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Qualitative properties of solutions to integro-differential elliptic problems

by

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A mis padres

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Summary

The thesis is devoted to the analysis of elliptic Partial Differential Equations (PDE) and related problems such as integro-differential equations. It is mainly focused on the study of qualitative and regularity properties of solutions to equations driven by integro-differential operators of the form

$$Lu(x) = \mathrm{P.\,V.} \int_{\mathbb{R}^n} \big(u(x) - u(y)\big) K(x - y)\, \mathrm{d}y.$$

The canonical example of such nonlocal operators is the fractional Laplacian, $(-\Delta)^s$, which corresponds to the radial and homogeneous kernel $K(z) = c_{n,s} |z|^{-n-2s}$ for $s \in (0, 1)$.

The study of equations involving nonlocal operators has attracted much attention recently since they arise naturally in finance, fluid dynamics, and other areas when dealing with processes where long range interaction phenomena appear. In addition, fractional equations may turn out to be a useful approximation of certain local equations for a limit of the parameter s (this could be the case for nonlocal and classical minimal surfaces).

The thesis is divided into three parts: Part I, which corresponds to Chapters 1 and 2, concerns the study of uniqueness and regularity properties of solutions to integro-differential linear problems; Part II, divided into Chapters 3, 4, and 5, is focused on the saddle-shaped solution to integro-differential Allen-Cahn equations; and Part III (Chapter 6) is devoted to construct a calibration for general nonlocal variational problems.

In Chapter 1 we treat uniqueness, up to multiplicative constant, of solutions to nonlocal linear equations of the form

$$L\varphi - c(x)\varphi = 0, \quad \text{in } \mathbb{R}.$$

Uniqueness of solutions to linear equations is a very important tool in the theory of PDEs. Indeed, there are many motivations to treat this problem. On the one hand, it is in the essence of Sturm-Liouville theory on eigenfunctions and eigenvalues. On the other hand, it has important consequences when studying qualitative properties of solutions to some nonlinear problems.

Unlike what happens in the local framework, where the one-dimensional problem is just a second order linear ODE whose solutions are very well understood, in this nonlocal setting there are many basic questions in dimension one which are still open problems.

In this scenario, we prove the uniqueness of solutions to the previous equation, where L is an elliptic integro-differential operator, in the presence of a positive solution or of an odd solution vanishing only at zero. As an application, we deduce the nondegeneracy of layer solutions (bounded and monotone solutions) to the semilinear problem $Lu = f(u)$ in $\mathbb{R}$, when the nonlinearity is of Allen-Cahn type. In order to prove the linear uniqueness result we first show, by using a maximum principle in the exterior of an interval, that the quotient of two solutions is bounded. Next, we follow a nonlocal Liouville-type method. A main point here is to find an equation for the quotient of two solutions, something that has

been accomplished by Hamel, Ros-Oton, Sire, and Valdinoci for general integro-differential operators. Then, one uses this equation to show that the quotient is constant. This requires an integral estimate for the function $K(x-y)$ in unbounded cross-shaped regions of the plane.

Chapter 2 concerns the boundary regularity for the Neumann problem associated to the fractional Laplacian

$$\left\{ \begin{array}{rcll} (-\Delta)^s u & = & f & \text{in } \Omega, \\ \mathcal{N}_s u & = & 0 & \text{in } \Omega^c, \end{array} \right.$$

where $\mathcal{N}_s$ is a "nonlocal normal derivative", given by

$$\mathcal{N}_s u(x) = c_{N,s} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \qquad x \in \Omega^c.$$

In this thesis, we establish the first boundary regularity result for the Neumann problem. We prove that weak solutions are C^α up to the boundary for some $\alpha > 0$. Moreover, in case $s > 1/2$, we show that weak solutions are $C^{2s-1+\alpha}(\overline{\Omega})$. Our methods allow us to treat, as well, the Neumann problem for the regional fractional Laplacian, for which we show the same boundary regularity result.

In the local case one can use the reflection method to reduce the boundary regularity to the interior one. Here, a completely different strategy is needed. In the Dirichlet problem for the fractional Laplacian, this difficulty was overcome by using methods that extend the boundary Harnack theory for local equations in non-divergence form. In the Neumann case, instead, the corresponding local theory for equations in divergence form (boundary Moser iteration) is the appropriate one.

In our strategy, we first transform the Neumann problem into a regional-type problem, with a kernel with logarithmic behavior on the boundary. Next, we develop a delicate Moser iteration on the boundary (for positive and negative powers) with logarithmic corrections in order to prove the L^∞ and C^α regularity of solutions. Finally, we establish a Neumann Liouville-type theorem in a half-space, which is used together with a blow-up argument to show higher regularity of solutions.

Part II of the thesis is focused on the study of the so-called saddle-shaped solution to the integro-differential Allen-Cahn equation $Lu = f(u)$ in $\mathbb{R}^{2m}$, where the nonlinearity f of bistable type. A crucial property of these solutions is that their zero level set is the Simons cone. Their importance comes from their role in a fractional version of a conjecture by De Giorgi about the one-dimensional symmetry of monotone solutions. Indeed, the saddle-shaped solution is expected to be the simplest minimizer which is not one-dimensional to the (local and nonlocal) Allen-Cahn equation (this will occur in high enough dimensions). It plays, thus, the same role as the Simons cone in the theory of minimal surfaces.

First, we study the saddle-shaped solution when $L = (-\Delta)^s$ by using the extension problem. We establish its uniqueness and, in dimensions $2m \geq 14$, its stability. As a byproduct, we give the first analytical proof of a stability result for the Simons cone in the nonlocal setting. We show that it is a stable nonlocal minimal surface in dimensions $2m \geq 14$. To prove these results we use a maximum principle for the linearized operator $(-\Delta)^s - f'(u)$. Such a maximum principle is used to establish some monotonicity properties of the saddle-shaped solution, that in turn are used to prove the stability of the saddle solution in dimensions $2m \geq 14$.

Next, we study saddle-shaped solutions when L is any rotation invariant and uniformly elliptic integro-differential operator. In this scenario, we need to develop some new nonlocal techniques, since here the extension technique is not available. To our knowledge, there is no previous work on saddle-shaped solutions for general operators. In this respect, our main contribution is a characterization of the kernels K for which one can develop a theory of existence and uniqueness of saddle-shaped solutions. It turns out that a sufficient condition is K being radially symmetric and $K(\sqrt{\cdot})$ strictly convex. Under these assumptions, we establish an energy estimate for doubly radial odd minimizers and some properties of the saddle-shaped solution, namely: existence, uniqueness, asymptotic behavior, and a maximum principle for the linearized operator.

Finally, in Part III we develop a nonlocal Weirstrass extremal field theory. A classical problem in the Calculus of Variations consists of giving sufficient conditions for an extremal of an energy functional, i.e., a solution of the Euler-Lagrange equation, to be minimizer. A very useful strategy to establish the minimality in the local framework goes through the theory of calibrations. In this setting, a calibration $\mathcal{C}$ associated to the energy functional $\mathcal{E}$ and the function u is a functional depending only on the boundary values of u and satisfying $\mathcal{C}(u) = \mathcal{E}(u)$ and $\mathcal{C}(v) \leq \mathcal{E}(v)$ for any v. Once the calibration is available, the minimality of u follows immediately, with no need to have an existence result of minimizers, neither to know their regularity.

In analogy to the local theory, we construct such a calibration for the nonlocal functional

$$\mathcal{E}_N(w) = \iint_{(\Omega^c\times\Omega^c)^c} G_N\Big(x, y, w(x), w(y)\Big)\, dx\, dy$$

in the presence of a foliation made of solutions (or an extremal field) when the nonlocal Lagrangian $G_N(x, y, a, b)$ satisfies the ellipticity condition $\partial^2_{ab} G_N \leq 0$. The model case in our setting corresponds to the energy functional for the fractional Laplacian, for which such a calibration was still unknown. In addition, the ellipticity condition is satisfied in the case of the fractional p-Dirichlet energy, the fractional s-area functional for graphs, as well as for convolution energies. The existence of such a calibration allows us to prove that any leaf of the foliation is automatically a minimizer for its own exterior datum.

Each of the chapters of the thesis is made up by one of the following articles or preprints:

- [111] J.C. Felipe-Navarro and T. Sanz-Perela, *Uniqueness and stability of the saddle-shaped solution to the fractional Allen-Cahn equation*, Rev. Mat. Iberoam. 36 (2020), 1887-1916.

- [110] J.C. Felipe-Navarro and T. Sanz-Perela, *Semilinear integro-differential equations I: odd solutions with respect to the Simons cone*, J. Funct. Anal. 278 (2020).

- [112] J.C. Felipe-Navarro and T. Sanz-Perela, *Semilinear integro-differential equations II: one-dimensional and saddle-shaped solutions to the Allen-Cahn equation*, Mathematics in Engineering, 2021, 3 (5): 1-36.

- [13] A. Audrito, J.C. Felipe-Navarro, and X. Ros-Oton, *The Neumann problem for the fractional Laplacian: regularity up to the boundary.* Preprint at arXiv:2006.10026.

- [109] J.C. Felipe-Navarro, *Uniqueness for linear integro-differential equations in the real line and applications.* Preprint at arXiv:2103.13081.

- [53] X. Cabré, I. U. Erneta, and J.C. Felipe-Navarro, *Null-Lagrangians and calibrations for nonlocal functionals.* In preparation.

Contents

Introduction

This thesis concerns the analysis of elliptic Partial Differential Equations (PDE) and related problems such as integro-differential equations. It is mainly focused on the study of qualitative and regularity properties of solutions to equations driven by integro-differential operators of the form

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The study of equations involving nonlocal operators has attracted much attention recently since they arise naturally in finance, fluid dynamics, and other areas when dealing with processes where nonlocal interaction phenomena appear. In addition, fractional equations may turn out to be a useful approximation of certain local equations for a limit of the parameter s (this could be the case for nonlocal and classical minimal surfaces).

In the following, we first introduce integro-differential operators (with special emphasis on the fractional Laplacian) and nonlocal minimal surfaces. They appear in many nonlocal models from different areas. Next, we focus on the Dirichlet and Neumann problems for the fractional Laplacian, recalling the available regularity results and the probabilistic interpretation. After that, we review an important qualitative property of solutions to variational equations: minimality. Then, we present some rigidity properties for linear and nonlinear elliptic equations. Finally, we describe the main results of the thesis.

Nonlocal problems

The study of models driven by an energy functional of the form

$$\mathcal{E}_N(u) = \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} G_N\Big(x, y, u(x), u(y)\Big)\,dx\,dy \tag{1}$$

has attracted much attention in the last decades. The structure of this energy takes into account interactions between each pair of points $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Thus, it is suitable when dealing with situations where long range interaction phenomena appear. For this reason these are called *nonlocal* models.

Nonlocal models have been used in Solid Mechanics and Elasticity since the first half of the XXth century. The most important example is the Peierls-Nabarro equation

$$(-\Delta)^{1/2} u = \sin u,$$

which arises when studying crystal dislocations; see [188, 121]. More recently, there has been an increased interest in nonlocal variational problems, mainly after Silling introduced the Peridynamics continuum model in [178]. In such a formulation, unlike the usual

Cauchy–Green elasticity theory, forces between separate points are taken into account. Hence, it seems to be adequate for studying deformations where discontinuities appear, as in the case of fractures; see [180, 142, 179].

It is also very natural to treat nonlocal problems in Epidemiology. The global pandemic of COVID-19 has showed that long range interactions are crucial when studying the spreading of diseases. The reason is that people are able to move long distances in a short time. In [40], an analysis of human movement based on bill tracking suggests that Lévy flights are appropriate models to describe these processes.

On the other hand, in Quantum Mechanics, it is well known that the behavior of soft matter does not follow from relativistic or quantum properties of elementary molecules. Thus, different models have been presented in recent years to treat such an anomalous behavior. For instance, Laskin [141] introduced fractional quantum mechanics as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. This is a nonlocal model that has been found to be a very useful tool to study the physical problem; see [69].

Turning to Fluid Dynamics, an important nonlocal model, which has a strong connection with the extension problem for the fractional Laplacian, is the Benjamin-Ono equation

$$(-\Delta)^{1/2}u = -u + u^2.$$

It arises in the theory of internal waves of irrotational, incompressible, and inviscid fluids in deep regions; see [25, 10].

Finally, in Finance, some relevant models, such as the fractional obstacle problem, have nonlocal nature. It is mainly motivated by the discontinuous behavior of asset prizes; see [148, 144, 160]. In such cases, Lévy flights are more suitable than Brownian motion to describe the processes.

Integro-differential operators

An important family of nonlocal operators that arises when modeling most of the previous examples is the one formed by linear integro-differential operators. These are of the form

$$Lu(x) = \text{P. V.} \int_{\mathbb{R}^n} \big(u(x) - u(y)\big)\, K(x,y)\, dy, \tag{2}$$

where $K \geq 0$ is the kernel of the operator and P. V. stands for principal value. It is clear that such linear operators have a nonlocal character since in order to evaluate Lu at x_0 one needs to know the values of u outside any small neighborhood around the point. Moreover, when the kernel K is positive, $Lu(x_0)$ depends on the values of u in the whole $\mathbb{R}^n$.

This kind of operators shares important properties with the classical Laplacian and other second order differential operators, such as the global maximum principle [78]. To illustrate this, if u has a global maximum at x_0 then $Lu(x_0) \geq 0$. Nevertheless, there are also some relevant differences between differential and integro-differential operators. For instance, being x_0 a local maximum of u is usually not enough to conclude that $Lu(x_0) \geq 0$. This fact introduces certain difficulties, as it occurs when studying odd solutions to equations driven by such operators (see Chapters 1, 4, and 5).

Integro-differential operators arise also naturally in probability, since they are the infinitesimal generators of Lévy flights, which are, roughly speaking, stochastic processes where long jumps are allowed. In this framework, $K(x,y)$ measures the probability of jumping from the point x to y (we refer to Chapter 2 in [43] and references therein for more details), and satisfies the condition

$$\int_{\mathbb{R}^n} \min\left\{1, |x-y|^2\right\} K(x,y)\, \mathrm{d}y < +\infty, \qquad \text{for every} \ \ x \in \mathbb{R}^n. \tag{3}$$

Another natural assumption from the probabilistic point of view is the kernel being symmetric in the variables x and y, i.e, $K(x,y) = K(y,x)$. It corresponds with the probability of jumping from x to y being equal to the probability of jumping from y to x. Moreover, this assumption ensures a variational structure which, apart from providing a very rich mathematical theory, is natural when dealing with problems coming from physical phenomena. When K is symmetric, the integro-differential operator is self-adjoint and appears as the Euler-Lagrange equation of the term

$$\iint |u(x) - u(y)|^2 K(x,y)\, \mathrm{d}x\, \mathrm{d}y$$

in an energy functional. The domain of integration will be made precise later.

In order to establish certain results involving integro-differential operators some additional assumptions are required. Among the most frequently adopted, we have:

- Translation invariance. We say that the operator L is *translation invariant* if it satisfies $L\Big(u(\cdot + h)\Big)(x) = Lu(x+h)$ for every function u and points $x, h \in \mathbb{R}^n$. In the case of integro-differential operators, this condition yields the kernel being of the form $K(x,y) = k(x-y)$. As a consequence, such operators can be seen as multiplier operators, whose Fourier symbol is given by

  $$m(\xi) = \int_{\mathbb{R}^n} \left(1 - \mathrm{e}^{i\xi\cdot z}\right) k(z)\, \mathrm{d}z.$$

  Within the nonlocal framework, these operators are the analogues of second order differential operators with constant coefficients.

- Rotation invariance: The operator L is said to be *rotation invariant* if it satisfies $L(u(R\cdot))(x) = Lu(Rx)$ for every function u, rotation $R \in O(n)$, and point $x \in \mathbb{R}^n$. For linear integro-differential operators, rotation invariance is equivalent to K satisfying $K(Rx, Ry) = K(x,y)$. This property appears naturally when modeling isotropic processes. In such phenomena the properties of different magnitudes are independent of the direction in which they are measured.

- Scale invariance: We say that the operator L is *scale invariant* if there is a parameter α such that $L(u(\lambda\cdot))(x) = \lambda^\alpha Lu(\lambda x)$ for every function u, point $x \in \mathbb{R}^n$, and scalar $\lambda \in \mathbb{R}$. When L is an integro-differential operator it is equivalent to the kernel being homogeneous, i.e., $K(\lambda x, \lambda y) = \lambda^{-n-\alpha} K(x,y)$. Let us point out that scale invariance is an important property which is present in many physical problems.

If we impose a linear integro-differential operator to be translation, rotation, and scale invariant and to satisfy the integrability condition (3), we end up with the kernel being

$$K(x,y) = C\,|x-y|^{-(n+2s)},$$

for some positive constant C and $s \in (0,1)$. In such a case the operator L corresponds to the fractional Laplacian, which we present below in this introduction.

When the integro-differential operator has a kernel that vanishes outside a domain Ω it is said to be of regional-type and can be written as

$$Lu(x) = \mathrm{P.V.} \int_{\Omega} \big(u(x) - u(y)\big)\, K(x,y)\, dy.$$

These operators are strongly related to censored processes in probability (see [33]). Moreover, they will play an important role in the present thesis. They are often useful to treat different problems involving general integro-differential operators of the form (2). For instance, they appear in Parts I and II of this thesis, both when studying odd solutions and when treating the Neumann problem.

In relation with fractional analogs of the p-Laplacian and minimal surfaces, problems involving quasilinear integro-differential operators of the form

$$Lu(x) = \mathrm{P.V.} \int_{\mathbb{R}^n} \Phi\big(u(x) - u(y)\big)\, K(x,y)\, dy$$

have been studied in the last years; see [133, 91, 138, 90, 42]. We also treat problems involving such operators in Part III of the thesis.

The fractional Laplacian

If we impose a linear integro-differential operator to be translation, rotation, and scale invariant, we end up with a multiple of the *fractional Laplacian*

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy, \qquad s \in (0,1),$$

where $c_{n,s} > 0$ is an explicit normalizing constant (see [56] for its precise value and [92] for its asymptotic properties).

In the nonlocal framework, this operator takes the role of the Laplacian for second order differential operators. For instance, a linear integro-differential operator is said to be (uniformly) elliptic if its kernel is comparable to the one of the fractional Laplacian, i.e.,

$$\frac{\lambda}{|x-y|^{n+2s}} \leq K(x,y) \leq \frac{\Lambda}{|x-y|^{n+2s}} \tag{4}$$

for some positive constants $0 < \lambda \leq \Lambda$. This is a crucial property to be assumed, especially when studying regularity of solutions. Roughly speaking, it ensures the operator being close to $(-\Delta)^s$.

The fractional Laplacian has been studied from different points of view (probability, Fourier analysis, potential theory, and functional analysis) since the second half of the XXth century, mainly when treating linear problems; see [16, 24, 31, 32, 34, 137, 140, 152, 175, 184, 183]. However, it has been in the last decade, after the work by Caffarelli and Silvestre [63], when it gained a lot of attention from the PDE community, by allowing to treat nonlinear problems.

The name "fractional Laplacian" comes from the fact that it is a multiplier operator with Fourier symbol $|\xi|^{2s}$, i.e.,

$$\mathcal{F}[(-\Delta)^s u](\xi) = |\xi|^{2s} \mathcal{F}[u](\xi),$$

in contrast with the Laplacian, whose symbol is $|\xi|^2$. It follows that

$$(-\Delta)^s \circ (-\Delta)^t = (-\Delta)^{s+t}.$$

Moreover, under certain regularity assumptions on u, it is satisfied that

$$\lim_{s\uparrow 1}(-\Delta)^s u = -\Delta u \quad \text{and} \quad \lim_{s\downarrow 0}(-\Delta)^s u = u.$$

The fundamental solution is an important tool when studying equations involving the Laplacian. In the fractional framework, an analogous theory is also available. Indeed, a solution to

$$(-\Delta)^s u = f \quad \text{in } \mathbb{R}^n,$$

is given by the Riesz potential

$$u(x) = \tilde{c}_{n,s} \int_{\mathbb{R}^n} \frac{f(z)}{|x-z|^{n-2s}}\, dz \quad n \neq 2s.$$

This identity, that can be checked by using Fourier analysis, is very useful since the Riesz potential is a well studied tool in harmonic analysis. For instance, it is known that it regularizes functions up to $2s$ derivatives. In particular, if $f \in C^\alpha$, then $u \in C^{2s+\alpha}$ as long as $2s+\alpha$ is not an integer.

A very powerful technique to treat problems involving the fractional Laplacian is the local extension problem. It was first introduced from a probabilistic perspective in 1968 by Molchanov and Ostrovskii [152]. In such a framework, they established that each symmetric stable process can be obtained as the trace of a degenerate Bessel diffusion process. A similar extension had been also used in Conformal Geometry. Later, in 2005, Caffarelli and Silvestre [63] retook it from a PDE point of view, proving that the fractional Laplacian can be recovered as a Dirichlet to Neumann operator. Let us explain this. Given $u : \mathbb{R}^n \to \mathbb{R}$, we define its s-harmonic extension $U : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ as the solution to

$$\begin{cases} \operatorname{div}(\lambda^{1-2s}\nabla U) &= 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)\,, \\ U &= u \quad \text{on } \mathbb{R}^n\,. \end{cases}$$

Note that although this problem is a degenerate PDE when $s < 1/2$ and singular when $s > 1/2$, the weight λ^{1-2s} belongs to the Muckenhoupt class A_2 and thus the extension function U is well defined [103]. In addition, it has an explicit expression in terms of the Poisson kernel of a half-space.

Now, we can define the Dirichlet to Neumann operator

$$T_s u(x) = -\lim_{\lambda\downarrow 0}\left(\lambda^{1-2s}\, U_\lambda(x,\lambda)\right).$$

It is clearly a nonlocal operator since the extension U depends on the values of u in the whole $\mathbb{R}^n$. Moreover, it turns out that

$$(-\Delta)^s u = d_s\, T_s u,$$

where $d_s > 0$ is an explicit normalizing constant (see [56] for the precise value).

Such a relation can be checked both by using the explicit expression of the Poisson kernel or the Fourier transform. Furthermore, one can obtain it from a heuristic point of view when $s = 1/2$. That is, in that case $T_{1/2}u(x) = -U_\lambda(x, 0)$. Indeed, it is clear that $-U_\lambda$ is the harmonic extension of $T_{1/2}u$. Hence,

$$T_{1/2}^2 u = T_{1/2}(T_{1/2}u) = T_{1/2}(-U_\lambda(\cdot, 0)) = U_{\lambda\lambda}(\cdot, 0) = -\Delta_x U(\cdot, 0) = -\Delta u,$$

which means that $T_{1/2} = -\partial_\lambda|_{\lambda=0}$ is a square root of the Laplacian.

The introduction of the extension method was a turning point in the field since it allowed to treat nonlocal problems by using purely and very well developed local techniques. Thus, many fractional problems, mainly nonlinear, have been studied in depth in recent years. Among others, we mention the regularity theory for the fractional obstacle problem [62], the uniqueness of ground states for Benjamin-Ono-type equations [117, 118], or a fractional version of De Giorgi's conjecture [48, 49, 182, 116]. Nevertheless, an important restriction of the extension technique is that it is not available for most of the integro-differential operators. Hence, alternative methods need to be developed in most cases.

Besides the integral, Fourier, and extension representations of the fractional Laplacian, a fourth useful way to define this operator is via the heat semigroup:

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(\mathrm{e}^{t\Delta} u - u \right) \frac{dt}{t^{1+s}}.$$

In fact, the method of semigroups has been used to define and study fractional powers of other general operators; see [185] and references therein.

Nonlocal minimal surfaces

The perimeter functional appears naturally when modeling certain physical phenomena, such as surface tension in soap bubbles or phase transitions. Given a bounded domain $E \subset \mathbb{R}^n$ we know that the definition of its perimeter, defined for smooth sets as the Hausdorff measure of the boundary, can be extended to less regular sets as

$$\mathcal{P}(E) = [\mathbb{1}_E]_{W^{1,1}(\mathbb{R}^n)}.$$

This definition becomes crucial when solving the Plateau problem in $\Omega \subset \mathbb{R}^n$, i.e., when proving the existence of a set $E \subset \mathbb{R}^n$ such that

$$\mathcal{P}(E \cap \Omega) \leq \mathcal{P}(F \cap \Omega)$$

for every $F \subset \mathbb{R}^n$ with $F \setminus \Omega = E \setminus \Omega$. Indeed, once a good definition of the perimeter functional for nonregular sets is introduced, the existence of minimizers follows from its lower semicontinuity and a compactness argument. The boundaries of minimizers of the perimeter functional are the so-called minimizing minimal surfaces. They have zero mean curvature (in a suitable viscosity sense).

Once the existence of minimizing minimal surfaces is established, it is natural to wonder about their regularity. If a minimizing minimal surface is the graph of a function, one can combine a gradient estimate by Bombieri, De Giorgi, and Miranda [39] with De Giorgi-Nash-Moser theory to conclude that it is smooth in any dimension. This is not true for general minimizing minimal hypersurfaces in $\mathbb{R}^n$. In fact, they are analytic when $n \leq 7$, while they may have singularities if $n \geq 8$. A key ingredient to obtain such a result is the classification of minimizing minimal cones; see [81] and references therein. The Simons cone, defined in $\mathbb{R}^{2m}$ by

$$\mathscr{C} = \{x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x'| = |x''|\} \, , \tag{5}$$

plays an important role in the classification. It has zero mean curvature at every regular point in every even dimension. In addition, it minimizes the perimeter functional in dimensions $2m \geq 8$, as established by Bombieri, De Giorgi, and Giusti in [38].

If we replace now the $W^{1,1}$-seminorm by a fractional one, nonlocal versions of the perimeter can be defined. That is, we consider the fractional perimeter

$$\mathcal{P}_\alpha(E) = [\mathbb{1}_E]_{W^{\alpha,1}(\mathbb{R}^n)} = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\mathbb{1}_E(x) - \mathbb{1}_E(y)|}{|x-y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y = 2 \iint_{E \times E^c} \frac{\mathrm{d}x \, \mathrm{d}y}{|x-y|^{n+\alpha}}.$$

Roughly speaking, $\mathcal{P}_\alpha$ gives an interpolation between the volume and the classical perimeter of the set (see Appendix A in [99]). Other nonlocal perimeters can be considered by choosing different kernels K as in the case of integro-differential operators.

By using integration by parts, the fractional perimeter can be rewritten as an integral over the boundary

$$\mathcal{P}_\alpha(E) = c_{\alpha,n} \iint_{\partial E \times \partial E} \frac{2 - |\nu(x) - \nu(y)|^2}{|x-y|^{n+\alpha-2}} \, \mathrm{d}x \, \mathrm{d}y.$$

Hence, it can be understood as a weighted measure of the variation of the normal vector on the boundary.

This fractional version of perimeter inherits some properties of the classical counterpart. For instance, it enjoys an isoperimetric inequality [119, 120]. That is, balls minimize the fractional perimeter among all sets with a prescribed volume. Furthermore, the fractional perimeter has important applications in different areas, such as digital image reconstruction (see [99]).

In [61], Caffarelli, Roquejoffre, and Savin introduced the notion of fractional perimeter of a set E inside a bounded domain Ω as

$$\mathcal{P}_\alpha(E;\Omega) = \iint_{(\Omega^c \times \Omega^c)^c} \frac{|\mathbb{1}_E(x) - \mathbb{1}_E(y)|}{|x-y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y$$

in order to develop a theory of nonlocal minimal surfaces. Remarkably, such a theory shares many analogies with the classical one. At the same time, important problems within the nonlocal theory are still open.

As in the classical perimeter, the existence of nonlocal minimal surfaces follows from the lower semicontinuity of $\mathcal{P}_\alpha$ and a compactness argument. Moreover, if the set is regular enough, it has zero nonlocal mean curvature, i.e, it satisfies $H_\alpha[E](x) = 0$ for every $x \in \partial E \cap \Omega$, where

$$H_\alpha[E](x) = \int_{\mathbb{R}^n} \frac{\mathbb{1}_{E^c}(y) - \mathbb{1}_E(y)}{|x-y|^{n+\alpha}} \, \mathrm{d}y.$$

Heuristically, this means that the interactions of each point on the boundary with both E and E^c are equal.

Thanks to a monotonicity formula which is also available in the fractional framework, the regularity of minimizing nonlocal minimal surfaces follows from classifying minimizing nonlocal minimal cones. As in the local case, they are expected to be hyperplanes only in low dimensions. Nevertheless, such a classification is far from complete. In dimension $n = 2$ and for every α, minimizing nonlocal minimal cones were proved to be flat by Savin and Valdinoci [171]. In higher dimensions, the main result in this direction is the one by Caffarelli and Valdinoci [64], establishing that they are hyperplanes in dimensions $2 \leq n \leq 7$ for α close to 1.

It is also an open problem to find, in high dimensions, an example of a nontrivial minimizing nonlocal minimal cone. The main candidate is the Simons cone $\mathscr{C}$ (see (5) to recall the definition). As in the local framework, it is easy to check that it is stationary for the nonlocal perimeter functional. However, it is an open problem to determine if it is a minimizer for some dimensions. In [86], Dávila, del Pino, and Wei gave the unique result in this direction. They showed that establishing the stability of the Simons cone is equivalent to checking an inequality involving two hypergeometric constants which depend only on α and n. By using numerical computations, they find that, in dimensions $2m \leq 6$ and for α close to zero, the Simons cone is unstable.

When the minimizing nonlocal minimal surface is the graph of a function, Cabré and Cozzi [52] showed through a gradient estimate that the surface is smooth in every dimension. It is important to point out that in the case of graphs, the nonlocal mean curvature being zero is a sufficient condition to be minimizer (see [82, 47]).

Integro-differential equations in bounded domains

Dirichlet and Neumann problems

For the Laplacian, existence and uniqueness of weak solutions to the Dirichlet problem

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{6}$$

follow from the Riesz representation theorem. Moreover, the solution is known to be the unique minimizer of the energy functional

$$\mathcal{E}_f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f\, u$$

among all functions $u \in H^1(\Omega)$ with the trace condition $u = 0$ on $\partial\Omega$.

On the other hand, if we look for free minimizers (no trace condition is assumed) of such an energy functional, we recover the Neumann problem for the Laplacian

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{7}$$

Note that the existence of free minimizers is not always guaranteed. Indeed, if $\int_\Omega f \neq 0$, given any function we can construct a sequence of functions, by adding/subtracting constants, whose energy decreases as much as we want. It turns out that $\int_\Omega f = 0$ is not only a necessary but sufficient condition to have existence of solutions. Moreover, when such a condition is assumed, uniqueness (up to additive constant) of solution can also be obtained by using Riesz representation theorem when restricting to the space of functions with zero mean.

We point out that similar Dirichlet and Neumann problems can be stated when replacing the Laplacian by any second order differential operator in divergence form.

In the fractional setting, we can follow a similar strategy. The Dirichlet problem for the fractional Laplacian is

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Omega^c, \end{cases} \tag{8}$$

where now the Dirichlet condition is imposed in the complement of Ω due to the nonlocal nature of the operator. We call it exterior condition.

Existence and uniqueness of solutions to (8) can also be treated by using variational techniques. That is, solutions can be found by minimizing the nonlocal energy functional

$$\mathcal{E}_f^s(u) = \frac{c_{n,s}}{4} \iint_{(\Omega^c\times\Omega^c)^c} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}z - \int_\Omega f\,u$$

among all functions $u \in H^s(\mathbb{R}^n)$ with $u = 0$ in Ω^c. Note that the set $\Omega^c \times \Omega^c$ is not included in the domain of integration of the energy functional, since functions vanish in such a set. Moreover, this is the suitable formulation when dealing with nonhomogeneous exterior conditions in order to work with well-defined energies.

If we look now for free minimizers of the energy functional $\mathcal{E}_f^s$ we obtain a nonlocal version of the Neumann problem as the Euler-Lagrange equation. It is the following:

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ \mathcal{N}_s u &= 0 \quad \text{in } \Omega^c, \end{cases} \tag{9}$$

where $\mathcal{N}_s$ is a "nonlocal Neumann derivative" defined as

$$\mathcal{N}_s u(x) = c_{n,s} \int_\Omega \frac{u(x)-u(z)}{|x-z|^{n+2s}}\,\mathrm{d}z.$$

This Neumann problem for the fractional Laplacian was first introduced by Dipierro, Ros-Oton, and Valdinoci [96] (see also [100]), and has been subsequently studied in several papers (see for example [1, 8, 77, 143]). It shares some interesting properties with its classical analogue, for instance, the probabilistic interpretation that we explain next or the conservation of mass for the associated heat equation.

As in the local framework, both the Dirichlet and Neumann problems can be extended to more general nonlocal operators with variational structure.

Heuristic interpretation through probabilities

Both the Dirichlet and Neumann problems have an interesting probabilistic interpretation in terms of random walks. Next, we present them from a heuristic point of view.

Let us consider Brownian motion in $\mathbb{R}^n$. We may think that this process represents a continuous random walk of a particle. One can restrict the movement inside the bounded domain Ω by finishing the process when the particle touches the boundary. Let us define $u(x)$ as the expected gain of the random walk starting at x and receiving a quantity proportional to f at each point of the path. It turns out that u solves the Dirichlet problem (6). Thanks to a similar interpretation for the heat equation, when $f \equiv 1$, the solution u can be understood as the expected exit time of the particle.

Alternatively, we can restrict the movement of the particle to stay inside the domain Ω by making it bounce (by reflecting) when it hits the boundary $\partial\Omega$. In this scenario, one can formally deduce that the large-time expected gain of the particle satisfies the Neumann problem (7). Since this process has no end, certain conditions seem natural to be needed in order to guarantee the convergence of the long-time expected gain. For instance, if $f \equiv 1$, it is clear that the accumulated gain grows up to infinity with time. In this way we can understand from a different point of view why some conditions need to be imposed on f in order to have existence of solution for the Neumann problem, namely $\int_\Omega f = 0$.

In the nonlocal setting, similar processes can be defined by replacing the Brownian motion by Lévy flights, but we need to decide what happens when the particle jumps to the exterior of the domain Ω. As in the case of the Laplacian, we recover the Dirichlet problem (8) if we impose the movement to finish when the particle goes out of the domain. On the other hand, the Neumann problem corresponds to reflecting the particles into the domain in order to continue the process. When the probability law that one uses to reflect them is proportional to the one of the Lévy flight, the Neumann problem for the fractional Laplacian (9) is recovered (see [96] and references therein for an extended explanation).

In both the local and nonlocal cases we can consider random processes that are neither isotropic nor symmetric. In such cases we end up with problems involving either second order or integro-differential operators which are different from the Laplacian or the fractional Laplacian, respectively.

Regularity of solutions

The regularity of solutions is one of the most basic and important questions in the theory of PDEs. In the classical framework, it is known that any function u weakly satisfying the equation $-\Delta u = f$ in B_1 is such that $u \in C^{1,\alpha}(B_1)$ for every $\alpha \in (0,1)$ when $f \in L^\infty(B_1)$. Moreover, it is satisfied

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}\right).$$

It immediately follows from the previous estimate that solutions to the Dirichlet and Neumann problem are $C^{1,\alpha}$ in the interior of Ω. Higher regularity can be deduced when the source term f is more regular. In addition, similar results can be proved when the Laplacian is replaced by other second order elliptic differential operators.

As it is natural, the boundary regularity of solutions depends on the smoothness of the domain and the boundary conditions. A usual strategy to establish the regularity both for solutions to the Dirichlet and Neumann problem consists of flattening the boundary and reflecting the domain through it. Let us explain this procedure by following the steps represented in Figure 1. If the domain is smooth, given any point $x_0 \in \partial\Omega$ we can find a regular diffeomorphism Φ mapping a neighborhood of that point into the half ball. Moreover, if we define $v(x) = u(\Phi(x))$ and $\widetilde{f}(x) = f(\Phi(x))$, they satisfy the linear equation $Lv = \widetilde{f}$ in $B_1^+ = B_1 \cap \{x_n > 0\} \subset \mathbb{R}^n$ for some elliptic second order differential operator L in divergence form with smooth coefficients. Next, we can extend the problem to the whole ball by an odd/even reflection depending on whether the boundary conditions are Dirichlet or Neumann. Thus, we have reduced the study of the boundary regularity for the Laplacian in a general smooth domain to the study of the interior regularity for a second order differential equation in a ball. In this way, we can deduce the boundary regularity of u from the interior one of the auxiliary function v.

In the fractional framework, interior regularity for functions satisfying $(-\Delta)^s u = f$ in B_1 is known since the work by Landkof [140] in the seventies. Indeed, once the fundamental solution and the fractional Poisson kernel for the ball are known explicitly, the strategy through the regularity results is analogous to the classical one for the Laplacian, and the following estimate can be deduced:

$$\|u\|_{C^{\alpha}(\overline{B_{1/2}})} \leq C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1)}\right).$$

for every $0 \leq \alpha \leq 2s$ provided that it is not an integer. Similar results for general elliptic integro-differential operators can be obtained by using different techniques (see [165]).

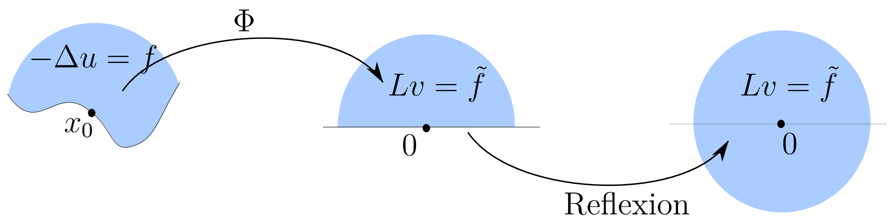

Figure 1: How to reduce the boundary regularity for the Laplacian to the interior one for an elliptic operator.

Unlike what happens in the local case, it turns out that the boundary regularity is much more delicate than the interior one. This is because in this nonlocal context it is not known how to use a reflection method to study solutions near the boundary, and a completely different strategy is needed.

In the Dirichlet problem for the fractional Laplacian, Ros-Oton and Serra [163] overcame this difficulty by using methods that extend the Krylov [139] boundary Harnack theory for local equations in non-divergence form. It is based on two ingredients: the interior regularity results for the fractional Laplacian and a suitable upper barrier. With such a strategy they prove that $u \in C^s(\mathbb{R}^n)$ when $f \in L^\infty(\Omega)$, i.e., u is C^s in the whole $\mathbb{R}^n$ and, in particular, up to the boundary.

Let us point out that this Hölder regularity is optimal. Indeed,

$$u(x) = \frac{2^{-2s}\Gamma(n/2)}{\Gamma\left(\frac{n+2s}{2}\right)\Gamma(1+s)}\left(1-|x|^2\right)_+^s$$

can be checked to be the solution of the Dirichlet problem

$$\begin{cases} (-\Delta)^s u &= 1 \quad \text{in } B_1, \\ u &= 0 \quad \text{in } B_1^c. \end{cases}$$

Hence, it is clear that even being $f \in C^\infty(B_1)$, the solution is C^s up to the boundary and not better. For this reason, it is important to study further the regularity of u/δ^s up to $\partial\Omega$, where

$$\delta(x) = \mathrm{dist}(x, \partial\Omega).$$

In [163], it is proved that u/δ^s is Hölder continuous up to the boundary. Moreover, when f is more regular the Hölder estimates for such a quotient can be improved.

While the Dirichlet problem for the fractional Laplacian is very well understood at this moment, much less is known for the Neumann case. To the best of our knowledge, the boundary regularity results that we establish here are the only ones in the literature. Indeed, even the boundedness of solutions was not known prior to this thesis.

Minimality of solutions

When studying linear problems with variational structure, uniqueness of solutions yields their minimality for the associated energy functional. We have previously noted this in the case of the Dirichlet and Neumann problems for the classical and fractional Laplacian. However, nonlinear problems usually do not enjoy uniqueness and admit solutions which are not minimizers.

A classical problem in the Calculus of Variation consists of finding conditions for a function to be a minimizer of a (local) energy functional of the form

$$\mathcal{E}_L(w) = \int_\Omega G_L\Big(x, w(x), \nabla w(x)\Big)\,\mathrm{d}x.$$

In this framework, the function $G_L(x, \lambda, q)$ is called the Lagrangian of the functional.

Let us recall that we say that $u \in C^1(\Omega)$ is a *minimizer* of the functional energy $\mathcal{E}_L$ if

$$\mathcal{E}_L(u) \le \mathcal{E}_L(w)$$

for all $w \in C^1(\Omega)$ such that $w = u$ on $\partial\Omega$.

The most basic necessary condition to be a minimizer of the functional is being an extremal. By computing the first variation, we know that extremals are weak solutions to the Euler-Lagrange equation

$$\mathcal{L}_L(u) = 0 \quad \text{in } \ \Omega,$$

where

$$\mathcal{L}_L(u)(x) = \partial_\lambda G_L\Big(x, u(x), \nabla u(x)\Big) - \operatorname{div}\Big(\partial_q G_L(x, u(x), \nabla u(x))\Big). \tag{10}$$

If the energy functional is convex, it is well known that being a critical point (extremal or solution) is a sufficient condition to be a minimizer. Many models that arise from physical phenomena exhibit such a convexity property. Nevertheless, there are also important models with nonconvex energy functionals. This is the case of the Allen-Cahn energy or of minimal surfaces, among many others. In this case, the associated energy functionals may have several extremals, with most of them not being minimizers.

In 1879, Weierstrass found a new necessary first order condition for ODEs, the so-called *Weierstrass necessary condition*. Later on it was extended to high dimensional problems as follows: if u is a minimizer of the energy functional $\mathcal{E}_L$, then it must satisfy

$$E(x, u(x), \nabla u(x), \xi) \ge 0 \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^n,$$

where E is the *Weierstrass excess function*

$$E(x, \lambda, q, \widetilde{q}) = G_L(x, \lambda, \widetilde{q}) - G_L(x, \lambda, q) - \partial_q G_L(x, \lambda, q) \cdot (\widetilde{q} - q).$$

Note that the Weierstrass excess function is always nonnegative when the Lagrangian $G_L(x, \lambda, q)$ is convex with respect to q.

In models where there is no uniqueness of extremal, an important type of solutions are the stable ones (sometimes known as local minimizers). These are solutions where the second variation of the energy is nonnegative (equivalently, the operator associated to the linearized equation has nonnegative spectrum). Hence, stability is a second order necessary condition for minimality. From a physical point of view, stable solutions are the observable states in nature since unstable solutions decay towards stable ones.

Regarding sufficient conditions, one effective strategy to establish the minimality of extremals goes through the theory of calibrations and null-Lagrangians:

Definition 1. A functional $\mathcal{C}_L$ is a *calibration* for $\mathcal{E}_L$ and u if the following conditions hold:

($\mathcal{C}1$) $\mathcal{C}_L(u) = \mathcal{E}_L(u)$,

($\mathcal{C}2$) $\mathcal{C}_L(w) \le \mathcal{E}_L(w)$ for all w,

($\mathcal{C}3$) $\mathcal{C}_L$ is a *null-Lagrangian*, i.e., $\mathcal{C}_L(w) = \mathcal{C}_L(\tilde{w})$ for all $w, \tilde{w}$ such that $w \equiv \tilde{w}$ on $\partial\Omega$.

Once the calibration is available, the minimality of the function u follows immediately. Indeed, if $\mathcal{C}_L$ is a calibration for $\mathcal{E}_L$ and u, then, for any w with $w \equiv u$ in $\partial\Omega$, applying ($\mathcal{C}1$), ($\mathcal{C}2$), and ($\mathcal{C}3$) in this order we obtain

$$\mathcal{E}_L(u) = \mathcal{C}_L(u) = \mathcal{C}_L(w) \leq \mathcal{E}_L(w),$$

and u is a minimizer. Here, an a priori existence and regularity result for minimizers is not needed.

A classical result in the Calculus of Variations asserts that if a solution u is embedded[1] in an *extremal field* $\{u^t\}_{t\in\mathbb{R}}$ (i.e., a one-parameter family of extremals of the energy functional $\mathcal{E}_L$ whose graphs give a foliation) where each leaf satisfies the Weierstrass necessary condition, then it is a minimizer. One can show it by giving an explicit calibration. That is, defining the *leaf–parameter function* $t : \Omega \times \mathbb{R} \to \mathbb{R}$ as the unique $t = t(x, \lambda) \in \mathbb{R}$ such that $u^t(x) = \lambda$, one can construct the functional

$$\begin{aligned}\mathcal{C}_L(w) = &\int_\Omega \Big(\partial_q G(x, u^t(x), \nabla u^t(x))\Big)\Big(\nabla w(x) - \nabla u^t(x)\Big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x \\ &+ \int_\Omega G(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x.\end{aligned} \tag{11}$$

It can be checked that $\mathcal{C}_L$ satisfies ($\mathcal{C}1$) and ($\mathcal{C}3$) with $u = u^t$, for all t. Moreover, it is well known that the energy functional $\mathcal{E}_L$ can be decomposed in terms of the calibration functional $\mathcal{C}_L$ and the Weierstrass excess function E as

$$\mathcal{E}_L(w) = \mathcal{C}_L(w) + \int_\Omega E(x, u^t(x), \nabla u^t(x), \nabla w(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x.$$

Hence, if each u^t satisfies the Weierstrass necessary condition, it follows that ($\mathcal{C}2$) is also satisfied, and $\mathcal{C}_L$ is a calibration for each u^t.

As an illustrative example, the energy functional

$$\mathcal{E}_F(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 - \int_\Omega F(w), \tag{12}$$

where $F \in C^1$, admits a calibration

$$\mathcal{C}_F(w) = \int_\Omega \left\{\nabla u^t \cdot \nabla w - \frac{1}{2}|\nabla u^t|^2\right\}\Big|_{t=t(x,w(x))} - \int_\Omega F(w),$$

when the functions u^t satisfy the semilinear equation $-\Delta u^t = F'(u^t)$ in Ω.

Calibrations and extremal fields have also been studied in relation to the theory of minimal surfaces. In this framework, once the existence of a foliation by minimal sets E^t is known, a calibration for the perimeter is given by

$$\mathcal{C}_{\mathcal{P}}(F) = \int_{\Omega\cap\partial F} \nu_{\partial E^t}\Big|_{t=\phi(x)} \cdot \nu_{\partial F}\,\mathrm{d}\mathcal{H}^{n-1}$$

[1] If a solution u of the Euler-Lagrange equation is stable, there is a natural way to embed it into an extremal field. Indeed, by the Implicit Function Theorem there is a small tubular neighborhood of u which is foliated by a family of solutions, see Proposition 6.3.4 in [125].

where $\nu_{\partial A}$ denotes the outward normal vector to the surface ∂A. In [88], a calibration argument was used by De Philippis and Paolini to give a simple proof of the minimality of the Simons cone in dimensions $n \geq 8$.

While the theory of calibrations for local energies is well known, there are almost no results when dealing with nonlocal energies. Even the form of a calibration in the simplest case of quadratic nonlocal energies was not known. To the best of our knowledge, prior to this thesis the unique result in this direction is due to Cabré [47] for the fractional perimeter

$$\mathcal{P}_K(F) = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} |\mathbb{1}_F(x) - \mathbb{1}_F(y)| \, K(x-y) \, \mathrm{d}x \, \mathrm{d}y.$$

In this case, a calibration functional is given by

$$\mathcal{C}_K(F) = \frac{1}{2} \iint_{(\Omega^c \times \Omega^c)^c} \left(\mathbb{1}_{(E^t)^c}(y) - \mathbb{1}_{E^t}(y) \right)\Big|_{t=\phi(x)} (\mathbb{1}_F(x) - \mathbb{1}_F(y)) K(x-y) \, \mathrm{d}x \, \mathrm{d}y.$$

Classification of solutions

The study of rigidity properties or classification of solutions in the whole space or in the half-space is well known to be in strong connection with the regularity theory of solutions in bounded domains. Moreover, this relation can be exploited in both directions.

Liouville theorems for linear problems

The most basic classification result in PDEs is the classical theorem of Liouville. It states that any bounded harmonic function in $\mathbb{R}^n$ is constant. It is also well known that the result remains true when only asking for the solution to be bounded from one side (from above or from below). Furthermore, this condition can be replaced (see Theorem 9.10 in [15]) by the much weaker one

$$\liminf_{|x| \to \infty} \frac{u(x)}{|x|} = 0,$$

without changing the conclusion of the theorem. Note that one can understand this condition as the solution cannot grow more than linear from below (or above). One proof of these results relies on the mean value property for harmonic functions.

Liouville theorems have been generalized to linear nonlocal operators, where one cannot use as simple arguments. In [37], Bogdan, Kulczycki, and Nowak used a gradient estimate to show that nonnegative s-harmonic functions must be constant. Later on, Chen, D'Ambrosio, and Lin [70] proved a Liouville theorem for the fractional Laplacian under the condition

$$\liminf_{|x| \to \infty} \frac{u(x)}{|x|^\gamma} = 0,$$

with $0 \leq \gamma \leq 1$ and $\gamma < 2s$. They used potential theory and Fourier analysis, together with the distributional notion of solutions. Finally, Fall [104] established through Cauchy-type estimates for the derivatives (from Poisson kernel representation) that every s-harmonic (in the distributional sense) function u such that $(1 + |\cdot|^{n+2s})^{-1} u \in L^1(\mathbb{R}^n)$ is affine.

Regarding more general integro-differential operators, Ros-Oton and Serra [165] proved that when L is a symmetric stable operator, i.e., with $K(z) = a(z/|z|)|z|^{-n-2s}$ for some $a \geq 0$ and $a \in L^1$, and the solution u satisfies the growth condition

$$||u||_{L^\infty(B_R)} \leq C R^\gamma, \quad \text{when} \quad R > 1,$$

for some $\gamma < 2s$, then u is affine. They obtained this result by iterating a C^α-estimate found using the heat kernel. Finally, Fall and Weth [106] have proved a similar result for distributional solutions. They were able to arrive at the same conclusion by using Fourier analysis when $a \in L^\infty$ and has certain regularity, but is not necessarily positive.

Nonlinear 1D problems

In some nonlinear settings, the analogue of the Liouville theorem is a rigidity result establishing the one-dimensional symmetry of solutions. Let us recall that we say that a function is one-dimensional if it depends only on one Euclidean variable. Equivalently, all its level sets are flat. For this reason, the classification of solutions in dimension one plays an important role when studying qualitative properties of solutions to some nonlinear problems in high dimensions.

In the case of semilinear local equations, the one dimensional problem is just the nonlinear second order ODE

$$-u'' = f(u) \quad \text{in } \mathbb{R}, \tag{13}$$

whose bounded solutions are completely classified. Indeed, there are only three types of nonconstant bounded solutions: *layer solutions* (monotone solutions), *ground state* solutions (even, with only one change of monotonicity), and *periodic solutions*. The proof of this classification result is based on Picard's existence and uniqueness theorem for ODEs. Moreover, the nonlinearities that admit a solution of each type can be characterized. That is, given the potential $F \in C^1([a,b])$ such that $F' = f$, then

- there exists a layer solution to (13) if and only if

$$F'(a) = F'(b) = 0 \quad \text{and } F < F(a) = F(b) \ \text{ in } (a,b); \tag{14}$$

- there exists a ground state solution to (13) if and only if

$$F'(a) = 0 < F'(b) \quad \text{and } F < F(a) = F(b) \ \text{ in } (a,b); \tag{15}$$

- there exists a periodic solution to (13) if and only if

$$F'(a) < 0 < F'(b) \quad \text{and } F < F(a) = F(b) \ \text{ in } (a,b). \tag{16}$$

Given L an integro-differential operator of the form (2), a similar classification for the equation

$$Lu = f(u) \quad \text{in } \mathbb{R} \tag{17}$$

is still open (even in the case of the fractional Laplacian), although there are some partial results in this direction.

In the 90s, Amick and Toland established the first classification results for the square root of the Laplacian, $L = (-\Delta)^{1/2}$, but only certain specific nonlinearities which make the problem "completely integrable". In [10], they gave an explicit expression for all the solutions to the Benjamin-Ono equation

$$(-\Delta)^{1/2} u = -u + u^2 \quad \text{in } \mathbb{R}.$$

They used the extension problem to reduce the nonlocal equation to a separable ODE in the complex variable. In particular, they proved that

$$u(x) = \frac{2}{1+x^2},$$

which was found by Benjamin [25], is the unique (up to translations) ground state solution. Moreover, they showed the existence of periodic solutions for each period greater than 2π.

Next, Toland [188] proved that the derivative of any bounded solution of the Peierls-Nabarro problem

$$(-\Delta)^{1/2}u = \sin(u) \quad \text{in} \;\; \mathbb{R}$$

is the difference of two bounded solutions to the Benjamin-Ono equation. As a consequence, the unique bounded solutions to the Peierls-Nabarro equation are the periodic ones and the layer solution

$$u(x) = 2\arctan(x),$$

first discovered by Peierls [159].

Later on, Cabré, Sire, and Solà-Morales [58, 57] studied layer solutions for general nonlinearities and fractional powers by using the extension problem for the fractional Laplacian. They found that (14) turn out to be also the necessary and sufficient conditions in order to have existence of layer solution to (17) when $L = (-\Delta)^s$. Moreover, they show that uniqueness (up to translations) follows once the condition $f'(\pm 1) = F''(\pm 1) < 0$ is assumed. In these works they also prove symmetry, asymptotic behavior, and variational properties of the layer solution. In [83], Cozzi and Passalacqua established the analogous results in the case of general elliptic integro-differential operators.

In relation to a conjecture by De Giorgi concerning monotone solutions of the Allen-Cahn equation, layer solutions play a crucial role. Moreover, the unique layer solution that vanishes at the origin will be of great interest in this thesis. Denoted by u_0, it satisfies

$$\begin{cases} Lu_0 &= f(u_0) & \text{in } \mathbb{R}\,, \\ \dot{u}_0 &> 0 & \text{in } \mathbb{R}\,, \\ u_0(0) &= 0\,, & \\ \lim_{x\to\pm\infty} u_0(x) &= \pm 1\,, & \end{cases} \tag{18}$$

where L is an integro-differential operator of the form (2) and f is a nonlinearity of Allen-Cahn type.

Now that conditions of existence and uniqueness for layer solution are understood in the nonlocal setting, further properties are of interest. For instance, nondegeneracy is one of them. Dávila, del Pino, and Musso [85] proved the nondegeneracy of the layer solution when $L = (-\Delta)^{1/2}$ (with the extension problem) in order to construct solutions to (17) that develop multiple transitions from -1 to 1. In [101], Du, Gui, Sire, and Wei generalized the nondegeneracy to $s \in (1/2, 1)$ and used it to show the existence of clustering-layered solutions for a fractional inhomogeneous Allen-Cahn equation.

Turning to ground state solutions, the most important result in the literature in the nonlocal framework is due to Frank and Lenzmann [117]. In this work they established the uniqueness of this kind of solutions when $f(u) = u^{p+1} - u$ for p subcritical. In order to prove this, they first established their nondegeneracy by developing a delicate spectral theory for fractional linear operators (where the local extension problem and the polynomial structure of the nonlinearity play a crucial role) as the ones we treat in Chapter 1. Finally, they used the nondegeneracy to show the uniqueness result by an implicit function argument and the well known result for the local case ($s = 1$).

Apart from the works by Amick and Toland, with particular nonlinearities that provide a "completely integrable" equation, periodic solutions are not very well studied in the nonlocal framework. Let us mention the work by Barrios, Garcia-Melian, and Quass [22] where periodic solutions to certain fractional problems are constructed by using variational methods. Further work on periodic solutions is now being carried out by Cabré, Csato, Mas, and Solà-Morales.

A conjecture by De Giorgi about the one-dimensional symmetry of monotone solutions

The Allen-Cahn equation

$$-\Delta u = u - u^3 \quad \text{in} \quad \mathbb{R}^n \tag{19}$$

is the semilinear PDE that appears as the Euler-Lagrange equation of the Ginzburg-Landau energy functional

$$\mathcal{E}_{AC}(u) = \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{2} + \frac{(1-u^2)^2}{4}.$$

It is an important reaction-diffusion model that was first proposed in 1972 to describe phase separation in metal alloys [6, 7]. Furthermore, it also arises when studying interfaces in binary fluids, crystal dislocations, pattern formation in polymers, and superconductivity (see [65, 159, 154, 157, 126]).

This model presents a competition between the potential and the kinetic energies. That is, on the one hand, the potential function has two minima at $u = -1$ and $u = +1$ that represent the pure phases. In this way, minimizers "prefer" to take values very close to ± 1. On the other hand, the gradient term penalizes sharp changes between the two phases, avoiding the formation of unnecessary interfaces. Therefore, typical minimizers of the energy essentially split the space into two regions that are separated by a moderated transition.

These rough arguments suggest that minimizers of the Allen-Cahn equation should be close to the characteristic function (with values ± 1) of a set with "small" boundary. This is indeed true and gives a deep relation between Allen-Cahn and the theory of minimal surfaces. In fact, in the 80s, Modica and Mortola [151] proved the Γ-convergence of a rescaled version of the energy functional $\mathcal{E}_{AC}$ to a multiple of the perimeter functional $\mathcal{P}$. As a consequence, the blow-down sequence of a global minimizer converges in some sense to the characteristic function of a set whose boundary is a minimizing minimal surface.

Such a connection between minimal surfaces and the Allen-Cahn equation, together with the classification of entire minimal graphs, motivated a conjecture by De Giorgi [87]. He stated in 1978 that the level sets of every bounded monotone solution to the Allen-Cahn equation are hyperplanes, at least if the space dimension satisfies $n \leq 8$. Such a classification result is the nonlinear analogue of the Liouville theorem in this framework.

Let us explain the intuition behind the statement. Once the monotonicity is assumed, the level sets of the solution, and hence those of the blow down sequence, are graphs converging to a minimal graph. From the Bernstein problem, minimal graphs are known to be hyperplanes up to dimension 7, thus, the level sets of the solution are flat when looking "from very far", which suggest the validity of the conjecture.

The conjecture was first proved to be true in dimensions $n = 2$ and $n = 3$ by Ghoussoub and Gui [123], and by Ambrosio and Cabré [9], respectively. For dimensions $4 \leq n \leq 8$, Savin [167] established the conjecture under the additional assumption

$$\lim_{x_n \to \pm\infty} u(\tilde{x}, x_n) = \pm 1 \quad \text{for all } \tilde{x} \in \mathbb{R}^{n-1}. \tag{20}$$

Later on, del Pino, Kowalczyk, and Wei [89] constructed a counterexample in dimensions $n \geq 9$ by using the gluing method. Thus, the conjecture is still nowadays not completely closed (see [108, 67] and references therein).

Let us point out that there is a strong connection between monotone solutions and minimizers to the Allen-Cahn equation. First, it is well known that the monotonicity assumption yields to the minimality in a certain class of compactly supported perturbations that can be extended to global minimality when the limiting condition (20) is assumed[2]. Furthermore, Jerison and Monneau [135] gave a different strategy to the one by [89] to build a counterexample to the conjecture by taking into account the connection between monotonicity and minimality. They proved the existence of a bounded monotone solution to the Allen-Cahn equation in $\mathbb{R}^n$ which is not a one-dimensional solution from the existence of a bounded global minimizer in $\mathbb{R}^{n-1}$ which is not one-dimensional and even with respect to each coordinate.

In order to complete the program proposed by Jerison and Monneau, a classification of global minimizers is needed. In this respect, Savin established in [167] that global minimizers are one-dimensional when $n \leq 7$. On the other hand, from the relation between the Allen-Cahn equation and minimal surfaces, it is natural to look for an analogue of the Simons cone, which is known to be the simplest nonplanar minimizing minimal surface. Thus, a natural candidate to be nontrivial minimizer is expected to be found in the class of functions whose zero level set is the Simons cone and share its symmetries. This property motivates the following definition (we use the notation $x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$):

Definition 2. We say that a bounded solution u to (19) in even dimension $n = 2m$ is a *saddle-shaped solution* (or simply *saddle solution*) if

1. u is doubly radial, i.e., $u = u(|x'|, |x''|)$.
2. u is odd with respect to the Simons cone, that is, $u(|x'|, |x''|) = -u(|x''|, |x'|)$.
3. $u > 0$ in $\mathcal{O} = \{|x'| > |x''|\}$.

Dang, Fife, and Peletier [84] studied saddle-shaped solutions for the first time in dimension $2m = 2$ by establishing the existence, uniqueness, and some qualitative properties. The instability of these solutions in dimension $2m = 2$ was shown by Schatzman [174]. Later, Cabré and Terra [59, 60] extended the existence, asymptotic behavior, and monotonicity properties to dimensions higher than 2. Moreover, they showed that saddle-shaped solutions are unstable in dimension $2m = 4$ and $2m = 6$. In [46], Cabré established the uniqueness in every even dimension $2m \geq 2$, as well as the stability in dimensions $2m \geq 14$. In the recent work [147], stability was established in the remaining dimensions $8 \leq 2m \leq 12$. It is still an open problem to determine the minimality of the saddle-shaped solution in dimensions $2m \geq 8$.

In the last decades, variants to the Ginzburg-Landau energy have been introduced to model different phenomena. One of them consists of studying the Dirichlet energy in a half space with a boundary reaction. Indeed, it appears as a model for crystal dislocations and boundary vortices in micromagnetism. Let us recall that the Dirichlet energy in a half space can be rewritten as the fractional seminorm of its trace. Hence, it is natural to study

[2] By the translation invariance of the equation, a monotone solution u can be embedded in the extremal field $u^t(x) = u(\widetilde{x}, x_n + t)$. Hence, the calibration argument gives the minimality among functions w such that graph $w \subset \mathcal{G}$ where

$$\mathcal{G} = \Big\{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} \ : \lim_{x_n \to -\infty} u(\widetilde{x}, x_n) \leq \lambda \leq \lim_{x_n \to +\infty} u(\widetilde{x}, x_n)\Big\}.$$

the nonlocal energy functional

$$\mathcal{E}^s_{AC}(u) = \frac{c_{n,s}}{4} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} + \int_{\mathbb{R}^n} \frac{(1 - u^2)^2}{4},$$

which takes into account long-range particle interactions. Furthermore, the Euler-Lagrange equation associated to this energy functional is the fractional Allen-Cahn equation

$$(-\Delta)^s u = u - u^3 \quad \text{in} \quad \mathbb{R}^n.$$

In this framework, there is also a deep connection between the fractional Allen-Cahn equation and (local and nonlocal) minimal surfaces. A rescaled version of the energy functionals Γ-converge to the classical perimeter if $s \in [1/2, 1)$, and to the fractional one if $s \in (0, 1/2)$ [5, 127, 170].

The connection between local and nonlocal minimal surfaces with the fractional Allen-Cahn equation has motivated the study of fractional versions of De Giorgis's conjecture in the last years. This conjecture was proved to be true by Cabré and Sire [57], and by Sire and Valdinoci [182] in dimension $n = 2$ (see Cabré and Solà-Morales [58] for $s = 1/2$). Later, the conjecture was established in dimension $n = 3$ by Cabré and Cinti [48, 49] for $1/2 \leq s < 1$, and by Dipierro, Farina, and Valdinoci [94] for $0 < s < 1/2$. Under the additional limiting assumption (20), Savin [168, 169], for $1/2 \leq s < 1$, and Dipierro, Serra, and Valdinoci [97], for s close to $1/2$, have established the validity of the conjecture in dimensions $4 \leq n \leq 8$. More recently, Figalli and Serra [116] have proved the conjecture in dimension $n = 4$ and $s = 1/2$ without assuming the additional condition (20). Let us point out that this last result has no analogue in the local framework. A counterexample to the fractional version of the conjecture in dimensions $n \geq 9$ for $s \in (1/2, 1)$ was announced by Chan, Liu, and Wei [66].

In the fractional setting, the regularity of nonlocal minimal surfaces is still a widely open problem. For instance, the Simons cone (which is stationary for the fractional perimeter) is not known to be a minimizer in any dimension. Thus, results on the fractional Allen-Cahn equation could provide some insights on this problem. Since the saddle-shaped solution is expected to be a nontrivial minimizer of the Allen-Cahn energy, it is an interesting object regarding the previous questions.

Previous to this thesis, saddle-shaped solutions to the fractional Allen-Cahn equation have only been studied by Cinti [75, 76]. In such works, their existence and some further properties, such as their instability in dimensions 4 and 6, were proved by using the extension problem for the fractional Laplacian.

Results of the thesis

Uniqueness for linear integro-differential equations in the real line and applications

In Chapter 1, which corresponds to [109], we study the uniqueness, up to a multiplicative constant, of solutions to the linear integro-differential equation

$$L\varphi - c(x)\varphi = 0, \quad \text{in } \mathbb{R}, \tag{21}$$

under certain assumptions on the nonlocal operator L and the potential function c, and in the presence of a positive solution or of an odd solution vanishing only at zero.

The main result of the chapter is the following:

Theorem 3 (see Theorems 1.1.1 and 1.1.4)**.** *Let L be an integro-differential operator of the form* (2) *with even kernel K satisfying the ellipticity condition* (4) *for some $s \in [1/2, 1)$. Assume the potential function $c \in L^\infty(\mathbb{R})$ satisfies*

$$c(x) \leq -c_0 < 0 \quad \text{in } \mathbb{R} \setminus [-R_0, R_0], \qquad \text{and} \qquad ||c||_{C^{\beta_0}(\mathbb{R})} < +\infty$$

for some positive constants c_0, R_0 and β_0.

Let $w, \tilde{w}$ be two bounded solutions of the linear equation (21) *such that*

- *either $w > 0$ in $\mathbb{R}$,*
- *or K is decreasing in $(0, +\infty)$, both w and $\widetilde{w}$ are odd, and $w > 0$ in $(0, \infty)$.*

Then

$$\frac{\widetilde{w}}{w} \equiv ctt.$$

To the best of our knowledge, this is the first uniqueness result for general integro-differential operators in dimension one. Previous analogous results could only cover the case of the fractional Laplacian by using potential theory, Fourier analysis, or the Caffarelli-Silvestre extension problem. Here, we follow a nonlocal Liouville-type method developed by Hamel, Ros-Oton, Sire, and Valdinoci [132] for integro-differential operators with compactly supported kernels in dimension two. We adapt the strategy in order to remove the compact support condition by taking advantage of the one dimensionality of the problem.

The first step is controlling the growth of the quotient of the solutions $\sigma = \widetilde{w}/w$, for which we need to overcome some difficulties. On the one hand, we need to control the quotient even if the positive solution is arbitrarily close to zero at infinity. On the other hand, we have to ensure that in the odd framework it can be extended up to the origin, where the denominator vanishes. The strategy consists of proving and applying a maximum principle in the exterior of an interval to compare both solutions by transferring the information from an interval (where we know the quotient is bounded) to the whole line.

The second step is establishing an integral estimate for the kernel K in the unbounded cross-shaped regions of the plane

$$S_R = (B_{2R} \times B_R^c) \cup (B_R^c \times B_{2R}) \subset \mathbb{R} \times \mathbb{R}.$$

Indeed, we obtain that given $0 \leq \gamma \leq \min(s, 1/2)$,

$$\int_{S_R} \min\left\{1, \frac{|x-y|}{R}\right\}^2 |x|^{2\gamma} K(x-y)\, dxdy \leq C\, R^{1+2\gamma-2s},$$

for a positive constant C not depending on R.

With these ingredients, we use that the quotient of two solutions to (21) satisfies

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} \big(\sigma(x) - \sigma(y)\big)^2 \big(\tau^2(x) + \tau^2(y)\big)\, w(x)\, w(y)\, K(x-y)\, dxdy \\ = -\int_{\mathbb{R}}\int_{\mathbb{R}} \big(\sigma^2(x) - \sigma^2(y)\big)\big(\tau^2(x) - \tau^2(y)\big) w(x)\, w(y)\, K(x-y)\, dxdy,\end{aligned}$$

for any given function $\tau \in C_c^\infty(\mathbb{R})$, to conclude that it is constant. We point out that finding such an identity for general integro-differential operators is the crucial contribution by Hamel, Ros-Oton, Sire, and Valdinoci [132].

As a consequence of the uniqueness result, we can prove qualitative properties of solutions to the semilinear problem (17). The first one is the nondegeneracy of layer solutions to semilinear Allen-Cahn type equations.

Corollary 4 (see Theorem 1.1.3)**.** *Let L be an integro-differential operator of the form (2) with even kernel K satisfying the ellipticity condition (4) for some $s \in [1/2, 1)$. For $\gamma > 0$, let $f \in C^{1,\gamma}([-1, 1])$ be any given nonlinearity such that $f'(\pm 1) < 0$.*

Assume that u is a bounded solution to the semilinear equation (17), satisfying $u' > 0$ and $\lim_{x\to\pm\infty} u(x) = \pm 1$.

Then, u is nondegenerate, i.e., up to a multiplicative constant u' is the unique bounded solution to the linearized equation $L\varphi - f'(u)\varphi = 0$ in $\mathbb{R}$.

Let us mention that condition $f'(\pm 1) < 0$, which corresponds to $c = f'(u)$ being negative at infinity, is a natural assumption. Indeed, it is the same hypothesis needed to prove uniqueness (up to translations) of the layer solutions (see Theorem 1.2 in [58] in the case of the half-Laplacian). Moreover, this is also the needed condition for ± 1 to be local minimizers of the associated energy.

Finally, we prove a partial nondegeneracy result for ground state solution to (17).

Corollary 5 (see Theorem 1.1.5)**.** *Let L be an integro-differential operator of the form (2) with kernel K being even and decreasing in $(0, +\infty)$, and satisfying the ellipticity condition (4) for some $s \in [1/2, 1)$. For $\gamma > 0$, let $f \in C^{1,\gamma}([0, 1])$ be any given nonlinearity such that $f'(0) < 0$.*

Assume that u is a bounded even solution to the semilinear equation (17), satisfying $u' < 0$ in $(0, +\infty)$ and $\lim_{x\to\pm\infty} u(x) = 0$.

Then, up to a multiplicative constant u' is the unique bounded odd solution to the linearized equation $L\varphi - f'(u)\varphi = 0$ in $\mathbb{R}$.

As in the nondegeneracy result for layer solutions, the condition $f'(0) < 0$ is a natural assumption since it is a necessary condition in order to be $v \equiv 0$ a local minimizer of the associated energy.

Boundary regularity for the fractional Neumann problem

In Chapter 2 (corresponding to [13]), we study the regularity up to the boundary of solutions to the Neumann problem for the fractional Laplacian (9).

While the Dirichlet problem is very well understood, much less is known for the Neumann case. Our main result is the following:

Theorem 6 (see Theorem 2.1.1)**.** *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain. Let $s \in (0, 1)$, and u be any weak solution of (9) with $f \in L^q(\Omega)$, with $q > \frac{n}{2s}$ and $\int_\Omega f = 0$.*

Then,

$$\|u\|_{C^\alpha(\overline{\Omega})} \le C \left(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)} \right),$$

for some $\alpha > 0$. Moreover, if $s > \frac{1}{2}$, $q > n$, and Ω is C^1, we then have

$$\|u\|_{C^{2s-1+\alpha}(\overline{\Omega})} \le C \left(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$

The constants C and α depend only on n, s, q, and Ω.

Prior to our result, the interior regularity for the Neumann problem was well understood, but near the boundary even the boundedness of solutions was open. In the fractional case, we do not know how to reduce the boundary regularity to the interior one by using reflections, as it occurs in the local framework. Hence, different strategies are needed. In the Dirichlet case, Ros-Oton and Serra succeeded by extending the boundary Harnack theory for local equations in non-divergence form, based on barriers and an oscillation lemma. In the Neumann case, instead, this approach does not work. Our main contribution shows that the corresponding local theory for equations in divergence form (boundary Moser iteration) is appropriate for the nonlocal Neumann problem.

The first difficulty we encounter when studying the Neumann problem is that solutions are unknown both inside and outside the domain Ω. Hence, a key point to establish the regularity result is using an equivalent formulation for the problem. In [1], Abatangelo first noticed that the Neumann problem (9) can be transformed into a regional-type problem in Ω for the new operator

$$L_\Omega u(x) = \mathrm{PV}\int_\Omega \big(u(x)-u(y)\big)K_\Omega(x,y)\,dy, \tag{22}$$

where

$$K_\Omega(x,y) = \frac{c_{n,s}}{|x-y|^{n+2s}} + c_{n,s}\int_{\Omega^c}\frac{dz}{|x-z|^{n+2s}|y-z|^{n+2s}\int_\Omega \frac{dw}{|z-w|^{n+2s}}}, \qquad x,y\in\Omega, \tag{23}$$

is a kernel with logarithmic singularity along $\partial\Omega$. In particular, we show and use the estimate

$$K_\Omega(x,y) \asymp \frac{1+\log^-\left(\frac{d_{x,y}}{|x-y|}\right)}{|x-y|^{n+2s}} \qquad \text{for all} \quad x,y\in\Omega,$$

where $\log^- t = \max\{0, -\log t\}$.

Thanks to this formulation, we are able to develop a delicate Moser iteration on the boundary (for positive and negative powers) to prove both the L^∞ and C^α regularity of solutions. The strategy follows that of Kassmann [136] for interior regularity. Nevertheless, the logarithmic singularity of the kernel introduces several difficulties.

Next, in order to show higher regularity we use a blow-up argument (in the spirit of [165]) together with the following Liouville theorem with nonlocal Neumann condition:

Theorem 7 (see Thereom 2.5.1). *Let $\Omega = \mathbb{R}^n_+ = \{x_n > 0\}$ and $s\in\left(\frac{1}{2},1\right)$. Let L_Ω and K_Ω be given by (22)-(23). Assume v is a weak solution to*

$$L_\Omega v = 0 \quad \text{in } \mathbb{R}^n_+$$

with Neumann condition on $\partial\mathbb{R}^n_+ = \{x_n = 0\}$. Let $\alpha > 0$ be given by Theorem 6, and assume that

$$||v||_{L^\infty(B_R^+)} \le C_0(1+R^{2s-1+\varepsilon}) \quad \text{for all} \quad R>0,$$

for some C_0 and $\varepsilon\in(0,\alpha)$. Then,

$$v(x) = a + b\cdot x$$

for some $a\in\mathbb{R}$ and $b\in\mathbb{R}^n$ with $b_n = 0$. Moreover, if $2s-1+\epsilon<1$ then $b=0$.

The proof of this result is not standard and does not follow from classical tools such as even reflection for harmonic functions. Moreover, the extension problem for the fractional Laplacian is of no use here, and therefore the proof must be also different from the Dirichlet case. That is, the first step in our strategy consists of showing, by using the C^α regularity and the translation invariance of the operator, that the solution is linear in the parallel directions to the hyperplane, i.e.,

$$v(x) = w_0(x_n) + \sum_{i=1}^{n-1} w_i(x_n)x_i.$$

In addition, each of the functions w_i is a weak solution of the Neumann problem in dimension one. Thus, the problem is reduced to proving the Liouville theorem in $\mathbb{R}^+ = (0, +\infty)$. In such a case, solutions to the Neumann problem are, up to an additive constant, also solutions to the Dirichlet problem. Hence, the result follows from a boundary Harnack inequality for that problem. Let us point out that even in 1D, we do not know how to prove the best Neumann Liouville theorem (allowing more growth on the solutions).

Finally, our methods allow us to treat as well the Neumann problem for the regional fractional Laplacian, and we establish the same boundary regularity result. In that case, we refer to the recent work by Fall [105] for further results concerning the regularity of solutions to the Neumann problem.

Saddle-shaped solutions to the integro-differential Allen-Cahn equation

In Chapter 3, corresponding to [111], we study saddle-shaped solutions to the fractional Allen-Cahn equation

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^{2m}, \tag{24}$$

where the nonlinearity $f \in C^{2,\alpha}\big((-1,1)\big)$, for some $\alpha \in (0,1)$ and satisfies

$$f \text{ is odd}, \;\; f(0) = f(1) = 0, \;\; \text{and } f'' < 0 \text{ in } (0,1). \tag{25}$$

In order to treat this problem we follow the same approach as Cinti in [75, 76], where the main tool is the extension problem for the fractional Laplacian. That is, we know that problem (24) is equivalent to

$$\begin{cases} \operatorname{div}(\lambda^{1-2s}\nabla u) &= 0 & \text{in } \mathbb{R}^{2m+1}_+, \\ -d_s \lim_{\lambda\downarrow 0} \lambda^{1-2s} u_\lambda(x,\lambda) &= f(u) & \text{on } \partial\mathbb{R}^{2m+1}_+ = \mathbb{R}^{2m}. \end{cases}$$

The first main result of this chapter establishes the uniqueness of the saddle-shaped solution.

Theorem 8 (see Theorem 3.1.3)**.** *Let $s \in (0,1)$ and let f be a function satisfying (25). Then, for every even dimension $2m \geq 2$, there exists a unique saddle-shaped solution to problem (24).*

The proof of the result follows the strategy developed by Cabré [46] for the classical problem. It is based on using a maximum principle for the linearized operator at a saddle-shaped solution as well as the asymptotic behavior of the saddle solution. While the second one was proved by Cinti [75, 76], we need to establish the maximum principle.

Proposition 9 (see Proposition 3.1.4)**.** *Let u be a saddle-shaped solution to* (24)*. Let $\Omega \subset \mathcal{O} \times (0, +\infty) \subset \mathbb{R}^{2m+1}_+$ be an open set such that $\partial_0\Omega$ is nonempty. Let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be bounded from above and such that $\lambda^{1-2s} v_\lambda \in C(\overline{\Omega})$.*

Consider the operator $\mathscr{L}_u$ defined by

$$\mathscr{L}_u v = -d_s \lim_{\lambda \downarrow 0} \lambda^{1-2s} v_\lambda(x, \lambda) - f'(u)v \quad \text{on} \quad \mathbb{R}^{2m} \times \{0\},$$

and assume that

$$\begin{cases} -\operatorname{div}(\lambda^{1-2s}\nabla v) & \leq & b(x,\lambda)v & \text{in } \Omega \subset \mathcal{O} \times (0, +\infty)\,, \\ v & \leq & 0 & \text{on } \partial_L\Omega := \overline{\partial\Omega \cap \{\lambda > 0\}}\,, \\ \mathscr{L}_u v & \leq & 0 & \text{on } \partial_0\Omega := \partial\Omega \setminus \partial_L\Omega \subset \mathcal{O}\,, \\ \limsup\limits_{x\in\partial_0\Omega,\ |x|\to+\infty} v(x,0) & \leq & 0\,, & \end{cases}$$

with $b \leq 0$. Then, $v \leq 0$ in Ω.

To establish such a result we use a maximum principle in "narrow" sets (see [45, 30]) as it is done in the local case. Nevertheless, since we use the extension problem, a new notion of narrowness needs to be introduced to carry out the same type of arguments.

The maximum principle for the linearized operator is used, apart from proving uniqueness, to obtain some monotonicity and convexity properties of the saddle-shaped solution, that in turn are crucial to show the following stability result:

Theorem 10 (see Theorem 3.1.6)**.** *Assume that f satisfies* (25)*. If $2m \geq 14$, then the saddle-shaped solution u to* (24) *is stable in $\mathbb{R}^{2m+1}_+$.*

The stability follows from the existence of a explicit positive supersolution of the linearized operator that is built by taking advantage of the monotonicity and convexity properties of the saddle-shaped solution.

An important consequence of our previous result concerns the stability of the Simons cone as a nonlocal minimal surface. In [86], Davila, del Pino, and Wei showed that establishing the stability of the Simons cone is equivalent to checking an inequality involving two hypergeometric constants which depend only on s and n. By using numerical computations, they found that, in dimensions $2m \leq 6$ and for s close to zero, the Simons cone is unstable. Here, we give the first analytical proof of the previous question in dimensions $2m \geq 14$ by using the saddle-shaped solution to the Allen-Cahn equation.

Corollary 11 (see Corollary 3.1.7)**.** *Let $2m \geq 14$. Then, the Simons cone $\mathscr{C} \subset \mathbb{R}^{2m}$ is a stable nonlocal minimal surface.*

Our proof uses a result from [50], stating that the blow-down limit of stable solutions to the fractional Allen-Cahn equation with $s \in (0, 1/2)$ is a stable set for the fractional perimeter.

Since the extension technique is not available when dealing with more general integro-differential operators, new strategies are needed to study saddle-shaped solutions to

$$Lu = f(u) \quad \text{in } \mathbb{R}^{2m} \tag{26}$$

where L is a general integro-differential operators of the form (2). For this reason, we develop in Chapters 4 and 5 some nonlocal techniques to treat them.

First, we study the integro-differential operator when acting on doubly radial functions that are odd with respect to the Simons cone, i.e., functions of the form $w(x) = w(|x'|, |x''|)$ with $w(|x''|, |x'|) = -w(|x'|, |x''|)$. In such a scenario, we are able to rewrite L as a regional-type operator plus a zeroth order term. That is,

$$Lw(x) = \int_{\{|y'|>|y''|\}} \{w(x) - w(y)\}\{\overline{K}(x, y) - \overline{K}(x, y^\star)\}\, \mathrm{d}y + 2w(x) \int_{\{|y'|>|y''|\}} \overline{K}(x, y^\star)\, \mathrm{d}y\,,$$

where $x^\star = (x', x'')^\star = (x'', x')$ and $\overline{K}$ is obtained as

$$\overline{K}(x, y) = \fint_{O(m)^2} K(|Rx - y|)\, \mathrm{d}R\,. \tag{27}$$

Furthermore, the zeroth order term satisfies

$$\frac{1}{C} \operatorname{dist}(x, \mathscr{C})^{-2s} \leq \int_{\{|y'|>|y''|\}} \overline{K}(x, y^\star)\, \mathrm{d}y \leq C \operatorname{dist}(x, \mathscr{C})^{-2s}, \tag{28}$$

with $C > 0$ depending only on m, s, and the ellipticity constants.

From this alternative expression, a crucial property when studying the saddle-shaped solution is characterizing the kernels K for which the regional-type operator has a positive kernel.

Theorem 12 (see Theorem 4.1.1)**.** *Let $K : (0, +\infty) \to (0, +\infty)$ and consider the radially symmetric kernel $K(|x - y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ by (27).*

If

$$K(\sqrt{\tau}) \text{ is a strictly convex function of } \tau\,, \tag{29}$$

then L has a positive kernel in $\mathcal{O}$ when acting on doubly radial functions which are odd with respect to the Simons cone $\mathscr{C}$. More precisely, it holds

$$\overline{K}(x, y) > \overline{K}(x, y^\star) \quad \text{for every } x, y \in \mathcal{O}\,. \tag{30}$$

In addition, if $K \in C^2((0, +\infty))$, then (29) is not only a sufficient condition for (30) to hold, but also a necessary one.

Note that the convexity condition in the previous result is satisfied by the kernel of the fractional Laplacian, the model integro-differential operator.

Next, we establish the existence and uniqueness of saddle-shaped solutions.

Theorem 13 (see Theorems 4.1.4 and 5.1.2)**.** *Let f satisfy (25). Let K be a radially symmetric kernel satisfying the convexity assumption (29) and the ellipticity condition (4). Then, for every even dimension $2m \geq 2$, there exists a unique saddle-shaped solution u to (26). In addition, u satisfies $|u| < 1$ in $\mathbb{R}^{2m}$.*

We prove the existence result by using two alternative strategies: variational methods or a monotone iteration scheme adapted to the odd setting. The first approach exploits the fact that (26) is the Euler-Lagrange equation associated to the energy functional

$$\mathcal{E}(w, \Omega) = \frac{1}{4}\bigg\{ \int_\Omega \int_\Omega |w(x) - w(y)|^2 K(x - y)\, \mathrm{d}x\, \mathrm{d}y + 2 \int_\Omega \int_{\mathbb{R}^{2m}\setminus\Omega} |w(x) - w(y)|^2 K(x - y)\, \mathrm{d}x\, \mathrm{d}y \bigg\} - \int_\Omega F(w)\, \mathrm{d}x\,.$$

Furthermore, it requires the following energy estimate for doubly radial odd minimizers.

Theorem 14 (see Theorem 4.1.3)**.** *Let K be a radially symmetric kernel satisfying the convexity assumption* (29) *and the ellipticity condition* (4)*. Assume that f satisfy* (25) *and $f = F'$. Let $S \geq 2$ and let u be a doubly radial odd minimizer of $\mathcal{E}$ in B_R, with $R > S + 4$. Then*

$$\mathcal{E}(u, B_S) \leq \begin{cases} C\ S^{2m-2s} & \text{if} \quad s \in (0, 1/2), \\ C\ S^{2m-1} \log S & \text{if} \quad s = 1/2, \\ C\ S^{2m-1} & \text{if} \quad s \in (1/2, 1), \end{cases}$$

where C is a positive constant depending only on m, s, F, and the ellipticity constants.

In order to prove the energy estimate we adapt the arguments from Savin and Valdinoci [173] to the doubly radial odd framework. They are based on constructing a suitable competitor.

The uniqueness of the saddle-shaped solution follows, as in the classical and fractional problems, from a maximum principle for the linearized operator $L - f'(u)$ and the asymptotic behavior of the saddle-shaped solution at infinity.

Proposition 15 (see Proposition 5.1.4)**.** *Let $\Omega \subset \mathcal{O}$ be an open set (not necessarily bounded) and let K be a radially symmetric kernel satisfying the convexity assumption* (29) *and the ellipticity condition* (4)*. Let u be a saddle-shaped solution to* (26)*, and let $v \in L^1_s(\mathbb{R}^{2m})$ be a doubly radial function which is C^α in Ω and continuous up to the boundary, for some $\alpha > 2s$. Assume that v satisfies*

$$\begin{cases} Lv - f'(u)v - c(x)v & \leq & 0 & \text{in } \Omega\,, \\ v & \leq & 0 & \text{in } \mathcal{O} \setminus \Omega\,, \\ -v(x^\star) & = & v(x) & \text{in } \mathbb{R}^{2m}, \\ \limsup\limits_{x\in\Omega,\ |x|\to\infty} v(x) & \leq & 0\,, \end{cases}$$

with $c \leq 0$ in Ω. Then, $v \leq 0$ in Ω.

The key tool to prove the result is again a maximum principle in "narrow" sets. Let us point out that in the nonlocal framework such a result turns to be much simpler than that of the classical Laplacian or that of the fractional Laplacian through the extension problem. In the proof, both the convexity of the kernel (29) and the bounds on the zeroth order term (28) are crucial.

The behavior of saddle-shaped solutions at infinity is stated in the following result.

Theorem 16 (see Theorem 5.1.3)**.** *Let $f \in C^2(\mathbb{R})$ satisfy* (25)*. Let K be a radially symmetric kernel satisfying the convexity assumption* (29) *and the ellipticity condition* (4)*. Let u be a saddle-shaped solution to* (26) *and let U be defined by*

$$U(x) = u_0\left(\frac{|x'| - |x''|}{\sqrt{2}}\right),$$

where u_0 is the layer solution to (26)*.*

Then,

$$||u - U||_{L^\infty(\mathbb{R}^n \setminus B_R)} + ||\nabla u - \nabla U||_{L^\infty(\mathbb{R}^n \setminus B_R)} + \left|\left|D^2 u - D^2 U\right|\right|_{L^\infty(\mathbb{R}^n \setminus B_R)} \to 0$$

as $R \to +\infty$.

Since $(|x'|-|x''|)/\sqrt{2}$ corresponds to the signed distance to the Simons cone, the function U is constructed by centering the layer solution u_0 at each point of the cone and orienting it in the normal direction.

The asymptotic behavior is established by using a compactness argument as in [60, 75, 76], together with two symmetry results for semilinear equations, not previously proved for general nonlocal operators L. The first one is a Liouville type principle for nonnegative solutions in the whole space while the second one is a one dimensional symmetry result in a half-space.

Null-Lagrangians and calibrations for nonlocal functionals

In Chapter 6, which corresponds to [53], we build a calibration for nonlocal energy functionals of the form (1) under a certain ellipticity assumption. The construction requires the existence of a family of solutions (or more generally of sub/supersolutions) to the Euler-Lagrange equation whose graphs give a foliation of the space.

Before our work, calibrations for nonlocal energy functionals were not known, with the exception of the fractional perimeter. Indeed, in the simplest case

$$\mathcal{E}^s_F(w) = \frac{c_{n,s}}{4}\iint_{(\Omega^c\times\Omega^c)^c} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x,$$

which is the fractional analogue of (12), the form of the calibration, if any, was even unknown. Let us recall that extremals of $\mathcal{E}^s_F$ satisfy the semilinear equation

$$(-\Delta)^s u = F'(u) \quad \text{ in } \Omega.$$

Our fist main result of the chapter is the following:

Theorem 17 (see Theorem 6.1.3)**.** *Let $\{u^t\}_{t\in\mathbb{R}}$ be a field (a one-parameter family of functions whose graphs give a foliation) such that $(x,t)\to u^t(x)$ is a bounded C^2 function, and let $\mathcal{C}^s_F$ be the functional*

$$\mathcal{C}^s_F(w) = c_{n,s}\,\mathrm{P.V.}\iint_{(\Omega^c\times\Omega^c)^c}\int_{u^0(x)}^{w(x)} \left.\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\right|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x + \frac{c_{n,s}}{4}\iint_{(\Omega^c\times\Omega^c)^c}\frac{|u^0(x)-u^0(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Then, it follows that:

(a) *$\mathcal{C}^s_F(u^0)=\mathcal{E}^s_F(u^0)$ and $\mathcal{C}^s_F(w)\le\mathcal{E}^s_F(w)$ for all w such that $w\equiv u^0$ in Ω^c.*

(b) *Assume in addition that the family $\{u^t\}_{t\in\mathbb{R}}$ satisfies*

$$\begin{aligned}(-\Delta)^s u^t - F'(u^t) &\ge 0 \quad \text{in } \Omega \quad \text{for } t\ge 0,\\ (-\Delta)^s u^t - F'(u^t) &\le 0 \quad \text{in } \Omega \quad \text{for } t\le 0.\end{aligned}$$

Then $\mathcal{C}^s_F(u^0)\le\mathcal{C}^s_F(w)$ for all w such that $w\equiv u^0$ in Ω^c. In particular, u^0 is a minimizer of $\mathcal{E}^s_F$ among all functions with the same exterior data.

(c) *Assume in addition that* $\{u^t\}_{t\in\mathbb{R}}$ *is an extremal field, that is,*

$$(-\Delta)^s u^t - F'(u^t) = 0 \quad \text{in } \Omega \quad \text{for all } t \in \mathbb{R}.$$

Then $\mathcal{C}_F^s$ *is a calibration for* $\mathcal{E}_F^s$ *and* u^t*. In particular, each* u^t *minimizes the energy functional* $\mathcal{E}_F^s$ *among functions* w *such that* $w \equiv u^t$ *in* Ω^c*.*

The key point to be able to find the calibration was to review, from a new point of view, the classical theory for local functionals. We realized, inspired by the structure of the calibration for the nonlocal perimeter, that the functional $\mathcal{C}_L$ given by (11) can be written alternatively as

$$\mathcal{C}_L(w) = \int_\Omega \int_{u^0(x)}^{w(x)} \mathcal{L}_L(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \int_{\partial\Omega} \int_{u^0(x)}^{w(x)} \mathcal{N}_L(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{E}_L(u^0).$$

Here $\mathcal{L}_L$ is the Euler-Lagrange operator (10) and $\mathcal{N}_L$ is the Neumann operator defined as

$$\mathcal{N}_L(w)(x) = \partial_q G_L\Big(x, w(x), \nabla w(x)\Big) \cdot \nu_\Omega(x).$$

To the best of our knowledge, it is the first time that the classical calibration is written in this form. The importance of this expression comes from the fact that such a structure can be extrapolated to the nonlocal setting, since each of the terms above has a clear nonlocal counterpart.

Next, we extend the result to general nonlocal functionals of the form (1).

Theorem 18 (see Theorem 6.1.4)**.** *Let* $\{u^t\}_{t\in\mathbb{R}}$ *be a* sufficiently regular *field. Assume that* G_N *is pairwise symmetric, i.e.,*

$$G_N(y,x,b,a) = G_N(x,y,a,b) \quad \text{for all} \quad (x,y,a,b) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R},$$

and satisfies the nonlocal ellipticity condition

$$\partial_b G_N(x,y,a,b) \quad \text{is nonincreasing in } a. \tag{31}$$

Let $\mathcal{C}_N$ *be the functional*

$$\mathcal{C}_N(w) = \iint_{(\Omega^c\times\Omega^c)^c} \int_{u^0(x)}^{w(x)} \partial_a G_N(x,y,u^t(x),u^t(y))\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y + \mathcal{E}_N(u^0),$$

and $\mathcal{L}_N$ *the operator*

$$\mathcal{L}_N(w)(x) = \int_{\mathbb{R}^n} \partial_a G_N(x,y,w(x),w(y))\,\mathrm{d}y.$$

Then, it follows that:

(a) $\mathcal{C}_N(u^0) = \mathcal{E}_N(u^0)$ *and* $\mathcal{C}_N(w) \le \mathcal{E}_N(w)$ *for all* w*.*

(b) *Assume in addition that the family satisfies*

$$\begin{aligned} \mathcal{L}_N(u^t) \ge 0 \quad &\text{in } \Omega \quad \text{for } t \ge 0,\\ \mathcal{L}_N(u^t) \le 0 \quad &\text{in } \Omega \quad \text{for } t \le 0, \end{aligned}$$

where

$$\mathcal{L}_N(w)(x) = \int_{\mathbb{R}^n} \partial_a G_N(x, y, w(x), w(y))\, dy$$

is the Euler-Lagrange operator associated to energy functional $\mathcal{E}_N$. Then $\mathcal{C}_N(u^0) \leq \mathcal{C}_N(w)$ for all w such that $w \equiv u^0$ in Ω^c. In particular, u^0 is a minimizer of $\mathcal{E}_N$ among functions with the same exterior data.

(c) *Assume in addition that $\{u^t\}_{t\in\mathbb{R}}$ is an extremal field, that is,*

$$\mathcal{L}_N(u^t) = 0 \quad \text{in } \Omega \quad \text{for all } t \in \mathbb{R}.$$

Then $\mathcal{C}_N$ is a calibration for $\mathcal{E}_N$ and u^t. In particular, each u^t minimizes $\mathcal{E}_N$ among functions w such that $w \equiv u^t$ in Ω^c.

Let us point out that in this thesis we only give a formal derivation of the previous result since it includes a very wide class of energy functionals, each with their own particularities. For instance, it includes the fractional p-Dirichlet energy, the fractional s-area functional for graphs, as well as convolution energies. Nevertheless, we believe that one can justify each of the steps in our proof by assuming the adequate regularity conditions for each particular case.

Finally, we give an interpretation of the nonlocal ellipticity condition (31), which turns out to have a strong connection with a comparison principle for the associated Euler-Lagrange equation. More specifically, we see that if a function v is below another function w and they touch at a point x_0, then the monotonicity in b of $\partial_a G_N(x, y, a, b)$ gives the inequality $\mathcal{L}_N(v)(x_0) \geq \mathcal{L}_N(w)(x_0)$.

Part I

Linear problems: Uniqueness and regularity of solutions

Chapter 1

Uniqueness for integro-differential equations in the real line and applications

In this chapter, which corresponds to [109], we prove the uniqueness of solutions to the nonlocal linear equation $L\varphi - c(x)\varphi = 0$ in $\mathbb{R}$, where L is an elliptic integro-differential operator, in the presence of a positive solution or of an odd solution vanishing only at zero. As an application, we deduce the nondegeneracy of layer solutions (bounded and monotone solutions) to the semilinear problem $Lu = f(u)$ in $\mathbb{R}$ when the nonlinearity is of Allen-Cahn type. To our knowledge, this is the first work where such uniqueness and nondegeneracy results are proven in the nonlocal framework when the Caffarelli-Silvestre extension technique is not available. Our proofs are based on a nonlocal Liouville-type method developed by Hamel, Ros-Oton, Sire, and Valdinoci for nonlinear problems in dimension two.

1.1 Introduction and main results

We study the uniqueness, up to a multiplicative constant, of solutions to the linear integro-differential equation

$$L\varphi - c(x)\varphi = 0 \quad \text{in } \mathbb{R}, \tag{1.1.1}$$

under certain assumptions on the nonlocal operator L and the potential function c, and in the presence of a positive solution or of an odd solution vanishing only at zero. Throughout the chapter, L will be assumed to be an elliptic integro-differential operator of order between one (included) and two.

The uniqueness of solutions to equations of the form (1.1.1) is a very important tool in the theory of PDEs. Indeed, there are many motivations (from both linear and nonlinear frameworks) to treat this problem. On the one hand, it is in the essence of Sturm-Liouville theory on eigenfunctions and eigenvalues. On the other hand, it has important consequences when studying qualitative properties of solutions to semilinear problems. For instance, in the context of nonlinear Schrödinger equations, the nondegeneracy of ground state solutions (which plays a very important role in the stability and blow up analysis of solitary waves to related time-dependent equations) is reduced to study the uniqueness of solution to equation (1.1.1) when L is replaced by the radial component of the Laplacian, i.e., $L = r^{1-n}(r^{n-1}u_r)_r$ (see [122]). Furthermore, in the framework of the Allen-Cahn equation, Berestycki, Caffarelli, and Nirenberg [26] realized that the uniqueness of solutions to equation (1.1.1) in dimension n for the local case (with L replaced by a general second order uniformly elliptic operator) leads to the resolution of a conjecture by De Giorgi for monotone solutions.

In the present chapter, equation (1.1.1) is driven by a translation invariant integro-differential operator of the form

$$Lu(x) = \int_{\mathbb{R}^n} \Big(u(x) - u(y)\Big)\, K(x-y)\, dy. \tag{1.1.2}$$

In this nonlocal setting there are lots of basic open problems concerning solutions in dimension one, unlike the case of local equations where the one dimensional problem (1.1.1) is just a second order linear ODE. For instance, a full understanding of the phase portrait of solutions in the nonlocal framework is missing.

Most of the works in the literature concerning uniqueness of solutions to (1.1.1)-(1.1.2) treat the simplest case $L = (-\Delta)^s$ (see [117, 118, 70, 165, 104, 58, 56] and the comments along this introduction). In such a scenario, the main analytic tools are potential theory, Fourier analysis, and the Caffarelli-Silvestre extension problem. Since they are not available when dealing with more general integro-differential operators, new techniques are needed. In [132], Hamel, Ros-Oton, Sire, and Valdinoci develop a purely nonlocal method (in contrast to the local extension problem) to treat these operators. They use it to establish a uniqueness result in dimension two (motivated by a nonlocal version of De Giorgi's conjecture) in the case of operators with compactly supported kernel and power-like behavior at the origin. In this chapter, their methodology is used in dimension one for the first time. It leads to uniqueness results for equations of the form (1.1.1)-(1.1.2). Working in dimension one allows us to get rid of the compact support assumption in [132].

Throughout the chapter, we assume that the kernel K of the integro-differential operators satisfies the positivity and symmetry conditions

$$K(z) > 0 \quad \text{and} \quad K(-z) = K(z), \tag{K1}$$

together with an ellipticity assumption. That is, to be bounded both from above and below by a multiple of the kernel of the fractional Laplacian, i.e.,

$$\frac{\lambda}{|z|^{n+2s}} \leq K(z) \leq \frac{\Lambda}{|z|^{n+2s}}, \tag{K2}$$

for some constants $\Lambda \geq \lambda > 0$ and $s \in [1/2, 1)$. Note that the operator L will be assumed to be of order between one (included) and two. Condition (K2) is one of the most frequently adopted when dealing with nonlocal operators of the form (1.1.2). It is known to yield Hölder regularity of solutions (see [162] and [176]).

In some results the lower bound will not be assumed, and the upper one can be relaxed to

$$K(z) \leq \frac{\Lambda_1}{|z|^{n+2\underline{s}}} + \frac{\Lambda_2}{|z|^{n+2\overline{s}}}, \tag{K3}$$

for some constants $\Lambda_1, \Lambda_2 \geq 0$ and $1/2 \leq \underline{s} \leq \overline{s} < 1$. This is the case of Theorem 1.5.2 and Corollary 1.5.3.

We will sometimes assume the potential function c to be negative at infinity. That is,

$$c(x) \leq -c_0 < 0 \quad \text{in } \mathbb{R} \setminus [-R_0, R_0], \tag{1.1.3}$$

for some positive constants c_0 and R_0.

The following is our first important result. It establishes the uniqueness of solution to (1.1.1) in the presence of a positive one (in addition to other assumptions).

Theorem 1.1.1. *Let L be an integro-differential operator of the form (1.1.2) satisfying the symmetry and ellipticity conditions (K1) and (K2) with $s \in [1/2, 1)$. For $\alpha > 2s - 1$, let w and $\widetilde{w}$ be two $C^{1,\alpha}$ solutions of the linear equation*

$$L\varphi - c(x)\varphi = 0 \quad \text{in } \mathbb{R},$$

with $w > 0$. Assume that

- *either both w and $\widetilde{w}$ are bounded and the potential function $c \in L^\infty(\mathbb{R})$ satisfies*

$$c(x) \le -c_0 < 0 \quad \text{in } \mathbb{R} \setminus [-R_0, R_0], \qquad \text{and} \qquad ||c||_{C^{\beta_0}(\mathbb{R})} < +\infty$$

  for some positive constants c_0, R_0, and β_0;

- *or w satisfies*

$$0 < C^{-1} \le w(x) \le C \quad \text{in } \mathbb{R}$$

  and $\widetilde{w}$ is such that

$$||\widetilde{w}||_{L^\infty(-R,R)} \le CR^{s-\frac{1}{2}} \ \text{for all} \ R > 1,$$

  for some positive constant C.

Then

$$\frac{\widetilde{w}}{w} \equiv ctt.$$

Let us point out that some assumptions concerning the kernel can be relaxed to include a bigger class of operators (see Theorem 1.5.2 for the precise statement) such as the sum of fractional Laplacians with different order (see Corollary 1.5.3). Nevertheless, for the sake of clarity and simplicity we prefer to state Theorem 1.1.1 here.

To the best of our knowledge, Theorem 1.1.1 is the first uniqueness result for general integro-differential operators in dimension one. Previous analogue results could only cover the case of the fractional Laplacian (see Remark 1.1.2 for comments on such works).

In order to prove uniqueness we follow a Liouville-type method. The main idea consists of finding an equation for the quotient of two solutions, which is the crucial contribution by Hamel, Ros-Oton, Sire, and Valdinoci [132] for general integro-differential operators, and then showing that such a quotient is constant. This requires a growth estimate in both the local and nonlocal cases.

Unlike [132], where a key point is assuming that the kernels have compact support, we adapt the strategy in order to remove such a condition by taking advantage of the one dimensionality of the problem. In our approach, the first step is controlling the growth of the quotient of the solutions. This comes for free when the positive solution is just bounded from below by a strictly positive constant. However, a finer analysis is needed when the positive solution can be arbitrarily close to zero at infinity. In that case, we prove the boundedness of the quotient by using condition (1.1.3) and the boundedness of the solutions. Here, we use a maximum principle in the exterior of an interval, proved in Section 1.3, in order to compare both solutions by transferring the information from the interval (where we know the quotient is bounded) to the whole line. The second ingredient to prove the uniqueness theorem is an integral estimate for the function $K(x - y)$ with respect to both variables x and y in unbounded cross-shaped regions of the plane. In fact, the validity of this estimate is what prevents us from extending our result to $s \in (0, 1/2)$. We show it in Section 1.4. Let us point out that both ingredients become trivial when working with kernels with compact support, as it is done in [132].

Remark 1.1.2. As it is natural, our result, which includes a big class of integro-differential operators, is not optimal when we apply it to the fractional Laplacian. In order to compare it with other similar results in the literature, let us distinguish two cases depending on whether the equation has a zeroth order term or not.

On the one hand, when $c \equiv 0$, in [37], Bogdan, Kulczycki, and Nowak used a gradient estimate to show that nonnegative s-harmonic functions are constant. Later on, Chen, D'Ambrosio, and Lin [70] proved, by using potential theory and Fourier analysis, a Liouville theorem for the fractional Laplacian with the growth condition

$$\liminf_{|x|\to\infty} \frac{u(x)}{|x|^\gamma} = 0,$$

if $0 \le \gamma \le 1$ and $\gamma < 2s$. In this scenario, our result, by taking $w \equiv 1$ as the positive solution, leads to solutions growing less or equal than $|x|^{s-1/2}$ at infinity being constant. Thus, we notice what we have previously announced, that our condition is not sharp for the fractional Laplacian.

On the other hand, when the potential function is not identically zero, it is known that the uniqueness result for the fractional Laplacian, with $s \in [1/2, 1)$, follows from Theorem 4.10 by Cabré and Sire in [56] (see also the work by Cabré and Solà-Morales [58] for the half-Laplacian) and the use of the local extension problem. In this case, unlike our result, no condition on the potential function (or the positive solution) needs to be assumed.

An important and direct application of Theorem 1.1.1 is the nondegeneracy of layer solutions to Allen-Cahn type equations. Let us recall that a bounded solution to the semilinear problem

$$Lu = f(u) \quad \text{in } \mathbb{R}, \tag{1.1.4}$$

is called *layer solution* if it is strictly increasing. In particular, it has limits at infinity, which (without loss of generality) we can consider to be ± 1.

When L is a second order differential operator, layer solutions to equation (1.1.4) are just particular cases of heteroclinic connections to nonlinear ODEs. Nevertheless, in the nonlocal setting, even the existence of such solutions is not an easy problem due to the lack of an analogous nonlocal ODE theory. In the fractional case $L = (-\Delta)^s$, existence and uniqueness are shown in [58, 56, 57] by using the extension problem. For more general integro-differential operators, we can refer to the work by Cozzi and Passalacqua [83] where they prove existence, uniqueness (up to translations), and some qualitative properties of layer solutions (see Chapter 5 for further properties). Here, we prove nondegeneracy:

Theorem 1.1.3. *Let L be an integro-differential operator of the form (1.1.2) satisfying the symmetry and ellipticity conditions (K1) and (K2) with $s \in [1/2, 1)$. For $\gamma > 0$, let $f \in C^{1,\gamma}([-1, 1])$ be any given nonlinearity such that $f'(\pm 1) < 0$.*

Assume that u is a bounded solution to the semilinear equation (1.1.4), satisfying $u' > 0$ and $\lim_{x\to\pm\infty} u(x) = \pm 1$.

Then, u is nondegenerate, i.e., up to a multiplicative constant u' is the unique bounded solution to the linearized equation $L\varphi - f'(u)\varphi = 0$ in $\mathbb{R}$.

Let us point out that condition $f'(\pm 1) < 0$, which corresponds to $c = f'(u)$ being negative at infinity, is a natural assumption. Indeed, it is the same hypothesis needed to prove uniqueness (up to translations) of the layer solutions (see Theorem 1.2 in [58] in the case of the half-Laplacian). Moreover, this is also the needed condition for ± 1 to be local minimizers of the associated energy.

The nondegeneracy of solutions plays a very relevant role in the stability and blow up analysis for time dependent versions of equation (1.1.1). Furthermore, it is also important in stationary problems, as in the construction of new solutions to the semilinear equation (1.1.4) around a nondegenerate one by using an implicit function argument. Indeed, Dávila, del Pino, and Musso [85] proved the nondegeneracy of the layer solution when $L = (-\Delta)^{1/2}$ (with the extension problem) in order to construct solutions to (1.1.4) that develop multiple transitions from -1 to 1. In [101], Du, Gui, Sire, and Wei generalize the nondegeneracy to $s \in (1/2, 1)$ and use it to show the existence of clustering-layered solutions for a fractional inhomogeneous Allen-Cahn equation.

Next, we present the third main result of this work: a uniqueness theorem in the odd setting. Let us point out that in such a case our strategy allows us to show uniqueness only among odd functions. Completely different arguments would be needed to establish uniqueness among all functions, as it occurs in [117] for a particular case involving the fractional Laplacian (see the end of the present introduction for more details).

Theorem 1.1.4. *Let L be an integro-differential operator of the form (1.1.2) with kernel K being decreasing in $(0, +\infty)$ and satisfying the symmetry and ellipticity conditions (K1) and (K2) for some $s \in [1/2, 1)$. Assume the potential function $c \in L^\infty(\mathbb{R})$ satisfies*

$$c(x) \leq -c_0 < 0 \quad \text{in } \mathbb{R} \setminus [-R_0, R_0], \qquad \text{and} \qquad ||c||_{C^{\beta_0}(\mathbb{R})} < +\infty$$

for some positive constants c_0, R_0 and β_0.

Let $w, \tilde{w}$ be two odd bounded solutions of the linear equation

$$L\varphi - c(x)\varphi = 0 \quad \text{in } \mathbb{R},$$

with $w > 0$ in $(0, +\infty)$.

Then

$$\frac{\widetilde{w}}{w} \equiv ctt.$$

Note that since the integro-differential operator L preserves the oddness of functions, the potential function c needs to be even if we assume the problem to have existence of odd solutions. On the other hand, the monotonicity of the kernel is a natural assumption when working with odd functions in the nonlocal setting. Indeed, for the validity of the maximum principle (see Lemma 1.3.2 and section 3 of [134]), this condition is the analogue in the odd framework to the positivity of the kernel in (K1) for general functions.

As in Theorem 1.1.3 for the case of functions without any symmetry, we can apply the previous uniqueness result to prove qualitative properties of solutions to semilinear problems. Let us recall that a bounded solution (without loss of generality we can consider it to be bounded by 1) to the semilinear equation (1.1.4) is called *ground state* if it is even, positive, and decreasing to zero at infinity. We refer to the work by Frank and Lenzmann [117] and references therein for existence results of such solutions. Here, we establish a partial nondegeneracy result (in the sense that we prove uniqueness for the linearized equation only among odd functions):

Theorem 1.1.5. *Let L be an integro-differential operator of the form (1.1.2) with kernel K being decreasing in $(0, +\infty)$ and satisfying the symmetry and ellipticity conditions (K1) and (K2), for some $s \in [1/2, 1)$. For $\gamma > 0$, let $f \in C^{1,\gamma}([0, 1])$ be any given nonlinearity such that $f'(0) < 0$.*

Assume that u is a bounded even solution to the semilinear equation (1.1.4), satisfying $u' < 0$ in $(0, +\infty)$ and $\lim_{x \to \pm\infty} u(x) = 0$.

Then, up to a multiplicative constant u' is the unique bounded odd solution to the linearized equation $L\varphi - f'(u)\varphi = 0$ in $\mathbb{R}$.

As in the nondegeneracy result for layer solutions, the condition $f'(0) < 0$ is a natural assumption. Indeed, it is a necessary condition in order for $v \equiv 0$ to be a local minimizer of the associated energy.

The most important result in the literature dealing with nondegeneracy of ground states in the nonlocal framework is due to Frank and Lenzmann [117]. Unlike us, they were able to establish the full nondegeneracy (uniqueness for the linearized equation among all functions) in the particular case $L = (-\Delta)^s$ and f being a polynomial nonlinearity (see Lemma C.3 from [117]) as we explain next. An important point in their strategy is to note that the operator $L - f'(u)$ preserves odd/even symmetry. Thus, both the odd and even parts of any given solution of the linearized problem are also solutions, and a separated analysis can be done for each one. First, they prove the uniqueness among odd functions by using the heat kernel for the fractional Laplacian. Next, they show that the unique even solution is the trivial one, which is the most difficult step. In order to do it, they develop a delicate spectral theory for fractional Schrödinger operators (where the local extension problem and the polynomial structure of the nonlinearity play a crucial role). Finally, the uniqueness among all functions follows from the previous results. The nondegeneracy of ground states turns out to be very important since they use it to prove their uniqueness result by using an implicit function argument and the well known result for the local case ($s = 1$).

Finally, let us comment that the strategy to prove Theorem 1.1.4 follows the same lines as the one of Theorem 1.1.1. Nevertheless, there are some difficulties we have to overcome. First, we need to take advantage of the odd symmetry to find an equation for the quotient of two solutions (see Corollary 1.2.2) which involves only the values of the functions in $(0, \infty)$, where the first solution w is known to be positive. Next, we need to assure the quotient to be well-defined at the origin, where the denominator vanishes. We can accomplish it by using a maximum principle in small domains around the origin and taking into account that the numerator also vanishes at this point.

The chapter is organized as follows. In Section 1.2 we present the equation satisfied by the quotient of two solutions to (1.1.1)-(1.1.2). Section 1.3 is devoted to show the maximum principles in the exterior of an interval. In Section 1.4 we give some estimates involving the integral of the kernel in cross-shaped unbounded domains. Finally, in Sections 1.5 and 1.6 we prove the main results of the chapter.

1.2 Preliminary results: An equation for the quotient of solutions

In this section we include a few preliminary algebraic computations that will be employed in the proof of the main theorems. They are inspired by the computations done by Hamel, Ros-Oton, Sire, and Valdinoci in [132].

In the local framework (see proof of Theorem 1.8 in Section 4 of [26]), it is well known that given a positive supersolution w and a solution $\tilde{w}$ to the linear equation $-\Delta\varphi - c(x)\varphi = 0$, the quotient $\sigma := \tilde{w}/w$ satisfies $\sigma \operatorname{div}(w^2 \nabla \sigma) \geq 0$. Thus, multiplying by τ^2, where τ is any cut-off function, and integrating in the whole space, one arrives at

$$2 \int_{\mathbb{R}^n} \tau^2(x) w^2(x) |\nabla \sigma(x)|^2 dx \leq - \int_{\mathbb{R}^n} w^2(x) \nabla(\tau^2(x)) \cdot \nabla(\sigma^2(x)) dx. \tag{1.2.1}$$

Similar computations can also be done, by using the extension problem, when the Laplacian is replaced by the fractional Laplacian (see [58, 56]).

In the general integro-differential case we establish the following:

Lemma 1.2.1. *Let L be an integro-differential operator of the form (1.1.2). Assume that w and σ are two smooth functions such that w and $\tilde{w} := \sigma w$ satisfy*

$$w\,(Lw - cw) \geq 0 \quad \text{in } \mathbb{R}$$

and

$$\tilde{w}\,(L\tilde{w} - c\tilde{w}) \leq 0 \quad \text{in } \mathbb{R},$$

respectively, for some potential function $c = c(x)$.

Then, given any function $\tau \in C_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\sigma(x) - \sigma(y)\Big)^2 \Big(\tau^2(x) + \tau^2(y)\Big)\, w(x)\, w(y)\, K(x-y)\, dxdy$$
$$\leq -\int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\sigma^2(x) - \sigma^2(y)\Big)\Big(\tau^2(x) - \tau^2(y)\Big) w(x)\, w(y)\, K(x-y)\, dxdy.$$

Moreover, if $w\,(Lw - cw) = \tilde{w}\,(L\tilde{w} - c\tilde{w}) = 0$, there is equality in the previous expression.

This result, which is a generalization of Lemma 2.1 from [132], is a nonlocal analogue to (1.2.1). In Section 1.5, we will use it to prove that the quotient of two solutions to the linear equation (1.1.1) is constant.

Proof. First, combining $w(Lw - cw) \geq 0$ and $\tilde{w}(L\tilde{w} - c\tilde{w}) \leq 0$, we can easily check that $\sigma(\tilde{w}Lw - wL\tilde{w}) \geq 0$. Then, multiplying by τ^2, where τ is any cut-off function, and repeating the algebraic computations done in [132] we find that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\sigma(x) - \sigma(y)\Big)^2 \tau^2(x)\, w(x)\, w(y)\, K(x-y)\, dxdy$$
$$\leq -\int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\sigma(x) - \sigma(y)\Big)\Big(\tau^2(x) - \tau^2(y)\Big)\sigma(x)\, w(x)\, w(y)\, K(x-y)\, dxdy.$$

Finally, symmetrizing in both x and y we conclude the proof. □

As a consequence of the previous lemma, we can also find a useful identity for the quotient of two odd solutions to the linear equation (1.1.1). In such a case, all the integrals can be written in $(0, +\infty)$ by taking advantage of the symmetry of the functions.

Corollary 1.2.2. *Let L be an integro-differential operator of the form (1.1.2). Assume that w and σ are two smooth functions such that both w and $\tilde{w} := \sigma w$ are odd solutions to the linear equation*

$$L\varphi - c(x)\varphi = 0, \quad \text{in } \mathbb{R},$$

for some even potential function $c = c(x)$.

Then, given any even function $\tau \in C_c^\infty(\mathbb{R})$,

$$\int_0^\infty\int_0^\infty \Big(\sigma(x) - \sigma(y)\Big)^2 \Big(\tau^2(x) + \tau^2(y)\Big)\, w(x)\, w(y)\, \Big\{K(x-y) - K(x+y)\Big\}\, dxdy$$
$$= -\int_0^\infty\int_0^\infty \Big(\sigma^2(x) - \sigma^2(y)\Big)\Big(\tau^2(x) - \tau^2(y)\Big) w(x)\, w(y)\, \Big\{K(x-y) - K(x+y)\Big\}\, dxdy.$$

Note that the previous identity is completely identical to the one in the general case but with integrals now computed in the half-line instead of the whole line, and with $K(x-y)-K(x+y)$ taking the role of $K(x-y)$.

Proof of Corollary 1.2.2. We use the symmetry properties of the functions (σ and τ are even while w is odd) to rewrite the identity from Lemma 1.2.1 in terms of integrals computed only in $\mathbb{R}^+$. That is,

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\big(\sigma(x)-\sigma(y)\big)^2\big(\tau^2(x)+\tau^2(y)\big)\,w(x)\,w(y)\,K(x-y)\,dxdy\\
&\quad=\int_{\mathbb{R}}\int_0^{\infty}\big(\sigma(x)-\sigma(y)\big)^2\big(\tau^2(x)+\tau^2(y)\big)\,w(x)\,w(y)\,[K(x-y)-K(x+y)]\,dxdy\\
&\quad=2\int_0^{\infty}\int_0^{\infty}\big(\sigma(x)-\sigma(y)\big)^2\big(\tau^2(x)+\tau^2(y)\big)\,w(x)\,w(y)\\
&\qquad\qquad\qquad\qquad\cdot[K(x-y)-K(x+y)]\,dxdy
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\big(\sigma^2(x)-\sigma^2(y)\big)\big(\tau^2(x)-\tau^2(y)\big)\,w(x)\,w(y)\,K(x-y)\,dxdy\\
&\quad=\int_{\mathbb{R}}\int_0^{\infty}\big(\sigma^2(x)-\sigma^2(y)\big)\big(\tau^2(x)-\tau^2(y)\big)\,w(x)\,w(y)\,[K(x-y)-K(x+y)]\,dxdy\\
&\quad=2\int_0^{\infty}\int_0^{\infty}\big(\sigma^2(x)-\sigma^2(y)\big)\big(\tau^2(x)-\tau^2(y)\big)\,w(x)\,w(y)\\
&\qquad\qquad\qquad\qquad\cdot[K(x-y)-K(x+y)]\,dxdy.
\end{aligned}$$

From this, we conclude the desired result by applying Lemma 1.2.1. □

1.3 Some maximum principles in the exterior of an interval

In this section we prove two maximum principles in the exterior of an interval for some linear equations driven by an integro-differential operator plus a zeroth order term. The first result applies to functions without any symmetry, while the second one concerns odd functions. They will be the fundamental tool in Section 1.5 and 1.6 to show that the quotient of two bounded solutions to equation (1.1.1) is also bounded.

Proposition 1.3.1. *Let L be an integro-differential operator of the form (1.1.2) satisfying conditions (K1) and (K3) for some $1/2\le \underline{s}\le\overline{s}<1$. Assume that the potential function $c=c(x)$ satisfies (1.1.3) for some positive constants R_0 and c_0.*

For $\alpha>2\overline{s}-1$, let φ be a bounded and C^1 function in $\mathbb{R}$ such that $[\varphi']_{C^\alpha(\mathbb{R})}<+\infty$,

$$L\varphi-c\varphi\ge 0\quad\text{in }\mathbb{R}\setminus[-R_0,R_0],$$

and

$$\varphi\ge 0\quad\text{in }[-R_0,R_0].$$

Then

$$\varphi\ge 0\quad\text{in }\mathbb{R}.$$

For simplicity, we are assuming $1/2 \le \underline{s} \le \overline{s} < 1$ since this is the range in which we are applying the result. However, the proof can be easily adapted to $0 < \underline{s} \le \overline{s} < 1$ and any dimension (with the ball taking the role of the interval). Moreover, we point out that the negativity of the potential function c at infinity, which is an assumption in some parts of Theorem 1.1.1, originates on this maximum principle.

Proof of Proposition 1.3.1. Assume the result to be false. Then, the infimum of φ is negative. In the case it is achieved, the contradiction comes directly from evaluating the operator $L\varphi - c\varphi$ at a point where such a minimum is attained. On the contrary, if the infimum is not achieved, we can construct a sequence of points $x_k \notin [-R_0, R_0]$ where φ takes negative values and approaches the infimum in the following way:

$$\varphi(x_k) - \varphi(x) \le \varphi(x_k) - \inf_{\mathbb{R}} \varphi \le \frac{1}{k} \quad \text{for all } \ x \in \mathbb{R}. \tag{1.3.1}$$

Next, we evaluate $L\varphi - c\varphi$ at that sequence of points. In order to do it, we split the integro-differential term of the operator into two parts, and we estimate each one separately. That is,

$$\begin{aligned} L\,\varphi(x_k) &= \int_{-\infty}^{\infty} \Big(\varphi(x_k) - \varphi(y)\Big) K(x_k - y)\, dy = \int_{-\infty}^{\infty} \Big(\varphi(x_k) - \varphi(x_k - z)\Big) K(z)\, dz \\ &= \int_{\delta}^{\infty} \Big(2\varphi(x_k) - \varphi(x_k - z) - \varphi(x_k + z)\Big) K(z)\, dz \\ &\quad + \int_{0}^{\delta} \Big(2\varphi(x_k) - \varphi(x_k - z) - \varphi(x_k + z)\Big) K(z)\, dz, \end{aligned}$$

where δ is a positive parameter to be chosen later. Here, we have used the odd symmetry of the kernel K to write the operator in terms of the second order differences.

Let us first estimate the term of the tails. If we use condition (1.3.1) and the ellipticity assumption (K3) we obtain

$$\begin{aligned} \int_{\delta}^{\infty} \Big(2\varphi(x_k) - \varphi(x_k - z) - \varphi(x_k + z)\Big) K(z)\, dz &\le \frac{2}{k} \int_{\delta}^{\infty} K(z)\, dz \\ &\le \frac{C}{k} \left(\int_{\delta}^{\infty} \frac{1}{z^{1+2\underline{s}}} dz + \int_{\delta}^{\infty} \frac{1}{z^{1+2\overline{s}}} dz \right) \le \frac{C}{k} \left(\delta^{-2\overline{s}} + \delta^{-2\underline{s}} \right). \end{aligned}$$

For the second integral we use the regularity of φ. Since φ' is globally Hölder with exponent $\alpha > 2\overline{s} - 1 \ge 2\underline{s} - 1$, the second order incremental quotients satisfy

$$|\varphi(x_k + z) + \varphi(x_k - z) - 2\varphi(x_k)| \le C|z|^{\alpha+1}.$$

Therefore, using this estimate and the ellipticity assumption (K3) we get

$$\begin{aligned} \int_{0}^{\delta} \Big(2\varphi(x_k) - \varphi(x_k - z) - \varphi(x_k + z)\Big) K(z)\, dz &\le C \int_{0}^{\delta} |z|^{\alpha+1}\, K(z)\, dz \\ &\le C \left(\int_{\delta}^{\infty} \frac{z^{1+\alpha}}{z^{1+2\underline{s}}} dz + \int_{\delta}^{\infty} \frac{z^{1+\alpha}}{z^{1+2\overline{s}}} dz \right) \le C \left(\delta^{\alpha+1-2\overline{s}} + \delta^{\alpha+1-2\underline{s}} \right). \end{aligned}$$

On the other hand, we use assumption (1.1.3) together with conditions $\varphi(x_k) < 0$ and $\varphi(x_k) \le \frac{1}{k} + \inf_{\mathbb{R}} \varphi$ to bound the zeroth order term as follows

$$-c(x_k)\,\varphi(x_k) \le c_0\,\varphi(x_k) \le \frac{c_0}{k} + c_0 \inf_{\mathbb{R}} \varphi.$$

Combining all this and taking $\delta = k^{-1/2}$, we find that

$$\begin{aligned} 0 &\leq L\varphi(x_k) - c(x_k)\,\varphi(x_k) \\ &\leq C\left(k^{\overline{s}-1} + k^{(2\overline{s}-1-\alpha)/2} + k^{\underline{s}-1} + k^{(2\underline{s}-1-\alpha)/2}\right) + \frac{c_0}{k} + c_0 \inf_{\mathbb{R}} \varphi \quad \text{for all } \ k \in \mathbb{Z}^+. \end{aligned}$$

Finally, by letting k tend to infinity and using the assumptions $1/2 \leq \underline{s} \leq \overline{s} < 1$ and $\alpha > 2\overline{s} - 1 \geq 2\underline{s} - 1$ we conclude

$$0 \leq c_0 \inf_{\mathbb{R}} \varphi,$$

which contradicts the infimum being negative. □

Odd functions are defined by their values in $(0, +\infty)$. We want to take advantage of this property to find an alternative and more useful expression for integro-differential operators when acting on such functions.

Lemma 1.3.2. *Let L be an integro-differential operator of the form (1.1.2), and let φ be an odd function. Then,*

$$L\varphi(x) = \int_0^\infty \big(\varphi(x) - \varphi(y)\big)\big(K(x-y) - K(x+y)\big)\,dy + \left(2\int_x^\infty K(z)\,dz\right)\varphi(x).$$

Note that this alternative expression consists on a regional-type integro-differential operator in $(0, +\infty)$ plus a zeroth order term. This structure is more suitable to work with, and it will be used to establish a maximum principle in the odd setting. As it occurs in Corollary 1.2.2, in the odd framework $K(x-y) - K(x+y)$ takes the role of $K(x-y)$. For this reason it is natural to impose the condition $K(x-y) - K(x+y) \geq 0$ for each $x, y \in (0, +\infty)$ when working with odd functions. Actually, such a condition turns out to be equivalent to K being nonincreasing in $(0, +\infty)$.

Proof of Lemma 1.3.2. If we split the integral into two terms and use the odd symmetry we arrive at

$$\begin{aligned} L\varphi(x) &= \int_{-\infty}^\infty \big(\varphi(x) - \varphi(y)\big)\,K(x-y)\,dy \\ &= \int_{-\infty}^0 \big(\varphi(x) - \varphi(y)\big)\,K(x-y)\,dy + \int_0^\infty \big(\varphi(x) - \varphi(y)\big)\,K(x-y)\,dy \\ &= \int_0^\infty \big(\varphi(x) - \varphi(-y)\big)\,K(x+y)\,dy + \int_0^\infty \big(\varphi(x) - \varphi(y)\big)\,K(x-y)\,dy \\ &= \int_0^\infty \big(\varphi(x) + \varphi(y)\big)\,K(x+y)\,dy + \int_0^\infty \big(\varphi(x) - \varphi(y)\big)\,K(x-y)\,dy \\ &= \int_0^\infty \big(\varphi(x) - \varphi(y)\big)\big(K(x-y) - K(x+y)\big)\,dy + \left(2\int_x^\infty K(z)\,dz\right)\varphi(x). \end{aligned}$$

□

Next, we establish an analogous maximum principle to Proposition 1.3.1 in the case of odd functions. In this scenario, conditions are only imposed in the half-line since the odd symmetry transfers the information to the whole space.

Proposition 1.3.3. *Let L be an integro-differential operator of the form (1.1.2) with nonincreasing kernel K satisfying conditions (K1) and (K2) for some $s \in [1/2, 1)$ and $0 < \lambda \leq \Lambda$. Assume the potential function $c = c(x)$ is even and satisfies (1.1.3) and*

$$||c||_{L^\infty(\mathbb{R})} < \frac{\lambda}{s\, r_0^{2s}}, \tag{1.3.2}$$

for some positive constants $R_0 > r_0 > 0$.

For $\alpha > 2s-1$, let φ be a bounded and C^1 odd function in $\mathbb{R}$ such that $[\varphi']_{C^\alpha(\mathbb{R})} < +\infty$,

$$L\varphi - c\varphi \geq 0 \quad \text{in} \ (0, r_0) \cup (R_0, +\infty),$$

and

$$\varphi \geq 0 \quad \text{in} \ [r_0, R_0].$$

Then,

$$\varphi \geq 0 \quad \text{in} \ [0, +\infty).$$

Note that (1.3.2) is a small domain condition, which is satisfied when r_0 is small enough depending on the integro-differential operator and the potential function. When applying this result in Section 1.6, such a condition will not impose any restriction since we will have enough freedom to choose $r_0 > 0$ as small as needed.

Proof of Proposition 1.3.3. We begin by noticing that using the previous lemma we can rewrite $L\varphi - c\varphi \geq 0$ as

$$\int_0^\infty \big(\varphi(x) - \varphi(y)\big)\big(K(x-y) - K(x+y)\big)\, dy - \left(c(x) - 2\int_x^\infty K(z)\, dz\right)\varphi(x) \geq 0.$$

Thus, it is clear that we can repeat the proof of Proposition 1.3.1 if we show that

$$\tilde{c}(x) := c(x) - 2\int_x^\infty K(z)\, dz,$$

satisfies

$$\tilde{c}(x) \leq -\tilde{c}_0 < 0 \quad \text{in} \quad (0, r_0) \cup (R_0, +\infty)$$

for some positive constant $\tilde{c}_0$.

On the one hand, by combining the positivity of the kernel K and condition (1.1.3), we deduce that given any $x \in (R_0, +\infty)$,

$$\tilde{c}(x) \leq -c_0 < 0.$$

On the other hand, by using the ellipticity assumption (K2), we obtain that given any $x \in (0, r_0)$,

$$\tilde{c}(x) \leq ||c||_{L^\infty(\mathbb{R})} - 2\lambda \int_x^\infty z^{-1-2s}\, dz = ||c||_{L^\infty(\mathbb{R})} - \frac{\lambda}{s} x^{-2s} \leq ||c||_{L^\infty(\mathbb{R})} - \frac{\lambda}{s} r_0^{-2s} < 0.$$

Hence, it is enough to take $\tilde{c}_0 = \min\left\{c_0, \frac{\lambda}{s} r_0^{-2s} - ||c||_{L^\infty(\mathbb{R})}\right\} > 0$. □

Let us remark that a maximum principle as in Proposition 1.3.3 cannot hold if we remove the odd symmetry of the function. In that case, having a negative minimum in $(0, +\infty)$ does not give any information about the sign of the operator at this point since the behavior of the function in $(-\infty, 0)$ is unknown.

1.4 Integrability bounds for the kernel

This section is devoted to presenting some integrability bounds that will be needed to establish Theorems 1.1.1 and 1.1.4. In fact, the validity of these bounds is what prevents us from extending our results to $s \in (0, 1/2)$.

In [132], Hamel, Ros-Oton, Sire, and Valdinoci work with compactly supported kernels in dimension 2. Once such a condition is assumed, the integrability bounds for the kernel follow immediately for free. In our case, when removing that assumption, some estimates become much more delicate. In order to control the integrals we define some auxiliary sets and prove certain relations between them that simplify the computations.

First, we show the following identity:

Lemma 1.4.1. *Let S_R, D_R, $\mathcal{T}_R^x$, and $\mathcal{T}_R^y$ be the sets*

$$S_R = (B_{2R} \times B_R^c) \cup (B_R^c \times B_{2R}) \subset \mathbb{R}^n \times \mathbb{R}^n,$$

$$D_R = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq 4R\} \subset \mathbb{R}^n \times \mathbb{R}^n,$$

$$\mathcal{T}_R^x = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \ \text{ s.t. } \ |x| < 2R \ \text{ and } \ |x - y| \geq 4R\} \subset \mathbb{R}^n \times \mathbb{R}^n,$$

and

$$\mathcal{T}_R^y = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \ \text{ s.t. } \ |y| < 2R \ \text{ and } \ |x - y| \geq 4R\} \subset \mathbb{R}^n \times \mathbb{R}^n.$$

Then, $\mathcal{T}_R^x$ and $\mathcal{T}_R^y$ are disjoint and satisfy

$$S_R \setminus D_R = \mathcal{T}_R^x \cup \mathcal{T}_R^y.$$

Proof. On the one hand, let $(x, y) \in S_R \setminus D_R$. By the symmetry of the set with respect to x and y we can assume without loss of generality that $(x, y) \in (B_{2R} \times B_R^c) \cap \{|x - y| > 4R\}$. Then, $(x, y) \in \mathcal{T}_R^x$ follows trivially.

On the other hand, given $(x, y) \in \mathcal{T}_R^x$, we can apply the triangle inequality to deduce that $|y| \geq 2R$. Therefore, we conclude that $(x, y) \in S_R \setminus D_R$.

Finally, in order to prove that the sets $\mathcal{T}_R^x$ and $\mathcal{T}_R^y$ are disjoint we only need to recall that given $(x, y) \in \mathcal{T}_R^x$, it satisfies $|y| \geq 2R$, and therefore $(x, y) \notin \mathcal{T}_R^y$. □

Next, we prove a useful inclusion of sets.

Lemma 1.4.2. *Let S_R and D_R be as in Lemma 1.4.1, and let $\mathcal{R}_R^x$ and $\mathcal{R}_R^y$ be the sets*

$$\mathcal{R}_R^x = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \ \text{ s.t. } \ |x| < R \ \text{ and } \ |x - y| \leq 2R\} \subset \mathbb{R}^n \times \mathbb{R}^n,$$

and

$$\mathcal{R}_R^y = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \ \text{ s.t. } \ |y| < R \ \text{ and } \ |x - y| \leq 2R\} \subset \mathbb{R}^n \times \mathbb{R}^n.$$

Then,

$$\mathcal{R}_{2R}^x \setminus \mathcal{R}_R^x \subseteq S_R \cap D_R \subseteq \left(\mathcal{R}_{2R}^x \setminus \mathcal{R}_{2/3R}^x\right) \cup \left(\mathcal{R}_{2R}^y \setminus \mathcal{R}_{2/3R}^y\right)$$

Proof. The proof of these inclusions is simple. As in Lemma 1.4.1, we only need to consider different cases and use the triangle inequality to relate $|x|$, $|y|$, and $|x - y|$.

For the first inclusion, let $(x, y) \in \mathcal{R}_{2R}^x \setminus \mathcal{R}_R^x$. We distinguish two cases: either $|x| \leq R$ and $2R \leq |x-y| \leq 4R$, or $R \leq |x| \leq 2R$ and $|x-y| \leq 4R$. In the first scenario, it is clear by using the triangle inequality that $|y| \geq R$, and therefore $(x, y) \in (B_{2R} \times B_R^c) \cap D_R \subset S_R \cap D_R$. In the second one, we only need to note that $(B_{2R} \setminus B_R) \times \mathbb{R}^n \subset (B_{2R} \times B_R^c) \cup (B_R^c \times B_{2R})$.

For the second inclusion, by taking advantage of the symmetry with respect to x and y of the sets S_R and D_R it is enough to prove that $(B_{2R}\times B_R^c)\cap D_R \subset (\mathcal{R}^x_{2R}\setminus\mathcal{R}^x_{2/3R})\cup(\mathcal{R}^y_{2R}\setminus\mathcal{R}^y_{2/3R})$. Then, given $(x,y)\in(B_{2R}\times B_R^c)\cap D_R$, if $4/3R\le|x-y|\le 4R$ or $2/3R\le|x|\le 2R$, it is clear that $(x,y)\in\mathcal{R}^x_{2R}\setminus\mathcal{R}^x_{2/3R}$. Therefore, we are left with proving the desired result for the case $|x|\le 2/3R$, $|y|\ge R$, and $|x-y|\le 4/3R$. By applying the triangle inequality we can deduce that in such a case $|y|\le 2R$ and we conclude that $(x,y)\in\mathcal{R}^y_{2R}\setminus\mathcal{R}^y_{2/3R}$. $\square$

Once we have established the previous relations of sets, we can proceed by proving the integral estimates. We first state them for the kernel of the fractional Laplacian. The case of general integro-differential operators will follow from them as a consequence of the ellipticity assumptions.

Lemma 1.4.3. *Let S_R and D_R be as in Lemma 1.4.1 and Lemma 1.4.2. Assume $s\in(0,1)$ and $0\le\gamma\le\min(s,1/2)$.*

Then,

$$\int_{S_R\cap D_R}\frac{|x|^{2\gamma}}{|x-y|^{n+2s-2}}\,dxdy\le C\,R^{2\gamma+n+2-2s},$$

and

$$\int_{S_R\setminus D_R}\frac{|x|^{2\gamma}}{|x-y|^{n+2s}}\,dxdy\le C\,R^{2\gamma+n-2s},$$

where C is a positive constant depending only on n, s, and γ.

We point out that analogous bounds from below can also be deduced. However, since we will not use such estimates in the present work, we skip them.

Proof of Lemma 1.4.3. To obtain the first estimate we use the inclusion of sets given by Lemma 1.4.2. That is,

$$\begin{aligned}\int_{S_R\cap D_R}\frac{|x|^{2\gamma}}{|x-y|^{n+2s-2}}\,dxdy &\le CR^{2\gamma}\int_{S_R\cap D_R}|x-y|^{2-n-2s}\,dxdy\\
&\le CR^{2\gamma}\left(\int_{\mathcal{R}^x_{2R}\setminus\mathcal{R}^x_{2/3R}}|x-y|^{2-n-2s}\,dxdy+\int_{\mathcal{R}^y_{2R}\setminus\mathcal{R}^y_{2/3R}}|x-y|^{2-n-2s}\,dxdy\right)\\
&\le CR^{2\gamma}\int_{\mathcal{R}^x_{2R}\setminus\mathcal{R}^x_{2/3R}}|x-y|^{2-n-2s}\,dxdy\\
&= CR^{2\gamma}\left(\int_{\mathcal{R}^x_{2R}}|x-y|^{2-n-2s}\,dxdy-\int_{\mathcal{R}^x_{2/3R}}|x-y|^{2-n-2s}\,dxdy\right)\\
&= CR^{2\gamma}\left((2R)^{n-2s+2}-(2/3R)^{n-2s+2}\right)\\
&= C\,R^{2\gamma+n+2-2s}.\end{aligned}$$

The second bound is more delicate. First we find that

$$\begin{aligned}\int_{\mathcal{T}^x_{2R}}\frac{|x|^{2\gamma}}{|x-y|^{n+2s}}\,dxdy &\le CR^{2\gamma}\int_{\mathcal{T}^x_{2R}}|x-y|^{-n-2s}\,dxdy,\\
&= CR^{2\gamma}\int_{B_{2R}}dw\int_{B^c_{4R}}|z|^{-n-2s}dz\\
&= C\,CR^{2\gamma+n}\int_{4R}^{\infty}r^{-n-2s}r^{n-1}\,dr\\
&= C\,R^{2\gamma+n-2s},\end{aligned}$$

where we have performed the change of variables: $z = x - y$ and $w = x$. Next, we obtain

$$\begin{aligned}\int_{\mathcal{T}_{2R}^y} \frac{|x|^{2\gamma}}{|x-y|^{n+2s}}\,dxdy &= \int_{B_{2R}} dw \int_{B_{4R}^c} dz \frac{|w+z|^{2\gamma}}{|z|^{n+2s}} \\ &\le \int_{B_{2R}} dw \int_{B_{4R}^c} dz \frac{|w|^{2\gamma}+|z|^{2\gamma}}{|z|^{n+2s}} \\ &\le C\,R^n \left(R^{2\gamma} \int_{4R}^{\infty} r^{-n-2s} r^{n-1}\,dr + \int_{4R}^{\infty} r^{-n-2s+2\gamma} r^{n-1}\,dr \right) \\ &= C\,R^{2\gamma+n-2s}.\end{aligned}$$

Finally, we conclude the proof by applying Lemma 1.4.1. Let us point out that it is crucial in the last estimate to assume $\gamma \le \min(s, 1/2)$ in order to ensure the integrability. □

Once we have established the previous bounds for the kernel of the fractional Laplacian, we can easily obtain the estimates we need, in cross-shaped domains, for the biggest class of operators satisfying condition (K3).

Corollary 1.4.4. *Let L be an integral operator of the form (1.1.2), with kernel K satisfying conditions (K1) and (K3) for some $0 < \underline{s} \le \overline{s} < 1$. Assume the set S_R is defined as in Lemma 1.4.1 and $0 \le \gamma \le \min(\underline{s}, 1/2)$.*

Then

$$\int_{S_R} \min\left\{1, \frac{|x-y|}{R}\right\}^2 |x|^{2\gamma}\, K(x-y)\, dxdy \le C\, R^{n+2\gamma-2\underline{s}},$$

for a positive constant C not depending on R.

In particular, if $n = 1$, $1/2 \le \underline{s} \le \overline{s} < 1$, and $\gamma \in [0, \underline{s} - 1/2]$, there is a positive constant C, independent of R, such that

$$\int_{S_R} \min\left\{1, \frac{|x-y|}{R}\right\}^2 |x|^{2\gamma}\, K(x-y)\, dxdy \le C,$$

for any $R \ge 1$.

Note that the uniform bound can only be established when $n + 2\gamma - 2\underline{s} \le 0$. Since the dimension n is an integer, it means that the previous condition is not satisfied unless $n = 1$, $1/2 \le \underline{s} \le \overline{s} < 1$, and $\gamma \in [0, \underline{s} - 1/2]$. This is the reason why we need to assume such dimension and range of fractional powers, in addition to a growth condition of order $s - 1/2$ in Theorem 1.1.1.

Proof of Corollary 1.4.4. First, note that

$$\min\left\{1, \frac{|x-y|}{R}\right\} = \begin{cases} \frac{|x-y|}{R} & \text{if } (x,y) \in D_R, \\ 1 & \text{otherwise,} \end{cases}$$

where D_R is the set defined in Lemma 1.4.3.

Then, by the linearity of the integral, the ellipticity assumption in the kernel (K3), and the relations of sets from Lemma 1.4.1 we get

$$
\begin{aligned}
&\int_{S_R} \min\left\{1, \frac{|x-y|}{R}\right\}^2 |x|^{2\gamma} K(x-y)\, dxdy \\
&\qquad = \int_{S_R \cap D_R} \frac{|x-y|^2}{R^2} |x|^{2\gamma} K(x-y)\, dxdy + \int_{S_R \setminus D_R} |x|^{2\gamma} K(x-y)\, dxdy \\
&\qquad \le \Lambda_1 \left(\int_{S_R \cap D_R} \frac{|x|^{2\gamma}}{R^2 |x-y|^{n+2\underline{s}-2}}\, dxdy + \int_{S_R \setminus D_R} \frac{|x|^{2\gamma}}{|x-y|^{n+2\underline{s}}}\, dxdy \right) \\
&\qquad\quad + \Lambda_2 \left(\int_{S_R \cap D_R} \frac{|x|^{2\gamma}}{R^2 |x-y|^{n+2\overline{s}-2}}\, dxdy + \int_{S_R \setminus D_R} \frac{|x|^{2\gamma}}{|x-y|^{n+2\overline{s}}}\, dxdy \right) \\
&\qquad \le \Lambda_1\, C_{n,\underline{s}}\, R^{n+2\gamma-2\underline{s}} + \Lambda_2\, C_{n,\overline{s}}\, R^{n+2\gamma-2\overline{s}} \le C\, R^{n+2\gamma-2\underline{s}}.
\end{aligned}
$$

□

Finally, we establish an analogue result in the odd setting.

Corollary 1.4.5. *Let L be an integral operator of the form (1.1.2), with kernel K being radially decreasing and satisfying conditions (K1) and (K2). Assume $n = 1$, $0 \le \gamma \le \min(s, 1/2)$, and the set*

$$
S_R^{++} = S_R \cap \left(\mathbb{R}^+ \times \mathbb{R}^+ \right)
$$

with S_R as in the previous results.

Then,

$$
\int_{S_R^{++}} \min\left\{1, \frac{|x-y|}{R}\right\}^2 |x|^{2\gamma} \left(K(x-y) - K(x+y) \right) dxdy \le C\, R^{1+2\gamma-2s},
$$

for a positive constant C not depending on R. In particular, in the case $s \in [1/2, 1)$ and $0 \le \gamma \le s - 1/2$

$$
\int_{S_R^{++}} \min\left\{1, \frac{|x-y|}{R}\right\}^2 |x|^{2\gamma} \left(K(x-y) - K(x+y) \right) dxdy \le C,
$$

for any $R \ge 1$.

Proof. By using Lemma 1.4.3, the ellipticity condition of the kernel and the symmetries of the domain S_R with respect to x and y we get

$$\begin{aligned}
\int_{S_R^{++}} \min\left\{1, \frac{|x-y|}{R}\right\}^2 |x|^{2\gamma} \Big(K(x-y) - K(x+y)\Big)\, dxdy \\
&\le \int_{S_R^{++}} \min\left\{1, \frac{|x-y|}{R}\right\}^2 |x|^{2\gamma} \Big(K(x-y) + K(x+y)\Big)\, dxdy \\
&= \int_{S_R^{++}} \min\left\{1, \frac{\big||x|-|y|\big|}{R}\right\}^2 |x|^{2\gamma} \left(K(x-y) + K(x+y)\right)\, dxdy \\
&= \frac{1}{2}\int_{S_R} \min\left\{1, \frac{\big||x|-|y|\big|}{R}\right\}^2 |x|^{2\gamma} K(x-y)\, dxdy \\
&\le \frac{1}{2}\int_{S_R} \min\left\{1, \frac{|x-y|}{R}\right\}^2 |x|^{2\gamma} K(x-y)\, dxdy \\
&\le \frac{\Lambda}{2}\int_{S_R} \min\left\{1, \frac{|x-y|}{R}\right\}^2 |x|^{2\gamma} K(x-y)\, dxdy \\
&\le \Lambda\, C_s\, R^{1+2\gamma-2s}.
\end{aligned}$$

□

1.5 Proof of Theorems 1.1.1 and 1.1.4

This section is devoted to proving the results presented in Section 1.1 where no symmetries are assumed.

In order to deal with the first scenario in Theorem 1.1.1 we first show that the quotient of two bounded solutions is also bounded:

Proposition 1.5.1. *Let L be an integro-differential operator of the form (1.1.2) satisfying the symmetry and ellipticity conditions (K1) and (K3) for some $1/2 \le \underline{s} \le \overline{s} < 1$. Assume that the potential function $c = c(x)$ satisfies condition (1.1.3) for some positive constant R_0.*

For $\alpha > 2\overline{s} - 1$, let w and $\widetilde{w}$ be two bounded and C^1 functions such that $[w']_{C^\alpha(\mathbb{R})}$ and $[\widetilde{w}']_{C^\alpha(\mathbb{R})}$ are finite. In addition, assume that

$$w > 0 \quad \text{in } [-R_0, R_0],$$

$$Lw - cw \ge 0 \quad \text{in } \mathbb{R} \setminus [-R_0, R_0],$$

and

$$L\widetilde{w} - c\widetilde{w} = 0 \quad \text{in } \mathbb{R} \setminus [-R_0, R_0].$$

Then, there exists a positive constant C such that

$$\left|\frac{\widetilde{w}}{w}\right| \le C \quad \text{in } \mathbb{R}.$$

Proof. First, by applying Proposition 1.3.1 and the strong maximum principle we deduce

$$w > 0 \quad \text{in } \mathbb{R}.$$

As a consequence, the quotient $\tilde{w}/w$ is well-defined and continuous in the whole real line.

Next, we prove that such a quotient is indeed bounded. This will follow after showing the positivity of the functions

$$\varphi_{\pm} = C\,w \pm \tilde{w},$$

where C is a nonnegative constant to be chosen. Note that these functions inherit the regularity of w and $\tilde{w}$ from being a linear combination of them.

Let us take $C \geq 0$ satisfying

$$C \geq \left|\left|\frac{\tilde{w}}{w}\right|\right|_{L^\infty(-R_0,R_0)}.$$

It is clear by definition that $\varphi_\pm \geq 0$ in $[-R_0, R_0]$. Moreover,

$$L\varphi_\pm - c\varphi_\pm = C\big(Lw - cw\big) \pm \big(L\tilde{w} - c\tilde{w}\big) = C\big(Lw - cw\big) \geq 0 \quad \text{in } \mathbb{R} \setminus [-R_0, R_0].$$

Hence, by applying Proposition 1.3.1 to $\varphi_\pm$ we conclude that

$$\varphi_\pm = C\,w \pm \tilde{w} \geq 0 \qquad \text{in } \mathbb{R},$$

which is equivalent to

$$\left|\frac{\tilde{w}}{w}\right| \leq C \quad \text{in } \mathbb{R}.$$

□

Next, we establish the uniqueness result for the linear equation (1.1.1). As already explained in the introduction, we present here a more general result from which we will deduce Theorem 1.1.1 among others. On the one hand, the ellipticity condition on the kernel is relaxed to (K3), which means the kernel being bounded only from above, even with different order at the origin and infinity. On the other hand, it is not needed the existence of a positive solution but a positive supersolution.

Theorem 1.5.2. *Let L be an integro-differential operator of the form (1.1.2) satisfying the symmetry and ellipticity conditions (K1) and (K3) for some $1/2 \leq \underline{s} \leq \overline{s} < 1$.*

For $\alpha > 2\overline{s} - 1$, let w and $\tilde{w}$ be $C^{1,\alpha}$ functions in $\mathbb{R}$. Assume that

- *either w and $\tilde{w}$ are both bounded and such that $[w']_{C^\alpha(\mathbb{R})}$ and $[\tilde{w}']_{C^\alpha(\mathbb{R})}$ are finite, $w > 0$, and the potential function $c = c(x)$ satisfies condition (1.1.3);*
- *or w is such that*

$$0 < C^{-1} \leq w(x) \leq C \quad \text{in } \mathbb{R},$$

and $\tilde{w}$ satisfies the growth condition

$$||\tilde{w}||_{L^\infty(-R,R)} \leq C R^{\underline{s}-1/2}, \quad \text{for every } \; R > 1$$

for some positive constant C.

In addition, assume that

$$Lw - cw \geq 0 \quad \text{in } \mathbb{R},$$

and

$$L\tilde{w} - c\tilde{w} = 0 \quad \text{in } \mathbb{R}.$$

Then

$$\frac{\tilde{w}}{w} \equiv ctt.$$

In the following proof, once the boundedness of $\sigma := \widetilde{w}/w$ (Proposition 1.5.1) and some integrability estimates (Lemma 1.4.3) are established, it will be enough to follow the strategy developed in [132] to conclude that such a quotient is constant.

Proof of Theorem 1.5.2. We begin by noticing that, using the bounds on w and $\widetilde{w}$, and applying Proposition 1.5.1, we immediately deduce that $\sigma = \widetilde{w}/w$ satisfies the growth condition $|\sigma(x)| \le C|x|^{\frac{s}{2}-1/2}$. This is the first step to show that σ is constant.

Let η be a C^∞ function on $[0, +\infty)$ such that $0 \le \eta \le 1$ and

$$\eta = \begin{cases} 1 & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t \ge 2. \end{cases}$$

For each $R > 1$, we take $\eta_R(x) = \eta\left(\frac{|x|}{R}\right)$. It is clear that it satisfies the pointwise estimate

$$|\eta_R(x) - \eta_R(y)| \le C \min\left\{1, \frac{|x-y|}{R}\right\} \quad \text{for every} \quad x, y \in \mathbb{R} \tag{1.5.1}$$

and some positive constant C depending only on η.

Next, we apply Lemma 1.2.1 with $\tau = \eta_R$ to deduce

$$\begin{aligned} 0 \le J_1 :&= \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\sigma(x) - \sigma(y)\Big)^2 \Big(\eta_R^2(x) + \eta_R^2(y)\Big)\, w(x)\, w(y)\, K(x-y)\; dxdy \\ &\le - \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\sigma^2(x) - \sigma^2(y)\Big)\Big(\eta_R^2(x) - \eta_R^2(y)\Big) w(x)\, w(y)\, K(x-y)\, dxdy \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} |\sigma(x) - \sigma(y)||\sigma(x) + \sigma(y)||\eta_R(x) - \eta_R(y)||\eta_R(x) + \eta_R(y)| \cdot \\ &\qquad\qquad \cdot w(x)\, w(y)\, K(x-y)\, dxdy \\ &= \int_{S_R} |\sigma(x) - \sigma(y)||\sigma(x) + \sigma(y)||\eta_R(x) - \eta_R(y)||\eta_R(x) + \eta_R(y)| \cdot \\ &\qquad\qquad \cdot w(x)\, w(y)\, K(x-y)\, dxdy \\ &=: J_2. \end{aligned}$$

Note that the last equality follows from the support of $|\eta_R(x) - \eta_R(y)|$ being the set S_R defined in Lemma 1.4.1.

Furthermore, by using Cauchy-Schwartz inequality we get

$$\begin{aligned} J_2^2 &\le \int_{S_R} \Big(\sigma(x) - \sigma(y)\Big)^2 \Big(\eta_R(x) + \eta_R(y)\Big)^2 w(x)\, w(y)\, K(x-y)\, dxdy \,\cdot \\ &\qquad \cdot \int_{S_R} \Big(\sigma(x) + \sigma(y)\Big)^2 \Big(\eta_R(x) - \eta_R(y)\Big)^2 w(x)\, w(y)\, K(x-y)\, dxdy \\ &\le 2\, J_1 \int_{S_R} \Big(\sigma(x) + \sigma(y)\Big)^2 \Big(\eta_R(x) - \eta_R(y)\Big)^2 w(x)\, w(y)\, K(x-y)\, dxdy. \end{aligned}$$

Now, by combining the boundedness of w, the growth condition on σ, the pointwise estimate (1.5.1) for η_R, and the integrability result from Corollary 1.4.4, we find

$$\begin{aligned} &\int_{S_R} \Big(\sigma(x) + \sigma(y)\Big)^2 \Big(\eta_R(x) - \eta_R(y)\Big)^2 w(x)\, w(y)\, K(x-y)\, dx\, dy \le \\ &\qquad\qquad \le C \int_{S_R} \Big(\eta_R(x) - \eta_R(y)\Big)^2 |\sigma(x)|^2\, K(x-y)\, dx\, dy \le C. \end{aligned}$$

Summarizing, we have

$$0 \le J_1^2 \le J_2^2 \le C\, J_1,$$

which leads to

$$J_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} \big(\sigma(x) - \sigma(y)\big)^2 \big(\eta_R^2(x) + \eta_R^2(y)\big)\, w(x)\, w(y)\, K(x-y)\, dx\, dy \le C.$$

In particular, since $\eta_R = 1$ in B_R, we deduce

$$\int_{B_R} \int_{B_R} \big(\sigma(x) - \sigma(y)\big)^2\, w(x)\, w(y)\, K(x-y)\, dx\, dy \le C,$$

where C is a positive constant not depending on R. From that estimate and monotone convergence theorem we obtain that $\big(\sigma(x) - \sigma(y)\big)^2\, w(x)\, w(y)\, K(x-y)$ belongs to $L^1(\mathbb{R}\times\mathbb{R})$. Hence, we conclude from the dominated convergence theorem that

$$\lim_{R\to\infty} \int_{S_R} \big(\sigma(x) - \sigma(y)\big)^2\, w(x)\, w(y)\, K(x-y) dx\, dy = 0.$$

Combining all this together, we arrive at

$$\begin{aligned}
&\left[\int_{\mathbb{R}} \int_{\mathbb{R}} \big(\sigma(x) - \sigma(y)\big)^2\, w(x)\, w(y)\, K(x-y) dx dy\right]^2 = \\
&\quad = \frac{1}{2} \lim_{R\to\infty} \left[\int_{\mathbb{R}} \int_{\mathbb{R}} \big(\sigma(x) - \sigma(y)\big)^2 \big(\eta_R^2(x) + \eta_R^2(y)\big)\, w(x)\, w(y)\, K(x-y) dx dy\right]^2 \\
&\quad \le C \lim_{R\to\infty} \int_{S_R} \big(\sigma(x) - \sigma(y)\big)^2 \big(\eta_R^2(x) + \eta_R^2(y)\big)\, w(x)\, w(y)\, K(x-y) dx dy \\
&\quad \le C \lim_{R\to\infty} \int_{S_R} \big(\sigma(x) - \sigma(y)\big)^2 w(x)\, w(y)\, K(x-y) dx dy = 0.
\end{aligned}$$

From this and the positivity of both w and K, we obtain that $\big(\sigma(x) - \sigma(y)\big)^2 = 0$ for almost every $(x, y) \in \mathbb{R}\times\mathbb{R}$. Thus, by continuity, we conclude that

$$\sigma = \frac{\tilde{w}}{w} \equiv ctt.$$

□

Using the previous result we can easily deduce Theorem 1.1.1. In fact, we only need to check that solutions from Theorem 1.1.1 have the required regularity to apply Theorem 1.5.2. Such property will follow thanks to the regularizing effect of operators satisfying the ellipticity assumption (K2).

Proof of Theorem 1.1.1. In order to prove Theorem 1.1.1 we will show that any bounded solution to the linear equation (1.1.1) with L being of the form (1.1.2) and satisfying (K1) and (K2) is globally Hölder continuous with exponent $\alpha + 1 > 2s$ (we use here the notation $C^\gamma = C^{\lfloor\gamma\rfloor, \gamma - \lfloor\gamma\rfloor}$ whenever $\gamma > 1$). From this and Theorem 1.5.2, the uniqueness result will follow.

The proof of the regularity is based on defining the auxiliary function $f(x) := c(x)u(x)$ and using the interior regularity results from [176] for the nonlocal equation

$$Lu = f \quad \text{in } B_1 \subset \mathbb{R}^n.$$

Let us first prove that any solution u satisfies $||u||_{C^\beta(\mathbb{R})} < \infty$ for each $\beta < 2s$. The boundedness of both u and c leads to $f \in L^\infty(\mathbb{R})$. Thus, we can apply Corollary 1.2 from [176] for each unitary ball in $\mathbb{R}$ to conclude that

$$\begin{aligned} ||u||_{C^\beta\left(B_{1/2}(x_0)\right)} &\leq C\left(||f||_{L^\infty(B_1(x_0))} + ||u||_{L^\infty(\mathbb{R})}\right) \\ &\leq C\left(||f||_{L^\infty(\mathbb{R})} + ||u||_{L^\infty(\mathbb{R})}\right) \end{aligned}$$

for any given point $x_0 \in \mathbb{R}$ and $\beta < 2s$.

In particular, we know that $||u||_{C^{\beta_0}(\mathbb{R})}$ is finite. Hence, we can use the fact that $||c||_{C^{\beta_0}(\mathbb{R})}$ is also finite to deduce that f inherits such a property and apply Theorem 1.1 from [176] to establish

$$\begin{aligned} ||u||_{C^{2s+\beta_0}\left(B_{1/2}(x_0)\right)} &\leq C\left(||f||_{C^{\beta_0}(B_1(x_0))} + ||u||_{C^{\beta_0}(\mathbb{R})}\right) \\ &\leq C\left(||f||_{C^{\beta_0}(\mathbb{R})} + ||u||_{C^{\beta_0}(\mathbb{R})}\right). \end{aligned}$$

Finally, if we take $\alpha := 2s+\beta_0-1$, we can apply Theorem 1.5.2 to deduce the uniqueness of solution, concluding the proof. □

We state now an interesting consequence of Theorem 1.5.2 which is not included in Theorem 1.1.1. It deals with sums of fractional Laplacians.

Corollary 1.5.3. *Let L be a nonlocal operator of the form*

$$Lu = \int_{\underline{s}}^{\overline{s}} (-\Delta)^s u \, d\mu(s),$$

with $1/2 \leq \underline{s} \leq \overline{s} < 1$, where μ is a probability measure supported in $[\underline{s}, \overline{s}]$, i.e.,

$$\mu \geq 0 \quad \text{and} \quad \mu\left([\underline{s}, \overline{s}]\right) = \mu(\mathbb{R}) = 1.$$

Assume that c is bounded in $\mathbb{R}$, satisfies condition (1.1.3), and $||c||_{C^{1,2\overline{s}-1}(\mathbb{R})} < +\infty$.

Let w and $\widetilde{w}$ be two bounded solutions of the linear equation

$$L\varphi - c(x)\varphi = 0 \quad \text{in } \mathbb{R},$$

with $w > 0$. Then

$$\frac{\widetilde{w}}{w} \equiv ctt.$$

Proof. In order to establish Corollary 1.5.3 we only need to show that the operator L is of the form (1.1.2) satisfying (K1) and (K3) and that bounded solutions of the linear equation are globally Hölder continuous with exponent grater than $2\overline{s}$.

First, let us rewrite the expression of L in an alternative way:

$$\begin{aligned} Lu &:= \int_{\underline{s}}^{\overline{s}} (-\Delta)^s u \, d\mu(s) = \int_{\underline{s}}^{\overline{s}} \left[\int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}} dy\right] d\mu(s) \\ &= \int_{\mathbb{R}} \Big(u(x)-u(y)\Big)\left(\int_{\underline{s}}^{\overline{s}} \frac{d\mu(s)}{|x-y|^{1+2s}}\right) dy. \end{aligned}$$

Thus, L is an integral operator of the form (1.1.2) with kernel

$$K(z) = \int_{\underline{s}}^{\overline{s}} \frac{d\mu(s)}{|z|^{1+2s}}.$$

Moreover, it satisfies conditions (K1) and (K3). Indeed,

$$\begin{aligned} K(z) &\le \int_{\underline{s}}^{\overline{s}} \frac{d\mu(s)}{|z|^{1+2\overline{s}}} \chi_{\{|z|\le 1\}}(z) + \int_{\underline{s}}^{\overline{s}} \frac{d\mu(s)}{|z|^{1+2\underline{s}}} \chi_{\{|z|\ge 1\}}(z) \\ &= \frac{1}{|z|^{1+2\overline{s}}} \chi_{\{|z|\le 1\}}(z) + \frac{1}{|z|^{1+2\underline{s}}} \chi_{\{|z|\ge 1\}}(z) \\ &\le \frac{1}{|z|^{1+2\overline{s}}} + \frac{1}{|z|^{1+2\underline{s}}}. \end{aligned}$$

Next, let us apply the regularity results from [55] to deduce the Hölder regularity of bounded solutions. Since c is a $C^{1,2\overline{s}-1}$ function we can use a standard bootstrap argument that leads to the desired regularity of the solution after using Lemma 2.1 from [55] $\lfloor 2\overline{s}/\alpha \rfloor + 1$ times, where α is a positive constant depending only $\underline{s}$ and n. Thus, we conclude that u belongs to $C^{1,2\overline{s}-1+\beta}$ in $\mathbb{R}$ with $\beta = \lfloor 2\overline{s}/\alpha \rfloor \alpha + \alpha - 2\overline{s} > 0$.

Combining all this, we can apply Theorem 1.5.2 to establish the uniqueness of solutions. □

Finally, we prove the nondegeneracy of layer solutions, Theorem 1.1.3.

Proof of Theorem 1.1.3. First, we know by Theorem 1 in [83] that the layer solution u is a $C^{2,2s-1+\gamma}$ function for some $\gamma > 0$ and u' is bounded in the whole line.

We need to show that u' is the unique bounded solution to

$$Lv - f'(u)v = 0 \quad \text{in } \mathbb{R}. \tag{1.5.2}$$

Let us take $c(x) = f'(u(x))$. We only need to check that the hypotheses in Theorem 1.1.1 are satisfied. Since $f' \in C^{\gamma}([-1,1])$ and u is a continuous and bounded function, it is clear that c is bounded and such that $[c]_{C^{\gamma}(\mathbb{R})}$ is finite. Furthermore

$$\lim_{x\to\pm\infty} c(x) = \lim_{z\to\pm 1} f'(z) = f'(\pm 1) < 0.$$

From this property and the continuity of c we deduce that condition (1.1.3) is satisfied.

Finally, since u' is a $C^{1,2s-1+\gamma}$ and positive (by definition of layer) bounded solution to (1.5.2) we can apply Theorem 1.1.1 to conclude the proof of the result. □

1.6 Odd solutions: Proof of Theorem 1.1.4 and Corollary 1.1.5

In this section we prove the main results of the chapter dealing with odd functions.

We begin by establishing that the quotient of an odd bounded solution and an odd bounded positive supersolution is also bounded.

Proposition 1.6.1. *Let L be an integro-differential operator of the form (1.1.2) with non-increasing kernel K satisfying the symmetry and ellipticity conditions (K1) and (K2) for some $s \in [1/2, 1)$ and $0 < \lambda \le \Lambda$. Assume the potential function $c = c(x)$ is bounded, even, and satisfies condition (1.1.3) for some positive constants R_0 and c_0.*

For $\alpha > 2s-1$, let w and $\tilde{w}$ be two odd bounded and C^1 functions such that $[w']_{C^{\alpha}(\mathbb{R})}$ and $[\tilde{w}']_{C^{\alpha}(\mathbb{R})}$ are finite and satisfy

$$w > 0 \quad in \ (0, R_0),$$

$$Lw - cw \geq 0 \quad \textit{in} \ [0, +\infty),$$

and

$$L\widetilde{w} - c\widetilde{w} = 0 \quad \textit{in} \ [0, +\infty).$$

Then, there exists a constant $C \geq 0$ such that

$$\left| \frac{\widetilde{w}}{w} \right| \leq C \quad \textit{in} \ \mathbb{R}.$$

Proof. First, by applying Proposition 1.3.3 and the strong maximum principle for odd functions we get

$$w > 0 \ \text{ in } \ (0, +\infty).$$

As a consequence, the quotient $\sigma := \tilde{w}/w$ is well-defined and continuous in $\mathbb{R} \setminus \{0\}$.

We will show that the quotient can be extended to be continuous and bounded in the whole real line. As in Proposition 1.5.1, this will follow after showing the positivity of the functions

$$\varphi_{\pm} = C\, w \pm \widetilde{w},$$

for some positive constant C in $[0, +\infty)$.

For this, let us take r_0 and C such that

$$0 < r_0 < \sqrt[2s]{\frac{\lambda}{s||c||_{L^\infty(\mathbb{R})}}},$$

and

$$C \geq \left|\left| \frac{\widetilde{w}}{w} \right|\right|_{L^\infty(r_0, R_0)}.$$

Note that the existence of such constants is guaranteed by the boundedness of the potential function c and the positivity of w.

Now, it is enough to check that the hypotheses of Proposition 1.3.3 are satisfied. By the choice of C, it is clear that $\varphi_{\pm} \geq 0$ in $[r_0, R_0]$ and

$$L\varphi_{\pm} - c\varphi_{\pm} = C\big(Lw - cw\big) \pm \big(L\widetilde{w} - c\widetilde{w}\big) = C\big(Lw - cw\big) \geq 0 \quad \text{in } \mathbb{R}^+.$$

Furthermore, both functions $\varphi_{\pm}$ are odd and inherit the regularity of w and $\widetilde{w}$ from being linear combinations of them.

Thus, Proposition 1.3.3 leads to

$$\varphi_{\pm} = C\, w \pm \widetilde{w} \geq 0 \qquad \text{in} \ \ [0, \infty),$$

which is equivalent to

$$\left| \frac{\widetilde{w}}{w} \right| \leq C \ \text{ in } \ (0, +\infty).$$

Finally, by continuity of both w and $\tilde{w}$ we can extend the result to the whole real line, concluding the proof. □

At this point we have all the ingredients to prove that the quotient of two odd solutions to (1.1.1), with one of them changing sign only once, is not only bounded but constant.

Proof of Theorem 1.1.4. The proof of this result is completely analogous to the one of Theorem 1.5.2, applying Proposition 1.6.1, Corollary 1.4.5, and Corollary 1.2.2 instead of Proposition 1.5.1, Corollary 1.4.4, and Lemma 1.2.1. □

Finally, we prove Corollary 1.1.5.

Proof of Corollary 1.1.5. First, let us point out that by the regularity theory for nonlocal equations and Proposition 1.1 in [117] we know that u is a $C^{2,2s-1+\gamma}$ function in $\mathbb{R}$ for some $\gamma > 0$. Furthermore, u is strictly decreasing in $\mathbb{R}^+ = (0, +\infty)$ with u' being bounded. Note that the even symmetry of u leads to the odd symmetry of u'.

We need to show that u' is the unique bounded odd solution to

$$Lv - f'(u)v = 0 \quad \text{in } \mathbb{R}. \tag{1.6.1}$$

Let us take $c(x) = f'(u(x))$. It is enough to check that the hypotheses in Theorem 1.1.4 are satisfied. Since $f' \in C^\gamma([0,1])$ and u is an even, continuous, and bounded function, we deduce that c is even, bounded, and such that $[c]_{C^\gamma(\mathbb{R})}$ is finite. Moreover, it satisfies

$$\lim_{x\to\pm\infty} c(x) = \lim_{z\to 0} f'(z) = f'(0) < 0.$$

Hence, (1.1.3) holds.

Finally, since $-u'$ is a positive bounded odd solution to (1.6.1) in $\mathbb{R}^+$ we can apply Theorem 1.1.4 to complete the proof. □

Chapter 2

The Neumann problem for the fractional Laplacian: regularity up to the boundary

In this chapter, which corresponds to [13] in collaboration with A. Audrito and X. Ros-Oton, we study the regularity up to the boundary of solutions to the Neumann problem for the fractional Laplacian. We prove that if u is a weak solution of $(-\Delta)^s u = f$ in Ω, $\mathcal{N}_s u = 0$ in Ω^c, then u is C^α up tp the boundary for some $\alpha > 0$. Moreover, in case $s > \frac{1}{2}$, we then show that $u \in C^{2s-1+\alpha}(\overline{\Omega})$. To prove these results we need, among other things, a delicate Moser iteration on the boundary with some logarithmic corrections.

Our methods allow us to treat as well the Neumann problem for the regional fractional Laplacian, and we establish the same boundary regularity result.

Prior to our results, the interior regularity for these Neumann problems was well understood, but near the boundary even the continuity of solutions was open.

2.1 Introduction and main results

We study the regularity of solutions to the Neumann problem

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega \\ \mathcal{N}_s u &= 0 \quad \text{in } \Omega^c, \end{cases} \tag{2.1.1}$$

where $\mathcal{N}_s$ is a "nonlocal normal derivative", given by

$$\mathcal{N}_s u(x) := c_{n,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy, \qquad x \in \Omega^c. \tag{2.1.2}$$

The constant $c_{n,s}$ is the one appearing in the definition the fractional Laplacian

$$(-\Delta)^s u(x) = c_{n,s}\, \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy. \tag{2.1.3}$$

The Neumann problem (2.1.1) was first introduced in [96, 100], and has been subsequently studied in several papers; see for example [1, 8, 77, 143, 190]. As explained in detail in [96], (2.1.1) is a natural Neumann problem for the fractional Laplacian, for several reasons:

• The problem has a variational structure, and weak solutions are obtained by minimizing the energy functional

$$\mathcal{E}(u) := \frac{c_{n,s}}{4} \int\int_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \int_\Omega f\,u. \tag{2.1.4}$$

Solutions exist if and only if $\int_\Omega f = 0$.

• The following integration by parts formulas hold for C^2 functions u, v:

$$\int_\Omega (-\Delta)^s u\,dx = -\int_{\Omega^c} \mathcal{N}_s u\,dx$$

and

$$\frac{c_{n,s}}{2} \int\int_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega v\,(-\Delta)^s u + \int_{\Omega^c} v\,\mathcal{N}_s u. \tag{2.1.5}$$

• The corresponding heat equation with homogeneous Neumann conditions possesses natural properties like conservation of mass inside Ω or convergence to a constant as $t \to \infty$.

• The problem has a natural probabilistic interpretation, heuristically described in [96], and rigorously studied in [190].

• As $s \uparrow 1$, we recover the classical Neumann problem for the Laplacian in Ω.

• The energy functional (2.1.4) is the same that yields solutions to the Dirichlet problem for the fractional Laplacian; see [163, 162].

The aim of this chapter is to study the boundary regularity of solutions to (2.1.1).

2.1.1 Main results

While the Dirichlet problem is very well understood [2, 14, 18, 36, 74, 114, 129, 128, 140, 162, 163], much less is known for the Neumann case. Our main result reads as follows:

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain. Let $s \in (0,1)$, and u be any weak solution of (2.1.1) with $f \in L^q(\Omega)$, with $q > \frac{n}{2s}$ and $\int_\Omega f = 0$.*

Then,

$$\|u\|_{C^\alpha(\overline{\Omega})} \le C\left(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

for some $\alpha > 0$. Moreover, if $s > \frac{1}{2}$, $q > n$, and Ω is C^1, we then have

$$\|u\|_{C^{2s-1+\alpha}(\overline{\Omega})} \le C\left(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

The constants C and α depend only on n, s, q, and Ω.

This is the first boundary regularity result for the Neumann problem (2.1.1), and even the continuity of solutions is new.

As in case of the Dirichlet problem [163], it turns out that the boundary regularity is much more delicate than the interior one, and does not follow easily by adapting the classical methods used for $s = 1$ [156, 149]. This is because in this nonlocal context one cannot use any even/odd reflection to study solutions near the boundary, and a completely different strategy is needed.

In [163], a key idea was to use the methods coming from equations with bounded measurable coefficients in non-divergence form. Here, instead, we will need to use methods coming from equations with bounded measurable coefficients in divergence form. More precisely, we will need (among other things) a delicate Moser iteration on the boundary involving some logarithmic corrections on $\partial\Omega$. This will be explained in more detail later on in the chapter.

In a sense, Theorem 2.1.1 can be seen as the Neumann version of the boundary regularity theory for the Dirichlet problem developed in [163].

Remark 2.1.2. It is important to remark that $2s-1$ is a natural critical exponent in this problem. This can be seen easily when $\Omega = \{x_n > 0\}$, in which the function $|x_n|^{2s-1}$ solves (2.1.1) pointwise, even though it is not a weak solution — nor it satisfies (2.1.5). Thus, $C^{2s-1+\alpha}(\overline{\Omega})$ is the minimum regularity needed in order to discard this kind of solutions. This will become even more clear in case of the regional fractional Laplacian, explained below.

2.1.2 Regional fractional Laplacian

The methods developed in this chapter allow us to treat as well the Neumann problem for the *regional* fractional Laplacian. This corresponds to a censored stochastic process; see [33].

Solutions to this problem are obtained by minimizing the energy

$$\mathcal{E}(u) := \frac{c_{n,s}}{4}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \int_\Omega f\,u, \tag{2.1.6}$$

and the operator is given by

$$(-\Delta)^s_\Omega u(x) = c_{n,s}\,\mathrm{PV}\int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy. \tag{2.1.7}$$

This problem shares many of the properties of (2.1.1) described above: it has a variational formulation, a nice probabilistic interpretation, convergence as $s\uparrow 1$ to the Neumann problem for the Laplacian, and conservation of mass for its parabolic version. The main difference is that the operator given by (2.1.7) depends on Ω, and that in this case $\mathbb{R}^n\setminus\overline{\Omega}$ plays no role.

The Dirichlet problem in this setting is obtained by considering (2.1.6) among all functions $u=0$ on $\partial\Omega$. Notice that, by trace theorems for $H^s(\Omega)$ spaces [93], this makes sense only when $s>\frac{1}{2}$. It turns out then that solutions to the Dirichlet problem are $C^{2s-1}(\overline{\Omega})$, and if $f>0$ they actually satisfy

$$u \asymp d^{2s-1} \quad \text{in} \quad \Omega;$$

see [33, 68, 73, 131].

However, as in case of the fractional Laplacian (2.1.1), the Neumann case is much less understood, and it is not even clear what is the right pointwise Neumann condition for solutions in this case.

An integration by parts formula found in [130] suggests that the right quantity in this context is given by[1]

$$\partial_\nu^{2s-1}u(z) := \lim_{t\downarrow 0}\frac{u(z+t\nu)-u(z)}{t^{2s-1}}, \qquad z\in\partial\Omega,$$

[1]Notice that when $u=0$ on $\partial\Omega$ (Dirichlet case), then this quantity is the same as $u/d^{2s-1}|_{\partial\Omega}$.

where ν is the (inward) unit normal to $\partial\Omega$. More precisely, it is proved in [130] that, if $u, v \in d^{2s-1}C^2(\overline{\Omega}) + C^2(\overline{\Omega})$ then[2]

$$\frac{c_{n,s}}{2}\int_\Omega\int_\Omega \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega v\,(-\Delta)^s_\Omega u + \kappa_{n,s}\int_{\partial\Omega} v\,\partial_\nu^{2s-1}u. \quad (2.1.8)$$

This is the analogue of (2.1.5) in this context, and suggests that the pointwise Neumann condition in this setting should be

$$\partial_\nu^{2s-1}u = 0 \quad \text{on} \quad \partial\Omega. \quad (2.1.9)$$

Our main result in this context answers positively this question, and reads as follows.

Theorem 2.1.3. *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain. Let $s \in (0,1)$, $f \in L^q(\Omega)$, with $q > \frac{n}{2s}$, be such that $\int_\Omega f = 0$, and u be any free minimizer of (2.1.6).*

Then,

$$\|u\|_{C^\alpha(\overline{\Omega})} \le C\left(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

for some $\alpha > 0$. Moreover, if $s > \frac{1}{2}$, $q > n$, and Ω is C^1, we then have

$$\|u\|_{C^{2s-1+\alpha}(\overline{\Omega})} \le C\left(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

In particular, for every $s \in (0,1)$ we have (2.1.9). The constants C and α depend only on n, s, q, and Ω.

In particular, thanks to Theorem 2.1.3, we find that the Neumann problem for the regional fractional Laplacian is actually

$$\begin{cases} (-\Delta)^s_\Omega u &= f \quad \text{in } \Omega \\ \partial_\nu^{2s-1}u &= 0 \quad \text{on } \partial\Omega. \end{cases} \quad (2.1.10)$$

Notice that our result also implies that solutions to the Neumann problem are more regular than those corresponding to the Dirichlet case, as expected.

Remark 2.1.4. Other Neumann problems for the fractional Laplacian $(-\Delta)^s$ have been introduced in [17, 19] and [128]. These different Neumann problems recover the classical Neumann problem as a limit case, and the one in [17, 19] has a probabilistic interpretation as well. We refer to [96] for a comparison between these different models, and related problems for the other operators.

2.1.3 Organization of the chapter

In Section 2.2 we transform the Neumann problem (2.1.1) into a regional-type operator inside Ω. In Section 2.3 we prove an L^∞ bound for solutions of (2.1.1) and (2.1.10). Then, in Section 2.4 we develop a Moser iteration (with logarithmic corrections), and deduce that solutions are C^α for some $\alpha > 0$. In Section 2.5 we establish a Neumann Liouville-type theorem in a half-space, and finally in Section 2.6 we use it to prove higher regularity of solutions.

[2] A function w belongs to $d^{2s-1}C^2(\overline{\Omega})+C^2(\overline{\Omega})$ if it can be written as $w = d^{2s-1}g+h$, with $g, h \in C^2(\overline{\Omega})$.

2.2 An equivalent problem in Ω

As first noticed in [1], problem (2.1.1) can be reformulated as a regional-type problem in Ω for a new operator

$$L_\Omega u(x) := \mathrm{PV}\int_\Omega \Big(u(x) - u(y)\Big) K_\Omega(x,y)\, dy, \tag{2.2.1}$$

with

$$K_\Omega(x,y) = \frac{c_{n,s}}{|x-y|^{n+2s}} + k_\Omega(x,y), \tag{2.2.2}$$

$$k_\Omega(x,y) = c_{n,s}\int_{\Omega^c} \frac{dz}{|x-z|^{n+2s}|y-z|^{n+2s}\int_\Omega \frac{dw}{|z-w|^{n+2s}}}, \qquad x,y\in\Omega. \tag{2.2.3}$$

Moreover, it was proved in [1] that, for every fixed $x\in\Omega$, the kernel $k_\Omega(x,y)$ has a logarithmic singularity along $\partial\Omega$. Here we need more precise estimates, with constants that are independent of $x,y\in\Omega$.

2.2.1 Fine estimates on the new kernel

Here, and throughout the chapter, we denote $A\asymp B$ whenever $C^{-1}A\le B\le CA$ for some positive constant C.

Proposition 2.2.1. *Let $\Omega\subset\mathbb{R}^n$ be any Lipschitz domain, let d be the distance function to the boundary, and denote*

$$d_{x,y} := \min\{d(x),\, d(y)\}, \qquad x,y\in\Omega.$$

Then, the kernel k_Ω satisfies

$$k_\Omega(x,y) \asymp \begin{cases} \dfrac{1+\left|\log\left(\frac{d_{x,y}}{|x-y|}\right)\right|}{|x-y|^{n+2s}} & \text{if} \quad d_{x,y}\le |x-y| \\ d_{x,y}^{-n-2s} & \text{if} \quad d_{x,y}\ge |x-y| \end{cases} \tag{2.2.4}$$

In particular, the kernel K_Ω satisfies

$$K_\Omega(x,y) \asymp \frac{1+\log^-\left(\frac{d_{x,y}}{|x-y|}\right)}{|x-y|^{n+2s}} \qquad \text{for all} \quad x,y\in\Omega, \tag{2.2.5}$$

where $\log^- t := \max\{0, -\log t\}$.

The constants in (2.2.4) and (2.2.5) depend only on Ω. Moreover, if $\Omega\cap B_2$ can be written as a Lipschitz graph, then (2.2.4) and (2.2.5) hold for $x,y\in\Omega\cap B_1$ with constants depending only on n and the Lipschitz norm of such graph.

Proof. Since (2.2.5) follows immediately from (2.2.4), it suffices to prove (2.2.4). Moreover, since any Lipschitz domain can be locally written as a Lipschitz graph, we will assume that $\Omega\cap B_2$ is a Lipschitz graph, and prove the estimate for $x,y\in\Omega\cap B_1$.

By [1, Lemma 2.1] we have that

$$\int_\Omega \frac{dw}{|z-w|^{n+2s}} \asymp \min\left\{d^{-2s}(z),\, d^{-n-2s}(z)\right\}$$

for $z \in \Omega^c$, so we deduce that

$$k_\Omega(x,y) \asymp \int_{\Omega^c} \frac{d^{2s}(z)\,dz}{|x-z|^{n+2s}|y-z|^{n+2s}\min\left\{1,\, d^{-n}(z)\right\}}, \qquad x,y\in\Omega\cap B_1.$$

On the other hand, notice that the kernel is scale invariant, in the sense that

$$k_\Omega(rx,ry) = r^{-n-2s}k_{r^{-1}\Omega}(x,y),$$

and it is symmetric in x, y. Moreover, the estimate we want to prove is also scale invariant and symmetric. Therefore, to prove the desired estimate, we may assume that

$$d(y)\le d(x) \qquad \text{and} \qquad \max\{d(x),\, |x-y|\} = 1.$$

Moreover, since for $x, y \in \Omega\cap B_1$ the contributions from $\Omega^c\cap B_2^c$ in (2.2.3) are bounded, we have

$$k_\Omega(x,y) \asymp \int_{\Omega^c\cap B_2} \frac{d^{2s}(z)\,dz}{|x-z|^{n+2s}|y-z|^{n+2s}}, \qquad x,y\in\Omega\cap B_1. \tag{2.2.6}$$

Now, notice that since such integral is obviously bounded when $d(x)\ge d(y)\ge \frac{1}{2}$, since $z\in\Omega^c$ and therefore the integrand is bounded. Further, notice that if $|x-y|\ge\frac{1}{2}$ then the singularities are well separated, and therefore we can split the integral into two pieces.

Because of this, we split the proof into different cases. First, assume that $|x-y|\le d(y)\le d(x) = \frac{1}{2}$. Then, by triangle inequality we have $d(y)+|x-y|\ge d(x)$, and therefore $d(y)\ge\frac{1}{2}$, which yields that the integrand in (2.2.6) is bounded. Hence, in this case, $k_\Omega\asymp 1$.

For the second case, assume that $d(y)\le |x-y|\le d(x) = 1$. By triangle inequality, we have $|x-y|\ge\frac{1}{2}$ in this case. The factor $|x-z|^{-n-2s}$ is bounded, and hence we have

$$k_\Omega(x,y) \asymp \int_{\Omega^c\cap B_2} \frac{d^{2s}(z)\,dz}{|y-z|^{n+2s}}.$$

Then, by doing a bi-Lipschitz transformation, it suffices to consider the case in which $\Omega\cap B_2$ is flat, i.e., $\Omega\cap B_2 = \{x_n>0\}\cap B_2$. (Notice that the estimates are invariant under a biLipschitz transformation, since all distances stay comparable.) Then, we get

$$k_\Omega(x,y) \asymp \int_{\{z_n<0\}\cap B_2} \frac{|z_n|^{2s}\,dz}{|y-z|^{n+2s}} \asymp 1+\left|\log d(y)\right|.$$

The last estimate can be proved as follows: denote $d(y) = y_n =: \delta > 0$, so that by a change of variables $z\mapsto \delta z$ we have

$$\int_{\{z_n<0\}\cap B_2} \frac{|z_n|^{2s}\,dz}{|y-z|^{n+2s}} \asymp \int_{\{z_n<0\}\cap B_{1/\delta}} \frac{|z_n|^{2s}}{1+|z|^{n+2s}}\,dz \asymp 1+\left|\log\delta\right|,$$

as claimed.

Finally, for the third case, assume that $d(y)\le d(x)\le |x-y| = 1$. Then, by the same argument we have

$$\begin{aligned} k_\Omega(x,y) &\asymp \int_{\Omega^c\cap B_{1/2}(x)} \frac{d^{2s}(z)\,dz}{|x-z|^{n+2s}} + \int_{\Omega^c\cap B_{1/2}(y)} \frac{d^{2s}(z)\,dz}{|y-z|^{n+2s}} + C \\ &\asymp 1+\left|\log d(y)\right|, \end{aligned}$$

where we used that $d(y)\le d(x)$. Thus, the result is proved. □

Thanks to these estimates, we will treat problem (2.1.1) as a problem inside Ω for an operator (2.2.1) with kernel satisfying (2.2.5). This will allow us to treat at the same time both problems (2.1.1) and (2.1.10).

More precisely, throughout the next two sections we assume that L_Ω is an operator of the form (2.2.1), with kernel K_Ω satisfying either

$$K_\Omega(x,y) \asymp \frac{1+\log^-\left(\frac{d_{x,y}}{|x-y|}\right)}{|x-y|^{n+2s}} \qquad \text{for} \quad x,y\in\Omega, \tag{2.2.7}$$

or

$$K_\Omega(x,y) \asymp \frac{1}{|x-y|^{n+2s}} \qquad \text{for} \quad x,y\in\Omega. \tag{2.2.8}$$

The first case covers the Neumann problem for the fractional Laplacian, while the second case covers the regional fractional Laplacian. The constants in (2.2.7) and (2.2.8) are given by Proposition 2.2.1.

The corresponding bilinear form is given by

$$B(u,v) := \int_\Omega\int_\Omega \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)K_\Omega(x,y)\,dx, \tag{2.2.9}$$

and the definition of weak solution to the Neumann problem is the following.

2.2.2 Weak solutions

Here, and throughout the chapter, we denote with $H_K(\Omega)$ the space of functions for which

$$||w||^2_{H_K(\Omega)} = ||w||^2_{L^2(\Omega)} + \int_\Omega\int_\Omega |w(x)-w(y)|^2\, K_\Omega(x,y)dxdy$$

is finite.

Similar, we denote with $H_{K,loc}(\Omega)$ the space of functions for which the quantity

$$||w||^2_{L^2(\Omega\cap B)} + \int_{\Omega\cap B}\int_{\Omega\cap B} |w(x)-w(y)|^2\, K_\Omega(x,y)dx\,dy$$

is finite for any ball $B\subset\mathbb{R}^n$.

Definition 2.2.2. Let $\Omega\subset\mathbb{R}^n$ be any Lipschitz domain, $B\subset\mathbb{R}^n$ be a ball, and $D := B\cap\Omega$. Let K_Ω be any kernel of the form either (2.2.7) or (2.2.8), and let L_Ω and B be given by (2.2.1) and (2.2.9), respectively. Let $\mu, f\in L^q(D)$ with $q\in\left(\frac{n}{2s},\infty\right]$.

We say that $u\in H_{K,loc}(\Omega)$ is a weak supersolution in D, with Neumann conditions on $\partial\Omega\cap B$, and we write

$$L_\Omega u \geq \mu u + f \quad \text{ in } D,$$

if

$$B(u,\eta) \geq \int_D \mu u\eta dx + \int_D f\eta dx \quad \text{ for all } \eta\in C_0^\infty(B),\ \eta\geq 0.$$

We say that $u\in H_{K,loc}(\Omega)$ is a weak subsolution in D, with Neumann conditions on $\partial\Omega\cap B$, and we write

$$L_\Omega u \leq \mu u + f \quad \text{ in } D,$$

if

$$B(u,\eta) \le \int_D \mu u \eta dx + \int_D f \eta dx \quad \text{for all } \eta \in C_0^\infty(B),\ \eta \ge 0.$$

We say that $u \in H_{K,loc}(\Omega)$ is a weak solution to

$$L_\Omega u = \mu u + f \quad \text{in } D,$$

with Neumann conditions on $\partial\Omega \cap B$, if it is both a weak supersolution and subsolution in D with Neumann conditions on $\partial\Omega \cap B$.

Finally, we say that u is a weak (sub/super)-solution in Ω if the previous definition holds for all balls $B \subset \mathbb{R}^n$.

We will also need the following.

Lemma 2.2.3. *Let Ω be a bounded Lipschitz domain and K_Ω, B, f, μ, as in Definition 2.2.2.*

Then, the following statements hold.

(i) Let u satisfy

$$L_\Omega u = \mu u + f \quad \text{in } D,$$

with Neumann condition on $\partial\Omega \cap B$. Then u_+ and u_- satisfy respectively

$$L_\Omega u_+ \le \mu u_+ + f_+ \quad \text{in } D,$$

and

$$L_\Omega u_- \le \mu u_- + f_- \quad \text{in } D,$$

with Neumann condition on $\partial\Omega \cap B$.

(ii) Let $\mu, f \ge 0$ and u a nonnegative function weakly satisfying

$$L_\Omega u \le \mu u + f \quad \text{in } D,$$

with Neumann condition on $\partial\Omega \cap B$. Then for any $l \ge 0$, the function $\underline{u} = \max\{u, l\}$ also satisfies

$$L_\Omega \underline{u} \le \mu \underline{u} + f \quad \text{in } D,$$

with Neumann condition on $\partial\Omega \cap B$.

Proof. We follow the proof of [136, Lemma 2.4]. The proof is very general and does not really use the explicit form of the kernel.

Let us first prove (i). Setting $p(x) = x_+$, we consider a sequence of smooth and convex functions $p_k : \mathbb{R} \to \mathbb{R}$, such that

$$p_k, p_k' \ge 0, \quad p_k(x) = p(x),\ x \in \mathbb{R} \setminus (-\tfrac{1}{k}, \tfrac{1}{k}), \quad \|p - p_k\|_{H^1(\mathbb{R})} \le \tfrac{1}{k}, \tag{2.2.10}$$

for all positive integer k. Using the convexity of p_k, it is not difficult to verify that

$$B(p_k(u), \eta) \le B(u, p_k'(u)\eta),$$

for all k and all $\eta \in H_K(\Omega)$, $\eta \ge 0$. Further, we notice that, thanks to the properties of p_k and the fact that $u \in H_K(\Omega)$, $p_k'(u)\eta$ is an admissible test, whenever $\eta \in H_K(\Omega)$ (by approximation it is always possible to test with functions belonging to $H_K(\Omega)$).

Consequently,

$$\begin{aligned}
B(p_k(u),\eta) - \int_\Omega \mu p_k(u)\eta dx - \int_\Omega f_+\eta dx \\
&\le B(u, p_k'(u)\eta) - \int_\Omega \mu p_k(u)\eta dx - \int_\Omega f_+\eta dx \\
&= \int_\Omega \mu u p_k'(u)\eta dx + \int_\Omega f p_k'(u)\eta dx - \int_\Omega \mu p_k(u)\eta dx - \int_\Omega f_+\eta dx,
\end{aligned}$$

for all k and all $\eta \in H_K(\Omega)$, $\eta \geq 0$. Finally, passing to the limit as $k \to +\infty$, and noticing that $\int_\Omega \mu u p_k'(u)\eta, \int_\Omega \mu p_k(u)\eta \to \int_\Omega \mu u_+\eta$, it follows

$$B(u_+,\eta) - \int_\Omega \mu u_+\eta dx - \int_\Omega f_+\eta dx \le \int_{\Omega\cap\{u>0\}} f\eta dx - \int_\Omega f_+\eta dx \le 0,$$

for all $\eta \in H_K(\Omega)$, $\eta \geq 0$, which proves the first part of our claim. To prove the second part, it is enough to notice that $-u$ is a solution with $-f$ and apply the first part of our statement. We obtain that $u_- = (-u)_+$ is a subsolution with $f_- = (-f)_+$, which is exactly what we wanted to prove.

To prove part (ii), we proceed as before. We fix $l \geq 0$ and we define $p(x) := \max\{x, l\}$. Then, we consider a sequence of smooth and convex functions p_k satisfying (2.2.10). Thus,

$$\begin{aligned}
B(p_k(u),\eta) - \int_\Omega \mu p_k(u)\eta dx - \int_\Omega f\eta dx \\
&\le B(u, p_k'(u)\eta) - \int_\Omega \mu p_k(u)\eta dx - \int_\Omega f\eta dx \\
&\le \int_\Omega \mu u p_k'(u)\eta dx + \int_\Omega f p_k'(u)\eta dx - \int_\Omega \mu p_k(u)\eta dx - \int_\Omega f\eta dx,
\end{aligned}$$

for all k and all $\eta \in H_K(\Omega)$, $\eta \geq 0$. Passing to the limit as $k \to +\infty$, we obtain

$$B(p(u),\eta) - \int_\Omega \mu p(u)\eta dx - \int_\Omega f\eta dx \le -l\int_{\Omega\cap\{u<l\}} \mu\eta dx - \int_{\Omega\cap\{u<l\}} f\eta dx \le 0,$$

for all $\eta \in H_K(\Omega)$, $\eta \geq 0$, and our statement follows.

□

2.3 L^∞ bounds

The aim of this section is to prove L^∞ bounds for solutions to the Neumann problems that we study. For this, we only need the lower bound $K_\Omega(x,y) \gtrsim |x-y|^{-n-2s}$.

We next prove the boundedness of solutions to (2.1.1) and (2.1.10). We start with the following.

Lemma 2.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $c \in L^q(\Omega)$ and $q > \frac{n}{2s}$. Let K_Ω be of the form either (2.2.7) or (2.2.8). Assume that u satisfies*

$$\begin{cases} L_\Omega u \le c(x)u & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega, \end{cases} \tag{2.3.1}$$

in the weak sense with Neumann conditions on $\partial\Omega$. Then

$$\|u\|_{L^\infty(\Omega)} \le C\left(1+\|c\|_{L^q(\Omega)}^{\frac{qn}{4qs-2n}}\right)\|u\|_{L^2(\Omega)},$$

for some constant $C>0$ depending only on n, s, q, and Ω.

Proof. Note that by scaling properties we can assume $\|c\|_{L^q(\Omega)} \le 1$. That is, we only need to work with the auxiliary function $w(x) = u\left(\|c\|_{L^q(\Omega)}^{\frac{q}{n-2qs}} x\right)$ in $\widetilde{\Omega} = \|c\|_{L^q(\Omega)}^{\frac{q}{2n-4qs}}\Omega \subset \Omega$ when $\|c\|_{L^q(\Omega)} > 1$. Given $\beta \ge 2$, the idea is to take $u^{\beta-1}$ as test function in the weak formulation and thanks to Sobolev inequality, improve iteratively the integrability of u. Since a priori we cannot guaranteed that $u^{\beta-1} \in H_K(\Omega)$ we need to truncate it in some sense in order to be an admissible test function. That is, let us consider the sequence

$$u_k := \min\{u, k\},$$

for all $k \in \mathbb{N}$, $k \ge 1$. We have $u_k \in H_K(\Omega)$, $0 \le u_k \le u_{k+1}$ and $u_k \to u$ a.e. in Ω. Testing the inequality with $\eta = u_k^{\beta-2}u$, we immediately deduce

$$B(u, u_k^{\beta-2}u) \le \int_\Omega c(x) u_k^{\beta-2} u^2 dx. \tag{2.3.2}$$

Note that the fact $u_k^{\beta-2}u \in H_K(\Omega)$, for $\beta \ge 2$, can be easily checked.

Now, setting $v := u_k^{\beta/2-1}u$ and applying [136, Lemma 2.3], we obtain

$$B(v,v) \le \beta B(u, u_k^{\beta-2}u) \tag{2.3.3}$$

for all $\beta \ge 2$. On the other hand, by Hölder inequality, we have

$$\int_\Omega c(x) u_k^{\beta-2}u^2 dx \le \|c\|_{L^q(\Omega)}\|v\|^2_{L^{2q'}(\Omega)} \le \|v\|^2_{L^{2q'}(\Omega)}. \tag{2.3.4}$$

Since $q > \frac{n}{2s}$, it follows that $2 < 2q' < 2_s^*$ and so, taking $\vartheta \in (0,1)$ satisfying

$$\frac{1}{2q'} = \frac{\vartheta}{2} + \frac{1-\vartheta}{2_s^*}, \qquad \text{i.e. } \vartheta = \frac{2qs-n}{2qs},$$

and using the interpolation and the Sobolev inequality, we obtain

$$\|v\|^2_{L^{2q'}(\Omega)} \le \|v\|^{2\vartheta}_{L^2(\Omega)}\|v\|^{2(1-\vartheta)}_{L^{2_s^*}(\Omega)} \le C\left(\|v\|^2_{L^2(\Omega)} + B(v,v)\right)^{1-\vartheta}\|v\|^{2\vartheta}_{L^2(\Omega)}. \tag{2.3.5}$$

Now, thanks to the fact that $\vartheta \in (0,1)$, we infer

$$\left(\|v\|^2_{L^2(\Omega)} + B(v,v)\right)^{1-\vartheta}\|v\|^{2\vartheta}_{L^2(\Omega)} \le \varepsilon B(v,v) + (1+\varepsilon^{-\frac{1-\vartheta}{\vartheta}})\|v\|^2_{L^2(\Omega)}, \tag{2.3.6}$$

for all $\varepsilon > 0$. Putting together (2.3.2), (2.3.3), (2.3.4), (2.3.5), (2.3.6) and choosing

$$\varepsilon = (C\,\beta)^{-1},$$

it follows by taking into account that $\beta \ge 2$ that

$$B(v,v) \le C\beta^{\frac{1}{\vartheta}}\|v\|^2_{L^2(\Omega)},$$

and, using Sobolev inequality again, we deduce

$$\left(\int_\Omega u^2 u_k^{\beta\gamma-2}dx\right)^{\frac{1}{\beta\gamma}} \le (C\beta)^{\frac{1}{\beta\vartheta}} \left(\int_\Omega u^2 u_k^{\beta-2}dx\right)^{\frac{1}{\beta}}, \tag{2.3.7}$$

for some new constant $C > 0$ depending only on n, s, q, and the Lipschitz norm of $\partial\Omega$. Here, $\gamma := 2_s^*/2 > 1$.

Now, taking $\beta_0 = 2$ and $\beta_i := \gamma\beta_{i-1} = \beta_0\gamma^i$ for all integers $i \ge 1$, and iterating (2.3.7), we obtain

$$\|u_k\|^{\vartheta}_{L^{2\gamma^j}(\Omega)} \le \|u\|^{\vartheta}_{L^2(\Omega)} \sum_{i=0}^{j-1} (C\gamma^i)^{\frac{1}{2\gamma^i}} \le \|u\|^{\vartheta}_{L^2(\Omega)} \sum_{i=0}^{\infty} (C\gamma^i)^{\frac{1}{2\gamma^i}} = C\,\|u\|^{\vartheta}_{L^2(\Omega)}.$$

Thus, passing to the limit as $j \to +\infty$, it follows

$$\|u_k\|^{\vartheta}_{L^\infty(\Omega)} \le C\|u\|^{\vartheta}_{L^2(\Omega)}. \tag{2.3.8}$$

Finally, since the previous inequality holds for any k with the same constant C, we conclude that

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{L^2(\Omega)}.$$

□

We now prove the following result, which gives the boundedness of solutions. We notice that, in case of (2.1.1), a similar result has been obtained in [95], with a different proof.

Proposition 2.3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $\mu, f \in L^q(\Omega)$, with $q > \frac{n}{2s}$. Let K_Ω be of the form either (2.2.7) or (2.2.8). Let u be a weak solution to*

$$L_\Omega u = \mu u + f \quad \text{in } \Omega,$$

with Neumann conditions on $\partial\Omega$. Then,

$$\|u\|_{L^\infty(\Omega)} \le C\left(\|u\|_{L^2(\Omega)} + \|f\|_{L^q(\Omega)}\right),$$

for some constant $C > 0$ depending only on n, s, q, $\|\mu\|_{L^q(\Omega)}$ and Ω.

Proof. Thanks to Lemma 2.2.3 (part (i)), we know that u_+ is a nonnegative subsolution with $\mu = \mu_+$ and $f = f_+$. Consequently, the function $v = \max\{u_+, 1\}$ is still a subsolution and, furthermore, $v \ge 1$ (Lemma 2.2.3 part (ii)). Consequently, v satisfies

$$L_\Omega v \le c(x)v \quad \text{in } \Omega$$

in the weak sense (with Neumann conditions on $\partial\Omega$), where $c = \mu_+ + f_+$.

Now, note that if $\|u_+\|_{L^2(\Omega)} \le 1$ then $\|v\|_{L^2(\Omega)} \le \sqrt{1 + |\Omega|}$ and so, under the assumptions $\|u_+\|_{L^2(\Omega)} \le 1$ and $\|f_+\|_{L^q(\Omega)} \le 1$, it follows by Lemma 2.3.1

$$\|u_+\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega)} \le C,$$

for some constant depending only on n, s, q, $\|\mu_+\|_{L^q(\Omega)}$ and Ω. Applying the above inequality to the subsolution

$$w = \frac{u_+}{\|u_+\|_{L^2(\Omega)} + \|f_+\|_{L^q(\Omega)}},$$

we deduce

$$\|u_+\|_{L^\infty(\Omega)} \le C\left(\|u_+\|_{L^2(\Omega)} + \|f_+\|_{L^q(\Omega)}\right),$$

for some constant depending only on n, s, q, $\|\mu_+\|_{L^q(\Omega)}$ and Ω. Finally, repeating the same procedure for the subsolution u_- (with $\mu = \mu_-$ and $f = f_-$), we complete the proof of the theorem. □

We will also need the following. Here, we denote $D_R(x_0) = \Omega \cap B_R(x_0)$.

Lemma 2.3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $R > 0$, $x_0 \in \Omega$ and $f \in L^q(D_{2R}(x_0))$ with $q > \frac{n}{2s}$. Let K_Ω be of the form either (2.2.7) or (2.2.8). Moreover, assume that $\partial\Omega \cap B_{3R}(x_0)$ is a Lipschitz graph. Then, there is a weak solution to*

$$\begin{cases} L_\Omega v = |f| & \text{in } D_{2R}(x_0), \\ v = 0 & \text{in } \Omega \setminus D_{2R}(x_0), \end{cases} \tag{2.3.9}$$

with Neumann conditions on $\partial\Omega \cap B_{2R}(x_0)$ in the sense of Definition 2.2.2. Furthermore, it satisfies

$$0 \le v \le \kappa_0\, R^{2s-\frac{n}{q}}\, ||f||_{L^q(D_{2R}(x_0))} \quad \text{in} \quad D_{2R}(x_0),$$

for some nonnegative constant κ_0 depending only on n, s, q, and the Lipschitz norm of $\partial\Omega \cap B_{3R}(x_0)$.

Proof. Since the general case comes by scaling, we take $R = 1$. First, let us notice that the existence (and uniqueness) of such solution v can be obtained by minimizing the functional

$$\mathcal{E}(w) = \frac{1}{4}\int_\Omega\int_\Omega |w(x) - w(y)|^2 K_\Omega(x-y)\, dx\, dy - \int_\Omega |f(x)| w(x)\, dx$$

among all functions $w \in H_K(\Omega)$ such that $w \equiv 0$ in $\Omega \setminus D_2(x_0)$. See [162, Section 3] for the details in case of the fractional Laplacian.

Next, in order to prove that the solution is nonnegative we can use the same argument of [162, Theorem 4.1], consisting on using v_- as a test function in the weak formulation, which yields $v_- \equiv 0$ in Ω. The bound from above is more delicate and we need to repeat the arguments from Lemma 2.3.1 and Proposition 2.3.2 adapted to this setting of mixed Dirichlet and Neumann conditions. In that way we obtain that

$$v \le C(||v||_{L^2(D_2(x_0))} + ||f||_{L^q(D_2(x_0))}) \quad \text{in} \quad D_2(x_0),$$

where C is a nonnegative constant depending only on n, s, q, and the Lipschitz norm of $\partial\Omega \cap B_3(x_0)$.

Finally, we need to estimate the L^2-norm of v in terms of the L^q-norm of f. In order to do that it is sufficient to use v as a test function in the weak formulation and applying the fractional Poincaré inequality in $D_3(x_0)$. That is,

$$\begin{aligned} ||v||^2_{L^2(D_2(x_0))} = ||v||^2_{L^2(D_3(x_0))} &\le C_P[v]^2_{H^s(D_3(x_0))} \le C_P[v]^2_{H^s(\Omega)} \le C\int_{D_2(x_0)} fv \\ &\le C||f||_{L^q(D_2(x_0))}\,||v||_{L^2(D_2(x_0))}. \end{aligned}$$

Let us remark that we apply the fractional Poincaré inequality in $D_3(x_0)$ since we need v to be zero in some subset of the domain of v. □

2.4 Moser-type iteration and Hölder regularity up to the boundary

The goal of this section is to develop a Moser-type iteration for our nonlocal problem with Neumann boundary conditions. The overall strategy follows that of Kassmann [136] for interior regularity but, as we will see, the logarithmic singularity of the kernel in (2.2.7) will introduce several difficulties.

From now on, for any $r > 0$ and $x_0 \in \Omega$ we denote

$$D_r(x_0) := B_r(x_0) \cap \Omega.$$

The main result of this section is the following.

Theorem 2.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $R > 0$, $x_0 \in \Omega$ and $f \in L^q(D_R(x_0))$ with $q > \frac{n}{2s}$. Assume that $\partial\Omega \cap B_R(x_0)$ is a Lipschitz graph. Let K_Ω be of the form either (2.2.7) or (2.2.8). Assume that u is a weak bounded solution to*

$$L_\Omega u = f \qquad \text{in } D_R(x_0),$$

with Neumann conditions on $\partial\Omega \cap B_R(x_0)$ in the sense of Definition 2.2.2.

Then there exist $\alpha \in (0, 1)$ and C depending only on the Lipschitz norm of $\partial\Omega \cap B_R(x_0)$, n, s, and q, such that

$$|u(x) - u(y)| \leq C \left(\frac{|x-y|}{R}\right)^\alpha \left[\|u\|_{L^\infty(\Omega)} + R^{2s-\frac{n}{q}} \|f\|_{L^q(D_R(x_0))}\right] \tag{2.4.1}$$

for a.e. $x, y \in D_{R/2}(x_0)$.

Theorem 2.4.1 will be obtained through several auxiliary results. The first step in the proof is the following.

Lemma 2.4.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $R > 0$ and $x_0 \in \Omega$. Let K_Ω be of the form either (2.2.7) or (2.2.8). Assume that $\partial\Omega \cap B_{2R}(x_0)$ is a Lipschitz graph.*

Then for any $c > 0$, $\delta_0 \in (0, 1/2)$ and $\vartheta > 1$, there exists $\gamma \in (0, 2s)$ depending only on the Lipschitz constant of $\partial\Omega \cap B_{2R}(x_0)$, n, s, c, δ_0 and ϑ such that for any $u \in L^\infty(\Omega)$ satisfying

$$\begin{cases} u(x) \geq 0 & \text{for a.e. } x \in D_R(x_0) \\ u(x) \geq c\left[1 - \left(\vartheta \frac{|x-x_0|}{R}\right)^\gamma\right] & \text{for a.e. } x \in \Omega \setminus B_R(x_0) \\ \frac{|\{u \geq 1\} \cap D_R(x_0)|}{|D_R(x_0)|} \geq \frac{1}{2}, \end{cases} \tag{2.4.2}$$

it holds

$$\int_{\Omega \setminus B_r(x_0)} u(x) K_\Omega(x, y)dx \geq 0 \qquad \text{for a.e. } y \in D_r(x_0), \tag{2.4.3}$$

for all $r < R$ such that

$$\frac{|\{u \geq 1\} \cap (D_R(x_0) \setminus D_r(x_0))|}{|D_R(x_0)|} \geq \delta_0. \tag{2.4.4}$$

Proof. Taking $u_R(x) = u(x_0 + Rx)$ instead of u, we may assume $R = 1$ and $x_0 = 0$. We prove the result for K of the form (2.2.7); the case (2.2.8) is simpler.

By the third assumption in (2.4.2), we deduce the existence of $r_0 \in (0,1)$ depending only $\delta_0 > 0$, n and the Lipschitz constant of $\partial\Omega$ such that (2.4.4) holds if $r \le r_0$.

Let us take $r \le r_0$ satisfying (2.4.4) and set $A_r := \{u \ge 1\} \cap (D_1 \setminus D_r)$. By assumption we have $|A_r| \ge \delta_0 |D_1|$, $u \ge 0$ in D_1 and so for a.e. $y \in D_r$, it follows

$$\int_{D_1 \setminus D_r} u(x) K_\Omega(x,y)dx \ge \int_{A_r} K_\Omega(x,y)dx \ge c \int_{A_r} \frac{1+\log^-\left(\frac{d_{x,y}}{|x-y|}\right)}{|x-y|^{n+2s}}dx$$

with $c > 0$, where $d_{x,y} = \min\{d(x), d(y)\}$. We have to find a suitable lower bound for the above integral. To do so, we first notice that for any fixed $d > 0$, the function

$$\varrho \to \frac{1+\log^-(d/\varrho)}{\varrho^{n+2s}}, \quad \varrho > 0$$

is decreasing and thus, since $|x-y| \le 2$, we find

$$\int_{A_r} \frac{1+\log^-\left(\frac{d_{x,y}}{|x-y|}\right)}{|x-y|^{n+2s}}dx \ge 2^{-n-2s} \int_{A_r} 1+\log^-\left(\frac{d_{x,y}}{2}\right)dx \ge c|A_r|\left(1+\log^-\left(\frac{d(y)}{2}\right)\right).$$

Consequently, whenever $d(y) \ge 1$, we have

$$\int_{D_1 \setminus D_r} u(x) K_\Omega(x,y)dx \ge c|A_r| \ge c\delta_0, \tag{2.4.5}$$

for some $c > 0$ depending only on n, s and Ω. Conversely, when $0 < d(y) < 1$, we obtain by the inequality above

$$\int_{D_1 \setminus D_r} u(x) K_\Omega(x,y)dx \ge C\delta_0 |D_1|(1+|\log d(y)|). \tag{2.4.6}$$

On the other hand, for a.e. $y \in D_r$, it holds

$$\int_{\Omega \setminus B_1} u(x) K_\Omega(x,y)dx \ge -c \int_{\Omega \setminus B_1} |1-(\vartheta|x|)^\gamma| K_\Omega(x,y)dx,$$

thanks to the second inequality in (2.4.2). Moreover,

$$\begin{aligned}\int_{\Omega \setminus B_1} |1-(\vartheta|x|)^\gamma| K_\Omega(x,y)dx &\le C \int_{\Omega \setminus B_1} |1-(\vartheta|x|)^\gamma| \frac{1+\log^-\left(\frac{d_{x,y}}{|x-y|}\right)}{|x-y|^{n+2s}}dx \\ &= C\int_{\Omega_1} \frac{|1-(\vartheta|x|)^\gamma|}{|x-y|^{n+2s}}\left|\log\left(\frac{d_{x,y}}{|x-y|}\right)\right|dx \\ &\quad + C\int_{\Omega \setminus B_1} \frac{|1-(\vartheta|x|)^\gamma|}{|x-y|^{n+2s}}dx := I_1(\gamma) + I_2(\gamma),\end{aligned} \tag{2.4.7}$$

where $\Omega_1 := (\Omega \setminus B_1) \cap \{d_{x,y} \le |x-y|\}$. Notice that

$$\begin{aligned}
I_1(\gamma) &= \int_{\Omega_1 \cap \{d_{x,y}=d(x)\}} \frac{|1-(\vartheta|x|)^\gamma|}{|x-y|^{n+2s}} \left|\log\left(\frac{d(x)}{|x-y|}\right)\right| dx \\
&\quad + \int_{\Omega_1 \cap \{d_{x,y}=d(y)\}} \frac{|1-(\vartheta|x|)^\gamma|}{|x-y|^{n+2s}} \left|\log\left(\frac{d(y)}{|x-y|}\right)\right| dx \\
&\le \int_{\Omega_1 \cap \{d_{x,y}=d(x)\}} \frac{|1-(\vartheta|x|)^\gamma|}{|x-y|^{n+2s}} \left|\log\left(\frac{d(x)}{|x-y|}\right)\right| dx \\
&\quad + \int_{\Omega_1 \cap \{d_{x,y}=d(y)\}} \frac{|1-(\vartheta|x|)^\gamma|}{|x-y|^{n+2s}} \log|x-y| dx \\
&\quad + |\log d(y)| \int_{\Omega_1 \cap \{d_{x,y}=d(y)\}} \frac{|1-(\vartheta|x|)^\gamma|}{|x-y|^{n+2s}} dx.
\end{aligned}$$

Further, $|1-(\vartheta|x|)^\gamma| \to 0$ for a.e. $x \in \Omega \setminus B_1$ as $\gamma \to 0^+$. So, since $|\log d(x)|$ is integrable near $\partial\Omega$ and recalling that $|x-y| \ge 1-r > 0$, we deduce the existence of $\delta_\gamma \to 0^+$ as $\gamma \to 0^+$ such that $I_1(\gamma) \le \delta_\gamma(1+|\log d(y)|)$ for all small $\gamma > 0$, by dominated convergence. Similar for $I_2(\gamma)$. Therefore, by (2.4.5), (2.4.6) and (2.4.7),

$$\begin{aligned}
\int_{\Omega\setminus B_r} u(x) K_\Omega(x,y) dx &= \int_{D_1 \setminus D_r} u(x) K_\Omega(x,y) dx + \int_{\Omega \setminus B_1} u(x) K_\Omega(x,y) dx \\
&\ge C\delta_0 |D_1| (1+|\log d(y)|) - \delta_\gamma(1+|\log d(y)|) \ge 0,
\end{aligned}$$

if $\gamma > 0$ is small enough and our statement follows. □

Using the previous lemma, we can now prove the following.

Lemma 2.4.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $R > 0$ and $x_0 \in \Omega$. Let K_Ω be of the form either (2.2.7) or (2.2.8). Assume that $\partial\Omega \cap B_{2R}(x_0)$ is a Lipschitz graph, and that u satisfies*

$$\begin{cases} L_\Omega u \ge 0 & \text{in } D_R(x_0) \\ u > 0 & \text{in } D_R(x_0), \end{cases}$$

with Neumann conditions on $\partial\Omega \cap B_R(x_0)$. Assume also that u satisfies (2.4.3) with $r = R$. Then,

$$\left(\fint_{D_R(x_0)} u(x)^{\beta_0} dx \right)^{1/\beta_0} \le C \left(\fint_{D_R(x_0)} u(x)^{-\beta_0} dx \right)^{-1/\beta_0},$$

for some $\beta_0 \in (0,1)$ and $C > 0$ depending only on the Lipschitz constant of $\partial\Omega \cap B_{2R}(x_0)$, n, and s.

Proof. The proof is basically the same for both classes of kernels, (2.2.7) and (2.2.8).

By scaling and translation we may assume $R = 1$ and $x_0 = 0$. Given any arbitrary $z_0 \in D_1$ and $\varrho > 0$ such that $B_{2\varrho}(z_0) \subset D_1$, we take $B_\varrho = B_\varrho(z_0)$. Then, exactly as in [136, Lemma 3.3] with $r = \varrho$ (here we use the assumption (2.4.3)), we find

$$\int_{B_\varrho \times B_\varrho} \frac{[\log u(x) - \log u(y)]^2}{|x-y|^{n+2s}} dx dy \le C \varrho^{n-2s},$$

for some constant $C > 0$ depending only on n, s and the constants in (2.2.7)-(2.2.8) (which depend only on the Lipschitz norm of the domain). This yields $\log u \in H^s(B_\varrho)$ and thus, by the Poincaré inequality,

$$\int_{B_\varrho} \left|\log u(x) - [\log u]_{B_\varrho}\right|^2 dx \leq C\varrho^n,$$

for some constant C depending only on n, s and the constants in (2.2.7)-(2.2.8), where $[\log u]_{B_\varrho} := \fint_{B_\varrho} \log u$. By Hölder inequality, it follows that

$$\int_{B_\varrho} \left|\log u(x) - [\log u]_{B_\varrho}\right| dx \leq C\varrho^n,$$

and therefore, thanks to the arbitrariness of z_0 and $\varrho > 0$, we deduce that $\log u \in \mathrm{BMO}(D_1)$ (see [41, Theorem 0.3]). Now, by the John-Nirenberg inequality (see [41, Theorem 0.3 and Theorem 0.4]), we deduce the existence of $\beta_0 \in (0, 1)$ and C, depending only on the Lipschitz constant of $\partial\Omega$, n, and s, such that

$$\int_{D_1} e^{\beta_0 |\log u(x) - [\log u]_{D_1}|} dx \leq C.$$

Finally, since

$$\begin{aligned}
&\left(\fint_{D_1} u(x)^{\beta_0} dx\right)^{1/\beta_0} \cdot \left(\fint_{D_1} u(x)^{-\beta_0} dx\right)^{1/\beta_0} \\
&= \left(\fint_{D_1} e^{\beta_0\{\log u(x) - [\log u]_{D_1}\}} dx\right)^{1/\beta_0} \cdot \left(\fint_{D_1} e^{-\beta_0\{\log u(x) - [\log u]_{D_1}\}} dx\right)^{1/\beta_0} \leq C,
\end{aligned}$$

the result follows. □

On the other hand, we next prove a key lemma for the Moser-type iteration.

Lemma 2.4.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $R > 0$, $x_0 \in \Omega$ and $\beta > 1$. Let K_Ω be of the form either (2.2.7) or (2.2.8). Assume that u satisfies*

$$\begin{cases} L_\Omega u \geq 0 & \text{in } D_R(x_0) \\ u > 0 & \text{in } D_R(x_0), \end{cases}$$

with Neumann conditions on $\partial\Omega \cap B_R(x_0)$, in the sense of Definition 2.2.2.

Then, there exists a constant C depending only on n, s, and the Lipschitz constant of $\partial\Omega \cap B_{2R}(x_0)$, such that

$$\begin{aligned}
&\int_{D_r(x_0)\times D_r(x_0)} \frac{\left[u(x)^{\frac{1-\beta}{2}} - u(y)^{\frac{1-\beta}{2}}\right]^2}{|x-y|^{n+2s}} dxdy \\
&\leq \frac{C\beta^2}{(R-r)^{2s}} \int_{D_R(x_0)} u(x)^{1-\beta} \left(1 + \left|\log \frac{d(x)}{R-r}\right|\right) dx,
\end{aligned} \tag{2.4.8}$$

for all $0 < r < R$. In case (2.2.8), the same estimate holds without the logarithmic term.

Proof. Since the kernels and (2.4.8) are scale-invariant, after a rescaling we may assume that $R-r=1$. We take a smooth cut-off function $0 \le \varphi \le 1$ satisfying

$$\varphi = 1 \quad \text{in } \overline{B}_r, \quad \text{supp}(\varphi) \subset B_R, \quad \sup|\nabla\varphi| \le c.$$

Testing $L_\Omega u \ge 0$ in D_R with $\eta := \varphi^{1+\beta}u^{-\beta}$ (notice that η is an admissible test since $u > 0$ in D_R and $\varphi = 0$ in $\Omega \setminus B_R$), it follows that

$$\int_\Omega \int_\Omega [u(x)-u(y)][\varphi^{1+\beta}(x)u^{-\beta}(x) - \varphi^{1+\beta}(y)u^{-\beta}(y)]K_\Omega(x,y)dxdy \ge 0.$$

In particular, for any $\varepsilon > 0$,

$$\begin{aligned}\int\int_{\substack{\Omega\times\Omega\\|x-y|>\varepsilon}} &[u(y)-u(x)][\varphi^{1+\beta}(x)u^{-\beta}(x) - \varphi^{1+\beta}(y)u^{-\beta}(y)]K_\Omega(x,y)dxdy \\ &\le - \int_{\substack{\Omega\times\Omega\\|x-y|\le\varepsilon}} [u(y)-u(x)][\varphi^{1+\beta}(x)u^{-\beta}(x) - \varphi^{1+\beta}(y)u^{-\beta}(y)]K_\Omega(x,y)dxdy.\end{aligned}$$

Now, we apply [136, Lemma 2.5] with $a = u(x)$, $b = u(y)$, $\tau_1 = \varphi(x)$, $\tau_2 = \varphi(y)$ and $p = \beta$, integrate on $(\Omega\times\Omega)\cap\{|x-y|>\varepsilon\}$ and use the above inequality to obtain

$$\begin{aligned}\int\int_{\substack{\Omega\times\Omega\\|x-y|>\varepsilon}} &\varphi(x)\varphi(y)\left[\left(\frac{u(x)}{\varphi(x)}\right)^{\frac{1-\beta}{2}} - \left(\frac{u(y)}{\varphi(y)}\right)^{\frac{1-\beta}{2}}\right]^2 K_\Omega(x,y)dxdy \\ &\le c_\beta \int_{\substack{\Omega\times\Omega\\|x-y|>\varepsilon}} [\varphi(x)-\varphi(y)]^2 \left[\left(\frac{u(x)}{\varphi(x)}\right)^{1-\beta} + \left(\frac{u(y)}{\varphi(y)}\right)^{1-\beta}\right] K_\Omega(x,y)dxdy \\ &\quad - (\beta-1)\int_{\substack{\Omega\times\Omega\\|x-y|\le\varepsilon}} [u(y)-u(x)][\varphi^{1+\beta}(x)u^{-\beta}(x) - \varphi^{1+\beta}(y)u^{-\beta}(y)]K_\Omega(x,y)dxdy,\end{aligned}$$

where $c_\beta := \max\{\frac{\beta-1}{2}, \frac{6(\beta-1)^2}{16}\} \le \beta^2$, since $\beta > 1$. Since $\eta = \varphi^{1+\beta}u^{-\beta} \in H_K(D_R)$, the last term converges to zero when we pass to limit as $\varepsilon \to 0$. Thus, we deduce

$$\begin{aligned}\int_\Omega\int_\Omega &\varphi(x)\varphi(y)\left[\left(\frac{u(x)}{\varphi(x)}\right)^{\frac{1-\beta}{2}} - \left(\frac{u(y)}{\varphi(y)}\right)^{\frac{1-\beta}{2}}\right]^2 K_\Omega(x,y)dxdy \\ &\le \beta^2 \int_\Omega\int_\Omega [\varphi(x)-\varphi(y)]^2 \left[\left(\frac{u(x)}{\varphi(x)}\right)^{1-\beta} + \left(\frac{u(y)}{\varphi(y)}\right)^{1-\beta}\right] K_\Omega(x,y)dxdy.\end{aligned}$$

Now, using that $\varphi \equiv 1$ in D_r, we bound from below the left hand side as

$$\begin{aligned}\int_\Omega\int_\Omega &\varphi(x)\varphi(y)\left[\left(\frac{u(x)}{\varphi(x)}\right)^{\frac{1-\beta}{2}} - \left(\frac{u(y)}{\varphi(y)}\right)^{\frac{1-\beta}{2}}\right]^2 K_\Omega(x,y)dxdy \\ &\ge \int_{D_r}\int_{D_r} \varphi(x)\varphi(y)\left[\left(\frac{u(x)}{\varphi(x)}\right)^{\frac{1-\beta}{2}} - \left(\frac{u(y)}{\varphi(y)}\right)^{\frac{1-\beta}{2}}\right]^2 K_\Omega(x,y)dxdy \\ &\ge c\int_{D_r}\int_{D_r} \frac{\left[u(x)^{\frac{1-\beta}{2}} - u(y)^{\frac{1-\beta}{2}}\right]^2}{|x-y|^{n+2s}}dxdy,\end{aligned}$$

where $c>0$ depends only on n, s and the Lipschitz constant of $\partial\Omega\cap B_{2R}(x_0)$. Here we have used (2.2.2) and that $k_\Omega \geq 0$.

On the other hand, by symmetry, we have

$$\begin{aligned}
&\int_\Omega\int_\Omega [\varphi(x)-\varphi(y)]^2\left[\left(\frac{u(x)}{\varphi(x)}\right)^{1-\beta}+\left(\frac{u(y)}{\varphi(y)}\right)^{1-\beta}\right]K_\Omega(x,y)dxdy\\
&=2\int_\Omega\int_\Omega \varphi(x)^{\beta-1}[\varphi(x)-\varphi(y)]^2u(x)^{1-\beta}K_\Omega(x,y)dxdy\\
&\leq 2\int_{D_R} u(x)^{1-\beta}\int_\Omega[\varphi(x)-\varphi(y)]^2K_\Omega(x,y)dydx.
\end{aligned}$$

Therefore, we have proved that

$$\int_{D_r}\int_{D_r}\frac{\left[u(x)^{\frac{1-\beta}{2}}-u(y)^{\frac{1-\beta}{2}}\right]^2}{|x-y|^{n+2s}}dxdy\leq C\beta^2\int_{D_R}u(x)^{1-\beta}\int_\Omega[\varphi(x)-\varphi(y)]^2K_\Omega(x,y)dydx.$$

To finish the proof, we have to estimate the integral

$$\begin{aligned}
\int_\Omega[\varphi(x)-\varphi(y)]^2K_\Omega(x,y)dy=&\int_{D_1(x)}[\varphi(x)-\varphi(y)]^2K_\Omega(x,y)dy\\
&+\int_{\Omega\setminus B_1(x)}[\varphi(x)-\varphi(y)]^2K_\Omega(x,y)dy:=J_1+J_2,
\end{aligned}$$

where $x\in D_R$ is fixed and $d:=d(x)<1$. In view of (2.2.2) and (2.2.4), have

$$\begin{aligned}
J_1&\leq C\int_{D_1(x)}\frac{|\log|x-y||+|\log d(x)|+|\log d(y)|}{|x-y|^{n+2s-2}}dy\\
&\leq C(1+|\log d(x)|)+C\int_{D_1(x)\cap\{d(x)/2\leq d(y)\leq 2\}}\frac{|\log d(y)|}{|x-y|^{n+2s-2}}dy\\
&\quad+C\int_{D_1(x)\cap\{d(y)\leq d(x)/2\}}\frac{|\log d(y)|}{|x-y|^{n+2s-2}}dy\\
&=C(1+|\log d(x)|)+I_1+I_2.
\end{aligned}\tag{2.4.9}$$

Now, taking into account that $|\log d(y)|\leq C(1+|\log d(x)|)$ when $d(x)/2\leq d(y)\leq 2$ we obtain that $I_1\leq C(1+|\log d(x)|)$. Next, in order to estimate I_2 it is enough to consider the case in which D_1 is flat since any other Lipschitz domain can be transform through a bi-Lipschitz transformation. In that case,

$$\begin{aligned}
I_2&=C\int_{D_1(x)\cap\{0\leq y_n\leq x_n/2\}}\frac{|\log y_n|}{|x-y|^{n+2s-2}}dy\\
&\leq -C\int_{x_n/2}^{x_n}\log(x_n-y_n)\left(\int_{B_1\subset\mathbb{R}^{n-1}}(y_n^2+|z|^2)^{\frac{-n-2s+2}{2}}dz\right)dy_n\\
&\leq -C\int_{x_n/2}^{x_n}\log(x_n-y_n)(1+y_n^{1-2s})dy_n\\
&\leq C(1+|\log x_n|)=C(1+|\log d(x)|).
\end{aligned}\tag{2.4.10}$$

Here, we have used the following estimate

$$\begin{aligned}\int_{B_1\subset\mathbb{R}^{n-1}}(y_n^2+|z|^2)^{\frac{-n-2s+2}{2}}dz &\le C\,y_n^{1-2s}\int_0^{1/y_n}\frac{r^{n-2}}{(1+r^2)^{\frac{n+2s-2}{2}}}dr\\ &\le C\,y_n^{1-2s}\left(\int_0^{1/2}r^{n-2}dr+\int_{1/2}^{1/y_n}r^{-2s}dr\right)\\ &\le C\,(1+y_n^{1-2s}).\end{aligned}$$

Putting together (2.4.9) and (2.4.10), we find

$$J_1\le C(1+|\log d(x)|),$$

for some constant $C>0$ depending on n, s and the Lipschitz constant of Ω.

To estimate J_2, we notice that

$$J_2\le 2\int_{\Omega\setminus B_1(x)}K_\Omega(x,y)dy\le C\int_{\Omega\setminus B_1(x)}\frac{1+\log^-\left(\frac{d_{x,y}}{|x-y|}\right)}{|x-y|^{n+2s}}dy,$$

for some universal $C>0$ and that the kernel is singular only near $\partial\Omega$, due to the fact that $|x-y|\ge 1$. Moreover, $y\to|\log d(y)|d(y)^{-n-2s}$ is integrable for $|y|$ large and thus repeating the arguments which have led to (2.4.9) and (2.4.10), we find

$$J_2\le C(1+|\log d(x)|),$$

for some $C>0$ depending on n, s and the Lipschitz constant of Ω, as wanted. □

Using the previous lemma, and a Moser-type iteration, we deduce the following.

Corollary 2.4.5. *Let $\Omega\subset\mathbb{R}^n$ be a domain, $R>0$, $x_0\in\Omega$ and $\beta>1$. Moreover, assume that $\partial\Omega\cap B_R(x_0)$ is a Lipschitz graph. Let K_Ω be of the form either (2.2.7) or (2.2.8). Let u satisfy*

$$\begin{cases}L_\Omega u\ge 0 & \text{in } D_R(x_0)\\ u>0 & \text{in } D_R(x_0),\end{cases}$$

with Neumann conditions on $\partial\Omega\cap B_R(x_0)$ in the sense of Definition 2.2.2.

Then, there exists a constant $C>0$ depending only on the Lipschitz constant of $\partial\Omega$, n, s, and $\beta>0$, such that

$$\operatorname*{ess\,inf}_{x\in D_{R/2}(x_0)}u(x)\ge C\left(⨍_{D_R(x_0)}u(x)^{-\beta}dx\right)^{-1/\beta}. \tag{2.4.11}$$

Proof. By scaling, we can assume $x_0=0$ and $R=1$.

Let $\{r_k\}_{k\in\mathbb{N}}$ be a decreasing sequence satisfying $r_0=1$ and $r_k\to 1/2$ as $k\to+\infty$. For a given $\beta>1$, we apply the Sobolev inequality to (2.4.8) to obtain

$$\left(\int_{D_{r_{k+1}}}u(x)^{(1-\beta)\gamma}dx\right)^{1/\gamma}\le\frac{C\beta^2}{(r_k-r_{k+1})^{2s}}\int_{D_{r_k}}u(x)^{1-\beta}\left(1+\left|\log\frac{d(x)}{r_k-r_{k+1}}\right|\right)dx,$$

where $\gamma:=2_s^*/2>1$ and where C depends only on the Lipschitz constant of $\partial\Omega$, n, and s.

Let $\varepsilon \in (0, \gamma - 1)$ and apply Hölder inequality to the right hand side:

$$\begin{aligned}\int_{D_{r_k}} u(x)^{1-\beta}\left(1+\left|\log\frac{d(x)}{r_k-r_{k+1}}\right|\right)dx &\le \left(\int_{D_{r_k}} u(x)^{(1-\beta)(1+\varepsilon)}dx\right)^{\frac{1}{1+\varepsilon}} \\ &\quad\times\left(\int_{D_{r_k}}\left(1+\left|\log\frac{d(x)}{r_k-r_{k+1}}\right|\right)^{\frac{1+\varepsilon}{\varepsilon}}dx\right)^{\frac{\varepsilon}{1+\varepsilon}} \\ &= C_k\left(\int_{D_{r_k}} u(x)^{(1-\beta)(1+\varepsilon)}dx\right)^{\frac{1}{1+\varepsilon}},\end{aligned}$$

where

$$C_k := \left(\int_{D_{r_k}}\left(1+\left|\log\frac{d(x)}{r_k-r_{k+1}}\right|\right)^{\frac{1+\varepsilon}{\varepsilon}}dx\right)^{\frac{\varepsilon}{1+\varepsilon}}.$$

Notice that, since $r_k - r_{k+1} \to 0$ and $r_k \to 1/2$, we have

$$\begin{aligned}C_k &\le C\left(\int_{D_{1/2}}\left(1+\left|\log\frac{d(x)}{r_k-r_{k+1}}\right|\right)^{\frac{1+\varepsilon}{\varepsilon}}dx\right)^{\frac{\varepsilon}{1+\varepsilon}} \\ &\le C\left[\left(\int_{D_{1/2}}(1+|\log d(x)|)^{\frac{1+\varepsilon}{\varepsilon}}dx\right)^{\frac{\varepsilon}{1+\varepsilon}}+|\log(r_k-r_{k+1})|\right] \le C|\log(r_k-r_{k+1})|,\end{aligned}$$

for some C. Further, for any fixed $\alpha \in (0, 1)$,

$$|\log(r_k - r_{k+1})| \le C_\alpha (r_k - r_{k+1})^{-\alpha},$$

for some C_α, and so

$$C_k \le C_\alpha (r_k - r_{k+1})^{-\alpha}, \tag{2.4.12}$$

for some C_α. Now, changing $1 - \beta \to -\beta$, we easily deduce

$$\left(\int_{D_{r_{k+1}}} u(x)^{-\beta\gamma}dx\right)^{-\frac{1}{\beta\gamma}} \ge \left[\frac{(r_k-r_{k+1})^{2s}}{CC_k(1+\beta)^2}\right]^{\frac{1}{\beta}}\left(\int_{D_{r_k}} u(x)^{-\beta(1+\varepsilon)}dx\right)^{-\frac{1}{\beta(1+\varepsilon)}}.$$

Further, setting $v := u^{1+\varepsilon}$, $\sigma := \frac{\gamma}{1+\varepsilon} > 1$, and using (2.4.12), it follows

$$\left(\int_{D_{r_{k+1}}} v(x)^{-\beta\sigma}dx\right)^{-\frac{1}{\beta\sigma}} \ge \left[\frac{(r_k-r_{k+1})^{2s+\alpha}}{C(1+\beta)^2}\right]^{\frac{1+\varepsilon}{\beta}}\left(\int_{D_{r_k}} v(x)^{-\beta}dx\right)^{-\frac{1}{\beta}}, \tag{2.4.13}$$

for some C. Thus, given $\beta_0 > 0$, we define $\beta_k := \beta_0\sigma^k$, $k \ge 1$. Iterating (2.4.13) with $\beta = \beta_0$, we obtain

$$\begin{aligned}\|v\|_{L^{-\beta_k}(D_{r_k})} &\ge \prod_{j=0}^{k-1}\left[\frac{(r_j-r_{j+1})^{2s+\alpha}}{C(1+\beta_j)^2}\right]^{\frac{1}{\beta_j}}\|v\|_{L^{-\beta_0}(D_{r_0})} \\ &= \prod_{j=0}^{k-1}\left[\frac{(r_j-r_{j+1})^{2s+\alpha}}{C(1+\beta_0\sigma^j)^2}\right]^{\frac{1}{\beta_0\sigma^j}}\|v\|_{L^{-\beta_0}(D_{r_0})},\end{aligned} \tag{2.4.14}$$

up to changing the constant $C > 0$, independently of $k \in \mathbb{N}$. Now, we notice that

$$\prod_{j=0}^{k-1} \left[\frac{(r_j - r_{j+1})^{2s+\alpha}}{C(1+\beta_0 \sigma^j)^2} \right]^{\frac{1}{\beta_0 \sigma^j}} = \exp \left\{ \frac{1}{\beta_0} \sum_{j=0}^{k-1} \frac{1}{\sigma^j} \log \left[\frac{(r_j - r_{j+1})^{2s+\alpha}}{C(1+\beta_0 \sigma^j)^2} \right] \right\},$$

for all $k \geq 1$, and so, choosing r_j such that $(r_j - r_{j+1})^{2s+\alpha} = C\beta_0^2 \sigma^{-2j}$ for $j \in \mathbb{N}$ large enough, we obtain

$$\sum_{j=0}^{\infty} \frac{1}{\sigma^j} \log \left[\frac{(r_j - r_{j+1})^{2s+\alpha}}{C(1+\beta_0 \sigma^j)^2} \right] \geq -C \sum_{j=0}^{\infty} \frac{j}{\sigma^j} > -\infty.$$

Consequently, we can pass to the limit in (2.4.14) and deduce (2.4.11), thanks to the fact that $\|v\|_{L^{-\beta_k}(D_{r_k})} \to \operatorname{ess\,inf}_{x \in D_{R/2}(x_0)} v(x)$ as $k \to +\infty$ and $v = u^{1+\varepsilon}$. □

Combining Lemma 2.4.3 and Corollary 2.4.5, we finally deduce the following.

Theorem 2.4.6. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $R > 0$, $x_0 \in \Omega$ and $\beta > 1$. Assume that $\partial\Omega \cap B_{3R}(x_0)$ is a Lipschitz graph. Let K_Ω be of the form either (2.2.7) or (2.2.8). Assume that u satisfies*

$$\begin{cases} L_\Omega u \geq 0 & \text{in } D_{2R}(x_0) \\ u > 0 & \text{in } D_{2R}(x_0), \end{cases}$$

with Neumann conditions on $\partial\Omega \cap B_{2R}(x_0)$ in the sense of Definition 2.2.2.

Then for any $c > 0$ and $\vartheta > 1$, there exist $\kappa > 0$ and $\gamma \in (0, 2s)$ depending only on the Lipschitz constant of $\partial\Omega$, n, s, c and ϑ, such that if

$$\begin{cases} u(x) \geq c \left[1 - \left(\vartheta \frac{|x - x_0|}{R} \right)^\gamma \right] & \text{for a.e. } x \in \Omega \setminus B_{2R}(x_0) \\ \frac{|\{u \geq 1\} \cap D_{2R}(x_0)|}{|D_{2R}(x_0)|} \geq \frac{1}{2}, \end{cases} \tag{2.4.15}$$

then

$$\operatorname*{ess\,inf}_{x \in D_{R/4}(x_0)} u(x) \geq \kappa. \tag{2.4.16}$$

Proof. By scaling, it is enough to consider the case $R = 1$ and $x_0 = 0$.

First, since $\partial\Omega \cap B_3$ is a Lipschitz graph, and $0 \in \Omega$, we can show that there exists $\omega \in (0, 1/2)$ such that

$$\frac{|D_{1/2}|}{|D_2|} \leq \omega.$$

Indeed, this follows from the pointwise inequality $\sqrt{4 - x^2} > 3\sqrt{1/4 - x^2}$, which shows that we can take $\omega = 2/5 < 1/2$.

Now we claim that the second condition in (2.4.15) guarantees the existence of $r_0 \in (1/2, 2)$ such that

$$\frac{|\{u \geq 1\} \cap D_{r_0}|}{|D_2|} \geq \frac{1 + 2\omega}{4}, \qquad \frac{|\{u \geq 1\} \cap (D_2 \setminus D_{r_0})|}{|D_2|} \geq \frac{1 - 2\omega}{4}.$$

Let us define the functions

$$h(\rho) := \frac{|\{u \geq 1\} \cap D_\rho|}{|D_2|}, \qquad \tilde{h}(\rho) := \frac{|\{u \geq 1\} \cap (D_2 \setminus D_\rho)|}{|D_2|}.$$

It is clear that they are both continuous. Moreover, the first one is nondecreasing and satisfies $h(1/2) \leq \omega$ and $h(2) \geq 1/2$ by hypothesis. This means that there exists $r_0 \in (1/2, 2)$ such that $h(r_0) = (1/2+\omega)/2 = (1+2\omega)/4$. If we now use that $h(\rho)+\tilde{h}(\rho) \geq 1/2$, the claim easily follows.

Applying Corollary 2.4.5 (with $R = r_0$), we obtain that for any $\beta > 0$

$$\operatorname*{ess\,inf}_{x\in D_{r_0/2}} u(x) \geq C \left(\fint_{D_{r_0}} u(x)^{-\beta} dx \right)^{-1/\beta}, \tag{2.4.17}$$

for some constant $C > 0$ depending only on the Lipschitz constant of $\partial\Omega$, n, s, and β. Now, by Lemma 2.4.2 with $R = 2$, $\delta_0 = (1-2\omega)/4$ and $r = r_0$, there is $\gamma \in (0, 2s)$ depending only on the Lipschitz constant of $\partial\Omega$, n, s, c and ϑ such that

$$\int_{\Omega\setminus B_{r_0}} u(x) K_\Omega(x,y) dx \geq 0 \qquad \text{for a.e. } y \in D_{r_0}.$$

On the other hand, by Lemma 2.4.3 (with $R = r_0$), there exists $\beta_0 \in (0, 1)$ depending only on the Lipschitz constant of $\partial\Omega$, n, and s such that

$$\left(\fint_{D_{r_0}} u(x)^{\beta_0} dx \right)^{1/\beta_0} \leq C \left(\fint_{D_{r_0}} u(x)^{-\beta_0} dx \right)^{-1/\beta_0},$$

and thus, choosing $\beta = \beta_0$ in (2.4.17), it follows

$$\begin{aligned}
\operatorname*{ess\,inf}_{x\in D_{r_0/2}} u(x) &\geq C \left(\fint_{D_{r_0}} u(x)^{\beta_0} dx \right)^{1/\beta_0} \geq C \left(\frac{1}{|D_{r_0}|} \int_{D_{r_0}\cap\{u\geq 1\}} u(x)^{\beta_0} dx \right)^{1/\beta_0} \\
&\geq C \left(\frac{|\{u \geq 1\} \cap D_{r_0}|}{|D_{r_0}|} \right)^{1/\beta_0} \geq C \left(\frac{|\{u \geq 1\} \cap D_{r_0}|}{|D_2|} \right)^{1/\beta_0} \\
&\geq C \left(\frac{1+2\omega}{4} \right)^{1/\beta_0} := \kappa.
\end{aligned}$$

Since $r_0 \geq 1/2$, the thesis follows. □

As a first consequence, we can prove a version of the above theorem that allows a right hand side f.

Theorem 2.4.7. *(Weak Harnack inequality) Let $\Omega \subset \mathbb{R}^n$ be a domain, $R > 0$, $x_0 \in \Omega$ and $f \in L^q(D_{2R}(x_0))$ with $q > \frac{n}{2s}$. Assume that $\partial\Omega \cap B_{3R}(x_0)$ is a Lipschitz graph. Let K_Ω be of the form either (2.2.7) or (2.2.8). Assume that u satisfies*

$$\begin{cases} L_\Omega u \geq f & \text{in } D_{2R}(x_0) \\ u > 0 & \text{in } D_{2R}(x_0), \end{cases}$$

with Neumann conditions on $\partial\Omega \cap B_{2R}(x_0)$, in the sense of Definition 2.2.2.

Then for any $c > 0$ and $\vartheta > 1$, there exist $\kappa_0 > 0$, $\kappa > 0$ and $\gamma \in (0, 2s)$ depending only on the Lipschitz constant of $\partial\Omega$, n, s, c and ϑ, such that if (2.4.15) holds, then

$$\operatorname*{ess\,inf}_{x\in D_{R/4}(x_0)} u(x) + \kappa_0 R^{2s-\frac{n}{q}} \|f\|_{L^q(D_{2R}(x_0))} \geq \kappa. \tag{2.4.18}$$

Proof. We assume $R = 1$, $x_0 = 0$. Let us consider the function $w := u + v$, where v satisfies (2.3.9) (with $R = 1$ and $x_0 = 0$). Then, w satisfies

$$\begin{cases} L_\Omega w \geq 0 & \text{in } D_2 \\ w > 0 & \text{in } D_2, \end{cases}$$

with Neumann conditions on $\partial\Omega \cap B_2$ in the sense of Definition 2.2.2. Notice that $w \geq u$ in Ω and thus it satisfies the assumptions in (2.4.15). Consequently, we can apply Theorem 2.4.6 to the function w and, since $v \leq \kappa_0 \|f\|_{L^q(D_1)}$ in $D_{1/2}$ (by Lemma 2.3.3), we deduce

$$\operatorname*{ess\,inf}_{x \in D_{1/4}} u(x) + \kappa_0 \|f\|_{L^q(D_1)} \geq \operatorname*{ess\,inf}_{x \in D_{1/4}} w(x) \geq \kappa,$$

which proves (2.4.18). □

We finally use the previous weak Harnack inequality to deduce the Hölder regularity of solutions.

Proof of Theorem 2.4.1. The result follows by iterating the previous weak Harnack inequality, with an argument similar to those in [136, 181]. By scaling and a covering argument as in [115, Remark 2.13], it is sufficient to assume that u is a weak bounded solution to

$$L_\Omega u = f \quad \text{ in } D_3(x_0),$$

with Neumann conditions on $\partial\Omega \cap B_3(x_0)$ (in the sense of Definition 2.2.2) and prove

$$|u(x) - u(y)| \leq C|x - y|^\alpha \left[\|u\|_{L^\infty(\Omega)} + \|f\|_{L^q(D_3(x_0))} \right]$$

for a.e. $x, y \in D_{1/2}(x_0)$.

Step 1. Let us take $\vartheta = 4$, $c = 2$, $\kappa \in (0,1)$, $\gamma \in (0, 2s)$ and $\kappa_0 > 0$ as in Theorem 2.4.7 (depending only on the Lipschitz constant of $\partial\Omega$, n, s, and q). We set $\overline{\kappa} := \kappa/2$.

Given any $z_0 \in D_1(x_0)$, we construct a non-decreasing sequence $(m_i)_{i\in\mathbb{Z}}$ and a non-increasing sequence $(M_i)_{i\in\mathbb{Z}}$ such that

$$\begin{aligned} & m_i \leq u(y) \leq M_i \quad \text{ for a.e. } y \in D_{\vartheta^{-i}}(z_0) \\ & M_i - m_i = K\vartheta^{-i\alpha}, \end{aligned} \tag{2.4.19}$$

for all $i \in \mathbb{Z}$, some $\alpha \in (0,1)$ and $K > 0$ to be determined (independently of z_0 and x_0). We choose

$$0 < \varepsilon_0 \leq \min\left\{\frac{1}{2}, \frac{\kappa}{4\kappa_0}\right\} \tag{2.4.20}$$

and

$$M_0 := \|u\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon_0}\|f\|_{L^q(D_3(x_0))}, \qquad m_0 := -\|u\|_{L^\infty(\Omega)},$$

so that

$$K := M_0 - m_0 = 2\|u\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon_0}\|f\|_{L^q(D_3(x_0))}.$$

Now, we assume that (2.4.19) holds and show how (2.4.1) follows. Since u is bounded, whenever $x, y \in D_1(x_0)$ satisfy $|x - y| \geq 1$, (2.4.1) follows with $C = 2$ and any $\alpha \in (0,1)$.

Thus it is enough to check the validity of (2.4.1) when $x \neq y$ and $|x - y| < 1$. In such case, we take $x = z_0$ and consider $l \in \mathbb{N}$ (depending on y) such that

$$\vartheta^{-(l+1)} \leq |x - y| < \vartheta^{-l}.$$

Consequently,

$$
\begin{aligned}
|u(x)-u(y)| &\le \mathrm{osc}_{B_{\vartheta^{-l}(x)}} u \le M_l - m_l = K\vartheta^{-l\alpha} \le K\vartheta^{\alpha}|x-y|^{\alpha}\\
&\le \frac{\vartheta^{\alpha}}{\varepsilon_0}|x-y|^{\alpha}\left[\|u\|_{L^\infty(\Omega)} + \|f\|_{L^q(D_3(x_0))}\right],
\end{aligned}
$$

which is exactly (2.4.1) with $C = \vartheta^{\alpha}/\varepsilon_0$. Using the arbitrariness of $x, y \in D_1(x_0)$ with $|x-y| < 1$ and $x \neq y$, the estimate (2.4.1) follows.

Step 2. Notice that, since u is bounded in Ω, the choice of K guarantees that (2.4.19) hold true for $i = 0$ and, moreover, setting $M_i = M_0$ and $m_i = m_0$ for all negative integers i, (2.4.19) hold true for any $i \in \mathbb{Z}$, $i < 0$.

Step 3. We construct the sequences $(m_i)_{i\in\mathbb{N}}$ and $(M_i)_{i\in\mathbb{N}}$ by induction on $i \in \mathbb{N}$. So, we assume that there exists $k \ge 1$ such that (2.4.19) hold for all $i \le k-1$, and we show how to choose m_k and M_k such that (2.4.19) hold for $i = k$.

We define

$$
\alpha := \min\left\{\gamma, \ln\left(\frac{2}{2-\kappa}\right)/\ln\vartheta\right\}, \tag{2.4.21}
$$

and we consider the function

$$
v(x) := \left(u(\vartheta^{-(k-1)}x + z_0) - \frac{M_{k-1}+m_{k-1}}{2}\right)\frac{2\vartheta^{(k-1)\alpha}}{K}.
$$

Notice that, in view of (2.4.19), we have

$$
|v| \le 1 \quad \text{in } \widetilde{D}_1,
$$

where $B_1 = B_1(0)$, $\widetilde{\Omega} := \{x \in \mathbb{R}^n : \vartheta^{-(k-1)}x + z_0 \in \Omega\}$ and $\widetilde{D}_1 := B_1 \cap \widetilde{\Omega}$. Note that since $\widetilde{\Omega}$ is a dilation, its Lipschitz constant does not increase. Now, we divide the proof in two cases. First, we assume

$$
\frac{|\{v \le 0\} \cap \widetilde{D}_1|}{|\widetilde{D}_1|} \ge \frac{1}{2}. \tag{2.4.22}
$$

In order to apply Theorem 2.4.7, we study the decaying of v in $\widetilde{\Omega}\backslash B_1$. So, for any $y \in \widetilde{\Omega}\backslash B_1$ we have $|y| \ge 1$ and thus there is $j \in \mathbb{N}$, $j \ge 1$ (depending on y) such that

$$
\vartheta^{j-1} \le |y| < \vartheta^{j}.
$$

Using that $(m_n)_{n\in\mathbb{N}}$ is non-decreasing, the fact that $y \in B_{\vartheta^j}$ and (2.4.19), we obtain

$$
\begin{aligned}
v(y) &= \frac{2\vartheta^{(k-1)\alpha}}{K}\left(u(\vartheta^{-(k-1)}y + z_0) - \frac{M_{k-1}+m_{k-1}}{2}\right)\\
&\le \frac{2\vartheta^{(k-1)\alpha}}{K}\left(M_{k-j-1} - m_{k-j-1} + m_{k-j-1} - \frac{M_{k-1}+m_{k-1}}{2}\right)\\
&\le \frac{2\vartheta^{(k-1)\alpha}}{K}\left(M_{k-j-1} - m_{k-j-1} - \frac{M_{k-1}-m_{k-1}}{2}\right)\\
&= \frac{2\vartheta^{(k-1)\alpha}}{K}\left(K\vartheta^{-(k-j-1)\alpha} - \frac{K}{2}\vartheta^{-(k-1)\alpha}\right) = 2\vartheta^{j\alpha} - 1 \le 2\vartheta^{\alpha}|y|^{\alpha} - 1,
\end{aligned}
$$

which, setting $w := 1 - v$, is equivalent to

$$
w(y) \ge 2\left[1 - (\vartheta|y|)^{\alpha}\right] \quad \text{for a.e. } y \in \widetilde{\Omega}\setminus B_1.
$$

Furthermore, w is a weak solution to

$$L_\Omega w = -\frac{2}{K}\vartheta^{(\alpha-2s)(k-1)} f \quad \text{in } \widetilde{D}_2,$$

and so, thanks to assumption (2.4.22) and the fact that $\alpha \le \gamma$ (see (2.4.21)), we can apply Theorem 2.4.7 (with $R = 1$) to deduce

$$\operatorname*{ess\,inf}_{x\in\widetilde{D}_{\vartheta^{-1}}} w(x) + \frac{2\kappa_0}{K}\vartheta^{(\alpha-2s)(k-1)}\|f\|_{L^q(\widetilde{D}_2)} \ge \kappa,$$

which implies

$$v(x) \le 1 - \kappa + \frac{2\kappa_0}{K}\vartheta^{(\alpha-2s)(k-1)}\|f\|_{L^q(\widetilde{D}_2)} \quad \text{for a.e. } x \in \widetilde{D}_{\vartheta^{-1}}.$$

Notice that, using the definition of K and that $\alpha \le 2s$ (cfr. with (2.4.21)) and $\vartheta > 1$, we have

$$\frac{2\kappa_0}{K}\vartheta^{(\alpha-2s)(k-1)}\|f\|_{L^q(\widetilde{D}_2)} \le 2\kappa_0 \frac{\varepsilon_0\|f\|_{L^q(\widetilde{D}_2)}}{\varepsilon_0\|u\|_{L^\infty(\widetilde{D}_2)} + \|f\|_{L^q(\widetilde{D}_2)}} \le 2\kappa_0\varepsilon_0 \le \frac{\kappa}{2},$$

thanks to the choice of $\varepsilon_0 > 0$ in (2.4.20). Consequently,

$$v(x) \le 1 - \frac{\kappa}{2} := 1 - \overline{\kappa} \quad \text{for a.e. } x \in \widetilde{D}_{\vartheta^{-1}}.$$

So, using the definition of v and the above inequality, we obtain

$$\begin{aligned} u(x) &\le \frac{1-\overline{\kappa}}{2} K\vartheta^{-(k-1)\alpha} + \frac{M_{k-1}+m_{k-1}}{2} = \frac{1-\overline{\kappa}}{2}(M_{k-1}-m_{k-1}) + \frac{M_{k-1}+m_{k-1}}{2} \\ &= m_{k-1} + \left(1 - \frac{\overline{\kappa}}{2}\right)(M_{k-1}-m_{k-1}) \end{aligned}$$

for a.e. $x \in D_{\vartheta^{-k}}(z_0)$. Finally, using (2.4.21), we have that $1 - \frac{\overline{\kappa}}{2} \le \vartheta^{-\alpha}$, and so from the definition of K, we deduce

$$u(x) \le m_{k-1} + K\vartheta^{-k\alpha} \quad \text{for a.e. } x \in D_{\vartheta^{-k}}(z_0).$$

Choosing $m_k := m_{k-1}$ and $M_k := m_{k-1} + K\vartheta^{-k\alpha}$, it follows that (2.4.19) is satisfied for $i = k$ and we complete the proof of the first case.

Finally, if (2.4.22) is not satisfied, it is sufficient to notice that it holds for $\tilde{v} := -v$ and repeat the above procedure working with $\tilde{v}$. □

2.5 A Neumann Liouville theorem in the half-space

The goal of this section is to prove the following Liouville-type theorem in a half-space with nonlocal Neumann boundary conditions.

Theorem 2.5.1. *Let $\Omega = \mathbb{R}^n_+ = \{x_n > 0\}$, and $s \in (\frac{1}{2}, 1)$. Let L_Ω and K_Ω be given by either (2.2.1)-(2.2.2)-(2.2.3), or (2.1.7). Assume v is a weak solution to*

$$L_\Omega v = 0 \quad \text{in } \mathbb{R}^n_+$$

with Neumann condition on $\partial\mathbb{R}^n_+ = \{x_n = 0\}$ *(in the sense of Definition 2.2.2). Let* $\alpha > 0$ *be given by Theorem 2.5.7, and assume that*

$$||v||_{L^\infty(B_R^+)} \leq C_0(1 + R^{2s-1+\varepsilon}) \quad \textit{for all} \quad R > 0,$$

for some C_0 *and* $\varepsilon \in (0, \alpha)$*. Then,*

$$v(x) = a + b \cdot x$$

for some $a \in \mathbb{R}$ *and* $b \in \mathbb{R}^n$ *with* $b_n = 0$*. Moreover, if* $2s - 1 + \epsilon < 1$ *then* $b = 0$*.*

The proof of this result is not standard and does not follow from classical tools such as even reflection for harmonic functions. Moreover, the extension problem for the fractional Laplacian is of no use here, and therefore the proof must be different from the Dirichlet case, too.

We stress that, even in 1D, we do not know how to prove a better Liouville theorem (allowing more growth on v). This seems a challenging open problem, which is strongly related to the higher boundary regularity of solutions to (2.1.1).

2.5.1 1D barriers

We need sub- and supersolutions for both problems (2.1.1) and (2.1.10). We start with the following.

Lemma 2.5.2. *(Supersolution for (2.1.1) and (2.1.10)) Let* $n = 1$*,* $\Omega = (0, \infty)$*, and* $s \in (\frac{1}{2}, 1)$*. Let* L_Ω *and* K_Ω *be given by either (2.2.1), (2.2.2)-(2.2.3) or (2.1.7). Given any* $r_0 > 0$*, let us consider* $\eta \in C_0^\infty([0, 2r_0))$ *satisfying* $0 \leq \eta \leq 1$ *and* $\eta = 1$ *in* $[0, r_0]$*.*

Then, there exists $\overline{c} > 0$ *(depending only on* r_0*) such that the function*

$$\overline{\varphi}(x) := \eta(x)x^{2s-1}$$

satisfies

$$L_\Omega\overline{\varphi} \geq \overline{c} \quad \textit{in} \quad (0, r_0).$$

Moreover, if L_Ω *and* K_Ω *are given by (2.2.1), (2.2.2)-(2.2.3), a logarithmic improvement can be done. That is,*

$$L_\Omega\overline{\varphi} \geq \overline{c}\left(1 + \log^-\left(\frac{x}{r_0}\right)\right) \quad \textit{in} \quad (0, r_0).$$

Proof. We prove the result for K_Ω of the form (2.2.1), (2.2.2)-(2.2.3); the case (2.1.7) is simpler.

By scaling, we may assume $r_0 = 1$. Given $x \in (0, 1)$ and using the definition of $\overline{\varphi}$, we compute

$$\begin{aligned} L_\Omega\overline{\varphi}(x) &= \int_0^\infty \left\{x^{2s-1} - \eta(y)y^{2s-1}\right\} K_\Omega(x, y)\, dy \\ &= \int_0^\infty \left\{x^{2s-1} - y^{2s-1}\right\} K_\Omega(x, y)\, dy + \int_0^\infty y^{2s-1}(1 - \eta(y))\, K_\Omega(x, y)\, dy := I_1 + I_2. \end{aligned}$$

Now, by the symmetry and the scaling of the kernel K_Ω (see Section 2.2), it is easy to check that $L_\Omega(x^{2s-1}) = 0$ in $\mathbb{R}_+$ and so $I_1(x) = 0$. On the other hand, we know that $\eta = 1$

in $[0,1]$ while $\eta = 0$ in $[2,\infty)$. Moreover, if we use that $1 \le y - x \le y$ for all $x < 1 < 2 \le y$, it follows

$$
\begin{aligned}
I_2(x) &= \int_1^2 y^{2s-1}(1-\eta(y))\, K_\Omega(x,y)\, dy + \int_2^\infty y^{2s-1}(1-\eta(y))\, K_\Omega(x,y)\, dy \\
&\ge \int_2^\infty y^{2s-1}(1-\eta(y))\, K_\Omega(x,y)\, dy = \int_2^\infty y^{2s-1}\, K_\Omega(x,y)\, dy \\
&\ge c \int_2^\infty y^{2s-1} \frac{1+\log^-\left(\frac{x}{y-x}\right)}{(y-x)^{1+2s}}\, dy \\
&= c \int_2^\infty \frac{y^{2s-1}}{(y-x)^{1+2s}}\, dy + c \int_2^\infty y^{2s-1} \frac{\log\left(\frac{y-x}{x}\right)}{(y-x)^{1+2s}} dy \\
&\ge c \int_2^\infty y^{-2} dy + c \int_2^\infty y^{-2} \log\left(\frac{y-x}{x}\right) dy \ge c \int_2^\infty y^{-2} dy + c \int_2^\infty y^{-2} \log\left(\frac{1}{x}\right) dy \\
&\ge \bar{c}\left(1 + \log^- x\right).
\end{aligned}
$$

□

We next show the following construction of subsolutions.

Lemma 2.5.3. *(Subsolution for (2.1.1) and (2.1.10)) Let $n = 1$, $\Omega = (0,\infty)$, and $s \in (\frac{1}{2}, 1)$. Let L_Ω and K_Ω be given by either (2.2.1), (2.2.2)-(2.2.3) or (2.1.7). Given any $r_0 > 0$, let us consider $\eta \in C_0^\infty([0, 2r_0))$ satisfying $0 \le \eta \le 1$, $\eta = 1$ in $[0, r_0]$ and $\zeta \in C_0^\infty((r_0, 2r_0))$ satisfying $0 \le \zeta \le 1$ and $\zeta \not\equiv 0$.*

Then, for any $\underline{c} \ge 0$, there exist $M > 0$ (depending on $\underline{c}$, s and r_0) such that the function

$$\underline{\varphi}(x) := \eta(x) x^{2s-1} + M\zeta(x)$$

satisfies

$$L_\Omega \underline{\varphi} \le -\underline{c} \quad \text{in} \quad (0, r_0).$$

Moreover, if L_Ω and K_Ω are given by (2.2.1), (2.2.2)-(2.2.3), a logarithmic improvement can be done. That is,

$$L_\Omega \underline{\varphi} \le -\underline{c}\left(1 + \log^-\left(\frac{x}{r_0}\right)\right) \quad \text{in} \quad (0, r_0).$$

Proof. We proceed as in the previous lemma, proving the result only in the case L_Ω and K_Ω are given by (2.2.1), (2.2.2)-(2.2.3) and $r_0 = 1$. Given $x \in (0,1)$ and using the properties of ζ and the identity $L_\Omega(x^{2s-1}) = 0$ in $\mathbb{R}_+$, we obtain

$$
\begin{aligned}
L_\Omega \underline{\varphi}(x) &= L_\Omega \overline{\varphi}(x) + M L_\Omega \zeta(x) \\
&= \int_1^2 y^{2s-1}(1-\eta(y))\, K_\Omega(x,y)\, dy + \int_2^\infty y^{2s-1}\, K_\Omega(x,y)\, dy - M \int_1^2 \zeta(y)\, K_\Omega(x,y)\, dy \\
&:= I_1(x) + I_2(x) - M\, I_3(x).
\end{aligned}
$$

Now, we consider separately each of the three terms. That is,

$$
\begin{aligned}
I_1(x) &= \int_1^2 y^{2s-1}(1-\eta(y))\, K_\Omega(x,y)\, dy \\
&\le C \int_1^2 y^{2s-1}(1-\eta(y))\, \frac{1+\log^-\left(\frac{x}{y-x}\right)}{(y-x)^{1+2s}}\, dy \\
&= C \int_1^2 \frac{y^{2s-1}(1-\eta(y))}{(y-x)^{1+2s}}\, dy + C \int_1^2 y^{2s-1}(1-\eta(y))\, \frac{\log\left(\frac{y-x}{x}\right)}{(y-x)^{1+2s}}\, \chi_{\{y>2x\}}\, dy \\
&:= I_{11}(x) + I_{12}(x).
\end{aligned}
$$

On the one hand, we know the existence of two positive constants δ and C, such that

$$
1-\eta(y) \le C(y-1)^2 \quad \text{for all } y \in [1, 1+\delta).
$$

This follows from the fact that $\eta'(1) = 0$ and that $\eta''(1)$ is bounded (notice that δ and C depend only on η''). Consequently, since $x \in (0,1)$, we have $y - x \ge y - 1$ and, moreover, when $y \in (1+\delta, 2)$ we have $y - x \ge \delta$. Thus,

$$
\begin{aligned}
I_{11}(x) &= C \int_1^{1+\delta} \frac{y^{2s-1}(1-\eta(y))}{(y-x)^{1+2s}} dy + C \int_{1+\delta}^2 \frac{y^{2s-1}(1-\eta(y))}{(y-x)^{1+2s}} dy \\
&\le C \int_1^{1+\delta} y^{2s-1}(y-1)^{1-2s} dy + C\, \delta^{-1-2s} \int_{1+\delta}^2 y^{2s-1} dy < C < +\infty.
\end{aligned}
$$

On the other hand, taking into account that $y - x \ge y - 1/2 \ge 1/2$ when $x \le 1/2$, whilst $y - x > x > 1/2$ when $x > 1/2$ and $y > 2x$ we arrive at

$$
\begin{aligned}
I_{11}(x) &\le C \int_1^2 \frac{\log\left(\frac{y-x}{x}\right)}{(y-x)^{1+2s}}\, \chi_{\{y>2x\}}\, dy \le C \int_1^2 \log\left(\frac{2}{x}\right)\, dy \\
&\le C \left(1 + \log\left(\frac{1}{x}\right)\right) \le C \left(1 + \log^- x\right).
\end{aligned}
$$

Thus, we obtain

$$
I_1(x) \le C_1 \left(1 + \log^- x\right).
$$

Next, we proceed with the estimate of the term I_2. That is, since $y - x \ge y/2$ when $x \le 1 < 2 \le y$ we get

$$
\begin{aligned}
I_2(x) &= \int_2^\infty y^{2s-1}\, K_\Omega(x,y)\, dy \le C \int_2^\infty y^{2s-1}\, \frac{1+\log^-\left(\frac{x}{y-x}\right)}{(y-x)^{1+2s}}\, dy \\
&\le C \int_2^\infty y^{-2} \left(1 + \log\left(\frac{y-x}{x}\right)\right) dy \\
&\le C \int_2^\infty y^{-2} \left(1 + \log\left(\frac{y}{x}\right)\right)\, dy \le C\,(1-\log x) \int_2^\infty y^{-2}\,(1+\log y)\, dy \\
&\le C_2 \left(1 + \log^- x\right).
\end{aligned}
$$

Finally, we consider I_3. By using again again that $1/2 \le y - x \le 2$ when $2x < y < 2$, we arrive at

$$\begin{aligned}
I_3(x) &= \int_1^2 \zeta(y)\, K_\Omega(x,y)\, dy \geq c \int_1^2 \zeta(y)\, \frac{1+\log^-\left(\frac{x}{y-x}\right)}{(y-x)^{1+2s}}\, dy \\
&\geq c \int_1^2 \zeta(y)\, \frac{1+\log\left(\frac{y-x}{x}\right)\chi_{\{y>2x\}}}{(y-x)^{1+2s}}\, dy \\
&\geq c \int_1^2 \zeta(y)\, \left(1+\log\left(\frac{y-x}{x}\right)\chi_{\{1>2x\}}\right) dy \\
&\geq c \int_1^2 \zeta(y)\, \left(1+\log\left(\frac{1}{2x}\right)\chi_{\{1>2x\}}\right) dy \geq c\left(1+\log\left(\frac{1}{2x}\right)\chi_{\{1>2x\}}\right) \\
&\geq C_3\left(1+\log^- x\right).
\end{aligned}$$

Therefore, as a consequence of the previous computations, for all $x \in (0,1)$ and all $\underline{c} \geq 0$, we obtain

$$L_\Omega \underline{\varphi}(x) = I_1 + I_2 - M I_3 \leq (C_1 + C_2 - M C_3)\left(1+\log^- x\right) := -\underline{c}\left(1+\log^- x\right),$$

if we take $M > 0$ large enough, depending only on s and $\underline{c}$. □

2.5.2 A 1D boundary Harnack

We now prove a boundary Harnack estimate in dimension 1, by using the previous sub/supersolutions and following the general steps from [163].

For any $R > 0$, we define

$$I_R := (0,R) \quad \text{and} \quad I_R^+ := (R/4, R/2).$$

The first step is the following.

Lemma 2.5.4. *Let $n = 1$, $\Omega = (0,\infty)$, $s \in (\frac{1}{2},1)$, and $K_0 \geq 0$. Assume that either L_Ω and K_Ω are given by (2.1.7) and u satisfies*

$$\begin{cases} L_\Omega u \geq -K_0 & \text{in } I_R, \\ u \geq 0 & \text{in } \mathbb{R}_+, \end{cases}$$

or L_Ω and K_Ω are given by (2.2.1)-(2.2.2)-(2.2.3) and u satisfies

$$\begin{cases} L_\Omega u \geq -K_0\left[1+\log^-\left(\frac{x}{R}\right)\right] & \text{in } I_R, \\ u \geq 0 & \text{in } \mathbb{R}_+. \end{cases}$$

Then, there exists $C > 0$ depending only on s, such that

$$\inf_{x\in I_R^+} \frac{u(x)}{x^{2s-1}} \leq C\left[\inf_{x\in I_{R/4}} \frac{u(x)}{x^{2s-1}} + K_0 R\right]. \tag{2.5.1}$$

Proof. We prove the result for K_Ω of the form (2.2.1), (2.2.2)-(2.2.3) since the case (2.1.7) is completely analogous.

By scaling properties we may assume $R = 1$. The general case is recovered by applying (2.5.1) (with $R = 1$) to the function $u_R(x) := R^{-2s}u(Rx)$, $R > 0$.

Step 1. Assume $K_0 = 0$. Let us define

$$m := \inf_{x \in I_1^+} \frac{u(x)}{x^{2s-1}} \geq 0.$$

If $m = 0$, the thesis follows immediately. So, assume $m > 0$. In this case, it holds

$$u(x) \geq m x^{2s-1} \geq m r_0^{2s-1} \quad \text{ in } I_1^+.$$

Now, for any $\varepsilon > 0$, we define

$$\varphi(x) := \varepsilon \underline{\varphi}(x),$$

where $\underline{\varphi}$ is the subsolution constructed in Lemma 2.5.3 for $r_0 = 1/4$ and $\underline{c} = 0$, satisfying $L_\Omega \underline{\varphi} \leq 0$ in $I_{1/4}$, and $\operatorname{supp}(\underline{\varphi}) \subset I_{1/2}$. Consequently, φ is a subsolution in $I_{1/4}$ for any $\varepsilon > 0$ and, furthermore,

$$\varphi(x) = \varepsilon[\eta(x)x^{2s-1} + M\zeta(x)] \leq \varepsilon(2^{1-2s} + M) \leq m4^{1-2s} \leq u(x),$$

for all $x \in [1/4, 1/2)$, whenever $0 < \varepsilon \leq \varepsilon_0 := m4^{1-2s}/(2^{1-2s} + M)$. Thus, choosing $\varepsilon = \varepsilon_0$ and recalling that u is nonnegative, it follows that $\varphi \leq u$ in $[1/4, +\infty)$ and so applying the comparison principle in $I_{1/4}$ we obtain

$$\varepsilon_0 x^{2s-1} = \varphi(x) \leq u(x) \quad \text{ in } I_{1/4}.$$

Taking $C = (2^{1-2s} + M)/4^{1-2s}$ and using the definition of ε_0, it easily follows

$$m \leq C \inf_{x \in I_{1/4}} \frac{u(x)}{x^{2s-1}},$$

and the proof in the case $K_0 = 0$ is completed.

Step 2. Assume $K_0 > 0$. For any $\kappa_0 > 0$, we define

$$v(x) := \kappa_0 \overline{\varphi}(x) + u(x) = \kappa_0 x^{2s-1} + u(x) \quad \text{ in } I_1,$$

where $\overline{\varphi}$ is the supersolution constructed in Lemma 2.5.2 (with $r_0 = 1$ and $r_1 = 2$), satisfying $L_\Omega \overline{\varphi} \geq \overline{c}\left(1 + \log^- x\right)$ in I_1, for some universal constant $\overline{c} > 0$, and $\operatorname{supp}(\underline{\varphi}) \subset I_2$. Thus, choosing $\kappa_0 = K_0/\overline{c}$ and recalling that $\overline{\varphi}$ is nonnegative, it follows

$$\begin{cases} L_\Omega v \geq 0 & \text{ in } I_1 \\ v \geq 0 & \text{ in } \mathbb{R}_+. \end{cases}$$

Hence, we can apply *Step 1* to the function v to conclude the existence of a constant $C > 0$ (depending on s) such that

$$\inf_{x \in I_1^+} \frac{v(x)}{x^{2s-1}} \leq C \inf_{x \in I_{1/4}} \frac{v(x)}{x^{2s-1}}.$$

Finally, (2.5.1) follows easily since $v(x) = \kappa_0 x^{2s-1} + u(x)$ in I_1. Notice that the constant $C > 0$ changes passing from v to u. □

We will also need the following, which follows from the interior Harnack inequality (see for instance [91]).

Lemma 2.5.5. *Let $n = 1$, $\Omega = (0, \infty)$, and $s \in (\frac{1}{2}, 1)$. Let L_Ω and K_Ω be given by either (2.2.1)-(2.2.2)-(2.2.3), or (2.1.7). Assume that*

$$\begin{cases} |L_\Omega u| \leq K_0 \left(1 + \log^- \left(\frac{x}{R}\right)\right) & \text{in } I_R \\ u \geq 0 & \text{in } \mathbb{R}_+, \end{cases}$$

for some $K_0 \geq 0$. Then there exists $C > 0$ depending only on s, such that

$$\sup_{x \in I_R^+} \frac{u(x)}{x^{2s-1}} \leq C \left[\inf_{x \in I_R^+} \frac{u(x)}{x^{2s-1}} + K_0 R \right]. \tag{2.5.2}$$

Proof. Again, it is enough to prove the case $R = 1$. Inequality (2.5.2) easily follows from the interior Harnack inequality (see (2.2)-(2.3) in [91])

$$\sup_{x \in I_1^+} u(x) \leq C \left[\inf_{x \in I_1^+} u(x) + K_0 \right],$$

and using that $x \in (1/4, 1/2)$, and that $\log^- x$ is bounded in $[1/8, 1]$. □

We can now prove the oscillation decay for the quotient u/x^{2s-1}.

Lemma 2.5.6. *Let $n = 1$, $\Omega = (0, \infty)$, $s \in (\frac{1}{2}, 1)$, and $K_0 \geq 0$. Assume that either L_Ω and K_Ω are given by (2.1.7) and u satisfies*

$$\begin{cases} |L_\Omega u| \leq K_0 & \text{in } I_2 \\ u(0) = 0, \end{cases}$$

or L_Ω and K_Ω are given by (2.2.1)-(2.2.2)-(2.2.3) and u satisfies

$$\begin{cases} |L_\Omega u| \leq K_0 \left(1 + \log^- x\right) & \text{in } I_2 \\ u(0) = 0. \end{cases}$$

Moreover, assume that u satisfies the growth condition

$$|u(y)| \leq c_0 (1 + y^{2s - \varepsilon_0}), \quad \text{for all } y > 0, \tag{2.5.3}$$

for some $c_0 > 0$, $\varepsilon_0 > 1$. Then there exist $\alpha \in (0, 1)$ and $C > 0$ (depending on s, c_0 and ε_0), such that

$$\sup_{x \in I_R} \frac{u(x)}{x^{2s-1}} - \inf_{x \in I_R} \frac{u(x)}{x^{2s-1}} \leq C R^\alpha \left[\|u\|_{L^\infty(I_2)} + K_0 \right], \tag{2.5.4}$$

for all $R \in (0, 1]$.

Proof. As in the previous results we are only proving it in the case L_Ω and K_Ω are given by (2.2.1)-(2.2.2)-(2.2.3).

Let us fix $\vartheta = 4$ and $R = 1$. Similar to the approach followed in the proof of Theorem 2.4.1, we construct a non-decreasing sequence $(m_i)_{i \in \mathbb{N}}$ and a non-increasing sequence $(M_i)_{i \in \mathbb{N}}$ such that

$$\begin{aligned} & m_i \leq \frac{u(y)}{y^{2s-1}} \leq M_i \quad \text{for a.e. } y \in I_{\vartheta^{-i}} \\ & M_i - m_i = K \vartheta^{-i\alpha}, \end{aligned} \tag{2.5.5}$$

for all $i \in \mathbb{N}$, some $\alpha \in (0,1)$ and $K > 0$ to be suitably determined. We proceed by induction on $i \in \mathbb{N}$.

Step 1. We prove the case $i = 0$. Let $\eta \in C_0^\infty([0,2))$ satisfying $0 \le \eta \le 1$ and $\eta = 1$ in $[0,1]$ and define

$$v(x) = \eta(x)u(x), \quad x \ge 0.$$

Notice that for any $x \in I_1 = (0,1)$, we have $v(x) = u(x)$ and, furthermore,

$$\begin{aligned}|L_\Omega v(x)| &\le |L_\Omega u(x)| + \int_1^2 \frac{|u(y)|[1-\eta(y)]}{(y-x)^{1+2s}}dy + \int_2^\infty \frac{|u(y)|}{(y-x)^{1+2s}}dy \\ &\le K_0(1+\log^- x) + K_1(1+\log^- x) + K_2(1+\log^- x) := \overline{K}_0(1+\log^- x),\end{aligned}$$

where $\overline{K_0}$ depends only on $c_0 > 0$, $\varepsilon_0 > 1$ and s. The above bounds follow by using that $x \in (0,1)$, $y > 1$ (and so $y - x > y - 1$), the regularity properties of η and (2.5.3).

Now, let $\overline{\varphi}$ be the supersolution constructed in Lemma 2.5.2 (with $r_0 = 1$, $r_1 = 2$) satisfying $L_\Omega\overline{\varphi} \ge \overline{c}(1+\log^- x)$ in I_1, and let $\psi(x) := A\overline{\varphi}(x)$, $A > 0$. Since, v is bounded and has support contained in I_2, we can choose A large enough (for instance, $A \ge \max\{\|u\|_{L^\infty(I_2)}, \overline{K}_0/\overline{c}\}$) so that

$$\begin{aligned}&\psi \ge v \quad \text{in } [1,\infty), \\ &L_\Omega\psi \ge A\overline{c}(1+\log^- x) \ge \overline{K}_0(1+\log^- x) \ge L_\Omega v \quad \text{in } I_1,\end{aligned} \tag{2.5.6}$$

and so, recalling that $\psi(0) = v(0) = 0$, it follows $\psi \ge v$ in I_1 by the maximum principle. In particular, $u(x) \le Ax^{2s-1}$ for all $x \in I_1$. Notice that the function $\varphi = -\psi$ works as a subsolution in I_1 with $\varphi \le -v$ in $[1,\infty)$ and so $|u(x)| \le Ax^{2s-1}$ for all $x \in I_1$.

Thus we can choose $M_0 = A$, $m_0 = -A$ and $K = M_0 - m_0 = 2A$. We anticipate that in the second part of the proof we will ask $K > 3CK_0$ (see (2.5.9)), where $C > 0$ is the constant appearing in Lemma 2.5.4 and Lemma 2.5.5. To guarantee this, it is enough to choose

$$K = 2A, \quad A = C_0\left(\|u\|_{L^\infty(I_2)} + K_0\right), \quad C_0 > \max\{1, 3C/2, \overline{K}_0/(\overline{c}K_0)\}. \tag{2.5.7}$$

Notice that this choice guarantees $A \ge \max\{\|u\|_{L^\infty(I_2)}, \overline{K}_0/\overline{c}\}$ and thus (2.5.6) is justified.

Step 2. We assume that (2.5.5) hold for all $i \le k$ and we prove the existence of m_{k+1} and M_{k+1} satifying (2.5.5), too. Define

$$u_k(x) := u(x) - m_k x^{2s-1},$$

and write $u_k = u_k^+ - u_k^-$. Notice that in view of (2.5.5) we have

$$u_k^+ = u_k \quad \text{in } I_{\vartheta^{-k}}.$$

Using the monotonicity of $(m_k)_{k\in\mathbb{N}}$ and $(M_k)_{k\in\mathbb{N}}$, we easily deduce that given $x \in I_{\vartheta^{-j}}$, it satisfies

$$\begin{aligned}u_k(x) = u(x) - m_k x^{2s-1} &\ge (m_j - m_k)x^{2s-1} \ge (m_j - M_j + M_k - m_k)x^{2s-1} \\ &= K(-\vartheta^{-j\alpha} + \vartheta^{-k\alpha})x^{2s-1} \ge -K\vartheta^{-j(2s-1)}(\vartheta^{-j\alpha} - \vartheta^{-k\alpha}),\end{aligned}$$

for all $j \le k$. Now, for any $x > \vartheta^{-k}$, there is $j \le k-1$ such that $\vartheta^{-j-1} < x \le \vartheta^{-j}$, and thus, if $x \in I_{\vartheta^{-j}} \setminus I_{\vartheta^{-k}}$, we have

$$\begin{aligned}u_k(x) &\ge -K\vartheta^{-j(2s-1)}(\vartheta^{-j\alpha} - \vartheta^{-k\alpha}) = -K\frac{\vartheta^{-j(2s-1)}}{\vartheta^{-k(2s-1)}}\vartheta^{-k(2s-1+\alpha)}\left(\frac{\vartheta^{-j\alpha}}{\vartheta^{-k\alpha}} - 1\right) \\ &\ge -K\vartheta^{-k(2s-1+\alpha)}\left(\frac{\vartheta x}{\vartheta^{-k}}\right)^{2s-1}\left[\left(\frac{\vartheta x}{\vartheta^{-k}}\right)^\alpha - 1\right], \quad x \in I_{\vartheta^{-j}} \setminus I_{\vartheta^{-k}}.\end{aligned} \tag{2.5.8}$$

Since the r.h.s. of the above inequality does not depend on j, we conclude that (2.5.8) holds for all $x \in \mathbb{R}_+ \setminus I_{\vartheta^{-k}}$. Now, let us take $x \in I_{\vartheta^{-k}/2}$. Using that $u_k^- = 0$ in $I_{\vartheta^{-k}}$ and (2.5.8), we obtain

$$
\begin{aligned}
0 \le -L_\Omega u_k^-(x) &= \int_0^\infty u_k^-(y)K(x,y)dy = \int_{\vartheta^{-k}}^\infty u_k^-(y)K(x,y)dy \\
&\le C_s \int_{\vartheta^{-k}-x}^\infty \frac{u_k^-(x+y)}{y^{1+2s}} \left(1 + \left|\log\left(\frac{x}{y}\right)\right|\right) dy \\
&\le C_s K \vartheta^{-k(2s-1+\alpha)} \int_{\vartheta^{-k}-x}^\infty \left(\frac{\vartheta(x+y)}{\vartheta^{-k}}\right)^{2s-1} \left[\left(\frac{\vartheta(x+y)}{\vartheta^{-k}}\right)^\alpha - 1\right] \frac{1 + \left|\log\left(\frac{x}{y}\right)\right|}{y^{1+2s}} dy \\
&\le C_s K \vartheta^{-k(2s-1+\alpha)} \int_{\vartheta^{-k}/2}^\infty \left(\frac{2\vartheta y}{\vartheta^{-k}}\right)^{2s-1} \left[\left(\frac{2\vartheta y}{\vartheta^{-k}}\right)^\alpha - 1\right] \frac{1 + \left|\log\left(\frac{x}{y}\right)\right|}{y^{1+2s}} dy \\
&\le C_s K \vartheta^{-k(\alpha-1)} \int_{1/2}^\infty (2\vartheta y)^{2s-1} \left[(2\vartheta y)^\alpha - 1\right] \frac{1 + |\log y| + |\log x|}{y^{1+2s}} dy \\
&\le \varepsilon_0(\alpha) C_s K \vartheta^{-k(\alpha-1)} (1 + \log^- x),
\end{aligned}
$$

where

$$
\varepsilon_0(\alpha) := \int_{1/2}^\infty (2\vartheta y)^{2s-1} \left[(2\vartheta y)^\alpha - 1\right] \frac{1 + |\log y|}{y^{1+2s}} dy.
$$

Notice that $\varepsilon_0(\alpha) \to 0$ as $\alpha \to 0$, since $(2\vartheta y)^\alpha \to 1$ as $\alpha \to 0$ for all $y > 1/2$ and Lebesgue dominated convergence theorem. Consequently, recalling that K has been fixed in (2.5.7), we choose $\alpha \in (0,1)$ in the following way: if $C > 0$ denotes the constant appearing in the statements of Lemma 2.5.4 and Lemma 2.5.5, we take α small such that

$$
\varepsilon_0(\alpha) < \frac{1}{3C}, \qquad \vartheta^{-\alpha} > 1 - \frac{1}{3C}. \tag{2.5.9}
$$

Notice that the second inequality above is guaranteed by (2.5.7). Now, writing $u_k^+ = u_k + u_k^-$ and using that $L_\Omega(x^{2s-1}) = 0$ in $(0,\infty)$, $\vartheta \ge 1$ and $\alpha \in (0,1)$, we estimate

$$
\begin{aligned}
|L_\Omega u_k^+(x)| \le |L_\Omega u(x)| + |L_\Omega u_k^-(x)| &\le K_0(1 + \log^- x) + \varepsilon_0(\alpha) C_s K \vartheta^{-k(\alpha-1)} (1 + \log^- x) \\
&\le \left[K_0 + \varepsilon_0(\alpha) C_s K\right] \vartheta^{-k(\alpha-1)} (1 + \log^- x),
\end{aligned}
$$

for all $x \in I_{\vartheta^{-k}/2}$. Consequently, we can apply Lemma 2.5.4 and Lemma 2.5.5 to u_k^+ and, recalling that $u_k^+ = u_k$ in $I_{\vartheta^{-k}}$, we deduce

$$
\begin{aligned}
\sup_{x \in I^+_{\vartheta^{-k}/2}} \left[\frac{u(x)}{x^{2s-1}} - m_k\right] &\le C \left\{ \inf_{x \in I^+_{\vartheta^{-k}/2}} \left[\frac{u(x)}{x^{2s-1}} - m_k\right] + (K_0 + \varepsilon_0(\alpha) K) \vartheta^{-k\alpha} \right\} \\
&\le C \left\{ \inf_{x \in I_{\vartheta^{-k}/4}} \left[\frac{u(x)}{x^{2s-1}} - m_k\right] + (K_0 + \varepsilon_0(\alpha) K) \vartheta^{-k\alpha} \right\}
\end{aligned}
$$

Now, defining

$$
u^k(x) := M_k x^{2s-1} - u(x),
$$

and repeating the above argument, we deduce

$$
\sup_{x \in I^+_{\vartheta^{-k}/2}} \left[M_k - \frac{u(x)}{x^{2s-1}}\right] \le C \left\{ \inf_{x \in I_{\vartheta^{-k}/4}} \left[M_k - \frac{u(x)}{x^{2s-1}}\right] + (K_0 + \varepsilon_0(\alpha) K) \vartheta^{-k\alpha} \right\}.
$$

Summing, it follows

$$\begin{aligned} M_k - m_k \le C\Big\{ &\inf_{x\in I^+_{\vartheta^{-k}/4}} \left[\frac{u(x)}{x^{2s-1}} - m_k\right] + \inf_{x\in I_{\vartheta^{-k}/4}} \left[M_k - \frac{u(x)}{x^{2s-1}}\right] \\ &+ (K_0 + \varepsilon_0(\alpha)K)\vartheta^{-k\alpha}\Big\} \\ = C\Big\{ &\inf_{x\in I_{\vartheta^{-k}/4}} \frac{u(x)}{x^{2s-1}} - \sup_{x\in I_{\vartheta^{-k}/4}} \frac{u(x)}{x^{2s-1}} + M_k - m_k \\ &+ (K_0 + \varepsilon_0(\alpha)K)\vartheta^{-k\alpha}\Big\}. \end{aligned}$$

In particular, we deduce

$$\begin{aligned} \sup_{x\in I_{\vartheta^{-(k+1)}}} \frac{u(x)}{x^{2s-1}} - \inf_{x\in I_{\vartheta^{-(k+1)}}} \frac{u(x)}{x^{2s-1}} &\le \frac{C-1}{C}(M_k - m_k) + (K_0 + \varepsilon_0(\alpha)K)\vartheta^{-k\alpha} \\ &= \left(\frac{C-1}{C} + \frac{K_0}{K} + \varepsilon_0(\alpha)\right) K\vartheta^{-k\alpha}, \end{aligned}$$

and so, thanks to (2.5.7) and (2.5.9), we find

$$\frac{C-1}{C} + \frac{K_0}{K} + \varepsilon_0(\alpha) \le \vartheta^{-\alpha}.$$

Consequently, choosing

$$M_{k+1} := \sup_{x\in I_{\vartheta^{-(k+1)}}} \frac{u(x)}{x^{2s-1}}, \qquad m_{k+1} := \inf_{x\in I_{\vartheta^{-(k+1)}}} \frac{u(x)}{x^{2s-1}},$$

the thesis follows. □

We can finally prove the following.

Theorem 2.5.7. *Let $n = 1$, $\Omega = (0,\infty)$, and $s \in (\frac{1}{2}, 1)$. Let L_Ω and K_Ω be given by either (2.2.1)-(2.2.2)-(2.2.3), or (2.1.7). Let $R > 0$ and $f \in L^\infty(I_{2R})$. Assume that*

$$\begin{cases} L_\Omega u = f & \text{in } I_{2R} \\ u(0) = 0, \end{cases}$$

and u satisfies (2.5.3) for some $c_0 > 0$, $\varepsilon_0 > 1$. Then the function

$$x \to \frac{u(x)}{x^{2s-1}}$$

can be continuously extended up to $x = 0$ and, furthermore, there exist $\alpha \in (0,1)$ and $C > 0$ (depending on s, c_0 and ε_0), such that

$$\left|\frac{u(x)}{x^{2s-1}} - \frac{u(y)}{y^{2s-1}}\right| \le CR^{1-2s}\left(\frac{|x-y|}{R}\right)^\alpha \left[\|u\|_{L^\infty(I_{2R})} + R^{2s}\|f\|_{L^\infty(I_{2R})}\right], \tag{2.5.10}$$

for all $x, y \in \overline{I_R}$.

Proof. We define $\delta(x) := x$, $v := u/\delta^{2s-1}$, $K_0 := \|f\|_{L^\infty(I_2)}$ and we set $R = 1$. First, from *Step 1* of the proof of Lemma 2.5.6, we have

$$\|v\|_{L^\infty(I_1)} \leq C_0 \left(\|u\|_{L^\infty(I_2)} + K_0 \right), \tag{2.5.11}$$

for some suitable $C_0 > 0$ depending only on s, c_0 and ε_0. Further, by Lemma 2.5.6, we have also (see (2.5.4))

$$\sup_{I_\varrho} v - \inf_{I_\varrho} v \leq C\varrho^\gamma \left[\|u\|_{L^\infty(I_2)} + K_0 \right], \tag{2.5.12}$$

for some $\gamma \in (0,1)$, $C > 0$ (depending only on s, c_0 and ε_0) and all $\varrho \in (0,1]$. In particular, notice that from (2.5.12) one can easily deduce that v can be continuously extended up to $x = 0$.

Now, for any $x \in I_1$, we set $J_r^+ := (x/2, 3x/2)$. Thus,

$$[u]_{C^{0,\beta}(\overline{J_r^+})} \leq Cr^{-\beta} \left[\|u\|_{L^\infty(I_2)} + K_0 \right],$$

for all $\beta \in (0,\beta_*)$ and some suitable $\beta_* \in (0,1)$ (cf. Theorem 2.4.1). On the other hand, it is not difficult to check that

$$\|\delta^{1-2s}\|_{L^\infty(\overline{J_r^+})} \leq C_s r^{1-2s}, \qquad [\delta^{1-2s}]_{C^{0,1}(\overline{J_r^+})} \leq C_s r^{-2s},$$

for some $C_s > 0$ depending only on s. As a consequence, by interpolation

$$[\delta^{1-2s}]_{C^{0,\beta}(\overline{J_r^+})} \leq C_s r^{1-2s-\beta},$$

for all $\beta \in (0,1)$. Thus, for any $\beta \in (0,\beta_*)$ and all $z, y \in J_r^+$ $(z \neq y)$, using the definition of v, it follows

$$\begin{aligned}
\frac{|v(z) - v(y)|}{|z-y|^\beta} &\leq \|\delta^{1-2s}\|_{L^\infty(\overline{J_r^+})} \frac{|u(z) - u(y)|}{|z-y|^\beta} + \|u\|_{L^\infty(\overline{I_2})} \frac{|\delta^{1-2s}(z) - \delta^{1-2s}(y)|}{|z-y|^\beta} \\
&\leq Cr^{1-2s-\beta} \left[\|u\|_{L^\infty(I_2)} + K_0 \right],
\end{aligned}$$

for some new constant $C > 0$, which implies

$$[v]_{C^{0,\beta}(\overline{J_r^+})} \leq Cr^{1-2s-\beta} \left[\|u\|_{L^\infty(I_2)} + K_0 \right], \tag{2.5.13}$$

for all $\beta \in (0,\beta_*)$. Now, we see how (2.5.11), (2.5.12), and (2.5.13) lead to

$$[v]_{C^{0,\alpha}(\overline{I_1})} \leq C \left[\|u\|_{L^\infty(I_2)} + K_0 \right],$$

for some $\alpha \in (0,1)$ depending only on s, c_0 and ε_0.

Given $x, y \in \overline{I_1}$, we suppose $x \geq y$, and set $\widetilde{\varrho} = x$, $\varrho = |x-y|$. Notice that thanks to (2.5.11), we can assume $\varrho \in (0,1)$. Finally, we fix

$$p > \frac{\beta + 2s - 1}{\beta},$$

where $\beta \in (0,\beta_*)$ as above. There are two possible cases:

Case 1. $\varrho \geq \tilde{\varrho}^p/2$. Then, thanks to (2.5.12),

$$
\begin{aligned}
|v(x)-v(y)| \leq |v(x)-v(0)| + |v(0)-v(y)| &\leq C\left[\|u\|_{L^\infty(I_2)} + K_0\right]\tilde{\varrho}^\gamma \\
&\leq C\varrho^{\gamma/p}\left[\|u\|_{L^\infty(I_2)} + K_0\right],
\end{aligned}
$$

and so it is enough to choose $\alpha = \gamma/p$.

Case 2. Assume $\varrho \leq \tilde{\varrho}^p/2$. Since $p > 1$, we see that $y \in J^+_{\tilde{\varrho}} = (x/2, 3x/2)$ and so, using (2.5.13), it follows

$$
|v(x)-v(y)| \leq C\tilde{\varrho}^{1-2s-\beta}\varrho^\beta\left[\|u\|_{L^\infty(I_2)} + K_0\right] \leq C\varrho^{\beta-\frac{\beta+2s-1}{p}}\left[\|u\|_{L^\infty(I_2)} + K_0\right],
$$

and so we complete the proof by choosing $\alpha := \min\left\{\frac{\gamma}{p}, \beta - \frac{\beta+2s-1}{p}\right\} > 0$. □

2.5.3 Proof of the Liouville theorem

First, as a consequence of the 1D boundary Harnack, we can deduce the following Neumann Liouville theorem in the half-line.

Corollary 2.5.8. *Let $n = 1$, $\Omega = (0,\infty)$, and $s \in (\frac{1}{2}, 1)$. Let L_Ω and K_Ω be given by either (2.2.1)-(2.2.2)-(2.2.3), or (2.1.7). Assume that*

$$
\begin{cases} L_\Omega u = 0 & \text{in } \mathbb{R}_+ \\ u(0) = 0, \end{cases} \tag{2.5.14}
$$

and u satisfies

$$
|u(y)| \leq c_0(1 + y^{2s-1+\varepsilon}), \quad y > 0, \tag{2.5.15}
$$

for some $c_0 > 0$ and $\varepsilon \in (0,\alpha)$, where $\alpha \in (0,1)$ is as in Theorem 2.5.7. Then,

$$
u(x) = Ax^{2s-1},
$$

for some $A \in \mathbb{R}$.

Furthermore, if in addition u satisfies (2.5.14) in the weak sense with Neumann condition (in the sense of Definition 2.2.2) at $x = 0$, then $u = 0$ in $\overline{\mathbb{R}_+}$.

Proof. From (2.5.15), we immediately see that

$$
\|u\|_{L^\infty(I_{2R})} \leq C_0(1 + R^{2s-1+\varepsilon}),
$$

for some $C_0 > 0$ depending only on s, c_0 and ε, and all $R > 0$. On the other hand, we notice that all the assumptions of Theorem 2.5.7 are satisfied (in particular, (2.5.15) implies (2.5.3)). Thus, setting $v(x) := u(x)/x^{2s-1}$, and combining (2.5.10) with the above inequality, it follows

$$
[v]_{C^{0,\alpha}(I_R)} \leq CR^{1-2s-\alpha}\|u\|_{L^\infty(I_{2R})} \leq CR^{\varepsilon-\alpha},
$$

for some new constant $C > 0$ and all $R > 0$. Since $\varepsilon \in (0,\alpha)$, we can pass to the limit as $R \to +\infty$ to deduce $[v]_{C^{0,\alpha}(\mathbb{R}_+)} = 0$, which trivially implies that $v = A$ for some $A \in \mathbb{R}$, i.e. the first part of our thesis.

To show the second part, we recall that u satisfies

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}[u(x)-u(y)][\eta(x)-\eta(y)]K_\Omega(x,y)dxdy=0,$$

for all $\eta\in C_0^\infty(\overline{\mathbb{R}_+})$ and, since $u\in C^\infty(\mathbb{R}_+)$ (see [153]), it satisfies $L_\Omega u=0$ in $\mathbb{R}_+$. Consequently, from the first part of the statement we deduce that $u(x)=Ax^{2s-1}$, for some $A\in\mathbb{R}$.

However, assume $A>0$ and take $\eta\in C_0^\infty((-\infty,1])$, with $\eta'\le 0$ and $\eta\not\equiv 0$. Using that $x\to x^{2s-1}$ is strictly increasing in $\mathbb{R}_+$, it follows

$$\begin{aligned}0&=A\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}[x^{2s-1}-y^{2s-1}][\eta(x)-\eta(y)]K_\Omega(x,y)dxdy\\&=A\int_{\{x<y\}}\underbrace{[x^{2s-1}-y^{2s-1}]}_{<0}\underbrace{[\eta(x)-\eta(y)]}_{\ge 0}K_\Omega(x,y)dxdy\\&\quad+A\int_{\{x\ge y\}}\underbrace{[x^{2s-1}-y^{2s-1}]}_{\ge 0}\underbrace{[\eta(x)-\eta(y)]}_{\le 0}K_\Omega(x,y)dxdy<0,\end{aligned}$$

since $\eta\not\equiv 0$ (similar if we assume $A<0$). This leads to a contradiction, unless $A=0$, and thus $u=0$. □

In order to extend the previous Neumann Liouville theorem to higher dimensions we need some preliminary lemmata. The first one concerns Hölder regularity of solutions in the half-space.

Lemma 2.5.9. *Let $\Omega=\mathbb{R}^n_+=\{x_n>0\}$, and $s\in(\frac{1}{2},1)$. Let L_Ω and K_Ω be given by either (2.2.1)-(2.2.2)-(2.2.3), or (2.1.7). Assume that v is a weak solution to*

$$L_\Omega v=0\qquad in\ \mathbb{R}^n_+$$

with Neumann condition on $\partial\mathbb{R}^n_+=\{x_n=0\}$ (in the sense of Definition 2.2.2). If

$$||v||_{L^\infty(B_R^+)}\le R^\sigma,\qquad R\ge 1,$$

for some $0<\sigma<2s$. Then

$$[v]_{C^\alpha(B_R^+)}\le CR^{\sigma-\alpha},\qquad R\ge 1,$$

for some constant $C>0$ depending only on n,s, and σ, and α as in Theorem 2.4.1.

Proof. As usual along this chapter, we are proving the result in the case L_Ω and K_Ω are given by (2.2.1)-(2.2.2)-(2.2.3). The other case is analogous, but without the logarithmic corrections.

The main idea is to apply Theorem 2.4.1 but, since v is not bounded, we first need to cut it in a suitable way in order to making use of the Hölder estimate. By scaling, it is enough to prove the result for the case $R=1$.

Let us define the auxiliary function $w(x)=v(x)\chi_{B_4}(x)$. It is clear, due to the growth condition on v, that this new function w is bounded in $\mathbb{R}^n_+$. Indeed,

$$||w||_{L^\infty(\mathbb{R}^n_+)}\le 4^\sigma.$$

First, we prove that w satisfies

$$L_\Omega w = f \quad \text{ in } B_2^+,$$

in the weak sense with Neumann condition on $\partial\mathbb{R}^n_+ \cap B_2$, where $f \in L^q(B_2^+)$ is a function which will be determined next. So, given any test function $\eta \in C_0^\infty(B_2)$ and using the equation satisfied by v we have

$$\begin{aligned}
B(w,\eta) &= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} (w(x)-w(y))(\eta(x)-\eta(y))\, K_\Omega(x,y)dxdy \\
&= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} (v(x)\chi_{B_4}(x)-v(y)\chi_{B_4}(y))(\eta(x)-\eta(y))\, K_\Omega(x,y)dxdy \\
&= \int_{B_4^+}\int_{B_4^+} (v(x)-v(y))(\eta(x)-\eta(y))\, K_\Omega(x,y)dxdy \\
&\quad + 2\int_{B_4^+} dx \int_{(B_4^c)^+} dy\, v(x)\eta(x)\, K_\Omega(x,y) \\
&= \int_{B_2^+}\left(2\int_{(B_4^c)^+} v(y)\, K_\Omega(x,y)dy\right)\eta(x)dx =: \int_{B_2^+} f(x)\eta(x)dx
\end{aligned}$$

Then, given any $x \in B_2^+$ we claim that f satisfies the following pointwise estimate

$$|f(x)| \le C\left(1+\log^-(x_n)\right),$$

for some positive constant C depending only on n, s and σ. In particular, it follows that $f \in L^q(B_2^+)$ for any $1 \le q < \infty$.

Now, if we apply Theorem 2.4.1 to w with $q = n/s$, and we take into account that $v \equiv w$ in B_2^+ we obtain

$$[v]_{C^\alpha(B_1^+)} = [w]_{C^\alpha(B_1^+)} \le C\left(||w||_{L^\infty(\mathbb{R}^n_+)} + ||f||_{L^q(B_2^+)}\right) \le C,$$

as we wanted.

Finally, let us prove the pointwise estimate for f. Letting $d = d_{x,y}$, using (2.2.5) and taking into account that $|y|/2 \le |x-y| \le 2|y|$ and $d \le |x-y|$ when $x \in B_2$ and $y \in B_4^c$, we have

$$\begin{aligned}
|f(x)| = 2\left|\int_{(B_4^c)^+} v(y)\, K_\Omega(x,y)dy\right| &\le C\int_{(B_4^c)^+} |y|^\sigma \frac{1+\log^-\left(\frac{d}{|x-y|}\right)}{|x-y|^{n+2s}}dy \\
&= C\int_{(B_4^c)^+}\frac{|y|^\sigma dy}{|x-y|^{n+2s}} + C\int_{(B_4^c)^+}|y|^\sigma \frac{\log\left(\frac{|x-y|}{d}\right)}{|x-y|^{n+2s}}dy \\
&\le C\int_{(B_4^c)^+}\frac{dy}{|y|^{n+2s-\sigma}} + C\int_{(B_4^c)^+}\frac{\log(2|y|)+|\log d|}{|y|^{n+2s}}dy \\
&\le C + C\int_{(B_4^c)^+\cap\{x_n \le y_n\}}\frac{\log|y|+|\log x_n|}{|y|^{n+2s}}dy \\
&\quad + C\int_{(B_4^c)^+\cap\{y_n < x_n\}}\frac{\log|y|+|\log y_n|}{|y|^{n+2s}}dy \le C\left(1+\log^-(x_n)\right),
\end{aligned}$$

for some positive constant C depending only on n, s and σ. Here, it is crucial the fact that $\sigma < 2s$ and $d(x) = x_n$ together with the integrability of $\log d$ close to $\partial\mathbb{R}^n_+$. □

Next step is proving that weak solutions to $L_\Omega v = 0$ in $\Omega = \mathbb{R}^n_+$ are linear functions.

Proposition 2.5.10. *Let* $\Omega = \mathbb{R}^n_+ = \{x_n > 0\}$, *and* $s \in (\frac{1}{2}, 1)$. *Let* L_Ω *and* K_Ω *be given by either* (2.2.1)-(2.2.2)-(2.2.3), *or* (2.1.7). *Assume* v *is a weak solution to*

$$L_\Omega v = 0 \quad \text{in } \mathbb{R}^n_+ = \{x_n > 0\}$$

with Neumann condition on $\partial\mathbb{R}^n_+ = \{x_n = 0\}$ *(in the sense of Definition 2.2.2). If*

$$||v||_{L^\infty(B_R^+)} \le c_0(1 + R^\sigma), \quad R > 0, \tag{2.5.16}$$

for some $c_0 > 0$ *and* $0 < \sigma < 2s$. *Then, there exist functions* $w_0, ..., w_{n-1}$ *such that*

$$v(x) = w_0(x_n) + \sum_{i=1}^{n-1} w_i(x_n)x_i.$$

Furthermore, $v(x) = w_0(x_n)$ *if* $\sigma < 1$.

Proof. Note that we can assume that $||v||_{L^\infty(B_R^+)} \le R^\sigma$ for every $R > 1$, after dividing by a suitable constant.

First, we prove that v is a polynomial in the first $n-1$ variables with coefficients depending on x_n, i.e.,

$$v(x) = \sum_{|j| \le n} a_j(x_n)\tilde{x}^j,$$

where $j = (j_1, ..., j_{n-1})$ is a multiindex and $\tilde{x}^j = x_1^{j_1} \cdots x_{n-1}^{j_{n-1}}$.

By Lemma 2.5.9 we know that $[v]_{C^\alpha(B_R)} \le CR^{\sigma-\alpha}$. Now, given any direction $e = (\tilde{e}, 0) \in S^{n-1}$ and any $h > 0$ we define the function

$$v_{h,1}^e(x) = \frac{v(x + he) - v(x)}{C|h|^\alpha},$$

where C is the positive constant appearing in the statement of Lemma 2.5.9. Then, since $e_n = 0$, it is clear that $v_{h,1}^e$ satisfies

$$\begin{cases} L_\Omega v_{h,1}^e = 0 & \text{in } \mathbb{R}^n_+ \\ ||v_{h,1}^e||_{L^\infty(B_R^+)} \le R^{\sigma-\alpha}, & R > 1. \end{cases}$$

Now, since $v_{h,1}^e$ satisfies the same equation of v and an "improved" growth condition, we can iterate this procedure and, defining recursively

$$v_{h,k}^e(x) = \frac{v_{h,k-1}^e(x + he) - v_{h,k-1}^e(x)}{C|h|^\alpha},$$

we obtain that $||v_{h,k}^e||_{L^\infty(B_R^+)} \le R^{\sigma-k\alpha}$. Therefore, if we choose $k \ge d+1 := \lceil \sigma/\alpha \rceil$ and take $R \to \infty$ we get that $v_{h,d+1}^e \equiv 0$ in $\mathbb{R}^n_+$. By definition, this means that the discrete differences of order d of v in every direction e are zero and thus v is a polynomial of degree d in the first $n-1$ variables. Furthermore, in view of (2.5.16) and that $\sigma < 2s < 2$, it follows $d = 1$ and therefore v has the form stated above. Indeed, since for any given $x_n > 0$, $v(\cdot, x_n)$ is a polynomial of degree d, then $||v(\cdot, x_n)||_{L^\infty(B_R^+)} \ge cR^d$, for some constant c depending on x_n and any $R > 1$. On the other hand, by (2.5.16) we obtain that $||v(\cdot, x_n)||_{L^\infty(B_R^+)} \le CR^\sigma$ with $\sigma < 2$. It thus follows that $d = 1$. Notice that when $\sigma < 1$ we get that $d = 0$ and so we conclude $v(x) = w_0(x_n)$. □

Lemma 2.5.11. *Let $\Omega = \mathbb{R}^n_+ = \{x_n > 0\}$, and $s \in (\frac{1}{2}, 1)$. Let B_Ω be given by (2.2.9) with K_Ω either of the form (2.2.2)-(2.2.3), or (2.1.7). Assume $v, \tilde{v} \in H_K(\mathbb{R}^n_+)$ and $\eta \in C_0^\infty(\mathbb{R}^n)$ are functions of the form $v(x) = x_i w(x_n)$ for some $i \in \{1, ..., n-1\}$, $\tilde{v}(x) = \tilde{w}(x_n)$ and $\eta(x) = \tilde{\eta}(\bar{x})\eta_n(x_n)$ with $x = (\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$. Then,*

$$B_{\mathbb{R}^n_+}(\tilde{v}, \eta) = \left(\int_{\mathbb{R}^{n-1}} \tilde{\eta}(z)\ dz \right) B_{\mathbb{R}_+}(\tilde{w}, \eta_n),$$

and

$$B_{\mathbb{R}^n_+}(v, \eta) = \left(\int_{\mathbb{R}^{n-1}} z_i \tilde{\eta}(z)\ dz \right) B_{\mathbb{R}_+}(w, \eta_n).$$

Proof. The proof comes from direct computation. On the one hand, if we use the form of $\tilde{v}$ and η, add and subtract the term $\eta_n(x_n)\tilde{\eta}(\bar{y})(\tilde{w}(x_n) - \tilde{w}(y_n))$ and rearrange them, we arrive at

$$\begin{aligned}
B_{\mathbb{R}^n_+}(\tilde{v}, \eta) &= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} (\tilde{v}(x) - \tilde{v}(y))(\eta(x) - \eta(y))\, K_{\mathbb{R}^n_+}(x, y)\ dxdy \\
&= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} (\tilde{w}(x_n) - \tilde{w}(y_n))(\tilde{\eta}(\bar{x})\eta_n(x_n) - \tilde{\eta}(\bar{y})\eta_n(y_n))\, K_{\mathbb{R}^n_+}(x, y)\ dxdy \\
&= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \eta_n(x_n)(\tilde{w}(x_n) - \tilde{w}(y_n))(\tilde{\eta}(\bar{x}) - \tilde{\eta}(\bar{y}))\, K_{\mathbb{R}^n_+}(x, y)\ dxdy \\
&\quad + \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \tilde{\eta}(\bar{y})(\tilde{w}(x_n) - \tilde{w}(y_n))(\eta_n(x_n) - \eta_n(y_n))\, K_{\mathbb{R}^n_+}(x, y)\ dxdy \\
&=: J_1 + J_2.
\end{aligned}$$

Now, one can conclude that $J_1 = 0$ due to the antisymmetry of the integrand with respect to the variables $\bar{x}$ and $\bar{y}$. Next, we can use the identity

$$\int_{\mathbb{R}^{n-1}} K_{\mathbb{R}^n_+}(\bar{x}, x_n, \bar{y}, y_n)\, d\bar{x} = K_{\mathbb{R}^+}(x_n, y_n), \tag{2.5.17}$$

which can be easily checked in both frameworks: K_Ω either of the form (2.2.2)-(2.2.3), or (2.1.7), in order to deduce that

$$J_2 = \left(\int_{\mathbb{R}^{n-1}} \tilde{\eta}(\bar{y})\ d\bar{y} \right) B_{\mathbb{R}_+}(\tilde{w}, \eta_n).$$

On the other hand, if we use the form of v and η, add and subtract again different terms and rearrange them, we arrive at

$$\begin{aligned}
B_{\mathbb{R}^n_+}(v,\eta) &= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{(v(x)-v(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\,dxdy\\
&= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} (x_i w(x_n) - y_i w(y_n))(\tilde{\eta}(\bar{x})\eta_n(x_n) - \tilde{\eta}(\bar{y})\eta_n(y_n))\, K_{\mathbb{R}^n_+}(x,y)\,dxdy\\
&= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \eta_n(x_n)(w(x_n)-w(y_n))y_i(\tilde{\eta}(\bar{x})-\tilde{\eta}(\bar{y}))\, K_{\mathbb{R}^n_+}(x,y)\,dxdy\\
&\quad + \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} w(x_n)(\eta_n(x_n)-\eta_n(y_n))(x_i-y_i)\tilde{\eta}(\bar{y})\, K_{\mathbb{R}^n_+}(x,y)\,dxdy\\
&\quad + \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} w(x_n)\eta_n(x_n)(x_i-y_i)(\tilde{\eta}(\bar{x})-\tilde{\eta}(\bar{y}))\, K_{\mathbb{R}^n_+}(x,y)\,dxdy\\
&\quad + \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} y_i\tilde{\eta}(\bar{y})(w(x_n)-w(y_n))(\eta_n(x_n)-\eta_n(y_n))\, K_{\mathbb{R}^n_+}(x,y)\,dxdy\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Now, we show that the first three integrals are zero while the last one give us the desired result. That is, by symmetrization with respect to the variables $\bar{x}$ and $\bar{y}$ and the translation invariance and odd symmetry of the kernel $K_{\mathbb{R}^n_+}(x,y)$ in the first $n-1$ variables, we get

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \eta_n(x_n)(w(x_n)-w(y_n))y_i(\tilde{\eta}(\bar{x})-\tilde{\eta}(\bar{y}))\, K_{\mathbb{R}^n_+}(x,y)\,dxdy\\
&= -\frac{1}{2}\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \eta_n(x_n)(w(x_n)-w(y_n))(x_i-y_i)(\tilde{\eta}(\bar{x})-\tilde{\eta}(\bar{y}))\, K_{\mathbb{R}^n_+}(x,y)\,dxdy\\
&= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \eta_n(x_n)(w(x_n)-w(y_n))(x_i-y_i)\tilde{\eta}(\bar{y})\, K_{\mathbb{R}^n_+}(x,y)\,dxdy\\
&= \int_{\mathbb{R}^n_+}\int_0^\infty\int_{\mathbb{R}^{n-1}} \eta_n(x_n)(w(x_n)-w(y_n))z_i\tilde{\eta}(\bar{y})\, K_{\mathbb{R}^n_+}(z+\bar{y},x_n,\bar{y},y_n)\,dzdx_ndy\\
&= 0
\end{aligned}$$

The computations of I_2 and I_3 are completely analogous, although we do not have to do the first symmetrization. Next, we proceed with I_4. By using again the identity (2.5.17) we arrive at

$$\begin{aligned}
I_4 &= \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} y_i\tilde{\eta}(\bar{y})(w(x_n)-w(y_n))(\eta_n(x_n)-\eta_n(y_n))\, K_{\mathbb{R}^n_+}(x,y)\,dxdy\\
&= \left(\int_{\mathbb{R}^{n-1}} y_i\tilde{\eta}(\bar{y})\,d\bar{y}\right) B_{\mathbb{R}_+}(w,\eta_n).
\end{aligned}$$

□

Finally we present the proof of Theorem 2.5.1.

Proof of Theorem 2.5.1. First, by applying Proposition 2.5.10 with $\sigma = 2s-1+\varepsilon$ we know that

$$v(x) = w_0(x_n) + \sum_{i=1}^{n-1} w_i(x_n)x_i.$$

Now, we are going to take advantage of Lemma 2.5.11 to prove that every w_i satisfies

$$L_\Omega w_i = 0 \quad \text{in } \mathbb{R}_+ \tag{2.5.18}$$

in the weak sense with Neumann boundary condition at 0. To do this, let us take any test function with separated variables, i.e., $\eta(z) = \tilde{\eta}(\bar{z})\eta_n(z_n)$. Then, by applying Lemma 2.5.11 and the fact that v is a weak solution of the problem $(B_{\mathbb{R}^n_+}(v,\eta) = 0)$, we obtain

$$B_{\mathbb{R}_+}(w_0,\eta_n)\int_{\mathbb{R}^{n-1}} \tilde{\eta}(z)\ dz + \sum_{i=1}^{n-1}\left(B_{\mathbb{R}_+}(w_i,\eta_n)\int_{\mathbb{R}^{n-1}} z_i\tilde{\eta}(z)\ dz\right) = 0,$$

for any given $\tilde{\eta} \in C_0^\infty(\mathbb{R}^{n-1})$ and $\eta_n \in C_0^\infty(\mathbb{R}_+)$.

We claim that this equality is equivalent to $B_{\mathbb{R}_+}(w_i,\eta_n) = 0$ for any $\eta_n \in C_0^\infty(\mathbb{R}_+)$ and therefore that w_i satisfies (2.5.18), as we wanted. In order to show that we only need to choose $\tilde{\eta}$ properly. On the one hand, by taking a radial $\tilde{\eta}$, we get that $B_{\mathbb{R}_+}(w_0,\eta_n) = 0$. On the other, if we choose the test function $\tilde{\eta}$ to be odd with respect to the i^{th}-variable and even with respect to the others we conclude $B_{\mathbb{R}_+}(w_i,\eta_n) = 0$ for $i > 0$.

Moreover, it is clear that each w_i satisfies the same growth condition as v, i.e., $||w_i||_{L^\infty(B_R^+)} \leq c_0(1 + R^{2s-1+\varepsilon})$ for any $R > 0$ and so, applying Corollary 2.5.8 to each w_i, we obtain the desired result:

$$v(x) = a + \sum_{i=1}^{n-1} b_i x_i,$$

as wanted. □

2.6 Higher regularity by blow-up

The aim of this final section is to establish a $C^{2s-1+\alpha}$ estimate (in case $s > \frac{1}{2}$), by combining the C^α estimate from Section 2.4, a blow-up argument in the spirit of [165], and the Liouville theorem with nonlocal Neumann conditions established in Section 2.5.

We will also need the following.

Lemma 2.6.1. *Let $\Omega \subset \mathbb{R}^n$ be any Lipschitz domain, $f \in L^2_{loc}(\Omega)$ and $x_0 \in \Omega$. Let L_Ω and K_Ω be given by either (2.2.1)-(2.2.2)-(2.2.3), or (2.1.7). Assume that u satisfies*

$$L_\Omega u = f \quad \text{in } \Omega$$

with Neumann conditions on $\partial\Omega$. Assume that

$$|u(x)| \leq M_0(1 + |x|^{s-\varepsilon}) \quad \text{in} \quad \mathbb{R}^n.$$

Then, for any $0 < r < R$ and any $x_0 \in \Omega$, we have

$$[u]^2_{H_K(D_r(x_0))} \leq C\left\{\|f\|^2_{L^2(D_R(x_0))} + M_0^2\right\},$$

with C depending only on n, s, x_0, ε, r and R.

Proof. Fix $x_0 \in \Omega$ and $0 < r < R$. Let $\varphi \in C_0^\infty(B_R(x_0))$, such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ in $B_r(x_0)$. Testing the weak formulation with $\eta = u\varphi$, we obtain

$$B(u,\eta) := \int_\Omega\int_\Omega [u(x) - u(y)][u(x)\varphi(x) - u(y)\varphi(y)]K_\Omega(x,y)dxdy = \int_\Omega fu\varphi dx.$$

Writing

$$u(x)\varphi(x) - u(y)\varphi(y) = [u(x) - u(y)]\varphi(x) + u(y)[\varphi(x) - \varphi(y)],$$

we deduce by symmetry

$$2[u(x)-u(y)][u(x)\varphi(x)-u(y)\varphi(y)] = [u(x)-u(y)]^2[\varphi(x)+\varphi(y)]+[u^2(x)-u^2(y)][\varphi(x)-\varphi(y)].$$

Consequently, using the symmetry of K_Ω and the definition of φ, it follows

$$\begin{aligned} 2B(u,\eta) &= \int_\Omega \int_\Omega [u(x) - u(y)]^2[\varphi(x) + \varphi(y)]K_\Omega(x,y)dxdy \\ &\quad + \int_\Omega \int_\Omega [u^2(x) - u^2(y)][\varphi(x) - \varphi(y)]K_\Omega(x,y)dxdy \\ &\geq 2[u]^2_{H_K(D_r(x_0))} - 2\int_\Omega u^2(x)|L_\Omega\varphi(x)|dx. \end{aligned}$$

Now, since $\varphi \in C_0^\infty(B_R(x_0))$, we claim hat

$$\int_\Omega u^2(x)|L_\Omega\varphi(x)|dx \leq CM_0^2\int_\Omega (1 + |x|^{2s-2\varepsilon})|L_\Omega\varphi(x)|dx \leq CM_0^2, \tag{2.6.1}$$

for some constant C depending on Ω, n, s, R, ε, and x_0. If (2.6.1) holds, then

$$[u]^2_{H^s(D_r(x_0))} \leq \int_{D_R(x_0)} fudx + CM_0^2,$$

and combining Young's inequality with the growth condition on u we complete the proof. Hence, it only remains to prove (2.6.1).

Let us estimate $|L_\Omega\varphi|$. For this, notice first that since φ is Lipschitz, then

$$|L_\Omega\varphi(x)| \leq C\int_\Omega |x - y|\, K_\Omega(x,y)dy,$$

which gives a universal bound whenever $s < \frac{1}{2}$. However, in case $s \geq \frac{1}{2}$ the bound is nontrivial, since we cannot immediately symmetrize the integral. In that case, we separate the proof into two cases.

- *Assume first that L_Ω is given by* (2.1.7). Let $x \in B_{2R}(x_0)$ and $d = d(x)$. Then,

$$\begin{aligned} L_\Omega\varphi(x) &= PV\int_{B_d(x)} (\varphi(x) - \varphi(y))K_\Omega(x,y)dy + \int_{\Omega\setminus B_d(x)} (\varphi(x) - \varphi(y))K_\Omega(x,y)dy \\ &:= I + J. \end{aligned}$$

By the regularity of φ and symmetry of K_Ω inside $B_d(x)$, it follows that

$$|I| \leq \int_{B_d} \frac{|2\varphi(x) - \varphi(x-y) - \varphi(x+y)|}{|y|^{n+2s}}dy \leq C\int_{B_d} \frac{dy}{|y|^{n+2s-2}}dy \leq C,$$

for some constant depending on n, s, Ω and φ. Further, since φ is Lipschitz, we obtain

$$|J| \leq \int_{\Omega\setminus B_d(x)} |\varphi(x) - \varphi(y)|K_\Omega(x,y)dy \leq C\int_{\mathbb{R}^n\setminus B_d(x)} \frac{dy}{|x-y|^{n+2s-1}} \leq Cd(x)^{1-2s},$$

with C depending only on n, s and φ.

Consequently, we have proved

$$|L_\Omega\varphi(x)| \leq C(1 + d^{1-2s}(x)), \quad x \in B_{2R}(x_0). \tag{2.6.2}$$

Now, since φ has compact support in $B_R(x_0)$, for all $x \in B_{2R}(x_0)^c$ we find

$$|L_\Omega\varphi(x)| \leq \int_\Omega |\varphi(y)|K_\Omega(x,y)dy \leq C\int_{\text{supp}\varphi} \frac{dy}{|x-y|^{n+2s}} \leq \frac{C}{(1+|x|)^{n+2s}}. \tag{2.6.3}$$

Thus, combining (2.6.2) and (2.6.3), (2.6.1) follows.

- *Assume now that L_Ω is given by* (2.2.1)-(2.2.2)-(2.2.3). For $x \in B_{2R}(x_0)$ we have

$$\begin{aligned} L_\Omega\varphi(x) &= PV\int_{B_{d/2}(x)} (\varphi(x)-\varphi(y))K_\Omega(x,y)dy + \int_{\Omega\setminus B_{d/2}(x)} (\varphi(x)-\varphi(y))K_\Omega(x,y)dy \\ &:= I + J, \end{aligned}$$

and

$$I = c_{n,s}PV\int_{B_{d/2}(x)} \frac{\varphi(x)-\varphi(y)}{|x-y|^{n+2s}}dy + PV\int_{B_{d/2}(x)} (\varphi(x)-\varphi(y))k_\Omega(x,y)dy.$$

Exactly as above, the first integral is bounded, by symmetry. Moreover, thanks to Proposition 2.2.1, in $B_{d/2}(x)$ we have $|k_\Omega(x,y)| \leq Cd^{-n-2s}$, and thus since φ is Lipschitz we deduce that

$$|I| \leq C(1 + d^{1-2s}(x)).$$

On the other hand, using (2.2.5) and the fact that φ is Lipschitz, it is not difficult to see that

$$|J| \leq C\int_{\Omega\setminus B_{d/2}(x)} |x-y|\,\frac{1+\log^-\left(\frac{d_{x,y}}{|x-y|}\right)}{|x-y|^{n+2s}}dy \leq C(1+|\log d(x)|)(1+d(x)^{1-2s}).$$

Therefore,

$$|L_\Omega\varphi(x)| \leq C(1+|\log d(x)|)(1+d^{1-2s}(x)), \quad x \in B_{2R}(x_0). \tag{2.6.4}$$

Finally, a similar computation shows that for $x \in B^c_{2R}(x_0)$ we have

$$|L_\Omega\varphi(x)| \leq \int_\Omega |\varphi(y)|K_\Omega(x,y)dy \leq C\int_{\text{supp}\varphi} K_\Omega(x,y)\,dy \leq \frac{C|\log d(x)|}{(1+|x|)^{n+2s}}, \tag{2.6.5}$$

and thus (2.6.1) follows. □

We can now proceed with the blow-up argument.

Proposition 2.6.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain, $s > \frac{1}{2}$, and $f \in L^q(\Omega)$ with $q > n$. Let L_Ω and K_Ω be given by either* (2.2.1)-(2.2.2)-(2.2.3), *or* (2.1.7). *Assume that $u \in H_K(\Omega)$ is a weak solution to*

$$L_\Omega u = f \quad \text{in } \Omega,$$

with Neumann conditions on $\partial\Omega$ in the sense of Definition 2.2.2.

Then, there exist C and $\gamma > 0$, depending only on n, s, q and Ω, such that for any $z \in \partial\Omega$ and $x \in \Omega$, we have

$$|u(x)-u(z)| \leq C|x-z|^{2s-1+\gamma}\left[\|u\|_{L^\infty(\Omega)} + \|f\|_{L^q(\Omega)}\right]. \tag{2.6.6}$$

In particular, for any $z \in \partial\Omega$,

$$\lim_{\lambda \to 0^+} \frac{u(z) - u(z - \lambda\nu(z))}{\lambda^{2s-1}} = 0, \tag{2.6.7}$$

where $\nu(z)$ denotes the exterior unit normal to $\partial\Omega$ at z.

Proof. Recall that, thanks to Proposition 2.3.2, we have $u \in L^\infty(\Omega)$. So, dividing u by a constant if necessary, we may assume that $\|u\|_{L^\infty(\Omega)} + \|f\|_{L^q(\Omega)} \leq 1$, and (2.6.6) can be written as

$$|u(x) - u(z)| \leq C|x - z|^{2s-1+\gamma}, \tag{2.6.8}$$

for all $x \in \Omega$ and $z \in \partial\Omega$. Now, we prove (2.6.8) with a blow-up and contradiction argument, for some $\gamma > 0$ small enough, to be chosen later.

Assume by contradiction that there are sequences:

- $(u_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ of weak solutions to $L_\Omega u_k = f_k$ in Ω with Neumann conditions on $\partial\Omega$, satisfying $\|u_k\|_{L^\infty(\Omega)} + \|f_k\|_{L^q(\Omega)} \leq 1$ for all $k \in \mathbb{N}$,
- $(x_k)_{k\in\mathbb{N}} \in \Omega$ and $(z_k)_{k\in\mathbb{N}} \in \partial\Omega$,
- and $C_k \to +\infty$ as $k \to +\infty$, such that

$$\frac{|u_k(x_k) - u_k(z_k)|}{|x_k - z_k|^\sigma} \geq C_k, \tag{2.6.9}$$

where $\sigma := 2s - 1 + \gamma$.

It follows $|x_k - z_k| \to 0$ as $k \to +\infty$ and so, up to passing to a subsequence, $x_k, z_k \to z_0$ as $k \to +\infty$, for some suitable $z_0 \in \partial\Omega$.

Now, the function

$$\vartheta(r) := \sup_{k\in\mathbb{N}} \vartheta_k(r) := \sup_{k\in\mathbb{N}} \max_{\varrho \geq r} \varrho^{-\sigma} \|u_k - u_k(z_k)\|_{L^\infty(B_\varrho(z_k))}$$

is clearly monotone non-increasing and, thanks to (2.6.9), it satisfies $\vartheta(r) \to +\infty$ as $r \to 0^+$, that is

$$\sup_{k\in\mathbb{N}} \sup_{r>0} r^{-\sigma} \|u_k - u_k(z_k)\|_{L^\infty(B_r(z_k))} = +\infty. \tag{2.6.10}$$

Indeed, choosing $r_k = |x_k - z_k|$, we have

$$\vartheta_k(r_k) \geq r_k^{-\sigma} \|u_k - u_k(z_k)\|_{L^\infty(B_{r_k}(z_k))} \geq \frac{|u_k(x_k) - u_k(z_k)|}{|x_k - z_k|^\sigma},$$

and thus, in view of (2.6.9), we can pass to the limit as $k \to +\infty$ and (2.6.10) follows.

Furthermore, by the definition of ϑ we deduce the existence of two sequences $r_j \to 0^+$ and $(k_j)_{j\in\mathbb{N}}$ such that

$$r_j^{-\sigma} \|u_{k_j} - u_{k_j}(z_{k_j})\|_{L^\infty(B_{r_j}(z_{k_j}))} \geq \frac{\vartheta(r_j)}{2}, \quad j \in \mathbb{N}. \tag{2.6.11}$$

Step 1: Blow-up sequence. Now, we introduce the blow-up sequence

$$v_j(x) := \frac{u_{k_j}(r_j x + z_{k_j}) - u_{k_j}(z_{k_j})}{r_j^\sigma \vartheta(r_j)}, \quad j \in \mathbb{N},$$

which satisfies $v_j(0) = 0$ for all $j \in \mathbb{N}$ and

$$\|v_j\|_{L^\infty(B_1)} \geq \frac{1}{2}, \quad \text{for all } j \in \mathbb{N}, \tag{2.6.12}$$

thanks to (2.6.11). Further, for any $R \geq 1$, we have

$$\|v_j\|_{L^\infty(B_R)} = \frac{1}{r_j^\sigma \vartheta(r_j)} \|u_{k_j} - u_{k_j}(z_{k_j})\|_{L^\infty(B_{r_j R}(z_{k_j}))} \leq \frac{1}{r_j^\sigma \vartheta(r_j)} (r_j R)^\sigma \vartheta(r_j R) \leq R^\sigma,$$

where we have used the definition of ϑ and its monotonicity: $\vartheta(r_j R) \leq \vartheta(r_j)$ for $j \in \mathbb{N}$ and all $R \geq 1$. Thus:

$$\|v_j\|_{L^\infty(B_R)} \leq R^\sigma, \quad j \in \mathbb{N}, \ R \geq 1. \tag{2.6.13}$$

On the other hand, each v_j satisfies

$$L_j v_j(x) = \frac{r_j^{2s-\sigma}}{\vartheta(r_j)} f(r_j x + z_{k_j}) := \widetilde{f}_j(x), \quad x \in \Omega_j := r_j^{-1}(z_{k_j} - \Omega), \tag{2.6.14}$$

in the weak sense with Neumann conditions on $\partial\Omega_j$, where $L_j := L_{\Omega_j}$, and

$$\|\widetilde{f}_j\|_{L^q(\Omega_j)} \leq \|f\|_{L^q(\Omega)} \frac{r_j^{2s-\frac{n}{q}-\sigma}}{\vartheta(r_j)}, \quad \text{for all } j \in \mathbb{N}. \tag{2.6.15}$$

Now, fix $R \geq 1$ and define $w_j := v_j \chi_{B_{4R}}$, $j \in \mathbb{N}$. Following the proof of Lemma 2.5.9 and setting $D_R^j := B_R \cap \Omega_j$, it is not difficult to verify that

$$L_{\Omega_j} w_j = \overline{f}_j \quad \text{in } D_{2R}^j,$$

where

$$\overline{f}_j := \widetilde{f}_j + 2 \int_{\Omega_j \setminus B_{4R}} v_j(y)\, K_{\Omega_j}(x,y) dy.$$

Using (2.6.15) and that $q > n$, we can choose $\gamma > 0$ small enough so that $2s - n/q - \sigma > 0$, and thus $\|\widetilde{f}_j\|_{L^q(\Omega_j)}$ is uniformly bounded. Further, using (2.6.13) and repeating the proof of Lemma 2.5.9, we find that also the second term in the definition of $\overline{f}_j$ is bounded in $L^q(D_{2R}^j)$, uniformly w.r.t. j (recall that we can reduce consider the case $\Omega_j = \mathbb{R}^n_+$ by using a local bi-Lipschitz transformation of Ω_j). In particular, $\overline{f}_j$ is bounded in $L^q(D_{2R}^j)$, uniformly w.r.t. j and thus Theorem 2.4.1 implies

$$[w_j]_{C^\alpha(D_R^j)} \leq C R^{-\alpha} \left[\|w_j\|_{L^\infty(\Omega_j)} + R^{2s-\frac{n}{q}} \|\overline{f}_j\|_{L^q(D_{2R}^j)} \right].$$

By the argument above and since $\|w_j\|_{L^\infty(\Omega_j)} = \|v_j\|_{L^\infty(\Omega_j \cap B_{4R})} \leq C R^\sigma$ (see (2.6.13)), it follows that $[w_j]_{C^\alpha(D_R^j)} \leq C_R$ for all $j \in \mathbb{N}$ and some constant $C_R > 0$ (independent of j). In particular, since $w_j = v_j$ in D_R^j, we obtain

$$[v_j]_{C^\alpha(D_R^j)} \leq C_R. \tag{2.6.16}$$

Moreover, choosing $\gamma > 0$ small enough so that $\sigma < s$, we combine Lemma 2.6.1, (2.6.13) and (2.6.15), to deduce

$$[v_j]^2_{H^s(D_R^j)} \leq C_R, \tag{2.6.17}$$

for any fixed $R \geq 1$ and some new constant $C_R > 0$ independent of $j \in \mathbb{N}$.

Step 2: Compactness. Using simultaneously (2.6.13), (2.6.16), the fact that Ω is of class C^1 together with $z_{k_j} \to z_0 \in \partial\Omega$, and the Ascoli-Arzelà theorem, it follows that for any $R \geq 1$ and any $\nu \in (0, \alpha)$,

$$v_j \to v,$$

uniformly in $\overline{B_R \cap H}$ (and in C^ν), where $H := \{e \cdot x > 0\}$, for some unit vector e depending on $z_0 \in \partial\Omega$. Moreover, $v \in C^\nu(\overline{B_R \cap H})$ and $v(0) = 0$. Further, in view of (2.6.17), the sequence $\{v_j\}_{j\in\mathbb{N}}$ is uniformly bounded in $H_{K,loc}(\overline{\Omega}_j)$ and so $v \in H_{K,loc}(\overline{H})$.

Notice also that by uniform convergence, we obtain that v satisfies

$$\|v\|_{L^\infty(B_1)} \geq \frac{1}{2}, \qquad \|v\|_{L^\infty(B_R)} \leq R^\sigma, \qquad \text{for all } R \geq 1, \tag{2.6.18}$$

once we pass to the limit in (2.6.12) and (2.6.13).

Step 3: Passage to the limit into the equation. Since the v_j's satisfy (2.6.14) in the weak sense with Neumann conditions on $\partial\Omega_j$, they satisfy the same equation in the distributional sense, that is

$$\int_{\Omega_j} v_j L_j \eta dx = \frac{1}{2}\int_{\Omega_j} \tilde{f}_j \eta, \tag{2.6.19}$$

for all $\eta \in C_0^\infty(\mathbb{R}^n)$, and all $j \in \mathbb{N}$. To justify this, we fix $\eta \in C_0^\infty(\mathbb{R}^n)$, $j \in \mathbb{N}$, $\varepsilon \in (0,1)$ and we notice that, by the symmetry of the kernel, we have

$$\begin{aligned}\int_{\Omega_j} v_j(x) \int_{\Omega_j \setminus B_\varepsilon(x)} [\eta(x) - \eta(y)] K_{\Omega_j}(x,y) dy dx &= \iint_{D_j^\varepsilon} v_j(x)[\eta(x) - \eta(y)] K_{\Omega_j}(x,y) dx dy \\ &= \frac{1}{2}\iint_{D_j^\varepsilon} [v_j(x) - v_j(y)][\eta(x) - \eta(y)] K_{\Omega_j}(x,y) dx dy,\end{aligned} \tag{2.6.20}$$

where $D_j^\varepsilon := \{(x,y) \in \Omega_j \times \Omega_j : |x-y| > \varepsilon\}$. For any $x \in \Omega_j$, we define

$$L_j^\varepsilon \eta(x) := \int_{\Omega_j \setminus B_\varepsilon(x)} [\eta(x) - \eta(y)] K_{\Omega_j}(x,y) dy.$$

Notice that $L_j^\varepsilon \eta \to L_j \eta$ a.e. in $\mathbb{R}^n$ as $\varepsilon \to 0^+$ and

$$|L_j^\varepsilon \eta(x)| \leq \frac{h_j(x)}{(1+|x|)^{n+2s}}, \tag{2.6.21}$$

for some $h_j \in L^1_{loc}(\mathbb{R}^n)$ independent of $\varepsilon \in (0,1)$; see (2.6.2)-(2.6.3) and (2.6.4)-(2.6.5) in the proof of Lemma 2.6.1. Noticing that the function $x \to (1+|x|)^{-n-\alpha} h_j(x)$ belongs to $L^1(\mathbb{R}^n)$ for any $\alpha > 0$, recalling (2.6.13) and that $\sigma < s$, we can pass to the limit into (2.6.20) to obtain

$$\int_{\Omega_j} v_j(x) \int_{\Omega_j \setminus B_\varepsilon(x)} [\eta(x) - \eta(y)] K_{\Omega_j}(x,y) dy dx \to \int_{\Omega_j} v_j L_j \eta dx$$

as $\varepsilon \to 0$, thanks to the dominated convergence theorem. On the other hand, since $D_j^\varepsilon \to \Omega_j \times \Omega_j$, we find

$$\iint_{D_j^\varepsilon} [v_j(x) - v_j(y)][\eta(x) - \eta(y)] K_{\Omega_j}(x,y) dx dy \to B_{\Omega_j}(v_j, \eta) = \int_{\Omega_j} \tilde{f}_j \eta,$$

and so, in view of (2.6.20), (2.6.19) is proved.

Now, we fix an arbitrary $\eta \in C_0^\infty(\mathbb{R}^n)$ and we pass to the limit as $j \to +\infty$ in (2.6.19). Using (2.6.15) and that $2s - n/q - \sigma > 0$, the right hand side of the equation converges to 0 as $j \to +\infty$. Further, using that $\chi_j \to \chi_H$ and $K_{\Omega_j} \to K_H$ a.e. in $\mathbb{R}^n$, we apply the Vitali's convergence theorem (here we use again (2.6.2)-(2.6.3) and (2.6.4)-(2.6.5)), to deduce $L_j\eta \to L_H\eta$ a.e. in $\mathbb{R}^n$. Writing

$$\left|\int_{\Omega_j} v_j L_j\eta dx - \int_H vL_H\eta dx\right| \le \left|\int_{\Omega_j} v_j(L_j\eta - L_H\eta)dx\right| + \left|\int_H (v_j - v)L_H\eta dx\right| := I_j + \overline{I}_j,$$

we easily see that both I_j and $\overline{I}_j$ go to 0 as $j \to +\infty$. Indeed, since $L_j\eta \to L_H\eta$, the v_j's satisfy (2.6.13) and $\sigma < 2s$, we obtain $I_j \to 0$ as $j \to +\infty$, applying the Vitali's convergence theorem again. Similar for $\overline{I}_j$, using that $v_j \to v$ uniformly on compact sets of $\mathbb{R}^n$.

As a consequence, we can pass to the limit and deduce that v satisfies

$$\int_H v\, L_H\eta dx = 0, \qquad \text{for all } \eta \in C_0^\infty(\mathbb{R}^n). \tag{2.6.22}$$

From interior regularity estimates and (2.6.17), we know that $v \in C^\infty(H) \cap H_{K,loc}(\overline{H})$ and thus v is a weak solution to

$$L_H v = 0 \quad \text{ in } H, \tag{2.6.23}$$

with Neumann conditions on ∂H in the sense of Definition 2.2.2. Indeed, let $\eta \in C_0^\infty(\mathbb{R}^n)$ and set

$$L_H^\varepsilon \eta(x) := \int_{H\setminus B_\varepsilon(x)} [\eta(x) - \eta(y)]K_H(x,y)\, dy.$$

By (2.6.20), we have

$$\int_H v(x)L_H^\varepsilon\eta(x)dx = \frac{1}{2}\iint_{D^\varepsilon} [v(x) - v(y)][\eta(x) - \eta(y)]K_H(x,y)dxdy, \tag{2.6.24}$$

where $D^\varepsilon := \{(x,y) \in H \times H : |x-y| > \varepsilon\}$. Now, proceeding as above, it follows

$$\int_H v(x)L_H^\varepsilon\eta(x)dx \to \int_H v(x)L_H\eta(x)dx,$$

as $\varepsilon \to 0^+$ and so, in view of (2.6.22) and the fact that $D^\varepsilon \to H\times H$ as $\varepsilon \to 0^+$, we obtain

$$\int_H\int_H [v(x) - v(y)][\eta(x) - \eta(y)]K_H(x,y)dxdy = 0.$$

Recalling that $v \in H_{K,loc}(\overline{H})$, (2.6.23) follows.

Step 4: Conclusion. In view of (2.6.18) and Theorem 2.5.1, we deduce that v is constant in H. On the other hand, recalling that $v(0) = 0$, it must be $v \equiv 0$ in H, a contradiction with (2.6.18). □

We will also need the following observation.

Lemma 2.6.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain, $\sigma \in (0, 2s)$, and assume that u satisfies:*

- $|u| \le C_0$ *in* Ω,
- $\mathcal{N}_s u = 0$ *in* Ω^c,
- $|u(x) - u(z)| \le C_0|x-z|^\sigma$ *for all* $z \in \partial\Omega$, $x \in \Omega$.

Then, we have

$$|u(x) - u(z)| \le CC_0|x - z|^\sigma \quad \textit{for all} \quad z \in \partial\Omega, \quad x \in \mathbb{R}^n. \tag{2.6.25}$$

The constant C depends only on Ω.

Proof. Notice that, since $\mathcal{N}_s u = 0$ in Ω^c, then

$$u(x) \int_\Omega \frac{dy}{|x-y|^{n+2s}} = \int_\Omega \frac{u(y)}{|x-y|^{n+2s}}\, dy,$$

for all $x \in \Omega^c$, and thus

$$(u(x) - u(z)) \int_\Omega \frac{dy}{|x-y|^{n+2s}} = \int_\Omega \frac{u(y) - u(z)}{|x-y|^{n+2s}}\, dy,$$

for any $z \in \partial\Omega$.

When $d(x) > 1$ the bound (2.6.25) holds trivially, so we will assume $d(x) \le 1$. In that case, by [1, Lemma 2.1] we have

$$\int_\Omega \frac{dy}{|x-y|^{n+2s}} \asymp d^{-2s}(x).$$

Moreover, since Ω is C^1, choosing z to be the projection of y onto $\partial\Omega$, we have

$$\int_\Omega \frac{|u(y) - u(z)|}{|x-y|^{n+2s}}\, dy \le C \int_\Omega \frac{|y-z|^\sigma}{|x-y|^{n+2s}}\, dy \le C \int_\Omega \frac{|y-z|^\sigma}{(d(x) + |y-z|)^{n+2s}}\, dy,$$

for some $C > 0$ depending on Ω. Since

$$\int_{\mathbb{R}^n} \frac{|y-z|^\sigma}{(A + |y-z|)^{n+2s}}\, dy \asymp A^{\sigma - 2s},$$

we deduce

$$\int_\Omega \frac{|u(y) - u(z)|}{|x-y|^{n+2s}}\, dy \le C d^{\sigma-2s}(x) = C|x-z|^{\sigma-2s}.$$

Combining the previous estimates, the result follows. □

Finally, to prove Theorems 2.1.1 and 2.1.3, we will also need the following interior regularity results. The first one is probably well known, we give a short proof for completeness.

Lemma 2.6.4. *Let $n \ge 2$ and $s > \frac{1}{2}$. Assume that $u \in L^\infty(B_1)$, $(1+|x|)^{-n-2s}u(x) \in L^1(\Omega)$, satisfies*

$$(-\Delta)^s u = f \quad \textit{in} \quad B_1,$$

for some $f \in L^q(B_1)$ with $q > n/(2s)$. Then, for any $\gamma \le 2s - n/q$,

$$\|u\|_{C^\gamma(B_{1/2})} \le C(||f||_{L^q(B_1)} + ||(1+|x|)^{-n-2s}u(x)||_{L^1(\mathbb{R}^n)} + ||u||_{L^\infty(B_1)}),$$

where C is a positive constant depending only on n, s, q and γ.

Proof. We can decompose $u = v + w$, where $v = (-\Delta)^{-s} f$ (in the sense that v is the Riesz potential of order $2s$ of the function f extended by zero outside B_1) and w satisfies $(-\Delta)^s w = 0$ in B_1. Then, if we apply the estimates in [164, Theorem 1.6 (ii)] and [163, Corollary 2.5], we get

$$[v]_{C^\gamma(\mathbb{R}^n)} \le C||f||_{L^q(B_1)}, \qquad ||(1+|x|)^{-n-2s} v(x)||_{L^1(\mathbb{R}^n)} \le C||f||_{L^q(B_1)},$$

and

$$[w]_{C^\gamma(B_{1/2})} \le C(||(1+|x|)^{-n-2s} w(x)||_{L^1(\Omega)} + ||w||_{L^\infty(B_2)}).$$

The result then follows from these estimates. □

The second one is for the regional fractional Laplacian.

Lemma 2.6.5. *Let $\Omega \subset \mathbb{R}^n$ be any domain with $n \ge 2$ and $s > \frac{1}{2}$. Let L_Ω be given by (2.1.7). Assume that $u \in L^\infty(B_2)$, $(1+|x|)^{-n-2s} u(x) \in L^1(\Omega)$ and satisfies*

$$L_\Omega u = f \quad \text{in } B_3 \subset \Omega,$$

for some $f \in L^q(B_3)$ with $q > n/(2s)$. Then, for any $\gamma \le 2s - n/q$,

$$[u]_{C^\gamma(B_{1/2})} \le C(||f||_{L^q(B_2)} + ||(1+|x|)^{-n-2s} u(x)||_{L^1(\Omega)} + ||u||_{L^\infty(B_2)}),$$

where C is a positive constant depending only on n, s, q and γ.

Proof. Extend u to be zero outside Ω. Then, for any $x \in B_2$, it is clear that

$$(-\Delta)^s u(x) = L_\Omega u(x) + u(x) \int_{\Omega^c} |x-y|^{-n-2s} = f(x) + u(x) \int_{\Omega^c} |x-y|^{-n-2s}\, dy =: g(x).$$

Moreover,

$$|g| \le |f| + C|u| \int_{B_3^c} |y|^{-n-2s}\, dy \le |f| + C|u|,$$

which means that $||g||_{L^q(B_2)} \le C(||f||_{L^q(B_2)} + ||u||_{L^\infty(B_2)})$.

Hence, u satisfies

$$(-\Delta)^s u = g \quad \text{in } B_2 \subset \Omega,$$

for some $g \in L^q(B_2)$ with norm depending only on n, s and f. The result then follows from Lemma 2.6.4. □

We can now give the:

Proof of Theorem 2.1.1. We divide the proof in two steps:

Step 1: C^α estimate. Since Ω is bounded and Lipschitz, it can be covered with a finite number of balls in such way that $\partial\Omega \cap B$ is a Lipschitz graph for any ball B. Consequently, combining the interior estimate of Lemma 2.6.4 and the boundary one of Theorem 2.4.1, we deduce

$$|u(x) - u(y)| \le C\left(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)}\right)|x-y|^\alpha$$

for every $x, y \in \overline{\Omega}$ with α and C depending only on n, s, q and Ω.

Step 2: $C^{2s-1+\alpha}$ estimate for $s > \frac{1}{2}$. Dividing u by a constant if needed, we may assume $\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)} \le 1$. Now, given $x, y \in \overline{\Omega}$, we define $r = |x-y|$ and $\rho = \min\{d(x), d(y)\}$ and, without loss of generality, we assume $\rho = d(x)$. We divide the proof in two cases.

On the one hand, when $\rho \le 2r$, we take $z \in \partial\Omega$ such that $|z-x| = \rho$ and, using Proposition 2.6.2, we conclude

$$\begin{aligned} |u(x)-u(y)| &\le |u(x)-u(z)| + |u(y)-u(z)| \le C\left(|x-z|^{2s-1+\alpha} + |y-z|^{2s-1+\alpha}\right) \\ &\le C\left(d(x)^{2s-1+\alpha} + (d(x)+r)^{2s-1+\alpha}\right) \le C\, r^{2s-1+\alpha} = C\,|x-y|^{2s-1+\alpha}, \end{aligned}$$

for some $\alpha > 0$ small enough.

On the other, if $\rho > 2r$ we have $B_{2r}(y) \subset \Omega$. We define the auxiliary function $u_r(x) = u(y+rx) - u(y)$ and the set $\Omega_r := (\Omega - x)/r$. Then, it is clear that u_r satisfies

$$L_{\Omega_r} u_r(x) = r^{2s} f(y+rx) =: f_r(x) \quad \text{ in } \ B_2,$$

with $||f_r||_{L^q(B_2)} \le Cr^{2s-n/q}$. Moreover, by using Proposition 2.6.2 and Lemma 2.6.3 we know that $|u_r(x)| < C|rx|^{2s-1+\alpha}$ for some α small enough, which yields

$$||u_r||_{L^\infty(B_2)} < Cr^{2s-1+\alpha} \qquad \text{and} \qquad ||(1+|x|)^{-n-2s} u_r(x)||_{L^1(\mathbb{R}^n)} < Cr^{2s-1+\alpha}.$$

Furthermore, since $q > n$, we can take α small enough such that $2s - n/q > 2s - 1 + \alpha$. Thus, applying Lemma 2.6.4 with $\gamma = 2s - 1 + \alpha$, we arrive at

$$\begin{aligned} [u_r]_{C^{2s-1+\alpha}(B_1)} &\le C(||f_r||_{L^q(B_2)} + ||(1+|x|)^{-n-2s} u_r(x)||_{L^1(\mathbb{R}^n)} + ||u_r||_{L^\infty(B_2)}) \\ &\le C(r^{2s-n/q} + r^{2s-1+\alpha} + r^{2s-1+\alpha}) \le Cr^{2s-1+\alpha}, \end{aligned}$$

which is equivalent to say

$$[u]_{C^{2s-1+\alpha}(B_r(y))} \le C,$$

for some constant independent of y and r. Consequently,

$$\begin{aligned} |u(x)-u(y)| &= r^{2s-1+\alpha} \frac{|u(x)-u(y)|}{|x-y|^{2s-1+\alpha}} \le r^{2s-1+\alpha} \sup_{x \in B_r(y)} \frac{|u(x)-u(y)|}{|x-y|^{2s-1+\alpha}} \\ &\le r^{2s-1+\alpha} [u]_{C^{0,2s-1+\alpha}(B_r(y))} \le Cr^{2s-1+\alpha} = C|x-y|^{2s-1+\alpha}. \end{aligned}$$

Since $x, y \in \overline{\Omega}$ have been arbitrarily chosen, the thesis follows. □

Finally, we give the:

Proof of Theorem 2.1.3. The proof is basically the same as the previous one, applying Lemma 2.6.5 instead of Lemma 2.6.4. □

2.7 Appendix: Equivalence for weak solutions

For completeness, we prove here the equivalence established in [1] for classical solutions, in the setting of weak solutions.

Proposition 2.7.1. *Let $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be such that*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \ \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \ \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

Then, it satisfies

$$\int_\Omega \{u(x) - u(y)\}\ K_\Omega(x,y)\, dy = f(x) \qquad \text{in } \ \Omega,$$

where K_Ω is given by (2.2.2)-(2.2.3).

Proof. Given any $z \in \Omega^c$, we have

$$0 = \mathcal{N}_s u(z) = \int_\Omega \frac{u(z)-u(y)}{|z-y|^{n+2s}}\, dy$$
$$= u(z)\int_\Omega |z-y|^{-n-2s}\, dy - \int_\Omega \frac{u(y)}{|z-y|^{n+2s}}\, dy,$$

and so

$$u(z) = \frac{\int_\Omega u(y)|z-y|^{-n-2s}\, dy}{\int_\Omega |z-\overline{z}|^{-n-2s}\, d\overline{z}} \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}.$$

Now, we substitute this identity in the fractional Laplacian. Given any $x \in \Omega$

$$\begin{aligned}
\frac{(-\Delta)^s u(x)}{c_{n,s}} &= \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy = \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy + \int_{\Omega^c} \frac{u(x)-u(z)}{|x-z|^{n+2s}}\, dz \\
&= \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy + \int_{\Omega^c} \frac{u(x) - \frac{\int_\Omega u(y)|z-y|^{-n-2s}\, dy}{\int_\Omega |z-\overline{z}|^{-n-2s}\, d\overline{z}}}{|x-z|^{n+2s}}\, dz \\
&= \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy + \int_{\Omega^c} \frac{\int_\Omega \frac{u(x)-u(y)}{|z-y|^{n+2s}}\, dy}{|x-z|^{n+2s}\int_\Omega |z-\overline{z}|^{-n-2s}\, d\overline{z}}\, dz \\
&= \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy + \int_\Omega \{u(x)-u(y)\} \int_{\Omega^c} \frac{|x-z|^{-n-2s}|y-z|^{-n-2s}}{\int_\Omega |z-\overline{z}|^{-n-2s}\, d\overline{z}}\, dz\, dy,
\end{aligned}$$

and the result follows. □

In what follows, we denote

$$||w||^2_{H^s_\Omega} := ||w||^2_{L^2(\Omega)} + \int\int_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}} dx\, dy.$$

Lemma 2.7.2. *Let $v, w : \mathbb{R}^n \to \mathbb{R}$ be such that $\mathcal{N}_s w = 0$ in $\mathbb{R}^n \setminus \Omega$. Then,*

$$\int_\Omega\int_\Omega \{v(x)-v(y)\}\{w(x)-w(y)\}\, K_\Omega(x,y) dx\, dy$$
$$= c_{n,s} \int\int_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{\{v(x)-v(y)\}\{w(x)-w(y)\}}{|x-y|^{n+2s}} dx\, dy,$$

where K_Ω is given by (2.2.2)-(2.2.3).

Proof. Note that adding and subtracting the terms $w(z)(v(x)+v(y)+v(z))$ and $v(z)(w(x)+$

$w(y)+w(z))$, and rearranging them, we obtain

$$\begin{aligned}
&\int_\Omega\int_\Omega \{v(x)-v(y)\}\{w(x)-w(y)\}\ K_\Omega(x,y)dx\,dy\\
&\qquad = c_{n,s}\int_\Omega\int_\Omega \frac{\{v(x)-v(y)\}\{w(x)-w(y)\}}{|x-y|^{n+2s}}dx\,dy\\
&\qquad\quad + c_{n,s}\int_\Omega dx\int_\Omega dy\int_{\Omega^c} dz\frac{\{v(x)-v(y)\}\{w(x)-w(y)\}}{|x-z|^{n+2s}|y-z|^{n+2s}\int_\Omega |z-\overline{z}|^{-n-2s}\,d\overline{z}}\\
&\qquad = c_{n,s}\int_\Omega\int_\Omega \frac{\{v(x)-v(y)\}\{w(x)-w(y)\}}{|x-y|^{n+2s}}dx\,dy\\
&\qquad\quad + c_{n,s}\int_\Omega dx\int_\Omega dy\int_{\Omega^c} dz\frac{\{v(x)-v(z)\}\{w(x)-w(z)\}}{|x-z|^{n+2s}|y-z|^{n+2s}\int_\Omega |z-\overline{z}|^{-n-2s}\,d\overline{z}}\\
&\qquad\quad + c_{n,s}\int_\Omega dx\int_\Omega dy\int_{\Omega^c} dz\frac{\{v(z)-v(y)\}\{w(x)-w(z)\}}{|x-z|^{n+2s}|y-z|^{n+2s}\int_\Omega |z-\overline{z}|^{-n-2s}\,d\overline{z}}\\
&\qquad\quad + c_{n,s}\int_\Omega dx\int_\Omega dy\int_{\Omega^c} dz\frac{\{v(x)-v(z)\}\{w(z)-w(y)\}}{|x-z|^{n+2s}|y-z|^{n+2s}\int_\Omega |z-\overline{z}|^{-n-2s}\,d\overline{z}}\\
&\qquad\quad + c_{n,s}\int_\Omega dx\int_\Omega dy\int_{\Omega^c} dz\frac{\{v(z)-v(y)\}\{w(z)-w(y)\}}{|x-z|^{n+2s}|y-z|^{n+2s}\int_\Omega |z-\overline{z}|^{-n-2s}\,d\overline{z}}\\
&\qquad = I_1+I_2+I_3+I_4+I_5.
\end{aligned}$$

By symmetry in the variables x and y it is clear that $I_2=I_5$ and $I_3=I_4$. Now, let us simplify them. On the one hand

$$\begin{aligned}
I_2=I_5 &= c_{n,s}\int_\Omega dx\int_\Omega dy\int_{\Omega^c} dz\frac{\{v(x)-v(z)\}\{w(x)-w(z)\}}{|x-z|^{n+2s}|y-z|^{n+2s}\int_\Omega |z-\overline{z}|^{-n-2s}\,d\overline{z}}\\
&= c_{n,s}\int_\Omega dx\int_{\Omega^c} dz\frac{\{v(x)-v(z)\}\{w(x)-w(z)\}}{|x-z|^{n+2s}\int_\Omega |z-\overline{z}|^{-n-2s}\,d\overline{z}}\left(\int_\Omega |y-z|^{-n-2s}\,dy\right)\\
&= c_{n,s}\int_\Omega dx\int_{\Omega^c} dz\frac{\{v(x)-v(z)\}\{w(x)-w(z)\}}{|x-z|^{n+2s}}.
\end{aligned}$$

On the other hand, using the condition $\mathcal{N}_s w=0$ in $\mathbb{R}^n\setminus\overline{\Omega}$ we obtain

$$\begin{aligned}
I_3=I_4 &= c_{n,s}\int_\Omega dx\int_\Omega dy\int_{\Omega^c} dz\frac{\{v(z)-v(y)\}\{w(x)-w(z)\}}{|x-z|^{n+2s}|y-z|^{n+2s}\int_\Omega |z-\overline{z}|^{-n-2s}\,d\overline{z}}\\
&= c_{n,s}\int_\Omega dy\int_{\Omega^c} dz\frac{v(z)-v(y)}{|y-z|^{n+2s}\int_\Omega |z-\overline{z}|^{-n-2s}\,d\overline{z}}\left(\int_\Omega \frac{w(x)-w(z)}{|x-z|^{n+2s}}dx\right)\\
&= c_{n,s}\int_\Omega dy\int_{\Omega^c} dz\frac{-\mathcal{N}_s w(z)\{v(z)-v(y)\}}{|y-z|^{n+2s}\int_\Omega |z-\overline{z}|^{-n-2s}\,d\overline{z}}\\
&= 0.
\end{aligned}$$

Putting all the terms together we finally arrive at

$$\begin{aligned}
&\int_\Omega\int_\Omega \{v(x)-v(y)\}\{w(x)-w(y)\}\ K_\Omega(x,y)dx\,dy\\
&\qquad = c_{n,s}\int\int_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{\{v(x)-v(y)\}\{w(x)-w(y)\}}{|x-y|^{n+2s}}dx\,dy,
\end{aligned}$$

as wanted. □

Finally, we prove:

Proposition 2.7.3. *Let $u \in H^s_\Omega$ be such that*

$$\frac{c_{n,s}}{2}\int\int_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{\{u(x)-u(y)\}\{v(x)-v(y)\}}{|x-y|^{n+2s}}dx\,dy = \int_\Omega f(x)v(x)\,dx \tag{2.7.1}$$

for all test function $v \in H^s_\Omega$. Then, $u \in H_K(\Omega)$ and it satisfies

$$\frac{1}{2}\int_\Omega\int_\Omega \{u(x)-u(y)\}\{\overline{v}(x)-\overline{v}(y)\}\ K_\Omega(x,y)dx\,dy = \int_\Omega f(x)\overline{v}(x)\,dx \tag{2.7.2}$$

for all $\overline{v} \in H_K(\Omega)$, where K_Ω is given by (2.2.2)-(2.2.3). Moreover, $\mathcal{N}_s u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$.

Proof. Given any test function $\overline{v} \in H_K(\Omega)$ we define $v : \mathbb{R}^n \to \mathbb{R}$ in the following way

$$v(x) = \begin{cases} \overline{v}(x) & \text{if } x \in \Omega, \\ \left(\int_\Omega \frac{\overline{v}(z)}{|x-z|^{n+2s}}\,dz\right)\left(\int_\Omega |x-z|^{-n-2s}\,dz\right)^{-1} & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

Indeed, this is the extension of v outside Ω that ensures $\mathcal{N}_s v = 0$ in Ω^c. Then, applying Lemma 2.7.2, we obtain

$$\begin{aligned} &\int\int_{\Omega\times\Omega} \{u(x)-u(y)\}\{\overline{v}(x)-\overline{v}(y)\}\ K_\Omega(x,y)dx\,dy \\ &\qquad = \int\int_{\Omega\times\Omega} \{u(x)-u(y)\}\{v(x)-v(y)\}\ K_\Omega(x,y)dx\,dy \\ &\qquad = c_{n,s}\int\int_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{\{u(x)-u(y)\}\{v(x)-v(y)\}}{|x-y|^{n+2s}}dx\,dy. \end{aligned}$$

Moreover, by using v as a test function in (2.7.1) we have

$$\begin{aligned} &\frac{c_{n,s}}{2}\int\int_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{\{u(x)-u(y)\}\{v(x)-v(y)\}}{|x-y|^{n+2s}}dx\,dy \\ &\qquad\qquad = \int_\Omega f(x)v(x)\,dx = \int_\Omega f(x)\overline{v}(x)\,dx. \end{aligned}$$

Thus, (2.7.2) follows by putting together the previous identities. Notice that applying Lemma 2.7.2 with $w = v$, we conclude that $v \in H^s_\Omega$. Thus, we can use it as a test function in (2.7.1).

Now, taking any $\varphi \in C^\infty_0(\mathbb{R}^n \setminus \overline{\Omega}) \subset H^s_\Omega$ and using it as a test function in (2.7.1), we deduce

$$\int_{\Omega^c} \varphi(y)\mathcal{N}_s u(y)\,dy = \int_{\Omega^c} \varphi(y)\left(\frac{u(x)-u(y)}{|x-y|^{n+2s}}dx\right)dy = 0,$$

and so we get that $\mathcal{N}_s u = 0$ in $\mathbb{R} \setminus \overline{\Omega}$. Furthermore, we can apply Lemma 2.7.2 with $v = w = u$ and, since $u \in H^s_\Omega$, we conclude that $u \in H_K(\Omega)$. $\square$

Part II

Semilinear problems: The saddle-shaped solution to the integro-differential Allen-Cahn equation

Chapter 3

Uniqueness and stability of the saddle-shaped solution to the fractional Allen-Cahn equation

In this chapter, which corresponds to [111] in collaboration with T. Sanz-Perela, we prove the uniqueness of the saddle-shaped solution to the semilinear nonlocal elliptic equation $(-\Delta)^\gamma u = f(u)$ in $\mathbb{R}^{2m}$, where $\gamma \in (0,1)$ and f is of Allen-Cahn type. Moreover, we prove that this solution is stable if $2m \geq 14$. As a consequence of this result and the connection of the problem with nonlocal minimal surfaces, we show that the Simons cone $\{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x'| = |x''|\}$ is a stable nonlocal (2γ)-minimal surface in dimensions $2m \geq 14$.

Saddle-shaped solutions of the fractional Allen-Cahn equation are doubly radial, odd with respect to the Simons cone, and vanish only in this set. It was known that these solutions exist in all even dimensions and are unstable in dimensions 2, 4 and 6. Thus, after our result, the stability remains an open problem only in dimensions 8, 10 and 12.

The importance of studying this type of solution is due to its relation with the fractional version of a conjecture by De Giorgi. Saddle-shaped solutions are the simplest non 1D candidates to be global minimizers in high dimensions, a property not yet established in any dimension.

3.1 Introduction

This chapter is devoted to the study of saddle-shaped solutions to the fractional Allen-Cahn equation

$$(-\Delta)^\gamma u = f(u) \quad \text{in } \mathbb{R}^n \,, \tag{3.1.1}$$

where $n = 2m$ is an even integer, f is of bistable type (see (3.1.2) below), and $(-\Delta)^\gamma$ is the fractional Laplacian, defined for $\gamma \in (0,1)$ by

$$(-\Delta)^\gamma u(x) := c_{n,\gamma} \, \mathrm{P.\,V.} \int_{\mathbb{R}^n} \frac{u(x) - u(\tilde{x})}{|x - \tilde{x}|^{n+2\gamma}} \, \mathrm{d}\tilde{x} \,.$$

Here $c_{n,\gamma} > 0$ is a normalizing constant depending only on n and γ, and P. V. stands for principal value. This problem is motivated by the fractional De Giorgi conjecture and it is closely related to the theory of nonlocal minimal surfaces, as we will explain later in this introduction.

Throughout the chapter we assume that $f \in C^{2,\alpha}\big((-1,1)\big)$, for some $\alpha \in (0,1)$, and that is of bistable type, i.e.,

$$f \text{ is odd}, \ \ f(0) = f(1) = 0, \ \text{ and } f'' < 0 \text{ in } (0,1). \tag{3.1.2}$$

Note that as a consequence we have $f > 0$ in $(0,1)$. A typical example of this kind of nonlinearity is $f(u) = u - u^3$.

An important role in this chapter is played by the Simons cone, which is defined in $\mathbb{R}^{2m}$ by

$$\mathscr{C} := \{x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x'| = |x''|\} \, .$$

It is well known that the Simons cone has zero mean curvature at every point $x \in \mathscr{C} \setminus \{0\}$, in every dimension $2m \geq 2$. However, it is only in dimensions $2m \geq 8$ that $\mathscr{C}$ is a minimizer of the area functional, as established by Bombieri, De Giorgi, and Giusti in [38]. Regarding the fractional setting, for every $\gamma \in (0, 1/2)$, $\mathscr{C}$ has zero *nonlocal* mean curvature in every even dimension but it is not known if, in addition, it is a minimizer of the fractional perimeter in dimensions $2m \geq 8$. We recall that it is only in dimension $2m = 2$ where we have a complete classification of minimizing nonlocal minimal cones, establishing that they must be flat (see [171]). The same classification result for *stable* nonlocal minimal cones holds also in $\mathbb{R}^2$ (see [172]), and in $\mathbb{R}^3$ and for γ close to $1/2$ (see [51]). Recall that by stability we understand that the second variation of the energy functional is nonnegative (and thus, it is a weaker property than minimality). In higher dimensions n, the classification of nonlocal minimal cones is widely open and the main result in this direction is the one in [64], establishing that minimizing nonlocal minimal cones are flat in dimensions $2 \leq n \leq 7$ for γ close to $1/2$. It is also an open problem to find, in high dimensions, an example of nonsmooth minimizing nonlocal minimal surface. A main candidate for this is, as in the local case, the Simons cone.

The only other result (apart from the previous ones) concerning the possible minimality of the Simons cone refers to its stability, and it is proved in [86] by Dávila, del Pino, and Wei. In that paper, the authors characterize the stability of Lawson cones through an inequality involving only two hypergeometric constants which depend only on γ and the dimension n. It is a hard task to verify the criterion analytically, and this has not been accomplished. It seems also delicate to check it numerically, but some cases are treated in [86]. With a numerical computation, [86] finds that, in dimensions $n \leq 6$ and for γ close to zero, no Lawson cone with zero nonlocal mean curvature is stable. The Simons cone is a particular case of Lawson cone corresponding to $C_m^m(2\gamma)$ in the notation of [86]. Numerics also shows that all Lawson cones in dimension 7 are stable if γ is close to zero. These results for small γ fit with the general belief that, in the fractional setting, the Simons cone should be stable (and even a minimizer) in dimensions $2m \geq 8$ (as in the local case), probably for all $\gamma \in (0, 1/2)$, though this is still an open problem.

In the present chapter, we make a first contribution to the previous question by showing that the Simons cone is a stable (2γ)-minimal cone in dimensions $2m \geq 14$. Our proof uses the so-called saddle-shaped solution to the Allen-Cahn equation. As we will see in more detail, by the fractional Modica-Mortola type Γ-convergence result, the remarks above on the stability of the Simons cone are expected to hold also for saddle-shaped solutions. Indeed, our proof proceeds by establishing the stability of such solution to the fractional Allen-Cahn equation in dimensions $2m \geq 14$ (see Theorem 3.1.6 below). Then, as a consequence of this and a recent result by Cabré, Cinti, and Serra in [50] (see also the comments in [51]) concerning the preservation of stability along a blow-down procedure for the fractional Allen-Cahn equation, we deduce the stability of the Simons cone as a nonlocal minimal surface in these dimensions (see Corollary 3.1.7).

To introduce saddle-shaped solutions, we define the following variables:

$$s := \sqrt{x_1^2 + \ldots + x_m^2} \quad \text{and} \quad t := \sqrt{x_{m+1}^2 + \ldots + x_{2m}^2}\,,$$

for which the Simons cone becomes $\mathscr{C} = \{s = t\}$. Through the chapter we will also use the letter $\mathcal{O}$ to denote one of the sets in which the cone divides the space:

$$\mathcal{O} := \{x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x'| > |x''|\} = \{s > t\}.$$

We define saddle-shaped solutions as follows.

Definition 3.1.1. We say that a bounded solution u to (3.1.1) is a *saddle-shaped solution* (or simply *saddle solution*) if

(i) u is a doubly radial function, that is, $u = u(s, t)$.

(ii) u is odd with respect to the Simons cone, that is, $u(s, t) = -u(t, s)$.

(iii) $u > 0$ in $\mathcal{O} = \{s > t\}$.

Saddle-shaped solutions for the classical Allen-Cahn equation involving the Laplacian were first studied by Dang, Fife, and Peletier in [84] in dimension $2m = 2$. They established the existence and uniqueness of this type of solutions, as well as some monotonicity properties and asymptotic behavior. In [174], Schatzman studied the instability property of saddle solutions in $\mathbb{R}^2$. Later, Cabré and Terra proved the existence of a saddle solution in every dimension $2m \geq 2$, and they established some qualitative properties such as asymptotic behavior, monotonicity properties, as well as instability in dimensions $2m = 4$ and $2m = 6$ (see [59, 60]). The uniqueness in dimensions higher than 2 was established by Cabré in [46], where he also proved that the saddle solution is stable in dimensions $2m \geq 14$.

In the nonlocal framework, there are only two works concerning saddle-shaped solutions to (3.1.1). In [75, 76], first for $\gamma = 1/2$ and then for $\gamma \in (0, 1)$, Cinti proved the existence of a saddle-shaped solution to (3.1.1) as well as some qualitative properties such as asymptotic behavior, monotonicity properties, and instability in low dimensions (see Theorem 3.1.2 below).

In the present chapter, we prove further properties of these solutions, the main ones being uniqueness and, when $2m \geq 14$, stability. Uniqueness is important since then the saddle-shaped solution becomes a canonical object associated to the Allen-Cahn equation and the Simons cone.

In [75, 76], the main tool used is the extension problem for the fractional Laplacian, due to Caffarelli and Silvestre [63] (see (3.1.3) below). This is also the approach of the present chapter. It should be remarked that the extension technique has the limitation that it only works for the fractional Laplacian, and therefore the same arguments cannot be carried out for more general integro-differential operators of the form

$$L_K u(x) = \mathrm{P.V.} \int_{\mathbb{R}^n} \{u(x) - u(\tilde{x})\} K(x - \tilde{x}) \, \mathrm{d}\tilde{x}.$$

In Chapters 4 and 5 we address this problem by studying saddle-shaped solutions to equation $L_K u = f(u)$ in $\mathbb{R}^{2m}$, where L_K is an elliptic integro-differential operator of the previous form with a radially symmetric kernel K. One of the most basic tools that we need is a maximum principle for the operator acting on functions which are odd with respect to the Simons cone. In Chapter 4 we find a necessary and sufficient condition to have such a maximum principle and, as we will see there, this will require a certain convexity property of the kernel K.

Let us now introduce the extension problem for the fractional Laplacian, which is the main tool used in this chapter. First we should settle the notation. We call $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,+\infty)$ and denote points by $(x,\lambda) \in \mathbb{R}^{n+1}_+$ with $x \in \mathbb{R}^n$ and $\lambda > 0$. As it is well known, see [63], if $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ solves $\operatorname{div}(\lambda^a \nabla u) = 0$ in $\mathbb{R}^{n+1}_+$ with $a = 1-2\gamma$, then

$$\frac{\partial u}{\partial \nu^a}(x) := -\lim_{\lambda \downarrow 0} \lambda^a u_\lambda(x,\lambda) = \frac{(-\Delta)^\gamma u(x,0)}{d_\gamma},$$

where d_γ is a positive constant depending only on γ. Therefore, problem (3.1.1) is equivalent to

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+\,, \\ d_\gamma \dfrac{\partial u}{\partial \nu^a} = f(u) & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n\,. \end{cases} \tag{3.1.3}$$

We will always consider functions defined in $\mathbb{R}^{n+1}_+$ and not only in $\mathbb{R}^n$, and we will use the same letter to denote both the function and its trace on $\mathbb{R}^n$. Regarding sets in $\mathbb{R}^{n+1}_+$, we use the following notation. If $\Omega \subset \mathbb{R}^{n+1}_+$, we define

$$\partial_L \Omega := \overline{\partial\Omega \cap \{\lambda > 0\}} \quad \text{and} \quad \partial_0 \Omega := \partial\Omega \setminus \partial_L\Omega \subset \{\lambda = 0\}\,. \tag{3.1.4}$$

We write

$$B_R^+ := \left\{(x,\lambda) \in \mathbb{R}^{n+1}_+ \;:\; |(x,\lambda)| < R\right\},$$

for half-balls in $\mathbb{R}^{n+1}_+$. If $x_0 \in \mathbb{R}^n$, $B_R^+(x_0) = (x_0,0) + B_R^+$.

A certain solution of problem (3.1.1) in dimension 1, the so-called *layer solution*, plays a crucial role through this chapter. It is the unique solution of the following problem:

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u_0) = 0 & \text{in } \mathbb{R}^2_+ = \mathbb{R} \times (0,+\infty)\,, \\ d_\gamma \dfrac{\partial u_0}{\partial \nu^a} = f(u_0) & \text{on } \partial\mathbb{R}^2_+ = \mathbb{R}\,, \\ \partial_x u_0 > 0 & \text{on } \partial\mathbb{R}^2_+ = \mathbb{R}\,, \\ u_0(0,0) = 0\,, & \\ \lim_{x\to\pm\infty} u_0(x,0) = \pm 1\,. & \end{cases} \tag{3.1.5}$$

Under the assumptions on f in (3.1.2), the existence and uniqueness of such solution are well known (see [56]).

The importance of the layer solution comes from the fact that the associated function

$$U(x,\lambda) := u_0\left(\frac{s-t}{\sqrt{2}}, \lambda\right) \quad \text{for } x \in \mathbb{R}^{2m} \text{ and } \lambda > 0, \tag{3.1.6}$$

which is odd with respect to the Simons cone and positive in $\mathcal{O} \times [0,+\infty)$, describes the asymptotic behavior of saddle-shaped solutions at infinity (as shown in [75, 76]; see Theorem 3.1.2 below). Note that from Lemma 4.2 in [59], we know that $|s-t|/\sqrt{2}$ is the distance to the Simons cone. Therefore, we can understand the function U as the layer solution centered at each point of the Simons cone and oriented in the normal direction to the cone. Moreover, in this chapter we show (see Proposition 3.1.5) that the saddle-shaped solution lies below U in $\mathcal{O}$, as it occurs in the local case (see Proposition 1.5 in [59]).

It is sometimes useful to consider also the following variables:

$$y := \frac{s+t}{\sqrt{2}} \quad \text{and} \quad z := \frac{s-t}{\sqrt{2}}\,,$$

which satisfy $y \geq 0$ and $-y \leq z \leq y$. In these variables, $\mathscr{C} = \{z = 0\}$ and $\mathcal{O} = \{z > 0\}$. Therefore, we can write $U(x, \lambda) = u_0(z, \lambda)$.

To study the minimality and stability of the saddle-shaped solution, we recall the energy functional associated to equation (3.1.3):

$$\mathcal{E}(w, \Omega) := \frac{d_\gamma}{2} \int_\Omega \lambda^a |\nabla w|^2 \,\mathrm{d}x \,\mathrm{d}\lambda + \int_{\partial_0 \Omega} G(w) \,\mathrm{d}x\,, \quad \text{where } \; G' = -f\,.$$

We say that u is a minimizer for problem (3.1.3) in $\Omega \subset \mathbb{R}^{2m+1}_+$ if

$$\mathcal{E}(u, \Omega) \leq \mathcal{E}(w, \Omega)$$

for every w such that $w = u$ on $\partial_L \Omega$. Observe that the admissible competitors do not have the boundary condition prescribed on $\partial_0 \Omega$. This is in correspondence with the Neumann condition in (3.1.3). We say that u is a global minimizer if it is a minimizer in every bounded domain Ω of $\mathbb{R}^{2m+1}_+$.

A bounded solution to (3.1.3) is said to be *stable* if the second variation of the energy with respect to perturbations ξ which have compact support in $\overline{\mathbb{R}^{2m+1}_+}$ is nonnegative. That is, if

$$\int_{\mathbb{R}^{2m}} f'(u)\, \xi^2 \,\mathrm{d}x \leq d_\gamma \int_0^\infty \int_{\mathbb{R}^{2m}} \lambda^a \, |\nabla \xi|^2 \,\mathrm{d}x \,\mathrm{d}\lambda \tag{3.1.7}$$

for every $\xi \in C^\infty_c(\overline{\mathbb{R}^{2m+1}_+})$.

In the following theorem we collect the known results concerning saddle-shaped solutions to (3.1.1).

Theorem 3.1.2 ([75, 76, 58, 57]). *Let $\gamma \in (0,1)$ and let $f \in C^{2,\alpha}\big((-1,1)\big)$ be a function satisfying (3.1.2).*

(i) For every even dimension $2m \geq 2$, there exists a saddle-shaped solution to problem (3.1.1) with $|u| < 1$.

(ii) For every even dimension $2m \geq 2$, every saddle-shaped solution to problem (3.1.1) satisfies

$$\Big|\Big| \, |u - U| + |\nabla_x (u - U)| \, \Big|\Big|_{L^\infty(\mathbb{R}^{2m} \setminus B_R)} \to 0, \qquad \text{as} \;\; R \to +\infty,$$

where U is defined in (3.1.6).

(iii) In dimension $2m$ with $2 \leq 2m \leq 6$, every saddle-shaped solution is unstable.

Here ∇_x denotes the gradient only in the horizontal variables $x \in \mathbb{R}^{2m}$, not to be confused with the gradient $\nabla = \nabla_{(x,\lambda)}$ in (3.1.3) or (3.1.7), for instance.

Points (i) and (ii) of Theorem 3.1.2 were proved by Cinti, first for $\gamma = 1/2$ in [75] and then extended to all powers $\gamma \in (0,1)$ in [76]. Instability in dimension $2m = 2$ follows from a general result on stable solutions established in [57] (previously proved for $\gamma = 1/2$ in [58]). Instead, instability in dimensions $2m = 4$ and $2m = 6$ was proved in [75, 76].

Our first main result is the uniqueness of the saddle-shaped solution. As a consequence, such solution to the fractional Allen-Cahn equation becomes a canonical object associated to the cone $\mathscr{C}$.

Theorem 3.1.3. *Let $\gamma \in (0,1)$ and let f be a function satisfying (3.1.2). Then, for every even dimension $2m \geq 2$, there exists a unique saddle-shaped solution to problem (3.1.3).*

As in the paper of Cabré [46] for the classical case, the proof of the uniqueness result follows from the asymptotic behavior of the saddle solution (point (ii) in Theorem 3.1.2) and a maximum principle in $\mathcal{O}$ for the linearized operator at a saddle-shaped solution. The maximum principle is the following.

Proposition 3.1.4. *Let u be a saddle-shaped solution of (3.1.3). Let $\Omega \subset \mathcal{O} \times (0,+\infty) \subset \mathbb{R}^{2m+1}_+$ be an open set such that $\partial_0\Omega$ is nonempty. Let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be bounded from above and such that $\lambda^a v_\lambda \in C(\overline{\Omega})$.*

Consider the operator $\mathscr{L}_u$ defined by

$$\mathscr{L}_u v := d_\gamma \frac{\partial v}{\partial \nu^a} - f'(u)v \quad \text{on} \quad \partial_0\Omega \subset \mathbb{R}^{2m} \times \{0\}, \tag{3.1.8}$$

and assume that

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla v) & \leq & b(x,\lambda)v & \text{in } \Omega \subset \mathcal{O} \times (0,+\infty)\,, \\ \mathscr{L}_u v & \leq & 0 & \text{on } \partial_0\Omega \subset \mathcal{O}\,, \\ v & \leq & 0 & \text{on } \partial_L\Omega\,, \\ \limsup\limits_{x\in\partial_0\Omega,\ |x|\to+\infty} v(x,0) & \leq & 0\,, & \end{cases}$$

with $b \leq 0$. Then, $v \leq 0$ in Ω.

To establish the previous maximum principle we follow the proof of the analogous result for the local case ($\gamma = 1$) in [46]. It involves a maximum principle in "narrow" sets (see also [45, 30]). The main difference between our proof and the one in [46] is that, since we are using the extension problem, a new notion of narrowness is needed to carry out the same type of arguments (see Section 3.2 for the details).

The second main result of this chapter is the following pointwise estimate for the saddle-shaped solution. We prove that the function $U(s,t,\lambda) := u_0((s-t)/\sqrt{2}, \lambda)$ is a barrier for the saddle-shaped solution. This result was established in the local setting ($\gamma = 1$) in [59], but in such case the proof is quite simple by using the so-called Modica estimate (see [59] for the details). In the fractional framework, this estimate is only available (in a nonlocal form) in dimension 1 (see [58, 56]) and therefore we need another type of argument. Our strategy is to use a maximum principle for the linearized operator at U, similar to the one in Proposition 3.1.4. The pointwise estimate we establish is the following.

Proposition 3.1.5. *Let u be the saddle-shaped solution of (3.1.3), let u_0 be the layer solution given by (3.1.5) and let U be defined by (3.1.6). Then,*

$$|u(x,\lambda)| \leq |U(x,\lambda)| = |u_0(\operatorname{dist}(x,\mathscr{C}),\lambda)| \quad \text{for every } (x,\lambda) \in \overline{\mathbb{R}^{2m+1}_+}\,. \tag{3.1.9}$$

The third main result of the present chapter establishes the stability of the saddle solution in high dimensions. This is an extension of Theorem 1.4 in [46] to the nonlocal case. For its proof, it is crucial to use the extension problem.

Theorem 3.1.6. *Assume that f satisfies (3.1.2). If $2m \geq 14$, then the saddle-shaped solution u of (3.1.3) is stable in $\mathbb{R}^{2m+1}_+$, i.e., (3.1.7) holds.*

Its stability is a consequence of the following fact. For every constant $b > 0$ satisfying $b(b - m + 2) \leq -(m - 1)$, the function

$$\varphi := t^{-b} u_s - s^{-b} u_t,$$

defined in $\mathbb{R}^{2m+1}_+ \setminus \{st = 0\}$, is even with respect to the Simons cone and is a positive supersolution of the linearized operator. More precisely, $-\operatorname{div}(\lambda^a \nabla \varphi) \geq 0$ in $\mathbb{R}^{2m+1}_+ \setminus \{st = 0\}$ and $\mathscr{L}_u \varphi \geq 0$ in $\mathbb{R}^{2m} \setminus \{st = 0\}$, where $\mathscr{L}_u$ is defined in (3.1.8).

An important consequence of this result is Corollary 3.1.7, stated next, on the stability of the Simons cone as a (2γ)-minimal surface in dimensions $2m \geq 14$. This is the first analytical proof of its stability for some γ and m. It follows directly from the convergence results proved in [50] for stable solutions to the Allen-Cahn equation after a blow-down, together with the preservation of the stability along this procedure (see also the comments at the end of this introduction).

Corollary 3.1.7. *Let $\gamma \in (0, 1/2)$ and $2m \geq 14$. Then, the Simons cone $\mathscr{C} \subset \mathbb{R}^{2m}$ is a stable (2γ)-minimal surface.*

The key ingredients to prove Theorem 3.1.6 are some monotonicity and second derivative properties for the saddle-shaped solution. In fact, φ being a positive supersolution will follow from such properties. More precisely, our arguments will use the following.

Proposition 3.1.8. *Let u be the saddle-shaped solution to (3.1.3). Then,*

(i) $u_y > 0$ *in* $\mathcal{O} \times [0, +\infty)$.

(ii) $-u_t > 0$ *in* $(\mathcal{O} \setminus \{t = 0\}) \times [0, +\infty)$.

(iii) $u_{st} > 0$ *in* $(\mathcal{O} \setminus \{t = 0\}) \times [0, +\infty)$.

As a consequence, for every direction $\partial_\eta = \alpha \partial_y - \beta \partial_t$, where α and β are nonnegative constants, $\partial_\eta u > 0$ in $\{s > t > 0,\ \lambda \geq 0\}$.

The monotonicity properties (i) and (ii) were proved in the papers of Cinti [75, 76] for the so-called maximal saddle solution —note that in those papers the uniqueness of the saddle-shaped solution was not known yet. From her result and our uniqueness theorem, (i) and (ii) in Proposition 3.1.8 follow. Nevertheless, we present here a new proof of them by applying the maximum principle for the linearized operator to certain equations satisfied by u_s and u_t. A similar argument will establish the new property (iii) for the crossed second derivative u_{st}.

To conclude this introduction, let us comment briefly on the importance of problem (3.1.1) and its relation with a conjecture of De Giorgi and the theory of minimal surfaces.

The interest on problem (3.1.1) originates from a famous conjecture of De Giorgi for the classical Allen-Cahn equation. It reads as follows. Let u be a bounded solution to $-\Delta u = u - u^3$ in $\mathbb{R}^n$ which is monotone in one direction, say $\partial_{x_n} u > 0$. Then, if $n \leq 8$, u is one dimensional, i.e., u depends only on one Euclidean variable. This conjecture was proved to be true in dimension $n = 2$ by Ghoussoub and Gui in [123], and in dimension $n = 3$ by Ambrosio and Cabré in [9]. For dimensions $4 \leq n \leq 8$, and under the additional assumption

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \quad \text{for all } x' \in \mathbb{R}^{n-1}. \tag{3.1.10}$$

the conjecture was established by Savin [167] (see also the previous work of Ghoussoub and Gui [124] in dimensions 4 and 5 for antisymmetric solutions). A counterexample to the conjecture in dimensions $n \geq 9$ was given by del Pino, Kowalczyk and Wei [89].

The corresponding conjecture in the nonlocal setting, where one replaces the operator $-\Delta$ by $(-\Delta)^\gamma$, has been widely studied in the last years. In this framework, the conjecture has been proven to be true in dimension $n = 2$ by Cabré and Solà-Morales in [58] for $\gamma = 1/2$, and extended to every power $0 < \gamma < 1$ by Cabré and Sire in [57] and also by Sire and Valdinoci in [182]. In dimension $n = 3$, the conjecture has been proved by Cabré and Cinti for $1/2 \leq \gamma < 1$ in [48, 49] and by Dipierro, Farina, and Valdinoci for $0 < \gamma < 1/2$ in [94]. Recently, in [168, 169] Savin has established the validity of the conjecture in dimensions $4 \leq n \leq 8$ and for $1/2 \leq \gamma < 1$, but assuming the additional hypothesis (3.1.10). Under the same extra assumption, the conjecture is true in the same dimensions for $0 < \gamma < 1/2$ and γ close to $1/2$, as proved by Dipierro, Serra, and Valdinoci in [97]. The most recent result concerning the proof of the conjecture is the one by Figalli and Serra in [116], where they have established the conjecture in dimension $n = 4$ and $\gamma = 1/2$ without requiring the additional limiting assumption (3.1.10). Note that, without (3.1.10), the analogous result for the Laplacian in dimension $n = 4$ is not known. In the forthcoming paper [50], Cabré, Cinti, and Serra prove the conjecture in dimension $n = 4$ for $0 < \gamma < 1/2$ and γ sufficiently close to $1/2$. A counterexample to the De Giorgi conjecture for fractional Allen-Cahn equation in dimensions $n \geq 9$ for $\gamma \in (1/2, 1)$ has been very recently announced in [66].

Coming back to the local Allen-Cahn equation, while studying this conjecture by De Giorgi, another question arose naturally: do global minimizers in $\mathbb{R}^n$ of the Allen-Cahn energy have one-dimensional symmetry? A deep result from Savin [167] states that in dimension $n \leq 7$ this is indeed true. On the other hand, Liu, Wang, and Wei [146] have constructed minimizers in dimensions $n \geq 8$ which are not one-dimensional. We should mention that the same question for stable solutions (instead of minimizers) is still largely open, only solved in dimension $n = 2$ (see [123, 26]).

The saddle-shaped solution is of special interest regarding the previous two questions. It is expected to be a simple example of non one-dinsional minimizer to the Allen-Cahn equation in high dimensions, having the same role as the Simons cone for the theory of minimal surfaces. In addition, regarding the conjecture by De Giorgi, if the saddle-shaped solution was proved to be a minimizer in some even dimension $2m$, we would automatically have a counterexample to the conjecture in higher dimensions. This is due to a result by Jerison and Monneau [135], where they show that such a counterexample in dimension $\mathbb{R}^{n+1}$ can be constructed with a rather natural procedure if there exists a non one-dimensional global minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$ which is bounded and even with respect to each coordinate. The saddle-shaped solution is of special interest in relation with the Jerison-Monneau program since it is even with respect to all the coordinate axis and it is expected to be a minimizer in high dimensions. If proved to be a minimizer, the saddle-shaped solution would provide an alternative construction of a counterexample to the original conjecture of De Giorgi, different from the one of [89].

Let us explain why the Allen-Cahn equation has a very strong connection with the theory of minimal surfaces. A deep result from the seventies by Modica and Mortola (see [150, 151]) states that considering an appropriately rescaled version of the Allen-Cahn equation, the corresponding energy functionals Γ-converge to the perimeter functional. Thus, the minimizers of the equation converge to the characteristic function of a set of minimal perimeter. This same fact holds for the equation with the fractional Laplacian, though we have two different scenarios depending on the parameter $\gamma \in (0,1)$. If $\gamma \geq 1/2$, the rescaled energy functionals associated to (3.1.1) Γ-converge to the classical perimeter (see [5, 127]), while in the case $\gamma \in (0,1/2)$ they Γ-converge to the fractional perimeter (see [170]). As a consequence, if the saddle-shaped solution was proved to be a minimizer in a certain dimension for some $\gamma \in (0,1/2)$, it would follow that the Simons cone $\mathscr{C}$ would be a minimizing nonlocal (2γ)-minimal surface in such dimensions. As mentioned before, this last statement is an open problem in any dimension. Our Corollary 3.1.7 on stability is related to this question, but for a weaker property than minimality.

By a result of Cabré, Cinti, and Serra in [50], also the stability is preserved in the blow-down limit when $\gamma \in (0,1/2)$. Therefore, a limit of stable solutions to (3.1.1) with $\gamma \in (0,1/2)$ will be a stable set for the (2γ)-perimeter. Thus, as a consequence of Theorem 3.1.6 we deduce Corollary 3.1.7.

The chapter is organized as follows. In section 3.2 we prove the maximum principle for the linearized operator in $\mathcal{O}$, Proposition 3.1.4. Section 3.3 is devoted to show Theorem 3.1.3 concerning the uniqueness of the saddle-shaped solution. In Section 3.4 we establish some monotonicity properties of the layer solution u_0, as well as the pointwise estimate for the saddle solution in terms of the layer u_0, stated in Proposition 3.1.5. In Section 3.5 we prove the monotonicity and second derivative properties of the saddle solution stated in Proposition 3.1.8. Finally, Section 3.6 concerns the proof of the stability results, Theorem 3.1.6 and Corollary 3.1.7.

3.2 Maximum principles for the linearized operator

In this section we establish Proposition 3.1.4, a maximum principle for the linearized operator. To prove it, we follow the ideas appearing in [46], where an analogous maximum principle is proved for the local case $\gamma = 1$. The proof for the Laplacian uses a maximum principle in "narrow" sets (see for instance [45, 30]). In our case, the use of the extension problem requires a similar maximum principle but in pairs of sets that we will call "extension-narrow", defined next.

Definition 3.2.1 ("Extension-narrow" pair of sets)**.** Let $\Omega \subset \mathbb{R}^{n+1}_+$ be an open set, not necessarily bounded, and let $\Gamma \subset \partial_0\Omega$ be nonempty —recall that $\partial_0\Omega$ is defined by (3.1.4). Given $\theta \in (0,1)$ and $a \in (-1,1)$, we define $R_a(\Omega,\Gamma,\theta) \in (0,+\infty]$ to be the smallest positive constant R for which

$$\frac{|B^+_R(x) \setminus \Omega|_a}{|B^+_R(x)|_a} \geq \theta \quad \text{for every } x \in \Gamma\,, \tag{3.2.1}$$

where

$$|E|_a := \int_E \lambda^a \,\mathrm{d}x\,\mathrm{d}\lambda\,.$$

We say that $R_a(\Omega,\Gamma,\theta) = +\infty$ if no such radius exists. From this definition, we will say that a pair (Ω,Γ) is "extension-narrow" if $R_a(\Omega,\Gamma,\theta)$ is small enough depending on certain quantities.

Note that if in (3.2.1) we consider $a = 0$ and full balls centered at every point $x \in \Omega$, we recover the usual definition of "narrow" set. Here, instead, we only consider half-balls centered at points $x \in \Gamma \subset \partial_0\Omega$.

Let us remark that both sets $\Omega \in \mathbb{R}^{n+1}_+$ and $\Gamma \subset \partial_0\Omega$ play an important role in this notion of "narrowness", as illustrated in the following examples in $\mathbb{R}^2_+$. On the one hand, let

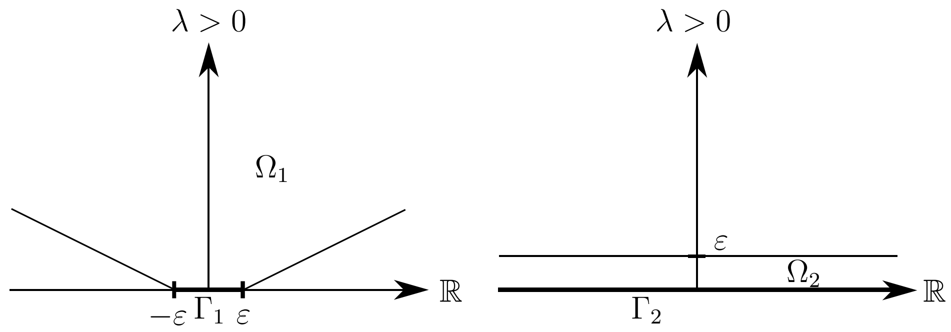

Figure 3.1: (a) A pair (Ω_1, Γ_1) satisfying $R_0(\Omega_1, \Gamma_1, 1/2) = +\infty$ but with Γ_1 being "narrow" in $\mathbb{R}$. (b) An "extension-narrow" narrow pair (Ω_2, Γ_2) with Γ_2 not "narrow" in $\mathbb{R}$.

$\Omega_1 = \{\lambda > (x-\varepsilon)/2\} \cap \{\lambda > -(x+\varepsilon)/2\} \cap \{\lambda > 0\} \subset \mathbb{R}^2_+$ and let $\Gamma_1 = \partial_0\Omega_1 = (-\varepsilon, \varepsilon) \subset \mathbb{R}$ —see Figure 3.1 (a). This pair has $R_0(\Omega_1, \Gamma_1, 1/2) = +\infty$ for all $\varepsilon > 0$ even though Γ_1 is "narrow" in $\mathbb{R}$ in the usual sense if ε is small enough. On the other hand, the pair consisting of $\Omega_2 = \{0 < \lambda < \varepsilon\}$ and $\Gamma_2 = \partial_0\Omega_2 = \mathbb{R}$ is "extension-narrow" if ε is small enough, while Γ_2 is not "narrow" in the usual sense in $\mathbb{R}$ —see Figure 3.1 (b).

Once the quantity $R_a(\Omega, \Gamma, \theta)$ is defined, we can state precisely the maximum principle in "extension-narrow" pairs.

Proposition 3.2.2 (Maximum principle in "extension-narrow" pairs)**.** *Let $\Omega \subset \mathbb{R}^{n+1}_+$ be an open set and let $\Gamma \subset \partial_0\Omega$ be nonempty. Assume that there exists a nonempty open cone $E \subset \partial_0\mathbb{R}^{n+1}_+ = \mathbb{R}^n$ such that $(E \times (0, +\infty)) \cap \Omega = \varnothing$.*

Let $a \in (-1, 1)$ and let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be a function bounded from above such that $\lambda^a v_\lambda \in C(\overline{\Omega})$, and assume that it satisfies

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla v) & \leq & b(x,\lambda)v & \text{in } \Omega\,, \\ \dfrac{\partial v}{\partial \nu^a} + c(x)v & \leq & 0 & \text{on } \Gamma\,, \\ v & \leq & 0 & \text{on } \partial\Omega \setminus \Gamma\,, \end{cases} \tag{3.2.2}$$

where $b \leq 0$ in Ω and c is bounded from below on Γ.

Then, for every $\theta \in (0,1)$ there exists a constant R^*, depending only on n, a, θ, and $||c_-||_{L^\infty(\Gamma)}$, such that $v \leq 0$ in Ω whenever $R_a(\Omega, \Gamma, \theta) \leq R^*$.

Before proving this result, let us explain why we need to introduce the notion of "extension-narrowness". In the proof of Proposition 3.1.4 we will use this maximum principle in a pair (Ω, Γ) with $\Omega \subset \mathcal{O} \times (0, +\infty)$, and $\Gamma \subset \partial_0\Omega$ in an ε-neighborhood in $\mathcal{O}$ of the cone $\mathscr{C}$. In this case, Ω could be very big (and not "narrow" in the usual sense) in $\mathbb{R}^{2m+1}_+$, as in Figure 3.2. However, $\mathcal{O}^c \times (0, +\infty)$ is contained in the complement of Ω — even if Ω filled all $\mathcal{O} \times (0, +\infty)$. Thus, it follows readily that (Ω, Γ) is "extension-narrow" by using that balls in this notion are centered in Γ (see Corollary 3.2.5 below for the details).

To prove Proposition 3.2.2 we need the following weak Harnack inequality.

Proposition 3.2.3 (Proposition 3.2 of [187])**.** *Let $v \in H^1(B^+_R, \lambda^a)$ be a nonnegative function that weakly satisfies*

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla v) & \geq & 0 & \text{in } B^+_R\,, \\ \dfrac{\partial v}{\partial \nu^a} & \geq & 0 & \text{on } \partial_0 B^+_R\,. \end{cases}$$

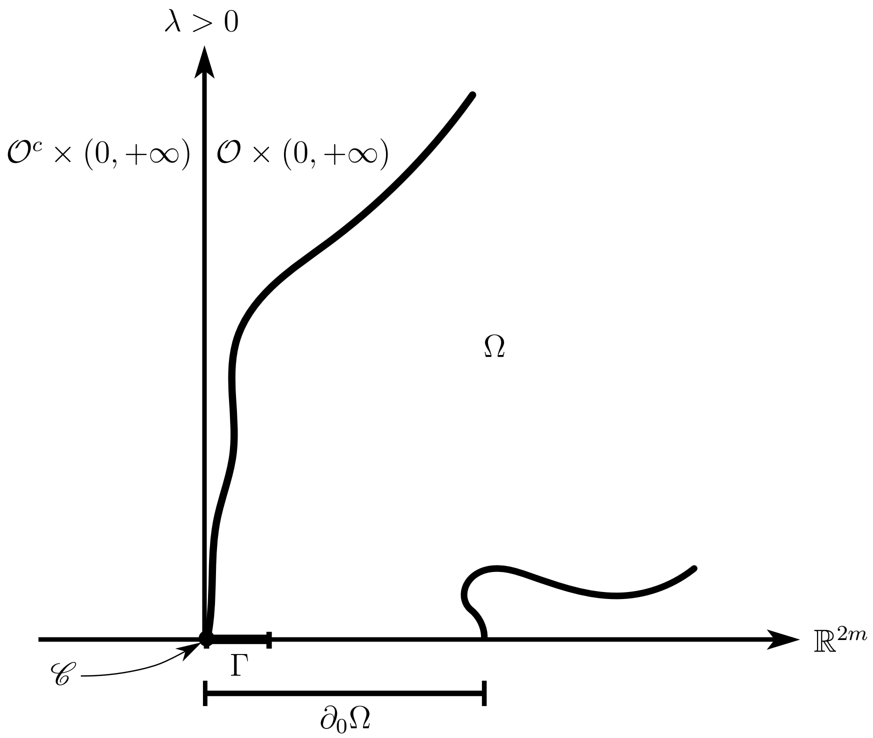

Figure 3.2: An example of a pair (Ω, Γ) which is "extension-narrow".

Then, there exists a constant $p_0 > 0$, depending only on n and a, such that for all $p \leq p_0$,

$$\left(\int_{B^+_{R/2}} \lambda^a v^p \, dx \, d\lambda \right)^{1/p} \leq C_h R^{\frac{n+1+a}{p}} \inf_{B^+_{R/4}} v \,, \tag{3.2.3}$$

for a positive constant C_h depending only on n and a.

With this result available, we can now present the proof of the maximum principle in "extension-narrow" pairs.

Proof of Proposition 3.2.2. Define the sets

$$\Omega_+ := \{(x, \lambda) \in \Omega \ : \ v(x, \lambda) > 0\} \qquad \text{and} \qquad \Gamma_+ := \partial\Omega_+ \cap \Gamma \,,$$

and by contradiction assume that Ω_+ is nonempty. Then, since $b \leq 0$, v satisfies

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla v) & \leq & 0 & \text{in } \Omega_+ \,, \\ \dfrac{\partial v}{\partial \nu^a} + c(x) v & \leq & 0 & \text{on } \Gamma_+ \text{ (if this set is nonempty)} \,, \\ v & \leq & 0 & \text{on } \partial\Omega_+ \setminus \Gamma_+ \,. \end{cases}$$

Now, we proceed in two steps in order to arrive at a contradiction.

Step 1. First, we claim that if Γ_+ is nonempty then $\sup_{\Omega_+} v = \sup_{\Gamma_+} v$. That is, if we call

$$\overline{v} := v - \sup_{\Gamma_+} v,$$

we then have $\overline{v} \leq 0$ in Ω_+. To prove this, we use a classical Phragmen-Lindelöf-type argument, as follows. Similar methods appear, among many others, in the proof of Theorem 1.2 of [27], or Section 2.4 of [58].

We now claim that, since the cone E is open, there exists a nonempty open cone $F \subset E$ satisfying

$$|x-y| \geq c_0 > 0 \quad \text{for every } x \in E^c \text{ and } y \in \overline{F}\,, \tag{3.2.4}$$

for some positive constant c_0.

Indeed, since E is an open cone (with vertex, say, $z \in \partial E$), there exists a circular cone $E' \subset E$ with the same vertex z. Then, by sliding this circular cone in the direction of its axis, which can be assumed to be $e_n = (0, ..., 0, 1)$, we obtain a new open cone $F \subset E$. Let us now show (3.2.4). Since $F \subset E' \subset E$, it is enough to prove (3.2.4) for $x \in \partial E'$ and $y \in \partial F$. Hence, we have

$$x_n - z_n = \omega|x' - z'| \quad \text{and} \quad y_n - z_n = \tau + \omega|y' - z'|,$$

for some positive constants ω and τ. Here, we are using the notation $z = (z', z_n)$. Now, if we call $\sigma = |x' - z'| - |y' - z'|$, we have $|x' - y'| \geq |\sigma|$ and thus

$$\begin{aligned} |x-y|^2 &= |x'-y'|^2 + |x_n - y_n|^2 \geq \sigma^2 + (\omega\sigma - \tau)^2 \\ &= \left(\sqrt{1+\omega^2}\sigma - \frac{\omega\tau}{\sqrt{1+\omega^2}}\right)^2 + \frac{\tau^2}{1+\omega^2} \geq \frac{\tau^2}{1+\omega^2}, \end{aligned}$$

where the last constant is in fact the minimum distance between points on $\partial E'$ and ∂F.

Now, without loss of generality, we may assume that the vertex of F is the origin. Let F' be an open cone with the same vertex as F, and such that $\overline{F' \cap \mathbb{S}^{n-1}} \subset F \cap \mathbb{S}^{n-1}$. Let ϕ be the first eigenfunction of the Laplace-Beltrami operator in $\mathbb{S}^{n-1} \setminus \overline{F'} \subset \mathbb{R}^n$ with zero Dirichlet boundary conditions on $\partial F' \cap \mathbb{S}^{n-1}$, and let $\mu > 0$ be its associated eigenvalue. Since $\partial F' \cap \mathbb{S}^{n-1}$ is contained in F, there exists a positive constant δ such that $\phi \geq \delta > 0$ in $\mathbb{S}^{n-1} \setminus \overline{F}$. Now, define the auxiliary function

$$\psi(x, \lambda) = (1 + \lambda^{2\gamma})|x|^\beta \phi(x/|x|),$$

where β is a positive real number and $\gamma = (1-a)/2 \in (0, 1)$. Then, $\phi(x/|x|) \geq \delta$ for each $(x, \lambda) \in \Omega_+$, since $x/|x| \in \mathbb{S}^{n-1} \setminus \overline{F}$. Moreover, by (3.2.4) with $y = 0$, we deduce that

$$\psi(x, \lambda) \geq \delta(1 + \lambda^{2\gamma})|x|^\beta \geq \delta c_0^\beta > 0 \quad \text{in } \Omega_+,$$

since 0 is the vertex of F. On the other hand, note that if we choose $\beta > 0$ solving $\beta(\beta + n - 2) = \mu$, we have that ψ satisfies

$$\left\{ \begin{array}{rcll} -\operatorname{div}(\lambda^a \nabla \psi) &=& 0 & \text{in } \Omega_+\,, \\ \displaystyle\lim_{(x,\lambda)\in\Omega_+,\ |(x,\lambda)|\to+\infty} \psi &=& +\infty. & \end{array} \right.$$

Thus, if we define

$$\overline{w} := \frac{\overline{v}}{\psi} = \frac{v - \sup_{\Gamma_+} v}{\psi}\,,$$

proving that $\overline{v} \leq 0$ in Ω_+ is equivalent to showing that $\overline{w} \leq 0$ in Ω_+, since ψ is positive. Now, since $\sup_{\Gamma_+} v \geq 0$, it is easy to show that $\overline{w}$ satisfies

$$\left\{ \begin{array}{rcll} -\operatorname{div}(\lambda^a \nabla \overline{w}) - 2\lambda^a \dfrac{\nabla\psi}{\psi} \cdot \nabla \overline{w} &\leq& 0 & \text{in } \Omega_+\,, \\ \overline{w} &\leq& 0 & \text{on } \partial\Omega_+\,, \\ \displaystyle\lim_{(x,\lambda)\in\Omega_+,\ |(x,\lambda)|\to+\infty} \overline{w} &\leq& 0\,. & \end{array} \right. \tag{3.2.5}$$

Then, by the classical maximum principle we deduce that $\overline{w} \leq 0$ in Ω_+, which yields $\overline{v} \leq 0$ in Ω_+.

Note that if Γ_+ is empty, the same argument applied to v instead of $\overline{v}$ yields a contradiction with the assumption that Ω_+ is nonempty. From now on in this proof, we will assume that $\Gamma_+ \neq \varnothing$.

Step 2. By Step 1 and the definition of Ω_+, we have that

$$M := \sup_{\Gamma_+} v > 0\,. \tag{3.2.6}$$

Therefore, since $v \leq 0$ on $\partial\Omega_+ \setminus \Gamma_+$, there exists a sequence $(x_k, 0) \in \Gamma_+$ such that

$$v(x_k) = v(x_k, 0) \geq M\left(1 - \frac{1}{k}\right),$$

where we are identifying v with its trace on $\mathbb{R}^n$ to simplify the notation.

Now, given any $R > 0$, let $\overline{c}_{n,\gamma}$ be the constant such that

$$(-\Delta)^\gamma\{\overline{c}_{n,\gamma}(R^2 - |x - x_k|^2)_+^\gamma\} = 1 \quad \text{in } B_R(x_k)\,,$$

(see [35] for its explicit value) and take $\phi = \phi(x, \lambda)$ to be the γ-harmonic extension of

$$\phi(x, 0) = c_1 M \overline{c}_{n,\gamma}(R^2 - |x - x_k|^2)_+^\gamma\,,$$

where c_1 is a positive constant to be chosen later. Thus, ϕ solves

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla\phi) &= 0 & \text{in } B_R^+(x_k)\,, \\ \dfrac{\partial\phi}{\partial\nu^a} &= \dfrac{c_1 M}{d_\gamma} & \text{on } \partial_0 B_R^+(x_k)\,. \end{cases}$$

Moreover, on $\partial_0 B_R^+(x_k) \cap \Gamma_+$ we have

$$\frac{\partial v}{\partial \nu^a} \leq -cv \leq ||c_-||_{L^\infty(\Gamma)}\, v \leq ||c_-||_{L^\infty(\Gamma)}\, M \leq \frac{\partial\phi}{\partial\nu^a}$$

if we choose $c_1 > d_\gamma\, ||c_-||_{L^\infty(\Gamma)}$.

Thus, $v - \phi$ is γ-subharmonic in $B_R^+(x_k) \cap \Omega_+$ and has a nonpositive flux on $\partial_0 B_R^+(x_k) \cap \Gamma_+$. In addition, $v - \phi \leq v \leq 0$ in $B_R^+(x_k) \cap (\partial\Omega_+ \setminus \Gamma_+)$. Therefore, its positive part $(v-\phi)_+$ extended to be zero in $B_R^+(x_k) \setminus \Omega_+$ is a continuous function which is γ-subharmonic in $B_R^+(x_k)$ and has a nonpositive flux on $\partial_0 B_R^+(x_k)$, both properties in a weak sense.

We define $w := M - (v - \phi)_+$, which is a continuous nonnegative function and satisfies in a weak sense

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla w) & \geq 0 & \text{in } B_R^+(x_k)\,, \\ \dfrac{\partial w}{\partial \nu^a} & \geq 0 & \text{on } \partial_0 B_R^+(x_k)\,. \end{cases}$$

Hence, w fulfills the hypotheses of Proposition 3.2.3 and thus (3.2.3) holds. As a consequence, if we take $R = 2\,R_a(\Omega,\Gamma,\theta)$ and p as in (3.2.3), we have

$$\begin{aligned}
\theta^{1/p} M &\le \left(\frac{|B^+_{R/2}(x_k)\setminus\Omega|_a}{|B^+_{R/2}(x_k)|_a} M^p \right)^{1/p} \\
&\le \left(\frac{|B^+_{R/2}(x_k)\setminus\Omega_+|_a}{|B^+_{R/2}(x_k)|_a} M^p \right)^{1/p} \\
&= \frac{1}{|B^+_{R/2}(x_k)|_a^{1/p}} \left(\int_{B^+_{R/2}(x_k)\setminus\Omega_+} \lambda^a M^p \,\mathrm{d}x\,\mathrm{d}\lambda \right)^{1/p} \\
&\le |B^+_1|_a^{-1/p} (R/2)^{-\frac{n+1+a}{p}} \left(\int_{B^+_{R/2}(x_k)} \lambda^a w^p \,\mathrm{d}x\,\mathrm{d}\lambda \right)^{1/p} \\
&\le 2^{\frac{n+1+a}{p}} |B^+_1|_a^{-1/p} C_h \inf_{B^+_{R/4}(x_k)} w \\
&\le 2^{\frac{n+1+a}{p}} |B^+_1|_a^{-1/p} C_h\, w(x_k).
\end{aligned}$$

Here we have used the definition of $R_a(\Omega,\Gamma,\theta)$, the fact that $w \equiv M$ in $B^+_R(x_k)\setminus\Omega_+$, the scaling properties of $|\cdot|_a$ and the weak Harnack inequality (3.2.3).

Now, if $c_1\bar{c}_{n,\gamma}R^{2\gamma} \le 1/2$, then $w(x_k) = M - v(x_k) + \phi(x_k)$ for k large enough. Therefore, for such indices k we conclude

$$\begin{aligned}
\theta^{1/p} M &\le 2^{\frac{n+1+a}{p}} |B^+_1|_a^{-1/p} C_h \{M - v(x_k) + \phi(x_k)\} \\
&\le 2^{\frac{n+1+a}{p}} |B^+_1|_a^{-1/p} C_h \{1/k + c_1\bar{c}_{n,\gamma}R^{2\gamma}\} M\,.
\end{aligned}$$

Hence, if we take $R_a(\Omega,\Gamma,\theta)$ small enough such that $c_1\bar{c}_{n,\gamma}(2R_a(\Omega,\Gamma,\theta))^{2\gamma} < 1$ and $2^{\frac{n+1+a}{p}}|B^+_1|_a^{-1/p}C_h c_1\bar{c}_{n,\gamma}($
$\theta^{1/p}$, we get that

$$M\left(1 - \frac{C}{k}\right) \le 0$$

for some positive constant C independent of k. Letting $k \to +\infty$, this leads to $M \le 0$, which contradicts (3.2.6).

Therefore, our initial assumption stating $\Omega_+ \neq \varnothing$ is false. This means that $v \le 0$ in Ω. □

Remark 3.2.4. It will be useful later to note that Proposition 3.2.2 (and as a consequence, Proposition 3.1.4) is also valid not requiring v to be C^2 in the whole Ω. Indeed, we only need to assume that $v \in C(\Omega)$, that the equation $\operatorname{div}(\lambda^a \nabla v) \le b(x,\lambda)v$ holds pointwise where v is regular, and that v cannot have a local maximum at a nonregular point.

This will be used in the proof of Proposition 3.1.5 with $v = u - CU$ in $\Omega = \mathcal{O}\times(0,+\infty)$, where u is a saddle-shaped solution, U is defined by (3.1.6), and C is a positive constant. Note that U is Lipschitz but not C^2 across $\{t = 0, \lambda \ge 0\}$. Therefore, as we will see in Section 3.4, U is only γ-superharmonic (pointwise) in $\Omega\setminus\{t = 0, \lambda \ge 0\}$. Nevertheless, by this remark, Proposition 3.2.2 will hold in this case thanks to the fact that the graph of $v = u - CU$ in its nonregular points makes the "good angle" for the maximum principle to hold (see the proof of Proposition 3.1.5 for the details).

As a consequence of Proposition 3.2.2, next we establish that the maximum principle holds in pairs (Ω,Γ) with $\Omega \subset \mathcal{O}\times(0,+\infty) \subset \mathbb{R}^{2m+1}_+$ and $\Gamma \subset \partial_0\Omega$ lying in an ε-neighborhood of the Simons cone.

Corollary 3.2.5. *Let $\Omega \subset \mathcal{O} \times (0,+\infty) \subset \mathbb{R}^{2m+1}_+$ and let $\Gamma \subset \partial_0\Omega$ be nonempty. Assume that $\Gamma \subset \mathcal{N}_\varepsilon := \{t < s < t+\varepsilon,\ \lambda = 0\}$.*

Then, if ε is small enough, depending only on n, γ, and $||c_-||_{L^\infty(\Gamma)}$, the maximum principle holds in Ω in the sense of Proposition 3.2.2. That is, if $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is bounded from above, $\lambda^a v_\lambda \in C(\overline{\Omega})$, and v satisfies (3.2.2), then $v \le 0$ in Ω.

To prove it, it is enough to realize that the Simons cone separates every ball centered at a point in the cone into two regions with comparable measure. In fact, it is interesting to note that these two regions have exactly the same measure, as stated next.

Lemma 3.2.6. *Let $x_0 \in \mathscr{C} \subset \mathbb{R}^{2m}$. Then,*

$$|B_r(x_0) \cap \mathcal{O}| = |B_r(x_0) \setminus \mathcal{O}| = \frac{1}{2}|B_r(x_0)| \quad \textit{for all} \ \ r > 0\,.$$

This result was stated in [46], but without a proof. For the sake of completeness, we include here a simple one.

Proof of Lemma 3.2.6. First, let us call $\mathcal{I} := \mathbb{R}^{2m} \setminus \overline{\mathcal{O}}$. Since $x_0 \in \mathscr{C}$, we have that $x_0 = (x_0', x_0'') \in \mathbb{R}^m \times \mathbb{R}^m$ satisfies that $|x_0'| = |x_0''|$. Therefore, there exists an orthogonal transformation $R \in O(m)$ such that $Rx_0' = x_0''$. Let us define $\overline{R} : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ by $\overline{R}(x', x'') = (Rx', x'')$, which is a linear isometry that keeps invariant $\mathcal{O}$ and $\mathcal{I}$. With these properties it is easy to check that for every $y \in \mathbb{R}^{2m}$ it holds

$$|B_r(y) \cap \mathcal{I}| = |\overline{R}\,(B_r(y) \cap \mathcal{I})\,| = |B_r(\overline{R}y) \cap \mathcal{I}|\,, \tag{3.2.7}$$

and the same replacing $\mathcal{I}$ with $\mathcal{O}$.

On the other hand, let us define $S : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ by $S(x', x'') = (x'', x')$, which is also a linear isometry and transforms $\mathcal{O}$ into $\mathcal{I}$ and vice versa. Therefore, for every $y \in \mathbb{R}^{2m}$ we have

$$|B_r(y) \cap \mathcal{I}| = |S\,(B_r(y) \cap \mathcal{I})\,| = |B_r(Sy) \cap \mathcal{O}|\,. \tag{3.2.8}$$

Finally, note that by the definition of $\overline{R}$, it is satisfied $S\overline{R}x_0 = \overline{R}x_0$. By combining this with (3.2.7) and (3.2.8) applied to $y = x_0$ and $y = \overline{R}x_0$ respectively, we obtain

$$|B_r(x_0) \cap \mathcal{I}| = |B_r(\overline{R}x_0) \cap \mathcal{I}| = |B_r(S\overline{R}x_0) \cap \mathcal{O}| = |B_r(\overline{R}x_0) \cap \mathcal{O}| = |B_r(x_0) \cap \mathcal{O}|\,.$$

□

With this lemma available we proceed with the proof of Corollary 3.2.5.

Proof of Corollary 3.2.5. Note that $\mathbb{R}^{2m} \setminus \overline{\mathcal{O}}$ is an open cone outside $\mathcal{O}$, and thus $\{(\mathbb{R}^{2m} \setminus \overline{\mathcal{O}}) \times (0,+\infty)\} \cap \Omega$ is empty. Hence, we can use Proposition 3.2.2 by noticing that, if we take $\theta = 2^{-4m-3-2a}$, then $R_a(\Omega, \Gamma, \theta) \le \varepsilon$. Indeed, recall first that by Lemma 4.2 in [59], $|s-t|/\sqrt{2}$ is the distance to the cone. Then, let $x \in \Gamma$ and let $\overline{x} \in \mathscr{C}$ a point realizing this distance. Since $x \in \Gamma \subset \mathcal{N}_\varepsilon$, we have that $|x - \overline{x}| \le \varepsilon/\sqrt{2} < 3\varepsilon/4$ and therefore

$$B^+_{\varepsilon/4}(\overline{x}) \setminus \Big(\mathcal{O} \times (0,+\infty)\Big) \subset B^+_{\varepsilon/4}(\overline{x}) \setminus \Omega \subset B^+_{\varepsilon}(x) \setminus \Omega\,.$$

Hence, by the scaling properties of $|\cdot|_a$ and Lemma 3.2.6 —used at each level $\{\lambda = \lambda_0\}$, with $\lambda_0 \in (0, \varepsilon/4)$—, we have

$$2^{-4m-3-2a}|B^+_\varepsilon(x)|_a = \frac{1}{2}|B^+_{\varepsilon/4}(\overline{x})|_a = |B^+_{\varepsilon/4}(\overline{x}) \setminus \Big(\mathcal{O} \times (0,+\infty)\Big)|_a \le |B^+_\varepsilon(x) \setminus \Omega|_a\,.$$

□

With this result at hand we can now establish the maximum principle for the linearized operator in $\mathcal{O} \times (0, +\infty)$ at a saddle-shaped solution.

Proof of Proposition 3.1.4. Let u be a saddle-shaped solution. A key point in the proof is that u is a positive supersolution in $\mathcal{O} \times (0, +\infty)$ of the linearized problem at u. Indeed, since $u > 0$ in $\partial_0 \Omega \subset \mathcal{O}$,

$$\mathscr{L}_u u = d_\gamma \frac{\partial u}{\partial \nu^a} - f'(u)u = f(u) - f'(u)u > 0 \quad \text{on } \partial_0 \Omega \,. \tag{3.2.9}$$

We have used that since $f'' < 0$ in $(0,1)$ and $f(0) = 0$, it satisfies $f'(\tau)\tau < f(\tau)$ for all $\tau \in (0,1)$.

Now, we define

$$w := \frac{v}{u} \,.$$

Note that w is well defined in Ω, since u is positive in such set. The usual strategy (see [30]) in some proofs of the maximum principle is to assume that the supremum of w in Ω is positive and then arrive at a contradiction. Nevertheless, a priori we do not know that $\sup_\Omega w < +\infty$, since u vanishes on $\mathscr{C} \times [0, +\infty)$ and $\partial \Omega$ could intersect this set. Thus, in the following arguments we will consider the supremum of w in a subset of $\partial_0 \Omega$ that is at a positive distance to the zero level set of u. Then, using the maximum principle in "extension-narrow" pairs we will see that, assuming this supremum to be positive, it will indeed agree with the supremum in the whole set Ω (see the details below). After some arguments, we will arrive at a contradiction. A similar strategy was used by Cabré in [46], to prove an analogous maximum principle in the local case $\gamma = 1$.

Les us proceed with the details. For $\varepsilon > 0$, set

$$\mathcal{O}_\varepsilon := \{t + \varepsilon < s, \ \lambda = 0\} \quad \text{and} \quad \mathcal{N}_\varepsilon := \{t < s < t + \varepsilon, \ \lambda = 0\} \,,$$

and take ε small enough such that for each set $\Gamma \subset \partial_0 \Omega$ satisfying $\Gamma \subset \mathcal{N}_\varepsilon$, the pair (Ω, Γ) is "extension-narrow". Hence, the maximum principle, as in Corollary 3.2.5, holds for the pair (Ω, Γ).

Next, we claim that

$$u \geq \delta > 0 \quad \text{in } \mathcal{O}_\varepsilon \tag{3.2.10}$$

for some positive constant δ. Indeed, thanks to the asymptotic behavior of u (see part (ii) of Theorem 3.1.2), and since $U(x) \geq u_0(\varepsilon/\sqrt{2})$ for $x \in \mathcal{O}_\varepsilon$, there exists a radius $R > 0$ such that $u(x) \geq u_0(\varepsilon/\sqrt{2})/2$ if $|x| > R$ and $x \in \mathcal{O}_\varepsilon$. Since u is positive in the compact set $\overline{\mathcal{O}_\varepsilon} \cap \overline{B_R}$, we conclude the claim.

We define

$$\Gamma := \partial_0 \Omega \cap \mathcal{N}_\varepsilon \,,$$

and let

$$S := \sup_{\partial_0 \Omega \cap \mathcal{O}_\varepsilon} w \,,$$

which is finite by the fact that u is bounded from below by $\delta > 0$ in $\mathcal{O}_\varepsilon$ and v is bounded from above. Assume by contradiction that $S > 0$.

First, we claim that $S = \sup_\Omega w$. To see this, we only need to show that $w \leq S$ in Ω. Define $\varphi := v - Su$ and note that since $S \geq 0$, φ satisfies

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla \varphi) & \leq & b(x,\lambda)\varphi & \text{in } \Omega \,, \\ \dfrac{\partial \varphi}{\partial \nu^a} & \leq & c(x)\varphi & \text{on } \Gamma \,, \\ \varphi & \leq & 0 & \text{on } \partial \Omega \setminus \Gamma \,, \end{cases}$$

with $c(x) = f'(u)/d_\gamma$. By the maximum principle in the "extension-narrow" pair (Ω, Γ), we have $\varphi \le 0$ in Ω, which yields $w = v/u \le S$ in Ω. Thus, the claim is proved.

Now, by the hypothesis on $\partial_L \Omega$ and at infinity on v, and the fact that $u > \delta$ in $\mathcal{O}_\varepsilon$, we have that S is attained at some point $(x_0, 0) \in \partial_0 \Omega \subset \mathcal{O}$. At this point we have

$$\frac{\partial w}{\partial \nu^a}(x_0) = -\lim_{\lambda \downarrow 0} \lambda^a w_\lambda(x_0, \lambda) = \lim_{\lambda \downarrow 0} \frac{w(x_0,0) - w(x_0,\lambda)}{\lambda^{2\gamma}} \ge 0\,, \tag{3.2.11}$$

since $w(x_0, 0)$ is the maximum.

On the other hand, observe that

$$d_\gamma u^2 \frac{\partial w}{\partial \nu^a} = d_\gamma \frac{\partial v}{\partial \nu^a} u - d_\gamma \frac{\partial u}{\partial \nu^a} v = u \mathscr{L}_u v - v \mathscr{L}_u u \le -v \mathscr{L}_u u \quad \text{ on } \partial_0 \Omega \subset \mathcal{O}\,,$$

since $u > 0$ in $\mathcal{O}$ and $\mathscr{L}_u v \le 0$ in $\partial_0 \Omega$. Therefore, at the point x_0 we have, using also (3.2.9),

$$\frac{\partial w}{\partial \nu^a}(x_0) \le -\frac{S}{d_\gamma u(x_0)} \mathscr{L}_u u(x_0) < 0\,,$$

which contradicts (3.2.11). Note that in this last argument is crucial the fact that $x_0 \in \partial_0 \Omega \subset \mathcal{O}$ and thus $u(x_0) > 0$ and $\mathscr{L}_u u(x_0) > 0$.

Hence, the assumption $S > 0$ is false and therefore $w \le 0$ in $\partial_0 \Omega \cap \mathcal{O}_\varepsilon$. Since $u > 0$ in $\mathcal{O}$, this yields that $v \le 0$ in $\partial_0 \Omega \cap \mathcal{O}_\varepsilon$. Finally, by the maximum principle in the "extension-narrow" pair (Ω, Γ) applied to v, it follows that $v \le 0$ in Ω. □

3.3 Uniqueness of the saddle-shaped solution

Thanks to the maximum principle in $\mathcal{O} \times (0, +\infty)$ for the linearized operator we can now establish the uniqueness of the saddle-shaped solution.

Proof of Theorem 3.1.3. Let u_1 and u_2 be two saddle-shaped solutions. Define $v := u_1 - u_2$, a function that depends only on s and t and that is odd with respect to $\mathscr{C}$. Then, $\operatorname{div}(\lambda^a \nabla v) = 0$ in $\mathcal{O} \times (0, +\infty)$, $v = 0$ on $\partial_L (\mathcal{O} \times (0, +\infty)) = \mathscr{C} \times [0, +\infty)$ and

$$d_\gamma \frac{\partial v}{\partial \nu^a} = f(u_1) - f(u_2) \le f'(u_2)(u_1 - u_2) = f'(u_2) v \quad \text{ on } \mathcal{O} \times \{0\}\,,$$

since f is concave in $(0,1)$. Moreover, by the asymptotic result (see Theorem 3.1.2), we have

$$\limsup_{x \in \mathcal{O},\ |x| \to +\infty} v(x, 0) = 0\,.$$

Finally, by the maximum principle for the linearized operator in $\mathcal{O} \times (0, +\infty)$, see Proposition 3.1.4, we deduce that $v \le 0$ in $\mathcal{O} \times [0, +\infty)$, which yields $u_1 \le u_2$ in $\mathcal{O} \times [0, +\infty)$. Interchanging u_1 and u_2, we obtain $u_1 \ge u_2$ in $\mathcal{O} \times [0, +\infty)$. Therefore, $u_1 = u_2$ in $\mathbb{R}^{2m+1}_+$. □

3.4 The layer solution and a pointwise estimate for the saddle-shaped solution

This section is devoted to establish some monotonicity properties of the layer solution u_0 and a pointwise estimate for the saddle-shaped solution (Proposition 3.1.5). We start with a maximum principle similar to Proposition 3.1.4, but for the linearized operator at u_0 in the set $\{u_0 > 0\}$, which plays the role that $\mathcal{O} \times (0, +\infty)$ had for the saddle-shaped solution.

Proposition 3.4.1. *Let $u_0 : \overline{\mathbb{R}^2_+} \to \mathbb{R}$ be the layer solution of (3.1.5) and let $\mathscr{L}_{u_0}$ be defined by*

$$\mathscr{L}_{u_0} v := d_\gamma \frac{\partial v}{\partial \nu^a} - f'(u_0) v \quad \text{on } \ \mathbb{R} = \partial_0 \mathbb{R}^2_+ .$$

Let $\Omega \subset (0, +\infty) \times (0, +\infty)$ be an open set such that $\partial_0 \Omega$ is nonempty.

Let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be bounded from above and satisfying $\lambda^a v_\lambda \in C(\overline{\Omega})$. Assume that

$$\begin{cases} -\operatorname{div}(\lambda^a \nabla v) \leq b(x, \lambda) v & \text{in } \Omega \subset (0, +\infty) \times (0, +\infty) , \\ \mathscr{L}_{u_0} v \leq 0 & \text{on } \partial_0 \Omega \subset (0, +\infty) , \\ v \leq 0 & \text{on } \partial_L \Omega , \\ \limsup_{x \in \partial_0 \Omega, \ |x| \to +\infty} v(x, 0) \leq 0 , & \end{cases}$$

with $b \leq 0$. Then, $v \leq 0$ in Ω.

Proof. Since it is analogous (and simpler) to the proof of Proposition 3.1.4, we just sketch it here pointing out what needs to be adapted. The key fact is that u_0 is a positive supersolution to the linearized problem. This is an analogous situation to that of Proposition 3.1.4 . That is, u_0 is γ-harmonic in $(0, +\infty) \times (0, +\infty)$, positive in $(0, +\infty) \times [0, +\infty)$, and

$$d_\gamma \frac{\partial u_0}{\partial \nu^a} = f(u_0) > f'(u_0) u_0 \quad \text{on } (0, +\infty) \times \{0\} , \tag{3.4.1}$$

where we have used that $f'' < 0$ in $(0, 1)$ and $f(0) = 0$.

Then, one defines $w := v/u_0$ and proceeds exactly as in the proof of Proposition 3.1.4, replacing u by u_0 in the whole argument, and also replacing $\mathcal{O}_\varepsilon$ and $\mathcal{N}_\varepsilon$ by $(\varepsilon, +\infty)$ and $(0, \varepsilon)$ respectively. In addition, (3.2.10) follows immediately from the fact that $u_0(x, 0)$ is increasing. The rest of the proof is completely analogous by using (3.4.1). □

With this maximum principle we can now prove the following monotonicity and concavity properties of the layer solution.

Lemma 3.4.2. *Let u_0 be the layer solution of (3.1.5). Then,*

$$\frac{\partial}{\partial x} u_0(x, \lambda) > 0 \quad \text{in} \quad \mathbb{R} \times [0, +\infty)$$

and

$$\frac{\partial^2}{\partial x^2} u_0(x, \lambda) < 0 \quad \text{in} \quad (0, +\infty) \times [0, +\infty) .$$

Proof. First of all, let us remark that u_0 has the required regularity to apply the following arguments by the results of [56] (see Section 3.5 for more details in the more involved setting of the saddle-shaped solution).

The monotonicity of the first derivative was already stated in Remark 4.7 of [56], but we include here the short proof for completeness. By differentiating (3.1.5) with respect to x, we obtain that $\operatorname{div}(\lambda^a \nabla(\partial_x u_0)) = 0$ in $\mathbb{R} \times (0, +\infty)$. Moreover, $\partial_x u_0(x, 0) > 0$ for $x \in \mathbb{R}$; see (3.1.5). Then, the result follows directly from the Poisson formula.

Next, we show the second statement. If we call

$$v(x, \lambda) := \partial_{xx} u_0(x, \lambda) ,$$

by differentiating (3.1.5) twice with respect to x, we get

$$\begin{cases} \operatorname{div}(\lambda^a \nabla v) &= 0 \quad \text{in } (0,+\infty)\times(0,+\infty)\,,\\ d_a \dfrac{\partial v}{\partial \nu^a} - f'(u_0)v = f''(u_0)(\partial_x u_0)^2 &\le 0 \quad \text{on } (0,+\infty)\times\{0\}\,,\\ v &= 0 \quad \text{on } \{0\}\times(0,+\infty)\,. \end{cases}$$

Notice that $v=0$ on $\{0\}\times(0,+\infty)$ since v is an odd function with respect to the first variable (recall that u_0 is odd in x).

Moreover, by repeating the argument of Lemma 4.8 in [56] for $\partial_{xx}u_0$, it is easy to see that $\partial_{xx}u_0(x,0)\to 0$ as $|x|\to+\infty$. Therefore, by Proposition 3.4.1 we deduce that $v\le 0$ in $[0,+\infty)\times[0,+\infty)$. Finally, we get that it is in fact negative in $(0,+\infty)\times[0,+\infty)$ by applying the strong maximum principle. □

Now we prove that the function

$$U(s,t,\lambda) := u_0\left(\frac{s-t}{\sqrt{2}},\lambda\right)$$

is a barrier for the saddle-shaped solution. To do it, we will use a maximum principle in $\mathcal{O}\times(0,+\infty)$ for the linearized problem at U.

Proof of Proposition 3.1.5. The idea is to repeat the arguments in the proof of Proposition 3.1.4, but using U instead of u as the positive supersolution to the linearized problem involving the operator

$$\mathscr{L}_U w := d_\gamma \frac{\partial w}{\partial \nu^a} - f'(U)w.$$

In order to do it, we need to point out several facts.

First, note that

$$U\in C^2\Big((\mathcal{O}\times(0,+\infty))\setminus\{t=0,\ \lambda>0\}\Big)\cap \operatorname{Lip}\Big(\mathbb{R}^{2m+1}_+\Big)\,,$$

and U cannot have a local minimum at $\{t=0,\ \lambda\ge 0\}$. Indeed, for every $\lambda\ge 0$,

$$\lim_{\tau\to 0^-}\partial_{x_{m+1}}U(x_1,\dots x_m,\tau,0,\dots,0,\lambda) = \frac{1}{\sqrt{2}}\partial_x u_0\left(\frac{s}{\sqrt{2}},\lambda\right)>0\,,$$

and

$$\lim_{\tau\to 0^+}\partial_{x_{m+1}}U(x_1,\dots x_m,\tau,0,\dots,0,\lambda) = -\frac{1}{\sqrt{2}}\partial_x u_0\left(\frac{s}{\sqrt{2}},\lambda\right)<0\,.$$

Note that the same property concerning a local minimum at $\{t=0,\ \lambda\ge 0\}$ holds if we add to U a regular function.

Next, we claim that U is a positive supersolution in $\mathcal{O}$ to the linearized problem for $\mathscr{L}_U$. Indeed, by the concavity of f, we have that

$$\mathscr{L}_U U = f(U)-f'(U)U>0 \quad \text{in } \mathcal{O}\,.$$

Moreover, a simple computation in the (s,t,λ) variables shows that

$$\operatorname{div}(\lambda^a\nabla U) = \lambda^a\frac{m-1}{\sqrt{2}}\frac{t-s}{st}\partial_x u_0\left(\frac{s-t}{\sqrt{2}},\lambda\right) \quad \text{in } \mathbb{R}^{2m+1}_+\setminus\{st=0,\ \lambda>0\}\,. \tag{3.4.2}$$

Therefore, U is γ-superharmonic in $(\mathcal{O} \times (0,+\infty)) \setminus \{t=0,\ \lambda>0\}$ —recall that $\partial_x u_0 > 0$ by Lemma 3.4.2.

Now, we define

$$v := u - U \quad \text{ and } \quad \Omega := \mathcal{O} \times (0,+\infty)\,,$$

and we want to see that $v \le 0$ in Ω. First, since u is γ-harmonic, we have that

$$-\operatorname{div}(\lambda^a \nabla v) \le 0 \quad \text{ in } \Omega \setminus \{t=0,\ \lambda>0\}$$

and that v cannot have a local maximum at $\{t=0,\ \lambda \ge 0\}$. In addition, both u and U vanish at $\mathscr{C} \times [0,+\infty)$ and by the asymptotic behavior of u (see Theorem 3.1.2), we have $\lim_{x\in\mathcal{O},\ |x|\to+\infty} v(x,0) = 0$. On the other hand, since f is concave in $(0,1)$, we get

$$d_\gamma \frac{\partial v}{\partial \nu^a} = f(u) - f(U) \le f'(U) v \quad \text{ on } \partial_0\Omega\,.$$

Collecting all these facts, we can repeat the proof of Proposition 3.1.4, using U instead of u as the positive supersolution to the linearized problem for $\mathscr{L}_U$ to see that $v \le 0$ in Ω. All the arguments are analogous, taking into account Remark 3.2.4 when using the maximum principles in "extension-narrow" pairs. Therefore, we conclude that $v \le 0$ in Ω and, by the odd symmetry of u and U, we get (3.1.9). □

3.5 Monotonicity properties

In this section we establish the monotonicity properties of u stated in Proposition 3.1.8. For this, we will apply the maximum principle of Proposition 3.1.4 to some derivatives of u. Therefore, we need some regularity results that we collect next.

Recall that we assume that $f \in C^{2,\alpha}$ for some $\alpha \in (0,1)$. Since u is a bounded solution to the first equation in (3.1.3), then $u \in C^\infty(\mathbb{R}^{2m+1}_+)$. Regarding the regularity on $\{\lambda = 0\}$, $u(\cdot,0) \in C^{2,\alpha}(\mathbb{R}^{2m})$ by applying Lemma 4.4 from [56]. Moreover, [56] also gives the following uniform bound:

$$||u||_{C^\alpha\left(\overline{\mathbb{R}^{2m+1}_+}\right)} + ||\nabla_x u||_{C^\alpha\left(\overline{\mathbb{R}^{2m+1}_+}\right)} + ||D^2_x u||_{C^\alpha\left(\overline{\mathbb{R}^{2m+1}_+}\right)} \le C,$$

for some $C>0$ depending only on m, γ , $||f||_{C^{2,\alpha}}$, and $||u||_{L^\infty(\mathbb{R}^{2m+1}_+)}$.

Next, since the horizontal first derivatives of u satisfy $\operatorname{div}(\lambda^a \nabla u_{x_i}) = 0$ and also $d_\gamma \partial_{\nu^a} u_{x_i} = f'(u)\, u_{x_i} \in C^\alpha(\mathbb{R}^{2m})$, and the horizontal second derivatives of u satisfy $\operatorname{div}(\lambda^a \nabla u_{x_i x_j}) = 0$ and also $d_\gamma \partial_{\nu^a} u_{x_i x_j} = f''(u)\, u_{x_i}\, u_{x_j} + f'(u)\, u_{x_i x_j} \in C^\alpha(\mathbb{R}^{2m})$ for all indices i and j from 1 to $2m$, we can apply Lemma 4.5 from [56] to obtain that

$$||\lambda^a\, u_\lambda||_{C^\beta(\mathbb{R}^{2m}\times[0,1])} + ||\lambda^a\,(u_{x_i})_\lambda||_{C^\beta(\mathbb{R}^{2m}\times[0,1])} + ||\lambda^a\,(u_{x_i x_j})_\lambda||_{C^\beta(\mathbb{R}^{2m}\times[0,1])} \le C,$$

for some $C>0$ and $\beta \in (0,1)$ depending only on m, γ, $||f||_{C^{2,\alpha}}$, and $||u||_{L^\infty(\mathbb{R}^{2m+1}_+)}$.

Now, since u depends only on s, t and λ, from the previous results we obtain

$$\begin{aligned}
u_s \in C^{2,\alpha}(\mathbb{R}^{2m+1}_+ \setminus \{s=0,\ \lambda\ge 0\}),\quad \lambda^a\,(u_s)_\lambda \in C^\alpha(\overline{\mathbb{R}^{2m+1}_+} \setminus \{s=0,\ \lambda\ge 0\}),\\
u_t \in C^{2,\alpha}(\mathbb{R}^{2m+1}_+ \setminus \{t=0,\ \lambda\ge 0\}),\quad \lambda^a\,(u_t)_\lambda \in C^\alpha(\overline{\mathbb{R}^{2m+1}_+} \setminus \{t=0,\ \lambda\ge 0\}),\\
u_{st} \in C^{2,\alpha}(\mathbb{R}^{2m+1}_+ \setminus \{st=0,\ \lambda\ge 0\}),\quad \lambda^a\,(u_{st})_\lambda \in C^\alpha(\overline{\mathbb{R}^{2m+1}_+} \setminus \{st=0,\ \lambda\ge 0\}).
\end{aligned}$$

Furthermore, as it is explained in Section 4 of [46], the regularity and the symmetry of u, in s and t, yield

$$u_s = 0 \text{ in } \{s = 0, \lambda \geq 0\}, \; u_t = 0 \text{ in } \{t = 0, \lambda \geq 0\}, \; u_{st} = 0 \text{ in } \{st = 0, \lambda \geq 0\},$$

and

$$u_s, \; u_t, \; u_{st} \in C(\overline{\mathbb{R}^{2m+1}_+}).$$

Before proceeding to the proof of Proposition 3.1.8, we first need the following asymptotic result for the second derivatives in x of u. This derivative was not included in the asymptotic theorem of [75, 76]. We will use it to show that $u_{st} > 0$ in $\{s > t > 0\} \times [0, +\infty)$.

Lemma 3.5.1. *Let f satisfy conditions (3.1.2), and let u be the saddle-shaped solution of (3.1.3). Then, denoting $U(x,\lambda) := u_0((s-t)/\sqrt{2}, \lambda) = u_0(z,\lambda)$, we have*

$$||D_x^2 u(\cdot,\lambda) - D_x^2 U(\cdot,\lambda)||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)} \to 0, \qquad \text{as} \;\; R \to +\infty,$$

for every $\lambda \in [0, +\infty)$.

Proof. The proof follows the ones of the analogous results in [76, 46, 60], where a compactness argument is used. Therefore, we only give here the main ideas, since the details can be found in those papers. Arguing by contradiction, we suppose that the asymptotic result does not hold. Hence, there exists an $\varepsilon > 0$ and a sequence $\{x_k\} \subset \mathcal{O}$ such that

$$|D_x^2 u(x_k,\lambda) - D_x^2 U(x_k,\lambda)| \geq \varepsilon \quad \text{and} \quad |x_k| \to +\infty. \tag{3.5.1}$$

Now we distinguish two cases, depending on whether the sequence $\{\text{dist}(x_k, \mathscr{C})\}$ is unbounded or bounded. In the first case, we show that, up to a subsequence, the function $u_k(x,\lambda) := u(x + x_k, \lambda)$ converges to a solution u_∞ of the semilinear Neumann problem in the half-space $\mathbb{R}^{2m+1}_+$ appearing in the statement of Theorem 5.3 in [76] (see [145] for the proof). Using this result and the stability of u_∞ we get that $u_\infty \equiv 1$. Thus, $|D_x^2 u(x_k,\lambda)| \to 0$, and since $|D_x^2 U(x_k,\lambda)| \to 0$, we arrive at a contradiction with (3.5.1).

In the second case, we have $\text{dist}(x_k, \mathscr{C}) = |x_k - x_k^0|$ bounded, where $x_k^0 \in \mathscr{C}$. Since the Simons cone converges to a hyperplane at infinity (see the details in [60]), it can be proved that, up to a subsequence and a rotation, the function $u_k(x,\lambda) := u(x + x_k^0, \lambda)$ converges to a positive solution u_∞ of an equation in the quarter-space $\mathbb{R}^{2m+1}_{++} = \mathbb{R}^{2m+1}_+ \cap \{x_{2m} > 0\}$ with zero Dirichlet boundary conditions, as in the statement of Theorem 5.5 in [76] (see [186] for the proof). Applying this last theorem and the stability again, we conclude that u_∞ must be the 2D solution u_0 depending only on x_{2m} and λ. Hence, $D_x^2(u-U)(x_k,\lambda)$ converges to zero, and we arrive at a contradiction with (3.5.1). □

With the help of the maximum principle of Proposition 3.1.4, the asymptotic result for the saddle-shaped solution, and the monotonicity properties of the layer solution, we can prove Proposition 3.1.8.

Proof of Proposition 3.1.8. We write (3.1.3) in (s,t,λ) variables:

$$\begin{cases} u_{ss} + u_{tt} + u_{\lambda\lambda} = -(m-1)\left(\dfrac{u_s}{s} + \dfrac{u_t}{t}\right) - \dfrac{a}{\lambda} u_\lambda & \text{in } \{st > 0, \; \lambda > 0\}, \\ u_s = 0 & \text{on } \{s = 0, \; \lambda \geq 0\}, \\ u_t = 0 & \text{on } \{t = 0, \; \lambda \geq 0\}, \\ d_\gamma \dfrac{\partial u}{\partial \nu^a} = f(u) & \text{on } \{\lambda = 0\}. \end{cases} \tag{3.5.2}$$

Differentiating the previous equation with respect to s we find that

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u_s) &= (m-1)\dfrac{\lambda^a}{s^2} u_s \quad \text{in } \{s>t,\ \lambda>0\}\,, \\ d_\gamma \dfrac{\partial u_s}{\partial \nu^a} &= f'(u) u_s \qquad\quad \text{on } \{s>t,\ \lambda=0\}\,. \end{cases}$$

Since $u=0$ on $\{s=t,\ \lambda\geq 0\}$ and $u>0$ in $\{s>t,\ \lambda\geq 0\}$, we have that $u_s\geq 0$ on $\partial_L\{s>t,\ \lambda>0\}=\{s=t,\ \lambda\geq 0\}$. Moreover, by the asymptotic result (point (ii) of Theorem 3.1.2) and the monotonicity properties of the layer solution u_0 (Lemma 3.4.2), we have

$$\liminf_{\{s>t\},\ |(s,t)|\to+\infty} u_s(s,t,0)\geq 0\,.$$

Indeed, if u_0 is the layer solution,

$$\partial_s U(x,0) = \frac{1}{\sqrt{2}}\partial_x u_0\left(\frac{s-t}{\sqrt{2}},0\right)\geq 0$$

and

$$\lim_{R\to+\infty} ||(u_s-\partial_s U)(\cdot,0)||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)}=0\,.$$

Thus, by the maximum principle for the linearized operator (Proposition 3.1.4) applied to $v=-u_s$, with $b(x,\lambda)=-(m-1)\lambda^a/s^2\leq 0$, we conclude that $u_s\geq 0$ in $\{s\geq t,\ \lambda\geq 0\}$.

Similarly, if we differentiate (3.5.2) with respect to t, we obtain

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u_t) &= (m-1)\dfrac{\lambda^a}{t^2} u_t \quad \text{in } \{s>t>0,\ \lambda>0\}\,, \\ d_\gamma \dfrac{\partial u_t}{\partial \nu^a} &= f'(u) u_t \qquad\quad \text{on } \{s>t>0,\ \lambda=0\}\,. \end{cases}$$

In the lateral boundary $\partial_L\{s>t>0,\ \lambda>0\}=\{s=t,\ \lambda\geq 0\}\cup\{t=0,\ \lambda\geq 0\}$ we have $-u_t\geq 0$. Indeed, $u_t=0$ on $\{t=0,\ \lambda\geq 0\}$, and since $u=0$ on $\{s=t,\ \lambda\geq 0\}$ and $u>0$ in $\{s>t,\ \lambda\geq 0\}$, it holds $-u_t\geq 0$ on $\{s=t,\ \lambda\geq 0\}$. Furthermore, the asymptotic behavior of u and the monotonicity properties of the layer solution u_0 yield

$$\limsup_{\{s>t>0\},\ |(s,t)|\to+\infty} u_t(s,t,0)\leq 0\,.$$

Indeed,

$$\partial_t U(x,0) = -\frac{1}{\sqrt{2}}\partial_1 u_0\left(\frac{s-t}{\sqrt{2}},0\right)\leq 0$$

and

$$\lim_{R\to+\infty} ||(u_t-\partial_t U)(\cdot,0)||_{L^\infty(\mathbb{R}^{2m}\setminus B_R)}=0\,.$$

Thus, using again the maximum principle for the linearized operator we find that $-u_t\geq 0$ in $\{s\geq t,\ \lambda\geq 0\}$.

By the odd symmetry of u, i.e., $u(s,t)=-u(t,s)$, we conclude that $u_s\geq 0$ and $u_t\leq 0$ in $\mathbb{R}^{2m}\times[0,+\infty)$. This fact and the strong maximum principle give that $u_s>0$ in $(\mathbb{R}^{2m}\setminus\{s=0\})\times[0,+\infty)$ and $-u_t>0$ in $(\mathbb{R}^{2m}\setminus\{t=0\})\times[0,+\infty)$.

Now we check the sign of the y-derivative. We use that $\partial_y=(\partial_s+\partial_t)/\sqrt{2}$ to see that

$$\operatorname{div}(\lambda^a\nabla u_y) = (m-1)\frac{\lambda^a}{\sqrt{2}}\left(\frac{u_s}{s^2}+\frac{u_t}{t^2}\right) = (m-1)\frac{\lambda^a}{s^2}u_y + (m-1)\frac{\lambda^a}{\sqrt{2}}\frac{s^2-t^2}{s^2t^2}u_t\,.$$

Hence, using that $u_t \leq 0$ in $\{s > t > 0,\ \lambda > 0\}$ we get

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u_y) & \leq & (m-1)\dfrac{\lambda^a}{s^2} u_y & \text{in } \{s > t > 0,\ \lambda > 0\}\,, \\ d_\gamma \dfrac{\partial u_y}{\partial \nu^a} & = & f'(u) u_y & \text{on } \{s > t > 0,\ \lambda = 0\}\,. \end{cases}$$

Note that, since u vanishes at $\mathscr{C} \times [0, +\infty)$, $u_y = 0$ on $\{s = t,\ \lambda \geq 0\}$. Moreover, $u_s \geq 0$ and $u_t = 0$ on $\{t = 0,\ \lambda \geq 0\}$. Therefore, $u_y \geq 0$ on $\partial_L \{s > t > 0,\ \lambda > 0\} = \{s = t,\ \lambda \geq 0\} \cup \{t = 0,\ \lambda \geq 0\}$. Furthermore, by the asymptotic behavior of u and the monotonicity properties of the layer solution u_0 we have

$$\liminf_{\{s>t>0\},\ |(s,t)| \to +\infty} u_y(s,t,0) = 0\,,$$

since

$$\partial_y U(x,0) = \partial_y u_0(z,0) = 0 \quad \text{and} \quad \lim_{R \to +\infty} ||(u_y - \partial_y U)(\cdot, 0)||_{L^\infty(\mathbb{R}^{2m} \setminus B_R)} = 0\,.$$

Again, by using the maximum principle of Proposition 3.1.4, we deduce that $u_y \geq 0$ in $\{s \geq t,\ \lambda \geq 0\}$, and the strong maximum principle yields $u_y > 0$ on $\{s > t,\ \lambda \geq 0\}$.

Finally, we prove the last statement concerning the crossed derivatives. By differentiating (3.5.2), first with respect to s and then with respect to t, we find

$$\begin{cases} \operatorname{div}(\lambda^a \nabla u_{st}) & = & (m-1)\lambda^a \left(\dfrac{1}{s^2} + \dfrac{1}{t^2}\right) u_{st} & \text{in } \{s > t > 0,\ \lambda > 0\}\,, \\ d_\gamma \dfrac{\partial u_{st}}{\partial \nu^a} & = & f'(u) u_{st} + f''(u) u_s u_t \geq f'(u) u_{st} & \text{on } \{s > t > 0,\ \lambda = 0\}\,. \end{cases}$$

Here we have used that $f''(\tau) \leq 0$ if $\tau \in (0,1)$ and that $u_s u_t \leq 0$ in $\{s > t > 0,\ \lambda = 0\}$. Note that, by symmetry, $u_{st} = 0$ on $\{s = t,\ \lambda \geq 0\}$. Moreover, since $u_t(s, 0, \lambda) = 0$ for every $s > 0$ and $\lambda \geq 0$, $u_{st} = 0$ on $\{t = 0,\ \lambda \geq 0\}$. Therefore, $u_{st} = 0$ on $\partial_L \{s > t > 0,\ \lambda > 0\}$. In addition, by the asymptotic result of Lemma 3.5.1 and the monotonicity properties of the layer solution u_0 (Lemma 3.4.2), we have

$$\liminf_{\{s>t>0\},\ |(s,t)| \to +\infty} u_{st}(s,t,0) \geq 0\,,$$

since

$$U_{st}(x,0) = -\frac{1}{2}\partial_1^2 u_0 \left(\frac{s-t}{\sqrt{2}}, 0\right) \geq 0$$

and

$$\lim_{R \to +\infty} ||(u_{st} - U_{st})(\cdot, 0)||_{L^\infty(\mathbb{R}^{2m} \setminus B_R)} = 0\,.$$

Hence, by the maximum principle for the linearized operator (Proposition 3.1.4), we deduce that $u_{st} \geq 0$ in $\{s \geq t,\ \lambda \geq 0\}$, and the strong maximum principle yields $u_{st} > 0$ in $\{s > t > 0,\ \lambda \geq 0\}$. □

3.6 Stability of the saddle-shaped solution and the Simons cone in dimensions $2m \geq 14$

In this last section we prove our stability results. The first one is Theorem 3.1.6 and it establishes the stability of the saddle-shaped solution in dimensions $2m \geq 14$. The proof follows the strategy of its analogue in [46] and it is based on finding a positive supersolution to the linearized problem.

Proof of Theorem 3.1.6. Let us show that $\varphi = t^{-b}u_s - s^{-b}u_t$, with $b(b-m+2)+m-1 \le 0$ and $b > 0$, is a positive supersolution of the linearized operator. That is, it satisfies

$$\varphi > 0 \quad \text{in } \overline{\mathbb{R}^{2m+1}_+} \setminus \{st = 0,\ \lambda > 0\}\,, \tag{3.6.1}$$

$$-\operatorname{div}(\lambda^a \nabla\varphi) \ge 0 \quad \text{in } \mathbb{R}^{2m+1}_+ \setminus \{st = 0,\ \lambda > 0\}\,, \tag{3.6.2}$$

and

$$\mathscr{L}_u \varphi \ge 0 \quad \text{on } \mathbb{R}^{2m} \setminus \{st = 0\}\,. \tag{3.6.3}$$

Indeed, note that $\varphi > 0$ in $\{s > t > 0,\ \lambda \ge 0\}$ by the monotonicity properties of u (Proposition 3.1.8). Since φ is even with respect to the Simons cone, i.e., $\varphi(t,s,\lambda) = \varphi(s,t,\lambda)$, it holds (3.6.1). Moreover, (3.6.3) follows readily, since φ satisfies

$$d_\gamma \frac{\partial \varphi}{\partial \nu^a} = f'(u)\,\varphi\,.$$

Let us now show (3.6.2). Since φ is even with respect to the Simons cone, it is enough to check that $\operatorname{div}(\lambda^a \nabla\varphi) \le 0$ in $\{s > t > 0,\ \lambda > 0\}$. By using that $\operatorname{div}(\lambda^a \nabla u) = 0$, we obtain by a direct computation that

$$\begin{aligned}
\lambda^{-a}\operatorname{div}(\lambda^a \nabla\varphi) = {} & b(b-m+2)\left(t^{-b-2}\,u_s - s^{-b-2}\,u_t\right) \\
& + (m-1)\left(t^{-b}s^{-2}u_s - s^{-b}t^{-2}u_t\right) \\
& + 2b\left(t^{-b-1} - s^{-b-1}\right)u_{st}\,.
\end{aligned}$$

Now, by using that $u_{st} > 0$, $u_y > 0$ and $-u_t > 0$ in $\{s > t > 0,\ \lambda > 0\}$, and the fact that $b > 0$ satisfies $b(b-m+2) \le -(m-1)$, we arrive at

$$\begin{aligned}
\lambda^{-a}\operatorname{div}(\lambda^a \nabla\varphi) \le {} & t^{-b}(u_s + u_t)\left((m-1)s^{-2} + b(b-m+2)t^{-2}\right) \\
& - t^{-b}u_t\left\{(m-1)s^{-2} + b(b-m+2)t^{-2}\right\} \\
& - s^{-b}u_t\left\{(m-1)t^{-2} + b(b-m+2)s^{-2}\right\} \\
= {} & \sqrt{2}t^{-b}u_y\left((m-1)s^{-2} + b(b-m+2)t^{-2}\right) \\
& + (-u_t)(m-1)\left(t^{-b}s^{-2} + s^{-b}t^{-2}\right) \\
& + (-u_t)b(b-m+2)\left(t^{-2-b} + s^{-2-b}\right) \\
= {} & \sqrt{2}(m-1)t^{-b}u_y\left(s^{-2} - t^{-2}\right) \\
& + (-u_t)(m-1)\left(t^{-b}s^{-2} + s^{-b}t^{-2} - t^{-2-b} - s^{-2-b}\right) \\
\le {} & (-u_t)(m-1)(s^{-b} - t^{-b})(t^{-2} - s^{-2}) \\
\le {} & 0\,.
\end{aligned}$$

Note that the existence of $b > 0$ such that $b(b-m+2) \le -(m-1)$ is guaranteed by the assumption $2m \ge 14$.

Finally, let us show that since we have a positive supersolution to the linearized operator on $\mathbb{R}^{2m} \setminus \{st = 0\}$, the stability of u follows. We must check that (3.1.7) holds. To do it, let us first take nonnegative functions $\zeta \in C^1(\overline{\mathbb{R}^{2m+1}_+})$ with compact support in $\{st > 0,\ \lambda \ge 0\}$. Multiply (3.6.2) by ζ and integrate by parts. Using (3.6.3) we obtain

$$\int_{\{st>0\}} f'(u)\,\varphi\,\zeta\,\mathrm{d}x \le d_\gamma \int_0^\infty \int_{\{st>0\}} \lambda^a \nabla\varphi \cdot \nabla\zeta\,\mathrm{d}x\,\mathrm{d}\lambda\,. \tag{3.6.4}$$

Now, let $\overline{\xi} \in C_c^\infty(\overline{\mathbb{R}^{2m+1}_+} \setminus \{st = 0,\ \lambda \geq 0\})$. Since $\varphi > 0$ in $\{st > 0, \lambda \geq 0\}$, taking $\zeta = \overline{\xi}^2/\varphi$ in (3.6.4) and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_{\{st>0\}} f'(u)\,\overline{\xi}^2\,\mathrm{d}x &= \int_{\{st>0\}} f'(u)\,\varphi\frac{\overline{\xi}^2}{\varphi}\,\mathrm{d}x \leq d_\gamma \int_0^\infty\int_{\{st>0\}} \lambda^a\nabla\varphi\cdot\nabla\left(\frac{\overline{\xi}^2}{\varphi}\right)\mathrm{d}x\,\mathrm{d}\lambda \\ &= d_\gamma\int_0^\infty\int_{\{st>0\}} \lambda^a\,\frac{2\overline{\xi}}{\varphi}\,\nabla\varphi\cdot\nabla\overline{\xi}\,\mathrm{d}x\,\mathrm{d}\lambda - d_\gamma\int_0^\infty\int_{\{st>0\}}\lambda^a\,\frac{\overline{\xi}^2}{\varphi^2}|\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}\lambda \\ &\leq d_\gamma\int_0^\infty\int_{\{st>0\}}\lambda^a\,|\nabla\overline{\xi}|^2\,\mathrm{d}x\,\mathrm{d}\lambda\,.\end{aligned}$$

To conclude the proof, let us show that the last inequality holds for every smooth function ξ with compact support in $\overline{\mathbb{R}^{2m+1}_+}$. This will yield the stability of u. Take $\eta_\varepsilon \in C^\infty(\mathbb{R})$ such that $0 \leq \eta_\varepsilon \leq 1$ and

$$\eta_\varepsilon = \begin{cases} 1 & \text{in} \quad [\varepsilon, +\infty)\,, \\ 0 & \text{in} \quad [0, \varepsilon/2)\,. \end{cases}$$

Then, since $\xi\,\eta_\varepsilon(s)\,\eta_\varepsilon(t)$ has compact support in $\{st > 0,\ \lambda \geq 0\}$, we can replace $\overline{\xi}$ by $\xi\,\eta_\varepsilon(s)\,\eta_\varepsilon(t)$ in the previous inequality to get

$$\frac{1}{d_\gamma}\int_{\mathbb{R}^{2m}} f'(u)\,\xi^2\,\eta_\varepsilon^2(s)\,\eta_\varepsilon^2(t)\,\mathrm{d}x \leq \int_{\mathbb{R}^{2m+1}_+}\lambda^a\,|\nabla(\xi\,\eta_\varepsilon(s)\,\eta_\varepsilon(t))|^2\,\mathrm{d}x\,\mathrm{d}\lambda\,.$$

Now, we compute the terms in the right-hand side of this inequality. By using Cauchy-Schwarz, we see that to deduce the stability condition

$$\frac{1}{d_\gamma}\int_{\mathbb{R}^{2m}} f'(u)\,\xi^2\,\mathrm{d}x \leq \int_{\mathbb{R}^{2m+1}_+}\lambda^a|\nabla\xi|^2\,\mathrm{d}x\,\mathrm{d}\lambda$$

by letting $\varepsilon \to 0$, it is enough to show that

$$\int_{\mathbb{R}^{2m+1}_+}\lambda^a|\nabla\eta_\varepsilon(s)|^2\,\mathrm{d}x\,\mathrm{d}\lambda \to 0 \quad \text{as} \quad \varepsilon \to 0\,,$$

and the same with $\eta_\varepsilon(s)$ replaced by $\eta_\varepsilon(t)$. To see this, let $R > 0$ be such that $\operatorname{supp}(\xi) \subset \overline{B_R^+}$. Then, since $m \geq 3$,

$$\begin{aligned}\int_{\mathbb{R}^{2m+1}_+}\lambda^a|\nabla\eta_\varepsilon(s)|^2\,\mathrm{d}x\,\mathrm{d}\lambda &\leq \frac{C}{\varepsilon^2}\int_0^R \mathrm{d}\lambda\,\lambda^a\int_0^\varepsilon \mathrm{d}s\,s^{m-1}\int_0^R \mathrm{d}t\,t^{m-1} \\ &\leq C\,R^{m+a+1}\,\varepsilon^{m-2} \to 0 \quad \text{as} \quad \varepsilon \to 0\,,\end{aligned}$$

The computation is analogous for $\eta_\varepsilon(t)$. □

Finally, we present the proof of the stability of the Simons cone as a nonlocal (2γ)-minimal surface whenever $2m \geq 14$ and $\gamma \in (0, 1/2)$.

Proof of Corollary 3.1.7. Let u be the saddle-shaped solution of (3.1.1) in dimension $2m \geq 14$. Consider the blow-down sequence $u_k(x) = u(kx)$ with $k \in \mathbb{N}$. On the one hand, since

u is stable in such dimensions and $\gamma \in (0,1/2)$, by Theorem 2.6 in [50] there exists a subsequence k_j such that

$$u_{k_j} \to \chi_\Sigma - \chi_{\mathbb{R}^{2m}\setminus\Sigma} \quad \text{in } L^1(B_1) \quad \text{as } k_j \to +\infty\,,$$

for some cone Σ that is a stable set for the fractional perimeter.

On the other hand, by the asymptotic behavior of u (point (ii) in Theorem 3.1.2) it is clear that

$$u_k \to \chi_{\mathcal{O}} - \chi_{\mathbb{R}^{2m}\setminus\mathcal{O}} \quad \text{a.e. as } k \to +\infty\,.$$

Putting all together we conclude that $\mathcal{O}$ is a stable set for the fractional perimeter if $2m \geq 14$ and $\gamma \in (0,1/2)$. This is the same as saying that the Simons cone is a stable nonlocal (2γ)-minimal surface in such dimensions. □

Chapter 4

Semilinear integro-differential equations, I: odd solutions with respect to the Simons cone

This chapter corresponds to [110], in collaboration with T. Sanz-Perela, and it is the first of two chapters concerning saddle-shaped solutions to the semilinear equation $L_K u = f(u)$ in $\mathbb{R}^{2m}$, where L_K is a linear elliptic integro-differential operator and f is of Allen-Cahn type.

Saddle-shaped solutions are doubly radial, odd with respect to the Simons cone $\{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m : |x'| = |x''|\}$, and vanish only on this set. By the odd symmetry, L_K coincides with a new operator $L_K^{\mathcal{O}}$ which acts on functions defined only on one side of the Simons cone, $\{|x'| > |x''|\}$, and that vanish on it. This operator $L_K^{\mathcal{O}}$, which corresponds to reflect a function oddly and then apply L_K, has a kernel on $\{|x'| > |x''|\}$ which is different from K.

In this first chapter, we characterize the kernels K for which the new kernel is positive and therefore one can develop a theory on the saddle-shaped solution. The necessary and sufficient condition for this turns out to be that K is radially symmetric and $\tau \mapsto K(\sqrt{\tau})$ is a strictly convex function.

Assuming this, we prove an energy estimate for doubly radial odd minimizers and the existence of saddle-shaped solution. In Chapter 3, further qualitative properties of saddle-shaped solutions will be established, such as their asymptotic behavior, a maximum principle for the linearized operator, and their uniqueness.

4.1 Introduction

In this chapter we study solutions to the semilinear integro-differential equation

$$L_K u = f(u) \quad \text{in } \mathbb{R}^{2m} \tag{4.1.1}$$

which are odd with respect to the Simons cone — defined in (4.1.5). The interest on these solutions, often called saddle-shaped solutions, is motivated by the nonlocal version of a conjecture by De Giorgi on the Allen-Cahn equation (see details below) with the aim of finding a counterexample in high dimensions. Moreover, this problem is related to the regularity theory of nonlocal minimal surfaces.

There are only three works in the literature concerning saddle-shaped solutions to (4.1.1) with L_K being the fractional Laplacian: [75, 76] by Cinti and [111], which corresponds to Chapter 3 of the present thesis, by the author and Sanz-Perela. In all of them the main tool is the extension problem. In the this chapter and the next one, we study for the first time (4.1.1) without the extension. For this reason our arguments are purely nonlocal and hold for a more general class of kernels.

Equation (4.1.1) is driven by an integro-differential operator L_K of the form

$$L_K u(x) = \int_{\mathbb{R}^n} \{u(x) - u(y)\} K(x-y)\,\mathrm{d}y, \tag{4.1.2}$$

where the kernel K satisfies

$$K \geq 0\,, \quad K(z) = K(-z) \quad \text{and} \quad \int_{\mathbb{R}^n} \min\left\{|z|^2, 1\right\} K(z)\,\mathrm{d}z < +\infty\,. \tag{4.1.3}$$

The integral in (4.1.2) has to be understood in the principal value sense. The most canonical example of such operators is the fractional Laplacian, defined for $\gamma \in (0,1)$ as

$$(-\Delta)^\gamma u = c_{n,\gamma} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\gamma}}\,\mathrm{d}y\,,$$

where $c_{n,\gamma}$ is a normalizing constant.

Recall that the fractional Laplacian has an associated extension problem (see [63]) that allows the use of local arguments to deal with equations such as (4.1.1). This is not the case for general operators L_K, and therefore some purely nonlocal techniques are developed along this work.

Throughout the chapter, we assume that L_K is uniformly elliptic, that is,

$$\lambda \frac{c_{n,\gamma}}{|z|^{n+2\gamma}} \leq K(z) \leq \Lambda \frac{c_{n,\gamma}}{|z|^{n+2\gamma}}\,, \tag{4.1.4}$$

where λ and Λ are two positive constants. This condition is frequently adopted since it yields Hölder regularity of solutions (see [162, 176]). The family of linear operators satisfying conditions (4.1.3) and (4.1.4) is the so-called $\mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ ellipticity class. For short we will usually write $\mathcal{L}_0$ and we will make explicit the parameters only when needed.

Moreover, for many purposes we will need the operators to be invariant under rotations. This is equivalent to saying that the kernel is radially symmetric, $K(z) = K(|z|)$.

The Simons cone will be a central object along this chapter. It is defined in $\mathbb{R}^{2m}$ by

$$\mathscr{C} := \left\{ x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m} \ : \ |x'| = |x''| \right\}. \tag{4.1.5}$$

This cone is of special importance in the theory of local and nonlocal minimal surfaces, and its variational properties are related to the conjecture of De Giorgi (see the end of this introduction for more details). Through the whole chapter we will use $\mathcal{O}$ and $\mathcal{I}$ to denote each of the parts in which $\mathbb{R}^{2m}$ is divided by the cone $\mathscr{C}$:

$$\mathcal{O} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| > |x''| \right\} \text{ and } \mathcal{I} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| < |x''| \right\}.$$

Both $\mathcal{O}$ and $\mathcal{I}$ belong to a family of sets in $\mathbb{R}^{2m}$ which are called of *double revolution*. These are sets that are invariant under orthogonal transformations in the first m variables, as well as under orthogonal transformations in the last m variables. That is, $\Omega \subset \mathbb{R}^{2m}$ is a set of double revolution if $R\Omega = \Omega$ for every given transformation $R \in O(m)^2 = O(m) \times O(m)$, where $O(m)$ is the orthogonal group of $\mathbb{R}^m$.

In this chapter we deal with functions that are *doubly radial*. These are functions $w : \mathbb{R}^{2m} \to \mathbb{R}$ that only depend on the modulus of the first m variables and on the modulus of the last m ones, i.e., $w(x) = w(|x'|, |x''|)$. Equivalently, $w(Rx) = w(x)$ for every $R \in O(m)^2$.

In order to define oddness and evenness of functions with respect to the Simons cone, we consider the following isometry, which will play a significant role in this chapter:

$$\begin{array}{cccc} (\cdot)^\star : & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m \\ & x = (x', x'') & \mapsto & x^\star = (x'', x') \,. \end{array} \tag{4.1.6}$$

Note that this isometry is actually an involution that maps $\mathcal{O}$ into $\mathcal{I}$ (and vice versa) and leaves the cone $\mathscr{C}$ invariant —although not all points in $\mathscr{C}$ are fixed points of $(\cdot)^\star$. Taking into account this transformation, we say that a doubly radial function w is *odd with respect to the Simons cone* if $w(x) = -w(x^\star)$. Similarly, we say that a doubly radial function w is *even with respect to the Simons cone* if $w(x) = w(x^\star)$.

Regarding the doubly radial symmetry we define the following variables

$$s := |x'| \quad \text{and} \quad t := |x''| \,.$$

They are specially useful when dealing with the Laplacian in these coordinates, since

$$\Delta w = w_{ss} + w_{tt} + \frac{m-1}{s} w_s + \frac{m-1}{t} w_t \tag{4.1.7}$$

becomes an expression suitable to work with. A similar formula appears in the case of the fractional Laplacian thanks to the local extension problem. Having a PDE in the two variables $(s,t) \in \mathbb{R}^2$ is useful to perform certain computations (see [59, 60, 46, 54] for the local case and [75, 76, 111] for the fractional framework).

If we try to follow the same strategy by writing a rotation invariant operator L_K in (s,t) variables, the expression of the new operator is quite complex. Indeed, if $w : \mathbb{R}^{2m} \to \mathbb{R}$ is doubly radial and we define $\widetilde{w}(s,t) := w(s, 0, ..., 0, t, 0, ..., 0)$, it holds

$$L_K w(x) = \widetilde{L}_K \widetilde{w}(|x'|, |x''|)$$

with

$$\widetilde{L}_K \widetilde{w}(s,t) := \int_0^{+\infty} \int_0^{+\infty} \sigma^{m-1} \tau^{m-1} \Big(\widetilde{w}(s,t) - \widetilde{w}(\sigma, \tau) \Big) J(s,t,\sigma,\tau) \, \mathrm{d}\sigma \, \mathrm{d}\tau \tag{4.1.8}$$

and

$$J(s,t,\sigma,\tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\Big(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\widetilde{\omega}_1} \Big) \, \mathrm{d}\omega \, \mathrm{d}\widetilde{\omega} \,.$$

Note that $\widetilde{L}_K$ is an integro-differential operator in $(0,+\infty) \times (0,+\infty)$, but the expression of its kernel is quite involved. Indeed, such an expression does not become simpler even when L_K is the fractional Laplacian. In this case, the kernel J involves hypergeometric functions of two variables, the so-called Appell functions (see Appendix 4.8 for more details on it), but this does not simplify computations.

Instead of working with the (s,t) variables, we follow another approach that we find more clear and concise. It consists on rewriting the operator L_K with a different kernel $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ that is doubly radial with respect to its both arguments, but in such a way that it still acts on functions defined in $\mathbb{R}^{2m}$ —and not in $(0,+\infty)^2$. As it is explained in detail in Section 4.2, if K is a radially symmetric kernel, then we can write L_K acting on a doubly radial function w as

$$L_K w(x) = \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \overline{K}(x,y)\, \mathrm{d}y\,, \tag{4.1.9}$$

where $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ is doubly radial in both arguments and is defined by

$$\overline{K}(x,y) := \fint_{O(m)^2} K(|Rx - y|)\, \mathrm{d}R\,. \tag{4.1.10}$$

Here, $\mathrm{d}R$ denotes integration with respect to the Haar measure on $O(m)^2$ (see Section 4.2 for the details).

This new expression (4.1.9) has some advantages compared with (4.1.8). First, the computations in this new setting are shorter and more transparent than the analogous ones using (s,t) variables. This also makes the notation more concise. Furthermore we avoid some issues of the (s,t) variables such as the special treatment of the set $\{st = 0\}$. Although in this chapter we do not work in (s,t) variables, we include an appendix at the end with some computations using them (see Appendix 4.8). We think that this could be useful in future works.

Once we have rewritten L_K with a doubly radial kernel $\overline{K}$, as in (4.1.9), we shall find a suitable expression of the operator when acting on odd functions with respect to the Simons cone. Note that such functions are defined by their values in $\mathcal{O}$ and therefore we want to rewrite L_K taking this into account. To this purpose, we define the new operator

$$\begin{aligned} L_K^{\mathcal{O}} w(x) &:= \int_{\mathcal{O}} \{w(x) - w(y)\} \overline{K}(x,y)\, \mathrm{d}y + \int_{\mathcal{O}} \{w(x) + w(y)\} \overline{K}(x,y^\star)\, \mathrm{d}y \\ &= \int_{\mathcal{O}} \{w(x) - w(y)\} \{\overline{K}(x,y) - \overline{K}(x,y^\star)\}\, \mathrm{d}y + 2w(x) \int_{\mathcal{O}} \overline{K}(x,y^\star)\, \mathrm{d}y\,, \end{aligned} \tag{4.1.11}$$

where $(\cdot)^\star$ is defined in (4.1.6). As we show in Section 4.2, $L_K^{\mathcal{O}}$ acting on a doubly radial function $w : \mathcal{O} \to \mathbb{R}$ coincides with L_K acting on the odd extension of w with respect to the Simons cone.

Our first main result concerns necessary and sufficient conditions on the original kernel K for this operator to have a positive kernel. As we will stress through this chapter, and also in the next one, the positivity of the kernel in (4.1.11) is crucial in order to develop a theory on the saddle-shaped solution. In particular, under this assumption a maximum principle for doubly radial odd functions will hold (see Proposition 4.1.2 below).

Theorem 4.1.1. *Let $K : (0,+\infty) \to (0,+\infty)$ and consider the radially symmetric kernel $K(|x-y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ by (4.1.10).*

If

$$K(\sqrt{\tau}) \text{ is a strictly convex function of } \tau\,, \tag{4.1.12}$$

then L_K has a positive kernel in $\mathcal{O}$ when acting on doubly radial functions which are odd with respect to the Simons cone $\mathscr{C}$. More precisely, it holds

$$\overline{K}(x,y) > \overline{K}(x,y^\star) \quad \text{for every } x,y \in \mathcal{O}\,. \tag{4.1.13}$$

In addition, if $K \in C^2((0,+\infty))$, then (4.1.12) is not only a sufficient condition for (4.1.13) to hold, but also a necessary one.

This theorem is proved in Section 4.2 (see Propositions 4.2.4 and 4.2.5). Its proof is based on breaking the integral defining $\overline{K}$ in four clever regions —see (4.2.6)— that allow to compare the integrands for $y \in \mathcal{O}$ and for its reflected $y^* \in \mathcal{I}$. We will use a result on convex functions proved in Appendix 4.6 (Proposition 4.6.1). In the previous statement, by strict convexity in (4.1.12) we mean that

$$K(\sqrt{\tau_1}) + K(\sqrt{\tau_2}) > 2K(\sqrt{(\tau_1 + \tau_2)/2})$$

for every τ_1, $\tau_2 \in (0, +\infty)$.

In [134], Jarohs and Weth study solutions to general integro-differential equations which are odd with respect to a hyperplane. Here the natural sufficient condition on K to have a positive kernel when acting on odd functions is that K is decreasing in the orthogonal direction to the hyperplane. That this suffices is readily deduced after making a change of variables given by the symmetry with respect to such hyperplane. In our case, since we deal with a more complex symmetry, the kernel K is required to satisfy further assumptions than just monotonicity. Moreover, the proof of Theorem 4.1.1 is quite involved and requires a finer argument. Indeed, if we simply make the change $y \mapsto y^\star$ in (4.1.2), following [134], we should prove that $K(|x - y|) > K(|x - y^\star|)$ for every x and y in $\mathcal{O}$, but this is false even in the easiest case $L_K = (-\Delta)^\gamma$ and $2m = 2$. Instead, if we write L_K in the form (4.1.9) with the kernel $\overline{K}$, the analogous positivity condition (4.1.13) holds if we assume $K(\sqrt{\cdot})$ to be convex. Here the use of the (s, t) variables would not simplify the proof of Theorem 4.1.1. As mentioned in Appendix 4.8, an analogous result can be established for the kernel J in (4.1.8), but its proof presents exactly the same difficulties as the one for $\overline{K}$.

The first direct consequence of the positivity condition (4.1.13) is the following maximum principle.

Proposition 4.1.2 (Maximum principle for odd functions with respect to $\mathscr{C}$)**.** *Let $\Omega \subset \mathcal{O}$ be an open set and let L_K be an integro-differential operator with a radially symmetric kernel K satisfying the positivity condition (4.1.13). Let $u \in C^\alpha(\Omega) \cap C^\gamma(\overline{\Omega}) \cap L^\infty(\mathbb{R}^{2m})$, with $\alpha > 2\gamma$, be a doubly radial function which is odd with respect to the Simons cone.*

(i) (Weak maximum principle) Assume that

$$\begin{cases} L_K u + c(x) u & \geq & 0 & \text{in } \Omega\,, \\ u & \geq & 0 & \text{in } \mathcal{O} \setminus \Omega\,, \end{cases}$$

with $c \geq 0$, and that either

$$\Omega \text{ is bounded} \quad \text{or} \quad \liminf_{x \in \mathcal{O},\, |x| \to +\infty} u(x) \geq 0\,.$$

Then, $u \geq 0$ in Ω.

(ii) (Strong maximum principle) Assume that $L_K u + c(x) u \geq 0$ in Ω, with c any continuous function, and that $u \geq 0$ in $\mathcal{O}$. Then, either $u \equiv 0$ in $\mathcal{O}$ or $u > 0$ in Ω.

This statement differs from the usual maximum principle for L_K in the fact that we only assume that u is nonpositive in $\mathcal{O} \setminus \Omega$, instead of in $\mathbb{R}^{2m} \setminus \Omega$ (an assumption that makes no sense for odd functions). This form of maximum principle is analogous to the ones in [71, 134], where similar statements are considered for functions that are odd with respect to a hyperplane.

Since in this chapter we will always consider doubly radial functions u which are odd with respect to the Simons cone, $L_K u = L_K^{\mathcal{O}} u$ in $\mathcal{O}$. Thus, to simplify the notation we will always write L_K for $L_K^{\mathcal{O}}$. To mean that Proposition 4.1.2 holds, we will say that L_K has a maximum principle in $\mathcal{O}$ when acting on doubly radial odd functions.

Let us now turn to the variational problem from which equation (4.1.1) arises. As it is well known, (4.1.1) is the Euler-Lagrange equation associated to the energy functional

$$\begin{aligned}\mathcal{E}(w,\Omega) := \frac{1}{4}\Big\{ &\int_\Omega \int_\Omega |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ 2\int_\Omega \int_{\mathbb{R}^{2m}\setminus\Omega} |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y \Big\} + \int_\Omega G(w)\,\mathrm{d}x\,,\end{aligned} \tag{4.1.14}$$

where G a C^2 function satisfying $G' = -f$. In this chapter, we assume the following conditions on G:

$$G \text{ is even and } G \geq G(\pm 1) = 0 \text{ in } \mathbb{R}\,. \tag{4.1.15}$$

Note that the previous conditions on G yield that f is a C^1 odd function with $f(0) = f(\pm 1) = 0$. In some cases, as in Theorem 4.1.4 below, we will further assume that $G(0) > 0$. In such situation, equation (4.1.1) can be seen as a model for phase transitions. The Allen-Cahn nonlinearity, $f(u) = u - u^3$, is the most typical example.

Using the same type of arguments as for the operator L_K, we can rewrite the energy of doubly radial odd functions with a suitable new expression that involves the kernel

$$\overline{K}(x,y) - \overline{K}(x,y^\star) > 0$$

and that only takes into account the values of the functions in $\mathcal{O}$. This will be extremely useful in many computations and estimates involving the nonlocal energy $\mathcal{E}$ (see Sections 4.3 and 4.4). To write this new expression, we introduce the following notation. For A, $B \subset \mathcal{O}$, two sets of double revolution, we define

$$\begin{aligned}I_w(A,B) := 2\int_A \int_B &|w(x)-w(y)|^2 \left\{\overline{K}(x,y) - \overline{K}(x,y^\star)\right\} \mathrm{d}x\,\mathrm{d}y \\ &+ 4\int_A \int_B \left\{w^2(x) + w^2(y)\right\} \overline{K}(x,y^\star)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Then, as proved in Section 4.3 (see Lemma 4.3.2), we can rewrite the energy of a doubly radial odd function w as

$$\mathcal{E}(w,\Omega) = \frac{1}{4}\Big\{I_w(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + 2I_w(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega)\Big\} + 2\int_{\Omega\cap\mathcal{O}} G(w)\,\mathrm{d}x\,. \tag{4.1.16}$$

Thanks to this new expression for the energy, we are able to establish the second main result of this chapter. It is the following energy estimate for doubly radial odd minimizers of $\mathcal{E}$. To define such minimizers properly, we denote by $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ the space of doubly radial odd functions that vanish outside B_R and for which the energy $\mathcal{E}$ is well defined (see Section 4.3 for the precise definition). We say that $u \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ is a doubly radial odd minimizers of $\mathcal{E}$ in B_R if

$$\mathcal{E}(u,B_R) \leq \mathcal{E}(w,B_R)$$

for every $w \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$.

Theorem 4.1.3. *Let K be a radially symmetric kernel satisfying the convexity assumption* (4.1.12)[1] *and such that $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$. Assume that G is a potential satisfying* (4.1.15)*. Let $S \geq 2$ and let $u \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ be a doubly radial odd minimizer of $\mathcal{E}$ in B_R, with $R > S + 4$. Then*

$$\mathcal{E}(u, B_S) \leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0, 1/2), \\ C\ S^{2m-1} \log S & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2, 1), \end{cases} \tag{4.1.17}$$

where C is a positive constant depending only on m, γ, λ, Λ, and $||G||_{C^2([-1,1])}$.

In the proof of this result, a first basic ingredient is that $-1 \leq u \leq 1$, as provided by Lemma 4.3.3. This information, $|u| \leq 1$, is also of importance for a solution of an Allen-Cahn equation, as in the existence Theorem 4.1.4 below. That $|u| \leq 1$ is proved with a variational cutting argument: cutting above 1 and below -1 reduces de energy. We believe that this property requires $\overline{K}(x, y) - \overline{K}(x, y^\star)$ to be nonnegative. In addition, the proof of Lemma 4.3.3 is a priori not simple since it involves a nonlocal energy of functions with symmetries. We succeeded to greatly simplify the computations by writing the energy as in (4.3.4), obtaining a short proof.

Note that Theorem 4.1.3 does not follow from the energy estimate for general minimizers stated in [173] by Savin and Valdinoci. The minimizers that they consider do not have any type of symmetry. In our case, the function u in the previous statement minimizes the energy in a smaller class of functions and the result in [173] cannot be applied. Nevertheless, we are able to adapt the arguments of Savin and Valdinoci to our setting. The strategy they follow is to compare the energy of u with the one of a suitable competitor which is constructed by taking the minimum between u and a radially symmetric auxiliary function —see (4.4.5) below. Such competitor is not permitted in our case, since it is not odd with respect to the Simons cone. Nevertheless, we show in Section 4.4 how to modify the auxiliary functions of [173] to carry out the same type of arguments. The assumption (4.1.12) will be crucial to guarantee that $0 \leq u \leq 1$ in $\mathcal{O}$.

The particular result of Theorem 4.1.3 for the fractional Laplacian has been proved by Cabré and Cinti [48] in the case of the half-Laplacian, and extended to all the powers $0 < \gamma < 1$ by Cinti [76] (see [49] for an extension to non-doubly radial minimizers). These papers use the local extension problem and therefore their proofs cannot be extended to general operators like L_K. Our proof, following [173], overcomes this issue.

As an application of the previous results, we prove, by using standard variational methods, the existence of saddle-shaped solution to (4.1.1) when f is of Allen-Cahn type. We say that a bounded solution u to (4.1.1) is a *saddle-shaped* solution if u is doubly radial, odd with respect to the Simons cone, and positive in $\mathcal{O}$.

Theorem 4.1.4 (Existence of saddle-shaped solution)**.** *Let G satisfy* (4.1.15)*, $G(0) > 0$, and let $f = -G'$. Let K be a radially symmetric kernel satisfying the convexity assumption* (4.1.12) *and such that $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$.*

Then, for every even dimension $2m \geq 2$, there exists a saddle-shaped solution u to (4.1.1)*. In addition, u satisfies $|u| < 1$ in $\mathbb{R}^{2m}$.*

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[1] In this theorem, as well as in Theorem 4.1.4, we assume (4.1.12) instead of the positivity condition (4.1.13) —recall that for C^2 kernels they are equivalent. The reason for this is that in the proofs we will make use of some estimates that require the Lipschitz regularity of the kernel K (see Remark 4.3.5 below). Such regularity for K holds if (4.1.12) is satisfied, but it is not clear if this happens, when $K \notin C^2$, assuming (4.1.13) instead.

We are interested in the study of this type of solutions since they are relevant in connection with a famous conjecture for the (classical) Allen-Cahn equation raised by De Giorgi, that reads as follows. Let u be a bounded monotone (in some direction) solution to $-\Delta u = u - u^3$ in $\mathbb{R}^n$, then, if $n \leq 8$, u depends only on one Euclidean variable, that is, all its level sets are hyperplanes. This conjecture is not completely closed (see [108] and references therein) but a counterexample in dimension $n = 9$ was build in [89] by using the so-called gluing method. Saddle-shaped solutions are natural objects to build a counterexample in a simpler way, as explained next. On the one hand, Jerison and Monneau [135] showed that a counterexample to the conjecture of De Giorgi in $\mathbb{R}^{n+1}$ can be constructed with a rather natural procedure if there exists a global minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$ which is bounded and even with respect to each coordinate, but is not one-dimensional. On the other hand, by the Γ-converge results from Modica and Mortola (see [150, 151]) and the fact that the Simons cone is the simplest nonplanar minimizing minimal surface, saddle-shaped solutions are expected to be global minimizers of the Allen-Cahn equation in dimensions $2m \geq 8$ (this is still an open problem).

Similar facts happen in the nonlocal setting (see the introduction of Chapter 3 for further details). For this reason, saddle-shaped solutions are of interest in the study of the nonlocal version of the conjecture of De Giorgi for equation (4.1.1).

Saddle-shaped solutions to the local Allen-Cahn equation involving the Laplacian were studied in [84, 174, 59, 60, 46]. In these works, it is established the existence, uniqueness, and some qualitative properties of this type of solutions, such as their instability when $2m \leq 6$ and their stability if $2m \geq 14$. Stability in dimensions $8, 10$, and 12 is still an open problem, as well as minimality in dimensions $2m \geq 8$.

In the fractional framework, there are only three works concerning saddle-shaped solutions to the equation $(-\Delta)^\gamma u = f(u)$. In [75, 76], Cinti proved the existence of saddle-shaped solution as well as some qualitative properties such as their asymptotic behavior, some monotonicity properties, and their instability in low dimensions. In Chapter 3, which corresponds to [111], further properties of these solutions have been established, the main ones being uniqueness and, when $2m \geq 14$, stability. In the present chapter and the next one we first study saddle-shaped solutions for general integro-differential equations of the form (4.1.1). In the three previous works [75, 76, 111], the main tool used is the extension problem for the fractional Laplacian (see [63]). As mentioned, this technique cannot be carried out for general integro-differential operators different from the fractional Laplacian. Therefore, some purely nonlocal techniques are developed through both works.

In Chapter 5, we study saddle-shaped solutions to (4.1.1) in more detail taking advantage of the setting for odd functions built in the present chapter. We give an alternative proof for the existence of a saddle-shaped solution by using monotone iteration and maximum principle techniques. As in the proof of Theorem 4.1.4, the assumtion (4.1.13) is crucial. Furthermore, we prove the asymptotic behaviour of this type of solutions by using some symmetry and Liouville type results for general integro-differential operators that we establish in the same chapter. Finally, we also show there the uniqueness of the saddle-shaped solution through a maximum principle for the linearized operator, which we also prove in that chapter.

Let us make some final remarks on the minimality and stability properties of the Simons cone. Recall that, in the classical theory of minimal surfaces, it is well known that the Simons cone has zero mean curvature at every point $x \in \mathscr{C} \setminus \{0\}$, in all even dimensions, and it is a minimizer of the perimeter functional when $2m \geq 8$. Concerning the nonlocal setting, $\mathscr{C}$ has also zero nonlocal mean curvature in all even dimensions, although it is not known if it is a minimizer of the nonlocal perimeter in any dimension. If $2m = 2$ it cannot be a minimizer since in [171] it is proven that all minimizing nonlocal minimal cones in $\mathbb{R}^2$ are flat. In higher dimensions, the only available results appear in [86, 111] but concern stability, a weaker property than minimality. In [86], Dávila, del Pino, and Wei characterize the stability of Lawson cones —a more general class of cones that includes $\mathscr{C}$— through an inequality involving only two hypergeometric constants which depend only on γ and the dimension n. This inequality is checked numerically in [86], finding that, in dimensions $n \leq 6$ and for γ close to zero, no Lawson cone with zero nonlocal mean curvature is stable. Numerics also shows that all Lawson cones in dimension 7 are stable if γ is close to zero. These results for small γ fit with the general belief that, in the fractional setting, the Simons cone should be stable (and even a minimizer) in dimensions $2m \geq 8$ (as in the local case), probably for all $\gamma \in (0, 1/2)$, though this is still an open problem. In [111], we proved, by using the saddle-shaped solution to the fractional Allen-Cahn equation and a Γ-convergence result of [50], that the Simons cone is a stable (2γ)-minimal cone in dimensions $2m \geq 14$. To the best of our knowledge, this is the first analytical proof of a stability result for the Simons cone in any dimension.

This chapter is organized as follows. Section 4.2 is devoted to study the operator L_K acting on doubly radial odd functions. We deduce the expression of the kernel $\overline{K}$ and rewrite the operator acting on doubly radial odd functions, finding the expression (4.1.11). We also show Theorem 4.1.1 and Proposition 4.1.2. In Section 4.3 we study the energy functional associated to (4.1.1) and in Section 4.4 we establish the energy estimate stated in Theorem 4.1.3. Finally, in Section 4.5 we prove the existence of a saddle-shaped solution to the integro-differential Allen-Cahn equation. At the end of the chapter there are three appendices. Appendix 4.6 is devoted to some results on convex functions, and Appendix 4.7 contains some auxiliary computations. Both are used in the proof of Theorem 4.1.1. In Appendix 4.8 we include some results and expressions in (s, t) variables for future reference.

4.2 Rotation invariant operators acting on doubly radial odd functions

This section is devoted to study rotation invariant operators of the class $\mathcal{L}_0$ when they act on doubly radial odd functions. First, we deduce an alternative expression for the operator in terms of a doubly radial kernel $\overline{K}$. Then, we present necessary and sufficient conditions on the kernel K in order to (4.1.13) hold (we establish Theorem 4.1.1). Finally, we show two maximum principles for doubly radial odd functions (Proposition 4.1.2).

4.2.1 Alternative expressions for the operator L_K

The main purpose of this subsection is to deduce an alternative expression for a rotation invariant operator $L_K \in \mathcal{L}_0$ acting on doubly radial functions. This expression is more suitable to work with and it will be used throughout the chapter. Our first remark is that if w is invariant by $O(m)^2$, the same holds for $L_K w$. Indeed, for every $R \in O(m)^2$,

$$\begin{aligned} L_K w(Rx) &= \int_{\mathbb{R}^{2m}} \{w(Rx) - w(y)\} K(|Rx - y|)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^{2m}} \{w(Rx) - w(R\tilde{y})\} K(|Rx - R\tilde{y}|)\, \mathrm{d}\tilde{y} \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(\tilde{y})\} K(|x - \tilde{y}|)\, \mathrm{d}\tilde{y} \\ &= L_K w(x)\,. \end{aligned}$$

Here we have used the change $y = R\tilde{y}$ and the fact that $w(R\cdot) = w(\cdot)$ for every $R \in O(m)^2$.

Next, we present an alternative expression for the operator L_K acting on doubly radial functions. This expression involves the new kernel $\overline{K}$, which is also doubly radial.

Lemma 4.2.1. *Let $L_K \in \mathcal{L}_0(2m, \gamma)$ have a radially symmetric kernel K, and let w be a doubly radial function such that $L_K w$ is well-defined. Then, $L_K w$ can be expressed as*

$$L_K w(x) = \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \overline{K}(x, y)\, \mathrm{d}y$$

where $\overline{K}$ is symmetric, invariant by $O(m)^2$ in both arguments, and it is defined by

$$\overline{K}(x, y) := \fint_{O(m)^2} K(|Rx - y|)\, \mathrm{d}R\,.$$

Here, $\mathrm{d}R$ denotes integration with respect to the Haar measure on $O(m)^2$.

Recall (see for instance [155]) that the Haar measure on $O(m)^2$ exists and it is unique up to a multiplicative constant. Let us state next the properties of this measure that will be used in the rest of the chapter. In the following, the Haar measure is denoted by μ. First, since $O(m)^2$ is a compact group, it is unimodular (see Chapter II, Proposition 13 of [155]). As a consequence, the measure μ is left and right invariant, that is, $\mu(R\Sigma) = \mu(\Sigma) = \mu(\Sigma R)$ for every subset $\Sigma \subset O(m)^2$ and every $R \in O(m)^2$. Moreover, it holds

$$\fint_{O(m)^2} g(R^{-1})\, \mathrm{d}R = \fint_{O(m)^2} g(R)\, \mathrm{d}R \tag{4.2.1}$$

for every $g \in L^1(O(m)^2)$ —see [155] for the details.

Proof of Lemma 4.2.1. Since $L_K w(x) = L_K w(Rx)$ for every $R \in O(m)^2$, by taking the mean over all the transformations in $O(m)^2$, we get

$$\begin{aligned} L_K w(x) &= \fint_{O(m)^2} L_K w(Rx)\, \mathrm{d}R = \fint_{O(m)^2} \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} K(|Rx - y|)\, \mathrm{d}y\, \mathrm{d}R \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \fint_{O(m)^2} K(|Rx - y|)\, \mathrm{d}R\, \mathrm{d}y \\ &= \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \overline{K}(x, y)\, \mathrm{d}y\,. \end{aligned}$$

Now, we show that $\overline{K}$ is symmetric. Using property (4.2.1), we get

$$\begin{aligned}\overline{K}(y,x) &= \fint_{O(m)^2} K(|Ry-x|)\,\mathrm{d}R = \fint_{O(m)^2} K(|R^{-1}(Ry-x)|)\,\mathrm{d}R \\ &= \fint_{O(m)^2} K(|R^{-1}x-y)|)\,\mathrm{d}R = \overline{K}(x,y)\,.\end{aligned}$$

It remains to show that $\overline{K}$ is invariant by $O(m)^2$ in its two arguments. By the symmetry, it is enough to check it for the first one. Let $\tilde{R}\in O(m)^2$. Then,

$$\overline{K}(\tilde{R}x,y) = \fint_{O(m)^2} K(|R\tilde{R}x-y|)\,\mathrm{d}R = \fint_{O(m)^2} K(|Rx-y|)\,\mathrm{d}R = \overline{K}(x,y)\,,$$

where we have used the right invariance of the Haar measure. □

In the following lemma we present some properties of the involution $(\cdot)^\star$ defined by (4.1.6) and its relation with the doubly radial kernel $\overline{K}$ and the transformations of $O(m)^2$. In particular, in the proof of Theorem 4.1.1 it will be useful to consider the following transformation. For every $R\in O(m)^2$, we define $R_\star\in O(m)^2$ by

$$R_\star := (R(\cdot)^\star)^\star\,. \tag{4.2.2}$$

Equivalently, if $R=(R_1,R_2)$ with $R_1,\,R_2\in O(m)$, then $R_\star=(R_2,R_1)$.

Lemma 4.2.2. *Let $(\cdot)^\star:\mathbb{R}^{2m}\to\mathbb{R}^{2m}$ be the involution defined by $x^\star=(x',x'')^\star=(x'',x')$ —see (4.1.6). Then,*

1. *The Haar integral on $O(m)^2$ has the following invariance:*
$$\int_{O(m)^2} g(R_\star)\,\mathrm{d}R = \int_{O(m)^2} g(R)\,\mathrm{d}R\,, \tag{4.2.3}$$
for every $g\in L^1(O(m)^2)$.
2. *$\overline{K}(x^\star,y)=\overline{K}(x,y^\star)$. As a consequence, $\overline{K}(x^\star,y^\star)=\overline{K}(x,y)$.*

Proof. The first statement is easy to check by using Fubini:

$$\begin{aligned}\int_{O(m)^2} g(R_\star)\,\mathrm{d}R &= \int_{O(m)}\mathrm{d}R_1\int_{O(m)}\mathrm{d}R_2\;\; g(R_2,R_1) = \int_{O(m)}\mathrm{d}R_2\int_{O(m)}\mathrm{d}R_1\;\; g(R_2,R_1)\\ &= \int_{O(m)}\mathrm{d}R_1\int_{O(m)}\mathrm{d}R_2\;\; g(R_1,R_2) = \int_{O(m)^2} g(R)\,\mathrm{d}R\,.\end{aligned}$$

To show the second statement, we use the definition of $R_\star$ and (4.2.3) to see that

$$\begin{aligned}\overline{K}(x^\star,y) &= \fint_{O(m)^2} K(|Rx^\star-y|)\,\mathrm{d}R = \fint_{O(m)^2} K(|(Rx^\star-y)^\star|)\,\mathrm{d}R\\ &= \fint_{O(m)^2} K(|(Rx^\star)^\star-y^\star|)\,\mathrm{d}R = \fint_{O(m)^2} K(|R_\star x-y^\star|)\,\mathrm{d}R\\ &= \fint_{O(m)^2} K(|Rx-y^\star|)\,\mathrm{d}R = \overline{K}(x,y^\star)\,.\end{aligned}$$

As a consequence, we have that $\overline{K}(x^\star,y^\star)=\overline{K}(x,(y^\star)^\star)=\overline{K}(x,y)$. □

To conclude this subsection, we present two alternative expressions for the operator L_K when it acts on doubly radial odd functions. These expressions are suitable in the rest of the present chapter and also in the next one, since the integrals appearing in the expression are computed only in $\mathcal{O}$, and this is important to prove maximum principle and other properties.

Lemma 4.2.3. *Let w be a doubly radial function which is odd with respect to the Simons cone. Let $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$ be a rotation invariant operator and let $L_K^{\mathcal{O}}$ be defined by (4.1.11).*

Then,

$$L_K w(x) = \int_{\mathcal{O}} \{w(x) - w(y)\} \overline{K}(x, y) \, dy + \int_{\mathcal{O}} \{w(x) + w(y)\} \overline{K}(x, y^\star) \, dy$$
$$= \int_{\mathcal{O}} \{w(x) - w(y)\} \{\overline{K}(x, y) - \overline{K}(x, y^\star)\} \, dy + 2w(x) \int_{\mathcal{O}} \overline{K}(x, y^\star) \, dy \, .$$

In particular, the second equality shows that $L_K w(x) = L_K^{\mathcal{O}} w(x)$. Moreover,

$$\frac{1}{C} \operatorname{dist}(x, \mathscr{C})^{-2\gamma} \le \int_{\mathcal{O}} \overline{K}(x, y^\star) \, dy \le C \operatorname{dist}(x, \mathscr{C})^{-2\gamma}, \tag{4.2.4}$$

where $C > 0$ is a constant depending only on $m, \gamma, \lambda,$ and Λ.

Proof. The first statement is just a computation. Indeed, using the change of variables $\bar{y} = y^\star$ and the odd symmetry of w, we see that

$$\int_{\mathcal{I}} \{w(x) - w(y)\} \overline{K}(x, y) \, dy = \int_{\mathcal{O}} \{w(x) - w(y^\star)\} \overline{K}(x, y^\star) \, dy$$
$$= \int_{\mathcal{O}} \{w(x) + w(y)\} \overline{K}(x, y^\star) \, dy \, .$$

Hence,

$$L_K w(x) = \int_{\mathbb{R}^{2m}} \{w(x) - w(y)\} \overline{K}(x, y) \, dy$$
$$= \int_{\mathcal{O}} \{w(x) - w(y)\} \overline{K}(x, y) \, dy + \int_{\mathcal{I}} \{w(x) - w(y)\} \overline{K}(x, y) \, dy$$
$$= \int_{\mathcal{O}} \{w(x) - w(y)\} \overline{K}(x, y) \, dy + \int_{\mathcal{O}} \{w(x) + w(y)\} \overline{K}(x, y^\star) \, dy \, .$$

By adding and subtracting $w(x)\overline{K}(x, y^\star)$ in the last integrand, we immediately deduce

$$L_K w(x) = \int_{\mathcal{O}} \{w(x) - w(y)\} \{\overline{K}(x, y) - \overline{K}(x, y^\star)\} \, dy + 2w(x) \int_{\mathcal{O}} \overline{K}(x, y^\star) \, dy \, .$$

Note that we can add and subtract the term $w(x)\overline{K}(x, y^\star)$ since it is integrable with respect to y in $\mathcal{O}$. This is a consequence of (4.2.4).

Let us show now (4.2.4). In the following arguments we will use the letters C and c to denote positive constants, depending only on $m, \gamma, \lambda,$ and Λ, that may change its value in each inequality.

On the one hand, for the upper bound in (4.2.4) we only need to use the ellipticity of the kernel and the inclusion $\mathcal{I} \subset \{y \in \mathbb{R}^{2m} : |x-y| \geq \operatorname{dist}(x,\mathscr{C})\}$ for every $x \in \mathcal{O}$. Indeed,

$$\begin{aligned}\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y &= \int_{\mathcal{O}} K(|x-y^\star|)\,\mathrm{d}y = \int_{\mathcal{I}} K(|x-y|)\,\mathrm{d}y \leq \int_{|x-y|\geq \operatorname{dist}(x,\mathscr{C})} K(|x-y|)\,\mathrm{d}y \\ &\leq C\int_{|x-y|\geq \operatorname{dist}(x,\mathscr{C})} |x-y|^{-2m-2\gamma}\,\mathrm{d}y = C\int_{\operatorname{dist}(x,\mathscr{C})}^{\infty} \rho^{-1-2s}\,\mathrm{d}\rho \\ &= C\operatorname{dist}(x,\mathscr{C})^{-2s}\,.\end{aligned}$$

On the other hand, for the lower bound in (4.2.4), let be $\overline{x} \in \mathscr{C}$ such that $|x-\overline{x}| = \operatorname{dist}(x,\mathscr{C})$. Then, given $y \in B_{\operatorname{dist}(x,\mathscr{C})}(\overline{x})$, it is clear that $|x-y| \leq |x-\overline{x}| + |\overline{x}-y| \leq 2\operatorname{dist}(x,\mathscr{C})$. Therefore, we have

$$\begin{aligned}\int_{\mathcal{O}} \overline{K}(x,y^\star)\,\mathrm{d}y &= \int_{\mathcal{I}} K(|x-y|)\,\mathrm{d}y \geq c\int_{\mathcal{I}} |x-y|^{-2m-2\gamma}\,\mathrm{d}y \\ &\geq c\int_{B_{\operatorname{dist}(x,\mathscr{C})}(\overline{x})\cap\mathcal{I}} |x-y|^{-2m-2\gamma}\,\mathrm{d}y \\ &\geq c\,(2\operatorname{dist}(x,\mathscr{C}))^{-2m-2\gamma}|B_{\operatorname{dist}(x,\mathscr{C})}(\overline{x})\cap\mathcal{I}| = c\operatorname{dist}(x,\mathscr{C})^{-2\gamma}.\end{aligned}$$

Here we have used a property of the Simons cone: $|B_R(z)\cap\mathcal{I}| = 1/2|B_R|$ for every $z \in \mathscr{C}$ (see Lemma 3.2.6 for the proof). □

4.2.2 Necessary and sufficient conditions for ellipticity

In this subsection, we establish Theorem 4.1.1. As we have mentioned in the introduction, the kernel inequality (4.1.13) is crucial in the rest of the results of this chapter, as well as in the ones in Chapter 5. We will see in the next subsection that this inequality guarantees that the operator L_K has a maximum principle for odd functions (see Proposition 4.1.2).

First, we give a sufficient condition on a radially symmetric kernel K so that $\overline{K}$ satisfies (4.1.13). It is the following result.

Proposition 4.2.4. *Let $K : (0,+\infty) \to \mathbb{R}$ define a positive radially symmetric kernel $K(|x-y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K} : \mathbb{R}^{2m}\times\mathbb{R}^{2m} \to \mathbb{R}$ by (4.1.10). Assume that $K(\sqrt{\cdot})$ is strictly convex in $(0,+\infty)$. Then, the associated kernel $\overline{K}$ satisfies*

$$\overline{K}(x,y) > \overline{K}(x,y^\star) \quad \text{for every } x,y \in \mathcal{O}\,. \tag{4.2.5}$$

Proof. Since $\overline{K}$ is invariant by $O(m)^2$, it is enough to choose a unitary vector $e \in \mathbb{S}^{m-1}$ and show (4.2.5) for points $x,y \in \mathcal{O}$ of the form $x = (|x'|e,|x''|e)$ and $y = (|y'|e,|y''|e)$.

Now, define

$$\begin{aligned} Q_1 &:= \left\{R = (R_1,R_2) \in O(m)^2 \;:\; e\cdot R_1 e > |e\cdot R_2 e|\right\}, \\ Q_2 &:= \left\{R = (R_1,R_2) \in O(m)^2 \;:\; e\cdot R_2 e > |e\cdot R_1 e|\right\} = (Q_1)_\star, \\ Q_3 &:= \left\{R = (R_1,R_2) \in O(m)^2 \;:\; e\cdot R_1 e < -|e\cdot R_2 e|\right\} = -Q_1, \\ Q_4 &:= \left\{R = (R_1,R_2) \in O(m)^2 \;:\; e\cdot R_2 e < -|e\cdot R_1 e|\right\} = -(Q_1)_\star. \end{aligned} \tag{4.2.6}$$

Recall that given $R = (R_1,R_2) \in O(m)^2$, then $R_\star = (R_2,R_1) \in O(m)^2$ —see (4.2.2). Moreover, note that the sets Q_i are disjoint, have the same measure and cover all $O(m)^2$ up to a set of measure zero.

Therefore,

$$\begin{aligned}4\overline{K}(x,y) &= 4\fint_{O(m)^2} K(|x-Ry|)\,\mathrm{d}R\\ &= \fint_{Q_1} K(|x-Ry|)\,\mathrm{d}R + \fint_{Q_2} K(|x-Ry|)\,\mathrm{d}R\\ &\quad + \fint_{Q_3} K(|x-Ry|)\,\mathrm{d}R + \fint_{Q_4} K(|x-Ry|)\,\mathrm{d}R\\ &= \fint_{Q_1} \{K(|x-Ry|) + K(|x+Ry|)\\ &\quad + K(|x-R_\star y|) + K(|x+R_\star y|)\}\,\mathrm{d}R\end{aligned}$$

and

$$\begin{aligned}4\overline{K}(x,y^\star) &= 4\fint_{O(m)^2} K(|x-Ry^\star|)\,\mathrm{d}R\\ &= \fint_{Q_1} \{K(|x-Ry^\star|) + K(|x+Ry^\star|)\\ &\quad + K(|x-R_\star y^\star|) + K(|x+R_\star y^\star|)\}\,\mathrm{d}R.\end{aligned}$$

Thus, if we prove

$$\begin{aligned}&K(|x-Ry|) + K(|x+Ry|) + K(|x-R_\star y|) + K(|x+R_\star y|)\\ &\quad \geq K(|x-Ry^\star|) + K(|x+Ry^\star|) + K(|x-R_\star y^\star|) + K(|x+R_\star y^\star|)\,,\end{aligned} \tag{4.2.7}$$

for every $R \in Q_1$, we immediately deduce (4.2.5) with a non strict inequality. To see that it is indeed a strict one, we will show that the inequality in (4.2.7) is strict for every $R \in Q_1$.

For a short notation, we call

$$\alpha := e \cdot R_1 e \quad \text{and} \quad \beta := e \cdot R_2 e\,. \tag{4.2.8}$$

Now, note that since $x = (|x'|e, |x''|e)$ and $y = (|y'|e, |y''|e)$, we have

$$\begin{aligned}|x \pm Ry|^2 &= |x' \pm R_1 y'|^2 + |x'' \pm R_2 y''|^2\\ &= |x'|^2 + |y'|^2 \pm 2x' \cdot R_1 y' + |x''|^2 + |y''|^2 \pm 2x'' \cdot R_2 y''\\ &= |x|^2 + |y|^2 \pm 2|x'||y'|\alpha \pm 2|x''||y''|\beta.\end{aligned}$$

Similarly,

$$|x \pm R_\star y|^2 = |x|^2 + |y|^2 \pm 2|x'||y'|\beta \pm 2|x''||y''|\alpha,$$

$$|x \pm Ry^\star|^2 = |x|^2 + |y|^2 \pm 2|x'||y''|\alpha \pm 2|x''||y'|\beta,$$

and

$$|x \pm R_\star y^\star|^2 = |x|^2 + |y|^2 \pm 2|x'||y''|\beta \pm 2|x''||y'|\alpha.$$

We define now

$$g(\tau) := K\left(\sqrt{|x|^2 + |y|^2 + 2\tau}\right) + K\left(\sqrt{|x|^2 + |y|^2 - 2\tau}\right).$$

Thus, proving (4.2.7) is equivalent to show that, for every α, $\beta \in [-1,1]$ such that $\alpha > |\beta|$, it holds

$$\begin{aligned} g\Big(|x'||y'|\alpha + |x''||y''|\beta\Big) + g\Big(|x'||y'|\beta + |x''||y''|\alpha\Big) \\ \geq g\Big(|x'||y''|\alpha + |x''||y'|\beta\Big) + g\Big(|x'||y''|\beta + |x''||y'|\alpha\Big) . \end{aligned} \tag{4.2.9}$$

Let

$$\begin{aligned} A_{\alpha,\beta} &:= |x'||y'|\alpha + |x''||y''|\beta\,, \quad B_{\alpha,\beta} := |x'||y''|\alpha + |x''||y'|\beta\,, \\ C_{\alpha,\beta} &:= |x''||y'|\alpha + |x'||y''|\beta\,, \quad D_{\alpha,\beta} := |x''||y''|\alpha + |x'||y'|\beta\,. \end{aligned}$$

With this notation and taking into account that g is even, (4.2.9) is equivalent to

$$g(|A_{\alpha,\beta}|) + g(|D_{\alpha,\beta}|) \geq g(|C_{\alpha,\beta}|) + g(|B_{\alpha,\beta}|)\,, \tag{4.2.10}$$

for every α, $\beta \in [-1,1]$ such that $\alpha > |\beta|$. Note that g is defined in the open interval $I = (-(|x|^2 + |y|^2)/2,\ (|x|^2 + |y|^2)/2)$ and that $A_{\alpha,\beta}$, $B_{\alpha,\beta}$, $C_{\alpha,\beta}$, $D_{\alpha,\beta} \in I$.

To show (4.2.10), we use Proposition 4.6.1 of the Appendix 4.6. There, it is stated that in order to establish (4.2.10) it is enough to check that

$$\begin{cases} |A_{\alpha,\beta}| \geq |B_{\alpha,\beta}|, \quad |A_{\alpha,\beta}| \geq |C_{\alpha,\beta}|, \quad |A_{\alpha,\beta}| \geq |D_{\alpha,\beta}|\,, \\ |A_{\alpha,\beta}| + |D_{\alpha,\beta}| \geq |B_{\alpha,\beta}| + |C_{\alpha,\beta}|\,. \end{cases}$$

The verification of these inequalities is a simple but tedious computation and it is presented in Appendix 4.7 —see point (1) of Lemma 4.7.1. Once this is proved, we deduce (4.2.10) by Proposition 4.6.1.

To finish, we see that the inequality in (4.2.10) is always strict for every α, $\beta \in [-1,1]$ such that $\alpha > |\beta|$ (that corresponds to Q_1). By contradiction, assume that equality holds in (4.2.10). Thus, by Proposition 4.6.1, if follows that the sets $\{|A_{\alpha,\beta}|, |D_{\alpha,\beta}|\}$ and $\{|B_{\alpha,\beta}|, |C_{\alpha,\beta}|\}$ coincide. This fact and point (2) of Lemma 4.7.1 yield $\alpha = \beta = 0$, a contradiction. Thus, the inequality in (4.2.10) is strict, as well as the inequality in (4.2.7). This leads to (4.2.5). □

Now, we give a necessary condition on the kernel K so that inequality (4.1.13) holds.

Proposition 4.2.5. *Let $K : (0,+\infty) \to \mathbb{R}$ define a positive radially symmetric kernel $K(|x-y|)$ in $\mathbb{R}^{2m}$. Define $\overline{K} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}$ by* (4.1.10).

If

$$\overline{K}(x,y) > \overline{K}(x,y^\star) \quad \textit{for almost every } x,y \in \mathcal{O}\,, \tag{4.2.11}$$

then $K(\sqrt{\cdot})$ cannot be concave in any open interval $I \subset [0,+\infty)$.

Proof. It suffices to show that if there exists an open interval where $K(\sqrt{\cdot})$ is concave, then we can find a nonempty open set in $\mathcal{O} \times \mathcal{O}$ where (4.2.11) is not satisfied.

Let $\ell_2 > \ell_1 > 0$ be such that $K(\sqrt{\cdot})$ is concave in (ℓ_1,ℓ_2) and define the set $\Omega_{\ell_1,\ell_2} \subset \mathbb{R}^{4m}$ as the points $(x,y) \in \mathcal{O} \times \mathcal{O}$ satisfying

$$\begin{cases} (|x'| - |y'|)^2 + (|x''| - |y''|)^2 & > & \ell_1, \\ (|x'| + |y'|)^2 + (|x''| + |y''|)^2 & < & \ell_2. \end{cases} \tag{4.2.12}$$

First, it is easy to see that Ω_{ℓ_1,ℓ_2} is a nonempty open set. In fact, points of the form $(x',0,y',0) \in (\mathbb{R}^m)^4$ such that $(|x'| - |y'|)^2 > \ell_1$ and $(|x'| + |y'|)^2 < \ell_2$ belong to Ω_{ℓ_1,ℓ_2}.

We need to prove that $\overline{K}(x,y) \le \overline{K}(x,y^\star)$ in Ω_{ℓ_1,ℓ_2} for any $(x,y) \in \mathcal{O} \times \mathcal{O}$ satisfying (4.2.12). For such points, we are going to show, as in the previous proof, that

$$\begin{aligned} &K(|x-Ry|) + K(|x+Ry|) + K(|x-R_\star y|) + K(|x+R_\star y|) \\ &\qquad \le K(|x-Ry^\star|) + K(|x+Ry^\star|) + K(|x-R_\star y^\star|) + K(|x+R_\star y^\star|)\,, \end{aligned} \tag{4.2.13}$$

for any $R \in Q_1$, where Q_1 is defined in (4.2.6) (see the proof of Proposition 4.2.4). As before, we can assume that x and y are of the form $x = (|x'|e, |x''|e)$ and $y = (|y'|e, |y''|e)$, with $e \in \mathbb{S}^{m-1}$ an arbitrary unitary vector. Then, by defining α and β as in (4.2.8), we see that proving (4.2.13) is equivalent to establish that

$$g(A_{\alpha,\beta}) + g(D_{\alpha,\beta}) \le g(B_{\alpha,\beta}) + g(C_{\alpha,\beta})\,, \tag{4.2.14}$$

for every $\alpha, \beta \in [-1,1]$ such that $\alpha > |\beta|$, where

$$\begin{aligned} A_{\alpha,\beta} &= |x'||y'|\alpha + |x''||y''|\beta\,, \quad & B_{\alpha,\beta} &= |x'||y''|\alpha + |x''||y'|\beta\,, \\ C_{\alpha,\beta} &= |x''||y'|\alpha + |x'||y''|\beta\,, \quad & D_{\alpha,\beta} &= |x''||y''|\alpha + |x'||y'|\beta\,. \end{aligned}$$

and

$$g(\tau) = K\left(\sqrt{|x|^2 + |y|^2 + 2\tau}\right) + K\left(\sqrt{|x|^2 + |y|^2 - 2\tau}\right).$$

Now, by (4.2.12), we have $\ell_1 < |x|^2 + |y|^2 < \ell_2$. As a consequence of this and the concavity of $K(\sqrt{\cdot})$ in (ℓ_1, ℓ_2), it is easy to see (by using Lemma 4.6.2 stated for $-h$, a concave function, instead of h) that g is concave in $\left(-\overline{\ell}, \overline{\ell}\right)$, and decreasing in $(0, \overline{\ell})$, where

$$\overline{\ell} := \min\left\{\frac{\ell_2 - |x|^2 - |y|^2}{2}, \frac{|x|^2 + |y|^2 - \ell_1}{2}\right\}.$$

Note that, since $\ell_1 < |x|^2 + |y|^2 < \ell_2$, we have $\overline{\ell} > 0$.

We claim that $A_{\alpha,\beta}, B_{\alpha,\beta}, C_{\alpha,\beta}$, and $D_{\alpha,\beta}$ belong to $(-\overline{\ell}, \overline{\ell})$ for every $\alpha, \beta \in [-1,1]$ such that $\alpha > |\beta|$. Indeed, it is easy to check that for every $\alpha, \beta \in [-1,1]$ such that $\alpha > |\beta|$, the numbers $A_{\alpha,\beta}, B_{\alpha,\beta}, C_{\alpha,\beta}$, and $D_{\alpha,\beta}$ belong to the open interval $(-|x'||y'| - |x''||y''|, |x'||y'| + |x''||y''|)$. Furthermore, since $x, y \in \Omega_{\ell_1,\ell_2}$, we obtain from (4.2.12) that

$$\begin{cases} |x'||y'| + |x''||y''| &< \dfrac{\ell_2 - |x|^2 - |y|^2}{2} \\ |x'||y'| + |x''||y''| &< \dfrac{|x|^2 + |y|^2 - \ell_1}{2} \end{cases}$$

and thus $|x'||y'| + |x''||y''| < \overline{\ell}$ and the claim is proved.

Finally, by applying Lemma 4.6.2 to the function $-g$ in $(0, \overline{\ell})$ (using again point (1) of Lemma 4.7.1), we obtain that inequality (4.2.14) is satisfied, which yields (4.2.13). Finally, by integrating (4.2.13) with respect to all the rotations $R \in Q_1$ we get

$$\overline{K}(x,y) \le \overline{K}(x,y^\star),$$

for every $(x,y) \in \Omega_{\ell_1,\ell_2}$, contradicting (4.2.11). □

From the two previous results, Theorem 4.1.1 follows immediately.

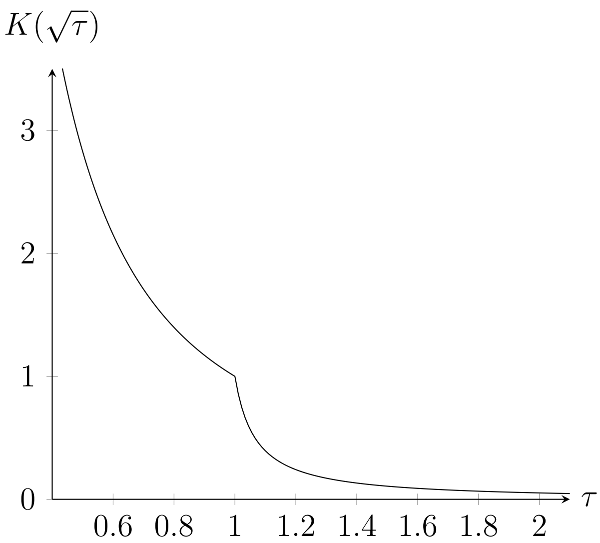

Figure 4.1: An example of kernel $K(\sqrt{\tau})$ ($m = 1$ and $\gamma = 1/2$) which is not strictly convex in $(0, +\infty)$ but does not have any interval of concavity.

Proof of Theorem 4.1.1. The first statement is exactly the same as Proposition 4.2.4. Assume now that K is a C^2 function and that (4.1.13) holds. Then, by Proposition 4.2.5, $h(\cdot) := K(\sqrt{\cdot})$ is not concave in any interval of $[0, +\infty)$. Therefore, we cannot have $h'' < 0$ at any point. Thus, $h'' \geq 0$ in $[0, +\infty)$ or, in other words, h' is nondecreasing. Using again that h is not concave in any interval, we deduce that h' must be, in fact, increasing. It follows that $h(\cdot) = K(\sqrt{\cdot})$ is strictly convex as defined after the statement of Theorem 4.1.1. □

Remark 4.2.6. Note that a priori we cannot relax the $K \in C^2$ assumption in the necessary condition of Theorem 4.1.1, since there are C^1 functions that are neither convex nor concave in any interval (they can be constructed as a primitive of a Weierstrass function, whose graph is a non rectifiable curve with fractal dimension). Besides these "exotic" examples, there are also simple radially symmetric kernels K that are not C^1 for which we do not know if the positivity condition (4.1.13) holds. For instance, given $0 < \gamma < 1$, if we consider the kernel

$$K(\tau) = \frac{1}{\tau^{2m+2\gamma}} \chi_{(0,1)}(\tau) + \frac{1}{10\tau^{2m+2\gamma} - 9} \chi_{[1,+\infty)}(\tau),$$

it is easy to check that K is continuous and decreasing but $K(\sqrt{\tau})$ is not convex in $(0, +\infty)$ even though it does not have any interval of concavity (see Figure 4.1).

4.2.3 Maximum principles for doubly radial odd functions

In this subsection we prove Proposition 4.1.2, a weak and a strong maximum principles for doubly radial functions that are odd with respect to the Simons cone. The formulation of these maximum principles is very suitable since all the hypotheses refer to the set $\mathcal{O}$ and not $\mathbb{R}^{2m}$. The key ingredient in the proofs is the kernel inequality (4.1.13).

Proof of Proposition 4.1.2. (*i*) By contradiction, suppose that u takes negative values in Ω. Under the hypotheses we are assuming, a negative minimum must be achieved. Thus, there exists $x_0 \in \Omega$ such that

$$u(x_0) = \min_{\Omega} u =: m < 0\,.$$

Then, using the expression of L_K for odd functions (see Lemma 4.2.3), we have

$$L_K u(x_0) = \int_{\mathcal{O}} \{m - u(y)\}\{\overline{K}(x_0, y) - \overline{K}(x_0, y^\star)\}\,\mathrm{d}y + 2m \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\,\mathrm{d}y\,.$$

Now, since $m - u(y) \le 0$ in $\mathcal{O}$, $m < 0$, $c \ge 0$, and $\overline{K}(x_0, y) \ge \overline{K}(x_0, y^\star) > 0$ —by (4.1.13)—, we get

$$0 \le L_K u(x_0) + c(x_0)u(x_0) \le m\left(2\int_{\mathcal{O}} \overline{K}(x_0, y^\star)\,\mathrm{d}y + c(x_0)\right) < 0\,,$$

a contradiction.

(ii) Assume that $u \not\equiv 0$ in $\mathcal{O}$. We shall prove that $u > 0$ in Ω. By contradiction, assume that there exists a point $x_0 \in \Omega$ such that $u(x_0) = 0$. Then, using the expression of L_K for odd functions given in Lemma 4.2.3, the kernel inequality (4.1.13), and the fact that $u \ge 0$ in $\mathcal{O}$, we obtain

$$0 \le L_K u(x_0) + c(x_0)u(x_0) = -\int_{\mathcal{O}} u(y)\Big\{\overline{K}(x_0, y) - \overline{K}(x_0, y^\star)\Big\}\,\mathrm{d}y < 0\,,$$

a contradiction. □

Remark 4.2.7. Note that since the operator L_K includes a term of order zero with positive coefficient in addition to the integro-differential part, the condition $c \ge 0$ in point (i) of the previous proposition can be slightly relaxed. Indeed, following the proof of the result, we see that

$$c(x) > -2\int_{\mathcal{O}} \overline{K}(x, y^\star)\,\mathrm{d}y$$

suffices. This hypothesis seems hard to be checked for applications apart from the case $c \ge 0$. Nevertheless, recall that by Lemma 4.2.3 we have an explicit lower bound for the quantity $\int_{\mathcal{O}} \overline{K}(x, y^\star)\,\mathrm{d}y$ in terms of the function $\operatorname{dist}(x, \mathscr{C})$. This fact will be crucial for establishing a maximum principle in "narrow" sets close to the Simons cone —see Proposition 5.6.1.

4.3 The energy functional for doubly radial odd functions

This section is devoted to the energy functional associated to the semilinear equation (4.1.1). We first define appropriately the functional spaces where we are going to apply classic techniques of calculus of variations. Next we rewrite the energy in terms of the new kernel $\overline{K}$ and we give an alternative expression for the energy of doubly radial odd functions. Finally, we establish some results that are useful when using variational techniques, and that will be exploited in the next section.

Let us start by defining the functional spaces that we are going to consider in the rest of the chapter. Given a set $\Omega \subset \mathbb{R}^n$ and a translation invariant and positive kernel K satisfying (4.1.3), we define the Hilbert space

$$\mathbb{H}^K(\Omega) := \Big\{ w \in L^2(\Omega) \ :\ [w]^2_{\mathbb{H}^K(\Omega)} < +\infty \Big\}\,,$$

where

$$[w]^2_{\mathbb{H}^K(\Omega)} := \frac{1}{2}\int\int_{(\mathbb{R}^n)^2 \setminus (\mathbb{R}^n \setminus \Omega)^2} |w(x) - w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

We also define

$$\mathbb{H}_0^K(\Omega) := \Big\{ w \in \mathbb{H}^K(\Omega) \;:\; w = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega \Big\}$$
$$= \Big\{ w \in \mathbb{H}^K(\mathbb{R}^n) \;:\; w = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega \Big\}.$$

Assume that $\Omega \subset \mathbb{R}^{2m}$ is a set of double revolution. Then, we define

$$\widetilde{\mathbb{H}}^K(\Omega) := \Big\{ w \in \mathbb{H}^K(\Omega) \;:\; w \text{ is doubly radial a.e.} \Big\}.$$

and

$$\widetilde{\mathbb{H}}_0^K(\Omega) := \Big\{ w \in \mathbb{H}_0^K(\Omega) \;:\; w \text{ is doubly radial a.e.} \Big\}.$$

We will add the subscript 'odd' and 'even' to these spaces to consider only functions that are odd (respectively even) with respect to the Simons cone.

Remark 4.3.1. If $\widetilde{\mathbb{H}}_0^K(\Omega)$ is equipped with the scalar product

$$\langle v, w \rangle_{\widetilde{\mathbb{H}}_0^K(\Omega)} := \frac{1}{2} \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \Big(v(x) - v(y) \Big) \Big(w(x) - w(y) \Big) K(x-y) \, \mathrm{d}x \, \mathrm{d}y \,,$$

then it is easy to check that $\widetilde{\mathbb{H}}_0^K(\Omega)$ can be decomposed as the orthogonal direct sum of $\widetilde{\mathbb{H}}_{0,\,\text{even}}^K(\Omega)$ and $\widetilde{\mathbb{H}}_{0,\,\text{odd}}^K(\Omega)$.

Note that when K satisfies (4.1.4), then $\mathbb{H}_0^K(\Omega) = \mathbb{H}_0^\gamma(\Omega)$, which is the space associated to the kernel of the fractional Laplacian, $K(z) = c_{n,\gamma} |z|^{-n-2\gamma}$. Furthermore, $\mathbb{H}^\gamma(\Omega) \subset H^\gamma(\Omega)$, where $\mathbb{H}^\gamma(\Omega)$ is the usual fractional Sobolev space where interactions of x and y are only computed in $\Omega \times \Omega$ (see [92]). For more comments on this, see [83], and the references therein.

Once presented the functional setting of our problem, we proceed with the study of the energy functional associated to equation (4.1.1).

Given a kernel K satisfying (4.1.3), a potential G, and a function $w \in \mathbb{H}^K(\Omega)$, with $\Omega \subset \mathbb{R}^n$, we write the energy defined in (4.1.14) as

$$\mathcal{E}(w, \Omega) = \mathcal{E}_{\mathrm{K}}(w, \Omega) + \mathcal{E}_{\mathrm{P}}(w, \Omega)\,,$$

where

$$\mathcal{E}_{\mathrm{K}}(w, \Omega) := \frac{1}{2} [w]^2_{\mathbb{H}^K(\Omega)} \quad \text{and} \quad \mathcal{E}_{\mathrm{P}}(w, \Omega) := \int_\Omega G(w) \, \mathrm{d}x\,.$$

We will call $\mathcal{E}_{\mathrm{K}}$ and $\mathcal{E}_{\mathrm{P}}$ the *kinetic* and *potential* energies respectively. Recall that sometimes it is useful to rewrite the kinetic energy as

$$\mathcal{E}_{\mathrm{K}}(w, \Omega) = \frac{1}{4} \Big\{ \int_\Omega \int_\Omega |w(x) - w(y)|^2 K(x-y) \, \mathrm{d}x \, \mathrm{d}y + 2 \int_\Omega \int_{\mathbb{R}^n \setminus \Omega} |w(x) - w(y)|^2 K(x-y) \, \mathrm{d}x \, \mathrm{d}y \Big\}. \tag{4.3.1}$$

Roughly speaking, we have $\mathcal{E}_{\mathrm{K}}$ split into two parts: "interactions inside-inside" and "interactions inside-outside".

Note that, for functions $w \in \mathbb{H}_0^K(\Omega)$, it holds $\mathcal{E}_{\mathrm{K}}(w, \Omega) = \mathcal{E}_{\mathrm{K}}(w, \mathbb{R}^n)$. Moreover, if $G \geq 0$, the energy satisfies $\mathcal{E}(w, \Omega) \leq \mathcal{E}(w, \Omega')$ whenever $\Omega \subset \Omega'$.

Our goal is to rewrite the kinetic energy of doubly radial odd functions in terms of the kernel $\overline{K}$ and with integrals computed only in $\mathcal{O}$, in the spirit of the previous section with the operator L_K. In particular, we are interested in finding an expression similar to (4.3.1), where the positive kernel $\overline{K}(x,y) - \overline{K}(x,y^\star)$ appears. To do this, we introduce the following notation for the interaction. Given A, $B \subset \mathcal{O}$ sets of double revolution, we define

$$
\begin{aligned}
I_w(A,B) := 2\int_A\int_B & |w(x)-w(y)|^2 \left\{\overline{K}(x,y) - \overline{K}(x,y^\star)\right\} \,dx\,dy \\
& + 4\int_A\int_B \left\{w^2(x)+w^2(y)\right\} \overline{K}(x,y^\star)\,dx\,dy\,.
\end{aligned}
\tag{4.3.2}
$$

Thanks to this notation, we rewrite the kinetic energy as follows.

Lemma 4.3.2. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{2m}$ be two sets of double revolution that are symmetric with respect to the Simons cone, i.e., $\Omega_i^\star = \Omega_i$, and let $w \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\mathbb{R}^n)$. Let K be a radially symmetric kernel satisfying (4.1.3). Then,*

$$
\int_{\Omega_1}\int_{\Omega_2} |w(x)-w(y)|^2 K(x-y) = I_w(\Omega_1\cap\mathcal{O}, \Omega_2\cap\mathcal{O}), \tag{4.3.3}
$$

where $I_w(\cdot,\cdot)$ is the interaction defined in (4.3.2).

As a consequence, given a doubly radial set $\Omega \subset \mathbb{R}^{2m}$ with $\Omega^\star = \Omega$, and a function $v \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)$, we can write the kinetic energy as

$$
\mathcal{E}_{\mathrm{K}}(v,\Omega) = \frac{1}{4}\Big\{I_v(\Omega\cap\mathcal{O},\Omega\cap\mathcal{O}) + 2I_v(\Omega\cap\mathcal{O},\mathcal{O}\setminus\Omega)\Big\}. \tag{4.3.4}
$$

Proof. First, note that equality (4.3.4) for the kinetic energy follows directly combining expressions (4.3.1) and (4.3.3). Hence, we only need to prove (4.3.3).

Now, since w is doubly radial and Ω_1, Ω_2 are sets of double revolution, we obtain

$$
\int_{\Omega_1}\int_{\Omega_2} |w(x)-w(y)|^2 K(x-y)\,dx\,dy = \int_{\Omega_1}\int_{\Omega_2} |w(x)-w(y)|^2 \overline{K}(x,y)\,dx\,dy\,,
$$

once we consider the change $y = R\tilde{y}$ and take the average among all $R \in O(m)^2$ as in Lemma 4.2.1.

Finally, we split Ω_i into $\Omega_i\cap\mathcal{O}$ and $\Omega_i\setminus\mathcal{O} = (\Omega_i\cap\overline{\mathcal{O}})^\star$. By using the change of variables given by $(\cdot)^\star$ and the symmetries of Ω_i and w, we get

$$
\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} |w(x)-w(y)|^2 \overline{K}(x,y)\,dx\,dy \\
&\qquad = 2\int_{\Omega_1\cap\mathcal{O}}\int_{\Omega_2\cap\mathcal{O}} |w(x)-w(y)|^2 \overline{K}(x,y) + |w(x)+w(y)|^2 \overline{K}(x,y^\star)\,dx\,dy \\
&\qquad = 2\int_{\Omega_1\cap\mathcal{O}}\int_{\Omega_2\cap\mathcal{O}} |w(x)-w(y)|^2 \left\{\overline{K}(x,y) - \overline{K}(x,y^\star)\right\} dx\,dy \\
&\qquad\quad + 4\int_{\Omega_1\cap\mathcal{O}}\int_{\Omega_2\cap\mathcal{O}} \left\{w^2(x)+w^2(y)\right\} \overline{K}(x,y^\star)\,dx\,dy \\
&\qquad = I_w(\Omega_1\cap\mathcal{O},\Omega_2\cap\mathcal{O}).
\end{aligned}
$$

Here we have used that $\overline{K}(x^\star,y^\star) = \overline{K}(x,y)$ —see Lemma 4.2.2. □

Using the previous expression for the energy, we can establish now the following lemma regarding the decrease of the energy under some operations. This result will be crucial in the next section, since it will allow us to assume that the minimizers of the energy are bounded by 1 by above (respectively by −1 by below) and that are nonnegative in $\mathcal{O}$.

Lemma 4.3.3. *Let $\Omega \subset \mathbb{R}^{2m}$ be a set of double revolution that is symmetric with respect to the Simons cone, and let K be a radially symmetric kernel satisfying the positivity condition (4.1.13). Given $u \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\Omega)$, we define*

$$v(x) = \begin{cases} |u(x)| & \text{if } \; x \in \mathcal{O}, \\ -|u(x)| & \text{if } \; x \in \mathcal{I}, \end{cases} \quad \text{and} \quad w(x) = \begin{cases} \min\{1, u(x)\} & \text{if } \; x \in \mathcal{O}, \\ \max\{-1, u(x)\} & \text{if } \; x \in \mathcal{I}. \end{cases}$$

If G satisfies (4.1.15), then

$$\mathcal{E}(v, \Omega) \leq \mathcal{E}(u, \Omega) \quad \text{and} \quad \mathcal{E}(w, \Omega) \leq \mathcal{E}(u, \Omega).$$

Proof. We first establish the result for v. Let us show that $\mathcal{E}_{\mathrm{K}}(v) \leq \mathcal{E}_{\mathrm{K}}(u)$. Note that $v \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\Omega)$. Thus, by using the expression of the kinetic energy given in (4.3.4) and the fact that $\overline{K}(x, y) > \overline{K}(x, y^\star) > 0$ if $x, y \in \mathcal{O}$ —see (4.1.13)—, we only need to check that $|v(x) - v(y)|^2 \leq |u(x) - u(y)|^2$ and $v^2(x) \leq u^2(x)$ whenever $x, y \in \mathcal{O}$. The first condition follows from the equivalence

$$\Big| |u(x)| - |u(y)| \Big|^2 \leq |u(x) - u(y)|^2 \iff u(x)u(y) \leq |u(x)u(y)|,$$

while the second one is trivial and it is in fact an equality. Concerning the potential energy, since G is an even function we have that $\mathcal{E}_{\mathrm{P}}(v) = \mathcal{E}_{\mathrm{P}}(u)$, and therefore we get the desired result for v by adding the kinetic and potential energies.

We show now the result for w. Let us show that $\mathcal{E}_{\mathrm{K}}(w) \leq \mathcal{E}_{\mathrm{K}}(u)$. As before, $w \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(\Omega)$ and thus, in view of (4.3.4) and the kernel inequality (4.1.13), we only need to check that $|w(x) - w(y)|^2 \leq |u(x) - u(y)|^2$ and $w^2(x) \leq u^2(x)$ whenever $x, y \in \mathcal{O}$. The first inequality is trivial whenever $u(x) \leq 1$ and $u(y) \leq 1$, or $u(x) \geq 1$ and $u(y) \geq 1$. If $u(x) \geq 1$ and $u(y) \leq 1$, then $|u(x) - u(y)|^2 - |w(x) - w(y)|^2 = |u(x) - u(y)|^2 - |1 - u(y)|^2 = (u(x) - 1))^2 + 2(u(x) - 1)(1 - u(y)) \geq 0$. The second inequality follows from the fact that $w^2(x) = u^2(x)$ when $u(x) \leq 1$, while $w^2(x) = 1 \leq u^2(x)$ if $u(x) \geq 1$. Concerning the potential energy, since G is such that $G(x) \geq G(1) = G(-1) = 0$ if $|x| \leq 1$, then clearly $\mathcal{E}_{\mathrm{P}}(w) \leq \mathcal{E}_{\mathrm{P}}(u)$, and therefore we get the desired result by adding the kinetic and potential energies. ☐

Next we present a result that will be used later, and concerns weak solutions to semilinear Dirichlet problems. Its main consequence is that a function $u \in \widetilde{\mathbb{H}}^K_0(\Omega)$ that minimizes the energy $\mathcal{E}$, but only among doubly radial functions, is actually a weak solution to a semilinear Dirichlet problem in Ω. We remark that to show the following result we do not need to use the kernel $\overline{K}$.

Proposition 4.3.4. *Let $\Omega \subset \mathbb{R}^{2m}$ be a bounded set of double revolution and let $L_K \in \mathcal{L}_0$ with kernel K radially symmetric. Let $u \in \widetilde{\mathbb{H}}^K_0(\Omega)$ be such that*

$$\int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \{u(x) - u(y)\}\{\xi(x) - \xi(y)\} K(|x - y|) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^{2m}} f(u(x))\xi(x) \, \mathrm{d}x$$

for every $\xi \in C^\infty_c(\Omega)$ that is doubly radial. Then, u is a weak solution to

$$\begin{cases} L_K u = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^{2m} \setminus \Omega, \end{cases} \tag{4.3.5}$$

i.e.,

$$\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y=\int_{\mathbb{R}^{2m}}f(u(x))\eta(x)\,\mathrm{d}x$$

for every $\eta\in C_c^\infty(\Omega)$ (not necessarily doubly radial).

As a consequence, if $u\in\widetilde{\mathbb{H}}_0^K(\Omega)$ is a doubly radial odd minimizer of the energy $\mathcal{E}(u,\Omega)$, then it is a weak solution to (4.3.5).

Proof. Let $\eta\in C_c^\infty(\Omega)$. We define an associated doubly radial function by

$$\overline{\eta}(x):=\fint_{O(m)^2}\eta(Rx)\,\mathrm{d}R\,.$$

Now, on the one hand, given $R\in O(m)^2$ and using the change $x=R\tilde{x}$, $y=R\tilde{y}$ and the fact that u is doubly radial, we get

$$\begin{aligned}&\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y=\\&\qquad=\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\{\eta(Rx)-\eta(Ry)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Taking the average in the previous equality among all $R\in O(m)^2$ we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y=\\&\qquad=\fint_{O(m)^2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\{\eta(Rx)-\eta(Ry)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}R\\&\qquad=\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\left\{\overline{\eta}(x)-\overline{\eta}(y)\right\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

On the other hand, using also the change $x=R\tilde{x}$, we have

$$\int_\Omega f(u(x))\eta(x)\,\mathrm{d}x=\int_\Omega f(u(R^{-1}x))\eta(x)\,\mathrm{d}x=\int_\Omega f(u(x))\eta(Rx)\,\mathrm{d}x\,.$$

Similarly as before, taking the average among all $R\in O(m)^2$, we get

$$\int_\Omega f(u(x))\eta(x)\,\mathrm{d}x=\fint_{O(m)^2}\int_\Omega f(u(x))\eta(Rx)\,\mathrm{d}x\,\mathrm{d}R=\int_\Omega f(u(x))\overline{\eta}(x)\,\mathrm{d}x\,.$$

Hence, since $\overline{\eta}\in C_c^\infty(\Omega)$ is doubly radial, we have

$$\begin{aligned}&\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\{\eta(x)-\eta(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega f(u(x))\eta(x)\,\mathrm{d}x\\&\quad=\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}}\{u(x)-u(y)\}\left\{\overline{\eta}(x)-\overline{\eta}(y)\right\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y-\int_\Omega f(u(x))\overline{\eta}(x)\,\mathrm{d}x\\&\quad=0\,,\end{aligned}$$

and thus the first result is proved.

We next show that if u is a doubly radial odd minimizer, then it is a weak solution to (4.3.5). To see this, we consider perturbations $u+\varepsilon\xi$ with $\varepsilon\in\mathbb{R}$ and $\xi\in\widetilde{\mathbb{H}}_0^K(\Omega)$. By

Remark 4.3.1, it suffices to consider only even and odd functions ξ. Let first $\xi \in \widetilde{\mathbb{H}}^K_{0,\,\text{odd}}(\Omega)$. Then, since u is a minimizer among functions in $\widetilde{\mathbb{H}}^K_{0,\,\text{odd}}(\Omega)$, we get

$$0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}(u+\varepsilon\xi,\Omega) = \langle u,\xi\rangle_{\widetilde{\mathbb{H}}^K_0(\Omega)} - \langle f(u),\xi\rangle_{L^2(\Omega)} .$$

Next, take $\xi \in \widetilde{\mathbb{H}}^K_{0,\,\text{even}}(\Omega)$. Since u is odd with respect to the Simons cone, the same holds for $f(u)$ —recall that f is odd. Thus,

$$\langle u,\xi\rangle_{\widetilde{\mathbb{H}}^K_0(\Omega)} = 0 \quad \text{and} \quad \langle f(u),\xi\rangle_{L^2(\Omega)} = 0 .$$

Therefore,

$$\langle u,\xi\rangle_{\widetilde{\mathbb{H}}^K_0(\Omega)} = \langle f(u_R),\xi\rangle_{L^2(\Omega)}$$

for every $\xi \in \widetilde{\mathbb{H}}^K_0(\Omega)$ with compact support in Ω. In particular,

$$\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \{u_R(x)-u_R(y)\}\{\xi(x)-\xi(y)\}K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^{2m}} f(u_R(x))\xi(x)\,\mathrm{d}x$$

for every $\xi \in C^\infty_c(\Omega)$ that is doubly radial. Finally, by the first statement of the proposition, that we just proved, we obtain that u is a weak solution to (4.3.5). □

The previous proposition, combined with the regularity results of the following remark, yields that bounded minimizers among doubly radial functions of the energy $\mathcal{E}(\cdot,\Omega)$ are classical solutions to $L_K u = f(u)$ in Ω.

Remark 4.3.5. Let us present here some interior estimates that will be used in the sequel. If $w \in L^\infty(\mathbb{R}^n)$ is a weak solution to $L_K w = h$ in $B_1 \subset \mathbb{R}^n$, with $L_K \in \mathcal{L}_0(n,\gamma,\lambda,\Lambda)$, then

$$||w||_{C^{2\gamma}(\overline{B_{1/2}})} \leq C\left(||h||_{L^\infty(B_1)} + ||w||_{L^\infty(\mathbb{R}^n)}\right). \tag{4.3.6}$$

If, in addition, $w \in C^\alpha(\mathbb{R}^n)$ with $\alpha+2\gamma$ not an integer, then

$$||w||_{C^{\alpha+2\gamma}(\overline{B_{1/2}})} \leq C\left(||h||_{C^\alpha(\overline{B_1})} + ||w||_{C^\alpha(\mathbb{R}^n)}\right), \tag{4.3.7}$$

where the previous two constants C depend only on n, γ, λ, and Λ (see [162, 176] and the references therein). Note that in some situations these estimates are not suitable enough to be applied repeatedly due to the term $||w||_{C^\alpha(\mathbb{R}^n)}$ in (4.3.7). Let us show how to overcome this difficulty in our setting, that is, when K is radially symmetric and $K(\sqrt{\cdot})$ is convex. In this case, using the ellipticity property (4.1.4) and the convexity assumption (4.1.12) for K, it is not difficult to show that K is locally Lipschitz in $\mathbb{R}^n \setminus \{0\}$ and

$$[K]_{\mathrm{Lip}(\mathbb{R}^n\setminus B_R)} \leq CR^{-n-2\gamma-1}, \qquad \text{for all } R>0, \tag{4.3.8}$$

with a positive constant C depending only on n, γ, λ, and Λ. Using for L_K the same cut-off argument as in Corollary 2.4 of [163] for the fractional Laplacian, and taking into account (4.3.8), one can modify (4.3.7) to obtain the estimate

$$||w||_{C^{\alpha+2\gamma}(\overline{B_{1/4}})} \leq C\left(||h||_{C^\alpha(\overline{B_1})} + ||w||_{C^\alpha(\overline{B_1})} + \left|\left|\frac{w(x)}{(1+|x|)^{n+2\gamma}}\right|\right|_{L^1(\mathbb{R}^n)}\right), \tag{4.3.9}$$

for all $\alpha \in (0,1)$ with $\alpha+2\gamma$ not an integer, and with C depending only on n, γ, λ, and Λ. This, combined with (4.3.6), will be used in the following section to obtain uniform Lipschitz interior estimates for the semilinear equation $L_K u = f(u)$.

4.4 An energy estimate for doubly radial odd minimizers

In this section we present an estimate for the energy in the ball B_S of minimizers in the space $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ with $R > S + 4$. That is, we prove Theorem 4.1.3. In order to establish this result, we follow the ideas of Savin and Valdinoci in [173], where they show the same estimate but for minimizers without any restriction on their symmetry.

First of all, let us comment briefly the strategy used in [173]. The argument is based on comparing the energy of the minimizer u in $B_R \subset \mathbb{R}^n$ with the energy of a suitable competitor v. This function v satisfies, in $B_{S+2} \subset B_R \subset \mathbb{R}^n$, the following properties:

(i) $-1 \le v \le 1$.

(ii) $v = u$ in ∂B_{S+2}.

(iii) The set $\{v \not\equiv -1\} \cap B_{S+2}$ has measure bounded by CS^{n-1} for some constant C.

(iv) $v \in \mathrm{Lip}(\overline{B_{S+2}})$ with a Lipschitz constant independent of R and S.

By the second property, v can be extended to coincide with u outside B_{S+2}, becoming an admissible competitor. Then, the desired estimate follows by finding precise bounds on the energy of v in B_{S+2}. The function v is constructed in B_{S+2} as $v = \min\{u, \phi_S\}$, where $\phi_S(x) = -1 + 2\min\{(|x| - S - 1)_+, 1\}$ —we will also use this function below, see (4.4.5).

In our case, the strategy will be the same but adapting some ingredients, namely, the competitor v. First, note that the previous construction for v cannot be used in our setting, since it would not produce a doubly radial odd function. To overcome this problem, we will construct a function w defined in $B_{S+2} \cap \mathcal{O}$ and satisfying the four previous assumptions on v. In addition, we will require w to be doubly radial and to vanish on the Simons cone (then we will consider its odd extension through $\mathscr{C}$).

To state the precise properties of w, we need to consider the Lipschitz constant of u in $\overline{B_{S+3}}$, namely

$$\mu := [u]_{\mathrm{Lip}(\overline{B_{S+3}})}\,. \tag{4.4.1}$$

By Proposition 4.3.4, we know that u solves $L_K u = f(u)$ in B_R with $R > S + 4$. Moreover, by Lemma 4.3.3 we know that u is bounded. Therefore, by applying repeatedly the estimates (4.3.6) and (4.3.9) in balls centered at points in B_{S+3}, it is easy to see that $\mu \le C$ with a positive constant C depending only on m, γ, λ, Λ, and $||f||_{C^1([-1,1])}$ (and thus, independent of R and S). Recall that $G' = -f$ and hence $||f||_{C^1([-1,1])} \le ||G||_{C^2([-1,1])}$.

We can now define the set

$$\Omega_S := \left(\overline{B_{S+2}} \setminus B_S\right) \cup \left(\overline{B_{S+2}} \cap \{\mu \mathrm{dist}(\cdot, \mathscr{C}) \le 1\}\right), \tag{4.4.2}$$

—see Figure 4.2 (a). It is easy to see that

$$|\Omega_S| \le C\, S^{2m-1}, \tag{4.4.3}$$

with a constant C depending only on m and μ. This can be checked following the computations in the proof of the energy estimate for the local equation in Theorem 1.3 of [59].

In the following lemma we state the precise properties for the competitor w that suffice to establish the energy estimate given by the right-hand side of (4.1.17) for $\mathcal{E}(w, B_{S+2})$.

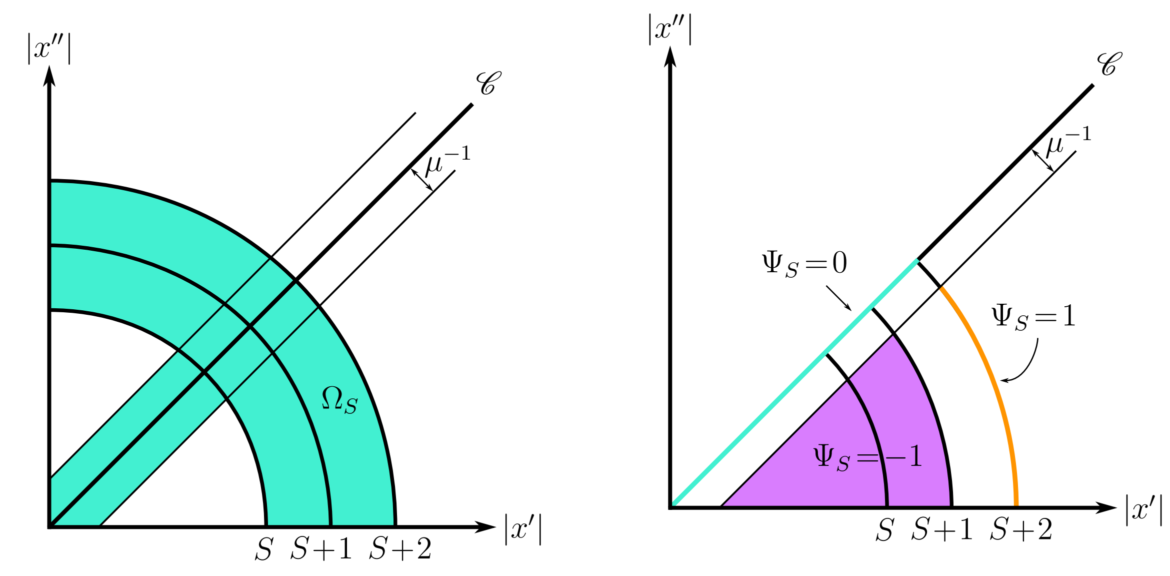

Figure 4.2: (a) The set Ω_S. (b) The 1 and -1 level sets of Ψ_S in $\overline{B_{S+2}} \cap \mathcal{O}$.

Lemma 4.4.1. *Let $S \geq 2$ and $R > S+4$. Let $u \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$ be a doubly radial odd minimizer of the energy* (4.1.14) *and let μ be defined by* (4.4.1)*. Then, there exists a function $w : \overline{B_{S+2} \cap \mathcal{O}} \to \mathbb{R}$ satisfying the following:*

(H1) $-1 \leq w \leq 1$.

(H2) *w doubly radial and $w = 0$ in $\mathscr{C}$.*

(H3) *$w = u$ on $\partial B_{S+2} \cap \mathcal{O}$.*

(H4) *$w \equiv -1$ on $(B_{S+2} \cap \mathcal{O}) \setminus \Omega_S = B_S \cap \{\mu\mathrm{dist}(\cdot, \mathscr{C}) > 1\}$.*

(H5) *$w \in \mathrm{Lip}(\overline{B_{S+2}})$ with a Lipschitz constant independent of R and S. In addition,*

$$|w(x) - w(y)| \leq \frac{C}{\mathrm{dist}(x, \mathscr{C})}|x - y| \tag{4.4.4}$$

whenever $x, y \in B_{S+1} \cap \mathcal{O}$, $\mu\mathrm{dist}(x, \mathscr{C}) \geq 1$ and $\mu\mathrm{dist}(y, \mathscr{C}) \leq 1$, and with C a constant independent of R and S.

Proof. To construct the function w we first define

$$\phi_S(x) := \begin{cases} -1 & \text{if } \ |x| \leq S+1\,, \\ -1 + 2(|x| - S - 1) & \text{if } \ S+1 \leq |x| \leq S+2\,, \\ 1 & \text{if } \ S+2 \leq |x|\,, \end{cases} \tag{4.4.5}$$

which is the function used in [173]. Now, we modify it in order to make it vanish on $\mathscr{C}$. We define

$$\Psi_S(x) := \begin{cases} \phi_S(x)\mu\mathrm{dist}(x, \mathscr{C}) & \text{if } \ \mu\mathrm{dist}(x, \mathscr{C}) \leq 1\,, \\ \phi_S(x) & \text{if } \ \mu\mathrm{dist}(x, \mathscr{C}) \geq 1\,, \end{cases}$$

—see Figure 4.2 (b) for an schematic representation.

With this function on hand, we construct the competitor $w : \overline{B_{S+2} \cap \mathcal{O}} \to \mathbb{R}$ as

$$w := \min\{u, \Psi_S\}.$$

We check next that (H1)-(H5) hold.

First of all, recall that by Lemma 4.3.3, $0 \le u \le 1$ in $\mathcal{O}$. Since $-1 \le \Psi_S \le 1$ in $B_{S+2} \cap \mathcal{O}$, (H1) holds trivially. Moreover, since both functions are doubly radial and vanish on $\mathscr{C}$, (H2) follows —recall that the distance to the cone, in $\mathcal{O}$, is the doubly radial function given by $(|x'| - |x''|)/\sqrt{2}$. The verification of (H4) is easy, since $\Psi_S \equiv -1 \le u$ in $(B_{S+2} \cap \mathcal{O}) \setminus \Omega_S$.

Now, we check that (H3) holds. On the one hand, if $x \in \partial B_{S+2} \cap \mathcal{O}$ and $\mu \operatorname{dist}(x, \mathscr{C}) \ge 1$, we have $\Psi_S(x) = \phi_S(x) = 1 \ge u(x)$, and therefore $w(x) = u(x)$. On the other hand, for $x \in \partial B_{S+2} \cap \mathcal{O}$ with $\mu \operatorname{dist}(x, \mathscr{C}) \le 1$, we have $\Psi_S(x) = \mu \operatorname{dist}(x, \mathscr{C})$. By (4.4.1),

$$|u(y) - u(z)| \le \mu |y - z| \quad \text{for every } y,\ z \in \overline{B_{S+3}},$$

and thus, by taking $y = x$ and $z \in \mathscr{C}$ to be a point realizing $\operatorname{dist}(x, \mathscr{C})$, we obtain that

$$u(x) = |u(x)| \le \mu |x - z| = \mu \operatorname{dist}(x, \mathscr{C}) = \Psi_S(x).$$

Thus, $w(x) = u(x)$ and (H3) holds.

Finally, we verify (H5). Obviously, w is Lipschitz in $\overline{B_{S+2}}$ since it is the minimum of two Lipschitz functions —with Lipschitz constants depending only on μ. From this it follows that (4.4.4) also holds, for a large constant C depending on μ, at points where $\operatorname{dist}(x, \mathscr{C}) \le 2/\mu$. Finally, assume that $\operatorname{dist}(x, \mathscr{C}) \ge 2/\mu$. Then, by using the triangular inequality and the definition of distance to the Simons cone, we have

$$|x - y| \ge \operatorname{dist}(x, \mathscr{C}) - \operatorname{dist}(y, \mathscr{C}) \ge \frac{1}{2} \operatorname{dist}(x, \mathscr{C}).$$

From this and (H1), we readily deduce that (4.4.4) holds for a large constant C. $\square$

To estimate the energy of w in B_{S+2}, it will be important to control the double integrals in the nonlocal energy first in the set where $|x - y| \ge d_S(x)$, and then in $\{|x - y| \le d_S(x)\}$, where

$$d_S(x) := \min\{\operatorname{dist}(x, \partial B_{S+1}), \mu \operatorname{dist}(x, \mathscr{C})\} \quad \text{for } x \in B_S\,.$$

A similar technicality was used by Savin and Valdinoci in [173] with the function $\operatorname{dist}(x, \partial B_{S+1})$, and it is the key point to get (4.1.17). We can now establish the energy estimate of Theorem 4.1.3.

Proof of Theorem 4.1.3. Take w constructed in Lemma 4.4.1 and extend it oddly through $\mathscr{C}$ and then to coincide with u outside B_{S+2}. Hence, since u is a doubly radial odd minimizer in B_R, and w an admissible competitor, $\mathcal{E}(u, B_R) \le \mathcal{E}(w, B_R)$. Moreover, $u \equiv w$ in $\mathbb{R}^{2m} \setminus B_{S+2}$, and thus it follows that

$$\mathcal{E}(u, B_{S+2}) \le \mathcal{E}(w, B_{S+2}).$$

By the monotonicity of the energy $\mathcal{E}$ by inclusions we get

$$\mathcal{E}(u, B_S) \le \mathcal{E}(w, B_{S+2}).$$

Therefore, to obtain the desired result it remains to estimate $\mathcal{E}(w, B_{S+2})$.

In the following inequalities, the letter C will be a constant depending only on m, γ, λ, Λ, and $||G||_{C^2([-1,1])}$. Recall that μ defined in (4.4.1) depends only on these quantities.

First, note that using the upper bound for the kernel K —(4.1.4)— and the change of variables given by $(\cdot)^\star$, it follows that

$$\mathcal{E}(w, B_{S+2}) \le C \int_{B_{S+2}\cap\mathcal{O}} \mathrm{d}x \int_{\mathbb{R}^{2m}} \mathrm{d}y \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} + \int_{B_{S+2}} G(w)\,\mathrm{d}x.$$

Now we estimate separately the potential and kinetic energies.

Estimate for the potential energy. Since, $w = \pm 1$ in $B_{S+2}\backslash\Omega_S$ by (H4), $-1 \le w \le 1$ by (H1), and $G(1) = G(-1) = 0$, it is clear that

$$\int_{B_{S+2}} G(w) = \int_{\Omega_S} G(w) \le C|\Omega_S| \le CS^{2m-1}.$$

Here we have used (4.4.3).

Estimate for the kinetic energy. We split the integral in three terms, as follows.

$$\begin{aligned}
\int_{B_{S+2}\cap\mathcal{O}} \mathrm{d}x \int_{\mathbb{R}^{2m}} \mathrm{d}y \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} &= \int_{\Omega_S\cap\mathcal{O}} \mathrm{d}x \int_{\mathbb{R}^{2m}} \mathrm{d}y\, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\quad + \int_{(B_S\backslash\Omega_S)\cap\mathcal{O}} \mathrm{d}x \int_{B_{S+1}\cap\mathcal{O}} \mathrm{d}y\, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\quad + \int_{(B_S\backslash\Omega_S)\cap\mathcal{O}} \mathrm{d}x \int_{(B_{S+1}\cap\mathcal{O})^c} \mathrm{d}y\, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Here $(\cdot)^c$ denotes the complementary set. Now we control each term separately.

We estimate the first integral:

$$\begin{aligned}
I_1 &= \int_{\Omega_S\cap\mathcal{O}} \mathrm{d}x \int_{\mathbb{R}^{2m}} \mathrm{d}y\, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&= \int_{\Omega_S\cap\mathcal{O}} \mathrm{d}x \int_{|x-y|\le 1} \mathrm{d}y\, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} + \int_{\Omega_S\cap\mathcal{O}} \mathrm{d}x \int_{|x-y|\ge 1} \mathrm{d}y\, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\
&\le C \int_{\Omega_S\cap\mathcal{O}} \mathrm{d}x \int_0^1 \mathrm{d}r\; r^{1-2\gamma} + C \int_{\Omega_S\cap\mathcal{O}} \mathrm{d}x \int_1^\infty \mathrm{d}r\; r^{-1-2\gamma} \\
&= C|\Omega_S| \le C\; S^{2m-1}.
\end{aligned}$$

We have used that w is Lipschitz in $\overline{B_{S+3}}$ —see (H5) and (4.4.1)— to bound the first integral, while the second one is controlled using only that w is bounded, by (H1).

Next, we estimate I_2. To do it, we first claim that, if $|x-y| \le d_S(x)$, then

$$|w(x) - w(y)| \le \frac{C}{d_S(x)}|x-y| \tag{4.4.6}$$

for every $x \in (B_S \setminus \Omega_S) \cap \mathcal{O}$ and $y \in B_{S+1} \cap \mathcal{O}$. Recall that d_S is defined as $d_S(x) = \min\{\operatorname{dist}(x, \partial B_{S+1}), \mu\operatorname{dist}(x, \mathscr{C})\}$, and therefore it suffices to show that

$$|w(x) - w(y)| \le \frac{C}{\operatorname{dist}(x,\mathscr{C})}|x-y|$$

for $x \in B_S \cap \mathcal{O}$ with $\mu\mathrm{dist}(x, \mathscr{C}) \geq 1$ and $y \in B_{S+1} \cap \mathcal{O}$ (recall that C may depend on μ). Now, if we assume that $\mu\mathrm{dist}(y, \mathscr{C}) \geq 1$, it follows that $w(x) = w(y) = -1$ and (4.4.6) is trivially true. On the other hand, if we assume that $\mu\mathrm{dist}(y, \mathscr{C}) \leq 1$, then (4.4.6) follows from (H5). Therefore, the claim is proved.

Using (4.4.6), we proceed as before splitting the integrals to obtain

$$\begin{aligned} I_2 &= \int_{(B_S \setminus \Omega_S)\cap\mathcal{O}} \mathrm{d}x \int_{B_{S+1}\cap\mathcal{O}} \mathrm{d}y \, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\ &\leq \int_{(B_S \setminus \Omega_S)\cap\mathcal{O}} \mathrm{d}x \int_{\{|x-y|\leq d_S(x)\}\cap B_{S+1}\cap\mathcal{O}} \mathrm{d}y \, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\ &\quad + \int_{(B_S \setminus \Omega_S)\cap\mathcal{O}} \mathrm{d}x \int_{|x-y|\geq d_S(x)} \mathrm{d}y \, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\ &\leq C \int_{(B_S \setminus \Omega_S)\cap\mathcal{O}} d_S(x)^{-2\gamma} \, \mathrm{d}x. \end{aligned}$$

Here we have used (4.4.6) to estimate the first term, while for the second one we have only used that w is bounded, by (H1). The last integral for $d_S(x)^{-2\gamma}$ will be bounded later on.

Next, we estimate I_3. To do it, we first claim that if $x \in (B_S \setminus \Omega_S) \cap \mathcal{O}$ and $y \in (B_{S+1} \cap \mathcal{O})^c = \mathcal{I} \cup B^c_{S+1}$, then $|x-y| \geq c d_S(x)$ for some constant $c > 0$ depending only on μ. Indeed, on the one hand it is clear that, if $y \in B^c_{S+1}$, then $|x-y| \geq \mathrm{dist}(x, \partial B_{S+1}) \geq d_S(x)$. On the other hand, if $y \in \mathcal{I}$, then $|x-y| \geq \mathrm{dist}(x, \mathscr{C}) \geq d_S(x)/\mu$.

By the previous claim, since w is bounded, we obtain

$$\begin{aligned} I_3 &= \int_{(B_S \setminus \Omega_S)\cap\mathcal{O}} \mathrm{d}x \int_{(B_{S+1}\cap\mathcal{O})^c} \mathrm{d}y \, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\ &\leq \int_{(B_S \setminus \Omega_S)\cap\mathcal{O}} \mathrm{d}x \int_{|x-y|\geq C d_S(x)} \mathrm{d}y \, \frac{|w(x)-w(y)|^2}{|x-y|^{2m+2\gamma}} \\ &\leq C \int_{(B_S \setminus \Omega_S)\cap\mathcal{O}} d_S(x)^{-2\gamma} \, \mathrm{d}x. \end{aligned}$$

Now, we add up I_1, I_2, and I_3 to get

$$\mathcal{E}(w, B_{S+2}) \leq C \left(\int_{(B_S \setminus \Omega_S)\cap\mathcal{O}} d_S(x)^{-2\gamma} \, \mathrm{d}x + S^{2m-1} \right). \tag{4.4.7}$$

We conclude the proof by estimating the integral of $d_S(x)^{-2\gamma}$, as follows.

$$\begin{aligned} \int_{B_{S+2}\setminus\Omega_S} d_S(x)^{-2\gamma} \, \mathrm{d}x &= \int_{B_S \cap\{\mu\mathrm{dist}(x,\mathscr{C})>1\}} \max\{\mathrm{dist}(x, \partial B_{S+1})^{-2\gamma}, (\mu\mathrm{dist}(x, \mathscr{C}))^{-2\gamma}\} \, \mathrm{d}x \\ &\leq \int_{B_S} (S+1-|x|)^{-2\gamma} \, \mathrm{d}x \\ &\quad + C \int_{B_S \cap\{\mu\mathrm{dist}(x,\mathscr{C})>1\}} \mathrm{dist}(x, \mathscr{C})^{-2\gamma} \, \mathrm{d}x. \end{aligned} \tag{4.4.8}$$

We next control these two integrals.

The first integral can be estimated by using spherical coordinates and the change $\tau = r/(S+1)$. Indeed,

$$\begin{aligned}\int_{B_S} (S+1-|x|)^{-2\gamma}\, \mathrm{d}x &= C\int_0^S \frac{r^{2m-1}}{(S+1-r)^{2\gamma}}\,\mathrm{d}r \\ &\leq C(S+1)^{2m-2\gamma}\int_0^{1-\frac{1}{S+1}} \frac{\tau^{2m-1}}{(1-\tau)^{2\gamma}}\,\mathrm{d}\tau \\ &\leq C(S+1)^{2m-2\gamma}\int_0^{1-\frac{1}{S+1}} (1-\tau)^{-2\gamma}\,\mathrm{d}\tau \\ &\leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma\in(0,1/2),\\ C\ S^{2m-1}\log S & \text{if} \quad \gamma = 1/2,\\ C\ S^{2m-1} & \text{if} \quad \gamma\in(1/2,1).\end{cases}\end{aligned}$$

To bound the second integral (note that it only appears in the proof when $1/\mu \leq S$), we write it in the $(\overline{s},\overline{t})$ variables in $\mathbb{R}^2$, where

$$\overline{s} := \frac{|x'|+|x''|}{\sqrt{2}} \qquad \text{and} \qquad \overline{t} := \frac{|x'|-|x''|}{\sqrt{2}}.$$

Note that $\overline{t}$ is the signed distance to the cone (see Lemma 4.2 in [59]). Thus, still denoting by B_S the ball of radius S in $\mathbb{R}^2$,

$$\begin{aligned}\int_{B_S\cap\{\mu \mathrm{dist}(x,\mathscr{C})>1\}} \mathrm{dist}(x,\mathscr{C})^{-2\gamma}\,\mathrm{d}x &\leq C\int\int_{B_S\cap\{\overline{s}\geq|\overline{t}|>1/\mu\}} |\overline{t}|^{-2\gamma}\,(\overline{s}^2-\overline{t}^2)^{m-1}\,\mathrm{d}\overline{s}\,\mathrm{d}\overline{t} \\ &\leq C\int\int_{B_S\cap\{\overline{s}\geq|\overline{t}|>1/\mu\}} |\overline{t}|^{-2\gamma}\,\overline{s}^{2m-2}\,\mathrm{d}\overline{s}\,\mathrm{d}\overline{t} \\ &\leq C\int_{1/\mu}^S \mathrm{d}\overline{t}\ \overline{t}^{-2\gamma}\int_0^S \mathrm{d}\overline{s}\ \overline{s}^{2m-2} \\ &\leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma\in(0,1/2),\\ C\ S^{2m-1}\log S & \text{if} \quad \gamma = 1/2,\\ C\ S^{2m-1} & \text{if} \quad \gamma\in(1/2,1).\end{cases}\end{aligned}$$

Using these two estimates, combined with (4.4.7) and (4.4.8), the desired result follows by noticing that the term CS^{2m-1} in (4.4.7) is of lower order when $\gamma \leq 1/2$. □

4.5 Existence of saddle-shaped solutions: variational method

In this section we establish the existence of saddle-shaped solutions to the integro-differential Allen-Cahn equation. The proof is based on the direct method of the calculus of variations, and it uses most of the results appearing in the previous sections.

Proof of Theorem 4.1.4. Since $\mathcal{E}(w, B_R)$ is bounded from below —by 0—, we can take a minimizing sequence in $\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)$, that we call u_R^j with $j\in\mathbb{Z}^+$. Note that, by Lemma 4.3.3 we can assume that $-1\leq u_R^j\leq 1$ and that $u_R^j \geq 0$ in $\mathcal{O}$ and $u_R^j\leq 0$ in $\mathcal{I}$.

Now, using (4.1.4), $G \geq 0$, and the fact that u_R^j is a minimizing sequence, we deduce using (4.1.14) that

$$[u_R^j]^2_{H^\gamma(B_R)} \leq \frac{c_{n,\gamma}}{\lambda}[u_R^j]^2_{\mathbb{H}^K(B_R)} \leq \frac{2c_{n,\gamma}}{\lambda}\mathcal{E}(u_R^j, B_R) \leq C$$

for a constant C that does not depend on j. Therefore, by combining this with the fractional Poincaré inequality (recall that $u_R^j \equiv 0$ in $\mathbb{R}^{2m} \setminus B_R$) we get that the sequence $\{u_R^j\}$ is bounded in $H^\gamma(B_R)$. Hence, by the compact embedding $H^\gamma(B_R) \subset\subset L^2(B_R)$ (see [3] and Theorem 7.1 of [92]), there exists a subsequence, still denoted by u_R^j, that converges to some doubly radial $u_R \in L^2(B_R)$, and thus, a.e. in B_R. By Fatou's lemma, we have

$$\mathcal{E}(u_R, B_R) \leq \liminf_{j\to\infty} \mathcal{E}(u_R^j, B_R) = \inf\left\{\mathcal{E}(w, B_R) \ : \ w \in \widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)\right\}.$$

Therefore, $u_R \in \widetilde{\mathbb{H}}^K(B_R)$. In addition, $u_R(x) = -u_R(x^\star)$ for every $x \in \mathbb{R}^{2m}$, and $u_R \equiv 0$ in $\mathbb{R}^{2m} \setminus B_R$. Thus, u_R is a minimizer of $\mathcal{E}(\cdot, B_R)$ in $\widetilde{\mathbb{H}}^K_{0,\mathrm{odd}}(B_R)$. Moreover, it satisfies $-1 \leq u_R \leq 1$ in B_R and $u_R \geq 0$ in $\mathcal{O}$. As a consequence, by Proposition 4.3.4 and the regularity for operators in L_K (see Remark 4.3.5), we have that u_R is a classical solution to

$$\begin{cases} L_K u_R &= f(u_R) \quad \text{in } B_R\,, \\ \quad u_R &= 0 \qquad\quad \text{in } \mathbb{R}^{2m} \setminus B_R. \end{cases}$$

The next step is to pass to the limit in R to obtain a solution in $\mathbb{R}^{2m}$. This is done using a compactness argument. Let $S > 0$ and consider the family $\{u_R\}$, for $R > S + 1$, of solutions to $L_K u_R = f(u_R)$ in B_S. Note first that, if w solves $L_K w = f(w)$ in B_ρ and $|w| \leq 1$ in $\mathbb{R}^{2m}$ with $f \in C^\alpha([-1, 1])$ for some $\alpha > 0$, the combination of the estimates (4.3.6) and (4.3.9) yields

$$||w||_{C^{2\gamma+\varepsilon}(B_{\rho/8})} \leq C\left(n,\ \gamma,\ \lambda,\ \Lambda,\ ||f||_{C^\alpha([-1,1])}\right).$$

for some $\varepsilon > 0$. By applying this to u_R in balls of radius $\rho = 1$ and centered at points in $\overline{B_S}$, we obtain a uniform $C^{2\gamma+\varepsilon}(\overline{B_S})$ bound for u_R. By the Arzelà-Ascoli theorem, as $R \to +\infty$, a subsequence of $\{u_R\}$ converges in $C^{2\gamma+\varepsilon/2}(\overline{B_S})$ to a (pointwise) solution in B_S. Taking now $S = 1, 2, 3, \ldots$ and using a diagonal argument, we obtain a sequence u_{R_j} converging uniformly on compacts in the $C^{2\gamma+\varepsilon/2}$ norm to a solution $u \in C^{2\gamma+\varepsilon/2}(\mathbb{R}^{2m})$ of (4.1.1).

Therefore, we have obtained a solution u to $L_K u = f(u)$ in $\mathbb{R}^{2m}$ which is doubly radial. Furthermore, u is odd with respect to the Simons cone $\mathscr{C}$, i.e., $u(x) = -u(x^\star)$ for $x \in \mathbb{R}^{2m}$, and $0 \leq u \leq 1$ in $\mathcal{O}$.

Finally, we show that $0 < u < 1$ in $\mathcal{O}$. This will ensure that u is a saddle-shaped solution. First, note that $|u| < 1$ by the usual strong maximum principle (since u vanishes at $\mathscr{C}$ and is continuous, we have $u \not\equiv 1$ and $u \not\equiv -1$ in $\mathbb{R}^{2m}$). Let us show now that $u \not\equiv 0$. To do this, we use the energy estimate of Theorem 4.1.3. That is, if we consider u_R the minimizer of $\mathcal{E}(\cdot, B_R)$ with $R > 8$, we have

$$\mathcal{E}(u_R, B_S) \leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0, 1/2), \\ C\ S^{2m-2\gamma}\log S & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2, 1), \end{cases}$$

for every $2 < S < R-5$ and with a constant C independent of R and S. By letting $R \to \infty$ we obtain the same estimate for u. By contradiction, assume $u \equiv 0$. Then, the previous estimate leads to

$$c_m G(0) S^{2m} = \mathcal{E}(0, B_S) \leq \begin{cases} C\ S^{2m-2\gamma} & \text{if} \quad \gamma \in (0, 1/2), \\ C\ S^{2m-2\gamma} \log S & \text{if} \quad \gamma = 1/2, \\ C\ S^{2m-1} & \text{if} \quad \gamma \in (1/2, 1), \end{cases}$$

and, since $G(0) > 0$, this is a contradiction for S large enough. Therefore, $u \not\equiv 0$ and the strong maximum principle for odd functions (see Proposition 4.1.2) yields that $u > 0$ in $\mathcal{O}$. □

4.6 Appendix: Some auxiliary results on convex functions

In this appendix we present some auxiliary results concerning convex functions. The main result, used in the proof of Theorem 4.1.1, is the following.

Proposition 4.6.1. *Let $K : (0, +\infty) \to (0, +\infty)$ be a measurable function. Then, the following statements are equivalent:*

i) *$K(\sqrt{\cdot})$ is strictly convex in $(0, +\infty)$.*

ii) *For every positive constants c_1 and c_2, the function $g : (0, 1/c_2) \to \mathbb{R}$ defined by*

$$g(z) := K(c_1\sqrt{1 + c_2 z}) + K(c_1\sqrt{1 - c_2 z}) \tag{4.6.1}$$

satisfies

$$g(A) + g(D) \geq g(B) + g(C) \tag{4.6.2}$$

whenever A, B, C, and D belong to $(0, 1/c_2)$ and satisfy

$$A = \max\{A,\ B,\ C,\ D\} \quad \text{and} \quad A + D \geq B + C.$$

In addition, still assuming $A = \max\{A,\ B,\ C,\ D\}$ and $A + D \geq B + C$, equality holds in (4.6.2) if and only if the sets $\{A, D\}$ and $\{B, C\}$ coincide.

To prove this proposition, we need a lemma on convex functions.

Lemma 4.6.2. *Let $0 < M \leq +\infty$ and let $h : (0, M) \to \mathbb{R}$ be a measurable nondecreasing function. Then, the following statements are equivalent.*

(a) *h is convex in $(0, M)$.*

(b) *For every $0 \leq L \leq 2M$, the function $h_L(x) := h(x) + h(L - x)$ is convex in $(\max\{L - M, 0\}, \min\{L, M\})$.*

(c) *For every A, B, C, D in the interval $(0, M)$ such that*

$$A = \max\{A,\ B,\ C,\ D\} \quad \text{and} \quad A + D \geq B + C\,,$$

it holds

$$h(A) + h(D) \geq h(B) + h(C)\,. \tag{4.6.3}$$

Proof. $(a) \Rightarrow (c)$. Since B and C are interchangeable and h is nondecreasing, we may assume that $A \geq B \geq C \geq D$. Now, let M_C be the maximum slope of the supporting lines of h at C, and let m_B be the minimum slope of the supporting lines of h at B. By the convexity and monotonicity of h, it holds $m_B \geq M_C \geq 0$ and also

$$h(x) \geq h(B) + m_B(x - B) \quad \text{and} \quad h(x) \geq h(C) + M_C(x - C)$$

for every $x \in (0, M)$.

Hence, since $A - B \geq C - D \geq 0$, we have

$$h(A) - h(B) \geq m_B(A - B) \geq M_C(C - D) \geq h(C) - h(D)\,.$$

$(c) \Rightarrow (b)$. Let $x,\, y \in (\max\{L - M, 0\}, \min\{L, M\})$ and assume that $x > y$. By taking $A = x$, $B = C = (x + y)/2$, and $D = y$ in (4.6.3), we get

$$\frac{h(x) + h(y)}{2} \geq h\left(\frac{x+y}{2}\right).$$

Similarly, by taking $A = L - y$, $B = C = L - (x + y)/2$, and $D = L - x$ in (4.6.2), we get

$$\frac{h(L-x) + h(L-y)}{2} \geq h\left(L - \frac{x+y}{2}\right).$$

By adding up the previous two inequalities we obtain

$$\frac{h_L(x) + h_L(y)}{2} \geq h_L\left(\frac{x+y}{2}\right).$$

$(b) \Rightarrow (a)$. Let $x_0,\, y_0 \in (0, M)$ and choose $L = x_0 + y_0 \leq 2M$. By (b) we have

$$\frac{h(x) + h(x_0 + y_0 - x) + h(y) + h(x_0 + y_0 - y)}{2} \geq h\left(\frac{x+y}{2}\right) + h\left(x_0 + y_0 - \frac{x+y}{2}\right),$$

for every x and y in the interval $(\max\{L - M, 0\}, \min\{L, M\})$. By choosing $x = x_0$ and $y = y_0$ we obtain

$$h(x_0) + h(y_0) \geq 2h\left(\frac{x_0+y_0}{2}\right).$$

□

Remark 4.6.3. We can replace convexity by strict convexity in (a) and (b), and then the inequality in (4.6.3) is strict unless the sets $\{A, D\}$ and $\{B, C\}$ coincide.

Remark 4.6.4. Note that the function h_L is even with respect to $L/2$. Thus, if it is convex, it is nondecreasing in $(L/2, \min\{L, M\})$.

Remark 4.6.5. The assumption of h being nondecreasing is only used to deduce (c) from (a). It is not required to show the equivalence between (a) and (b), neither to deduce (a) from (c).

With this result available we can show now Proposition 4.6.1

Proof. $i) \Rightarrow ii)$ We take $M = +\infty$ and $h(\cdot) = K(\sqrt{\cdot})$ in Lemma 4.6.2. Since h is strictly convex, the function h_L is strictly convex in $(0, L)$ for every $L > 0$ (recall that we do not need to assume that h is monotone to deduce this, see Remark 4.6.5). Moreover, by Remark 4.6.4, h_L is nondecreasing in $(L/2, L)$. Thus, the function $\phi(\cdot) = h_L(\cdot + L/2)$

is strictly convex in $(-L/2, L/2)$ and nondecreasing in $(0, L/2)$. If we choose $L = 2c_1^2$, we have that $\phi((L/2)c_2\cdot) = g(\cdot)$, where g is defined by (4.6.1). Therefore, g is strictly convex in $(-1/c_2, 1/c_2)$ and nondecreasing in $(0, 1/c_2)$. Thus, the result follows by applying Lemma 4.6.2 to g in $(0, 1/c_2)$ (taking into account Remark 4.6.3).

ii) $\Rightarrow$ *i)* By Lemma 4.6.2 applied to g we deduce that g is strictly convex and nondecreasing in $(0, 1/c_2)$ —take $C = D$ to see that g is monotone. Thus, since g is even and nondecreasing, g is strictly convex in $(-1/c_2, 1/c_2)$ and $\varphi(\cdot) = g(\cdot/(c_1^2 c_2))$ is strictly convex in $(-c_1^2, c_1^2)$. Hence, if we call $h(\cdot) := K(\sqrt{\cdot})$ and $L := 2c_1^2$, we have that $\varphi(\cdot - c_1^2) = h(\cdot) + h(L - \cdot) =: h_L(\cdot)$, and thus h_L is strictly convex in $(0, L)$. Note that since $c_1 > 0$ is arbitrary, h_L is strictly convex in $(0, L)$ for all $L > 0$. Therefore, by Lemma 4.6.2, with $M = +\infty$, we conclude that $h(\cdot) = K(\sqrt{\cdot})$ is strictly convex in $(0, +\infty)$. □

4.7 Appendix: An auxiliary computation

In this appendix we present an auxiliary computation that is needed in Section 4.2 in order to complete the proof of Proposition 4.2.4.

Lemma 4.7.1. *Let α, β be two real numbers satisfying $\alpha \geq |\beta|$. Let $x = (x', x'')$, $y = (y', y'') \in \mathcal{O} \subset \mathbb{R}^{2m}$. Define*

$$A = |x'||y'|\alpha + |x''||y''|\beta\,, \qquad B = |x'||y''|\alpha + |x''||y'|\beta\,,$$
$$C = |x''||y'|\alpha + |x'||y''|\beta\,, \qquad D = |x''||y''|\alpha + |x'||y'|\beta\,.$$

Then,

1. *It holds*
$$\begin{cases} |A| \geq |B|, \ |A| \geq |C|, \ |A| \geq |D|\,, \\ |A| + |D| \geq |B| + |C|\,. \end{cases}$$
2. *If the sets $\{|A|, |D|\}$ and $\{|B|, |C|\}$ coincide, then necessarily $\alpha = \beta = 0$.*

Proof. The proof is elementary but requires to check some cases. In all of them we will use the following inequalities. Since $\alpha \geq |\beta|$,

$$\alpha \geq 0 \quad \text{and} \quad -\alpha \leq \beta \leq \alpha\,.$$

Moreover, since $x, y \in \mathcal{O}$, it holds

$$|x'| > |x''| \quad \text{and} \quad |y'| > |y''|\,.$$

We start establishing the first statement. We show next that $A \geq 0$ and that

$$A \geq |B|, \ A \geq |C|, \ A \geq |D|\,.$$

- $A \geq 0$:
$$A = |x'||y'|\alpha + |x''||y''|\beta \geq (|x'||y'| - |x''||y''|)\alpha \geq 0\,.$$
- $A \geq |B|$:
$$A \pm B = (|x'|\alpha - |x''|\beta)(|y'| \pm |y''|) \geq 0\,.$$
- $A \geq |C|$:
$$A \pm C = (|y'|\alpha - |y''|\beta)(|x'| \pm |x''|) \geq 0\,.$$

- $A \geq |D|$:
$$A \pm D = (|x'||y'| \pm |x''||y''|)(\alpha \pm \beta) \geq 0\,.$$

It remains to show
$$A + |D| \geq |B| + |C|\,.$$

The proof of this fact is just a computation considering all the eight possible configurations of the signs of B, C, and D. Since the roles of B and C are completely interchangeable, we may assume that $B \geq C$ and we only need to check six cases. To do it, note first that

$$A + D - B - C = (|x'| - |x''|)(|y'| - |y''|)(\alpha + \beta) \geq 0\,, \tag{4.7.1}$$

$$A - D - B + C = (|x'| + |x''|)(|y'| - |y''|)(\alpha - \beta) \geq 0\,, \tag{4.7.2}$$

and

$$A + D + B + C = (|x'| + |x''|)(|y'| + |y''|)(\alpha + \beta) \geq 0\,, \tag{4.7.3}$$

With these three relations at hand we check the six cases.

- If $B \geq 0$, $C \geq 0$, and $D \geq 0$, then by (4.7.1) we have
$$A + |D| - |B| - |C| = A + D - B - C \geq 0\,.$$
- If $B \geq 0$, $C \geq 0$, and $D \leq 0$, we use the sign of D and (4.7.1) to see that
$$A + |D| - |B| - |C| = A - D - B - C = (A + D - B - C) + (-2D) \geq 0\,.$$
- If $B \geq 0$, $C \leq 0$, and $D \geq 0$, we use the sign of D and (4.7.2) to see that
$$A + |D| - |B| - |C| = A + D - B + C = (A - D - B + C) + 2D \geq 0\,.$$
- If $B \geq 0$, $C \leq 0$, and $D \leq 0$, then by (4.7.2) we have
$$A + |D| - |B| - |C| = A - D - B + C \geq 0\,.$$
- If $B \leq 0$, $C \leq 0$, and $D \geq 0$, then by (4.7.3) we have
$$A + |D| - |B| - |C| = A + D + B + C \geq 0\,.$$
- If $B \leq 0$, $C \leq 0$, and $D \leq 0$, we use the sign of D and (4.7.3) to see that
$$A + |D| - |B| - |C| = A - D + B + C = (A + D + B + C) + (-2D) \geq 0\,.$$

This concludes the proof of the first statement.

We prove now the second point of the lemma. Since the roles of B and C are completely interchangeable, we only need to show the result in the case $|A| = |B|$ and $|C| = |D|$.

Recall that $A \geq 0$. Hence, since $A = |B|$ and $|C| = |D|$, a simple computation shows that
$$\alpha = \operatorname{sign}(B)\frac{|x''|}{|x'|}\beta \quad \text{and} \quad \beta = \operatorname{sign}(C)\operatorname{sign}(D)\frac{|x''|}{|x'|}\alpha\,.$$
Hence, combining both equalities we obtain
$$\alpha = \operatorname{sign}(B)\operatorname{sign}(C)\operatorname{sign}(D)\frac{|x''|^2}{|x'|^2}\alpha.$$
Finally, if we assume $\alpha \neq 0$, then necessarily $\operatorname{sign}(B)\operatorname{sign}(C)\operatorname{sign}(D) = 1$ and $|x'| = |x''|$, but this is a contradiction with $x \in \mathcal{O}$. Therefore, $\alpha = 0$ and thus $\beta = 0$. □

4.8 Appendix: The integro-differential operator L_K in the (s,t) variables

The goal of this appendix is to take advantage of the doubly radial symmetry of the functions we are dealing with to write equation (4.1.1) in (s,t) variables, passing from an equation in $\mathbb{R}^{2m}$ to an equation in $(0,+\infty)\times(0,+\infty)\subset\mathbb{R}^2$. Although we do not use these computations in this chapter, we include them here to show the usefulness of having introduced the $\overline{K}$ kernel obtained after integration with respect to the Haar measure on $O(m)^2$. Moreover, the following expressions could be useful for future reference. In the case of the fractional Laplacian, the kernel that we obtain involves essentially an hypergeometric function which is the so-called Appell function F_2 (see [12] for its definition).

Lemma 4.8.1. *Let $m\geq 1$, $\gamma\in(0,1)$, and let $w\in C^\alpha(\mathbb{R}^{2m})$, with $\alpha>2\gamma$, be a doubly radial function, i.e., depending only on the variables s and t. Let L_K be a rotation invariant operator, that is, $K(z)=K(|z|)$, of the form (4.1.2). Then, if we define $\tilde{w}:(0,+\infty)\times(0,+\infty)\to\mathbb{R}$ by $\tilde{w}(s,t)=w(s,0,...,0,t,0,...,0)$, it holds*

$$L_K w(x)=\tilde{L}_K\tilde{w}(|x'|,|x''|),$$

with

$$\tilde{L}_K\tilde{w}(s,t):=\int_0^{+\infty}\int_0^{+\infty}\sigma^{m-1}\tau^{m-1}\Big(\tilde{w}(s,t)-\tilde{w}(\sigma,\tau)\Big)J(s,t,\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau\,,$$

where:

1. *If $m=1$,*

$$J(s,t,\sigma,\tau):=\sum_{i=0}^{1}\sum_{j=0}^{1}K\left(\sqrt{s^2+t^2+\sigma^2+\tau^2-2s\sigma(-1)^i-2t\tau(-1)^j}\right). \tag{4.8.1}$$

2. *If $m\geq 2$,*

$$J(s,t,\sigma,\tau):=c_m^2\int_{-1}^{1}\int_{-1}^{1}(1-\theta^2)^{\frac{m-2}{2}}(1-\overline{\theta}^2)^{\frac{m-2}{2}} K\left(\sqrt{s^2+t^2+\sigma^2+\tau^2-2s\sigma\theta-2t\tau\overline{\theta}}\right)\mathrm{d}\theta\,\mathrm{d}\overline{\theta}\,, \tag{4.8.2}$$

with

$$c_m=\frac{2\pi^{\frac{m-1}{2}}}{\Gamma(\frac{m-1}{2})}.$$

Proof. Let $x=(sx_s,tx_t)$ with $x_s,\,x_t\in\mathbb{S}^{m-1}$ and $y=(\sigma y_\sigma,\tau y_\tau)$ with $y_\sigma,\,y_\tau\in\mathbb{S}^{m-1}$. Then, decomposing $\mathbb{R}^{2m}=\mathbb{R}^m\times\mathbb{R}^m$ and using spherical coordinates in each $\mathbb{R}^m$ we obtain

$$\begin{aligned}L_K u(x)&=\int_{\mathbb{R}^{2m}}\Big(u(x)-u(y)\Big)K(|x-y|)\,\mathrm{d}y\\&=\int_0^{+\infty}\int_0^{+\infty}\sigma^{m-1}\tau^{m-1}\Big(u(s,t)-u(\sigma,\tau)\Big)\\&\qquad\left(\int_{\mathbb{S}^{m-1}}\int_{\mathbb{S}^{m-1}}K\left(\sqrt{|sx_s-\sigma y_\sigma|^2+|tx_t-\tau y_\tau|^2}\right)\mathrm{d}y_\sigma\,\mathrm{d}y_\tau\right)\mathrm{d}\sigma\,\mathrm{d}\tau.\end{aligned}$$

Now, we define the kernel

$$J(x_s, x_t, s, t, \sigma, \tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\left(\sqrt{|sx_s - \sigma y_\sigma|^2 + |tx_t - \tau y_\tau|^2}\right) \mathrm{d}y_\sigma \, \mathrm{d}y_\tau \,. \tag{4.8.3}$$

First of all, it is easy to see that J does not depend on x_s nor x_t. Indeed, consider a different point $(z_s, z_t) \in \mathbb{S}^{m-1} \times \mathbb{S}^{m-1}$ and let M_s and M_t be two orthogonal transformations such that $M_s(x_s) = z_s$ and $M_t(x_t) = z_t$. Then, making the change of variables $y_\sigma = M_s(\tilde{y}_\sigma)$ and $y_\tau = M_t(\tilde{y}_\tau)$, and using that $M_s(\mathbb{S}^{m-1}) = M_t(\mathbb{S}^{m-1}) = \mathbb{S}^{m-1}$, we find out that

$$\begin{aligned}
&J(z_s, z_t, s, t, \sigma, \tau) = \\
&\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\left(\sqrt{|sM_s(x_s) - \sigma y_\sigma|^2 + |tM_t(x_t) - \tau y_\tau|^2}\right) \mathrm{d}y_\sigma \, \mathrm{d}y_\tau \\
&\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\left(\sqrt{|sM_s(x_s) - \sigma M_s(\tilde{y}_\sigma)|^2 + |tM_t(x_t) - \tau M_t(\tilde{y}_\tau)|^2}\right) \mathrm{d}\tilde{y}_\sigma \, \mathrm{d}\tilde{y}_\tau \\
&\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\left(\sqrt{|M_s(sx_s - \sigma \tilde{y}_\sigma)|^2 + |M_t(tx_t - \tau \tilde{y}_\tau)|^2}\right) \mathrm{d}\tilde{y}_\sigma \, \mathrm{d}\tilde{y}_\tau \\
&\quad = \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\left(\sqrt{|sx_s - \sigma \tilde{y}_\sigma|^2 + |tx_t - \tau \tilde{y}_\tau|^2}\right) \mathrm{d}\tilde{y}_\sigma \, \mathrm{d}\tilde{y}_\tau \\
&\quad = J(x_s, x_t, s, t, \sigma, \tau) \,.
\end{aligned}$$

Therefore, we can replace x_s and x_t in (4.8.3) by $e = (1, 0, \ldots, 0) \in \mathbb{S}^{m-1}$. Thus, we have

$$J(s, t, \sigma, \tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\left(\sqrt{|se - \sigma y_\sigma|^2 + |te - \tau y_\tau|^2}\right) \mathrm{d}y_\sigma \, \mathrm{d}y_\tau \,.$$

For an easier notation, we rename $\omega = y_\sigma$ and $\tilde{\omega} = y_\tau$, and thus we have

$$\begin{aligned}
|se - \sigma y_\sigma|^2 + |te - \tau y_\tau|^2 &= |se - \sigma\omega|^2 + |te - \tau\tilde{\omega}|^2 \\
&= s^2 + \sigma^2 - 2s\sigma e \cdot \omega + t^2 + \tau^2 - 2t\tau e \cdot \tilde{\omega} \\
&= s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1 \,.
\end{aligned}$$

Then, we can rewrite J as

$$J(s, t, \sigma, \tau) := \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\left(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\right) \mathrm{d}\omega \, \mathrm{d}\tilde{\omega} \,.$$

At this point we have to distinguish the cases $m = 1$ and $m \geq 2$. For the first one, since $\mathbb{S}^0 = \{-1, 1\}$ we directly obtain (4.8.1). For the second one, since the integrand only depends on ω_1 and $\tilde{\omega}_1$, defining $\rho(\cdot) = \sqrt{1 - |\cdot|^2}$ we proceed as follows

$$\begin{aligned}
J(s, t, \sigma, \tau) &= \int_{\mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} K\left(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\right) \mathrm{d}\omega \, \mathrm{d}\tilde{\omega} \\
&= \int_{-1}^{1} \mathrm{d}\omega_1 \int_{\partial B_{\rho(\omega_1)}} \mathrm{d}\omega_2 \cdots \mathrm{d}\omega_m \int_{-1}^{1} \mathrm{d}\tilde{\omega}_1 \int_{\partial B_{\rho(\tilde{\omega}_1)}} \mathrm{d}\tilde{\omega}_2 \cdots \mathrm{d}\tilde{\omega}_m \\
&\qquad\qquad K\left(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\right) \\
&= \int_{-1}^{1} \int_{-1}^{1} |\partial B_{\rho(\omega_1)}| |\partial B_{\rho(\tilde{\omega}_1)}| \\
&\qquad\qquad K\left(\sqrt{s^2 + \sigma^2 - 2s\sigma\omega_1 + t^2 + \tau^2 - 2t\tau\tilde{\omega}_1}\right) \mathrm{d}\omega_1 \, \mathrm{d}\tilde{\omega}_1.
\end{aligned}$$

Finally, we obtain (4.8.2) once we replace $|\partial B_{\rho(\cdot)}| = c_m\,\rho(\cdot)^{m-2} = c_m(1-|\cdot|^2)^{\frac{m-2}{2}}$, where c_m is the measure of the boundary of the unitary ball in $\mathbb{R}^{m-1}$. □

If the operator L_K is the fractional Laplacian, the previous expression of the kernel J can be rewritten in terms of a hypergeometric function of two variables, the so-called Appell function F_2 (see [12]). This expression does not simplify any of the arguments of this chapter. Nevertheless, we think that it is worthy to point out the relation between J and F_2, since the known properties of the last one could provide some information about the kernel J.

Lemma 4.8.2. *Let F_2 be the Appell hypergeometric function defined in [12]. If $L_K = (-\Delta)^\gamma$ and $m \geq 2$, then*

$$J(s,t,\sigma,\tau) = \frac{c_{2m,\gamma}\pi^m\Gamma\left(\frac{m}{2}\right)^2}{\Gamma\left(\frac{m-1}{2}\right)^2\Gamma\left(\frac{m+1}{2}\right)^2}\frac{F_2\left(m+\gamma;\frac{m}{2},m;\frac{m}{2},m;\frac{4s\sigma}{(s+\sigma)^2+(t+\tau)^2},\frac{4t\tau}{(s+\sigma)^2+(t+\tau)^2}\right)}{[(s+\sigma)^2+(t+\tau)^2]^{m+\gamma}}. \tag{4.8.4}$$

Proof. If we take $K(z) = c_{2m,\gamma}|z|^{-2m-2\gamma}$ in (4.8.2) we get

$$J(s,t,\sigma,\tau) = c_{2m,\gamma}c_m^2\int_{-1}^{1}\int_{-1}^{1}\frac{(1-\theta^2)^{\frac{m-2}{2}}(1-\overline{\theta}^2)^{\frac{m-2}{2}}}{(s^2+t^2+\sigma^2+\tau^2-2s\sigma\theta-2t\tau\overline{\theta})^{m+\gamma}}\,d\theta\,d\overline{\theta}\,.$$

Then, if we make the change of variables $\theta = 2\varpi_1 - 1$ and $\overline{\theta} = 2\varpi_2 - 1$ we arrive at

$$\begin{aligned}J(s,t,\sigma,\tau) &= \frac{c_{2m,\gamma}2^{2m-4}c_m^2}{[(s+\sigma)^2+(t+\tau)^2]^{m+\gamma}}\cdot\\ &\qquad\int_0^1\int_0^1\frac{\varpi_1^{\frac{m-2}{2}}(1-\varpi_1)^{\frac{m-2}{2}}\varpi_2^{\frac{m-2}{2}}(1-\varpi_2)^{\frac{m-2}{2}}}{\left(1-\frac{4s\sigma}{(s+\sigma)^2+(t+\tau)^2}\varpi_1-\frac{4t\tau}{(s+\sigma)^2+(t+\tau)^2}\varpi_2\right)^{m+\gamma}}\,d\varpi_1\,d\varpi_2\\ &= \frac{c_{2m,\gamma}2^{2m-4}c_m^2}{[(s+\sigma)^2+(t+\tau)^2]^{m+\gamma}}\frac{\Gamma\left(\frac{m}{2}\right)^4}{\Gamma(m)^2}\cdot\\ &\qquad F_2\left(m+\gamma;\frac{m}{2},m;\frac{m}{2},m;\frac{4s\sigma}{(s+\sigma)^2+(t+\tau)^2},\frac{4t\tau}{(s+\sigma)^2+(t+\tau)^2}\right).\end{aligned}$$

We finally obtain (4.8.4) by using the duplication formula for the Γ-function. □

To conclude the appendix, we rewrite the kernel inequality (4.1.13) in (s,t) variables and in terms of the kernel J. We do not present a proof of this result since it is identical to the one of Proposition 4.2.4 but changing the notation.

Lemma 4.8.3. *Let $m \geq 1$ and let J the kernel defined in (4.8.1)-(4.8.2) with $K(\sqrt{\cdot})$ strictly convex. Then, if $s > t$ and $\sigma > \tau$, we have*

$$J(s,t,\sigma,\tau) > J(s,t,\tau,\sigma)\,.$$

Chapter 5

Semilinear integro-differential equations, II: one-dimensional and saddle-shaped solutions to the Allen-Cahn equation

This chapter, which corresponds to [112] in collaboration with T. Sanz-Perela, addresses saddle-shaped solutions to the semilinear equation $L_K u = f(u)$ in $\mathbb{R}^{2m}$, where L_K is a linear elliptic integro-differential operator with a radially symmetric kernel K, and f is of Allen-Cahn type. Saddle-shaped solutions are doubly radial, odd with respect to the Simons cone $\{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m : |x'| = |x''|\}$, and vanish only in this set.

We establish the uniqueness and the asymptotic behavior of the saddle-shaped solution. For this, we prove a Liouville type result, the one-dimensional symmetry of positive solutions to semilinear problems in a half-space, and maximum principles in "narrow" sets. The existence of the solution was already proved in Chapter 4.

5.1 Introduction

In this chapter we study saddle-shaped solutions to the semilinear equation

$$L_K u = f(u) \quad \text{in } \mathbb{R}^{2m}, \tag{5.1.1}$$

where L_K is a linear integro-differential operator of the form (5.1.2) and f is of Allen-Cahn type. These solutions (see Definition 5.1.1 below) are particularly interesting in relation to the nonlocal version of a conjecture by De Giorgi, with the aim of finding a counterexample in high dimensions. Moreover, this problem is related to the regularity theory of nonlocal minimal surfaces (see Subsection 5.1.3).

Previous to our results from Chapter 4 and the present one, there are only three works devoted to saddle-shaped solutions to the equation (5.1.1) with L_K being the fractional Laplacian. In [75, 76], Cinti proved the existence of a saddle-shaped solution as well as some qualitative properties, such as asymptotic behavior, monotonicity properties, and instability in even dimensions $2m \leq 6$. In Chapter 3, further properties of these solutions were proved, the main ones being uniqueness and, when $2m \geq 14$, stability. The possible stability in dimensions 8, 10, and 12 is still an open problem, as well as the possible minimality of this solution in dimensions $2m \geq 8$. Concerning saddle-shaped solutions to the classical Allen-Cahn equation $-\Delta u = f(u)$, the same results are established; see [46] and the references therein. The stability of the saddle-shaped solution to $-\Delta u = u - u^3$ in dimensions $n = 8$, 10, and 12 has been recently announced [147].

The present chapter together with the previous one are the first works in the literature studying saddle-shaped solutions for general integro-differential equations of the form (5.1.1). In the previous ones [75, 76, 111], the extension problem for the fractional Laplacian was a key tool. Since this technique cannot be carried out for general integro-differential operators, some purely nonlocal techniques were developed in Chapter 4 and we exploit them in the present one.

In part I, we established an appropriate setting to study solutions to (5.1.1) that are doubly radial and odd with respect to the Simons cone, a property that is satisfied by saddle-shaped solutions (see Subsection 5.1.1). We found an alternative and useful expression for the operator L_K when acting on doubly radial odd functions —see (5.1.5). This was used to establish maximum principles for odd functions under a convexity assumption on the kernel K of the operator L_K —see (5.1.8). Moreover, we proved an energy estimate for doubly radial and odd minimizers of the energy associated to the equation, as well as the existence of saddle-shaped solutions to (5.1.1).

In the current chapter, we further study saddle-shaped solutions to (5.1.1), by proving their uniqueness and asymptotic behavior. To establish the uniqueness (Theorem 5.1.2) we use a maximum principle for the linearized operator $L_K - f'(u)$ (Proposition 5.1.4). To prove the asymptotic behavior (Theorem 5.1.3), we use two ingredients: a Liouville type theorem (Theorem 5.1.5) and a one-dimensional symmetry result (Theorem 5.1.6), both for semilinear equations of the form (5.1.1) under some hypotheses on f. The first of these results is obtained by adapting the ideas of Berestycki, Hamel, and Nadirashvili [29] to the nonlocal framework, and requires a Harnack inequality and a parabolic maximum principle. The second one requires the sliding method and the moving planes argument, extended to a general integro-differential setting.

In addition to the previous results, in this chapter we establish further properties of the so-called *layer solution* u_0 (see Section 5.5). We also include an alternative proof of the existence of the saddle-shaped solution using the monotone iteration method (as an alternative to the proof in Chapter 4 where we used variational techniques).

Equation (5.1.1) is driven by a linear integro-differential operator L_K of the form

$$L_K w(x) = \int_{\mathbb{R}^n} \Big(w(x) - w(y)\Big) K(x-y)\, \mathrm{d}y. \tag{5.1.2}$$

The most canonical example of such operators is the fractional Laplacian, which corresponds to the kernel $K(z) = c_{n,\gamma}|z|^{-n-2\gamma}$, where $\gamma \in (0,1)$ and $c_{n,\gamma}$ is a normalizing positive constant —see (5.5.2). Note that some of the results in this chapter are new even for the fractional Laplacian (namely Proposition 5.1.4 and the statement on odd solutions of Theorem 5.1.6), while others are only proved in the literature using the extension problem (in contrast with our proofs).

Throughout the chapter, we assume that K is symmetric, i.e.,

$$K(z) = K(-z), \tag{5.1.3}$$

and that L_K is uniformly elliptic, that is,

$$\lambda \frac{c_{n,\gamma}}{|z|^{n+2\gamma}} \leq K(z) \leq \Lambda \frac{c_{n,\gamma}}{|z|^{n+2\gamma}}\,, \tag{5.1.4}$$

where λ and Λ are two positive constants. Conditions (5.1.3) and (5.1.4) are frequently adopted since they yield Hölder regularity of solutions (see [162, 176]). The family of linear operators satisfying these two conditions is the so-called $\mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ ellipticity class. For short we will usually write $\mathcal{L}_0$ and we will make explicit the parameters only when needed.

When dealing with doubly radial functions we will assume that the operator L_K is rotation invariant, that is, K is radially symmetric. This extra assumption allows us to rewrite the operator in a suitable form when acting on doubly radial odd functions, as explained next.

5.1.1 Integro-differential setting for odd functions with respect to the Simons cone

In this subsection we present the basic definitions and terminology used along the chapter. We also recall the setting established in Chapter 4.

First, we present the Simons cone, a central object along this chapter. It is defined in $\mathbb{R}^{2m}$ by

$$\mathscr{C} := \left\{ x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m} \ : \ |x'| = |x''| \right\} .$$

This cone is of importance in the theory of (local and nonlocal) minimal surfaces (see Subsection 5.1.3). We will use the letters $\mathcal{O}$ and $\mathcal{I}$ to denote each of the parts in which $\mathbb{R}^{2m}$ is divided by the cone $\mathscr{C}$:

$$\mathcal{O} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| > |x''| \right\} \quad \text{and} \quad \mathcal{I} := \left\{ x = (x', x'') \in \mathbb{R}^{2m} \ : \ |x'| < |x''| \right\} .$$

Both $\mathcal{O}$ and $\mathcal{I}$ belong to a family of sets in $\mathbb{R}^{2m}$ which are called of *double revolution*. These are sets that are invariant under orthogonal transformations in the first m variables, as well as under orthogonal transformations in the last m variables. Related to this concept, we say that a function $w : \mathbb{R}^{2m} \to \mathbb{R}$ is *doubly radial* if it depends only on the modulus of the first m variables and on the modulus of the last m ones, i.e., $w(x) = w(|x'|, |x''|)$.

We recall now the definition of $(\cdot)^\star$, an isometry that played a significant role in part I. It is defined by

$$\begin{array}{rccc} (\cdot)^\star : & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m \\ & x = (x', x'') & \mapsto & x^\star = (x'', x') \, . \end{array}$$

Note that this isometry is actually an involution that maps $\mathcal{O}$ into $\mathcal{I}$ (and vice versa) and leaves the cone $\mathscr{C}$ invariant —although not all points in $\mathscr{C}$ are fixed points of $(\cdot)^\star$, for instance, $x = (1, 0, \dots, 0, 1)$. Taking into account this transformation, we say that a doubly radial function w is *odd with respect to the Simons cone* if $w(x) = -w(x^\star)$.

Now we can define saddle-shaped solutions.

Definition 5.1.1. We say that a bounded solution u to (5.1.1) is a *saddle-shaped solution* (or simply *saddle solution*) if

1. u is doubly radial.

2. u is odd with respect to the Simons cone.

3. $u > 0$ in $\mathcal{O} = \{|x'| > |x''|\}$.

Note that these solutions are even with respect to the coordinate axes and that their zero level set is the Simons cone $\mathscr{C} = \{|x'| = |x''|\}$.

In part I, we developed a purely nonlocal theory regarding the integro-differential operator L_K when acting on odd solutions with respect to the Simons cone. First, recall that if K is a radially symmetric kernel we can rewrite the operator acting on a doubly radial function w as

$$L_K w(x) = \int_{\mathbb{R}^{2m}} \big(w(x) - w(y)\big) \overline{K}(x, y) \, \mathrm{d}y \, ,$$

where $\overline{K}$ is doubly radial in both variables and is defined by

$$\overline{K}(x, y) := \fint_{O(m)^2} K(|Rx - y|) \, \mathrm{d}R \, .$$

Here, $\mathrm{d}R$ denotes integration with respect to the Haar measure on $O(m)^2$, where $O(m)$ is the orthogonal group of $\mathbb{R}^m$. It is important to notice that, in contrast with $K = K(x-y)$, the kernel $\overline{K}$ is no longer translation invariant (i.e., it is a function of x and y but not of the difference $x - y$).

If we consider doubly radial functions that are, in addition, odd with respect to the Simons cone, we can use the involution $(\cdot)^\star$ to find that

$$L_K w(x) = \int_{\mathcal{O}} \big(w(x) - w(y)\big)\big(\overline{K}(x, y) - \overline{K}(x, y^\star)\big) \, \mathrm{d}y + 2w(x) \int_{\mathcal{O}} \overline{K}(x, y^\star) \, \mathrm{d}y \, . \qquad (5.1.5)$$

Furthermore,

$$\frac{1}{C} \mathrm{dist}(x, \mathscr{C})^{-2\gamma} \leq \int_{\mathcal{O}} \overline{K}(x, y^\star) \, \mathrm{d}y \leq C \mathrm{dist}(x, \mathscr{C})^{-2\gamma}, \qquad (5.1.6)$$

with $C > 0$ depending only on m, γ, λ, and Λ.

Note that the expression (5.1.5) has an integro-differential part plus a term of order zero with a positive coefficient. Thus, the most natural assumption to make in order to have an elliptic operator (when acting on doubly radial odd functions) is that the kernel of the integro-differential term is positive. That is,

$$\overline{K}(x, y) - \overline{K}(x, y^\star) > 0 \quad \text{for every } x, y \in \mathcal{O} \, . \qquad (5.1.7)$$

One of the main results in part I established a necessary and sufficient condition on the original kernel K for L_K to have a positive kernel when acting on doubly radial odd functions. It turns to be

$$K(\sqrt{\tau}) \text{ is a strictly convex function of } \tau \, . \qquad (5.1.8)$$

The positivity of the kernel of L_K when acting on doubly-radial odd functions was crucial in order to obtain the existence of the saddle-shaped solution. As we will see, it is essential as well to establish the uniqueness. Therefore, (5.1.8) will be a key assumption in some of our results.

5.1.2 Main results

Through all the chapter we will assume that f, the nonlinearity in (5.1.1), is a C^1 function satisfying

$$f \text{ is odd}, \quad f(\pm 1) = 0, \quad \text{and} \quad f \text{ is strictly concave in } (0, 1). \qquad (5.1.9)$$

It is easy to see that these properties yield $f > 0$ in $(0, 1)$, $f'(0) > 0$ and $f'(\pm 1) < 0$.

In some statements in this chapter, we will denote by $L^1_\gamma(\mathbb{R}^n)$ the space of measurable functions w satisfying

$$\int_{\mathbb{R}^n} \frac{|w(x)|}{1+|x|^{n+2\gamma}}\,\mathrm{d}x < +\infty\,.$$

This regularity will be required on a function w (in addition to C^α Hölder continuity, with $\alpha > 2\gamma$) to ensure that $L_K w$ is well-defined.

The first main result of this chapter concerns uniqueness of saddle-shaped solution.

Theorem 5.1.2. *Let f satisfy (5.1.9). Let K be a radially symmetric kernel satisfying the convexity assumption (5.1.8) and such that $L_K \in \mathcal{L}_0(2m,\gamma,\lambda,\Lambda)$. Then, for every even dimension $2m \geq 2$, there exists a unique saddle-shaped solution u to (5.1.1). In addition, u satisfies $|u| < 1$ in $\mathbb{R}^{2m}$.*

To establish the uniqueness of the saddle-shaped solution we will need two ingredients: the asymptotic behavior of saddle-shaped solutions and a maximum principle for the linearized operator in $\mathcal{O}$. Both results will be described below. The existence of saddle-shaped solutions was already proved in Chapter 4 using variational techniques. Here, we show that it can also be established using, instead, the monotone iteration procedure. Let us remark that, in both methods, having the convexity assumption (5.1.8) is crucial.

The second main result of this chapter is Theorem 5.1.3 below, on the asymptotic behavior of a saddle-shaped solution at infinity. To state it, let us introduce an important type of solutions in the study of the integro-differential Allen-Cahn equation: the layer solutions.

We say that a solution v to $L_K v = f(v)$ in $\mathbb{R}^n$ is a *layer solution* if v is increasing in one direction, say $e \in \mathbb{S}^{n-1}$, and $v(x) \to \pm 1$ as $x\cdot e \to \pm\infty$ (not necessarily uniform). When $n = 1$, a result of Cozzi and Passalacqua (Theorem 1 in [83]) establishes the existence and uniqueness (up to translations) of a layer solution. In addition, this solution is odd with respect to some point. They assume the kernel to be in the ellipticity class $\mathcal{L}_0(1,\gamma,\lambda,\Lambda)$ and the nonlinearity satisfying (5.1.9). In the case of the fractional Laplacian this result was proved in [58, 57] using the extension problem.

Given K a translation invariant kernel in $\mathbb{R}^n$, we define a new kernel K_1 in $\mathbb{R}$ as

$$K_1(\tau) := \int_{\mathbb{R}^{n-1}} K(\theta,\tau)\,\mathrm{d}\theta = |\tau|^{n-1}\int_{\mathbb{R}^{n-1}} K(\tau\sigma,\tau)\,\mathrm{d}\sigma.$$

Then, we denote by u_0 the (unique) layer solution in $\mathbb{R}$ associated to L_{K_1} that vanishes at the origin. That is,

$$\begin{cases} L_{K_1} u_0 &= f(u_0) & \text{in } \mathbb{R}\,,\\ \dot{u}_0 &> 0 & \text{in } \mathbb{R}\,,\\ u_0(x) &= -u_0(-x) & \text{in } \mathbb{R}\,,\\ \lim\limits_{x\to\pm\infty} u_0(x) &= \pm 1. \end{cases} \tag{5.1.10}$$

This solution will play an important role to establish the asymptotic behavior of saddle-shaped solutions. Indeed, its importance lies in that the associated function

$$U(x) := u_0\left(\frac{|x'|-|x''|}{\sqrt{2}}\right) \tag{5.1.11}$$

will describe the asymptotic behavior of saddle solutions at infinity. Note that $(|x'| - |x''|)/\sqrt{2}$ is the signed distance to the Simons cone (see Lemma 4.2 in [60]). Therefore, the function U consists of "copies" of the layer solution u_0 centered at each point of the Simons cone and oriented in the normal direction to the cone.

The precise statement on the asymptotic behavior of saddle-shaped solutions at infinity is the following.

Theorem 5.1.3. *Let $f \in C^2(\mathbb{R})$ satisfy (5.1.9). Let K be a radially symmetric kernel satisfying the convexity assumption (5.1.8) and such that $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$. Let u be a saddle-shaped solution to (5.1.1) and let U be the function defined by (5.1.11). Then,*

$$||u - U||_{L^\infty(\mathbb{R}^n \setminus B_R)} + ||\nabla u - \nabla U||_{L^\infty(\mathbb{R}^n \setminus B_R)} + \left|\left|D^2 u - D^2 U\right|\right|_{L^\infty(\mathbb{R}^n \setminus B_R)} \to 0$$

as $R \to +\infty$.

Let us now describe some of the main ingredients that are used to prove Theorems 5.1.2 and 5.1.3. Concerning the uniqueness of the saddle-shaped solution, besides the asymptotic behavior described in Theorem 5.1.3 we also need the following maximum principle in $\mathcal{O}$ for the linearized operator $L_K - f'(u)$.

Proposition 5.1.4. *Let $\Omega \subset \mathcal{O}$ be an open set (not necessarily bounded) and let K be a radially symmetric kernel satisfying the convexity assumption (5.1.8) and such that $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$. Let u be a saddle-shaped solution to (5.1.1), and let $v \in L^1_\gamma(\mathbb{R}^{2m})$ be a doubly radial function which is C^α in Ω and continuous up to the boundary, for some $\alpha > 2\gamma$. Assume that v satisfies*

$$\begin{cases} L_K v - f'(u)v - c(x)v & \leq & 0 & \text{in } \Omega\,, \\ v & \leq & 0 & \text{in } \mathcal{O} \setminus \Omega\,, \\ -v(x^\star) & = & v(x) & \text{in } \mathbb{R}^{2m}, \\ \limsup\limits_{x\in\Omega,\ |x|\to\infty} v(x) & \leq & 0\,, \end{cases}$$

with $c \leq 0$ in Ω. Then, $v \leq 0$ in Ω.

To establish it, the key tool is to use a maximum principle in "narrow" sets, also proved in Section 5.6. Our proof of this result is much simpler than that of the analogue maximum principle for the classical Laplacian. This is an example of how the nonlocality of the operator can make some arguments easier and less technical (informally speaking, the reason would be that L_K "sees more", or "further", than the Laplacian). It is also interesting to notice that the proof of Proposition 5.1.4 is by far simpler than the one using the extension problem in the case of the fractional Laplacian (Proposition 3.1.4). In the proof, the positivity condition (5.1.7) —guaranteed by the convexity of the kernel— is crucial, together with the bounds (5.1.6).

Regarding the proof of Theorem 5.1.3, to establish the asymptotic behavior of saddle-shaped solutions we use a compactness argument as in [60, 75, 76], together with two important results presented next and established in Section 5.4. The first one, Theorem 5.1.5, is a Liouville type principle for nonnegative solutions to a semilinear equation in the whole space. This result, in contrast with the previous ones, does not require the kernel K to be radially symmetric, but only to satisfy (5.1.3) and (5.1.4).

Theorem 5.1.5. *Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let v be a bounded solution to*

$$\begin{cases} L_K v & = & f(v) & \text{in } \mathbb{R}^n\,, \\ v & \geq & 0 & \text{in } \mathbb{R}^n\,, \end{cases} \tag{5.1.12}$$

with a nonlinearity $f \in C^1$ satisfying

- $f(0) = f(1) = 0$,
- $f'(0) > 0$,
- $f > 0$ *in* $(0, 1)$, *and* $f < 0$ *in* $(1, +\infty)$.

Then, $v \equiv 0$ *or* $v \equiv 1$.

Similar classification results have been proved for the fractional Laplacian in [72, 145] (either using the extension problem or not) with the method of moving spheres, which uses crucially the scale invariance of the operator $(-\Delta)^\gamma$. To the best of our knowledge, there is no similar result available in the literature for general kernels in the ellipticity class $\mathcal{L}_0$ (which are not necessarily scale invariant). Thus, we present here a proof based on the techniques introduced by Berestycki, Hamel, and Nadirashvili [29] for the local equation with the classical Laplacian. It relies on a maximum principle for a nonlinear heat equation, the translation invariance of the operator, a Harnack inequality, and a stability argument.

The second ingredient needed to prove the asymptotic behavior of saddle-shaped solutions is a symmetry result for equations in a half-space, stated next. Here and in the rest of the chapter we use the notation $\mathbb{R}^n_+ = \{(x_H, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \ : \ x_n > 0\}$.

Theorem 5.1.6. *Let* $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ *and let* v *be a bounded solution to one of the following two problems: either to*

$$\begin{cases} L_K v &= f(v) & \text{in } \mathbb{R}^n_+, \\ v &> 0 & \text{in } \mathbb{R}^n_+, \\ v(x_H, x_n) &= -v(x_H, -x_n) & \text{in } \mathbb{R}^n, \end{cases} \tag{P1}$$

or to

$$\begin{cases} L_K v &= f(v) & \text{in } \mathbb{R}^n_+, \\ v &> 0 & \text{in } \mathbb{R}^n_+, \\ v &= 0 & \text{in } \mathbb{R}^n \setminus \mathbb{R}^n_+. \end{cases} \tag{P2}$$

Assume that, in $\mathbb{R}^n_+$*, the kernel* K *of the operator* L_K *is decreasing in the direction of* x_n*, i.e., it satisfies*

$$K(x_H - y_H, x_n - y_n) \geq K(x_H - y_H, x_n + y_n) \quad \text{for all} \ \ x, y \in \mathbb{R}^n_+.$$

Suppose that $f \in C^1$ *and*

- $f(0) = f(1) = 0$,
- $f'(0) > 0$, *and* $f'(\tau) \leq 0$ *for all* $\tau \in [1 - \delta, 1]$ *for some* $\delta > 0$,
- $f > 0$ *in* $(0, 1)$, *and*
- f *is odd in the case of* (P1).

Then, v *depends only on* x_n *and it is increasing in this direction.*

The result for (P2) has been proved for the fractional Laplacian under some assumptions on f (weaker than the ones in Theorem 5.1.6) in [161, 98, 20, 21, 107]. Instead, no result was available for general integro-differential operators. To the best of our knowledge, problem (P1) on odd solutions with respect to a hyperplane has not been treated even for the fractional Laplacian. In our case, the fact that f is of Allen-Cahn type allows us to use rather simple arguments that work for both problems (P1) and (P2) —moving planes and sliding methods, similarly as done in [98]. Moreover, the fact that the kernel of the operator is $|\cdot|^{-n-2\gamma}$ or a general K satisfying uniform ellipticity bounds does not affect significantly the proof. Although (P2) will not be used in this chapter, we include it here for future reference since the proof for this problem is analogous to the one for (P1).

5.1.3 Saddle-shaped solutions in the context of a conjecture by De Giorgi and the theory of nonlocal minimal surfaces

To conclude this introduction, let us make some comments on the importance of problem (5.1.1) and its relation with the theory of (classical and nonlocal) minimal surfaces and a famous conjecture raised by De Giorgi.

A main open problem (even in the local case) is to determine whether the saddle-shaped solution is a minimizer of the energy functional associated to the equation, depending on the dimension $2m$. This question is deeply related to the regularity theory of local and nonlocal minimal surfaces, as explained next.

It is well-known that, for powers $\gamma \in [1/2, 1]$, the rescaled energy functionals associated to the equation $(-\Delta)^{\gamma} u = f(u)$ Γ-converge to the classical perimeter functional (see [5, 127]), while in the case $\gamma \in (0, 1/2)$, they Γ-converge to the *fractional perimeter* functional (see [170]). Thus, a blow-down sequence of minimizers of the Allen-Cahn energy converges to the characteristic function of a set whose classical or fractional perimeter (depending on the power γ) is minimal.

In the recent years there has been an increasing interest in developing a regularity theory for nonlocal minimal surfaces, although very few results are known for the moment. It is beyond the scope of this work to describe all of them in detail, and we refer the interested reader to [81, 43] and the references therein. Let us just make some comments on the scarce available results concerning the possible minimality of the Simons cone as a nonlocal minimal surface, since this is connected to our work on saddle-shaped solutions. Note first that, due to all its symmetries, it is easy to check that the Simons cone $\mathscr{C}$ is stationary for the fractional perimeter. If $2m = 2$, a purely geometric argument shows that it cannot be a minimizer (see [189]). Note indeed that in [171] Savin and Valdinoci proved that all minimizing nonlocal minimal cones in $\mathbb{R}^2$ are flat, and that dimension 2 is the only one where a complete classification of minimizing nonlocal minimal cones is available. In higher dimensions, the only available results regarding the possible minimality of $\mathscr{C}$ appear in [86] and in our paper [111] (Chapter 3), but they concern stability, a weaker property than minimality.

In [86], Dávila, del Pino, and Wei found a very interesting characterization of the stability of the Simons cone. It consists of an inequality involving two hypergeometric constants which depend only on γ and the dimension. This inequality is checked numerically in [86], finding that, in dimensions $2m \leq 6$ and for γ close to zero, the Simons cone is not stable. Numerics also show that the Simons cone should be stable in dimension 8 if γ is close to zero. These two facts for small γ fit with the general belief that, in the fractional setting, the Simons cone should be stable (and even a minimizer) in dimensions $2m \geq 8$ (as in the local case), probably for all $\gamma \in (0, 1/2)$, though this is still an open problem.

In contrast with the numeric computations in [86], our arguments in [111] (Chapter 3) establishing the stability of $\mathscr{C}$ in dimensions $2m \geq 14$ are the first analytical proof of a stability result for the Simons cone in any dimension (in the nonlocal setting). Our approach, which is completely different from theirs, relies on establishing the stability of the saddle-shaped solution and using that this property is preserved along a blow-down limit. This shows that the saddle-shaped solution does not only have its interest in the context of the Allen-Cahn equation, but it can also provide strategies to prove stability and minimality results in the theory of nonlocal minimal surfaces.

In addition to all this, saddle-shaped solutions are natural objects to build a counterexample to a famous conjecture raised by De Giorgi, asking whether bounded monotone solutions to $-\Delta u = u - u^3$ in $\mathbb{R}^n$ are one-dimensional if $n \leq 8$. This conjecture is still nowadays not completely closed (see [108] and references therein), but a counterexample in dimensions $n \geq 9$ was given in [89] by using the gluing method. An alternative approach to the one of [89] to construct a counterexample to the conjecture was given by Jerison and Monneau in [135]. They showed that a counterexample in $\mathbb{R}^{n+1}$ can be constructed with a rather natural procedure if there exists a global minimizer of $-\Delta u = f(u)$ in $\mathbb{R}^n$ which is bounded and even with respect to each coordinate but is not one-dimensional. The saddle-shaped solution is of special interest in search of this counterexample, since it is even with respect to all the coordinate axis and it is canonically associated to the Simons cone, which in turn is the simplest nonplanar minimizing minimal surface. Therefore, by proving that the saddle solution to the classical Allen-Cahn equation is a minimizer in some dimension $2m$, one would obtain automatically a counterexample to the conjecture in $\mathbb{R}^{2m+1}$.

For a more complete account on the available results concerning the conjecture by De Giorgi in the nonlocal setting, as well as to related conjectures on minimizers and stable solutions (in which the saddle-shaped solution is expected to have a role as a counterexample), we refer the interested reader to [166] and the references therein.

5.1.4 Plan of the chapter

The chapter is organized as follows. In Section 5.2 we present some preliminary results that will be used in the rest of the chapter. Section 5.3 contains the proof of the uniqueness of a saddle-shaped solution, as well as the alternative proof of existence —via the monotone iteration method. In Section 5.4 we establish the Liouville type and symmetry results, Theorems 5.1.5 and 5.1.6. Section 5.5 is devoted to the layer solution u_0 of problem (5.1.1), and to the proof of the asymptotic behavior of saddle-shaped solutions, Theorem 5.1.3. Finally, Section 5.6 concerns the proof of a maximum principle in $\mathcal{O}$ for the linearized operator $L_K - f'(u)$ (Proposition 5.1.4).

5.2 Preliminaries

In this section we collect some preliminary results that will be used in the rest of this chapter. First, we summarize the regularity results needed in the forthcoming sections. Then, we state a remark on stability that will be used later in this chapter, and finally we recall the basic maximum principles for doubly radial odd functions proved in Chapter 4.

5.2.1 Regularity theory for nonlocal operators in the class $\mathcal{L}_0$

In this subsection we present the regularity results that will be used in the chapter. For further details, see [162, 176, 83] and the references therein.

We first give a result on the interior regularity for linear equations.

Proposition 5.2.1 ([162, 176]). *Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let $w \in L^\infty(\mathbb{R}^n)$ be a weak solution to $L_K w = h$ in B_1. Then,*

$$||w||_{C^{2\gamma}(B_{1/2})} \leq C \left(||h||_{L^\infty(B_1)} + ||w||_{L^\infty(\mathbb{R}^n)} \right). \tag{5.2.1}$$

Moreover, let $\alpha > 0$ and assume additionally that $w \in C^\alpha(\mathbb{R}^n)$. Then, if $\alpha + 2\gamma$ is not an integer,

$$||w||_{C^{\alpha+2\gamma}(B_{1/2})} \leq C \left(||h||_{C^\alpha(B_1)} + ||w||_{C^\alpha(\mathbb{R}^n)} \right), \tag{5.2.2}$$

where C is a constant that depends only on n, γ, λ, and Λ.

Throughout the chapter we consider u to be a saddle solution to (5.1.1) that satisfies $|u| \leq 1$ in $\mathbb{R}^n$. Hence, by applying (5.2.1) we find that for any $x_0 \in \mathbb{R}^n$,

$$\begin{aligned}||u||_{C^{2\gamma}(B_{1/2}(x_0))} &\leq C\left(||f(u)||_{L^\infty(B_1(x_0))} + ||u||_{L^\infty(\mathbb{R}^n)}\right)\\ &\leq C\left(1 + ||f||_{L^\infty([-1,1])}\right).\end{aligned}$$

Note that the estimate is independent of the point x_0, and thus since the equation is satisfied in the whole $\mathbb{R}^n$,

$$||u||_{C^{2\gamma}(\mathbb{R}^n)} \leq C\left(1 + ||f||_{L^\infty([-1,1])}\right).$$

Then, we use estimate (5.2.2) repeatedly and the same kind of arguments yield that, if $f \in C^k([-1,1])$, then $u \in C^\alpha(\mathbb{R}^n)$ for all $\alpha < k + 2\gamma$. Moreover, the following estimate holds:

$$||u||_{C^\alpha(\mathbb{R}^n)} \leq C,$$

for some constant C depending only on n, γ, λ, Λ, k, and $||f||_{C^k([-1,1])}$.

Let us now state a result on the boundary regularity of solutions to a Dirichlet problem for an operator $L_K \in \mathcal{L}_0$.

Proposition 5.2.2 ([83, 162]). *Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let $w \in L^\infty(\mathbb{R}^n)$ be a weak solution to*

$$\begin{cases} L_K w &= h \quad \text{in } \Omega, \\ \quad w &= \varphi \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

with $h \in L^\infty(\Omega)$ and $\varphi \in C^{2\gamma+\eta}(\mathbb{R}^n \setminus \Omega)$ for some $\eta \in (0, 2-2\gamma)$. Assume that Ω is a bounded $C^{1,1}$ domain. Then, there exists an $\alpha_0 \in (0, \gamma)$, depending only on n, γ, λ, Λ, and η, such that

$$||w||_{C^{\alpha_0}(\overline{\Omega})} \leq C\left(||h||_{L^\infty(\Omega)} + ||\varphi||_{C^{2\gamma+\eta}(\mathbb{R}^n\setminus\Omega)}\right),$$

where C is a constant that depends only on n, γ, λ, Λ, η, and Ω.

Note that this result can be combined with the interior estimate (5.2.2) to prove that weak solutions are indeed classical solutions.

5.2.2 A remark on stability

Recall that we say that a bounded solution w to $L_K w = f(w)$ in $\Omega \subset \mathbb{R}^n$ is *stable* in Ω if the second variation of the energy at w is nonnegative. That is, if

$$\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\xi(x) - \xi(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega f'(w)\xi^2\,\mathrm{d}x \geq 0$$

for every $\xi \in C_c^\infty(\Omega)$.

The following fact regarding stability will be used in Sections 5.4 and 5.5. Let $w \leq 1$ be a positive solution to $L_K w = f(w)$ in a set $\Omega \subset \mathbb{R}^n$, with f satisfying (5.1.9). Then w is stable in Ω. The proof of this fact is standard and rather simple, and it is a consequence of the fact that w is a positive supersolution of the linearized operator $L_K - f'(w)$. We present it here for completeness (a more detailed discussion can be found in [132]). On the one hand, since f is strictly concave in $(0,1)$ and $f(0) = 0$, then $f'(w)w < f(w)$ in Ω

(recall that w is positive there). On the other hand, it is easy to check that the following pointwise inequality holds for all functions φ and ξ, with $\varphi > 0$:

$$\big(\varphi(x) - \varphi(y)\big)\left(\frac{\xi^2(x)}{\varphi(x)} - \frac{\xi^2(y)}{\varphi(y)}\right) \leq |\xi(x) - \xi(y)|^2 . \tag{5.2.3}$$

Using these two facts and the symmetry of K, for every $\xi \in C_c^\infty(\Omega)$ we have

$$\begin{aligned}\int_\Omega f'(w)\xi^2 \,\mathrm{d}x &\leq \int_\Omega \frac{\xi^2}{w} f(w)\,\mathrm{d}x = \int_\Omega \frac{\xi^2}{w} L_K w \,\mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \big(w(x) - w(y)\big)\left(\frac{\xi^2(x)}{w(x)} - \frac{\xi^2(y)}{w(y)}\right) K(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\xi(x) - \xi(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y .\end{aligned}$$

Thus, w is stable in Ω.

5.2.3 Maximum principles for doubly radial odd functions

In this last subsection, we state the basic maximum principles for doubly radial odd functions. Note that in the following result we only need assumptions on the functions at one side of the Simons cone thanks to their symmetry. This was proved in part I and follows readily from the expression (5.1.5) by using the key inequality (5.1.7) for the kernel $\overline{K}$.

Proposition 5.2.3 (Maximum principle for odd functions with respect to $\mathscr{C}$ [110])**.** *Let $\Omega \subset \mathcal{O}$ be an open set and let L_K be an integro-differential operator with a radially symmetric kernel K satisfying the positivity condition (5.1.7) and such that $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$. Let $w \in C^\alpha(\Omega) \cap C(\overline{\Omega}) \cap L^\infty(\mathbb{R}^{2m})$, with $\alpha > 2\gamma$, be a doubly radial function which is odd with respect to the Simons cone.*

(i) (Weak maximum principle) Assume that

$$\begin{cases} L_K w + c(x) w \geq 0 & \text{in } \Omega , \\ w \geq 0 & \text{in } \mathcal{O} \setminus \Omega , \end{cases}$$

with $c \geq 0$, and that either

$$\Omega \text{ is bounded} \quad \text{or} \quad \liminf_{x\in\mathcal{O},\, |x|\to+\infty} w(x) \geq 0 .$$

Then, $w \geq 0$ in Ω.

(ii) (Strong maximum principle) Assume that $L_K w + c(x) w \geq 0$ in Ω, with c any continuous function, and that $w \geq 0$ in $\mathcal{O}$. Then, either $w \equiv 0$ in $\mathcal{O}$ or $w > 0$ in Ω.

Remark 5.2.4. Following the proof of this result in part I, it is easy to see that the interior regularity assumptions on w in the previous statement can be weakened. Indeed, we are assuming that $w \in C^\alpha(\Omega)$ with $\alpha > 2\gamma$ in order to guarantee that $L_K w$ is finite everywhere in Ω. Instead of this, we can simply assume that w is Hölder continuous in Ω (with Hölder exponent arbitrarily small), as long as $L_K w = +\infty$ at the points of Ω where w is not regular enough for $L_K w$ to be finite. In such case, $L_K w + c(x) w \geq 0$ holds as well and we can proceed with the argument as done in part I. Proposition 5.2.3 with these weaker assumptions on w will used later in the proof of Theorem 5.1.2 (see Remark 5.3.3 below): we will apply it to a function w being no more regular than C^{α_0} at some points in the interior of Ω, where α_0 is given by Proposition 5.2.2.

5.3 Existence and uniqueness of the saddle-shaped solution: monotone iteration method

In this section we prove the existence and uniqueness result of Theorem 5.1.2. The proof of the existence is based on the maximum principle and the first ingredient that we need is a version of the monotone iteration procedure for doubly radial functions which are odd with respect to the Simons cone $\mathscr{C}$. In order to prove the uniqueness we will use the asymptotic behavior result of Theorem 5.1.3 together with the maximum principle for the linearized operator $L_K - f'(u)$, given in Proposition 5.1.4; both results will be proved in the subsequent sections.

We next present the monotone iteration method for doubly radial odd functions. In this result and along the section, we will call odd sub/supersolutions to problem (5.3.2) the functions that are doubly radial, odd with respect to the Simons cone, and satisfy the corresponding problem in (5.3.1).

Proposition 5.3.1. *Let $\gamma \in (0,1)$ and let K be a radially symmetric kernel satisfying the convexity assumption (5.1.8) and such that $L_K \in \mathcal{L}_0$. Assume that $\underline{v} \leq \overline{v}$ are two bounded functions which are doubly radial, odd with respect to the Simons cone, and belonging to $C^{2\gamma+\varepsilon}(B_R)$ for some $\varepsilon > 0$. Furthermore, assume that $\underline{v} \in C^{\varepsilon}(\overline{B_R})$ and that $\underline{v}$ and $\overline{v}$ satisfy respectively*

$$\begin{cases} L_K \underline{v} & \leq & f(\underline{v}) & \text{in } B_R \cap \mathcal{O}\,, \\ \underline{v} & \leq & \varphi & \text{in } \mathcal{O} \setminus B_R\,, \end{cases} \quad \text{and} \quad \begin{cases} L_K \overline{v} & \geq & f(\overline{v}) & \text{in } B_R \cap \mathcal{O}\,, \\ \overline{v} & \geq & \varphi & \text{in } \mathcal{O} \setminus B_R\,, \end{cases} \tag{5.3.1}$$

with f a C^1 odd function and $\varphi \in C^{2\gamma+\varepsilon}(\mathbb{R}^n)$ a bounded doubly radial odd function. Then, there exists a classical solution v to the problem

$$\begin{cases} L_K v & = & f(v) & \text{in } B_R\,, \\ v & = & \varphi & \text{in } \mathbb{R}^{2m} \setminus B_R\,, \end{cases} \tag{5.3.2}$$

such that $v \in C^{2\gamma+\tilde{\varepsilon}}(B_R) \cap C^{\tilde{\varepsilon}}(\overline{B_R})$ for some $\tilde{\varepsilon} > 0$, it is doubly radial, odd with respect to the Simons cone, and $\underline{v} \leq v \leq \overline{v}$ in $\mathcal{O}$.

In the previous statement we required $C^{2\gamma+\varepsilon}$ regularity on $\underline{v}$ and $\overline{v}$ in order to L_K be finite when applied to them. In view of Remark 5.2.4, we can relax this assumption, since we do not need the operator to be finite in the whole set B_R when applied to a subsolution (respectively supersolution), it can take the value $-\infty$ (respectively $+\infty$) at some points. Note, however, that we cannot drop the assumption $\underline{v} \in C^{\varepsilon}(\overline{B_R})$ if we want v to have the desired regularity.

Proof of Proposition 5.3.1. The proof follows the classical monotone iteration method for elliptic equations (see for instance [102]). We just give here a sketch of the proof. First, let $M \geq 0$ be such that $-M \leq \underline{v} \leq \overline{v} \leq M$ and set

$$b := \max \left\{ 0, -\min_{[-M,M]} f' \right\} \geq 0\,.$$

Then one defines

$$\widetilde{L}_K w := L_K w + bw \quad \text{and} \quad g(\tau) := f(\tau) + b\tau\,.$$

Therefore, our problem is equivalent to find a solution to

$$\begin{cases} \widetilde{L}_K v &= g(v) \quad \text{in } B_R\,, \\ v &= \varphi \quad \text{in } \mathbb{R}^{2m}\setminus B_R\,, \end{cases}$$

such that v is doubly radial, odd with respect to the Simons cone and $\underline{v} \leq v \leq \overline{v}$ in $\mathcal{O}$. Here the main point is that g is also odd but satisfies $g'(\tau) \geq 0$ for $\tau \in [-M, M]$. Moreover, since $b \geq 0$, $\widetilde{L}_K$ satisfies the maximum principle for odd functions in $\mathcal{O}$ (as in Proposition 5.2.3). We define $v_0 = \underline{v}$ and, for $k \geq 1$, let v_k be the solution to the linear problem

$$\begin{cases} \widetilde{L}_K v_k &= g(v_{k-1}) \quad \text{in } B_R\,, \\ v_k &= \varphi \quad \text{in } \mathbb{R}^{2m}\setminus B_R\,. \end{cases}$$

It is easy to see by induction and the regularity results from Proposition 5.2.1 that $v_k \in L^\infty(\mathbb{R}^n) \cap C^{2\gamma+2\tilde{\varepsilon}}(B_R) \cap C^{2\tilde{\varepsilon}}(\overline{B_R})$ for some $\tilde{\varepsilon} > 0$. Moreover, given $\Omega \subset B_R$ a compact set, then $||v_k||_{C^{2\gamma+2\tilde{\varepsilon}}(\Omega)}$ is uniformly bounded in k. Then, using the maximum principle it is not difficult to show by induction that

$$\underline{v} = v_0 \leq v_1 \leq \ldots \leq v_k \leq v_{k+1} \leq \ldots \overline{v} \quad \text{in } \mathcal{O}\,,$$

and that each function v_k is doubly radial and odd with respect to $\mathscr{C}$. Finally, by the Arzelà-Ascoli theorem and the compact embedding of Hölder spaces we see that, up to a subsequence, v_k converges to the desired solution $v \in C^{2\gamma+\tilde{\varepsilon}}(B_R) \cap C^{\tilde{\varepsilon}}(\overline{B_R})$. □

In order to construct a positive subsolution to (5.3.2) with zero exterior data, we also need a characterization and some properties of the first odd eigenfunction and eigenvalue for the operator L_K, which are presented next. This eigenfunction is obtained though a minimization of the Rayleigh quotient in the appropriate space, defined next.

Given a set $\Omega \subset \mathbb{R}^{2m}$ and a translation invariant and positive kernel K, we define the space

$$\mathbb{H}_0^K(\Omega) := \left\{ w \in L^2(\Omega) \ : \ w = 0 \quad \text{a.e. in } \mathbb{R}^{2m}\setminus\Omega \quad \text{and } [w]^2_{\mathbb{H}^K(\mathbb{R}^{2m})} < +\infty \right\},$$

where

$$[w]^2_{\mathbb{H}^K(\mathbb{R}^{2m})} := \frac{1}{2}\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{5.3.3}$$

Recall also that when K satisfies the ellipticity assumption (5.1.4), then $\mathbb{H}_0^K(\Omega) = \mathbb{H}_0^\gamma(\Omega)$, which is the space associated to the kernel of the fractional Laplacian, $K(y) = c_{n,\gamma}|y|^{-n-2\gamma}$. We also define, for Ω doubly radial and symmetric with respect to $\mathscr{C}$, the space

$$\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega) := \left\{ w \in \mathbb{H}_0^K(\Omega) \ : \ w \text{ is doubly radial a.e. and odd with respect to } \mathscr{C} \right\}.$$

Recall that when K is radially symmetric and w is doubly radial, we can replace the kernel $K(x-y)$ in the definition (5.3.3) by the kernel $\overline{K}(x,y)$. This is readily deduced after a change of variables and taking the mean among all $R \in O(m)^2$ (see the details in Section 3 of Chapter 4.

Lemma 5.3.2. *Let $\Omega \subset \mathbb{R}^{2m}$ be a bounded set of double revolution and let K be a radially symmetric kernel satisfying the positivity condition (5.1.7) and such that $L_K \in \mathcal{L}_0(2m, \gamma, \lambda, \Lambda)$. Let us define*

$$\lambda_{1,\,\mathrm{odd}}(\Omega, L_K) := \inf_{w \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega)} \frac{\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |w(x)-w(y)|^2\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_\Omega w(x)^2\,\mathrm{d}x}\,.$$

Then, such infimum is attained at a function $\phi_1 \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(\Omega) \cap L^\infty(\Omega)$ *which solves*

$$\begin{cases} L_K\phi_1 &= \lambda_{1,\,\mathrm{odd}}(\Omega, L_K)\phi_1 & \text{in } \Omega\,, \\ \phi_1 &= 0 & \text{in } \mathbb{R}^{2m}\setminus\Omega\,, \end{cases}$$

and satisfies that $\phi_1 > 0$ *in* $\Omega\cap\mathcal{O}$*. We call this function* ϕ_1 *the* first odd eigenfunction of L_K in Ω*, and* $\lambda_{1,\,\mathrm{odd}}(\Omega, L_K)$*, the* first odd eigenvalue*. Moreover, in the case* $\Omega = B_R$*, there exists a constant* C *depending only on* n*,* γ*, and* Λ*, such that*

$$\lambda_{1,\,\mathrm{odd}}(B_R, L_K) \le CR^{-2\gamma}\,.$$

Proof. The first two statements are deduced exactly as in Proposition 9 of [177], using the same arguments as in Lemma 4.3.3 to guarantee that ϕ_1 is nonnegative in $\mathcal{O}$. The fact that $\phi_1 > 0$ in $\Omega\cap\mathcal{O}$ follows from the strong maximum principle (see Proposition 5.2.3). We show the third statement. Let $\widetilde{w}(x) := w(Rx)$ for every $w \in \widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)$. Then,

$$\begin{aligned}
&\min_{w\in\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_R)} \frac{\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |w(x)-w(y)|^2\overline{K}(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_R} w(x)^2\,\mathrm{d}x} \\
&\qquad\le \min_{\widetilde{w}\in\widetilde{\mathbb{H}}^K_{0,\,\mathrm{odd}}(B_1)} \frac{\dfrac{c_{n,\gamma}\Lambda}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\widetilde{w}(x/R)-\widetilde{w}(y/R)|^2|x-y|^{-n-2\gamma}\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_R} \widetilde{w}(x/R)^2\,\mathrm{d}x} \\
&\qquad= R^{-2\gamma}\min_{\widetilde{w}\in\widetilde{\mathbb{H}}^s_{0,\,\mathrm{odd}}(B_1)} \frac{\dfrac{c_{n,\gamma}\Lambda}{2}\displaystyle\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} |\widetilde{w}(x)-\widetilde{w}(y)|^2|x-y|^{-n-2\gamma}\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{B_1} \widetilde{w}(x)^2\,\mathrm{d}x} \\
&\qquad= \lambda_{1,\,\mathrm{odd}}(B_1, (-\Delta)^\gamma)\Lambda R^{-2\gamma}\,.
\end{aligned}$$

□

Remark 5.3.3. Note that, by the regularity results for L_K stated in Section 5.2, we have that $\phi_1 \in C^{\alpha_0}(\overline{\Omega})\cap C^{\alpha_0+2\gamma}(\Omega)$ for some $0 < \alpha_0 < \gamma$, and the regularity up to the boundary is optimal. Due to this and the fact that $\phi_1 > 0$ in $\Omega\cap\mathcal{O}$ while $\phi_1 = 0$ in $\mathbb{R}^{2m}\setminus\Omega$, it is easy to check by using (5.1.5) that $-\infty < L_K\phi_1 < 0$ in $\mathcal{O}\setminus\overline{\Omega}$ and that $L_K\phi_1 = -\infty$ on $\partial\Omega\cap\mathcal{O}$.

With these ingredients, we can proceed with the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. We divide it into two parts. ***i)*** **Existence:** The strategy is to build a suitable solution u_R of

$$\begin{cases} L_K u_R &= f(u_R) & \text{in } B_R\,, \\ u_R &= 0 & \text{in } \mathbb{R}^{2m}\setminus B_R\,, \end{cases} \tag{5.3.4}$$

and then let $R \to +\infty$ to get a saddle-shaped solution. Let $\phi_1^{R_0}$ be the first odd eigenfunction of L_K in $B_{R_0} \subset \mathbb{R}^{2m}$, given by Lemma 5.3.2, and let $\lambda_1^{R_0} := \lambda_{1,\,\mathrm{odd}}(B_{R_0}, L_K)$. We claim that for R_0 big enough and $\varepsilon > 0$ small enough, $\underline{u}_R := \varepsilon\phi_1^{R_0}$ is an odd subsolution of (5.3.4) for every $R \ge R_0$. To see this, note first that, without loss of generality, we can assume that $\left|\left|\phi_1^{R_0}\right|\right|_{L^\infty(B_R)} = 1$. Now, since f is strictly concave in $(0,1)$ and $f(0) = 0$, we have that $f'(\tau)\tau < f(\tau)$ for all $\tau > 0$. Thus, using that $\varepsilon\phi_1^{R_0} > 0$ in $B_{R_0} \cap \mathcal{O}$, it follows that for every $x \in B_{R_0} \cap \mathcal{O}$,

$$\frac{f(\varepsilon\phi_1^{R_0}(x))}{\varepsilon\phi_1^{R_0}(x)} > f'(\varepsilon\phi_1^{R_0}(x)) \geq f'(0)/2$$

if ε is small enough, independently of x (recall that we assumed $|\phi_1| \leq 1$). Therefore, since $f'(0) > 0$, taking R_0 big enough so that $\lambda_1^{R_0} < f'(0)/2$ (this can be achieved thanks to the last statement of Lemma 5.3.2), we have that for every $x \in B_{R_0} \cap \mathcal{O}$, $f(\varepsilon\phi_1^{R_0}(x)) > \lambda_1^{R_0}\varepsilon\phi_1^{R_0}(x)$. Thus,

$$L_K \underline{u}_R = \lambda_1^{R_0}\varepsilon\phi_1^{R_0} < f(\varepsilon\phi_1^{R_0}) = f(\underline{u}_R) \quad \text{in } B_{R_0} \cap \mathcal{O}\,.$$

In addition, if $x \in (B_R \setminus B_{R_0}) \cap \mathcal{O}$, by Remark 5.3.3 we have that

$$L_K \underline{u}_R < 0 = f(0) = f(\underline{u}_R) \quad \text{in } (B_R \setminus B_{R_0}) \cap \mathcal{O}\,.$$

Note that in ∂B_{R_0} we have $L_K \underline{u}_R = -\infty$. Hence, the claim is proved. Now, if we define $\overline{u}_R := \chi_{\mathcal{O}\cap B_R} - \chi_{\mathcal{I}\cap B_R}$, a simple computation shows that it is an odd supersolution to (5.3.4). Therefore, using the monotone iteration procedure given in Proposition 5.3.1 (taking into account Remarks 5.2.4 and 5.3.3 when using the maximum principle), we obtain a solution u_R to (5.3.4) such that it is doubly radial, odd with respect to the Simons cone, and $\varepsilon\phi_1^{R_0} = \underline{u}_R \leq u_R \leq \overline{u}_R$ in $\mathcal{O}$. Note that, since $\underline{u}_R > 0$ in $\mathcal{O} \cap B_{R_0}$, the same holds for u_R. Using a standard compactness argument, we let $R \to +\infty$ to obtain a sequence u_{R_j} converging on compacts in $C^{2\gamma+\eta}(\mathbb{R}^{2m})$ norm, for some $\eta > 0$, to a solution $u \in C^{2\gamma+\eta}(\mathbb{R}^{2m})$ of $L_K u = f(u)$ in $\mathbb{R}^{2m}$. Note that u is doubly radial, odd with respect to the Simons cone and $0 \leq u \leq 1$ in $\mathcal{O}$. Let us show that $0 < u < 1$ in $\mathcal{O}$, which will yield that u is a saddle-shaped solution. By the usual strong maximum principle it follows readily that $u < 1$ in $\mathcal{O}$. Moreover, since $u_R \geq \varepsilon\phi_1^{R_0} > 0$ in $\mathcal{O} \cap B_{R_0}$ for $R > R_0$, this holds also the limit, that is, $u \geq \varepsilon\phi_1^{R_0} > 0$ in $\mathcal{O} \cap B_{R_0}$. Therefore, by applying the strong maximum principle for odd functions (see Proposition 5.2.3) we obtain that $0 < u < 1$ in $\mathcal{O}$. ***ii*) Uniqueness:** Let

u_1 and u_2 be two saddle-shaped solutions. Define $v := u_1 - u_2$, which is a doubly radial function that is odd with respect to $\mathscr{C}$. Then,

$$L_K v = f(u_1) - f(u_2) \leq f'(u_2)(u_1 - u_2) = f'(u_2)v \quad \text{in } \mathcal{O}\,,$$

since f is concave in $(0,1)$. Moreover, by the asymptotic result (see Theorem 5.1.3), we have

$$\limsup_{x\in\mathcal{O},\ |x|\to\infty} v(x) = 0\,.$$

Then, by the maximum principle in $\mathcal{O}$ for the linearized operator $L_K - f'(u_2)$ (see Proposition 5.1.4), it follows that $v \leq 0$ in $\mathcal{O}$, which means $u_1 \leq u_2$ in $\mathcal{O}$. Repeating the argument with $-v = u_2 - u_1$ we deduce $u_1 \geq u_2$ in $\mathcal{O}$. Therefore, $u_1 = u_2$ in $\mathbb{R}^{2m}$. □

Remark 5.3.4. Since the saddle-shaped solution u is positive in $\mathcal{O}$, it follows that u is stable in this set, as explained in Section 5.2. This fact will be used in Section 5.5.

5.4 Symmetry and Liouville type results

This section is devoted to prove the Liouville type result of Theorem 5.1.5 and the one-dimensional symmetry result of Theorem 5.1.6. Both of them will be needed in the following section to establish the asymptotic behavior of the saddle-shaped solution.

5.4.1 A Liouville type result for positive solutions in the whole space

In the proof of Theorem 5.1.5 we will need two main ingredients, that we present next. The first one is a Harnack inequality for solutions to the semilinear problem (5.1.12). This inequality follows readily from the results of Cozzi in [79], although the precise result that we need is not stated there. For the reader's convenience and for future reference, we present the result here and indicate how to deduce it from the results in [79].

Proposition 5.4.1. *Let $L_K \in \mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ and let w be a solution to (5.1.12) with f a Lipschitz nonlinearity such that $f(0)=0$. Then, for every $x_0 \in \mathbb{R}^n$ and every $R>0$, it holds*

$$\sup_{B_R(x_0)} w \leq C \inf_{B_R(x_0)} w,$$

with $C>0$ depending only on n,γ,λ,Λ, and R.

Proof. Following the notation of [79], since f is Lipschitz and $f(0)=0$, we have

$$|f(u)| \leq d_1 + d_2|u|^{q-1} \quad \text{in } \mathbb{R}^n,$$

with $d_1=0$, $d_2=||f||_{\mathrm{Lip}}$ and $q=2$. With this choice of the parameters, we only need to repeat the proof of Proposition 8.5 in [79] (with $p=2$ and $\Omega=\mathbb{R}^n$) in order to obtain that u belongs to the fractional De Giorgi class $\mathrm{DG}^{\gamma,2}(\mathbb{R}^n,0,H,-\infty,2\gamma/n,2\gamma,+\infty)$ for some constant $H>0$ (see [79] for the precise definition of these classes). Therefore, the Harnack inequality follows from Theorem 6.9 in [79]. □

The second ingredient that we need in the proof of Theorem 5.1.5 is the following parabolic maximum principle in the unbounded set $\mathbb{R}^n \times (0,+\infty)$.

Proposition 5.4.2. *Let $L_K \in \mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ and let v be a bounded function, C^α with $\alpha > 2\gamma$ in space and C^1 in time, such that*

$$\begin{cases} \partial_t v + L_K v + c(x)\, v & \leq & 0 & \text{in } \mathbb{R}^n \times (0,+\infty)\,, \\ v(x,0) & \leq & 0 & \text{in } \mathbb{R}^n\,, \end{cases}$$

with $c(x)$ a continuous and bounded function. Then,

$$v(x,t) \leq 0 \quad \text{in} \quad \mathbb{R}^n \times [0,+\infty).$$

This result can be deduced from the usual parabolic maximum principle in a bounded (in space and time) set with a rather simple argument. Since we have not found a specific reference where such result is stated, let us present its proof with full detail for the sake of clarity. First of all, we present the usual parabolic maximum principle in a bounded set in $\mathbb{R}^n \times (0,+\infty)$. The proof for cylindrical sets $\Omega \times (0,T)$ can be found for instance in [23]. Although the argument for general bounded sets is essentially the same, we include here a short proof for the sake of completeness.

Lemma 5.4.3. *Let $\Omega \subset B_R \times (0,T) \subset \mathbb{R}^n \times (0,+\infty)$ be a bounded open set. Let L_K be an integro-differential operator of the form (5.1.2) with a symmetric kernel satisfying (5.1.4), and let v be a bounded function, C^α with $\alpha > 2\gamma$ in space and C^1 in time, satisfying*

$$\begin{cases} \partial_t v + L_K v & \leq & 0 & \text{in } \Omega \subset B_R \times (0,T)\,, \\ v(x,0) & \leq & 0 & \text{in } \overline{\Omega} \cap \{t=0\} \subset B_R\,, \\ v & \leq & 0 & \text{in } (\mathbb{R}^n \times (0,T)) \setminus \Omega\,. \end{cases}$$

Then, $v \leq 0$ in $\mathbb{R}^n \times [0,T]$.

Proof. By contradiction, for every small $\varepsilon > 0$ assume that

$$M := \sup_{\mathbb{R}^n \times (0,T-\varepsilon)} v > 0.$$

By the sign of the initial condition and since $v \leq 0$ in $(\mathbb{R}^n \times (0,T)) \setminus \Omega$, v attains this positive value M at a point $(x_0,t_0) \in \Omega$ with $t_0 \leq T-\varepsilon$. If $t_0 \in (0,T-\varepsilon)$, then (x_0,t_0) is an interior global maximum (in $\mathbb{R}^n \times (0,T-\varepsilon)$) and it must satisfy $v_t(x_0,t_0) = 0$ and $L_K v(x_0,t_0) > 0$, which contradicts the equation. If $t_0 = T-\varepsilon$, then $v_t(x_0,t_0) \geq 0$ and $L_K v(x_0,t_0) > 0$, which is also a contradiction with the equation. Thus, $v \leq 0$ in $\mathbb{R}^n \times [0,T-\varepsilon)$ and since this holds for $\varepsilon > 0$ arbitrarily small, we deduce $v \leq 0$ in $\mathbb{R}^n \times [0,T)$, and by continuity, in $\mathbb{R}^n \times [0,T]$. □

To establish Proposition 5.4.2 from Lemma 5.4.3, we need to introduce an auxiliary function enjoying certain properties (see Lemma 5.4.5 below). Before presenting it, we need the following result.

Lemma 5.4.4. *There is no bounded solution to $L_K v = 1$ in $\mathbb{R}^n$ for any $L_K \in \mathcal{L}_0$.*

Proof. Assume by contradiction that such solution exists. Then, by interior regularity (see Section 5.2) $v \in C^1(\mathbb{R}^n)$ and $|\nabla v| \leq C$ in $\mathbb{R}^n$. For every $i = 1, \ldots, n$, we differentiate the equation with respect to x_i to obtain

$$\begin{cases} L_K v_{x_i} & = & 0 & \text{in } \mathbb{R}^n\,, \\ |v_{x_i}| & \leq & C & \text{in } \mathbb{R}^n\,. \end{cases}$$

By the Liouville theorem for the operator L_K (it is proved exactly as in [165], see also [176]), v_{x_i} is constant. Hence, ∇v is constant, and thus v is affine. But since v is bounded, v must be constant, and we arrive at a contradiction with $L_K v = 1$. □

With this result we can introduce the auxiliary function that we will use to prove the parabolic maximum principle of Proposition 5.4.2.

Lemma 5.4.5. *Let $L_K \in \mathcal{L}_0(n,\gamma,\lambda,\Lambda)$. Then, for every $R > 0$ there exists a constant $M_R > 0$ and a continuous function $\psi_R \geq 0$ solution to*

$$\begin{cases} L_K \psi_R & = & -1/M_R & \text{in } B_R\,, \\ \psi_R & = & 1 & \text{in } \mathbb{R}^n \setminus B_R\,, \end{cases} \tag{5.4.1}$$

satisfying

$$\psi_R \to 0 \ \text{ pointwise and } \quad M_R \to +\infty \qquad \text{as } R \to +\infty\,.$$

Proof. First, consider ϕ_R the solution to

$$\begin{cases} L_K\phi_R &= 1 \quad \text{in } B_R\,, \\ \phi_R &= 0 \quad \text{in } \mathbb{R}^n\setminus B_R\,. \end{cases}$$

Note that the existence of a weak solution to the previous problem is given by the Riesz representation theorem. Moreover, by standard regularity results (see Section 5.2.1), ϕ_R is in fact a classical solution and by the maximum principle, $\phi_R > 0$ in B_R. Define $M_R := \sup_{B_R}\phi_R$. Since M_R is increasing (to check this, use the maximum principle to compare ϕ_R and $\phi_{R'}$ with $R > R'$), it must have a limit $M \in \mathbb{R}\cup\{+\infty\}$. Assume by contradiction that $M < +\infty$ and consider the new function $\varphi_R := \phi_R/M_R$, which satisfies

$$\begin{cases} L_K\varphi_R &= 1/M_R \quad \text{in } B_R\,, \\ \varphi_R &= 0 \qquad\quad \text{in } \mathbb{R}^n\setminus B_R\,, \\ \varphi_R &\leq 1\,. \end{cases} \tag{5.4.2}$$

By a standard compactness argument, we deduce that as $R\to+\infty$, φ_R converges (up to a subsequence) to a function φ that solves $L_K\varphi = 1/M$ in $\mathbb{R}^n$ and satisfies $|\varphi| \leq 1$. This contradicts Lemma 5.4.4 and therefore, $M_R\to+\infty$ as $R\to+\infty$. Define now $\psi_R := 1-\phi_R/M_R = 1-\varphi_R$, which solves trivially (5.4.1). Thus, it only remains to show that $\psi_R\to 0$ as $R\to+\infty$. We will see that $\varphi_R\to 1$ as $R\to+\infty$. Recall that φ_R solves problem (5.4.2), and by the previous arguments, by letting $R\to+\infty$ we have that a subsequence of φ_R converges uniformly in compact sets to a bounded function $\varphi\geq 0$ that solves $L_K\varphi = 0$ in $\mathbb{R}^n$. By the Liouville theorem, φ must be constant, and since its L^∞ norm is 1 and $\varphi\geq 0$, we conclude $\varphi\equiv 1$. □

With these ingredients, we establish now the parabolic maximum principle in $\mathbb{R}^n\times(0,+\infty)$.

Proof of Proposition 5.4.2. First of all, note that with the change of function $\tilde v(x,t) = \mathrm{e}^{-\alpha t}v(x,t)$ we can reduce the initial problem in the statement of Proposition 5.4.2 to

$$\begin{cases} \partial_t\tilde v + L_K\tilde v &\leq 0 \quad \text{in } \Omega\subset\mathbb{R}^n\times(0,+\infty)\,, \\ \tilde v &\leq 0 \quad \text{in } (\mathbb{R}^n\times(0,+\infty))\setminus\Omega\,, \\ \tilde v(x,0) &\leq 0 \quad \text{in } \mathbb{R}^n\,, \end{cases}$$

if we take $\alpha > ||c||_{L^\infty}$ and $\Omega := \{(x,t)\in\mathbb{R}^n\times(0,+\infty)\ :\ v(x,t) > 0\}$. Now, consider the function

$$w_R(x,t) := ||\tilde v||_{L^\infty(\mathbb{R}^n\times(0,+\infty))}\left(\psi_R + \frac{t}{M_R}\right)\,,$$

where ψ_R and M_R are defined in Lemma 5.4.5. Then, it is easy to check that w_R satisfies

$$\begin{cases} \partial_t w_R + L_K w_R &= 0 \qquad\qquad\qquad\qquad\quad \text{in } B_R\times(0,T)\,, \\ w_R(x,0) &\geq 0 \qquad\qquad\qquad\qquad\quad \text{in } B_R\,, \\ w_R(x,t) &\geq ||\tilde v||_{L^\infty(\mathbb{R}^n\times(0,+\infty))} \quad \text{in } (\mathbb{R}^n\setminus B_R)\times(0,T)\,, \end{cases}$$

for every $T > 0$ and $R > 0$. Since $w_R\geq 0\geq\tilde v$ in $(\mathbb{R}^n\times(0,+\infty))\setminus\Omega$, by the maximum principle in $(B_R\times(0,T))\cap\Omega$ (see Lemma 5.4.3) we can easily deduce that $w_R\geq\tilde v$ in $B_R\times(0,T)$. Finally, given an arbitrary point $(x_0,t_0)\in\Omega$, take $R_0 > 0$ and $T > 0$ such that $(x_0,t_0)\in B_{R_0}\times(0,T)$. Thus,

$$\tilde v(x_0,t_0)\leq w_R(x_0,t_0) = ||\tilde v||_{L^\infty(\mathbb{R}^n\times(0,+\infty))}\left(\psi_R(x_0) + \frac{t_0}{M_R}\right), \quad \text{for every} \quad R\geq R_0.$$

Letting $R \to +\infty$ and using that $\psi_R(x_0) \to 0$ and $M_R \to +\infty$ (see Lemma 5.4.5), we conclude $\tilde{v}(x_0, t_0) \le 0$, and therefore $v(x_0, t_0) = \mathrm{e}^{\alpha t_0}\,\tilde{v}(x_0, t_0) \le 0$. □

By using the Harnack inequality and the parabolic maximum principle we can now establish Theorem 5.1.5. The proof follows the ideas of Berestycki, Hamel, and Nadirashvili from Theorem 2.2 in [29] but adapted to the whole space and with an integro-differential operator.

Proof of Theorem 5.1.5. Assume $v \not\equiv 0$. Then, by the strong maximum principle $v > 0$. Our goal is to show that $v \equiv 1$, and this will be accomplished in two steps. **Step 1: We show that** $m := \inf_{\mathbb{R}^n} v > 0$**.** By contradiction, we assume $m = 0$. Then, there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $v(x_k) \to 0$ as $k \to +\infty$. On the one hand, by the Harnack inequality of Proposition 5.4.1, given any $R > 0$ we have

$$\sup_{B_R(x_k)} v \le C_R \inf_{B_R(x_k)} v \le C_R\, v(x_k) \to 0 \text{ as } k \to +\infty. \tag{5.4.3}$$

Moreover, since $f(0) = 0$ and $f'(0) > 0$, it is easy to show that $f(t) \ge f'(0)t/2$ if t is small enough. Therefore, from this and (5.4.3) we deduce that there exists $M(R) \in \mathbb{N}$ such that

$$L_K v - \frac{f'(0)}{2} v \ge 0 \quad \text{in} \quad B_R(x_{M(R)})\,. \tag{5.4.4}$$

On the other hand, let us define

$$\lambda_R^{x_0} = \inf_{\substack{\varphi\in C_c^1(B_R(x_0))\\ \varphi\not\equiv 0}} \frac{\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\varphi(x)-\varphi(y)|^2\, K(x-y)\,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_{\mathbb{R}^n} \varphi(x)^2\,\mathrm{d}x},$$

which decreases to zero uniformly in x_0 as $R \to +\infty$ from being $L_K \in \mathcal{L}_0$ (see the proof of Lemma 5.3.2 and also Proposition 9 of [177]). Therefore, there exists $R_0 > 0$ such that $\lambda_R^x < f'(0)/2$ for all $x \in \mathbb{R}^n$ and $R \ge R_0$. In particular, by choosing $x = x_{M(R_0)}$ there exists $w \in C_c^1(B_{R_0}(x_{M(R_0)}))$ such that $w \not\equiv 0$ and

$$\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |w(x)-w(y)|^2\, K(x-y)\,\mathrm{d}x\,\mathrm{d}y < \frac{f'(0)}{2}\int_{\mathbb{R}^n} w^2\,\mathrm{d}x. \tag{5.4.5}$$

Finally, to get the contradiction, multiply (5.4.4) by $w^2/v \ge 0$ and integrate in $\mathbb{R}^n$. After symmetrizing the integral involving L_K we get

$$\begin{aligned}
0 &\le \int_{\mathbb{R}^n} \frac{w^2}{v} L_K v\,\mathrm{d}x - \frac{f'(0)}{2}\int_{\mathbb{R}^n} w^2\,\mathrm{d}x\\
&= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(v(x)-v(y)\big)\left(\frac{w^2(x)}{v(x)} - \frac{w^2(y)}{v(y)}\right) K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \frac{f'(0)}{2}\int_{\mathbb{R}^n} w^2\,\mathrm{d}x\\
&\le \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |w(x)-w(y)|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \frac{f'(0)}{2}\int_{\mathbb{R}^n} w^2\,\mathrm{d}x,
\end{aligned}$$

which contradicts (5.4.5). Here we have used that the kernel is positive and symmetric and the inequality (5.2.3). Therefore, $\inf_{\mathbb{R}^n} v > 0$. **Step 2: We show that** $v \equiv 1$**.** Choose $0 < \xi_0 < \min\{1, m\}$, which is well defined by Step 1, and let $\xi(t)$ be the solution of the ODE

$$\begin{cases} \dot{\xi}(t) &= f(\xi(t)) \quad \text{in } (0,+\infty)\,,\\ \xi(0) &= \xi_0\,. \end{cases}$$

Since $f > 0$ in $(0, 1)$ and $f(1) = 0$ we have that $\dot{\xi}(t) > 0$ for all $t \geq 0$, and $\lim_{t\to+\infty} \xi(t) = 1$. Now, note that both $v(x)$ and $\xi(t)$ solve the parabolic equation

$$\partial_t w + L_K w = f(w) \quad \text{in } \mathbb{R}^n \times (0, +\infty)\,,$$

and satisfy

$$v(x) \geq m \geq \xi_0 = \xi(0).$$

Thus, by the parabolic maximum principle (Proposition 5.4.2) applied to $v - \xi$, taking $c(x) = -\big(f(v) - f(\xi)\big)/(v - \xi)$, we deduce that $v(x) \geq \xi(t)$ for all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$. By letting $t \to +\infty$ we obtain

$$v(x) \geq 1 \quad \text{in } \mathbb{R}^n\,.$$

In a similar way, taking $\tilde{\xi}_0 > ||v||_{L^\infty} \geq 1$, using $f < 0$ in $(1, +\infty)$, $f(1) = 0$ and the parabolic maximum principle, we obtain the upper bound $v \leq 1$. □

5.4.2 A one-dimensional symmetry result for positive solutions in a half-space

In this subsection we establish Theorem 5.1.6. To do it, we proceed in three steps. First, we show that the solution is monotone in the x_n direction by using a moving planes argument (see Proposition 5.4.6 below). Once this is shown, we can deduce that the solution v has uniform limits as $x_n \pm \to \infty$. Finally, by using the sliding method (see Proposition 5.4.12 below), we deduce the one-dimensional symmetry of the solution.

We proceed now with the details of the arguments. As we have said, the first step is to show that the solution is monotone. We establish the following result.

Proposition 5.4.6. *Let v be a bounded solution to one of the problems (P1) or (P2), with $L_K \in \mathcal{L}_0$ such that the kernel K is nonincreasing in the direction of x_n in $\mathbb{R}^n_+$, that is,*

$$K(x_H - y_H, x_n - y_n) \geq K(x_H - y_H, x_n + y_n) \quad \textit{for all} \quad x, y \in \mathbb{R}^n_+.$$

Let f be a Lipschitz nonlinearity such that $f > 0$ in $(0, ||v||_{L^\infty(\mathbb{R}^n_+)})$. Then,

$$\frac{\partial v}{\partial x_n} > 0 \quad \textit{in} \quad \mathbb{R}^n_+.$$

To prove this monotonicity result, we use a moving planes argument, and for this reason we need a maximum principle in "narrow" sets for odd functions with respect to a hyperplane (see Proposition 5.4.10). Recall that for a set $\Omega \subset \mathbb{R}^n$, we define the quantity $R(\Omega)$ as the smallest positive R for which

$$\frac{|B_R(x) \setminus \Omega|}{|B_R(x)|} \geq \frac{1}{2} \qquad \text{for every } x \in \Omega. \tag{5.4.6}$$

If no such radius exists, we define $R(\Omega) = +\infty$. We say that a set Ω is "narrow" if $R(\Omega)$ is small depending on certain quantities.

An important result needed to establish the maximum principle in "narrow" sets is the following ABP-type estimate. It is proved in [161] for the fractional Laplacian, following the arguments in [44] (see also [45]). The proof for a general operator L_K does not differ significantly from the one for the fractional Laplacian. Nevertheless, we include it here for the sake of completeness.

Theorem 5.4.7. *Let $\Omega \subset \mathbb{R}^n$ with $R(\Omega) < +\infty$. Let $L_K \in \mathcal{L}_0(n, \gamma, \lambda, \Lambda)$ and let $v \in L^1_\gamma(\mathbb{R}^n) \cap C^\alpha(\Omega)$, with $\alpha > 2\gamma$, such that $\sup_\Omega v < +\infty$ and satisfying*

$$\begin{cases} L_K v - c(x)v & \leq & h & \text{in } \Omega\,, \\ v & \leq & 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

with $c(x) \leq 0$ in Ω and $h \in L^\infty(\Omega)$. Then,

$$\sup_\Omega v \leq CR(\Omega)^{2\gamma} \|h\|_{L^\infty(\Omega)}\,,$$

where C is a constant depending on n, γ, and Λ.

The only ingredient needed to show Theorem 5.4.7 is the following weak Harnack inequality proved in [80].

Proposition 5.4.8 (see Corollary 4.4 of [80])**.** *Let $\Omega \subset \mathbb{R}^n$ and $L_K \in (n, \gamma, \lambda, \Lambda)$. Let $w \in L^1_\gamma(\mathbb{R}^n) \cap C^\alpha(\Omega)$, with $\alpha > 2\gamma$, such that $w \geq 0$ in $\mathbb{R}^n$. Assume that w satisfies weakly $L_K w \geq h$ in Ω, for some $h \in L^\infty(\Omega)$. Then, there exists an exponent $\varepsilon > 0$ and a constant $C > 1$, both depending on n, γ and Λ, such that*

$$\left(\fint_{B_{R/2}(x_0)} w^\varepsilon \, dx \right)^{1/\varepsilon} \leq C \left(\inf_{B_R(x_0)} w + R^{2\gamma} \|h\|_{L^\infty(\Omega)} \right)$$

for every $x_0 \in \Omega$ and $0 < R < \operatorname{dist}(x_0, \partial\Omega)$.

With the previous weak Harnack inequality we can now establish the ABP estimate.

Proof of Theorem 5.4.7. First, note that it is enough to show it for $v > 0$ in Ω satisfying

$$\begin{cases} L_K v & \leq & h & \text{in } \Omega\,, \\ v & \leq & 0 & \text{in } \mathbb{R}^n \setminus \Omega\,. \end{cases}$$

Indeed, if we consider $\Omega_0 = \{x \in \Omega \ : \ v > 0\}$, then since $c \leq 0$ we have $L_K v \leq L_K v - c(x)v \leq h$ in Ω_0. Define $M := \sup_\Omega v$. Then, for every $\delta > 0$ there exists a point $x_\delta \in \Omega$ such that $v(x_\delta) \geq M - \delta$. Consider now the function $w := M - v^+$. Note that $0 \leq w \leq M$, $w(x_\delta) \leq \delta$, and $w \equiv M$ in $\mathbb{R}^n \setminus \Omega$. If we extend h to be 0 outside Ω, we can easily verify that $L_K w \geq -h$ in $B_R(x_\delta)$. Now, by choosing $R = 2R(\Omega)$, and using the weak Harnack inequality of Proposition 5.4.8, we get

$$\begin{aligned} M \left(\frac{1}{2}\right)^{1/\varepsilon} &\leq \left(M^\varepsilon \frac{|B_{R/2}(x_\delta) \setminus \Omega|}{|B_{R/2}(x_\delta)|} \right)^{1/\varepsilon} = \left(\frac{1}{|B_{R/2}(x_\delta)|} \int_{B_{R/2}(x_\delta)\setminus\Omega} w^\varepsilon \, dx \right)^{1/\varepsilon} \\ &\leq \left(\fint_{B_{R/2}(x_\delta)} w^\varepsilon \, dx \right)^{1/\varepsilon} \leq C \left(\inf_{B_R(x_\delta)} w + R^{2\gamma} \|h\|_{L^\infty(\Omega)} \right) \\ &\leq C \left(\delta + R^{2\gamma} \|h\|_{L^\infty(\Omega)} \right). \end{aligned}$$

The conclusion follows from letting $\delta \to 0$. □

As a consequence of this result, one can deduce easily a general maximum principle in "narrow" sets.

Corollary 5.4.9. *Let $\Omega \subset \mathbb{R}^n$ with $R(\Omega) < +\infty$. Let $L_K \in \mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ and let $v \in L^1_\gamma(\mathbb{R}^n) \cap C^\alpha(\Omega)$, with $\alpha > 2\gamma$, such that $\sup_\Omega v < +\infty$ and satisfying*

$$\begin{cases} L_K v + c(x) v & \leq & 0 & \textit{in } \Omega\,, \\ v & \leq & 0 & \textit{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

with $c(x)$ bounded by below. Then, there exists a number $\overline{R} > 0$ such that $v \leq 0$ in Ω whenever $R(\Omega) < \overline{R}$.

Proof. We write $c = c^+ - c^-$, and therefore $L_K v - (-c^+)v \leq c^- v^+$. By Theorem 5.4.7 we get

$$\sup_\Omega v \leq C R(\Omega)^{2\gamma} \left|\left| c^- v^+ \right|\right|_{L^\infty(\Omega)} \leq C R(\Omega)^{2\gamma} \left|\left| c^- \right|\right|_{L^\infty(\Omega)} \sup_\Omega v\,.$$

Hence, if $CR(\Omega)^{2\gamma} \left|\left|c^-\right|\right|_{L^\infty(\Omega)} < 1$, we deduce that $v \leq 0$ in Ω. □

The previous maximum principle in "narrow" sets is not suitable enough to apply the moving planes method, and we need to adapt it to the setting of odd functions with respect to a hyperplane (see Proposition 5.4.10 below, which will be deduced from Corollary 5.4.9). The reason why we need it is the following. In the moving the argument, we would want to use a maximum principle in a "narrow" band and applied to an odd function with respect to a hyperplane. However, odd functions cannot have a constant sign in the exterior of a band, and in the hypotheses of Corollary 5.4.9 there is a prescribed constant sign of a function outside the set Ω. Thus, we need another version of a maximum principle in "narrow" sets that applies to odd functions and only requires a constant sign of the function at one side of a hyperplane (in the spirit of the maximum principles of Proposition 5.2.3). This is accomplished with the following result.

Proposition 5.4.10. *Let H be a half-space in $\mathbb{R}^n$, and denote by $x^\#$ the reflection of any point x with respect to the hyperplane ∂H. Let $L_K \in \mathcal{L}_0$ with a positive kernel K satisfying*

$$K(x-y) \geq K(x-y^\#), \quad \textit{for all } \; x,y \in H. \tag{5.4.7}$$

Assume that $v \in L^1_\gamma(\mathbb{R}^n) \cap C^\beta(\Omega)$, with $\beta > 2\gamma$, satisfies

$$\begin{cases} L_K v & \geq & c(x)\, v & \textit{in } \Omega \subset H, \\ v & \geq & 0 & \textit{in } H \setminus \Omega, \\ v(x) & = & -v(x^\#) & \textit{in } \mathbb{R}^n, \end{cases}$$

with $c(x)$ bounded below. Then, there exist a number $\overline{R}$ such that $v \geq 0$ in H whenever $R(\Omega) \leq \overline{R}$.

Proof. Let us begin by defining $\Omega_- = \{x \in \Omega \,:\, v < 0\}$. We shall prove that Ω_- is empty. Assume by contradiction that it is not empty. Then, we split $v = v_1 + v_2$, where

$$v_1(x) = \begin{cases} v(x) & \text{in } \Omega_-, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega_-, \end{cases} \qquad \text{and} \qquad v_2(x) = \begin{cases} 0 & \text{in } \Omega_-, \\ v(x) & \text{in } \mathbb{R}^n \setminus \Omega_-. \end{cases}$$

We first show that $L_K v_2 \leq 0$ in Ω_-. To see this, take $x \in \Omega_-$ and thus

$$L_K v_2(x) = \int_{\mathbb{R}^n \setminus \Omega_-} -v_2(y) K(x-y)\, \mathrm{d}y = -\int_{\mathbb{R}^n \setminus \Omega_-} v(y) K(x-y)\, \mathrm{d}y.$$

Now, we split $\mathbb{R}^n \setminus \Omega_-$ into

$$A_1 = \Omega_-^{\#}, \qquad \text{and} \qquad A_2 = (H \setminus \Omega_-) \cup (H \setminus \Omega_-)^{\#},$$

and we compute the previous integral in these two sets separately using that v is odd. On the one hand, since $v \leq 0$ in Ω_- and $K \geq 0$ in $\mathbb{R}^n$, we have

$$-\int_{A_1} v(y)K(x-y)\,dy = -\int_{\Omega_-} v(y^{\#})K(x-y^{\#})\,dy = \int_{\Omega_-} v(y)K(x-y^{\#})\,dy \leq 0.$$

On the other hand, by the kernel inequality (5.4.7)

$$\begin{aligned} -\int_{A_2} v(y)K(x-y)\,dy &= -\int_{H\setminus\Omega_-} v(y)K(x-y)\,dy - \int_{H\setminus\Omega_-} v(y^{\#})K(x-y^{\#})\,dy \\ &= -\int_{H\setminus\Omega_-} v(y)\left(K(x-y) - K(x-y^{\#})\right)\,dy \leq 0. \end{aligned}$$

Thus, we get $L_K v_2 \leq 0$ in Ω_-. Finally, since $L_K v_2 \leq 0$ in Ω_-, it holds

$$L_K v_1 = L_K v - L_K v_2 \geq L_K v \geq c(x)\, v = c(x)\, v_1 \quad \text{in} \ \ \Omega_-.$$

Therefore v_1 solves

$$\begin{cases} L_K v_1 & \geq & c(x)\, v_1 & \text{in } \Omega_-, \\ v_1 & = & 0 & \text{in } \mathbb{R}^n \setminus \Omega_-, \end{cases}$$

and we can apply the usual maximum principle for "narrow" sets (Corollary 5.4.9) to v_1 in Ω_-. We deduce that $v_1 \geq 0$ in all $\mathbb{R}^n$ whenever $R(\Omega) \leq \overline{R}$. This contradicts the definition of v_1 since we assumed that Ω_- was not empty. Thus, $\Omega_- = \varnothing$ and this yields $v \geq 0$ in Ω. □

Remark 5.4.11. A maximum principle such as Proposition 5.4.10 was already proved for the fractional Laplacian in [71], but with the additional hypothesis that either Ω is bounded or $\liminf_{x\in\Omega,\ |x|\to\infty} v(x) \geq 0$. In the proof of Theorem 3.1 in [161], Quaas and Xia use a suitable argument (the truncation used in the previous proof, previously used by Felmer and Wang in [113]) to avoid the requirement of such additional hypotheses on Ω or v.

With the maximum principle in "narrow" sets for odd functions with respect to a hyperplane we can use the moving plane argument. Now we establish Proposition 5.4.6.

Proof of Proposition 5.4.6. The proof is based on the moving planes method, and is exactly the same as the analogue proof of Theorem 3.1 in [161], where Quaas and Xia establish an equivalent result for the fractional Laplacian. For this reason, we give here just a sketch. As usual, for $\lambda > 0$ we define $w_\lambda(x) = v(x_H, 2\lambda - x_n) - v(x_H, x_n)$ (recall that $x_H \in \mathbb{R}^{n-1}$) and since the nonlinearity is Lipschitz, w_λ solves, in both cases —(P1) or (P2)—, the following problem:

$$\begin{cases} L_K w_\lambda & = & c_\lambda(x)\, w_\lambda & \text{in } \Sigma_\lambda \subset H_\lambda, \\ w_\lambda & \geq & 0 & \text{in } H_\lambda \setminus \Sigma_\lambda, \\ w_\lambda(x_H, 2\lambda - x_n) & = & -w_\lambda(x_H, x_n) & \text{in } \mathbb{R}^n, \end{cases}$$

where $\Sigma_\lambda := \{x = (x_H, x_n) \ : \ 0 < x_n < \lambda\}$ and $H_\lambda := \{x = (x_H, x_n) \ : \ x_n < \lambda\}$ and c_λ is a bounded function. Note that w_λ is odd with respect to ∂H_λ. Then, using the maximum principle in "narrow" sets for odd functions (Proposition 5.4.10) we deduce that, if λ is small enough, $w_\lambda > 0$ in Σ_λ. To conclude the proof, we define

$$\lambda^* := \sup\{\lambda \ : \ w_\eta > 0 \ \text{ in } \ \Sigma_\lambda \ \text{ for all } \ \eta < \lambda\}.$$

Note that λ^* is well defined (but may be infinite) by the previous argument. To conclude the proof, one has to show that $\lambda^* = \infty$. This can be done by proving that, if λ^* is finite, then there exists a small $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0]$ we have

$$w_{\lambda^*+\delta}(x) > 0 \quad \text{ in } \Sigma_{\lambda^*-\varepsilon} \setminus \Sigma_\varepsilon$$

for some small ε. This can be established using a compactness argument exactly as in Lemma 3.1 of [161] and thus we omit the details. In the argument a Harnack inequality is needed, one can use for instance Proposition 5.4.1. Finally, by the maximum principle in "narrow" sets we deduce that $w_{\lambda^*+\delta}(x) > 0$ in $\Sigma_{\lambda^*+\delta}$ if δ is small enough, contradicting the definition of λ^*. □

Now, we present the other important ingredient needed in the proof of Theorem 5.1.6. It is the following symmetry result.

Proposition 5.4.12. *Let $L_K \in \mathcal{L}_0$ and let v be a bounded solution to one of the following problems:*

$$\begin{cases} L_K v = f(v) & \text{in } \mathbb{R}^n, \\ \lim\limits_{x_n \to \pm\infty} v(x_H, x_n) = \pm 1 & \text{uniformly.} \end{cases} \tag{P3}$$

$$\begin{cases} L_K v = f(v) & \text{in } \mathbb{R}^n_+ = \{x_n > 0\}, \\ v = 0 & \text{in } \mathbb{R}^n \setminus \mathbb{R}^n_+ = \{x_n \le 0\}, \\ \lim\limits_{x_n \to +\infty} v(x_H, x_n) = 1 & \text{uniformly.} \end{cases} \tag{P4}$$

Assume that there exists a $\delta > 0$ such that

$$f' \le 0 \quad \text{in} \quad [-1, -1+\delta] \cup [1-\delta, 1],$$

for problem (P3) and

$$f' \le 0 \quad \text{in} \quad [1-\delta, 1]$$

for problem (P4). Then, v depends only on x_n and is increasing in that direction.

Proof. It is based on the sliding method, exactly as in the proof of Theorem 1 in [28]. The idea is, as usual, to define $v^\tau(x) := v(x + \nu\tau)$ for every $\nu \in \mathbb{R}^n$ with $|\nu| = 1$ and $\nu_n > 0$, and the aim is to show that $v^\tau(x) - v(x) \ge 0$ for all $\tau \ge 0$. Despite the fact that L_K is a nonlocal operator, the proof is exactly the same as the one in [28] —it only relies on the maximum principle, the translation invariance of the operator and the Liouville type result of Theorem 5.1.5. Therefore, we do not include here the details. □

Finally, we can proceed with the proof of Theorem 5.1.6.

Proof of Theorem 5.1.6. Note that by Proposition 5.4.12 we only need to prove that

$$\lim_{x_n \to +\infty} v(x_H, x_n) = 1$$

uniformly. Therefore we divide the proof in two steps: first, we prove that the limit exists and is 1, and then we prove that it is uniform. **Step 1:** Given $x_H \in \mathbb{R}^{n-1}$, then

$\lim_{x_n \to +\infty} v(x_H, x_n) = 1$. By Proposition 5.4.6 we know that v is strictly increasing in the direction x_n. Since v is also bounded by hypothesis, we know that, given $x_H \in \mathbb{R}^{n-1}$, the one variable function $v(x_H, \cdot)$ has a limit as $x_n \to +\infty$, which we call $\overline{v}(x_H)$. Note that, since $v(x_H, 0) = 0$ and $v_{x_n} > 0$, it follows that $\overline{v}(x_H) > 0$. Let x_n^k be any increasing sequence tending to infinity. Define $v_k(x_H, x_n) := v(x_H, x_n + x_n^k)$. By the regularity theory of the operator L_K (see Section 5.2) and a standard compactness argument, we see that, up to a subsequence, v_k converge uniformly on compact sets to a function v_∞ which is a classical solution to

$$\begin{cases} L_K v_\infty &= f(v_\infty) \quad \text{in } \mathbb{R}^n, \\ v_\infty &\geq 0 \quad \text{in } \mathbb{R}^n. \end{cases} \tag{5.4.8}$$

By Theorem 5.1.5, either $v_\infty \equiv 0$ or $v_\infty \equiv 1$. But, by construction,

$$v_\infty(x_H, 0) = \lim_{k \to +\infty} v_k(x_H, 0) = \lim_{k \to +\infty} v(x_H, x_n^k) = \overline{v}(x_H) > 0,$$

and therefore the only possibility is

$$\lim_{x_n \to \infty} v(x_H, x_n) = 1 \quad \text{for all} \quad x_H \in \mathbb{R}^{n-1}.$$

Step 2: The limit is uniform in x_H. Let us proceed by contradiction. Suppose that the limit is not uniform. This means that given any $\varepsilon > 0$ small enough, there exists a sequence of points (x_H^k, x_n^k) with $x_n^k \to +\infty$ such that $v(x_H^k, x_n^k) = 1 - \varepsilon$. Similarly as before, the sequence of functions $\tilde{v}_k(x_H, x_n) = v(x_H + x_H^k, x_n + x_n^k)$ converge uniformly on compact sets to a function $\tilde{v}_\infty$ that also solves (5.4.8). By Theorem 5.1.5, either $\tilde{v}_\infty \equiv 0$ or $\tilde{v}_\infty \equiv 1$. But, by construction

$$\tilde{v}_\infty(0, 0) = \lim_{k \to +\infty} \tilde{v}_k(0, 0) = \lim_{k \to +\infty} v(x_H^k, x_n^k) = 1 - \varepsilon,$$

which is a contradiction for $\varepsilon > 0$ small enough. Thus, the limit is uniform. Finally, by applying Proposition 5.4.12, we get that v depends only on x_n and is increasing in that direction. □

5.5 Asymptotic behavior of a saddle-shaped solution

In this section, we show Theorem 5.1.3, concerning the asymptotic behavior of the saddle-shaped solution.

In order to establish the result, it is important to study one-dimensional layer solutions in $\mathbb{R}^n$. Actually, in relation with the available results concerning a conjecture by De Giorgi, in low dimensions all layer solutions are one-dimensional (see Subsection 5.1.3).

One-dimensional layer solutions in $\mathbb{R}^n$ are in correspondence with the ones in $\mathbb{R}$. This comes for free when dealing with the local case, since if v is a solution to $-\ddot{v} = f(v)$ in $\mathbb{R}$, then $w(x) = v(x \cdot e)$ solves $-\Delta w = f(w)$ in $\mathbb{R}^n$ for every unitary vector $e \in \mathbb{R}^n$. The same fact also happens for the fractional Laplacian, that is, if v is a solution to $(-\Delta)^\gamma v = f(v)$ in $\mathbb{R}$, then $w(x) = v(x \cdot e)$ solves the same equation in $\mathbb{R}^n$. We can easily see this relation via the local extension problem.

Nevertheless, for a general operator L_K this is not true anymore and we need a way to relate a solution to a one-dimensional problem with a one-dimensional solution to a n-dimensional problem. This is given in the next result. Some of its points appear in [83] with a different notation but we state and prove them here for completeness.

Proposition 5.5.1. *Let $L_K \in \mathcal{L}_0(n,\gamma,\lambda,\Lambda)$ be a symmetric and translation invariant integro-differential operator of the form (5.1.2) with kernel $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$. Define the one dimensional kernel $K_1 : \mathbb{R} \setminus \{0\} \to (0,+\infty)$ by*

$$K_1(\tau) := \int_{\mathbb{R}^{n-1}} K(\theta,\tau)\, \mathrm{d}\theta = |\tau|^{n-1} \int_{\mathbb{R}^{n-1}} K(\tau\sigma,\tau)\, \mathrm{d}\sigma. \tag{5.5.1}$$

(i) Let $v : \mathbb{R} \to \mathbb{R}$ and consider $w : \mathbb{R}^n \to \mathbb{R}$ defined by $w(x) = v(x_n)$. Then, $L_K w(x) = L_{K_1} v(x_n)$. If we assume moreover that K is radially symmetric, then the same happens with $w(x) = v(x \cdot e)$ for every unitary vector $e \in \mathbb{S}^{n-1}$. That is, $L_K w(x) = L_{K_1} v(x \cdot e)$.

(ii) If K is nonincreasing/decreasing in the x_n-direction in $\{x_n > 0\}$, then $K_1(\tau)$ is nonincreasing/decreasing in $(0,+\infty)$.

(iii) $L_{K_1} \in \mathcal{L}_0(1,\gamma,\lambda,\Lambda)$, and moreover, if L_K is the fractional Laplacian in dimension n, then L_{K_1} is the fractional Laplacian in dimension 1.

Proof. We start proving point (i). We write $y = (y_H, y_n)$, with $y_H \in \mathbb{R}^{n-1}$.

$$\begin{aligned} L_K w(x) &= \int_{\mathbb{R}^n} \big(w(x) - w(y)\big) K(x-y)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^n} \big(v(x_n) - v(y_n)\big) K(x_H - y_H, x_n - y_n)\, \mathrm{d}y_H\, \mathrm{d}y_n. \end{aligned}$$

Now we make the change of variables $\theta = x_H - y_H$. That is,

$$\begin{aligned} L_K w(x) &= \int_{\mathbb{R}} \big(v(x_n) - v(y_n)\big) \int_{\mathbb{R}^{n-1}} K(\theta, x_n - y_n)\, \mathrm{d}\theta\, \mathrm{d}y_n \\ &= \int_{\mathbb{R}} \big(v(x_n) - v(y_n)\big) K_1(x_n - y_n)\, \mathrm{d}y_n = L_{K_1} v(x_n). \end{aligned}$$

This shows the first equality in (5.5.1). The alternative expression of the kernel K_1, that is useful in some cases, can be obtained from the change of variables $\theta = \tau\sigma$. Furthermore, in the case of K radially symmetric, the result is valid for $u(x) = v(x \cdot e)$ for every unitary vector $e \in \mathbb{S}^{n-1}$ after a change of variables in the previous computations. The proof of point (ii) follows directly from the first expression of the unidimensional kernel K_1. That is,

$$K_1(\tau_2) - K_1(\tau_1) = \int_{\mathbb{R}^{n-1}} \big(K(\theta,\tau_2) - K(\theta,\tau_1)\big)\, \mathrm{d}\theta \geq 0 \quad \text{for any} \quad \tau_2 > \tau_1 > 0.$$

We establish now point (iii). To do it, we bound the kernel K_1 using the ellipticity condition on K:

$$\begin{aligned} K_1(\tau) &= |\tau|^{n-1} \int_{\mathbb{R}^{n-1}} K(\tau(\sigma,1))\, \mathrm{d}\sigma \geq |\tau|^{n-1} \int_{\mathbb{R}^n} c_{n,\gamma} \frac{\lambda}{|\tau|^{n+2\gamma}(|\sigma|^2+1)^{\frac{n+2s}{2}}}\, \mathrm{d}\sigma \\ &= c_{n,\gamma} \frac{\lambda}{|\tau|^{1+2\gamma}} \int_{\mathbb{R}^{n-1}} \frac{\mathrm{d}\sigma}{(|\sigma|^2+1)^{\frac{n+2\gamma}{2}}} = c_{n,\gamma} \frac{\lambda}{|\tau|^{1+2\gamma}} \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^\infty \frac{r^{n-2}}{(r^2+1)^{\frac{n+2\gamma}{2}}}\, \mathrm{d}r \\ &= c_{n,\gamma} \frac{\lambda}{|t|^{1+2\gamma}} \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{1}{2}+\gamma)}{\Gamma(\frac{n}{2}+\gamma)} = c_{n,\gamma} \frac{\lambda}{|t|^{1+2\gamma}} \frac{c_{1,\gamma}}{c_{n,\gamma}} = c_{1,\gamma} \frac{\lambda}{|t|^{1+2\gamma}}, \end{aligned}$$

where we have used the explicit value of the normalizing constant for the fractional Laplacian,

$$c_{n,\gamma} = \gamma \frac{2^{2\gamma}\Gamma(\frac{n}{2}+\gamma)}{\pi^{n/2}\Gamma(1-\gamma)}, \tag{5.5.2}$$

and the definition of the Beta and Gamma functions. The upper bound for K_1 is obtained in the same way. Note that the previous computation is an equality with $\lambda = 1$ in the case of the fractional Laplacian. □

In the proof of Theorem 5.1.3 we will use some properties of the layer solution u_0, defined in (5.1.10). First, in [83] it is proved that there exists a constant C such that

$$|u_0(x) - \operatorname{sign}(x)| \le C|x|^{-2\gamma} \quad \text{and} \quad |\dot{u}_0(x)| \le C|x|^{-1-2\gamma} \quad \text{for large } |x|. \tag{5.5.3}$$

In our arguments we need also to show that the second derivative of the layer goes to zero at infinity. This is the first statement of the following lemma.

Lemma 5.5.2. *Let $K_1 : \mathbb{R}\setminus\{0\} \to (0,+\infty)$ be a symmetric kernel satisfying (5.1.4) and assume that it is decreasing in $(0,+\infty)$. Let u_0 be the layer solution associated to the kernel K_1, that is, u_0 solving (5.1.10). Then,*

(i) $\ddot{u}_0(x) \to 0$ *as* $x \to \pm\infty$.

(ii) $\ddot{u}_0(x) < 0$ *in* $(0,+\infty)$.

We prove here the first statement of this lemma, and we postpone the proof of the second one until the next section, since we need to use a maximum principle for the linearized operator $L_{K_1} - f'(u_0)$.

Proof of point (i) of Lemma 5.5.2. By contradiction, suppose that there exists an unbounded sequence $\{x_j\}$ satisfying $|\ddot{u}_0(x_j)| > \varepsilon$ for some $\varepsilon > 0$. Note that by the symmetry of u_0 we may assume that $x_j \to +\infty$. Now define $w_j(x) := \ddot{u}_0(x + x_j)$. By differentiating twice the equation of the layer solution, we see that $\ddot{u}_0$ solves

$$L_{K_1}\ddot{u}_0 = f''(u_0)\dot{u}_0^2 + f'(u_0)\ddot{u}_0 \quad \text{in } \mathbb{R}.$$

Hence, as $x_j \to +\infty$ a standard compactness argument combined with the asymptotic behavior given by (5.5.3) yields that w_j converges on compact sets to a function w that solves

$$L_{K_1}w = f'(1)w \quad \text{in } \mathbb{R}.$$

In addition, since $|\ddot{u}_0(x_j)| > \varepsilon$ we have $|w(0)| \ge \varepsilon$. At this point we use Lemma 4.3 of [83] to deduce that, since $f'(1) < 1$, then $w \to 0$ as $|x| \to +\infty$. Therefore, if w is not identically zero, it has either a positive maximum or a negative minimum, but this contradicts the maximum principle (recall that $f'(1) < 1$). We conclude that $w \equiv 0$ in $\mathbb{R}$, but this is a contradiction with $|w(0)| \ge \varepsilon$. □

Now we have all the ingredients to establish the asymptotic behavior of the saddle-solution. The proof follows exactly the same compactness arguments used to prove the analogous result in the local case (see [60]) and for the fractional Laplacian using the extension problem (see [75, 76]). Thus we will omit some details. The main ingredients too establish this results are the translation invariance of the operator, the Liouville type and symmetry results of Theorems 5.1.5 and 5.1.6 and a stability argument (recall the comments in Section 5.2).

Proof of Theorem 5.1.3. By contradiction, assume that the result does not hold. Then, there exists an $\varepsilon > 0$ and an unbounded sequence $\{x_k\}$, such that

$$|u(x_k) - U(x_k)| + |\nabla u(x_k) - \nabla U(x_k)| + |D^2 u(x_k) - D^2 U(x_k)| > \varepsilon. \tag{5.5.4}$$

By the symmetry of u, we may assume without loss of generality that $x_k \in \overline{\mathcal{O}}$, and by continuity we can further assume $x_k \notin \mathscr{C}$. Let $d_k := \mathrm{dist}(x_k, \mathscr{C})$. We distinguish two cases: **Case 1: $\{d_k\}$ is an unbounded sequence.** In this situation, we may assume that $d_k \geq 2k$. Define

$$w_k(x) := u(x + x_k),$$

which satisfies $0 < w_k < 1$ in $\overline{B_k}$ and

$$L_K w_k = f(w_k) \ \text{ in } B_k.$$

Letting $k \to +\infty$, by standard estimates for the operators of the class $\mathcal{L}_0$ (see Section 5.2) and the Arzelà-Ascoli theorem, we have that, up to a subsequence, w_k converges on compact sets to a function w which is a pointwise solution to

$$\begin{cases} L_K w &= f(w) \quad \text{in } \mathbb{R}^n\,, \\ \quad w &\geq 0 \qquad \text{in } \mathbb{R}^n\,. \end{cases}$$

Then, by Theorem 5.1.5, either $w \equiv 0$ or $w \equiv 1$. First, note that w cannot be zero. Indeed, since w_k are stable with respect to perturbations supported in B_k (see the comments in Section 5.2 and Remark 5.3.4), w is stable in $\mathbb{R}^n$, which means that the linearized operator $L_K - f'(w)$ is a positive operator. Nevertheless, if $w \equiv 0$, then the linearized operator $L_K - f'(w) = L_K - f'(0)$ is negative for sufficiently large balls, since $f'(0) > 0$ and the first eigenvalue of L_K is of order $R^{-2\gamma}$ in balls of radius R (as in Lemma 5.3.2, see Proposition 9 of [177]). Therefore $w \equiv 1$. On the other hand, since $d_k \to +\infty$ and $U(x_k) = u_0(d_k)$, we get by the properties of the layer solution that $U(x_k) \to 1$, $\nabla U(x_k) \to 0$ and $D^2 U(x_k) \to 0$ —see (5.5.3) and Lemma 5.5.2. From this and condition (5.5.4) we get

$$|u(x_k) - 1| + |\nabla u(x_k)| + |D^2 u(x_k)| > \varepsilon/2,$$

for k big enough. This yields that

$$|w_k(0) - 1| + |\nabla w_k(0)| + |D^2 w_k(0)| > \varepsilon/2,$$

and this contradicts $w \equiv 1$. **Case 2: $\{d_k\}$ is a bounded sequence.** In this situation, at least for a subsequence, we have that $d_k \to d$. Now, for each x_k we define x_k^0 as its projection on $\mathscr{C}$. Therefore, we have that $\nu_k^0 := (x_k - x_k^0)/d_k$ is the unit normal to $\mathscr{C}$. Through a subsequence, $\nu_k^0 \to \nu$ with $|\nu| = 1$. We define

$$w_k(x) := u(x + x_k^0),$$

which solves

$$L_K w_k = f(w_k) \text{ in } \mathbb{R}^n.$$

Similarly as before, by letting $k \to +\infty$, up to a subsequence w_k converges on compact sets to a function w which is a pointwise solution to

$$\begin{cases} L_K w &= f(w) \quad \text{in } H := \{x \cdot \nu > 0\}\,, \\ \quad w &\geq 0 \qquad \text{in } H\,, \\ w \text{ is odd with respect to } H. \end{cases}$$

For the details about the fact that $\mathcal{O} + x_k^0 \to H$, see [60]. As in the previous case, by stability w cannot be zero, and thus $w > 0$ in H (by the strong maximum principle for odd functions with respect to a hyperplane, see [71]). Hence, by Theorem 5.1.6, w only depends on $x \cdot \nu$ and is increasing. Finally, by the uniqueness of the layer solution, $w(x) = u_0(x \cdot \nu)$ and

$$\begin{aligned} u(x_k) &= w_k(x_k - x_k^0) = w(x_k - x_k^0) + \mathrm{o}(1) \\ &= u_0((x_k - x_k^0) \cdot \nu) + \mathrm{o}(1) = u_0((x_k - x_k^0) \cdot \nu_k^0) + \mathrm{o}(1) \\ &= u_0(d_k |\nu_k^0|^2) + \mathrm{o}(1) = u_0(d_k) + \mathrm{o}(1) = U(x_k) + \mathrm{o}(1), \end{aligned}$$

contradicting (5.5.4). The same is done for ∇u and $D^2 u$. □

Remark 5.5.3. The previous result yields that, for $\varepsilon > 0$ the saddle-shaped solution satisfies $u \geq \delta$ in the set $\mathcal{O}_\varepsilon := \{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x''| + \varepsilon < |x'|\}$, for some positive constant δ. That is, thanks to the asymptotic result, and since $U(x) \geq u_0(\varepsilon/\sqrt{2})$ for $x \in \mathcal{O}_\varepsilon$, there exists a radius $R > 0$ such that $u(x) \geq U(x)/2 \geq u_0(\varepsilon/\sqrt{2})/2$ if $x \in \mathcal{O}_\varepsilon \setminus B_R$. Moreover, since u is positive in the compact set $\overline{\mathcal{O}_\varepsilon} \cap \overline{B_R}$ it has a positive minimum in this set, say $m > 0$. Therefore, if we choose $\delta = \min\{m, u_0(\varepsilon/\sqrt{2})/2\}$ we obtain the desired result.

5.6 Maximum principles for the linearized operator

In this section we show that the linearized operator $L_K - f'(u)$ satisfies the maximum principle in $\mathcal{O}$. This, combined with the asymptotic result of Theorem 5.1.3, yields the uniqueness of the saddle-shaped solution.

In order to prove the maximum principle of Proposition 5.1.4, we need a maximum principle in "narrow" sets, stated next.

Proposition 5.6.1. *Let $\varepsilon > 0$ and let*

$$\mathcal{N}_\varepsilon \subset \{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \ : \ |x''| < |x'| < |x''| + \varepsilon\} \subset \mathcal{O}$$

be an open set (not necessarily bounded). Let K be a radially symmetric kernel satisfying the positivity condition (5.1.7) and such that $L_K \in \mathcal{L}_0$. Let $v \in C(\overline{\mathcal{N}_\varepsilon}) \cap C^\alpha(\mathcal{N}_\varepsilon) \cap L^1_\gamma(\mathbb{R}^{2m})$, for some $\alpha > 2\gamma$, be a doubly radial function satisfying

$$\begin{cases} L_K v + c(x) v \leq 0 & \text{in } \mathcal{N}_\varepsilon\,, \\ v \leq 0 & \text{in } \mathcal{O} \setminus \mathcal{N}_\varepsilon\,, \\ -v(x^\star) = v(x) & \text{in } \mathbb{R}^{2m}, \\ \limsup_{x \in \mathcal{N}_\varepsilon,\ |x| \to \infty} v(x) \leq 0\,, \end{cases} \tag{5.6.1}$$

with c a function bounded by below. Under these assumptions there exists $\overline{\varepsilon} > 0$ depending only on λ, m, γ and $||c_-||_{L^\infty}$ such that, if $\varepsilon < \overline{\varepsilon}$, then $v \leq 0$ in $\mathcal{N}_\varepsilon$.

Proof. Assume, by contradiction, that

$$M := \sup_{\mathcal{N}_\varepsilon} v > 0\,.$$

Under the assumptions (5.6.1), M must be attained at an interior point $x_0 \in \mathcal{N}_\varepsilon$. Then,

$$0 \geq L_K v(x_0) + c(x_0) v(x_0) \geq L_K v(x_0) - ||c_-||_{L^\infty(\mathcal{N}_\varepsilon)} M\,. \tag{5.6.2}$$

Now, we compute $L_K v(x_0)$. Since v is doubly radial and odd with respect to the Simons cone, we can use the expression (5.1.5) to write

$$\begin{aligned} L_K v(x_0) &= \int_{\mathcal{O}} \big(M - v(y)\big)\big(\overline{K}(x_0, y) - \overline{K}(x_0, y^\star)\big)\,\mathrm{d}y + 2M \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\,\mathrm{d}y \\ &\geq 2M \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\,\mathrm{d}y, \end{aligned}$$

where the inequality follows from being M the supremum of v in $\mathcal{O}$ and the kernel inequality (5.1.7). Combining this last inequality with (5.6.2), we obtain

$$0 \geq L_K v(x_0) + c(x_0)v(x_0) \geq M \left(2 \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\,\mathrm{d}y - ||c_-||_{L^\infty(\mathcal{N}_\varepsilon)}\right).$$

Finally, if we use the lower bound of (5.1.6) and the fact that $\mathrm{dist}(x_0, \mathscr{C}) \leq \varepsilon/\sqrt{2}$, we get

$$\begin{aligned} 0 &\geq M \left(2 \int_{\mathcal{O}} \overline{K}(x_0, y^\star)\,\mathrm{d}y - ||c_-||_{L^\infty(\mathcal{N}_\varepsilon)}\right) \geq M \left(\frac{1}{C}\mathrm{dist}(x_0, \mathscr{C})^{-2\gamma} - ||c_-||_{L^\infty(\mathcal{N}_\varepsilon)}\right) \\ &\geq M \left(\frac{1}{C}\varepsilon^{-2\gamma} - ||c_-||_{L^\infty(\mathcal{N}_\varepsilon)}\right). \end{aligned}$$

Therefore, for ε small enough, we arrive at a contradiction that follows from assuming that the supremum is positive. □

Remark 5.6.2. Using same arguments as in the proof of Proposition 5.4.10, the previous result can be extended to general doubly radial "narrow" sets (that is, assuming that the set $\mathcal{N}_\varepsilon$ in the statement of Proposition 5.6.1 satisfies (5.4.6), instead of just being contained in an ε-neighborhood of the cone). Indeed, we only need to replace the symmetry with respect to a hyperplane by the symmetry with respect to the Simons cone and use the kernel inequality (5.1.7) —note that in this case, the assumption at infinity in (5.6.1) is not needed. Nevertheless, we preferred to present the result for sets that are contained in an ε-neighborhood of the Simons cone, since we are only going to use the maximum principle in such sets. In addition, the crucial fact that the sets are contained in $\{(x', x'') \in \mathbb{R}^m \times \mathbb{R}^m \;:\; |x''| < |x'| < |x''| + \varepsilon\}$ makes the argument rather simple.

Once this maximum principle in "narrow" sets is available, we can proceed with the proof of Proposition 5.1.4.

Proof of Proposition 5.1.4. For the sake of simplicity, we will denote

$$\mathscr{L}w := L_K w - f'(u)w - cw\,.$$

A crucial point in this proof is that u is a positive supersolution of the operator $\mathscr{L}$. Indeed, since f is strictly concave in $(0,1)$ and $f(0) = 0$, then $f'(\tau)\tau < f(\tau)$ for all $\tau > 0$, and thus

$$\mathscr{L}u = L_K u - f'(u)u - cu \geq f(u) - f'(u)u > 0 \quad \text{in } \Omega \subset \mathcal{O}\,, \tag{5.6.3}$$

where in the first inequality we have used that $u > 0$ in $\mathcal{O}$ and that $c \leq 0$. By contradiction, assume that there exists $x_0 \in \Omega$ such that $v(x_0) > 0$. We will show next that, if we assume this, we deduce $v \leq 0$ in Ω, arriving at a contradiction. Let $\varepsilon > 0$ be such that the maximum principle of Proposition 5.6.1 is valid and define the following sets:

$$\Omega_\varepsilon := \Omega \cap \{|x'| > |x''| + \varepsilon\} \quad \text{and} \quad \mathcal{N}_\varepsilon := \Omega \cap \{|x''| < |x'| < |x''| + \varepsilon\}\,.$$

Define also, for $\tau \geq 0$,

$$w := v - \tau u.$$

First, we claim that $w \leq 0$ in Ω if τ is big enough. To see this, note first that by the asymptotic behavior of the saddle-shaped solution, we have

$$u \geq \delta > 0 \quad \text{in } \overline{\Omega}_\varepsilon\,, \tag{5.6.4}$$

for some $\delta > 0$ (see Remark 5.5.3). Therefore, $w < 0$ in $\overline{\Omega}_\varepsilon$ if τ is big enough. Moreover, since $v \leq 0$ in $\mathcal{O} \setminus \Omega$, we have

$$w \leq 0 \quad \text{in } \mathcal{O} \setminus \mathcal{N}_\varepsilon\,.$$

Furthermore, it also holds

$$\limsup_{x\in\mathcal{N}_\varepsilon,\ |x|\to\infty} w(x) \leq 0$$

and, by (5.6.3),

$$\mathscr{L} w = \mathscr{L} v - \tau \mathscr{L} u \leq 0 \text{ in } \mathcal{N}_\varepsilon\,.$$

Thus, since w is odd with respect to $\mathscr{C}$, we can apply Proposition 5.6.1 in $\mathcal{N}_\varepsilon$ to deduce that

$$w \leq 0 \quad \text{in } \Omega\,,$$

if τ is big enough. Now, define

$$\tau_0 := \inf\{\tau > 0 \ : \ v - \tau u \leq 0 \ \text{ in } \Omega\}\,.$$

By the previous claim, τ_0 is well defined. Moreover, it is easy to see that $\tau_0 > 0$. Indeed, it is obvious $v - \tau_0 u \leq 0$ in Ω and thus, since $v(x_0) > 0$, we have $-\tau_0 u(x_0) < v(x_0) - \tau_0 u(x_0) \leq 0$. Using that $u(x_0) > 0$, it follows that $\tau_0 > 0$. We claim that $v - \tau_0 u \not\equiv 0$. Indeed, if $v - \tau_0 u \equiv 0$ then $v = \tau_0 u$ and thus, by using (5.6.3), the equation for v, and the fact that $\tau_0 > 0$, we get

$$0 \geq \mathscr{L} v(x_0) = \tau_0 \mathscr{L} u(x_0) > 0\,,$$

which is a contradiction. Then, since $v - \tau_0 u \not\equiv 0$, the strong maximum principle for odd functions (see Proposition 5.2.3) yields

$$v - \tau_0 u < 0 \quad \text{in } \Omega\,.$$

Therefore, by continuity, the assumption on v at infinity and (5.6.4), there exists $0 < \eta < \tau_0$ such that

$$\tilde{w} := v - (\tau_0 - \eta) u < 0 \quad \text{in } \overline{\Omega}_\varepsilon\,.$$

Note that here we used crucially (5.6.4), and this is the reason for which we needed to introduce the sets Ω_ε and $\mathcal{N}_\varepsilon$. Using again the maximum principle in "narrow" sets with $\tilde{w}$ in $\mathcal{N}_\varepsilon$, we deduce that

$$v - (\tau_0 - \eta) u \leq 0 \quad \text{in } \Omega\,,$$

and this contradicts the definition of τ_0. Hence, $v \leq 0$ in Ω and, as we said, this contradicts our initial assumption on the existence of a point x_0 where $v(x_0) > 0$. □

Note that if in the previous result we assume that $\partial\Omega \cap \mathscr{C}$ is empty, then Ω is at a positive distance to the cone and the lower bound on u in (5.6.4) holds in Ω. In this case no maximum principle in "narrow" sets is required in the previous argument. Instead, if we want to consider sets with $\partial\Omega \cap \mathscr{C} \neq \varnothing$, we need to introduce the set Ω_ε to have the uniform lower bound (5.6.4) and be able to carry out the proof.

The same argument used in the previous proof can be used to establish the remaining statement of Lemma 5.5.2.

Proof of point (ii) of Lemma 5.5.2. Let $v = \ddot{u}_0$. First we show that $v \leq 0$ in $(0, +\infty)$. To see this, note that since f is concave and by point (i) of Lemma 5.5.2, we have that

$$\begin{cases} L_{K_1} v - f'(u_0) v & \leq & 0 & \text{in } (0, +\infty)\,. \\ v(x) & = & -v(-x) & \text{for every } x \in \mathbb{R}\,, \\ \limsup\limits_{x \to +\infty} v(x) & = & 0\,. \end{cases}$$

Now, we follow the proof of Proposition 5.1.4 but with the previous problem, replacing u by u_0 and using that

$$L_{K_1} u_0 - f'(u_0) u_0 > 0 \quad \text{in } (0, +\infty)\,.$$

All the arguments are the same, using the maximum principle of Proposition 5.4.10 in the set $(0, \varepsilon)$, and yield that $v \leq 0$ in $(0, +\infty)$. The fact that $\ddot{u}_0 = v < 0$ in $(0, +\infty)$ can be readily deduced from the strong maximum principle for odd functions in $\mathbb{R}$, as follows. Suppose by contradiction that there exists a point $x_0 \in (0, +\infty)$ such that $v(x_0) = 0$. Then,

$$\begin{aligned} 0 \geq L_{K_1} v(x_0) &= -\int_{-\infty}^{+\infty} v(y) K_1(x_0 - y)\, \mathrm{d}y \\ &= -\int_{-\infty}^{+\infty} v(y) \Big(K_1(x_0 - y) - K_1(x_0 + y) \Big)\, \mathrm{d}y > 0\,, \end{aligned}$$

arriving at a contradiction. Here we have used that $v \not\equiv 0$ and the fact that K_1 is decreasing in $(0, +\infty)$, which yields $K_1(x - y) \geq K_1(x + y)$ for every $x > 0$ and $y > 0$. □

Part III

General variational problems: Foliations and minimality

Chapter 6

Null-Lagrangians and calibrations for nonlocal functionals

In this chapter we build a null-Lagrangian and a calibration for nonlocal energy functionals assuming the existence of a foliation formed by graphs of solutions to the Euler-Lagrange equation. This result requires a certain ellipticity assumption on the nonlocal Lagrangian. As an application we deduce the minimality of each leaf in the foliation. Our results extend for the first time the classical theory of extremal fields in the Calculus of Variations to the nonlocal framework. The model case in our setting corresponds to the energy functional for the fractional Laplacian, for which such null-Lagrangian was still unknown.

6.1 Introduction

In this chapter we extend certain features of the classical Weierstrass theory of extremal fields to the nonlocal setting. Our principal result is the construction of a *calibration* (see definition below) for the fractional energy

$$\mathcal{E}_F^s(w) := \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x.$$

where $s \in (0,1)$, $c_{n,s}$ is a normalizing constant, and

$$Q(\Omega) := (\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c) = (\Omega \times \Omega) \cup \Big(\Omega \times (\mathbb{R}^n \setminus \Omega)\Big) \cup \Big((\mathbb{R}^n \setminus \Omega) \times \Omega\Big).$$

The functional $\mathcal{E}_F^s$ is known to be the energy associated to the semilinear equation

$$(-\Delta)^s u = F'(u) \quad \text{ in } \Omega,$$

where

$$(-\Delta)^s u = c_{n,s}\,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y$$

is the fractional Laplacian.

An interesting feature of our construction is that it does not use the Caffarelli-Silvestre extension technique for the fractional Laplacian and extends naturally to general nonlocal energy functionals of the form

$$\mathcal{E}_N(w) := \frac{1}{2} \iint_{Q(\Omega)} G_N(x,y,w(x),w(y))\,\mathrm{d}x\,\mathrm{d}y. \tag{6.1.1}$$

Our construction requires the nonlocal Lagrangian $G_N(x, y, a, b)$ to satisfy a certain ellipticity condition, as well as the existence of a family of critical points of $\mathcal{E}_N$ (or rather, of sub/supersolutions to the Euler-Lagrange equation) whose graphs make up a foliation, in some sense.

6.1.1 Examples

The functional (6.1.1) includes several important examples of nonlocal energies:

- The case
$$G_N(x, y, a, b) = \frac{|a - b|^p}{2p|x - y|^{n+ps}}$$
corresponds to the fractional p-Dirichlet energy, which gives rise to the fractional p-Laplace equation. Furthermore, nonlinearities can be added to this example, for instance, by letting
$$G_N(x, y, a, b) = \frac{|a - b|^p}{2p|x - y|^{n+ps}} - \frac{1}{2|\Omega|}\mathbb{1}_{\Omega\times\Omega}(x, y)(F(a) + F(b)).$$
In particular, for $p = 2$ this is the energy functional associated to the semilinear equation $(-\Delta)^s u = F'(u)$ in Ω.

- The Lagrangian
$$G_N(x, y, a, b) = \frac{\mathfrak{g}\left(\frac{a-b}{|x-y|}\right)}{|x - y|^{n+s-1}},$$
where
$$\mathfrak{g}''(\tau) = \frac{1}{(1 + \tau^2)^{\frac{n+s+1}{2}}} \quad \text{and} \quad \mathfrak{g}'(0) = \mathfrak{g}(0) = 0,$$
gives the fractional area functional for graphs; see [82].

- The case
$$G_N(x, y, a, b) = K(x - y)\, a\, b\ \mathbb{1}_{\Omega\times\Omega}(x, y)$$
includes convolution operators.

- The structure
$$G_N(x, y, a, b) = G(x - y, a - b)\ \mathbb{1}_{\Omega\times\Omega}(x, y)$$
appears naturally in Peridynamics; see [178].

6.1.2 The notion of calibration

A classical problem in the Calculus of Variations consists of finding conditions for a function to be a minimizer of an energy functional. Abstractly, given $\mathcal{E}\colon \mathcal{A} \to \mathbb{R}$ defined on some set of functions $\mathcal{A}$, and given $u \in \mathcal{A}$, we ask whether u minimizes the energy functional $\mathcal{E}$ among competitors in $\mathcal{A}$ with the same Dirichlet conditions.

In classical local problems, the Dirichlet condition refers to the value of u on the boundary of a bounded domain Ω, while in nonlocal problems one must consider the value in the whole exterior of Ω, i.e., in $\Omega^c = \mathbb{R}^n \setminus \Omega$.

One effective strategy to establish the minimality of a function u consists of constructing a calibration functional:

Definition 6.1.1. A functional $\mathcal{C}\colon \mathcal{A} \to \mathbb{R}$ is a *calibration* for $\mathcal{E}$ and $u \in \mathcal{A}$ if the following conditions hold:

($\mathcal{C}1$) $\mathcal{C}(u) = \mathcal{E}(u)$.

($\mathcal{C}2$) $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$.

($\mathcal{C}3$) $\mathcal{C}(w) = \mathcal{C}(\tilde{w})$ for all $w, \tilde{w} \in \mathcal{A}$ with the same Dirichlet conditions.

Once a calibration is available, the minimality of u follows immediately. Indeed, if $\mathcal{C}$ is a calibration for $\mathcal{E}$ and $u \in \mathcal{A}$, then, for any $w \in \mathcal{A}$ with the same Dirichlet conditions as u, applying ($\mathcal{C}1$), ($\mathcal{C}2$), and ($\mathcal{C}3$) in this order we obtain

$$\mathcal{E}(u) = \mathcal{C}(u) = \mathcal{C}(w) \leq \mathcal{E}(w),$$

and u is a minimizer. Here, an a priori existence and regularity result for minimizers is not needed.

Functionals satisfying ($\mathcal{C}3$) are known as *null-Lagrangians* in the literature. The null-Lagrangian property imposes strong restrictions on the form of the functional, leading to a complete characterization of such functionals in the local case; see [125, Chapter 1.4.2]. Instead, if we are only interested in proving the minimality of u, it is clear from the argument above that conditions ($\mathcal{C}2$) and ($\mathcal{C}3$) can be relaxed to the less stringent

($\mathcal{C}2'$) $\mathcal{C}(w) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$ with the same Dirichlet conditions as u.

($\mathcal{C}3'$) $\mathcal{C}(u) \leq \mathcal{E}(w)$ for all $w \in \mathcal{A}$ with the same Dirichlet conditions as u.

Indeed, these conditions appear naturally in the applications (for instance, see [88, 47]).

6.1.3 The classical theory of calibrations

Historically, a fundamental question in the classical theory of the Calculus of Variations was to determine necessary and sufficient conditions for a function to be a minimizer. Satisfactory answers have been obtained for local functionals of the form

$$\mathcal{E}_L(w) := \int_\Omega G_L\big(x, w(x), \nabla w(x)\big)\, \mathrm{d}x. \tag{6.1.2}$$

In this framework, the function $G_L(x, \lambda, q)$ is called the Lagrangian of the functional.

Concerning necessary conditions, the most elementary is the vanishing of the first variation

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}_L(u + \varepsilon\eta) = \int_\Omega \eta(x)\mathcal{L}_L(u)(x)\, \mathrm{d}x = 0 \quad \text{for all} \quad \eta \in C_c^\infty(\Omega),$$

which leads to the function satisfying the Euler-Lagrange equation $\mathcal{L}_L(u) = 0$ in Ω (in a weak sense). We call these critical points *extremals* of the energy functional $\mathcal{E}_L$.

Further necessary conditions can be derived depending on the topology with respect to which the functional is being minimized. If $u \in C^1(\overline{\Omega})$ is a minimizer of $\mathcal{E}_L$ with respect to small $C_c(\Omega)$ perturbations[1], then it is well-known that it must satisfy the *Weierstrass necessary condition*

$$E(x, u(x), \nabla u(x), \xi) \geq 0 \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{6.1.3}$$

[1]This type of local minimizers are often referred to as *strong minimizers* in the classical literature.

where E is the *Weierstrass excess function* defined as

$$E(x,\lambda,q,\tilde{q}) := G_N(x,\lambda,\tilde{q}) - G_N(x,\lambda,q) - \partial_q G_N(x,\lambda,q)\cdot(\tilde{q}-q).$$

Turning to sufficient conditions, a basic one in known for energy functionals that are convex, since in this case every extremal is already a minimizer. Many models from Physics exhibit such a convexity property, however, there are also numerous nonconvex energies that appear in the applications. This is the case of the Allen-Cahn energy or of minimal surfaces, among many others. These energy functionals may have several extremals, with only a few of them being minimizers.

In order to discuss further sufficient conditions we give the following definition:

Definition 6.1.2. We say that a family of functions $\{u^t\}_{t\in\mathbb{R}}$ is a *field* if

- the functions $t \mapsto u^t(x)$ are increasing for each x
- the map $(x,t) \mapsto u^t(x)$ is continuous

Moreover, we always assume that the u^t are defined on the same domain, which will be $\overline{\Omega}$ or $\mathbb{R}^n$, depending on the local or nonlocal nature of the problem.

A classical result first proven for ODEs by Weierstrass asserts that if u is embedded in an *extremal field* $\{u^t\}_{t\in\mathbb{R}}$, that is, a field such that each of the functions u^t is an extremal of the energy functional $\mathcal{E}_L$, and moreover each of the functions u^t satisfies the Weierstrass necessary condition, then u is a minimizer.

The Weierstrass theory of extremal fields is strongly related to the construction of calibrations for $\mathcal{E}_L$. If $\{u^t\}_{t\in\mathbb{R}}$ is a field, the region

$$\mathcal{G} = \{(x,\lambda)\in\mathbb{R}^n\times\mathbb{R}\colon \lambda = u^t(x) \text{ for some } t\in\mathbb{R}\}$$

has a foliation whose leaves are the graphs of the u^t. In particular, we can define a *leaf–parameter function*

$$t\colon \mathcal{G}\to\mathbb{R}$$

as the unique $t = t(x,\lambda)\in\mathbb{R}$ such that

$$u^t(x) = \lambda.$$

From the continuity assumption in Definition 6.1.2 it follows that the leaf–parameter function t is also continuous.

Given an extremal field $\{u^t\}_{t\in\mathbb{R}}$, one can define the functional $\mathcal{C}_L$ on functions w with graph $w\subset\mathcal{G}$ as

$$\begin{aligned}\mathcal{C}_L(w) &:= \int_\Omega \partial_q G_L(x,u^t(x),\nabla u^t(x))\cdot\Big(\nabla w(x)-\nabla u^t(x)\Big)\Big|_{t=t(x,w(x))}\,\mathrm{d}x \\ &\quad + \int_\Omega G_L(x,u^t(x),\nabla u^t(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x.\end{aligned} \tag{6.1.4}$$

It can be checked that $\mathcal{C}_L$ satisfies $(\mathcal{C}1)$ and $(\mathcal{C}3)$ with $u := u^t$, for all t. Moreover, it is well-known that the energy functional $\mathcal{E}_L$ can be decomposed in terms of $\mathcal{C}_L$ and the Weierstrass excess function E as

$$\mathcal{E}_L(w) = \mathcal{C}_L(w) + \int_\Omega E(x,w(x),\nabla u^t(x),\nabla w(x))\Big|_{t=t(x,w(x))}\,\mathrm{d}x. \tag{6.1.5}$$

Hence, if each u^t satisfies the Weierstrass necessary condition, it follows that $(\mathcal{C}2)$ is also satisfied, and $\mathcal{C}_L$ is a calibration for each u^t.

An important class of local functionals are those for which the Lagrangian $G_L(x, \lambda, q)$ is convex with respect to q, which can be understood as an ellipticity condition. In this case, the Weierstrass excess function is always nonnegative, and the necessary condition (6.1.3) is always satisfied.

As an illustrative example, the energy functional

$$\mathcal{E}_F(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 - \int_\Omega F(w),$$

where $F \in C^1$, admits a calibration

$$\mathcal{C}_F(w) = \int_\Omega \left\{ \nabla u^t \cdot \nabla w - \frac{1}{2}|\nabla u^t|^2 \right\}\Big|_{t=t(x,w(x))} - \int_\Omega F(w).$$

Calibrations and extremal fields have also found important applications in the theory of minimal surfaces. For instance, in [88], a simple calibration argument was used to show the minimality of the Simons cone in dimensions $n \geq 8$. This is a fundamental result in the classification of minimal surfaces.

6.1.4 Nonlocal calibrations

While the theory of calibrations for local energies is well understood, there are almost no results when dealing with nonlocal energies. To the best of our knowledge, the only result in this direction is due to Cabré [47] for the fractional perimeter. He gave a calibration for such a particular nonlocal energy as explained in Section 6.2. Independently, Pagliari [158] investigated the analytical structure of calibrations for the nonlocal total variation and succeeded in constructing a calibration tailored to half-space solutions, although extremal fields are not mentioned in this approach.

Even when the nonlocal energy functional is as simple as $\mathcal{E}_F^s$, which is the energy associated to the fractional Laplacian and nonlocal counterpart of $\mathcal{E}_F$, the form of a calibration was not known prior to this thesis. The following is our main result, which builds a calibration for $\mathcal{E}_F^s$ in the presence of a field. Let us point out that the regularity assumptions on the field can be significantly weakened as explained in Section 6.4:

Theorem 6.1.3. *Let $\{u^t\}_{t\in\mathbb{R}}$ be a field, $u := u^0$, and $\mathcal{A} = \{w \in C(\mathbb{R}^n) : \operatorname{graph} w \subset \mathcal{G}\}$. Assume that $(x,t) \to u^t(x)$ is a bounded C^2 function, and let $\mathcal{C}_F^s$ be the functional*

$$\mathcal{C}_F^s(w) = c_{n,s}\,\mathrm{P.V.} \iint_{(\Omega^c\times\Omega^c)^c} \int_{u^0(x)}^{w(x)} \frac{u^t(x) - u^t(y)}{|x-y|^{n+2s}}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x$$
$$+ \frac{c_{n,s}}{4}\iint_{(\Omega^c\times\Omega^c)^c} \frac{|u^0(x)-u^0(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Then, it follows that:

(a) *$\mathcal{C}_F^s$ satisfies $(\mathcal{C}1)$ and $(\mathcal{C}2')$.*

(b) *Assume in addition that the family $\{u^t\}_{t\in\mathbb{R}}$ satisfies*

$$\begin{aligned} (-\Delta)^s u^t - F'(u^t) &\geq 0 \quad \text{in } \Omega \quad \text{for } t \geq 0,\\ (-\Delta)^s u^t - F'(u^t) &\leq 0 \quad \text{in } \Omega \quad \text{for } t \leq 0. \end{aligned}$$

Then $\mathcal{C}_F^s$ satisfies (C3′). In particular, u minimizes $\mathcal{E}_F^s$ among functions w in $\mathcal{A}$ such that $w \equiv u$ in Ω^c.

(c) *Assume in addition that $\{u^t\}_{t\in\mathbb{R}}$ is an extremal field, that is,*

$$(-\Delta)^s u^t - F'(u^t) = 0 \quad \text{in } \Omega \quad \text{for all } t \in \mathbb{R}.$$

Then $\mathcal{C}_F^s$ satisfies (C3). In particular, each u^t minimizes $\mathcal{E}_F^s$ among functions w in $\mathcal{A}$ such that $w \equiv u^t$ in Ω^c.

The construction of the calibration functional was not evident since the expression for $\mathcal{C}_F$ has not clear analogues from the fractional point of view. A natural approach consists of replacing the gradient terms appearing in the expression of $\mathcal{C}_F$ by fractional ones. Nevertheless, such a construction leads to a functional that fails to be a calibration. The use of the extension problem for the fractional Laplacian was an alternative approach, but we could not arrive to the desired results.

The key to finding the calibration is to review, from a PDE point of view, the classical theory for local functionals. Inspired by the structure of the calibration for the nonlocal perimeter, we observed that the functional $\mathcal{C}_L$ given by (6.1.4) can be written in terms of the Euler-Lagrange and Neumann operators (see Section 6.3) as

$$\mathcal{C}_L(w) = \int_\Omega \int_{u^0(x)}^{w(x)} \mathcal{L}_L(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x + \int_{\partial\Omega} \int_{u^0(x)}^{w(x)} \mathcal{N}_L(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{E}_L(u^0).$$

To the best of our knowledge, this is the first time that the classical calibration is written in this form. The importance of this expression comes from the fact that its structure can be extrapolated to the nonlocal setting, since each of the terms above has a clear nonlocal counterpart. Indeed, this structure gives us, after simple manipulations, the right definition of the calibration functional $\mathcal{C}_F^s$ from the statement of the theorem.

The theorem above can be extended to nonlocal functionals $\mathcal{E}_N$ of the form (6.1.1).

Theorem 6.1.4. *Let $\{u^t\}_{t\in\mathbb{R}}$ be a* sufficiently regular *field, $u := u^0$, and $\mathcal{A}$ an* appropriate space of functions *$w\colon \mathbb{R}^n \to \mathbb{R}$ with* graph $w \subset \mathcal{G}$. *Assume that G_N is pairwise symmetric, i.e.,*

$$G_N(y,x,b,a) = G_N(x,y,a,b) \quad \text{for all} \quad (x,y,a,b) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R},$$

and satisfies the nonlocal ellipticity condition

$$\partial_b G_N(x,y,a,b) \quad \text{is nonincreasing in } a.$$

Let $\mathcal{C}_N$ be the functional

$$\mathcal{C}_N(w) := \iint_{Q(\Omega)} \int_{u^0(x)}^{w(x)} \partial_a G_N(x,y,u^t(x),u^t(y))\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y + \mathcal{E}_N(u^0)$$

and $\mathcal{L}_N$ be the Euler-Lagrange operator

$$\mathcal{L}_N(w)(x) := \int_{\mathbb{R}^n} \partial_a G_N(x,y,w(x),w(y))\,\mathrm{d}y.$$

Then, it follows that:

(a) *$\mathcal{C}_N$ satisfies (C1) and (C2).*

(b) *Assume in addition that the family satisfies the inequalities*

$$\begin{aligned} \mathcal{L}_N(u^t) &\geq 0 \quad \text{in } \Omega \quad \text{for } t \geq 0, \\ \mathcal{L}_N(u^t) &\leq 0 \quad \text{in } \Omega \quad \text{for } t \leq 0, \end{aligned}$$

where $\mathcal{L}_N$ is the Euler-Lagrange operator associated to $\mathcal{E}_N$. Then $\mathcal{C}_N$ satisfies ($\mathcal{C}3'$). In particular, u minimizes $\mathcal{E}_N$ among functions w in $\mathcal{A}$ such that $w \equiv u$ in Ω^c.

(c) *Assume in addition that $\{u^t\}_{t\in\mathbb{R}}$ is an extremal field, that is,*

$$\mathcal{L}_N(u^t) = 0 \quad \text{in } \Omega \quad \text{for all } t \in \mathbb{R}.$$

Then $\mathcal{C}_N$ is a calibration. In particular, each u^t minimizes $\mathcal{E}_N$ among functions w in $\mathcal{A}$ such that $w \equiv u^t$ in Ω^c.

Let us point out that in this chapter we only give a formal derivation of the previous result since it includes a very wide class of energy functionals, each with their own particularities. For instance, it includes the fractional p-Dirichlet energy, the fractional s-area functional for graphs, as well as convolution energies. Nevertheless, we could give rigorous theorems for specific families, as done for the fractional Dirichlet energy, by assuming the adequate regularity conditions for each particular case.

Finally, we give an interpretation of the nonlocal ellipticity condition, which turns out to have a strong connection with a comparison principle for the associated Euler-Lagrange equation. More specifically, we see that if a function v is below another function w, and they touch at a point x_0, then the monotonicity in b of $\partial_a G_N(x, y, a, b)$ gives the inequality $\mathcal{L}_N(v)(x_0) \geq \mathcal{L}_N(w)(x_0)$. We elaborate on this further in Appendix 6.7.

6.1.5 Organization of the chapter

The chapter is organized as follows. In Section 6.2 we review the work of Cabré on calibrations for the nonlocal perimeter [47], underlining the ideas that led us to the correct result for the fractional Laplacian. In Section 6.3 we recall some known facts about the classical theory for calibrations in extremal fields. The most important result in this section is Proposition 6.3.1, where we rewrite the calibration in a new way, the structure of which can be extrapolated to the nonlocal setting. In Section 6.4 we study the case for the fractional Laplacian, where we investigate the properties of $\mathcal{C}_F^s$ under minimal assumptions on the field and prove Theorem 6.1.3. In Section 6.5 we study the general nonlocal case and exhibit formal arguments that justify Theorem 6.1.4. In Section 6.6 we show how to combine local and nonlocal calibrations to yield calibrations for compound energies. In Appendix 6.7 we relate the ellipticity condition for the nonlocal Lagrangian appearing in Theorem 6.1.4 with a comparison principle for nonlocal nonlinear operators.

6.2 The nonlocal perimeter

In this section we briefly revisit the work by Cabré [47] concerning the construction of a calibration for the fractional perimeter. Identifying the key feature that makes the functional a calibration will give us a candidate to extend the theory to other nonlocal problems.

Consider the $K-$nonlocal perimeter of a set $F \subset \mathbb{R}^n$ inside Ω defined by

$$\mathcal{P}(F) := \frac{1}{2} \iint_{Q(\Omega)} |\mathbb{1}_F(x) - \mathbb{1}_F(y)|\, K(x-y)\, \mathrm{d}x\, \mathrm{d}y.$$

It is well-known that the nonlocal mean curvature H_K defined for sets $A \subset \mathbb{R}^n$ at boundary points $x \in \partial A$ by

$$H_K[A](x) := \int_{\mathbb{R}^n} \Big(\mathbb{1}_{A^c}(y) - \mathbb{1}_A(y)\Big) K(x-y)\, \mathrm{d}y$$

is the Euler-Lagrange operator associated to the energy functional $\mathcal{P}$; see [61]. In particular, if E minimizes the functional $\mathcal{P}$ with respect to sets with the same exterior condition and is regular enough, then $H_K[E](x) = 0$ for $x \in \partial E \cap \Omega$.

In [47], Cabré showed that, given a measurable function $\phi\colon \mathbb{R}^n \to \mathbb{R}$, the functional

$$\mathcal{C}_{\mathcal{P}}(F) := \frac{1}{2} \iint_{Q(\Omega)} \operatorname{sign}(\phi(x) - \phi(y))(\mathbb{1}_F(x) - \mathbb{1}_F(y)) K(x-y)\, \mathrm{d}x\, \mathrm{d}y \tag{6.2.1}$$

is a calibration for the nonlocal perimeter $\mathcal{P}$ when the superlevel sets $\{x \in \mathbb{R}^n : \phi(x) > \lambda\}$ have zero nonlocal mean curvature. As a consequence, each superlevel set is a minimizer of the functional $\mathcal{P}$ with respect to sets with their own exterior data.

Let us point out that Definition 6.1.1 can be easily modified to treat subsets of $\mathbb{R}^n$ instead of functions. For instance, in reference to property ($\mathcal{C}3$), we say that two sets E and F have the same Dirichlet/exterior condition if $E \setminus \Omega = F \setminus \Omega$.

Properties ($\mathcal{C}1$) and ($\mathcal{C}2$) are easy to check from expression (6.2.1) directly, without any additional assumptions on the superlevel sets. In order to show the null-Lagrangian property ($\mathcal{C}3$), however, it requires to find an alternative expression for $\mathcal{C}_{\mathcal{P}}$. For this, let us introduce a different notation for the superlevel sets that better resembles the one for extremal fields mentioned at the beginning of the chapter. The notation will also be useful to tie the ideas from this section together with our developments in later parts of the section.

For $t \in \mathbb{R}$ we define

$$E^t := \{x \in \mathbb{R}^n \colon \phi(x) > t\}.$$

We assume for simplicity that ϕ is smooth and $\nabla\phi(x) \neq 0$ for all x. Hence, the level sets are regular and

$$\partial E^t = \{x \in \mathbb{R}^n \colon \phi(x) = t\}$$

has zero Lebesgue measure in $\mathbb{R}^n$. Then, it can be readily checked that

$$\operatorname{sign}(\phi(x) - \phi(y)) = \Big(\mathbb{1}_{(E^t)^c}(y) - \mathbb{1}_{E^t}(y)\Big)\Big|_{t=\phi(x)} \quad \text{for a.e. } (x,y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Substituting this identity in the definition of the calibration functional (6.2.1), we see that it can be written as

$$\mathcal{C}_{\mathcal{P}}(F) = \frac{1}{2} \iint_{Q(\Omega)} \Big(\mathbb{1}_{(E^t)^c}(y) - \mathbb{1}_{E^t}(y)\Big)\Big|_{t=\phi(x)} (\mathbb{1}_F(x) - \mathbb{1}_F(y)) K(x-y)\, \mathrm{d}x\, \mathrm{d}y. \tag{6.2.2}$$

Using the skew-symmetry of the function $\operatorname{sign}(\phi(x) - \phi(y))$ and splitting the domain of integration into $\Big(\Omega \times \mathbb{R}^n\Big) \cup \Big(\Omega^c \times \Omega\Big)$, we arrive at

$$\begin{aligned} \mathcal{C}_{\mathcal{P}}(F) = &\int_\Omega \mathbb{1}_F(x)\, H_K[E^t](x)\Big|_{t=\phi(x)} \mathrm{d}x \\ &+ \int_{\Omega^c} \mathbb{1}_F(x) \left\{ \int_\Omega \Big(\mathbb{1}_{(E^t)^c}(y) - \mathbb{1}_{E^t}(y)\Big) K(x-y)\, \mathrm{d}y \right\}\Bigg|_{t=\phi(x)} \mathrm{d}x, \end{aligned} \tag{6.2.3}$$

Passing from (6.2.1) to (6.2.3) is the crucial step in [47]. The structure of the alternative expression (6.2.3) for the calibration functional $\mathcal{C}_{\mathcal{P}}$ is what we find most interesting, since the functional appears decomposed into two distinct terms: the first one involves the Euler-Lagrange equation of $\mathcal{P}$ at each of the superlevel sets E^t, while the second one depends only on the set F outside Ω. Thus, if for all t we have $H_K[E^t] = 0$ in Ω, then the quantity $\mathcal{C}_{\mathcal{P}}(F)$ depends only on the exterior condition $F \setminus \Omega$, which makes it a null-Lagrangian.

As mentioned after defining the concept of calibration, to show minimality one does not actually need the null-Lagrangian property $(\mathcal{C}3)$ but rather the weaker condition $(\mathcal{C}3')$. For instance, to prove that the set E^0 minimizes $\mathcal{P}$, from identity (6.2.3) it can be shown that it is sufficient that the sets E^t "above" E^0 are supersolutions and the E^t "below" subsolutions. For more details see [47] and compare with Theorem 6.1.3 and Propositions 6.3.4, 6.5.1, and 6.6.1 in the following sections.

Remark 6.2.1. It is interesting to compare the calibration functional for the *fractional perimeter* (that is, the $K-$nonlocal perimeter with $K(z) = |z|^{-n-2s}$ for some $s \in (0,1)$) with the one for the classical perimeter functional $\mathcal{P}_L$. Indeed, given

$$\mathcal{P}_L(F) := \mathcal{H}^{n-1}(\Omega \cap \partial F),$$

it is well known that it admits the following calibration:

$$\mathcal{C}_{\mathcal{P}_L}(F) := \int_{\Omega\cap\partial F} \nu_{\partial E^t}\Big|_{t=\phi(x)} \cdot \nu_{\partial F} \, \mathrm{d}\mathcal{H}^{n-1},$$

where $\nu_{\partial A}$ denotes the outward normal vector to the surface ∂A. Let us point out that this expression is the local analogue to (6.2.2).

Our goal in the next section will be to rewrite the calibration from the classical theory in a way that shares the same structure with formula (6.2.3). A calibration functional for nonlocal equations will be then easy to derive.

6.3 The local theory of calibrations

In this section we review the classical theory of extremal fields and calibrations for local functionals from a PDE point of view. Inspired by the structure of the calibration (6.2.3) for the nonlocal perimeter, we find an alternative expression for classical calibrations which involves the Euler-Lagrange and Neumann operators on the field. This expression will allow us to extrapolate part of the local theory to the nonlocal setting as explained in the following sections. To the best of our knowledge, it is the first time that the classical calibration has been written in this way.

Consider an energy functional of the form

$$\mathcal{E}_L(w) := \int_\Omega G_L\Big(x, w(x), \nabla w(x)\Big) \, \mathrm{d}x,$$

where the Lagrangian $G_L(x, \lambda, q)$ is bounded from below and of class C^2 in all arguments. Assume moreover that G_L is convex with respect to q. This assumption still includes many interesting examples, such as the p-Dirichlet energy and the area functional for minimal graphs.

Given a field of functions $u^t : \overline{\Omega} \to \mathbb{R}$ (in the sense of Definition 6.1.2) we let

$$\mathcal{G} := \{(x, \lambda) \in \overline{\Omega} \times \mathbb{R} : \lambda = u^t(x) \text{ for some } t \in \mathbb{R}\}$$

and consider the set of admissible functions

$$\mathcal{A} := \{w \in C^1(\overline{\Omega}) : \operatorname{graph} w \subset \mathcal{G}\}.$$

If the field is regular enough, it is a classical result that the functional $\mathcal{C}_L \colon \mathcal{A} \to \mathbb{R}$ defined through the Legendre transform of the Lagrangian G_L as

$$\begin{aligned} \mathcal{C}_L(w) := &\int_\Omega G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,w(x))} \,\mathrm{d}x \\ &+ \int_\Omega \partial_q G_L(x, u^t, \nabla u^t) \cdot \Big(\nabla w(x) - \nabla u^t(x)\Big)\Big|_{t=t(x,w(x))} \,\mathrm{d}x \end{aligned} \tag{6.3.1}$$

is a calibration for $\mathcal{E}_L$ and each leaf u^t under the additional assumption that $\{u^t\}_{t\in\mathbb{R}}$ is a field of extremals; see [125, 11, 4]. As an example, the p-Dirichlet energy

$$\mathcal{E}_{p-Dir}(w) = \frac{1}{p}\int_\Omega |\nabla w|^p \,\mathrm{d}x$$

has a calibration

$$\mathcal{C}_{p-Dir}(w) = \int_\Omega \Big\{|\nabla u^t|^{p-2}\nabla u^t \cdot \nabla w - \frac{p-1}{p}|\nabla u^t|^p\Big\}\Big|_{t=t(x,w(x))} \,\mathrm{d}x.$$

The key point in our approach is to rewrite the calibration functional $\mathcal{C}_L$ in an alternative form involving only those operators which are of interest to the theory of PDE. We compute the first variation of $\mathcal{E}_L$ at $u \in C^2(\overline{\Omega})$ in the direction of $\eta \in C^\infty(\overline{\Omega})$ as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}_L(w+\varepsilon\eta)\Big|_{\varepsilon=0} \\ &= \int_\Omega \partial_\lambda G_L(x, w(x), \nabla w(x))\eta(x)\,\mathrm{d}x + \int_\Omega \partial_q G_L(x, w(x), \nabla w(x)) \cdot \nabla\eta(x)\,\mathrm{d}x \\ &= \int_\Omega \eta(x)\Big\{\partial_\lambda G_L(x, w(x), \nabla w(x)) - \operatorname{div}\Big(\partial_q G_L(x, w(x), \nabla w(x))\Big)\Big\}\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} \eta(x)\partial_q G_L(x, w(x), \nabla w(x)) \cdot \nu_\Omega(x)\,\mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

where ν_Ω denotes the outward unit normal vector to $\partial\Omega$. Introducing the Euler-Lagrange operator

$$\mathcal{L}_L(w)(x) := \partial_\lambda G_L\Big(x, w(x), \nabla w(x)\Big) - \operatorname{div}\Big(\partial_q G_L(x, w(x), \nabla w(x))\Big)$$

and the Neumann operator

$$\mathcal{N}_L(w)(x) := \partial_q G_L\Big(x, w(x), \nabla w(x)\Big) \cdot \nu_\Omega(x),$$

we can write the first variation more succinctly as

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}_L\Big(w+\varepsilon\eta\Big)\Big|_{\varepsilon=0} = \int_\Omega \eta(x)\mathcal{L}_L(w)\,\mathrm{d}x + \int_{\partial\Omega}\eta(x)\mathcal{N}_L(w)\,\mathrm{d}\mathcal{H}^{n-1}.$$

In particular, if u is a minimizer of the energy functional $\mathcal{E}_L$ among functions with the same boundary conditions, then it must weakly satisfy $\mathcal{L}_L(u) = 0$ in Ω.

We claim that the calibration functional in (6.3.1) can be expressed as

$$\mathcal{C}_L(w) = \int_\Omega \int_{u^0(x)}^{w(x)} \mathcal{L}_L(u^t)\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x + \int_{\partial\Omega} \int_{u^0(x)}^{w(x)} \mathcal{N}_L(u^t)\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{E}_L(u^0). \tag{6.3.2}$$

This identity is a consequence of the following explicit formula for the difference of $\mathcal{C}_L$ on two arbitrary functions:

Proposition 6.3.1. *Let $\{u^t\}_{t\in\mathbb{R}}$ be a field such that $(x,t) \to u^t(x)$ is C^2 in $\overline{\Omega}\times\mathbb{R}$. Then, for all w and $\tilde{w}$ in $\mathcal{A}$ we have*

$$\mathcal{C}_L(w) = \mathcal{C}_L(\tilde{w}) + \int_\Omega \int_{\tilde{w}(x)}^{w(x)} \mathcal{L}_L(u^t)\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}x + \int_{\partial\Omega} \int_{\tilde{w}(x)}^{w(x)} \mathcal{N}_L(u^t)\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\,\mathrm{d}\mathcal{H}^{n-1}.$$

The proof of the result follows the usual strategy when establishing that $\mathcal{C}_L$ is a calibration. It consists of writing the difference $\mathcal{C}_L(w) - \mathcal{C}_L(\tilde{w})$ as the integral of a directional derivative, see [4]. While in the literature the functions w and $\tilde{w}$ are assumed to have the same boundary conditions, here we do not impose such a restriction.

Proof of Proposition 6.3.1. We let $\zeta := w - \tilde{w}$ and $w_\theta := (1-\theta)w + \theta\tilde{w} = \tilde{w} + \theta\zeta$. Then, on the one hand,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\theta}\Big\{G_L(x,u^t,\nabla u^t)\Big|_{t=t(x,w_\theta(x))}\Big\}\\ &\quad= \partial_\lambda G_L(x,u^t,\nabla u^t)\Big|_{t=t(x,w_\theta(x))}\zeta + \partial_q G_L(x,u^t,\nabla u^t)\Big|_{t=t(x,w_\theta(x))} \cdot \frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\nabla u^t|_{t=t(x,w_\theta(x))}\Big),\end{aligned}$$

while, on the other hand,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\theta}\Big\{\partial_q G_L(x,u^t,\nabla u^t)\cdot\Big(\nabla w_\theta - \nabla u^t\Big)\Big|_{t=t(x,w_\theta(x))}\Big\}\\ &\qquad= \partial_t\Big[\partial_q G_L(x,u^t,\nabla u^t)\Big]\cdot\Big(\nabla w_\theta - \nabla u^t\Big)\Big|_{t=t(x,w_\theta(x))}\frac{\mathrm{d}}{\mathrm{d}\theta}\Big[t(x,w_\theta(x))\Big]\\ &\qquad\quad+ \partial_q G_L(x,u^t,\nabla u^t)\cdot\nabla\zeta\Big|_{t=t(x,w_\theta(x))}\\ &\qquad\quad- \partial_q G_L(x,u^t,\nabla u^t)\Big|_{t=t(x,w_\theta(x))}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\nabla u^t|_{t=t(x,w_\theta(x))}\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{C}_L(w_\theta) &= \int_\Omega \frac{\mathrm{d}}{\mathrm{d}\theta}\Big\{G_L(x,u^t,\nabla u^t) + \partial_q G_L(x,u^t,\nabla u^t)\cdot\Big(\nabla w_\theta - \nabla u^t\Big)\Big|_{t=t(x,w_\theta(x))}\Big\}\,\mathrm{d}x\\ &= \int_\Omega \partial_\lambda G_L(x,u^t,\nabla u^t)\Big|_{t=t(x,w_\theta(x))}\zeta\,\mathrm{d}x + \int_\Omega \partial_q G_L(x,u^t,\nabla u^t)\cdot\nabla\zeta\Big|_{t=t(x,w_\theta(x))}\mathrm{d}x\\ &\quad+ \int_\Omega \partial_t\Big[\partial_q G_L(x,u^t,\nabla u^t)\Big]\cdot\Big(\nabla w_\theta - \nabla u^t\Big)\Big|_{t=t(x,w_\theta(x))}\partial_\lambda t(x,w_\theta(x))\zeta\,\mathrm{d}x.\end{aligned} \tag{6.3.3}$$

The second term in (6.3.3) can be integrated by parts to yield

$$\begin{aligned}
&\int_\Omega \partial_q G_L(x,u^t,\nabla u^t)\cdot \nabla\zeta\Big|_{t=t(x,w_\theta(x))}\,\mathrm{d}x \\
&= \int_{\partial\Omega} \partial_q G_L(x,u^t,\nabla u^t)\cdot \nu_\Omega\Big|_{t=t(x,w_\theta(x))}\zeta\,\mathrm{d}\mathcal{H}^{n-1} - \int_\Omega \zeta\,\mathrm{div}\Big(\partial_q G_L(x,u^t,\nabla u^t)\Big)\Big|_{t=t(x,w_\theta(x))} \\
&\quad + \int_\Omega \zeta\partial_t\Big[\partial_q G_L(x,u^t,\nabla u^t)\Big]\Big|_{t=t(x,w_\theta(x))}\cdot\nabla\Big[t(x,w_\theta(x))\Big]\,\mathrm{d}x.
\end{aligned} \tag{6.3.4}$$

Next, we use the definition of the leaf-parameter function t to simplify the expressions above. Taking the gradient of the identity $u^{t(x,w_\theta(x))}(x)=w_\theta(x)$ gives

$$\nabla u^t(x)\Big|_{t=t(x,w_\theta(x))} + \partial_t u^t(x)\Big|_{t=t(x,w_\theta(x))}\nabla\Big[t(x,w_\theta(x))\Big] = \nabla w_\theta(x),$$

and differentiating $u^t(x)\Big|_{t=t(x,\lambda)}=\lambda$ with respect to λ, we also have

$$\partial_t u^t\Big|_{t=t(x,\lambda)}\partial_\lambda t(x,\lambda)=1.$$

Together, the above yield the relation

$$\nabla\Big[t(x,w_\theta(x))\Big] = \partial_\lambda t(x,w_\theta(x))\Big(\nabla w_\theta(x)-\nabla u^t(x)\Big)\Big|_{t=t(x,w_\theta(x))}. \tag{6.3.5}$$

Substituting (6.3.5) in (6.3.4) leads to the following expression for (6.3.3)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{C}_L(w_\theta) &= \int_\Omega \Big\{\partial_\lambda G_L(x,u^t,\nabla u^t) - \mathrm{div}\Big(\partial_q G_L(x,u^t,\nabla u^t)\Big)\Big\}\Big|_{t=t(x,w_\theta(x))}\zeta(x)\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} \partial_q G_L(x,u^t,\nabla u^t)\cdot\nu_\Omega\Big|_{t=t(x,w_\theta(x))}\zeta(x)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_\Omega \mathcal{L}_L(u^t)\Big|_{t=t(x,w_\theta(x))}\zeta(x)\,\mathrm{d}x + \int_{\partial\Omega}\mathcal{N}_L(u^t)\Big|_{t=t(x,w_\theta(x))}\zeta(x)\,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}$$

Finally, integrating in θ and applying the change of variables $\lambda=\tilde{w}(x)+\theta\zeta(x)$ for each x, we obtain

$$\begin{aligned}
\mathcal{C}_L(w)-\mathcal{C}_L(\tilde{w}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{C}_L(w_\theta)\,\mathrm{d}\theta = \\
&= \int_\Omega\int_0^1 \mathcal{L}_L(u^t)\Big|_{t=t(x,w_\theta(x))}\zeta(x)\,\mathrm{d}\theta\,\mathrm{d}x + \int_{\partial\Omega}\int_0^1 \mathcal{N}_L(u^t)\Big|_{t=t(x,w_\theta(x))}\zeta(x)\,\mathrm{d}\theta\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_\Omega\int_{\tilde{w}(x)}^{w(x)} \mathcal{L}_L(u^t)\Big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x + \int_{\partial\Omega}\int_{\tilde{w}(x)}^{w(x)} \mathcal{N}_L(u^t)\Big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}\mathcal{H}^{n-1},
\end{aligned}$$

which was the claim. □

Remark 6.3.2. Proposition 6.3.1 has a nice geometric interpretation via the divergence theorem in $\mathbb{R}^{n+1}$. Consider the vector field $X\colon \overline{\Omega}\times\mathbb{R}\subset\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ given by

$$X(x,\lambda)=\Big(X^x(x,\lambda),X^\lambda(x,\lambda)\Big),$$

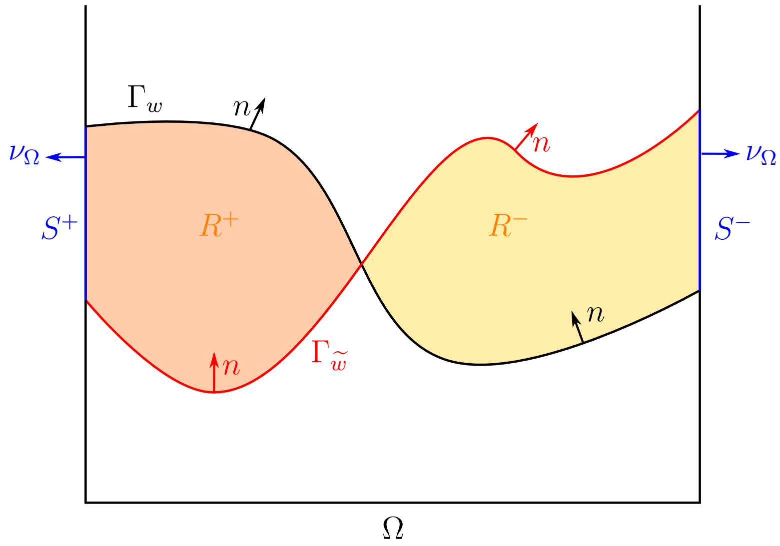

Figure 6.1: Geometric interpretation of Proposition 6.3.1

where

$$X^x(x,\lambda) := -\partial_q G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,\lambda)}$$
$$X^\lambda(x,\lambda) := -\partial_q G_L(x, u^t(x), \nabla u^t(x)) \cdot \nabla u^t(x)\Big|_{t=t(x,\lambda)} + G_L(x, u^t(x), \nabla u^t(x))\Big|_{t=t(x,\lambda)}.$$

Then, it is easy to check that

$$\operatorname{div} X(x,\lambda) = \mathcal{L}_L(u^t)\Big|_{t=t(x,\lambda)},$$

where div is the divergence in $\mathbb{R}^{n+1}$, that is,

$$\operatorname{div} X(x,\lambda) := \operatorname{div}_x X^x(x,\lambda) + \partial_\lambda X^\lambda(x,\lambda).$$

From the definition of the vector field X, it can be checked that the calibration functional $\mathcal{C}_L$ can be written in the compact form

$$\mathcal{C}_L(w) = \int_{\Gamma_w} X \cdot \mathrm{n} \, \mathrm{d}\mathcal{H}^n,$$

where $\Gamma_w \subset \mathbb{R}^{n+1}$ is the graph of w and n is the unit vector normal to Γ_w pointing "upwards". In coordinates, n reads $\mathrm{n}(x, w(x)) = (1 + |\nabla w(x)|^2)^{-1/2}(-\nabla w(x), 1)$.

Consider now the region between the graphs, distinguishing the parts that are above or below each function

$$R^+ = \{(x,\lambda) \in \Omega \times \mathbb{R} \colon \tilde{w}(x) < \lambda < w(x)\}$$

$$R^- = \{(x,\lambda) \in \Omega \times \mathbb{R} \colon w(x) < \lambda < \tilde{w}(x)\},$$

as well as their boundaries on $\partial\Omega$

$$S^+ = \{(x,\lambda) \in \partial\Omega \times \mathbb{R} \colon \tilde{w}(x) < \lambda < w(x)\}$$

$$S^- = \{(x,\lambda) \in \partial\Omega \times \mathbb{R} \colon w(x) < \lambda < \tilde{w}(x)\},$$

see Figure 6.1.

Applying the divergence theorem to the field X separately in each of the regions R^+ and R^-, one can obtain

$$\mathcal{C}_L(w) = \mathcal{C}_L(\tilde{w}) + \int_{R^+} \operatorname{div} X \, \mathrm{d}\mathcal{H}^{n+1} - \int_{R^-} \operatorname{div} X \, \mathrm{d}\mathcal{H}^{n+1} - \int_{S^+} X \cdot \nu_\Omega \, \mathrm{d}\mathcal{H}^n + \int_{S^-} X \cdot \nu_\Omega \, \mathrm{d}\mathcal{H}^n,$$

where ν_Ω is the unit normal vector pointing outwards of $R^+ \cup R^-$ on $S^+ \cup S^-$, see Figure 6.1. It is also easy to check that

$$X \cdot \nu_\Omega = \mathcal{N}_L(u^t)\Big|_{t=t(x,\lambda))}$$

on $S^+ \cup S^-$. Thus, we obtain the passage from formula (6.3.1) to (6.3.2) as a simple application of the divergence theorem.

An important property of $\mathcal{C}_L$ is that it coincides with the energy functional $\mathcal{E}_L$ on each of the leaves, while being smaller on all other functions.

Proposition 6.3.3. *Let $\{u^t\}_{t\in\mathbb{R}}$ be a field such that $(x,t) \to u^t(x)$ is C^2 in $\overline{\Omega} \times \mathbb{R}$. Then, for all $t \in \mathbb{R}$, we have*

$$\mathcal{C}_L(u^t) = \mathcal{E}_L(u^t).$$

Moreover, for all w in $\mathcal{A}$ we have

$$\mathcal{C}_L(w) \leq \mathcal{E}_L(w).$$

Proof. On the one hand, given $\tau \in \mathbb{R}$, we deduce from the definition of the leaf-parameter function that $t(x, u^\tau(x)) = \tau$. In particular, it follows that $\nabla u^\tau = \nabla u^t\Big|_{t=t(x,u^\tau(x))}$. Hence, from the definition of the calibration functional $\mathcal{C}_L$ in (6.3.1) we deduce that

$$\mathcal{C}_L(u^\tau) = \int_\Omega G_L(x, u^\tau(x), \nabla u^\tau(x)) \, \mathrm{d}x = \mathcal{E}_L(u^\tau).$$

On the other hand, by convexity of the lagrangian $G_L(x, \lambda, \cdot)$, for all $x \in \Omega$, $\lambda \in \mathbb{R}$ and $q, \tilde{q} \in \mathbb{R}^n$ we have

$$G_L(x,\lambda,q) + \partial_q G_L(x,\lambda,q) \cdot (\tilde{q} - q) \leq G_L(x,\lambda,\tilde{q}).$$

For each $x \in \Omega$ we now take $\lambda = w(x) = u^t(x)|_{t=t(x,w(x))}$, $q = \nabla u^t(x)|_{t=t(x,w(x))}$, and $\tilde{q} = \nabla w(x)$. The claim follows after integrating the inequality in Ω. □

The following property follows readily from the alternative expression (6.3.2) for $\mathcal{C}_L$:

Proposition 6.3.4. *Let $\{u^t\}_{t\in\mathbb{R}}$ be a field such that $(x,t) \to u^t(x)$ is C^2 in $\overline{\Omega} \times \mathbb{R}$.*

Assume that the leaves satisfy

$$\begin{aligned} \mathcal{L}_L(u^t) &\geq 0 \quad \text{in } \Omega \quad \text{for } t \geq 0, \\ \mathcal{L}_L(u^t) &\leq 0 \quad \text{in } \Omega \quad \text{for } t \leq 0. \end{aligned} \tag{6.3.6}$$

Then, for all w in $\mathcal{A}$ such that $w \equiv u^0$ on $\partial\Omega$, we have

$$\mathcal{C}_L(u^0) \leq \mathcal{C}_L(w).$$

Assume, moreover, that the functions $\{u^t\}_{t\in\mathbb{R}}$ satisfy the Euler-Lagrange equation in Ω, that is,

$$\mathcal{L}_L(u^t) = 0 \quad \text{in } \Omega \quad \text{for all } t. \tag{6.3.7}$$

Then, for all w as above, we have

$$\mathcal{C}_L(w) = \mathcal{C}_L(u^0).$$

Proof. We use identity (6.3.6). Since $\mathcal{C}_L(u^0) = \mathcal{E}_L(u^0)$ and the last term in (6.3.2) depends only on the boundary datum, it suffices to show that

$$\int_{u^0(x)}^{w(x)} \mathcal{L}_L(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \geq 0.$$

This is clear by (6.3.6) and the fact that u^t are increasing with respect to t. If we additionally have (6.3.7), then the integral above is zero and the proof is complete. □

6.4 The calibration for the fractional Laplacian

In this section we construct a calibration for the functional

$$\mathcal{E}_F^s(w) := \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y - \int_\Omega F(w(x))\,\mathrm{d}x,$$

where $F \in C^1(\mathbb{R})$, $s \in (0,1)$, and $c_{n,s}$ is a normalizing constant.

If u is a minimizer of the energy functional $\mathcal{E}_F^s$ with respect to functions with the same exterior data, then it is well known that it must satisfy the nonlocal semilinear equation

$$(-\Delta)^s u - F'(u) = 0 \quad \text{in } \Omega. \tag{6.4.1}$$

Given a field of functions $u^t : \mathbb{R}^n \to \mathbb{R}$ we let

$$\mathcal{G} := \{(x,\lambda) \in \mathbb{R}^n \times \mathbb{R} : \lambda = u^t(x) \text{ for some } t \in \mathbb{R}\}$$

and consider the set of admissible functions

$$\mathcal{A} := \{w \in C(\mathbb{R}^n) : \operatorname{graph} w \subset \mathcal{G}\}.$$

In the statement of Theorem 6.1.3 in the introduction we have assumed by simplicity that the joint function $(x,t) \mapsto u^t(x)$ belongs to $C^2(\mathbb{R}^n \times \mathbb{R}) \cap L^\infty(\mathbb{R}^n \times \mathbb{R})$. Nevertheless, the regularity assumption on $(x,t) \mapsto u^t(x)$ can be significantly weakened. The following assumptions on the field are enough to establish the result:

(A1) For each $x \in \mathbb{R}^n$, the map $t \mapsto u^t(x)$ is locally Lipschitz in $\mathbb{R}$

(A2) For each compact interval $I \subset \mathbb{R}$ and compact set $K \subset \mathbb{R}^n$, we have

$$\sup_{t\in I} \left|\left|u^t\right|\right|_{C^{2s+\delta}(K)} + \sup_{t\in I} \left|\left|u^t\right|\right|_{L^1_s} < +\infty$$

for some small $\delta > 0$. Here, L^1_s denote the space of functions such that

$$||u||_{L^1_s} := \int_{\mathbb{R}^n} \frac{|u(y)|}{1+|y|^{n+2s}} \,\mathrm{d}y < +\infty.$$

Essentially, one needs a reasonable regularity of the joint function as well as some further regularity separately in each of the variables, but uniformly in the other parameter.

For instance, by (A1) the paths $t \mapsto u^t(x)$ are differentiable for almost every $t \in \mathbb{R}$ and, moreover, for each compact interval $I \subset \mathbb{R}$ and $x \in \mathbb{R}^n$ we have

$$\operatorname*{ess\,sup}_{t\in I} |\partial_t u^t(x)| < +\infty.$$

By (A2), the fractional Laplacian $(-\Delta)^s u^t$ is locally bounded uniformly in t.

We let $u := u^0$ and introduce the functional $\mathcal{C}_F^s$, defined on $\mathcal{A}$ as

$$\begin{aligned}\mathcal{C}_F^s(w) := c_{n,s} \lim_{\varepsilon\to 0} \iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} \int_{u(x)}^{w(x)} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}\lambda\, \mathrm{d}x\, \mathrm{d}y \\ - \int_\Omega F(w(x))\, \mathrm{d}x + \frac{c_{n,s}}{4} \iint_{Q(\Omega)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, \mathrm{d}x\, \mathrm{d}y.\end{aligned} \tag{6.4.2}$$

Note that given $w \in \mathcal{A}$, it is possible that $\mathcal{E}_F^s(w) = +\infty$ since functions in $\mathcal{A}$ are no more than continuous. Nevertheless, this will not bother us in the definition of the functional $\mathcal{C}_F^s$, which will always be finite by the regularity assumptions on the filed $\{u^t\}_{t\in\mathbb{R}}$.

Next, we present different expressions for $\mathcal{C}_F^s$ when acting on functions $w \in \mathcal{A}$ such that $w \equiv u$ in Ω^c. These alternative representations will be crucial to deduce that $\mathcal{C}_F^s$ is a calibration for the energy functional $\mathcal{E}_F^s$ and the function u.

First, we find that the functional in (6.4.2) can be cast in an alternative form which involves the Euler-Lagrange equation of the field $\{u^t\}_{t\in\mathbb{R}}$:

Lemma 6.4.1. *Let $\{u^t\}_{t\in\mathbb{R}}$ be a field satisfying (A1) and (A2). Then, for all $w \in \mathcal{A}$ such that $w \equiv u$ in Ω^c we have*

$$\mathcal{C}_F^s(w) = \int_\Omega \int_{u(x)}^{w(x)} \{(-\Delta)^s u^t(x) - F'(u^t(x))\}\Big|_{t=t(x,\lambda)} \mathrm{d}\lambda\, \mathrm{d}x + \mathcal{E}_F^s(u). \tag{6.4.3}$$

Proof. Splitting the domain $Q(\Omega) = (\Omega\times\mathbb{R}^n)\cup(\Omega^c\times\Omega)$ and using that $w(x) = u(x)$ for $x \in \Omega^c$, the integral in (6.4.2) depending on ε can be written by Fubini as

$$\begin{aligned} c_{n,s} \iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} \int_{u(x)}^{w(x)} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}\lambda\, \mathrm{d}x\, \mathrm{d}y \\ = c_{n,s} \int_\Omega \int_{u(x)}^{w(x)} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}y\, \mathrm{d}\lambda\, \mathrm{d}x\end{aligned}$$

and by symmetry, for ε small,

$$\begin{aligned} \int_{u(x)}^{w(x)} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}y\, \mathrm{d}\lambda \\ = \int_{u(x)}^{w(x)} \int_{B_1\setminus B_\varepsilon} \frac{2u^t(x)-u^t(x+z)-u^t(x-z)}{|z|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}z\, \mathrm{d}\lambda \\ + \int_{u(x)}^{w(x)} \int_{\mathbb{R}^n\setminus B_1(x)} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \mathrm{d}y\, \mathrm{d}\lambda.\end{aligned}$$

Since u, w, and the map $t(x,\lambda)$ are continuous, we have that

$$T = \sup\Big\{|t(x,\lambda)| : x \in \overline{\Omega},\quad u(x) \le \lambda \le w(x) \text{ or } w(x) \le \lambda \le u(x)\Big\} < +\infty,$$

from which it follows that the integrands are dominated by

$$\begin{cases} \frac{1}{2}|2u^t(x)-u^t(x+z)-u^t(x-z)|\,|z|^{-(n+2s)} \le \sup\limits_{t\in[-T,T]} \left\|u^t\right\|_{C^{2s+\delta}(\overline{B_1(x)})} |z|^{-n+\delta} \\ \displaystyle\int_{\mathbb{R}^n\setminus B_1(x)} \frac{|u^t(x)-u^t(y)|}{|x-y|^{n+2s}}\, \mathrm{d}y \le C \sup_{t\in[-T,T]} \left\|u^t\right\|_{C(\overline{\Omega})} + C \sup_{t\in[-T,T]} \left\|u^t\right\|_{L^1_s} \end{cases}$$

for $x\in\Omega$ and $|z|\geq 1$.

Applying dominated convergence, from the estimates above it is now clear that

$$c_{n,s}\lim_{\varepsilon\to 0}\iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}}\int_{u(x)}^{w(x)}\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y$$

$$=\int_\Omega\int_{u(x)}^{w(x)}(-\Delta)^s u^t(x)\Big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x.$$

Writing the nonlinearity as

$$F(w(x))=F(u(x))+\int_{u(x)}^{w(x)}F'(\lambda)\,\mathrm{d}\lambda$$

and using the relation $u^{t(x,\lambda)}(x)=\lambda$, the claim follows. □

From identity (6.4.3) it is clear that $\mathcal{C}_F^s(w)$ is well defined and finite for all $w\in\mathcal{A}$ such that $w\equiv u$ in Ω^c.

Before we prove the main theorem of this section, it is convenient to rewrite the functional $\mathcal{C}_F^s$ in yet another way:

Lemma 6.4.2. *Let $\{u^t\}_{t\in\mathbb{R}}$ be a field satisfying (A1) and (A2). Then, for all $w\in\mathcal{A}$ such that $w\equiv u$ in Ω^c we have*

$$\begin{aligned}\mathcal{C}_F^s(w)=&-\frac{c_{n,s}}{2}\lim_{\varepsilon\to 0}\iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}}\int_{t(x,w(x))}^{t(y,w(y))}\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\partial_t u^t(y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\\&+\frac{c_{n,s}}{4}\iint_{Q(\Omega)}\frac{|u^t(x)-u^t(y)|^2}{|x-y|^{n+2s}}\bigg|_{t(x,w(x))}\mathrm{d}x\,\mathrm{d}y-\int_\Omega F(w(x))\,\mathrm{d}x.\end{aligned}\tag{6.4.4}$$

Proof. For each $x\in\mathbb{R}^n$, we apply the change of variables $\lambda\mapsto t$ with $u^t(x)=\lambda$ in the $\varepsilon-$dependent integral in (6.4.2), which yields

$$\begin{aligned}c_{n,s}&\iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}}\int_{u(x)}^{w(x)}\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)}\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y\\&=c_{n,s}\iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}}\int_0^{t(x,w(x))}\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\partial_t u^t(x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{6.4.5}$$

By the symmetry of the domain $Q(\Omega)$, we can write the last integral as

$$\begin{aligned}c_{n,s}&\iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}}\int_0^{t(x,w(x))}\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\partial_t u^t(x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\\&=\frac{c_{n,s}}{2}\iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}}\int_0^{t(x,w(x))}\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\partial_t u^t(x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\\&\quad-\frac{c_{n,s}}{2}\iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}}\int_0^{t(y,w(y))}\frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\partial_t u^t(y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{6.4.6}$$

Now, splitting the integral $\int_0^{t(y,w(y))} \mathrm{d}t$ into $\int_0^{t(x,w(x))} \mathrm{d}t + \int_{t(x,w(x))}^{t(y,w(y))} \mathrm{d}t$ and rearranging terms, this expression is equal to

$$\begin{aligned} &\frac{c_{n,s}}{2} \iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} \int_0^{t(x,w(x))} \frac{(u^t(x)-u^t(y))(\partial_t u^t(x) - \partial_t u^t(y))}{|x-y|^{n+2s}} \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad - \frac{c_{n,s}}{2} \iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} \int_{t(x,w(x))}^{t(y,w(y))} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}} \partial_t u^t(y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{6.4.7}$$

Moreover, since

$$(u^t(x)-u^t(y))(\partial_t u^t(x) - \partial_t u^t(y)) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u^t(x)-u^t(y)|^2,$$

we can integrate the first term in (6.4.7) explicitly as

$$\begin{aligned} \frac{c_{n,s}}{2} \iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} & \int_0^{t(x,w(x))} \frac{(u^t(x)-u^t(y))(\partial_t u^t(x) - \partial_t u^t(y))}{|x-y|^{n+2s}} \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{c_{n,s}}{4} \iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} \frac{|u^t(x)-u^t(y)|^2}{|x-y|^{n+2s}}\bigg|_{t=t(x,w(x))} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \frac{c_{n,s}}{4} \iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Combining (6.4.5), (6.4.6), and (6.4.7) we find that

$$\begin{aligned} c_{n,s} \iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} & \int_{u(x)}^{w(x)} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}}\bigg|_{t=t(x,\lambda)} \,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{c_{n,s}}{4} \iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} \frac{|u^t(x)-u^t(y)|^2}{|x-y|^{n+2s}}\bigg|_{t=t(x,w(x))} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \frac{c_{n,s}}{4} \iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \frac{c_{n,s}}{2} \iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}} \int_{t(x,w(x))}^{t(y,w(y))} \frac{u^t(x)-u^t(y)}{|x-y|^{n+2s}} \partial_t u^t(y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

which substituted in (6.4.2) gives the claim. □

We can finally give the proof of Theorem 6.1.3. We show that the functional $\mathcal{C}_F^s$ is a calibration for $\mathcal{E}_F^s$ and each u^t when the family $\{u^t\}_{t\in\mathbb{R}}$ is an extremal field, that is, when each u^t solves the semilinear equation (6.4.1). In particular, each of the u^t is a minimizer. More generally, we show that u^0 minimizes $\mathcal{E}_F^s$ if the u^t above u^0 are supersolutions of (6.4.1) and the u^t below are subsolutions.

Proof of Theorem 6.1.3. (a) From (6.4.2) it is clear that $\mathcal{C}_F^s(u) = \mathcal{E}_F^s(u)$, which gives $(\mathcal{C}1)$.

In order to obtain $(\mathcal{C}2')$ we will use Lemma 6.4.2. Let $w \in \mathcal{A}$ such that $w \equiv u$ in Ω^c and consider the expression

$$\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x)-u^t(y))\partial_t u^t(y)\,\mathrm{d}t$$

appearing inside the last integral in (6.4.4).

Assume first that $t(x,w(x)) \le t(y,w(y))$. Since u^t is increasing in t, we have $u^t(x) \ge u^{t(x,w(x))}(x) = w(x)$ for $t \in [t(x,w(x)), t(y,w(y))]$, as well as $\partial_t u^t(y) \ge 0$. Therefore,

$$\int_{t(x,w(x))}^{t(y,w(y))} (u^t(x) - u^t(y))\partial_t u^t(y)\,\mathrm{d}t \ge \int_{t(x,w(x))}^{t(y,w(y))} (w(x) - u^t(y))\partial_t u^t(y)\,\mathrm{d}t. \tag{6.4.8}$$

A similar argument shows that (6.4.8) is also satisfied when $t(y,w(y)) \le t(x,w(x))$.

Now, we observe that the LHS in (6.4.8) can be integrated explicitly as

$$\begin{aligned}\int_{t(x,w(x))}^{t(y,w(y))} (w(x) - u^t(y))\partial_t u^t(y)\,\mathrm{d}t &= -\frac{1}{2}\int_{t(x,w(x))}^{t(y,w(y))} \frac{\mathrm{d}}{\mathrm{d}t}|w(x) - u^t(y)|^2\,\mathrm{d}t \\ &= -\frac{1}{2}|w(x) - w(y)|^2 + \frac{1}{2}|u^t(x) - u^t(y)|^2\Big|_{t=t(x,w(x))}.\end{aligned}$$

Hence,

$$\begin{aligned}&-\frac{c_{n,s}}{2}\lim_{\varepsilon\to 0}\iint_{Q(\Omega)\setminus\{|x-y|<\varepsilon\}}\int_{t(x,w(x))}^{t(y,w(y))} \frac{u^s(x) - u^s(y)}{|x-y|^{n+2s}}\partial_s u^s(y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\le \frac{c_{n,s}}{4}\iint_{Q(\Omega)} \frac{|w(x) - w(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y - \frac{c_{n,s}}{4}\iint_{Q(\Omega)} \frac{|u^t(x) - u^t(y)|^2}{|x-y|^{n+2s}}\Big|_{t(x,w(x))}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

which substituted in the expression (6.4.4) yields $\mathcal{C}^s_F(w) \le \mathcal{E}^s_F(w)$.

(b) By identity (6.4.3), it suffices to shows that

$$\int_{u(x)}^{w(x)} \{(-\Delta)^s u^t(x) - F'(u^t(x))\}\Big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda \ge 0$$

for $x \in \Omega$, which is already clear from the hypothesis (6.1.3).

We have already seen in the introduction how properties $(\mathcal{C}1)$, $(\mathcal{C}2')$, and $(\mathcal{C}3')$ yield the minimality of u.

(c) By identity (6.4.3) and using (6.1.3) we have

$$\mathcal{C}^s_F(w) - \mathcal{E}^s_F(u) = \int_\Omega\int_{u(x)}^{w(x)} \{(-\Delta)^s u^t(x) - F'(u^t(x))\}\Big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x = 0.$$

Thus, $\mathcal{C}^s_F(w) = \mathcal{E}^s_F(u) = \mathcal{C}^s_F(u)$ for all $w \in \mathcal{A}$ such that $w \equiv u$ in Ω^c.

For $w \in \mathcal{A}$ such that $w \equiv u^t$ in Ω^c we can simply take $\mathcal{C}^s_F$ with $u \equiv u^t$ in (6.4.2). By the above, it is then clear that this functional is a calibration for $\mathcal{E}^s_F$ and u^t, and the minimality of each u^t follows. □

6.5 Calibration for nonlocal functionals

Having obtained a calibration for the semilinear problem involving the fractional Laplacian, we are now interested in extending this construction to a general class of nonlocal functionals. This way, we plan to obtain a picture similar to that of the general local theory treated in Section 6.3. We will give a functional $\mathcal{C}_N$ that, at least at the formal level, is a calibration for the nonlocal energy functional. In general, the appropriate regularity assumptions on the extremal field will depend on the problem under investigation.

Consider the nonlocal energy functional

$$\mathcal{E}_N(w) := \frac{1}{2} \iint_{Q(\Omega)} G_N(x, y, w(x), w(y))\, \mathrm{d}x\, \mathrm{d}y,$$

where we write $G_N(x, y, a, b)$ with $(x, y) \in Q(\Omega)$, $a, b \in \mathbb{R}$.

Using that $Q(\Omega)$ is invariant with respect to the reflection $(x, y) \mapsto (y, x)$, we may assume without loss of generality that the nonlocal Lagrangian G_N is *pairwise symmetric*, i.e.,

$$G_N(y, x, b, a) = G_N(x, y, a, b) \quad \text{for all } (x, y) \in Q(\Omega) \text{ and } a, b \in \mathbb{R}. \tag{6.5.1}$$

In particular, from the pairwise symmetry it follows that

$$\partial_b G_N(x, y, a, b) = \partial_a G_N(y, x, b, a) \quad \text{for all } (x, y) \in Q(\Omega) \text{ and } a, b \in \mathbb{R}. \tag{6.5.2}$$

The first variation of the energy functional $\mathcal{E}_N$ at u in the direction of $\eta \in C_c^\infty(\mathbb{R}^n)$ (notice that it is not necessarily supported in Ω) can be computed as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathcal{E}_N(u + \varepsilon\eta)\Big|_{\varepsilon=0} \\
&\quad = \frac{1}{2} \iint_{Q(\Omega)} \partial_a G_N(x, y, u(x), u(y))\eta(x)\, \mathrm{d}x\, \mathrm{d}y + \frac{1}{2} \iint_{Q(\Omega)} \partial_b G_N(x, y, u(x), u(y))\eta(y)\, \mathrm{d}x\, \mathrm{d}y \\
&\quad = \frac{1}{2} \iint_{Q(\Omega)} \partial_a G_N(x, y, u(x), u(y))\eta(x)\, \mathrm{d}x\, \mathrm{d}y + \frac{1}{2} \iint_{Q(\Omega)} \partial_b G_N(y, x, u(y), u(x))\eta(x)\, \mathrm{d}x\, \mathrm{d}y \\
&\quad = \iint_{Q(\Omega)} \partial_a G_N(x, y, u(x), u(y))\ \eta(x)\, \mathrm{d}x\, \mathrm{d}y,
\end{aligned}$$

where we have used the symmetry of $Q(\Omega)$ and the identity (6.5.2).

Writing the domain $Q(\Omega)$ as the disjoint union $Q(\Omega) = (\Omega \times \mathbb{R}^n) \cup (\Omega^c \times \Omega)$, we can split the last integral to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathcal{E}_N(u + \varepsilon\eta)\Big|_{\varepsilon=0} = \int_\Omega \eta(x)\mathcal{L}_N(u)\, \mathrm{d}x + \int_{\Omega^c} \eta(x)\mathcal{N}_N(u)\, \mathrm{d}x,$$

where we have introduced the nonlinear operators

$$\mathcal{L}_N(u)(x) := \int_{\mathbb{R}^n} \partial_a G_N(x, y, u(x), u(y))\, \mathrm{d}y$$

and

$$\mathcal{N}_N(u)(x) := \int_\Omega \partial_a G_N(x, y, u(x), u(y))\, \mathrm{d}y.$$

As in the local case, we refer to $\mathcal{L}_N$ as the *Euler-Lagrange operator* associated to $\mathcal{E}_N$, while $\mathcal{N}_N$ is the nonlocal *Neumann operator*.

Since we are interested in minimization problems with respect to functions with the same exterior data, from now on we only consider variations that are compactly supported in Ω. Thus, an extremal u of $\mathcal{E}_N$ will satisfy the Euler-Lagrange equation

$$\mathcal{L}_N(u) = 0 \quad \text{in } \Omega. \tag{6.5.3}$$

Let $u^t \colon \mathbb{R}^n \to \mathbb{R}$ be a field (in the sense of Definition 6.1.2) which covers the region

$$\mathcal{G} := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : \lambda = u^t(x) \text{ for some } t \in \mathbb{R}\},$$

and let us also consider the class of admissible functions

$$\mathcal{A} := \{w\colon \mathbb{R}^n \to \mathbb{R} : w \text{ is } \textit{regular enough} \text{ and } \operatorname{graph} w \subset \mathcal{G}\}.$$

Since we are not making any growth or structure assumption on G_N, the class of functions w for which $\mathcal{E}_N(w)$ is well-defined is not clear a priori. This admissible class must be chosen according to each nonlinear functional under investigation. We do not pursue the question of finding appropriate regularity assumptions in this work.

We define the functional $\mathcal{C}_N$ on $\mathcal{A}$ as

$$\mathcal{C}_N(w) := \iint_{Q(\Omega)} \int_{u^0(x)}^{w(x)} \partial_a G_N(x, y, u^t(x), u^t(y))\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x \,\mathrm{d}y + \mathcal{E}_N(u^0). \tag{6.5.4}$$

In is clear by splitting the domain into $Q(\Omega) = (\Omega \times \mathbb{R}^n) \cup (\Omega^c \times \Omega)$ that we can rewrite (6.5.4) as

$$\mathcal{C}_N(w) = \int_\Omega \int_{u^0(x)}^{w(x)} \mathcal{L}_N(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x + \int_{\Omega^c} \int_{u^0(x)}^{w(x)} \mathcal{N}_N(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x + \mathcal{E}_N(u^0). \tag{6.5.5}$$

Note that (6.5.5) is the "canonical" nonlocal analogue of identity (6.3.2).

Next, we show the analogue of Proposition 6.3.4 in the general nonlocal case, that is, if the field $\{u^t\}_{t\in\mathbb{R}}$ is made up of supersolutions above u^0 and subsolutions below, then u^0 minimizes $\mathcal{C}_N$ among functions in $\mathcal{A}$ with the same exterior data. Furthermore, if *all* the functions u^t satisfy the Euler-Lagrange equation (i.e. u^t is an extremal field), then $\mathcal{C}_N$ is a null-Lagrangian and its value depends only on the exterior datum. This property follows readily from the identity (6.5.5) for $\mathcal{C}_N$:

Proposition 6.5.1. *Let $\{u^t\}_{t\in\mathbb{R}}$ be a* sufficiently regular *field. Assume that the leaves satisfy the inequalities*

$$\begin{aligned} \mathcal{L}_N(u^t) &\ge 0 \quad \text{in } \Omega \quad \text{for } t \ge 0, \\ \mathcal{L}_N(u^t) &\le 0 \quad \text{in } \Omega \quad \text{for } t \le 0. \end{aligned} \tag{6.5.6}$$

Then, for all w in $\mathcal{A}$ such that $w \equiv u^0$ in Ω^c, we have

$$\mathcal{C}_N(u^0) \le \mathcal{C}_N(w).$$

Assume in addition that the leaves satisfy the Euler-Lagrange equation (6.5.3)*, that is,*

$$\mathcal{L}_N(u^t) = 0 \quad \text{in } \Omega \quad \text{for all } t. \tag{6.5.7}$$

Then, for all w in $\mathcal{A}$ such that $w \equiv u^0$ in Ω^c, we have

$$\mathcal{C}_N(u^0) = \mathcal{C}_N(w).$$

Proof. First, notice that $\mathcal{C}_N(u^0) = \mathcal{E}_N(u^0)$. Since $w \equiv u^0$ in Ω^c, by (6.5.5) we have

$$\mathcal{C}_N(w) - \mathcal{C}_N(u^0) = \int_\Omega \int_{u^0(x)}^{w(x)} \mathcal{L}_N(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x.$$

Assuming (6.5.6), it suffices to show that for all $x \in \Omega$ we have

$$\int_{u^0(x)}^{w(x)} \mathcal{L}_N(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \ge 0. \tag{6.5.8}$$

If $w(x) \geq u^0(x)$, using that the functions $\{u^t\}_{t\in\mathbb{R}}$ are increasing in t, we have $t(x,\lambda) \geq 0$ for $\lambda \in [u^0(x), w(x)]$. Then, by assumption we find that $\mathcal{L}_N(u^t)\big|_{t=t(x,\lambda)} \geq 0$, whence (6.5.8) follows in this case. The case $w(x) \leq u^0(x)$ is similar.

If we further assume (6.5.7), then the integral in (6.5.8) above is zero and the claim follows. □

The functional $\mathcal{C}_N$ can be rewritten in an alternative form that will be helpful when dealing with the calibration properties below:

Lemma 6.5.2. *For all w in $\mathcal{A}$, we have*

$$\begin{aligned}\mathcal{C}_N(w) = \frac{1}{2}\iint_{Q(\Omega)} & G_N(x,y,w(x),u^t(y))\big|_{t=t(x,w(x))}\,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{2}\iint_{Q(\Omega)}\int_{t(x,w(x))}^{t(y,w(y))} \partial_b G_N(x,y,u^s(x),u^s(y))\partial_s u^s(y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{6.5.9}$$

Proof. Looking at the integral in the expression (6.5.4) for $\mathcal{C}_N$, if for each x we apply the change of variables $\lambda \mapsto s$ with $u^s(x) = \lambda$, we obtain

$$\begin{aligned}\iint_{Q(\Omega)}\int_{u^0(x)}^{w(x)} & \partial_a G_N(x,y,u^t(x),u^t(y))\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{Q(\Omega)}\int_0^{t(x,w(x))} \partial_a G_N(x,y,u^s(x),u^s(y))\partial_s u^s(x)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

which we can be then symmetrized in (x,y) and, recalling the identity (6.5.2), yields

$$\begin{aligned}\iint_{Q(\Omega)}\int_{u^0(x)}^{w(x)} & \partial_a G_N(x,y,u^t(x),u^t(y))\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\iint_{Q(\Omega)}\int_0^{t(x,w(x))} \partial_a G_N(x,y,u^s(x),u^s(y))\partial_s u^s(x)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{1}{2}\iint_{Q(\Omega)}\int_0^{t(y,w(y))} \partial_b G_N(y,x,u^s(y),u^s(x))\partial_s u^s(y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{6.5.10}$$

If we now split the integral $\int_0^{t(y,w(y))}\mathrm{d}s$ into $\int_0^{t(x,w(x))}\mathrm{d}s + \int_{t(x,w(x))}^{t(y,w(y))}\mathrm{d}s$, (6.5.10) becomes

$$\begin{aligned}\iint_{Q(\Omega)}\int_{u^0(x)}^{w(x)} & \partial_a G_N(x,y,u^t(x),u^t(y))\big|_{t=t(x,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\iint_{Q(\Omega)}\int_0^{t(x,w(x))} \frac{\mathrm{d}}{\mathrm{d}s}\Big\{G(x,y,u^s(x),u^s(y))\Big\}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \frac{1}{2}\iint_{Q(\Omega)}\int_{t(x,w(x))}^{t(y,w(y))} \partial_b G_N(x,y,u^s(x),u^s(y))\partial_s u^s(y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{6.5.11}$$

where we have used that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}&\Big\{G(x,y,u^s(x),u^s(y))\Big\} \\ &= \partial_a G_N(x,y,u^s(x),u^s(y))\partial_s u^s(x) + \partial_b G_N(x,y,u^s(x),u^s(y))\partial_s u^s(y).\end{aligned}$$

Integrating the $s-$integral of the RHS of (6.5.11) explicitly and using that $u^{t(x,w(x))}(x) = w(x)$ by definition of the leaf–parameter function, we obtain

$$\iint_{Q(\Omega)} \int_{u^0(x)}^{w(x)} \partial_a G_N(x,y,u^t(x),u^t(y))\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x \,\mathrm{d}y$$
$$= \frac{1}{2} \iint_{Q(\Omega)} G(x,y,w(x),u^{t(x,w(x))}(y)) \,\mathrm{d}x \,\mathrm{d}y - \mathcal{E}_N(u^0)$$
$$+ \frac{1}{2} \iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \partial_b G_N(x,y,u^s(x),u^s(y)) \partial_s u^s(y) \,\mathrm{d}s \,\mathrm{d}x \,\mathrm{d}y. \tag{6.5.12}$$

Finally, the claim follows by adding $\mathcal{E}_N(u^0)$ to both sides of (6.5.12). □

The following property now follows directly from Lemma 6.5.2:

Proposition 6.5.3. *Let* $\{u^t\}_{t\in\mathbb{R}}$ *be a* sufficiently regular *field.*
Then, for all $t \in \mathbb{R}$*, we have*

$$\mathcal{C}_N(u^t) = \mathcal{E}_N(u^t).$$

Proof. Let $\tau \in \mathbb{R}$. Choosing $w = u^\tau$ in (6.5.9), since $t(x,w(x)) = \tau$ for all x, we find that

$$\mathcal{C}_N(u^\tau) = \frac{1}{2} \iint_{Q(\Omega)} G_N(x,y,u^\tau(x),u^\tau(y)) \,\mathrm{d}x \,\mathrm{d}y = \mathcal{E}_N(u^\tau).$$

□

We now state our fundamental nonlocal "ellipticity" assumption, which will be crucial in establishing the next proposition below. We say that nonlocal Lagrangian G_N is elliptic if

$$\partial_b G_N(x,y,a,b) \quad \text{is nonincreasing in } a. \tag{6.5.13}$$

This ellipticity condition turns out to have a strong connection with a comparison principle for the associated Euler-Lagrange equation as explained in Appendix 6.7.

As explained in Section 6.3, in the case of local functionals, the convexity assumption of the Lagrangian ensures that the Weierstrass excess function is always nonnegative. The next proposition says that our ellipticity assumption (6.5.13) gives a nonlocal analogue of this fact. We have:

Proposition 6.5.4. *Let* $\{u^t\}_{t\in\mathbb{R}}$ *be a* sufficiently regular *field. Assume that the ellipticity condition* (6.5.13) *holds.*
Then, for all w *in* $\mathcal{A}$ *we have*

$$\mathcal{C}_N(w) \le \mathcal{E}_N(w).$$

Proof. If we compute the difference $\mathcal{E}_N(w) - \mathcal{C}_N(w)$, using the alternative expression (6.5.9) for $\mathcal{C}_N$, we have

$$\mathcal{E}_N(w) - \mathcal{C}_N(w)$$
$$= \frac{1}{2} \iint_{Q(\Omega)} \left\{ G_N(x,y,w(x),w(y)) - G_N(x,y,w(x),u^{t(x,w(x))}(y)) \right\} \mathrm{d}x \,\mathrm{d}y$$
$$- \frac{1}{2} \iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \partial_b G_N(x,y,u^s(x),u^s(y)) \partial_s u^s(y) \,\mathrm{d}s \,\mathrm{d}x \,\mathrm{d}y. \tag{6.5.14}$$

Recalling that $u^{t(y,w(y))}(y) = w(y)$, we can write the first integral on the RHS of (6.5.14) as

$$\begin{aligned}\frac{1}{2}\iint_{Q(\Omega)} &\Big\{G_N(x,y,w(x),w(y)) - G_N(x,y,w(x),u^{t(x,w(x))}(y))\Big\}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2}\iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \frac{\mathrm{d}}{\mathrm{d}s}\Big\{G_N(x,y,w(x),u^s(y))\Big\}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \qquad (6.5.15)\\ &= \frac{1}{2}\iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \partial_b G_N(x,y,w(x),u^s(y))\partial_s u^s(y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Plugging (6.5.15) into (6.5.14), we see that

$$\begin{aligned}&\mathcal{E}_N(w) - \mathcal{C}_N(w)\\ &= \iint_{Q(\Omega)} \int_{t(x,w(x))}^{t(y,w(y))} \Big\{\partial_b G_N(x,y,w(x),u^s(y)) - \partial_b G_N(x,y,u^s(x),u^s(y))\Big\}\,\mathrm{d}s\,\partial_s u^s(y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Hence, it suffices to show that

$$\int_{t(x,w(x))}^{t(y,w(y))} \Big\{\partial_b G_N(x,y,w(x),u^s(y)) - \partial_b G_N(x,y,u^s(x),u^s(y))\Big\}\,\mathrm{d}s\,\partial_s u^s(y) \geq 0 \qquad (6.5.16)$$

for all $(x,y) \in Q(\Omega)$.

Let $(x,y) \in Q(\Omega)$ and assume first that $t(x,w(x)) \leq t(y,w(y))$. For $s \in [t(x,w(x)), t(y,w(y))]$, by the monotonicity of u^t in t we have

$$w(x) = u^{t(x,w(x))}(x) \leq u^s(x),$$

and by property (6.5.13),

$$\partial_b G_N(x,y,w(x),u^s(y))\partial_s u^s(y) \geq \partial_b G_N(x,y,u^s(x),u^s(y))\partial_s u^s(y),$$

whence (6.5.16) follows. The case $t(x,w(x)) \geq t(y,w(y))$ is similar. □

Finally, combining Propositions 6.5.1, 6.5.3, and 6.5.4, we immediately obtain Theorem 6.1.4:

Proof of Theorem 6.1.4. (a) Property $(\mathcal{C}1)$ follows from Proposition 6.5.3 and property $(\mathcal{C}2)$ follows from Proposition 6.5.4.

(b) This follows from the first part of Proposition 6.5.1.

(c) This follows from the second part of Proposition 6.5.1. □

6.6 Calibration for functionals involving both local and nonlocal terms

The results derived in Section 6.5 may be combined with those in Section 6.3 to yield a theory that applies to functionals involving both local and nonlocal interactions. Indeed, let us consider the compound energy

$$\begin{aligned}\mathcal{E}_T(w) &:= \mathcal{E}_N(w) + \mathcal{E}_L(w)\\ &= \frac{1}{2}\iint_{Q(\Omega)} G_N(x,y,w(x),w(y))\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega G_L(x,w(x),\nabla w(x))\,\mathrm{d}x \qquad (6.6.1)\end{aligned}$$

where $\mathcal{E}_L$, G_L, $\mathcal{E}_N$, and G_N are defined as in Sections 6.3 and 6.5. Then, we can prove that

$$\mathcal{C}_T(w) := \mathcal{C}_N(w) + \mathcal{C}_L(w), \tag{6.6.2}$$

where $\mathcal{C}_N$ and $\mathcal{C}_L$ are defined as in the sections above is a calibration functional. Let us point out that as in Section 6.5, the functional $\mathcal{C}_T$ is defined in a set $\mathcal{A}$ of sufficiently regular functions.

By combining equations (6.5.5) and (6.3.2), the functional $\mathcal{C}_T$ may be written equivalently as

$$\begin{aligned}\mathcal{C}_T(w) = &\int_\Omega \int_{u^0(x)}^{w(x)} \mathcal{L}_T(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x \\ &+ \int_{\Omega^c} \int_{u^0(x)}^{w(x)} \mathcal{N}_N(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}x + \int_{\partial\Omega} \int_{u^0(x)}^{v(x)} \mathcal{N}_L(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &+ \mathcal{E}_T(u^0),\end{aligned} \tag{6.6.3}$$

where

$$\mathcal{L}_T(w) := \mathcal{L}_N(w) + \mathcal{L}_L(w),$$

with $\mathcal{L}_L$, $\mathcal{N}_L$, $\mathcal{L}_N$, and $\mathcal{N}_N$ are defined as in Sections 6.3 and 6.5.

The following results are the analogues of Propositions 6.5.1, 6.5.3, and 6.5.4 for the compound energy. The proofs do not differ from the ones above, and they also use the properties of the local functionals from Section 6.3.

Proposition 6.6.1. *Let* $\{u^t\}_{t\in\mathbb{R}}$ *be a* sufficiently regular *field. Assume that the leaves satisfy the inequalities*

$$\begin{aligned}\mathcal{L}_T(u^t) &\geq 0 \quad \text{in } \Omega \quad \text{for } t \geq 0,\\ \mathcal{L}_T(u^t) &\leq 0 \quad \text{in } \Omega \quad \text{for } t \leq 0.\end{aligned} \tag{6.6.4}$$

Then, for all w in $\mathcal{A}$ such that $w \equiv u^0$ in Ω^c, we have

$$\mathcal{C}_T(u^0) \leq \mathcal{C}_T(w).$$

Assume in addition that the leaves satisfy the Euler-Lagrange equation in Ω, that is,

$$\mathcal{L}_T(u^t) = 0 \quad \text{in } \Omega \quad \text{for all } t. \tag{6.6.5}$$

Then, for all w as above, we have

$$\mathcal{C}_T(w) = \mathcal{C}_T(u^0).$$

Proof. First, notice that $\mathcal{C}_T(u^0) = \mathcal{E}_T(u^0)$. The last two integrals in (6.6.3) depend only on the boundary datum and must be zero if $w \equiv u^0$ in Ω^c. Thus, assuming (6.6.4), it suffices to show that for all $x \in \Omega$ we have

$$\int_{u^0(x)}^{w(x)} \mathcal{L}_T(u^t)\Big|_{t=t(x,\lambda)} \,\mathrm{d}\lambda \geq 0.$$

This is now clear by (6.6.4).

If we further have (6.6.5), then the integral above is zero and the second claim follows. □

Proposition 6.6.2. *Let $\{u^t\}_{t\in\mathbb{R}}$ be a* sufficiently regular *field. Then, for all $t \in \mathbb{R}$, we have*

$$\mathcal{C}_T(u^t) = \mathcal{E}_T(u^t).$$

Proof. Let $\tau \in \mathbb{R}$. By Proposition (6.5.1) we have $\mathcal{C}_N(u^\tau) = \mathcal{E}_N(u^\tau)$, and by Proposition (6.3.4) we have $\mathcal{C}_L(u^\tau) = \mathcal{E}_L(u^\tau)$. Combining both results, by definitions (6.6.1) and (6.6.2) we obtain the claim. □

Proposition 6.6.3. *Let $\{u^t\}_{t\in\mathbb{R}}$ be a* sufficiently regular *field. Assume that the nonlocal ellipticity condition (6.5.13) holds and that G_L is convex with respect to q.*

Then, for all w in $\mathcal{A}$ we have

$$\mathcal{C}_T(w) \le \mathcal{E}_T(w).$$

Proof. By assumption, we may apply Propositions 6.5.4 and 6.3.3, whence $\mathcal{C}_N(w) \le \mathcal{E}_N(w)$ and $\mathcal{C}_L(w) \le \mathcal{E}_L(w)$. Combining these inequalities, the claim follows. □

6.7 Appendix: Ellipticity boils down to a comparison principle

In this appendix we try to understand the consequences of having an ellipticity condition such as (6.5.13) for the nonlocal Lagrangian G_N described in Section 6.5. Essentially, this condition will result in the Euler-Lagrange operator $\mathcal{L}_N$ satisfying a comparison principle.

Given G_N a pairwise symmetric nonlocal Lagrangian it is convenient to consider a stronger condition than ellipticity. We say that G_N is uniformly elliptic if

$$\partial_b G_N(x, y, a, b) \quad \text{is strictly decreasing in } a. \tag{6.7.1}$$

Let us recall that the Euler-Lagrange operator $\mathcal{L}_N$ is defined as an integral

$$\mathcal{L}_N(w)(x) = \int_{\mathbb{R}^n} \partial_a G_N(x, y, w(x), w(y))\, dy.$$

Since we are not making any concrete assumptions on the regularity or growth of G_N, the expression above must be understood in a formal sense. In the applications, $\mathcal{L}_N(w)$ will make sense for sufficiently regular functions w.

First, we establish the following result:

Lemma 6.7.1. *Let $u, v \in C(\mathbb{R}^n)$ be* sufficiently regular *functions such that*

$$u(x) \le v(x) \quad \text{for } x \in \mathbb{R}^n$$

and

$$u(x_0) = v(x_0) \quad \text{for some } x_0 \in \mathbb{R}^n.$$

Assume that G_N satisfies the ellipticity condition (6.5.13). Then, we have

$$\mathcal{L}_N(u)(x_0) \ge \mathcal{L}_N(v)(x_0).$$

Assume in addition that $u \ne v$ and the uniform ellipticity condition (6.7.1) holds. Then, we have

$$\mathcal{L}_N(u)(x_0) > \mathcal{L}_N(v)(x_0).$$

Proof. By the monotonicity of $\partial_a G_N(x, y, a, b)$ in b, letting $a = u(x_0) = v(x_0)$ and using that $u(y) \le v(y)$, we find that

$$\partial_a G_N(x_0, y, v(x_0), v(y)) \ge \partial_a G_N(x_0, y, u(x_0), u(y)) \quad \text{for all } y \in \mathbb{R}^n. \tag{6.7.2}$$

Integrating (6.7.2) with respect to y in $\mathbb{R}^n$ yields the first claim of the proposition.

If $u \neq v$, then by continuity there is an open ball $B \subset \mathbb{R}^n$ such that $u(y) < v(y)$ for $y \in B$. By the strict monotonicity of $\partial_a G_N(x, y, a, b)$ in b, we then have

$$\partial_a G_N(x_0, y, v(x_0), v(y)) > \partial_a G_N(x_0, y, u(x_0), u(y)) \quad \text{for all } y \in B. \tag{6.7.3}$$

Integrating (6.7.3) in B and (6.7.2) in $\mathbb{R}^n \setminus B$ now yields the second claim of the proposition. □

Lemma 6.7.1 suggests an alternative way of proving the minimality of a solution that is embedded in an extremal field or, more generally, in a field made of sub/supersolutions. The method does not use the concept of a calibration, although it relies on the existence of regular enough minimizers for the nonlocal energy $\mathcal{E}_N$. For instance, in the case of semilinear equations driven by the fractional Laplacian, the direct method of the Calculus of Variations yields a weak solution for subcritical nonlinearities, which is then regular enough for the operator to make sense; see [163]. For linear problems, existence results are available for integro-differential operators involving a range of kernels; see [114]. Note that the method of calibrations does not require the *a priori* existence of a minimizer.

We have the following corollary:

Corollary 6.7.2. *Assume that G_N satisfies the uniform ellipticity condition* (6.7.1)*. Let $\{u^t\}_{t\in\mathbb{R}}$ be a* sufficiently regular *field (in the sense of Definition 6.1.2) such that the leaves satisfy the inequalities*

$$\begin{aligned} \mathcal{L}_N(u^t) &\ge 0 \quad \text{in } \Omega \quad \text{for } t \ge 0, \\ \mathcal{L}_N(u^t) &\le 0 \quad \text{in } \Omega \quad \text{for } t \le 0. \end{aligned}$$

Assume that there exists a sufficiently regular *function v such that $v \equiv u^0$ in Ω^c and*

$$\mathcal{E}_N(v) \le \mathcal{E}_N(w) \quad \text{for all functions } w \text{ such that } w \equiv u^0 \text{ in } \Omega^c.$$

Then $v = u^0$ and it is the unique minimizer of $\mathcal{E}_N$ among functions with the same exterior datum.

Proof. Assume the result to be false. Therefore, u^0 is not a minimizer with respect to functions with the same exterior condition. In particular, we have that $v \neq u^0$ and there is a first leaf u^{t_0} that touches v from above or below at an interior point $x_0 \in \Omega$. Assume without loss of generality that u^{t_0} touches v from above (and hence $t_0 > 0$). By Lemma 6.7.1 we deduce that

$$0 = \mathcal{L}_N(v)(x_0) > \mathcal{L}_N(u^{t_0})(x_0) \ge 0,$$

which is a contradiction.

Thus, u^0 is a minimizer and, by the argument above, must be equal to v. Since v was an arbitrary minimizer, we see that u^0 is actually the unique minimizer. □

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