\alpha, g^*_{1/\beta}, 0) = L_p(g_\alpha, g^*_{1/\beta}, 1) \pmod{L^\times}.$$

Define $\mathcal{S}_{\mathrm{Hida}} \subset \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}$ as the surface for which the collection of crystalline points (y, z, s) of weights (ℓ, m, m) is Zariski dense. The Euler-like factor at p showing-up in the interpolation formula of Hida–Rankin's p-adic L-function turns out to be a rigid-analytic Iwasawa function when restricted to $\mathcal{S}_{\mathrm{Hida}}$. In §2.3 we recall how this observation allows to conclude thanks to [Das16] that

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) := \alpha'_{\mathbf{g}}(y_0) = I_p(g) \pmod{L^\times}. \tag{3.11}$$

Computing the latter derivative is a question which can be reduced to a problem in Galois deformation theory as in [DLR15b], although the methods in loc. cit. do not apply directly and need to be adapted in order to cover our setting. This is carried out in Section 3.4, and allows us to prove Theorems A and A', as we spell out in detail in Section 3.5.

(II) The main result of Section 3.3 is Theorem 3.3.7, which we state here in slightly rough terms (as the Bloch–Kato logarithm map appearing below has not been specified; cf. loc. cit. for more details):

Theorem C. (Theorem 3.3.7) *The derived Beilinson–Flach element satisfies*

$$\log_p(\kappa'(g_\alpha, g^*_{1/\beta})) = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) \cdot L_p(g_\alpha, g^*_{1/\beta}, 0) \pmod{L^\times}. \tag{3.12}$$

We may see this as an *exceptional zero formula*, reminiscent of the main theorem of [GS94], which asserts that for an elliptic curve $E/\mathbb{Q}$ with split multiplicative reduction at p, we have

$$L'_p(E, 1) = \mathcal{L}_p(E) \cdot L_{\mathrm{alg}}(E, 1).$$

We thus may invoke Theorem A, which combined with (3.12) implies that

$$\log_p(\kappa'(g_\alpha, g^*_{1/\beta})) = \Big(\frac{\log_p(u_1)\log_p(v_{\alpha\otimes\bar\beta}) - \log_p(u_{\alpha\otimes\bar\beta})\log_p(v_1)}{\log_p(u_{\alpha\otimes\bar\beta})} \Big)^2 \pmod{L^\times}.$$

In light of the general properties satisfied by Beilinson–Flach elements established in [KLZ17], this allows us to prove in Section 3.5 that the following equality holds in $\mathcal{O}_H[1/p]^\times[\mathrm{ad}^0(g)] \otimes L_p$:

$$\kappa'(g_\alpha, g^*_{1/\beta}) = \frac{\log_p(u_1)\log_p(v_{\alpha\otimes\bar\beta}) - \log_p(u_{\alpha\otimes\bar\beta})\log_p(v_1)}{\log_p(u_{\alpha\otimes\bar\beta})^2} \times \Big(\log_p(v_{\alpha\otimes\bar\beta})u - \log_p(u_{\alpha\otimes\bar\beta})v \Big)$$

up to a factor in $L^\times$, as claimed.

We close the chapter in Section 3.6 by pointing out to the connection between our derived Beilinson–Flach classes, Darmon–Dasgupta units and the Artin p-adic L-functions associated to the motives in play. We prove a factorization theorem in the CM case, and explain how our main results shed some light in the much more intriguing RM setting.

During the realization of this thesis, different works around the subjects discussed in this chapter have appeared, showing an increasing interest in the study of exceptional zeros. As it is discussed in 5, Theorem A may be understood as a p-adic version of the intriguing Harris–Venkatesh conjecture [HV19]. Our study of exceptional zeros is also related to the recent work of Benois and Horte [BH20], which considers precisely the case of two modular forms (g, h) which are not self-dual. It would be interesting to understand the relationship between both approaches and look for further applications.

Finally, and as a concluding remark, we hope the approach introduced in this chapter may be adapted to prove other instances of variants of the Elliptic Stark Conjecture in presence of exceptional zeros. We refer the reader to Chapter 7 for an exploration of a similar situation in the setting of diagonal cycles and triple product p-adic L-functions.

3.2 Preliminary concepts

Hida families

Fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. For a number field K, let $G_K := \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ denote its absolute Galois group. Fix a prime p and an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, and let ord_p denote the resulting p-adic valuation on $\bar{\mathbb{Q}}^\times$, normalized in such a way that $\mathrm{ord}_p(p) = 1$.

Let g be a newform of weight $k \geq 1$, level N and character χ, with Fourier coefficients in a finite extension L of $\mathbb{Q}$. Label the roots of the p-th Hecke polynomial of g as α_g, β_g with $\mathrm{ord}_p(\alpha_g) \leq \mathrm{ord}_p(\beta_g)$. By enlarging L if necessary, we shall assume throughout that L contains α_g, β_g, the N-th roots of unity and the pseudo-eigenvalue $\lambda_N(g)$ with respect to the Atkin-Lehner operator W_N (cf. [AL78]).

Let L_p denote the completion of L in $\bar{\mathbb{Q}}_p$ and let V_g denote the two-dimensional representation of $G_{\mathbb{Q}}$ with coefficients in L_p associated to g as defined e.g. in [KLZ17, Section 2.8], where it is denoted $M_{L_p}(g)^*$. If g is ordinary at p, there is an exact sequence of $G_{\mathbb{Q}_p}$-modules

$$0 \to V_g^+ \to V_g \to V_g^- \to 0, \qquad V_g^+ \simeq L_p(\varepsilon_{\mathrm{cyc}}^{k-1}\chi\psi_g^{-1}), \quad V_g^- \simeq L_p(\psi_g), \tag{3.13}$$

where ψ_g is the unramified Galois character of $G_{\mathbb{Q}_p}$ sending Fr_p to α_g.

The formal spectrum $\mathcal{W} = \mathrm{Spf}(\Lambda)$ of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is called the weight space attached to Λ, and their A-valued points over a p-adic ring A are given by

$$\mathcal{W}(A) = \mathrm{Hom}_{\mathrm{alg}}(\Lambda, A) = \mathrm{Hom}_{\mathrm{grp}}(\mathbb{Z}_p^\times, A^\times).$$

Weight space is equipped with a distinguished class of *arithmetic points* $\nu_{s,\varepsilon}$ indexed by integers $s \in \mathbb{Z}$ and Dirichlet characters $\varepsilon : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ of p-power conductor. The point $\nu_{s,\varepsilon} \in \mathcal{W}$ is defined by the rule

$$\nu_{s,\varepsilon}(n) = \varepsilon(n)n^s.$$

We let $\mathcal{W}^{\mathrm{cl}}$ denote the subset of $\mathcal{W}$ formed by such arithmetic points. When $\varepsilon = 1$ is the trivial character, we denote the point $\nu_{s,1}$ simply as ν_s or even s by a slight abuse of notation.

If $\tilde{\Lambda}$ is a finite flat algebra over Λ, there is a natural finite map $\tilde{\mathcal{W}} := \mathrm{Spf}(\tilde{\mathcal{W}}) \xrightarrow{\mathrm{w}} \mathcal{W}$, and we say that a point $x \in \tilde{\mathcal{W}}$ is arithmetic of weight s and character ε if $\mathrm{w}(x) = \nu_{s,\varepsilon}$. As in the introduction, let $\varepsilon_{\mathrm{cyc}}$ denote the p-adic cyclotomic character and $\underline{\varepsilon}_{\mathrm{cyc}}$ be the composition of $\varepsilon_{\mathrm{cyc}}$ with the natural inclusion $\mathbb{Z}_p^\times \subset \Lambda^\times$ taking z to the group-like element $[z]$ in $\Lambda^\times$.

Let $N \geq 1$ be an integer not divisible by p, and let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}_p^\times$ be a Dirichlet character. Let $\mathbf{g}$ be a Hida family of tame level N and tame character χ as defined and normalized in [KLZ17, Section 7.2]. Let $\Lambda_{\mathbf{g}}$ the associated Iwasawa algebra (cf. Def. 7.2.5 of loc. cit.), which is finite and flat over Λ. As in [KLZ17, Section 7.3], we may specialize $\mathbf{g}$ at any arithmetic point $x \in \mathcal{W}_{\mathbf{g}} = \mathrm{Spf}(\Lambda_{\mathbf{g}})$ of weight $k \geq 2$ and character $\varepsilon : (\mathbb{Z}/p^r\mathbb{Z})^\times \longrightarrow L_p^\times$ and obtain a classical p-ordinary eigenform g_x in the space $S_k(Np^r, \chi\varepsilon)$ of cusp forms of weight k, level Np^r and nebentype $\chi\varepsilon$.

Definition 3.2.1. Let $x \in \mathcal{W}_{\mathbf{g}}$ be an arithmetic point of weight $k \geq 1$ and character ε. We say x is crystalline if $\varepsilon = 1$ and there exists an eigenform g_x° of level N such that g_x is the ordinary p-stabilization of g_x° (which given the previous condition, is automatic if $k > 2$, but not necessarily if $k \leq 2$). We denote by $\mathcal{W}_{\mathbf{g}}^\circ$ the set of crystalline arithmetic points of $\mathcal{W}_{\mathbf{g}}$.

As shown by Wiles [Wi88], [KLZ17, Section 7.2] and already recalled in the introduction, a Hida family $\mathbf{g}$ as above comes equipped with a free $\Lambda_{\mathbf{g}}[G_{\mathbb{Q}}]$-module $\mathbb{V}_{\mathbf{g}}$ of rank two, yielding a Galois representation $\varrho_{\mathbf{g}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\Lambda_{\mathbf{g}})$, all whose classical specializations recover the p-adic Galois representation associated to the classical specialization of $\mathbf{g}$. We assume throughout that the mod p residual representation $\bar{\varrho}_{\mathbf{g}}$ associated to any of its classical specializations is irreducible, and that the semi-simplification of $\bar{\varrho}_{|G_{\mathbb{Q}_p}}$ is non-scalar, as in hypotheses (H1-H2) in the introduction.

Similarly as in (3.13), the restriction to $G_{\mathbb{Q}_p}$ of $\mathbb{V}_{\mathbf{g}}$ admits a filtration

$$0 \to \mathbb{V}_{\mathbf{g}}^{+} \to \mathbb{V}_{\mathbf{g}} \to \mathbb{V}_{\mathbf{g}}^{-} \to 0, \tag{3.14}$$

where $\mathbb{V}_{\mathbf{g}}^{+}$ and $\mathbb{V}_{\mathbf{g}}^{-}$ are flat $\Lambda_{\mathbf{g}}[G_{\mathbb{Q}_p}]$-modules, free of rank one over $\Lambda_{\mathbf{g}}$. If we let $\psi_{\mathbf{g}}$ denote the unramified character of $G_{\mathbb{Q}_p}$ taking the arithmetic Frobenius element Fr_p to $a_p(\mathbf{g})$, then

$$\mathbb{V}_{\mathbf{g}}^{+} \simeq \Lambda_{\mathbf{g}}(\psi_{\mathbf{g}}^{-1}\chi\varepsilon_{\mathrm{cyc}}^{-1}\underline{\varepsilon}_{\mathrm{cyc}}), \qquad \mathbb{V}_{\mathbf{g}}^{-} \simeq \Lambda_{\mathbf{g}}(\psi_{\mathbf{g}}). \tag{3.15}$$

Let $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}$ be the ring of integers of the completion of the maximal unramified extension of $\mathbb{Q}_p$. Given a finite-dimensional $G_{\mathbb{Q}_p}$-module V with coefficients over a finite extension L_p of $\mathbb{Q}_p$, the de Rham Dieudonné module associated to V is defined as

$$D(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_{\mathbb{Q}_p}},$$

where B_{dR} is Fontaine's field of de Rham periods. If V is unramified there is a further canonical isomorphism

$$D(V) \simeq \left(V_{\mathrm{int}}\hat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}_p^{\mathrm{ur}}}\right)^{\mathrm{Fr}_p=1}[\frac{1}{p}],$$

where V_{int} is an integral lattice in V. Similarly, if $\mathbb{V}$ is a free module over an Iwasawa algebra equipped with an unramified action of $G_{\mathbb{Q}_p}$, we may define

$$\mathbb{D}(\mathbb{V}) := \left(\mathbb{V}\hat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}_p^{\mathrm{ur}}}\right)^{\mathrm{Fr}_p=1}. \tag{3.16}$$

Given a newform $g \in S_k(N, \chi)$ as at the beginning of this section, $D(V_g)$ is an L_p-filtered vector space of rank 2. Set $g^* = g \otimes \bar{\chi} \in S_k(N, \bar{\chi})$. Poincaré duality induces a perfect pairing

$$\langle\, , \,\rangle : D(V_g(-1)) \times D(V_{g^*}) \to L_p. \tag{3.17}$$

If g is ordinary at p, then (3.13) gives rise to an exact sequence of Dieudonné modules

$$0 \to D(V_g^{+}) \xrightarrow{i} D(V_g) \xrightarrow{\pi} D(V_g^{-}) \to 0$$

where $D(V_g^{+})$ and $D(V_g^{-})$ have rank 1 over L_p.

If $k \geq 2$, Faltings' comparison theorem allows to associate to g a regular differential form $\omega_g \in \mathrm{Fil}(D(V_g))$, which induces a linear form

$$\omega_g : D(V_{g^*}^{+}) \to L_p, \quad \eta \mapsto \langle \omega_g, \eta \rangle.$$

There is also the differential form η_g, characterized by the properties that it spans the line $D(V_g^{+})$ and $\langle \eta_g, \omega_{g^*} \rangle = 1$. It again gives rise to a linear functional

$$\eta_g : D(V_{g^*}^{-}) \to L_p, \quad \omega \mapsto \langle \pi^{-1}(\omega), \eta_g \rangle.$$

As shown in [Oh00] and [KLZ17], the differential forms (or linear functionals) ω_g and η_g vary in families. In order to recall this more precisely, let $\mathbf{g}$ be a Hida family of tame level N and tame character χ as above. Set $\mathbf{g}^* = \mathbf{g} \otimes \bar{\chi}$. Let $\mathcal{Q}_{\mathbf{g}}$ denote the fraction field of $\Lambda_{\mathbf{g}}$, and set $\mathbb{U}_{\mathbf{g}}^{+} := \mathbb{V}_{\mathbf{g}}^{+}(\chi^{-1}\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})$; denote by $U_{g_y}^{+}$ its specialization at a point $y \in \mathcal{W}_{\mathbf{g}}^{\circ}$. By [KLZ17, Proposition 10.1.1], there exist

- A homomorphism of $\Lambda_{\mathbf{g}}$-modules

$$\langle\, , \omega_{\mathbf{g}} \rangle : \mathbb{D}(\mathbb{U}_{\mathbf{g}^*}^{+}) \to \Lambda_{\mathbf{g}} \tag{3.18}$$

such that for every $y \in \mathcal{W}_{\mathbf{g}}^{\circ}$, the specialization of $\omega_{\mathbf{g}}$ at y is the linear form

$$y \circ \langle\, , \omega_{\mathbf{g}} \rangle = \langle\, , \mathrm{Pr}^{\alpha *}(\omega_{g_y^\circ}) \rangle : D(U^+_{g_y^*}) \to L_p$$

where $\mathrm{Pr}^{\alpha *}$ is the p-stabilization pull-back isomorphism defined in loc. cit.. Note that this makes sense: since L_p is assumed to contain the N-th roots of unity, $D(U^+_{g_y^*})$ and $D(V^+_{g_y^*})$ are the same module up to a shift in their filtration.

- A homomorphism of $\Lambda_{\mathbf{g}}$-modules

$$\langle\, , \eta_{\mathbf{g}} \rangle : \mathbb{D}(\mathbb{V}^-_{\mathbf{g}*}) \to \mathcal{Q}_{\mathbf{g}} \tag{3.19}$$

such that for every $y \in \mathcal{W}_{\mathbf{g}}^{\circ}$ we have

$$y \circ \langle\, , \eta_{\mathbf{g}} \rangle = \frac{\mathrm{Pr}^{\alpha *}(\eta_{g_y^\circ})}{\lambda_N(g_y^\circ)\mathcal{E}_0(g_y^\circ)\mathcal{E}_1(g_y^\circ)} : D(V^-_{g_y^*}) \to L_p,$$

where $\lambda_N(g_y^\circ)$ stands for the pseudo-eigenvalue of g_y°, and

$$\mathcal{E}_0(g_y^\circ) = 1 - \chi^{-1}(p)\beta^2_{g_y^\circ} p^{1-k}, \quad \mathcal{E}_1(g_y^\circ) = 1 - \chi(p)\alpha^{-2}_{g_y^\circ} p^{k-2}.$$

Hida–Rankin's three-variable p-adic L-function

Let $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$, $\mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$ be a pair of Hida families of tame level N and tame characters χ_g and χ_h respectively. As in the introduction, set $\Lambda_{\mathbf{gh}} := \Lambda_{\mathbf{g}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda_{\mathbf{h}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda$ and $\mathcal{W}_{\mathbf{gh}} := \mathrm{Spf}(\Lambda_{\mathbf{gh}})$. Let $\mathcal{Q}_{\mathbf{gh}}$ denote the fraction field of $\Lambda_{\mathbf{gh}}$, and set $\mathcal{W}_{\mathbf{gh}}^{\circ} := \mathcal{W}_{\mathbf{g}}^{\circ} \times \mathcal{W}_{\mathbf{h}}^{\circ} \times \mathcal{W}^{\mathrm{cl}}$.

Definition 3.2.2. The *critical* range is the set of points $(y, z, \sigma) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$ of weights (ℓ, m, s) such that $\ell, m \geq 2$ and $m \leq s < \ell$.

The *geometric* range is defined to be the set of points $(y, z, \sigma) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$ of weights (ℓ, m, s) such that $\ell, m \geq 2$ and $1 \leq s < \min(\ell, m)$.

Hida constructed [Hi85], [Hi88] a three-variable p-adic Rankin L-function $L_p(\mathbf{g}, \mathbf{h})$ on $\mathcal{W}_{\mathbf{gh}}$, interpolating the algebraic parts of the critical values $L(g_y^\circ, h_z^\circ, s)$ for every triple of classical points (y, z, s) in $\mathcal{W}_{\mathbf{gh}}^{\circ}$ lying in the *critical* range. More precisely, [Hi88, Theorem 5.1d] asserts the following.

Theorem 3.2.3. *(Hida) There exists a unique element $L_p(\mathbf{g}, \mathbf{h}) \in \mathcal{Q}_{\mathbf{gh}}$ whose value at any $(y, z, s) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$ of weights (ℓ, m, s) in the* critical *range is well-defined and equal to*

$$L_p(\mathbf{g}, \mathbf{h})(y, z, s) = \frac{C \cdot \mathcal{E}(y, z, s)}{(2\pi i)^{2s-m+1}\langle g_y^\circ, g_y^\circ \rangle} \times L(g_y^\circ, h_z^\circ, s)$$

where C is a non-zero algebraic number in the finite extension $\mathbb{Q}(g_y^\circ, h_z^\circ)$ generated by the Fourier coefficients of g_y° and h_z°, $\langle g_y^\circ, g_y^\circ \rangle$ is the Petersson norm as normalized in loc. cit., and

$$\mathcal{E}(y, z, s) = \Big(1 - \frac{p^{s-1}}{\alpha_{g_y^\circ}\alpha_{h_z^\circ}}\Big)\Big(1 - \frac{p^{s-1}}{\alpha_{g_y^\circ}\beta_{h_z^\circ}}\Big)\Big(1 - \frac{\beta_{g_y^\circ}\alpha_{h_z^\circ}}{p^s}\Big)\Big(1 - \frac{\beta_{g_y^\circ}\beta_{h_z^\circ}}{p^s}\Big). \tag{3.20}$$

Let g and h be classical specializations of the families $\mathbf{g}$ and $\mathbf{h}$ at some point $(y_0, z_0) \in \mathcal{W}_{\mathbf{g}}^{\mathrm{cl}} \times \mathcal{W}_{\mathbf{h}}^{\mathrm{cl}}$ of weight (ℓ, m). We denote by $L_p(g, h, s)$ the restriction of $L_p(\mathbf{g}, \mathbf{h})(y, z, s)$ to the line (y_0, z_0, s). As quoted e.g. in [Das16, §9.2], this p-adic L-function satisfies the functional equation

$$L_p(g, h, \ell + m - 1 - s) = \epsilon(g, h, s) L_p(g^*, h^*, s) \tag{3.21}$$

where $\epsilon(g, h, \psi, s) = A \cdot B^s$, with $A \in \mathbb{Q}(g_y, h_z)^\times$ and $B \in \mathbb{Q}^\times$.

Improved p-adic L-functions

As before, let $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$ be a Hida family of tame level N and tame character $\chi = \chi_g$, and set $\mathbf{h} = \mathbf{g}^* := \mathbf{g} \otimes \chi^{-1}$. As in the introduction, define the surface

$$\mathcal{S}_{\mathrm{Hida}} := \mathcal{S}_{\ell,m,m} := \{(y, z, \sigma) \in \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W} : \mathrm{w}(z) = \sigma\}. \tag{3.22}$$

Note that this is a sub-variety of $\mathcal{W}_{\mathbf{gg}^*}$ all whose crystalline arithmetic points have weights (ℓ, m, m) for some $\ell, m \geq 1$.

The restriction to $\mathcal{S}_{\mathrm{Hida}}$ of the *second multiplier* in the Euler-like factor (3.20) appearing in the interpolation formula for Hida–Rankin's p-adic L-function is

$$1 - \frac{p^{s-1}}{\alpha_{g_y^\circ}\beta_{h_z^\circ}} = 1 - \frac{\alpha_{g_z^\circ}}{\alpha_{g_y^\circ}}.$$

This expression interpolates to an Iwasawa function in $\Lambda_{\mathbf{g}} \times \Lambda_{\mathbf{g}}$, which by abuse of notation we continue to denote with the same symbol. One naturally expects $1 - \frac{\alpha_{g_z^\circ}}{\alpha_{g_y^\circ}}$ should divide the two-variable p-adic L-function $L_p(\mathbf{g}, \mathbf{g}^*)(y, z, s)$, where (y, z) vary in $\mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{g}}$ and $s = \mathrm{w}(z)$ is determined by the weight of z. Hida proved in [Hi88] the following stronger statement:

Theorem 3.2.4 (Hida)**.** *Let $\hat{L}_p(\mathbf{g}, \mathbf{g}^*)$ be the unique element in the fraction field of $\Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{g}}$ such that*

$$L_p(\mathbf{g}, \mathbf{g}^*)(y, z, \mathrm{w}(z)) = \left(1 - \frac{\alpha_{g_z^\circ}}{\alpha_{g_y^\circ}}\right) \cdot \hat{L}_p(\mathbf{g}, \mathbf{g}^*)(y, z).$$

Then, fixing $y_0 \in \mathcal{W}_{\mathbf{g}}^\circ$, the one-variable meromorphic function

$$\hat{L}_p(\mathbf{g}, \mathbf{g}^*)(y_0, z)$$

on $\mathcal{W}_{\mathbf{g}}$ has a simple pole at $z = y_0$ whose residue is a non-zero explicit rational number.

Associated to the adjoint representation attached to any classical specialization $g = g_{y_0}$ of the Hida family $\mathbf{g}$ at some arithmetic point $y_0 \in \mathcal{W}_{\mathbf{g}}^{\mathrm{cl}}$, Hida defined an analytic $\mathcal{L}$-invariant, which can be recast in several equivalent ways (cf. the works of Hida, Harron, Citro and Dasgupta (cf. [Hi04], [Ci08], [Das16]). We may define it for instance as:

$$\mathcal{L}(\mathrm{ad}^0(g_{y_0})) := \frac{-\alpha'_{\mathbf{g}}(y_0)}{\alpha_{\mathbf{g}}(y_0)}, \tag{3.23}$$

where recall $\alpha_{\mathbf{g}} = a_p(\mathbf{g}) \in \Lambda_{\mathbf{g}}$ is the Iwasawa function given by the eigenvalue of the Hecke operator U_p acting on $\mathbf{g}$, and $\alpha'_{\mathbf{g}}$ is its derivative.

Let $L'_p(\mathrm{ad}^0(g_{y_0}), s)$ denote Hida-Schmidt's p-adic L-function associated to the adjoint of the ordinary eigenform g_{y_0} (cf. [Sc88], [Hi04]). The argument below is mainly due to Citro and Dasgupta, but since in loc. cit. they often assume that $\ell \geq 2$, we include it in order to ensure that it holds as well at weight 1, which is the case we mostly focus on. The main point is that the objects in play all vary in Hida families.

Proposition 3.2.5. *For a crystalline classical point $y_0 \in \mathcal{W}_{\mathbf{g}}^\circ$ of weight $\ell \geq 1$, we have*

$$\mathcal{L}(\mathrm{ad}^0(g_{y_0})) = L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, \ell) = L'_p(\mathrm{ad}^0(g_{y_0}), \ell),$$

up to a non-zero rational constant.

Proof. The first equality follows from Theorem 3.2.4, which amounts to say that

$$L_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,\ell)=\lim_{z\to y_0}\left(1-\frac{\alpha_{g_z^\circ}}{\alpha_{g_{y_0}^\circ}}\right)\cdot \hat{L}_p(\mathbf{g},\mathbf{g}^*)(y_0,z).$$

Since $\left(1-\frac{\alpha^\circ_{g_z}}{\alpha_{g^\circ_{y_0}}}\right)$ vanishes at $z=y_0$ and $\hat{L}_p(\mathbf{g},\mathbf{g}^*)(y_0,z)$ has a pole at $z=y_0$ given by a non-zero rational number, the value of the previous limit agrees, modulo $L^\times$, with the derivative of the first factor, i.e.,

$$L_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,\ell)=\frac{-\alpha'_{\mathbf{g}}(y_0)}{\alpha_{\mathbf{g}}(y_0)}.$$

In addition, Dasgupta's factorization proved in [Das16] asserts that $L_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,s)=\zeta_p(s-\ell+1)L_p(\mathrm{ad}^0(g_{y_0}),s)$. Here $\zeta_p(s)$ is the p-adic zeta function, which has a pole at $s=1$ with non-zero rational residue. The second factor vanishes at $s=\ell$ and it follows that $L_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,\ell)=L'_p(\mathrm{ad}^0(g_{y_0}),\ell)\ (\mathrm{mod}\,\mathbb{Q}^\times)$. □

3.3 Derived Beilinson–Flach elements

The three-variable Euler system of Kings, Lei, Loeffler and Zerbes

Let $\mathbf{g}$ and $\mathbf{h}$ be a pair of p-adic cuspidal Hida families of tame conductor N and tame nebentype χ_g and χ_h as in Section 3.2. As in the introduction, and keeping the notations of the previous section, define the $\Lambda_{\mathbf{gh}}$-module

$$\mathbb{V}_{\mathbf{gh}}:=\mathbb{V}_{\mathbf{g}}\hat{\otimes}_{\mathbb{Z}_p}\mathbb{V}_{\mathbf{h}}\hat{\otimes}_{\mathbb{Z}_p}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}). \tag{3.24}$$

This Λ-adic Galois representation is characterized by the property that for any $(y,z,\sigma)\in\mathcal{W}^\circ_{\mathbf{gh}}$ with $\mathrm{w}(\sigma)=\nu_s$ with $s\in\mathbb{Z}$, (3.24) specializes to

$$\mathbb{V}_{\mathbf{gh}}(y,z,\sigma)=V_{g_y}\otimes V_{h_z}(1-s),$$

the $(1-s)$-th Tate twist of the tensor product of the Galois representations attached to g_y and h_z.

Fix $c\in\mathbb{Z}_{>1}$ such that $(c,6pN_gN_h)=1$. [KLZ17, Theorem A] yields a three-variable Λ-adic global Galois cohomology class

$$\kappa^c(\mathbf{g},\mathbf{h})\in H^1(\mathbb{Q},\mathbb{V}_{\mathbf{gh}})$$

that is referred to as the Euler system of Beilinson–Flach elements associated to $\mathbf{g}$ and $\mathbf{h}$. We denote by $\kappa_p^c(\mathbf{g},\mathbf{h})\in H^1(\mathbb{Q}_p,\mathbb{V}_{\mathbf{gh}})$ the image of $\kappa^c(\mathbf{g},\mathbf{h})$ under the restriction map.

Since c is fixed throughout, we may sometimes drop it from the notation. This constant does make an appearance in fudge factors accounting for the interpolation properties satisfied by the Euler system, but in all cases we are interested in these fudge factors do not vanish and hence do not pose any problem for our purposes.

Given a crystalline arithmetic point $(y,z,s)\in\mathcal{W}^\circ_{\mathbf{gh}}$ of weights (ℓ,m,s), set for notational simplicity throughout this section $g=g_y^\circ$, $h=h_z^\circ$. With these notations, g_y (resp. h_z) is the p-stabilization of g (resp. h) with U_p-eigenvalue α_g (resp. α_h).

Define

$$\kappa(g_y,h_z,s):=\kappa(\mathbf{g},\mathbf{h})(y,z,s)\in H^1(\mathbb{Q},V_{g_y}\otimes V_{h_z}(1-s)) \tag{3.25}$$

as the specialisation of $\kappa(\mathbf{g},\mathbf{h})$ at (y,z,s).

If one further assumes that (y,z,s) lies in the geometric range, Kings, Loeffler and Zerbes showed in [KLZ20] that the cohomology group appearing in (3.25) also hosts a canonical *Rankin-Eisenstein class*, denoted

$$\mathrm{Eis}_{\mathrm{et}}^{[g,h,s]}\in H^1(\mathbb{Q},V_g\otimes V_h(1-s)). \tag{3.26}$$

This class is attached to the classical pair (g,h) and can be constructed purely by geometric methods, without appealing to the variation of (g_y,h_z) in p-adic families. It is for this reason that in fact the classes $\mathrm{Eis}_{\mathrm{et}}^{[g,h,s]}$ lie in the Bloch–Kato Selmer subgroup

$$H^1_{\mathrm{f}}(\mathbb{Q},V_g\otimes V_h(1-s))\subset H^1(\mathbb{Q},V_g\otimes V_h(1-s)).$$

We refer to [KLZ20, Section 5] and [KLZ17, Definition 3.3.2] for the precise statements.

Since both $\kappa(g_y,h_z,s)$ and $\mathrm{Eis}_{\mathrm{et}}^{[g,h,s]}$ live in the same space, it makes sense to ask whether they are related. This is the content of [KLZ17, Theorem A (8.1.3)]:

Theorem 3.3.1. *Assume $(y,z,s)\in\mathcal{W}^{\circ}_{\mathbf{gh}}$ lies in the geometric range. Then*

$$\kappa(g_y,h_z,s)=\mathcal{E}(g,h,s)\cdot\mathrm{Eis}_{\mathrm{et}}^{[g,h,s]}\tag{3.27}$$

where

$$\mathcal{E}(g,h,s)=\frac{\left(1-\frac{p^{s-1}}{\alpha_g\alpha_h}\right)\left(1-\frac{\alpha_g\beta_h}{p^s}\right)\left(1-\frac{\beta_g\alpha_h}{p^s}\right)\left(1-\frac{\beta_g\beta_h}{p^s}\right)(c^2-c^{2s-\ell-m+2})}{(-1)^{s-1}(s-1)!\binom{\ell-2}{s-1}\binom{m-2}{s-1}}.\tag{3.28}$$

In particular $\kappa(g_y,h_z,s)$ lies in $H^1_{\mathrm{f}}(\mathbb{Q},V_g\otimes V_h(1-s))$.

The following proposition recalls the existence of the so-called Perrin-Riou big logarithm, interpolating the Bloch–Kato logarithm $\log_{\mathrm{BK}}$ and dual exponential map $\exp^*_{\mathrm{BK}}$ associated to the classical specializations of a Λ-adic representation of $G_{\mathbb{Q}_p}$. We refer to [BK93] and [Bel09] for an introduction to p-adic Hodge theory and the definitions of these maps.

Recall the unramified character $\psi_{\mathbf{g}}$ of $G_{\mathbb{Q}_p}$ taking a Frobenius element Fr_p to $a_p(\mathbf{g})$, and as before, let $\varepsilon_{\mathbf{g}}$ be the composition of the cyclotomic character $\underline{\varepsilon}_{\mathrm{cyc}}$ with the natural inclusion $\Lambda^\times\subset\Lambda_{\mathbf{g}}^\times$. Define the $G_{\mathbb{Q}_p}$-subquotient

$$\mathbb{V}^{-+}_{\mathbf{gh}}:=\mathbb{V}^-_{\mathbf{g}}\hat{\otimes}\mathbb{V}^+_{\mathbf{h}}$$

of $\mathbb{V}_{\mathbf{g}}\hat{\otimes}\mathbb{V}_{\mathbf{h}}$ of rank one over the two-variable Iwasawa algebra $\Lambda_{\mathbf{g}}\hat{\otimes}\Lambda_{\mathbf{h}}$. In light of (3.15), the Galois action on $\mathbb{V}^{-+}_{\mathbf{gh}}$ is given by the character

$$\eta^{\mathbf{g}}_{\mathbf{h}}:=\varepsilon_{\mathrm{cyc}}^{-1}\chi_h\cdot\psi_{\mathbf{g}}\otimes\psi_{\mathbf{h}}^{-1}\varepsilon_{\mathbf{h}}.\tag{3.29}$$

It follows that $\mathbb{U}^{-+}_{\mathbf{gh}}:=\mathbb{V}^{-+}_{\mathbf{gh}}(\varepsilon_{\mathrm{cyc}}\varepsilon_{\mathbf{h}}^{-1})$ is an unramified $G_{\mathbb{Q}_p}$-module and we can thus invoke its Λ-adic Dieudonné module as defined in (3.16).

Proposition 3.3.2. *[KLZ17, Theorem 8.2.8] There is an injective morphism of $\Lambda_{\mathbf{gh}}$-modules*

$$\mathcal{L}^{-+}_{\mathbf{gh}}:H^1(\mathbb{Q}_p,\mathbb{V}^{-+}_{\mathbf{gh}}\hat{\otimes}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}))\to\mathbb{D}(\mathbb{U}^{-+}_{\mathbf{gh}})\hat{\otimes}\Lambda$$

such that for all $\kappa_p\in H^1(\mathbb{Q}_p,\mathbb{V}^{-+}_{\mathbf{gh}}\hat{\otimes}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$ and all $(y,z,s)\in\mathcal{W}^{\circ}_{\mathbf{gh}}$ of weights (ℓ,m,s):

- *if $s<m$, $(\mathcal{L}^{-+}_{\mathbf{gh}}(\kappa_p))_{y,z,s}=\left(1-\frac{p^{s-1}}{\alpha_g\beta_h}\right)\left(1-\frac{\alpha_g\beta_h}{p^s}\right)^{-1}\cdot\frac{(-1)^{m-s+1}}{(m-s+1)!}\cdot\log_{\mathrm{BK}}(\kappa_p(y,z,s))$;*
- *if $s\geq m$, $(\mathcal{L}^{-+}_{\mathbf{gh}}(\kappa_p))_{y,z,s}=\left(1-\frac{p^{s-1}}{\alpha_g\beta_h}\right)\left(1-\frac{\alpha_g\beta_h}{p^s}\right)^{-1}(s-m)!\cdot\exp^*_{\mathrm{BK}}(\kappa_p(y,z,s))$.*

Here, $\log_{\mathrm{BK}}$ and $\exp^*_{\mathrm{BK}}$ stand for the Bloch–Kato logarithm (resp. dual exponential) associated to the Dieudonné module of the p-adic representation $V^-_{g_y}\otimes U^+_{h_z}(1-s)$.

Remark 3.3.3. In the case where $\alpha_g\beta_h=p^s$, we implicitly understand that the Euler factor in the denominator appears on the left hand side of the equality.

As shown in [KLZ17, Theorem 8.1.7], there is an injection

$$H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^- \hat{\otimes} \mathbb{V}_{\mathbf{h}}^+ \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1})) \hookrightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^- \hat{\otimes} \mathbb{V}_{\mathbf{h}} \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1})).$$

If we denote by $\kappa_p^{--}(\mathbf{g}, \mathbf{h})$ the projection of $\kappa_p(\mathbf{g}, \mathbf{h})$ to $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^- \hat{\otimes} \mathbb{V}_{\mathbf{h}}^- \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$, it is further shown in loc. cit. that

$$\kappa_p^{--}(\mathbf{g}, \mathbf{h}) = 0. \tag{3.30}$$

As a consequence, the projection of $\kappa_p(\mathbf{g}, \mathbf{h})$ to $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^- \hat{\otimes} \mathbb{V}_{\mathbf{h}} \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$ actually lies in $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^- \hat{\otimes} \mathbb{V}_{\mathbf{h}}^+ \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$ and we may hence denote it $\kappa_p^{-+}(\mathbf{g}, \mathbf{h})$.

Let $\lambda_N(\mathbf{g})$ denote the Λ-adic pseudo-eigenvalue of $\mathbf{g}$ as defined in [KLZ17, Section 10], interpolating the Atkin-Lehner pseudo-eigenvalues of the classical specializations of $\mathbf{g}$. Recall from (3.18) and (3.19) Ohta's families of differential forms $\eta_{\mathbf{g}} \in \mathbb{D}(\mathbb{U}_{\mathbf{g}}^+)$ and $\omega_{\mathbf{h}} \in \mathbb{D}(\mathbb{V}_{\mathbf{h}}^-)$. As it follows from the properties recalled in loc. cit., there exists a homomorphism of $\Lambda_{\mathbf{gh}}$-modules

$$\langle \, , \eta_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle : \mathbb{D}(\mathbb{U}_{\mathbf{gh}}^{-+}) \hat{\otimes} \Lambda \to \mathcal{Q}_{\mathbf{gh}} \otimes \mathbb{Q}_p(\mu_N)$$

such that for all $\delta \in \mathbb{D}(\mathbb{U}_{\mathbf{gh}}^{-+}) \hat{\otimes} \Lambda$ and all $(y, z, s) \in \mathcal{W}_{\mathbf{gh}}^\circ$,

$$\begin{aligned} \nu_{y,z,s}(\langle \delta, \eta_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle) &= \frac{1}{\lambda_N(g)\mathcal{E}_0(g)\mathcal{E}_1(g)} \cdot \langle \nu_{y,z,s}(\delta), \mathrm{Pr}^{\alpha *}(\eta_{g_y^\circ}) \otimes \mathrm{Pr}^{\alpha *}(\omega_{h_z^\circ}) \rangle \\ &= \frac{1}{\lambda_N(g)\mathcal{E}_0(g)\mathcal{E}_1(g)} \cdot \langle \mathrm{Pr}_*^{\alpha}(\nu_{y,z,s}(\delta)), \eta_{g_y^\circ} \otimes \omega_{h_z^\circ} \rangle, \end{aligned} \tag{3.31}$$

where, recall again,

$$\mathcal{E}_0(g) = 1 - \chi_g^{-1}(p) \beta_g^2 p^{1-\ell}, \qquad \mathcal{E}_1(g) = 1 - \chi_g(p) \alpha_g^{-2} p^{\ell-2}.$$

The following explicit reciprocity law is [KLZ17, Theorem 10.2.2].

Theorem 3.3.4. *Define the Iwasawa function*

$$\mathcal{A}(\mathbf{g}, \mathbf{h}) := \lambda_N(\mathbf{g})^{-1} (-1)^s (c^2 - c^{-(\ell+m-2-2s)} \varepsilon_{\mathbf{g}}(c)^{-1} \varepsilon_{\mathbf{h}}(c)^{-1}) \tag{3.32}$$

in $\Lambda_{\mathbf{gh}}$. *Then*

$$\langle \mathcal{L}_{\mathbf{gh}}^{-+}(\kappa_p^{-+}(\mathbf{g}, \mathbf{h})), \eta_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle = \mathcal{A}(\mathbf{g}, \mathbf{h}) \cdot L_p(\mathbf{g}, \mathbf{h}). \tag{3.33}$$

The self-dual case

Let $\mathbf{g}$ be a Hida family of tame conductor N and tame nebentype χ, and set again $\mathbf{h} = \mathbf{g}^* = \mathbf{g} \otimes \bar{\chi}$. Define the curve

$$\mathcal{C} := \mathcal{C}_{\ell,\ell,\ell-1} = \{(y, z, \sigma) \in \mathcal{W}_{\mathbf{gg}^*} : y = z, \quad \alpha_{g_y} \neq \beta_{g_y}, \quad \mathrm{w}(z) = \sigma \cdot \varepsilon_{\mathrm{cyc}}\}. \tag{3.34}$$

Note that $\mathcal{C}$ is a finite cover of the line in $\mathcal{W}^3$ given as the set of regular points in the Zariski closure of the set of points of weights $(\ell, \ell, \ell - 1)$ for some $\ell \geq 1$.

Theorem 3.3.5. *The restriction of* $\kappa_p(\mathbf{g}, \mathbf{g}^*)$ *to* $\mathcal{C}$ *is zero.*

Proof. Recall firstly from (3.30) that $\kappa_p^{--}(\mathbf{g}, \mathbf{g}^*) = 0$. We also claim that

$$\kappa_p^{-+}(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}} = \kappa_p^{+-}(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}} = 0.$$

To see this, observe that the $\Lambda_{\mathbf{g}}[G_{\mathbb{Q}_p}]$-module $\mathbb{V}_{\mathbf{gg}^*}^{-+} \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1})_{|\mathcal{C}}$ is isomorphic to $\Lambda_{\mathbf{g}}(1)$. This follows directly from (3.29) and (3.34), because $\frac{\alpha_{g_y^\circ}}{\alpha_{g_y^\circ}} \cdot \varepsilon_{\mathrm{cyc}}^{\ell-1} \cdot \varepsilon_{\mathrm{cyc}}^{2-\ell} = \varepsilon_{\mathrm{cyc}}$. Hence

$$H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*}^{-+} \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1})_{|\mathcal{C}}) \simeq H^1(\mathbb{Q}_p, \Lambda_{\mathbf{g}}(1)) \simeq H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \hat{\otimes} \Lambda_{\mathbf{g}} \overset{(\mathrm{ord}_p, \log_p)}{\simeq} \Lambda_{\mathbf{g}} \oplus \Lambda_{\mathbf{g}} \tag{3.35}$$

and $\kappa_p^{-+}(\mathbf{g},\mathbf{g}^*)_{|\mathcal{C}}$ vanishes if and only if infinitely many of its specializations are zero. But this is true for any crystalline classical point $(y,y,\ell-1)$ on $\mathcal{C}$ with $\ell>1$. Indeed, the factor $\mathcal{E}(g_y,h_z,s)$ of Theorem 3.3.1 vanishes, as $\alpha_g\beta_g=\chi(p)p^{\ell-1}$, and hence

$$\alpha_h=\alpha_g\cdot\chi^{-1}(p)=\frac{\alpha_g p^{\ell-1}}{\alpha_g\beta_g}=\frac{p^{\ell-1}}{\beta_g}. \tag{3.36}$$

By (3.27) this shows that the specialization of the global cohomology class $\kappa(\mathbf{g},\mathbf{g}^*)(y,y,\ell-1)$ is zero for all $y\in\mathcal{W}_{\mathbf{g}}^{\circ}$ of weight $\ell>1$, and a fortiori $\kappa_p^{-+}(\mathbf{g},\mathbf{g}^*)(y,y,\ell-1)=0$. We conclude that $\kappa_p^{-+}(\mathbf{g},\mathbf{g}^*)_{|\mathcal{C}}=0$ and likewise $\kappa_p^{+-}(\mathbf{g},\mathbf{g}^*)_{|\mathcal{C}}=0$ by a symmetric reasoning.

Finally, note that there is an exact sequence

$$0\to H^1(\mathbb{Q}_p,\mathbb{V}_{\mathbf{gg}^*}^{++}\hat{\otimes}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})_{|\mathcal{C}})\to H^1(\mathbb{Q}_p,\mathbb{V}_{\mathbf{gg}^*|\mathcal{C}})\to H^1(\mathbb{Q}_p,\mathbb{V}_{\mathbf{gg}^*}/(\mathbb{V}_{\mathbf{gg}^*}^{++}\hat{\otimes}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}))_{|\mathcal{C}}). \tag{3.37}$$

The first map above is injective because $H^0(\mathbb{Q}_p,\mathbb{V}_{\mathbf{gg}^*}/(\mathbb{V}_{\mathbf{gg}^*}^{++}\hat{\otimes}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}))_{|\mathcal{C}})=0$, as it follows again from the description of $\mathbb{V}_{\mathbf{gg}^*}^{-+}$ given in (3.29) (and similarly for $\mathbb{V}_{\mathbf{gg}^*}^{+-}$ and $\mathbb{V}_{\mathbf{gg}^*}^{--}$).

Since we have already shown that $\kappa_p^{--}(\mathbf{g},\mathbf{g}^*)_{|\mathcal{C}}=\kappa_p^{-+}(\mathbf{g},\mathbf{g}^*)_{|\mathcal{C}}=\kappa_p^{+-}(\mathbf{g},\mathbf{g}^*)_{|\mathcal{C}}=0$, this implies that the image of $\kappa(\mathbf{g},\mathbf{g}^*)_{|\mathcal{C}}$ in the right-most term of (3.37) vanishes. Hence, $\kappa(\mathbf{g},\mathbf{g}^*)_{|\mathcal{C}}$ lies in $H^1(\mathbb{Q}_p,\mathbb{V}_{\mathbf{g}}^{+}\hat{\otimes}\mathbb{V}_{\mathbf{g}^*}^{+}\hat{\otimes}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})_{|\mathcal{C}})$. It follows from [KLZ17, Theorem 8.2.3, Remark 8.2.4] that the latter space is isomorphic to $\Lambda_{\mathbf{g}}$, and thus $\kappa(\mathbf{g},\mathbf{g}^*)_{|\mathcal{C}}$ is zero if and only if infinitely many of its specializations are, which is the case as already argued above.

□

A derived system of Beilinson–Flach elements

Keep the notations and assumptions as in previous sections. Theorem 3.3.5 above establishes the vanishing of the local cohomology class $\kappa_p(\mathbf{g},\mathbf{g}^*)$ along $\mathcal{C}$ and it is thus natural to ask about the existence of a *derived* cohomology class $\kappa_p'(\mathbf{g},\mathbf{g}^*)$ on a proper subspace of three-dimensional weight space $\mathcal{W}_{\mathbf{gg}^*}$ containing $\mathcal{C}$, bearing a reciprocity law with Hida–Rankin's improved p-adic L-function.

The purpose of this section is making this construction explicit. Consider the surface

$$\mathcal{S}:=\mathcal{S}_{\ell,m,m-1}=\{(y,z,\sigma)\in\mathcal{W}_{\mathbf{gg}^*}\,:\,\mathrm{w}(z)=\sigma\cdot\varepsilon_{\mathrm{cyc}}\},$$

which is a finite cover of the plane in $\mathcal{W}^3$ arising as the Zariski closure of points of weights $(\ell,m,m-1)$ for some $\ell,m\geq1$. Note that $\mathcal{S}$ obviously contains the curve $\mathcal{C}$.

Let $\mathbb{V}_{\mathbf{gg}^*|\mathcal{S}}$ denote the restriction of $\mathbb{V}_{\mathbf{gg}^*}$ to the surface $\mathcal{S}$ and $\mathbb{V}_{\mathbf{gg}^*|\mathcal{C}}$ denote its restriction to $\mathcal{C}$. The following proposition establishes the existence of a class $\kappa_p'(\mathbf{g},\mathbf{g}^*)\in H^1(\mathbb{Q}_p,\mathbb{V}_{\mathbf{gg}^*|\mathcal{S}})$ that may be regarded as the derivative of $\kappa_p(\mathbf{g},\mathbf{g}^*)$ along the z-direction.

We shrink weight space $\mathcal{W}$ to a rigid-analytic open disk $\mathcal{U}\subset\mathcal{W}$ centered at 1 at which the finite cover $\mathrm{w}:\mathcal{W}_{\mathbf{g}}\to\mathcal{W}$ restricts to an isomorphism $\mathrm{w}:\mathcal{U}_{\mathbf{g}}\xrightarrow{\sim}\mathcal{U}$ with $y_0\in\mathcal{U}_{\mathbf{g}}$. Let $\Lambda_{\mathcal{U}_{\mathbf{g}}}=\mathcal{O}(\mathcal{U}_{\mathbf{g}})$ denote the Iwasawa algebra of analytic functions on $\mathcal{U}_{\mathbf{g}}$ whose supremum norm is bounded by 1. Shrink likewise $\mathcal{C}$ and $\mathcal{S}$ so that projection to weight space restricts to an isomorphism with $\mathcal{U}$ and $\mathcal{U}\times\mathcal{U}$ respectively. Having done that, their associated Iwasawa algebras are respectively $\mathcal{O}(\mathcal{C})=\Lambda_{\mathcal{U}_{\mathbf{g}}}\simeq\mathbb{Z}_p[[Z]]$ and $\mathcal{O}(\mathcal{S})=\Lambda_{\mathcal{U}_{\mathbf{g}}}\hat{\otimes}\Lambda_{\mathcal{U}_{\mathbf{g}}}\simeq\mathbb{Z}_p[[Y,Z]]$. The isomorphism $\Lambda_{\mathcal{U}_{\mathbf{g}}}\simeq\mathbb{Z}_p[[Z]]$ is not canonical and depends on the choice of an element $\gamma\in\Lambda_{\mathcal{U}_{\mathbf{g}}}^{\times}$ which is sent to $1+Z$.

Consider the short exact sequence of $\mathbb{Z}_p$-modules

$$0\to\mathbb{Z}_p[[Y,Z]]\xrightarrow{\cdot(Z-Y)}\mathbb{Z}_p[[Y,Z]]\to\mathbb{Z}_p[[Z]]\to0.$$

Under the above identifications, the previous exact sequence may be recast as

$$0\to\mathcal{O}_{\mathcal{S}}\xrightarrow{\delta}\mathcal{O}_{\mathcal{S}}\to\mathcal{O}_{\mathcal{C}}\to0$$

with $\delta=1\otimes(\gamma-1)-(\gamma-1)\otimes1$ in $\mathcal{O}_{\mathcal{S}}\simeq\Lambda_{\mathcal{U}_{\mathbf{g}}}\hat{\otimes}\Lambda_{\mathcal{U}_{\mathbf{g}}}$.

Proposition 3.3.6. *There exists a unique local class* $\kappa'_{p,\gamma}(\mathbf{g},\mathbf{g}^*) \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}})$ *such that*

$$\kappa_p(\mathbf{g},\mathbf{g}^*)_{|\mathcal{S}} = \delta \cdot \kappa'_{p,\gamma}(\mathbf{g},\mathbf{g}^*).$$

Proof. The short exact sequence of $G_{\mathbb{Q}_p}$-modules

$$0 \to \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}} \xrightarrow{\delta} \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}} \to \mathbb{V}_{\mathbf{gg}^*|\mathcal{C}} \to 0$$

gives rise to the long exact sequence

$$H^0(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{C}}) \to H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}}) \xrightarrow{\delta} H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}}) \to H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{C}}).$$

Since $H^0(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{C}}) = 0$ as already argued in the proof of Theorem 3.3.5, the vanishing of $\kappa_p(\mathbf{g},\mathbf{g}^*)_{|\mathcal{C}}$ implies the existence of a unique element $\kappa'_{p,\gamma}(\mathbf{g},\mathbf{g}^*) \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}})$ satisfying the claim. □

We are interested in the restriction of $\kappa'_{p,\gamma}(\mathbf{g},\mathbf{g}^*)$ to $\mathcal{C}$; although it depends on the choice of the topological generator γ, the class $\kappa'_p(\mathbf{g},\mathbf{g}^*) \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{C}})$ defined by

$$\kappa'_p(\mathbf{g},\mathbf{g}^*) = \frac{\kappa'_{p,\gamma}(\mathbf{g},\mathbf{g}^*)}{\log_p(\gamma)}$$

is independent of γ.

Since in our setting the Euler factor $1 - \frac{\alpha_{g_y^\circ}\beta_{h_z^\circ}}{p^s}$ appearing in Proposition 3.3.2 is equal to $1 - \frac{\alpha_{g_y^\circ}}{\alpha_{g_z^\circ}}$, it is natural to introduce a modified p-adic L-function on the surface $\mathcal{S}$, defined as

$$\tilde{L}_p(\mathbf{g},\mathbf{g}^*)(y,z,s) = \left(1 - \frac{\alpha_{g_y^\circ}}{\alpha_{g_z^\circ}}\right) \times L_p(\mathbf{g},\mathbf{g}^*)(y,z,s). \tag{3.38}$$

Fix a point $(y,z,s) \in \mathcal{W}^\circ_{\mathbf{gh}} \cap \mathcal{S}$. Set $L = \mathbb{Q}(g_y^\circ, h_z^\circ, \lambda_N(g_y^\circ))$ and let L_p denote the p-adic completion of L. Define

$$\log^{-+} : H^1(\mathbb{Q}_p, \mathbb{V}_{g_y} \otimes V_{h_z}(1-s)) \xrightarrow{\mathrm{pr}^{-+}} H^1(\mathbb{Q}_p, \mathbb{V}_{g_y}^- \otimes V_{h_z}^+(1-s)) \to L_p, \tag{3.39}$$

where the first map is the projection onto $H^1(\mathbb{Q}_p, \mathbb{V}_{g_y}^- \otimes V_{h_z}^+(1-s))$ and the last one is the composition of the Bloch–Kato logarithm with the pairing with the differential $\eta_{g_y^\circ} \otimes \omega_{h_z^\circ}$.

It follows from Theorems 3.3.2 and 3.3.4 that for all $(y,z,s) \in \mathcal{W}^\circ_{\mathbf{gh}} \cap \mathcal{S}$:

$$\tilde{L}_p(\mathbf{g},\mathbf{g}^*)(y,z,s) = \left(1 - \frac{\alpha_{g_z^\circ}}{p\alpha_{g_y^\circ}}\right) \cdot \log^{-+}(\kappa_p(\mathbf{g},\mathbf{g}^*)(y,z,s)), \tag{3.40}$$

up to multiplication by the c-factor we have described in Theorem 3.3.4 and which does not affect to our discussion since we always work modulo $L^\times$.

Observe that the function $\tilde{L}_p(\mathbf{g},\mathbf{g}^*)$ vanishes along the curve $\mathcal{C}$, so the restriction of its derivative to that line is exactly zero. Recall that $\tilde{L}_p(\mathbf{g},\mathbf{g}^*)$ is a two-variable function, determined by the values of y and z, since the third variable σ comes automatically determined by z. Hence, one has that

$$\frac{\partial}{\partial y}\tilde{L}_p(\mathbf{g},\mathbf{h}) + \frac{\partial}{\partial z}\tilde{L}_p(\mathbf{g},\mathbf{h}) = 0.$$

Recall that $\alpha_{\mathbf{g}}$ is an analytic function defined over $\Lambda_{\mathbf{g}}$; we denote by $\alpha'_{\mathbf{g}}$ its derivative. We may consider as before the $\mathcal{L}$-invariant attached to the adjoint representation of $\mathbf{g}$,

$$\mathcal{L}(\mathrm{ad}^0(\mathbf{g})) = -\frac{\alpha'_{\mathbf{g}}}{\alpha_{\mathbf{g}}}.$$

Observe that its specializations at classical points agree with the definitions given before in (3.23).

Now, we can compute the partial derivatives of $\tilde{L}_p(\mathbf{g}, \mathbf{h})$ at a crystalline point $(y, y, \ell-1)$ of the curve $\mathcal{C}$. Using (3.38), one gets

$$\frac{\partial}{\partial y} \tilde{L}_p(\mathbf{g}, \mathbf{h})(y, y, \ell-1) = \mathcal{L}(\mathrm{ad}^0(g_y)) \cdot L_p(\mathbf{g}, \mathbf{g}^*)(y, y, \ell-1). \tag{3.41}$$

Then, using (3.40), we first observe that $\kappa_p(\mathbf{g}, \mathbf{g}^*)(y, z, s)$ vanishes at $(y, y, \ell-1)$ and its derivative in the z-direction is precisely the logarithm of the derived cohomology class we have previously computed in Proposition 3.3.6; hence, one gets that

$$\frac{\partial}{\partial z} \tilde{L}_p(\mathbf{g}, \mathbf{h})(y, y, \ell-1) = (1-p^{-1}) \cdot (\log^{-+}(\kappa_p'(\mathbf{g}, \mathbf{g}^*)(y, y, \ell-1))). \tag{3.42}$$

We have then proved the following result:

Theorem 3.3.7. *For any crystalline point $(y, y, \ell-1)$ on $\mathcal{C}$, it holds that*

$$\mathcal{L}(\mathrm{ad}^0(g_y)) \cdot L_p(\mathbf{g}, \mathbf{g}^*)(y, y, \ell-1) = \log^{-+}(\kappa_p'(\mathbf{g}, \mathbf{g}^*)(y, y, \ell-1)) \pmod{L^\times}.$$

Weight one modular forms

Let $g \in S_1(N, \chi_g)$ and $h \in S_1(N, \chi_h)$ be two cuspidal eigenforms of weight one. Let V_g and V_h denote the Artin representations over a finite extension L of $\mathbb{Q}$ attached to g and h. Let α_g, β_g (resp. α_h, β_h) denote the roots of the p-th Hecke polynomial of g (resp. of h). We assume throughout that $\alpha_g \neq \beta_g$ and $\alpha_h \neq \beta_h$. We also assume L is large enough as specified in 3.2.

Definition 3.3.8. Let $\mathbf{g}$ and $\mathbf{h}$ be Hida families passing through p-stabilizations g_α, h_α of g, h at some point $(y_0, z_0) \in \mathcal{W}_{\mathbf{g}}^\circ \times \mathcal{W}_{\mathbf{h}}^\circ$ of weights $(1,1)$. Define

$$\kappa(g_\alpha, h_\alpha) := \kappa(\mathbf{g}, \mathbf{h})(y_0, z_0, 0) \in H^1(\mathbb{Q}, V_{gh} \otimes L_p(1))$$

as the specialization of $\kappa(\mathbf{g}, \mathbf{h})$ at the point $(y_0, z_0, 0)$.

This procedure yields four a priori different global cohomology classes:

$$\kappa(g_\alpha, h_\alpha), \quad \kappa(g_\alpha, h_\beta), \quad \kappa(g_\beta, h_\alpha), \quad \kappa(g_\beta, h_\beta), \tag{3.43}$$

one for each choice of pair of roots of the p-th Hecke polynomials of g and h.

Given a p-adic representation V of $G_{\mathbb{Q}_p}$ with coefficients in $\mathbb{Q}_p$, Bloch and Kato introduced in [BK93] a collection of subspaces of the local Galois cohomology group $H^1(\mathbb{Q}_p, V)$, denoted respectively

$$0 \subset H^1_e(\mathbb{Q}_p, V) \subset H^1_{\mathrm{f}}(\mathbb{Q}_p, V) \subset H^1_g(\mathbb{Q}_p, V) \subset H^1(\mathbb{Q}_p, V).$$

Definition 3.3.9. Let V be a representation of $G_{\mathbb{Q}}$ with coefficients in $\mathbb{Q}_p$. The group of classes that are de Rham at p (i.e. the restriction to $\mathbb{Q}_p$ lies in $H^1_g(\mathbb{Q}_p, V)$) and unramified at all primes $\ell \neq p$ is denoted as $H^1_{\mathrm{f},p}(\mathbb{Q}, V)$.

The group of classes that are crystalline at p and unramified at any other prime $q \neq p$ is denoted as $H^1_{\mathrm{f}}(\mathbb{Q}, V)$.

Proposition 3.3.10. *The four classes in* (3.43) *lie in $H^1_{\mathrm{f},p}(\mathbb{Q}, V_{gh} \otimes L_p(1))$.*

Proof. The dimensions of $H^1_g(\mathbb{Q}_p, V_{gh}(1))$ and $H^1(\mathbb{Q}_p, V_{gh}(1))$ are equal, according to the discussion of [DR20b, Section 1.4]; hence, the classes are de Rham at p. Furthermore, the restriction of these classes to $\mathbb{Q}_q$ is 0 for $q \neq p$. In fact a stronger fact holds true: the local Λ-adic classes $\kappa_q(\mathbf{g}, \mathbf{h}) \in H^1(\mathbb{Q}_q, \mathbb{V}_{\mathbf{gh}})$ are 0 at all $q \neq p$. This can be argued for instance by fixing weights (ℓ, m, s)

large enough so that for every triple $(\epsilon_1, \epsilon_2, \epsilon_3)$ of characters of arbitrary p-power conductor and for any point (x, y, z) above $(\nu_{\ell,\epsilon_1}, \nu_{m,\epsilon_2}, \nu_{s,\epsilon_3})$, $\mathbb{V}(\mathbf{g}, \mathbf{h})(x, y, z)$ contains no sub-quotient isomorphic to neither $\mathbb{Q}_p$ nor $\mathbb{Q}_p(1)$. It then follows from Tate's local Euler characteristic formula (cf. [Nek98, 2.5]) that $H^1(\mathbb{Q}_q, \mathbb{V}_{\mathbf{gh}}(x, y, z)) = 0$. From this it follows that $H^1(\mathbb{Q}_q, \mathbb{V}_{\mathbf{gh}}) = 0$ arguing as in e.g. [KLZ17, Prop. 8.1.7 or Lemma 8.2.6]. □

Recall that we let H denote the Galois extension of $\mathbb{Q}$ cut out by $V_g \otimes V_h$.

Proposition 3.3.11. *There are natural identifications*

$$\begin{aligned} H^1_{\mathrm{f}}(\mathbb{Q}, V_{gh} \otimes_L L_p(1)) &= (\mathcal{O}_H^\times \otimes_{\mathbb{Z}} V_{gh} \otimes_L L_p)^{G_{\mathbb{Q}}}, \\ H^1_{\mathrm{f},p}(\mathbb{Q}, V_{gh} \otimes_L L_p(1)) &= (\mathcal{O}_H[1/p]^\times \otimes_{\mathbb{Z}} V_{gh} \otimes_L L_p)^{G_{\mathbb{Q}}}. \end{aligned} \tag{3.44}$$

Proof. This follows from the same arguments as in e.g. [Bel09, Prop. 2.9]. □

Since $\alpha_g \neq \beta_g$, V_g decomposes as a $G_{\mathbb{Q}_p}$-module as $V_g = V_g^{\alpha_g} \oplus V_g^{\beta_g}$, where $V_g^{\alpha_g}$ and $V_g^{\beta_g}$ are the $G_{\mathbb{Q}_p}$-invariant lines on which Fr_p acts with eigenvalue α_g and β_g respectively. We similarly have $V_h = V_h^{\alpha_h} \oplus V_h^{\beta_h}$, and we may define

$$V_{gh}^{\alpha\alpha} := V_g^{\alpha_g} \otimes V_h^{\alpha_h}, \quad ..., \quad V_{gh}^{\beta\beta} := V_g^{\beta_g} \otimes V_h^{\beta_h}.$$

Note that these four $G_{\mathbb{Q}_p}$-invariant lines in V_{gh} are linearly independent even though some of the eigenvalues $\alpha_g\alpha_h$, $\alpha_g\beta_h$, $\beta_g\alpha_h$, $\beta_g\beta_h$ might be equal. Hence there is a decomposition of $G_{\mathbb{Q}_p}$-modules

$$V_{gh} := V_g \otimes V_h = V_{gh}^{\alpha\alpha} \oplus ... \oplus V_{gh}^{\beta\beta}. \tag{3.45}$$

It follows that the local class $\kappa_p(g_\alpha, h_\alpha)$ may be decomposed as

$$\kappa_p(g_\alpha, h_\alpha) = \kappa_p^{++}(g_\alpha, h_\alpha) + \kappa_p^{+-}(g_\alpha, h_\alpha) + \kappa_p^{-+}(g_\alpha, h_\alpha) + \kappa_p^{++}(g_\alpha, h_\alpha),$$

where

$$\begin{aligned} &\kappa_p^{++}(g_\alpha, h_\alpha) \in H^1(\mathbb{Q}_p, V_{gh}^{\beta\beta} \otimes L_p(1)), \quad \kappa_p^{+-}(g_\alpha, h_\alpha) \in H^1(\mathbb{Q}_p, V_{gh}^{\beta\alpha} \otimes L_p(1)) \\ &\kappa_p^{-+}(g_\alpha, h_\alpha) \in H^1(\mathbb{Q}_p, V_{gh}^{\alpha\beta} \otimes L_p(1)), \quad \kappa_p^{--}(g_\alpha, h_\alpha) \in H^1(\mathbb{Q}_p, V_{gh}^{\alpha\alpha} \otimes L_p(1)). \end{aligned} \tag{3.46}$$

As before, we are specially interested in the case where $h = g^* = g \otimes \chi^{-1}$, and we impose on g the assumptions (H1-H2-H3) listed in the introduction; if we denote by $\{\alpha, \beta\}$ the p-th Hecke eigenvalues of g, the p-th Hecke eigenvalues of h are $\{1/\beta, 1/\alpha\}$.

Let $\mathbf{g}$ and $\mathbf{h} = \mathbf{g}^* := \mathbf{g} \otimes \chi^{-1}$ denote the Hida families over $\mathcal{W}_{\mathbf{g}} = \mathcal{W}_{\mathbf{h}}$ passing through g_α and $(g_\alpha)^* = (g^*)_{1/\beta}$ respectively at some point $y_0 \in \mathcal{W}_{\mathbf{g}}$ in weight space. Thanks to our running assumptions, the main theorem of [BeDi16] ensures that the weight map $\mathcal{W}_{\mathbf{g}} \longrightarrow \mathcal{W}$ is étale at the point associated to g_α. It is therefore possible to fix an open subset in $\mathcal{W}_{\mathbf{g}}$ around g_α on which the weight map is an isomorphism. This way we are entitled to work under the simplifying assumptions posed in 3.3 and the results in loc. cit. and 3.3 may be applied.

Proposition 3.3.12. *The global cohomology classes $\kappa(g_\alpha, g^*_{1/\beta})$ and $\kappa(g_\beta, g^*_{1/\alpha})$ are zero.*

Proof. The restriction map

$$\mathrm{res}_p : H^1(\mathbb{Q}, V_{gh} \otimes L_p(1)) \to H^1(\mathbb{Q}_p, V_{gh} \otimes L_p(1))$$

is injective. This follows from [Bel09, Proposition 2.12], which asserts that there are natural isomorphisms

$$H^1(\mathbb{Q}, V_{gh} \otimes L_p(1)) \simeq (H^\times \otimes V_{gh})^{G_{\mathbb{Q}}}, \quad H^1(\mathbb{Q}_p, V_{gh} \otimes L_p(1)) \simeq (H_p^\times \otimes V_{gh})^{G_{\mathbb{Q}_p}},$$

and hence the restriction map corresponds to the natural inclusion $H \hookrightarrow H_p$. From Theorem 3.3.5, it follows that $\kappa_p(g_\alpha, g^*_{1/\beta})$ and $\kappa_p(g_\beta, g^*_{1/\alpha})$ are both zero, and the result follows. □

Define the curve

$$\mathcal{D} := \{(y,z,\sigma) \in \mathcal{W}_{\mathbf{gg}^*} : y = y_0, \quad \mathrm{w}(z) = \sigma \cdot \varepsilon_{\mathrm{cyc}}\} \subset \mathcal{S}$$

Proposition 3.3.13. *There exists a unique global class* $\kappa'_\gamma(\mathbf{g},\mathbf{g}^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}})$ *such that*

$$\kappa'_\gamma(\mathbf{g},\mathbf{g}^*)_{|\mathcal{D}} = (\gamma-1)\cdot \kappa'_\gamma(\mathbf{g},\mathbf{g}^*).$$

Proof. The short exact sequence of $G_{\mathbb{Q}}$-modules

$$0 \to \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}} \xrightarrow{\delta} \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}} \to V_{gh} \otimes L_p(1) \to 0$$

gives rise to the long exact sequence

$$H^0(\mathbb{Q}, V_{gh}\otimes L_p(1)) \to H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}}) \xrightarrow{\delta} H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}}) \to H^1(\mathbb{Q}, V_{gh}\otimes L_p(1)).$$

Since $H^0(\mathbb{Q}, V_{gh}\otimes L_p(1)) = 0$, the vanishing of $\kappa(g_\alpha, g^*_{1/\beta})$ proved in Proposition 3.3.12 implies the existence of a unique element $\kappa'_\gamma(\mathbf{g},\mathbf{g}^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}})$ satisfying the claim. □

Remark 3.3.14. The *global* cohomology class in Proposition 3.3.13 only makes sense along the curve $\mathcal{D}$. Besides, the *local* cohomology class constructed in Proposition 3.3.6 exists along the whole surface $\mathcal{S}$. Hence one can not define the latter on the surface $\mathcal{S}$ as the restriction at p of the former, although this is indeed true after restricting to $\mathcal{D}$, a fact that we shall apply right below.

Set

$$\kappa'(g_\alpha, g^*_{1/\beta}) = \frac{1}{\log_p(\gamma)} \kappa'_\gamma(\mathbf{g},\mathbf{g}^*)(y_0,y_0,0). \tag{3.47}$$

Since the construction of this class coincides with the one performed in the previous section once we localize at p and restrict to the curve $\mathcal{D}$, Theorem 3.3.7 applies and we deduce that

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) \cdot L_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,0) = \log^{-+}(\kappa'_p(g_\alpha, g^*_{1/\beta})) \pmod{L^\times}. \tag{3.48}$$

Recall from (3.21) that $L_p(\mathbf{g},\mathbf{g}^*)$ satisfies a functional equation relating the values at $s=0$ and $s=1$ up to a simple non-zero rational constant. Together with Proposition 3.2.5 this implies that

$$L_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,1)^2 = \mathcal{L}(\mathrm{ad}^0(g_{y_0}))^2 = \log^{-+}(\kappa'_p(g_\alpha, g^*_{1/\beta})) \pmod{L^\times}. \tag{3.49}$$

This formula is the key input for deriving Theorem B, the second main result of this chapter.

3.4 Derivatives of Fourier coefficients via Galois deformations

As in previous sections, let $g \in S_1(N,\chi)$ satisfying the hypothesis of the introduction. Let

$$\varrho_g : \mathrm{Gal}\,(H_g/\mathbb{Q}) \hookrightarrow \mathrm{GL}(V_g) \simeq \mathrm{GL}_2(L), \quad \varrho_{\mathrm{ad}^0(g)} : \mathrm{Gal}\,(H/\mathbb{Q}) \hookrightarrow \mathrm{GL}(\mathrm{ad}^0(g)) \simeq \mathrm{GL}_3(L)$$

denote the Artin representations associated to g and its adjoint, respectively. Here L is a finite extension of $\mathbb{Q}$ and $H_g \supseteq H$ denote the finite Galois extensions of $\mathbb{Q}$ cut out by these representations. Let $\mathcal{P}$ denote the set of primes of H lying above p, and fix once for all a prime $\wp \in \mathcal{P}$, thus determining an embedding $H \subset H_p \subset \bar{\mathbb{Q}}_p$ of H into its completion H_p at $\wp$, and an arithmetic Frobenius $\mathrm{Fr}_p \in \mathrm{Gal}\,(H_p/\mathbb{Q}_p)$.

As it occurred in [DLR16], the regularity assumptions we have imposed on g imply by e.g. in [Das99, Prop. 3.2.2] that

$$\dim_L(\mathcal{O}_H^\times \otimes V_{gg^*})^{G_{\mathbb{Q}}} = 1, \quad \dim_L(\mathcal{O}_H[1/p]^\times \otimes V_{gg^*})^{G_{\mathbb{Q}}} = 3,$$

and thus
$$\dim_L(\mathcal{O}_H[1/p]^\times/p^{\mathbb{Z}} \otimes V_{gg^*})^{G_{\mathbb{Q}}} = \dim_L(\mathcal{O}_H[1/p]^\times/p^{\mathbb{Z}} \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}}} = 2.$$

Fix two linearly independent global cohomology classes

$$u \in H^1_{\mathrm{f}}(\mathbb{Q}, V_{gg^*} \otimes L_p(1)), \quad v \in H^1_{\mathrm{f},p}(\mathbb{Q}, V_{gg^*} \otimes L_p(1))$$

such that under the identifications provided by Proposition 3.3.11, project to a basis of the two-dimensional space $(\mathcal{O}_H[1/p]^\times/p^{\mathbb{Z}} \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}}}$. By a slight abuse of notation, we continue to denote

$$u \in (\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}}}, \quad v \in (\mathcal{O}_H[1/p]^\times/p^{\mathbb{Z}} \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}}}$$

the resulting elements.

Recall from (3.45) that V_{gh} admits a natural decomposition as $G_{\mathbb{Q}_p}$-module as the direct sum of the four different lines $V_{gg^*}^{\alpha\alpha}$, ..., $V_{gg^*}^{\beta\beta}$. Since $\mathrm{ad}^0(V_g)$ is the quotient of V_{gg^*} by the trivial representation, (3.45) descends to a decomposition of $\mathrm{ad}^0(V_g)$ as $G_{\mathbb{Q}_p}$-module as

$$\mathrm{ad}^0(g) = \mathrm{ad}^0(g)^1 \oplus \mathrm{ad}^0(g)^{\alpha\otimes\bar{\beta}} \oplus \mathrm{ad}^0(g)^{\beta\otimes\bar{\alpha}} = L \cdot e_1 \oplus L \cdot e_{\alpha\otimes\bar{\beta}} \oplus L \cdot e_{\beta\otimes\bar{\alpha}},$$

where $\mathrm{Fr}_p(e_1) = e_1$, $\mathrm{Fr}_p(e_{\alpha\otimes\bar{\beta}}) = \frac{\alpha}{\beta} \cdot e_{\alpha\otimes\bar{\beta}}$, $\mathrm{Fr}_p(e_{\beta\otimes\bar{\alpha}}) = \frac{\beta}{\alpha} \cdot e_{\beta\otimes\bar{\alpha}}$. Note that $\alpha/\beta \neq 1$ thanks to the regularity assumption. It could be that $\alpha = -\beta$ and hence $\alpha/\beta = \beta/\alpha = -1$, but the above decomposition is still available as explained in (3.45).

Restriction to the decomposition group at p allows us to regard u and v as elements in $H^1(\mathbb{Q}_p, \mathrm{ad}^0(g) \otimes L_p(1)) = (H_p^\times \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}_p}}$, and as such we may write u and v as

$$u = u_1 \otimes e_1 + u_{\alpha\otimes\bar{\beta}} \otimes e_{\alpha\otimes\bar{\beta}} + u_{\beta\otimes\bar{\alpha}} \otimes e_{\beta\otimes\bar{\alpha}},\ v = v_1 \otimes e_1 + v_{\alpha\otimes\bar{\beta}} \otimes e_{\alpha\otimes\bar{\beta}} + v_{\beta\otimes\bar{\alpha}} \otimes e_{\beta\otimes\bar{\alpha}}, \quad (3.50)$$

where $u_1, v_1, u_{\alpha\otimes\bar{\beta}}, v_{\alpha\otimes\bar{\beta}}, u_{\beta\otimes\bar{\alpha}}, v_{\beta\otimes\bar{\alpha}} \in H_p^\times$ satisfy

$$\mathrm{Fr}_p(u_1) = u_1, \quad \mathrm{Fr}_p(u_{\alpha\otimes\bar{\beta}}) = \frac{\beta}{\alpha} \cdot u_{\alpha\otimes\bar{\beta}}, \quad \mathrm{Fr}_p(u_{\beta\otimes\bar{\alpha}}) = \frac{\alpha}{\beta} \cdot u_{\beta\otimes\bar{\alpha}}$$

and similarly for v.

We can now provide the last step in our proof of Theorems A, A' and B in the introduction. As it was shown in Corollary 3.2.5, Theorem A may be reduced to the computation of the derivative of the Fourier coefficient $a_p(\mathbf{g})$ at y_0, where $\mathbf{g}$ stands for the unique Hida family passing through g_α.

Let

$$\tilde{\varrho}_g : G_{\mathbb{Q}} \to \mathrm{GL}_2(L_p[\varepsilon])$$

be the unique first order α-ordinary deformation of ϱ_g such that

$$\det \tilde{\varrho}_g = \chi_g(1 + \log_p \chi_{\mathrm{cyc}} \cdot \varepsilon).$$

This representation, whose existence follows from [BeDi16], satisfies

$$\tilde{\varrho}_g = (1 + \varepsilon \cdot \kappa_g) \cdot \varrho_g,$$

for some cohomology class $\kappa_g : G_{\mathbb{Q}} \to \mathrm{ad}(\varrho)$. Considering a diagonal basis for the Frobenius action (where we take the first vector to have eigenvalue α), the matrix form of $\kappa(\sigma)$ can be expressed as

$$\kappa(\sigma) = \begin{pmatrix} \kappa_1(\sigma) & \kappa_2(\sigma) \\ \kappa_3(\sigma) & \kappa_4(\sigma) \end{pmatrix}.$$

We denote by $\tilde{\varrho}_{g,p}$ the restriction of $\tilde{\varrho}_g$ to the decomposition group at p; in the same way, the restriction of κ to the decomposition group at p is denoted by κ_p, and similarly we denote by $\kappa_{i,p}$ the restriction of κ_i to $G_{\mathbb{Q}_p}$.

In [BeDi16, Lemmas 2.3 and 2.5] the authors determine the tangent space to a deformation problem which can be seen to be equivalent to ours. They conclude that there is a natural bijection between this tangent space and a certain subspace of $H^1(\mathbb{Q}, \mathrm{ad}(\varrho_g))$; in this case, it consists on those classes κ whose matrix representation satisfies

$$\kappa_{3,p}(\sigma) = 0, \qquad \kappa_{1,p}(\sigma)|_{I_p} = 0, \tag{3.51}$$

being I_p the inertia group at p. In [DLR18], the restriction of κ to the inertia group at p was determined, and the identifications of class field theory allow us to extend this to the whole decomposition group. A similar setting, where also a quite related deformation problem arises, is exploded in [BDP19] to treat the case of weight one Eisenstein points.

Let V_g^α be the étale subspace on which the action of the Frobenius is unramified for the Λ-adic representation. Restricting to the decomposition group at p, we have that

$$\tilde{\varrho}_{g,p}|_{V_g^\alpha} = (1 + \varepsilon \cdot \kappa_{1,p}) \cdot \alpha_g,$$

where $\kappa_{1,p} \in H^1(\mathbb{Q}_p, \mathrm{Hom}(V_g^\alpha, V_g^\alpha))$. If g_α' stands for the derivative of $\mathbf{g}$ evaluated at y_0, we then have

$$a_p(g_\alpha) + \varepsilon \cdot a_p(g_\alpha') = \alpha_g + \varepsilon \cdot \kappa_{1,p}(\mathrm{Fr}_p) \cdot \alpha_g,$$

and consequently we have the following.

Proposition 3.4.1. *Let $\mathbf{g}$ be the Hida family through g_α. Then, it holds that*

$$a_p(g_\alpha') = \kappa_{1,p}(\mathrm{Fr}_p) \pmod{L^\times}.$$

Taking into account the identifications provided by class field theory, one can make $\kappa_1(\mathrm{Fr}_p)$ explicit.

From [BeDi16, Section 3.2], there is an exact sequence

$$0 \to \mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}((\mathcal{O}_H \otimes \mathbb{Q}_p)^\times, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}_H^\times \otimes \mathbb{Q}_p, \bar{\mathbb{Q}}_p).$$

Similarly, one has another exact sequence

$$0 \to \mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}((H \otimes \mathbb{Q}_p)^\times, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}_p, \bar{\mathbb{Q}}_p).$$

Consequently, we have identifications

$$H^1(\mathbb{Q}, \bar{\mathbb{Q}}_p) \otimes \mathrm{ad}(\varrho) \simeq (H^1(H, \bar{\mathbb{Q}}_p) \otimes \mathrm{ad}(\varrho))^{G_{\mathbb{Q}}} \simeq (\mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p) \otimes \mathrm{ad}(\varrho))^{G_{\mathbb{Q}}},$$

and this corresponds with the subspace of homomorphisms of

$$(\mathrm{Hom}((H \otimes \mathbb{Q}_p)^\times, \bar{\mathbb{Q}}_p) \otimes \mathrm{ad}(\varrho))^{G_{\mathbb{Q}}}$$

vanishing at $\mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}_p$.

We now recall some results which allow us to determine each of the $\kappa_{i,p}$. Since there are isomorphisms

$$(\mathcal{O}_H \otimes \mathbb{Q}_p)^\times \simeq \prod_{\mathfrak{q} \in \mathcal{P}} \mathcal{O}_{H_\mathfrak{q}}^\times, \qquad (H \otimes \mathbb{Q}_p)^\times \simeq \prod_{\mathfrak{q} \in \mathcal{P}} H_\mathfrak{q}^\times,$$

we may write elements in $(\mathcal{O}_H \otimes \mathbb{Q}_p)^\times$ (or $(H \otimes \mathbb{Q}_p)^\times$) as tuples $(x_i)_{i \in \mathcal{P}}$. The action of the Galois group $G = \mathrm{Gal}\,(H/\mathbb{Q})$ is transitive on $\mathcal{P}$, so any Galois equivariant homomorphism from $(H \otimes \mathbb{Q}_p)^\times$ (resp. $(\mathcal{O}_H \otimes \mathbb{Q}_p)^\times$) to $\bar{\mathbb{Q}}_p$ is completely determined by its values on $H_p^\times$ (resp. $\mathcal{O}_{H_p}^\times$).

From class field theory, one has two distinguished elements in $H^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)$:

1. The class κ_{nr}, which is the unique homomorphism
$$\kappa_{\mathrm{nr}} \in \mathrm{Hom}(\mathrm{Gal}\,(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p), \bar{\mathbb{Q}}_p)$$
taking Fr_p to 1.

2. The restriction to $G_{\mathbb{Q}_p}$ of the logarithm of the cyclotomic character
$$\kappa_{\mathrm{cyc}} := \log_p(\varepsilon_{\mathrm{cyc}}),$$
which gives a ramified element of $H^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)$.

Furthermore, observe that $H^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)$ is identified with $\mathrm{Hom}(G_{H_p}, \bar{\mathbb{Q}}_p)^{\mathrm{Gal}\,(H_p/\mathbb{Q}_p)}$ as explained above. In the latter space, we denote by κ_{nr} the morphism that takes Fr_p to 1, and by κ_{cyc} the ramified element defined by $\log_p(\varepsilon_{\mathrm{cyc}})$.

Now we can determine explicitly the element $\kappa \in H^1(\mathbb{Q}, \mathrm{ad}(\varrho))$, by constructing a homomorphism
$$\Phi_g(x) : (H \otimes \mathbb{Q}_p)^\times \to H_p \otimes \mathrm{ad}(\varrho)$$
corresponding to κ via the previous identifications. In particular, it vanishes when evaluated at the basis $\{u, v\}$ of units for the adjoint. Moreover, $\Phi_g(\pi_p^{-1})$ leaves invariant the one-dimensional space V_g^α, being π_p^{-1} the idèle which is equal to the inverse of a local uniformiser of H_p for the fixed prime $\wp$ above p, and to 1 everywhere else. The eigenvalue for the action of $\Phi_g(\pi_p^{-1})$ on that subspace is precisely $\kappa_1(\mathrm{Fr}_p)$.

Let $u_g^\times$ be any generator of $(\mathcal{O}_H^\times \otimes \mathrm{ad}(\varrho))^{G_\mathbb{Q}}$, and let $v_g^\times$ be any element of the space $(\mathcal{O}_H[1/p]^\times \otimes \mathrm{ad}(\varrho))^{G_\mathbb{Q}}$ such that $\{u_g^\times, v_g^\times\}$ is a basis of $(\mathcal{O}_H[1/p]^\times/p^{\mathbb{Z}} \otimes \mathrm{ad}(\varrho))^{G_\mathbb{Q}}$. Let
$$u_g := (\log_p \otimes \mathrm{Id})(u_g^\times), \quad v_g := (\log_p \otimes \mathrm{Id})(v_g^\times), \quad \tilde{v}_g := (\mathrm{ord}_p \otimes \mathrm{Id})(v_g^\times) \quad \in H_p \otimes \mathrm{ad}(\varrho).$$

Consider the element
$$A_g \in (H_p \otimes \mathrm{ad}(\varrho))^{G_{\mathbb{Q}_p}},$$
which on an eigenbasis for the Frobenius takes the form
$$A_g : \begin{pmatrix} 0 & \frac{\log_p(u_1)}{\log_p(u_{\alpha\otimes\bar{\beta}})} \\ 0 & 1 \end{pmatrix}.$$
Consider also $J \in (H_p \otimes \mathrm{ad}(\varrho))^{G_{\mathbb{Q}_p}}$, which in the same basis is expressed as
$$J = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$
where $a, b, c \in H_p$.

These choices give rise to the $G_{\mathbb{Q}_p}$-equivariant homomorphism $\Phi_g : H^\times \to H_p \otimes \mathrm{ad}(\varrho)$ given by the rule
$$\Phi_g(x) := \sum_{\sigma \in G} \log_p({}^\sigma x) \cdot (\sigma^{-1} \cdot A_g) + \sum_{\sigma \in G} \mathrm{ord}_p({}^\sigma x) \cdot (\sigma^{-1} \cdot J). \tag{3.52}$$

Here, $\sigma^{-1} \cdot A_g$ denotes the action of σ^{-1} by conjugation on the second factor of the tensor product, $\mathrm{ad}(\varrho)$; then, this homomorphism can be extended to the whole $(H \otimes \mathbb{Q}_p)^\times$. The aim is to determine a suitable J such that $\Phi_g(x)$ corresponds to κ via class field theory. The following result explains the behavior of the terms coming from the log-*part*.

Proposition 3.4.2. *Let* $\mathrm{ad}(\varrho)^{\mathrm{ord}} := \mathrm{Hom}(V_g/V_g^\alpha, V_g)$. *The homomorphism* Φ_g *vanishes on* $\mathcal{O}_H^\times \otimes \mathbb{Q}_p$, *and* $\Phi_g(x) \subset H_p \otimes \mathrm{ad}(\varrho)^{\mathrm{ord}}$ *for any* x *of the form* $(x_p, 1, \ldots, 1)$, *where* $x_p \in \mathcal{O}_{H_p}^\times$. *Moreover,* $\Phi_g(x)$ *fixes* V_g^α *for all* x *of the form* $(x_p, 1, \ldots, 1)$, *with* $x_p \in H_p^\times$.

Proof. This follows from [DLR18, Lemma 1.6], where Φ_g is defined in the same way but without the ord-terms. It is clear that the behavior at the $\mathcal{O}_H$-units is not affected by the presence of these extra terms. The last part of the statement follows from the definition of A_g and J. □

To determine the remaining parameters of the matrix J (a, b and c), we first observe that the homomorphism Φ_g must vanish at $\mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}_p$. The endomorphism v_g is represented by a matrix of the form

$$v_g : \begin{pmatrix} \log_p(v_1) & \log_p(v_{\beta\otimes\bar\alpha}) \\ \log_p(v_{\alpha\otimes\bar\beta}) & -\log_p(v_1) \end{pmatrix},$$

and hence

$$\mathrm{Tr}(A_g v_g) = \frac{\log_p(u_1)\cdot\log_p(v_{\alpha\otimes\bar\beta}) - \log_p(u_{\alpha\otimes\bar\beta})\cdot\log_p(v_1)}{\log_p(u_{\alpha\otimes\bar\beta})}. \tag{3.53}$$

Furthermore, observe that the restrictions of both $\kappa_{1,p}$ and $\kappa_{4,p}$ to the decomposition group at p belong to

$$H^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p) \simeq H^1(H_p, \bar{\mathbb{Q}}_p)^{G_{\mathbb{Q}_p}};$$

this space is two-dimensional and it is generated by κ_{cyc} and κ_{nr}; in the same way, $\kappa_{2,p}$ is a cohomology class in the one-dimensional space

$$H^1(H_p, \bar{\mathbb{Q}}_p(\beta/\alpha))^{G_{\mathbb{Q}_p}}.$$

It is clear that

$$\kappa_1|_{I_p} = 0, \quad \kappa_2|_{I_p} = \frac{\log_p(u_1)}{\log_p(u_{\alpha\otimes\bar\beta})}\kappa_{\mathrm{cyc}}, \qquad \kappa_4|_{I_p} = \kappa_{\mathrm{cyc}}.$$

Then, to determine the restriction of the cohomology classes to the whole decomposition group, we impose these three conditions:

1. The fact that the trace of the representation is prescribed (and is equal to κ_{cyc}) forces that

$$\kappa_1 = \lambda\cdot\kappa_{\mathrm{nr}}, \qquad \kappa_4 = \kappa_{\mathrm{cyc}} - \lambda\cdot\kappa_{\mathrm{nr}}.$$

2. The fact that $H^1(H_p, \bar{\mathbb{Q}}_p(\beta/\alpha))^{G_{\mathbb{Q}_p}}$ is one-dimensional (since $\alpha \neq \beta$), makes that

$$\kappa_2 = \frac{\log_p(u_1)}{\log_p(u_{\alpha\otimes\bar\beta})}\kappa_{\mathrm{cyc}}$$

(i.e., there is no contribution coming from the Frobenius).

3. The remaining parameter λ can be determined by considering the associated matrix J giving rise to the endomorphism $\Phi_g(x)$. This is the content of the following proposition.

Proposition 3.4.3. *There exists a unique* λ *such that the homomorphism* $\Phi_g(x)$ *vanishes at* $\mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}_p$.

Proof. Picking $w \in \mathcal{O}_H[1/p]^\times$ and an arbitrary $B \in \mathrm{ad}(\varrho)$, it is enough to see that

$$\mathrm{Tr}(\Phi_g(w)\cdot B) = 0,$$

due to the non-degeneracy of the H_p-valued trace pairing on $H_\wp \otimes \mathrm{ad}(\varrho)$. Set

$$w_g^\times := \sum_{\sigma \in G} {}^\sigma w \otimes (\sigma \cdot B) \in (\mathcal{O}_H^\times \otimes \mathrm{ad}(\varrho))^{G_\mathbb{Q}}.$$

Observe that $w_g^\times$ can be expressed in terms of the basis $u_g^\times$ and $v_g^\times$. In particular, let

$$w_g := (\log_p \otimes \mathrm{Id})(w_g^\times) = \lambda \cdot u_g + \mu \cdot v_g; \quad \tilde{w}_g := (\mathrm{ord}_p \otimes \mathrm{Id})(w_g^\times) = \mu \cdot \tilde{v}_g.$$

Then,

$$\mathrm{Tr}(\Phi_g(w) \cdot B) = \lambda \cdot \mathrm{Tr}(A_g \cdot u_g) + \mu \cdot \mathrm{Tr}(A_g v_g - J \cdot \tilde{v}_g).$$

Observe that

$$J = \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

and from (3.53), one sees that

$$\lambda = \frac{\log_p(u_1) \cdot \log_p(v_{\alpha\otimes\bar{\beta}}) - \log_p(v_1) \cdot \log_p(u_{\alpha\otimes\bar{\beta}})}{2 \cdot \log_p(u_{\alpha\otimes\bar{\beta}}) \cdot \mathrm{ord}_p(v_1)}.$$

□

Hence, we conclude that

$$\kappa_1 = -\frac{\log_p(u_1) \cdot \log_p(v_{\alpha\otimes\bar{\beta}}) - \log_p(v_1) \cdot \log_p(u_{\alpha\otimes\bar{\beta}})}{2 \cdot \log_p(u_{\alpha\otimes\bar{\beta}}) \cdot \mathrm{ord}_p(v_1)} \cdot \kappa_{\mathrm{nr}},$$

and evaluating at Fr_p we finally obtain the formula we anticipated below

$$a_p(g'_\alpha) = \frac{\log_p(u_1) \cdot \log_p(v_{\alpha\otimes\bar{\beta}}) - \log_p(v_1) \cdot \log_p(v_{\alpha\otimes\bar{\beta}})}{\log_p(u_{\alpha\otimes\bar{\beta}})} \pmod{L^\times}, \tag{3.54}$$

as claimed.

3.5 Proof of main results

Proof of Theorems A and A'

Recall that by [DLR16, Proposition 4.2], the p-adic iterated integral of the statement agrees, up to multiplication by a scalar in $L^\times$, with the special value $L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 1)$. Further, applying the relation between $L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 1)$ and $a_p(g'_\alpha)$ as described in Corollary 3.2.5, we have that

$$L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 1) = L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 0) = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) = a_p(g'_\alpha) \pmod{L^\times}.$$

The derivative of the Fourier coefficient was computed in the previous section, and it follows from (3.54) that

$$L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 1) = \frac{\log_p(u_1) \cdot \log_p(v_{\alpha\otimes\bar{\beta}}) - \log_p(v_1) \cdot \log_p(u_{\alpha\otimes\bar{\beta}})}{\log_p(u_{\alpha\otimes\bar{\beta}})} \pmod{L^\times}. \tag{3.55}$$

This proves Theorems A and A'.

Proof of Theorem B

We may combine (3.49) with the above result to deduce that

$$\log^{-+}(\kappa'_p(g_\alpha, g^*_{1/\beta})) = \Big(\frac{\log_p(u_1)\log_p(v_{\alpha\otimes\bar\beta}) - \log_p(u_{\alpha\otimes\bar\beta})\log_p(v_1)}{\log_p(u_{\alpha\otimes\bar\beta})} \Big)^2 \pmod{L^\times}.$$

Recall that in (3.39) we defined the map $\log^{-+}$ as the composition of the Perrin-Riou big logarithm of Proposition 3.3.2 specialized at weight $(y_0, y_0, 0)$ and the pairing with the class $\eta_{g_\alpha} \otimes \omega_{g^*_{1/\beta}}$ introduced in 3.2. These differential classes satisfy

$$\langle \eta_{g_\alpha}, \omega_{g^*_{1/\beta}} \rangle = \frac{1}{\lambda_N(g_\alpha)\mathcal{E}_0(g_\alpha)\mathcal{E}_1(g_\alpha)} \in L^\times.$$

under the perfect pairing $D(V_g) \times D(V_{g^*}) \to L_p$. We may take a decomposition of V_g and V_{g^*} as $G_{\mathbb{Q}_p}$-modules

$$V_g = L \cdot e^g_\alpha \oplus L \cdot e^g_\beta, \qquad V_{g^*} = L \cdot e^{g^*}_{1/\alpha} \oplus L \cdot e^{g^*}_{1/\beta},$$

respectively, where $\{e^g_\alpha, e^g_\beta\}$ and $\{e^{g^*}_{1/\alpha}, e^{g^*}_{1/\beta}\}$ are basis of V_g and V_{g^*}, one dual of each other, and compatible with the choice of the basis for the tensor product V_{gg^*} considered at the previous section. As explained in [DR16, Section 2], one may define p-adic periods

$$\Xi_{g_\alpha} \in H_p^{\mathrm{Fr}_p=\beta^{-1}}, \qquad \Omega_{g^*_{1/\beta}} \in H_p^{\mathrm{Fr}_p=\beta} \tag{3.56}$$

satisfying that

$$\Xi_{g_\alpha} \otimes e^g_\beta = \eta_{g_\alpha}, \qquad \Omega_{g^*_{1/\beta}} \otimes e^{g^*}_{1/\beta} = \omega_{g^*_{1/\beta}}.$$

The natural pairing between $D(V_g) = (H_p \otimes V_g)^{G_{\mathbb{Q}_p}}$ and $D(V_{g^*}) = (H_p \otimes V_{g^*})^{G_{\mathbb{Q}_p}}$ induces a duality between V_g and V_{g^*}, and hence the quantity

$$\langle \eta_{g_\alpha}, \omega_{g^*_{1/\beta}} \rangle = \Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\beta}} \cdot \langle e^g_\beta, e^{g^*}_{1/\beta} \rangle = \Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\beta}}$$

belongs to $L^\times$. Consequently, the class

$$\kappa_\circ = \frac{\log_p(u_1)\log_p(v_{\alpha\otimes\bar\beta}) - \log_p(u_{\alpha\otimes\bar\beta})\log_p(v_1)}{\log_p(u_{\alpha\otimes\bar\beta})^2} \times \Big(\log_p(v_{\alpha\otimes\bar\beta})u - \log_p(u_{\alpha\otimes\bar\beta})v \Big)$$

satisfies $\log^{-+}(\kappa'_p(g_\alpha, g^*_{1/\beta})) = \log^{-+}(\mathrm{res}_p(\kappa_\circ))$.

We may write the cohomology class $\kappa'(g_\alpha, g^*_{1/\beta})$ as a linear combination

$$\kappa'(g_\alpha, g^*_{1/\beta}) = a \cdot u + b \cdot v + c \cdot p,$$

where $a, b, c \in L_p$.

The condition for an element to lie in the kernel of the map $\log^{-+}$ is

$$a \cdot \log_p(u_{\alpha\otimes\bar\beta}) + b \cdot \log_p(v_{\alpha\otimes\bar\beta}) = 0.$$

Hence, we have that

$$\kappa'(g_\alpha, g^*_{1/\beta}) = \kappa_\circ + \lambda(\log_p(v_{\alpha\otimes\bar\beta}) \cdot u - \log_p(u_{\alpha\otimes\bar\beta}) \cdot v) + \mu \cdot p.$$

Consider now the map

$$\log^{--} : H^1(\mathbb{Q}_p, V_{gh}(1)) \xrightarrow{\mathrm{pr}^{--}} H^1(\mathbb{Q}_p, V_{gh}^{--}(1)) \simeq H^1(\mathbb{Q}_p, \mathbb{Q}_p(\alpha/\beta)(1)) \xrightarrow{\log_{\mathrm{BK}}} L_p.$$

According to (3.30), the class $\kappa_p'(g_\alpha, g^*_{1/\beta})$ also lies in the kernel of $\log^{--}$. Hence, taking equalities up to periods,

$$\log^{--}(\kappa_p'(g_\alpha, g^*_{1/\beta})) = \log^{--}(\mathrm{res}_p(\kappa_\circ)) + \lambda(\log_p(v_{\alpha\otimes\bar\beta})\cdot\log_p(u_1) - \log_p(u_{\alpha\otimes\bar\beta})\cdot\log_p(v_1))$$

$$= \lambda\cdot(\log_p(v_{\alpha\otimes\bar\beta})\cdot\log_p(u_1) - \log_p(u_{\alpha\otimes\bar\beta})\cdot\log_p(v_1)) = \lambda\cdot\log_p(u_{\alpha\otimes\bar\beta})\cdot L_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,1) = 0.$$

The assumption $\mathcal{L}(\mathrm{ad}^0(g_\alpha)) \neq 0$ in Theorem B implies that $L_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,1)\neq 0$ by the first display in this section. Hence $\lambda = 0$ and Theorem B follows.

Non-vanishing of the $\mathcal{L}$-invariant

We conclude this section by noting that the non-vanishing of the $\mathcal{L}$-invariant can be proved in most dihedral cases, because the expression (3.55) simplifies considerably. Indeed, let K be a real or imaginary quadratic field of discriminant D and let $\psi : G_K \longrightarrow L^\times$ be a finite order character of conductor $\mathfrak{c}\subset\mathcal{O}_K$ (and of mixed signature at the two archimedean places if K is real). Then the theta series $g = \theta(\psi)$ attached to ψ is an eigenform of weight 1, level $N_g = |D|\cdot\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{c})$ and nebentype $\chi_g = \chi_K\chi_\psi$, where χ_K is the quadratic character associated to $K/\mathbb{Q}$ and χ_ψ is the central character of ψ.

Let ψ' denote the $\mathrm{Gal}\,(K/\mathbb{Q})$-conjugate of ψ defined by the rule $\psi'(\sigma) = \psi(\sigma_0\sigma\sigma_0^{-1})$ for any choice of $\sigma_0\in\Gamma_\mathbb{Q}\setminus G_K$. If $\psi\neq\psi'$ then g is cuspidal.

Let $p\nmid N_g$ be a prime number and fix an embedding $\bar{\mathbb{Q}}\subset\bar{\mathbb{Q}}_p$. In line with the introduction, suppose that hypotheses (H1-H2-H3) are fulfilled. This amounts to asking that

(i) $\psi\neq\psi' \pmod p$, so that g is cuspidal even residually at p.

(ii) If K is imaginary and $p = \wp\bar\wp$ splits in K, then $\psi(\wp)\neq\psi'(\wp)\pmod p$.

(iii) If K is real, p does not split. If K is imaginary and the field $H = \bar K^{\ker(\psi)}$ cut out by ψ has Galois group $\mathrm{Gal}\,(H/\mathbb{Q}) = D_4$, the dihedral group of order 8, then p does not split in the single real quadratic field contained in H.

Proposition 3.5.1. *If K is imaginary with p split, or K is real with p inert, then*

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) = \log_p(v_1) \qquad (\mathrm{mod}\ L^\times). \tag{3.57}$$

In particular $\mathcal{L}(\mathrm{ad}^0(g_\alpha))\neq 0$.

Proof. If K is real (and thus p remains inert in it) then u_1 is the norm of the fundamental unit of K and hence its p-adic logarithm vanishes. If K is imaginary and p splits, then v is a p-unit in $K^\times$ and hence $\log_p(v_{\alpha\otimes\bar\beta}) = 0$. □

Note that Theorem B simplifies considerably in the setting of the above proposition. The simplest scenario where it is not a priori obvious that the $\mathcal{L}$-invariant is non-zero arises when K is imaginary but p remains inert. Fix a prime $\wp$ in H above p and set $\psi_{\mathrm{ad}} = \psi/\psi'$. Note that ψ_{ad} is a ring class character, regardless of whether ψ is so or not. Let $u_{\psi_{\mathrm{ad}}}$ (resp. $v_{\psi_{\mathrm{ad}}}$) denote any element spanning the 1-dimensional L-vector space

$$\mathcal{O}_H^\times[\psi_{\mathrm{ad}}] := \{x\in\mathcal{O}_H^\times\otimes L : \sigma(x) = \psi_{\mathrm{ad}}(\sigma)x, \quad \sigma\in\mathrm{Gal}\,(H/K)\},$$

respectively $\frac{\mathcal{O}_H[1/\wp]^\times[\psi_{\mathrm{ad}}]}{\mathcal{O}_H^\times[\psi_{\mathrm{ad}}]}$. Set also $u'_{\psi_{\mathrm{ad}}} = \mathrm{Fr}_\wp u_{\psi_{\mathrm{ad}}}$ and $v'_{\psi_{\mathrm{ad}}} = \mathrm{Fr}_\wp v_{\psi_{\mathrm{ad}}}$. Then it is a straight-forward computation to check that the $\mathcal{L}$-invariant appearing in Theorems A and A' is

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) = \frac{\log_\wp(u_{\psi_{\mathrm{ad}}})\log_\wp(v'_{\psi_{\mathrm{ad}}}) - \log_\wp(u'_{\psi_{\mathrm{ad}}})\log_\wp(v_{\psi_{\mathrm{ad}}})}{\log_\wp(u_{\psi_{\mathrm{ad}}}) - \log_\wp(u'_{\psi_{\mathrm{ad}}})} \qquad (\mathrm{mod}\ L^\times). \tag{3.58}$$

3.6 Darmon–Dasgupta units and p-adic L-functions

Darmon–Dasgupta units and Gross' conjecture

Let us place ourselves again in the setting of 3.5, where K is real and p remains inert in it. In this scenario Darmon and Dasgupta [DD06] associated to the ring class character ψ_{ad} a local unit $v_{\mathrm{DD}}[\psi_{\mathrm{ad}}] \in K_p^\times$ and conjectured that $v_{\mathrm{DD}}[\psi_{\mathrm{ad}}]$ actually belongs to $\mathcal{O}[1/\wp]^\times[\psi_{\mathrm{ad}}]$. We refer to the introductory chapters for a more exhaustive introduction of the preceding objects. The combination of [Park10, Theorem 4.4] and Darmon–Dasgupta-Pollack's [DDP11, Theorem 2] provides strong evidence for this conjecture, as putting these results together it follows that, in our notations,

$$\log_p(\mathbf{N}_{K_p/\mathbb{Q}_p}(v_{\mathrm{DD}}[\psi_{\mathrm{ad}}])) = \log_p(v_1) \qquad (\mathrm{mod}\ L^\times). \tag{3.59}$$

The above equality together with Theorems B and C yield a formula relating the derived Beilinson–Flach elements of this chapter with Darmon–Dasgupta units, much in the spirit of [BSV20a, Theorem A] and [DR20b, Theorem C] for diagonal cycles versus Stark-Heegner points. Taking into account the decomposition introduced in (3.50), the element $\kappa'(g_\alpha, g^*_{1/\beta})_1$ belongs to $\mathbb{Q}_p^\times \otimes L_p$.

Corollary 3.6.1. *Let $g = \theta(\psi)$ be the theta series associated to a finite order character ψ of mixed signature of a real quadratic field K. Let p be a prime that remains inert in K. Then,*

$$\kappa'(g_\alpha, g^*_{1/\beta})_1 = \log_p(\mathbf{N}_{K_p/\mathbb{Q}_p}(v_{\mathrm{DD}}[\psi_{\mathrm{ad}}])) \cdot \mathbf{N}_{K_p/\mathbb{Q}_p}(v_{\mathrm{DD}}[\psi_{\mathrm{ad}}]) \quad (\mathrm{mod}\ L^\times).$$

In spite of the ostensible parallelism between the above formula and [BSV20a, Theorem A] and [DR20b, Theorem C], note that the proof of Corollary 3.6.1 follows quite a different route from [BSV20a] and [DR20b]. The main reason is that in the latter two references it was crucially exploited a factorization of p-adic L-functions, which follows from a comparison of critical values.

In our setting here one still expects to have an analogous factorization, but proving it appears to be far less trivial. Since this issue poses intriguing questions, and our results shed some light on them, we discuss it in more detail below.

Artin p-adic L-functions

Let $g = \theta(\psi) \in S_1(N, \chi)$ be a theta series of a quadratic field K and p be a prime which is split (resp. inert) if K is imaginary (resp. real). Keep the assumptions of Proposition 3.5.1, and set $h = g^* = \theta(\psi^{-1})$ and $\psi_{\mathrm{ad}} = \psi/\psi'$ as usual.

Let $L_p(g_\alpha, g^*_{1/\beta}, s)$ denote the cyclotomic p-adic Rankin-Hida L-function associated to the pair $(g_\alpha, g^*_{1/\beta})$ of p-stabilizations of g and h, as in 3.2. Note that this p-adic L-function has no critical points.

Nevertheless, since

$$V_g \otimes V_{g^*} \simeq 1 \oplus \chi_K \oplus \mathrm{Ind}_{\mathbb{Q}}^K(\psi_{\mathrm{ad}}), \tag{3.60}$$

one may still wonder whether $L_p(g_\alpha, g^*_{1/\beta}, s)$ admits a factorization mirroring the one satisfied by its classical counterpart:

$$L(g, h, s) = \zeta(s) \cdot L(\chi_K, s) \cdot L(K, \psi_{\mathrm{ad}}, s). \tag{3.61}$$

While (3.61) follows directly from (3.60) by Artin formalism, a putative analogous factorization of $L_p(g_\alpha, g^*_{1/\beta}, s)$ is far less trivial. In the CM case one can prove it by a method which is nowadays standard, but rather deep as one needs firstly to extend $L_p(g_\alpha, g^*_{1/\beta}, s)$ to a two-variable p-adic L-function, prove a factorization in this scenario, and then invoke Gross's theorem [Gro80]. Since we did not find it in the literature and the output is not precisely what one would naïvely expect[2], we provide it below.

As a piece of notation, we say that an element $\mathfrak{f}$ in a finite algebra over $\Lambda^{\otimes n}$ for some $n \geq 0$, is an *L-rational fudge factor* if it is a rational function with coefficients in L which extends to an Iwasawa function with neither poles nor zeros at crystalline classical points. Let also L_p^{Katz} denote Katz's p-adic L-function on the space of Hecke characters of an imaginary quadratic field. If ξ is one such a character of K, we write $L_p^{\mathrm{Katz}}(\xi, s) = L_p^{\mathrm{Katz}}(\xi \cdot \mathbb{N}^s)$ where $\mathbb{N}$ stands for the Hecke character of infinity type $(1,1)$ induced by the norm from K to $\mathbb{Q}$.

Theorem 3.6.2. *Assume K is imaginary and p splits in it. Then there exists an L-rational fudge factor $\mathfrak{f} \in \Lambda$ such that*

$$L_p(g, g^*, s) = \frac{\mathfrak{f}(s)}{\log_p(u_{\psi_{\mathrm{ad}}})} \cdot \zeta_p(s) \cdot L_p(\chi_K \omega, s) \cdot L_p^{\mathrm{Katz}}(\psi_{\mathrm{ad}}, s).$$

Proof. We follow the notations and normalizations adopted in [DLR15a, Section 3] and [DLR16, Section 4]. Fix a prime $\wp$ of K above p. Take a Hecke character λ with image in $\mathbb{Z}_p^\times$ of infinity type $(0,1)$ and conductor $\bar{\wp}$. For every integer $\ell \geq 1$ define $\psi_{g,\ell-1}^{(p)} = \psi_g \langle \lambda \rangle^{\ell-1}$ and let $\psi_{g,\ell-1}$ be the Hecke character given by

$$\psi_{g,\ell-1}(\mathfrak{q}) = \begin{cases} \psi_{g,\ell-1}^{(p)}(\mathfrak{q}) & \text{if } \mathfrak{q} \neq \bar{\wp}, \\ \chi(p) p^{\ell-1} / \psi_{g,\ell-1}^{(p)}(\wp) & \text{if } \mathfrak{q} = \bar{\wp}. \end{cases}$$

As explained in loc. cit. there is a p-adic family $\psi_{\mathbf{g}}$ of Hecke characters whose weight ℓ specialization is $\psi_{g,\ell-1}^{(p)}$ and such that the Hida family $\mathbf{g}$ passing through g_α satisfies $g_\ell^\circ = \theta(\psi_{g,\ell-1})$.

Given a pair of classical weights (ℓ, s), define the Hecke characters

$$\Psi_{gh}(\ell, s) = \psi_{g,\ell-1}^{-1} \cdot \psi_g' \cdot \mathbb{N}^s, \qquad \Psi_{gh'}(\ell, s) = \psi_{g,\ell-1}^{-1} \cdot \psi_g \cdot \mathbb{N}^s, \qquad \Psi_g(\ell) = \psi_{g,\ell-1}^{-2} \chi \mathbb{N}^\ell.$$

All pairs (ℓ, s) such that $\ell > s \geq 1$ belong to the region of interpolation of both Rankin-Hida's p-adic L-function $L_p(\mathbf{g}, g_{1/\beta}^*, s)$ and Katz's p-adic L-functions $L_p^{\mathrm{Katz}}(\psi_{\mathbf{g}h}(\ell, s))$ and $L_p^{\mathrm{Katz}}(\psi_{\mathbf{g}h'}(\ell, s))$. At such critical pairs, it is readily verified that the following factorization of classical L-values occurs up to an L-rational fudge factor:

$$L(g_\ell, h, s) = L(\psi_{gh}(\ell, s)^{-1}, 0) L(\psi_{gh'}(\ell, s)^{-1}, 0).$$

Using this identity, the same computations as in [DLR16, Theorem 4.2] show that there is an L-rational fudge factor $\mathfrak{f}(\ell, s) \in \Lambda^{\otimes 2}$ such that

$$L_p(\mathbf{g}, h)(\ell, s) \cdot L_p^{\mathrm{Katz}}(\psi_g(\ell)) = \mathfrak{f}(\ell, s) \cdot L_p^{\mathrm{Katz}}(\psi_{gh}(\ell, s)) \cdot L_p^{\mathrm{Katz}}(\psi_{gh'}(\ell, s)). \tag{3.62}$$

If we now restrict to $\ell = 1$ and invoke Katz's p-adic analogue of Kronecker limit formula which asserts that $L_p^{\mathrm{Katz}}(\psi) = \log_p(u_{\psi_{\mathrm{ad}}}) \pmod{L^\times}$, it follows that

$$L_p(g, g^*, s) = \frac{\mathfrak{f}(s)}{\log_p(u_{\psi_{\mathrm{ad}}})} \cdot L_p^{\mathrm{Katz}}(\mathbb{N}^s) \cdot L_p^{\mathrm{Katz}}(\psi_{\mathrm{ad}}, s). \tag{3.63}$$

Finally, Gross's main theorem in [Gro80] together with the functional equation for Kubota-Leopoldt's p-adic L-function asserts that

$$L_p^{\mathrm{Katz}}(s) = \zeta_p(s) \cdot L_p(\chi_K \omega, s) \tag{3.64}$$

up to a rational fudge factor. This yields the theorem. $\square$

[2]Indeed, the formula $L_p(g, g^*, s) = \mathfrak{f}(s) \cdot \zeta_p(s) \cdot L_p(\chi_K \omega, s) \cdot L_p^{\mathrm{Katz}}(\psi_{\mathrm{ad}} \cdot \mathbb{N}^s)$ is not correct, in spite of being the direct analogue of (3.61). For one thing, this formula would enter in contradiction with Theorem A.

Assume now that $g = \theta(\psi)$ is the theta series of a character of a real quadratic field in which p is inert. In light of Theorem 3.6.2 it is natural to pose the following question:

Question 3.6.3. Assume K is real and p remains inert in K. Let u_K be a fundamental unit of K and let $L_p(\psi_{\mathrm{ad}}\omega, s)$ denote the Deligne-Ribet p-adic L-function attached to $\psi_{\mathrm{ad}}\omega$. Is it true that

$$L_p(g, g^*, s) \stackrel{?}{=} \frac{1}{\log_p(u_K)} \cdot \zeta_p(s) \cdot L_p(\chi_K, s) \cdot L_p(\psi_{\mathrm{ad}}\omega, s) \tag{3.65}$$

up to an L-rational fudge factor?

Note that the results of this chapter, combined with Darmon–Dasgupta-Pollack's [DDP11, Theorem 2] prove that the above factorization holds when evaluated at $s = 0$ and $s = 1$. Indeed, Theorem A in this setting takes the simple form

$$L_p(g, g^*, 1) = \log_p(v_1) \pmod{L^\times},$$

while [DDP11, Theorem 2] asserts that $L_p(\psi_{\mathrm{ad}}\omega, s)$ vanishes at $s = 0$ and

$$L_p'(\psi_{\mathrm{ad}}\omega, 0) = \log_p(v_1) \pmod{L^\times}.$$

Moreover, $\zeta_p(s)$ has a simple pole at $s = 0$ whose residue is a non-zero rational number, and Leopoldt's formula asserts that $L_p(\chi_K, 0) = \log_p(u_K) \pmod{L^\times}$. Putting all together shows that (3.65) is true at $s = 0$. The functional equations satisfied by each of the p-adic L-functions in play ensure that the same is true at $s = 1$. This of course falls short from establishing (3.65).

Chapter 4

Beilinson–Flach elements, Stark units and p-adic iterated integrals

We study again one specializations of the Euler systems of Beilinson–Flach elements introduced by Kings, Loeffler and Zerbes [KLZ17], with a view towards the main conjecture formulated by Darmon, Lauder and Rotger in [DLR16]. In this framework, we show how the latter conjecture follows from expected properties of Beilinson–Flach elements, and prove the analogue of the main theorem of [CH20] in our setting.

The results presented at this chapter are the content of the research article [RR19], which is a joint work with Victor Rotger.

4.1 Introduction

As we have already pointed out, in the last decade there has been substantial progress in the theory of Euler systems of Garrett–Rankin type associated to triples (f, g, h) of modular forms. This framework includes the original scenario of Kato's Euler system [Ka04] and also encompasses the Euler systems of Beilinson–Flach elements and diagonal cycles. When f is a weight two cusp form, associated say to an elliptic curve $E/\mathbb{Q}$, and (g, h) is a pair of modular forms of weight 1, this approach yielded new results on the Birch and Swinnerton-Dyer conjecture for twists of E by an Artin representation (see [BDR15b], [DR14], [DR17] and [KLZ17]).

These results, together with extensive numerical computations performed with the algorithm [Lau14] of A. Lauder, led to the formulation of the *Elliptic Stark Conjecture* in [DLR15a], relating the value of a p-adic iterated integral (that may be also recast as a special value of a triple-product p-adic L-function at a point lying outside the region of interpolation) to a regulator defined in terms of logarithms of global points on E. The authors of loc. cit. proved their conjecture in the case where g, h are theta series attached to an imaginary quadratic field in which the prime p splits, but the general case remains open.

While no Euler systems are invoked at all in [DLR15a], it was clear that they were behind the scenes, and the connection was made explicit in [DR16], where it was proved how the Elliptic Stark conjecture of [DLR15a] is implied by a precise (but so far unproved at the time of writing this thesis) recipe for the weight $(2, 1, 1)$ specializations of the Euler system of diagonal cycles of [DR17], [DR20b].

There is a parallel story when one replaces the cusp form f and its associated abelian variety with an Eisenstein series and the multiplicative group $\mathbb{G}_m$. The article [DLR16] proposed a conjecture of the same flavor as the Elliptic Stark conjecture of [DLR15a], where the entries of the regulator are p-adic logarithms of Stark units in the number field cut out by the Artin representations associated to g and h. As in loc. cit., this conjecture was proved when g and h are theta series attached to an imaginary quadratic field in which the prime p splits.

One of the aims of the present chapter is to describe the connection between the Euler system of Beilinson–Flach elements and the arithmetic of unit groups of number fields, showing how expected properties of the former imply the main conjecture of [DLR16].

In order to state more precisely our results, let

$$g = \sum_{n\geq 1} a_n q^n \in S_1(N_g, \chi_g), \quad h = \sum_{n\geq 1} b_n q^n \in S_1(N_h, \chi_h)$$

be two normalized newforms, and let V_g and V_h denote the Artin representations attached to them by Serre and Deligne, with coefficients in a finite extension $L/\mathbb{Q}$.

Consider also the tensor product $V_{gh} := V_g \otimes V_h$, and let H be the smallest number field cut out by this representation.

Fix a prime number p which does not divide $N_g N_h$ and label the roots of the p-th Hecke polynomial of g and h as

$$X^2 - a_p(g)X + \chi_g(p) = (X - \alpha_g)(X - \beta_g) \qquad X^2 - a_p(h)X + \chi_h(p) = (X - \alpha_h)(X - \beta_h).$$

Let $g_\alpha(q) = g(q) - \beta_g g(q^p)$ and $h_\alpha(q) = h(q) - \beta_h h(q^p)$ denote the p-stabilization of g (resp. h) on which the Hecke operator U_p acts with eigenvalue α_g (resp. α_h).

Let g^* denote the twist of g by the inverse of its nebentype, i.e., $g^* := g \otimes \chi_g^{-1}$. Note that the U_p-eigenvalues of g^* are $1/\alpha$ and $1/\beta$, and $(g_\alpha)^* = g^*_{1/\beta}$.

By enlarging it if necessary, assume throughout that L contains both the Fourier coefficients of g and h and the roots of their p-th Hecke polynomials. Define

$$U_{gh} = \mathcal{O}_H^\times \otimes L, \quad U_{gh}[1/p] = \mathcal{O}_H[1/p]^\times \otimes L.$$

In order to lighten notation, assume that the prime p splits completely in $L/\mathbb{Q}$, so that L is equipped with an embedding into $\mathbb{Q}_p$, which will be fixed from now on.

We assume throughout that

(H1) The reduction of V_g and V_h mod p are irreducible;

(H2) g and h are both p-distinguished, i.e. $\alpha_g \neq \beta_g \pmod p$ and $\alpha_h \neq \beta_h \pmod p$;

(H3) V_g is not induced from a character of a real quadratic field in which p splits;

(H4) $h_\alpha \neq g^*_{1/\beta}$.

Assumption (H4) splits naturally into two different settings, namely the case where $h \neq g^*$ and the case where $h = g^*$ *and* $\alpha_h = 1/\alpha_g$. Case $h_\alpha = g^*_{1/\beta}$, excluded here, presents remarkable differences and we refer to the previous chapter for a thorough study of this scenario.

The results of [KLZ17] imply that there exists a *Beilinson–Flach class*

$$\kappa(g_\alpha, h_\alpha) \in H^1(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$$

that can be identified, via Kummer theory, with an element of

$$\mathrm{Hom}_{G_\mathbb{Q}}(V_{gh} \otimes \mathbb{Q}_p, U_{gh}[1/p]) = (U_{gh}[1/p] \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}.$$

With a slight abuse of notation, we shall still denote $\kappa(g_\alpha, h_\alpha)$ the projection of the cohomology class to the space $(U_{gh}[1/p]/p^\mathbb{Z} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$. Note that there also exist three other classes $\kappa(g_\alpha, h_\beta)$, $\kappa(g_\beta, h_\alpha)$ and $\kappa(g_\beta, h_\beta)$ attached to the different p-stabilizations of g and h.

As in [DLR16], we impose throughout the following:

Assumption 4.1.1. $\dim_L (U_{gh}[1/p]/p^\mathbb{Z} \otimes V_{gh}^\vee)^{G_\mathbb{Q}} = 2.$

When $h \neq g^*$ this amounts to asking that none of the Frobenius eigenvalues of V_{gh} is equal to 1, that is to say:

$$\alpha_g\alpha_h,\ \alpha_g\beta_h,\ \beta_g\alpha_h,\ \beta_g\beta_h \neq 1.$$

Under this assumption, [DLR16, Lemma 1.1] also implies that

$$\dim_L(U_{gh} \otimes V_{gh}^\vee)^{G_\mathbb{Q}} = \dim_L(U_{gh}[1/p] \otimes V_{gh}^\vee)^{G_\mathbb{Q}} = 2.$$

When $h_\alpha = g^*_{1/\alpha}$, the regularity assumption (H2) directly grants Assumption 4.1.1 and we have

$$\dim_L(U_{gh} \otimes V_{gh}^\vee)^{G_\mathbb{Q}} = 1, \quad \dim_L(U_{gh}[1/p] \otimes V_{gh}^\vee)^{G_\mathbb{Q}} = 3.$$

In either case, fix elements $\{u, v\}$ of $(U_{gh}[1/p] \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$ such that they project to a basis of the two-dimensional space $(U_{gh}[1/p]/p^\mathbb{Z} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$. When $h_\alpha = g^*_{1/\alpha}$ we impose the additional condition that u spans the line $(U_{gh} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$.

Fix a prime ideal $\wp$ of H lying above p, thus determining an embedding $H \subset H_p \subset \bar{\mathbb{Q}}_p$ of H into its completion H_p at $\wp$, and an arithmetic Frobenius $\mathrm{Fr}_p \in \mathrm{Gal}\,(H_p/\mathbb{Q}_p)$. Thanks to (H2), the $\mathrm{Gal}\,(H_p/\mathbb{Q}_p)$-modules V_g, V_h decompose as

$$V_g := V_g^\alpha \oplus V_g^\beta, \quad V_h := V_h^\alpha \oplus V_h^\beta,$$

where Fr_p acts on V_g^α with eigenvalue α_g, and similarly for the remaining summands. The tensor product V_{gh} decomposes then as $G_{\mathbb{Q}_p}$-module as the direct sum of four different lines $V_{gh}^{\alpha\otimes\alpha} := V_g^{\alpha_g} \otimes V_h^{\alpha_h}$, ..., $V_{gh}^{\beta\otimes\beta}$. After choosing a basis, we may write this decomposition as

$$V_{gh} = L \cdot e_{\alpha\alpha} \oplus L \cdot e_{\alpha\beta} \oplus L \cdot e_{\beta\alpha} \oplus L \cdot e_{\beta\alpha},$$

where

$$\mathrm{Fr}_p(e_{\lambda\mu}) = \lambda\mu \cdot e_{\lambda\mu}, \quad \text{for any } \lambda \in \{\alpha_g, \beta_g\}, \mu \in \{\alpha_h, \beta_h\}.$$

We denote by $\{e_{\alpha\alpha}^\vee, e_{\alpha\beta}^\vee, e_{\beta\alpha}^\vee, e_{\beta\beta}^\vee\}$ the dual basis of $V_{gh}^\vee = \mathrm{Hom}(V_{gh}, L)$, where

$$\mathrm{Fr}_p(e_{\alpha\alpha}^\vee) = \chi_{gh}^{-1}(p)\beta_g\beta_h \cdot e_{\alpha\alpha}^\vee\,, \,\ldots\,, \,\mathrm{Fr}_p(e_{\beta\beta}^\vee) = \chi_{gh}^{-1}(p)\alpha_g\alpha_h \cdot e_{\beta\beta}^\vee.$$

Restriction to the decomposition group at p allows us to regard u and v as elements in $H^1(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1)) = (H_p^\times \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_{\mathbb{Q}_p}}$, and as such we may decompose u and v as

$$\begin{aligned} u &= u_{\beta\beta} \otimes e_{\alpha\alpha}^\vee + u_{\beta\alpha} \otimes e_{\alpha\beta}^\vee + u_{\alpha\beta} \otimes e_{\beta\alpha}^\vee + u_{\alpha\alpha} \otimes e_{\beta\beta}^\vee, \\ v &= v_{\beta\beta} \otimes e_{\alpha\alpha}^\vee + v_{\beta\alpha} \otimes e_{\alpha\beta}^\vee + v_{\alpha\beta} \otimes e_{\beta\alpha}^\vee + v_{\alpha\alpha} \otimes e_{\beta\beta}^\vee, \end{aligned} \tag{4.1}$$

where $u_{\alpha\alpha} \in H_p^\times \otimes \mathbb{Q}_p$ satisfies $\mathrm{Fr}_p(u_{\alpha\alpha}) = \beta_g\beta_h \cdot u_{\alpha\alpha}$ and similarly for the other terms.

Define the regulator

$$\mathrm{Reg}_{g_\alpha}(V_{gh}) = \log_p(u_{\alpha\alpha}) \cdot \log_p(v_{\alpha\beta}) - \log_p(u_{\alpha\beta}) \cdot \log_p(v_{\alpha\alpha}). \tag{4.2}$$

In the body of the chapter we introduce a p-adic avatar of the second derivative of $L(g \otimes h, s)$ at $s = 1$, denoted by $\mathscr{L}_p{}^{g_\alpha}(g, h)$ and which can be defined in terms of special values of the Hida–Rankin p-adic L-function; alternatively, it can also be recast as a Coleman p-adic *iterated integral*. The non-vanishing of $\mathscr{L}_p{}^{g_\alpha}(g, h)$ implies the non-vanishing of $\mathrm{Reg}_{g_\alpha}(V_{gh})$.

The following result is proved in Section 4.3. Recall that we are identifying cohomology classes with their projection to the space $(U_{gh}[1/p]/p^\mathbb{Z} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$.

Theorem 4.1.2. *We have*

$$\kappa(g_\alpha, h_\alpha) = \Omega \cdot \Big(\log_p(v_{\alpha\alpha}) \cdot u - \log_p(u_{\alpha\alpha}) \cdot v \Big) \tag{4.3}$$

for some $\Omega \in H_p$. Moreover, if $\mathscr{L}_p{}^{g_\alpha} \neq 0$, then $\kappa(g_\alpha, h_\alpha) \neq 0$. In this case, if we additionally impose that $h \neq g^*$, the two cohomology classes

$$\kappa(g_\alpha, h_\alpha), \quad \kappa(g_\alpha, h_\beta)$$

span the whole group $(U_{gh} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$.

The computations of [DLR16] suggest the following refinement of Theorem 4.1.2, which is discussed in Section 4.3. Let u_{g_α} be the Stark unit attached to g_α, as defined in [DLR15a]. The choice of a basis for V_{gh} determines elements

$$\Xi_{g_\alpha} \in H_p^{\mathrm{Fr}_p = \beta_g^{-1}}, \qquad \Omega_{h_\alpha} \in H_p^{\mathrm{Fr}_p = \alpha_h^{-1}},$$

which are properly defined in [DR16, equation (8)] and which we later recall.

Conjecture 4.1.3. *The Beilinson–Flach class $\kappa(g_\alpha, h_\alpha)$ satisfies*

$$\kappa(g_\alpha, h_\alpha) = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{1}{\log_p(u_{g_\alpha})} \cdot \Big(\log_p(v_{\alpha\alpha}) \cdot u - \log_p(u_{\alpha\alpha}) \cdot v \Big) \pmod{L^\times}. \tag{4.4}$$

Theorem 4.1.4. *Conjecture A implies the main conjecture of [DLR16]. If we further assume that* $\mathrm{Reg}_{g_\alpha}(V_{gh}) \neq 0$*, the converse also holds.*

As an application of our results, we are able to prove the analogue of the main theorem of Castella and Hsieh [CH20, Theorem 1] in the setting of units in number fields and Beilinson–Flach elements. Assume now that $h_\alpha = g^*_{1/\alpha}$ and recall the four global cohomology classes

$$\kappa(g_\alpha, g^*_{1/\alpha}),\ \kappa(g_\beta, g^*_{1/\beta}),\ \kappa(g_\alpha, g^*_{1/\beta}),\ \kappa(g_\beta, g^*_{1/\alpha}) \in H^1(\mathbb{Q}, V_{gg^*} \otimes \mathbb{Q}_p(1)) \tag{4.5}$$

arising from the various p-stabilizations of g and g^*.

Again, since these classes are unramified at primes $\ell \neq p$, they belong to the subspace which is identified with $(U_{gg^*}[1/p] \otimes V_{gg^*}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$ under the Kummer map.

It follows from Proposition 3.3.12 that $\kappa(g_\alpha, g^*_{1/\beta}) = \kappa(g_\beta, g^*_{1/\alpha}) = 0$. It is thus natural to wonder whether one can determine the remaining two classes $\kappa(g_\alpha, g^*_{1/\alpha})$ and $\kappa(g_\beta, g^*_{1/\beta})$.

Theorem 4.1.5. *Assume that* $\mathrm{Reg}_{g_\alpha}(V_{gg^*}) \neq 0$*. Then, $\kappa(g_\alpha, g^*_{1/\alpha})$ and $\kappa(g_\beta, g^*_{1/\beta})$ are non-zero and Conjecture A holds for them. Moreover,*

$$\langle \kappa(g_\alpha, g^*_{1/\alpha}) \rangle = \langle \kappa(g_\beta, g^*_{1/\beta}) \rangle.$$

Remark 4.1.6. When g is the theta series attached to an imaginary quadratic field where p splits or to a real quadratic field where p remains inert, we prove that $\mathrm{Reg}_{g_\alpha}(V_{gg^*}) \neq 0$ and hence the above statement holds unconditionally.

The organization of the chapter is as follows. In Section 2, we recover the formulation of the main conjecture of [DLR16], both in terms of iterated integrals and of the Hida–Rankin p-adic L-function. Section 3 is devoted to prove Theorems 4.1.2 and 4.1.4, exploring some properties of Beilinson–Flach classes. Section 4 provides the proof of the analogue of the main theorem of Castella and Hsieh in the setting of Beilinson–Flach elements. Finally, Section 5 analyzes some particular cases where the representation V_{gh} is reducible, in connection with the more classical Euler systems of circular and elliptic units.

4.2 The main conjecture of Darmon, Lauder and Rotger

The aim of this section is to recall briefly the Elliptic Stark conjecture formulated in [DLR16] for units in number fields. We keep the same notations and assumptions of the introduction. Throughout this section we further assume

$$h \neq g^*,$$

which in particular implies, as recalled above, that

$$\alpha_g\alpha_h, \alpha_g\beta_h, \beta_g\alpha_h, \beta_g\beta_h \neq 1. \tag{4.6}$$

We leave the self-dual case $h_\alpha = g^*_{1/\alpha}$ for Section 4.4.

Let

$$f := E_2(1, \chi_{gh}^{-1}) \in M_2(N, \chi_{gh}^{-1})$$

be the weight two Eisenstein series for the character χ_{gh}^{-1}, and consider also

$$F := d^{-1}f = E_0^{[p]}(\chi_{gh}^{-1}, 1),$$

the overconvergent Eisenstein series of weight zero attached to the pair $(\chi_{gh}^{-1}, 1)$ of Dirichlet characters.

As shown in [DLR15a], the above hypothesis ensures that any generalized overconvergent modular form associated to g_α is simply a multiple of g_α. We denote by e_{ord} Hida's ordinary projection on the space of overconvergent modular forms of weight one and by $e_{g_\alpha^*}$ the Hecke equivariant projection to the generalized eigenspace attached to the system of Hecke eigenvalues for the dual form g_α^* of g_α.

We attach to g_α a two-dimensional subspace of the representation V_{gh}, namely

$$V_{gh}^\beta := V_g^\beta \otimes V_h.$$

Remark 4.2.1. In the Eisenstein case the conjecture also makes sense, as it is emphasized in [DLR16]. In this case, if $g = E_1(\chi^+, \chi^-)$, the classicality assumption asserts that $\chi^+(p) = \chi^-(p)$ and the role of the two dimensional space V_{gh}^β is played by $W \otimes V_h$, where W is any line in V_g which is not stable under $G_{\mathbb{Q}}$.

Recall the unit u_{g_α} in $\mathcal{O}_H^\times \otimes L$ attached to the p-stabilized eigenform g_α, as it is defined in [DLR15a, 1.2]. It belongs to the $\mathrm{ad}^0(g)$-isotypic part of $\mathcal{O}_H^\times \otimes L$ and is an eigenvector for Fr_p with eigenvalue β_g/α_g. For a Dirichlet character of conductor m, we can consider

$$\mathfrak{g}(\chi) := \sum_{a=1}^{m} \chi^{-1}(a)e^{2\pi i a/m}, \tag{4.7}$$

the usual Gauss sum, on which $G_{\mathbb{Q}}$ acts through χ and thus Fr_p acts with eigenvalue $\chi_{gh}(p)$.

Since g_α is new at level Np, the L-dual space $(S_1(Np, \chi_g^{-1})_L^\vee[g_\alpha^*])^\vee$ is one-dimensional and we may fix a basis, say γ_α. As before, we consider a regulator attached to V_{gh}^β given in terms of the p-units u and v, defined in the introduction of the chapter and which admits a Frobenius decomposition as in (4.1):

$$\mathrm{Reg}_{g_\alpha}(V_{gh}) = \log_p(u_{\alpha\alpha}) \cdot \log_p(v_{\alpha\beta}) - \log_p(u_{\alpha\beta}) \cdot \log_p(v_{\alpha\alpha}).$$

The following question is the main conjecture of [DLR16].

Conjecture 4.2.2. *It holds that*

$$\gamma_\alpha(e_{g_\alpha^*} e_{\mathrm{ord}}(Fh)) = \frac{\mathrm{Reg}_{g_\alpha}(V_{gh})}{\mathfrak{g}(\chi_{gh}) \cdot \log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{4.8}$$

We can reformulate the previous conjecture in the language of special values of p-adic L-functions. Let $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$ and $\mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$ be Hida families through g_α and h_α with coefficients in finite flat extensions $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, respectively. Write $\mathcal{W} = \mathrm{Spf}(\Lambda)$, $\mathcal{W}_{\mathbf{g}} = \mathrm{Spf}(\Lambda_{\mathbf{g}})$ and $\mathcal{W}_{\mathbf{h}} = \mathrm{Spf}(\Lambda_{\mathbf{h}})$ for the associated weight spaces. Let y_0 and z_0 be weight one points of $\mathcal{W}_{\mathbf{g}}$ and $\mathcal{W}_{\mathbf{h}}$ such that $\mathbf{g}_{y_0} = g_\alpha$ and $\mathbf{h}_{z_0} = h_\alpha$. Associated to the two cuspidal Hida families $\mathbf{g}$ and $\mathbf{h}$, Hida constructed in [Hi85] and [Hi88] a three-variable p-adic Rankin L-function $L_p(\mathbf{g}, \mathbf{h})$ on $\mathcal{W}_{\mathbf{gh}} := \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}} \times \mathcal{W}$ interpolating the algebraic parts of the critical values $L(g_y, h_z, s)$. See the previous chapter for more details on the notations and normalizations we adopt. The next result follows from [DLR16, Lemma 4.2].

Proposition 4.2.3. *Up to multiplication by a scalar in $L^\times$, we have*

$$L_p(\mathbf{g}, \mathbf{h})(y_0, z_0, 1) = \mathfrak{g}(\chi_{gh}) \times \gamma_\alpha(e_{g_\alpha^*} e_{\mathrm{ord}}(Fh)). \tag{4.9}$$

Hence, as pointed out already in [DLR16], the above conjecture may be recast as

$$L_p(\mathbf{g}, \mathbf{h})(y_0, z_0, 1) \stackrel{?}{=} \frac{\mathrm{Reg}_{g_\alpha}(V_{gh})}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{4.10}$$

4.3 Beilinson–Flach elements and the main conjecture

The Euler system of Beilinson–Flach elements

We begin this section with a quick review of the main results of [KLZ17], which are crucially used to study the conjecture we have discussed along the previous section. We also refer the reader to the previous chapter for a expanded description of the results of [KLZ17] with the same notations and normalizations adopted here. The purpose of this digression is to make this chapter more self-contained.

Let

$$\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]], \quad \mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$$

be Hida families of tame level N and tame characters $\chi_{\mathbf{g}}$ and $\chi_{\mathbf{h}}$ respectively, and let $\Lambda_{\mathbf{gh}} := \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}} \hat{\otimes} \Lambda$. Let also $\underline{\varepsilon}_{\mathrm{cyc}}$ denote the Λ-adic cyclotomic character. Let $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{h}}$ stand for the Λ-adic representations attached to $\mathbf{g}$ and $\mathbf{h}$, respectively, endowed with the filtration

$$0 \to \mathbb{V}_{\mathbf{g}}^+ \to \mathbb{V}_{\mathbf{g}} \to \mathbb{V}_{\mathbf{g}}^- \to 0$$

described in [DR16, Section 2], and similarly for $\mathbf{h}$. We also consider the canonical differentials $\eta_{\mathbf{g}}$ and $\omega_{\mathbf{g}}$ as introduced in [Oh00] and [KLZ17, Section 10.1], and denote by η_{g_y}, ω_{g_y} the corresponding specializations at weight y. As it has been extensively discussed in loc. cit. and recalled in the introductory sections of the previous chapter, this induces homomorphisms of $\Lambda_{\mathbf{gh}}$-modules given by the pairings with these differentials; these pairings are denoted as $\langle \cdot, \cdot \rangle$. Finally, let α_g and β_g stand for the roots of the p-th Hecke polynomial of g, ordered in such a way that $\mathrm{ord}_p(\alpha_g) \leq \mathrm{ord}_p(\beta_g)$. We also consider the same objects for the family $\mathbf{h}$.

We say that a weight $y \in \mathcal{W}_{\mathbf{g}}$ is crystalline when there exists an eigenform g_y° of level N such that g_y is the ordinary p-stabilization of g_y°. We denote by $\mathcal{W}_{\mathbf{g}}^\circ$ (resp. $\mathcal{W}_{\mathbf{h}}^\circ$, $\mathcal{W}_{\mathbf{gh}}^\circ$) the set of crystalline points of $\mathcal{W}_{\mathbf{g}}$ (resp. $\mathcal{W}_{\mathbf{h}}$, $\mathcal{W}_{\mathbf{gh}}$). A point in the latter space is identified with a triple (y, z, s), where the weights are referred to as (ℓ, m, s). Note that for a matter of simplicity we are just assuming that the points corresponding to the third variable have trivial nebentype.

The following result recovers the existence of a Perrin-Riou map (also referred in the literature as *big regulator*) interpolating both the Bloch–Kato logarithm and the dual exponential map. Although it had already appeared in the previous chapter, we recall it here in the form it will be used in forthcoming sections.

Proposition 4.3.1. *There exists an injective homomorphism of $\Lambda_{\mathbf{gh}}$-modules*

$$\mathcal{L}_{\mathbf{gh}}^{-+} : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^{-}\hat{\otimes}\mathbb{V}_{\mathbf{h}}^{+}\hat{\otimes}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})) \to \Lambda_{\mathbf{gh}}$$

satisfying the following interpolation property: for every $(y,z,s) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$, set $g := g_y^{\circ}$ and $h = h_z^{\circ}$. Then, the specialization of $\mathcal{L}_{\mathbf{gh}}^{-+}$ at (y,z,s) is the homomorphism

$$\mathcal{L}_{\mathbf{gh}}^{-+}(y,z,s) : H^1(\mathbb{Q}_p, V_g^{-} \otimes V_h^{+}(1-s)) \to \mathbb{C}_p$$

given by

$$\mathcal{L}_{\mathbf{gh}}^{-+}(y,z,s) = \frac{(1-p^{s-1}\alpha_g^{-1}\beta_h^{-1})(1-\alpha_h^{-1}\beta_h)}{(1-p^{-s}\alpha_g\beta_h)(1-p^{-1}\alpha_g^{-1}\beta_g)} \times \begin{cases} \frac{(-1)^{m-s-1}}{(m-s-1)!} \times \langle \log_{\mathrm{BK}}, \eta_g \otimes \omega_h \rangle & \text{if } s < m \\ (s-m)! \times \langle \exp_{\mathrm{BK}}^{*}, \eta_g \otimes \omega_h \rangle & \text{if } s \geq m, \end{cases}$$

where $\log_{\mathrm{BK}}$ is the Bloch-Kato logarithm and $\exp_{\mathrm{BK}}^{*}$, the dual exponential map (see [BK93] for proper definitions of these morphisms).

Proof. This follows from [KLZ17, Theorem 8.2.8, Proposition 10.1.1] and the relations

$$\eta_{g_y} = \Big(1 - \frac{\beta_{g_y^{\circ}}}{\alpha_{g_y^{\circ}}}\Big)\eta_{g_y^{\circ}}, \quad \omega_{h_z} = \Big(1 - \frac{\beta_{h_z^{\circ}}}{\alpha_{h_z^{\circ}}}\Big)\omega_{h_z}^{\circ}.$$

□

We can now formulate the main results of [KLZ17], which assert that there exists a family of cohomology classes indexed by points of $\mathcal{W}_{\mathbf{gh}}$ and whose image under the previous Perrin-Riou map agrees with the Hida–Rankin p-adic L-function.

Theorem 4.3.2. *Fix an integer $c > 1$ relatively prime to $6pN$. Then, there exists a global cohomology class*

$${}_c\kappa(\mathbf{g},\mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{g}}\hat{\otimes}\mathbb{V}_{\mathbf{h}}\hat{\otimes}\Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$$

such that:

1. *The projection of the local class $\mathrm{res}_p({}_c\kappa(\mathbf{g},\mathbf{h}))$ to $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^{-}\hat{\otimes}\mathbb{V}_{\mathbf{h}}(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$ lands in*
$$H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^{-}\hat{\otimes}\mathbb{V}_{\mathbf{h}}^{+}(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})).$$
2. *Letting ${}_c\kappa_p^{-+}(\mathbf{g},\mathbf{h})$ denote the local cohomology class in the above space, we have*
$$\mathcal{L}_{\mathbf{gh}}^{-+}({}_c\kappa_p^{-+}(\mathbf{g},\mathbf{h})) = \frac{(-1)^s}{\lambda_{\mathbf{g}}} \cdot (c^2 - c^{2s-\ell-m+2}) \times L_p(\mathbf{g},\mathbf{h}),$$
where $\lambda_{\mathbf{g}}$ denotes the pseudo-eigenvalue of $\mathbf{g}$, an Iwasawa function interpolating the pseudo-eigenvalue at N of the crystalline classical specializations of $\mathbf{g}$.

Proof. The global cohomology class ${}_c\kappa(\mathbf{g},\mathbf{h})$ is introduced in [KLZ17, Definition 8.1.1]. The first part of the result is just [KLZ17, Proposition 8.1.7], while the second part is Theorem B of [KLZ17]. □

Since c is fixed throughout, we may sometimes drop it from the notation. The constant does make an appearance in fudge factors accounting for the interpolation properties satisfied by the Euler system, but in the case we are interested in these fudge factors do not vanish. We typically refer to this class as the *Beilinson–Flach class* or the *Beilinson–Flach element attached to* $\mathbf{g}$ *and* $\mathbf{h}$.

An explicit description of the cohomology classes

From now on, we retain the setting of Section 2, where $g \in S_1(N, \chi_g)$ and $h \in S_1(N, \chi_h)$ are two cuspidal eigenforms satisfying hypotheses (H1)-(H3) and $h \neq g^*$. Let L_p denote the completion of L in $\bar{\mathbb{Q}}_p$ under the embedding $L \subset \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ fixed at the outset. Under our running hypothesis, we actually have $L_p = \mathbb{Q}_p$, although recall this was only assumed for simplicity of exposition.

Definition 4.3.3. Let $\mathbf{g}$ and $\mathbf{h}$ be Hida families passing through p-stabilizations g_α, h_α of g, h at some point $(y_0, z_0) \in \mathcal{W}_{\mathbf{g}}^\circ \times \mathcal{W}_{\mathbf{h}}^\circ$ of weights $(1, 1)$. Define

$$\kappa(g_\alpha, h_\alpha) := \kappa(\mathbf{g}, \mathbf{h})(y_0, z_0, 0) \in H^1(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$$

as the specialization of $\kappa(\mathbf{g}, \mathbf{h})$ at the point $(y_0, z_0, 0)$.

This procedure yields four a priori different global cohomology classes:

$$\kappa(g_\alpha, h_\alpha), \quad \kappa(g_\alpha, h_\beta), \quad \kappa(g_\beta, h_\alpha), \quad \kappa(g_\beta, h_\beta), \tag{4.11}$$

one for each choice of pair of roots of the p-th Hecke polynomials of g and h.

Let $H^1_{\mathrm{f}}(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$ denote the *finite* Bloch-Kato Selmer group, which is the subspace of $H^1(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$ which consists on those classes which are crystalline at p and unramifed at all $\ell \neq p$

Proposition 4.3.4. *The cohomology classes in* (4.11) *belong in fact to* $H^1_{\mathrm{f}}(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$.

Proof. The two cohomology spaces $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1))$ and $H^1(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1))$ are equal according to the discussion in [DR20b, Section 1.4] combined with the results of [Bel09, Proposition 2.8 and Exercise 2.21]. Then, the restrictions to $\mathbb{Q}_\ell$, for $\ell \neq p$, are unramified because of the results established in [Nek98, Section 2.4]. □

By standard results in Kummer theory (see for example [Bel09, Prop.2.12]), there exists an isomorphism between $H^1_{\mathrm{f}}(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$ and $(U_{gh} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$.

As we have already mentioned, the units $u, v \in (U_{gh} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$ introduced in the introduction can be also identified with elements in $\mathrm{Hom}(V_{gh} \otimes \mathbb{Q}_p, U_{gh})^{G_\mathbb{Q}}$. In the lemma below, we regard the local class $\kappa_p(g_\alpha, h_\alpha)$ as a homomorphism in $\mathrm{Hom}(V_{gh} \otimes \mathbb{Q}_p, H_p^\times \otimes L)^{G_\mathbb{Q}}$.

Lemma 4.3.5. $\kappa_p(g_\alpha, h_\alpha)(e_{\beta\beta}) = 0$.

Proof. This follows after specializing the content of [KLZ17, Proposition 8.1.7] (also rephrased here in the first part of Theorem 4.3.2), at the point $(y_0, z_0, 0)$. It asserts that the component of the Beilinson–Flach class corresponding to the projection in the quotient $\mathbb{V}_{\mathbf{g}}^- \hat{\otimes} \mathbb{V}_{\mathbf{h}}^-$ vanishes. Combining the natural dualities with the above referred identifications between the spaces of homomorphisms and the cohomology groups, the result follows. □

Consider again the special p-adic L-value $\mathscr{L}_p{}^{g_\alpha} = L_p(\mathbf{g}, \mathbf{h})(y_0, z_0, 1)$. Recall that this value is the same if $\mathbf{h}$ is chosen to be the Hida family through h_β. This value can be understood as a p-adic avatar of the second derivative of the classical Hida–Rankin L-function, because of the following result.

Proposition 4.3.6. *The order of vanishing of* $L(V_{gh}, s)$ *at* $s = 1$ *is two.*

Proof. According to [Das99, Section 3.2], we know that

$$\mathrm{ord}_{s=0} L(V_{gh}, s) = 2 - \dim_L (V_{gh})^{G_\mathbb{Q}},$$

and the order of vanishing at $s = 1$ can be derived via a functional equation relating the values at $s = 0$ and $s = 1$, where some gamma factors arise.

Besides, the assumptions we have fixed imply that $\dim_L (V_{gh})^{G_\mathbb{Q}} = 0$, and since the functional equation introduces no extra zero or pole at $s = 1$ (see [Das16]), we conclude that the order of vanishing at $s = 1$ is also 2. □

The following result was stated in the introduction as Theorem 4.1.2.

Theorem 4.3.7. *There exists a period* $\Omega \in H_p$ *such that*

$$\kappa(g_\alpha, h_\alpha) = \Omega \cdot (\log_p(v_{\alpha\alpha}) \cdot u - \log_p(u_{\alpha\alpha}) \cdot v).$$

Moreover, if $\mathscr{L}_p{}^{g_\alpha} \neq 0$*, then* $\kappa(g_\alpha, h_\alpha) \neq 0$ *and the two global classes*

$$\kappa(g_\alpha, h_\alpha),\ \kappa(g_\alpha, h_\beta)$$

are linearly independent in the Selmer group $H^1(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$*.*

Proof. The running assumptions imply that $\alpha_g \alpha_h \neq 1$ and then $H^1(\mathbb{Q}_p, V_{gh}^{\alpha\alpha} \otimes \mathbb{Q}_p(1))$ is one-dimensional. Observe that $V_{gh}^{\alpha\alpha} \simeq \mathbb{Q}_p(\alpha_g\alpha_h)$ as $G_{\mathbb{Q}_p}$-modules, where the latter stands for the unramified character sending Fr_p to $\alpha_g\alpha_h$. Hence, the Bloch-Kato logarithm associated to this p-adic representation gives rise to an isomorphism

$$\log_{\mathrm{BK}} : H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh}^{\alpha\alpha} \otimes \mathbb{Q}_p(1)) \xrightarrow{\sim} H_p.$$

Since $\{u, v\}$ forms a basis of $H^1(\mathbb{Q}, V_{gh} \otimes \mathbb{Q}_p(1))$, we may write

$$\kappa(g_\alpha, h_\alpha) = \lambda u + \mu v,$$

with $\lambda, \mu \in \mathbb{Q}_p$. The preceding lemma implies that

$$0 = \lambda \cdot u_{\alpha\alpha} \otimes e_{\beta\beta}^\vee + \mu \cdot v_{\alpha\alpha} \otimes e_{\beta\beta}^\vee,$$

and taking logarithms, we conclude that (λ, μ) is a scalar multiple of $(\log_p(v_{\alpha\alpha}), -\log_p(u_{\alpha\alpha}))$. In particular,

$$\kappa(g_\alpha, h_\alpha) = \Omega \cdot (\log_p(v_{\alpha\alpha}) \cdot u - \log_p(u_{\alpha\alpha}) \cdot v),$$

for some $\Omega \in H_p$.

For the second part of the statement, observe that since $\kappa_p(g_\alpha, h_\alpha)$ and $\kappa_p(g_\alpha, h_\beta)$ may be regarded as elements in $\mathrm{Hom}(V_{gh} \otimes \mathbb{Q}_p, U_{gh})^{G_{\mathbb{Q}}}$, it suffices to prove that the action over two different vectors of V_{gh} gives rise to an invertible matrix. By Lemma 4.3.5 we have

$$\begin{pmatrix} \kappa_p(g_\alpha, h_\alpha)(e_{\beta\beta}) & \kappa_p(g_\alpha, h_\alpha)(e_{\beta\alpha}) \\ \kappa_p(g_\alpha, h_\beta)(e_{\beta\beta}) & \kappa_p(g_\alpha, h_\beta)(e_{\beta\alpha}) \end{pmatrix} = \begin{pmatrix} 0 & ? \\ ? & 0 \end{pmatrix}$$

and hence we must show that the two entries off the diagonal are non-zero. Combining the injectivity of the Perrin-Riou map introduced in Proposition 4.3.1 with the reciprocity law for the Beilinson–Flach classes as recalled in the second part of Theorem 4.3.2, this is equivalent to the non-vanishing of $\mathscr{L}_p{}^{g_\alpha}(g, h)$, as claimed. □

Remark 4.3.8. The assumption $h \neq g^*$ is necessary, since we need to guarantee that the Euler-like factors at p appearing in the description of the Perrin-Riou map of [KLZ17, Theorem 10.2.2] do not vanish. In fact, when $h_\alpha = g^*_{1/\beta}$ we have $\kappa(g_\alpha, g^*_{1/\beta}) = 0$ (see the previous chapter). Similarly, if $h_\alpha = g^*_{1/\alpha}$ then $\kappa(g_\alpha, g^*_{1/\alpha}) = 0$.

A conjecture in terms of Beilinson–Flach elements

We now come back to the main question of [DLR16].

Fix eigenbasis $\{e_g^\alpha, e_g^\beta\}$ and $\{e_h^\alpha, e_h^\beta\}$ of V_g and V_h respectively, which are compatible with the choice of the basis for V_{gh}, i.e.,

$$e_{\alpha\alpha} = e_g^\alpha \otimes e_h^\alpha, \quad e_{\alpha\beta} = e_g^\alpha \otimes e_h^\beta, \quad e_{\beta\alpha} = e_g^\beta \otimes e_h^\alpha, \quad e_{\beta\beta} = e_g^\beta \otimes e_h^\beta.$$

As before, let $\eta_{g_\alpha} \in (H_p \otimes V_g^\beta)^{G_{\mathbb{Q}_p}}$ and $\omega_{h_\alpha} \in (H_p \otimes V_h^\alpha)^{G_{\mathbb{Q}_p}}$ stand for the canonical periods arising as the weight one specialization of the Λ-adic periods $\eta_{\mathbf{g}}$ and $\omega_{\mathbf{h}}$.

We can now follow [DR16] and define p-adic periods $\Xi_{g_\alpha} \in H_p^{\mathrm{Fr}_p=\beta_g^{-1}}$ and $\Omega_{h_\alpha} \in H_p^{\mathrm{Fr}_p=\alpha_h^{-1}}$ by setting

$$\Xi_{g_\alpha} \otimes e_g^\beta = \eta_{g_\alpha}, \qquad \Omega_{h_\alpha} \otimes e_h^\alpha = \omega_{h_\alpha}.$$

Hence, we have

$$\Xi_{g_\alpha} \cdot \Omega_{h_\alpha} \otimes e_{\beta\alpha} = \eta_{g_\alpha} \otimes \omega_{h_\alpha}.$$

We now apply the Perrin-Riou big logarithm described in Proposition 4.3.1 to the *local* cohomology class $\kappa_p(g_\alpha, h_\alpha) \in H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1)) \simeq (H_p^\times \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_{\mathbb{Q}_p}}$.

Indeed, let

$$\log^{-+} : H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1)) \xrightarrow{\mathrm{pr}^{-+}} H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh}^{\alpha\beta} \otimes \mathbb{Q}_p(1)) \xrightarrow{\mathcal{L}^{-+}} \mathbb{Q}_p,$$

where here $\mathcal{L}^{-+}$ must be understood as the composition of the map of [KLZ17, Theorem 8.2.8] specialized at $(y_0, z_0, 0)$ with the pairing with the differentials $\eta_{g_\alpha} \otimes \omega_{h_\alpha}$.

Under the identification of $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh}^{\alpha\beta} \otimes \mathbb{Q}_p(1)) \simeq H_p^\times \otimes e_{\beta\alpha}^\vee$, the map $\mathcal{L}^{-+}$ corresponds to the usual p-adic logarithm in $H_p^\times$, followed by the pairing with $\Xi_{g_\alpha}\Omega_{h_\alpha} \otimes e_{\beta\alpha}$. Alternatively, and via the identification of $H^1_{\mathrm{f}}(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1))$ with $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(V_{gh} \otimes \mathbb{Q}_p, H_p^\times \otimes L)$, the map $\log^{-+}$ is $\phi \mapsto \Xi_{g_\alpha} \cdot \Omega_{h_\alpha} \cdot \log_p(\phi(e_{\beta\alpha}))$. Then, combining the reciprocity law of Theorem 4.3.2 with Theorem 4.3.7, we have

$$L_p(\mathbf{g}, \mathbf{h})(y_0, z_0, 1) = \Omega \cdot \Xi_{g_\alpha} \cdot \Omega_{h_\alpha} \cdot (\log_p(v_{\alpha\alpha}) \cdot \log_p(u_{\alpha\beta}) - \log_p(u_{\alpha\alpha}) \cdot \log_p(v_{\alpha\beta})). \tag{4.12}$$

Hence, Conjecture 4.2.2 in the form given in (4.10) suggests that

$$\Omega = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{1}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{4.13}$$

Moreover, if we assume [DR16, Conjecture 2.1] this reduces to

$$\Omega = \frac{1}{\Omega_{g_\alpha} \cdot \Omega_{h_\alpha}} \pmod{L^\times}. \tag{4.14}$$

Remark 4.3.9. These periods we have described are completely non-canonical and depend on the choice of an L-basis. It is possible to formulate an analogue conjecture to [DR16, Conjecture 3.12], which only involves the so-called *enhanced* regulator as well as the differentials $\omega_{g_\alpha}, \omega_{h_\alpha}$, and the Beilinson–Flach class. This formulation has the advantage that it overcomes the period dependence by giving an equality in $D(V_{gh}^{\alpha\alpha}) \otimes (U_{gh} \otimes V_{gh} \otimes \mathbb{Q}_p)^{G_{\mathbb{Q}}}$ up to multiplication in $L^\times$; in particular, it would state that

$$\omega_{g_\alpha}\omega_{h_\alpha} \otimes \kappa(g_\alpha, h_\alpha) = (\log(v_{\alpha\alpha}) \cdot u - \log(u_{\alpha\alpha}) \cdot v) \otimes e_{\alpha\alpha} \pmod{L^\times}. \tag{4.15}$$

This conjecture itself is not directly equivalent to the main conjecture of [DLR16] and also relies on [DR16, Conjecture 2.1].

In any case, and under the assumptions of the introduction on g and h, we can formulate the following conjecture (Conjecture A in the introduction of the chapter).

Conjecture 4.3.10. *The Beilinson–Flach element $\kappa(g_\alpha, h_\alpha)$ satisfies the following equality in $(U_{gh} \otimes V_{gh}^\vee \otimes \mathbb{Q}_p)^{G_{\mathbb{Q}}}$.*

$$\kappa(g_\alpha, h_\alpha) = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{1}{\log_p(u_{g_\alpha})} \cdot \Big(\log_p(v_{\alpha\alpha}) \cdot u - \log_p(u_{\alpha\alpha}) \cdot v \Big) \pmod{L^\times}. \tag{4.16}$$

Under a quite general non-vanishing assumption, it turns out that the previous conjecture is equivalent to Conjecture 4.2.2. The following is what we anticipated as Theorem 4.1.4.

Proposition 4.3.11. *Conjecture 4.3.10 implies Conjecture 4.2.2. If we assume that* $\mathrm{Reg}_{g_\alpha}(V_{gh}) \neq 0$*, then the converse also holds.*

Proof. As before, let

$$\log^{-+} : H^1(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1)) \xrightarrow{\mathrm{pr}^{-+}} H^1(\mathbb{Q}_p, V_{gh}^{\alpha\beta} \otimes \mathbb{Q}_p(1)) \xrightarrow{\mathcal{L}^{-+}} \mathbb{Q}_p.$$

Applying this map to both sides of (4.16) and using the reciprocity law which relates the Beilinson–Flach class with the Hida–Rankin p-adic L-function, as recalled in Theorem 4.3.2, we get the main result of [DLR16] as stated in (4.10).

Conversely, assuming Conjecture 4.2.2 and using again the reciprocity law presented as Theorem 4.3.2, we have

$$\log^{-+}(\kappa_p(g_\alpha, h_\alpha)) = \frac{\log_p(v_{\alpha\alpha}) \cdot \log_p(u_{\alpha\beta}) - \log_p(u_{\alpha\alpha}) \cdot \log_p(v_{\alpha\beta})}{\log_p(u_{g_\alpha})} \pmod{L^\times}.$$

The class

$$\kappa_\circ = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{1}{\log_p(u_{g_\alpha})} \cdot \Big(\log_p(v_{\alpha\alpha}) \cdot u - \log_p(u_{\alpha\alpha}) \cdot v \Big) \pmod{L^\times}$$

satisfies

$$\log^{-+}(\kappa_p(g_\alpha, h_\alpha)) = \log^{-+}(\mathrm{res}_p(\kappa_\circ)).$$

We may write the cohomology class $\kappa(g_\alpha, h_\alpha)$ as a linear combination

$$\kappa(g_\alpha, h_\alpha) = \kappa_\circ + a \cdot u + b \cdot v,$$

where $a, b \in \mathbb{Q}_p$. Since $\kappa_p(g_\alpha, h_\alpha) - \mathrm{res}_p(\kappa_\circ)$ lies in the kernel of $\log^{-+}$, one must have

$$a \cdot \log_p(u_{\alpha\beta}) + b \cdot \log_p(v_{\alpha\beta}) = 0. \tag{4.17}$$

Consider also the map

$$\log^{--} : H^1(\mathbb{Q}_p, V_{gh} \otimes \mathbb{Q}_p(1)) \xrightarrow{\mathrm{pr}^{--}} H^1(\mathbb{Q}_p, V_{gh}^{\alpha\alpha} \otimes \mathbb{Q}_p(1)) \xrightarrow{\log_{\mathrm{BK}}} \mathbb{Q}_p.$$

The class $\kappa_p(g_\alpha, h_\alpha) - \mathrm{res}_p(\kappa_\circ)$ is also in the kernel of this map according to the first part of Theorem 4.3.2, and hence

$$a \cdot \log_p(u_{\alpha\alpha}) + b \cdot \log_p(v_{\alpha\alpha}) = 0. \tag{4.18}$$

However, if both (4.17) and (4.18) are satisfied and $(a, b) \neq (0, 0)$, then $\mathrm{Reg}_{g_\alpha}(V_{gh})$ is zero, contradicting the hypothesis. □

Remark 4.3.12. We expect that the non-vanishing of the regulator $\mathrm{Reg}_{g_\alpha}(V_{gh})$ could follow from results on transcendental number theory.

4.4 The self-dual case

Throughout this section we assume that $h = g^*$ and $\alpha_h = 1/\alpha_g$, which amounts to saying that $h_\alpha = g^*_{1/\alpha}$.

As usual, there exist four global cohomology classes, that we denote

$$\kappa(g_\alpha, g^*_{1/\alpha}), \quad \kappa(g_\beta, g^*_{1/\beta}), \quad \kappa(g_\alpha, g^*_{1/\beta}), \quad \kappa(g_\beta, g^*_{1/\alpha}) \in H^1(\mathbb{Q}, V_{gg^*} \otimes \mathbb{Q}_p(1)). \tag{4.19}$$

It was proved in Section 3.3 that $\kappa(g_\alpha, g^*_{1/\beta}) = \kappa(g_\beta, g^*_{1/\alpha}) = 0$, using an argument involving the *exceptional* vanishing of some Euler factors. The aim of this section is to describe the two remaining classes.

An explicit description of the cohomology classes

Under our running assumptions, $(U_{gg^*} \otimes V_{gg^*}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$ has dimension 1, while $(U_{gg^*}[1/p] \otimes V_{gg^*}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$ is a three-dimensional space. Fix now a basis $\{u, v, p\}$ of the latter, with the element u spanning the line $(U_{gh} \otimes V_{gg^*}^\vee \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}$. Write the Frobenius decomposition of these units as

$$u = u_{\beta,1/\alpha} \otimes e_{\alpha,1/\beta}^\vee + u_{\beta,1/\beta} \otimes e_{\alpha,1/\alpha}^\vee + u_{\alpha,1/\alpha} \otimes e_{\beta,1/\beta}^\vee + u_{\alpha,1/\beta} \otimes e_{\beta,1/\alpha}^\vee \tag{4.20}$$
$$v = v_{\beta,1/\alpha} \otimes e_{\alpha,1/\beta}^\vee + v_{\beta,1/\beta} \otimes e_{\alpha,1/\alpha}^\vee + v_{\alpha,1/\alpha} \otimes e_{\beta,1/\beta}^\vee + v_{\alpha,1/\beta} \otimes e_{\beta,1/\alpha}^\vee. \tag{4.21}$$

Observe that the above cohomology classes are no longer crystalline at p, and according to the discussion of Proposition 4.3.4, they belong to $H^1_{\mathrm{f},p}(\mathbb{Q}, V_{gg^*} \otimes \mathbb{Q}_p(1))$, the subspace of $H^1(\mathbb{Q}, V_{gg^*} \otimes \mathbb{Q}_p(1))$ formed by those classes which are unramified at any prime $\ell \neq p$ and de Rham at p. The one-dimensional subspace $H^1_{\mathrm{f}}(\mathbb{Q}, V_{gg^*} \otimes \mathbb{Q}_p(1))$ is spanned by the unit u.

The representation $V_{gg^*}^\vee$ is no longer irreducible, as

$$V_{gg^*}^\vee \simeq \mathrm{ad}^0(V_g^\vee) \oplus \mathrm{Id}, \tag{4.22}$$

where $\mathrm{ad}^0(V_g)$ stands for the adjoint representation of g and Id for the trivial representation.

The isomorphism (4.22) can be explicitly described as follows: fixing a basis $\{e_1^\vee, e_{\alpha\otimes\bar\beta}^\vee, e_{\beta/\otimes\bar\alpha}^\vee\}$ of $\mathrm{ad}^0(V_g^\vee)$ and also a basis $\{e_{\mathrm{Id}}^\vee\}$ for Id, it is given by the rule

$$\begin{gathered} e_{\alpha,1/\alpha}^\vee + e_{\beta,1/\beta}^\vee \mapsto (0, e_{\mathrm{Id}}^\vee), \quad e_{\alpha,1/\alpha}^\vee - e_{\beta,1/\beta}^\vee \mapsto (e_1^\vee, 0), \\ e_{\alpha,1/\beta} \mapsto (e_{\alpha\otimes\bar\beta}^\vee, 0), \quad e_{\beta,1/\alpha} \mapsto (e_{\beta\otimes\bar\alpha}^\vee, 0). \end{gathered} \tag{4.23}$$

Considering the decomposition

$$H^1_{\mathrm{f},p}(\mathbb{Q}, V_{gg^*}^\vee \otimes \mathbb{Q}_p(1)) = H^1_{\mathrm{f},p}(\mathbb{Q}, \mathrm{ad}^0(V_g^\vee) \otimes \mathbb{Q}_p(1)) \oplus H^1_{\mathrm{f},p}(\mathbb{Q}, \mathbb{Q}_p(1)), \tag{4.24}$$

we observe that according to [Bel09, Proposition 2.12], the space $H^1_{\mathrm{f},p}(\mathbb{Q}, \mathbb{Q}_p(1)) \simeq (\mathbb{Z}[1/p]^\times) \otimes \mathbb{Q}_p$ has dimension 1 and is spanned by p, while there is a canonical identification

$$H^1_{\mathrm{f},p}(\mathbb{Q}, \mathrm{ad}^0(V_g^\vee) \otimes \mathbb{Q}_p(1)) = (U_{gh}[1/p] \otimes \mathrm{ad}^0(V_g^\vee) \otimes \mathbb{Q}_p)^{G_\mathbb{Q}}. \tag{4.25}$$

Since $u, v \in (U_{gg^*}[1/p] \otimes \mathrm{ad}^0(V_g^\vee))^{G_\mathbb{Q}}$, it follows from the first equation of (4.23) that

$$u_{\alpha,1/\alpha} \cdot u_{\beta,1/\beta} = v_{\alpha,1/\alpha} \cdot v_{\beta,1/\beta} = 1.$$

Set $u_1 := u_{\alpha,1/\alpha} = u_{\beta,1/\beta}^{-1}$ and $v_1 := v_{\alpha,1/\alpha} = v_{\beta,1/\beta}^{-1}$. Making a slight abuse of notation, we still denote by $\kappa(g_\alpha, g_{1/\alpha}^*)$ the projection of the cohomology class to the space $H^1_{\mathrm{f},p}(\mathbb{Q}, \mathrm{ad}^0(V_g^\vee)/p^\mathbb{Z} \otimes \mathbb{Q}_p(1))$. The following result corresponds to Theorem 4.1.2 when $h_\alpha = g_{1/\alpha}^*$.

Proposition 4.4.1. *There exists a period $\Omega \in H_p$ such that the equality*

$$\kappa(g_\alpha, g_{1/\alpha}^*) = \Omega \cdot (\log_p(u_1) \cdot v - \log_p(v_1) \cdot u) \pmod{L^\times} \tag{4.26}$$

holds in $H^1_{\mathrm{f},p}(\mathbb{Q}, \mathrm{ad}^0(V_g^\vee)/p^\mathbb{Z} \otimes \mathbb{Q}_p(1))$. Moreover, if

$$\Omega = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{g_{1/\alpha}^*}} \cdot \frac{1}{\log_p(u_{\alpha\otimes\bar\beta})},$$

Conjecture 4.2.2 is true, and under the assumption that $\mathrm{Reg}_{g_\alpha}(V_{gg^*}) \neq 0$ the converse also holds.

Proof. The first part of the statement follows the same argument used in the proof of Theorem 4.3.7. Now, we may write

$$\kappa(g_\alpha, g^*_{1/\alpha}) = \lambda u + \mu v + \nu p$$

for some p-adic scalars λ, μ and ν. Next, we project to $H^1(\mathbb{Q}_p, V_g^\alpha \otimes V_{g^*}^{1/\alpha} \otimes \mathbb{Q}_p(1))$, which is no longer one-dimensional, but isomorphic to the two-dimensional space $H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p^\times \hat{\otimes} \mathbb{Q}_p \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p$. This amounts to saying that both the p-adic valuation and p-adic logarithm are zero. In particular,

$$\lambda \log_p(u_1) + \mu \log_p(v_1) = 0,$$

and the equality in (4.26) follows.

In the same way, Proposition 4.3.11 is equally valid once we have considered the quotient by the trivial representation and we can write the cohomology class as a combination of the units u and v. $\square$

Remark 4.4.2. We have previously proven that when $h \neq g^*$, the non-vanishing of the special value $\mathscr{L}_p{}^{g_\alpha}$ allows us to conclude that the classes $\kappa(g_\alpha, h_\alpha)$ and $\kappa(g_\alpha, h_\beta)$ are linearly independent. The same argument implies that, under the same non-vanishing hypothesis, the class $\kappa(g_\alpha, g^*_{1/\alpha})$ and the derived class $\kappa'(g_\alpha, g^*_{1/\beta})$ constructed in the previous chapter are linearly independent.

Assuming again that we know that the regulator does not vanish, we can prove that $\Omega \neq 0$ in (4.26) and can provide a formula for Ω in $H_p^\times / L^\times$. Furthermore, it clearly follows from Proposition 4.4.1 that the two classes $\kappa(g_\alpha, g^*_{1/\alpha})$ and $\kappa(g_\beta, g^*_{1/\beta})$ are linearly dependent.

Proof of Theorem 4.1.5

We now move to the proof of Theorem 4.1.5 of the introduction. Although we stated the result in terms of the unit group $(U_{gh}[1/p]/p^{\mathbb{Z}} \otimes V_{gh}^\vee)^{G_\mathbb{Q}}$, the identifications of Kummer theory allow us to consider an equivalent formulation in terms of cohomology groups.

Proposition 4.4.3. *If we assume that* $\mathrm{Reg}_{g_\alpha}(V_{gg^*}) \neq 0$*, then*

$$\kappa(g_\alpha, g^*_{1/\alpha}) = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\alpha}}} \cdot \frac{\log_p(u_1) \cdot v - \log_p(v_1) \cdot u}{\log_p(u_{\alpha \otimes \bar{\beta}})}, \tag{4.27}$$

$$\kappa(g_\beta, g^*_{1/\beta}) = \frac{1}{\Xi_{g_\beta} \cdot \Omega_{g^*_{1/\beta}}} \cdot \frac{\log_p(u_1) \cdot v - \log_p(v_1) \cdot u}{\log_p(u_{\beta \otimes \bar{\alpha}})}, \tag{4.28}$$

hold in the space $H^1_{\mathrm{f},p}(\mathbb{Q}, \mathrm{ad}^0(V_g^\vee)/p^{\mathbb{Z}} \otimes \mathbb{Q}_p(1))$*, modulo* $L^\times$*:*

Proof. According to Proposition 4.4.1, this is equivalent to Conjecture 4.2.2, concerning the value of $L_p(\mathbf{g}, \mathbf{h})$ at $(y_0, y_0, 1)$. However, observe that this value does not depend on the p-stabilization of h, and hence this follows from Theorem A in the previous chapter, where it was proved that

$$L_p(\mathbf{g}, \mathbf{h})(y_0, y_0, 1) = \frac{\log_p(u_1) \cdot v - \log_p(v_1) \cdot u}{\log_p(u_{\alpha \otimes \bar{\beta}})} \pmod{L^\times}.$$

$\square$

In particular, the non-vanishing of the regulator implies that these classes are non-zero, since the vanishing of both $\log_p(u_1)$ and $\log_p(v_1)$ would automatically imply that $\mathrm{Reg}_{g_\alpha}(V_{gg^*}) = 0$. Furthermore, it is clear from this description that the two classes span a one-dimensional subspace, as asserted in Theorem 4.1.5.

Theta series of quadratic fields

When g is the theta series of an imaginary quadratic field where p splits, or the theta series of a real quadratic field where p remains inert, we can give a more explicit description of the classes. Furthermore, in these cases we know that the regulator does not vanish.

Quadratic imaginary case, with p split. Let $\psi : G_K \to \mathbb{C}^\times$ be a ring class character of conductor prime to p, and write $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ splits in K. Set $\alpha := \psi(\bar{\mathfrak{p}})$ and $\beta := \psi(\mathfrak{p})$. Let g and h be the weight 1 theta series of ψ and ψ^{-1}, respectively.

In this case,

$$V_{gh} \simeq \mathrm{Ind}_K^{\mathbb{Q}} \psi_{\mathrm{triv}} \oplus \mathrm{Ind}_K^{\mathbb{Q}} \psi^2,$$

where ψ_{triv} stands for the trivial character of K. The theory of elliptic units allows us to attach a canonical unit u_{ψ^2} (resp. $u_{\psi^{-2}}$) to the character ψ^2 (resp. ψ^{-2}), where Fr_p acts with eigenvalue α/β (resp. β/α). Let $u = u_{\psi^2} \otimes e^\vee_{\alpha\otimes\bar{\beta}} + u_{\psi^{-2}} \otimes e^\vee_{\beta\otimes\bar{\alpha}}$.

Let h_K denote the class number of K, and write $v_{\mathfrak{p}} \in K^\times$ for any p-unit satisfying $(v_{\mathfrak{p}}) = \mathfrak{p}^{h_K}$. Let $v = v_{\mathfrak{p}} \otimes e_1^\vee$.

From the description of u and v we see that $u_1 = 0$ and hence we have

$$\kappa(g_\alpha, g^*_{1/\alpha}) = \tilde{\Omega}_1 \cdot u, \quad \kappa(g_\beta, g^*_{1/\beta}) = \tilde{\Omega}_2 \cdot u, \tag{4.29}$$

where $\tilde{\Omega}_1, \tilde{\Omega}_2 \in H_p$. We must prove that these numbers are both non-zero. Projecting to the (β,α)-component of V_{gh} and applying the Perrin-Riou map described in Proposition 4.3.1, the explicit reciprocity law in the form of Theorem 4.3.2 gives that

$$\log^{-+}(\kappa(g_\alpha, g^*_{1/\alpha})) = \tilde{\Omega}_1 \cdot \log(u_{\psi^{-2}}) = L_p(\mathbf{g},\mathbf{h})(y_0,y_0,1) \pmod{L^\times}.$$

Although $L_p(\mathbf{g},\mathbf{h})(y_0,y_0,1)$ depends on the chosen of a p-stabilization for g, this is not the case for h. Moreover, according to [Theorem 4.2, DLR2],

$$L_p(\mathbf{g},\mathbf{h})(y_0,y_0,1) = \frac{\log_p(v_{\mathfrak{p}}) \cdot \log_p(u_{\psi^{-2}})}{\log_p(u_{\psi^{-2}})} = \log_p(v_{\mathfrak{p}}) \pmod{L^\times},$$

and in particular

$$\tilde{\Omega}_1 = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\alpha}}} \cdot \frac{\log_p(v_{\mathfrak{p}})}{\log_p(u_{\psi^{-2}})}, \qquad \tilde{\Omega}_2 = \frac{1}{\Xi_{g_\beta} \cdot \Omega_{g^*_{1/\beta}}} \cdot \frac{\log_p(v_{\mathfrak{p}})}{\log_p(u_{\psi^2})} \pmod{L^\times}.$$

Real quadratic case, with p inert. In this case, u is the fundamental unit ϵ_K attached to K and v is a p-unit in the field H cut out by the character. Writing v^+ for the norm of v and keeping the same notations as in the previous case, we get that the periods $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are given by

$$\tilde{\Omega}_1 = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\alpha}}} \cdot \frac{\log_p(v^+)}{\log_p(\epsilon_K)}, \quad \tilde{\Omega}_2 = \frac{1}{\Xi_{g_\beta} \cdot \Omega_{g^*_{1/\beta}}} \cdot \frac{\log_p(v^+)}{\log_p(\epsilon_K)} \pmod{L^\times}.$$

As a consequence of this discussion, the following theorem is proved.

Theorem 4.4.4. *Let g be a theta series of an imaginary (resp. real) quadratic field K where p splits (remains inert). Then, the classes of (4.19) span a line in the space $H^1_{\mathrm{f},p}(\mathbb{Q}, \mathrm{ad}^0(V_g^\vee)/p^{\mathbb{Z}} \otimes \mathbb{Q}_p(1))$. To be more precise, $\kappa(g_\alpha, g^*_{1/\beta}) = \kappa(g_\beta, g^*_{1/\alpha}) = 0$, and the remaining classes can be described as follows:*

(a) In the imaginary quadratic case, with p a prime which splits in K,

$$\kappa(g_\alpha, g^*_{1/\alpha}) = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\alpha}}} \cdot \frac{\log_p(v_{\mathfrak{p}})}{\log_p(u_{\psi^{-2}})} \cdot u, \quad \kappa(g_\beta, g^*_{1/\beta}) = \frac{1}{\Xi_{g_\beta} \cdot \Omega_{g^*_{1/\beta}}} \cdot \frac{\log_p(v_{\mathfrak{p}})}{\log_p(u_{\psi^2})} \cdot u \pmod{L^\times}.$$

(b) *In the real quadratic case, with p an inert prime,*

$$\kappa(g_\alpha, g^*_{1/\alpha}) = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\alpha}}} \cdot \frac{\log_p(v^+)}{\log_p(\epsilon_K)} \cdot u, \quad \kappa(g_\beta, g^*_{1/\beta}) = \frac{1}{\Xi_{g_\beta} \cdot \Omega_{g^*_{1/\beta}}} \cdot \frac{\log_p(v^+)}{\log_p(\epsilon_K)} \cdot u \pmod{L^\times}.$$

Remark 4.4.5. The proof of our result is ostensibly easier than that of Castella and Hsieh. The main reason is that, while the order of vanishing of the theta element $\Theta_{f/K}(T)$ is unknown, we do know that the Katz two-variable p-adic L-function $L_p^{\mathrm{Katz}}(K)$ attached to the trivial character vanishes at order zero at $s = 0$. To overcome this difficulty, Castella and Hsieh use a bound for this order of vanishing coming from Iwasawa theory and later develop the theory of derived heights in the case of elliptic curves. Although this treatment would make sense here, it would not yield any new result.

4.5 Particular cases of the conjecture

In this section we discuss in more detail some of the reducible cases that are considered in [DLR16]. These are scenarios where the Beilinson–Flach classes appearing in the main body of the chapter can be recast in terms of circular units, elliptic units or Beilinson–Kato elements, and our statements admit an ostensibly simpler formulation.

As in Section 4.3, and to ease the exposition, we assume that $h \neq g^*$ throughout this section.

Eisenstein series and circular units

Let

$$g = E_1(\chi_g^+, \chi_g^-), \quad h = E_1(\chi_h^+, \chi_h^-)$$

denote two Eisenstein series attached to pairs of Dirichlet characters, with the assumption that χ_g^+ is even and χ_g^- is odd, and likewise for χ_h^+ and χ_h^-. In the Eisenstein case, as discussed in [DLR15a], the classicality hypothesis on g reads as

$$\chi_g^+(p) = \chi_g^-(p).$$

Let $\chi_g := \chi_g^+ \chi_g^-$ and $\chi_h := \chi_h^+ \chi_h^-$. The representation V_{gh} decomposes as a direct sum of four one-dimensional characters,

$$V_{gh} = \chi_{gh}^{++} \oplus \chi_{gh}^{--} \oplus \chi_{gh}^{+-} \oplus \chi_{gh}^{-+}.$$

Given an even character χ factoring through a finite abelian extension H_χ of $\mathbb{Q}$ and taking values in L, denote by $u(\chi) \in L \otimes \mathcal{O}_{H_\chi}^\times$ the fundamental unit in the χ-eigenspace for the $G_\mathbb{Q}$-action (this could be seen as the circular unit attached to χ, following the terminology of [DLR16, Section 3]). The pair (χ_g^+, χ_g^-) corresponds to a genus character ψ_g of the imaginary quadratic field K cut out by the odd Dirichlet character $\chi_g := \chi_g^+ \chi_g^-$. The weight one Eisenstein series $E_1(\chi_g^+, \chi_g^-)$ is equal to the theta series $\theta_K(\psi_g)$, and hence the modular form g admits three natural ordinary deformations, since

$$E_1(\chi_g^+, \chi_g^-) = E_1(\chi_g^-, \chi_g^+) = \theta_K(\psi_g).$$

Let $\mathbf{g}$ (resp. $\mathbf{h}$) denote the cuspidal family passing through the p-stabilization of g (resp. h).

Instead of going through all the possible cases we just focus on the generic situation where χ_{gh}^{++} and χ_{gh}^{--} are both non-trivial and $\chi_{gh}^{\bullet\circ} \neq 1$ for $\bullet, \circ \in \{\pm\}$.

Then, [DLR16, Theorem 3.1] asserts that

$$L_p(\mathbf{g}, \mathbf{h})(y_0, z_0, 1) = \log^{-+}(\kappa_p(g_\alpha, h_\alpha)) = \frac{\log_p(u(\chi_{gh}^{++})) \log_p(u(\chi_{gh}^{--}))}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{4.30}$$

From here, we see that

$$\kappa(g_\alpha, h_\alpha) = \mathfrak{C} \cdot u \pmod{L^\times},$$

where u is the unit $u = u(\chi_{gh}^{--}) \otimes e_{\beta\alpha}^\vee + u(\chi_{gh}^{-+}) \otimes e_{\alpha\alpha}^\vee$, whose (β,α)-component agrees with $u(\chi_{gh}^{--})$, and $\mathfrak{C}$ is an explicit constant in terms of the periods we have previously defined, namely

$$\mathfrak{C} = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{\log_p(u(\chi_{gh}^{++}))}{\log_p(u(g_\alpha))} \pmod{L^\times}. \tag{4.31}$$

Remark 4.5.1. When some of the characters χ_{gh}^{++} or χ_{gh}^{--} are trivial, the regulator also involves the fundamental p-units in the χ-eigenspace for the Galois action, for an appropriate χ. See Case 2 and Case 3 of [DLR16, Section 3] for more details.

Theta series attached to imaginary quadratic fields and elliptic units

Let

$$g = \theta_{\psi_g} \in M_1(N_g, \chi_K\chi_g), \quad h = \theta_{\psi_h} \in M_1(N_h, \chi_K\chi_h)$$

be the theta series associated to two arbitrary finite order characters ψ_g, ψ_h of the imaginary quadratic field K. Let $\psi_1 = \psi_g\psi_h$ and $\psi_2 = \psi_g\psi_h'$, where ψ' denotes the character given by $\psi'(\sigma) = \psi(\sigma_0\sigma\sigma_0^{-1})$, for any choice of $\sigma_0 \in G_{\mathbb{Q}}\backslash G_K$. Let $V_g = \mathrm{Ind}_K^{\mathbb{Q}}(\psi_g)$ and $V_h = \mathrm{Ind}_K^{\mathbb{Q}}(\psi_h)$ denote the two-dimensional induced representations of ψ_g and ψ_h. Then,

$$V_{gh} = V_{\psi_g} \otimes V_{\psi_h} \simeq V_{\psi_1} \oplus V_{\psi_2}.$$

For any character ψ, we define u_ψ as the corresponding elliptic unit attached to it, as recalled in [DLR16, Section 4].

Following the same argument as in Theorem 3.6.2, we have that for any $s \in \mathbb{Z}_p$,

$$L_p(g,h,s) = \frac{1}{\log_p(u_{\psi_{\mathrm{ad}}})} \cdot L_p^{\mathrm{Katz}}(\psi_{gh}^{-1}, s) \cdot L_p^{\mathrm{Katz}}(\psi_{gh'}^{-1}, s) \pmod{L^\times}, \tag{4.32}$$

where $\psi_{gh} = \psi_g \cdot \psi_h$, $\psi_{gh'} = \psi_g \cdot \psi_{h'}$, and $L_p^{\mathrm{Katz}}(\psi, s)$ stands for the evaluation of the two-variable Katz p-adic L-function attached to ψ at the character $\mathbb{N}^s$. In particular,

$$\log^{-+}(\kappa(g_\alpha, h_\alpha)) = \frac{\log_p(u_{\psi_1'}) \log_p(u_{\psi_2'})}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{4.33}$$

As we have done before, we can express $\kappa(g_\alpha, h_\alpha)$ as an explicit constant multiplied by the elliptic unit $u := u_{\psi_2} \otimes e_{\alpha\beta}^\vee + u_{\psi_2'} \otimes e_{\beta\alpha}^\vee$, making use of the periods we have introduced before, as

$$\kappa(g_\alpha, h_\alpha) = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{\log_p(u_{\psi_1'})}{\log_p(u_{g_\alpha})} \cdot u \pmod{L^\times}$$

Eisenstein series and Beilinson–Kato elements

In the case where exactly one of the modular forms is Eisenstein, the representation V_{gh} also decomposes as a sum of two irreducible representations of dimension two. Let $g \in S_1(N_g, \chi_g)$ and $h = E_1(\chi_h^+, \chi_h^-)$; then,

$$V_{gh} = V_g(\chi_h^+) \oplus V_g(\chi_h^-),$$

which gives via Artin formalism a decomposition

$$L(V_{gh}, s) = L(g, \chi_h^+, s) \cdot L(g, \chi_h^-, s),$$

where $L(g, \chi_h^\bullet, s)$ is the L-function attached to g twisted by the finite order character $\chi_h^\bullet$.

Let $L_p(\mathbf{g}, E_1(\chi_h^+, \chi_h^-))$ denote the two-variable p-adic L-function of [BDR15a, Section 2.2.2] (when the second modular form does not vary in a Hida family it is allowed to be Eisenstein). Let χ be a Dirichlet character, and consider $L_p(\mathbf{g}, \chi)$, the two-variable Mazur–Kitagawa p-adic L-function attached to $\mathbf{g}$ and χ. With the notations of [BD14], let $\lambda^\pm(\ell) \in \mathbb{C}_p$ stand for the canonical periods involved in the construction of the Mazur–Kitagawa p-adic L-function. We adopt as in loc. cit. the normalization

$$L_p(\mathbf{g}, \chi)(y, s) = \lambda^\pm(\ell) \cdot \Big(1 - \frac{p^{s-1}}{\chi(p)\alpha_{g_y^\circ}}\Big) \cdot \Big(1 - \frac{\chi(p)\beta_{g_y^\circ}}{p^s}\Big) \times L^*(g_y, \chi, s),$$

where $L^*(g_y, \chi, s)$ is the algebraic part of $L(g_y, \chi, s)$, defined in [BD14, equation (22)].

Theorem 4.5.2. *There exists a rigid analytic function $\mathfrak{f}(y, s)$ such that the following equality holds in $\Lambda_{\mathbf{g}} \otimes \Lambda$*

$$L_p(\mathbf{g}, E_1(\chi_h^+, \chi_h^-))(y, s) = \mathfrak{f}(y, s) \cdot L_p(\mathbf{g}, \chi_h^+)(y, s) \cdot L_p(\mathbf{g}, \chi_h^-)(y, s). \tag{4.34}$$

Here, $\mathfrak{f}(y, s) = (C_{g_y, \chi_h^+, \chi_h^-} \cdot \mathcal{E}(g_y) \cdot \mathcal{E}^*(g_y) \cdot \lambda^+(\ell) \cdot \lambda^-(\ell))^{-1}$, being $C_{g_y, \chi_h^+, \chi_h^-}$ an explicit non-zero algebraic number and

$$\mathcal{E}(g_y) = 1 - \beta_{g_y^\circ}^2 p^{-\ell}, \qquad \mathcal{E}^*(g_y) := 1 - \beta_{g_y^\circ}^2 p^{1-\ell}.$$

Proof. In the range of classical interpolation we have an equality of the corresponding L-values. The result follows after gathering together the different factors appearing in the interpolation process, combined with the observation that $\Omega^+_{\mathbf{g}_y, \mathbb{C}} \cdot \Omega^-_{\mathbf{g}_y, \mathbb{C}} = 4\pi^2 \langle \mathbf{g}_y^\circ, \mathbf{g}_y^\circ \rangle$ ☐

It may be instructive to compare this result with [BD14, Theorem 3.4]: there, a two-variable p-adic L-function attached to a cuspidal Hida family and an Eisenstein family which interpolates central critical points is expressed as the product of two Mazur–Kitagawa p-adic L-functions. Unfortunately, our result is not as useful as one would expect: specialization at weights $(y_0, 1)$ establishes a connection between the Hida–Rankin p-adic L-function and the product of $L_p(g, \chi_h^+, 1)$ with $L_p(g, \chi_h^-, 1)$, but up to multiplication by the quantity $\mathfrak{f}(y_0, 1)$. Recall that a similar question arises at [DR20b, Proposition 2.6], since the factor $\mathfrak{f}(y, s)$ is essentially (up to multiplication by some explicit factors) the function $\mathscr{L}_p(\mathrm{Sym}^2(\mathbf{g}))(y)$. This multiplier is expected to be related with the logarithm of the Gross-Stark unit u_{g_α}. Further, this also connects with [DR16, Conjecture 2.1], since we hope $\lambda^+(1)$ to be eventually related with Ω_{g_α} and $\lambda^-(1)$ with $\Xi_{g_\alpha}^{-1}$.

As it is proved in [Och03], there exists a two-variable Euler system (usually referred to as Beilinson–Kato Euler system), $\kappa(\mathbf{g}, \chi)$, satisfying that $L_p(\mathbf{g}, \chi)$ is the image under a suitable Perrin-Riou map of $\kappa(\mathbf{g}, \chi)$. This allows us to obtain a connection between Beilinson–Flach and Beilinson–Kato elements, and also an expression for the regulator involving special values of the Mazur–Kitagawa p-adic L-function. Observe that this is the counterpart of [DLR15a, Sections 6,7], where this same situation was considered in the case of rational points over elliptic curves.

Chapter 5

Cyclotomic derivatives of Beilinson–Flach classes and a new proof of a Gross–Stark formula

We give a new proof of a conjecture of Darmon, Lauder and Rotger regarding the computation of the $\mathcal{L}$-invariant of the adjoint of a weight one modular form in terms of units and p-units. While in Chapter 3 the essential ingredient was the use of Galois deformations techniques following the computations of Bellaïche and Dimitrov, we propose a new approach exclusively using the properties of Beilinson–Flach classes. One of the key ingredients is the computation of a cyclotomic derivative of a cohomology class in the framework of Perrin-Riou theory, which can be seen as a counterpart to the earlier work of Loeffler, Venjakob and Zerbes. We hope that this method could be applied to other scenarios regarding exceptional zeros, and illustrate how this could lead to a better understanding of this setting by conjecturally introducing a new p-adic L-function whose special values involve information just about the unit of the adjoint (and not also the p-unit), in the spirit of the conjectures of Harris and Venkatesh.

The results presented at this chapter are the content of the research article [Ri20c], which is currently a work in progress and will be available soon.

5.1 Introduction

In our series of works with Rotger, presented here as part of Chapters 3 and 4, we proposed a systematic study of the conjecture of Darmon, Lauder and Rotger [DLR16] on p-adic iterated integrals in terms of certain cohomology classes constructed from the p-adic interpolation of Beilinson–Flach elements. This conjecture may be subsumed in a broader programme, comprising both the Gross–Stark conjectures and also the celebrated *Elliptic Stark conjectures*, which shed some light on the arithmetic of elliptic curves of rank 2.

Let χ be a Dirichlet character of level $N \geq 1$, and let $S_1(N,\chi)$ stand for the space of cuspidal modular forms of weight 1, level N and nebentype χ. Let $g = \sum_{n\geq 1} a_n q^n \in S_1(N,\chi)$ be a normalized newform and let $g^* = g \otimes \bar{\chi}$ denote its twist by the inverse of its nebentype. Let

$$\varrho_g : \mathrm{Gal}\,(H_g/\mathbb{Q}) \hookrightarrow \mathrm{GL}(V_g) \simeq \mathrm{GL}_2(L), \quad \varrho_{\mathrm{ad}^0(g)} : \mathrm{Gal}\,(H/\mathbb{Q}) \hookrightarrow \mathrm{GL}(\mathrm{ad}^0(g)) \simeq \mathrm{GL}_3(L)$$

denote the Artin representations associated to g and its adjoint, respectively. Here $H_g \supseteq H$ denote the finite Galois extensions of $\mathbb{Q}$ cut out by these representations, and L is a sufficiently large finite extension of $\mathbb{Q}$ containing their traces.

Label and order the roots of the p-th Hecke polynomial of g as $X^2 - a_p(g)X + \chi(p) = (X - \alpha)(X - \beta)$. We assume throughout that

(H1) The reduction of ϱ_g mod p is irreducible;

(H2) g is p-distinguished, i.e. $\alpha \neq \beta \,(\mathrm{mod}\, p)$, and

(H3) ϱ_g is not induced from a character of a real quadratic field in which p splits.

Hence, following the ideas of previous chapters, there are *four* (a priori distinct!) cohomology classes

$$\kappa(g_\alpha, g^*_{1/\alpha}), \quad \kappa(g_\beta, g^*_{1/\beta}), \quad \kappa(g_\alpha, g^*_{1/\beta}), \quad \kappa(g_\beta, g^*_{1/\alpha}) \in H^1(\mathbb{Q}, \mathrm{ad}^0(g)(1)). \tag{5.1}$$

However, and as it was proved in Chapter 3 when we dealt with Beilinson–Flach classes, it happens that

$$\kappa(g_\alpha, g^*_{1/\beta}) = \kappa(g_\beta, g^*_{1/\alpha}) = 0. \tag{5.2}$$

To overcome this situation, in Section 3.3 we had constructed certain derivatives of those classes, but it turns out that the definition we had used there is not completely useful for our purposes. Roughly speaking, we had taken the derivative along one of the weight directions associated to the Hida family interpolating one of the modular forms, while towards obtaining a more flexible and arithmetically interesting setting we need to consider also the *cyclotomic* derivative. This is an analogue situation to the scenario of [Buy16] and [Ven16], where the computation of the derivative of the Mazur–Kitagawa p-adic L-function along a certain direction of the weight space was relatively easy using the classical theory of Heegner points (and had already been carried out by Bertolini and Darmon [BD07]), but the computation of the *cyclotomic* derivative required new ideas. Hence, this work may be thought as a counterpart to the approach of Büyükboduk and Venerucci to the exceptional zero phenomenon, but in the easier case where elliptic curves are replaced by unit groups (and hence one can circumvent the technical complications introduced by the use of Nekovar's height theory). Similar results had been obtained by Loeffler, Venjakob and Zerbes [LVZ15], and one can see our computations as the *dual* of Proposition 2.5.5 and Theorem 3.1.2 of loc. cit. We refer also to the seminal works of Benois [Ben14a], [Ben14b] where similar questions are addressed.

Our main result in Chapter 3 was the computation of a special value formula for the Hida–Rankin p-adic L-function at weight one (alternatively, the derivative of the adjoint of the modular form). This is specially intriguing since that function, that we denote as $L_p(g, g^*, s)$, cannot be directly defined in terms of an interpolation property, and requires to consider the p-adic variation of the modular forms (g, g^*) along a Hida family. Indeed, it depends on the choice of a p-stabilization for g. Along this chapter, we sometimes write $L_p^{g_\alpha}(g, g^*, s)$ to emphasize this dependence. In [DLR16, Section 1] it is shown that

$$\dim_L(\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}} = 1, \quad \dim_L(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}} = 2.$$

Fix a generator u of $(\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$ and also an element v of $(\mathcal{O}_H^\times[1/p]^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$ in such a way that $\{u, v\}$ is a basis of $(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$. The element v may be chosen to have p-adic valuation $\mathrm{ord}_p(v) = 1$, and we do so.

Viewed as a $G_{\mathbb{Q}_p}$-module, $\mathrm{ad}^0(g)$ decomposes as $\mathrm{ad}^0(g) = L \oplus L^{\alpha\otimes\bar\beta} \oplus L^{\beta\otimes\bar\alpha}$, where each line is characterized by the property that the arithmetic Frobenius Fr_p acts on it with eigenvalue 1, α/β and β/α, respectively. Let H_p denote the completion of H in $\bar{\mathbb{Q}}_p$ and let

$$u_1,\ u_{\alpha\otimes\bar\beta},\ u_{\beta\otimes\bar\alpha},\ v_1,\ v_{\alpha\otimes\bar\beta},\ v_{\beta\otimes\bar\alpha} \in H_p^\times \otimes_\mathbb{Q} L \pmod{L^\times}$$

denote the projection of the elements u and v in $(H_p^\times \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}_p}}$ to the above lines.

Then, we had proven the following theorem, which was one of the main results of Chapter 3.

Theorem 5.1.1. *Assume that hypothesis (H1)-(H3) hold. Then, the following equality holds up to multiplication by a scalar in* $L^\times$

$$L_p^{g_\alpha}(g, g^*, 1) = \frac{\log_p(u_{\alpha\otimes\bar{\beta}}) \log_p(v_1) - \log_p(v_{\alpha\otimes\bar{\beta}}) \log_p(u_1)}{\log_p(u_{\alpha\otimes\bar{\beta}})}.$$

The proof we had given in Chapter 3 was lengthy and made use of the results of Bellaiche–Dimitrov computing the tangent space of a deformation problem, together with some techniques taken from the earlier work [DLR18]. In a certain way, that proof mimicked the approach of Greenberg–Stevens [GS94] to the exceptional zero phenomenon for elliptic curves with split multiplicative reduction at p. However, we had observed a tantalizing connection with the theory of Beilinson–Flach elements, that were affected by a similar exceptional zero phenomenon. This allowed us to interpret *derived* classes of Beilinson–Flach elements in terms of the units $\{u, v\}$, but does not give any new insight into the proof of Theorem 5.1.1, as we had initially expected. This work may be seen as the culmination of one of the initial objectives of this dissertation, that was proving the Gross–Stark conjecture of Darmon, Lauder and Rotger using just the properties of Beilinson–Flach elements and the flexibility provided by the notion of *derivatives*.

We can give, with these ideas at hand, a different proof of the previous theorem. This can be seen as the counterpart to the approach of Kobayashi [Ko06] to the Mazur–Tate–Teitelbaum conjecture in rank 0, since he reproves the result of Greenberg and Stevens [GS94] using the properties of Kato's cohomology classes.

Our proof is a combination of four main ideas (together with the same starting point coming from Hida's theory of improved p-adic L-functions):

(0) The results of Hida [Hi85], [Hi88], which reduce the conjecture to the computation of the derivative of the Frobenius eigenvalue along the weight direction.

(1) The local properties at p of Beilinson–Flach elements, which give an expression, *up to multiplication by a p-adic scalar*, for the derived class $\kappa'(g_\alpha, g^*_{1/\beta})$ in terms of the units u and v, where here the derivative is taken along any arbitrary direction of the weigh space.

(2) The observation that knowing two *weight* derivatives, together with the vanishing of the class $\kappa(\mathbf{g}, \mathbf{g}^*)$ along the line $(\ell, \ell, \ell-1)$, allows us to determine the cyclotomic derivative of the class.

(3) An explicit reciprocity law for the Λ-adic class $\kappa(\mathbf{g}, \mathbf{g}^*)$, obtained when g and g^* vary over Hida families $\mathbf{g}$ and $\mathbf{g}^*$, respectively. This was proved in [KLZ17]. In our situation, there is an exceptional vanishing, and hence we may consider a *derived* reciprocity law, in the sense of Chapter 3. This gives an expression for the *weight derivative* of the Beilinson–Flach class in terms of an unknown p-adic period and involving also the $\mathcal{L}$-invariant of the adjoint of g_α.

(4) The results of Büyükboduk [Buy12], [Buy16] and Venerucci [Ven16] around Coleman maps, which allow us to relate the *cyclotomic derivative* of the Beilinson–Flach class to the Hida–Rankin p-adic L-function. This part can be also recast, by duality, in terms of the computations developed in [LVZ15].

Observe that the study of universal norms has also allowed Roset, Rotger and Vatsal [RRV19] to reinterpret the $\mathcal{L}$-invariant of Theorem 5.1.1 in terms of an algebraic avatar initially defined by Greenberg [Gre91].

However, Theorem 5.1.1 is not completely satisfactory towards the understanding of the arithmetic of the adjoint of a weight one modular form, since it involves both the unit and the p-unit attached to the Galois representation. It is natural to expect a putative refinement of the previous result in the spirit of the conjectures of Harris–Venkatesh [HV19], with a p-adic L-function whose special values encode information just about the unit u. The last section of this chapter is devoted to discuss the following conjecture in the framework provided by Perrin-Riou maps.

Conjecture 5.1.2. *There exists an analytic p-adic L-function $L_p^{\mathrm{Eis}}(g, g^*, s)$, appropriately related with the cohomology class $\kappa(g_\alpha, g^*_{1/\alpha})$ via the theory of Perrin-Riou maps, and such that*

$$L_p^{\mathrm{Eis}}(g, g^*, 1) = \log_p(u_1) \pmod{L^\times}.$$

The organization of the chapter is as follows. Section 2 discusses the motivational case of circular units, where these same phenomena arise and that can serve as a motivation for our later work. Section 3 recalls the notations and results of the previous chapters around Beilinson–Flach elements which are needed in the proof, following Chapters 3 and 4. Section 4 contains the main results of the chapter and discuss the new approach to the proof of Theorem 5.1.1 using the notion of *derived* Belinson–Flach elements. Section 5 proposes an alternative interpretation of the previous results in terms of deformation theory. Finally, Section 6 discusses the p-adic L-function of Conjecture 5.1.2 and its relationship with the periods of weight one modular forms.

5.2 Analogy with the case of circular units

The situation we want to deal with has a clear parallelism with the case of circular units, which has been discussed in the introductory chapters as part of the background material. We fix a Dirichlet character χ, and write H for the field cut out by the Artin representation attached to it, and L for its coefficient field. The case where χ is odd gives rise to an exceptional vanishing of the Deligne–Ribet p-adic L-function $L_p(\chi\omega, s)$ at $s = 0$ when $\chi(p) = 1$. Under the assumption that χ is odd, $(\mathcal{O}_H^\times \otimes L)^\chi$ is a zero-dimensional L-vector space, while $(\mathcal{O}_H[1/p]^\times \otimes L)^\chi$ has dimension 1 if $\chi(p) = 1$. In this case, choosing a non-zero element v_χ of the latter space, we may define an $\mathcal{L}$-invariant

$$\mathcal{L}(\chi) = -\frac{\log_p(v_\chi)}{\mathrm{ord}_p(v_\chi)}, \tag{5.3}$$

where we are implicitly choosing a prime of H above p. Then,

$$L_p'(\chi\omega, s) = \mathcal{L}(\chi) \cdot L(\chi, 0). \tag{5.4}$$

See [DDP11] for more details on that and for a broader treatment in the setting of totally real number fields.

In the case where χ is even, the situation is ostensibly different. Then, $(\mathcal{O}_H^\times \otimes L)^\chi$ is one-dimensional and we may fix a generator c_χ of it, that we call a *circular* unit. We defined it, as usual, as a weighted combination of cyclotomic units

$$c_\chi = \prod_{a=1}^{N-1} (1 - \zeta_N^a)^{\chi^{-1}(a)},$$

where ζ_N is a fixed primitive N-th root of unity and the notation $(1 - \zeta_N^a)^{\chi^{-1}(a)}$ means $(1 - \zeta_N^a) \otimes \chi^{-1}(a)$. Moreover, if we further assume that $\chi(p) = 1$, $(\mathcal{O}_H[1/p]^\times \otimes L)^\chi$ has dimension 2, and we may consider a basis of the form $\{c_\chi, v_\chi\}$, with the convention that $\mathrm{ord}_p(v_\chi) = 1$.

Given any even, non trivial Dirichlet character, one always have Leopoldt's formula, which asserts that

$$L_p(\chi, 1) = -\frac{(1 - \chi(p)p^{-1})}{\mathfrak{g}(\bar\chi)} \cdot \log_p(c_\chi). \tag{5.5}$$

Let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. By adding p-power conductors to χ and considering a family of coherent units along the cyclotomic tower, one may construct a Λ-adic class $\kappa(\chi, s)$, whose bottom layer vanishes when $\chi(p) = 1$. We keep this assumption on the character χ all along this section and write

$$\kappa(\chi, s) \in H^1(\mathbb{Q}, L_p(\bar\chi) \otimes \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}))$$

for the Λ-adic class, where $\underline{\varepsilon}_{\mathrm{cyc}}$ stands for the Λ-adic cyclotomic character. Given $k \in \mathbb{Z}$, let $\nu_k : \Lambda(\underline{\varepsilon}_{\mathrm{cyc}}) \to \mathbb{Z}_p$ be the ring homomorphism sending the group-like element $a \in \mathbb{Z}_p^\times$ to a^{k-1}. This induces a $G_{\mathbb{Q}}$-equivariant specialization map

$$\nu_k : \Lambda(\underline{\varepsilon}_{\mathrm{cyc}}) \longrightarrow \mathbb{Z}_p(k-1)$$

and gives rise to a collection of global cohomology classes

$$\kappa(\chi, k) \in H^1(\mathbb{Q}, L_p(\bar{\chi})(k)).$$

The Perrin-Riou formalism allows us to understand $L_p(\chi, s)$ as the image under a Coleman map (also named as Perrin-Riou map, or Perrin-Riou regulator) of the local class $\kappa_p(\chi, s)$

$$\mathcal{L}_\chi : H^1(\mathbb{Q}_p, L_p(\bar{\chi}) \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}})) \longrightarrow I^{-1}\Lambda, \quad \mathcal{L}_\chi(\kappa_p(\chi, s)) = L_p(\chi, s),$$

where I is the ideal of Λ corresponding to the specialization at $s = 1$ (see [KLZ17, Section 8.2]). This map interpolates either the dual exponential map (for $s \leq 0$) or the Bloch–Kato logarithm (for $s \geq 1$). Unfortunately, the bottom layer $\kappa(\chi, 1)$ vanishes when $\chi(p) = 1$. Following the construction of [Buy12, Section 3], there is a derived class $\kappa'(\chi, s)$, defined as the unique class satisfying that

$$\kappa(\chi, s) = \frac{\gamma - 1}{\log_p(\gamma)} \cdot \kappa'(\chi, s), \tag{5.6}$$

where γ is a fixed topological generator of $\mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$. It is also proved in [Buy12] that $\kappa'(\chi, 1)$ belongs to an extended Selmer group, which in this case may be identified with the group of p-units where the Galois group acts via χ (we insist that when χ is even this space is two-dimensional).

In the cases where $\chi(p) = 1$, one can also define an *improved* map

$$\widetilde{\mathcal{L}_\chi} = \frac{\gamma - 1}{\frac{1}{p}\log_p(\gamma)} \times \mathcal{L}_\chi : H^1(\mathbb{Q}_p, L_p(\bar{\chi})(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}})) \longrightarrow I^{-1}\Lambda,$$

where $\varepsilon_{\mathrm{cyc}}$ is the usual cyclotomic character. Therefore,

$$\widetilde{\mathcal{L}_\chi}(\kappa_p'(\chi, s)) = p \cdot L_p(\chi, s).$$

The computations done in [Buy12, Section 6.2], in particular Remark 6.5, show that the map $\widetilde{\mathcal{L}_\chi}$, when specialized at $s = 1$, is given by the order map (applied in this case to the derived class). The crucial point is a computation of the universal norms over the cyclotomic tower, as well as the use of Lemma 6.4 of loc. cit. (see also [Ven16, Section 3]). Hence, we have the following (identifying as usual the cohomology classes with the corresponding units via the standard Kummer map).

Proposition 5.2.1. *The element $\kappa'(\chi, 1) \in (\mathcal{O}_H[1/p]^\times \otimes L)^\chi$ satisfies that*

$$L_p(\chi, 1) = -\frac{1 - p^{-1}}{\mathfrak{g}(\bar{\chi})} \cdot \mathrm{ord}_p(\kappa'(\chi, 1)).$$

Proof. This follows after combining the results of [Buy12, Section 6.2] on the properties of the map $\widetilde{\mathcal{L}_\chi}$ with Solomon's computations, showing that the p-adic valuation of the derived class (sometimes referred as the *wild* cyclotomic unit) agrees with the logarithm of the usual circular unit (see also Section 4 of loc. cit.). $\square$

5.3 Beilinson–Flach elements

The three variable cohomology classes

We begin by recalling some of the notations that had already been introduced both in Chapter 3 and 4. Let $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$ and $\mathbf{g}^* \in \Lambda_{\mathbf{g}}[[q]]$ be two Hida families of tame conductor N and tame nebentype χ and $\bar{\chi}$, where $\Lambda_{\mathbf{g}}$ is a finite flat extension of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. Let $\Lambda_{\mathbf{gg}^*} = \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda$, and consider also the $\Lambda_{\mathbf{g}}$-modules afforded by the Hida families attached to $\mathbf{g}$ and $\mathbf{g}^*$, that we denote $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{g}^*}$, respectively. Finally, consider the $\Lambda_{\mathbf{gg}^*}$-module

$$\mathbb{V}_{\mathbf{gg}^*} := \mathbb{V}_{\mathbf{g}} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{V}_{\mathbf{g}^*} \hat{\otimes}_{\mathbb{Z}_p} \Lambda(\varepsilon_{\rm cyc} \underline{\varepsilon}_{\rm cyc}^{-1}), \tag{5.7}$$

where $\Lambda(\varepsilon_{\rm cyc}\underline{\varepsilon}_{\rm cyc}^{-1})$ stands for the twist of Λ by the inverse of the Λ-adic cyclotomic character, with the conventions adopted in Section 3.2. The formal spectrum of $\Lambda_{\mathbf{gg}^*}$ is endowed with certain distinguished points, the so-called crystalline points, denoted as $\mathcal{W}_{\mathbf{gh}}^\circ$ and indexed by triples (y, z, σ); we refer the reader to Section 2 of loc. cit. for the definitions.

The Λ-adic Galois representation $\mathbb{V}_{\mathbf{gg}^*}$ is characterized by the property that for $(y, z, \sigma) \in \mathcal{W}_{\mathbf{gg}^*}^\circ$ with $\mathrm{w}(\sigma) = \nu_s$ and $s \in \mathbb{Z}$, (5.7) specializes to

$$\mathbb{V}_{\mathbf{gg}^*}(y, z, \sigma) = V_{g_y} \otimes V_{g_z^*}(1-s),$$

the $(1-s)$-th Tate twist of the tensor product of the Galois representations attached to g_y and g_z^*.

Fix $c \in \mathbb{Z}_{>1}$ such that $(c, 6pN) = 1$. [KLZ17, Theorem A] yields a three-variable Λ-adic global Galois cohomology class

$$\kappa^c(\mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gg}^*})$$

that is referred to as the Euler system of Beilinson–Flach elements associated to $\mathbf{g}$ and $\mathbf{g}^*$. We denote by $\kappa_p^c(\mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*})$ the image of $\kappa^c(\mathbf{g}, \mathbf{g}^*)$ under the restriction map. Since c is fixed throughout, we may sometimes drop it from the notation. This constant does make an appearance in fudge factors accounting for the interpolation properties satisfied by the Euler system, but in all cases we are interested in these fudge factors do not vanish and hence do not pose any problem for our purposes.

Given a crystalline arithmetic point $(y, z, s) \in \mathcal{W}_{\mathbf{gg}^*}^\circ$ of weights (ℓ, m, s), set for notational simplicity throughout this section $g = g_y^\circ$, $g^* = (g_z^*)^\circ$. With these notations, g_y (resp. g_z^*) is the p-stabilization of g (resp. g^*) with U_p-eigenvalue α_g (resp. α_{g^*}).

Define

$$\kappa(g_y, g_z^*, s) := \kappa(\mathbf{g}, \mathbf{g}^*)(y, z, s) \in H^1(\mathbb{Q}, V_{g_y} \otimes V_{g_z^*}(1-s)) \tag{5.8}$$

as the specialisation of $\kappa(\mathbf{g}, \mathbf{g}^*)$ at (y, z, s).

As explained in [DR16, Section 2] and as we have already recalled in Chapter 3, the spaces $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{g}^*}$, as $G_{\mathbb{Q}_p}$-modules, are endowed with a stable filtration

$$0 \longrightarrow \mathbb{V}_{\mathbf{g}}^+ \longrightarrow \mathbb{V}_{\mathbf{g}} \longrightarrow \mathbb{V}_{\mathbf{g}}^- \longrightarrow 0,$$

where $\mathbb{V}_{\mathbf{g}}^+$ and $\mathbb{V}_{\mathbf{g}}^-$ are flat $\Lambda_{\mathbf{g}}$-modules with a $G_{\mathbb{Q}_p}$-action, locally free of rank one over $\Lambda_{\mathbf{g}}$, and such that the quotient $\mathbb{V}_{\mathbf{g}}^-$ is unramified. Define the $G_{\mathbb{Q}_p}$-subquotient $\mathbb{V}_{\mathbf{gg}^*}^{-+} := \mathbb{V}_{\mathbf{g}}^- \hat{\otimes} \mathbb{V}_{\mathbf{g}^*}^+$ of $\mathbb{V}_{\mathbf{gg}^*}$, which is of rank one over the two-variable Iwasawa algebra $\Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{g}^*}$ (this quotient makes sense because of [KLZ17, Proposition 8.1.7]).

Then, one may consider a Perrin-Riou map

$$\langle \mathcal{L}_{\mathbf{gg}^*}^{-+}, \eta_{\mathbf{g}} \otimes \omega_{\mathbf{g}^*} \rangle : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*}^{-+} \hat{\otimes} \Lambda(\varepsilon_{\rm cyc}\underline{\varepsilon}_{\rm cyc}^{-1})) \longrightarrow I^{-1}\Lambda_{\mathbf{gg}^*} \otimes \mathbb{Q}_p(\mu_N). \tag{5.9}$$

This application satisfies an explicit reciprocity law, which is the content of [KLZ17, Theorem B], and which asserts that

$$\langle \mathcal{L}_{\mathbf{gg}^*}^{-+}(\kappa_p^{-+}(\mathbf{g}, \mathbf{g}^*)), \eta_{\mathbf{g}} \otimes \omega_{\mathbf{g}^*} \rangle = \mathcal{A}(\mathbf{g}, \mathbf{g}^*) \cdot L_p(\mathbf{g}, \mathbf{g}^*), \tag{5.10}$$

where $\mathcal{A}(\mathbf{g},\mathbf{g}^*)$ is the Iwasawa function of [KLZ17, Theorem 10.2.2] and $\kappa_p^{-+}(\mathbf{g},\mathbf{g}^*)$ stands for the composition of the localization-at-p map with the projection $\mathbb{V}_{\mathbf{gg}^*} \to \mathbb{V}_{\mathbf{gg}^*}^{-+}$ in local cohomology.

The different specializations of the map $\langle \mathcal{L}_{\mathbf{gg}^*}^{-+}, \eta_{\mathbf{g}} \otimes \omega_{\mathbf{g}^*}\rangle$ can be expressed in terms of the Bloch–Kato logarithm or the dual exponential map. In particular, we are interested in the specializations of the class $\kappa(\mathbf{g},\mathbf{g}^*)$ at weights $(1,1,0)$, and more generally, weights $(\ell,\ell,\ell-1)$, where the Perrin-Riou map interpolates, up to some explicit Euler factors, the Bloch–Kato logarithm. Unfortunately, these factors may vanish in the self-dual case, and one must recast to the concept of derivatives of Euler systems.

Derivatives of Beilinson–Flach elements

We keep the notations fixed in the introduction of the chapter regarding weight one modular forms and units for the adjoint representation. Further, we fix a point of weight one $y_0 \in \mathrm{Spf}(\Lambda_{\mathbf{g}})$ such that $\mathbf{g}_{y_0} = g_\alpha$ and $\mathbf{g}^*_{y_0} = g^*_{1/\beta}$.

In Section 3.3, we proved that the class $\kappa(\mathbf{g},\mathbf{g}^*)$ was zero over the line corresponding to the Zariski closure of points of weights $(\ell,\ell,\ell-1)$. In loc. cit. we also constructed a derivative along the y-direction (alternatively, keeping y fixed and varying at a time the other two variables). However, since the weight space is three-dimensional, it makes sense to ask about the derivative along any other direction. Although this is perfectly legit, it was not the approach of loc. cit., where we restricted only to analytic directions, that is, preserving the condition

$$s = m-1.$$

In particular, this excludes the option of considering *cyclotomic* derivatives (that is, keeping the weight fixed and varying the cyclotomic twists).

A first observation we make is that we can determine the derivative of the class $\kappa(\mathbf{g},\mathbf{g}^*)$ along any direction. Since along the line $(\ell,\ell,\ell-1)$ the class is identically zero, the derivative also vanishes. Hence, by an elementary argument in linear algebra, it suffices to determine the derivative along any other two *independent directions* to capture all the *first-order* information. These derivatives will be given as weighted combinations of the units $\{u,v\}$.

Remark 5.3.1. Observe that we are using the results of [LZ17] which assert that the Beilinson–Flach elements lie in the part corresponding to the adjoint in the decomposition $V_{gg^*} = \mathrm{ad}^0(V_g) \oplus 1$ (the same proof works for the case or families). Alternatively, one can consider the projection to the alternate part for weight greater than one and then apply a limit argument. Moreover, and following the discussion of Section 4.4, one has that the projection of the subspace $p^{\mathbb{Z}}$ to the adjoint component is trivial.

For the sake of simplicity, and since this suffices for our purposes, we restrict to the local classes at p. We closely follow the ideas of Chapter 3, showing that the specialization at weight 1 of the Λ-adic class is a linear combination of the unit u and the p-unit v, as described in the introduction.

Lemma 5.3.2. *The derivative along the y-direction (keeping fixed both z and s) satisfies the following equality in $H^1(\mathbb{Q}_p, V_{gg^*}(1))$*

$$\frac{\partial \kappa_p(\mathbf{g},\mathbf{g}^*)}{\partial y}\Big|_{(y_0,y_0,0)} = \Omega \cdot (\log_p(v_{\alpha\otimes\bar\beta})u - \log_p(u_{\alpha\otimes\bar\beta})v), \tag{5.11}$$

where $\Omega \in H_p$ and we have made use of the usual notations for writing directional derivatives.

Proof. According to the properties of the cohomology classes established in Chapters 3 and 4, the left hand side may be written as a combination of the units u and v. Then, the result follows by applying [KLZ17, Proposition 8.1.7] to $\kappa(\mathbf{g},\mathbf{g}^*)$ in order to conclude that its projection to $\mathbb{V}_{\mathbf{gg}^*}^{--}$ is identically zero, and therefore the same is true for its derivative. Specializing at $(y_0,y_0,0)$, the result automatically follows. $\square$

Lemma 5.3.3. *The derivative along the z-direction (keeping fixed both y and s) satisfies the following equality in the space $H^1(\mathbb{Q}_p, V_{gg^*}(1))$, up to a factor in $L^\times$*

$$\frac{\partial \kappa_p(\mathbf{g}, \mathbf{g}^*)}{\partial z}\Big|_{(y_0,y_0,0)} = \Omega \cdot (\log_p(v_{\alpha\otimes\bar{\beta}})u - \log_p(u_{\alpha\otimes\bar{\beta}})v). \tag{5.12}$$

Proof. This follows by applying the same result to the p-adic L-value associated to $g^*_{1/\beta}$, which agrees with the former since it holds that $L_p^{g_\alpha}(g_\alpha, g^*_{1/\beta}, 1) = L_p^{g^*_{1/\beta}}(g^*_{1/\beta}, g_\alpha, 1)$. Note that the product of the periods arising when pairing with the differentials is a rational quantity (see Section 3.5), and so it does not affect the result. □

Therefore, we may determine the derivative along the direction cyclotomic direction (keeping the weights fixed) by a linear algebra argument.

Proposition 5.3.4. *Assume that the derivative of $\kappa_p(\mathbf{g}, \mathbf{g}^*)$ along the cyclotomic derivative is non-vanishing. Then, up to multiplication by a scalar, the cyclotomic derivative of the Beilinson–Flach class is*

$$\frac{\partial \kappa_p(\mathbf{g}, \mathbf{g}^*)}{\partial s}\Big|_{(y_0,y_0,0)} = \Omega \cdot (\log_p(v_{\alpha\otimes\bar{\beta}})u - \log_p(u_{\alpha\otimes\bar{\beta}})v) \pmod{L^\times},$$

where $\Omega \in H_p$ is the period of equation (5.11) and the equality holds in $H^1(\mathbb{Q}_p, V_{gg^*}(1))$.

Proof. Recall that the class vanishes along the line $(\ell, \ell, \ell-1)$. Hence, the result follows from equations (5.11) and (5.12) combined with the fact that

$$(0,0,1) = (1,1,1) - (1,0,0) - (0,1,0).$$

□

We will see in the next section that the vanishing of the cyclotomic derivative is equivalent to the vanishing of the special value $L_p^{g_\alpha}(g, g^*, 1)$ and also of the regulator corresponding to the $\mathcal{L}$-invariant. Observe that the previous results show that the different derived classes, which are elements living in a two-dimensional space, span the same line!

We can now mimic the approach of Chapter 3 when proving the derived reciprocity law and obtain an expression for $L_p^{g_\alpha}(g, g^*, 1)$ involving the period Ω. In particular, considering the derivative along the analytic direction $(1,0,0)$ we have the following.

Proposition 5.3.5. *Up to multiplication by an element in $L^\times$, the following equality holds*

$$L_p^{g_\alpha}(g, g^*, 1) \cdot \Big(\frac{-\alpha'_{\mathbf{g}}}{\alpha_g}\Big)_{|y_0} = \Omega \cdot (\log_p(u_{\alpha\otimes\bar{\beta}}) \cdot \log_p(v_1) - \log_p(v_{\alpha\otimes\bar{\beta}}) \cdot \log_p(u_1)).$$

As usual $\alpha_{\mathbf{g}}$ stands for the derivative of the U_p-eigenvalue along the weight direction.

Proof. This follows from making explicit the Euler factors in the explicit reciprocity law of [KLZ17, §10] and taking derivatives along the y-direction. □

In the next section, our aim is determining the value of the period Ω appearing in Proposition 5.3.4, which would complete the proof of our main theorem.

5.4 Cyclotomic derivatives and proof of the main theorem

Cyclotomic derivatives

Along this section, we assume that g and g^* do not move along Hida families and we just consider the cyclotomic variation. As an abuse of notation, write $\kappa(g, g^*, s) := \kappa(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, s)$ to emphasize the dependence on s. The image of this class under the Perrin-Riou map recovers the p-adic L-function $L_p^{g_\alpha}(g, g^*, s)$, that is,

$$\langle \mathcal{L}_{gg^*}^{-+}(\kappa_p(g, g^*, s)), \eta_g \otimes \omega_{g^*} \rangle = L_p^{g_\alpha}(g, g^*, s). \tag{5.13}$$

The preceding discussion shows that $\kappa_p(g, g^*, 0)$ is zero, but we do not expect that $L_p^{g_\alpha}(g, g^*, 0) = 0$ in general. This is the same situation we previously found in the setting of circular units: the Kubota–Leopoldt p-adic L-function of a non-trivial, even Dirichlet character $L_p(\chi, s)$ is seen as the image of a Λ-adic cohomology class $\kappa(\chi, s)$ under a Perrin-Riou map; unfortunately, it happens that $\kappa(\chi, 1) = 0$ when $\chi(p) = 1$ and an Euler factor also vanishes, so we cannot assert (and indeed it is false!) that $L_p(\chi, 1) = 0$.

Along this section, and since there is no possible confusion, we write $\kappa'(g, g^*, s)$ for the cyclotomic derivative. Define the improved Perrin-Riou map as

$$\langle \widetilde{\mathcal{L}_{gg^*}^{-+}}, \eta_g \otimes \omega_{g^*} \rangle = \frac{\gamma - 1}{\frac{1}{p} \log_p(\varepsilon_{\mathrm{cyc}}(\gamma))} \times \langle \mathcal{L}_{gg^*}^{-+}, \eta_g \otimes \omega_{g^*} \rangle \; : \; H^1(\mathbb{Q}_p, V_{gg^*}^{-+}(1-s)) \longrightarrow \Lambda. \tag{5.14}$$

Note that we have

$$\langle \widetilde{\mathcal{L}_{gg^*}^{-+}}(\kappa_p'(g, g^*, s)), \eta_g \otimes \omega_{g^*} \rangle = p \cdot L_p^{g_\alpha}(g, g^*, s). \tag{5.15}$$

Hence, the value of $\langle \mathcal{L}_{gg^*}^{-+}(\kappa_p(g, g^*, s)), \eta_g \otimes \omega_{g^*} \rangle$ agrees with

$$\frac{p}{\log_p(\varepsilon_{\mathrm{cyc}}(\gamma))} \cdot \langle \mathcal{L}_{gg^*}^{-+}((\gamma - 1) \cdot \kappa_p'(g, g^*, s)), \eta_g \otimes \omega_{g^*} \rangle = \langle \widetilde{\mathcal{L}_{gg^*}^{-+}}(\kappa_p'(g, g^*, s)), \eta_g \otimes \omega_{g^*} \rangle.$$

For the following result, consider the identification

$$H^1(\mathbb{Q}_p, \mathrm{ad}^0(V_g)(1)) \simeq H_p^\times[\mathrm{ad}^0(g)] \otimes L_p, \tag{5.16}$$

and take the element $\kappa_p'(g, g^*, 0)$, which belongs to the latter space (and which may be therefore identified with a local unit in $H_p^\times$). The same study of [Buy12, Remark 6.5] works verbatim in our setting, where he argues that the improved Perrin-Riou map is a multiple of the order map applied to the derived class. However, we want to find out this explicit multiple (at least, up to multiplication by a rational constant). Compare for example this setting with the computations of [LVZ15, Proposition 2.5.5] and the discussions of Section 3 of loc. cit., showing that the improved exponential map they consider is indeed the order map (up to sign). We come back to this issue in the last section.

Proposition 5.4.1. *Identifying $\kappa_p'(g, g^*, 0)$ with an element in $(H_p^\times \otimes L)^{G_{\mathbb{Q}_p}}$, one has*

$$L_p^{g_\alpha}(g, g^*, 0) = (1 - p^{-1}) \cdot \mathrm{ord}_p(\kappa_p'(g, g^*, 0)).$$

Proof. We can rephrase the statement in terms of the well-known theory of Coleman's power series. Then, $\kappa_p(g, g^*, s)$ may be seen as a compatible system of units varying over the cyclotomic p-tower, but whose bottom layer is trivial. Similarly, $\kappa_p'(g, g^*, s)$ corresponds to a power series which can be written as $f_{\kappa'} = x^a g(x)$, where $g(x)$ is invertible in the ring of power series. Hence, using the well-known properties of universal norms (see for instance [Ven16, Section 3] and specially the proof of Proposition 3.6), we have that $g(x) = 1$ and the image under the Coleman map of $\kappa_p'(g, g^*, 0)$ agrees with

$$(1 - p^{-1}) \cdot \log_p(x^a) = a(1 - p^{-1}) \cdot \log_p(x).$$

Then, and with the usual identifications, we conclude that

$$\langle \widetilde{\mathcal{L}_{gg^*}^{-+}}(\kappa_p'(g,h,0)), \eta_g \otimes \omega_{g^*}\rangle = (p-1)\cdot \mathrm{ord}_p(\kappa_p'(g,g^*,0)),$$

and the statement follows. □

Roughly speaking, the previous theorem says that the derivative of the logarithm is the order (which can be seen as the dual of the result which interprets the derivative of the dual exponential as a logarithm).

Observe that Venerucci considers a rather similar situation in the setting of elliptic curves. Although he considers a point where the Perrin-Riou map interpolates the dual exponential map, we can adapt his approach.

Corollary 5.4.2. *With the notations introduced along the previous section, and up to multiplication by $L^\times$,*

$$L_p^{g_\alpha}(g,g^*,1) = \Omega \cdot \log_p(u_{\alpha\otimes\bar\beta}).$$

This can be connected again with the case of circular units, that is, $L_p(g,g^*,s)$ is also the order of the derivative of $\kappa(g,g^*,s)$ along the s-direction.

As usual, let

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) := \frac{-\alpha_{\mathbf{g}}'(y_0)}{\alpha_{\mathbf{g}}(y_0)}, \tag{5.17}$$

where recall $\alpha_{\mathbf{g}} = a_p(\mathbf{g}) \in \Lambda_{\mathbf{g}}$ is the Iwasawa function given by the eigenvalue of the Hecke operator U_p acting on $\mathbf{g}$, and $\alpha_{\mathbf{g}}'$ is its derivative.

Proposition 5.4.3. *Assume that the $\mathcal{L}$-invariant $\mathcal{L}(\mathrm{ad}^0(g_\alpha))$ is non-zero. Then, it may be written as*

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) = \frac{\log_p(u_{\alpha\otimes\bar\beta})\cdot\log_p(v_1) - \log_p(v_{\alpha\otimes\bar\beta})\cdot\log_p(u_1)}{\log_p(u_{\alpha\otimes\bar\beta})} \pmod{L^\times}.$$

Proof. Combining Proposition (5.3.5) with Proposition (5.4.1), we have that

$$\frac{\Omega}{\mathcal{L}(\mathrm{ad}^0(g_\alpha))}\cdot\Big(\frac{\log_p(u_{\alpha\otimes\bar\beta})\cdot\log_p(v_1) - \log_p(v_{\alpha\otimes\bar\beta})}{\log_p(u_{\alpha\otimes\bar\beta})}\Big) = \Omega \pmod{L^\times}.$$

Dividing by Ω (provided that this value is non zero!), the result follows. □

Improved p-adic L-functions

We finish the proof with the same argument invoked in Chapter 3, involving Hida's improved p-adic L-function. Then, the main theorem is automatically proved by virtue of the following result, which had already been discussed in loc. cit.

Proposition 5.4.4. *For a crystalline classical point $y_0 \in \mathcal{W}_{\mathbf{g}}^\circ$ of weight $\ell \geq 1$, we have*

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) = L_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,\ell) = L_p'(\mathrm{ad}^0(g_{y_0}),\ell),$$

up to a non-zero rational constant, and Theorem 5.1.1 holds.

5.5 A reinterpretation of the special value formula

The result we have proven along the last two sections was presented in the introduction of the chapter as a special value formula for the Hida–Rankin p-adic L-function. Alternatively, we emphasized in our earlier work how it can be regarded as the computation of the $\mathcal{L}$-invariant for the adjoint of a weight one modular form, and following the original formulation given by Darmon, Lauder, and Rotger, it also admits a reinterpretation in terms of p-adic iterated integral (and this point of view is specially useful towards computational experiments). In order to give a more conceptual view of our results, and how they fit in the theory of exceptional zeros and Galois deformations of modular forms, we would like two discuss two other interpretations which were already behind the scenes both in Chapter 3 and in Chapter 4.

Deformations of weight one modular forms

We fix a p-stabilization g_α of the weight one modular form $g \in S_1(N,\chi)$. We discuss a reinterpretation of the main results in terms of deformations of modular forms, in a striking analogy with the different works around the Gross–Stark conjecture, and which may be useful towards generalizations of the main results to totally real fields, following the recent approach of Dasgupta, Kakde, and Ventullo [DKV18].

Let E_k denote the weight k Eisenstein series, whose Fourier expansion is given by

$$E_k = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{5.18}$$

There are two possible ways of considering its p-adic variation in families, either by taking the ordinary p-stabilization E_k^{ord} or the critical one E_k^{crit}. For the sake of simplicity, we restrict to the ordinary p-stabilization, and after further normalizing by $\zeta(1-k)/2$, we have the usual Eisenstein series $G_k^{(p)}$, given by

$$G_k^{(p)} = 1 + 2\zeta_p(1-k)^{-1} \sum_{n=1}^{\infty} \sigma_{k-1}^{(p)}(n) q^n.$$

Since $\zeta_p(1-k)$ has a pole at $k=0$, it turns out that $G_0^{(p)} = 1$, and it makes sense to consider its derivative with respect to the weight variable

$$(G_0^{(p)})' := 2(1-p^{-1})^{-1} \cdot \sum_{n=1}^{\infty} \sigma_{-1}^{(p)} q^n.$$

Compare this with the analogue situation described by Darmon, Pozzi and Vonk [DPV20] in Theorem A and in the subsequent discussion. Therefore, it is possible to take the infinitesimal deformation $G_0^{(p)} + \varepsilon (G_0^{(p)})'$, and multiplying by g_α we obtain

$$(G_0^{(p)} + \varepsilon (G_0^{(p)})') \cdot g_\alpha = g_\alpha + \varepsilon (G_0^{(p)})' g_\alpha. \tag{5.19}$$

We regard this expression as a modular form of weight $1+\varepsilon$ corresponding to an infinitesimal deformation of g_α.

There is another natural deformation of g_α we want to consider, which is precisely the one behind the scenes in previous chapters and which also appeared in [DLR18]. This is defined as

$$g_\alpha' := \Big(\frac{d}{dy} \mathbf{g}_\alpha \Big)_{|y=y_0}.$$

Then, we may take a second deformation of the modular form g_α, given by

$$g_\alpha + \varepsilon g_\alpha'. \tag{5.20}$$

When we subtract the deformations in (5.19) and (5.20), we obtain $g_\alpha(E_0^{(p)})' - g_\alpha'$, which is overconvergent because of the results of Bellaïche–Dimitrov on the geometry of the eigencurve [BeDi16]. If we furthermore take the ordinary projection and project to the g_α-component, we obtain a multiple of g_α, that is

$$e_{g_\alpha} e_{\mathrm{ord}}(g_\alpha(E_0^{(p)})') = g_\alpha' + \mathcal{L} \cdot g_\alpha. \tag{5.21}$$

Let e_{ord} stand for the ordinary projector, and e_{g_α} for the projector onto the g_α-isotypic component.

Proposition 5.5.1. *Under the running assumptions,*

$$e_{g_\alpha} e_{\mathrm{ord}}(g_\alpha E_0^{[p]}) = (1 - \alpha_g U_p^{-1}) g_\alpha' \pmod{L^\times}.$$

Proof. Substracting the deformations in (5.19) and (5.20), we obtain $g_\alpha(G_0^{(p)})' - g_\alpha'$. If we furthermore take the ordinary projection and project to the g_α-component, we obtain a multiple of g_α, that is

$$e_{g_\alpha} e_{\mathrm{ord}}(g_\alpha(G_0^{(p)})') = g_\alpha' + \mathcal{L} \cdot g_\alpha. \tag{5.22}$$

Next, if we apply the operator $1 - \alpha_g U_p^{-1}$ to both sides of the previous equation, the left hand side becomes just the p-depletion

$$e_{g_\alpha} e_{\mathrm{ord}}(g_\alpha(G_0^{[p]})'), \tag{5.23}$$

while in the right hand side the operator $1 - \alpha_g U_p^{-1}$ annihilates g_α. We have thus proved the result. □

Note that the left hand side is an explicit multiple of the p-adic L-function $L_p^{g_\alpha}(g, g^*, 1)$, so the proposition asserts that the $\mathcal{L}$-invariant which governs the arithmetic of the adjoint may be read as a generalized eigenvalue attached to the deformation g_α', that is,

$$(1 - \alpha_g U_p^{-1}) g_\alpha' = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) \cdot g_\alpha \pmod{L^\times}.$$

Spaces of generalized eigenvectors

When discussing circular units, we have seen that the condition $\chi(p) = 1$ provides us with a p-unit in an extended Selmer group, and we have discussed how to mimic this approach in the case of Beilinson–Flach elements. But more generally, given two modular forms g and h of weights ℓ and m, respectively, and an integer s, there is a geometric construction of the so-called Eisenstein classes $\mathrm{Eis}^{[g,h,s]}$ whenever the triple (ℓ, m, s) satisfies the *weight condition* of [KLZ17, Section 7], that is,

$$1 \leq s < \min\{\ell, m\}.$$

This includes all the points of weights $(\ell, \ell, \ell-1)$ when $\ell \geq 2$, and in particular the most well-known case of $\ell = 2$.

The proof of the explicit reciprocity law for Beilinson–Flach classes rests on an explicit connection between the p-depleted Eisenstein series (which encodes values of the p-adic L-function) and the p-stabilized one (which encodes values of the regulator of a geometric cycle). In this chapter we recovered the expressions for the logarithm of the derived class in terms of p-adic L-values, but it is natural to look for a reciprocity law involving $\mathrm{Eis}^{[g,h,s]}$, whenever $h = g^*$ and $s = \ell - 1$. Note that this was the only case excluded by [KLZ20, Theorem 6.5.9]. Let us discuss the limitations for a result like that and that one may find natural in this framework.

According to [DR14, Lemma 4.10] (see also [KLZ20, Lemma 6.5.8]), one has

$$E_0^{[p]} g = (1 - \chi(p)\alpha_h \cdot U_p^{-1}) \cdot (E_0^{(p)} g_\alpha), \tag{5.24}$$

where $E_0^{[p]}$ (resp. $E_0^{(p)}$) stands for the p-depletion (resp. p-stabilization) of the weight 0 Eisenstein series E_0. In the non self-dual case, the corresponding operator acting on the space of generalized eigenforms $S_1(Np)[[g_\alpha]]$ is invertible, and we obtain a straightforward linear relation. But when $h = g^*$, the connection is more involved. In this case, consider a generalized eigenbasis $\{e_1, \dots, e_n\}$ for the U_p-operator acting on the space of generalized space of (non-necessarily overconvergent) modular forms $S_1(Np)[[g_\alpha]]$, that is,

$$U_p \cdot e_1 = \alpha_h \cdot e_1, \quad U_p \cdot e_2 = e_1 + \alpha_h \cdot e_2, \quad \dots, \quad U_p \cdot e_n = e_{n-1} + \alpha_h \cdot e_n.$$

Hence, the matrix corresponding to the operator $1 - \alpha_g \cdot U_p^{-1}$ acting on this space has the quantity $-1/\alpha_g$ all over the upper diagonal and zeros elsewhere. If we now apply this operator to $E_0^{(p)} g_\alpha$, written in this basis as $\sum \lambda_i e_i$, what we get in the first non-zero component is $-1/\alpha_g \cdot \lambda_2$. That is, the second vector of the generalized eigenbasis is the one which encodes the p-adic L-value. Therefore, one may consider two different classes.

(a) The class $\mathrm{Eis}^{[g,g^*,\ell-1]}$, where ℓ is the weight of ℓ, is related with the first coefficient in the expansion in the generalized eigenbasis (see [KLZ17, Corollary 6.5.7]). This controls the p-stabilization of the Eisenstein series.

(b) The derived class $\kappa'(g, g^*)$, constructed in Chapter 3, is related with the p-adic L-value, and hence with second coefficient in the generalized eigenbasis. This measures the p-depletion of the Eisenstein series.

Hence, when $g \in S_\ell(N, \chi_g)$ is an ordinary modular form of weight $\ell \geq 2$, the two classes

$$\{\mathrm{Eis}^{[g,g^*,\ell-1]}, \kappa'(g, g^*)\}$$

are a priori unrelated.

Question 5.5.2. Can we interpret the class $\mathrm{Eis}^{[g,g^*,\ell-1]}$ in terms of some p-adic L-value? (While the p-depleted class is connected with the usual p-adic L-value, a priori there is no natural p-adic avatar encoding the value of the p-stabilized class).

Observe that in the setting of diagonal cycles of [BSV20a], the authors take a different approach to the vanishing phenomenon, defining an *improved* class which is a putative geometric refinement to the analogue of the Eisenstein class, and which agrees up to some $\mathcal{L}$-invariant with the derived class.

5.6 A conjectural p-adic L-function

It is a somewhat vexing fact that our computations regarding the $\mathcal{L}$-invariant of the adjoint of a weight one modular form only captures a 2×2 regulator encoding information about both a unit and a p-unit, while the most natural object to work would be the unit itself, in the spirit of the Gross–Stark conjectures. Similarly, one would expect to be able to construct an *Eisenstein family*, in such a way that appropriate special values of it also capture information about the Beilinson–Flach classes, in a way that we now make precise. This section is purely conjectural, and must be regarded as a failure in our current work, where we have not succeeded in studying these aspects.

Consider the most general setting in which g and h are two weight one modular forms. As we have already recalled, there are four Beilinson–Flach classes attached to the choice of p-stabilizations of g and h,

$$\kappa(g_\alpha, h_\alpha), \quad \kappa(g_\alpha, h_\beta), \quad \kappa(g_\beta, h_\alpha), \quad \kappa(g_\beta, h_\beta). \tag{5.25}$$

We know that different components of it are related to special values of p-adic L-functions. Take for instance the case of $\kappa(g_\alpha, h_\alpha)$. Considering its restriction to a decomposition group at p, we may consider a decomposition of $\kappa_p(g_\alpha, h_\alpha)$ of the form

$$\kappa_p^{--}(g_\alpha, h_\alpha) \otimes e_{\beta\beta}^\vee \oplus \kappa_p^{-+}(g_\alpha, h_\alpha) \otimes e_{\beta\alpha}^\vee \oplus \kappa_p^{+-}(g_\alpha, h_\alpha) \otimes e_{\alpha\beta}^\vee \oplus \kappa_p^{++}(g_\alpha, h_\alpha) \otimes e_{\alpha\alpha}^\vee, \tag{5.26}$$

where $\{e_{\alpha\alpha}^\vee, e_{\alpha\beta}^\vee, e_{\beta\alpha}^\vee, e_{\beta\beta}^\vee\}$ is a basis of $V_{gh}^\vee$ with the conventions fixed in Chapter 4.

According to [KLZ17, Proposition 8.2.6], the component $\kappa_p^{--}(g_\alpha, h_\alpha) = 0$ vanishes. In the same way, the components $\kappa_p^{-+}(g_\alpha, h_\alpha)$ and $\kappa_p^{+-}(g_\alpha, h_\alpha)$ are related to the special values of the Hida–Rankin p-adic L-functions $\mathscr{L}_p{}^{g_\alpha}$ and $\mathscr{L}_p{}^{h_\alpha}$, respectively. Hence, it is natural to expect that the remaining component $\kappa_p^{++}(g_\alpha, h_\alpha)$ could arise as the special value of some p-adic L-function. Following the analogy with the case of diagonal cycles and triple product p-adic L-functions, it would be attached to the triple $(E_2(1, \chi_{gh}^{-1}), g, h)$, but varying over the region where the Eisenstein family is dominant. Of course this is not possible, but let us work formally recasting to the theory of Perrin-Riou maps. In particular, we may consider the three-variable cohomology class $\kappa(\mathbf{g}, \mathbf{h})$, take the restriction to the line where both g and h are fixed and take the image under the Perrin-Riou map. That way we would get an element over the Iwasawa algebra that we may denote $L_p^{\mathrm{Eis}}(g, h, s)$. It may be instructive to compare this with the scenario of triple products, where the existence of a third p-adic L-function $\mathscr{L}_p{}^f$, which at points of weight $(2, 1, 1)$ interpolates classical L-values, provides a richer framework and draws a more complete picture.

To simplify things, let us focus again on the case where both g and h are self dual, that is $h = g^*$. Recall that this situation naturally splits in two scenarios, namely $h_\alpha = g_{1/\beta}^*$ and $h_\alpha = g_{1/\alpha}^*$. As we have mentioned before, we expect the previous *cyclotomic* p-adic L-function to encode information about the logarithm of the unit u, and not about the apparently complicated regulator of our main result.

More concretely, assume that $\alpha_g \alpha_h = 1$, and take the class $\kappa(g_\alpha, g_{1/\alpha}^*)$, although the same works verbatim for $\kappa(g_\beta, g_{1/\beta}^*)$. Consider the map

$$\langle \mathcal{L}_{gg^*}^{++}, \omega_g \otimes \omega_{g^*} \rangle \,:\, H^1(\mathbb{Q}_p, V_{gg^*}^{++}(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1})) \longrightarrow I^{-1} \Lambda_{\mathbf{g}}, \tag{5.27}$$

whose specializations are given by

$$\nu_s(\langle \mathcal{L}_{gg^*}^{++}, \omega_g \otimes \omega_{g^*} \rangle) \,:\, H^1(\mathbb{Q}_p, V_{gg^*}^{++}(1-s)) \longrightarrow \mathbb{C}_p,$$

where

$$\nu_s(\langle \mathcal{L}_{gg^*}^{++}, \omega_g \otimes \omega_{g^*} \rangle) = \frac{1 - p^{s-1}}{1 - p^{-s}} \cdot \begin{cases} \langle \frac{(-1)^s}{(-s)!} \cdot \langle \log_{\mathrm{BK}}, \omega_g \otimes \omega_{g^*} \rangle & \text{if } s < 0 \\ (s-1)! \cdot \langle \exp_{\mathrm{BK}}^*, \omega_g \otimes \omega_{g^*} \rangle & \text{if } s > 1. \end{cases}$$

Observe that we have not said anything about the specializations at $s = 0$ and at $s = 1$.

When $s = 0$ (resp. $s = 1$), we are still in the region of the Bloch–Kato logarithm (resp. dual exponential map), but the expression $1 - p^{-s}$ (resp. $1 - p^{s-1}$) vanishes. Hence, and following [LVZ15, Proposition 2.5.5] (see also the computations of [Ven16, Section 3.1] and [Buy12, Section 6.3]), we expect to be able to establish the following result.

Expected Lemma 5.6.1. The map

$$\nu_0(\langle \mathcal{L}_{gg^*}^{++}, \omega_g \otimes \omega_{g^*} \rangle) \,:\, H^1(\mathbb{Q}_p, V_{gg^*}^{++}(1)) \longrightarrow \mathbb{C}_p \tag{5.28}$$

is given by

$$\nu_0(\langle \mathcal{L}_{gg^*}^{++}, \omega_g \otimes \omega_{g^*} \rangle) = (1 - p^{-1}) \cdot \langle \mathrm{ord}_p, \omega_g \otimes \omega_{g^*} \rangle. \tag{5.29}$$

Remark 5.6.2. The situation is slightly different to that of circular units: there, the fact of taking the derived class was related to the fact that the Coleman map was connected to an imprimitive p-adic L-function, vanishing at the point of interest and whose derivative there corresponds to the special value of the Kubota–Leopoldt p-adic L-function.

Define

$$L_p^{\mathrm{Eis}}(g,h,s) = \langle \mathcal{L}_{gg^*}^{++}(\kappa_p^{++}(g_\alpha, g^*_{1/\alpha}, s)), \omega_g \otimes \omega_{g^*} \rangle, \tag{5.30}$$

where $\kappa(g_\alpha, g^*_{1/\alpha}, s)$ is the restriction of the 3-variable cohomology class to the cyclotomic line.

As a piece of notation for the following result, let $\mathcal{L}_{g_\alpha}$ stand for the period ratio introduced in [DR16, Section 2]. This quantity will be extensively studied in Chapter 7.

Proposition 5.6.3. *Assume that Lemma 5.6.1 holds. Then, the special value of the derivative of $L_p^{\mathrm{Eis}}(g_\alpha, g^*_{1/\alpha}, 1)$ satisfies that*

$$L_p^{\mathrm{Eis}}(g_\alpha, g^*_{1/\alpha}, 1) = \frac{\mathcal{L}_{g_\alpha}}{\log_p(u_{\alpha\otimes\bar\beta})} \times \log_p(u_1) \pmod{L^\times}.$$

Proof. When $s=0$, the denominator of the Perrin-Riou map $\mathcal{L}_{gg^*}^{++}$ vanishes and we are in the setting discussed before. Then, the Perrin-Riou map is given by the order followed by the pairing with the canonical differentials, as in (5.29). Since according to the results of Chapter 4

$$\kappa(g_\alpha, g^*_{1/\alpha}) = \frac{1}{\Xi_{g_\alpha}\Omega_{g^*_{1/\alpha}}} \frac{\log_p(u_1)\cdot v - \log_p(v_1)\cdot u}{\log_p(u_{\alpha\otimes\bar\beta})} \pmod{L^\times},$$

the image of $\kappa_p(g_\alpha, g^*_{1/\alpha})$ under the map (5.29) agrees with

$$\log_p(u_1)\cdot \frac{\mathcal{L}_{g_\alpha}}{\log_p(u_{\alpha\otimes\bar\beta})} \pmod{L^\times}.$$

□

Further, recall that according to [DR16, Conjecture 2.3], we expect that $\mathcal{L}_{g_\alpha}$ agrees with $\log_p(u_{\alpha\otimes\bar\beta})$ up to multiplication by $L^\times$, and this would give just $\log_p(u_1)$ in the previous formula.

As a final comment, observe that the class $\kappa(g_\alpha, g^*_{1/\beta})$ vanishes, while neither the numerator nor the denominator of the Perrin-Riou map do. Hence, we expect the special value to be zero. However, it would be licit to take the derivative of both the class and the p-adic L-function. We hope to be able to expand this discussion to really find out until which extent the techniques developed in this chapter could lead to significant results for the study of the conjectures of Harris and Venkatesh, and understand the relation with the approach developed on the ongoing work of Darmon, Harris, Rotger and Venkatesh [DHRV20].

Chapter 6

The exceptional zero phenomenon for elliptic units

In this chapter we focus on the elliptic units of an imaginary quadratic field and study this exceptional zero phenomenon, proving an explicit formula relating the logarithm of a *derived* elliptic unit either to special values of the Katz's two variable p-adic L-function or to its derivative. Further, we interpret this fact in terms of an $\mathcal{L}$-invariant and relate this result to other approaches to the exceptional zero phenomenon concerning Heegner points and Beilinson–Flach elements.

The results presented at this chapter are the content of the research article [Ri20a].

6.1 Introduction

Since the introduction of the exceptional zero phenomenon for the Kubota–Leopoldt p-adic L-function by Ferrero and Greenberg [FG78] and for the p-adic L-function attached to an elliptic curve by Mazur, Tate and Teitelbaum [MTT86], a lot of progress has been made in the study of this topic. The main goal of this chapter is to study exceptional zero phenomena for Katz's two-variable p-adic L-function at points lying *outside* the region of classical interpolation, where the Euler system of elliptic units vanishes. Hence, our setting departs notably from loc. cit. and is closer in spirit to the study of e. g. [Cas18a] and [RR20a], sharing some points in common with earlier work of Solomon [Sol92] and Bley [Ble04], and also with the recent preprint of Büyükboduk and Sakamoto [BS19].

Fix once and for all a prime p and a quadratic imaginary field K in which p splits, and fix embeddings $\mathbb{C} \hookleftarrow \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ be the two prime ideals lying over p, with $\mathfrak{p}$ the prime above p induced by the previously fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ (assume in the introduction for notational simplicity that both ideals are principal, and with a slight abuse of notation we denote by $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ their generators). Let K_∞^{cyc} and K_∞^{ac} be the cyclotomic and anticyclotomic $\mathbb{Z}_p$-extensions of K, respectively, and set $K_\infty = K_\infty^{\mathrm{ac}} K_\infty^{\mathrm{cyc}}$. Denote $\Gamma_K = \mathrm{Gal}\,(K_\infty/K)$ and $\Lambda_K = \mathbb{Z}_p[[\Gamma_K]]$. The latter is a Galois module with an appropriate *tautological action* that we later recall. The weight space is the formal spectrum $\mathrm{Spf}(\Lambda_K)$ of the two-variable Iwasawa algebra Λ_K. Let ψ stand for a Hecke character of finite order, conductor $\mathfrak{n}$ and taking values in a number field L, and let $\mathcal{N}$ denote the norm character of K. We denote by N the norm of $\mathfrak{n}$, and assume that $(N,p) = 1$. Finally, let L_p denote a completion of L at the prime p. The Hecke character ψ may be also understood as a Galois character $\psi : G_K \to L^\times$; the notation ψ' will be used to designate the composition of ψ with conjugation by the non-trivial element in $\mathrm{Gal}\,(K/\mathbb{Q})$: $\psi'(\sigma) = \psi(\tau\sigma\tau^{-1})$, where τ is any element of $G_{\mathbb{Q}}$ which acts non-trivially on K.

As an additional piece of notation, let H_∞ denote the unique $\mathbb{Z}_p$-extension of K in which $\bar{\mathfrak{p}}$ is unramified (therefore, the prime $\mathfrak{p}$ is ramified in H_∞/K). We choose a Galois character λ of Γ_K, so that it factors through $\mathrm{Gal}\,(H_\infty/K)$, defining an isomorphism

$$\mathrm{Gal}\,(H_\infty/K) \longrightarrow 1 + p\mathbb{Z}_p.$$

The choice of λ is unique once we require that it is the Galois representation corresponding to a Grössencharacter for K of infinity type $(1,0)$. We define in the same way an extension H'_∞ and a character λ', exchanging the roles of $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. Although characters of Γ_K are elements of $\mathrm{Hom}_{\mathrm{cont}}(G_K, \bar{\mathbb{Q}}_p^\times)$, we are interested in the restriction to the subspace of $\mathrm{Hom}_{\mathrm{cont}}(G_K, \bar{\mathbb{Q}}_p^\times)$ given by

$$\Sigma_\psi := \{\psi\xi\mathcal{N}^s\lambda^t \text{ such that } (s,t) \in \mathbb{Z}_p^2\},$$

where ξ is a finite order character of p-power conductor.

One can naturally associate, via the theory of elliptic units of Robert [Rob71] (see also de Shalit and Yager's approaches [deS87], [Yag82]) and Kummer maps, a global cohomology class to ψ

$$\kappa_{\psi,\infty} \in H^1(K, \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N})). \tag{6.1}$$

We denote by $\kappa_\psi \in H^1(K, L_p(\psi^{-1}))$ and $\kappa_{\psi\mathcal{N}} \in H^1(K, L_p(\psi^{-1})(\mathcal{N}))$ the specializations of $\kappa_{\psi,\infty}$ at ψ and $\psi\mathcal{N}$, respectively. In section 4, we prove that κ_ψ never vanishes, while $\kappa_{\psi\mathcal{N}}$ vanishes if and only if $\psi(\mathfrak{p}) = 1$ or $\psi(\bar{\mathfrak{p}}) = 1$. In these cases, there exists a notion of *derived* cohomology class along the subvariety $\mathcal{C}'$ of weight space, that we define as the Zariski closure of the points $\mathcal{N}\bar{\lambda}^t$ with $t \in \mathbb{Z}^{\geq 0}$ in an appropriate, sense we make precise later on,

$$\kappa'_{\psi,\infty} \in H^1(K, \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N})_{|\mathcal{C}'}).$$

The specialization of this cohomology class at $\psi\mathcal{N}$ encodes the arithmetic information which is given by $\kappa_{\psi\mathcal{N}}$ in a non-exceptional situation. To be more explicit, denote by $u_\mathfrak{n}$ a fixed choice of an elliptic unit of conductor $\mathfrak{n}$, and to lighten notations, define

$$u_\psi = \prod_{\sigma \in \mathrm{Gal}\,(K_\mathfrak{n}/K)} (\sigma u_\mathfrak{n})^{\psi^{-1}(\sigma)} \in (\mathcal{O}_{K_\mathfrak{n}}^\times \otimes L)^\psi, \tag{6.2}$$

where $K_\mathfrak{n}$ is the field cut out by ψ. We expect a relation between $\kappa'_{\psi\mathcal{N}}$ and u_ψ, since they both lie in the same space (after applying a Kummer map), and this is the content of the main result of this chapter, which we now state.

Assume that $\psi(\mathfrak{p}) = 1$, and let us define the following $\mathcal{L}$-invariant

$$\mathcal{L}(\psi) = (1 - \psi^{-1}(\bar{\mathfrak{p}})) \cdot \log_p(\mathfrak{p}), \tag{6.3}$$

where $\log_p$ stands for the usual p-adic Iwasawa logarithm.

Then, we have the following result, proved in Section 6.4.

Theorem 6.1.1. *Suppose that $\psi(\mathfrak{p}) = 1$. Then,*

$$\kappa'_{\psi\mathcal{N}} = \mathcal{L}(\psi) \cdot u_\psi. \tag{6.4}$$

Although the previous result does not require any explicit mention to the theory of p-adic L-functions, it is fair to say that Katz's two variable p-adic L-function plays a prominent role in our results. The interplay between the Euler system of elliptic units and Katz's two variable p-adic L-function can be set as a very particular case of a wider theory. One may distinguish two main approaches to *construct* a p-adic L-function.

(a) Firstly, interpolating the algebraic parts of the special values of the classical L-function along the so-called *critical* region. This requires, as a starting point, the proof of certain algebraicity results.

(b) Secondly, as the image under a certain Perrin-Riou map of a family of cohomology classes, constructed along the so-called *geometric* region. These classes are typically obtained as the image under certain regulators of distinguished elements arising from the geometry of algebraic varieties.

In both approaches, the p-adic L-function is completely characterized by the value at the points lying either at the *critical* or at the *geometric* region. Moreover, some Euler factors arise, measuring the discrepancy between the interpolation of L-values in the critical region and the interpolation of cohomology classes in the geometric region. The vanishing of these factors lead us to study exceptional zero phenomena. In the case of the Perrin-Riou map, the shape of these factors is $\frac{1-p^j\phi}{1-p^{-1-j}\phi^{-1}}$, where j is related to the Hodge–Tate type of the character at which we are specializing, and ϕ refers to a Frobenius eigenvalue. As it is suggested for instance in [KLZ17, Section 8] or [LZ14], there are two kinds of Euler factors in the usual Perrin-Riou maps: those appearing in the numerator (which typically lead to an exceptional vanishing of the p-adic L-function via explicit reciprocity laws) and those appearing in the denominator (which lead to an exceptional vanishing of the cohomology class). While the former phenomenon has been widely studied, as far as we know the latter has only been discussed with the tools from Perrin-Riou theory in the setting of Heegner points in [Cas18a] and for Beilinson–Flach elements in [RR20a] (see also Chapter 3 of this memoir). Nevertheless, similar results have been obtained by Bley [Ble04], although there are some differences we later discuss.

Katz's two-variable p-adic L-function $L_{\mathfrak{p}}(K)(\cdot)$ is defined on the domain $\mathrm{Hom}_{\mathrm{cont}}(G_K, \bar{\mathbb{Q}}_p^\times)$, but we can consider its restriction to

$$\Sigma_\psi = \{\psi\xi\mathcal{N}^s\lambda^t \text{ such that } (s,t) \in \mathbb{Z}_p^2\}, \tag{6.5}$$

where ξ is a finite order character of p-power conductor. This allows us to make use of the techniques relative to p-adic variation, sharing some points in common with Hida theory. We write $L_{\mathfrak{p}}(K,\psi)(\cdot)$ for the restriction of $L_{\mathfrak{p}}(K)(\cdot)$ to the subspace of characters given in (6.5), and denote $L_{\mathfrak{p}}(K,\psi)(\chi_{\mathrm{triv}}) := L_{\mathfrak{p}}(K)(\psi)$ and $L_{\mathfrak{p}}(K,\psi)(\mathcal{N}) := L_{\mathfrak{p}}(K)(\psi\mathcal{N})$. In our case, and because of the dualities involved in the Perrin-Riou formalism, we are also interested in the function $L_{\mathfrak{p}}(K,(\psi')^{-1})$.

We can now describe the main ingredients involved in the proof of Theorem 6.1.1.

(a) An **explicit reciprocity law** for Katz's two-variable p-adic L-function due to Yager. This expresses the special value $L_{\mathfrak{p}}(K,(\psi')^{-1})(\mathcal{N})$ in terms of the image under a Perrin-Riou map of the cohomology class $\kappa_{\psi\mathcal{N}}$, and directly gives us that $\mathrm{loc}_{\mathfrak{p}}(\kappa_{\psi\mathcal{N}}) = 0$ when $\psi(\mathfrak{p}) = 1$, due to the vanishing of an Euler factor. Here, $\mathrm{loc}_{\mathfrak{p}}$ stands for the localization at $\mathfrak{p}$. The explicit description of the localization-at-$\mathfrak{p}$ map shows that we can conclude that $\kappa_{\psi\mathcal{N}} = 0$ and consider the *derived* cohomology class. We refer the reader to Sections 6.3 and 6.4 for a proper definition of *derived* class and for more details on that.

(b) A **derived reciprocity law**, expressing the Bloch–Kato logarithm of the *derived* class in terms of $L_{\mathfrak{p}}(K,(\psi')^{-1})(\mathcal{N})$. This requires an explicit description of the Perrin-Riou map, which at the norm character interpolates the Bloch–Kato logarithm and gives a map

$$\log_{\mathrm{BK}} : H^1(K_{\mathfrak{p}}, L_p(\psi^{-1})(\mathcal{N})) \longrightarrow \mathbb{D}_{\mathrm{dR}}(L_p(\psi^{-1})) \simeq L_p.$$

Under the identification induced by the Kummer morphism, this map corresponds, in a sense that we later make precise, to the usual p-adic logarithm. Then, we have the following result, whose proof is given in Section 6.4.

Proposition 6.1.2. *Assume that* $\psi(\mathfrak{p}) = 1$. *Then,*

$$\log_p(\mathfrak{p}) \cdot L_{\mathfrak{p}}(K,(\psi')^{-1})(\mathcal{N}) = -(1-p^{-1}) \cdot \log_p(\mathrm{loc}_{\mathfrak{p}}(\kappa'_{\psi\mathcal{N}})).$$

(c) The **functional equation** for Katz's two-variable p-adic L-function (see [Gro80, p.90–91]), which asserts that

$$L_{\mathfrak{p}}(K,(\psi')^{-1})(\mathcal{N}) = L_{\mathfrak{p}}(K,\psi)(\chi_{\mathrm{triv}}).$$

(d) **Katz's p-adic version of Kronecker limit formula**, expressing the special value of $L_{\mathfrak{p}}(K,\psi)$ at the trivial character in terms of the elliptic unit u_ψ

$$L_{\mathfrak{p}}(K,\psi)(\chi_{\mathrm{triv}}) = -(1-\psi^{-1}(\bar{\mathfrak{p}}))(1-\psi(\mathfrak{p})p^{-1}) \cdot \log_p(u_\psi).$$

In Section 6.3 we properly discuss the main features of Katz's two-variable p-adic L-function.

As a by-product of the previous discussion, along the text we also deal with other instances of the exceptional zero phenomenon. The results of Section 4 encompass two main situations: the exceptional vanishing of $\kappa_{\psi\mathcal{N}}$; and the exceptional vanishing of the p-adic L-function $L_{\mathfrak{p}}(K,\psi)$, which is a more well-established phenomenon that has been widely studied in the literature and already appears in Katz's original work.

Once these results have been developed, the last section of the chapter serves to analyze how our results fit with similar statements concerning exceptional zero phenomena. In particular, we emphasize the parallelism, but also the differences, with the theory of Heegner points, as well as the fact that these elliptic units may be seen as a particular case inside the theory of Beilinson–Flach elements, where different instances of the exceptional zero phenomena also appear. When g is a theta series of an imaginary quadratic field where p splits and we take the pair of modular forms (g,g^*), Chapter 3 describes a connection between a *derived* Beilinson–Flach element, an elliptic unit and an special value of the Hida–Rankin p-adic L-function attached to (g,g^*). The assumptions considered in loc. cit. (we had imposed that the Galois representation attached to g was p-distinguished) excluded the possibility of elliptic units presenting an *exceptional zero*, so in a certain way the results of this chapter regarding exceptional zeros of elliptic units can be thought as a degenerate case inside the theory of Beilinson–Flach elements. While our main theorem can be seen as the counterpart of [RR20a, Theorem B] (see also Chapter 3) in the framework of elliptic units, we point out that there is another exceptional zero phenomenon related to the vanishing of the numerator of the Perrin-Riou map, which in this case leads to a trivial zero of the Katz's two variable p-adic L-function (see Section 4.1) and which in the setting of Beilinson–Flach elements has been studied in [LZ17].

6.2 Circular units

Circular units constitute one of the first examples of Euler systems, and they play a key role in the proof of the classical Iwasawa main conjecture. We recall here some of their most relevant features because of the parallelism they keep with the theory of elliptic units. We discuss what the exceptional zero phenomenon represents in this case, and then we will compare this setting with that of elliptic units.

Leopoldt's formula

In this section, we denote by $\Lambda := \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, and let $\mathcal{W} := \mathrm{Spf}(\Lambda)$. We fix a primitive, non-trivial even Dirichlet character of conductor N,

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to L^\times$$

where L is a number field and $(p,N)=1$. We write L_p for its completion at a prime lying above p. For our applications to exceptional zero phenomena, we are interested in the case in which $\chi(p)=1$.

The Kubota–Leopoldt p-adic L-function attached to χ, $L_p(\chi, s)$, can be defined as the p-adic analytic function satisfying the interpolation property

$$L_p(\chi, n) = (1 - \chi(p)p^{-n})L(\chi, n), \quad \text{for all } n \leq 0.$$

Alternatively, we may understand it as a function defined over an appropriate rigid analytic space, sometimes called the *weight space.*

Definition 6.2.1. A *classical point* of $\mathcal{W}$ is a pair (k, ξ), where k is an integer and ξ is a Dirichlet character of p-power conductor, corresponding to the homomorphism

$$z \mapsto z^{k-1}\xi(z).$$

Then, the Kubota–Leopoldt p-adic L-function can be seen as an application

$$L_p(\chi, \cdot) : \quad \mathcal{W} \longrightarrow \mathbb{C}_p$$

defined in terms of an interpolation property for a subset of classical points. We warn the reader that there are several possible conventions regarding this function. Here, we closely follow the approach of [PR94, Section 3], and the p-adic L-function we have considered satisfies the interpolation property of Proposition 3.1.4 of loc. cit. See also [Tale14] for a reformulation of those ideas in our language. Another standard way of presenting this p-adic L-function is discussed in [Das99, Section 3], and we will come back to this issue later on; there, the interpolation property involves appropriate twists by powers of the Teichmüller character, but both approaches are closely connected as shown in [PR94, Section 3.1.5]: in particular, the p-adic L-values at integers n with $n \equiv 1$ modulo $p-1$ agree.

Let H denote the field cut out by χ, and for a choice of a primitive p^n-th root of unity ζ_{p^n}, let $H_n = H(\zeta_{p^n})$. Define the units

$$c_{\chi,n} := \prod_{a=1}^{N-1} (1 - \zeta_{Np^n}^a)^{\chi^{-1}(a)} \in (\mathcal{O}_{H_n}^\times \otimes L)^\chi, \tag{6.6}$$

that behave under the norm maps as dictated by the theory of Euler systems:

$$\mathcal{N}_{H_n}^{H_{n+1}}(c_{\chi,n+1}) = \begin{cases} c_{\chi,n} & \text{if } n \geq 1, \\ c_\chi \otimes (\chi(p) - 1) & \text{if } n = 0. \end{cases},$$

where $c_\chi = c_{\chi,0}$. As a word of caution, note that we have used the standard multiplicative notation, where the exponentiation $(1 - \zeta_{Np^n}^a)^{\chi^{-1}(a)}$ means $(1 - \zeta_{Np^n}^a) \otimes \chi^{-1}(a)$.

Hence, one can construct a norm compatible family of cohomology classes taking the image under the Kummer map δ. More precisely, we consider

$$\kappa_{\chi,n} := \delta(c_{\chi,n}) \in H^1(H_n, L_p(1))^\chi = H^1(H_n, L_p(\chi^{-1})(1)). \tag{6.7}$$

As in previous chapters, let

$$\underline{\varepsilon}_{\text{cyc}} : G_{\mathbb{Q}} \to \Lambda^\times$$

denote the Λ-adic cyclotomic character, sending a Galois element σ to the group-like element $[\varepsilon_{\text{cyc}}(\sigma)]$. Recall that it interpolates the powers of the $\mathbb{Z}_p$-cyclotomic character, in the sense that for any arithmetic point $\nu_{r,\xi} \in \mathcal{W}^{\text{cl}}$,

$$\nu_{r,\xi} \circ \underline{\varepsilon}_{\text{cyc}} = \xi \cdot \varepsilon_{\text{cyc}}^{r-1}. \tag{6.8}$$

These classes can be patched all together taking the projective limit for $n \geq 1$, resulting in an element $\kappa_{\chi,\infty}$

$$\kappa_{\chi,\infty} \in \varprojlim H^1(H_n, L_p(\chi^{-1})(1)) = H^1(\mathbb{Q}, L_p(\chi) \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}})). \tag{6.9}$$

Let $L_{\xi,p}$ stand for the compositum of L_p with the field of values of ξ. The specialization maps $\nu_{k,\xi} : \Lambda \to L_{\xi,p}$ are ring homomorphisms sending the group-like element $a \in \mathbb{Z}_p^\times$ to $a^{k-1}(\chi\xi)^{-1}(a)$, and induce $G_{\mathbb{Q}}$-equivariant specialization maps

$$\nu_{k,\xi} : \Lambda(\underline{\varepsilon}_{\mathrm{cyc}}) \to L_{\xi,p}(\xi^{-1})(k-1).$$

This gives rise to a collection of global cohomology classes

$$\kappa_{k,\chi\xi} := \nu_{k,\xi}(\kappa_{\chi,\infty}) \in H^1(\mathbb{Q}, L_{\xi,p}(\chi\xi)^{-1}(k)). \tag{6.10}$$

In order to state the following result, recall that the Gauss sum associated to a Dirichlet character η of conductor m and with values in a number field L is defined by

$$\mathfrak{g}(\eta) = \sum_{a=1}^{m-1} \zeta_m^a \otimes \eta(a) \in \mathcal{O}_{\mathbb{Q}(\zeta_m)}^\times \otimes L. \tag{6.11}$$

From now on, $\exp^*_{\mathrm{BK}}$ stands for the Bloch–Kato dual exponential map and $\log_{\mathrm{BK}}$ for the Bloch–Kato logarithm. The following proposition is a reformulation of a classical result by Coleman [Co79], using the formalism of Perrin-Riou regulators developed in [PR94].

Proposition 6.2.2 (Coleman, Perrin-Riou)**.** *There exists a morphism of Λ-modules (referred to as the Coleman or the Perrin-Riou map)*

$$\mathcal{L}_p : H^1(\mathbb{Q}_p, L_p(\chi) \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}})) \longrightarrow I^{-1}\Lambda$$

satisfying that for all classical points (k,ξ), the specialization map $\nu_{k,\chi\xi}(\mathcal{L}_p)$ is the homomorphism

$$\nu_{k,\chi\xi}(\mathcal{L}_p) : H^1(\mathbb{Q}_p, L_{\xi,p}(\chi\xi)^{-1}(k)) \longrightarrow \mathbb{D}_{\mathrm{dR}}(L_{\xi,p}((\chi\xi)^{-1})(k)) \simeq L_{\xi,p}$$

given by

$$\nu_{k,\chi\xi}(\mathcal{L}_p) = \frac{1}{\mathfrak{g}((\chi\xi)^{-1})} \cdot \frac{1-\chi\xi(p)p^{-k}}{1-(\chi\xi)^{-1}(p)p^{k-1}} \cdot \begin{cases} \frac{(-t)^k}{(k-1)!}\log_{\mathrm{BK}} & \text{if } k \geq 1 \\ (-k)!t^k \exp^*_{\mathrm{BK}} & \text{if } k < 1, \end{cases}$$

where t is Fontaine's p-adic analogue of $2\pi i$, and the target of both the Bloch–Kato logarithm and the dual exponential map is identified with $L_{\xi,p}$. Here, I is the kernel of the specialization at $\nu_{1,\chi}$.

We finally relate the image of the previously introduced class $\kappa_{\chi,\infty}$ under the Perrin-Riou regulator with the Kubota–Leopoldt p-adic L-function. See for instance [PR94] for a more detailed treatment of this result. Here, loc_p stands for the localization at p of a global cohomology class.

Theorem 6.2.3. *Let χ stand for a non-trivial and even Dirichlet character. Then, the cohomology class $\kappa_{\chi,\infty} \in H^1(\mathbb{Q}, L_p(\chi) \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}))$ introduced in* (6.9) *satisfies*

$$L_p(\chi,\cdot) = \mathcal{L}_p(\mathrm{loc}_p(\kappa_{\chi,\infty})).$$

The previous theorem can be seen as an equality in Λ, and we may apply the specialization maps to both sides at any $\mathbb{C}_p$-valued point. From the previous results, and using Kummer's identifications again, it turns out that one has the equality

$$L_p(\chi\xi, 1) = -\frac{(1-\chi\xi(p)p^{-1})}{(1-(\chi\xi)^{-1}(p))} \cdot \frac{\log_p(\mathrm{loc}_p(\kappa_{1,\chi\xi}))}{\mathfrak{g}((\chi\xi)^{-1})}, \tag{6.12}$$

whenever $(\chi\xi)(p) \neq 1$; if $(\chi\xi)(p) = 1$, both the Euler factor in the denominator and the cohomology class in the numerator vanish. Recall that here we have identified $t \cdot \log_{\mathrm{BK}}$ with the Iwasawa p-adic logarithm.

Since χ is non-trivial, one has $L_p(\chi, 1) \in \bar{\mathbb{Q}}_p^\times$. This suggests the existence of a *derived* cohomology class $\kappa'_{1,\chi}$ related with $L_p(\chi, 1)$, which is the content of the following section. We recover this idea along the chapter, but anyway it is good to keep in mind that in this setting one also has a p-adic Kronecker's limit formula expressing the value of $L_p(\chi, 1)$ in terms of a unit in the number field cut out by the character

$$L_p(\chi, 1) = -\frac{(1 - \chi(p)p^{-1})}{\mathfrak{g}(\chi^{-1})} \cdot \log_p \Big(\prod_{a=1}^{N-1} (1 - \zeta^a)^{\chi^{-1}(a)} \Big). \tag{6.13}$$

This result is generally due to Leopoldt (see also [PR94]), and is often called in the literature *Leopoldt's formula*. The quantity $\prod_{a=1}^{N-1}(1 - \zeta^a)^{\chi^{-1}(a)}$ is typically referred as the *circular unit* attached to χ and we have denoted it by c_χ.

As we have pointed out, we may instead consider a slightly different p-adic L-function, that we denote $L_{p,1}(\chi, n)$, and defined in terms of the interpolation property

$$L_{p,1}(\chi, n) = (1 - \chi\omega^{n-1}(p)p^{-n}) \cdot L(\chi\omega^{n-1}, n) \quad \text{for all } n \leq 0,$$

where ω stands for the modulo p cyclotomic character (Teichmüller). A very interesting object of study in the theory of $\mathcal{L}$-invariants is $L'_{p,1}(\chi, 0)$, in the case where $\chi\omega^{-1}(p) = 1$ and therefore $L_{p,1}(\chi, 0) = 0$ due to an exceptional vanishing. Washington [Was81] provides a formula for the value of the derivative in terms of Morita's p-adic Gamma function, $\Gamma_p(x)$:

$$L'_{p,1}(\chi, 0) = \log_p \Big(\prod_{a=1}^{N} \Gamma_p(a/N)^{\chi\omega^{-1}(a)} \Big) + L_{p,1}(\chi, 0) \log_p(N). \tag{6.14}$$

Hence, in the situation of *exceptional vanishing* $\chi\omega^{-1}(p) = 1$, there is an exceptional zero for $L_{p,1}(\chi, s)$ at $s = 0$ and one has that

$$L'_{p,1}(\chi, 0) = \log_p(v_\chi), \tag{6.15}$$

where

$$v_\chi = \prod_{a=1}^{N} \Gamma_p(a/N)^{\chi\omega^{-1}(a)}.$$

In the case where one considers instead an *odd* Dirichlet character η with $\eta(p) = 1$, the determination of $L'_{p,1}(\eta\omega, 0)$ is a particular case of Gross' conjectures, as studied first by Ferrero–Greenberg [FG78], and then by Darmon–Dasgupta–Pollack (among others!) for arbitrary totally real fields. Here, this derivative is expressed in terms of the logarithm of a p-unit in the field cut out by the character.

Exceptional zeros and circular units

Suppose from now on that χ is a non-trivial, even, Dirichlet character of conductor N with $\chi(p) = 1$ and $(p, N) = 1$. Then, the arguments of the previous section show that the specialization of the Λ-adic class $\kappa_{\chi,\infty}$ at χ vanishes, that is, $\kappa_{1,\chi} = 0$. Of course, this can be interpreted in terms of the vanishing of the denominator of the Perrin-Riou map. This section, where no claim of originality is made, explains how to obtain a formula for $L_p(\chi, 1)$ involving $\kappa_{\chi,\infty}$, following for that the work of Solomon [Sol92] and Büyükboduk [Buy12], and also discusses how the vanishing of the numerator of the Perrin-Riou regulator at $s = 0$ can be studied inside the framework developed in [Ven16].

With the previous notations, let $T = L_p(\chi^{-1})(1)$ and $T^* = L_p(\chi)$, viewed as representations of $G_{\mathbb{Q}}$. We single out one of the $(p-1)$ connected components of $\mathcal{W}$, which corresponds to the choice of the residue class of 1 modulo $p-1$ and of the Iwasawa algebra $\Lambda_0 = \mathbb{Z}_p[[1+p\mathbb{Z}_p]] \subset \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. After fixing a topological generator γ of $1+p\mathbb{Z}_p$, one may consider the short exact sequence of $\mathbb{Z}_p$-modules

$$0 \to \Lambda_0(\underline{\varepsilon}_{\mathrm{cyc}}) \otimes T \xrightarrow{\gamma-1} \Lambda_0(\underline{\varepsilon}_{\mathrm{cyc}}) \otimes T \to T \to 0$$

which induces a long exact sequence in cohomology. Since $H^0(\mathbb{Q}, T) = 0$,

$$0 \to H^1(\mathbb{Q}, \Lambda_0(\underline{\varepsilon}_{\mathrm{cyc}}) \otimes T) \xrightarrow{\gamma-1} H^1(\mathbb{Q}, \Lambda_0(\underline{\varepsilon}_{\mathrm{cyc}}) \otimes T) \xrightarrow{\mathcal{N}} H^1(\mathbb{Q}, T).$$

The image of $\kappa_{\chi,\infty}$ under the map $\mathcal{N}$ vanishes since $\chi(p) = 1$, and hence there exists a unique

$$\kappa'_{\chi,\infty} \in H^1(\mathbb{Q}, \Lambda_0(\underline{\varepsilon}_{\mathrm{cyc}}) \otimes T)$$

such that

$$\frac{\gamma - 1}{\log_p(\gamma)} \cdot \kappa'_{\chi,\infty} = \kappa_{\chi,\infty}.$$

The reason of normalizing by $\log_p(\gamma)$ is, as discussed in [Buy12, Section 3], that the *derived* class does not longer depend on the choice of the topological generator γ.

Summing all up, we have the following result. We refer the reader to [Buy12, Proposition 3.4] for a more detailed discussion.

Proposition 6.2.4. *If $\chi(p) = 1$, the class $\kappa_{\chi,\infty}$ vanishes at the character χ and there exists a derived cohomology class $\kappa'_{\chi,\infty} \in H^1(\mathbb{Q}, \Lambda_0(\underline{\varepsilon}_{\mathrm{cyc}}) \otimes T)$ such that*

$$\kappa_{\chi,\infty} = \frac{\gamma - 1}{\log_p(\gamma)} \cdot \kappa'_{\chi,\infty}.$$

Remark 6.2.5. It may be tempting to look for a relation between $\kappa'_{1,\chi}$ and the special value $L_p(\chi, 1)$. However, the fact that the Euler factor $1 - p^{k-1}$ is not analytic in the variable k precludes the possibility of directly taking derivatives in the reciprocity law of Proposition 6.2.2. However, this can be remedied invoking Solomon's results, as we later see, connecting the order of the derived class with the special value at $s = 1$. In the following sections we discuss how in a bigger weight space certain derivatives are related with the p-adic logarithm and others with the p-adic valuation.

Let us provide a more explicit description of the previous result. In [Buy12] the author makes a connection between the value of $L_p(\chi, 1)$ and Nekovar's pairings. In [Buy12, Corollary 2.11] it is shown that

$$H^1_{\mathrm{f},p}(\mathbb{Q}, T) = (\mathcal{O}_H^\times[1/p])^\chi \otimes L_p,$$

is a two-dimensional space where we may explicitly construct a basis. Here, $\tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, T)$ stands for the Bloch–Kato Selmer group of classes which are unramified outside p and de Rham at p. As before, we have written H for the field cut out by χ. The fact that this space is two-dimensional reflects the exceptional zero coming from the condition $\chi(p) = 1$, which gives rise to an *extra* p-unit in the field cut out by the character.

Define

$$c_n = \mathcal{N}_{\mathbb{Q}(\zeta_{Nn})/H(\zeta_n)}(1 - \zeta_{Nn}) \in (\mathcal{O}^\times_{H(\zeta_n)} \otimes L),$$

and consider its χ-part, $c_{\chi,n}$. The element $c_\chi := c_{\chi,1}$ is called the *tame cyclotomic unit*, and agrees with the definition given in the previous section. For a finite abelian extension H' of $\mathbb{Q}$ of conductor m we also define

$$\xi_{H'} = \mathcal{N}_{\mathbb{Q}(\zeta_{mp})/H'}(1 - \zeta_{mp}).$$

With a slight abuse of notation, we may identify the units with the cohomology classes obtained via the Kummer map. Then, it turns out that the collection

$$\xi = \xi_{\chi,\infty} := \{e_\chi \xi_{H_n} \text{ for } n \geq 1\} \in \varprojlim H^1(H_n, T),$$

where for the sake of simplicity we have written e_χ for the χ-projector, satisfies the Euler system distribution relation, and moreover $\xi_H = 1$. Proceeding as before, we obtain an element $z_{\chi,\infty}$ satisfying

$$\frac{\gamma - 1}{\log_p(\gamma)} \times z_{\chi,\infty} = \xi.$$

We call its bottom layer $z_\chi := z_{0,\chi} \in H^1(\mathbb{Q}, T)$ the *cyclotomic p-unit*, and $\{c_\chi, z_\chi\}$ is a basis of $H^1_{\mathrm{f},p}(\mathbb{Q}, T)$. In [Sol92], it is proved that $\log_p(c_\chi) = \mathrm{ord}_p(z_\chi) \in L_p$, where ord_p is the usual p-adic valuation. Of course, this depends on the choice of a prime of L lying above p.

The interesting fact appears when the denominator of the Perrin-Riou map vanishes. To circumvent that problem, [Buy12, Section 6.1] recasts to the principle of improved Perrin-Riou map, which allows to introduce a *primitive* p-adic L-function $\tilde{L}_p(\chi, s)$ vanishing at $s = 1$. The main result of [Buy12] is the computation, via the theory of Nekovar's pairings, of a formula for $L_p(\chi, 1)$, which asserts that

$$\tilde{L}_p'(\chi, 1) = p \cdot L_p(\chi, 1) = \frac{1-p}{\mathfrak{g}(\chi^{-1})} \times \log_p(c_\chi) = \frac{1-p}{\mathfrak{g}(\chi^{-1})} \times \mathrm{ord}_p(z_\chi). \tag{6.16}$$

This also works for the case of an imaginary quadratic field when one only consider the $\mathbb{Z}_p$-extension which is ramified just over a fixed prime $\mathfrak{p}$ above p.

For the sake of completeness, we finish the section by analyzing what happens for $L_{p,1}(\chi, 0)$, where we may follow the approach of [Ven16, Section 3] to analyze the vanishing of the numerator in the Perrin-Riou map. To ease notations, let $\psi = \chi\omega^{-1}$ and write again L for its field of values. In particular, we know that when $\psi(p) = 1$, $L_{p,1}(\chi, 0) = 0$.

Since the numerator of the Perrin-Riou regulator vanishes, we can consider its derivative. Let I stand for the augmentation ideal of Λ. Then, we define the *derivative* of the Perrin-Riou map $\mathcal{L}_p$ of Proposition 6.2.2 as the application

$$\mathcal{L}_p' : H^1(\mathbb{Q}_p, \Lambda(\underline{\varepsilon}_{\mathrm{cyc}}) \otimes_{\mathbb{Z}_p} L_p(\psi^{-1})(1)) \to I/I^2,$$

i. e., the composition of $\mathcal{L}_p$ with the projection $\{\cdot\} : I \to I/I^2$.

Let $\kappa_\psi = (\kappa_{n,\psi})$ be the cohomology class we have previously introduced in (6.10). Following the same strategy as in [Ven16, Prop. 3.6], and identifying $\kappa_{0,\psi}$ with an element in $\mathrm{Hom}(\mathbb{Q}_p^\times, \mathbb{Q}_p) \otimes L_p(\psi^{-1})$, one has that

$$\mathcal{L}_p'(\kappa_\psi) = -\mathfrak{g}(\psi^{-1})^{-1}(1 - p^{-1})^{-1} \cdot \frac{\exp^*_{\mathrm{BK}}(\mathrm{loc}_p(\kappa_{0,\psi}))}{\log_p(\gamma)} \cdot \{\gamma\},$$

where we have identified I/I^2 with the multiplicative group $1 + p\mathbb{Z}_p$. As in [Ven16, Section 5], we can relate the derivative of $\mathcal{L}_p$ with the derivative of $L_{p,1}(\chi, s)$ and obtain this way a formula for $L'_{p,1}(\chi, 0)$ in terms of $\exp^*_{\mathrm{BK}}(\mathrm{loc}_p(\kappa_{0,\psi}))$.

6.3 Elliptic units

In this section we introduce Katz's two-variable p-adic L-function and the theory of elliptic units, following mainly [deS87] and [Yag82], but adapting their results to the framework discussed before. We also recall the Perrin-Riou big logarithm and recast Yager's theorem, which gives an explicit reciprocity law analogue to Theorem 6.2.3 in this setting. We recover the notations of the introduction, where K is an imaginary quadratic field and we fix a prime p which splits on K, i.e. $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. We also fix an identification of $\mathbb{C}_p$ with $\mathbb{C}$ and embeddings of $\bar{\mathbb{Q}}$ to either of these fields, which are compatible with these identifications. Let h denote the class number of K. Then, let $\pi_{\mathfrak{p}} \in \mathcal{O}_K$ be such that $\mathfrak{p}^h = \pi_{\mathfrak{p}}\mathcal{O}_K$, and define $\varpi_{\mathfrak{p}} = \pi_{\mathfrak{p}}/\pi_{\bar{\mathfrak{p}}}$. For simplicity, we assume that $\mathcal{O}_K^\times = \pm 1$ and that the discriminant of K is an odd number $D < 0$.

Consider also a non-trivial Hecke character of finite order ψ, of conductor $\mathfrak{n}$, where $(\mathfrak{n}, p) = 1$. In the particular case that χ is a Dirichlet character of conductor $N := \mathcal{N}_{K/\mathbb{Q}}(\mathfrak{n})$, the Dirichlet character may be seen by restriction as an example of the kind of Hecke characters we are interested in, provided that K is a quadratic field where all primes dividing N split. As before, let L stand for the field cut out by the character and L_p for its completion.

Elliptic units

Elliptic units are the result of evaluating modular units at CM points. They give rise to units in abelian extensions of the imaginary quadratic field K and are the counterpart of circular units for cyclotomic fields. They also constitute one of the key ingredients for the proof of the Iwasawa main conjecture for imaginary quadratic fields [Rub92].

For the general construction of elliptic units, we refer the reader to the seminal work of Coates and Wiles [CW78], or alternatively to Robert's original paper [Rob71]. Let us give an explicit description in the special setting where the conductor $\mathfrak{n}$ of ψ satisfies that there exists a rational integer N such that $\mathcal{O}_K/\mathfrak{n} \simeq \mathbb{Z}/N\mathbb{Z}$, and ψ can be interpreted as a Dirichlet character of conductor N. We closely follow the survey [Tale14] for that purpose.

Special values of L-series are encoded in terms of the so-called Siegel units $g_a \in \mathcal{O}_{Y_1(N)}^\times \otimes \mathbb{Q}$ attached to a fixed choice of primitive N-th root of unity ζ_N and a parameter $1 \leq a \leq N-1$. Its q-expansion is given by

$$g_a(q) = q^{1/12}(1-\zeta_N^a)\prod_{n>0}(1-q^n\zeta_N^a)(1-q^n\zeta_N^{-a}). \tag{6.17}$$

Let $\tau_{\mathfrak{n}} = \frac{b+\sqrt{D}}{2N}$, where $\mathfrak{n} = \mathbb{Z}N + \mathbb{Z}\frac{b+\sqrt{D}}{2}$. The classical and p-adic elliptic units are defined by

$$u_{a,\mathfrak{n}} := g_a(\tau_{\mathfrak{n}}), \qquad u_{a,\mathfrak{n}}^{(p)} := g_a^{(p)}(\tau_{\mathfrak{p}\mathfrak{n}}), \tag{6.18}$$

being g_a the infinite product of (6.17) and $g_a^{(p)} := g_{pa}(q^p)g_a(q)^{-p}$. As we did with circular units, we may define

$$u_\psi := \prod_{\sigma \in \mathrm{Gal}\,(K_{\mathfrak{n}}/K)} (\sigma u_{1,\mathfrak{n}})^{\psi^{-1}(\sigma)}, \tag{6.19}$$

where $K_{\mathfrak{n}}$ is the ray class field of K of conductor $\mathfrak{n}$ and $u_{1,\mathfrak{n}} \in \mathcal{O}_{K_{\mathfrak{n}}}^\times$. In additive notation, this corresponds to $(\sigma u_{1,\mathfrak{n}}) \otimes \psi^{-1}(\sigma)$. This construction works in greater generality and one can always define the element u_ψ (see [Rob71]). These units are the bottom layer of a norm compatible family of elliptic units over the two-variable $\mathbb{Z}_p$-extension K_∞ of K.

Performing a similar construction to that of (6.7) and (6.9), the work of Katz [Katz76] and de Shalit [deS87] gives a cohomology class

$$\kappa_{\psi,\infty} \in H^1(K, \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N})), \tag{6.20}$$

where Λ_K is the two-variable Iwasawa algebra of the introduction endowed with the tautological Galois action. In particular, if η is a Hecke character of infinity type (κ_1, κ_2), the global class obtained by specializing $\kappa_{\psi,\infty}$ at η, although it arises from elliptic units, encodes information about a Galois representation of K attached to a Hecke character.

Katz's two-variable p-adic L-function of an imaginary quadratic field

The classical two-variable L-function attached to K and a Hecke ψ is defined by

$$L(K,\psi,\kappa_1,\kappa_2) := \sum_{\alpha\in\mathcal{O}_K}{}' \psi(\alpha)\alpha^{-\kappa_1}\bar{\alpha}^{-\kappa_2},$$

where the sum is over the set of non-zero ideals of $\mathcal{O}_K$. This L-series allows us to recover the more familiar L-function attached to a character ψ of an imaginary quadratic field, via the relation

$$L(K,\psi,s) = \frac{1}{2}L(K,\psi,s,s). \tag{6.21}$$

We follow [DLR15a, Section 3] for the construction of Katz's two variable p-adic L-function. Let $\mathfrak{c} \subset \mathcal{O}_K$ be an integral ideal of K, and let Σ be the set of Hecke characters of K of conductor dividing $\mathfrak{c}$. Define $\Sigma_K = \Sigma_K^{(1)} \cup \Sigma_K^{(2)} \subset \Sigma$ to be the disjoint union of the sets

$$\Sigma_K^{(1)} = \{\psi \in \Sigma \text{ of infinity type } (\kappa_1,\kappa_2), \text{ with } \kappa_1 \le 0, \kappa_2 \ge 1\},$$

$$\Sigma_K^{(2)} = \{\psi \in \Sigma \text{ of infinity type } (\kappa_1,\kappa_2), \text{ with } \kappa_1 \ge 1, \kappa_2 \le 0\}.$$

For all $\psi \in \Sigma_K$, the complex argument $s=0$ is a critical point for $L(\psi^{-1},s)$, and Katz's p-adic L-function is constructed interpolating the algebraic part of $L(\psi^{-1},0)$, as ψ ranges over $\Sigma_K^{(2)}$.

Let $\hat{\Sigma}_K$ be the completion of $\Sigma_K^{(2)}$ with respect to the compact open topology on the space of $\mathcal{O}_{L_p}$-valued functions on a subset of $\mathbb{A}_K^\times$. By the work of Katz, there exists a p-adic analytic function

$$L_{\mathfrak{p}}(K) : \hat{\Sigma}_K \longrightarrow \mathbb{C}_p,$$

uniquely determined by the interpolation property that for all $\xi \in \Sigma_K^{(2)}$ of infinity type (κ_1,κ_2),

$$L_{\mathfrak{p}}(K)(\xi) = \mathfrak{a}(\xi) \times \mathfrak{e}(\xi) \times \mathfrak{f}(\xi) \times \frac{\Omega_p^{\kappa_1-\kappa_2}}{\Omega^{\kappa_1-\kappa_2}} \times L_{\mathfrak{c}}(\xi^{-1},0), \tag{6.22}$$

where

1. $\mathfrak{a}(\xi) = (\kappa_1-1)!\pi^{-\kappa_2}$,
2. $\mathfrak{e}(\xi) = (1-\xi(\mathfrak{p})p^{-1})(1-\xi^{-1}(\bar{\mathfrak{p}}))$,
3. $\mathfrak{f}(\xi) = D_K^{\kappa_2/2}2^{-\kappa_2}$,
4. $\Omega_p \in \mathbb{C}_p^\times$ is a p-adic period attached to K,
5. $\Omega \in \mathbb{C}^\times$ is the complex period associated to K,
6. $L_{\mathfrak{c}}(\xi^{-1},s)$ is Hecke's L-function associated to ξ^{-1} with the Euler factors at primes dividing $\mathfrak{c}$ removed.

We have followed the conventions of [BDP12, Proposition 3.1], which in turn follows from [Katz76, Section 5.3.0]. Observe that the definition is not symmetric with respect to the primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ above p, and hence we can also consider the function $L_{\bar{\mathfrak{p}}}(K)(\cdot)$.

The p-adic L-function $L_{\mathfrak{p}}(K)(\cdot)$ satisfies a functional equation

$$L_{\mathfrak{p}}(K)(\xi) = L_{\mathfrak{p}}(K)((\xi')^{-1}\mathcal{N}), \tag{6.23}$$

where again ξ' is the composition of ξ with the complex conjugation (see [Gro80, pages 90–91]). We remark that since our characters are unramified at p, the Gauss sum that sometimes appears in the interpolation formula is equal to 1. Finally, and according to this definition, the interpolation is over the special values of the form $L_{\mathfrak{c}}(\xi^{-1}, 0)$; this explains some *discrepancies* regarding certain conventions with the case of circular units.

It is possible to obtain an expression for the value of $L_{\mathfrak{p}}(K)(\psi)$ at finite order characters. This is usually referred to as the *p-adic Kronecker's limit formula*, and is due to Katz:

$$L_{\mathfrak{p}}(K)(\psi) = \begin{cases} \frac{1}{2}\left(\frac{1}{p} - 1\right) \cdot \log_p(\pi_{\mathfrak{p}}^{1/h}) & \text{if } \psi = 1; \\ -(1 - \psi^{-1}(\bar{\mathfrak{p}}))(1 - \psi(\mathfrak{p})p^{-1}) \cdot \log_p(u_\psi) & \text{if } \psi \neq 1. \end{cases} \tag{6.24}$$

Here, h stands for the class number of K and $\pi_{\mathfrak{p}}$ for a generator of the $\mathcal{O}_K$-ideal $\mathfrak{p}^h$. Via the functional equation, we also have an expression for the value at the $\psi\mathcal{N}$

$$L_{\mathfrak{p}}(K)(\psi\mathcal{N}) = \begin{cases} \frac{1}{2}\left(\frac{1}{p} - 1\right) \cdot \log_p(\pi_{\mathfrak{p}}^{1/h}) & \text{if } \psi = 1; \\ -(1 - \psi(\mathfrak{p}))(1 - \psi^{-1}(\bar{\mathfrak{p}})p^{-1}) \cdot \log_p(u_{(\psi')^{-1}}) & \text{if } \psi \neq 1. \end{cases} \tag{6.25}$$

Definition 6.3.1. Let $L_{\mathfrak{p}}(K, \psi)(\cdot)$ stand for the restriction of the p-adic L-function of (6.22) to characters of the form $\psi\xi\lambda^{\kappa_1}(\lambda')^{\kappa_2}$, where ξ is a character of p-power conductor and λ is the character of infinity type $(1, 0)$ presented in the introduction.

In particular, write $L_{\mathfrak{p}}(K, \psi)(\chi_{\mathrm{triv}}) := L_{\mathfrak{p}}(K)(\psi)$ and $L_{\mathfrak{p}}(K, \psi)(\mathcal{N}) := L_{\mathfrak{p}}(K)(\psi\mathcal{N})$.

Remark 6.3.2. Depending on the normalization we choose for the two variable p-adic L-function, the value $L_{\mathfrak{p}}(K)(\psi)$ may be affected by multiplication by a non-zero explicit rational number. Further, this number can depend on the conductor of ψ; however, since we are restricting the function to characters of the form $\psi\xi$, where ξ has p-power conductor, we can adopt a suitable normalization such that our special value formulas always work.

Further, observe that the p-adic L-function of [Buy12, Theorem 6.3] also differs from this one in the factor $(1 - \xi^{-1}(\bar{\mathfrak{p}}))$.

We finish this description of Katz's two-variable p-adic L-function by discussing its relation with the theory of improved p-adic L-functions. We say that a character is *analytic* if it is of the form $\psi\lambda^t$, with $t \in \mathbb{Z}^{\geq 0}$. The reason for this terminology is that they correspond to the subvariety of the weight space along which the Euler factors appearing in the Perrin-Riou map are analytic as functions in the variable t. Katz constructed in [Katz76, Section 7.2] a one-variable p-adic L-function $L_{\mathfrak{p}}^*(K, \psi, k)$, such that the restriction of $L_{\mathfrak{p}}(K, \psi)$ to analytic characters yields the equation

$$L_{\mathfrak{p}}(K, \psi)(\lambda^k) = (1 - \psi^{-1}(\bar{\mathfrak{p}})\pi_{\bar{\mathfrak{p}}}^{-k/h})L_{\mathfrak{p}}^*(K, \psi, k). \tag{6.26}$$

The ratio of the two p-adic L-series is a p-adic analytic function of k, since $\pi_{\bar{\mathfrak{p}}}$ belongs to $\mathcal{O}_{K_{\mathfrak{p}}}^{\times}$. This ratio measures the difference between working with the ordinary p-stabilization of the Eisenstein series, $E_{k,\psi}^{(p)}$, and the p-depletion, $E_{k,\psi}^{[p]}$. Further, by a result of Katz [Katz76, Section 7.2], one has a relation between the p-adic L-function of a quadratic imaginary field and elliptic units, given by

$$L_{\mathfrak{p}}^*(K, \psi, 0) = -(1 - \psi(\mathfrak{p})p^{-1}) \cdot \log_p(u_\psi). \tag{6.27}$$

We refer the reader to [BeDi19, Section 4.3] for a more detailed exposition of this material.

A reciprocity law for elliptic units

In this section, we recall the existence of a Perrin-Riou map interpolating both the dual exponential map and the Bloch–Kato logarithm, as we did with circular units. We closely follow the treatment of [deS87], recalling the main properties of this regulator map, whose source is the Iwasawa cohomology of the representation induced by a Hecke character, and which interpolates the Bloch–Kato logarithm and the dual exponential map, depending on the Hodge–Tate type of the character at which we specialize.

Although this is part of a rather general theory, we are interested in a more down-to-earth version of these results, which have been recovered by Loeffler and Zerbes in [LZ14] in the setting of *two-variable Perrin-Riou regulators.* In particular, Theorem 4.15 of loc. cit. gives an analogue to Proposition 6.2.2 in the setting of elliptic units. We restrict to characters of K of the form $\psi\lambda^{\kappa_1}(\lambda')^{\kappa_2}$, where λ is the character of infinity type $(1,0)$ of the introduction and λ' is its complex conjugate. Of course we may also consider twists by characters ξ of p-power order, but we neglect this possibility so as to ease the exposition.

Recall that Λ_K is the two-variable Iwasawa algebra attached to K with the tautological Galois action. As with circular units, for any character $\eta = \lambda^{\kappa_1}(\lambda')^{\kappa_2}$ as above there exists a specialization map that we normalize, to ease the exposition, as the one inducing specializations of the form

$$\nu_{\kappa_1,\kappa_2} : \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N}) \to L_p(\psi^{-1})(\lambda^{\kappa_2}(\lambda')^{\kappa_1}).$$

We identify again the target of both the dual exponential map and the Bloch–Kato logarithm with L_p. The following result has been established in [LZ14, Section 6.4, Appendix B], and here is presented in the language of [KLZ17, Section 8].

Proposition 6.3.3. *There exists a morphism*

$$\mathcal{L}_{\mathfrak{p}} : H^1(K_{\mathfrak{p}}, \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N})) \longrightarrow \Lambda_K$$

interpolating both the dual exponential map and the Bloch–Kato logarithm, and such that for any point η of infinity type (κ_1,κ_2) with $\xi = 1$, the specialization of $\mathcal{L}$ at η is the homomorphism

$$\nu_{\kappa_1,\kappa_2}(\mathcal{L}_{\mathfrak{p}}) : H^1(K_{\mathfrak{p}}, L_p(\psi^{-1})(\lambda^{\kappa_2}(\lambda')^{\kappa_1})) \longrightarrow \mathbb{D}_{\mathrm{dR}}(L_p(\psi^{-1})(\lambda^{\kappa_2}(\lambda')^{\kappa_1})) \simeq L_p$$

given by

$$\nu_{\kappa_1,\kappa_2}(\mathcal{L}_{\mathfrak{p}}) = \frac{1-\psi(\mathfrak{p})\pi_{\mathfrak{p}}^{-\kappa_2/h}\pi_{\bar{\mathfrak{p}}}^{-\kappa_1/h}}{1-\frac{\psi^{-1}(\mathfrak{p})\pi_{\mathfrak{p}}^{\kappa_2/h}\pi_{\bar{\mathfrak{p}}}^{\kappa_1/h}}{p}} \begin{cases} \frac{(-t)^{\kappa_2}}{\kappa_2!}\log_{\mathrm{BK}} & \text{if } \kappa_2 > 0 \\ (-\kappa_2)!t^{\kappa_2}\exp^*_{\mathrm{BK}} & \text{if } \kappa_2 \le 0. \end{cases}$$

Remark 6.3.4. It is interesting to analyze the shape of the Euler factors and compare it with those of [Cas18a, Theorem 3.5]. To follow this parallelism, let $\kappa = \kappa_1 + \kappa_2$ and $r = -\kappa_2$. Then,

$$\frac{1-\psi(\mathfrak{p})\pi_{\mathfrak{p}}^{-\kappa_2/h}\pi_{\bar{\mathfrak{p}}}^{-\kappa_1/h}}{1-\frac{\psi^{-1}(\mathfrak{p})\pi_{\mathfrak{p}}^{\kappa_2/h}\pi_{\bar{\mathfrak{p}}}^{\kappa_1/h}}{p}} = \frac{1-\psi(\mathfrak{p})\pi_{\mathfrak{p}}^{r/h}\pi_{\bar{\mathfrak{p}}}^{(-\kappa-r)/h}}{1-\frac{\psi^{-1}(\mathfrak{p})\pi_{\mathfrak{p}}^{-r/h}\pi_{\bar{\mathfrak{p}}}^{(\kappa+r)/h}}{p}} = \frac{1-\psi(\mathfrak{p})\varpi_{\mathfrak{p}}^{r/h}\pi_{\bar{\mathfrak{p}}}^{-\kappa/h}}{1-\frac{\psi^{-1}(\mathfrak{p})\varpi_{\mathfrak{p}}^{-r/h}\pi_{\bar{\mathfrak{p}}}^{\kappa/h}}{p}}. \tag{6.28}$$

Our results concerning elliptic units can be seen as a counterpart of those for Heegner points when the cuspidal Hida family is replaced by an Eisentein series.

Remark 6.3.5. As we will discuss in the last section, this also fits well with [KLZ17, Theorem 8.1.7]; with the notations of loc. cit., if we fix $s = 0$ and identify the modular forms g and h with two weight one theta series attached to the imaginary quadratic field K, we recover the map of Proposition 6.3.3. Further, observe that the numerology is coherent, and the condition $m > s$ defining the region of interpolation of the Bloch–Kato logarithm becomes in this case $\kappa_2 > 0$.

The following result is the main theorem of [Yag82].

Proposition 6.3.6. *The cohomology class $\kappa_{\psi,\infty} \in H^1(K, \Lambda_K \otimes L_p(\psi^{-1}))$ satisfies*

$$L_{\mathfrak{p}}(K,(\psi')^{-1}) = \mathcal{L}_{\mathfrak{p}}(\mathrm{loc}_{\mathfrak{p}}(\kappa_{\psi,\infty})).$$

Again, the denominator of the Perrin-Riou regulator may vanish. Assume that $\psi(\mathfrak{p}) = 1$. Then, we have the following:

(i) If $\kappa_1 = \kappa_2 = 0$, then the numerator vanishes and the denominator equals $1 - p^{-1}$.

(ii) If $\kappa_1 = \kappa_2 = 1$, the numerator equals $1 - p^{-1}$ and the denominator vanishes.

Remark 6.3.7. For a fixed κ_2, both the numerator and the denominator are analytic functions on the variable κ_1.

6.4 Exceptional zeros and elliptic units

We analyze different instances of exceptional zero phenomena and discuss the existence of *derived* cohomology classes and some of their properties. Our main result, stated as Theorem 6.1.1 in the introduction, is about the exceptional vanishing of $\kappa_{\psi\mathcal{N}}$, but for the sake of convenience we also study the exceptional vanishing of Katz's two variable p-adic L-functon at ψ in Section 6.4. Then, in Section 6.4 we discuss the different cases of exceptional zeros at $\psi\mathcal{N}$ and prove the main result of the chapter. Finally, Section 6.4 discusses the special case where ψ is the restriction of a Dirichlet character, suggesting a tantalizing connection with the theory of circular units.

Specialization at the character ψ

We assume that the condition $\psi(\bar{\mathfrak{p}}) = 1$ is satisfied. We begin this section by discussing the vanishing of Katz's two-variable p-adic L-function at the character ψ under this hypothesis. Indeed, from (6.26), it is straightforward that $\psi(\bar{\mathfrak{p}}) = 1$ is a necessary and sufficient condition for the vanishing of $L_{\mathfrak{p}}(K, \psi)(\chi_{\mathrm{triv}})$ when the classical value is non-zero.

Until otherwise stated, and so as to shorten the notation, derivatives of p-adic L-functions are considered along the character λ (that is, along the H_∞ direction).

Proposition 6.4.1. *Assume that $\psi(\bar{\mathfrak{p}}) = 1$. Then, $L_{\mathfrak{p}}(K, \psi)(\chi_{\mathrm{triv}}) = 0$, and*

$$L'_{\mathfrak{p}}(K, \psi)(\chi_{\mathrm{triv}}) = \log_p(\pi_{\mathfrak{p}}^{1/h}) \cdot (1 - \psi(\mathfrak{p})p^{-1}) \cdot \log_p(u_\psi). \tag{6.29}$$

Proof. This follows by taking derivatives in (6.26) and applying the special value formula of (6.27). Observe that we have also used that $\log_p(\pi_{\mathfrak{p}}^{1/h}) = -\log_p(\pi_{\bar{\mathfrak{p}}}^{1/h})$. □

The Euler factors arising in the Perrin-Riou map are analytic in the variable κ_1 once the value of κ_2 is fixed. In particular, we may consider the Perrin-Riou map with $\kappa_2 = 0$ fixed. Writing $\kappa_{\kappa_1,\psi}$ for the corresponding specialization of the Λ-adic class, one has

$$(1 - \pi_{\bar{\mathfrak{p}}}^{\kappa_1/h}/p) \cdot L_{\mathfrak{p}}(K, \psi)(\chi_{\mathrm{triv}}) = (1 - \pi_{\bar{\mathfrak{p}}}^{-\kappa_1/h}) \cdot \exp^*_{\mathrm{BK}}(\mathrm{loc}_{\mathfrak{p}}(\kappa_{\kappa_1,(\psi')^{-1}})),$$

for all $\kappa_1 \geq 0$. Taking derivatives with respect to κ_1 at both sides and evaluating at $\kappa_1 = 0$, we get that

$$(1 - p^{-1}) L'_{\mathfrak{p}}(K, \psi)(\chi_{\mathrm{triv}}) = -\log_p(\pi_{\mathfrak{p}}^{1/h}) \cdot \exp^*_{\mathrm{BK}}(\mathrm{loc}_{\mathfrak{p}}(\kappa_{(\psi')^{-1}})), \tag{6.30}$$

where $\kappa_{(\psi')^{-1}}$ stands for the specialization at the trivial character.

Here, by class field theory,

$$H^1(K_{\mathfrak{p}}, L_p(\psi^{-1})) \simeq \mathrm{Hom}_{\mathrm{cont}}(K_{\mathfrak{p}}^\times, K_{\mathfrak{p}}) \otimes L_p(\psi^{-1}),$$

and $\mathrm{loc}_{\mathfrak{p}}(\kappa_{(\psi')^{-1}})$ corresponds to the evaluation at the inverse of a local uniformizer, $\pi_{\mathfrak{p}}^{-1/h}$. In particular, there is a non-canonical isomorphism with L_p. Under this isomorphism, the dual exponential map corresponds to the ord map. Then, we may identify an element of the space

$$\mathrm{Hom}_{\mathrm{cont}}(K_{\mathfrak{p}}^\times, K_{\mathfrak{p}}) \otimes L_p(\psi^{-1})$$

with its image under evaluation at $\pi_{\mathfrak{p}}^{-1/h}$.

Therefore, combining Proposition 6.4.1 with equation (6.30), we get the following.

Proposition 6.4.2. *Assume that $\psi(\bar{\mathfrak{p}}) = 1$. Then, with the previous identifications,*

$$\mathrm{loc}_{\mathfrak{p}}(\kappa_{(\psi')^{-1}}) = -(1-p^{-1})(1-\psi(\mathfrak{p})p^{-1}) \times \log_p(u_\psi). \tag{6.31}$$

Specialization at the character $\psi\mathcal{N}$

The motivation for this section comes from the following lemma.

Lemma 6.4.3. *The following relation between the unit u_ψ and the specialization of $\kappa_{\psi,\infty}$ at the character $\psi\mathcal{N}$ holds:*

$$\log_p(\kappa_{\psi\mathcal{N}}) = (1-\psi^{-1}(\mathfrak{p})) \cdot (1-\psi^{-1}(\bar{\mathfrak{p}})) \cdot \log_p(u_\psi). \tag{6.32}$$

In particular, $\log_p(\kappa_{\psi\mathcal{N}})$ vanishes if and only if $\psi(\mathfrak{p}) = 1$ or $\psi(\bar{\mathfrak{p}}) = 1$.

Proof. This follows by combining the p-adic Kronecker limit formula given in (6.25), now applied to $(\psi')^{-1}$, with Proposition 6.3.6. Observe that the factor $1 - \psi(\mathfrak{p})/p$ cancels out, and the same occurs for the two minus signs (the one coming from the Perrin-Riou map and that of the p-adic Kronecker limit formula). □

Further, since both $\kappa_{\psi\mathcal{N}}$ and u_ψ are ψ-units, the p-adic logarithm defines an isomorphism and we may upgrade the previous lemma to an equality in $(\mathcal{O}_{K_{\mathfrak{n}}}^\times \otimes L)^\psi$. Observe that we are implicitly using that the localization map corresponds to the injection of global units inside local units, and this allows us to conclude that the global class $\kappa_{\psi\mathcal{N}}$ also vanishes when $\psi(\mathfrak{p}) = 1$ or $\psi(\bar{\mathfrak{p}}) = 1$.

We distinguish now three different situations.

(a) If $\psi(\mathfrak{p}) = 1$, the cohomology class $\kappa_{\psi\mathcal{N}} = 0$. One can construct a *derived* class whose derivative along the direction λ is computed and expressed in terms of the unit u_ψ. If $\psi(\bar{\mathfrak{p}}) \neq 1$, the special value $L_{\mathfrak{p}}(K,\psi)(\mathcal{N})$ does not vanish and it is related with the Bloch–Kato logarithm of the *derived* class.

(b) If $\psi(\bar{\mathfrak{p}}) = 1$, both $\kappa_{\psi\mathcal{N}} = 0$ and $L_{\mathfrak{p}}(K,\psi)(\mathcal{N}) = 0$. The logarithm of the *derived* local class $\mathrm{loc}_{\mathfrak{p}}(\kappa'_{\psi\mathcal{N}})$, once the value of κ_2 is fixed, can be expressed in terms of $L'_{\mathfrak{p}}(K,\psi)(\mathcal{N})$.

(c) When $\psi(\mathfrak{p}) = \psi(\bar{\mathfrak{p}}) = 1$, both $\kappa_{\psi\mathcal{N}} = 0$ and $L_{\mathfrak{p}}(K,\psi)(\mathcal{N}) = 0$. Observe that this can be understood as an extension of the subcases (a) and (b). Then, $\kappa'_{\psi\mathcal{N}} = 0$ too, and there is a notion of *second derivative* of the cohomology class, whose logarithm is related with $L'_{\mathfrak{p}}(K,\psi)(\mathcal{N})$.

Case (a). Suppose that $\psi(\mathfrak{p}) = 1$. Let $\mathcal{C}$ be the line of weight space obtained by taking the Zariski closure of all the points of the form $\mathcal{N}\lambda^t$ (that is, fixing $\kappa_2 = 1$). We are considering the $\mathbb{C}_p$-points as a rigid analytic space, and realizing all these characters as elements of this space. Analogously, define $\mathcal{C}'$ as the closure of the points of the form $\mathcal{N}\bar{\lambda}^t$. At the level of p-adic L-functions, and under the map of Proposition 6.3.6, the point $\mathcal{N}\lambda^t$ in $\mathcal{C}$ corresponds to $L_{\mathfrak{p}}(K,(\psi')^{-1})(\mathcal{N}\bar{\lambda}^t)$, or alternatively and via the functional equation, to $L_{\mathfrak{p}}(K,\psi)(\lambda^t)$.

In this situation of *exceptional zeros*, the Euler factor in the denominator of the Perrin-Riou map of Proposition 6.3.3 vanishes, and hence one must carry out some new constructions to obtain a formula relating the special value of the p-adic L-function with an appropriate cohomology class. For that purpose, we can argue the existence of a *derived* cohomology class arising from elliptic units, which is directly related with the special value of the derivative of Katz's two variable p-adic L-function. Observe that derivatives of cohomology classes must be taken along $\mathcal{C}'$, since elsewhere the Euler factors are not analytic (and not even continuous!).

Proposition 6.4.4. *It holds that $\kappa_{\psi\mathcal{N}} = 0$, and there exists a derived cohomology class along $\mathcal{C}'$*

$$\kappa'_{\gamma,\psi,\infty} \in H^1(K, \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N})_{|\mathcal{C}'})$$

satisfying that

$$\kappa_{\psi|\mathcal{C}'} = (\gamma - 1)\kappa'_{\gamma,\psi,\infty},$$

where γ is a fixed topological generator of $1 + p\mathbb{Z}_p$.

Proof. The vanishing of the local class $\mathrm{loc}_{\mathfrak{p}}(\kappa_{\psi\mathcal{N}})$ directly follows from Lemma 6.4.3 and the discussion after it.

The construction of the derived class follows the same argument than in the case of circular units explained in Section 2, and which is already present in the work of Bley [Ble04]. After fixing a topological generator γ of $1 + p\mathbb{Z}_p$, one may consider the short exact sequence of $\mathbb{Z}_p$-modules

$$0 \to \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N})_{|\mathcal{C}'} \xrightarrow{\gamma-1} \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N})_{|\mathcal{C}'} \to L_p(\psi^{-1})(\mathcal{N}) \to 0$$

which induces a long exact sequence in cohomology. Since $H^0(K, L_p(\psi^{-1})(\mathcal{N})) = 0$,

$$0 \to H^1(K, \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N})_{|\mathcal{C}'}) \xrightarrow{\gamma-1} H^1(K, \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N})_{|\mathcal{C}'}) \xrightarrow{\mathcal{N}} H^1(K, L_p(\psi^{-1})(\mathcal{N})).$$

As we have just seen, the image of $\kappa_{\psi,\infty}$ under the map $\mathcal{N}$ vanishes, and hence there exists a unique

$$\kappa'_{\gamma,\psi,\infty} \in H^1(K, \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N})_{|\mathcal{C}'})$$

such that

$$\frac{\gamma - 1}{\log_p(\gamma)} \times \kappa'_{\gamma,\psi,\infty} = \kappa_{\gamma,\psi,\infty}.$$

$\square$

Remark 6.4.5. If we normalize dividing by $\log_p(\gamma)$ the specialization of the resulting class at the character $\psi\mathcal{N}$ does not depend on γ (see [Buy12, Section 3]). We define $\kappa'_{\psi,\infty} := \frac{\kappa'_{\gamma,\psi,\infty}}{\log_p(\gamma)}$.

We can relate the cohomology class with $L_{\mathfrak{p}}(K)(\cdot)$ at $\psi\mathcal{N}$.

Definition 6.4.6. Let $\mathcal{E}_{\mathfrak{p}}(K, (\psi')^{-1})(\cdot)$ stand for the function defined over the set of characters of the form $\lambda^{\kappa_1}\bar{\lambda}$ (and then extended to their Zariski closure over weight space), and given by

$$\mathcal{E}_{\mathfrak{p}}(K, (\psi')^{-1})(\lambda^{\kappa_1}\bar{\lambda}) = \Big(1 - \frac{\pi_{\mathfrak{p}}^{1/h}\pi_{\bar{\mathfrak{p}}}^{\kappa_1/h}}{p}\Big) L_{\mathfrak{p}}(K, (\psi')^{-1})(\lambda^{\kappa_1}\bar{\lambda}). \tag{6.33}$$

Lemma 6.4.7. *The function $\mathcal{E}_{\mathfrak{p}}(K, (\psi')^{-1})$ satisfies that for a character η of infinity type $(\kappa_1, 1)$*

$$\mathcal{E}_{\mathfrak{p}}(K)(\psi\eta) = -\Big(1 - \pi_{\mathfrak{p}}^{-1/h}\pi_{\bar{\mathfrak{p}}}^{-\kappa_1/h}\Big) \cdot t \log_{\mathrm{BK}}(\mathrm{loc}_{\mathfrak{p}}(\kappa_{\psi\eta})). \tag{6.34}$$

In particular, the function $\mathcal{E}_{\mathfrak{p}}(K, (\psi')^{-1})$ is zero at the norm character.

Proof. The first part follows from Proposition 6.3.6. The second statement is due to the vanishing of the specialization of the cohomology class at $\psi\mathcal{N}$ (recall that for characters of infinity type $(\kappa_1, 1)$ the Bloch–Kato logarithm interpolates the usual logarithm map). $\square$

This is sufficient to prove Theorem 6.1.1. Recall again that we may identify the Bloch–Kato logarithm with the usual p-adic logarithm under Kummer's isomorphism.

Remark 6.4.8. For the following result, we need to use that $(\log_{\mathrm{BK}}(\kappa))' = \log_{\mathrm{BK}}(\kappa')$. This can be easily seen by considering the natural isomorphism between the weight space and $\mathbb{Z}_p[[X]]$. Then, the class κ corresponds to a function f vanishing at 0, and hence there is another function g such that $f = X \cdot g$. The Bloch–Kato logarithm is a linear morphism between a $\mathbb{Z}_p[[X]]$-module and a field embedded in $\mathbb{C}_p$, that we may denote with the letter Φ. Then,

$$\Phi(f)'|_{X=0} = \Phi(g)|_{X=0},$$

as desired.

Theorem 6.4.9. *It holds that*

$$\log_p(\pi_{\mathfrak{p}}^{1/h}) \cdot L_{\mathfrak{p}}(K, (\psi')^{-1})(\mathcal{N}) = -(1-p^{-1}) \cdot \log_p(\mathrm{loc}_{\mathfrak{p}}(\kappa'_{\psi\mathcal{N}})).$$

Moreover,

$$\kappa'_{\psi\mathcal{N}} = \log_p(\pi_{\mathfrak{p}}^{1/h}) \cdot (1 - \psi^{-1}(\bar{\mathfrak{p}})) \cdot u_\psi,$$

where $\mathrm{loc}_{\mathfrak{p}}$ *stands here for the composition of the Kummer map with localization at* $\mathfrak{p}$.

Proof. The first part follows by considering the derivative of the function $\mathcal{E}_{\mathfrak{p}}(K, (\psi')^{-1})$ using both (6.33) and (6.34). That is, the left hand side is the result of deriving (6.33) and evaluating at $\kappa_1 = 1$, where the Euler factor vanishes; similarly, the right hand side is the result of deriving (6.34) and evaluating again at $\kappa_1 = 1$, where the class vanishes. Observe that we have identified $t \log_{\mathrm{BK}}$ with the p-adic logarithm, as usual.

The second part directly follows after comparing the first statement with the special value formula given in (6.25). Further, since we are taking a derivative along the $\bar{\mathfrak{p}}$-ramified extension H'_∞/K, the resulting *derived* class could only have non-zero valuation at the prime $\bar{\mathfrak{p}}$. This cannot be the case when $\psi(\bar{\mathfrak{p}}) \neq 1$ since there are no extra $\bar{\mathfrak{p}}$-units, and the p-adic logarithm is therefore an isomorphism for the ψ-component of the unit group. If $\psi(\mathfrak{p}) = \psi(\bar{\mathfrak{p}}) = 1$ the same conclusion follows by recasting to the arguments of [Blo86]. We discuss this situation in more detail in the following sections. □

This result slightly differs from the work of Bley [Ble04]. There, the author takes the derivative along the direction $\mathcal{C}$, which corresponds to the $\mathbb{Z}_p$-extension which is ramified at the prime $\mathfrak{p}$. In that case, the derivative of the Perrin-Riou map agrees with the *order* (and not with the logarithm), which is coherent with the fact that when $\psi(\mathfrak{p}) = 1$ there is a $\mathfrak{p}$-unit in the ψ-component. Hence, our results are coherent with the computations of [Buy12, Section 6.4] relating the valuation of the extra $\mathfrak{p}$-unit with the logarithm of the elliptic unit u_ψ.

Case (b). Assume now that $\psi(\bar{\mathfrak{p}}) = 1$. Observe that via the functional equation for Katz's two-variable L-function, this leads to

$$L_{\mathfrak{p}}(K, (\psi')^{-1})(\mathcal{N}) = 0,$$

and $L'_{\mathfrak{p}}(K, (\psi')^{-1})(\mathcal{N})$ can be related with the *derived* cohomology class constructed in the previous section. We recall that the derivative of the p-adic L-function always means derivative along the H_∞ direction. As we did in Proposition 6.4.4, we may take the class

$$\kappa'_{\psi,\infty} \in H^1(K, \Lambda_K \otimes L_p(\psi^{-1})(\mathcal{N})_{|\mathcal{C}'}).$$

Proposition 6.4.10. *Assume that* $\psi(\bar{\mathfrak{p}}) = 1$. *Then,*

$$(1 - \psi^{-1}(\mathfrak{p})) \cdot L'_{\mathfrak{p}}(K, (\psi')^{-1})(\mathcal{N}) = -(1 - \psi(\mathfrak{p})p^{-1}) \cdot \log_p(\mathrm{loc}_{\mathfrak{p}}(\kappa'_{\psi\mathcal{N}})). \tag{6.35}$$

Proof. This directly follows by considering the derivative with respect to the variable κ_1 in the reciprocity law of Proposition 6.3.6 when κ_2 is fixed. $\square$

The derivative $L'_{\mathfrak{p}}(K,\psi)(\mathcal{N})$ is related, via the funcional equation, with the derivative at the trivial character along the direction λ', and a priori we do not know any expression for that value in terms of u_ψ, which would allow us to prove an analogue of Theorem 6.1.1 in this setting. Further, in this case the derived class $\kappa'_{\psi\mathcal{N}}$ is no longer a unit, but a $\bar{\mathfrak{p}}$-unit, and the p-adic logarithm is not sufficient to characterize $\kappa'_{\psi\mathcal{N}}$ (one needs to use the information about its p-adic valuation). It would be nice to understand these results in the framework provided by [BeDi19, Section 4], and we hope to come back to this issue in forthcoming work.

In the framework of circular units, Gross' factorization formula [Gro80], combined with the results of the previous section, allows us to express, after considering the identifications provided by Kummer's isomorphisms, the class $\kappa'_{\psi\mathcal{N}}$ as a linear combination of a circular unit and an elliptic unit. We discuss this in Section 6.4

Remark 6.4.11. Following Proposition 6.4.10, we can extend our computations for the derivative of the cohomology class to arbitrary directions. Indeed, we may consider the function $L_{\bar{\mathfrak{p}}}(K)(\cdot)$, thus complementing our picture.

- When $\psi(\mathfrak{p})=1$, $L_{\bar{\mathfrak{p}}}(K)(\cdot)$ vanishes at $\psi\mathcal{N}$ and in this case we can relate *the derivative of* $\mathrm{loc}_{\mathfrak{p}}(\kappa_{\psi\mathcal{N}})$ *along the direction* $\mathcal{C}$ with *the derivative of* $L_{\bar{\mathfrak{p}}}(K)$ *at* ψ *along the* $(-1,0)$ *direction.*
- When $\psi(\bar{\mathfrak{p}})=1$, *the derivative of* $\mathrm{loc}_{\mathfrak{p}}(\kappa_{\psi\mathcal{N}})$ *along the direction* $\mathcal{C}$ can be explicitly expressed in terms of $L_{\bar{\mathfrak{p}}}(\psi\mathcal{N})$.

Case (c). When $\psi(\bar{\mathfrak{p}})=\psi(\mathfrak{p})=1$, we expect that the cohomology class $\kappa'_{\psi\mathcal{N}}$ vanishes since $L_{\mathfrak{p}}(K,\psi)(\mathcal{N})=0$. More precisely, and due to the symmetry between both directions λ and λ', we may combine the results of previous sections with [Buy12, Theorem 6.13] to conclude that the class $\kappa'_{\psi\mathcal{N}}$ is zero.

When this happens, we may take a *second* derivative of the cohomology class

$$\kappa''_{\gamma,\psi,\infty}\in H^1(K,\Lambda_K\otimes L_p(\psi^{-1})(\mathcal{N})_{|\mathcal{C}'}).$$

Now, we normalize the class dividing by $\log_p(\gamma)^2$ in such a way that its value at the character $\psi\mathcal{N}$, $\kappa''_{\psi\mathcal{N}}$, does not depend on γ. Proceeding in the same way as before, we may obtain the following result.

Proposition 6.4.12. *Assume that* $\psi(\bar{\mathfrak{p}})=\psi(\mathfrak{p})=1$. *Then,*

$$\log_p(\pi_{\mathfrak{p}}^{1/h})\cdot L'_{\mathfrak{p}}(K,(\psi')^{-1})(\mathcal{N})=-2(1-p^{-1})\cdot\log_p(\mathrm{loc}_{\mathfrak{p}}(\kappa''_{\psi\mathcal{N}})). \tag{6.36}$$

Proof. Consider again the reciprocity law of Proposition 6.3.3 and fix again $\kappa_2=1$. When evaluating at the norm character, it happens that both the Euler factor of the denominator and the special value $L_{\mathfrak{p}}(K,(\psi')^{-1})(\mathcal{N})$ are zero. Hence, multiplying both sides by $1-\pi_{\mathfrak{p}}^{-1/h}\pi_{\bar{\mathfrak{p}}}^{-\kappa_1/h}$ and taking twice the derivative with respect to κ_1, we obtain the result. $\square$

Remark 6.4.13. We point out, just for the sake of completeness, that another interesting instance of the *exceptional zero phenomenon* can be observed in [BDP12, Proposition 3.5], which asserts that a *self-dual* character of infinity type $(1+j,-j)$ with $j\geq 0$ satisfies that the evaluation of Katz's two-variable p-adic L-function at ν agrees, up to multiplication by some periods and gamma factors, with a classical L-value times $(1-\nu^{-1}(\bar{\mathfrak{p}}))^2$. Again, if $\nu(\bar{\mathfrak{p}})=0$ we observe the presence of an *exceptional zero.*

In the special case where we consider characters of infinity type $(1,0)$, Agboola [Agb07] studies a variant of the p-adic BSD conjecture for CM elliptic curves concerning special values of Katz's two-variable p-adic L-function. Here, we are again in a situation where our same condition leads to an exceptional vanishing.

Interactions with the theory of circular units

Observe that in the case where $\psi(\mathfrak{p}) = 1$ we have described the *derived* cohomology class $\kappa'_{\psi\mathcal{N}}$ as an explicit multiple of the elliptic unit u_ψ. However, when $\psi(\bar{\mathfrak{p}}) = 1$ this is no longer possible, since we cannot express in terms of u_ψ the derivative of the Katz's two variable p-adic L-function at ψ along the λ'-direction.

In general, one may wish to determine the derivatives of the p-adic L-function along the different directions of the weight space. We know the derivative along the λ-direction, and it turns out that in some particular cases we can further determine the derivative along the *norm* direction.

This is the case when ψ is a finite order Hecke character which comes from the restriction to G_K of a Dirichlet character (that is, a $G_{\mathbb{Q}}$ character), and hence one can invoke Gross' factorization formula, which is the main result of [Gro80]. To begin with, consider that ψ comes from a Dirichlet character and that $\psi(\bar{\mathfrak{p}}) = 1$. In this case, we have determined the derivative of $L_{\mathfrak{p}}(K, \psi)$ at χ_{triv} along the direction λ, and this is

$$\frac{\partial L_{\mathfrak{p}}(K, \psi)}{\partial \lambda} = \log_p(\pi_{\mathfrak{p}}^{1/h}) \cdot (1 - \psi(\mathfrak{p})p^{-1}) \times \log_p(u_\psi).$$

Similarly, using Gross' factorization with the conventions about Gauss sums followed in the chapter, we have that

$$\frac{\partial L_{\mathfrak{p}}(K, \psi)}{\partial \mathcal{N}} = -(1 - \psi(\mathfrak{p})p^{-1}) \cdot L'_{p,1}(\psi^{-1}\chi_K\omega, 0) \times \log_p(c_\psi),$$

where χ_K stands for the quadratic character attached to K and ω is the Teichmüller character. Observe that here $L_p(\psi^{-1}\chi_K\omega, 0) = 0$ due to the running assumptions.

Then, one can determine the derivative along any direction: for $\eta = \psi\lambda^a\mathcal{N}^b$, the derivative of $L_{\mathfrak{p}}(K, \psi)$ at the trivial character along the direction η is given by

$$a \cdot \log_p(\pi_{\mathfrak{p}}^{1/h}) \cdot (1 - \psi(\mathfrak{p})p^{-1}) \cdot \log_p(u_\psi) - b \cdot L'_{p,1}(\psi^{-1}\chi_K\omega, 0) \cdot (1 - \psi(\mathfrak{p})p^{-1}) \cdot \log_p(c_\psi). \tag{6.37}$$

Remark 6.4.14. There is only one direction along which this value is zero; as discussed in the inspiring presentation [Gre12], this has significant applications towards Iwasawa theory.

Now, the functional equation gives the derivative of $L_{\mathfrak{p}}(K, (\psi')^{-1})$ at the norm character along any direction, and in particular, for the direction λ, we have to set $a = 1$ and $b = -1$ (this is the direction $-(\lambda')$ at the trivial character). Then, Proposition 6.4.10 yields that, up to an element in the kernel of the logarithm,

$$\kappa'_{\psi\mathcal{N}} = \alpha u_\psi + \beta c_\psi$$

where c_ψ is the circular unit introduced in Section 6.2, and α and β can be determined combining (6.37) with (6.13):

$$\alpha = -\log_p(\pi_{\mathfrak{p}}^{1/h})(1 - \psi^{-1}(\mathfrak{p})),$$

$$\beta = -L'_{p,1}(\psi^{-1}\chi_K\omega, 0)(1 - \psi^{-1}(\mathfrak{p})).$$

If we further assume that $\psi(\mathfrak{p}) = 1$, Proposition 6.4.12 gives a new expression for the second derivative of the cohomology class, again up to an element in the kernel of the logarithm,

$$\kappa''_{\psi\mathcal{N}} = \alpha' u_\psi + \beta' c_\phi,$$

where now

$$\alpha' = -\log_p(\pi_{\mathfrak{p}}^{1/h})^2/2,$$

$$\beta' = -\log_p(\pi_{\mathfrak{p}}^{1/h})L'_{p,1}(\psi^{-1}\chi_K\omega, 0)/2.$$

6.5 Beilinson–Flach elements and beyond

In this last section, we emphasize the interplay between the results we have presented until now and other related works in this direction. In particular, we study how the phenomena we have discussed arise in some of the other Euler systems presented in the survey [Tale14], focusing on two main aspects:

(a) Elliptic units can be seen as the natural substitute of Heegner points when instead of considering a cusp form, one takes an Eisenstein series.

(b) Elliptic units can be recast in terms of Beilinson–Flach elements, when we take two weight one modular forms corresponding to theta series of the same imaginary quadratic field where the prime p splits.

As it has been extensively discussed in the literature, there is a striking parallelism between the theory of Heegner points and that of elliptic units. Following this analogy, this chapter may be read as the counterpart of [Cas18a] when the cusp form f is replaced by an Eisenstein series. With the notations introduced in loc. cit., where the weight space is modeled by a weight variable (that we denote with the letter k) and an anticyclotomic variable (denoted with the letter t), an exceptional zero arises at the point $(k,t)=(2,0)$ when $\mathbf{a}_p(\mathbf{f})=1$ (the associated elliptic curve has split multiplicative reduction at p) and we specialize at the character $\psi\mathcal{N}$. There is a clear interplay between that setting and ours, but we would like to point out some of the differences:

- In [Cas18a] the author extends the p-adic Gross–Zagier formula of [BDP13] and finds an explicit expression for its value at the norm character, which is different from zero. However, in our setting it may occur that the p-adic L-function vanishes both at $\psi\mathcal{N}$ and at ψ. This simultaneous vanishing of the Euler factor and the p-adic L-function gives rise to a higher order vanishing of the *derived* class $\kappa_{\psi,\infty}$ at the character $\psi\mathcal{N}$. This can be understood via [Cas18a, Eq. 0.2], where the specialization of the higher dimensional Heegner cycle (whose role is now played by the cohomology class coming from the elliptic unit) and the *Heegner class* (whose role is now played by the unit u_ψ) are related by the factor
$$\Big(1-\frac{p^{k/2-1}}{\nu_k(\mathbf{a}_p)}\Big)^2.$$
In our case, however, the link is via the factor
$$(1-\psi^{-1}(\bar{\mathfrak{p}})\mathfrak{p}^{(\kappa_1-1)/h}\bar{\mathfrak{p}}^{(\kappa_2-1)/h})\cdot(1-\psi^{-1}(\mathfrak{p})\mathfrak{p}^{(\kappa_2-1)/h}\bar{\mathfrak{p}}^{(\kappa_1-1)/h}),$$
and hence there are two possible (and independent) sources of vanishing.

- In our setting there are two points where the exceptional zero phenomenon emerges: the character ψ and the character $\psi\mathcal{N}$. In [Cas18a] the vanishing of the numerator in the Perrin-Riou map would occur at $(k,t)=(0,-1)$, where there is not a clear *geometric meaning* of this phenomenon.

In any case, the similitude between his main result and ours is evident, expressing a *derived* cohomology class as a certain $\mathcal{L}$-invariant times a more classical avatar. Obviously, his $\mathcal{L}$-invariant encodes information both about the elliptic curve and the imaginary quadratic field K.

Elliptic units and Beilinson–Flach elements

Elliptic units can be understood as a degenerate case of the theory of Beilinson–Flach elements. To make this statement more precise, let
$$g=\sum_{n\geq 1}a_nq^n\in S_1(N_g,\chi_g),\quad h=\sum_{n\geq 1}b_nq^n\in S_1(N_h,\chi_h)$$

be two normalized newforms, and let V_g and V_h denote the Artin representations attached to them. Consider also $V_{gh} := V_g \otimes V_h$, and let H be the smallest number field cut out by this representation. We fix a prime number p which does not divide $N_g N_h$. Label the roots of the p-th Hecke polynomial of g and h as

$$X^2 - a_p(g)X + \chi_g(p) = (X - \alpha_g)(X - \beta_g) \quad X^2 - a_p(h)X + \chi_h(p) = (X - \alpha_h)(X - \beta_h).$$

Let

$$g_\alpha(q) = g(q) - \beta_g q(q^p), \qquad h_\alpha(q) = h(q) - \beta_h h(q^p)$$

denote the p-stabilization of g (resp. h) on which the Hecke operator U_p acts with eigenvalue α_g (resp. α_h). Let L be a number field containing both the Fourier coefficients of g and h and the eigenvalues for the p-th Hecke polynomials. We can attach in a natural way two *canonical* differentials ω_{g_α} and η_{g_α} to the weight one modular form g, as it is recalled in Sections 2 and 3 of Chapter 3. The reinterpretation of the main results of [KLZ17] in previous chapters establishes the existence of cohomology classes

$$\kappa(g_\alpha, h_\alpha), \kappa(g_\alpha, h_\beta), \kappa(g_\beta, h_\alpha), \kappa(g_\beta, h_\beta) \in H^1(\mathbb{Q}, V_{gh} \otimes L_p(1)),$$

and also gives an explicit reciprocity law which in slightly rough terms asserts that

$$\Big(1 - \frac{1}{p\alpha_g\beta_h}\Big) \cdot \log^{-+}(\kappa_p(g_\alpha, h_\alpha)) = (1 - \alpha_g\beta_h) \cdot L_p(g, h, 1) \pmod{L^\times}.$$

Here, $L_p(g, h, s)$ stands for the Hida–Rankin p-adic L-function attached to the convolution $g \otimes h$, $\kappa_p(g_\alpha, h_\alpha)$ is the restriction of the cohomology class to a decomposition group at p, and $\log^{-+}$ is the result of applying the Bloch–Kato logarithm to a certain projection of the local class followed by the pairing with the canonical differentials. The proof of this result is based on considering Hida families $\mathbf{g}$, $\mathbf{h}$ interpolating g_α and h_α, and on proving the corresponding equality over a dense set of points of the weight space. We refer the reader to Chapter 3 for a complete discussion of the results.

Now, let g^* stand for the twist of g by the inverse of its nebentype. When $h_\alpha = g^*_{1/\beta}$, the Euler factor $1 - \alpha_g\beta_h$ is zero, and Proposition 3.12 of loc. cit. establishes that both $\kappa(g_\alpha, g^*_{1/\beta})$ and $\kappa(g_\beta, g^*_{1/\alpha})$ vanish and moreover the authors prove the existence of a *derived* cohomology class $\kappa'(g_\alpha, g^*_{1/\beta}) \in H^1(\mathbb{Q}, V_{gg^*} \otimes L_p(1))$ satisfying

$$\log^{-+}(\kappa'_p(g_\alpha, g^*_{1/\beta})) = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) \cdot L_p(g, g^*, 1) \pmod{L^\times}, \tag{6.38}$$

being $\mathcal{L}(\mathrm{ad}^0(g_\alpha))$ the $\mathcal{L}$-invariant of the adjoint of the weight one modular form g_α.

Other interesting cases arise when both g and h are theta series attached to the same quadratic imaginary field where the prime p splits. In Section 6 of Chapter 3, we proved a formula establishing an explicit connection between the Hida–Rankin p-adic L-function attached to the pair of modular forms (g, g^*) and Katz's two variable p-adic L-function. Indeed, let $g = \theta(\psi)$, the theta series attached to the character ψ. Then, Theorem 6.2 of Chapter 3 asserts that for any $s \in \mathbb{Z}_p$ the following equality holds up to multiplication by $L^\times$:

$$L_p(g, g^*, s) = \frac{1}{\log_p(u_{\psi_{\mathrm{ad}}})} \cdot \zeta_p(s) \cdot L_p(\chi_K\omega, s) \cdot L_{\mathfrak{p}}(K, \psi_{\mathrm{ad}})(\mathcal{N}^s), \tag{6.39}$$

being $\psi_{\mathrm{ad}} = \psi/\psi'$. Note that ψ_{ad} is a ring class character, regardless of whether ψ is so or not. Then, according to the results of Chapter 3,

$$L_p(g, g^*, 0) = \log_p(v_1) \pmod{L^\times}, \tag{6.40}$$

where v_1 is the norm of a generator v of the one-dimensional space $(\mathcal{O}_H^\times[1/p] \otimes L)^{G_{\mathbb{Q}}}$.

This suggests a link between Beilinson–Flach elements and the cohomology classes coming from elliptic units via the Kummer map, expressed as

$$\kappa'(g_\alpha, g_{1/\beta}^*) = \log_p(v_1) \cdot v \pmod{L^\times}. \tag{6.41}$$

Additionally, the derived Beilinson–Flach element is also related with the cohomology class $\kappa_{\psi\mathcal{N}}$ via the factorization formula (6.39) and the results of the preceding sections.

As sketched in Section 5.2 of 4, the factorization formula (6.39) admits a counterpart in the case where g and h are no longer self-dual. In this case,

$$L_p(g,h,0) = \log^{-+}(\kappa_p(g_\alpha, h_\alpha)) = \frac{\log_p(u_{\psi_1}) \cdot \log_p(u_{\psi_2})}{\log_p(u_{g_\alpha})} \pmod{L^\times}, \tag{6.42}$$

where $\psi_1 = \psi_g \psi_h$ and $\psi_2 = \psi_g \psi_h'$, u_{ψ_i} is the elliptic unit attached to ψ_i and u_{g_α} is the Stark unit attached to the adjoint representation of g_α.

Then, and following Chapter 4,

$$\kappa(g_\alpha, h_\alpha) = \mathfrak{C} \cdot u_2, \tag{6.43}$$

with $\mathfrak{C}$ an explicit constant involving u_{ψ_1}, u_{g_α} and certain periods explicitly described in loc. cit., and $u_2 = u_{\psi_2} u_{\psi_2'}$, where as usual ψ_2' is the composition of ψ_2 with the complex conjugation.

An interesting observation is that the case where the Euler system of elliptic units presents an exceptional zero never arises in the setting of Chapter 3, due to the regularity assumptions which are assumed in loc. cit (the fact of g being p-distinguished). Hence, our results may be seen as a degenerate case of the theory of Beilinson–Flach elements for weight one modular forms.

Let us be more precise in this last sentence. Let $\mathbf{g}$ stand for the Hida family of CM theta series whose weight κ_1 specialization has characteristic Hecke polynomial at p given by

$$(x - \mathfrak{p}^{(\kappa_1 - 1)/h})(x - \bar{\mathfrak{p}}^{(\kappa_1 - 1)/h}).$$

Similarly, let $\mathbf{h}$ be the canonical Hida family of CM forms, such that its weight κ_2 specialization has characteristic Hecke polynomial at p

$$(x - \psi(\mathfrak{p})\mathfrak{p}^{(\kappa_2 - 1)/h})(x - \psi(\bar{\mathfrak{p}})\bar{\mathfrak{p}}^{(\kappa_2 - 1)/h}).$$

Then, as we had already anticipated, Proposition 6.3.3 may be seen as a degenerate case of Proposition 3.2 in Chapter 3, where we do not consider the twist by the cyclotomic character (we fix $s = 0$). Observe that there, the role played by Katz's two-variable p-adic L-function is done not exactly by the Hida–Rankin p-adic L-function, but by its product with the c-factor

$$c^2(1 - \chi_g(c)^{-1}\chi_h(c)^{-1}), \tag{6.44}$$

where c is a fixed integer number relatively prime to $6pN_gN_h$.

Remark 6.5.1. The connection between Beilinson–Flach elements and units (in this case circular units) is also exploited in [Das99], where the proof of the main result, a factorization formula for the Rankin–Selberg p-adic L-function, rests on a explicit comparison between a certain unit constructed via the theory of Beilinson–Flach elements and a circular unit. However, the approach used in loc. cit. is quite different, since the unit is constructed via the specialization of the Beilinson–Flach class at a point of weight $(2, 2, 1)$.

Beilinson–Flach elements and exceptional zeros

As we have pointed out, elliptic units may be understood as a special case inside the theory of Beilinson–Flach elements, where the two modular forms are theta series attached to the same imaginary quadratic field. Hence, it is reasonable to expect that the two phenomena we have described concerning exceptional zeros also arise in this setting. This section serves to recall the main characteristics of the exceptional zero phenomenon for Beilinson–Flach elements, following closely the discussions of [LZ17].

With the notations of the previous section, let $\mathbf{g}$ and $\mathbf{h} = \mathbf{g}^*$ Hida families interpolating two self-dual modular forms g_α and $h_\alpha = g^*_{1/\beta}$, respectively. We assume that $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$, where $\Lambda_{\mathbf{g}}$ is a finite flat extensions of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. Write $\mathcal{W} = \mathrm{Spf}(\Lambda)$ and $\mathcal{W}_{\mathbf{g}} = \mathrm{Spf}(\Lambda_{\mathbf{g}})$. Let y_0 be a weight one point of $\Lambda_{\mathbf{g}}$ such that $\mathbf{g}_{y_0} = g_\alpha$ and $\mathbf{g}^*_{y_0} = g^*_{1/\beta}$.

The work of [KLZ17] attaches to $(\mathbf{g}, \mathbf{g}^*)$ a three-variable family of cohomology classes $\kappa(\mathbf{g}, \mathbf{g}^*)$ parameterized by points $(y, z, s) \in \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}$ of weights (ℓ, m, s). More precisely, if $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{g}^*}$ stand for Hida's Λ-adic Galois representations afforded by $\mathbf{g}$ and $\mathbf{g}^*$, respectively, and $\underline{\varepsilon}_{\mathrm{cyc}}$ is the Λ-adic cyclotomic character,

$$\kappa(\mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{g}} \hat{\otimes} \mathbb{V}_{\mathbf{g}^*} \hat{\otimes} \Lambda(\underline{\varepsilon}_{\mathrm{cyc}}^{-1})(1)).$$

Observe that here, and as a matter of convention, we have used the inverse of the tautological action over Λ, and this is why we have written $\Lambda(\underline{\varepsilon}_{\mathrm{cyc}}^{-1})$ (we may avoid this by invoking the appropriate functional equation).

As it follows from the discussion of [KLZ17, Sections 8,10], the three-variable Hida–Rankin p-adic L-function $L_p(\mathbf{g}, \mathbf{g}^*)$ is the image of the class $\kappa(\mathbf{g}, \mathbf{g}^*)$ under a Perrin-Riou map. Again, the numerator or the denominator may vanish in some *exceptional* cases.

For the precise statements concerning Beilinson–Flach classes, we refer the reader to the notations of previous chapters. As a first observation, we have that in this self-dual case, and according to [Das99, Theorem 9.4], one has that $L_p(g, g^*, 0) = L_p(g, g^*, 1)$. The main results we want to discuss here are the following ones:

(i) When we specialize both $\mathbf{g}$ and $\mathbf{g}^*$ at a fixed weight one point y_0, and the cyclotomic variable s is set as $s = 0$, the denominator of the Perrin-Riou map is zero and the cohomology class $\kappa(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 0)$ vanishes. Then, the explicit reciprocity law of [KLZ17] is substituted by a *derived* reciprocity law relating the *derived* cohomology class with $L_p(g, g^*, 0)$, up to multiplication by an $\mathcal{L}$-invariant. This is precisely equation (6.38).

(ii) When we specialize both $\mathbf{g}$ and $\mathbf{g}^*$ at weight one, and the cyclotomic variable s is set as $s = 1$ the numerator of the Perrin-Riou map is zero **but** $L_p(g, g^*, 1)$ does not vanish (at least generically). This is because [KLZ17, Thm. B] contains the correction factor of (6.44) at the L-function side, which vanishes in this case.

Exceptional vanishing of the denominator of the Perrin-Riou big logarithm. The denominator of the Perrin-Riou regulator introduced in Proposition 3.2 of Chapter 3 vanishes at all points $(y, y, \ell - 1)$ of weight $(\ell, \ell, \ell - 1)$. In particular, specializing $\mathbf{g}$ and $\mathbf{g}^*$ at the weight one modular forms g and g^* respectively, it turns out that

$$\log^{-+}(\kappa_p'(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 0)) = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) \cdot L_p(g, g^*, 0) = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) \cdot L_p(g, g^*, 1) \pmod{L^\times}.$$

To shorten our notations, we have written $\kappa_p'(\mathbf{g}, \mathbf{g}^*)$ for the localization at p of the class.

We would like to emphasize the similitude with our main results for elliptic units, which also relate the logarithm of the *derived* cohomology class with a special value of Katz's two-variable p-adic L-function, up to multiplication by a certain $\mathcal{L}$-invariant.

It turns out that the special value $L_p(\mathbf{g},\mathbf{g}^*)(y,y,0)$ is related via the functional equation with $L_p(\mathbf{g},\mathbf{g}^*)(y,y,1)$ and here, to determine its value, we can follow the same approach than in this chapter: over the line corresponding to those points of weight (ℓ,ℓ,ℓ), the p-adic L-function factors due to the analyticity of an Euler factor (this is properly developed in [Hi04]), and this allows us to obtain an explicit expression of the special value $L_p(g,g^*,1)$ via Galois deformation techniques. This expression involves units and p-units in the field cut out by the Galois representation V_{gg^*}.

Exceptional vanishing of the numerator of the Perrin-Riou big logarithm. The results we present now closely follow [LZ17] and are the counterpart of those developed in previous chapters. We include it here for the sake of completeness, and to illustrate how this exceptional phenomenon arises in a setting which is germane to ours.

Consider specializations of $\kappa(\mathbf{g},\mathbf{g}^*)$ at weights (y,z,m), where $\mathrm{w}(z)=m$. Then, if α_{g_y} and β_{g_y} stand for the eigenvalues of the p-th Hecke polynomial of g_y, with $\mathrm{ord}(\alpha_{g_y})\leq \mathrm{ord}(\beta_{g_y})$, the Euler factor in the numerator of the Perrin-Riou map is

$$1-\frac{\alpha_{g_z}}{\alpha_{g_y}}.$$

This factor is zero for all points of weight (ℓ,ℓ,ℓ). But this does not mean that the p-adic L-function vanishes at those points, since the explicit reciprocity law of [KLZ17, Thm. B] contains the factor

$$c^2-c^{2s+2-\ell-m}=c^2(1-c^{m-\ell})$$

multiplying the value of $L_p(\mathbf{g},\mathbf{g}^*)$, where c is a fixed positive integer coprime with both $6p$ and the level of g. At the points where $\mathrm{w}(z)=m$, the Perrin-Riou map interpolates the Bloch–Kato dual exponential map, and

$$c^2(1-c^{m-\ell})\Big(1-\frac{\alpha_{g_y}}{p\alpha_{g_z}}\Big)L_p(\mathbf{g},\mathbf{g}^*)(y,z,m)=\Big(1-\frac{\alpha_{g_z}}{\alpha_{g_y}}\Big)\cdot \exp_{\mathrm{BK}}^{*-+}(\kappa_p(\mathbf{g},\mathbf{g}^*)(y,z,s)),$$

where $\exp_{\mathrm{BK}}^{*-+}$ stands for the composition of the projection to a certain subspace of $V_{gg^*}\otimes L_p(1)$ followed by the dual exponential map and the pairing with the canonical differentials.

Since both sides of the previous equation vanish along the line $y=z$, $\mathrm{w}(z)=m$, we may consider the derivative at a point (y,y,ℓ), obtaining the expression

$$c^2(1-p^{-1})\dot{\log}_p(c)L_p(\mathbf{g},\mathbf{g}^*)(y,y,\ell)=\Big(\frac{-\alpha'_{\mathbf{g}_y}}{\alpha_{\mathbf{g}_y}}\Big)\cdot \exp_{\mathrm{BK}}^{*-+}(\kappa_p(\mathbf{g},\mathbf{g}^*)(y,y,\ell)),$$

up to multiplication by $L^\times$. Here, $\alpha'_{\mathbf{g}}$ stands for the derivative of the Iwasawa function $\alpha_{\mathbf{g}}$ along the weight direction.

Additionally, invoking Hida's result on the existence of an improved p-adic L-function [Hi04], we get that, whenever the weight of y is 1,

$$\log_p(c)=\exp_{\mathrm{BK}}^{*-+}(\kappa_p(\mathbf{g},\mathbf{g}^*)(y,y,\ell))\pmod{L^\times}.$$

Observe now that

$$L_p(\mathbf{g},\mathbf{g}^*)(y,y,\ell)=L_p(\mathbf{g},\mathbf{g}^*)(y,y,\ell-1)\pmod{L^\times},$$

and hence

$$L_p(\mathbf{g},\mathbf{g}^*)(y,y,\ell-1)=\mathcal{L}(\mathrm{ad}^0(g_\alpha))^{-1}\cdot \log^{-+}(\kappa'_p(\mathbf{g},\mathbf{g}^*)(y,y,\ell-1))\pmod{L^\times}.$$

Consequently, up to multiplication by $L^\times$, we have the equality

$$\frac{\log_p(c)}{\mathcal{L}(\mathrm{ad}^0(g_\alpha))^2}\cdot \log^{-+}(\kappa'_p(\mathbf{g},\mathbf{g}^*)(y,y,\ell-1))=\exp_{\mathrm{BK}}^{*-+}(\kappa_p(\mathbf{g},\mathbf{g}^*)(y,y,\ell))\tag{6.45}$$

In particular, modulo $L^\times$, one has

$$\frac{\log_p(c)}{\mathcal{L}(\mathrm{ad}^0(g_\alpha))^2} \cdot \log^{-+}(\kappa_p'(\mathbf{g},\mathbf{g}^*)(y_0,y_0,0)) = \exp_{\mathrm{BK}}^{*-+}(\kappa_p(\mathbf{g},\mathbf{g}^*)(y_0,y_0,1)). \tag{6.46}$$

Observe that this is coherent with the computations of Chapter 3, from where it follows that, up to multiplication by a scalar in $L^\times$,

$$\log^{-+}(\kappa_p'(\mathbf{g},\mathbf{g}^*)(y,y,\ell-1)) = \mathcal{L}(\mathrm{ad}^0(g_\alpha))^2.$$

This is a consequence of Hida's improved factorization and the *derived* version of the explicit reciprocity law of [KLZ17].

Chapter 7

Generalized Kato classes and exceptional zero conjectures

We study different instances of the *Elliptic Stark Conjectures* of Darmon, Lauder and Rotger, in a situation where the elliptic curve attached to the modular form f has split multiplicative reduction at p. For that purpose, we resort once more to the principle of improved p-adic L-functions and study their $\mathcal{L}$-invariants. We further interpret these results in terms of cohomology classes coming from the setting of diagonal cycles. This allows us to reduce, in a multiplicative situation, the conjecture of Darmon, Lauder and Rotger [DLR15a] to a more familiar statement about higher order derivatives of a triple product p-adic L-function at a point lying *inside* the region of classical interpolation, in the realm of the more well-known *exceptional zero conjectures*.

The results presented at this chapter are the content of the research article [Ri20b].

7.1 Introduction

The Elliptic Stark Conjecture was first formulated by Darmon, Lauder and Rotger in [DLR15a] as a "more constructive alternative to the Birch and Swinnerton-Dyer conjecture, since it often allows the efficient analytic computation of p-adic logarithms of global points". As pointed out by the authors, "it also yields conjectural constructions and explicit formulae, in situations of rank one and two, for global points over cyclotomic fields, abelian extensions of quadratic fields which are not necessarily anticyclotomic, and extensions of $\mathbb{Q}$ with Galois group a central extension of A_4, S_4 or A_5". The conjecture relates a p-adic iterated integral attached to a triple (f, g, h) of cuspidal modular forms with a regulator given in terms of points in an elliptic curve, in a rank 2 situation. Until the moment, not too much work towards the proof of the conjecture has been done: most of the results are restricted to situations where there exists a factorization of p-adic L-functions, which allows to interpret the conjecture in terms of the more familiar objects of Bertolini–Darmon–Prasanna [BDP13]. A multiplicative setting of the conjecture had already been studied in [CR19], but restricted to the case of theta series of imaginary quadratic fields, where a factorization formula for the triple product L-series is also available. This is based on the results of Castella [Cas18a], which extend the work of [BDP13] to the split multiplicative situation.

However, recent works of Bertolini–Seveso–Veneruci [BSV20a], [BSV20b] and Darmon–Rotger [DR20a], [DR20b] suggest an alternative conjecture also in terms of triple product p-adic L-functions: while the first formulation of [DLR15a] is concerned with the p-adic value at a point lying *outside* the region of classical interpolation, the *new* version we discuss is about higher order derivatives at a point which belongs to the classical interpolation region. This setting is germane to that explored firstly by Greenberg–Stevens [GS94] and then by Bertolini–Darmon [BD07] or Venerucci [Ven16]. We propose an alternative conjecture in the split multiplicative setting, and one of the main results of this chapter is the discussion of the equivalence between both formulations, using for that purpose the setting of generalized cohomology classes. This relies, however, on an apparently deep fact about periods of weight one modular forms, stated in [DR16] as Conjecture 2.1. We believe that this *translation* of the conjecture to a more well understood setting provides new evidence for a better understanding of the problem.

The genesis of this project comes from a parallel story where a new conjecture, formulated in [DLR16], arises; this gives a formula for the p-adic iterated integral when the modular form f is no longer cuspidal, but an Eisenstein series. In Chapter 3 we propose a method of proof for this conjecture when the two modular forms (g, h) are self-dual: this was based on Hida's improved factorization theorem for the Hida–Rankin p-adic L-function and allowed us to study the conjecture in terms of a question concerning Galois deformations.

The discussion of our results in this chapter also leads us to the study of an exceptional vanishing of the generalized cohomology classes of [DR16] and [CH20], proposing a putative refinement in terms of some *derived* generalized cohomology classes.

Setting and notations. Fix once for all a prime number $p \geq 3$ and three positive integers N_f, N_g, N_h. Let $N = \mathrm{lcm}(N_f, N_g, N_h)$ and assume that $p \nmid N$. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Let

$$f \in S_2(pN_f), \quad g \in M_1(N_g, \chi), \quad h \in M_1(N_h, \bar{\chi})$$

be a triple of newforms of weights $(2, 1, 1)$, levels (pN_f, N_g, N_h) and nebentype characters $(1, \chi, \bar{\chi})$, where $\bar{\chi}$ stands for the character obtained by composing χ with complex conjugation. Further, we denote by V_g and by V_h the Artin representations attached to g and h, respectively, and write $V_{gh} := V_g \otimes V_h$. Let H be the number field cut out by this representation, and L for the field over which it is defined. To simplify the exposition, we assume that f has rational Fourier coefficients and that is attached via modularity to an elliptic curve E with split multiplicative reduction at p. Under the assumption that $(pN_f, N_gN_h) = 1$, the global sign of the functional equation of $L(E, V_{gh}, s)$ is $+1$. We keep this assumption from now on.

Label and order the roots of the p-th Hecke polynomial of g as

$$X^2 - a_p(g)X + \chi(p) = (X - \alpha_g)(X - \beta_g)$$

and do the same for those of h. Let $g_\alpha(q) = g(q) - \beta_g(q^p)$ denote the p-stabilization of g with U_p-eigenvalue α_g; it is defined by the q-expansion $g_\alpha(q) = g(q) - \beta_g g(q^p)$. We want to deal with a situation of exceptional zeros, that is, where one or several of the Euler factors involved in the interpolation formula of the p-adic L-function vanish (alternatively, and as we will see later on, this can be understood in terms of the eigenvalues for the Frobenius action). This naturally splits into two different settings, namely

(a) the case where $\alpha_g\alpha_h = 1$ (and therefore $\beta_g\beta_h = 1$); and

(b) the case where $\alpha_g\beta_h = 1$ (and therefore $\beta_g\alpha_h = 1$).

In both cases, if we denote the roots of the p-th Hecke polynomial of g by $\{\alpha_g, \beta_g\}$, those of h are $\{1/\alpha_g, 1/\beta_g\}$. As a piece of notation, we write $h_{1/\alpha}$ and $h_{1/\beta}$ for the p-stabilizations of h with eigenvalues $1/\alpha_g$ and $1/\beta_g$, respectively. Along this work, we refer to these settings as *Case (a)* and *Case (b)*. In the framework of Beilinson–Flach elements and Hida–Rankin p-adic L-functions, the second case has been studied in Chapter 3, and the former has been worked out in the last section of Chapter 4.

To prove our main results, we also need a classicality property for g. Hence, we assume throughout that

(H1) the reduction of both V_g and V_h modulo p is irreducible (this requires the choice of integral lattices T_g and T_h, but the fact of being irreducible or not is independent of this choice);

(H2) g and h are p-distinguished, i.e, $\alpha_g \neq \beta_g, \alpha_h \neq \beta_h \pmod p$; and

(H3) V_g is not induced from a character of a real quadratic field in which p splits.

Enlarge L if necessary so that it contains all Fourier coefficients of g_α. As shown in [DLR15a], the above hypotheses ensure that any generalized overconvergent modular form with the same generalized eigenvalues as g_α is classical, and hence simply a multiple of g_α.

In order to describe our results more precisely, let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ be the Iwasawa algebra and denote by $\mathcal{W} = \mathrm{Spf}(\Lambda)$ the weight space. Hida's theory associates the following data to f:

- a finite flat extension $\Lambda_{\mathbf{f}}$ of Λ, giving rise to a covering $\mathrm{w} : \mathcal{W}_{\mathbf{f}} = \mathrm{Spf}(\Lambda_{\mathbf{f}}) \longrightarrow \mathcal{W}$;
- a family of overconvergent p-adic ordinary modular forms $\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]]$ specializing to f at some point $x_0 \in \mathcal{W}_{\mathbf{f}}$ of weight $\mathrm{w}(x_0) = 2$.
- a representation of the absolute Galois group $G_{\mathbb{Q}}$, $\varrho_{\mathbf{f}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}(\mathbb{V}_{\mathbf{f}}) \simeq \mathrm{GL}_2(\Lambda_{\mathbf{f}})$ characterized by the property that all its classical specializations coincide with the Galois representation associated by Deligne to the corresponding specialization of the Hida family.

The same occurs with g_α and h_α thanks to the work of Bellaiche and Dimitrov [BeDi16] on the geometry of the eigencurve for points of weight one; we denote by $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ the corresponding extensions of Λ over which the Hida families $\mathbf{g}$ and $\mathbf{h}$ are defined, and by $y_0 \in \mathcal{W}_{\mathbf{g}}$, $z_0 \in \mathcal{W}_{\mathbf{h}}$ the weight one points for which the specializations agree with g_α and h_α, respectively.

For each of the settings (a) and (b) presented above, we discuss three different objects which are expected to encode arithmetic information regarding the convolution of the three Galois representations attached to the modular forms f, g and h. We denote by (x, y, z) a triple of points in $\mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}}$, whose weights are referred as (k, ℓ, m).

(i) The cohomology classes $\kappa(f, g_\alpha, h_\alpha)$ studied for instance in [DR16] and [CH20], arising as the specialization at weights $(2, 1, 1)$ of the three-variable family $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})$ constructed as the image under a p-adic Abel-Jacobi map of certain diagonal cycles. In general, one may construct four different classes

$$\kappa(f, g_\alpha, h_\alpha), \quad \kappa(f, g_\alpha, h_\beta), \quad \kappa(f, g_\beta, h_\alpha), \quad \kappa(f, g_\beta, h_\beta),$$

one for each p-stabilization of g and h. Further, when some of these classes vanish, we are lead to consider their derivatives.

(ii) The special value $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at weights $(2, 1, 1)$ and its derivatives. Here, $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ stands for the three-variable p-adic L-function attached to three Hida families, characterized by an interpolation property regarding the classical values of the triple product L-function at the region where $k \geq \ell + m$. When this function vanishes at the point $(2, 1, 1)$, the derivatives along different directions of the weight space may encode *interesting* arithmetic information.

(iii) The special value $\mathscr{L}_p{}^{g_\alpha}(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at weights $(2, 1, 1)$, denoted $\mathscr{L}_p{}^{g_\alpha}$. This p-adic L-function is defined in an analogue way to the previous one, but now the region of interpolation is characterized by the inequality $\ell \geq k + m$ so the point $(2, 1, 1)$ is outside the region of classical interpolation. Similarly, we may also take $\mathscr{L}_p{}^{h_\alpha}(\mathbf{f}, \mathbf{g}, \mathbf{h})$, whose region of interpolation concerns those points for which $m \geq k + \ell$. Observe that the first value depends on the choice of p-stabilizations for the weight one form g_α.

(i) Cohomology classes coming from the theory of diagonal cycles. We begin by recalling the results concerning cohomology classes. Results of this kind had already been explored in [BSV20b] and [DR20b] when $\alpha_g \alpha_h = 1$. In that case, the cohomology class is not expected to vanish, but the numerator of the (Perrin-Riou) regulator in the reciprocity law for $\mathscr{L}_p{}^f$ does, which is coherent with the fact that the p-adic L-function $\mathscr{L}_p{}^f(f,g,h)$ is zero (this can be seen, of course, as an exceptional zero coming from the vanishing of an Euler factor).

Here we are mostly interested in the case where the denominator of the Perrin-Riou regulator in the reciprocity law for $\mathscr{L}_p{}^{g_\alpha}$ vanishes due to another exceptional zero phenomenon. This occurs when $\alpha_g\beta_h = 1$ and leads us to recover the ideas of [Cas18a], [RR20a] and [Ri20a], where this same phenomenon was studied for Heegner points, Beilinson–Flach elements and elliptic units, respectively. In those cases, the reciprocity laws linking Euler systems and p-adic L-functions were updated to *derived reciprocity laws.* A different approach is taken also in [BSV20a, Section 8], where the authors introduce certain *improved* cohomology classes, which in this case we may compare in an explicit way with appropriate *derived* elements.

Define the three-variable Iwasawa algebra $\Lambda_{\mathbf{fgh}} := \Lambda_{\mathbf{f}}\hat{\otimes}_{\mathbb{Z}_p}\Lambda_{\mathbf{g}}\hat{\otimes}_{\mathbb{Z}_p}\Lambda_{\mathbf{h}}$ and the $\Lambda_{\mathbf{fgh}}[G_{\mathbb{Q}}]$-module

$$\mathbb{V}_{\mathbf{fgh}} := \mathbb{V}_{\mathbf{f}}\hat{\otimes}_{\mathbb{Z}_p}\mathbb{V}_{\mathbf{g}}\hat{\otimes}_{\mathbb{Z}_p}\mathbb{V}_{\mathbf{h}}.$$

We work with $\mathbb{V}_{\mathbf{fgh}}^\dagger$, a certain twist of it by an appropriate power of the Λ-adic cyclotomic character defined for instance in [DR20b, Section 5.1] and that is needed to satisfy the self-dual assumption.

The works [BSV20a] and [DR20b] attach to $(\mathbf{f},\mathbf{g},\mathbf{h})$ a Λ-adic global cohomology class

$$\kappa(\mathbf{f},\mathbf{g},\mathbf{h}) \in H^1(\mathbb{Q},\mathbb{V}_{\mathbf{fgh}}^\dagger)$$

parameterized by the triple product of the weight space $\mathcal{W}_{\mathbf{fgh}} := \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}}$.

Consider the specialization of the class at weights (x_0,y_0,z_0),

$$\kappa(f,g_\alpha,h_{1/\beta}) \in H^1(\mathbb{Q},V_{fgh}),$$

where V_{fgh} is the tensor product $V_f\otimes V_g\otimes V_h$ of the Galois representations attached to the modular forms f, g and h. This class can be shown to be trivial and hence we are placed to work with an appropriate *derived* class $\kappa'(f,g_\alpha,h_{1/\beta})$.

As it occurred in the setting of Beilinson–Flach classes, the notion of derivative is rather flexible. We consider here a derivative along an analytic direction, and keeping fixed the weight of h. Rather informally, this may be thought as the line $(\ell+1,\ell,1)$ of the weight space. Note that at least in the self-dual case, where we may argue that the corresponding class vanishes all along the line $(2,\ell,\ell)$, we may consider the derivative along any direction of the weight space.

Let $\alpha_{\mathbf{f}}$ (resp. $\alpha_{\mathbf{g}}$, $\alpha_{\mathbf{h}}$) stand for the Iwasawa function corresponding to the root of the p-th Hecke polynomial of $\mathbf{f}$ (resp. $\mathbf{g}$, $\mathbf{h}$) with smallest p-adic valuation. As an additional piece of notation, let

$$\mathcal{L} := \frac{\alpha_g'}{\alpha_g} - \frac{\alpha_f'}{\alpha_f}, \tag{7.1}$$

where α_f' (resp. α_g', α_h') stands for the derivative of the Frobenius eigenvalues at x_0 (resp. y_0, z_0) when seen as an Iwasawa function along the Hida family $\Lambda_{\mathbf{f}}$ (resp. $\Lambda_{\mathbf{g}}$, $\Lambda_{\mathbf{h}}$). Observe that we can give explicit formulas for $\mathcal{L}$, involving both some units and p-units in the field cut out by the representation V_{gh} and the Tate uniformizer of the elliptic curve E. Hence, the $\mathcal{L}$-invariant governing the arithmetic of the triple (f,g,h) is related both with the $\mathcal{L}$-invariant of the elliptic curve (the logarithm of the Tate uniformizer) and also with the regulator attached to the adjoint representation $\mathrm{ad}^0(V_g)$, expressed in Chapters 3 and 5 as a combination of logarithms of units and p-units. Compare for instance this result with the main theorem of [Cas18a], where he interprets the $\mathcal{L}$-invariant attached to a modular form f and an anticyclotomic character as the sum of the two $\mathcal{L}$-invariants. Our first main result is the following (see Theorem 7.3.9 for the precise formulation), relating an appropriate logarithm of the *derived* class with the special value $\mathscr{L}_p{}^{g_\alpha}$.

Theorem 7.1.1. *The derived cohomology class satisfies*

$$\langle \log_{\mathrm{BK}}(\kappa_p'(f, g_\alpha, h_{1/\beta})^g), \eta_f \otimes \omega_{g_\alpha} \otimes \omega_{h_{1/\beta}} \rangle = \mathcal{L} \cdot \mathscr{L}_p{}^{g_\alpha}(\mathbf{f}, \mathbf{g}, \mathbf{h})(x_0, y_0, z_0) \pmod{L^\times},$$

where the superindex g stands for an appropriate projection of κ_p' that we later introduce, and $\log_{\mathrm{BK}}$ *refers to the Bloch–Kato logarithm, followed by the pairing $\langle -, - \rangle$ with certain canonical differentials.*

Remark 7.1.2. In [BSV20a] the authors take a different approach to this exceptional zero phenomenon, and construct an *improved* cohomology class $\kappa_g^*(f, g_\alpha, h_{1/\beta})$. As we will later show, there is a connection between both constructions and one may prove (under mild conditions!) that the following equality holds in $H^1(\mathbb{Q}, (V_f \otimes V_g \otimes V_h)_{|\mathcal{S}})$, where $\mathcal{S}$ stands for the subvariety of the weight space along which the derived and the improved class are defined, corresponding to the set of weights $k + m = \ell + 2$:

$$\kappa'(f, g_\alpha, h_{1/\beta}) = \mathcal{L} \cdot \kappa_g^*(f, g_\alpha, h_{1/\beta}). \tag{7.2}$$

(ii) The special value $\mathscr{L}_p{}^f$ and derivatives of the triple product p-adic L-function. In subsequent parts of the chapter we use the previous cohomology classes to study different instances of the Elliptic Stark Conjecture. Section 4 is devoted to analyze higher order derivatives of $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at (x_0, y_0, z_0). The presence of an Euler factor which vanishes at weights $(2, 1, 1)$ automatically forces the vanishing of that value. Therefore, it is natural to formulate several conjectures for the value of the derivatives of $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$.

When $\alpha_g \alpha_h = 1$ and $L(f \otimes g \otimes h, 1) \neq 0$, the results of [BSV20a] relying on the existence of an improved p-adic L-function allow us to state the following result. Although this can be seen as a straightforward corollary of the results developed in loc. cit., we want to point out that the $\mathcal{L}$-invariants attached to both g and h have a strong connection with the arithmetic of number fields. This reveals that in the rank 0 situation the quantity $\mathscr{L}_p{}^f$ is also a putative refinement of the more well-known $\mathcal{L}$-invariants of Greenberg–Stevens, where not only the Tate period q_E appears. This result follows from [BSV20a, Proposition 8.2].

Proposition 7.1.3 (Bertolini–Seveso–Venerucci)**.** *Let I denote the ideal of functions in $\Lambda_{\mathbf{fgh}}$ which vanish at (x_0, y_0, z_0). Assume that $L(f \otimes g \otimes h, 1) \neq 0$, and let $\mathcal{L}_\xi := \alpha_\xi'/\alpha_\xi$, for $\xi \in \{f, g, h\}$. Then, up to a constant in $L^\times$,*

$$\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h}) = (\mathcal{L}_g - \mathcal{L}_f)(\ell - 1) + (\mathcal{L}_h - \mathcal{L}_f)(m - 1) \pmod{I^2}.$$

Moreover, the quantities $\mathcal{L}_\chi$ are explicitly computable in terms of the arithmetic of number fields and elliptic curves.

Observe for example that the derivative along the y-direction agrees with the $\mathcal{L}$-invariant that also arises as the derivative of the diagonal class discussed before.

However, the most interesting case appears when $L(f \otimes g \otimes h, 1) = 0$. Let us put ourselves in the setting of [DLR15a] and assume that $(E(H) \otimes V_{gh}^\vee)^{\mathrm{Gal}\,(H/\mathbb{Q})}$ is two-dimensional, where $V_{gh}^\vee$ stands for the contragradient representation of V_{gh}. This group is equipped with an inclusion in the p-adic Selmer group corresponding to the group of extensions of $\mathbb{Q}_p$ by V_{fgh} in the category of $\mathbb{Q}_p$-linear representations of $G_{\mathbb{Q}}$ that are crystalline at p. This group is denoted by $H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh})$, and we also assume that is two-dimensional (the latter would follow from the Birch and Swinnerton-Dyer conjecture for the pair (E, V_{gh}) and the finiteness of the corresponding Tate–Shafarevich group).

Let $\{P, Q\}$ denote generators of $(E(H) \otimes V_{gh}^\vee)^{G_{\mathbb{Q}}}$, and fix a basis $\{e_{\alpha\alpha}^\vee, e_{\alpha\beta}^\vee, e_{\beta\alpha}^\vee, e_{\beta\beta}^\vee\}$ of $V_{gh}^\vee$ as a $G_{\mathbb{Q}_p}$-module with the Frobenius action. This allows us to write

$$P = P_{\alpha\alpha} \otimes e_{\beta\beta}^\vee + P_{\alpha\beta} \otimes e_{\beta\alpha}^\vee + P_{\beta\alpha} \otimes e_{\alpha\beta}^\vee + P_{\beta\beta} \otimes e_{\alpha\alpha}^\vee,$$

and similarly for Q. Here, the arithmetic Frobenius Fr_p acts on $P_{\alpha\alpha}$ with eigenvalue $\beta_g \beta_h$ and analogously for the remaining components. In this case, we can conjecture the following result, that we extensively discuss in Section 7.5.

Conjecture 7.1.4. *Assume that the L-dimension of* $(E(H) \otimes V_{gh}^{\vee})^{G_{\mathbb{Q}}}$ *is two. Then, under the running assumptions, the p-adic L-function* $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ *satisfies*

$$\frac{\partial^2 \mathscr{L}_p{}^f(\mathbf{f}_x, g_\alpha, h_{1/\alpha})}{\partial x^2}\Big|_{x=x_0} = \log_p(P_{\alpha\alpha}) \cdot \log_p(Q_{\beta\beta}) - \log_p(Q_{\alpha\alpha}) \cdot \log_p(P_{\beta\beta}) \pmod{L^\times}.$$

If the L-dimension of $(E(H) \otimes V_{gh}^{\vee})^{G_{\mathbb{Q}}}$ *is greater than two, then the left hand side vanishes.*

There are other interesting lines along weight space to take derivatives. For example, in [CH20] the study is concerned with the line $(2, \ell, \ell)$, where the derivatives are connected with appropriate *derived* heights of the points P and Q.

The work of Bertolini–Seveso–Venerucci and Darmon–Rotger establishes the conjecture for the case where g and h are theta series of a quadratic field where p is inert, which leads to a decomposition $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$. In the imaginary case, we can extend their computations to the split case, observing that here one has a trivial equality of the form $0 = 0$. We expect that the same occurs for the adjoint case, that is, when $h = g^*$.

Therefore, we may establish that Conjecture 7.1.4 holds in some dihedral cases. The first part of this Proposition follows from [BSV20b, Theorem A], and the second is established as part of Proposition 7.4.7.

Proposition 7.1.5. *Conjecture 7.1.4 holds in the following cases:*

(a) CM or RM series with p inert in K and at least one of ψ_1 *or* ψ_2 *being a genus characters;*

(b) CM series with p split in K.

We must say that in all these cases the proof is based on a factorization formula, so we expect that new ideas would be required for the proof in the general case.

(iii) The special value $\mathscr{L}_p{}^{g_\alpha}$. In the last section, we discuss a way to connect the previous conjecture with the Elliptic Stark Conjecture of [DLR15a] when $\alpha_g \alpha_h = 1$. Recall that the conjecture predicts that

$$\mathscr{L}_p{}^{g_\alpha} = \frac{\log_p(P_{\alpha\beta}) \log_p(Q_{\alpha\alpha}) - \log_p(P_{\alpha\alpha}) \log_p(Q_{\alpha\beta})}{\log_p(u_{g_\alpha})} \pmod{L^\times}, \tag{7.3}$$

with u_{g_α} being a Gross–Stark unit whose characterization we later recall. In particular, it is expected that this unit could be expressed as a ratio of periods attached to weight one forms. These two periods, denoted by Ω_{g_α} and Ξ_{g_α}, will play a prominent role in the last part of the work. More precisely, in [DR16, eq. (9)], the authors introduce a p-adic period, $\mathcal{L}_{g_\alpha} = \Omega_{g_\alpha}/\Xi_{g_\alpha}$ and conjecture (see Conjecture 2.1 of loc. cit.)

$$\mathcal{L}_{g_\alpha} = \log_p(u_{g_\alpha}). \tag{7.4}$$

In Section 7.5 we consider the following three conjectures:

(i) the Elliptic Stark Conjecture for $\mathscr{L}_p{}^{g_\alpha}$;

(ii) the conjecture for the second derivative along the f-direction for $\mathscr{L}_p{}^f$, i.e., Conjecture 7.1.4;

(iii) [DR16, Conjecture 2.1] about periods of weight one modular forms. Proposition 7.5.1 can be seen as an extra piece of theoretical evidence towards this conjecture, showing that

$$\frac{\mathcal{L}_{g_\alpha}}{\mathcal{L}_{g_\beta}} = \frac{\log_p(u_{g_\alpha})}{\log_p(u_{g_\beta})}.$$

Under certain non-vanishing hypothesis, we prove that if two of the previous conjectures are true, the third one automatically holds. In particular, we establish the following in Corollary 7.5.5.

Theorem 7.1.6. *Let g and h be theta series of a quadratic field (either real or imaginary) where p is inert. Write $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$, and assume that either ψ_1 or ψ_2 is a genus character. Then, under the given assumptions, the equality* (7.4) *is equivalent to the Elliptic Stark Conjecture of Darmon, Lauder and Rotger* (7.3).

All the previous results are based on the interaction of the different arithmetic actors when $\alpha_g\alpha_h = 1$. The case where $\alpha_g\beta_h = 1$ is more subtle, since here the cohomology class $\kappa(f, g_\alpha, h_{1/\beta})$ vanishes and we cannot extract the same arithmetic information. In any case, we expect that a similar result must hold in this setting. The reason is that the value of $\mathscr{L}_p{}^{g_\alpha}$ does not depend on the choice of a p-stabilization for h, and hence we can also give a conjectural expression for the derived cohomology class in terms of points, in complete analogy with Theorem B of Chapter 3.

Conjecture 7.1.7. *The following equality holds in $H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh})$:*

$$\kappa'(f, g_\alpha, h_{1/\beta}) = \frac{\mathcal{L}}{\Xi_{g_\alpha} \cdot \Omega_{h_{1/\beta}}} \cdot \frac{\log_p(P_{\alpha\alpha}) \cdot Q - \log_p(Q_{\alpha\alpha}) \cdot P}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{7.5}$$

Proceeding as in [DR16] and [RR19] (see Chapter 4) we may also obtain expressions (at least conjecturally) for the three remaining cohomology classes, $\kappa'(f, g_\beta, h_{1/\alpha})$, $\kappa(f, g_\alpha, h_{1/\alpha})$ and $\kappa(f, g_\beta, h_{1/\beta})$.

7.2 Preliminaries

This section aims to give an overview of the setting we present, concerned with triple product p-adic L-functions, and also recalls some known results in other related scenarios coming from the theory of elliptic curves and weight one modular forms.

Hsieh's triple product p-adic L-function

Fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. For a number field K, let $G_K := \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ denote its absolute Galois group. Fix also an odd prime p and an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

The formal spectrum $\mathcal{W} = \mathrm{Spf}(\Lambda)$ of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is called the weight space attached to Λ. The weight space is equipped with a distinguished class of *arithmetic points* $\nu_{s,\varepsilon}$ indexed by integers $s \in \mathbb{Z}$ and Dirichlet characters $\varepsilon : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ of p-power conductor. The point $\nu_{s,\varepsilon} \in \mathcal{W}$ is defined by the rule

$$\nu_{s,\varepsilon}(n) = \varepsilon(n)n^s.$$

Let $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ be a triple of p-adic Hida families of tame levels $N_{\mathbf{f}}$, $N_{\mathbf{g}}$, $N_{\mathbf{h}}$ and tame characters $\chi_{\mathbf{f}}$, $\chi_{\mathbf{g}}$, $\chi_{\mathbf{h}}$. Let also $(\mathbf{f}^*, \mathbf{g}^*, \mathbf{h}^*)$ denote the conjugate triple, and assume that $\chi_{\mathbf{f}}\chi_{\mathbf{g}}\chi_{\mathbf{h}} = 1$ (this is referred to as the self-duality assumption). Set $N = \mathrm{lcm}(N_{\mathbf{f}}, N_{\mathbf{g}}, N_{\mathbf{h}})$, and suppose that $p \nmid N$.

Let $\Lambda_{\mathbf{f}}$, $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ be the finite extensions of Λ generated by the coefficients of the Hida families $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$, respectively. The weight space attached to $\Lambda_{\mathbf{f}}$ is $\mathcal{W}_{\mathbf{f}} := \mathrm{Spf}(\Lambda_{\mathbf{f}})$. Since $\Lambda_{\mathbf{f}}$ is a finite flat algebra over Λ, there is a natural finite map

$$\pi : \mathcal{W}_{\mathbf{f}} := \mathrm{Spf}(\mathcal{W}_{\mathbf{f}}) \xrightarrow{\mathrm{w}} \mathcal{W},$$

and we say that a point $x \in \mathcal{W}_{\mathbf{f}}$ is arithmetic of weight s and character ε if $\pi(x) = \nu_{s,\varepsilon}$.

A point $x \in \mathcal{W}_{\mathbf{f}}$ of weight $k \geq 1$ and character ε is said to be crystalline if $\varepsilon = 1$ and there exists an eigenform f_x° of level N such that f_x is the ordinary p-stablization of f_x°. We denote by $\mathcal{W}_{\mathbf{f}}^\circ$ the set of crystalline arithmetic points of $\mathcal{W}_{\mathbf{f}}$.

Finally, set $\Lambda_{\mathbf{fgh}} = \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}}$ and let $\mathcal{W}^\circ_{\mathbf{fgh}} := \mathcal{W}^\circ_{\mathbf{f}} \times \mathcal{W}^\circ_{\mathbf{g}} \times \mathcal{W}^\circ_{\mathbf{h}} \subset \mathcal{W}_{\mathbf{fgh}} = \mathrm{Spf}(\Lambda_{\mathbf{fgh}})$ be the set of triples of crystalline classical points, at which the three Hida families specialize to modular forms with trivial nebentype at p. This set admits the natural partition

$$\mathcal{W}^\circ_{\mathbf{fgh}} = \mathcal{W}^f_{\mathbf{fgh}} \sqcup \mathcal{W}^g_{\mathbf{fgh}} \sqcup \mathcal{W}^h_{\mathbf{fgh}} \sqcup \mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}},$$

where

- $\mathcal{W}^f_{\mathbf{fgh}}$ denotes the set of points $(x,y,z) \in \mathcal{W}^\circ_{\mathbf{fgh}}$ of weights (k,ℓ,m) such that $k \geq \ell + m$.
- $\mathcal{W}^g_{\mathbf{fgh}}$ and $\mathcal{W}^h_{\mathbf{fgh}}$ are defined similarly, replacing the role of $\mathbf{f}$ by $\mathbf{g}$ (resp. $\mathbf{h}$).
- $\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$ is the set of balanced triples, consisting of points (x,y,z) of weights (k,ℓ,m) such that each of the weights is strictly smaller than the sum of the other two.

Recall from [DR20b, Section 1.4] the notion of test vector. As proved in Section 3.5 of loc. cit. following [Hs20], there is a canonical choice of test vectors for which there exists a *square-root* p-adic L-function

$$\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h}) : \mathcal{W}_{\mathbf{fgh}} \to \mathbb{C}_p,$$

characterized by an interpolation property relating its values at classical points $(x,y,z) \in \mathcal{W}^f_{\mathbf{fgh}}$ to the square root of the central critical value of Garrett's triple-product complex L-function $L(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z,s)$ associated to the triple of classical eigenforms $(\mathbf{f}_x,\mathbf{g}_y,\mathbf{h}_z)$. For the following proposition, let $\alpha_{\mathbf{f}_x}$ and $\beta_{\mathbf{f}_x}$ be the roots of the p-th Hecke polynomial of $\mathbf{f}_x$, ordered in such a way that $\mathrm{ord}_p(\alpha_{\mathbf{f}_x}) \leq \mathrm{ord}_p(\beta_{\mathbf{f}_x})$. The following result is [DR20b, Proposition 5.1].

Proposition 7.2.1. *Fix test vectors $(\tilde{\mathbf{f}},\tilde{\mathbf{g}},\tilde{\mathbf{h}})$ as in [Hs20, Section 3]. Then $\mathscr{L}_p{}^f(\tilde{\mathbf{f}},\tilde{\mathbf{g}},\tilde{\mathbf{h}})$ lies in $\Lambda_{\mathbf{fgh}}$ and for every $(x,y,z) \in \mathcal{W}^f_{\mathbf{fgh}}$ of weights (k,ℓ,m) we have*

$$\mathscr{L}_p{}^f(\tilde{\mathbf{f}},\tilde{\mathbf{g}},\tilde{\mathbf{h}})^2(x,y,z) = \frac{\mathfrak{a}(k,\ell,m)}{\langle \mathbf{f}^\circ_x, \mathbf{f}^\circ_x\rangle^2} \cdot \mathfrak{e}^2(x,y,z) \times L(\mathbf{f}^\circ_x,\mathbf{g}^\circ_y,\mathbf{h}^\circ_z,c),$$

where

1. $c = \frac{k+\ell+m-2}{2}$.
2. $\mathfrak{a}(k,\ell,m) = (2\pi i)^{-2k} \cdot \left(\frac{k+\ell+m-4}{2}\right)! \cdot \left(\frac{k+\ell-m-2}{2}\right)! \cdot \left(\frac{k-\ell+m-2}{2}\right)! \cdot \left(\frac{k-\ell-m}{2}\right)!,$
3. $\mathfrak{e}(x,y,z) = \mathcal{E}(x,y,z)/\mathcal{E}_0(x)\mathcal{E}_1(x)$ *with*

$$\begin{aligned}
\mathcal{E}_0(x) &:= 1 - \chi_f^{-1}(p)\beta^2_{\mathbf{f}_x}p^{1-k},\\
\mathcal{E}_1(x) &:= 1 - \chi_f(p)\alpha^{-2}_{\mathbf{f}_x}p^{k-2},\\
\mathcal{E}(x,y,z) &:= \left(1 - \chi_f(p)\alpha^{-1}_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\right) \times \left(1 - \chi_f(p)\alpha^{-1}_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\beta_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\right)\\
&\quad \times\left(1 - \chi_f(p)\alpha^{-1}_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\right) \times \left(1 - \chi_f(p)\alpha^{-1}_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z}p^{\frac{k-\ell-m}{2}}\right).
\end{aligned}$$

There is an ostensible parallelism between this p-adic L-function and the so-called Hida–Rankin p-adic L-function attached to a pair of Hida families $(\mathbf{g},\mathbf{h})$, but where the cyclotomic variable s is allowed to move freely. It may be instructive to keep in mind this analogy for the subsequent results.

Some of the easiest cases to understand these triple product p-adic L-functions arise when the representation attached to V_{gh} is irreducible. In particular, assume that g is a weight one theta series attached to a quadratic field K (either real of imaginary) where p remains inert. Then,

$V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$, and under the assumption that at least one between ψ_1 or ψ_2 is a genus (quadratic) character, the works [BSV20b] and [DR20b] show that

$$\mathscr{L}_p{}^f(\mathbf{f}, g, h)^2 = \mathfrak{f}(k) \cdot L_p(\mathbf{f}/K, \psi_1) \cdot L_p(\mathbf{f}/K, \psi_2), \tag{7.6}$$

where $\mathfrak{f}(k)$ is a bounded analytic function on $\Lambda_{\mathbf{f}}$ such that $\mathfrak{f}(x_0) \in L^\times$. Here, $\mathscr{L}_p(\mathbf{f}/K, \psi)$ is the two-variable p-adic L-function attached to a Hida family $\mathbf{f}$ and a character ψ of a quadratic field.

As a word of caution, observe that there are *three* different p-adic L-functions, depending on the region of classical interpolation (associated to the dominant weight).

Improved p-adic L-functions

It is a natural phenomenon in the study of p-adic L-functions that some of the Euler factors arising in the interpolation process are analytic along a subvariety of the weight space (recall that this idea already appeared in Chapter 3). When this happens, one is tempted to define *improved* p-adic L-functions, that is, functions over the corresponding subvariety characterized by the same interpolation property, but with these Euler factors removed. This is a quite well-known phenomenon, which dates back to Greenberg–Stevens [GS94] and their study of the Mazur–Kitagawa p-adic L-function. This was one of the key ingredients in the proof of our main results in previous chapter, and we would like to stress the limitations of the method in this triple product setting. The interest of this study is that we also want to discuss later on its applicability from the Euler system side in order to construct *improved cohomology classes.*

For the sake of simplicity, assume that χ_f is trivial. In the setting of triple product p-adic L-functions we have just discussed, one of the Euler factors appearing in the interpolation property of $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ is

$$1 - \frac{\alpha_{\mathbf{g}_y} \alpha_{\mathbf{h}_z}}{\alpha_{\mathbf{f}_x}} p^{\frac{k-\ell-m}{2}},$$

which is an Iwasawa function along the surface $\mathcal{S}_{k=\ell+m}$ defined by

$$\mathcal{S}_{k=\ell+m} = \{(x, y, z) \in \mathcal{W}_{\mathbf{f}}^\circ \times \mathcal{W}_{\mathbf{g}}^\circ \times \mathcal{W}_{\mathbf{h}}^\circ \text{ such that } k = \ell + m\}.$$

The definitions given in [DR14, Def. 4.4] can be adapted to yield an *improved* p-adic L-function $\mathscr{L}_p(\mathbf{f}, \mathbf{g}, \mathbf{h})^*$ on $\mathcal{S}_{k=\ell+m}$, by replacing the family $\mathbf{h} \times d^t \mathbf{g}^{[p]}$ with the family $\mathbf{h} \times \mathbf{g}$, whose coefficients vary analytically because $t = 0$ on $\mathcal{S}_{k=\ell+m}$.

Proposition 7.2.2. *There exists an analytic p-adic L-function over the surface $\mathcal{S}_{k=\ell+m}$, written as $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})^*$, and such that the following equality holds in $\mathcal{S}_{k=\ell+m}$:*

$$\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h}) = (1 - \alpha_{\mathbf{f}}^{-1} \alpha_{\mathbf{g}} \alpha_{\mathbf{h}}) \mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})^*. \tag{7.7}$$

Proof. This follows from the proof of [BSV20a, Proposition 8.2] and the discussion after it. □

We point out that the improved p-adic L-function we have considered, $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})^*$, interpolates classical L-values, in the same way than $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$, but with the vanishing Euler factor removed. Therefore, its value at (x_0, y_0, y_0) is given by an explicit non-zero multiple of the square root of the algebraic part of $L(f, g, h, 1)$. In particular, $L(f, g, h, 1) \neq 0$ if and only if the improved p-adic L-function does not vanish at (x_0, y_0, y_0).

Observe however that we may also consider other Euler factors. Take for example

$$1 - \frac{\bar{\chi}_h(p) \alpha_{\mathbf{h}_z}}{\alpha_{\mathbf{g}_y} \alpha_{\mathbf{f}_x}} p^{\frac{k+\ell-m-2}{2}},$$

which is analytic along $k + \ell = m + 2$.

We would expect that one can establish that these factors (each along its respective region) divide the p-adic L-function and yield other improved p-adic L-functions satisfying *mild* analytic and interpolation properties.

Exceptional zeros and L-invariants

The situations we study in this chapter are mostly concerned with the so-called exceptional zero phenomenon. We now recall several results which appear in the literature around that, mainly in [GS94], [Ven16] and [RR20a]. As anticipated before, the point is that the $\mathcal{L}$-invariant governing the arithmetic of $V_f \otimes V_{gh}$ is a combination of the $\mathcal{L}$-invariants attached to f and the adjoint of g (or the adjoint of h, according to the *direction* we choose). For a brief summary of the usual definition and the main properties of the adjoint representation in this scenario, see [DLR15a, Section 1.2].

The aim of this section is to give an arithmetic description of the different $\mathcal{L}$-invariants that later appear in the setting of triple products, to have a complete description of our picture.

Let us define, to ease notations,

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) := \frac{\alpha_g'}{\alpha_g}, \qquad \mathcal{L}(E) := \frac{\alpha_f'}{\alpha_f}, \tag{7.8}$$

where the derivative is taken along the unique Hida family passing through g_α and f, respectively, and then evaluating at the points corresponding to g_α and f.

I. The $\mathcal{L}$-invariant of the adjoint of a modular form. One of the main results of Chapter 3 was the computation of the $\mathcal{L}$-invariant associated to the adjoint of a modular form.

As shown in [DLR16, Lemma 1.1], we have

$$\dim_L(\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}} = 1, \quad \dim_L(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}} = 2.$$

Fix a generator u of $(\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$ and also an element v of $(\mathcal{O}_H^\times[1/p]^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$ such that $\{u, v\}$ is a basis of $(\mathcal{O}_H[1/p]^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$. The element v may be chosen to have p-adic valuation $\mathrm{ord}_p(v) = 1$, and we do so. Viewed as a $G_{\mathbb{Q}_p}$-module, $\mathrm{ad}^0(g)$ decomposes as $\mathrm{ad}^0(g) = L \oplus L^{\alpha\otimes\bar\beta} \oplus L^{\beta\otimes\bar\alpha}$, where all the summands are 1-dimensional subspaces characterized by the property that the arithmetic Frobenius Fr_p acts on it with eigenvalue 1, α/β and β/α, respectively. Let H_p denote the completion of H in $\bar{\mathbb{Q}}_p$ and let

$$u_1,\ u_{\alpha\otimes\bar\beta},\ u_{\beta\otimes\bar\alpha},\ v_1,\ v_{\alpha\otimes\bar\beta},\ v_{\beta\otimes\bar\alpha} \in H_p^\times \otimes_\mathbb{Q} L \qquad (\mathrm{mod}\ L^\times)$$

denote the projection of the elements u and v in $(H_p^\times \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}_p}}$ to the above lines. By construction we have $u_1, v_1 \in \mathbb{Q}_p^\times$ and

$$\mathrm{Fr}_p(u_{\alpha\otimes\bar\beta}) = \frac{\beta}{\alpha} u_{\alpha\otimes\bar\beta}, \quad \mathrm{Fr}_p(v_{\alpha\otimes\bar\beta}) = \frac{\beta}{\alpha} v_{\alpha\otimes\bar\beta}, \quad \mathrm{Fr}_p(u_{\beta\otimes\bar\alpha}) = \frac{\alpha}{\beta} u_{\beta\otimes\bar\alpha}, \quad \mathrm{Fr}_p(v_{\beta\otimes\bar\alpha}) = \frac{\alpha}{\beta} v_{\beta\otimes\bar\alpha}.$$

Let

$$\log_p : H_p^\times \otimes L \longrightarrow H_p \otimes L$$

denote the usual p-adic logarithm.

Then, $\mathcal{L}(\mathrm{ad}^0(g_\alpha))$ can be expressed as

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) = -\frac{\log_p(v_1)\log_p(u_{\alpha\otimes\bar\beta}) - \log_p(u_1)\log_p(v_{\alpha\otimes\bar\beta})}{2\mathrm{ord}_p(v_1)\cdot \log_p(u_{\alpha\otimes\bar\beta})}. \tag{7.9}$$

II. The $\mathcal{L}$-invariant of an elliptic curve (rank 0). In [GS94], the authors prove a conjecture of Mazur, Tate and Teitelbaum [MTT86] expressing the quantity $L_p(E, 1)$ in terms of the derivative of $L(E, 1)$ when the rank is zero. As a consequence of this, they show that an elliptic curve with split multiplicative reduction at p satisfies

$$\mathcal{L}(E) = -\frac{\log_p(q_E)}{2\mathrm{ord}_p(q_E)}, \tag{7.10}$$

where q_E is Tate's uniformizer for the elliptic curve E. We write $L_p(\mathbf{f})(x,s)$ for the usual two-variable Mazur–Kitagawa p-adic L-function, and x_0 for the weight two point satisfying $\mathbf{f}_{x_0} = f$, with f the modular form attached to E by modularity.

As recalled for instance in the discussion of [BD07, Remark 1.13], there exists an improved p-adic L-function along $s=1$, that we denote here as $L_p^*(\mathbf{f})(x)$ and which is characterized by

$$L_p(\mathbf{f})(x,1) = (1 - a_p(\mathbf{f}_x)^{-1}) \cdot L_p^*(\mathbf{f})(x).$$

Observe that in a rank 0 situation $L_p^*(\mathbf{f}_{x_0})$ is a non-zero algebraic number which agrees (up to constant) with the algebraic part of the classical L-value.

III. The $\mathcal{L}$-invariant of an elliptic curve (rank 1). In a rank 1 situation, Venerucci relates the second derivatives of the Mazur–Kitagawa p-adic L-function with certain heights of Heegner points. Observe that in this setting, $L_p(\mathbf{f})(x_0,1) = 0$ and the same happens for its first derivatives. To determine the second order derivatives, he recast in [Ven16] to the theory of Selmer complexes and Nekovář's Selmer groups, as introduced in [Nek06].

Following the conventions used in loc. cit., let $\tilde{H}^1_{\mathrm{f}}$ be Nekovar's extended Selmer group. It is a $\mathbb{Q}_p$-module, equipped with a natural inclusion of the extended Mordell-Weil group of E, that we denote by $E^\dagger(\mathbb{Q}) \otimes \mathbb{Q}_p$. In general,

$$\tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, V_p(E)) = H^1_{\mathrm{f}}(\mathbb{Q}, V_p(E)) \oplus \mathbb{Q}_p \cdot q_E,$$

where $H^1_{\mathrm{f}}(\mathbb{Q}, V_p(E))$ is the Bloch–Kato p-adic Selmer group. Using Nekovar and Venerucci's results, there is a canonical $\mathbb{Q}_p$-bilinear form

$$\langle \cdot, \cdot \rangle : \tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, V_p(E)) \otimes_{\mathbb{Q}_p} \tilde{H}^1_{\mathrm{f}}(\mathbb{Q}, V_p(E)) \to I/I^2,$$

where I stands for the augmentation ideal of the cyclotomic Iwasawa algebra, and which may be thought as the ring of functions vanishing at the point $(x,s) = (x_0,1)$, that with a slight abuse of notation we denote by $(2,1)$. This is the so-called *height-weight* pairing, which decomposes as

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_p^{\mathrm{cyc}} \cdot \{s-1\} + \langle \cdot, \cdot \rangle_p^{\mathrm{wt}} \cdot \{k-2\},$$

where $\langle \cdot, \cdot \rangle_p^{\mathrm{cyc}}$ and $\langle \cdot, \cdot \rangle_p^{\mathrm{wt}}$ are canonical $\mathbb{Q}_p$-valued pairings on the extended Selmer group. Finally, the Schneider height is defined by

$$\langle \cdot, \cdot \rangle_p^{\mathrm{Sch}} = \langle x, y \rangle_p^{\mathrm{cyc}} - \frac{\log_p(\mathrm{res}_p(x)) \cdot \log_p(\mathrm{res}_p(y))}{\log_p(q_E)},$$

where $\mathrm{res}_p(x)$ is the localization-at-p map. The following result provides expressions for the second derivative of $L_p(\mathbf{f})$ along different directions of the weight space.

Proposition 7.2.3. *The following formulas hold, where P is a generator of the Mordell–Weil group $E(\mathbb{Q})$.*

(a)

$$\left.\frac{d^2 L_p(\mathbf{f})(k,k/2)}{dk^2}\right|_{k=2} = \log_E(P)^2 \pmod{L^\times};$$

(b)

$$\left.\frac{d^2 L_p(\mathbf{f})(k,1)}{dk^2}\right|_{k=2} = \mathcal{L}(E) \cdot \langle P, P \rangle^{\mathrm{cyc}} \pmod{L^\times};$$

(c)

$$\left.\frac{d^2 L_p(\mathbf{f})(x_0,s)}{ds^2}\right|_{s=1} = \mathcal{L}(E) \cdot \langle P, P \rangle^{\mathrm{Sch}} \pmod{L^\times}.$$

Proof. The first part follows from the main result of [BD07], and the other two are [Ven16, Theorems D and E]. We refer the reader to loc. cit. for a definition of the corresponding pairings. □

IV. Results beyond modular forms of weight 2. The main result of Bertolini and Darmon [BD07] was generalized by Seveso [Se14] to modular forms of even weight. Let us recall here his main result for the sake of completeness and to illustrate that most of our results generalize to the situation of weights $(k, 1, 1)$, by replacing the points over the elliptic curve by the corresponding Heegner cycles. Let $f_k \in S_k(N)$, where $N = pN^+N^-$ and N^- is the squarefree product of an odd number of prime factors. The modular form corresponds, via the Jacquet-Langlands correspondence, to a modular form on a certain Shimura curve $X = X_{N^+,pN^-}$ uniformized by the p-adic upper half-plane. In this framework, Iovita and Spiess [IS03] constructed a Chow motive $\mathcal{M}_{k-2}$ attached to modular forms on X. Let $m = k/2 - 1$.

We fix $K/\mathbb{Q}$ a quadratic imaginary field extension, of discriminant D_K prime to pN, such that N^+ is a product of primes that are split in K, while pN^- is a product of primes that are inert in K; we further fix an order of K of conductor c prime to ND_K. Hence, one may consider a higher weight analogue of Heegner points, the Heegner cycles $y_\psi^{(n)} \in \mathrm{CH}^{m+1}(\mathcal{M}_n)$ attached to a character ψ. The p-adic étale Abel-Jacobi map takes the form

$$\mathrm{AJ}_p : \mathrm{CH}^{m+1}(\mathcal{M}_n) \to M_k^\vee.$$

For this result, the Mazur–Kitagawa p-adic L-function is replaced by the p-adic L-function attached to the quadratic field and the character ψ, that we denote by $\mathcal{L}(\mathbf{f}/K, \psi)(k, s)$ following the notations of [Se14].

Proposition 7.2.4 (Seveso)**.** *The first derivative of $\mathcal{L}(f/K, \psi)(k, s)$ in the weight direction is given by*

$$2\frac{d}{dx}\Big(\mathcal{L}(\mathbf{f}/K, \psi)(x, x/2)\Big)\Big|_{x=k} = \mathrm{AJ}_p(y_\psi^{(n)})(f) + (-1)^m \mathrm{AJ}_p(y_{\bar{\psi}}^{(n)})(f).$$

This suggests that some of our results can be transposed to a higher weight situation, replacing the points over the elliptic curve by the corresponding Heegner cycles. More precisely, the results relying on the work of Darmon, Lauder and Rotger on the Elliptic Stark Conjecture [DLR15a] can be adapted following the generalizations of Gatti and Guitart to higher weights [GG20]. Similarly, the construction of *derived* cohomology classes, anticipated in the introduction and developed in Section 7.3, can be also carried out for general weights $(k, 1, 1)$.

7.3 Derived diagonal cycles and an explicit reciprocity law

Diagonal cycles and an explicit reciprocity law

Darmon and Rotger constructed in [DR20b] an element

$$\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{fgh}})$$

arising from the interpolation of diagonal cycles along the balanced region. An alternative construction has been given by Bertolini, Seveso and Venerucci [BSV20a, Section 3]. This class is symmetric in all three variables. Let

$$\mathrm{res}_p : H^1(\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{fgh}}) \to H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{fgh}})$$

denote the restriction map to the local cohomology at p, and set

$$\kappa_p(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \mathrm{res}_p(\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})) \in H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{fgh}}).$$

One of the main results of both [BSV20a] and [DR20b] is the proof of an explicit reciprocity law. As showed in loc. cit., the Galois representation $\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ is endowed with a four-step filtration

$$0 \subset \mathbb{V}^{++}_{\mathbf{fgh}} \subset \mathbb{V}^{+}_{\mathbf{fgh}} \subset \mathbb{V}^{-}_{\mathbf{fgh}} \subset \mathbb{V}^{\dagger}_{\mathbf{fgh}} \tag{7.11}$$

by $G_{\mathbb{Q}_p}$-stable $\Lambda_{\mathbf{fgh}}$-submodules of ranks 0, 1, 4, 7 and 8, respectively. Moreover,

$$\mathbb{V}^{+}_{\mathbf{fgh}}/\mathbb{V}^{++}_{\mathbf{fgh}} = \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}} \oplus \mathbb{V}^{\mathbf{hf}}_{\mathbf{g}} \oplus \mathbb{V}^{\mathbf{fg}}_{\mathbf{h}}.$$

We discuss now the definition of $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}$. Let $\Theta^{\mathbf{gh}}_{\mathbf{f}}$ be the $\Lambda_{\mathbf{fgh}}$-adic cyclotomic character whose specialization at a point of weight (k,ℓ,m) is $\varepsilon^{t}_{\mathrm{cyc}}$, with $t := (-k+\ell+m)/2$, and let $\psi^{\mathbf{gh}}_{\mathbf{f}}$ be the unramified character of $G_{\mathbb{Q}_p}$ sending Fr_p to $\chi_f^{-1}(p)\mathbf{a}_p(\mathbf{f})\mathbf{a}_p(\mathbf{g})^{-1}\mathbf{a}_p(\mathbf{h})^{-1}$. Define $\mathbb{U}$ as the unramified $\Lambda_{\mathbf{fgh}}$-adic representation of $G_{\mathbb{Q}_p}$ given by several copies of the character $\psi^{\mathbf{gh}}_{\mathbf{f}}$, and let

$$\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}} = \mathbb{U}(\Theta^{\mathbf{gh}}_{\mathbf{f}}).$$

We finally introduce the Λ-adic Dieudonné module

$$\mathbb{D}(\mathbb{U}) := (\mathbb{U}\hat{\otimes}\mathbb{Z}_p^{\mathrm{nr}})^{G_{\mathbb{Q}_p}}.$$

Then, one may construct a Perrin-Riou regulator map whose source is $H^1(\mathbb{Q}_p, \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}) \to \Lambda_{\mathbf{fgh}}$ and which interpolates either the Bloch–Kato logarithm or the dual exponential map, according to the value of a certain Hodge–Tate weight. In order to state their main properties, we need to introduce more terminology. Let $c = \frac{k+\ell+m-2}{2}$, and with the previous notations, define

$$\mathcal{E}^{\mathrm{PR}}(x,y,z) = \frac{1-p^{-c}\beta_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{1-p^{-c}\alpha_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z}}.$$

The following result is discussed e.g. in [DR20b, Proposition 5.6].

Proposition 7.3.1. *There is a homomorphism (usually named Perrin-Riou regulator)*

$$\mathcal{L}_{\mathbf{f},\mathbf{gh}} : H^1(\mathbb{Q}_p, \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}) \to \mathbb{D}(\mathbb{U})$$

such that for all $\kappa_p \in H^1(\mathbb{Q}_p, \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}})$ the image $\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\kappa_p)$ satisfies the following interpolation properties:

1. *For all points $(x,y,z) \notin \mathcal{W}^{f}_{\mathbf{fgh}}$,*

$$\nu_{x,y,z}(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\kappa_p)) = \frac{(-1)^t}{t!}\mathcal{E}^{\mathrm{PR}}(x,y,z)\cdot\langle \log_{\mathrm{BK}}(\nu_{x,y,z}(\kappa_p)), \eta_f\otimes\omega_g\otimes\omega_h\rangle,$$

2. *For all points $(x,y,z) \in \mathcal{W}^{f}_{\mathbf{fgh}}$,*

$$\nu_{x,y,z}(\mathcal{L}_{\mathbf{f},\mathbf{gh}}(\kappa_p)) = (-1)^t\cdot(1-t)!\cdot\mathcal{E}^{\mathrm{PR}}(x,y,z)\cdot\langle \exp^{*}_{\mathrm{BK}}(\nu_{x,y,z}(\kappa_p)), \eta_f\otimes\omega_g\otimes\omega_h\rangle.$$

Following [DR20b], one can define

$$\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})^f \in H^1(\mathbb{Q}_p, \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}) \tag{7.12}$$

as the projection of the local class $\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})$ to $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}$. Let $\eta_{\mathbf{f}^*}$, $\omega_{\mathbf{g}^*}$ and $\omega_{\mathbf{h}^*}$ be the canonical differentials attached to Hida families as introduced for instance in [KLZ17, Section 10]. The following result has been independently established in [BSV20a, Thoerem A] and [DR20b, Theorem 10].

Proposition 7.3.2. *For any triplet of Λ-adic test vectors $(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}})$, the following equality holds in the ring of fractions of $\Lambda_{\mathbf{fgh}}$:*

$$\langle \mathcal{L}_{\mathbf{f},\mathbf{gh}}(\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})^f), \eta_{\tilde{\mathbf{f}}^*} \otimes \omega_{\tilde{\mathbf{g}}^*} \otimes \omega_{\tilde{\mathbf{h}}^*} \rangle = \mathcal{L}_p^f(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}}).$$

Remark 7.3.3. There exist analogue reciprocity laws for $\mathscr{L}_p{}^g$ and $\mathscr{L}_p{}^h$.

We can also formulate an explicit reciprocity law for the improved p-adic L-function. Since along the region $k = \ell + m$ the Perrin-Riou map interpolates the dual exponential, we have that

$$\frac{1}{1 - p^{-k+1}\alpha_{\mathbf{f}_x}\beta_{\mathbf{g}_y}\beta_{\mathbf{h}_z}} \cdot \langle \exp^*_{\mathrm{BK}}(\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})^f(x,y,z)), \eta_{\tilde{\mathbf{f}}^*_x} \otimes \omega_{\tilde{\mathbf{g}}^*_y} \otimes \omega_{\tilde{\mathbf{h}}^*_z} \rangle = \mathcal{L}_p^f(\tilde{\mathbf{f}}_x, \tilde{\mathbf{g}}_y, \tilde{\mathbf{h}}_z)^*, \tag{7.13}$$

and in particular the dual exponential map vanishes at (x_0, y_0, y_0) (i.e. the class is crystalline) if and only if the improved p-adic L-function is zero at that point.

Remark 7.3.4. In [GGMR20], the authors study the cohomology classes in a generic rank zero situation, where they are non-crystalline. This yields a formula for the special value $\mathscr{L}_p{}^g$ in terms of $\mathscr{L}_p{}^f$ in absence of exceptional zeros. Again, the key point is that each component of the cohomology class encodes information about a different p-adic L-function.

Vanishing of cohomology classes

In [BSV20a, Section 8.2], the authors deal with a situation where the numerator of the Perrin-Riou map $\mathcal{L}_{\mathbf{f},\mathbf{gh}}$ vanishes, defining an improved map whose derivatives may be explicitly computed. We come back to this question later on. Let us analyze, firstly, the vanishing of the denominator of the Perrin-Riou map, but in the case of the Perrin-Riou map $\mathcal{L}_{\mathbf{g},\mathbf{hf}}$, that is:

$$1 - p^{-c}\beta_{\mathbf{f}_x}\alpha_{\mathbf{g}_y}\beta_{\mathbf{h}_z} = 0. \tag{7.14}$$

Since we have placed ourselves in the ordinary setting, a necessary condition for this to happen is $k + m = \ell + 2$, which moreover suffices to guarantee the analyticity of the Euler factor in the denominator.

Hence, when f is of weight 2 with split multiplicative reduction at p, and g and h are self-dual of the same weight ($h = g \otimes \chi_g^{-1}$), the denominator of the Perrin-Riou map vanishes. This means that we expect

$$\log_{\mathrm{BK}}(\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})^g(x,y,z)) = 0.$$

However, the self-duality condition is not necessary for this vanishing, and following the conventions of the introduction in the case of weights $(2,1,1)$ (again, with E of split multiplicative reduction), it suffices to impose that $\alpha_g\beta_h = 1$. This encompasses for example the case of theta series of quadratic fields where the prime p is inert. Nevertheless, there are certain phenomena which are exclusive from the self-dual case: indeed, the fact that the Hida families interpolating both g and h keep the self-duality condition gives us a vanishing along the whole line $(2,\ell,\ell)$. We treat both the self-dual and the non self-dual case, emphasizing the main differences between them.

We begin by showing that when $\alpha_g\beta_h = 1$ and g and h are self-dual, the local class $\kappa_p(f, g_\alpha, h_\alpha)$ vanishes, using the techniques discussed in Chapter 3. Although this is not strictly necessary since we will later see that the *whole* global class is zero, we believe that it may be instructive for the reader to compare the formalism of Chapter 3, which relies on the basic properties of the Perrin-Riou maps, with the more conceptual proof of [BSV20a, Section 8], based on the geometric construction of an *improved* cohomology class.

Proposition 7.3.5. *With the running assumptions, the specialization of the Λ-adic cohomology class $\kappa_p(\mathbf{f},\mathbf{g},\mathbf{g}^*)$ at (x_0,y_0,y_0) vanishes, that is, $\kappa_p(f, g_\alpha, g^*_{1/\beta}) = 0$.*

Proof. We will follow the same strategy used in Theorem 3.3.5. First of all we show, invoking [BSV20a, Theorem 7.1], that any specialization of the three-variable Λ-adic class at a point of weights $(2, \ell, \ell)$, with $\ell \geq 2$, is zero. In order to achieve this, we just use the comparison provided by the aforementioned result with the twisted class $\kappa^\dagger$, twisting now in the g-variable, that is, applying the operator $\mathrm{Id} \otimes w_p' \otimes \mathrm{Id}$ according to the definitions given at the beginning of Section 7.2 of loc. cit., where w_p stands for the Atkin–Lehner involution. As we later discuss, this class may be understood as an *improved* cohomology class, since it agrees with the former up to multiplication by the Euler factor

$$1 - \frac{\bar{\chi}(p)\alpha_{\mathbf{g}_y}}{\alpha_{\mathbf{f}_x}\alpha_{\mathbf{h}_z}} p^{\frac{k-\ell+m-2}{2}}.$$

This factor is zero over the line $(2, \ell, \ell)$ when we take Hida families such that $\mathbf{h} = \mathbf{g}^*$, since $\bar{\chi}(p)\alpha_{\mathbf{g}_y} = \alpha_{\mathbf{h}_y}$. Observe that we are implicitly using Lemma 8.4 of loc. cit., which assert that the class $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})$ is symmetric in all three variables.

The second part of the proof consists on applying a limit argument componentwise, via the corresponding Perrin-Riou maps, to conclude that the limit when ℓ goes to one is also zero. For this last step, we look at the four different components of the local class $\kappa_p(f, g_\alpha, g^*_{1/\beta})$ corresponding to the *balanced* subspace $\mathbb{V}^+_{fgg^*}$. This suffices according to the results established in [BSV20a, Corollary 7.2] and following the notations of Section 6.2 in loc. cit., which asserts that the three-variable cohomology class lies in the balanced subspace. The components of the balanced subspace are denoted by $V_f^{gg^*}$, $V_g^{g^*f}$, $V_{g^*}^{fg}$ and $V_{fgg^*}^{++}$, where $V_f^{gg^*}$ stands for the specialization of $\mathbb{V}_{\mathbf{f}}^{\mathbf{gg}^*}$ and similarly for the other factors (recall the filtration of (7.11)).

- We first prove that the component associated to the rank one subspace $V_f^{gg^*}$ is zero. Observe that along the line $(2, \ell, \ell)$, the specialization of the module $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{f}}^{\mathbf{gg}^*})$ agrees with $H^1(\mathbb{Q}_p, \mathbb{Z}_p(\psi_{g_y}^{-2})(\ell - 1))$, where y is a point of weight ℓ. Then, the Perrin-Riou map is an application
$$H^1(\mathbb{Q}_p, \Lambda_{\mathbf{g}}(\psi_{\mathbf{g}}^{-2}) \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}})) \to \mathbb{D}(\Lambda_{\mathbf{g}}(\psi_{\mathbf{g}}^{-2})) \hat{\otimes} \Lambda. \tag{7.15}$$
Since $\psi_{\mathbf{g}}^{-2} \neq 1$, we have $H^0(\mathbb{Q}_p, \Lambda_{\mathbf{g}}(\psi_{\mathbf{g}}^{-2})) = 0$ and it follows from [KLZ17, Theorem 8.2.3] that the above map is an isomorphism. Moreover, using the same argument of the proof of the last step of Theorem 3.3.5, we conclude that the Λ-module of (7.15) is non-canonically isomorphic to $\Lambda_{\mathbf{g}}$. Therefore, and since infinitely many specializations vanish according to the previously quoted result of [BSV20a], the corresponding H^1 is zero.
- The components associated to $V_g^{g^*f}$ and $V_{g^*}^{fg}$ are zero; this is because
$$H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^{\mathbf{g}^*\mathbf{f}}|_{(2,\ell,\ell)}) \simeq H^1(\mathbb{Q}_p, \Lambda_{\mathbf{g}}(1)) \simeq \Lambda_{\mathbf{g}} \oplus \Lambda_{\mathbf{g}},$$
and although the Perrin-Riou map only kills one of the above two components, the restriction of the class is zero since again infinitely many specializations are zero.
- For the remaining component, the one corresponding to $\mathbb{V}_{fgg^*}^{++}$, the same argument used in the first step works once we have established that the other projections vanish.

□

Consider now the surface

$$\mathcal{S} = \mathcal{S}_{k,k+m-2,m} := \{(x, y, z) \in \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}} : \mathrm{w}(x) + \mathrm{w}(z) = \mathrm{w}(y) + 2\},$$

and also the line

$$\mathcal{C} := \{(x, y, z) \in \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}} : \mathrm{w}(x) = 2, \quad \mathrm{w}(y) = \mathrm{w}(z)\}.$$

Observe that the surface $\mathcal{S}$ is just a finite cover of the plane in $\mathcal{W}^3$ arising as the Zariski closure of weights $(k, k+m-2, m)$.

Using the results of [BSV20a, Section 8.2], we may upgrade Proposition 7.3.5 to the vanishing of the global class $\kappa(f, g_\alpha, h_\alpha)$ when $\alpha_g \beta_h = 1$ (and hence we can work beyond the setting of the adjoint, covering for example the case of theta series of quadratic fields where the prime p remains inert).

In particular, we have the following result.

Proposition 7.3.6. *The global class $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})$ vanishes along the line $\mathcal{C}$ in the self dual case. Moreover, the class $\kappa(f, g_\alpha, h_{1/\beta})$ is zero when $\alpha_g \beta_h = 1$.*

Proof. Following again [BSV20a, Section 8.2], there is an *improved* class $\kappa_g^*(\mathbf{f}, \mathbf{g}, \mathbf{h})$ along the surface $\mathcal{S}$ satisfying

$$\kappa(\mathbf{f}, \mathbf{g}, \mathbf{h})_{|\mathcal{S}} = \Big(1 - \frac{\bar{\chi}(p)\alpha_{\mathbf{g}_y}}{\alpha_{\mathbf{f}_x}\alpha_{\mathbf{h}_z}}\Big)\kappa_g^*(\mathbf{f}, \mathbf{g}, \mathbf{h}). \tag{7.16}$$

Hence, the vanishing of $\kappa(f, g_\alpha, h_{1/\beta})$ follows from the vanishing of the corresponding Euler factor. $\square$

Derived classes and reciprocity laws

Following the analogy with Chapter 3, let us focus firstly on the self-dual case to discuss the notion of *derived* classes. We shrink the weight space $\mathcal{W}$ to a rigid-analytic open disk $\mathcal{U} \subset \mathcal{W}$ centered at 2 at which the finite cover $w : \mathcal{W}_{\mathbf{f}} \to \mathcal{W}$ restricts to an isomorphism $w : \mathcal{U}_{\mathbf{f}} \xrightarrow{\sim} \mathcal{U}$ with $x_0 \in \mathcal{U}_{\mathbf{f}}$. Let $\Lambda_{\mathcal{U}_{\mathbf{f}}} = \mathcal{O}(\mathcal{U}_{\mathbf{f}})$ denote the Iwasawa algebra of analytic functions on $\mathcal{U}_{\mathbf{f}}$ whose supremum norm is bounded by 1. Shrink likewise $\mathcal{C}$ and $\mathcal{S}$ so that projection to the weight space restricts to an isomorphism with $\mathcal{U}$ and $\mathcal{U} \times \mathcal{U}$ respectively. Having done that, their associated Iwasawa algebras are respectively $\mathcal{O}(\mathcal{C}) = \Lambda_{\mathcal{U}_{\mathbf{f}}} \simeq \mathbb{Z}_p[[X]]$ and $\mathcal{O}(\mathcal{S}) = \Lambda_{\mathcal{U}_{\mathbf{f}}} \hat{\otimes} \Lambda_{\mathcal{U}_{\mathbf{h}}} \simeq \mathbb{Z}_p[[X, Z]]$. The isomorphism $\Lambda_{\mathcal{U}_{\mathbf{f}}} \simeq \mathbb{Z}_p[[X]]$ is not canonical and depends on the choice of an element $\gamma \in \Lambda_{\mathcal{U}_{\mathbf{f}}}^\times$ which is sent to $1 + X$.

Then, consider the short exact sequence of $\mathbb{Z}_p$-modules

$$0 \to \mathbb{Z}_p[[X, Z]] \xrightarrow{\cdot X} Z_p[[X, Z]] \to \mathbb{Z}_p[[Z]] \to 0.$$

Under the usual isomorphisms, $\Lambda_{\mathbf{f}}$ may be identified with $\mathbb{Z}_p[[X]]$ after fixing a topological generator γ of $\Lambda_{\mathcal{U}_{\mathbf{f}}}^\times$ and sending $[\gamma]$ to $1+X$. Then, $\Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{h}}$ becomes isomorphic to $\mathbb{Z}_p[[X, Z]]$ and the previous exact sequence may be recast as

$$0 \to \mathcal{O}_{\mathcal{S}} \xrightarrow{\delta} \mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{C}} \to 0 \tag{7.17}$$

with $\delta = (\gamma - 1) \otimes 1$ in $\mathcal{O}_{\mathcal{S}} \simeq \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{h}}$.

Proposition 7.3.7. *In the self-dual case, there is a unique class $\kappa'_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}\mathbf{g}\mathbf{g}^*|\mathcal{S}})$ such that*

$$\kappa(\mathbf{f}, \mathbf{g}, \mathbf{g}^*)|_{\mathcal{S}} = \delta \cdot \kappa'_\gamma(\mathbf{f}, \mathbf{g}, \mathbf{g}^*).$$

Proof. This follows by considering the long exact sequence in cohomology attached to (7.17):

$$H^0(\mathbb{Q}, \mathbb{V}_{\mathbf{f}\mathbf{g}\mathbf{g}^*|\mathcal{C}}) \to H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}\mathbf{g}\mathbf{g}^*|\mathcal{S}}) \to H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}\mathbf{g}\mathbf{g}^*|\mathcal{S}}) \to H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}\mathbf{g}\mathbf{g}^*|\mathcal{C}}).$$

Since the restriction of $\kappa(\mathbf{f}, \mathbf{g}, \mathbf{g}^*)$ to $H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{f}\mathbf{g}\mathbf{g}^*|\mathcal{C}})$ is zero by Proposition 7.3.6, one may assure the existence of a *derived* class as in the statement, which is moreover unique due to the vanishing of the H^0 for weight reasons (the Hodge–Tate weights corresponding to the balanced part cannot be zero, as shown in [DR20b, Corollary 5.3]). $\square$

Normalizing by $\log_p(\gamma)$, the specializations of this class over the line $(2,\ell,\ell)$ can be proved to be independent of the choice of γ.

In general, if we are no longer in the self-dual case but the condition $\alpha_g\beta_h = 1$ still holds, the notion of derived class makes sense at the point (x_0, y_0, z_0). For that purpose, let $\mathcal{D}$ stand for the codimension two subvariety

$$\mathcal{D} := \{(x,y,z) \in \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}} : \mathrm{w}(x) = \mathrm{w}(y) + 1, \quad z = z_0\}.$$

The following result is the analogue of Proposition 3.3.13 and its proof follows from the same argument of Proposition 7.3.7.

Proposition 7.3.8. *Assume that $\alpha_g\beta_h = 1$. Then, there exists a unique class $\kappa'_\gamma(\mathbf{f},\mathbf{g},\mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgh}|\mathcal{D}})$ such that*

$$\kappa(\mathbf{f},\mathbf{g},\mathbf{h})|_{\mathcal{D}} = \delta \cdot \kappa'_\gamma(\mathbf{f},\mathbf{g},\mathbf{h}).$$

Let $\mathcal{L} = \frac{\alpha'_g}{\alpha_g} - \frac{\alpha'_f}{\alpha_f}$, and consider the normalization of $\kappa'_\gamma(\mathbf{f},\mathbf{g},\mathbf{h})$ by γ, that is,

$$\kappa'(\mathbf{f},\mathbf{g},\mathbf{h}) = \frac{\kappa'_\gamma(\mathbf{f},\mathbf{g},\mathbf{h})}{\log_p(\gamma)}.$$

Theorem 7.3.9. *The logarithm of the derived class satisfies the following*

$$\langle \log_{\mathrm{BK}}(\kappa'_p(\mathbf{f},\mathbf{g},\mathbf{h})^g(x_0,y_0,z_0)), \eta_f \otimes \omega_g \otimes \omega_h \rangle = \mathcal{L} \cdot \mathscr{L}_p{}^{g_\alpha}(\mathbf{f},\mathbf{g},\mathbf{h})(x_0,y_0,z_0) \pmod{L^\times},$$

where as in (7.12) *the superindex g refers to the projection to $\mathbb{V}_{\mathbf{g}}^{\mathbf{hf}}$.*

Proof. Consider the reciprocity law of Proposition 7.3.2, now for $\mathscr{L}_p{}^{g_\alpha}(\mathbf{f},\mathbf{g},\mathbf{h})$, restricted to $\mathcal{D}$, and multiply both sides by the Euler factor in the denominator of the Perrin-Riou map. Then, we have an equality of the form

$$\Big(1 - \frac{\bar\chi(p)\alpha_{\mathbf{g}_y}}{p\alpha_{\mathbf{f}_x}\alpha_{\mathbf{h}_z}}\Big) \cdot \langle \log_{\mathrm{BK}}(\kappa_p(\mathbf{f},\mathbf{g},\mathbf{h})^g), \eta_{\mathbf{f}} \otimes \omega_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle = \Big(1 - \frac{\alpha_{\mathbf{f}_x}\alpha_{\mathbf{h}_z}}{\bar\chi(p)\alpha_{\mathbf{g}_y}}\Big) \cdot \mathscr{L}_p{}^{g_\alpha}(\mathbf{f},\mathbf{g},\mathbf{h}),$$

since along $\mathcal{D}$ the Perrin-Riou interpolates the Bloch–Kato logarithm. At the point (x_0, y_0, z_0) both the cohomology class at the left hand side and the Euler factor at the right are zero. Taking derivatives along the direction $(k+1, k, 1)$, and evaluating then at the point (x_0, y_0, z_0), the result follows. □

An analogue formula holds for any point over the line $(2,\ell,\ell)$ in the self-dual case, but of course the description of the $\mathcal{L}$-invariant is not so explicit and relies on the results of [Se14].

It may be instructive to compare this *derived* cohomology class with the *improved* cohomology class considered by Bertolini, Seveso and Venerucci. We can prove the following.

Proposition 7.3.10. *Consider the map given by the projection*

$$\phi_g^{hf} : H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{fgh}|\mathcal{S}}) \to H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^{\mathbf{hf}}|_{\mathcal{S}}).$$

Then, there is a relation between the improved class $\phi_g^{hf}(\kappa_g^*(\mathbf{f},\mathbf{g},\mathbf{h}))$ and $\phi_g^{hf}(\kappa'(\mathbf{f},\mathbf{g},\mathbf{h}))$, given by

$$\phi_g^{hf}(\kappa'(\mathbf{f},\mathbf{g},\mathbf{h})) = \mathcal{L} \cdot \phi_g^{hf}(\kappa_g^*(\mathbf{f},\mathbf{g},\mathbf{h})) \pmod{L^\times}.$$

Proof. This is proved by applying the map $\langle \log_{\mathrm{BK}}(\cdot), \eta_{\mathbf{f}} \otimes \omega_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle$ to both sides, and then comparing the results. For that purpose, we use that the Euler factors involved in the Perrin-Riou map are analytic along $\mathcal{S}$ and can be cancelled out. That way, we obtain an *improved* reciprocity law

$$\mathscr{L}_p{}^{g_\alpha}(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z) = \langle \log_{\mathrm{BK}}(\phi_g^{hf}(\kappa_g^*(\mathbf{f},\mathbf{g},\mathbf{h})^g(x,y,z))), \eta_{\mathbf{f}_x} \otimes \omega_{\mathbf{g}_y} \otimes \omega_{\mathbf{h}_z} \rangle \pmod{L^\times},$$

which holds for all the points (x,y,z) of $\mathcal{S}$. □

Finally, we point out that we may expect a relation between $\kappa_p'(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z)$ and the Gross–Kudla–Schoen cycle of [DR14], that we denote by $\Delta_{k,\ell,m} \in H^1(\mathbb{Q}, V_{fgh}((4-k-\ell-m)/2)$. In particular, we expect the following result to be true (or at least, a slight variant of it). Here, loc_p stands for the localization at p-map.

Question 7.3.11. Can we establish that, up to multiplication by a non-zero constant in $L^\times$ and for any point (x,y,z) of weights $(2,\ell,\ell)$ with $\ell \geq 2$, we have the equality

$$\kappa_p'(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z) = \mathcal{L} \cdot \mathrm{loc}_p(\Delta_{2,\ell,\ell})?$$

Of course, this would require the proof of an analogue result to [DR14, Theorem 5.1] in a situation where f has split multiplicative reduction.

7.4 Derivatives of triple product p-adic L-functions

In this section, we discuss a variant of the Elliptic Stark Conjecture for the derivative of the triple product p-adic L-function $\mathscr{L}_p{}^f$ in a situation of exceptional zeros. As before, we keep the assumption that f has split multiplicative reduction at p and that an exceptional zero condition occurs.

There are two main instances we want to consider: the rank zero situation and the rank two situation. While the former is quite well understood after the results developed in [BSV20a] and [BSV20b], the latter is more subtle and we will propose a conjectural formula in this scenario. Along this section, by the word *rank*, we refer to the rank of the V_{gh}-isotypic component of $E(H)$. According to our general assumptions on the local signs, the rank is always even. The V_{gh}-component of $E(H)$ is endowed with an inclusion in the Selmer group, that is,

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(E(H), V_{gh}) \simeq (E(H) \otimes V_{gh}^\vee)^{G_{\mathbb{Q}}} \subset H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh}),$$

where $H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh})$ is the group of extensions of $\mathbb{Q}_p$ by V_{fgh} in the category of $\mathbb{Q}_p$-linear representations of $G_{\mathbb{Q}}$ which are crystalline at p.

Recall that for higher ranks the computations performed in [DLR15a] lead us to expect that the special value $\mathscr{L}_p{}^{g_\alpha}$ presented in the introduction is zero, and that the second derivative of $\mathscr{L}_p{}^f$ along the f-direction vanishes, too. The odd rank situation is equally interesting, and we hope to come back to this question in a further work. We keep the notations of the previous section.

Proposition 7.4.1. *The value $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})(x_0,y_0,z_0)$ is zero. Moreover, the jacobian matrix of $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})$ at the point (x_0,y_0,z_0) is given by*

$$(0 \quad \mathcal{L}_{g_\alpha} - \mathcal{L}_f \quad \mathcal{L}_{h_\alpha} - \mathcal{L}_f) \cdot \mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})^*.$$

Proof. This directly follows from [BSV20a, Proposition 8.2]. □

Remark 7.4.2. Observe that, towards the rationality conjectures we are interested in, the value $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})^*$ is an algebraic number, and it is non-zero if and only if the class $\kappa(f,g_\alpha,h_\alpha)$ is non-crystalline.

In particular, the derivative along the direction $(2+k,1,1)$ vanishes and along the direction $(2,1+\ell,1+\ell)$ is given by $\mathcal{L}_{g_\alpha} + \mathcal{L}_{h_\alpha} - 2\mathcal{L}_f$, up to an explicit algebraic number in the number field L.

Suppose from now on that $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})^*$ vanishes at (x_0,y_0,z_0). Therefore, the cohomology class $\kappa(f,g_\alpha,h_\alpha)$ is crystalline, and following [BSV20b, Section 2.1] we may define a *new* Bloch–Kato logarithm, denoted by $\log_{\beta\beta}$ in loc. cit. Roughly speaking, it can be understood as a projection to the rank one subspace V_{fgh}^{++} arising in the filtration (7.11), followed by the Bloch–Kato logarithm and the pairing with $\omega_f \otimes \omega_g \otimes \omega_h$. To be coherent with the other notations we will need later on, write $\log^{++}$ for this map. Alternatively, we may consider the local class $\kappa_p(f, g_\alpha, h_\alpha)$ and take its decomposition according to the action of the Frobenius element, in such a way that $\kappa_{\beta\beta}$ is the part corresponding to the (β_g, β_h) component.

Assume further that $\alpha_g \alpha_h = 1$ (in particular, this also implies that $\beta_g \beta_h = 1$). The following result is the content of [BSV20b, Section 2.1].

Proposition 7.4.3. *Under the given conditions, the value $\mathscr{L}_p{}^f(\mathbf{f}, g_\alpha, h_\alpha)$ vanishes and*

$$\frac{d^2}{dx^2} \mathscr{L}_p{}^f(\mathbf{f}, g_\alpha, h_\alpha)|_{x=x_0} = \frac{1}{2\mathrm{ord}_p(q_E)} \cdot (1 - p^{-1})^{-1} \cdot \log^{++}(\kappa_p(f, g_\alpha, h_\alpha)).$$

Remark 7.4.4. In the adjoint case, when we take $h_{1/\alpha} = g^*_{1/\beta}$ we do have a relation between $\mathcal{L}_g$ and $\mathcal{L}_h$: indeed

$$\frac{(1/\alpha_g)'}{1/\alpha_g} = -\frac{\alpha_g'}{\alpha_g};$$

however when $h_{1/\alpha} = g^*_{1/\alpha}$ both quantities are a priori unrelated.

A conjecture for the second derivative

As we have discussed before, the improved p-adic L-function $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})^*$ interpolates an explicit non-zero multiple of $L(f \otimes g \otimes h, 1)$, and we expect this value to be zero when the rank of the corresponding isotypic component of the Selmer group is two. In those cases, we would like to compare the Kato class with a basis of $(E(H) \otimes V_{gh}^\vee)^{G_\mathbb{Q}}$, that we write as $\{P, Q\}$. We also assume that $H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh})$ has dimension 2.

To fix notations, observe that V_{gh} decomposes as a $G_{\mathbb{Q}_p}$-module as the direct sum of four different lines $V_{gh}^{\alpha\alpha} := V_g^{\alpha_g} \otimes V_h^{\alpha_h}, \ldots, V_{gh}^{\beta\beta}$. After choosing a basis of $V_{gh}^\vee$, we may write this decomposition as

$$V_{gh}^\vee = L \cdot e_{\alpha\alpha}^\vee \oplus L \cdot e_{\alpha\beta}^\vee \oplus L \cdot e_{\beta\alpha}^\vee \oplus L \cdot e_{\beta\beta}^\vee,$$

where

$$\mathrm{Fr}_p(e_{\lambda\mu}^\vee) = a_{\lambda\mu} \cdot e_{\lambda\mu}^\vee \quad \text{for any} \quad \lambda, \mu \in \{\alpha, \beta\}.$$

Here, $a_{\lambda\mu} = \beta_g \beta_h$ if $(\lambda, \mu) = (\alpha, \alpha)$ and similarly in the other three cases.

In particular, restricting the elements $\{P, Q\}$ to a decomposition group at p gives expressions

$$P = P_{\alpha\alpha} \otimes e_{\beta\beta}^\vee + P_{\alpha\beta} \otimes e_{\beta\alpha}^\vee + P_{\beta\alpha} \otimes e_{\alpha\beta}^\vee + P_{\beta\beta} \otimes e_{\alpha\alpha}^\vee,$$

and similarly for Q. As pointed out in the introduction, the arithmetic Frobenius Fr_p acts on $P_{\alpha\alpha}$ with eigenvalue $\beta_g \beta_h$ and analogously for the remaining components.

Conjecture 7.4.5. *The following equality holds:*

$$\frac{d^2}{dx^2} \mathscr{L}_p{}^f(\mathbf{f}, g_\alpha, h_\alpha)|_{x=x_0} = \log_p(P_{\alpha\alpha}) \log_p(Q_{\beta\beta}) - \log_p(Q_{\alpha\alpha}) \log_p(P_{\beta\beta}) \pmod{L^\times}.$$

This conjecture can be seen as a quite natural analogue for the first part of Proposition 7.2.3; that is, we are proposing an expression for the second derivative of the p-adic L-function along the line $(k, k/2)$ since the central critical point corresponding to $(k, 1, 1)$ is precisely $k/2$. It would be interesting to understand the derivatives along different directions; we expect that they would be related with appropriate height pairings. See [CH20] for an approximation to that question when g and h are theta series attached to the same quadratic imaginary field where the prime p splits.

Some reducible cases

We continue by recalling some factorization formulas in special cases where the representation V_{gh} becomes reducible. See [DLR16, Section 2] for a complete discussion of the different cases where this may occur.

A first case occurs when g and h are theta series of the same quadratic field K, but the behavior is ostensibly different depending on whether K is real or imaginary, and on whether p is inert or split in K. While the inert case was worked out in [BSV20b], the split case was not considered in loc. cit. However, it turns out that it is not specially interesting, at least when K is imaginary: the second derivative of $\mathscr{L}_p{}^f$ along the x-direction is 0 for trivial reasons.

Remark 7.4.6. It may be tempting to prove a factorization formula for $\mathscr{L}_p{}^f$ as in [CR19], or even when all three variables (k, ℓ, m) are allowed to move along a Hida family. However, the two-variable Castella's p-adic L-functions considered in loc. cit. would have infinity types

$$\Big(\frac{k+\ell+m}{2}-1, \frac{k+\ell+m}{2}-\ell-m+1\Big), \quad \Big(\frac{k+\ell+m}{2}-m, \frac{k+\ell+m}{2}-\ell\Big).$$

This precludes the possibility of comparing the different p-adic L-values along the region of classical interpolation, since they are disjoint.

Finally, in the case where $h = g^*$, the situation is also quite simple and the right hand of the conjecture is zero. For details on that, see the case by case analysis, completely analogue to our situation, of [DR16]. In particular, there are three possibilities according to the values of the ranks of $E(H)$ and $E(H) \otimes \mathrm{ad}^0(V_g)$, whose sum is two.

Proposition 7.4.7. *Conjecture 7.4.5 holds whenever (a) g is the theta series of an imaginary quadratic field where p splits; (b) g is the theta series of a quadratic field where p is inert, $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$, and either ψ_1 or ψ_2 is a genus character.*

Proof. Consider first the case of imaginary quadratic fields, where we can prove that both the left and the right hand side are zero for trivial reasons. For that purpose, recall the notations introduced in the discussion before Proposition 7.4.3. In order to see that the second derivative vanishes, it is enough to conclude that the component $\kappa_{\beta\beta} = 0$, and this follows after adapting the results of [DR16, Section 4.3] to the multiplicative situation, where one may invoke the discussion of [CR19]. In particular, if we assume without loss of generality that $P_{\beta\alpha} \neq 0$, then $P_{\alpha\alpha} = P_{\beta\beta} = 0$, and similarly $Q_{\alpha\beta} = Q_{\beta\alpha} = 0$. See [GGMR20, Section 4] for a completely analogue treatment of an analogue situation.

The case of theta series for quadratic fields where the prime is inert follows from the main results of Bertolini–Seveso–Venerucci [BSV20b, Section 3], taking into account the identifications among the different eigenspaces for the Frobenius action of e.g. [DLR15a, Section 3.3] and [DR16, Sections 4.3, 4.4]. □

The conjecture in other settings

We would like to make some comments regarding the case $\alpha_g\beta_h = 1$. Observe that the previous Euler factor that gave rise to the improved p-adic L-function does not vanish, but the factors

$$1 - \frac{\chi_f(p)\alpha_{\mathbf{g}_y}\beta_{\mathbf{h}_z}}{\alpha_{\mathbf{f}_x}} p^{\frac{k-\ell-m}{2}} \quad \text{and} \quad 1 - \frac{\chi_f(p)\beta_{\mathbf{g}_y}\alpha_{\mathbf{h}_z}}{\alpha_{\mathbf{f}_x}} p^{\frac{k-\ell-m}{2}}$$

do. The first one is analytic along the region $\mathcal{S}_{k+m=\ell+2}$, while the second is analytic along the region $\mathcal{S}_{k+\ell=m+2}$. In this case we cannot assure the existence of an improved p-adic L-function, but at least we can guarantee that $\mathscr{L}_p{}^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$ vanishes.

We get indeed a very similar result.

Proposition 7.4.8. *The value* $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})(x_0,y_0,z_0) = 0$. *Moreover, the jacobian matrix of* $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})$ *at* (x_0,y_0,z_0) *is given by an L-multiple of*

$$(0 \quad \mathcal{L}_{g_\alpha} - \mathcal{L}_f \quad \mathcal{L}_{h_\alpha} - \mathcal{L}_f).$$

Observe that Conjecture 7.4.5 still makes sense in this framework. And again, we can also take the derivative along the line $(2+k, 1+k, 1)$ and we would expect to relate it with an explicit multiple of an appropriate height pairing $\langle P, P\rangle$.

7.5 Applications to the Elliptic Stark Conjecture

Interplay between both settings and a conjecture of Darmon–Rotger

Let H denote the smallest number field cut out by the representation V_{gh}, with coefficients in a finite extension $L/\mathbb{Q}$. By enlarging it if necessary, assume throughout that L contains both the Fourier coefficients of g and h, and the roots of their p-th Hecke polynomials. Fix a prime ideal $\wp$ of H lying above p, thus determining an embedding $H \subset H_p \subset \bar{\mathbb{Q}}_p$ of H into its completion H_p at $\wp$, and an arithmetic Frobenius $\mathrm{Fr}_p \in \mathrm{Gal}\,(H_p/\mathbb{Q}_p)$. Due to our regularity assumptions, V_g and V_h decompose as

$$V_g = V_g^\alpha \oplus V_g^\beta, \quad V_h = V_h^\alpha \oplus V_h^\beta,$$

where Fr_p acts on V_g^α with eigenvalue α_g, and similarly for the remaining summands.

Fix eigenbases $\{e_g^\alpha, e_g^\beta\}$ and $\{e_h^\alpha, e_h^\beta\}$ of V_g and V_h, respectively, which are compatible with the choice of the basis for V_{gh} of the previous section, i.e.,

$$e_{\alpha\alpha} = e_g^\alpha \otimes e_h^\alpha, \quad e_{\alpha\beta} = e_g^\alpha \otimes e_h^\beta, \quad e_{\beta\alpha} = e_g^\beta \otimes e_h^\alpha, \quad e_{\beta\beta} = e_g^\beta \otimes e_h^\beta$$

(recall that in previous sections we were using the dual basis). Let $\eta_{g_\alpha} \in (H_p \otimes V_g^\beta)^{G_{\mathbb{Q}_p}}$ and $\omega_{h_\alpha} \in (H_p \otimes V_g^\alpha)^{G_{\mathbb{Q}_p}}$ denote the canonical periods arising as the weight one specializations of the Λ-adic periods $\eta_{\mathbf{g}}$ and $\omega_{\mathbf{h}}$ introduced in [KLZ17, Section 10.1]. Then, we can define p-adic periods $\Xi_{g_\alpha} \in H_p^{\mathrm{Fr}_p=\beta_g^{-1}}$ and $\Omega_{h_\alpha} \in H_p^{\mathrm{Fr}_p=\alpha_h^{-1}}$ by setting

$$\Xi_{g_\alpha} \otimes e_g^\beta = \eta_{g_\alpha}, \quad \Omega_{h_\alpha} \otimes e_h^\alpha = \omega_{h_\alpha},$$

and

$$\mathcal{L}_{g_\alpha} := \frac{\Omega_{g_\alpha}}{\Xi_{g_\alpha}} \in (H_p)^{\mathrm{Fr}_p = \frac{\beta_g}{\alpha_g}}. \tag{7.18}$$

At the same time, recall that u_{g_α} is the Stark unit attached to the adjoint representation of g_α, which arises as a normalization term in the conjectures of [DLR15a] and [DLR16] involving a *second-order* regulator. Then, it was conjectured by Darmon and Rotger [DR16] that

$$\mathcal{L}_{g_\alpha} = \log_p(u_{g_\alpha}) \pmod{L^\times}. \tag{7.19}$$

This relation gives a relatively easy interpretation of the apparently mysterious unit u_{g_α}. This suggests that more natural descriptions of this object should be available, involving only the arithmetic of the modular form g. However, this conjecture seems to be hard to prove, even in cases where the Elliptic Stark Conjecture is known (theta series of quadratic imaginary fields where the prime p splits). The main difficulty is the lack of an explicit description of the periods Ω_{g_α} and Ξ_{g_α}: in weights greater than one, these periods can be understood as certain algebraic numbers and be explicitly described, but in weight one this description is no longer available and Ω_{g_α} and Ξ_{g_α} are p-adic transcendental numbers.

The main point of this section is that the knowledge of different conjectures involving these periods can be enough to determine the value of the ratio $\mathcal{L}_{g_\alpha}$. Indeed, the generalized cohomology classes described in Section 7.3 can be decomposed as the sum of different components, each one encoding information about different p-adic L-functions. When combining these results, we may relate the different periods which are involved.

As a first application of this technique, let us prove a result of this kind using the theory of Beilinson–Flach elements. This corresponds to the limit case where the modular form f is Eisenstein and the arithmetic governing the triple product are ostensibly different. For the following discussion, the notations are the same of Chapter 4. Let $U_{gg^*} = \mathcal{O}_H^\times \otimes L$ and $U_{gg^*}[1/p] = \mathcal{O}_H[1/p]^\times \otimes L$, and assume that the hypothesis (H1)-(H3) of the introduction of Chapter 4 hold. Fix a basis $\{u, v\}$ of the two dimensional space $(U_{gg^*}[1/p]/p^{\mathbb{Z}} \otimes \mathrm{ad}^0(V_g))^{G_{\mathbb{Q}}}$ such that $u \in (\mathcal{O}_H^\times \otimes \mathrm{ad}^0(V_g))^{G_{\mathbb{Q}}}$. As in the case of elliptic curves, these unit groups are endowed with a Frobenius action, since the restriction to a decomposition group allows us to decompose $\mathrm{ad}^0(V_g) = L \oplus L^{\alpha\otimes\bar\beta} \oplus L^{\beta\otimes\bar\alpha}$ and we may take the projection of u and v to each of those components. Let

$$\begin{aligned} R_{g_\alpha} &= \log_p(u_1)\log_p(v_{\alpha\otimes\bar\beta}) - \log_p(v_1)\log_p(u_{\alpha\otimes\bar\beta}), \\ R_{g_\beta} &= \log_p(u_1)\log_p(v_{\beta\otimes\bar\alpha}) - \log_p(v_1)\log_p(u_{\beta\otimes\bar\alpha}) \end{aligned}$$

be the regulators which appear in the formulation of the main conjecture of [DLR16] and [RR19].

Proposition 7.5.1. *Assume that R_{g_α} and R_{g_β} are both non-zero. Then,*

$$\frac{\mathcal{L}_{g_\alpha}}{\mathcal{L}_{g_\beta}} = \frac{\log_p(u_{g_\alpha})}{\log_p(u_{g_\beta})} \pmod{L^\times}.$$

Proof. Recall the maps $\log^{+-}$ and $\log^{-+}$ introduced in Section 4.3 as the composition of the corresponding projection maps from V_{gh}, the Bloch–Kato logarithm, and the pairing with the canonical differentials. Apply then Proposition 4.4.3 twice, first with the map $\log^{-+}$ (and hence taking the $\beta\otimes\bar\alpha$ component of both u and v), and then with the map $\log^{+-}$ (taking the $\alpha\otimes\bar\beta$ component of both u and v). Then, comparing both expressions we have that

$$\Xi_{g_\alpha}\cdot\Omega_{g^*_{1/\alpha}}\cdot\log_p(u_{g_\alpha}) = \Omega_{g_\alpha}\cdot\Xi_{g^*_{1/\alpha}}\cdot\log_p(u_{g_\beta}) \pmod{L^\times}.$$

We now proceed as in Section 3.5 (see the discussion after display (3.56)), observing that

$$\Omega_{g^*_{1/\alpha}} = \Xi_{g_\beta}^{-1}, \quad \Xi_{g^*_{1/\alpha}} = \Omega_{g_\beta}^{-1} \pmod{L^\times},$$

and we are done. □

We would like to go a step beyond and aim for stronger results, so in a certain way we would like to keep the period attached to h fixed and vary just the one attached to g, which would yield the desired equality.

We do this by analyzing first the prototypical case of the Elliptic Stark Conjecture, where the Selmer group is two-dimensional and we may fix a basis $\{P, Q\}$ of the L-vector space

$$(E(H)\otimes V_{gh}^\vee)^{G_{\mathbb{Q}}}.$$

For the following Proposition we assume the hypothesis discussed in the introduction of [DLR15a]. Recall the decomposition

$$P = P_{\alpha\alpha}\otimes e_{\beta\beta}^\vee + P_{\alpha\beta}\otimes e_{\beta\alpha}^\vee + P_{\beta\alpha}\otimes e_{\alpha\beta}^\vee + P_{\beta\beta}\otimes e_{\alpha\alpha}^\vee,$$

and similarly for Q.

Define the regulators

$$\mathrm{Reg}_{g_\alpha}(V_{gh}) = \log_p(P_{\alpha\alpha})\log_p(Q_{\alpha\beta}) - \log_p(Q_{\alpha\alpha})\log_p(P_{\alpha\beta})$$

and
$$\mathrm{Reg}_f(V_{gh}) = \log_p(P_{\alpha\alpha})\log_p(Q_{\beta\beta}) - \log_p(Q_{\alpha\alpha})\log_p(P_{\beta\beta}).$$

To shorten notations, write
$$\log^{-+}(\kappa) = \langle \log_{\mathrm{BK}}(\kappa_p^g), \omega_f \otimes \eta_g \otimes \omega_h \rangle,$$
and whenever κ is crystalline, write $\log^{++}$ for the Bloch–Kato logarithm of [BSV20b, Section 2.1], as recalled before the proof of Proposition 7.4.3.

Proposition 7.5.2. *Assume that* $\mathrm{Reg}_{g_\alpha}(V_{gh})$, $\mathrm{Reg}_f(V_{gh}) \neq 0$. *Suppose that* two *of the following* three *equalities are true modulo* $L^\times$. *Then, the third automatically holds.*

(a)
$$\mathscr{L}_p{}^{g_\alpha}(f, g_\alpha, h_\alpha) = \frac{\log_p(P_{\alpha\alpha})\log_p(Q_{\alpha\beta}) - \log_p(Q_{\alpha\alpha})\log_p(P_{\alpha\beta})}{\log_p(u_{g_\alpha})}.$$

(b)
$$\frac{\partial^2 \mathscr{L}_p{}^f(\mathbf{f}_x, g_\alpha, h_\alpha)}{\partial x^2}\Big|_{x=x_0} = \log_p(P_{\alpha\alpha})\log_p(Q_{\beta\beta}) - \log_p(Q_{\alpha\alpha})\log_p(P_{\beta\beta}).$$

(c)
$$\mathcal{L}_{g_\alpha} = \log_p(u_{g_\alpha}).$$

Proof. The proof is based on the study of the local cohomology class $\kappa_p(f, g_\alpha, h_\alpha)$ introduced in the preceding sections.

Observe that (a) and (b) are equivalent to
$$\log^{-+}(\kappa_p(f, g_\alpha, h_\alpha)) = \frac{\log_p(P_{\alpha\alpha})\log_p(Q_{\alpha\beta}) - \log_p(Q_{\alpha\alpha})\log_p(P_{\alpha\beta})}{\log_p(u_{g_\alpha})} \pmod{L^\times}$$
and
$$\log^{++}(\kappa_p(f, g_\alpha, h_\alpha)) = \log_p(P_{\alpha\alpha})\log_p(Q_{\beta\beta}) - \log_p(Q_{\alpha\alpha})\log_p(P_{\beta\beta}) \pmod{L^\times},$$
respectively, by virtue of the explicit reciprocity laws of [BSV20a] (both in the usual version and *improved* version based on the techniques of Venerucci).

Let us define the local class
$$\kappa_0 = \frac{1}{\Xi_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot \frac{1}{\log_p(u_{g_\alpha})} \cdot (\log_p(P_{\alpha\alpha}) \cdot Q - \log_p(Q_{\alpha\alpha}) \cdot P), \tag{7.20}$$
where we have implicitly identified a point over the elliptic curve with its image under the Kummer map; take then $\tilde{\kappa} = \kappa - \kappa_0$. The element $\tilde{\kappa}$ clearly belongs to the kernel of the Bloch–Kato logarithm $\log^{-+}$, that we have defined by
$$\log^{-+} : H^1(\mathbb{Q}_p, V_{fgh}) \xrightarrow{\mathrm{pr}^{-+}} H^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\alpha\beta}) \to \mathbb{C}_p,$$
the last map being the composition of the Perrin-Riou map and the pairing with the differentials $\omega_f \otimes \eta_{g_\alpha} \otimes \omega_{h_\alpha}$. Then, $\tilde{\kappa} = \lambda(\log_p(P_{\alpha\beta}) \cdot Q - \log_p(Q_{\alpha\beta}) \cdot P)$. But observe that by [BSV20a, Corollary 7.2] we know that the cohomology class κ lies in the balanced part for the filtration attached to $H^1(\mathbb{Q}_p, V_{fgh})$ and hence $\tilde{\kappa}$ lies in the kernel of the map $\log^{--}$
$$\log^{--} : H^1(\mathbb{Q}_p, V_{fgh}) \xrightarrow{\mathrm{pr}^{--}} H^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\alpha\alpha}) \to \mathbb{C}_p.$$

Hence, the non-vanishing of the regulator $\mathrm{Reg}_{g_\alpha}(V_{gh})$, implies that $\tilde{\kappa} = 0$ and therefore $\kappa = \kappa_0$.

From the same argument and under the assumption that $\mathrm{Reg}_f(V_{gh})$, the second equation yields

$$\kappa = \frac{1}{\Omega_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot (\log_p(P_{\alpha\alpha}) \cdot Q - \log_p(Q_{\alpha\alpha}) \cdot P) \pmod{L^\times}, \tag{7.21}$$

where again we have identified the points with their image under the Kummer map.

Now the statement is clear. For instance, if both (a) and (b) are true, comparing the previous expressions, we get

$$\log_p(u_{g_\alpha}) = \frac{\Xi_{g_\alpha}}{\Omega_{g_\alpha}} \pmod{L^\times}.$$

The proof of the other implications is equally straightforward. $\square$

It would be interesting to prove an analogue result in a more general situation, beyond the case of split multiplicative reduction. The discussion around cohomology classes is still valid, but the point is that one needs a replacement for the results expressing the second derivative of $\mathscr{L}_p{}^f$ in terms of the Bloch–Kato logarithm of the cohomology class. While we can assure that the special value $\mathscr{L}_p{}^f$ is zero, it is not clear how to proceed with its derivatives.

Question 7.5.3. Is there a reciprocity law relating the second derivative of $\mathscr{L}_p{}^f$ (or some *variation* of it) with the logarithm $\log^{--}$ of the cohomology class $\kappa(f,g,h)$ in a *generic* situation (non exceptional zeros)?

Case (a)

We assume first that $\alpha_g\alpha_h = 1$. The results we have proved until now showing a deep interaction between the value of the derivatives of $\mathscr{L}_p{}^f(\mathbf{f},\mathbf{g},\mathbf{h})$ and the value of $\mathscr{L}_p{}^{g_\alpha}(\mathbf{f},\mathbf{g},\mathbf{h})$ may be applied to study new instances of the Elliptic Stark Conjecture.

Let us analyze some particular cases describing the exact shape of the generalized cohomology classes. For example, according to the results of [BSV20a], when g is a theta series attached to a quadratic field where the prime p is inert and $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$ with ψ_1 being a genus character, we have

$$\frac{d^2\mathscr{L}_p{}^f(\mathbf{f}, g_\alpha, h_\alpha)}{dx^2}\Big|_{x=x_0} = \log^{++}(\kappa_p(f, g_\alpha, h_\alpha)) = \log(P^+_{\psi_1}) \cdot \log(P^+_{\psi_2}) \pmod{L^\times},$$

where $P^+_{\psi_i} = P_{\psi_i} + \sigma_p P_{\psi_i}$, being $\sigma_p \in \mathrm{Gal}\,(H/\mathbb{Q})$ a Frobenius element at p.

Remark 7.5.4. This situation occurs in general when at least one of ψ_1 or ψ_2 is a genus character. See for example the discussion after [DLR15a, Lemma 3.10] where the authors explain how the regulator of the Elliptic Stark Conjecture admits a particularly simple expression in this case.

However, from the results we already know around the Elliptic Stark Conjecture, one obtains that

$$\mathscr{L}_p{}^{g_\alpha} = \frac{\log(P^+_{\psi_1}) \cdot \log(P^-_{\psi_2})}{\mathcal{L}_{g_\alpha}} \pmod{L^\times}, \tag{7.22}$$

where with the previous notations, $P^-_{\psi_i} = P_{\psi_i} - \sigma_p P_{\psi_i}$. This is quite significant, since it establishes the Elliptic Stark Conjecture only up to a conjecture about periods of weight one modular forms.

Corollary 7.5.5. *Let g be a theta series attached to a quadratic field (either real or imaginary) where the prime p remains inert, with $V_{gh} = V_{\psi_1} \oplus V_{\psi_2}$ and at least one of ψ_1 or ψ_2 being a genus character. Then, the Elliptic Stark Conjecture of [DLR15a] is equivalent to the conjecture about periods of [DR16].*

Proof. This follows from the fact that part (b) of Proposition 7.5.2 holds in this setting. $\square$

Moreover, we expect conjectural expressions for the generalized Kato classes. In particular, the previous result suggests the following conjecture.

Proposition 7.5.6. *In the setting of Proposition 7.5.2, if the formulas which appear in that statement are satisfied, then the equality*

$$\kappa(f, g_\alpha, h_{1/\alpha}) = \frac{1}{\Omega_{g_\alpha} \cdot \Omega_{h_\alpha}} \cdot (\log_p(P_{\alpha\alpha}) \cdot Q - \log_p(Q_{\alpha\alpha}) \cdot P),$$

holds in $H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh})$ *up to multiplication by* $L^\times$.

Proof. This follows verbatim the proof of Proposition 7.5.2, using the third statement to simplify the different period relations. □

The same result holds for $\kappa(f, g_\beta, h_{1/\alpha})$.

Case (b)

In the case where $\alpha_g\beta_h = 1$, the explicit reciprocity law gives a connection between $\mathscr{L}_p{}^{g_\alpha}$ and the Bloch–Kato logarithm of $\kappa(f, g_\alpha, h_{1/\beta})$, but unfortunately both the latter class and one of the Euler factors involved in the equality vanish. Therefore, that result is meaningless in this setting.

In previous sections we saw how to overcome that difficulty, proving a *derived reciprocity law* after having observed that certain Euler factors are analytic along the line $k+m = \ell+2$. There are two *natural* directions for considering the derivative over that plane (although of course it makes sense to take any combination of them): the line $(2, \ell, \ell)$ and the line $(k+1, k, 1)$; the former is not quite interesting since both the class $\kappa(f, g_\alpha, g^*_{1/\beta})$ and the Euler factor in the denominator of the Perrin-Riou map vanish identically. Hence, we may take derivative along $(k+1, k, 1)$ and we get an equality of the form

$$\mathcal{L} \cdot \mathscr{L}_p{}^{g_\alpha} = \log^{-+}(\kappa'_p(f, g_\alpha, h_{1/\beta})) \pmod{L^\times},$$

where $\mathcal{L}$ is the $\mathcal{L}$-invariant which already appeared in previous sections. Hence, if the Elliptic Stark Conjecture for $\mathscr{L}_p{}^{g_\alpha}$ were true, the class $\kappa'_p(f, g_\alpha, h_\alpha)$ could be expressed as a combination of points, normalized by appropriate $\mathcal{L}$-invariants. In particular, this would yield an equality of the form

$$\kappa'(f, g_\alpha, h_{1/\beta}) = \frac{\mathcal{L}}{\Xi_{g_\alpha} \cdot \Omega_{h_{1/\beta}}} \cdot \frac{\log_p(P_{\alpha\alpha}) \cdot Q - \log_p(Q_{\alpha\alpha}) \cdot P}{\log_p(u_{g_\alpha})} \pmod{L^\times}. \tag{7.23}$$

One may obtain a symmetric expression for $\kappa'(f, g_\beta, h_{1/\alpha})$. Recall that this is the analogue of Theorem B of Chapter 3.

Conjecture 7.5.7. *The equality*

$$\kappa'(f, g_\alpha, h_{1/\beta}) = \frac{\mathcal{L}}{\Xi_{g_\alpha} \cdot \Omega_{h_{1/\beta}}} \cdot \frac{\log_p(P_{\alpha\alpha}) \cdot Q - \log_p(Q_{\alpha\alpha}) \cdot P}{\log_p(u_{g_\alpha})} \pmod{L^\times}$$

holds in $H^1_{\mathrm{f}}(\mathbb{Q}, V_{fgh})$.

As it was pointed out before, in the self dual case the product $\Xi_{g_\alpha}\Omega_{h_{1/\beta}}$ is an element of $L^\times$.

We finish our work with the following result.

Proposition 7.5.8. *Assume that Conjecture 7.5.7 is true. Then, the special value* $\mathscr{L}_p{}^{g_\alpha}$ *satisfies*

$$\mathscr{L}_p{}^{g_\alpha}(f, g_\alpha, h_\alpha) = \frac{\log_p(P_{\alpha\alpha})\log_p(Q_{\alpha\beta}) - \log_p(Q_{\alpha\alpha})\log_p(P_{\alpha\beta})}{\log_p(u_{g_\alpha})} \pmod{L^\times}.$$

Proof. This follows by applying the Bloch–Kato logarithm $\log^{-+}$ to the class $\kappa'(f, g_\alpha, h_{1/\beta})$, and using the derived reciprocity law of Theorem 3.3.4. □

The converse can also be established with some extra assumptions, including the conjecture about periods of [DR16].

Chapter 8

Future research plans: Eisenstein congruences between circular units and Beilinson–Kato elements

Let f be a cuspidal eigenform of weight two and level N, and let $p \nmid N$ be a prime at which f is congruent to an Eisenstein series. The Beilinson–Kato cohomology class κ_f associated to f gives rise, under appropriate hypotheses, to two different components modulo p. In this chapter we discuss congruence relations connecting those components to explicit expressions involving circular units. The proof of the first congruence relation invokes a mod p factorization formula due to Mazur and Greenberg–Vatsal and reciprocity laws due to Coleman and Kato recasting Kubota–Leopoldt's and Mazur–Tate–Teitelbaum's p-adic L-functions in terms of the Euler systems of circular units and Beilinson–Kato elements respectively. The second congruence relation is more subtle and it crucially relies on the ideas developed by Fukaya–Kato in their work on Sharifi's conjectures.

We warn the reader that this chapter is part of some works in progress, specially [RR20b], and hence the results we discuss are conditional to several results we expect to develop in the future.

8.1 Introduction

Let $N > 1$ be a positive integer, $\theta : (\mathbb{Z}/N\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ an even Dirichlet character and $f \in S_2(N, \theta)$ a normalized cuspidal eigenform of level N, weight 2 and nebentype θ. Fix a prime $p \nmid 6N$. Let T_f denote the integral p-adic Galois representation given as the f-isotypical quotient of the first étale cohomology group $H^1_{\mathrm{et}}(\bar{Y}_1(N), \mathbb{Z}_p(1))$ of the open modular curve $Y_1(N)$; cf. Section 8.2 for the particular model of this curve over $\mathbb{Q}$ we employ in this chapter and (8.9) for the precise definition of T_f.

Let F be the finite extension of $\mathbb{Q}$ generated by the field of coefficients of f, the N-th roots of unity and the values of all Dirichlet characters of conductor N; let $\mathcal{O}$ be its ring of integers. Fix algebraic closures $\bar{\mathbb{Q}}$, $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$ respectively, and an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. This singles out a prime ideal $\mathfrak{p}$ of $\mathcal{O}$ lying above p and we let $\mathcal{O}_\mathfrak{p} \subset \bar{\mathbb{Q}}_p$ denote the completion of $\mathcal{O}$ at $\mathfrak{p}$. We assume throughout the following.

Assumption 8.1.1. The localization of the Hecke algebra of level N at the Eisenstein ideal is Gorenstein. In particular, this implies that T_f is a free $\mathcal{O}_\mathfrak{p}$-module of rank 2.

Kato introduced in [Ka04] a global Galois cohomology class

$$\kappa_f = \kappa_f(\chi_1, \chi_2) \in H^1(\mathbb{Q}, T_f(1)) \tag{8.1}$$

which depends on auxiliary data (cf. loc. cit. and [BD14], [Han16], [KLZ17, Section 9], [Sch10] for several presentations of the subject in the literature). The way we normalize here κ_f depends on the choice of two auxiliary Dirichlet characters χ_1 and χ_2 of the same parity (see Section 8.3 for more details) and it is straightforward to relate it to the other equivalent conventions adopted in loc. cit.

Assume that f is congruent to an Eisenstein series modulo (a power of) $\mathfrak{p}$. Up to replacing f with a twist of it, we may assume without loss of generality that

$$f \equiv E_2(\theta, 1) \bmod \mathfrak{p}^t \tag{8.2}$$

for some $t \geq 1$, where $E_2(\theta, 1)$ is the Eisenstein series defined in (8.8). If we let $f^* := f \otimes \bar{\theta}$ denote the twist of f by the inverse of its nebentype, then congruence (8.2) is equivalent to $f^* \equiv E_2(1, \bar{\theta})$, and this in turn implies that $\mathfrak{p}^t$ divides the generalized Bernoulli number $B_2(\bar{\theta})$ (or equivalently the L-value $L(\bar{\theta}, -1)$).

Let $\widetilde{T_f}$ denote the integral p-adic Galois representation given as the f-isotypical quotient of the first étale cohomology group $H^1_{\mathrm{et}}(\bar{X}_1(N), \mathcal{O}_\mathfrak{p}(1))$. In [FK12, Section 7.1.11] the authors establish the existence of a short exact sequence

$$0 \to \mathcal{O}/\mathfrak{p}^t(\theta) \to \widetilde{T_f} \otimes \mathcal{O}/\mathfrak{p}^t \xrightarrow{\pi^+} \mathcal{O}/\mathfrak{p}^t(1) \to 0. \tag{8.3}$$

From the relations between the lattices attached to the open and the closed modular curve, and motivated by [FK12, Section 6.3, Remark 6.3.3], we make the following assumption (which we hope to remove in the near future by showing that it holds unconditionally, using for that purpose the existence of the exact sequence of (8.3)).

Working Assumption 8.1.2. There is an exact sequence of $\mathcal{O}/\mathfrak{p}^t[G_\mathbb{Q}]$-modules

$$0 \to \mathcal{O}/\mathfrak{p}^t(1) \to \bar{T}_f := T_f \otimes \mathcal{O}/\mathfrak{p}^t \xrightarrow{\pi^-} \mathcal{O}/\mathfrak{p}^t(\theta) \to 0. \tag{8.4}$$

Observe that this sequence always exists as an exact sequence of $\mathcal{O}/\mathfrak{p}^t[G_{\mathbb{Q}_p}]$-modules, as recalled e.g. in [FK12, Section 1.7.2] (and even without considering it modulo $\mathfrak{p}^t$).

The choice of the maps π^+ and π^- in (8.3) and (8.4) is non-canonical, but we exploit the work of Ohta [Oh99, Oh00] and Fukaya-Kato [FK12] to rigidify them in a canonical way, in the sense that π^+ and π^- only depend on canonical periods naturally associated to f; cf. (8.28) and (8.35) for more details.

Associated to $\bar{\kappa}_f = \kappa_f \mod \mathfrak{p}^t$ there exist thus two natural classes. First of all, we define

$$\bar{\kappa}_{f,1} = \pi^-_*(\kappa_f \mod \mathfrak{p}^t) \in H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\theta)(1)).$$

Assume further that κ_f may be lifted to an element in the cohomology of the closed modular curve, that is, that there exists an element $\widetilde{\kappa_f} \in H^1(\mathbb{Q}, \widetilde{T_f}(1))$ mapping to κ_f under the natural map

$$H^1_{\mathrm{et}}(\bar{X}_1(N), \mathcal{O}_\mathfrak{p}(1)) \longrightarrow H^1_{\mathrm{et}}(\bar{Y}_1(N), \mathcal{O}_\mathfrak{p}(1)).$$

From here, and applying the projection map induced from (8.3)

$$\widetilde{T_f}(1) \otimes \mathcal{O}/\mathfrak{p}^t \xrightarrow{\pi^+} \mathcal{O}/\mathfrak{p}^t(2),$$

we get a class

$$\bar{\kappa}_{f,2} = \pi^+_*(\widetilde{\kappa_f} \mod \mathfrak{p}^t) \in H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(2)).$$

The aim of this chapter is providing an explicit description of these two mod p^t Galois cohomology classes in terms of circular units. In order to state our results, let

$$\mathbb{Z}[\mu_N]^\times[\theta] = (\mathbb{Z}[\mu_N]^\times \otimes \mathcal{O}_\mathfrak{p}(\bar{\theta}))^{\mathrm{Gal}\,(\mathbb{Q}(\mu_N)/\mathbb{Q})} \simeq \mathrm{Hom}(\mathcal{O}_\mathfrak{p}(\theta), \mathbb{Z}[\mu_N]^\times \otimes \mathcal{O}_\mathfrak{p}) \tag{8.5}$$

denote the θ-isotypic component of $\mathbb{Z}[\mu_N]^\times \otimes \mathcal{O}_\mathfrak{p}$ on which $\mathrm{Gal}\,(\mathbb{Q}(\mu_N)/\mathbb{Q})$ acts through θ, which may be naturally identified with a $\mathcal{O}_\mathfrak{p}$-submodule of $\mathbb{Z}[\mu_N]^\times \otimes \mathcal{O}_\mathfrak{p}$ of rank 1 when $\theta \neq 1$ (resp. rank 0 when $\theta = 1$). Kummer theory gives rise to an injective homomorphism

$$\mathbb{Z}[\mu_N]^\times[\bar\theta] \to \mathrm{Hom}(G_{\mathbb{Q}(\mu_N)}, \mathcal{O}_\mathfrak{p}(1))[\bar\theta] \to H^1(\mathbb{Q}, \mathcal{O}_\mathfrak{p}(\theta)(1)).$$

Fix a primitive N-th root of unity ζ_N and define the circular unit

$$c_\theta := \prod_{a=1}^{N-1} (1-\zeta_N^a)^{\theta(a)} \in \mathbb{Z}[\mu_N]^\times[\bar\theta]. \tag{8.6}$$

Let

$$c_\theta \in H^1(\mathbb{Q}, \mathcal{O}_\mathfrak{p}(\theta)(1))$$

denote, with the same symbol by a slight abuse of notation, its image under the identification provided by the Kummer map. Write $\bar c_\theta = c_\theta \pmod{p^t} \in H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\theta)(1))$.

Let $k = \mathbb{Q}(\mu_N)^+ = \mathbb{Q}(\zeta_N + \zeta_N^{-1})$ denote the maximal totally real subfield of $\mathbb{Q}(\mu_N)$ and set $d_N = [k:\mathbb{Q}]$. Let $\mathrm{Cl}(k)$ denote its class group. As in (8.5) let $\mathrm{Cl}(k)[\theta]$ denote its θ-eigencomponent. It follows[1] from the work of G. Gras [Gra82, Théorème I2] that

$$\text{(Gr)} \quad \mathrm{rank}_{\mathbb{Z}/p\mathbb{Z}}\,\mathrm{Cl}(k)[\bar\theta] \otimes \mathbb{Z}/p\mathbb{Z} \le \mathrm{rank}_{\mathbb{Z}/p\mathbb{Z}}\,\mathrm{Cl}(k(\mu_p))[\bar\theta\omega] \otimes \mathbb{Z}/p\mathbb{Z},$$

where $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ is the Teichmüller character. This inequality may be regarded as an instance of Leopoldt's spiegelungssatz.

The statement of our first main theorem is conditional on the following hypotheses:

(H1) *Non-trivial zeroes mod* $\mathfrak{p}$*:*

$$\theta(p)-1,\ \chi_1\bar\chi_2(p)-1,\ \theta\chi_1\bar\chi_2(p)-1 \neq 0 \pmod{\mathfrak{p}}.$$

(H2) $\bar\theta$*-regularity:* (Gr) is an equality.

Let $R_p(k)$ denote the p-adic regulator of k. As it is explained in e.g. [Gra16, Def. 2.3], one always has $\mathrm{ord}_p R_p(k) \ge d_N - 1$. It is shown in loc. cit. that (Gr) is an equality for all non-trivial even Dirichlet characters of conductor N if and only if $\mathrm{ord}_p R_p(k) = d_N - 1$. We refer to [Gra16, Section 7.3] for conjectures predicting that such an equality is expected to hold for all primes p away from a set of density 0.

We thus expect both hypotheses (H1)-(H2) to hold very often; in those cases where at least one of these fails, then we expect the following result to become the trivial congruence $0 \equiv 0$.

Define the algebraic L-value

$$L^{\mathrm{alg}}(f^*, \bar\chi_1\chi_2, 1) = L(f^*, \bar\chi_1\chi_2, 1)/\Omega_f^+ \in \mathcal{O}$$

where Ω_f^+ is Shimura's complex period associated to f, chosen in a specific way that we later recall in Section 8.3 (we have chosen the period Ω_f^+ since χ_1 and χ_2 have the same parity). Let also

$$\mathfrak{g}(\chi) = \sum_{a=1}^{N-1} \chi(a)\zeta_N^a$$

denote the Gauss sum attached to a Dirichlet character χ of conductor N.

Then, the following result relies on several assumptions we have made along the text. We expect in the near future to prove the following statement or a suitable variation of it.

[1] [Gra82, Théorème I2] applies because Leopoldt's conjecture is known for (k, p) by the work of Brumer, primes in k above p are totally ramified in $k(\mu_p)$ and therefore the ω-component of the $\mathrm{Gal}\,(k(\mu_p)/k)$-submodule of $\mathrm{Cl}(k(\mu_p))$ generated by ideals above p is trivial. [Gra82, Théorème I2] thus asserts that $\mathrm{rank}_{\mathbb{Z}/p\mathbb{Z}}\,\mathrm{Cl}(k(\mu_p))[\bar\theta\omega] \otimes \mathbb{Z}/p\mathbb{Z}$ is equal to the rank of the $\bar\theta$-component of the p-torsion of the Galois group $\mathrm{Gal}\,(H_p/k)$ of the maximal p-abelian extension of k unramified away from p. Hence (Gr) follows because the Hilbert class field H/k is contained in H_p and $\mathrm{Gal}\,(H/k) = \mathrm{Cl}(k)$.

Expected Theorem 8.1.3. Assume (H1) and (H2). Then in $H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\theta)(1))$ we have

$$\bar{\kappa}_{f,1} \equiv \frac{iN}{12} \cdot \frac{\mathfrak{g}(\theta\chi_1\bar{\chi}_2)}{\mathfrak{g}(\theta)} \cdot L^{\mathrm{alg}}(f^*, \bar{\chi}_1\chi_2, 1) \times \bar{c}_\theta \pmod{\mathfrak{p}^t}.$$

As we describe in more detail in Section 8.3, Kato's class is constructed as

$$\kappa_f = \pi_{f*}(u \cup v),$$

namely the push-forward to the f-isotypic component of the cup product of two modular units

$$u = u_{\chi_1,\chi_2} \quad \text{and} \quad v = u_{\bar{\chi}_1,\theta\bar{\chi}_2}$$

whose logarithmic derivative are respectively the Eisenstein series $E_2(\chi_1, \chi_2)$ and $E_2(\bar{\chi}_1, \theta\bar{\chi}_2)$ given in (8.8). The q-expansion of the modular units u_{χ_1,χ_2} can be written down explicitly. Given a pair of integers (a, b) between 0 and $N-1$, not both equal to 0, define the Siegel unit

$$u_{a,b;N} = q^w \prod_{n\geq 0} \left(1 - q^{n+a/N}\zeta_N^b\right) \prod_{n\geq 1} \left(1 - q^{n-a/N}\zeta_N^{-b}\right),$$

where $w = \frac{1}{12} - \frac{a}{N} + \frac{a^2}{2N^2}$. Then the q-expansion of the modular unit $u_{\chi_1,1}$ is given by

$$u_{\chi_1,1} = \frac{-1}{2\mathfrak{g}(\bar{\chi}_1)} \sum_{b=1}^{N-1} \bar{\chi}_1(b) \otimes u_{0,b;N}, \tag{8.7}$$

where here N stands for the conductor of χ_1. Although we will not use them here in this note, similar expressions can be given for u_{χ_1,χ_2} for arbitrary χ_2 by averaging $u_{a,b;N}$ and choosing an appropriate uniformizer.

Note that u_{χ_1,χ_2} are determined by their logarithmic derivative only up to a multiplicative constant, and therefore the first non-vanishing coefficient in the Laurent expansion of u_{χ_1,χ_2} at ∞, which we simply denote $u_{\chi_1,\chi_2}(\infty)$ as in [FK12, Section 5], may be chosen arbitrarily. Since $\mathrm{Gal}\,(\mathbb{Q}(\mu_N)/\mathbb{Q})$ acts on $E_2(\chi_1, \chi_2)$ via χ_1 [St82, Theorem 1.3.1], it is natural to normalize u_{χ_1,χ_2} likewise, so that $u_{\chi_1,\chi_2}(\infty)$ may be any power of the circular unit c_{χ_1}. In the literature one finds different normalizations, typically either $u_{\chi_1,\chi_2}(\infty) = 1$ or c_{χ_1}. In the statement below we have chosen to normalize the modular units above so that

$$u_{\chi_1,\chi_2}(\infty) = c_{\chi_1}, \quad u_{\bar{\chi}_1,\theta\bar{\chi}_2}(\infty) = c_{\bar{\chi}_1}$$

but any other choice would be perfectly fine, upon replacing accordingly the two circular units appearing in the cup-product below.

For the second result we make the following assumption, that we had already anticipated. The theory developed in [FK12, Sections 3, 4] suggests that imposing this condition is rather reasonable; for instance, in 4.4.1, the authors establish that a *similar* Kato class belongs to the cohomology of the closed modular curve.

Working Assumption 8.1.4 (Cohomological condition)**.** The Beilinson–Kato element in the cohomology group $H^1_{\mathrm{et}}(\bar{Y}_1(N), \mathcal{O}_{\mathfrak{p}}(1))$ may be lifted to a class in $H^1_{\mathrm{et}}(\bar{X}_1(N), \mathcal{O}_{\mathfrak{p}}(1))$.

Although for the moment we do not know until which extent the previous hypothesis is reasonable, we believe that the discussion of the following result introduces certain tools which can be easily adapted to slightly different settings where the condition does hold. We now formulate the second theorem. The precise statement may suffer variations as we make progress on the pending assumptions.

Expected Theorem 8.1.5. Assume θ is primitive of conductor N and $p \nmid \varphi(N)$. Suppose further that Assumption 8.1.4 holds. Let $L_p(\bar\theta, s)$ denote the Kubota–Leopoldt p-adic L-function attached to $\bar\theta$ and assume $L_p'(\bar\theta, -1)$ is a p-adic unit. The following equality holds in $H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(2))$:

$$\bar\kappa_{f,2} \equiv \frac{L_p'(\bar\theta,-1)}{1-p^{-1}} \cdot \frac{\bar c_{\bar\chi_1} \cup \bar c_{\chi_1}}{\cup \log_p(\varepsilon_{\mathrm{cyc}})} \pmod{\mathfrak{p}^t}.$$

Here $\varepsilon_{\mathrm{cyc}}$ is the cyclotomic character and $1/\cup \log_p(\varepsilon_{\mathrm{cyc}})$ denotes the inverse of the map

$$H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(2)) \to H^2(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(2)), \quad \kappa \mapsto \kappa \cup \log_p(\varepsilon_{\mathrm{cyc}}),$$

which is invertible under our assumptions.

Theorem 8.1.3 may be regarded as providing a *Jochnowitz congruence* between the first derivative of the Dirichlet L-function $L(\theta, s)$ and the first derivative of the Hasse-Weil L-function $L(f, s)$ at the critical point $s = 2$. The discussion of this result occupies Section 8.3 and exploits a mod p factorization of p-adic L-functions due to Mazur and Greenberg-Vatsal, and Ohta's work on p-adic families of modular forms.

As for Theorem 8.1.5, note that our running assumptions imply that $L_p(\bar\theta, -1) \equiv 0 \pmod{\mathfrak{p}^t}$ and it is thus natural that the first derivative of the Kubota-Leopoldt p-adic L-function makes an appearance. This result is discussed in Section 8.4 by a rather different method, as it invokes in a crucial way the ideas introduced by Fukaya and Kato in their work [FK12] on Sharifi's conjectures [Sha11]. For this reason it may be interpreted as a form (or consequence) of Sharifi's conjecture which could potentially lead to analogous formulations of the latter in other scenarios (including Katz's p-adic L-function associated to imaginary quadratic fields or Hida's Rankin p-adic L-function associated to a pair of modular forms); we hope to tackle this approach elsewhere.

Finally, we hope that the results developed along this chapter may be adapted to other settings regarding the Euler systems discussed in this memoir, specially to prove congruence relations between diagonal cycles and Beilinson–Flach classes.

8.2 Modular curves, modular units and Eisenstein series

Given a variety $Y/\mathbb{Q}$ and a field extension $F/\mathbb{Q}$, let $Y_F = Y \times F$ denote the base change of Y to F and set $\bar Y = Y_{\bar{\mathbb{Q}}}$. Fix an integer $N \geq 3$ and let $Y_1(N) \subset X_1(N)$ denote the canonical models over $\mathbb{Q}$ of the (affine and projective, respectively) modular curves classifying pairs (A, i) where A is a (generalized) elliptic curve and $i : \mu_N \to A$ is an embedding of group schemes. It is important to recall that this is not the model used by Fukaya and Kato, as they consider the one which classifies pairs (A, P), where A is a (generalized) elliptic curve and P is an N-torsion point of it. In any case, the model of [FK12] can be obtained from ours just taking the twist by the cocyle

$$\mathrm{Gal}\,(\mathbb{Q}(\mu_N)/\mathbb{Q}) \to \mathrm{Aut}(Y), \quad s \mapsto \langle s^{-1} \rangle,$$

where $\langle s \rangle$ stands for the diamond operator associated to $s \in (\mathbb{Z}/N\mathbb{Z})^\times$.

Let $C_N := X_1(N) \setminus Y_1(N)$ denote the finite scheme of cusps; among them one may distinguish the cusp $\infty \in C_N(\mathbb{Q})$ associated to Tate's elliptic curve over $\mathbb{Z}((q))$, which is rational over $\mathbb{Q}$ in this choice of model (cf. e.g. [St82, Section 1.3], [St85]). (Again, note that in the model of [FK12, §1.3.3], cusp ∞ is not defined over $\mathbb{Q}$ but over $\mathbb{Q}(\mu_N)$.)

Assume now that F contains the values of all Dirichlet characters of conductor N. Then a basis of $\mathrm{Eis}_2(\Gamma_1(N), F)$ is indexed by triples (χ_1, χ_2, r) where χ_1 and χ_2 are primitive Dirichlet characters of conductors N_1 and N_2 with $N_1 \cdot N_2 \mid N$, $\chi_1(-1) = \chi_2(-1)$, and r is a positive integer with $1 < rN_1N_2 \mid N$, provided by the Eisenstein series (cf. e.g. [DS05, Theorem 4.6.2], [St82, Def. 3.4.1]):

$$E_2(\chi_1, \chi_2, r) = a_0 + \sum_{n=1}^{\infty} \Big(\sum_{d \mid n} \chi_1(n/d)\chi_2(d)d \Big) q^{rn}, \quad a_0 = \begin{cases} \frac{L(\chi_2,-1)}{2} & \text{if } \chi_1 = 1 \\ 0 & \text{if } \chi_1 \neq 1 \end{cases} \tag{8.8}$$

unless $\chi_1 = \chi_2 = 1$, in which case $E_2(1,1,r) = \sum_{n=1}^{\infty}\Big(\sum_{d|n} d\Big)q^n - r\sum_{n=1}^{\infty}\Big(\sum_{d|n} d\Big)q^{rn}$.

When $r = 1$ we shall simply denote $E_2(\chi_1,\chi_2) := E_2(\chi_1,\chi_2,1)$.

When $\chi_1 = 1$, the constant term may also be recast as a generalized Bernoulli number: setting $B_2(x) = x^2 - x + 1/6$, define

$$B_2(\chi) := N\sum_{a=1}^{N-1}\chi(a)\cdot B_2(a/N)$$

for any Dirichlet character χ of conductor N. One then has $-2L(\chi_2,-1) = B_2(\chi_2)$.

Let $\mathbb{T} \subset \mathrm{End}\, H^1_{\mathrm{et},c}(\overline{Y}_1(N),\mathcal{O}_{\mathfrak{p}}(1))$ denote the Hecke algebra acting on the compactly-supported cohomology of the open modular curve generated by the standard Hecke operators T_ℓ for every (good or bad) prime ℓ let–commonly denoted U_ℓ at primes $\ell \mid N$. Let also $\mathbb{T}^* \subset \mathrm{End}\, H^1_{\mathrm{et}}(\overline{Y}_1(N),\mathcal{O}_{\mathfrak{p}}(1))$ denote the Hecke algebra acting on the cohomology of the open modular curve generated by the dual Hecke operators T^*_ℓ as defined in [Oh99, Section 3.4], [KLZ17, Def. 2.4.3] for every prime ℓ.

Given a newform $\varphi \in S_2(N,\theta)$, let $\varphi(q) = \sum a_n(\varphi)q^n$ denote its q-expansion at the cusp ∞ and let $\mathcal{O}_{\mathfrak{p}}$ be a finite ring extension of $\mathbb{Z}_p$ containing the eigenvalues $\{a_n(\varphi)\}_{n\geq 1}$.

Let $I_\varphi = (T_\ell - a_\ell(\varphi)) \subset \mathbb{T}$ and $I^*_\varphi = (T^*_\ell - a_\ell(\varphi)) \subset \mathbb{T}^*$ denote the ideals associated to the system of eigenvalues of φ with respect to the standard (resp. dual) Hecke operators. Define the $\mathcal{O}_{\mathfrak{p}}$-modules

$$M_\varphi = H^1_{\mathrm{et},c}(\overline{Y}_1(N),\mathcal{O}_{\mathfrak{p}})[I_\varphi] := \cap_\ell\, \mathrm{K}er(T_\ell - a_\ell(\varphi)) \subset H^1_{\mathrm{et},c}(\overline{Y}_1(N),\mathcal{O}_{\mathfrak{p}}(1)) \tag{8.9}$$

and

$$T_\varphi = H^1_{\mathrm{et}}(\overline{Y}_1(N),\mathcal{O}_{\mathfrak{p}}(1))/I^*_\varphi.$$

Poincaré duality yields a perfect pairing of finitely generated free $\mathbb{Z}_p$-modules

$$\langle\,,\rangle : H^1_{\mathrm{et},c}(\overline{Y}_1(N),\mathbb{Z}_p) \times H^1_{\mathrm{et}}(\overline{Y}_1(N),\mathbb{Z}_p(1)) \longrightarrow \mathbb{Z}_p$$

which in turn induces a perfect pairing of $\mathcal{O}_{\mathfrak{p}}$-modules

$$\langle\,,\rangle : M_\varphi \times T_\varphi \longrightarrow \mathcal{O}_{\mathfrak{p}}(1). \tag{8.10}$$

Recall that Assumption 8.1.1 allows us to say that M_φ and T_φ are $\mathcal{O}_{\mathfrak{p}}[G_{\mathbb{Q}}]$-modules, free of rank two over $\mathcal{O}_{\mathfrak{p}}$.

If φ is ordinary at $\mathfrak{p}$, there are exact sequences of $\mathcal{O}_{\mathfrak{p}}[G_{\mathbb{Q}_p}]$-modules

$$\begin{gathered} 0 \to M^{\mathrm{sub}}_\varphi \to M_\varphi \to M^{\mathrm{quo}}_\varphi \to 0 \\ 0 \to T^{\mathrm{sub}}_\varphi \to T_\varphi \to T^{\mathrm{quo}}_\varphi \to 0 \end{gathered} \tag{8.11}$$

such that

(i) M^{sub}_φ and T^{quo}_φ are unramified as $G_{\mathbb{Q}_p}$-modules.

(ii) M^{quo}_φ and T^{sub}_φ are free of rank 1 as $\mathcal{O}_{\mathfrak{p}}$-modules.

(iii) Poincaré duality induces a perfect pairing between M^{quo}_φ and T^{sub}_φ, and likewise between M^{sub}_φ and T^{quo}_φ.

8.3 First congruence relation

Keep the notations and assumptions fixed in the introduction concerning the first congruence relation. We begin by recalling more precisely the definition of Kato classes. Choose auxiliary Dirichlet characters χ_1, χ_2 as in the introduction and set $\xi_1 = \bar{\chi}_1$, $\xi_2 = \theta\bar{\chi}_2$.

Define the group of modular units $U(N)$ as the subgroup of rational functions of $X_1(N)_{\mathbb{Q}(\mu_N)}$ with zeroes and poles concentrated at the cusps, that is to say

$$U(N) = \mathcal{O}(Y_1(N)_{\mathbb{Q}(\mu_N)})^{\times}.$$

Similarly as in (8.5), let $U(N)[\chi]$ denote the χ-isotypic component of $U(N) \otimes \mathcal{O}_{\mathfrak{p}}$ on which the Galois group $\mathrm{Gal}\,(\mathbb{Q}(\mu_N)/\mathbb{Q})$ acts through the character χ. In light of [St82, Theorem 1.3.1], there exists a modular unit $u_{\chi_1,\chi_2} \in U(N)[\chi_1]$ satisfying

$$\mathrm{dlog}(u_{\chi_1,\chi_2}) = E_2(\chi_1,\chi_2)\frac{dq}{q} \quad \text{and} \quad u_{\chi_1,\chi_2}(\infty) = c_{\chi_1}. \tag{8.12}$$

Kummer theory induces a morphism

$$\delta : U(N)[\chi] \to H^1_{\mathrm{et}}(Y_1(N), \mathcal{O}_{\mathfrak{p}}(\bar{\chi})(1)). \tag{8.13}$$

By [Nek98, (1.2)], together with the fact that $H^j_{\mathrm{et}}(\bar{V}, \mathcal{O}_{\mathfrak{p}})$ vanishes for any smooth affine variety V of dimension d and any $j > d$, the Hochschild–Serre's spectral sequence gives rise to an isomorphism

$$H^2_{\mathrm{et}}(Y_1(N), \mathcal{O}_{\mathfrak{p}}(2)) \simeq H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\overline{Y_1(N)}, \mathcal{O}_{\mathfrak{p}}(2))). \tag{8.14}$$

In view of (8.9) there is a $G_{\mathbb{Q}}$-equivariant projection

$$\pi_f : H^1_{\mathrm{et}}(\overline{Y}_1(N), \mathcal{O}_{\mathfrak{p}}(1)) \to T_f,$$

which in turn gives rise to a homomorphism

$$\pi_{f*} : H^2_{\mathrm{et}}(Y_1(N), \mathcal{O}_{\mathfrak{p}}(2)) \longrightarrow H^1(\mathbb{Q}, T_f(1)). \tag{8.15}$$

It thus makes sense to define

$$\kappa_f := \pi_{f*}(\delta(u_{\xi_1,\xi_2}) \cup \delta(u_{\chi_1,\chi_2})) \in H^1(\mathbb{Q}, T_f(1)).$$

Note that f is ordinary at p because of (8.2). Let $L_p(\chi, s)$ denote the Kubota–Leopoldt p-adic L-function associated to a Dirichlet character χ, and $L_p(f, \chi, s)$ the Mazur–Tate–Teitelbaum p-adic L-function associated to (f, χ) (cf. e.g. [Ki94] or [MTT86]).

Label the roots of the p-th Hecke polynomial of f as α_f, β_f so that $\mathrm{ord}_p(\alpha_f) = 0$ and $\mathrm{ord}_p(\beta_f) = 1$, and define the Euler-like factor

$$\mathcal{E}_f = (1-\alpha_f)(1-\beta_f)(1-\bar{\theta}\bar{\chi}_1\chi_2(p)\beta_f p^{-1})(1-\chi_1\bar{\chi}_2(p)\beta_f p^{-1}). \tag{8.16}$$

Define also the p-adic L-value

$$\ell = -iN \cdot \frac{\mathfrak{g}(\theta\chi_1\bar{\chi}_2)}{\mathfrak{g}(\theta)\mathfrak{g}(\chi_1\bar{\chi}_2)} \cdot (1-\theta(p)) \cdot \zeta_p(-1) \cdot L_p(f^*, \bar{\chi}_1\chi_2, 1),$$

Recall from the introduction the class $\bar{\kappa}_{f,1} \in H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\theta)(1))$. This class always exists locally, and to make sense of it globally one needs to impose Assumption 8.1.2. We hope to be able to establish this fact by relating the exact sequence of [FK12, Section 7.1.11] with the explicit description of the canonical inclusion of the lattice corresponding to the closed modular curve in the lattice attached to the open modular curve.

Expected Theorem 8.3.1. (First congruence relation) Assuming (H1) and (H2) we have

$$\mathcal{E}_f \cdot \bar{\kappa}_{f,1} \equiv \ell \cdot \bar{c}_\theta \quad \text{in } H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\theta)(1)).$$

Theorem 8.1.3 in the introduction readily follows from the above statement. Indeed, as shown in [MTT86] the p-adic L-value $L_p(f^*, \bar{\chi}_1\chi_2, 1)$ may be written as

$$L_p(f^*, \bar{\chi}_1\chi_2, 1) = (1 - \bar{\theta}\bar{\chi}_1\chi_2(p)\beta_f p^{-1})(1 - \chi_1\bar{\chi}_2(p)\beta_f p^{-1}) \times \mathfrak{g}(\chi_1\bar{\chi}_2) \times \frac{L(f^*, \bar{\chi}_1\chi_2, 1)}{\Omega_f^+}.$$

Since $\zeta_p(-1) = (1-p)\cdot\zeta(-1) = (p-1)\cdot\frac{B_2}{2} = \frac{p-1}{12}$, the Euler factors in the above theorem cancel out in light of (H1) and the congruence $(\alpha_f, \beta_f) \equiv (\theta(p), p) \pmod{\mathfrak{p}^t}$. Then, Theorem 8.1.3 follows.

The remainder of this section is devoted to the proof of Theorem 8.3.1, which is a combination of the following four ingredients:

(a) A factorization formula for the p-adic L-function $L_p(f^*, \chi, s)$.

(b) Coleman's power series and the p-adic Kronecker limit formula.

(c) Kato's explicit reciprocity law.

(d) Congruences among Ohta's periods.

Mazur's factorization formula

The arithmetic of the Beilinson–Kato elements is governed by the so-called Hida–Rankin p-adic L-function. More precisely, [Hi88, Section 4] constructed a p-adic L-function associated to a pair of Hida families $(\mathbf{f}, \mathbf{g})$, where $\mathbf{g}$ is allowed to be either cuspidal or Eisenstein, and which depends on three variables (k, ℓ, s), where (k, ℓ) stand for the weight variables and s for the cyclotomic one. Recall that this had already been introduced as Theorem 3.2.3. Following [BD14], we assume that $\mathbf{g} = \mathbf{E}(\chi_1, \chi_2)$ is the Eisenstein family passing through $E_2(\chi_1, \chi_2)$, and we further assume that $s = k/2 + \ell - 1$. In this case, Theorem 3.4 of loc. cit. provides a factorization formula for such a two-variable Hida–Rankin p-adic L-function in terms of two Mazur–Kitagawa p-adic L-functions, using the same choices of periods as Hida in [Hi88].

For our purposes, f can be kept fixed and we may set $k = 2$. That way, we have a one-variable p-adic L-function associated to $f \otimes \mathbf{E}(\chi_1, \chi_2)$ and restricted to the central cyclotomic variable $s = \ell$. Let $\mathcal{I}_f$ be the congruence ideal associated to f, which is an integral ideal of $\mathcal{O}_\mathfrak{p}$. After fixing such a choice, it follows from Hida's construction that the restriction of the Hida–Rankin p-adic L-function to the variable ℓ belongs to $\mathcal{I}_f^{-1} \otimes_{\mathbb{Z}_p} \Lambda$, where Λ is the usual Iwasawa algebra.

Since we are interested in studying congruences, it is more convenient for our purposes to use a different normalization of the idoneous p-adic L-functions, rendering them integral. To do that, fix a uniformizer $\varpi_\mathfrak{p}$ of the maximal ideal of $\mathcal{O}_\mathfrak{p}$. In Hida's terminology, the appropriate power of $\varpi_\mathfrak{p}$ generating $\mathcal{I}_f$ is sometimes called a congruence divisor. Note that $\varpi_\mathfrak{p}$ is uniquely defined only up to multiplication by a $\mathfrak{p}$-adic unit.

In [Ki94] and [Va99], the authors work with a choice of the pair of periods (Ω_f^+, Ω_f^-) such that, according to [Va03, Remark 2.7], the following holds:

Proposition 8.3.2. *Let (Ω_f^+, Ω_f^-) stand for the pair of complex periods defined in [Va99]. Then,*

$$\varpi_\mathfrak{p}^r \cdot \Omega_f^+\Omega_f^- = 4\pi^2\langle f, f\rangle. \tag{8.17}$$

Remark 8.3.3. As we shall explain below, there is a canonical choice of period Ω^-; let us thus assume that the choice of Ω^- is already fixed throughout.

On the other hand, the choice of congruence divisor is well-defined only up to a p-adic unit. According to the previous proposition, the choices of ϖ^r and Ω^- prescribe the value of Ω_f^+. This is analogous to [BD14, eq. (21)]; in our setting however we work integrally and the ambiguity of our choices is only up to a p-adic unit.

Let us describe the extent to which this ambiguity in our choice of congruence divisor affects the statements below. If ϖ^r is replaced with $u\varpi_{\mathfrak{p}}^r$ for some $u \in \mathcal{O}_{\mathfrak{p}}^\times$, period Ω_f^+ is thus replaced with $u^{-1} \cdot \Omega_f^+$ and the values of the Mazur–Kitagawa p-adic L-function $L_p(f^*, \psi, s)$ at characters ψ satisfying $(-1)^{s-1}\psi(-1) = 1$ are affected accordingly. For our purposes this is harmless, because the values of the p-adic L-function at characters ψ with $(-1)^s\psi(-1) = 1$ remain untouched as Ω_f^- is chosen canonically.

As we have anticipated, we choose and fix Ω_f^- as specifically discussed in [FK12, 6.3.18, 7.1.11]. This permits to obtain an appropriate congruence with the product of two Kubota–Leopoldt p-adic L-functions (see Section 8.2 of loc. cit., and also [FK12, Theorem 3.12]). The choice of Ω_f^- automatically determines Ω_f^+ from (8.17) once we have fixed the generator of the congruence module.

In the rest of this section we are implicitly assuming the following.

Working Assumption 8.3.4. The canonical period introduced in [Va99], which is only well-defined up to p-adic unit, can be chosen to be equal to that of [FK12, Section 8.2].

Remark 8.3.5. As a minor observation, note that the ideal $\mathfrak{p}^r$ measures the existence of congruences between modular forms of level N and nebentype θ. In general, $r \geq t$, and the number may be larger when there exist other sources of congruences. See also [PW11] for a discussion of the arithmetic meaning of the discrepancy between the different periods involved in (8.17).

We now discuss a factorization formula for $L_p(f^*, \psi, s)$ in terms of two Kubota–Leopoldt p-adic L-functions. For the latter, we take the weight space as $\mathcal{O}_{\mathfrak{p}}[[\mathbb{Z}_p^\times]]$ and not as $\mathcal{O}_{\mathfrak{p}}[[1 + p\mathbb{Z}_p]]$; this avoids working with annoying powers of the Teichmüller character. See for instance the discussion of [Das16, Section 3], where this same approach is also considered.

The factorization formula we present is essentially due to Mazur [Maz79] and Greenberg–Vatsal [GV00, Theorem 3.12]. The precise formulation adopted here is borrowed from [FK12, Proposition 8.2.4], where the result is presented in terms of classical L-values. The following version immediately follows by invoking the interpolation formula a in [Das16, eq. (26)] and a density argument.

Proposition 8.3.6. *Let $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ be a Dirichlet character. Then, for any integer s satisfying $(-1)^s\psi(-1) = 1$, we have that*

$$L_p(f^*, \psi, s) \equiv 2 \cdot L_p(\bar{\psi}, 1 - s) \cdot L_p(\bar{\theta}\psi, s - 1) \pmod{\mathfrak{p}^t}. \tag{8.18}$$

In particular, if ψ is even, it holds that

$$L_p(f^*, \psi, 2) = 2 \cdot L_p(\bar{\psi}, -1) \cdot L_p(\bar{\theta}\psi, 1) \equiv -B_2(\bar{\psi}) \cdot L_p(\bar{\theta}\psi, 1) \pmod{\mathfrak{p}^t}. \tag{8.19}$$

Remark 8.3.7. The previous result depends on the choice of the period Ω_f^-, which we have chosen in the same way than in [FK12, Proposition 8.2.4]. However, it does not depend on Ω_f^+, and therefore it is also independent of the choice of the congruence divisor.

Dieudonné modules and congruences among Ohta's periods

Given a p-adic de Rham representation V of $G_{\mathbb{Q}_p}$ with coefficients in $F_{\mathfrak{p}}$, let $D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}$ denote its Dieudonné module and let $\log_{\mathrm{BK}}$ (resp. $\exp^*_{\mathrm{BK}}$) stand for the Bloch–Kato logarithm (resp. dual exponential map) attached to V as defined in [BK93], [Bel09].

Let $\hat{\mathbb{Z}}_p^{\mathrm{ur}}$ denote the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$. Given an unramified $\mathcal{O}_{\mathfrak{p}}[\mathrm{Gal}\,(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)]$-module T, define as in [Oh00, Theorem 2.1.11]

$$D(T) := (T \hat{\otimes}_{\mathbb{Z}_p} \hat{\mathbb{Z}}_p^{\mathrm{ur}})^{\mathrm{Fr}_p = 1}.$$

One has $D_{\mathrm{dR}}(T \otimes F_{\mathfrak{p}}) = D(T) \otimes F_{\mathfrak{p}}$.

A Dirichlet character χ of conductor N as in the introduction gives rise to an unramified character of $G_{\mathbb{Q}_p}$ that we continue to denote with the same symbol. Given a positive integer $s \geq 1$ (and assuming $\chi(p) \neq 1$ if $s = 1$), Bloch-Kato's logarithm associated to $V = F_{\mathfrak{p}}(\chi)(r)$ gives rise to an isomorphism of rank 1 $F_{\mathfrak{p}}$-vector spaces

$$\log_{\mathrm{BK}} : H^1(\mathbb{Q}_p, F_{\mathfrak{p}}(\chi)(s)) \to D_{\mathrm{dR}}(F_{\mathfrak{p}}(\chi)(s)). \tag{8.20}$$

As explained e.g. in [FK12, Prop. 1.7.6], there is a functorial isomorphism of $\mathcal{O}_{\mathfrak{p}}$-modules (forgetting the Galois structure) given by

$$T \xrightarrow{\sim} D(T). \tag{8.21}$$

This map is not canonical as it depends on a choice of root of unity; for $T = \mathcal{O}_{\mathfrak{p}}(\chi)$ we take it to be given by the rule $1 \mapsto \mathfrak{g}(\chi)$.

There are perfect pairings

$$\langle\, , \rangle \,:\, D_{\mathrm{dR}}(F_{\mathfrak{p}}(\chi)(s)) \times D_{\mathrm{dR}}(F_{\mathfrak{p}}(\bar{\chi})(-s)) \longrightarrow F_{\mathfrak{p}}$$

and

$$\langle\, , \rangle \,:\, D(\mathcal{O}_{\mathfrak{p}}(\chi)) \times D(\mathcal{O}_{\mathfrak{p}}(\bar{\chi})) \longrightarrow \mathcal{O}_{\mathfrak{p}}.$$

Since a canonical generator of $D_{\mathrm{dR}}(F_{\mathfrak{p}}(\bar{\chi})(-s))$ is given by $t^s\mathfrak{g}(\chi)^{-1}$, where t is Fontaine's p-adic analogue of $2\pi i$, the above pairing yields an isomorphism

$$D_{\mathrm{dR}}(F_{\mathfrak{p}}(\chi)(s)) \to F_{\mathfrak{p}} \quad c \mapsto \langle c, \frac{t^s}{\mathfrak{g}(\chi)}\rangle. \tag{8.22}$$

The de Rham Dieudonné module $D_{\mathrm{dR}}(V_\varphi)$ is a $F_{\mathfrak{p}}$-filtered free module of rank 2. As discussed in [KLZ17, Section 2.8], Poincaré duality induces a perfect pairing

$$\langle\, ,\, \rangle : D_{\mathrm{dR}}(V_\varphi(-1)) \times D_{\mathrm{dR}}(V_{\varphi^*}) \to F_{\mathfrak{p}},$$

and there is an exact sequence of Dieudonné modules

$$0 \to D_{\mathrm{dR}}(V_\varphi^{\mathrm{sub}}) \to D_{\mathrm{dR}}(V_\varphi) \to D_{\mathrm{dR}}(V_\varphi^{\mathrm{quo}}) \to 0, \tag{8.23}$$

where $D_{\mathrm{dR}}(V_\varphi^{\mathrm{sub}})$ and $D_{\mathrm{dR}}(V_\varphi^{\mathrm{quo}})$ have both rank 1. Falting's theorem associates to φ a regular differential form $\omega_\varphi \in \mathrm{Fil}(D_{\mathrm{dR}}(V_\varphi))$, which gives rise to an element in $D_{\mathrm{dR}}(V_\varphi^{\mathrm{quo}})$ via the right-most map in (8.23) and in turn induces a linear form

$$\omega_\varphi \,:\, D_{\mathrm{dR}}(V_{\varphi^*}^{\mathrm{sub}}(-1)) \to F_{\mathfrak{p}}, \quad \eta \mapsto \langle \omega_\varphi, \eta\rangle \tag{8.24}$$

that we continue to denote with same symbol by a slight abuse of notation.

There is also a differential η_φ, which is characterized by the property that it spans the line $D_{\mathrm{dR}}(V_\varphi^{\mathrm{sub}}(-1))$ and

$$\langle \eta_\varphi, \omega_{\varphi^*}\rangle = 1. \tag{8.25}$$

Again, it induces a linear form

$$\eta_\varphi \,:\, D_{\mathrm{dR}}(V_{\varphi^*}^{\mathrm{quo}}) \to F_{\mathfrak{p}}, \quad \omega \mapsto \langle \eta_\varphi, \omega\rangle \tag{8.26}$$

The image of the restriction of (8.26) to $D(T_{\varphi^*}^{\mathrm{quo}})$ is given in terms of the congruence ideal attached to f, and more precisely, making use of Assumption 8.1.1 and according to [KLZ17, Section 10.1.2], there is an isomorphism

$$\eta_\varphi \,:\, D(T_{\varphi^*}^{\mathrm{quo}}) \longrightarrow \varpi_{\mathfrak{p}}^{-r}\mathcal{O}_{\mathfrak{p}}, \quad \omega \mapsto \langle \eta_\varphi, \omega\rangle.$$

In particular, this means that $\omega_{\varphi^*}/\varpi_{\mathfrak{p}}^r$ is a generator of $D(T_{\varphi^*}^{\mathrm{quo}})$, which is a free rank one module over $\mathcal{O}_{\mathfrak{p}}$. We may reformulate this by saying that $\varpi_{\mathfrak{p}}^r \cdot \eta_\varphi$ induces an isomorphism

$$\varpi_{\mathfrak{p}}^r \cdot \eta_\varphi \,:\, D(T_{\varphi^*}^{\mathrm{quo}}) \longrightarrow \mathcal{O}_{\mathfrak{p}}, \quad \omega \mapsto \langle \varpi_{\mathfrak{p}}^r \eta_\varphi, \omega \rangle.$$

If φ is cuspidal, let $\alpha_\varphi \in \mathcal{O}_{\mathfrak{p}}^\times$ denote the unit root of the p-th Hecke polynomial of f; if $\varphi = E_2(\chi_1, \chi_2)$ is Eisenstein, set $\alpha_\varphi = \chi_1(p)$. Let $\psi_\varphi : G_{\mathbb{Q}_p} \longrightarrow \mathcal{O}_{\mathfrak{p}}^\times$ denote the unramified character characterized by $\psi_\varphi(\mathrm{Fr}_p) = \alpha_\varphi$, so that there is an isomorphism of $\mathcal{O}_{\mathfrak{p}}[G_{\mathbb{Q}_p}]$-modules $T_\varphi^{\mathrm{quo}} \simeq \mathcal{O}_{\mathfrak{p}}(\psi_\varphi)$.

As already mentioned in (8.4), congruence $f \equiv E_2(\theta, 1) \bmod \mathfrak{p}^t$ implies that there is an isomorphism

$$\pi^- \,:\, T_f^{\mathrm{quo}} \otimes \mathcal{O}/\mathfrak{p}^t \simeq \mathcal{O}/\mathfrak{p}^t(\theta). \tag{8.27}$$

Since (8.27) is equivalent to the congruence $\psi_f \equiv \theta \,(\mathrm{mod}\, \mathfrak{p}^t)$ as unramified characters of $G_{\mathbb{Q}_p}$, the choice of π^- may be regarded as the class $\pmod{\mathfrak{p}^t}$ of an isomorphism $T_f^{\mathrm{quo}} \simeq \mathcal{O}_{\mathfrak{p}}(\psi_f)$. Note that our running hypothesis (H1) implies that $\theta_{|G_{\mathbb{Q}_p}}$ is non-trivial. Hence (8.27) induces an isomorphism $D(\pi^-) : D(T_f^{\mathrm{quo}}) \otimes \mathcal{O}/\mathfrak{p}^t \simeq D(\mathcal{O}_{\mathfrak{p}}(\theta)) \otimes \mathcal{O}/\mathfrak{p}^t$ and conversely π^- is uniquely determined by $D(\pi^-)$.

Although π^- is non-canonical a priori, it can be rigidified by asking (8.27) to be the single map $\pi^- : T_f^{\mathrm{quo}} \otimes \mathcal{O}/\mathfrak{p}^t \simeq \mathcal{O}/\mathfrak{p}^t(\theta)$ making the following diagram commutative:

$$\begin{tikzcd} D(T_f^{\mathrm{quo}}) \otimes \mathcal{O}/\mathfrak{p}^t \arrow[r, "\langle\,,\varpi_{\mathfrak{p}}^r \eta_{f^*}\rangle"] \arrow[d, "D(\pi^-)"] & \mathcal{O}/\mathfrak{p}^t \\ D(\mathcal{O}_{\mathfrak{p}}(\theta)) \otimes \mathcal{O}/\mathfrak{p}^t \arrow[ur, "\cdot 1/\mathfrak{g}(\theta)"'] & \end{tikzcd} \tag{8.28}$$

Indeed, since both $\langle\,, \varpi_{\mathfrak{p}}^r \eta_{f^*}\rangle$ and $\cdot 1/\mathfrak{g}(\theta)$ are isomorphisms, it follows that such a map $D(\pi^-)$ exists and is unique, and this in turn pins down η in light of (8.21).

Coleman's power series and the Kubota-Leopoldt p-adic L-function

We now recast the Kubota–Leopoldt p-adic L-function as envisioned by Coleman in [Co79].

Let $\mathcal{W}$ denote weight space, namely the formal spectrum of the Iwasawa algebra $\Lambda = \mathcal{O}_{\mathfrak{p}}[[\mathbb{Z}_p^\times]]$. Let $\mathcal{W}^{\mathrm{cl}}$ denote the set of arithmetic points of $\mathcal{W}$ given by homomorphisms of the form $\nu_{s,\xi}(z) = \xi(s)z^{s-1}$ where $s \in \mathbb{Z}$ is an integer and ξ is a Dirichlet character of p-power conductor; if ξ is trivial we just write ν_s. Let $\mathcal{W}^\circ$ further denote the set of arithmetic points with $\xi = 1$; we shall often write s in place of ν_s. Let

$$\underline{\varepsilon}_{\mathrm{cyc}} : G_{\mathbb{Q}} \to \Lambda^\times$$

denote the Λ-adic cyclotomic character which sends a Galois element σ to the group-like element $[\varepsilon_{\mathrm{cyc}}(\sigma)]$. It interpolates the powers of the $\mathbb{Z}_p$-cyclotomic character, in the sense that for any arithmetic point $\nu_{s,\xi} \in \mathcal{W}^{\mathrm{cl}}$,

$$\nu_{s,\xi} \circ \underline{\varepsilon}_{\mathrm{cyc}} = \xi \cdot \varepsilon_{\mathrm{cyc}}^{s-1}. \tag{8.29}$$

The following result follows from the general theory of Perrin-Riou maps (see for instance [KLZ17, Section 8]).

Proposition 8.3.8. *There exists a morphism of Λ-modules*

$$\mathcal{L}_\chi : H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\chi) \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}})) \to \Lambda$$

satisfying that for all integers r, the specialization of $\mathcal{L}_\chi$ at $s \in \mathcal{W}^\circ$ is the homomorphism

$$\mathcal{L}_{\chi,s} : H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\chi)(s)) \to \mathcal{O}_{\mathfrak{p}}$$

given by

$$\mathcal{L}_{\chi,s} = \frac{1-\bar{\chi}(p)p^{-s}}{1-\chi(p)p^{s-1}} \cdot \begin{cases} \frac{(-1)^s}{(s-1)!} \cdot \langle \log_{\mathrm{BK}}, \frac{1}{\mathfrak{g}(\chi)} \rangle & \text{if } s \geq 1 \\ (-s)! \cdot \langle \exp^*_{\mathrm{BK}}, \frac{1}{\mathfrak{g}(\chi)} \rangle & \text{if } s < 1, \end{cases}$$

As a piece of notation, and for any p-adic representation V, we write $H^1_{\mathrm{f}}(\mathbb{Q}, V)$ for the *finite* Bloch–Kato Selmer group, which is the subspace of $H^1(\mathbb{Q}, V)$ which consists on those classes which are crystalline at p and unramified at $\ell \neq p$.

The following result is a reformulation of Coleman and Perrin-Riou's reciprocity law ([Co79], [PR94]), with the normalizations used for instance in [Tale14].

Proposition 8.3.9. *There exists a* Λ*-adic cohomology class*

$$\kappa_{\chi,\infty} \in H^1(\mathbb{Q}, \mathcal{O}_{\mathfrak{p}}(\chi) \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}))$$

such that:

(a) *Its image under restriction at* p *followed by the Perrin–Riou regulator gives the Kubota–Leopoldt* p*-adic* L*-function:*
$$\mathcal{L}_{\chi}(\mathrm{res}_p(\kappa_{\chi,\infty})) = L_p(\bar{\chi}, s).$$

(b) *The bottom layer* $\kappa_{\chi}(1) := \nu_1(\kappa_{\chi,\infty})$ *lies in* $H^1_{\mathrm{f}}(\mathbb{Q}, \mathcal{O}_{\mathfrak{p}}(\chi)(1))$ *and satisfies*
$$\kappa_{\chi}(1) = (1-\chi(p)) \cdot c_{\chi}.$$

Kato's explicit reciprocity law

Proposition 8.3.10. *There exists a homomorphism of* Λ*-modules*

$$\mathcal{L}_f^- : H^1(\mathbb{Q}_p, T_f^{\mathrm{quo}} \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}})) \to \mathcal{I}_f$$

satisfying the following interpolation property: for $s \in \mathcal{W}^{\circ}$*, the specialization of* $\mathcal{L}_f^-$ *at* s *is the homomorphism*

$$\mathcal{L}_{f,s}^- : H^1(\mathbb{Q}_p, T_f^{\mathrm{quo}}(s)) \longrightarrow \mathcal{O}_{\mathfrak{p}}$$

given by

$$\mathcal{L}_{f,s}^- = \frac{1-\bar{\theta}(p)\beta_f p^{-s-1}}{1-\theta(p)\beta_f^{-1}p^s} \times \begin{cases} \frac{(-1)^s}{(s-1)!} \times \langle \log_{\mathrm{BK}}, \eta_{f^*} \rangle & \text{if } s \geq 1 \\ (-s)! \times \langle \exp^*_{\mathrm{BK}}, \eta_{f^*} \rangle & \text{if } s < 1, \end{cases}$$

where $\log_{\mathrm{BK}}$ *is the Bloch-Kato logarithm and* $\exp^*_{\mathrm{BK}}$*, the dual exponential map.*

Proof. This follows from Coleman and Perrin-Riou's theory of Λ-adic logarithm maps as extended by Loeffler and Zerbes in [LZ14]. This is recalled for instance in [KLZ17, Sections 8,9]. More precisely, [KLZ17, Theorem 8.2.3] and, more particularly, the second displayed equation in [KLZ17, p. 82] yields an injective map

$$H^1(\mathbb{Q}_p, T_f^{\mathrm{quo}} \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}})) \longrightarrow D(T_f^{\mathrm{quo}}) \otimes \Lambda,$$

since $H^0(\mathbb{Q}_p, T_f^{\mathrm{quo}}(1)) = 0$ because of the assumption that $\alpha_f \equiv \theta(p) \neq 1$ modulo $\mathfrak{p}$.

This map is characterized by the interpolation property formulated in [LZ14, Appendix B]. Next we apply the pairing of (8.26) and the result follows. □

Theorem 8.3.11. *There exists a* Λ*-adic cohomology class*

$$\kappa_{f,\infty} \in H^1(\mathbb{Q}, T_f \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}))$$

such that:

(a) *There is an explicit reciprocity law*

$$\mathcal{L}_f^-(\mathrm{res}_p(\kappa_{f,\infty})^-) = \frac{N\mathfrak{g}(\theta\chi_1\bar{\chi}_2)\cdot L_p(f^*,\bar{\chi}_1\chi_2,1)}{2i\varpi^r\mathfrak{g}(\chi_1\bar{\chi}_2)} \times L_p(f^*,1+s),$$

where res_p *stands for the map corresponding to localization at* p *and* $\mathrm{res}_p(\kappa_{f,\infty})^-$ *is the map induced in cohomology from the projection map* $T_f \to T_f^{\mathrm{quo}}$ *of* (8.11).

(b) *The bottom layer* $\kappa_f(1)$ *lies in* $H^1_{\mathrm{f}}(\mathbb{Q},T_f(1))$ *and satisfies*

$$\kappa_f(1) = \mathcal{E}_f \cdot \kappa_f,$$

where $\mathcal{E}_f$ *is the Euler factor introduced in* (8.16).

Proof. This is due to Kato [Ka04] and has been reported in many other places in the literature. See [Och06] and, more specifically, [BD14, Theorems 4.4 and 5.1] combined with Besser's [Bes00, Proposition 9.11 and Corollary 9.10] showing that the p-adic regulator can be recast as the compostion of the p-adic étale regulator followed by the Bloch–Kato logarithm. □

Remark 8.3.12. The value $L_p(f^*,\bar{\chi}_1\chi_2,1)$ depends on the period Ω_f^+, fixed around the discussion of Remark 8.3.3.

Recall our running assumption that $f \equiv E_2(\theta,1) \pmod{\mathfrak{p}^t}$.

Corollary 8.3.13. *The following equality holds in* $\Lambda/\mathfrak{p}^t\Lambda$*:*

$$\varpi_{\mathfrak{p}}^r\mathcal{L}_f^-(\mathrm{res}_p\,\kappa_{f,\infty}^-) \equiv (-i)N\frac{\mathfrak{g}(\theta\chi_1\bar{\chi}_2)}{\mathfrak{g}(\chi_1\bar{\chi}_2)}\zeta_p(-1)\cdot L_p(f^*,\bar{\chi}_1\chi_2,1)\cdot\mathcal{L}_\theta(\mathrm{res}_p\,\kappa_{\theta,\infty}) \pmod{\mathfrak{p}^t}.$$

Proof. By Proposition 8.3.6,

$$L_p(f^*,1+s) \equiv 2\cdot\zeta_p(-1)\cdot L_p(\bar{\theta},1+s) \pmod{\mathfrak{p}^t}.$$

Applying now part (a) of Theorem 8.3.11 and Proposition 8.3.9 to the left and right hand sides respectively, the result follows. □

Discussion of Theorem 8.3.1

We discuss a path to the proof of Theorem 8.3.1 under the different assumptions we have made along the chapter. Note that since $\theta_{|G_{\mathbb{Q}_p}}$ is a non-trivial unramified character, it follows from e.g. [Bel09, §2.2] that

$$H^1(\mathbb{Q}_p,\mathcal{O}_{\mathfrak{p}}(\theta)(1)) = H^1_{\mathrm{f}}(\mathbb{Q}_p,\mathcal{O}_{\mathfrak{p}}(\theta)(1)).$$

We have seen in Proposition 8.3.8 that there is a homomorphism

$$\mathcal{L}_{\theta,1}: H^1(\mathbb{Q}_p,\mathcal{O}_{\mathfrak{p}}(\theta)(1)) \to L_{\mathfrak{p}}. \tag{8.30}$$

Lemma 8.3.14. *The map* $\mathcal{L}_{\theta,1}$ *induces an isomorphism*

$$\mathcal{L}_{\theta,1}: H^1(\mathbb{Q}_p,\mathcal{O}_{\mathfrak{p}}(\theta)(1)) \to \mathcal{O}_{\mathfrak{p}}.$$

Proof. According to Proposition 8.3.8, the map (8.30) is given by

$$\mathcal{L}_{\theta,1} = \frac{\bar{\theta}(p)p^{-1}-1}{1-\theta(p)}\cdot\langle\log_{\mathrm{BK}},\frac{t}{\mathfrak{g}(\theta)}\rangle.$$

Given a place v of $\mathbb{Q}(\mu_N)$ above p, let $\mathbb{Z}[\mu_N]_v$ denote the completion of $\mathbb{Z}[\mu_N]$ at v. Define the module of local units $U_p(N) = \prod_{v|p}\mathbb{Z}[\mu_N]_v^\times$, where $v = v_1,\dots,v_r$ ranges over all places of $\mathbb{Q}(\mu_N)$

above p. Note that $G = \mathrm{Gal}\,(\mathbb{Q}(\mu_N)/\mathbb{Q})$ acts on $U_p(N)$ by permuting the places v, and hence it makes sense to pick the eigen-component of $U_p(N)$ with respect to a character of G. In particular, we have

$$U_p(N)[\bar\theta] := (U_p(N) \otimes \mathcal{O}_{\mathfrak{p}}(\theta))^G.$$

Kummer theory identifies $H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\theta)(1))$ with $U_p(N)[\bar\theta]$, which is a $\mathcal{O}_{\mathfrak{p}}$-module of rank one.

Since $\mathbb{Q}(\mu_N)_v$ is an unramified extension of $\mathbb{Q}_p$, the maximal ideal of $\mathbb{Z}[\mu_N]_v$ is $p\mathbb{Z}[\mu_N]_v$ and the logarithm defines an isomorphism, as recalled for instance in [Con, §8]

$$\log_v\,:\,\mathbb{Z}[\mu_N]_v^\times \otimes \mathcal{O}_{\mathfrak{p}} \longrightarrow p\mathbb{Z}[\mu_N]_v \otimes \mathcal{O}_{\mathfrak{p}}. \tag{8.31}$$

Note that $\prod_v \mathbb{Z}[\mu_N]_v$ is naturally a G-module isomorphic to the regular representation and hence $(\prod_v \mathbb{Z}[\mu_N]_v)[\bar\theta]$ is again a free module of rank 1 over $\mathcal{O}_{\mathfrak{p}}$. Define

$$\log_{\bar\theta} := \sum_{\sigma\in G} \theta(\sigma)\log_{\sigma(v_1)}\,:\,U_p(N)[\bar\theta] \longrightarrow p(\prod_v \mathbb{Z}[\mu_N]_v)[\bar\theta].$$

A natural generator of the target may be taken to be the Gauss sum $\mathfrak{g}(\theta)$ diagonally embedded in $\prod_v \mathbb{Z}[\mu_N]_v$ and this yields an identification $\frac{1}{\mathfrak{g}(\theta)}(\prod_v \mathbb{Z}[\mu_N]_v)[\bar\theta] = \mathcal{O}_{\mathfrak{p}}$. Under these identifications, Bloch-Kato's logarithm may be recast classically as

$$\langle \log_{\mathrm{BK}}, \frac{t}{\mathfrak{g}(\theta)}\rangle = \frac{1}{\mathfrak{g}(\theta)}\log_{\bar\theta} : H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\theta)(1)) = U_p(N)[\bar\theta] \longrightarrow \mathcal{O}_{\mathfrak{p}},$$

and we already argued that this yields an isomorphism onto $p\mathcal{O}_{\mathfrak{p}}$.

Since $\mathrm{ord}_p(\frac{\bar\theta(p)p^{-1}-1}{1-\theta(p)}) = -1$, it follows that $\mathcal{L}_{\theta,1}$ is an isomorphism onto $\mathcal{O}_{\mathfrak{p}}$, as claimed. $\square$

Recall from Proposition 8.3.10 the map

$$\varpi_{\mathfrak{p}}^r \cdot \mathcal{L}_{f,1}^- : H^1(\mathbb{Q}_p, T_f^{\mathrm{quo}}(1)) \to D(T_f^{\mathrm{quo}}(1)) \overset{\cdot t\varpi_{\mathfrak{p}}^r\eta_{f^*}}{\to} \mathcal{O}_{\mathfrak{p}}.$$

Recall the isomorphism $T_f^{\mathrm{quo}} \otimes \mathcal{O}/\mathfrak{p}^t \simeq \mathcal{O}/\mathfrak{p}^t(\theta)$ fixed in (8.28) above and use it to identify the source of $\varpi_{\mathfrak{p}}^r \cdot \mathcal{L}_{f,1}^- \otimes \mathcal{O}_{\mathfrak{p}^t}$ with $H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^t(\theta)(1))$.

Lemma 8.3.15. *As homomorphisms $H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^t(\theta)(1)) \longrightarrow \mathcal{O}/\mathfrak{p}^t$ we have the congruence*

$$\varpi_{\mathfrak{p}}^r \cdot \mathcal{L}_{f,1}^- \equiv \mathcal{L}_{\theta,1} \pmod{\mathfrak{p}^t}.$$

Proof. This follows by comparing the maps $\mathcal{L}_{\theta,1}$ and $\varpi_{\mathfrak{p}}^r \cdot \mathcal{L}_{f,1}^-$ described respectively in Proposition 8.3.8 and 8.3.10. Note firstly that the Euler factors involved in the latter agree modulo $\mathfrak{p}^t$ with those of the former, since $\alpha_f\beta_f = \theta(p)p$.

Next, observe that in Proposition 8.3.8 the pairing takes place against $t\mathfrak{g}(\chi)^{-1}$, while in Proposition 8.3.10 this pairing is with $t\eta_{f^*}$. The lemma follows from the commutativity of the diagram (8.28). $\square$

Lemma 8.3.16. *Assuming hypothesis (H2) in the introduction, the global-to-local restriction map*

$$H^1_{\mathrm{f}}(\mathbb{Q}, \mathcal{O}_{\mathfrak{p}}(\theta)(1)) \to H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\theta)(1))$$

is an isomorphism.

Proof. Recall from the proof of Lemma 8.3.14 the definition of the group $U_p(N)$ of local units. Consider the following commutative diagram, where vertical arrows are isomorphisms induced from Kummer theory and the upper horizontal arrow stands for the map corresponding to localization at p:

$$\begin{tikzcd} H^1_{\mathrm{f}}(\mathbb{Q}, \mathcal{O}_{\mathfrak{p}}(\theta)(1)) \arrow[r] & H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\theta)(1)) \\ \mathbb{Z}[\mu_N]^\times[\bar\theta] \otimes \mathcal{O}_{\mathfrak{p}} \arrow[r] \arrow[u] & U_p(N)[\bar\theta] \otimes \mathcal{O}_{\mathfrak{p}} \arrow[u] \end{tikzcd}$$

The bottom horizontal arrow is injective because it is induced by the natural inclusion $\mathbb{Z}[\mu_N]^\times \hookrightarrow U_p(N)$. Moreover, since θ is even and nontrivial, both $\mathbb{Z}[\mu_N]^\times[\bar\theta] \otimes \mathcal{O}_{\mathfrak{p}}$ and $U_p(N)[\bar\theta]$ are $\mathcal{O}_{\mathfrak{p}}$-modules of rank 1. The cokernel

$$Q[\bar\theta] = U_p(N)[\bar\theta] \otimes \mathcal{O}_{\mathfrak{p}} / \mathbb{Z}[\mu_N]^\times[\bar\theta] \otimes \mathcal{O}_{\mathfrak{p}}$$

is thus a finite group.

In order to prove the lemma it thus suffices to show that $Q[\bar\theta]$ is trivial. Write $k = \mathbb{Q}(\mu_N)^+$ (resp. $\mathbb{Z}[\mu_N]^+$) for the maximal totally real subfield of $\mathbb{Q}(\mu_N)$ (resp. its ring of integers). Let $U_p^1(N) = \prod_{v|p} (\mathbb{Z}[\mu_N]_v^+)^1$, where $(\mathbb{Z}[\mu_N]_v^+)^1$ stands for the set of local units in $\mathbb{Z}_p[\mu_N]_v^+$ which are congruent to 1 modulo v. Let $U^+(N)$ denote the closure of the set of units of $\mathbb{Z}[\mu_N]^+$ congruent to 1 modulo each place above p, diagonally embedded in $U_p^1(N)$. Note that $Q[\bar\theta] = U_p^1(N)/U^+(N)$.

According to [Neu69, Chapter 4, Theorem 7.8], $Q[\bar\theta] \simeq \mathrm{Gal}\,(H_p/H)[\bar\theta]$, where H_p (resp. H) is the maximal p-abelian extension of k unramified away from primes above p (resp. everywhere unramified). Here the θ-eigencomponent on the Galois group is taken with respect to the natural action of $\mathrm{Gal}\,(k/\mathbb{Q})$ by conjugation on $\mathrm{Gal}\,(H_p/H)$. The lemma hence follows from the running hypothesis (H2) –see also the footnote in loc. cit. $\square$

We are finally in position to provide the *proof of Theorem 8.3.1:* After specializing Corollary 8.3.13 at $s = 1$ we obtain

$$\varpi_{\mathfrak{p}}^r \cdot \mathcal{L}_{f,1}^-(\mathrm{res}_p\, \kappa_f(1)^-) \equiv -iN \frac{\mathfrak{g}(\theta\chi_1\bar\chi_2)}{\mathfrak{g}(\chi_1\bar\chi_2)} \zeta_p(-1) \cdot L_p(f^*, \bar\chi_1\chi_2, 1) \cdot \mathcal{L}_{\theta,1}(\mathrm{res}_p\, \kappa_\theta(1)) \pmod{\mathfrak{p}^t}$$

Recall that Proposition 8.3.9 and Theorem 8.3.11 assert that

$$\kappa_\theta(1) = (1 - \theta(p)) \cdot c_\theta, \quad \kappa_f(1) = \mathcal{E}_f \cdot \kappa_f$$

and hence

$$\varpi_{\mathfrak{p}}^r \cdot \mathcal{E}_f \mathcal{L}_{f,1}^-(\mathrm{res}_p\, \kappa_f^-) \equiv -iN \frac{\mathfrak{g}(\theta\chi_1\bar\chi_2)}{\mathfrak{g}(\chi_1\bar\chi_2)} \zeta_p(-1)(1 - \theta(p)) \cdot L_p(f^*, \bar\chi_1\chi_2, 1) \cdot \mathcal{L}_{\theta,1}(\mathrm{res}_p\, c_\theta) \pmod{\mathfrak{p}^t}$$

Recall we have set

$$\ell = -iN \cdot \frac{\mathfrak{g}(\theta\chi_1\bar\chi_2)}{\mathfrak{g}(\theta)\mathfrak{g}(\chi_1\bar\chi_2)} \cdot (1 - \theta(p)) \cdot \zeta_p(-1) \cdot L_p(f^*, \bar\chi_1\chi_2, 1).$$

Using Lemma 8.3.15 together with Lemma 8.3.14, we deduce the equality of local classes

$$\mathcal{E}_f \cdot \mathrm{res}_p\, \kappa_f^- \equiv \ell \cdot \mathrm{res}_p\, c_\theta \pmod{\mathfrak{p}^t} \tag{8.32}$$

in $H^1(\mathbb{Q}_p, \mathcal{O}/\mathfrak{p}^t(\theta)(1))$. Observe that $\mathrm{res}_p(\kappa_f)^-$ is the local class obtained in cohomology by push-forward under the map induced by the projection $\bar T_f \to \bar T_f^{\mathrm{quo}}$ of (8.11), as already introduced in Theorem 8.3.11. This corresponds, modulo $\mathfrak{p}^t$, to what we have called $\bar\kappa_{f,1}$.

Lemma 8.3.16 allows us to upgrade (8.32) to an equality of global classes in $H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\theta)(1))$, namely

$$\mathcal{E}_f \cdot \kappa_{f,1} \equiv \ell \cdot c_\theta \pmod{\mathfrak{p}^t}.$$

Theorem 8.3.1 follows.

8.4 Second congruence relation

As in the introduction, let $N > 1$ be a positive integer, $\theta : (\mathbb{Z}/N\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ an even Dirichlet character and $f \in S_2(N, \theta)$ a normalized cuspidal eigenform of level N, weight 2 and nebentype θ. Fix a prime $p \nmid 6N\varphi(N)$ and assume as in (8.2) that $f \equiv E_2(\theta, 1) \bmod \mathfrak{p}^t$ for some $t \geq 1$. Recall this implies that $L_p'(\bar{\theta}, -1) \equiv 0 \pmod{\mathfrak{p}^t}$.

We keep the notations introduced along Section 8.2 and Section 8.3. Recall that the value of the modular unit u_{χ_1,χ_2} at ∞ is some power of the circular unit c_{χ_1}, and likewise for u_{ξ_1,ξ_2}. For the sake of concreteness, in the statement below we normalize them so that $u_{\chi_1,\chi_2}(\infty) = c_{\chi_1}$ and $u_{\xi_1,\xi_2}(\infty) = c_{\xi_1}$ although any other normalization would work.

For the following result, we suppose that Assumption 8.1.4 in the introduction holds, since the ideas we discuss clearly rely on the results established in [FK12, Section 9], which are developed for the cohomology of the closed modular curve. The validity of this hypothesis is discussed in 4.4.1 of loc. cit. for a different kind of Beilinson–Kato elements, which are essentially those of the form $u_{\psi,1}$ (see also the discussion of Section 3.3, and specially Proposition 3.3.14).

Expected Theorem 8.4.1. (Second congruence relation) Suppose $L_p'(\bar{\theta}, -1) \not\equiv 0 \pmod{\mathfrak{p}}$ and that Assumption 8.1.4 holds. Then the following equality holds in $H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(2))$:

$$\bar{\kappa}_{f,2} = \frac{L_p'(\bar{\theta}, -1)}{1 - p^{-1}} \cdot \frac{\bar{c}_{\xi_1} \cup \bar{c}_{\chi_1}}{\cup \log_p(\varepsilon_{\mathrm{cyc}})} \pmod{\mathfrak{p}^t}.$$

Here, $1/\cup \log_p(\varepsilon_{\mathrm{cyc}})$ denotes the inverse of the map

$$H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(2)) \to H^2(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(2)), \quad \kappa \mapsto \kappa \cup \log_p(\varepsilon_{\mathrm{cyc}}),$$

which is invertible under our running assumptions.

Cohomology and Eisenstein quotients

For any $r \geq 1$ and $j \in \mathbb{Z}$ let

$$H_r(j) = H^1_{\mathrm{et}}(\bar{X}_1(Np^r), \mathcal{O}_\mathfrak{p}(j))^{\mathrm{ord}}$$

denote the ordinary component of the étale cohomology group $H^1_{\mathrm{et}}(\bar{X}_1(Np^r), \mathcal{O}_\mathfrak{p}(j))$ with respect to the Hecke operator U_p. This is naturally an $\mathcal{O}_\mathfrak{p}[G_\mathbb{Q}]$-module and we may simply denote it H_r when the Galois action is understood or irrelevant. Let $\mathfrak{h}_r$ be the subring of $\mathrm{End}_{\mathcal{O}_\mathfrak{p}}(H_r)$ spanned over $\mathcal{O}_\mathfrak{p}$ by the Hecke operators T_n, $(n, N) = 1$. The Eisenstein ideal $I_r = I_{\mathrm{Eis},r} \subset \mathfrak{h}_r$ is the $\mathcal{O}_\mathfrak{p}$-submodule of $\mathfrak{h}_r$ generated by $U_p - 1$ and $T_\ell - \ell\langle \ell \rangle - 1$, for primes $\ell \nmid Np$; here, $\langle \ell \rangle$ stands for the usual diamond operator.

Passing to the projective limit we may define:

$$H(j) := H^1_{\mathrm{et}}(\bar{X}_1(Np^\infty), \mathcal{O}_\mathfrak{p}(j))^{\mathrm{ord}} = \varprojlim H_r(j), \quad \mathfrak{h} = \varprojlim \mathfrak{h}_r, \quad I = \varprojlim I_r \subset \mathfrak{h}.$$

The ideal I is a height one ideal contained in the maximal ideal $\mathfrak{M} = (I, \mathfrak{p})$; for any $t \geq 1$ we shall denote $\mathfrak{M}^{(t)} = (I, \mathfrak{p}^t)$, so that $\mathfrak{M} = \mathfrak{M}^{(1)}$. The ideal I is the intersection of a finite number of height one prime ideals $P \subset \mathfrak{M}$, each of which corresponds to a weight two eigenform that is congruent to an Eisenstein series mod $\mathfrak{p}$, like the modular form f of the introduction.

Let $\Lambda_N := \lim_\leftarrow \mathcal{O}_\mathfrak{p}[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$ denote the Iwasawa algebra of tame level N. Any Dirichlet character $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathcal{O}_\mathfrak{p}^\times$ may be extended by linearity to yield a homomorphism

$$\psi : \Lambda_N \longrightarrow \mathcal{O}_\mathfrak{p}[(\mathbb{Z}/Np\mathbb{Z})^\times] \longrightarrow \mathcal{O}_\mathfrak{p}$$

that we continue to denote with the same symbol.

For any Λ_N-module M let $M_\psi = M \otimes_{\Lambda_N,\psi} \mathcal{O}_\mathfrak{p}$ stand for the associated ψ-isotypical component. Note that $\Lambda_N = \oplus \Lambda_{N,\psi}$ where ψ ranges over all characters of $(\mathbb{Z}/Np\mathbb{Z})^\times$ and $\Lambda_{N,\psi} \simeq \mathcal{O}_\mathfrak{p}[[1+p\mathbb{Z}_p]]$.

We begin by rephrasing the results of [FK12, §9] on Sharifi's conjecture in a convenient way for our purposes. As we have seen at the beginning of Section 8.3, and more precisely in the discussion before equation (8.14), the Hochschild–Serre spectral sequence in étale cohomology yields the commutative diagram of $\Lambda_{N,\theta}[G_\mathbb{Q}]$-modules:

$$\begin{tikzcd} H^2_{\mathrm{et}}(X_1(Np^\infty), \mathcal{O}_\mathfrak{p}(2))^{\mathrm{ord}}_\theta \arrow[r] \arrow[d] & H^2_{\mathrm{et}}(X_1(Np), \mathcal{O}_\mathfrak{p}(2))^{\mathrm{ord}}_\theta \arrow[d] \\ H^1(\mathbb{Q}, H(2)_\theta) \arrow[r] & H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\bar{X}_1(Np), \mathcal{O}_\mathfrak{p}(2))^{\mathrm{ord}}_\theta). \end{tikzcd} \tag{8.33}$$

The module $H(2)$ is endowed with an action of complex conjugation, yielding the decomposition $H(2) = H(2)^+ \oplus H(2)^-$; in the sequel we shall employ a similar notation for any $\mathcal{O}_\mathfrak{p}$-module acted on by complex conjugation. It follows from [FK12, Prop. 6.3.2] that the quotient

$$(H(2)/\mathfrak{M}^{(t)})^+_\theta \tag{8.34}$$

is still endowed with a compatible action of $G_\mathbb{Q}$.

Our running assumptions imply that the (mod $\mathfrak{p}^t$) Galois representation $\bar{T}_f(1) = T_f(1) \otimes \mathcal{O}/\mathfrak{p}^t$ arises as a quotient of $H(2)/\mathfrak{M}^{(t)}$ as $\mathcal{O}/\mathfrak{p}^t[G_\mathbb{Q}]$-modules. Since the nebentype of f is θ, it belongs to the θ-isotypical component of the latter. Denote

$$\pi_f : (H(2)/\mathfrak{M}^{(t)})^+_\theta \longrightarrow \bar{T}_f(1)$$

the resulting projection.

Recall also from (8.4) that there is an isomorphism $\bar{T}_f(1) \simeq \mathcal{O}/\mathfrak{p}^t(\theta)(1) \oplus \mathcal{O}/\mathfrak{p}^t(2)$ as $\mathcal{O}/\mathfrak{p}^t[G_\mathbb{Q}]$-modules. Since θ is even and the cyclotomic character is odd, it follows that the $+$-component cuts out the second factor, that is to say: $\bar{T}_f(1)^+ \simeq \mathcal{O}/\mathfrak{p}^t(2)$ and this is naturally a quotient of (8.34). Henceforth we fix the canonical isomorphism provided by [FK12, 6.3.18 and 7.1.11] in order to identify

$$\bar{T}_f(1)^+ = \mathcal{O}/\mathfrak{p}^t(2) \tag{8.35}$$

as $\mathcal{O}/\mathfrak{p}^t[G_\mathbb{Q}]$-modules.

Summing up there is a commutative diagram of $G_\mathbb{Q}$-modules, where the horizontal arrows arise from specializing to $r = 1$, i.e. level Np:

$$\begin{tikzcd} H^1(\mathbb{Q}, H(2)_\theta) \arrow[r] \arrow[d] & H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\bar{X}_1(Np), \mathcal{O}_\mathfrak{p}(2))^{\mathrm{ord}}_\theta) \arrow[d] \\ H^1(\mathbb{Q}, (H(2)/\mathfrak{M}^{(t)})^+_\theta) \arrow[r] \arrow[dr] & H^1(\mathbb{Q}, (H^1_{\mathrm{et}}(\bar{X}_1(Np), \mathcal{O}_\mathfrak{p}(2))^{\mathrm{ord}}/(I_1, \mathfrak{p}^t))^+_\theta) \arrow[d, "\pi_f^+"] \\ & H^1(\mathbb{Q}, \bar{T}_f(1)^+) = H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(2)) \end{tikzcd} \tag{8.36}$$

Fukaya–Kato maps

Define the module $\mathcal{Q}$ as in [FK12, §6.3.1], with a twist by the square of the cyclotomic character, namely

$$\mathcal{Q} := (H(2)/IH(2))^+_\theta.$$

Recall from Section 8.3 (cf. e.g. Prop. 8.3.9) the Kubota–Leopoldt p-adic L-function $L_p(\bar\theta) \in \Lambda_{N,\theta}$ attached to the Dirichlet character $\bar\theta$. As shown in [FK12, §6.1.7], there is an isomorphism of Galois modules

$$(\mathfrak{h}/I)_\theta \simeq \Lambda_{N,\theta}/(L_p(\bar\theta))(\underline{\varepsilon}_{\mathrm{cyc}}),$$

where $\underline{\varepsilon}_{\mathrm{cyc}}$ is the Λ-adic cyclotomic character introduced in (8.29) and the identification follows from [FK12, §6.3], with the conventions recalled in §9.1.2 and 9.1.4. Moreover, and as a consequence of the proof of the Iwasawa main conjecture by Mazur and Wiles, in §6.3.18 of loc. cit. the authors show that there are isomorphisms of Galois modules

$$\mathcal{Q} \xrightarrow{\simeq} \Lambda_{N,\theta}/(L_p(\bar\theta))(\underline{\varepsilon}_{\mathrm{cyc}})(2) \simeq (\mathfrak{h}(2)/I\mathfrak{h}(2))_\theta. \tag{8.37}$$

Let $H^i(\mathbb{Z}[1/Np], M) \subset H^i(\mathbb{Q}, M)$ stand for the set of classes which are unramified at primes dividing Np. Recall that Shapiro's lemma gives an isomorphism

$$\varprojlim H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathcal{O}_{\mathfrak{p}}(2)) \simeq H^2(\mathbb{Z}[1/Np], \Lambda_N(\underline{\varepsilon}_{\mathrm{cyc}})(2)).$$

As a piece of notation, and following the definition of [FK12, §5.2.6], set

$$\mathcal{S} = \varprojlim H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathcal{O}_{\mathfrak{p}}(2))^+ \simeq H^2(\mathbb{Z}[1/Np], \Lambda_N(\underline{\varepsilon}_{\mathrm{cyc}})(2))^+,$$

where again the plus sign stands for the $(+1)$-eigenspace under the action of complex conjugation.

In [FK12, §9.1] Fukaya and Kato established the existence of isomorphisms

$$\mathrm{FK}_1 : H^1(\mathbb{Z}[1/Np], \mathcal{Q}) \simeq \mathcal{S}_\theta, \quad \mathrm{FK}_2 : \mathcal{S}_\theta \simeq H^2(\mathbb{Z}[1/Np], \mathcal{Q})$$

arising from the long exact sequence in cohomology induced by the short exact sequence

$$0 \to \Lambda_{N,\theta}(\underline{\varepsilon}_{\mathrm{cyc}})(2) \xrightarrow{\cdot L_p(\bar\theta)} \Lambda_{N,\theta}(\underline{\varepsilon}_{\mathrm{cyc}})(2) \to \mathcal{Q} \to 0.$$

stemming from (8.37).

In particular, the map we have denoted as FK_2 is just the $+$-component of the homomorphism

$$H^2(\mathbb{Z}[1/Np], \Lambda_{N,\theta}(\underline{\varepsilon}_{\mathrm{cyc}})(2)) \longrightarrow H^2(\mathbb{Z}[1/Np], \Lambda_{N,\theta}/(L_p(\bar\theta))(\underline{\varepsilon}_{\mathrm{cyc}})(2)) \tag{8.38}$$

induced by (a twist of) the natural projection $\Lambda_{N,\theta} \longrightarrow \Lambda_{N,\theta}/L_p(\bar\theta)$. For an explicit description of FK_1, see [FK12, §9.1].

The main result of §9.2 of loc. cit. asserts that the map

$$\mathrm{ev}_\infty : H^2_{\mathrm{et}}(X_1(Np^\infty), \mathcal{O}_{\mathfrak{p}}(2))_\theta \longrightarrow \varprojlim H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathcal{O}_{\mathfrak{p}}(2))_\theta \tag{8.39}$$

induced by evaluation at the cusp ∞ factors through the Eisenstein quotient, as stated below.

Proposition 8.4.2 (Fukaya–Kato)**.** *The map* ev_∞ *of* (8.39) *agrees with the composition*

$$H^2_{\mathrm{et}}(X_1(Np^\infty), \mathcal{O}_{\mathfrak{p}}(2))_\theta \to H^1(\mathbb{Z}[1/Np], H(2)_\theta) \to H^1(\mathbb{Z}[1/Np], \mathcal{Q}) \simeq \mathcal{S}_\theta$$

where:

- *the first map is the composition of* $H^2(X_1(Np^\infty), \mathcal{O}_{\mathfrak{p}}(2)) \to H^2(X_1(Np^\infty), \mathcal{O}_{\mathfrak{p}}(2))^{\mathrm{ord}}_\theta$ *and the left vertical arrow in* (8.33)*, both restricted to the subspace of classes unramified at the primes dividing* Np*;*
- *the second map is induced by the projection* $H(2)_\theta \to \mathcal{Q}$*;*
- *the last isomorphism is* FK_1.

In [FK12, §9.3] Fukaya and Kato further introduced two distinguished morphisms

$$a, b : H^1(\mathbb{Z}[1/Np], \mathcal{Q}) \to H^2(\mathbb{Z}[1/Np], \mathcal{Q}) \tag{8.40}$$
$$a = \mathrm{FK}_2 \circ \mathrm{FK}_1,$$
$$b = \cup\, (1 - p^{-1}) \log_p(\varepsilon_{\mathrm{cyc}})$$

where $\varepsilon_{\mathrm{cyc}} \in H^1(\mathbb{Q}, \mathcal{O}_{\mathfrak{p}}^{\times})$ stands for the cyclotomic character. Note that $(1 - p^{-1}) \log_p$ takes values in $\mathbb{Z}_p$ and hence b is indeed well-defined. Under these conditions, they show the following.

Proposition 8.4.3 (Fukaya–Kato)**.** *Let $L_p'(\bar{\theta}) \in \Lambda_{N,\theta}$ denote the derivative of $L_p(\bar{\theta})$. Then*

$$b = L_p'(\bar{\theta}) \cdot a. \tag{8.41}$$

Proof. This follows from [FK12, Proposition 9.3.1]; in particular, we just need the restriction of θ to $(\mathbb{Z}/N\mathbb{Z})^{\times}$ being primitive, since Lemma 9.1.3 of loc. cit. also works in this setting. □

Corollary 8.4.4. *Assume $L_p'(\bar{\theta}, -1)$ is a p-adic unit. Then the map*

$$H^1(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(2)) \longrightarrow H^2(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(2)), \quad \kappa \mapsto \kappa \cup (1 - p^{-1}) \log_p(\varepsilon_{\mathrm{cyc}})$$

is invertible.

Proof. Observe that FK_1 and FK_2 are Λ-adic isomorphisms, as it has been proved in [FK12, Section 9.1]. Hence, once we consider the specialization map at the trivial character, we still have isomorphisms of $\mathcal{O}_{\mathfrak{p}}$-modules. The same must be true for their composition multiplied by the p-adic unit $L_p'(\bar{\theta}, -1)$, and according to Proposition 8.4.3 and the definitions provided in [FK12, Section 4.1.3], this is precisely the above map. □

After applying the Fukaya–Kato map FK_1 to the bottom row of diagram (8.36), restricting to the subspace of unramified classes at primes dividing Np, we reach the commutative diagram

$$\begin{tikzcd} H^1(\mathbb{Z}[1/Np], (H(2)/\mathfrak{M}^{(t)})^+_{\theta}) \arrow[r] \arrow[d, "\bar{\mathrm{FK}}_1"] & H^1(\mathbb{Z}[1/Np], \bar{T}_f(1)^+) = H^1(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(2)) \arrow[d, "\bar{\mathrm{FK}}_1(r=1)"] \\ H^2(\mathbb{Z}[1/Np], \Lambda_{N,\theta}/\mathfrak{p}^t(\varepsilon_{\mathrm{cyc}})(2)) \arrow[r] & H^2(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(2)), \end{tikzcd} \tag{8.42}$$

where the left-most vertical map is $\bar{\mathrm{FK}}_1 = \mathrm{FK}_1 \pmod{\mathfrak{p}^t}$. As in (8.36), the horizontal arrows are specialization at $r = 1$, and the right-most vertical arrow is accordingly the specialization of $\bar{\mathrm{FK}}_1$ at $r = 1$.

We may further apply now Fukaya–Kato's map $\bar{\mathrm{FK}}_2 = \mathrm{FK}_2 \pmod{\mathfrak{p}^t}$ to the above diagram and obtain the following one:

$$\begin{tikzcd} H^2(\mathbb{Z}[1/Np], \Lambda_{N,\theta}/\mathfrak{p}^t(\varepsilon_{\mathrm{cyc}})(2)) \arrow[r] \arrow[d, "\bar{\mathrm{FK}}_2"] & H^2(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(2)) \arrow[d, "||"] \\ H^2(\mathbb{Z}[1/Np], (H(2)/\mathfrak{M}^{(t)})^+_{\theta}) \arrow[r] & H^2(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(2)) \end{tikzcd} \tag{8.43}$$

Again the horizontal maps are specialization in level Np at $r = 1$ and the right-most vertical map is the specialization of $\bar{\mathrm{FK}}_2$ at $r = 1$. In view of (8.38) the latter may be identified with the identity map: according to the definitions provided in [FK12, Sections 6.1.6, 4.1.3] and with our current conventions, the specialization of $L_p(\bar{\theta})$ at $r = 1$ is

$$L_p(\bar{\theta}, -1) = (1 - \bar{\theta}(p)p) \cdot L(\bar{\theta}, -1) = -(1 - \bar{\theta}(p)p) \cdot \frac{B_{2,\bar{\theta}}}{2},$$

which vanishes $\pmod{\mathfrak{p}^t}$ in light of our assumptions.

Discussion of Theorem 8.4.1

We can finally discuss how to prove Theorem 8.4.1. With a slight abuse of notation, we identify global units with their image in cohomology under the Kummer map.

According to Proposition 8.4.2, we have

$$\mathrm{FK}_1(\bar{\kappa}_{f,2}) = \mathrm{ev}_\infty(u_{\xi_1,\xi_2} \cup u_{\chi_1,\chi_2}) = \bar{c}_{\xi_1} \cup \bar{c}_{\chi_1} \pmod{\mathfrak{p}^t}, \tag{8.44}$$

where the circular units involved in the cup product are those resulting from the evaluation at infinity of the modular units u_{ξ_1,ξ_2} and u_{χ_1,χ_2}, respectively. Recall we are assuming that $\xi_1 = \bar{\chi}_1$, and it was proved in Theorem 8.3.11 that κ_f is unramified everywhere.

Next, we apply FK_2 to both sides of (8.44). Proposition 8.4.3 together with the commutativity of (8.43) allow us to establish that

$$\bar{\kappa}_{f,2} \cup (1 - p^{-1}) \log_p(\varepsilon_{\mathrm{cyc}}) = L_p'(\bar{\theta}, -1) \cdot (\bar{c}_{\xi_1} \cup \bar{c}_{\chi_1}).$$

The expect theorem 8.4.1 finally follows from Corollary 8.4.4.

Chapter 9

Summaries in Catalan and Galician

9.1 Resum extens en català

Aquesta tesi estudia algunes aplicacions aritmètiques dels sistemes d'Euler de Beilinson–Flach i cicles diagonals, tot i que la interacció amb altres construccions semblants també hi és present. Els sistemes d'Euler constitueixen un instrument cabdal en l'estudi de la teoria d'Iwasawa i dels grups de Selmer. De forma grollera, són col·leccions de classes de cohomologia galoisiana que satisfan relacions de compatibilitat entre elles, i que es construeixen típicament en la cohomologia étale de varietats algebraiques. La gènesi del concepte neix del treball de Kolyvagin, que els emprà per provar la conjectura de Birch i Swinnerton-Dyer en rang analític 1, i també de la recerca de Rubin, que proposà un context prou general on desenvolupar aquesta teoria. En els últims anys han aparegut moltes noves construccions i resultats al voltant d'aquests sistemes d'Euler, que mostren la seva gran aplicabilitat per l'estudi de diferents problemes matemàtics. L'objectiu d'aquesta monografia és treballar algunes de les seves aplicacions aritmètiques cap a la teoria de zeros excepcionals, fórmules de valors especials i resultats de congruències amb formes modulars d'Eisenstein.

Qualsevol motivació històrica dels problemes que s'estudien en aquesta tesi ha de començar necessàriament per la conjectura de Birch i Swinnerton-Dyer, un dels sis problemes del mil·lenni encara oberts. Sigui E una corba el·líptica definida sobre el cos dels nombres racionals, i consideri's la seva funció L de Hasse–Weil, $L(E,s)$, definida en termes d'un producte de factors d'Euler locals i que convergeix per a $\Re(s) > 3/2$. És conegut pel treball de Wiles i Taylor–Wiles que E és modular, i per tant la funció L té continuació analítica a tot el pla complex, i a més satisfà una equació funcional que relaciona els valors a s i a $2-s$. Per tant, té sentit considerar el seu ordre d'anul·lació a $s=1$, $\mathrm{ord}_{s=1}L(E,s)$. La conjectura de Birch i Swinnerton-Dyer (BSD per abreujar), tal i com la formulà Tate, s'acostuma presentar de la següent manera.

Conjectura 9.1.1. *Sigui E una corba el·líptica definida sobre els racionals i sigui r el rang dels seus punts $\mathbb{Q}$-racionals, amb l'estructura habitual de $\mathbb{Z}$-mòdul finitament generat. Llavors, les següents propietats són certes:*

1. $r = \mathrm{ord}_{s=1}L(E,s)$.
2. *El terme r-èssim de l'expansió de Taylor, $L^{(r)}(E,1)$ satisfà que*

$$\frac{L^{(r)}(E,1)}{r!\cdot\Omega_E\cdot\mathrm{Reg}_E} = \frac{|\mathrm{Sha}(E)|\cdot\prod_{p|N}c_p}{|E_{\mathrm{tors}}|^2}.$$

Aquí, Ω_E és el període canònic associat a la corba el·líptica; Reg_E és el regulador de l'aparellament de Néron–Tate en E; $\mathrm{Sha}(E)$, el grup de Shafarevich, que conjecturalment és finit; i c_p són els anomenats nombres de Tamagawa en p, que només depenen del comportament local de la corba el·líptica sobre $\mathbb{Q}_p$. S'ha denotat la cardinalitat del grup finit G per $|G|$.

El grup de Shafarevich (de vegades anomenat de Tate–Shafarevich) és, de forma grollera, el conjunt de classes de cohomologia global que són trivials a cada plaça local, i per tant es pot pensar com una mesura de la no aplicabilitat del principi de Hasse–Minkowski per a corbes el·líptiques. La seva finitud s'enuncia de vegades com a part de la conjectura de BSD.

Recordi's que la funció L també es pot entendre en termes d'un sistema compatible de representacions de Galois. Més en concret, per a cada primer ℓ, es pot introduir el mòdul de Tate $V_\ell(E)$ com

$$V_\ell(E) = \left(\lim_{\leftarrow} E[\ell^n] \right) \otimes \mathbb{Q}_\ell.$$

Aquest objecte ens permet construir una representació del grup de Galois absolut $G_\mathbb{Q}$, que es denota

$$\rho_{E,\ell} \,:\, G_\mathbb{Q} \longrightarrow \mathrm{Aut}(V_\ell(E)) \simeq \mathrm{GL}_2(\mathbb{Q}_\ell).$$

La família $\{V_\ell(E)\}_\ell$ és un sistema compatible de representacions de Galois, en el sentit que per a qualsevol primer $p \neq \ell$, el polinomi característic de Fr_p, l'element de Frobenius en p, té coeficients enters que no depenen de ℓ. Qualsevol sistema així dona lloc a una funció L, $L(\{V_\ell(E)\}_\ell, s)$, definida com un producte de factors locals.

Més en general, sigui $H/\mathbb{Q}$ una extensió de Galois finita i sigui $\rho \,:\, \mathrm{Gal}\,(H/\mathbb{Q}) \longrightarrow \mathrm{GL}_n(L)$ una representació d'Artin de grau n (on $L/\mathbb{Q}$ és una extensió finita). La funció L de E torçada per ρ és

$$L(E, \rho, s) = L(\{V_\ell(E) \otimes \rho\}_\ell, s).$$

De forma semblant, es pot definir la component ρ-isotípica del grup de Mordell–Weil $E(H)$ com

$$E(H)[\rho] = \mathrm{Hom}_{G_\mathbb{Q}}(V_\rho, E(H) \otimes L),$$

on V_ρ és l'L-espai vectorial associat a la representació ρ. Per tant, podem formular la següent versió de la conjectura de BSD.

Conjectura 9.1.2. *La funció L associada a E i torçada per ρ, $L(E, \rho, s)$, admet continuació analítica a tot el pla complex i satisfà una equació funcional que lliga els valors $L(E, \rho, s)$ i $L(E, \rho^\vee, 2 - s)$. A més,*

$$\dim_L E(H)[\rho] = \mathrm{ord}_{s=1} L(E, \rho, s).$$

Aquí, $\rho^\vee$ és la representació dual de ρ (també anomenada contragradient).

A mode de notació, ens referirem a l'ordre d'anul·lació de $L(E, \rho, s)$ a $s = 1$ com el *rang analític*, i escriurem $r_{\mathrm{an}}(E, \rho)$. De la mateixa manera, el *rang algebraic* serà el valor de $\dim_L E(H)[\rho]$ i en aquest cas posarem $r_{\mathrm{alg}}(E, \rho)$.

Aquestes conjectures són part d'un programa molt més general, que explorem en la primera secció del primer capítol. Això inclou les conjectures de Beilinson i Bloch–Kato, que relacionen l'anul·lació de les funcions L amb l'existència de cicles racionals sobre varietats algebraiques.

No hi ha gaires resultats coneguts al voltant de la conjectura de BSD. Coates i Wiles [CW77], al 1977, van ser els primers en trobar evidència teòrica cap a la conjectura quan E té multiplicació complexa per un cos quadràtic imaginari i $L(E, 1) \neq 0$. El punt clau de la seva prova fou l'ús del sistema d'unitats el·líptiques, que en aquesta tesi es discuteix en el context general dels sistemes d'Euler. Durant els vuitanta, Gross i Zagier [GZ86] varen esbrinar una manera de demostrar la conjectura quan el rang analític és 1, establint una relació entre la derivada $L'(E, 1)$ i l'aparellament de Néron–Tate d'un punt de Heegner. Aquests punts de Heegner es poden entendre com els substituts de les unitats el·líptiques quan la corba el·líptica no té multiplicació complexa. Aquest resultat fou usat després per Kolyvagin [Kol88a], [Kol88b] per donar una prova completa de la conjectura de BSD quan el rang analític és com a molt 1. Kolyvagin mostrà com l'existència d'aquest sistema compatible de classes de cohomologia dona també una fita superior per la mida del grup de Selmer (i per tant del grup de Mordell–Weil).

Teorema 9.1.1 (Gross–Zagier, Kolyvagin)**.** *Sigui $K = \mathbb{Q}(\sqrt{-D})$ un cos quadràtic imaginari i consideri's un caràcter $\psi : \mathrm{Gal}\,(H/K) \longrightarrow L^\times$, on H/K és abeliana i $H/\mathbb{Q}$ és de Galois i diedral. Sigui*

$$\rho_\psi = \mathrm{Ind}(\psi) : \mathrm{Gal}\,(H/\mathbb{Q}) \longrightarrow \mathrm{GL}(V_\psi) \simeq \mathrm{GL}_2(L).$$

Aleshores, si $r_{\mathrm{an}}(E, \rho_\psi) = r \in \{0, 1\}$, se satisfà que $r_{\mathrm{alg}}(E, \rho_\psi) = r$.

Aquesta mena de resultats aviat es van estendre a altres casos mitjançant la teoria dels sistemes d'Euler. Un dels aspectes més importants d'aquests objectes és que es poden veure com una mena de realització geomètrica d'una funció L p-àdica. Aquesta frase, d'aparença críptica, juga un paper clau al llarg d'aquest treball i és el que popularment anomenem *formalisme de Perrin-Riou.* Les funcions L p-àdiques són una de les eines més importants d'aquesta memòria, i es poden entendre també com un anàleg p-àdic de les funcions L complexes, que venen de la interpolació p-àdica de certs valors L clàssics. A la literatura hom pot trobar un ampli ventall de construccions: automorfes, amb cohomologia coherent... Però alternativament admeten una construcció més algebraica, que neix amb Iwasawa i la seva escola i que les connecta amb l'aritmètica dels cossos ciclotòmics i dels anomenats grups de Selmer. Això forma part també d'unes de les conjectures més estudiades, que connecten novament un objecte analític (la funció L p-àdica) amb un altre de caire algebraic (el grup de Selmer).

L'exemple de sistema d'Euler més senzill és el de les unitats circulars i el·líptiques. Les primeres van ser claus, per exemple, en la prova de la conjectura principal d'Iwasawa per Mazur i Wiles, i les segones formen part de la seva generalització per Rubin a cossos quadràtics imaginaris. Els sistemes d'Euler però que fan un paper més important en aquesta tesis són els que apareixen quan s'estudien diferents casos de la conjectura equivariant de Birch i Swinnerton-Dyer. Això es remunta al treball de Kato [Ka04], que provà el següent resultat.

Teorema 9.1.2 (Kato)**.** *Sigui $\rho : \mathrm{Gal}\,(H/\mathbb{Q}) \longrightarrow L^\times$ un caràcter de Dirichlet. Si $r_{\mathrm{an}}(E, \rho) = 0$, aleshores*

$$\mathrm{Hom}_{G_\mathbb{Q}}(V_\rho, E(H) \otimes L) = 0.$$

La prova d'aquest resultat es basa en la construcció de classes en la cohomologia galoisiana de les corbes el·líptiques, i de manera que es belluguin de forma compatible al llarg de la torre ciclotòmica. Aquesta idea es va generalitzar a dos altres contexts que seran especialment significatius per a nosaltres. El primer és el cas on ρ és una representació d'Artin senar, irreductible i de dimensió 2. Aquest és l'anomenat cas dels elements de Beilinson–Flach, que admet un tractament força més general: es pot parlar d'un sistema de Beilinson–Flach associat a dues formes modulars (g, h) de pesos arbitraris, incorporant a més torcedures per potències del caràcter ciclotòmic. Aquest context va ser treballat per nombrosos autors: Bertolini–Darmon–Rotger primer [BDR15a], [BDR15b], i després Kings–Lei–Loeffler–Zerbes en un context més general [LLZ14], [KLZ20], [KLZ17]. L'altre cas interessant és el donat per $\rho = \rho_1 \otimes \rho_2$, on ρ_1 i ρ_2 són dues representacions de Galois de dimensió dos, senars i irreductibles de manera que són autoduals. Aquest és el context dels anomenats cicles diagonals, que va ser explorat principalment per Darmon i Rotger [DR14], [DR17]. De totes maneres, aquest cas és força més complex perquè no s'ha pogut construir un sistema d'Euler en el sentit habitual.

Aquests darrers cicles però, van permetre obtenir aplicacions aritmètiques que van més enllà de la conjectura de BSD. Darmon i Rotger per una banda [DR20b], i Bertolini, Seveso i Venerucci per altra [BSV20a], van veure com això donava evidència teòrica cap a la racionalitat dels anomenats punts de Stark–Heegner (també coneguts com a punts de Darmon), que són substituts dels punts de Heegner quan el cos quadràtic imaginari es canvia per un cos quadràtic real. Això mostra com l'aplicabilitat d'aquestes eines va força més enllà de les que hom podria suposar de bon començament, i aquesta tesi n'és un petit exemple.

Fem ara un repàs dels resultats més rellevants d'aquesta memòria.

1. Una fórmula de Gross–Stark per a la convolució de dues formes modulars. El treball de Mazur, Tate i Teitelbaum [MTT86] és el primer estudi rellevant on es planteja com estendre la conjectura de Birch i Swinnerton-Dyer al context p-àdic. Si E és una corba el·líptica sobre $\mathbb{Q}$, és possible associar-li una funció L p-àdica, que anomenarem $L_p(E,s)$. Aquesta funció es pot definir en termes d'una propietat d'interpolació corresponent a una sèrie de valors anomenats crítics, i és analítica per a $s \in \mathbb{Z}_p$. En particular,

$$L_p(E,1) = (1-\alpha_p^{-1})(1-\beta_p p^{-1}) \cdot \frac{L(E,1)}{\Omega_E},$$

on α_p és l'arrel unitat del polinomi de Hecke en p; $\beta_p = p/\alpha_p$ si E és ordinària i 0 si té reducció multiplicativa desplegada; i Ω_E és el període canònic associat a E. Pot resultar suggeridor formular una conjectura de BSD p-àdica dient que l'ordre de la funció L p-àdica coincideix amb el rang d'E. Malauradament, això no és cert: si $L(E,1) \neq 0$ però $\alpha_p = 1$ (és a dir, E té reducció multiplicativa desplegada), la fórmula anterior mostra que $L_p(E,1) = 0$. En aquest cas, la teoria d'uniformització p-àdica de Tate mostra que existeix un enter $q_E \in p\mathbb{Z}_p$ i un isomorfisme analític

$$E(\bar{\mathbb{Q}}_p) \simeq \bar{\mathbb{Q}}_p^\times / q_E^{\mathbb{Q}}$$

que està definit sobre $\mathbb{Q}_p$. Sigui $\log_p$ el logaritme p-àdic de $\mathbb{Z}_p^\times$, estès a $\mathbb{Q}_p^\times$ posant $\log_p(p) = 0$; i sigui ord_p la seva valoració normalitzada. Definim

$$\mathcal{L}_p(E) := \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}.$$

Llavors, Greenberg i Stevens [GS94] provaren que per a un primer $p \geq 5$ pel qual la corba el·líptica tingui reducció multiplicativa desplegada,

$$L_p'(E,1) = \mathcal{L}_p(E) \cdot \frac{L(E,1)}{\Omega_E}.$$

S'ha conjecturat que en aquest cas de corbes el·líptiques amb reducció multiplicativa desplegada se satisfà

$$\mathrm{ord}_{s=1} L_p(E,s) = 1 + \mathrm{ord}_{s=1} L(E,s).$$

La demostració de Greenberg–Stevens ha estat una inspiració per a la prova dels resultats desenvolupats al llarg d'aquesta tesi. Ens conformem aquí amb donar una idea d'algun dels seus punts claus. En primer lloc, consideren l'anomenada funció L p-àdica de Mazur–Kitagawa, on enlloc de treballar només amb la variable ciclotòmica $s \in \mathbb{Z}_p$, també hi surt una altra variable k anomenada *pes* i que pertany a l'espectre formal d'una extensió finita i plana de l'àlgebra d'Iwasawa $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. Això es troba al centre del que s'anomena *teoria de Hida*, una eina intrínsecament p-àdica: una forma modular f (sota certes hipòtesis) es pot interpolar mitjançant una família (analítica en la topologia p-àdica) $\mathbf{f}$ indexada per un conjunt d'enters de manera que les seves especialitzacions $\mathbf{f}_k$ es corresponen a formes modulars de pes k. D'aquesta funció de dues variables $L_p(\mathbf{f})(k,s)$ ens interessaran diversos aspectes: les seves propietats d'interpolació; que satisfaci una equació funcional; i molt especialment la possibilitat de definir el que s'anomena una *funció L p-àdica millorada* en una subvarietat de codimensió 1 de l'espai de pesos de dimensió 2, eliminant així un dels factors d'Euler que surt a la propietat d'interpolació i que és el responsable del zero excepcional. En tornarem a parlar al tractar els nostres resultats.

Hi ha altres contexts on l'anul·lació d'un factor d'Euler dona lloc a fenòmens aritmètics interessants. Sigui η un caràcter de Dirichlet primitiu mòdul N, prenent valors en un cos de nombres L, i sigui $p \nmid N$ un primer fixat. La funció L p-àdica de Kubota–Leopoldt $L_p(\eta\omega, s)$ satisfà la propietat d'interpolació

$$L_p(\eta\omega, 1-j) = (1-(\eta\omega^{1-j})(p)p^{j-1})L(\eta\omega^{1-j}, 1-j), \quad j \geq 1.$$

Sota la hipòtesi $\eta(p) = 1$ i suposant a més que η és senar, $L(\eta, 0) \neq 0$, però el factor d'Euler s'anul·la, i això dona lloc novament a una anul·lació de $L_p(\eta\omega, s)$ at $s = 0$. Si H és el cos retallat per η, aquest zero excepcional té a veure amb el fet que el grup de les p-unitats $(\mathcal{O}_H[1/p]^\times \otimes L)^{\eta^{-1}}$ és de dimensió 1, per la qual cosa podem prendre un generador v_η, i fixar a més un primer $\mathfrak{P}$ de H per sobre de p. Això determina dos $\mathbb{Z}$-mòduls homomorfismes

$$\mathrm{ord}_{\mathfrak{P}} : \mathcal{O}_H[1/p]^\times \to \mathbb{Z}, \quad \mathrm{Log}_{\mathfrak{P}} : \mathcal{O}_H[1/p]^\times \to \mathbb{Z}_p,$$

on el darrer morfisme es defineix com

$$\mathrm{Log}_{\mathfrak{P}}(u) = \log_p(\mathbb{N}_{H_p/\mathbb{Q}_p}(u)).$$

En aquest cas, definint

$$\mathcal{L}(\eta) := -\frac{\mathrm{Log}_{\mathfrak{P}}(u_\eta)}{\mathrm{ord}_{\mathfrak{P}}(u_\eta)},$$

se satisfà

$$L_p'(\eta\omega, 0) = \mathcal{L}(\eta) \cdot L(\eta, 0).$$

Compari's aquest resultat amb la coneguda fórmula de Leopoldt, que quan χ és un caràcter parell no trivial, expressa el valor de $L_p(\chi, 1)$ en termes d'una unitat circular associada al caràcter χ (allà però no hi ha cap anul·lació de la funció L p-àdica).

La situació de zeros excepcionals per la que començarem el nostre estudi comparteix alguns fenòmens en comú amb les anteriors, ja que està relacionada amb la convolució de dues formes modulars de pes 1, (g, h). Més en concret, començarem suposant que $g \in S_1(N, \chi)$ i que $h = g^* \in S_1(N, \bar{\chi})$ és la seva torcedura per l'invers del caràcter central. Llavors, es pot considerar la funció L p-àdica de Hida–Rankin $L_p(g, g^*, s)$, que depèn però de la tria d'una p-estabilització de g. Si escrivim

$$x^2 - a_p(g) + \chi(p) = (x - \alpha)(x - \beta),$$

ens referirem al valor especial $L_p(g, g^*, 1)$ associat a la p-estabilització amb valor propi α com a $\mathscr{L}_p{}^{g_\alpha}$. Per descriure-ho, ens caldrà introduir certes unitats i p-unitats u i v.

Sigui H el cos retallat per la representació d'Artin associada a la representació adjunta de g, i sigui L el cos de coeficients per a g, que es pot allargar de manera que contingui α i β. Sigui $V_{gg^*} = V_g \otimes V_{g^*}$ el producte tensorial de les representacions associades a g i g^*; de forma semblant, sigui $\mathrm{ad}^0(g)$ la representació adjunta de g, que es pot interpretar com el quocient de V_{gg^*} per la representació trivial.

Sota certes hipòtesis de regularitat que es detallen al primer capítol, es té que

$$\dim_L(\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}} = 1, \quad \dim_L(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}} = 2,$$

i es pot fixar una base $\{u, v\}$ del darrer espai, de manera que $u \in (\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$. Com a $G_{\mathbb{Q}_p}$-mòdul, $\mathrm{ad}^0(g)$ descompon com $\mathrm{ad}^0(g) = L \oplus L^{\alpha\otimes\bar{\beta}} \oplus L^{\beta\otimes\bar{\alpha}}$, on cada línia vé caracteritzada per la propietat que el Frobenius aritmètic Fr_p hi actua amb valors propis 1, α/β, β/α, respectivament. Sigui H_p la completació d'H a $\bar{\mathbb{Q}}_p$, i siguin

$$u_1,\ u_{\alpha\otimes\bar{\beta}},\ u_{\beta\otimes\bar{\alpha}},\ v_1,\ v_{\alpha\otimes\bar{\beta}},\ v_{\beta\otimes\bar{\alpha}} \in H_p^\times \otimes_\mathbb{Q} L$$

les projeccions dels elements u i v de $(H_p^\times \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}_p}}$ a les línies precedents.

Tenim doncs el següent resultat, que forma part del treball [RR20a] i que aquí discutim al capítol 2.

Teorema 9.1.3. *Amb les notacions anteriors, sigui $\mathscr{L}_p{}^{g_\alpha}$ el valor especial $L_p(g, g^*, 1)$ associat a la p-estabilització de g corresponent al valor propi α. Aleshores, la següent igualtat se satisfà mòdul $L^\times$:*

$$\mathscr{L}_p{}^{g_\alpha} = \frac{\log_p(v_1) \cdot \log_p(u_{\alpha\otimes\bar{\beta}}) - \log_p(u_1) \cdot \log_p(v_{\alpha\otimes\bar{\beta}})}{\log_p(u_{\alpha\otimes\bar{\beta}})}.$$

En aquesta tesi donem dues proves d'aquest resultat. La primera fa servir la teoria de famílies de Hida i deformacions de Galois. Algunes idees claus són les següents:

1. Les formes modulars g i g^* es poden interpolar en famílies de Hida $\mathbf{g}$ i $\mathbf{g}^*$. A més, Hida [Hi85], [Hi88] construí una funció L p-àdica de tres variables $L_p(\mathbf{g}, \mathbf{g}^*)$ indexada per variables (y, z, s), on (y, z) són els pesos de $(\mathbf{g}, \mathbf{g}^*)$ i s és una variable ciclotòmica.
2. Hida [Hi04] provà l'existència d'una funció L p-àdica millorada, amb bones propietats d'interpolació i que a més ens permet eliminar un dels factors d'Euler que s'anul·la en aquest cas autodual.
3. Els resultats anteriors ens permeten reduir la prova a un problema de deformacions de Galois, que es pot resoldre amb les tècniques desenvolupades principalment per Bellaïche i Dimitrov [BeDi16].

2. Classes de cohomologia derivades. El segon apropament que fem al problema anterior fa servir els elements de Beilinson–Flach, i serveix com a leitmotiv per desenvolupar una teoria de classes de cohomologia derivades en diferents contextos. Com ja hem comentat, donades dues formes modulars (g, h) i un enter s, satisfent certes relacions entre els pesos, es pot construir el que s'anomena una classe d'Eisenstein $\mathrm{Eis}^{[g,h,s]}$. Això però no val per a pes 1 i un ha de procedir de forma més indirecta: es consideren famílies de Hida $(\mathbf{g}, \mathbf{h})$ i és possible obtenir classes Λ-àdiques que quan s'especialitzen als pesos geomètrics ens permeten recuperar, llevat factors d'Euler apropiats, les construccions anteriors (això és anàleg a la interpolació de valors crítics amb funcions L). Però podem mirar més enllà i preguntar-nos quina informació codifiquen aquestes classes en pesos no-geomètrics, com ara pes 1. Potser no és gaire sorprenent dir que aquestes classes, associades a la tria de p-estabilitzacions per a g i h i que escriurem $\kappa(g_\alpha, h_\alpha)$, reprodueixen fenòmens semblants relacionats amb la presència de zeros excepcionals. En particular, si $h = g^*$, els valors propis de g^* són $\{1/\alpha, 1/\beta\}$ i
$$\kappa(g_\alpha, g^*_{1/\beta}) = \kappa(g_\beta, g^*_{1/\alpha}) = 0.$$
En aquests casos, podem obtenir el que anomenarem *classes derivades* i *lleis de reciprocitat derivades.* Aquest concepte és força subtil i està present a diferents parts de la monografia.

- A la secció 3 del capítol 3 es construeixen classes derivades al llarg d'una direcció pes. A més, es prova una llei de reciprocitat que lliga aquesta classe de derivada amb la funció L p-àdica, mitjançant el mateix invariant $\mathcal{L}$ que s'obté quan es treballa amb les funcions L.
- Al capítol 5, en canvi, considerem derivades al llarg de direccions ciclotòmiques, la qual cosa dona més flexibilitat i ens permet obtenir més informació. Grosso modo, el que passa és que si treballem sobre un espai d'unitats p-àdiques, algunes derivades direccionals recuperaran el logaritme i altres, la valoració p-àdica. Aquest joc ens permet a més donar una prova alternativa del teorema de valors especials.

En el capítol 4 analitzem un cas més general del problema, quan h ja no satisfà la condició d'autodualitat. En aquest context formulem una nova conjectura sobre aquestes classes de cohomologia, que és equivalent sota certes hipòtesis al resultat sobre funcions L que plantejen Darmon, Lauder i Rotger [DLR16]. Recentment, Castellà i Hsieh [CH20] han obtingut un important resultat cap a la conjectura de BSD en rang 2, mitjançant un estudi de les classes de Kato generalitzades (construïdes per Darmon–Rotger [DR16] a partir de cicles diagonals). Aquí veurem que podem obtenir un anàleg al seu resultat en el context d'unitats, fent servir per a aquest propòsit novament les propietats dels elements de Beilinson–Flach.

Els capítols 6 i 7 estudien altres situacions on el fenomen de zeros excepcionals també hi apareix. La primera té a veure amb el sistema de les unitats el·líptiques, que ja hem mencionat. El resultat principal del capítol 6 és una fórmula que relaciona una unitat el·líptica amb una classe de cohomologia derivada, i on també hi surt un invariant $\mathcal{L}$ associat a la representació. Això es pot interpretar com una traducció cohomològica del resultat de Katz sobre l'existència d'una funció L *millorada*. Veiem a més com aquests resultats es poden entendre com un cas degenerat dels resultats al voltant d'elements de Beilinson–Flach.

Els resultats del capítol 7 tenen a veure amb l'aritmètica dels cicles diagonals, on la presència dels zeros excepcionals apareix en pesos $(2, 1, 1)$ quan la forma modular de pes 2 està associada a un fenomen de reducció multiplicativa dividida i el producte de dos valors propis per les altres dues formes modulars és 1. Aquí fem un lligam entre dos tipus de conjectures de zeros excepcionals diferents:

- Conjectures de zeros excepcionals que es relacionen amb derivades d'ordre superior a la regió d'interpolació clàssica.
- Conjectures que tenen a veure amb valors especials de funcions L per punts fora de la regió d'interpolació.

3. Congruències entre sistemes d'Euler. Altre dels temes que explorem és la relació entre els diferents sistemes d'Euler. Siguin V_1 i V_2 dues representacions del grup de Galois absolut $G_{\mathbb{Q}}$, i siguin $L(V_1, s)$ i $L(V_2, s)$ les funcions L complexes corresponents. No és gaire difícil comprovar que la funció L corresponent a la suma directa $V_1 \oplus V_2$, que escriurem $L(V_1 \oplus V_2, s)$, factoritza com

$$L(V_1 \oplus V_2, s) = L(V_1, s)L(V_2, s).$$

Això és part de l'anomenat *formalisme d'Artin*, un fenómen clàssic força estudiat a la literatura. Un problema recorrent quan es treballa amb mètodes p-àdics és la possibilitat d'obtenir fórmules semblants quan les funcions L complexes se substitueixen pels seus anàlegs p-àdics. Hi ha relativament pocs exemples a la literatura. Un exemple és la fórmula de factorització de Gross per a caràcters de cossos quadràtics imaginaris, i un altre és la fórmula de factorització de Dasgupta per l'adjunta. Un tema força relacionat amb l'anterior té a veure amb les reduccions mòdul p. Quan hom té una forma cuspidal que és Eisenstein mòdul p, és natural cercar factoritzacions d'aquest tipus mòdul p. Mazur primer, i després Greenberg i Vatsal [GV00], van treballar aquesta qüestió en alguns casos, que ha estat estudiada més recentment en el context anticiclotòmic per Kriz.

El formalisme de Perrin-Riou que lliga les funcions L p-àdiques i els sistems d'Euler suggereix l'existència d'un formalisme d'Artin que ens permeti descompondre un sistema d'Euler associat a una representació p-àdica V com la suma d'altres dos sistemes d'Euler. Aquest context és força interessant per nosaltres i aquí explorem aquesta qüestió en alguns casos. Sigui f una forma pròpia cuspidal de pes 2 i nivell N, i sigui $p \nmid N$ un primer pel qual f és congruent a una sèrie d'Eisenstein. La classe de cohomologia de Beilinson–Kato κ_f associada a f dona lloc, mòdul p, a classes de cohomologia associades a representacions de Galois de dimensió 1. En aquesta tesi discutim congruències relacionant aquestes components amb expressions explícites que involucren unitats circulars. La prova de la primera relació de congruències fa servir una factorització mòdul p de Mazur i Greenberg–Vatsal i les lleis de reciprocitat de Perrin-Riou provades en aquest context per Coleman i Kato, i que involucren les funcions L p-àdiques de Kubota–Leopoldt i Mazur–Tate–Teitelbaum, respectivament. La prova de la segona congruència, en canvi, fa servir de forma clau les idees de Fukaya–Kato desenvolupades al seu treball al voltant de les conjectures de Sharifi. Referim el lector a la introducció del capítol 8 per a un enunciat precís d'aquests resultats, ja que la seva formulació és lleugerament feixuga perquè es pugui fer sense la introducció de la notació adient.

El nostre estudi suggereix també altres tipus de congruències entre els anomenats sistemes d'Euler de tipus Garrett–Rankin–Selberg. Això inclou els casos d'elements de Beilinson–Kato, Beilinson–Flach i cicles diagonals, aquests dos darrers ja molt presents a la primera part de la tesi.

De fet, ens hem aprofitat en nombroses ocasions de la interacció entre tots dos, i de les analogies existents entre les funcions L p-àdiques de Hida–Rankin i la funció L p-àdica triple, així com entre l'aritmètica d'unitats en cossos de nombres i la de punts racionals en corbes el·líptiques. Aquesta similitud suggereix que quan hom comença amb un triplet (f, g, h) de formes cuspidals, de manera que h és congruent a una sèrie d'Eisenstein mòdul p, la classe de cohomologia associada als cicles diagonals descompongui com a suma de dues classes mòdul p, una d'elles associada a l'element de Beilinson–Flach construït a partir de la parella (f, g). Aquest mateix fenomen s'hauria d'observar quan g és congruent mòdul p a una sèrie d'Eisenstein i el sistema de Beilinson–Flach degenera en una classe de Beilinson–Kato (prèviament lligat, a la vegada, amb el sistema d'unitats circulars). Esperem desenvolupar aquesta línia de recerca en treballs futurs.

9.2 Resumo extenso en galego

Esta tese estuda certos problemas aritméticos relacionados cos sistemas de Euler e coas funcións L p-ádicas, centrándose especialmente nos casos dos sistemas de Beilinson–Flach e ciclos diagonais. Os sistemas de Euler constitúen un instrumento esencial no estudo da teoría de Iwasawa e dos grupos de Selmer. Sen entrar en detalles, poderiamos dicir que son clases de cohomoloxía galoisiana que cumpren certas relacións de compatibilidade entre elas, e que se constrúen tipicamente na cohomoloxía étale de variedades alxébricas. A xénese do concepto remóntase aos traballos de Kolyvagin, que usou os sistemas de Euler para probar a conxectura de Birch e Swinnerton-Dyer en rango analítico 1, e tamén á investigación de Rubin, que propuxo un contexto xeral onde desenvolver esta teoría. Nos últimos anos teñen aparecido moitas construcións novas e resultados en torno a estes sistemas de Euler, que mostran a súa gran aplicabilidade para o estudo de diferentes problemas matemáticos. O obxectivo desta monografía é traballar algunhas das súas aplicacións aritméticas cara á teoría de ceros excepcionais, fórmulas de valores especiais e resultados de congruencias con formas modulares de Eisenstein.

Calquera presentación histórica destes temas debe comezar forzosamente coa conxectura de Birch e Swinnerton-Dyer, un dos seis problemas do milenio aínda sen resolver. Sexa E unha curva elíptica definida sobre os números racionais, e vamos considerar a súa función L de Hasse–Weil, $L(E, s)$. Esta función está definida en termos dun produto de factores de Euler locais que converxe para $\Re(s) > 3/2$. Coñécese, a partir do traballo de Wiles e Taylor–Wiles, que E é modular, e polo tanto a función L ten continuación analítica a todo o plano complexo, e ademais cumpre unha ecuación funcional que relaciona os valores en s e $2 - s$. Polo tanto, podemos considerar a orde de anulación en $s = 1$, $\mathrm{ord}_{s=1} L(E, s)$. A conxectura de Birch e Swinnerton-Dyer (BSD para abreviar), tal e como a formulou Tate, acostúmase presentar do seguinte xeito.

Conxectura 9.2.1. *Sexa E unha curva elíptica e sexa r o rango dos seus puntos $\mathbb{Q}$-racionais, coa estrutura habitual de $\mathbb{Z}$-módulo de xeración finita. Entón, as seguintes propiedades son certas:*

1. $r = \mathrm{ord}_{s=1} L(E, s)$.
2. *O termo r-ésimo da expansión de Taylor, $L^{(r)}(E, 1)$ cumpre que*

$$\frac{L^{(r)}(E, 1)}{r! \cdot \Omega_E \cdot \mathrm{Reg}_E} = \frac{|\mathrm{Sha}(E)| \cdot \prod_{p|N} c_p}{|E_{\mathrm{tors}}|^2}.$$

Aquí, Ω_E é o período canónico asociada á curva elíptica; Reg_E é o regulador do aparellamento de Néron–Tate en E; $\mathrm{Sha}(E)$, o grupo de Shafarevich, que conxecturalmente é finito; e c_p son os números de Tamagawa en p, que só dependen do comportamento local da curva elíptica sobre $\mathbb{Q}_p$. Por último, a cardinalidade do grupo finito G representámola como $|G|$.

O grupo de Shafarevich (ás veces chamado de Tate–Shafarevich) é, grosso modo, o conxunto de clases de cohomoloxía global que son triviais en cada praza local, e polo tanto explica a maneira na que falla o principio de Hasse–Minkowski para curvas elípticas. O feito de que sexa finito enúnciase ás veces como parte da conxectura de BSD.

Recordemos que a función L tamén se pode entender en termos dun sistema compatible de representacións de Galois. Máis en concreto, para cada primo ℓ podemos introducir o módulo de Tate $V_\ell(E)$ como

$$V_\ell(E) = \left(\lim_{\leftarrow} E[\ell^n]\right) \otimes \mathbb{Q}_\ell.$$

Este obxecto permítenos construír unha representación do grupo de Galois absoluto $G_\mathbb{Q}$, que se escribe como

$$\rho_{E,\ell} : G_\mathbb{Q} \longrightarrow \mathrm{Aut}(V_\ell(E)) \simeq \mathrm{GL}_2(\mathbb{Q}_\ell).$$

A familia $\{V_\ell(E)\}_\ell$ é un sistema compatible de representacións de Galois, no sentido de que para cada calquera primo $p \neq \ell$, o polinomio característico de Fr_p, o elemento de Frobenius en p, ten coeficientes enteiros que non dependen de ℓ. Calquera sistema así dá lugar a unha función L, $L(\{V_\ell(E)\}_\ell, s)$, definida como un produto de factores locais.

Máis en xeral, sexa $H/\mathbb{Q}$ unha extensión de Galois finita e sexa $\rho : \mathrm{Gal}\,(H/\mathbb{Q}) \longrightarrow \mathrm{GL}_n(L)$ unha representación de Artin de grao n (onde $L/\mathbb{Q}$ é unha extensión finita). A función L de E torcida por ρ é

$$L(E, \rho, s) = L(\{V_\ell(E) \otimes \rho\}_\ell, s).$$

De forma semellante, podemos definir a compoñente ρ-isotípica do grupo de Mordell–Weil $E(H)$ como

$$E(H)[\rho] = \mathrm{Hom}_{G_\mathbb{Q}}(V_\rho, E(H) \otimes L),$$

onde V_ρ é o L-espazo vectorial asociado á representación ρ. Polo tanto, podemos formular a seguinte versión da conxectura de BSD.

Conxectura 9.2.2. *A función L asociada a E e torcida por ρ, $L(E, \rho, s)$, admite continuación analítica a todo o plano complexo e cumpre unha ecuación funcional que relaciona os valores $L(E, \rho, s)$ e $L(E, \rho^\vee, 2 - s)$. Ademais,*

$$\dim_L E(H)[\rho] = \mathrm{ord}_{s=1} L(E, \rho, s).$$

Aquí, $\rho^\vee$ é a representación dual de ρ (tamén chamada contragradiente).

A modo de notación, referirémonos á orde de anulación de $L(E, \rho, s)$ en $s = 1$ como o *rango analítico*, e escribiremos $r_{\mathrm{an}}(E, \rho)$. Do mesmo xeito, o *rango alxébrico* será o valor de $\dim_L E(H)[\rho]$ e neste caso poremos $r_{\mathrm{alg}}(E, \rho)$.

Estas conxecturas son parte dun programa máis amplo e xeral, que exploramos na primeira sección do primeiro capítulo. Isto inclúe as conxecturas de Beilinson e Bloch–Kato, que relacionan a anulación das funcións L coa existencia de ciclos racionais sobre variedades alxébricas.

Non hai moitos resultados coñecidos en torno á conxectura de BSD. Coates e Wiles [CW77], no 1977, foron os primeiros que acharon evidencias cara á conxectura cando E ten multiplicación complexa por un corpo cuadrático imaxinario e $L(E, 1) \neq 0$. O punto esencial da súa proba foi o uso do sistema de unidades elípticas, que nesta tese discutimos no contexto máis xeral dos sistemas de Euler. Durante os oitenta, Gross e Zagier [GZ86] atoparon unha maneira de demostrar a conxectura cando o rango analítico é 1, establecendo unha relación entre a derivada $L'(E, 1)$ e o aparellamento de Néron–Tate dun punto de Heegner. Estes puntos de Heegner pódense entender como os substitutos das unidades elípticas cando a curva elíptica non ten multiplicación complexa. Este resultado foi posteriormente usado por Kolyvagin [Kol88a], [Kol88b] para dar unha proba completa da conxectura de BSD cando o rango analítico é ao sumo 1. Kolyvagin probou que a existencia deste sistema compatible de clases de cohomoloxía dá tamén unha cota superior para o tamaño do grupo de Selmer (e por tanto do grupo de Mordell–Weil).

Teorema 9.2.1 (Gross–Zagier, Kolyvagin). *Sexa $K = \mathbb{Q}(\sqrt{-D})$ un corpo cuadrático imaxinario e tomemos un carácter $\psi : \mathrm{Gal}(H/K) \longrightarrow L^\times$, onde H/K é abeliana e $H/\mathbb{Q}$ é de Galois e diedral. Sexa*

$$\rho_\psi = \mathrm{Ind}(\psi) : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{GL}(V_\psi) \simeq \mathrm{GL}_2(L).$$

Entón, se $r_{\mathrm{an}}(E, \rho_\psi) = r \in \{0, 1\}$, cúmprese que $r_{\mathrm{alg}}(E, \rho_\psi) = r$.

Este tipo de resultados pronto se estenderon a outros casos usando a teoría dos sistemas de Euler. Un dos aspectos máis importantes destes obxectos é que se poden ver como unha especie de realización xeométrica dunha función L p-ádica. Esta frase, de aparencia críptica, fai un papel clave ao longo deste traballo e é o que popularmente coñecemos como *formalismo de Perrin-Riou*. As funcións L p-ádicas son unha das ferramentas máis importantes desta memoria e pódense entender tamén como un análogo p-ádico das funcións L complexas, que veñen da interpolación p-ádica das funcións L clásicas (complexas). Na literatura é posíbel atopar diferentes construcións: automorfas, con cohomoloxía coherente... Pero alternativamente admiten unha construción máis alxébrica, que nace con Iwasawa e a súa escola e que as conecta coa aritmética dos corpos ciclotómicos. Isto forma parte tamén dunha das conxecturas máis estudadas, que conectan novamente un obxecto analítico (a función L p-ádica) con outro de natureza alxébrica (o grupo de Selmer).

Os exemplos máis sinxelos de sistemas de Euler son o das unidades circulares e o das unidades elípticas. As primeiras foron clave, por exemplo, na proba da conxectura principal de Iwasawa por Mazur e Wiles, e as segundas forman parte da súa xeneralización por Rubin a corpos cuadráticos imaxinarios. Porén, os sistemas de Euler que fan un papel máis importante nesta tese son os que aparecen ao estudar diferentes casos da conxectura equivariante de Birch e Swinnerton-Dyer. Isto vén xa do traballo de Kato [Ka04], que probou o seguinte resultado.

Teorema 9.2.2 (Kato). *Sexa $\rho : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow L^\times$ un carácter de Dirichlet. Se $r_{\mathrm{an}}(E, \rho) = 0$, entón*

$$\mathrm{Hom}_{G_\mathbb{Q}}(V_\rho, E(H) \otimes L) = 0.$$

A proba deste resultado baséase na construción de clases na cohomoloxía galoisiana das curvas elípticas, de forma que varían de forma compatible ao longo da torre ciclotómica. Esta idea xeneralizouse a outros dous contextos que serán especialmente significativos para nós. O primeiro é o caso no que ρ é unha representación de Artin impar, irredutible e de dimensión 2. Este é o denominado caso dos elementos de Beilinson–Flach, que admite un tratamento bastante máis xeral: é posible falar dun sistema de Beilinson–Flach asociado a dúas formas modulares (g, h) de pesos arbitrarios, incorporando ademais torceduras por potencias do carácter ciclotómico. Este concepto foi traballado por varios autores: Bertolini–Darmon–Rotger primeiro [BDR15a], [BDR15b], e despois Kings–Lei–Loeffler–Zerbes nun contexto máis xeral [LLZ14], [KLZ20], [KLZ17]. O outro caso interesante é o dado por $\rho = \rho_1 \otimes \rho_2$, onde ρ_1 e ρ_2 son dúas representacións de Galois de dimensión dous, impares e irredutibles, de maneira que sexan autoduais. Este é o contexto dos *ciclos diagonais*, que foi explorado principalmente por Darmon e Rotger [DR14], [DR17]. De calquera xeito, este é un caso bastante máis complexo, porque non se puido construír un sistema de Euler no sentido habitual.

Estes últimos ciclos permitiron, porén, obter aplicacións aritméticas interesantes que van máis aló da conxectura de BSD. Darmon e Rotger por un lado [DR20b] e Bertolini, Seveso e Venerucci por outro [BSV20a], viron como esta construción daba evidencia teórica cara á racionalidade dos chamados puntos de Stark–Heegner (ou puntos de Darmon), que son substitutos dos puntos de Heegner cando o corpo cuadrático imaxinario se cambia por un corpo cuadrático real. Isto mostra como o rango de aplicacións destas ferramentas vai máis aló do que un podería esperar de entrada, e esta tese é un pequeno exemplo.

Faremos agora unha pequena andaina por algúns dos resultados máis relevantes desta memoria.

1. Unha fórmula de Gross–Stark para a convolución de dúas formas modulares. O traballo de Mazur, Tate e Teitelbaum [MTT86] é o primeiro estudo relevante no que se intenta estender a conxectura de Birch e Swinnerton-Dyer ao contexto p-ádico. Se E é unha curva elíptica sobre $\mathbb{Q}$, é posíbel asociarlle unha función L p-ádica, que chamaremos $L_p(E,s)$. Esta función pódese definir en termos dunha propiedade de interpolación correspondente a unha serie de valores chamados críticos, e é analítica para $s \in \mathbb{Z}_p$. En particular,

$$L_p(E,1) = (1-\alpha_p^{-1})(1-\beta_p p^{-1}) \cdot \frac{L(E,1)}{\Omega_E},$$

onde α_p é a raíz unidade do polinomio de Hecke en p; $\beta_p = p/\alpha_p$ se E é ordinaria e 0 se ten redución multiplicativa despregada; e Ω_E é o período canónico asociado a E. Pode resultar suxerinte formular unha conxectura de BSD p-ádica dicindo que a orde da función L p-ádica coincide co rango de E. Desafortunadamente, iso non é certo: se $L(E,1) \neq 0$ pero $\alpha_p = 1$ (é dicir, E ten redución multiplicativa despregada), entón $L_p(E,1) = 0$. Neste caso, a teoría da uniformización p-ádica de Tate mostra que existe un enteiro $q_E \in p\mathbb{Z}_p$ e un isomorfismo analítico

$$E(\bar{\mathbb{Q}}_p) \simeq \bar{\mathbb{Q}}_p^\times / q_E^{\mathbb{Q}}$$

que está definido sobre $\mathbb{Q}_p$. Sexa $\log_p$ o logaritmo p-adico de $\mathbb{Z}_p^\times$, estendido a $\mathbb{Q}_p^\times$ pondo $\log_p(p) = 0$; e sexa ord_p a súa valoración normalizada. Definimos

$$\mathcal{L}_p(E) := \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}.$$

Entón, Greenberg e Stevens [GS94] probaron que para un primo $p \geq 5$ para o que a curva elíptica teña redución multiplicativa despregada,

$$L_p'(E,1) = \mathcal{L}_p(E) \cdot \frac{L(E,1)}{\Omega_E}.$$

Conxecturouse que para as curvas elípticas con redución multiplicativa despregada se cumpre

$$\mathrm{ord}_{s=1} L_p(E,s) = 1 + \mathrm{ord}_{s=1} L(E,s).$$

A demostración de Greenberg–Stevens foi unha inspiración para a proba dos resultados desenvolvidos ao longo desta tese. Conformámonos aquí con dar unha idea dalgún dos puntos clave. En primeiro lugar, consideramos o que se chama función L p-ádica de Mazur–Kitagawa, onde non se traballa só coa variable ciclotómica $s \in \mathbb{Z}_p$, senón tamén con outra variable k chamada *peso* e que pertence ao espectro formal dunha extensión finita e plana da álxebra de Iwasawa $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. Iso encóntrase no centro do que se chama *teoría de Hida*, unha ferramenta intrinsecamente p-ádica: unha forma modular f (baixo certas hipóteses) pódese interpolar usando unha familia (que é analítica na topoloxía p-ádica) $\mathbf{f}$ indexada por un conxunto de enteiros, de maneira que as súas especializacións $\mathbf{f}_k$ correspóndense con formas modulares de peso k. Desta función de dúas variables $L_p(\mathbf{f})(k,s)$ interésannos varios aspectos: as súas propiedades de interpolación; que cumpre unha ecuación funcional; e moi especialmente a posibilidade de definir o que se chama unha *función L p-ádica mellorada* nunha subvariedade de codimensión 1 do espazo de pesos de dimensión 2, eliminando así un dos factores de Euler que aparece na propiedade de interpolación e que é o responsable do cero excepcional. Volveremos a falar disto ao tratar os nosos resultados.

Hai outros contextos onde a anulación dun factor de Euler dá lugar a fenómenos aritméticos interesantes. Sexa η un carácter de Dirichlet primitivo módulo N e que toma valores nun corpo de números L, e sexa $p \nmid N$ un primo fixado. A función L p-ádica de Kubota–Leopoldt $L_p(\eta\omega,s)$ cumpre a propiedade de interpolación

$$L_p(\eta\omega, 1-j) = (1-(\eta\omega^{1-j})p^{j-1})L(\eta\omega^{1-j}, 1-j), \quad j \geq 1.$$

Baixo a hipótese $\eta(p) = 1$ e asumiendo tamén que η é impar, $L(\eta, 0) \neq 0$, pero o factor de Euler anúlase, o que dá lugar de novo a unha anulación de $L_p(\eta\omega, s)$ en $s = 0$. Se H é corpo recortado por η, este cero excepcional ten que ver co feito de que o grupo das p-unidades $(\mathcal{O}_H[1/p]^\times \otimes L)^{\eta^{-1}}$ é de dimensión 1, e polo tanto podemos coller un xerador v_η, e fixar tamén un primo $\mathfrak{P}$ de H por riba de p. Isto determina dous homomorfismos de $\mathbb{Z}$-módulos

$$\mathrm{ord}_{\mathfrak{P}} : \mathcal{O}_H[1/p]^\times \to \mathbb{Z}, \quad \mathrm{Log}_{\mathfrak{P}} : \mathcal{O}_H[1/p]^\times \to \mathbb{Z}_p,$$

onde o último morfismo se define como

$$\mathrm{Log}_{\mathfrak{P}}(u) = \log_p(\mathbb{N}_{H_p/\mathbb{Q}_p}(u)).$$

Neste caso, definindo

$$\mathcal{L}(\eta) := -\frac{\mathrm{Log}_{\mathfrak{P}}(v_\eta)}{\mathrm{ord}_{\mathfrak{P}}(v_\eta)},$$

cúmprese

$$L_p'(\eta\omega, 0) = \mathcal{L}(\eta) \cdot L(\eta, 0).$$

Existe un paralelismo evidente coa coñecida fórmula de Leopoldt, que cando χ é un carácter par non trivial, expresa o valor de $L_p(\chi, 1)$ en termos dunha unidade circular asociada ao carácter χ (nese caso porén non hai ningunha anulación da función L p-ádica).

A situación de ceros excepcionais pola que comezaremos o noso estudo comparte algúns fenómenos en común coas anteriores, xa que está relacionada coa convolución de dúas formulares de peso 1, (g, h). Máis en concreto, comezaremos supondo que $g \in S_1(N, \chi)$ e que $h = g^* \in S_1(N, \bar{\chi})$ é a súa torcedura polo inverso do carácter central. Entón, pódese considerar a función L p-ádica de Hida–Rankin $L_p(g, g^*, s)$, que depende da elección dunha p-estabilización de g. Se escribimos

$$x^2 - a_p(g) + \chi(p) = (x - \alpha)(x - \beta),$$

referirémonos ao valor especial $L_p(g, g^*, 1)$ asociado á p-estabilización con valor propio α como $\mathscr{L}_p{}^{g_\alpha}$. Para describilo, cumprirá introducir unha unidade u e unha p-unidade v.

Sexa H o corpo recortado pola representación de Artin asociada á adxunta de g, e sexa L o corpo de coeficientes de g, que se pode estender de maneira que tamén conteña α e β. Sexa $V_{gg^*} = V_g \otimes V_{g^*}$ o produto tensorial das representacións asociadas a g e g^*. De forma semellante, sexa $\mathrm{ad}^0(g)$ a representación adxunta de g, que se pode interpretar como o cociente de V_{gg^*} pola representación trivial. Baixo certas hipóteses de regularidade que se detallan no primeiro capítulo, temos que

$$\dim_L(\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}} = 1, \quad \dim_L(\mathcal{O}_H[1/p]^\times / p^\mathbb{Z} \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}} = 2,$$

e podemos fixar unha base $\{u, v\}$ do último espazo, de maneira que $u \in (\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$. Como $G_{\mathbb{Q}_p}$-módulo, $\mathrm{ad}^0(g)$ descompón como $\mathrm{ad}^0(g) = L \oplus L^{\alpha\otimes\bar{\beta}} \oplus L^{\beta\otimes\bar{\alpha}}$, onde cada liña vén caracterizada pola propiedade de que o Frobenius aritmètico Fr_p actúa con valores propios 1, α/β, β/α, respectivamente. Sexa H_p a completación de H en $\bar{\mathbb{Q}}_p$, e sexan

$$u_1,\, u_{\alpha\otimes\bar{\beta}},\, u_{\beta\otimes\bar{\alpha}},\, v_1,\, v_{\alpha\otimes\bar{\beta}},\, v_{\beta\otimes\bar{\alpha}} \in H_p^\times \otimes_\mathbb{Q} L$$

as proxeccións dos elementos u e v de $(H_p^\times \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}_p}}$ ás liñas precedentes.

Temos entón o seguinte resultado, que forma parte do traballo [RR20a] e que se discute en profundidade no capítulo 2.

Teorema 9.2.3. *Coas notacións anteriores, sexa $\mathscr{L}_p{}^{g_\alpha}$ o valor especial $L_p(g, g^*, 1)$ asociado á p-estabilización de g correspondente ao valor propio α. Entón, a seguinte igualdade dáse módulo $L^\times$:*

$$\mathscr{L}_p{}^{g_\alpha} = \frac{\log_p(v_1) \cdot \log_p(u_{\alpha\otimes\bar{\beta}}) - \log_p(u_1) \cdot \log_p(v_{\alpha\otimes\bar{\beta}})}{\log_p(u_{\alpha\otimes\bar{\beta}})}.$$

Nesta tese damos dúas probas do resultado. A primeira emprega a teoría de familias de Hida e deformacións de Galois. Algunhas ideas importantes son as seguintes:

1. As formas modulares g e g^* pódense interpolar en familias de Hida $\mathbf{g}$ e $\mathbf{g}^*$. Ademais, Hida [Hi85], [Hi88] construíu unha función L p-ádica $L_p(\mathbf{g}, \mathbf{g}^*)$ indexada por tres variables (y, z, s), onde (y, z) son os pesos de $(\mathbf{g}, \mathbf{g}^*)$ e s é unha variable ciclotómica.

2. Hida [Hi04] probou a existencia dunha función L p-ádica mellorada, con boas propiedades de interpolación e que ademais nos permite eliminar un dos factores de Euler que se anula neste caso autodual.

3. Os resultados anteriores permítennos reducir a proba a un problema de deformacións de Galois, que se pode resolver coas técnicas desenvolvidas principalmente por Bellaïche e Dimitrov [BeDi16] e estendidas logo por Darmon, Lauder e Rotger [DLR18].

2. Clases de cohomoloxía derivadas. O segundo acercamento que propomos ao problema anterior usa os elementos de Beilinson–Flach, e serve como leitmotiv para desenvolver unha teoría de clases de cohomoloxía derivadas en diferentes contextos. Como xa comentamos, dadas dúas formas modualres (g, h) e un enteiro s cumprindo certas relacións cos pesos, pódese construír o que se chama unha clase de Eisenstein $\mathrm{Eis}^{[g,h,s]}$. Porén, iso non é posible para peso 1 e hai que proceder dun xeito máis indirecto: considéranse familias de Hida $(\mathbf{g}, \mathbf{h})$ e é posible obter clases Λ-ádicas que cando se especializan nos pesos xeométricos permiten recuperar, multiplicando polos factores de Euler axeitados, as construcións anteriores (iso é análogo á interpolación de valores críticos con funcións L). Podemos mirar máis aló e preguntarnos que información codifican estas clases en pesos non xeométricos, por exemplo peso 1. Quizais non resulta moi sorprendente dicir que estas clases, asociadas á elección de p-estabilizacións para g e h e que escribiremos como $\kappa(g_\alpha, h_\alpha)$, reproducen fenómenos similares relacionados coa presenza de ceros excepcionais. En particular, cando $h = g^*$, os valores propios de g^* son $\{1/\alpha, 1/\beta\}$ e

$$\kappa(g_\alpha, g^*_{1/\beta}) = \kappa(g_\beta, g^*_{1/\alpha}) = 0.$$

Nestes casos, podemos fabricar o que chamaremos *clases derivadas* e *leis de reciprocidade derivadas*. Este concepto é bastante sutil e está presente en diferentes partes do traballo.

- Na sección 3 do capítulo 3 construímos clases derivadas ao longo dunha dirección peso. Ademais, probamos unha lei de reciprocidade que conecta esta clase derivada coa función L p-ádica, usando o mesmo invariante $\mathcal{L}$ que se obtén cando se traballa coas funcións L.

- No capítulo 5, en cambio, consideramos derivadas ao longo de direccións ciclotómicas, o que nos dá máis flexibilidade e nos permite obter máis información. Grosso modo, o que pasa é que se traballamos sobre un espazo de unidades p-ádicas, algunhas derivadas direccionais recuperan o logaritmo e outras, a valoración p-ádica. Este xogo coas derivadas permítenos dar unha proba alternativa do teorema anterior sobre valores especiais.

No capítulo 4 analizamos un caso máis xeral do problema, cando a parella (g, h) non cumpre a condición de autodualidade. Neste contexto formulamos unha nova conxectura sobre estas clases de cohomoloxía, que é equivalente baixo algunhas hipóteses ao resultado sobre funcións L que formulan Darmon, Lauder e Rotger [DLR16]. Recentemente, Castellà e Hsieh [CH20] obtiveron un importante resultado cara á conxectura de BSD en rango 2, baseándose nun estado das clases de Kato xeneralizadas construídas por Darmon e Rotger [DR16] usando a teoría dos ciclos diagonais. Aquí veremos que podemos obter un análogo ao seu resultado no contexto de unidades, usando novamente as propiedades dos elementos de Beilinson–Flach.

Os capítulos 6 e 7 estudan outras situacións onde o fenómeno dos ceros excepcionais tamén aparece. A primeira ten que ver co sistema das unidades elípticas, que xa mencionamos. O resultado principal do capítulo 6 é unha fórmula que relaciona unha unidade elíptica cunha clase de cohomoloxía derivada, e onde tamén sae un invariante $\mathcal{L}$ asociado á representación. Iso pódese interpretar como unha tradución cohomolóxica do resultado de Katz sobre a existencia dunha función L mellorada. Vemos ademais como estes resultados se poden entender como un caso dexenerado dos nosos resultados anteriores sobre elementos de Beilinson–Flach.

Os resultados do capítulo 7 teñen que ver coa aritmética dos ciclos diagonais, onde a presenza dos ceros excepcionais aparece en pesos $(2,1,1)$ cando a forma modular de peso 2 está asociada a un fenómeno de redución multiplicativa dividida e o produto dos dous valores propios para as outras dúas formas modulares é 1. Aquí relacionamos dous tipos de conxecturas diferentes:

- Conxecturas de ceros excepcionais relacionadas con derivadas de orde superior na rexión de interpolación clásica.
- Conxecturas relacionadas con valores especiais de funcións L para puntos fóra da rexión de interpolación.

3. Congruencias entre sistemas de Euler. Outro dos temas que exploramos nesta tese é a relación entre os diferentes sistemas de Euler. Sexan V_1 e V_2 dúas representacións do grupo de Galois absoluto $G_{\mathbb{Q}}$, e sexan $L(V_1,s)$ e $L(V_2,s)$ as funcións L complexas asociadas a elas. Non é complicado probar que a función L correspondente á suma directa $V_1 \oplus V_2$, que escribiremos como $L(V_1 \oplus V_2, s)$, factoriza como

$$L(V_1 \oplus V_2, s) = L(V_1, s) \cdot L(V_2, s).$$

Iso é parte do *formalismo de Artin*, un fenómeno clásico moi estudado na literatura. Un problema recorrente cando se traballa con métodos p-ádicos é a posibilidade de obter fórmulas similares cando as funcións L complexas se substitúen polos seus análogos p-ádicos. Hai relativamente poucos exemplos na literatura: un deles é a fórmula de Gross [Gro80] para caracteres de corpos cuadráticos imaxinarios, e outro é a fórmula de factorización de Dasgupta [Das99] para a adxunta dunha forma modular. Un tema moi relacionado co anterior ten que ver coas reducións módulo p. Cando se ten unha forma cuspideal que é Eisenstein módulo p, é natural buscar factorizacións deste tipo módulo p. Mazur primeiro [Maz79], e logo Greenberg e Vatsal [GV00], traballaron esta cuestión nalgúns casos, que tamén foi tratada recentemente no caso anticiclotómico por Kriz [Kr16].

O formalismo de Perrin-Riou que liga as funcións L p-ádicas e os sistemas de Euler, suxire a existencia dun formalismo de Artin que nos permita descompor un sistema de Euler asociado a unha representación p-ádica $V = V_1 \oplus V_2$ como a suma doutros dous sistemas de Euler asociados a V_1 e V_2. Este contexto é bastante interesante para nós, e aquí exploramos esta cuestión nalgúns casos. Sexa f uha forma propia cuspidal de peso 2 e nivel N, e sexa $p \nmid N$ un primo para o que f é congruente cunha serie de Eisenstein. A clase de cohomoloxía de Beilinson–Kato κ_f asociada a f relaciónase de xeito natural con outras clases asociadas a representacións de Galois de dimensión 1, e que aquí relacionamos con expresións explícitas que involucran unidades circulares. A proba da primeira relación de congruencia baséase nunha factorización módulo p de Mazur e Greenberg–Vatsal, e tamén nas leis de reciprocidade de Perrin-Riou demostradas neste contexto por Coleman e Kato, e que involucran as funcións L p-ádicas de Kubota–Leopoldt e Mazur–Tate–Teitelbaum, respectivamente. A proba da segunda congruencia, en cambio, deberíase relacionar de forma clave coas ideas de Fukaya–Kato desenvolvidas no seu traballo sobre as conxecturas de Sharifi. Referimos ao lector á introdución do capítulo 7 para un enunciado preciso destes resultados.

Isto suxire tamén outro tipo de congruencias, que son as que se deberían observar entre os chamados sistemas de Euler de tipo Garrett–Rankin–Selberg. Este contexto inclúe os casos de elementos de Beilinson–Kato, Beilinson–Flach e ciclos diagonais, os dous últimos moi presentes na primeira parte da tese. De feito, usamos en varias ocasións a interacción entre ambos, así como as analoxías existentes entre as funcións L p-ádicas de Hida–Rankin e a función L p-ádica tripla, ou entre a aritméticas de unidades en corpos de números e a de puntos en curvas elípticas. Esta similitude suxire que cando un comeza cun triplete (f, g, h) de formas cuspidais, con h congruente cunha serie de Eisenstein módulo p, a clase de cohomoloxía asociada aos ciclos diagonais debería descompor como suma de dúas clases módulo p, unha das cales asociada ao elemento de Beilinson–Flach construído a partir da parella (f, g). Este mesmo fenómeno observaríase cando g é congruente módulo p cunha serie de Eisenstein e o sistema de Beilinson–Flach dexenera nunha clase de Beilinson–Kato (previamente conectada, á súa vez, co sistema de unidades ciculares). Esperamos desenvolver con máis detalle esta liña de investigación en futuros traballos.

9.3 Tra bufalo e locomotiva: una visió personal

Tra bufalo e locomotiva la differenza salta agli occhi: la locomotiva ha la strada segnata, il bufalo può scartare di lato e cadere.

Aquesta memòria reflecteix el treball de més de tres anys, i m'agradaria tancar-la amb un petit comentari de caire més personal, en la línia de la frase amb la que he començat el treball, un vers del magnífic cantautor italià Francesco de Gregori. El procés de realització d'una tesi doctoral és sovint mitificat, i la gent normalment ho descriu com enriquidor, gratificant i apassionant. Jo sé que tots ells menteixen. És però una mentida social fortament acceptada que ningú no gosa revelar. En el meu cas, m'he endinsat en un tema que penso que és molt maco, però els moments de plaer han estat escassos. Ha predominat la frustració d'enfrontar-se a resultats que molts cops no entenia, la solitud de l'investigador i la incertesa per una carrera laboral on tothom ha acceptat la precarietat i la misèria com a norma, amb l'excusa d'aquest romanticisme i caràcter vocacional que la gent imprimeix a la vida acadèmica. He gaudit per moments fent matemàtiques, però també ho he passat malament, i molts cops he volgut fugir. He estat un búfal que massa sovint ha tingut aspiracions de locomotora, però finalment he arribat al final del camí i no me'n penedeixo ni imagino la meva vida lluny de les matemàtiques i la teoria de nombres.

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