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Hitting probabilities for *g*-Gaussian processes

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Uno, me aventuro, es los libros que ha leído, la pintura que ha visto, la música escuchada y olvidada, las calles recorridas. Uno es su niñez, su familia, unos cuantos amigos, algunos amores, bastantes fastidios. Uno es una suma mermada por infinitas restas.

Sergio Pitol, *El arte de la fuga*

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Introduction

This work is a contribution to the study of hitting probabilities for Gaussian random fields. The motivation arises from applications to systems of linear stochastic partial differential equations.

Let $\boldsymbol{X} = \{\boldsymbol{X}(x), x \in \mathbb{R}^d\}$ be a $\mathbb{R}^D$-valued Gaussian process with independent and identically distributed components. The study of hitting probabilities for $\boldsymbol{X}$ consists mainly in obtaining upper and lower bounds on the probabilities of the random sets

$$F_{I,A} := \{\boldsymbol{X}^{-1}(A) \cap I \neq \emptyset\} = \{\boldsymbol{X}(I) \cap A \neq \emptyset\}, I \subset \mathbb{R}^d, A \subset \mathbb{R}^D.$$

In this thesis, a real valued Gaussian process X is termed *anisotropic* on I a compact subset of $\mathbb{R}^d$ if up to non null multiplicative constants its canonical metric

$$\mathfrak{d}(x,y) = \|X(x) - X(y)\|_{L^2(\Omega)}, x, y \in I,$$

is bounded below and above, up to a multiplicative constant, by the function

$$G(x-y) = \sum_{j=1}^{d} |x_j - y_j|^{H_j}, x, y \in I, H_j \in (0,1].$$

If $H_1 = ... = H_d$, X is termed *isotropic*.

Hitting probabilities estimates for the following examples of Gaussian anisotropic processes are find in: Fractional Brownian motion [Xia99], Brownian sheet [KS99], Funaki's random string [MT02], Stochastic heat equation driven by white noise [DKN07], Fractional stochastic heat equation on the circle [NV09], Stochastic wave equation driven by fractional colored noise [dlCT14], Stochastic Poisson equation driven by white noise [SSV18].

Additionally for centered Gaussian anisotropic processes abstract results on hitting probabilities have been proved in [Xia09, Thm. 7.6], [BLX09, Thm. 2.1]. Such results establish conditions which imply the existence of positive constants c, C such that for any Borel set $A \in \mathcal{B}_b(\mathbb{R}^D)$,

$$c\mathrm{Cap}_{D-D_0}(A) \leq P(\boldsymbol{X}(I) \cap A \neq \emptyset) \leq C\mathcal{H}_{D-D_0}(A),$$

for $D_0 = \sum_{j=1} H_j^{-1}$. Cap_γ and $\mathcal{H}_\beta$ denotes the γ-Bessel-Riesz capacity and the β-dimensional Hausdorff measure, respectively.

Although this topic will not be covered in this work, hitting probabilities for solution of systems of non linear stochastic differential equations have been subject of study in [DN04], [DKN09], [DKN13] and [DSS15]. Such solutions are non Gaussian stochastic processes with continuous trajectories. General criteria for hitting probabilities to non Gaussian processes are proved in [DSS10, Thms. 2.1 & 2.4].

Consider the solution of the linear stochastic heat equation with null conditions, introduced in [BT08],

$$\frac{\partial v}{\partial t} - \Delta v = \dot{W}^{H,\alpha}, (t,x) \in (0,T] \times \mathbb{R}^d,$$

where $\dot{W}^{H,\alpha}$ is a noise fractional in time and colored in space. In [TX17, Thm. 4] Tudor and Xiao prove that under the constraint $4H - (d-\alpha) = 2$, for any fixed $t \in (0,T]$, and $M > 0$,

$$c|\log|x-y||^{1/2}|x-y| \leq \|v(t,x) - v(t,y)\|_{L^2(\Omega)} \leq C|\log|x-y||^{1/2}|x-y|,$$

$x, y \in [-M,M]^D$. This result suggests the use of more general notions of Hausdorff measures and capacities than the classical γ-dimensional Hausdorff measure and the β-Bessel-Riesz capacity.

Building on this fundamental idea, we develop the main ideas of this work in two parts. The first part is devoted to the study of hitting probabilities for a class of Gaussian random fields with canonical metric such that

$$\mathfrak{d}(x,y) \asymp g(|x-y|), x, y \in I,$$

with g a gauge function (See Definition 1.1 for the definition). We term such generalization as g-Gaussian processes.

In Chapter 1, we prove two main results of hitting probabilities for g-Gaussian processes:

- Theorem 1.2 which establishes a criteria for upper bounds of hitting probabilities. In this case the q_g-Hausdorff measure $\mathcal{H}_{q_g}$ with

  $$q_g(\tau) = \tau^D/(g^{-1}(\tau))^d$$

  is the suitable choice for upper bounds of the hitting probabilities. We follow an approach close to the proof of [DSS10, Thm. 2.1].

- Theorem 1.3 which establishes a criteria for lower bounds of hitting probabilities, in terms of the $(q_g)^{-1}$ capacity denoted by $\mathrm{Cap}_{(q_g)^{-1}}$. The proof combines the approach of [BLX09, Thm.2.1],[Xia09, Thm. 7.6] based on weak approximations of measures, and the results of [DSS10, Sec. 3].

We note that our results hold for processes with continuous mean function, removing the constraint of being centered from prior research.

As an example of application of the results obtained in the previous chapter, Chapter 2 is about the study of hitting probabilities for the solution of the linear system of stochastic Poisson equations with boundary conditions given by

$$\begin{cases} -\Delta v_j(x) = \dot{W}_j(x), & x \in B_1, \\ \quad v_j(x) = v_0(x), & x \in \mathbb{S}^{d-1}, \end{cases}$$

$j = 1, ..., D$ where $(W_j, j = 1, ..., D)$ are independent white noises.

A version of this problem with null initial conditions $v_0 \equiv 0$ is studied in [SSV18][Sec. 5]. Such system has a random field solution for $d = 1, 2, 3$. If $d = 1, 3$, in [SSV18, Lem. 5.4 & Lem. 5.7] is proved that the solution is isotropic. In contrast, if $d = 2$ in [SSV18, Lem. 5.5] upper and lower estimates for the canonical metric which suggest that the solution may be a g-Gaussian process are find, but they are not sharp.

In Lemma 2.1 we prove that v is a g-Gaussian process in B_{ρ_0}, $\rho_0 \in (0,1)$ with $g(\tau) = [\log(2\rho_0/\tau)]^{1/2}\,\tau$ for $d = 2$. This lead us to Theorem 2.3 which establishes lower and upper bounds for hitting probabilities for $d = 1, 2, 3$. This improves the estimations in [SSV18, Thm. 5.10 & Thm. 5.11] for $d = 2$.

In the second part of this work, we study hitting probabilities for $\boldsymbol{g}$-Gaussian processes. Let $I \subset \mathbb{R}^{d_1}$, $J \subset \mathbb{R}^{d_2}$ if the canonical metric of a real valued Gaussian process Y is $\boldsymbol{g}$ Gaussian on $I \times J$ with $\boldsymbol{g} = (g_1, g_2)$, g_1, g_2 gauge functions, if its canonical metric satisfies

$$\mathfrak{d}((t,x),(s,y)) \asymp g_1(|t-s|) + g_2(|x-y|), (s,t),(x,y) \in I \times J.$$

In Chapter 3, we generalize the results of Chapter 1 for $\boldsymbol{g}$-Gaussian processes:

- Theorem 3.1 which establishes a criteria for upper bounds of hitting probabilities in terms of the $q_{\boldsymbol{g}}$-Hausdorff measure $\mathcal{H}_{q_{\boldsymbol{g}}}$ with
$$q_{\boldsymbol{g}}(\tau) = \tau^D/[(g^{-1}(\tau))^{d_1}(g^{-1}(\tau))^{d_1}].$$
- Theorem 3.2 which establishes a criteria for lower bounds of hitting probabilities, in terms of the $(q_{\boldsymbol{g}})^{-1}$ capacity $\mathrm{Cap}_{(q_{\boldsymbol{g}})^{-1}}$.

Finally, as an application of the criteria obtained in Chapter 3, we analyze two examples of $\boldsymbol{g}$-Gaussian processes. Chapter 4 is about the study of hitting probabilities for the solution of the system of stochastic heat equations with initial conditions given by

$$\begin{cases} \frac{\partial v_j}{\partial t} - \Delta v_j = \dot{W}_j^{H,\alpha},\ (t,x) \in (0,T] \times \mathbb{R}^d, \\ \quad v_j(0,x) = v_0(x),\ x \in \mathbb{R}^d, \end{cases}$$

$j = 1, ..., D$ with $(W_j^{H,\alpha}, j = 1, ..., D)$ independent copies of a fractional-colored noise with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $\alpha \in [0, d)$.

By following a similar approach that in [TX17], where the case $\alpha \in (0, d)$ is investigated, in Theorem 4.2 we prove that the canonical metric of the coordinates of the solution of such system satisfies that for any $t, s \in [t_0, T]$ and $x, y \in [-M, M]^d$,

$$\mathfrak{d}((t,x),(s,y)) \asymp |t-s|^{H-\frac{d-\alpha}{4}} + \left(\log\frac{2e\sqrt{d}M}{|x-y|}\right)^{\frac{\beta}{2}} |x-y|^{1\wedge(2H-\frac{d-\alpha}{2})},$$

where $\beta = 1$, if $4H - (d-\alpha) = 2$, and $\beta = 0$, otherwise.

In Chapter 5, we make a similar analysis for the solution for the solution of the system of stochastic biharmonic heat equations with initial conditions given by

$$\begin{cases} (\frac{\partial}{\partial t} + (-\Delta)^2) v_j = \dot{W}_j, (t,x) \in (0,T] \times \mathbb{T}^d, \\ \qquad v_j(0,x) = v_0(x), x \in \mathbb{T}^d, \end{cases}$$

$j = 1, ..., D$ with $(W_j, j = 1, ..., D)$ independent copies of a white noise. According to Theorem 5.2 each coordinate entry of such system turns out to be a $\boldsymbol{g}$-Gaussian process with

$$g_1(\tau) = \tau^{\frac{4-d}{8}}, \quad g_2(\tau) = \left(\log\frac{C(d)}{\tau}\right)^{\frac{\beta}{2}} \tau^{1\wedge\frac{4-d}{2}}, \quad \beta = 1_{\{d=2\}}, \quad d = 1, 2, 3.$$

Most of the research carried out in this work has been recently published in two articles: [HCSS21] which contains the main results obtained in Chapters 1, 3 and 4, [HCSS22] which is based in Chapter 5. Some of the proofs are given in more detail in the thesis.

Chapter 1

g-Gaussian random fields

This chapter is devoted to find hitting probabilities criteria for g-Gaussian processes. We prove two main results relative to g-Gaussian processes: Theorem 1.2 which gives upper bounds of hitting probabilities in terms of the q-Hausdorff measure and Theorem 1.3 which gives lower bounds of hitting probabilities in terms of the $\mathfrak{g}$-capacity.

Finally, as an example of g-Gaussian processes we introduce the class of isotropic Gaussian processes. We discuss the obtained results in the context of such processes.

We will use the following notations. Let K be a set in a metric space (S, d). For $\rho > 0$, $K^{(\rho)}$ denotes the set of points such that $d(x, K) \leq \rho$, and $\varnothing_K = \sup_{x,\bar{x}\in K} d(x, \bar{x})$ is the diameter of K. For $x \in \mathbb{R}^d$ and $r \geq 0$, $B_r(x)$ denotes the closed Euclidean ball centered at x with radius r, if $x = 0$ we abbreviate $B_r(0)$ by B_r. Positive real constants are generically denoted by the letter C, or variants, like $\bar{C}$, $\tilde{C}$, c, etc. If we want to make explicit the dependence on some parameters $a_1, a_2, \ldots$, we write $C(a_1, a_2, \ldots)$ or $C_{a_1,a_2,\ldots}$. The symbol $\asymp$ between two mathematical expressions means equivalence up to positive multiplicative constants.

1.1 Definition

A real valued *Gaussian random field* or *Gaussian process* is a real valued random field X on a parameter set $\mathcal{X}$ for which the distributions of

$$(X(x_1), ..., X(x_n)), n \in \mathbb{N}^*, (x_1, ..., x_n) \in \mathcal{X}^n,$$

are multivariate Gaussian. The functions

$$m_x := E(X(x)), \sigma_x^2 := E[(X(x) - m_x)^2], x \in \mathcal{X},$$

are called the *mean* and *variance* functions of X. The functions

$$\sigma_{x,\bar{x}} := E[(X(x) - m_x)(X(\bar{x}) - m_{\bar{x}})], \rho_{x,\bar{x}} := \frac{\sigma_{x,\bar{x}}}{\sigma_x \sigma_{\bar{x}}}; x, \bar{x} \in \mathcal{X},$$

are called the *covariance* and *correlation* functions of X.

Define a pseudometric $\mathfrak{d}$ on $\mathcal{X}$ by

$$\mathfrak{d}(x, \bar{x}) \equiv \mathfrak{d}_{x,\bar{x}} := \|X(x) - X(\bar{x})\|_{L^2(\Omega)}, \ x, \bar{x} \in \mathcal{X}.$$

$\mathfrak{d}$ is not a metric, since although it satisfies all the other demands of a metric, $\mathfrak{d}(x, \bar{x}) = 0$ does not necessarily imply that $x = \bar{x}$. Nevertheless, we shall abuse terminology by calling $\mathfrak{d}$ the *canonical metric* of X.

Definition 1.1. A *gauge function* is a strictly increasing continuous function $g : [0,\tau_0] \to \mathbb{R}_+$, $\tau_0 > 0$ satisfying $g(0) = 0$.

Example 1.1. The following are examples of gauge functions:

1. $g(\tau) = \tau^\nu, \tau, \nu > 0$.
2. $g(\tau) = |\log\tau|^\gamma \tau^\nu, \tau \in [0, e^{-\frac{\gamma}{\nu}}]$, $\gamma, \nu > 0$.

Definition 1.2. Fix K a compact set of $\mathbb{R}^d$ and assume that

$$X = \{X(x) : x \in K\}$$

is a Gaussian process. We say that X is *g-Gaussian* on K if there exists a gauge function g such that

$$\mathfrak{d}(x,\bar{x}) \asymp g(|x-\bar{x}|), x, \bar{x} \in K. \tag{1.1}$$

If X only satisfies the upper bound in (1.1) we say that it is *$\hat{g}$-Gaussian* on K.

Remark 1.1. The class of g-Gaussian processes where g is a gauge function as in Example 1.1 (1.) corresponds to isotropic Gaussian processes (see Section 1.3 bellow). Chapter 2 is devoted to study a g-Gaussian process with a g as in Example 1.1(2.) (see Theorem 2.2).

Write

$$B_{\mathfrak{d},\varepsilon}(x) := \{\bar{x} \in K : \mathfrak{d}(x,\bar{x}) \le \varepsilon\}$$

for the $\mathfrak{d}$ ball centered at $x \in K$ and radius ε. We denote by $N_{\mathfrak{d}}(K,\varepsilon) \equiv N(\varepsilon)$ the minimum number of such balls needed to cover K.

If X is a $\hat{g}$-Gaussian process on K, it is not hard to prove that

$$N(\varepsilon) \le C\left(\frac{\varnothing_K}{g^{-1}(\varepsilon)}\right)^d, \varepsilon \in (0, g(\varnothing_K)]. \tag{1.2}$$

[AT07, Thm. 1.3.5] and (1.2) implies that there exists a universal constant C and positive random variable η such that

$$\sup_{\substack{x,\bar{x}\in K,\\ \mathfrak{d}(x,\bar{x})\le\delta}} |X(x) - X(\bar{x})| \le C(K)\int_0^\delta d\varepsilon\sqrt{d\log\left(\frac{\varnothing_K}{g^{-1}(\varepsilon)}\right)}, \delta \in (0,\eta). \tag{1.3}$$

This estimate gives a criterion for sample path continuity of $\hat{g}$-Gaussian processes:

Example 1.2. Let X be a centered $\hat{g}$-Gaussian process on K a compact subset of $\mathbb{R}^d$ such that

$$g(\tau) \le h(\tau) := \left[\log\left(\frac{2\varnothing_K}{\tau}\right)\right]^{-\nu} \tag{1.4}$$

for some $\nu > 1/2$. We note that $h^{-1}(\tau) = 2\varnothing_K \exp(-\tau^{-\frac{1}{\nu}})$, thus we can assert that for any $\varepsilon > 0$ small enough

$$\log\left(\frac{\varnothing_K}{h^{-1}(\varepsilon)}\right) \le \varepsilon^{-\frac{1}{\nu}}.$$

This implies that

$$\int_0^\delta d\varepsilon\sqrt{\log\left(\frac{\varnothing_K}{h(\varepsilon)}\right)} \le \int_0^\delta d\varepsilon^{-\frac{1}{2\nu}} = \left(1-\frac{1}{2\nu}\right)^{-1}\delta^{1-\frac{1}{2\nu}} < \infty.$$

Then (1.3) implies that X has a modification with a.s. continuous sample paths.

Remark 1.2. The continuity criterion in Example 1.2 is sharp. According to Corollary 1.5.5 in [AT07], if X is a centered, stationary $\hat{g}$-Gaussian process with

$$g(\tau) = \left[\log\left(\frac{2\varnothing_K}{\tau}\right)\right]^{-\nu},$$

for some $\nu \in (0, 1/2)$, then X has discontinuous sample paths.

1.2 Criteria for hitting probabilities

Let K be a compact subset of $\mathbb{R}^d$ and

$$\boldsymbol{X} = \{\boldsymbol{X}(x) = (X_1(x), \ldots, X_D(x)),\ x \in K\},$$

be a D-dimensional process with i.i.d coordinates. The probability that the process $\boldsymbol{X}$ hits A a Borel subset of $\mathbb{R}^D$ is

$$P(\boldsymbol{X}(K) \cap A \neq \emptyset). \tag{1.5}$$

We devote this section to estimate (1.5) when X_1 is a g-Gaussian process. The main results are Theorems 1.2 and 1.3 which yield to upper and lower bounds in terms of the notions of q-Hausdorff measure and $\mathfrak{g}$-capacity, respectively.

1.2.1 Upper bounds for hitting probabilities

The aim of this subsection is to prove extensions of Theorem 2.6 in [DSS10] on sufficient conditions for upper bound estimates of hitting probabilities of Gaussian processes.

We start with a technical lemma which is a generalized version of [DSS10, Lem. 2.5].

Lemma 1.1. *[HCSS21, Lem. 3.1] Let g be a differentiable gauge function. Fix $x_0 \in \mathbb{R}^d$ and let*

$$\boldsymbol{M} = \{\boldsymbol{M}(x) = (M_1(x), \ldots, M_D(x)),\ x \in B_1(x_0)\},$$

be a D-dimensional stochastic process with a.s. continuous sample paths. Suppose that for all $\varepsilon > 0$ small enough,

$$E\left(\int_{B_\varepsilon(x_0)} dy \int_{B_\varepsilon(x_0)} d\bar{y}\ \exp\left(\frac{|\boldsymbol{M}(y) - \boldsymbol{M}(\bar{y})|}{g(|y - \bar{y}|)}\right)\right) \leq c_{1,1}\varepsilon^{2d}, \tag{1.6}$$

for some constant $c_{1,1}$. Set $S_\varepsilon(x_0) = B_{\frac{g^{-1}(\varepsilon)}{2}}(x_0)$. Then, the following statements hold.

1. *For all $p \geq 1$, there exist a constant $c_{1,2}$ depending on d and $c_{1,1}$ such that for $\varepsilon > 0$ small enough,*

$$E\left(\sup_{y \in S_\varepsilon(x_0)} |\boldsymbol{M}(y) - \boldsymbol{M}(x_0)|^p\right) \leq 10^p\ \varepsilon^{p-1} g^{-1}(\varepsilon) \int_0^1 \log^p\left(1 + \frac{c_{1,2}}{\tau^{2d}}\right) \dot{g}\left(g^{-1}(\varepsilon)\tau\right)\ d\tau, \tag{1.7}$$

where $\dot{g}$ denotes the derivative of g.

2. Assume that g is such that, for any $\rho, \tau \in [0, \tau_0]$, with $\tau_0 > 0$ sufficiently small,

$$g(\rho\tau) \le \varphi(\tau) g(\rho), \quad \dot{g}(\rho\tau) \le \frac{1}{\rho}\psi(\tau) g(\rho\tau), \tag{1.8}$$

where φ and ψ are Borel functions such that, denoting $\Phi(\tau) = \varphi(\tau)\psi(\tau)$, we have

$$\int_0^1 \log^p\left(1 + \frac{c_{1,2}}{\tau^{2d}}\right) \Phi(\tau)\ d\tau < \infty. \tag{1.9}$$

Then, for all $p \ge 1$, there exists a constant $c_{1,3}(c_{1,1}, p, d, \Phi)$ such that for all $\varepsilon > 0$ small enough,

$$E\Bigg(\sup_{y \in S_\varepsilon(x_0)} |\boldsymbol{M}(y) - \boldsymbol{M}(x_0)|^p\Bigg) \le c_{1,3}\varepsilon^p. \tag{1.10}$$

Proof. 1. Let

$$\mathcal{C}_\varepsilon(\omega) = \int_{S_\varepsilon(x_0)} dy \int_{S_\varepsilon(x_0)} d\bar{y}\ \exp\left(\frac{|\boldsymbol{M}(y,\omega) - \boldsymbol{M}(\bar{y},\omega)|}{g(|y-\bar{y}|)}\right). \tag{1.11}$$

From (1.6), we deduce $\mathcal{C}_\varepsilon(\omega) < \infty$, a.s. Notice that for almost all ω,

$$\mathcal{C}_\varepsilon(\omega) \ge |S_\varepsilon(x_0)|^2 \ge C_2\left(g^{-1}(\varepsilon)\right)^{2d}, \tag{1.12}$$

for some constant $C_2 > 0$ depending on d.

Applying [DKN07, Prop. A.1, (A.3)] to $S := S_\varepsilon(x_0)$ endowed with the Euclidean distance ρ, μ the Lebesgue measure, $\Psi(\tau) := e^\tau - 1$ and $p(\tau) := g(\tau)$, implies that for any $\delta > 0$

$$\sup_{y,\bar{y} \in S_\varepsilon(x_0), |y-\bar{y}| \le \delta} |\boldsymbol{M}(y) - \boldsymbol{M}(\bar{y})| \le 10 \int_0^{2\delta} \Psi^{-1}\left(\frac{C_1\mathcal{C}_\varepsilon(\omega)}{\tau^{2d}}\right) \dot{g}(\tau)\ d\tau,$$

with C_1 depending on d. Here, we have used that the volume of the d-dimensional Euclidean ball of radius r equals a multiple constant times r^d. By writing $\bar{y} = x_0$ and $\delta = \frac{g^{-1}(\varepsilon)}{2}$ on the last inequality, we deduce

$$\sup_{y \in S_\varepsilon(x_0)} |\boldsymbol{M}(y) - \boldsymbol{M}(x_0)| \le 10 \int_0^{g^{-1}(\varepsilon)} \Psi^{-1}\left(\frac{C_1\mathcal{C}_\varepsilon(\omega)}{\tau^{2d}}\right) \dot{g}(\tau)\ d\tau,$$

Therefore, for any $p \ge 1$,

$$\begin{aligned}
E\left(\sup_{y \in S_\varepsilon(x_0)} |\boldsymbol{M}(y) - \boldsymbol{M}(x_0)|^p\right) &\le 10^p E\left(\left|\int_0^{g^{-1}(\varepsilon)} \Psi^{-1}\left(\frac{C_1\mathcal{C}_\varepsilon(\omega)}{\tau^{2d}}\right) \dot{g}(\tau)\ d\tau\right|^p\right) \\
&\le 10^p \left(g\left(g^{-1}(\varepsilon)\right)\right)^{p-1} E\left(\int_0^{g^{-1}(\varepsilon)} \log^p\left(1 + \frac{C_1\mathcal{C}_\varepsilon(\omega)}{\tau^{2d}}\right) \dot{g}(\tau)\ d\tau\right) \\
&= 10^p \varepsilon^{p-1} \int_0^{g^{-1}(\varepsilon)} E\left[\log^p\left(1 + \frac{C_1\mathcal{C}_\varepsilon(\omega)}{\tau^{2d}}\right)\right] \dot{g}(\tau)\ d\tau, \qquad (1.13)
\end{aligned}$$

where in the second inequality, we have applied Hölder's inequality with respect to the measure $\dot{g}(\tau)d\tau$. Observe that we may take C_1 as large as we want.

The function $x \mapsto \log^p(1+x)$ is concave on $[e^{p-1}-1,\infty)$. By (1.12) $\mathcal{C}_\varepsilon(\omega) \geq C_2\tau^{2d}, \tau \in [0, g^{-1}(\varepsilon)]$ a.s. Hence, by taking $C_1 \geq (e^{p-1}-1)C_2^{-1}$, we can apply Jensen's inequality and (1.6) to estimate

$$E\left[\log^p\left(1+\frac{C_1\mathcal{C}_\varepsilon(\omega)}{\tau^{2d}}\right)\right] \leq \log^p\left(1+c_{1,2}\left(\frac{g^{-1}(\varepsilon)}{\tau}\right)^{2d}\right),$$

with some constant $c_{1,2}(c_{1,1},d)$. Use the last inequality on the right-hand side of (1.13), and write the change of variables $\tau \mapsto \left(g^{-1}(\varepsilon)\right)^{-1}\tau$, to obtain,

$$E\left(\sup_{y\in S_\varepsilon(x_0)} |\boldsymbol{M}(y)-\boldsymbol{M}(x_0)|^p\right) \leq 10^p\ \varepsilon^{p-1}g^{-1}(\varepsilon)\int_0^1 \log^p\left(1+\frac{c_{1,2}}{\tau^{2d}}\right)\dot{g}\left(g^{-1}(\varepsilon)\tau\right)\,d\tau. \tag{1.14}$$

This ends the proof of (1.7).

2. The conditions (1.8) imply $\rho\dot{g}(\rho\tau) \leq \Phi(\tau)g(\rho)$. For $\rho := g^{-1}(\varepsilon)$ this yields

$$g^{-1}(\varepsilon)\dot{g}(g^{-1}(\varepsilon)\tau) \leq \Phi(\tau)\varepsilon.$$

Thus, the right-hand side of (1.7) is less or equal to $(10\varepsilon)^p\int_0^1 \log^p\left(1+\frac{c_{1,2}}{\tau^{2d}}\right)\Phi(\tau)\,d\tau$ and therefore, assuming (1.9), we obtain (1.10).

□

Remark 1.3. Fix x_0 in $\mathbb{R}^d$ and assume that X is a $\hat{g}$-Gaussian centered process on $B_1(x_0)$. Then X satisfies the condition (1.6). Indeed, the upper bound in (1.1) implies that for $x,y \in B_\varepsilon(x_0)$

$$\exp\left(\frac{|X(x)-X(y)|}{g(|x-y|)}\right) \leq \exp\left(\frac{|X(x)-X(y)|}{C\mathfrak{d}(x,y)}\right) = \exp(c|Z|),$$

where $c = 1/C$ and Z is a standard Gaussian random variable. Since, $E([\exp(c|Z|)])$ is finite, by Fubini's theorem (1.6) holds.

Example 1.3. We verify conditions (1.8) and (1.9) of Lemma 1.1 for the gauge functions in Example 1.1.

1. $g(\tau) = \tau^\nu, \tau > 0, \nu > 0$. (1.8) hold for any $\rho,\tau > 0$, with $\varphi(\tau) = \tau^\nu$, $\psi(\tau) = \frac{\nu}{\tau}$. Since $\int_0^1 \log^p\left(1+\frac{C}{\tau^{2d}}\right)\tau^{\nu-1}d\tau < \infty$, for any $C > 0$ and $p \geq 1$, (1.9) holds.

2. $g(\tau) = |\log\tau|^\gamma\tau^\nu, \tau \in [0, e^{-\frac{\gamma}{\nu}}]$, $\nu,\gamma > 0$. Then,

$$\begin{aligned} g(\rho\tau) = \rho^\nu\tau^\nu|\log\rho\tau|^\gamma &\leq (|\log\rho| + |\log\tau|)^\gamma\,\rho^\nu\tau^\nu \\ &\leq C(\gamma,\nu)|\log\rho|^\gamma\,(1+|\log\tau|)^\gamma\,\rho^\nu\tau^\nu, \rho,\tau \in [0, e^{-\frac{\gamma}{\nu}}]. \end{aligned}$$

Hence,

$$g(\rho\tau) \leq \varphi(\tau)g(\rho), \text{ with } \varphi(\tau) = C(\gamma,\nu)\,(1+|\log\tau|)^\gamma\,\tau^\nu.$$

The derivative of g is $\dot{g}(\tau) = |\log\tau|^{\gamma-1}(\nu|\log\tau|-\gamma)\tau^{\nu-1}$, and it follows that g is increasing on $[0, e^{-\frac{\gamma}{\nu}}]$. Therefore $\dot{g}(\tau) \leq \nu|\log\tau|^\gamma\tau^{\nu-1} \leq \frac{\nu}{\tau}g(\tau)$. Consequently,

$$\dot{g}(\rho\tau) \leq \frac{1}{\rho}\psi(\tau)g(\rho\tau), \text{ with } \psi(\tau) = \frac{\nu}{\tau}.$$

Since $\Phi(\tau) = C(\gamma,\nu)(1+|\log\tau|)^\gamma\tau^{\nu-1}$, we see that condition (1.9) holds.

For any $\varepsilon \in (0,1)$, $j \in \mathbb{Z}^d$, $j = (j_1, \dots, j_d)$, set

$$R_j^\varepsilon = \prod_{i=1}^{d} \left[\frac{g^{-1}(\varepsilon)}{\sqrt{d}} j_i, \frac{g^{-1}(\varepsilon)}{\sqrt{d}} (j_i + 1) \right], \tag{1.15}$$

and for $x \in R_j^\varepsilon$, define $x_j^\varepsilon := \left(\frac{g^{-1}(\varepsilon)}{\sqrt{d}} j_i \right)_{i=1,\dots,d}$. Observe that $\operatorname{diam}(R_j^\varepsilon) = g^{-1}(\varepsilon)$ and $R_j^\varepsilon \subset B_{\frac{g^{-1}(\varepsilon)}{2}}(\bar{x}_j^\varepsilon)$, where $\bar{x}_j^\varepsilon = \left(\frac{g^{-1}(\varepsilon)}{\sqrt{d}} (j_i + \frac{1}{2}) \right)_{i=1,\dots,d}$. Moreover, by the triangle inequality,

$$\sup_{x \in R_j^\varepsilon} \left(\left| \boldsymbol{M}(x) - \boldsymbol{M}(x_j^\varepsilon) \right| \right) \leq 2 \sup_{x \in B_{\frac{g^{-1}(\varepsilon)}{2}}(\bar{x}_j^\varepsilon)} \left(\left| \boldsymbol{M}(x) - \boldsymbol{M}(\bar{x}_j^\varepsilon) \right| \right). \tag{1.16}$$

The next statement provides an extension of [DSS10, Thm. 2.6, (26)] to non necessarily centered Gaussian processes.

Theorem 1.1. *[HCSS21, Thm. 3.1] Fix K a compact set of $\mathbb{R}^d$ and $\eta > 0$. Let*

$$\boldsymbol{X} = \{ \boldsymbol{X}(x) = (X_1(x), ..., X_D(x)), x \in K^{(\eta)} \} \tag{1.17}$$

be a D-dimensional stochastic process with a.s. continuous sample paths and i.i.d. coordinates that are distributed as Gaussian random fields. Fix $\varepsilon > 0$ small enough, $j \in \mathbb{Z}^d$, and let R_j^ε be as in (1.15). Assume that $R_j^\varepsilon \subset K^{(\eta)}$ and let $\sigma^2_{K^{(\eta)}} := \inf_{x \in K^{(\eta)}} \sigma_x^2 > 0$.

1. Let $\boldsymbol{m}_x = E(\boldsymbol{X}(x))$ and $\tilde{\boldsymbol{X}}(x) = \boldsymbol{X}(x) - \boldsymbol{m}_x$. We assume that for some constant $C(d, D)$,

$$E \left(\sup_{x \in R_j^\varepsilon} \left| \tilde{\boldsymbol{X}}(x) - \tilde{\boldsymbol{X}}(x_j^\varepsilon) \right|^2 \right) \leq C(d, D) \varepsilon^2. \tag{1.18}$$

Then there exists a constant $C(\sigma^2_{K^{(\eta)}}, d, D)$ such that, for every $z \in \mathbb{R}^D$,

$$P \left(\boldsymbol{X}(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset \right) \leq C(\sigma^2_{K^{(\eta)}}, d, D) \varepsilon^D. \tag{1.19}$$

2. Suppose that for some constant $\bar{C}(d, D)$

$$E \left(\sup_{x \in R_j^\varepsilon} \left| \boldsymbol{X}(x) - \boldsymbol{X}(x_j^\varepsilon) \right|^2 \right) \leq \bar{C}(d, D) \varepsilon^2. \tag{1.20}$$

Then there exists a constant $\bar{C}(\sigma^2_{K^{(\eta)}}, d, D)$ such that, for every $z \in \mathbb{R}^D$,

$$P \left(\boldsymbol{X}(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset \right) \leq \bar{C}(\sigma^2_{K^{(\eta)}}, d, D) \varepsilon^D. \tag{1.21}$$

Proof. 1. We follow the approach of [DSS10, Thm. 2.6] with some modifications due to the fact that the process $\boldsymbol{X}$ is not centered.

Because $\boldsymbol{X}$ is continuous, for any $z \in \mathbb{R}^D$

$$P \left(\boldsymbol{X}(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset \right) = P \left(\inf_{x \in R_j^\varepsilon} |\boldsymbol{X}(x) - z| \leq \varepsilon \right).$$

Assume we can prove that there exists a constant $c(\sigma^2_{K(\eta)}, d, D)$ such that for any $z_1 \in \mathbb{R}$,

$$P\left(\inf_{x\in R^\varepsilon_j} |X_1(x) - z_1| \leq \varepsilon\right) \leq c(\sigma^2_{K(\eta)}, d, D)\ \varepsilon. \tag{1.22}$$

Observe that

$$\left\{\inf_{x\in R^\varepsilon_j} |\boldsymbol{X}(x) - z| \leq \varepsilon\right\} \subset \bigcap_{i=1}^{D} \left\{\inf_{x\in R^\varepsilon_j} |X_i(x) - z| \leq \varepsilon\right\}.$$

Then, because the components of $\boldsymbol{X}(x)$ are i.i.d, (1.22) yields (1.19) with $C = [c(\sigma^2_{K(\eta)}, d, D)]^D$.

For the proof of (1.22), we fix $x \in R^\varepsilon_j$ and since X_1 is a Gaussian processes, its conditional expectation es given by

$$E\left(X_1(x)|\tilde{X}_1(x^\varepsilon_j)\right) = m_x + E\left(\tilde{X}_1(x)|\tilde{X}_1(x^\varepsilon_j)\right) = m_x + c^\varepsilon_j(x)\tilde{X}_1(x^\varepsilon_j), \tag{1.23}$$

where

$$m_x = E(X_1(x)), \quad c^\varepsilon_j(x) = \frac{\operatorname{Cov}\left(\tilde{X}_1(x), \tilde{X}_1(x^\varepsilon_j)\right)}{\operatorname{Var}\left(\tilde{X}_1(x^\varepsilon_j)\right)}.$$

Define

$$Y^\varepsilon_j = \inf_{x\in R^\varepsilon_j} \left|E\left(X_1(x)|\tilde{X}_1(x^\varepsilon_j)\right) - z_1\right|, \quad Z^\varepsilon_j = \sup_{x\in R^\varepsilon_j} \left|X_1(x) - E\left(X_1(x)|\tilde{X}_1(x^\varepsilon_j)\right)\right|.$$

We claim that these are independent random variables satisfying

$$P\left(\inf_{x\in R^\varepsilon_j} |X_1(x) - z_1| \leq \varepsilon\right) \leq P\left(Y^\varepsilon_j \leq \varepsilon + Z^\varepsilon_j\right). \tag{1.24}$$

Indeed, let

$$\begin{aligned}
\tilde{Y}^\varepsilon_j &:= \left\{\tilde{Y}^\varepsilon_j(x) = E\left(X_1(x)|\tilde{X}_1(x^\varepsilon_j)\right) - z_1, x \in R^\varepsilon_j\right\},\\
\tilde{Z}^\varepsilon_j &:= \left\{\tilde{Z}^\varepsilon_j(x) = X_1(x) - E\left(X_1(x)|\tilde{X}_1(x^\varepsilon_j)\right), x \in R^\varepsilon_j\right\}.
\end{aligned}$$

By (1.23), $\tilde{Y}^\varepsilon_j$ and $\tilde{Z}^\varepsilon_j$ are Gaussian processes that for any $x, \bar{x} \in R^\varepsilon_j$,

$$\begin{aligned}
E(\tilde{Z}^\varepsilon_j(x)) &= 0,\\
E(\tilde{Y}^\varepsilon_j(x)\tilde{Z}^\varepsilon_j(\bar{x})) &= E\left(\left[c^\varepsilon_j(x)\tilde{X}_1(x^\varepsilon_j) + m_x - z_1\right]\left[\tilde{X}_1(\bar{x}) - c^\varepsilon_j(\bar{x})\tilde{X}_1(x^\varepsilon_j)\right]\right)\\
&= c^\varepsilon_j(x)E\left(\tilde{X}_1(x^\varepsilon_j)\left[\tilde{X}_1(\bar{x}) - c^\varepsilon_j(\bar{x})\tilde{X}_1(x^\varepsilon_j)\right]\right) = 0.
\end{aligned}$$

Thus Y^ε_j and Z^ε_j are independent. Notice that $\tilde{Y}^\varepsilon_j - \tilde{Z}^\varepsilon_j = X_1 - z_1$ implying (1.24) and finishing the proof of the claim

We next prove that, for any $r \geq 0$,

$$P(Y^\varepsilon_j \leq r) \leq C(\sigma^2_{K(\eta)}, d, D)r. \tag{1.25}$$

As an auxiliary result, we first check that for all $\varepsilon > 0$ and $x \in R_j^\varepsilon$,

$$|c_j^\varepsilon(x) - 1| \le C(\sigma^2_{K(\eta)}, d, D)\varepsilon, \tag{1.26}$$

implying that, for all $\varepsilon > 0$ small enough, say $\varepsilon \le [2C(\sigma^2_{K(\eta)}, d, D)]^{-1}$, and for all $x \in R_j^\varepsilon$, we have

$$c_j^\varepsilon(x) \ge \frac{1}{2}. \tag{1.27}$$

Indeed, because $\mathrm{Var}(\tilde{X}_1(x_j^\varepsilon)) \ge \sigma^2_{K(\eta)} > 0$, using Cauchy-Schwartz inequality and (1.18), we deduce

$$\left|c_j^\varepsilon(x) - 1\right| \le \left(\frac{E\left[\tilde{X}_1(x_j^\varepsilon) - \tilde{X}_1(x)\right]^2}{\mathrm{Var}(\tilde{X}_1(x_j^\varepsilon))} \right)^{\frac{1}{2}} \le C(\sigma^2_{K(\eta)}, d, D)\varepsilon.$$

We continue with the proof of (1.25). By (1.23),

$$\{Y_j^\varepsilon \le r\} = \left\{ \inf_{x \in R_j^\varepsilon} |m_x + E(\tilde{X}_1(x)|\tilde{X}_1(x_j^\varepsilon)) - z_1| \le r \right\}$$

and the inequality $\left|m_x + E\left(\tilde{X}_1(x)|\tilde{X}_1(x_j^\varepsilon)\right) - z_1\right| \le r$ is equivalent to

$$\frac{z_1 - m_x}{c_j^\varepsilon(x)} - \frac{r}{c_j^\varepsilon(x)} \le \tilde{X}_1(x_j^\varepsilon) \le \frac{z_1 - m_x}{c_j^\varepsilon(x)} + \frac{r}{c_j^\varepsilon(x)}.$$

Since by (1.27), $\inf_{x \in R_j^\varepsilon} c_j^\varepsilon(x) \ge \frac{1}{2}$, the above remarks yield

$$P(Y_j^\varepsilon \le r) \le \sup_{s \in \mathbb{R}} P\left(s - 2r \le \tilde{X}_1(x_j^\varepsilon) \le s + 2r\right) = \sup_{s \in \mathbb{R}} P\left(\tilde{X}_1(x_j^\varepsilon) \in B_{2r}(s)\right). \tag{1.28}$$

Because the density of $\tilde{X}_1(x_j^\varepsilon)$ is bounded by $(\mathrm{Var}(X_1(x_j^\varepsilon))2\pi)^{-1/2} \le C(\sigma^2_{K(\eta)}, d, D)$, we have

$$P(Y_j^\varepsilon \le r) \le \sup_{s \in \mathbb{R}} P\left(\tilde{X}_1(x_j^\varepsilon) \in B_{2r}(s)\right) \le C(\sigma^2_{K(\eta)}, d, D)\ r.$$

This proves (1.25).

We now address the last step in the proof of (1.22). Let $\mu_{Z_j^\varepsilon}$ be the distribution function of Z_j^ε, because Y_j^ε and Z_j^ε are independent, by (1.25) we obtain,

$$\begin{aligned} P\left(Y_j^\varepsilon \le \varepsilon + Z_j^\varepsilon\right) &= \int_{\mathbb{R}} \mu_{Z_j^\varepsilon}(dz) P\left(Y_j^\varepsilon \le \varepsilon + z\right) \\ &\le C(\sigma^2_{K(\eta)}, d, D) \int_{\mathbb{R}} \mu_{Z_j^\varepsilon}(dz)(\varepsilon + z) \\ &= C(\sigma^2_{K(\eta)}, d, D)\left[\varepsilon + E\left(Z_j^\varepsilon\right)\right]. \end{aligned}$$

(1.24) together with the last inequality, implies that

$$P\left(\inf_{x \in R_j^\varepsilon} |X_1(x) - z| \le \varepsilon\right) \le c(\sigma^2_{K(\eta)}, d, D)\left[\varepsilon + E\left(Z_j^\varepsilon\right)\right]. \tag{1.29}$$

Since $X_1(x) - E\left(X_1(x)|\tilde{X}_1(x_j^\varepsilon)\right) = \tilde{X}_1(x) - c_j^\varepsilon(x)\tilde{X}_1(x_j^\varepsilon)$ (see (1.23)), by the triangle inequality $Z_j^\varepsilon \le Z_{j,1}^\varepsilon + Z_{j,2}^\varepsilon$, with

$$Z_{j,1}^\varepsilon = \sup_{x\in R_j^\varepsilon} \left|\tilde{X}_1(x) - \tilde{X}_1(x_j^\varepsilon)\right|, \quad Z_{j,2}^\varepsilon = \sup_{x\in R_j^\varepsilon} \left|1 - c_j^\varepsilon(x)\right| \left|\tilde{X}_1(x_j^\varepsilon)\right|.$$

Apply (1.18) and Jensen's inequality to obtain $E(Z_{j,1}^\varepsilon) \le C(d, D)\varepsilon$. Also, as a consequence of (1.26), we have that $E(Z_{j,2}^\varepsilon) \le C(\sigma^2_{K(\eta)}, d, D)\varepsilon$. This yields $E\left(Z_j^\varepsilon\right) \le C(\sigma^2_{K(\eta)}, d, D)\varepsilon$. Along with (1.29), this implies (1.22) and, as was argued above, the proof of claim 1 is complete.

2. By going trough the proof of (1.19) one can check that hypothesis (1.18) was only used for proving the inequalities

$$E\left[\tilde{X}_1(x_j^\varepsilon) - \tilde{X}_1(x)\right]^2 \le C(d, D)\varepsilon^2,$$

and $E(Z_{j,1}^\varepsilon) \le C(d, D)\varepsilon$. We claim that those inequalities are valid if we assume (1.20) instead of (1.18). This implies (1.21).

Indeed, the first inequality is a direct consequence of (1.20), since

$$\begin{aligned} E\left[\tilde{X}_1(x_j^\varepsilon) - \tilde{X}_1(x)\right]^2 &= E\left[X_1(x_j^\varepsilon) - m_{x_j^\varepsilon} - (X_1(x) - m_x)\right]^2 \\ &= E\left[X_1(x_j^\varepsilon) - X_1(x)\right]^2 - (m_{x_j^\varepsilon} - m_x)^2 \\ &\le E\left[X_1(x_j^\varepsilon) - X_1(x)\right]^2, \end{aligned}$$

where m_x denotes the first coordinate of the vector $\boldsymbol{m}_x$. A similar argument together with Jensen's inequality implies that $E(Z_{j,1}^\varepsilon) \le C(d, D)\varepsilon$.

□

Definition 1.3. Let $q : \mathbb{R}_+ \to \mathbb{R}_+$ be monotone increasing and right-continuous. Assume that on a small non empty interval $[0, \varepsilon_0]$, q is strictly increasing. The *q-Hausdorff measure* of a Borel set $A \subset \mathbb{R}^D$ is defined by

$$\mathcal{H}_q(A) = \liminf_{\varepsilon\downarrow 0} \left\{\sum_{i=1}^\infty q(2r_i) : A \subset \bigcup_{i=1}^\infty B_{r_i}(x_i),\ \sup_{i\ge 1} r_i \le \varepsilon\right\} \tag{1.30}$$

(see e.g. [Rog98]). In the particular case $q(\tau) = \tau^\gamma$, with $\gamma > 0$, $\mathcal{H}_q(A)$ is the γ-dimensional Hausdorff measure, usually denoted by $\mathcal{H}_\gamma(A)$.

For a gauge function g define

$$q_g(\tau) = \frac{\tau^D}{\left(g^{-1}(\tau)\right)^d}, \quad \tau \in \mathbb{R}_+. \tag{1.31}$$

Fix $\rho_0 > 0$, and assume that g is differentiable in $(0, \rho_0)$. It is not hard to proof that q_g is strictly increasing on $(0, \rho_0)$ if and only if

$$D > d\, \frac{\tau}{g^{-1}(\tau)\dot{g}(g^{-1}(\tau))}, \quad \tau \in (0, \rho_0) \tag{1.32}$$

or equivalently

$$D > d\, \frac{g(\tau)}{\tau \dot{g}(\tau)}, \quad \tau \in (0, g^{-1}(\rho_0)), \tag{1.33}$$

this means that if (1.32) is satisfied, $\mathcal{H}_{q_g}$ is a well defined Hausdorff measure. The q_g-Haussdorf measure will naturally arise in the study of upper bounds for hitting probabilities for g-Gaussian random fields.

Example 1.4. We study the map q_g in the case of the gauge functions in Example 1.1.

1. $g(\tau) = \tau^\nu, \tau, \nu > 0$. We have that $q_g(\tau) = \tau^{D-d/\nu}$. Thus q_g is strictly increasing on $\mathbb{R}^+$ if and only if $D > d/\nu$. In this case we recover the γ-dimensional Hausdorff measure $\mathcal{H}_\gamma$ with $\gamma = D - d/\nu$.

2. $g(\tau) = |\log \tau|^\gamma \tau^\nu, \tau \in [0, e^{-\frac{\gamma}{\nu}}]$, $\gamma, \nu > 0$. The inverse function of g satisfies

$$\tau = [g^{-1}(\tau)]^\nu |\log g^{-1}(\tau)|^\gamma, \tau \in \left[0, \left(\frac{\gamma}{e\nu}\right)^\gamma\right],$$

or equivalently

$$-\frac{\nu}{\gamma}\tau^{-1/\gamma} = -\frac{\nu}{\gamma}\frac{\tau^{1/\gamma}}{[g^{-1}(\tau)]^{\frac{\nu}{\gamma}}} \exp\left[-\frac{\nu}{\gamma}\frac{\tau^{1/\gamma}}{[g^{-1}(\tau)]^{\frac{\nu}{\gamma}}}\right], \tau \in \left[0, \left(\frac{\nu}{e\gamma}\right)^\gamma\right]. \qquad (1.34)$$

Let W_{-1}, be the lower real valued branch of the Lambert W function, defined as the only solution of the following inequalities system

$$W_{-1}(\tau)e^{W_{-1}(\tau)} = \tau, \quad W_{-1}(\tau) \leq -1, \tau \in [-e^{-1}, 0]. \qquad (1.35)$$

(See [Cha13] for example). The last equality implies that

$$W_{-1}(\tau e^\tau) = \tau, \quad \tau \leq 1. \qquad (1.36)$$

By (1.35), (1.34) and (1.36),

$$-\frac{\nu}{\gamma}\tau^{1/\gamma} \exp\left[-W_{-1}\left(-\frac{\nu}{\gamma}\tau^{\frac{1}{\gamma}}\right)\right] = W_{-1}\left(-\frac{\nu}{\gamma}\tau^{\frac{1}{\gamma}}\right) = -\frac{\nu}{\gamma}\frac{\tau^{1/\gamma}}{[g^{-1}(\tau)]^{\frac{\nu}{\gamma}}}.$$

Clear away g^{-1} in the last equation to deduce that the inverse function g^{-1} is

$$g^{-1}(\tau) = \exp\left[\frac{\gamma}{\nu}W_{-1}\left(-\frac{\nu}{\gamma}\tau^{\frac{1}{\gamma}}\right)\right], \tau \in \left[0, e^{-\gamma}\left(\frac{\gamma}{\nu}\right)^\gamma\right]. \qquad (1.37)$$

According to [Cha13, Thm. 1],

$$-1 - \sqrt{2\tau} - \tau < W_{-1}(-e^{-\tau-1}) < -1 - \sqrt{2\tau} - \frac{2}{3}\tau, \quad \tau \in [-e^{-1}, 0].$$

Applying this result, we see that

$$g^{-1}(\tau) \asymp \tau^{\frac{1}{\nu}} \exp\left(-\frac{\gamma}{\nu}\left|2\log\left(c\tau^{\frac{1}{\gamma}}\right)\right|^{\frac{1}{2}}\right), \qquad (1.38)$$

where c is a constant depending on γ, ν. Consequently,

$$q_g(\tau) \asymp \tau^{D-\frac{d}{\nu}} \exp\left(\frac{d\gamma}{\nu}\left|2\log\left(c_1\tau^{\frac{1}{\gamma}}\right)\right|^{\frac{1}{2}}\right). \qquad (1.39)$$

Thus q_g is strictly increasing on an interval around zero if and only if $D > d/\nu$.

Theorem 1.2. *[HCSS21, Thm. 3.2] Consider a Gaussian process with continuous sample paths* $\boldsymbol{X}$ *as in* (1.17) *and such that* $\sigma^2_{K^{(\eta)}} := \inf_{x\in K^{(\eta)}} \sigma^2_x > 0$. *Let* g *be a gauge function and assume that the process* $\boldsymbol{X}$ *satisfies the condition* (1.6) *for any* $x \in K^{(\eta)}$. *Suppose also that* g *fulfills the hypothesis of Lemma 1.1(2.) and that the function* q_g *given in* (1.31) *is strictly increasing on a small interval* $(0, \rho_0)$. *Then there exists a constant* $C(K, \sigma^2_{K^{(\eta)}}, d, D)$ *such that for any Borel set* $A \subset \mathbb{R}^D$,

$$P\left(\boldsymbol{X}(K) \cap A \neq \emptyset\right) \leq C(K, \sigma^2_{K^{(\eta)}}, d, D)\ \mathcal{H}_{q_g}(A). \tag{1.40}$$

Proof. Let q_g be the function defined in (1.31) and assume that on a sufficiently small interval $(0, \rho_0)$, q_g is strictly increasing. Fix ε small enough. By the definition of the Hausdorff q_g-measure $\mathcal{H}_{q_g}(A)$, there exists a sequence of balls $(B_i, i \geq 1)$ with radii $r_i \in (0, \varepsilon)$, such that $B_i \cap A \neq \emptyset$, $A \subset \cup_{i\geq 1} B_i$, and $\sum_{i\geq 1} q_g(2r_i) \leq \mathcal{H}_{q_g}(A) + \varepsilon$. Then from (1.43) in Lemma 1.2 bellow, we deduce that for any Borel set $A \subset \mathbb{R}^D$,

$$\begin{aligned} P\left(\boldsymbol{X}(K) \cap A \neq \emptyset\right) &\leq \sum_{i\geq 1} P\left(\boldsymbol{X}(K) \cap B_i \neq \emptyset\right) \\ &\leq C(K, \sigma^2_{K^{(\eta)}}, d, D) \sum_{i\geq 1} q_g(2r_i) \leq \mathcal{H}_{q_g}(A) + \varepsilon. \end{aligned} \tag{1.41}$$

Letting ε tend to zero, we obtain

$$P\left(\boldsymbol{X}(K) \cap A \neq \emptyset\right) \leq C(K, \sigma^2_{K^{(\eta)}}, d, D)\ \mathcal{H}_{q_g}(A).$$

□

Remark 1.3 implies the following Corollary relative for upper bounds of hitting probabilities for $\hat{g}$-Gaussian processes.

Corollary 1.1. *Consider a Gaussian process with continuous sample paths* $\boldsymbol{X}$ *as in* (1.17), *and such that* $\sigma^2_{K^{(\eta)}} := \inf_{x\in K^{(\eta)}} \sigma^2_x > 0$. *Assume that* X_1 *is* $\hat{g}$*-Gaussian on* $K^{(\eta)}$ *with* g *a gauge function satisfying the hypothesis of Lemma 1.1(2.) and such that the function* q_g *given in* (1.31) *is strictly increasing on a small interval* $(0, \rho_0)$. *Then there exists a constant* $C(K, \sigma^2_{K^{(\eta)}}, d, D)$ *such that for any Borel set* $A \subset \mathbb{R}^D$,

$$P\left(\boldsymbol{X}(K) \cap A \neq \emptyset\right) \leq C(K, \sigma^2_{K^{(\eta)}}, d, D)\ \mathcal{H}_{q_g}(A). \tag{1.42}$$

The following Lemma which derive upper bounds for hitting probabilities of small balls in terms of the function q_g, was used in the proof of Theorem 1.2.

Lemma 1.2. *[HCSS21, Lem. 3.2] Fix* $z \in \mathbb{R}^D$ *and* $\varepsilon > 0$. *Consider a Gaussian process with continuous sample paths* $\boldsymbol{X}$ *as in* (1.17) *and such that* $\sigma^2_{K^{(\eta)}} := \inf_{x\in K^{(\eta)}} \sigma^2_x > 0$. *Let* g *be a gauge function and assume that the process* $\boldsymbol{X}$ *satisfies the condition* (1.6) *for any* $x \in K^{(\eta)}$. *Suppose also that* g *fulfills the hypothesis of Lemma 1.1(2.).*

Then, there exists a constant $C(K, \sigma^2_{K^{(\eta)}}, d, D)$ *such that,*

$$P\left(\boldsymbol{X}(K) \cap B_\varepsilon(z) \neq \emptyset\right) \leq C(K, \sigma^2_{K^{(\eta)}}, d, D) q_g(\varepsilon). \tag{1.43}$$

Proof. Since K is compact, there is a finite number of sets R^ε_j (defined in (1.15)) satisfying $K \cap R^\varepsilon_j \neq \emptyset$; this number is a constant (depending on the dimension d

and K) multiple of $\left(\frac{g^{-1}(\varepsilon)}{\sqrt{d}}\right)^{-d}$. Moreover, by Lemma 1.1 and the inequality (1.16), we see that the condition (1.20) holds for any R_j^ε such that $K \cap R_j^\varepsilon \neq \emptyset$ and this implies (1.21). Thus,

$$\begin{aligned} P\left(\boldsymbol{X}(K) \cap B_\varepsilon(z) \neq \emptyset\right) &\leq \sum_{j \in \mathbb{Z}^d : K \cap R_j^\varepsilon \neq \emptyset} P\left(\boldsymbol{X}(R_j^\varepsilon) \cap B_\varepsilon(z) \neq \emptyset\right) \\ &\leq C(K, \sigma^2_{K^{(\eta)}}, d, D)\ \varepsilon^D \left(\frac{g^{-1}(\varepsilon)}{\sqrt{d}}\right)^{-d} \\ &= C(K, \sigma^2_{K^{(\eta)}}, d, D)\ q_g(\varepsilon), \end{aligned} \tag{1.44}$$

where the last inequality is valid due to (1.21). □

From Lemma 1.2 we deduce conditions for points to be polar, as follows.

Corollary 1.2. *[HCSS21, Cor. 3.1] The hypotheses are as in Lemma 1.2. Assume further that*

$$\lim_{\tau \downarrow 0} q_g(\tau) := q_g(0) = 0. \tag{1.45}$$

Then, for any $z \in \mathbb{R}^D$, $P\left(\boldsymbol{X}(K) \cap \{z\} \neq \emptyset\right) = 0$, that is $\{z\}$ is polar for the process X restricted to K.

Proof. For any $\varepsilon > 0$, we have $P\left(\boldsymbol{X}(K) \cap \{z\} \neq \emptyset\right) \leq P\left(\boldsymbol{X}(K) \cap B_\varepsilon(z) \neq \emptyset\right)$. Applying (1.44) and using (1.45) yields the result. □

Example 1.5. We analyze condition (1.45) for the maps q_g in Example 1.4.

1. $g(\tau) = \tau^\nu, q_g(\tau) = \tau^{D-d/\nu}, \tau, \nu > 0$. In this case

$$q_g(0) = \begin{cases} 0 \text{ if } D > d/\nu, \\ 1 \text{ if } D = d/\nu, \\ \infty \text{ if } D < d/\nu. \end{cases} \tag{1.46}$$

   And (1.45) is satisfied if and only $D > d/\nu$.

2. $g(\tau) = |\log \tau|^\gamma \tau^\nu, \tau \in [0, e^{-\frac{\gamma}{\nu}}], \gamma, \nu > 0, q_g(\tau) = \tau^D \exp\left[-\frac{d\nu}{\gamma} W_{-1}\left(-\frac{\nu}{\gamma}\tau^{\frac{1}{\gamma}}\right)\right]$. (1.39) implies that

$$q_g(0) = \begin{cases} 0 \text{ if } D > d/\nu, \\ \infty \text{ if } D \leq d/\nu. \end{cases} \tag{1.47}$$

   And (1.45) is satisfied if and only if $D > d/\nu$.

1.2.2 Lower bounds for hitting probabilities

The goal of this subsection is to prove Theorem 1.3, which gives lower bounds for hitting probabilities for a $\hat{g}$-Gaussian process.

Let $X = \{X(x) : x \in K\}$ be a Gaussian process in K, a compact subset of $\mathbb{R}^d$, and $g : [0, \varnothing_K] \to \mathbb{R}^+$ be a gauge function. We will make use of the following conditions on X:

(CX1) $\sigma_x^2 \asymp 1$ for $x \in K$.

(CX2) There exist a positive constant C such that for all $x, \bar{x} \in K$

$$|m_x - m_{\bar{x}}| \leq C g(|x - \bar{x}|).$$

(CX3) There exists a positive constant c such that for all $x, \bar{x} \in K$

$$E(\mathrm{Var}(X(x)|X(\bar{x}))) \geq c g^2(|x - \bar{x}|).$$

$\mathrm{Var}(X(x)|X(\bar{x}))$ is the conditional variance of $X(x)$ given $X(\bar{x})$ this means

$$\mathrm{Var}(X(x)|X(\bar{x})) = E\left([X(x) - E(X(x)|X(\bar{x}))]^2 | X(\bar{x})\right).$$

Remark 1.4. It is a well known property of Gaussian processes that (see [Xia09, (41)] for example):

$$E(\mathrm{Var}\left(X(x)|X(\bar{x})\right)) = \frac{\left(\mathfrak{d}_{x,\bar{x}}^2 - (\sigma_x - \sigma_{\bar{x}})^2\right)\left((\sigma_x + \sigma_{\bar{x}})^2 - \mathfrak{d}_{x,\bar{x}}^2\right)}{4\sigma_{\bar{x}}^2} \tag{1.48}$$

$$= \sigma_x^2(1 - \rho_{x,\bar{x}}^2) \tag{1.49}$$

$$= \frac{\det \Gamma_{x,\bar{x}}}{\sigma_{\bar{x}}^2} = \frac{(\sigma_x \sigma_{\bar{x}})^2 - \sigma_{x,\bar{x}}^2}{\sigma_{\bar{x}}^2}, \tag{1.50}$$

where $\Gamma_{x,\bar{x}}$ is the covariance matrix of the vector $(X(x), X(\bar{x}))$.

Proposition 1.1 relates the canonical metric and the conditional variance of a $\hat{g}$-Gaussian process. It will be useful in the following chapters for checking that some examples of $\hat{g}$-Gaussian process satisfy (CX3). We follow a similar procedure than the proof of [DSS10, Lem. 3.2, (1)].

Proposition 1.1. *Let X be a Gaussian process on K a compact subset of $\mathbb{R}^d$ satisfying (CX1). Then*

$$E(\mathit{Var}(X(x)|X(\bar{x}))) \leq C \mathfrak{d}_{x,\bar{x}}^2, x, \bar{x} \in K. \tag{1.51}$$

Additionally assume that

1. *X is a $\hat{g}$-Gaussian process in K.*
2. $\lim_{\varepsilon \downarrow 0} \sup_{\substack{x,\bar{x} \in K \\ |x - \bar{x}| \leq \varepsilon}} \frac{|\sigma_x - \sigma_{\bar{x}}|}{\mathfrak{d}_{x,\bar{x}}} = 0.$
3. *$\rho_{x,\bar{x}} < 1$ for all $x, \bar{x} \in K$, $x \neq \bar{x}$.*

Then

$$E(\mathit{Var}(X(x)|X(\bar{x}))) \asymp \mathfrak{d}_{x,\bar{x}}^2, x, \bar{x} \in K. \tag{1.52}$$

Proof. We have that

$$\begin{aligned}
&\left(\mathfrak{d}_{x,\bar{x}}^2 - (\sigma_x - \sigma_{\bar{x}})^2\right)\left((\sigma_x + \sigma_{\bar{x}})^2 - \mathfrak{d}_{x,\bar{x}}^2\right) \\
&\qquad = \mathfrak{d}_{x,\bar{x}}^2\left[(\sigma_x + \sigma_{\bar{x}})^2 + (\sigma_x - \sigma_{\bar{x}})^2\right] - \left(\mathfrak{d}_{x,\bar{x}}^4 + [(\sigma_x + \sigma_{\bar{x}})(\sigma_x - \sigma_{\bar{x}})]^2\right) \\
&\qquad \leq \mathfrak{d}_{x,\bar{x}}^2\left[(\sigma_x + \sigma_{\bar{x}})^2 + (\sigma_x - \sigma_{\bar{x}})^2\right].
\end{aligned}$$

This, together with (CX1) and (1.48) implies (1.51).

Now we prove the lower bound in (1.52). (CX1) and 1. implies that for x, near $\bar{x}$, $(\sigma_x + \sigma_{\bar{x}})^2 - \mathfrak{d}^2_{x,\bar{x}}$ is bounded by bellow by a positive constant. Furthermore, by 2.,

$$\mathfrak{d}^2_{x,\bar{x}} - (\sigma_x - \sigma_{\bar{x}})^2 = \mathfrak{d}^2_{x,\bar{x}} \left(1 - \left[\frac{\sigma_x^2 - \sigma_{\bar{x}}^2}{\mathfrak{d}_{x,\bar{x}}}\right]^2\right) \geq c\mathfrak{d}^2_{x,\bar{x}},$$

for x near $\bar{x}$. This proves that there exists a $\delta > 0$ such that for any $x, \hat{x} \in K$ with $|x - \bar{x}| < \delta$ the lower bound in (1.52) is valid.

(1.49), (CX1) and 3. implies that $E(\mathrm{Var}(X(x)|X(\bar{x}))) > 0$ for $|x - \bar{x}| \geq \delta, x, \bar{x} \in K$. Since X is $\hat{g}$-Gaussian process on K, by (1.51) it follows that the map $x, \bar{x} \mapsto E(\mathrm{Var}(X(x)|X(\bar{x})))$ is continuous. Thus

$$E(\mathrm{Var}(X(x)|X(\bar{x}))) \geq c \geq \bar{c}g^2(|x - \bar{x}|) \geq \tilde{c}\mathfrak{d}^2_{x,\bar{x}},$$

for any $x, \hat{x} \in K$ that $|x - \bar{x}| \geq \delta$ finishing the proof of (1.52). □

Definition 1.4. A function $\mathfrak{g} : \mathbb{R}^D \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ is a *symmetric potential kernel* if:

(a) $\mathfrak{g}$ is symmetric.

(b) $\mathfrak{g}(z) > 0$, for all $z \neq 0$.

(c) $\mathfrak{g}(0) = \infty$.

(d) $\mathfrak{g}$ is continuous on $\mathbb{R}^D \setminus \{0\}$.

The *energy* of a measure μ on $\mathbb{R}^D$ relative to $\mathfrak{g}$ is given by the expression

$$\mathcal{E}_{\mathfrak{g}}(\mu) = \int_{\mathbb{R}^D \times \mathbb{R}^D} \mathfrak{g}(y - \bar{y})\ \mu(dy)\mu(d\bar{y}).$$

The $\mathfrak{g}$-*capacity* of a Borel set $A \subset \mathbb{R}^D$ is defined by

$$\mathrm{Cap}_{\mathfrak{g}}(A) = \left[\inf_{\mu \in \mathbb{P}(A)} \mathcal{E}_{\mathfrak{g}}(\mu)\right]^{-1}, \tag{1.53}$$

where $\mathbb{P}(A)$ denotes the set of probability measures on A. Since $\mathfrak{g}$ is symmetric, this defines a Choquet capacity (see e.g. [Kho02, Thm. 2.1.1, p. 533]).

When $\mathfrak{g}(z) = |z|^{-\gamma}$, $\gamma > 0$, the $\mathfrak{g}$-*capacity* is the Bessel-Riesz capacity usually denoted by $\mathrm{Cap}_\gamma(A)$ (see e.g. [Kho02, p. 376]).

Associated with a gauge function g we define

$$v_g(\tau) = \int_{g^{-1}(\tau)}^{\varnothing_K} [g(\rho)]^{-D} \rho^{d-1}\ d\rho,\ \tau \in \mathbb{R}_+. \tag{1.54}$$

We are ready to state and prove the main result of this subsection.

Theorem 1.3. *Fix a compact set K of $\mathbb{R}^d$ with positive Lebesgue measure. Let*

$$\boldsymbol{X} = \{\boldsymbol{X}(x) = (X_1(x), ..., X_D(x)), x \in K\}$$

be a D-dimensional stochastic process with i.i.d. coordinates. Fix $N > 0$ and let $A \subset B_N \subset \mathbb{R}^D$ be a Borel set. Assume that X_1 is a $\hat{g}$-Gaussian process on K satisfying conditions (CX1)-(CX3).

1. *[HCSS21, Thm. 3.4] Suppose that*

$$\lim_{\tau\downarrow 0} v_g(\tau)q_g(\tau) \in (0,\infty), \tag{1.55}$$

where q_g is defined in (1.31)*. Then there exists a positive constant C depending on m,K,N,d,D, such that*

$$P(\boldsymbol{X}(K)\cap A \neq \emptyset) \geq C\ \mathit{Cap}_{(q_g)^{-1}}(A). \tag{1.56}$$

2. *Instead of* (1.55)*, assume that*

$$v_g(0) < \infty. \tag{1.57}$$

Then there exists a constant $C := C(m,K,N,d,D) > 0$ such that

$$P(\boldsymbol{X}(K)\cap A \neq \emptyset) \geq C[v_g(0)]^{-1}. \tag{1.58}$$

Remark 1.5. (1.55) and (1.57) are excluding conditions: Assume that there exists a gauge function g satisfying (1.55) and (1.57). This implies $\lim_{\tau\downarrow 0} q_g(\tau) \in (0,\infty)$, and thus

$$q_g(g(\tau)) \leq C, \tau \in (0, \varnothing_K].$$

By the last inequality

$$v_g(0) = \int_0^{\varnothing_K} [q_g(g(\rho))]\rho^{-1}d\rho \geq c\int_0^{\varnothing_K} \rho^{-1}d\rho = \infty.$$

Implying a contradiction to (1.57).

Definition 1.5. For reasons explained bellow (see Remark 1.6) if neither (1.55) nor (1.57) are satisfied, we say that we are on a *critical dimension* type case.

Proof. We adapt the method used for example in [BLX09, Thm 2.1] inspired in [KK93, pp. 204-206].

Proof of 1. For any $x \in K$ and a probability measure μ on A, define

$$\begin{aligned}\bar{\nu}_n(x,\omega) &= \int_A (2\pi n)^{D/2} \exp\left(-\frac{n|\boldsymbol{X}(x)-z|^2}{2}\right)\ \mu(dz) \\ &= \int_A \mu(dz)\int_{\mathbb{R}^D} d\xi \exp\left(-\frac{|\xi|^2}{2n} + i\langle \xi, \boldsymbol{X}(x)-z\rangle\right). \end{aligned} \tag{1.59}$$

Consider the sequence of random measures on K, $(\nu_n, n\geq 1)$, with corresponding densities $(\bar{\nu}_n(x,\omega), n \geq 1)$. Set $\nu_n(K)(\omega) = \int_K \bar{\nu}_n(x,\omega)\ dx$. We aim to prove:

(i) There exists $C_1 > 0$ such that for any $n \geq 1$, $E\left(\nu_n(K)\right) \geq C_1$.

(ii) There exists $C_2 > 0$ such that for any $n \geq 1$, $E\left[\left(\nu_n(K)\right)^2\right] \leq C_2\ \mathcal{E}_{(q_g)^{-1}}(\mu)$.

We claim that (i) and (ii) implies (1.56). Indeed, Markov's and Cauchy-Schwartz inequalities together with (ii), leads to

$$P\left(\nu_n(K) \geq \frac{\sqrt{C_2\mathcal{E}_{(q_g)^{-1}}(\mu)}}{\varepsilon}\right) \leq \varepsilon, \varepsilon > 0, n \geq 1.$$

Thus, $(\mu_n(K))_{n\geq 1}$ is a uniformly tight sequence of r.v. and by Prohorov's theorem there exists a subsequence $n_j, j \geq 1$ and a nonnegative r.v. ν such that $\nu_{n_j}(K)$ converges in law to ν.

By using the first equality in (1.59) we deduce that the event $\{\boldsymbol{X}(K) \cap A = \emptyset\}$, implies that ν_{n_j} converges to zero in law ,and in consequence

$$P(\boldsymbol{X}(K) \cap A \neq \emptyset) \geq P(\nu > 0).$$

Additionally, by the Paley Zygmund inequality, (i) and (ii),

$$P(\nu_{n_j}(K) > 0) \geq \frac{\left[E\left(\nu_{n_j}(K)\right)\right]^2}{E\left[\left(\nu_{n_j}(K)\right)^2\right]} \geq \frac{C_1}{C_2 \mathcal{E}_{(q_g)^{-1}}(\mu)}.$$

We deduce (1.56), by letting $j \to \infty$ in the last inequality and then using the definition of q_g-capacity.

Proof of (i). Since $\boldsymbol{X}(x)$ is a random vector with i.i.d. normally distributed coordinates, its characteristic function is given by,

$$E\left(\exp(i\langle \xi, \boldsymbol{X}(x)\rangle)\right) = \exp\left(i\langle \xi, \boldsymbol{m}_x\rangle - \frac{\sigma_x^2 |\xi|^2}{2}\right).$$

Thus Fubini's theorem yields to

$$\begin{aligned} E\left(\nu_n(K)\right) &= \int_K dx \int_A \mu(dz) \int_{\mathbb{R}^D} d\xi \exp\left(-i\langle \xi, z - \boldsymbol{m}_x\rangle - \frac{|\xi|^2}{2}\left(\frac{1}{n} + \sigma_x^2\right)^2\right) \\ &= \int_K dx \int_A \mu(dz) \left(\frac{2\pi}{1/n + \sigma_x^2}\right)^{D/2} \exp\left(-\frac{|z - \boldsymbol{m}_x|^2}{2[1/n + \sigma_x^2]}\right). \end{aligned}$$

Let $N_0 = N + \sup_{x\in K} |\boldsymbol{m}_x|$. Applying (CX1), and since on the set A, $|z - \boldsymbol{m}_x| \leq N_0$, the above computations yield

$$\begin{aligned} E\left(\nu_n(K)\right) &\geq \int_K dx \int_A \mu(dz) \left(\frac{2\pi}{1 + \sigma_x^2}\right)^{D/2} \exp\left(-\frac{N_0^2}{2\sigma_x^2}\right) \\ &\geq |K| \left(\frac{2\pi}{1 + c_2}\right)^{D/2} \exp\left(-\frac{N_0^2}{2c_1}\right) := C_1. \end{aligned}$$

This ends the proof of (i). Notice that $C_1 := C_1(m, K, N, D)$.

Proof of (ii). For any $x, \bar{x} \in K$, $z, \bar{z} \in A$, set

$$\begin{aligned} &I(x, \bar{x}, z, \bar{z}) \\ &\quad = \int_{\mathbb{R}^D \times \mathbb{R}^D} e^{-i\langle(\xi,\bar{\xi}),(z,\bar{z})\rangle} \exp\left(-\frac{|(\xi, \bar{\xi})|^2}{2n}\right) \exp\left(i\langle(\xi, \bar{\xi}), (\boldsymbol{X}(x), \boldsymbol{X}(\bar{x}))\rangle\right) d\xi\ d\bar{\xi}. \end{aligned}$$

Using (1.59), the definition of $\nu_n(K)$ and Fubini's theorem, we see that

$$E\left[(\nu_n(K))^2\right] = \int_{K\times K} dx\ d\bar{x} \int_{A\times A} \mu(dy)\ \mu(d\bar{y}) E\left(I(x, \bar{x}, z, \bar{z})\right). \tag{1.60}$$

Observe that $I(x, \bar{x}, z, \bar{z}) = \prod_{j=1}^{D} I_j(x, \bar{x}, z, \bar{z})$, with

$I_j(x, \bar{x}, z, \bar{z})$

$$= \int_{\mathbb{R}^2} d\xi_j \; d\bar{\xi}_j \; e^{-i\langle(\xi_j,\bar{\xi}_j),(z_j,\bar{z}_j)\rangle} \exp\left(-\frac{|(\xi_j,\bar{\xi}_j)|^2}{2n}\right) \exp\left(i\langle(\xi_j,\bar{\xi}_j),(X_j(x),X_j(\bar{x}))\rangle\right)$$

Since the factors in the product above are i.i.d. random variables, from (1.60) we obtain

$$E\left[(\nu_n(K))^2\right] = \int_{K\times K} dx \; d\bar{x} \int_{A\times A} \mu(dz) \; \mu(d\bar{z}) \prod_{j=1}^{D} \left[E\left(I_j(x,\bar{x},z,\bar{z})\right)\right]. \tag{1.61}$$

Let $\Gamma_{x,\bar{x}}$ denote the covariance matrix of the 2-dimensional Gaussian random vector $(X_j(x), X_j(\bar{x}))$,

$$\Gamma_{x,\bar{x}} = \begin{pmatrix} \sigma_x^2 & \sigma_{x,\bar{x}} \\ \sigma_{x,\bar{x}} & \sigma_{\bar{x}}^2 \end{pmatrix},$$

and set $\Gamma^n_{x,\bar{x}} = \frac{1}{n}\mathrm{Id}_2 + \Gamma_{x,\bar{x}}$. The characteristic function of such random vector is given by

$$\begin{aligned} &E\left(\exp\left(i\langle(\xi_j,\bar{\xi}_j),(X_j(x),X_j(\bar{x}))\rangle\right)\right) \\ &\quad = \exp\left(i\langle(\xi_j,\bar{\xi}_j),(m_x,m_x)\rangle - \frac{1}{2}(\xi_j,\bar{\xi}_j)\Gamma_{x,\bar{x}}(\xi_j,\bar{\xi}_j)^{\intercal}\right), \end{aligned}$$

and we deduce,

$$\begin{aligned} &E\left(I_j(x,\bar{x},z,\bar{z})\right) \\ &\quad = \int_{\mathbb{R}^2} d\xi_j \; d\bar{\xi}_j \; e^{-i\langle(\xi_j,\bar{\xi}_j),(z_j-m_x,\bar{z}_j-m_{\bar{x}})\rangle} \exp\left(-\frac{1}{2}(\xi_j,\bar{\xi}_j)\Gamma^n_{x,\bar{x}}(\xi_j,\bar{\xi}_j)^{\intercal}\right). \end{aligned} \tag{1.62}$$

(1.50), (CX1) and (CX3) implies that

$$\det\Gamma^n_{x,\bar{x}} \geq \det\Gamma_{x,\bar{x}} = \sigma_{\bar{x}}^2 E(\mathrm{Var}(X(x)|X(\bar{x}))) \geq c_1c_4g^2(|x-\bar{x}|). \tag{1.63}$$

Thus $\det\Gamma^n_{x,\bar{x}} > 0$ (the case $x=\bar{x}$ can be proved by direct computations) and $\Gamma^n_{x,\bar{x}}$ is invertible.

The function inside the integrals in (1.62) is the characteristic function of a 2-dimensional Gaussian random vector with covariance matrix $\Gamma^n_{x,\bar{x}}$ and mean vector $(m_x, m_{\bar{x}})$ times the complex exponential function $e^{-i\langle(\xi_j,\bar{\xi}_j),(z_j,\bar{z}_j)\rangle}$. Then, by the Fourier inversion formula

$$\begin{aligned} &E\left(I_j(x,\bar{x},z,\bar{z})\right) \\ &\quad = \frac{2\pi}{\left(\det\Gamma^n_{x,\bar{x}}\right)^{1/2}} \exp\left(-\frac{1}{2}(z_j-m_x,\bar{z}_j-m_{\bar{x}})\left(\Gamma^n_{x,\bar{x}}\right)^{-1}(z_j-m_x,\bar{z}_j-m_{\bar{x}})^{\intercal}\right), \end{aligned} \tag{1.64}$$

where

$$(\Gamma^n_{x,\bar{x}})^{-1} = \frac{1}{\det\Gamma^n_{x,\bar{x}}}\begin{pmatrix} \frac{1}{n}+\sigma_{\bar{x}}^2 & -\sigma_{x,\bar{x}} \\ -\sigma_{x,\bar{x}} & \frac{1}{n}+\sigma_x^2 \end{pmatrix},$$

We find upper estimates for (1.64). Let $\tilde{X}_j(x) = X_j(x) - m_x$, then

$$(a,b)\left(\Gamma^n_{x,\bar{x}}\right)^{-1}(a,b)^{\intercal} = \frac{1}{\det\Gamma^n_{x,\bar{x}}}\left[(a^2\left(\frac{1}{n}+\sigma_{\bar{x}}^2\right) + b^2\left(\frac{1}{n}+\sigma_x^2\right) - 2ab\sigma_{x,\bar{x}}\right]$$

$$\geq \frac{1}{\det \Gamma^n_{x,\bar{x}}} \left(a^2\sigma^2_{\bar{x}} + b^2\sigma^2_x - 2ab\sigma_{x,\bar{x}}\right)$$
$$= \frac{E\left[\left(a\tilde{X}_j(x) - b\tilde{X}_j(\bar{x})\right)^2\right]}{\det \Gamma^n_{x,\bar{x}}},$$

for any $a, b \in \mathbb{R}$. Due to (CX2), $\tilde{X}_j$ is also a $\hat{g}$-Gaussian process on K, and it also satisfies (CX1) and (CX3) on K. Hence, applying Lemma 1.3 we deduce

$$E\left(I_j(x,\bar{x},z,\bar{z})\right) \leq C \frac{1}{\left(\det \Gamma^n_{x,\bar{x}}\right)^{1/2}} \exp\left(-\frac{c|(z_j - \bar{z}_j) - (m_x - m_{\bar{x}})|^2}{\det \Gamma^n_{x,\bar{x}}}\right). \tag{1.65}$$

Using this estimate in (1.61), we obtain

$$E\left[(\nu_n(K))^2\right] \leq C \int_{K\times K} dx\, d\bar{x} \int_{A\times A} \mu(dy)\, \mu(d\bar{z}) \frac{1}{\left(\det \Gamma^n_{x,\bar{x}}\right)^{D/2}} \times \exp\left(-\frac{c|(z-\bar{z}) - (\boldsymbol{m}_x - \boldsymbol{m}_{\bar{x}})|^2}{\det \Gamma^n_{x,\bar{x}}}\right). \tag{1.66}$$

(1.63) along with (CX2) implies

$$\sup_{x,\bar{x}\in K} \frac{|\boldsymbol{m}_x - \boldsymbol{m}_{\bar{x}}|^2}{\det \Gamma^n_{x,\bar{x}}} \leq C < \infty. \tag{1.67}$$

Apply the inequality $|(z-\bar{z}) - (\boldsymbol{m}_x - \boldsymbol{m}_{\bar{x}})|^2 \geq \frac{1}{2}|z-\bar{z}|^2 - |\boldsymbol{m}_x - \boldsymbol{m}_{\bar{x}}|^2$ and (1.67) on the right-hand side of (1.66) to deduce,

$$E\left[(\nu_n(K))^2\right] \leq C \int_{K\times K} dx\, d\bar{x} \int_{A\times A} \mu(dy)\, \mu(d\bar{z}) \frac{1}{\left(\det \Gamma^n_{x,\bar{x}}\right)^{D/2}} \exp\left(-\frac{c|z-\bar{z}|^2}{\det \Gamma^n_{x,\bar{x}}}\right). \tag{1.68}$$

If $\det \Gamma^n_{x,\bar{x}} \geq |z-\bar{z}|^2$, the integrand is bounded from above by $\left(\det \Gamma^n_{x,\bar{x}}\right)^{-D/2}$. If on the contrary, $\det \Gamma^n_{x,\bar{x}} < |z-\bar{z}|^2$, the integrand is bounded (up to a multiplicative constant) by $|z-\bar{z}|^{-D}$, because the function $\tau \mapsto \tau^{D/2}e^{-c\tau}$ is bounded over $\mathbb{R}_+$. In this way,

$$E\left[(\nu_n(K))^2\right] \leq C \int_{K\times K} dx\, d\bar{x} \int_{A\times A} \mu(dz)\, \mu(d\bar{z}) \frac{1}{\max\left(\left(\det \Gamma^n_{x,\bar{x}}\right)^{D/2}, |z-\bar{z}|^D\right)}$$
$$\leq C \int_{K\times K} dx\, d\bar{x} \int_{A\times A} \mu(dz)\, \mu(d\bar{z})\, \frac{1}{\max\left(g^D(|x-\bar{x}|), |z-\bar{z}|^D\right)}, \tag{1.69}$$

where in the second inequality we have applied (1.63).

Our next goal is to prove

$$\int_{K\times K} \frac{dx\, d\bar{x}}{\max\left(g^D(|x-\bar{x}|), |z-\bar{z}|^D\right)} \leq C(K,d)[q_g(|z-\bar{z}|)]^{-1}. \tag{1.70}$$

Indeed,

$$\int_{(K\times K)\cap\{g(|x-\bar{x}|)\leq|z-\bar{z}|\}} \frac{dx\, d\bar{x}}{\max\left(g^D(|x-\bar{x}|), |z-\bar{z}|^D\right)}$$

$$\leq C(K,d)|z-\bar{z}|^{-D}\int_0^{g^{-1}(|z-\bar{z}|)}\rho^{d-1}d\rho = C(K,d)\ [q_g(|z-\bar{z}|)]^{-1},$$

and

$$\begin{aligned}&\int_{(K\times K)\cap\{g(|x-\bar{x}|)>|z-\bar{z}|\}}\frac{dx\ d\bar{x}}{\max\left(g^D(|x-\bar{x}|),|z-\bar{z}|^D\right)}\\ &\leq C(K,d)\int_{g^{-1}(|z-\bar{z}|)}^{\varnothing_K}[g(\rho)]^{-D}\rho^{d-1}d\rho\\ &= C(K,d)v_g(|z-\bar{z}|)\leq \tilde{C}(K,d)[q_g(|z-\bar{z}|)]^{-1},\end{aligned}$$

where the last equality holds because (1.55) is equivalent to $\sup_{\tau\in[0,\varnothing_K]} v_g(\tau)q_g(\tau)\in(0,\infty)$.

Hence,

$$E\left[(\nu_n(K))^2\right]\leq C(K,d)\mathcal{E}_{(q_g)^{-1}}(\mu),$$

and the right-hand side does not depend of n. Finishing the proof of (1.56).

Proof of 2. The procedure is the same that in 1. But now instead of (ii) we claim that there exists $C_2>0$ such that for any $n\geq 1$, $E\left[(\nu_n(K))^2\right]\leq C_2$. Indeed, all the arguments given in the proof of 1. are also valid in this case until (1.69). Hence, by (1.57) and (1.69)

$$\begin{aligned}E\left[(\nu_n(K))^2\right]&\leq C\int_{K\times K}dx\ d\bar{x}\int_{A\times A}\mu(dz)\mu(d\bar{z})g^{-D}(|x-\bar{x}|)\\ &\leq C(d,K)v_g(0)<\infty.\end{aligned}$$

From the last inequality we conclude the validity of (1.58) instead of (1.56).

The proof of the theorem is complete.

□

Theorem 1.3 gives us a condition for non polarity of points:

Corollary 1.3. *Let* $\boldsymbol{X}$ *be as in Theorem 1.3 and assume* (1.57). *Then, for any* $z\in\mathbb{R}^D$, $P(\boldsymbol{X}(K)\cap\{z\}\neq\emptyset)>0$, *that is* $\{z\}$ *is non polar for the process restricted to* K.

Example 1.6. We compute v_g and analyze conditions (1.55) and (1.57) for the gauge functions in Example 1.4.

1. $g(\tau)=\tau^\nu$,$\tau,\nu>0$. We have that

$$\begin{aligned}v_g(\tau)&=\int_{\tau^\nu}^{\varnothing_K}\rho^{d-\nu D-1}d\rho\\ &=\begin{cases}(\nu D-d)^{-1}\left[\tau^{-(D-d/\nu)}-\varnothing_K^{-(D-d/\nu)}\right], & \text{if } d/\nu\neq D,\\ \nu^{-1}\log\left(\frac{\varnothing_K^\nu}{\tau}\right), & \text{otherwise.}\end{cases}\end{aligned}$$

Since $q_g(\tau)=\tau^{D-d/v}$, we deduce that

$$\lim_{\tau\downarrow 0}v_g(\tau)q_g(\tau)=\begin{cases}(\nu D-d)^{-1}, & \text{if } D>d/\nu,\\ \infty, & \text{otherwise,}\end{cases}$$

and

$$v_g(0) = \begin{cases} (d-\nu D)^{-1}\varnothing_K^{d/\nu - D}, & \text{if } d/\nu > D, \\ \infty, & \text{otherwise.} \end{cases}$$

Thus (1.55), (1.57) is satisfied if and only if $D > d/\nu$, $d/\nu > D$, respectively.

2. $g(\tau) = \tau^\nu |\log\tau|^\gamma, \tau \in [0, e^{-\frac{\gamma}{\nu}}]$, $\gamma, \nu > 0$. Due to (1.39)

$$\lim_{\tau\downarrow 0} q_g(\tau) = \begin{cases} \infty, & \text{if } D \leq d/\nu \\ 0, & \text{if } , D > d/\nu \end{cases} \tag{1.71}$$

We have that

$$v_g(\tau) = \int_{g^{-1}(\tau)}^{\varnothing_K} |\log\rho|^{-\gamma D}\rho^{-\nu D + d - 1} d\rho.$$

We claim that

$$v_g(0) = \begin{cases} C(\nu,\gamma,d,D), & \text{if either } D < d/\nu, \text{ or } D = d/\nu, \gamma D > 1 \\ \infty, & \text{if either } D > d/\nu, \text{ or } D = d/\nu, \gamma D \leq 1. \end{cases} \tag{1.72}$$

Indeed, since $g(0) = 0$ it is clear that the claim is valid when $D \neq d/\nu$. Now assume that $D = d/\nu$. By the change of coordinates $\rho \mapsto -\log\rho$,

$$\begin{aligned} v_g(0) &\asymp \int_0^1 |\log\rho|^{-\gamma D}\rho^{-1} d\rho = \int_1^\infty \rho^{-\gamma D} d\rho \\ &= \begin{cases} C(\nu,\gamma,d,D), & \text{if } \gamma D > 1 \\ \infty, & \text{if } \gamma D \leq 1. \end{cases} \end{aligned}$$

Finishing the proof the claim.

Now, suppose that $D > d/\nu$. By the definitions of v_g and q_g,

$$v_g(g(\tau))q_g(g(\tau)) = \left[\int_\tau^{g(\varnothing_K)} |\log\rho|^{-\gamma D}\rho^{-\nu D+d-1} d\rho\right]\left[\frac{\tau^d}{(g(\tau))^D}\right]^{-1}.$$

Then, since $g(0) = 0$, and by applying the L'Hôpital's rule which is valid in this case because of (1.71) and (1.72), we obtain that

$$\begin{aligned} \lim_{\tau\downarrow 0} v_g(\tau)q_g(\tau) &= \lim_{\tau\downarrow 0} v_g(g(\tau))q_g(g(\tau)) \\ &= \lim_{\tau\downarrow 0} \frac{-|\log\tau|^{-\gamma D}\tau^{-\nu D+d-1}}{\tau^{-\nu D+d-1}[(d-\nu D)|\log\rho|^{-\gamma D} - \gamma D|\log\tau|^{-\gamma D-1}]} \\ &= (D\nu - d)^{-1}. \end{aligned}$$

Summarizing the discussion above, (1.55) holds if and only if $D > d/\nu$. And (1.57) is valid if and only if either $D < d/\nu$ or $D = d/\nu, \gamma D > 1$.

We end this section with a technical lemma which proof is similar to Lemma 3.4 in [HCSS21].

Lemma 1.3. *Let $X = \{X(x) : x \in K\}$ be a $\hat{g}$-Gaussian process in K a compact subset of $\mathbb{R}^d$ satisfying (CX3) and $\sigma_K^2 := \inf_{x\in K} \sigma_x^2 > 0$. Then, there exists a positive constant c such that for any $a, b \in \mathbb{R}$ and $x, \bar{x} \in K$,*

$$E[(aX(x) - bX(\bar{x}))^2] \geq c(a-b)^2. \tag{1.73}$$

Proof. (1.73) is equivalent to say that the quadratic form

$$Q(a,b) = a^2(\sigma_x^2 - c) - 2ab(\sigma_{x,\bar{x}} - c) + b^2(\sigma_{\bar{x}}^2 - c), a, b \in \mathbb{R},$$

is positive semi-definite for any $x, \bar{x}$ in K. Hence, it is enough to proof that there exist $c > 0$ that for any $x, \bar{x} \in K$,

(i) $\sigma_x^2 - c \geq 0$,

(ii) $(\sigma_x^2 - c)(\sigma_{\bar{x}}^2 - c) - (\sigma_{x,\bar{x}} - c)^2 \geq 0$.

(i) is valid for any $c \in (0, \sigma_K^2]$. (ii) is equivalent to

$$\det \Gamma_{x,\bar{x}} \geq c\mathfrak{d}(x,\bar{x}), x, \bar{x} \in K, \tag{1.74}$$

where $\Gamma_{x,\bar{x}}$ is the covariance matrix of the random vector $(X(x), X(\bar{x}))$ and $\mathfrak{d}$ the canonical metric of X. By (1.50) and (CX3)

$$\det \Gamma_{x,\bar{x}} \geq c\,(\sigma_K)\, g^2(|x-\bar{x}|), x, \bar{x} \in K.$$

Since X is a $\hat{g}$-Gaussian process in K, the last inequality implies that there exists $\tilde{c}$ a positive constant satisfying (1.74). We conclude that $c = \sigma_K^2 \wedge \tilde{c}$ fulfills (1.73).

□

1.3 Isotropic Gaussian processes

Definition 1.6. Fix $\eta > 0$ and let $\boldsymbol{X}$ be a D-dimensional Gaussian process with i.i.d coordinates,

$$\boldsymbol{X} = \{\boldsymbol{X}(x) = (X_1(x), \ldots, X_D(x)),\ x \in K^{(\eta)}\},$$

where K is a compact subset of $\mathbb{R}^d$, such that X_1 is g-Gaussian with $g(\tau) = \tau^\nu, \nu \in (0,1]$, i.e,

$$\mathfrak{d}_{X_1}(x,\bar{x}) \asymp |x-\bar{x}|^\nu, x, \bar{x} \in K^{(\eta)}. \tag{1.75}$$

A process satisfying such kind of condition is called *isotropic* Gaussian process. By Example 1.2 the sample paths of $\boldsymbol{X}$ are continuous a.s.

In Example 1.3(1.) we proved that g satisfies the hypothesis of Lemma 1.1(ii). Additionally, from Example 1.4 (1.), $q_g(\tau) = \tau^{D-\frac{d}{\nu}}$ and q_g is increasing if and only if $D > d/\nu$. From Corollary 1.1, we deduce that if $\sigma^2_{K^{(\eta)}} > 0$ and $D > d/\nu$, for any Borel set $A \subset \mathbb{R}^D$,

$$P\left(\boldsymbol{X}(K) \cap A \neq \emptyset\right) \leq C\,\mathcal{H}_{D-\frac{d}{\nu}}(A), \tag{1.76}$$

with $C := C(K, \sigma^2_{K^{(\eta)}}, d, D)$.

Now, suppose that X_1 satisfies (CX1)-(CX3). Fix a bounded Borel set $A \subset B_N \subset \mathbb{R}^D$. According to Example 1.6 (1.55), (1.57) are satisfied if and only if $D > d/\nu$, $d/\nu > D$ respectively. Then, applying Theorem 1.3 we obtain,

1. If $D > d/\nu$,
$$P(\boldsymbol{X}(K) \cap A \neq \emptyset) \geq c \, \mathrm{Cap}_{D-\frac{d}{\nu}}(A), \tag{1.77}$$
with $c := c(K, N, d, D)$.

2. If $d/\nu > D$,
$$P(\boldsymbol{X}(K) \cap A \neq \emptyset) \geq c > 0 \tag{1.78}$$
with $c := c(K, N, d, D)$.

Remark 1.6. Fix $z \in \mathbb{R}^D$ and assume that (1.76), (1.77) and (1.78) are valid. Hence

$$P(\boldsymbol{X}(K) \cap \{z\} \neq \emptyset) = \begin{cases} \mathcal{H}_{D-\frac{d}{\nu}}(\{z\}) = 0, & \text{if } D > d/\nu, \\ P(\boldsymbol{X}(K) \cap \{z\} \neq \emptyset) \geq c > 0, & \text{if } D > d/\nu. \end{cases}$$

Thus, singletons $\{z\}$ are polar if $D > d/\nu$ and non polar if $D < d/\nu$.

We identify the value $D_0 = d/\nu$ as critical, usually called the *critical dimension* for polarity of points. Observe that if $D = d/\nu$ (1.55) and (1.57) are both false. Thus it makes sense in the general context of g-Gaussian process to say that under the same conditions we are on a critical dimension type case. (See Remark 1.5).

An open problem is to find a criteria to determine whether a g-Gaussian process hits or not points in the critical dimension. Dalang, Mueller and Xiao studied this problem in the case of Gaussian isotropic random fields in [DMX17]. We believe its procedure could be adapted to a more general context. It is expected that a g-Gaussian random field does not hit points in the critical dimension.

The following examples provide illustrations of the preceding results:

Example 1.7. *Fractional Brownian motion.* Let

$$\boldsymbol{B}^H = \{\boldsymbol{B}^H(x) = (B_1^H(x), ..., B_D^H(x)), x \in \mathbb{R}_+^D\}$$

a (d, D)- fractional Brownian motion of Hurst parameter $H \in (0, 1)$, i.e. a centered Gaussian random field with i.i.d. coordinates that

$$E(B_j^H(x)B_j^H(y)) = \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right), x, y \in \mathbb{R}_+^D.$$

If $d = 1, H = 1/2$, $\boldsymbol{B}^H$ is the D-dimensional Brownian motion.

Let $B^H = \{B_1^H(x), x \in \mathbb{R}_+^d\}$, then

$$\mathfrak{d}_{B^H}^2(x, y) = |x-y|^{2H}$$

so B^H is g-Gaussian on any compact subset of $\mathbb{R}_+^d$, with $g(\tau) = \tau^H$.

Fix $K = [a, b] \subset \mathbb{R}_+^d$, $0 < a < b < \infty$ a compact box. B^H satisfies that

- $\sigma_x^2 \asymp 1$ for $x \in K$.
- [Pit78, Lem. 7.1] For all $x, \bar{x} \in \mathbb{R}_+^d$,
$$E(\mathrm{Var}(B^H(x) \mid B^H(\bar{x}))) \geq |x - \bar{x}|^{2H}.$$

Let $D_0 = dH^{-1}$. By (1.76), (1.77) and (1.78) we deduce the following hitting probabilities results relative to the fractional Brownian motion.

1. Let $D > D_0$.

   (a) There exists a constant $C := C(d, \sigma^2_{K(\eta)}, D, K)$ such that for any Borel set $A \in \mathcal{B}(\mathbb{R}^D)$,

   $$P(\boldsymbol{B}^H(K) \cap A \neq \emptyset) \leq C\mathcal{H}_{D-D_0}(A).$$

   (b) Fix $N > 0$. Let $A \in \mathcal{B}(\mathbb{R}^D)$ be such that $A \subset B_N$. There exists a constant $c := c(d, D, K, N)$ such that

   $$P(\boldsymbol{B}^H(K) \cap A \neq \emptyset) \geq c\text{Cap}_{D-D_0}(A).$$

2. Assume that $D < D_0$. Let $A \in \mathcal{B}(\mathbb{R}^D)$ be such that $A \subset B_N$. There exists a constant $c := c(d, D, K, N) > 0$ such that

   $$P(\boldsymbol{B}^H(K) \cap A \neq \emptyset) \geq c > 0.$$

And D_0 is the critical dimension.

As stated in [DMX17, Thm. 6.1] points are polar for fractional Brownian motion in the critical dimension, that is, if $D = d/H$ for all $z \in \mathbb{R}^D$

$$P(\boldsymbol{B}^H(K) \cap \{z\} \neq \emptyset) = 0.$$

Example 1.8. *Fractional Brownian sheet.* Let

$$\boldsymbol{B}^H = \{\boldsymbol{B}^H(x) = (B^H_1(x), ..., B^H_D(x)), x \in \mathbb{R}^d_+\}$$

a (d, D)- fractional Brownian sheet of Hurst parameter $H \in (0, 1)$, i.e a centered Gaussian random field with i.i.d. coordinates that

$$E(B^H_j(x)B^H_j(y)) = \prod_{i=1}^{d} \frac{1}{2} \left(|x_i|^{2H} + |y_i|^{2H} - |x_i - y_i|^{2H}\right), x, y \in \mathbb{R}^d_+.$$

If $d = 1$, the fractional Brownian sheet and the fractional Brownian motion are the same process. If $H = 1/2$, $\boldsymbol{B}^H$ is the Brownian sheet.

Let $B^H = \{B^H_1(x), x \in \mathbb{R}^d_+\}$ and $K = [a, b] \subset \mathbb{R}^d_+$, $0 < a < b < \infty$ be a compact box. According to [AX05, Lem. 8],

$$\mathfrak{d}^2_{B^H}(x, y) \asymp |x - y|^{2H}, x, y, \in K,$$

so B^H is g-Gaussian on any compact subset of $\mathbb{R}^d_+$, with $g(\tau) = \tau^H$. Additionally, B^H satisfies that

- $\sigma^2_x \asymp 1$ for $x \in K$.

- [WX07, Lem. 7.1] For all $x, \bar{x} \in K$,

  $$E(\text{Var}(B^H(x) \mid B^H(\bar{x})) \geq c(K)|x - \bar{x}|^{2H}.$$

If we compare with the fractional Brownian motion of parameter H, the bounds of the canonical metric and the conditional variance are in terms of the same exponents. This leads to similar hitting probabilities results in both cases. For avoiding repetitions we don't explicitly state them for the fractional Brownian sheet.

Fix $N > 0$ and $K = [a, b] \subset \mathbb{R}^d_+$, $0 < a < b < \infty$, a compact box. In [KS99, Thm. 1.1] is proved that there exists constants depending on N and K, such that

$$P(\boldsymbol{B}^{1/2}(K) \cap A \neq \emptyset) \asymp \mathrm{Cap}_{D-2d}(A), A \in \mathcal{B}(\mathbb{R}^D) \cap B_N,$$

identifying the capacity as an optimal estimation for hitting probabilities. The method used for proving the result is based in the Markovian properties of the Brownian sheet, and it cannot be applied in the context of more general Gaussian processes as the fractional Brownian sheet when $H \neq \frac{1}{2}$.

Chapter 2

The linear stochastic Poisson equation

We apply the results of Chapter 1 to the linear stochastic Poisson equation with boundary conditions

$$\begin{cases} -\Delta v_j(x) = \dot{W}_j(x), & x \in B_1, \\ \quad v_j(x) = v_0(x), & x \in \mathbb{S}^{d-1}, \end{cases}$$

$j = 1, ..., D$, with $(W_j, j = 1, ..., D)$ independent copies of a white noise, B_1 the d-dimensional unit ball centered at the origin, and $\mathbb{S}^{d-1}$ the $d-1$-dimensional sphere.

We first prove that such equation has a random field solution if and only if $d \in \{1, 2, 3\}$. By Theorem 2.2 the solution turns out to be a g-Gaussian process. Section 2.3 is devoted to analyze additional second order properties required for apply the hitting probabilities criteria of Chapter 1. Finally, in Theorem 2.3 we find upper and lower bounds for hitting probabilities.

2.1 The solution

Definition 2.1. Let $W = \{W(A), A \in \mathcal{B}_b(\mathbb{R}^d)\}$ be a centered Gaussian random field with covariance

$$E(W(A)W(B)) = |A \cap B|,$$

where $|\cdot|$ denotes the Lebesgue measure, we say that W is a *white noise*.

Remark 2.1. The process $B = \{B(x) := W([0, x]), x \in \mathbb{R}^d_+\}$, is the Brownian sheet from Example 1.8 in Chapter 1.

The solution to the linear stochastic Poisson equation investigated in this Chapter, is given in terms of the Wiener stochastic integral with respect to white noise. We recall how to construct this integral.

We consider the case of deterministic integrands. For $\varphi = 1_A$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, an indicator function, we define

$$\int_{\mathbb{R}^d} 1_A W(dx) := W(A),$$

which yields to

$$E\left(\int_{\mathbb{R}^d} 1_A W(dx) \int_{\mathbb{R}^d} 1_B W(dx)\right) = |A \cap B| = \langle 1_A, 1_B \rangle_{L^2(\mathbb{R}^d)}, A, B \in B_b(\mathbb{R}^d). \quad (2.1)$$

Let

$$\mathcal{E} = \left\{ \varphi : \varphi = \sum_{i=1}^{n} a_i 1_{A_i}, a_i \in \mathbb{R}, A_i \in \mathcal{B}_b(\mathbb{R}^d), \right\}$$

be the vector space of step functions in $\mathbb{R}^d$. We extend the definition of the integral as

$$\int_{\mathbb{R}^d} \varphi(x) W(dx) := \sum_{i=1}^{d} a_i W(A_i), \quad \varphi \in \mathcal{E}.$$

It can be proved that this extension is well defined, and as in (2.1)

$$E\left(\int_{\mathbb{R}^d} \varphi(x) W(dx) \int_{\mathbb{R}^d} \psi(x) W(dx) \right) = \langle \varphi, \psi \rangle_{L^2(\mathbb{R}^d)}, \varphi, \psi \in \mathcal{E}.$$

Thus, the map $\varphi \mapsto \int \varphi(x) W(dx)$ is an isometry between $\mathcal{E}$ and a linear subspace of $L^2(\Omega)$. Such isometry can be extended to $L^2(\mathbb{R}^d)$ in a unique way, since step functions are dense in this space. We denote this extension by

$$\int_{\mathbb{R}^d} \varphi(x) W(dx), \varphi \in L^2(\mathbb{R}^d).$$

And we define the stochastic integral with respect to white noise of a function $\varphi \in L^2(\mathbb{R}^d)$, which is a centered normal random variable with variance $|\varphi|^2_{L^2(\mathbb{R}^d)}$.

The following property, consequence of the construction, is called the *(Itô) isometry* of the stochastic integral:

$$E\left(\int_{\mathbb{R}^d} \varphi(x) W(dx) \int_{\mathbb{R}^d} \psi(x) W(dx) \right) = \langle \psi, \varphi \rangle_{L^2(\mathbb{R}^d)}, \psi, \varphi \in L^2(\mathbb{R}^d). \tag{2.2}$$

The Green's function of the Laplace operator $\mathcal{L} = -\Delta$ on B_1 the d-dimensional unit ball centered at zero, is given by

$$G(x,y) = \begin{cases} \Gamma(|x-y|) - \Gamma\left(|y| \left| x - \frac{y}{|y|^2} \right| \right), & y \neq 0, \\ \Gamma(|x-y|) - \Gamma(1), & y = 0, \end{cases} \tag{2.3}$$

with

$$\Gamma(|z|) = \begin{cases} \frac{|z|^{2-d}}{d(d-2)\omega_d}, & d \neq 2, \\ -\frac{1}{2\pi} \log(|z|), & d = 2, \end{cases}$$

where ω_d is the d-volume of B_1. (see for example [Eva98, p. 40]).

Remark 2.2. According to [BP90, p. 221], for $d = 2, 3$, the Green function in (2.3) has the following alternative expression

$$G(x,y) = \Gamma(|x-y|) - E_x(\Gamma(|B_\tau - y|)), x, y \in B_1,$$

where B is a d-dimensional Brownian motion that starts from x at time zero and τ denotes the first time it hits $\mathbb{S}^{d-1}$.

Remark 2.3. We have that

$$\left(|y| \left| x - \frac{y}{|y|^2} \right| \right)^2 = |x|^2|y|^2 - 2x \cdot y + 1 = \left(|x| \left| y - \frac{x}{|x|^2} \right| \right)^2,$$

implying the symmetry of the Green function $G(x,y) = G(y,x)$, and that

$$1 - |x||y| \leq |y| \left| x - \frac{y}{|y|^2} \right| \leq 1 + |x||y|, x, y \in B_1 \setminus \mathbb{S}^{d-1}. \tag{2.4}$$

Consider the linear stochastic Poisson equation

$$\begin{cases} -\Delta v(x) = \dot{W}(x), & x \in B_1 \\ \quad v(x) = v_0(x), & x \in \mathbb{S}^{d-1}. \end{cases} \tag{2.5}$$

where W is a space white noise on B_1 and $v_0 : B_1 \to \mathbb{R}$ a measurable function.

Definition 2.2. Assume that for any $x \in B_1$, $1_{B_1} G(x, *) \in L^2(\mathbb{R}^d)$. The *random field solution* to (2.5) is the Gaussian stochastic process

$$v(x) = I_0(x) + u(x), x \in B_1, \tag{2.6}$$

where

$$I_0(x) = \frac{1-|x|^2}{d\omega_d} \int_{\mathbb{S}^{d-1}} \frac{v_0(y)}{|x-y|^d} dS(y), \quad u(x) = \int_{B_1} G(x,y) W(dy). \tag{2.7}$$

(See for example [MBS10, Thm. 3.3] for the definition of the hypersurface integral in the left of (2.7)).

Remark 2.4. Assume that $v_0 \in C(\mathbb{S}^{d-1})$. Then the function I_0 is in $C^\infty(B_1)$ and it is a solution to the classical Poisson equation

$$\begin{cases} -\Delta v(x) = 0, & x \in B_1, \\ \quad v(x) = v_0(x), & x \in \mathbb{S}^{d-1}. \end{cases}$$

(see [Eva98, Thm. 15, p.41]).

It turns the linear stochastic Poisson equation has a random field solution if and only if $d \in \{1, 2, 3\}$.

Theorem 2.1. *The stochastic process* $(u(x), x \in B_1)$ *in* (2.7) *is well defined if and only if* $d \in \{1, 2, 3\}$. *In this case,*

$$\sup_{x \in B_1} E(|u(x)|^2) < \infty. \tag{2.8}$$

Proof. By the stochastic integral isometry (2.2), and (2.7)

$$\sigma_x^2 = E(|u(x)|^2) = \int_{B_1} dy \left(\Gamma(|x-y|) - \Gamma\left(|y| \left| x - \frac{y}{|y|^2} \right| \right) \right)^2, x \in B_1. \tag{2.9}$$

We first claim that the integral on the right side of (2.9) is not finite when $d > 3$. Indeed, we have that $B_{r_x}(x) \subset B_1$ for $r_x := \frac{1-|x|}{2}$. Then, for $d > 3$

$$\sigma_x^2 \geq c(d) \int_{B_{r_x}(x)} dy \left[|x-y|^{2(2-d)} - \frac{1}{2} \left(|y| \left| x - \frac{y}{|y|^2} \right| \right)^{2(2-d)} \right] \tag{2.10}$$

By the lower bound in (2.4), for $x \in B_1 \setminus \mathbb{S}^{d-1}$

$$\int_{B_{r_x}(x)} dy \left(|y| \left| x - \frac{y}{|y|^2} \right| \right)^{2(2-d)} \leq (1-|x|)^{2(2-d)} \int_{B_{r_x}(x)} dy < \infty. \tag{2.11}$$

Additionally, since $d > 3$,

$$\int_{B_{r_x}(x)} dy|x-y|^{2(2-d)} = \int_{B_{r_x}} dy|y|^{2(2-d)} = c_d \int_0^{r_x} d\rho\rho^{3-d} = \infty. \tag{2.12}$$

The claim follows by (2.10)-(2.12).

Now assume that $d = 1, 2, 3$. We prove (2.8) by distinguishing cases:

Case 1. $d = 1$. We have that

$$\sigma_x^2 = \frac{1}{4}\int_{-1}^{1} dy\,(|x-y| - |1-xy|)^2 = \frac{1}{6}(1-x^2)^2 \leq \frac{1}{6}, \tag{2.13}$$

implying (2.8).

Case 2. $d = 2$. Since for any $x \in B_1$, $B_1(x) \subset B_2$,

$$\begin{aligned}\int_{B_1} dy\Gamma(|x-y|)^2 &= \int_{B_1(x)} dy\Gamma(|y|)^2 \\ &\leq \int_{B_2} dy\Gamma(|y|)^2 = c_d \int_0^2 d\rho\rho(\log\rho)^2 < \infty. \end{aligned} \tag{2.14}$$

Additionally, by (2.4)

$$\begin{aligned}\int_{B_1} dy\Gamma\left(|y|\left|x - \frac{y}{|y|^2}\right|\right)^2 &\leq \int_{B_1} dy\left(\log\left(1-|y|\right)^2 + \log\left[1+|y|\right)^2\right] \\ &= c_d\int_0^1 d\rho\rho\left[\log\left(1-\rho\right)^2 + \log\left(1+\rho\right)^2\right] < \infty. \end{aligned} \tag{2.15}$$

The identities (2.9), together with (2.14) and (2.15) implies (2.8).

Case 3. $d = 3$ The arguments for proving (2.8) are similar to those given in the proof of the Case $d = 2$. We omit the details.

□

In the rest of this chapter we will assume that $d \in \{1, 2, 3\}$.

2.2 Equivalence for the canonical metric

For the process u of Theorem 2.1, we denote by

$$\mathfrak{d}_u(x,y) = \|u(x) - u(y)\|_{L^2(\Omega)}. \tag{2.16}$$

the canonical metric associated with u. This section is devoted to prove Theorem 2.2 where we establish an equivalent pseudo-distance for $\mathfrak{d}_u$. This result implies that u is a g-Gaussian process. The cases $d = 1, 3$, are studied in Lemmas 5.4 and 5.7 of [SSV18]. In Lemma 2.1 we find sharp bounds for $d = 2$ this is an improvement of Lemma 5.5 in [SSV18].

Theorem 2.2. *Let $(u(x), x \in B_1)$ be the stochastic process defined in* (2.7). *Fix $\rho_0 \in (0,1)$. There exists positive constants c, C depending on ρ_0, d such that for any $x, y \in B_{\rho_0}$,*

$$\mathfrak{d}_u^2(x,y) \asymp \left[\log\left(\frac{2\rho_0}{|x-y|}\right)\right]^\beta |x-y|^{2\wedge(4-d)}, \tag{2.17}$$

where $\beta = 1$ if $d = 2$, and $\beta = 0$ otherwise.

Proof. The cases $d = 1, 3$ are Lemmas 5.4 and 5.7 in [SSV18]. The case $d = 2$ is Proposition 2.1 bellow. □

Proposition 2.1. *Assume that $d = 2$ and fix $\rho_0 \in (0, 1)$. Let $(u(x), x \in B_1)$ the stochastic process defined in (2.7). There exists positive constants c, C depending on ρ_0 such that for any $x, y \in B_{\rho_0}$,*

$$\mathfrak{d}_u^2(x, y) \asymp \log\left(\frac{2\rho_0}{|x-y|}\right)|x-y|^2. \tag{2.18}$$

Proof. Fix $x, y \in B_{\rho_0}$. From (2.7), (2.3) and the isometry property (2.2) we deduce

$$\mathfrak{d}_u^2(x, y) = \int_{B_1} dz \left[\log\left(\frac{|x-z|}{|y-z|}\frac{||z|^2 y - z|}{||z|^2 x - z|}\right)\right]^2. \tag{2.19}$$

Thus, since $|x - y| \le 2\rho_0$,

$$\frac{1}{2}\mathcal{I} - \mathcal{J} \le \mathfrak{d}_u^2(x, y) \le 2(\mathcal{I} + \mathcal{J}), \tag{2.20}$$

for

$$\mathcal{I} := \int_{B_1} dz\left[\log\left(\frac{|x-z|}{|y-z|}\right)\right]^2, \mathcal{J} := \int_{B_1} dz\left[\log\left(\frac{||z|^2 x - z|}{||z|^2 y - z|}\right)\right]^2.$$

Let θ_0 be the angle of the vector $x - y$ in polar coordinates. Writing the change of variables $z \mapsto y - z$ and then switching to polar coordinates,

$$\begin{aligned}\mathcal{I} &= \int_{B_1(y)} dz\left[\log\left(\frac{|x-y+z|}{|z|}\right)\right]^2 \\ &= \frac{1}{4}\int_A d(\theta, \rho)\rho\left[\log\left(1 + 2\frac{|x-y|}{\rho}\cos(\theta - \theta_0) + \frac{|x-y|^2}{\rho^2}\right)\right]^2,\end{aligned} \tag{2.21}$$

with $A = \{(\theta, \rho) \in [0, 2\pi) \times \mathbb{R}_+ : |\rho(\cos\theta, \sin\theta) - y| \le 1\}$.

[SSV18, (55)-(56), p.1877] implies that

$$\mathcal{J} \le \frac{|x-y|^2}{8\pi(1-\rho_0)^4}. \tag{2.22}$$

Upper bound. We have that $A \subset [0, 2\pi) \times [0, 1 + \rho_0]$. Thus,

$$\mathcal{I} \le \frac{1}{4}(\mathcal{I}_1 + \mathcal{I}_2) \tag{2.23}$$

where

$$\begin{aligned}\mathcal{I}_1 &:= \int_0^{2\pi} d\theta \int_0^{\frac{1+\rho_0}{2\rho_0}|x-y|} d\rho\rho\left[\log\left(1 + 2\frac{|x-y|}{\rho}\cos\theta + \frac{|x-y|^2}{\rho^2}\right)\right]^2, \\ &= |x-y|^2 \int_0^{2\pi} d\theta \int_0^{\frac{1+\rho_0}{2\rho_0}} d\rho\rho[\log(1 + 2\rho^{-1}\cos\theta + \rho^{-2})]^2 \\ &= c(\rho_0)|x-y|^2,\end{aligned} \tag{2.24}$$

$$\mathcal{I}_2 := \int_0^{2\pi} d\theta \int_{\frac{1+\rho_0}{2\rho_0}|x-y|}^{1+\rho_0} d\rho\rho \left[\log\left(1+2\frac{|x-y|}{\rho}\cos\theta + \frac{|x-y|^2}{\rho^2}\right)\right]^2$$
$$= |x-y|^2 \int_0^{2\pi} d\theta \int_{\frac{1+\rho_0}{2\rho_0}}^{\frac{1+\rho_0}{|x-y|}} d\rho\rho[\log(1+2\rho^{-1}\cos\theta+\rho^{-2})]^2. \tag{2.25}$$

We bound the integral on the r.h.s of (2.25). We distinguish two cases according to the inequalities

$$|\log(\tau_1)| \le |\log(\tau_2)| \text{ if } 1 \le \tau_1 \le \tau_2, \quad |\log(\tau_2)| \le |\log(\tau_1)| \text{ if } 0 < \tau_2 \le \tau_1 \le 1.$$

Case 1. $2\rho^{-1}\cos\theta + \rho^{-2} \ge 0$. Thus,

$$\int_{\frac{1+\rho_0}{2\rho_0}}^{\frac{1+\rho_0}{|x-y|}} d\rho\rho[\log(1+2\rho^{-1}\cos\theta+\rho^{-2})]^2$$
$$\le \int_{\frac{1+\rho_0}{2\rho_0}}^{\frac{1+\rho_0}{|x-y|}} d\rho\rho[\log(1+2\rho^{-1}+\rho^{-2})]^2 = 4\int_{\frac{1+\rho_0}{2\rho_0}}^{\frac{1+\rho_0}{|x-y|}} d\rho\rho[\log(1+\rho^{-1})]^2$$
$$\le 4\int_{\frac{1+\rho_0}{2\rho_0}}^{\frac{1+\rho_0}{|x-y|}} d\rho\rho^{-1} = 4\log\left(\frac{2\rho_0}{|x-y|}\right), \tag{2.26}$$

the last inequality is valid since $0 \le \log(1+\tau) \le \tau, \tau \ge 0$.

Case 2. $2\rho^{-1}\cos\theta + \rho^{-2} \le 0$. Thus,

$$\int_{\frac{1+\rho_0}{2\rho_0}}^{\frac{1+\rho_0}{|x-y|}} d\rho\rho[\log(1+2\rho^{-1}\cos\theta+\rho^{-2})]^2$$
$$\le 4\int_{\frac{1+\rho_0}{2\rho_0}}^{\frac{1+\rho_0}{|x-y|}} d\rho\rho[\log(1-\rho^{-1})]^2 = 4\int_{\frac{|x-y|}{1+\rho_0}}^{\frac{2\rho_0}{1+\rho_0}} d\rho\rho^{-1}\left[\frac{\log(1-\rho)}{\rho}\right]^2$$
$$\le 2\left[\frac{(1+\rho_0)}{\rho_0}\log\left(\frac{1+\rho_0}{1-\rho_0}\right)\right]^2 \int_{\frac{|x-y|}{1+\rho_0}}^{\frac{2\rho_0}{1+\rho_0}} d\rho\rho^{-1}$$
$$= 2\left[\frac{(1+\rho_0)}{\rho_0}\log\left(\frac{1+\rho_0}{1-\rho_0}\right)\right]^2 |\log\left(\frac{2\rho_0}{|x-y|}\right), \tag{2.27}$$

where in the second line we wrote the change of coordinates $\rho \mapsto \rho^{-1}$, and the third line follows from the fact that the function $\rho \mapsto (\rho^{-1}\log(1-\rho))^2$ is increasing in $[0,1)$.

By following (2.23)-(2.27) we deduce

$$\mathcal{I} \le c(\rho_0)\log\left(\frac{2\rho_0}{|x-y|}\right)|x-y|^2. \tag{2.28}$$

The r.h.s in (2.20), together with (2.22) and (2.28) implies the upper bound in (2.17).

Lower bound. We have that $[0,2\pi)\times[0,1-\rho_0] \subset A$. Thus, since $\cos(\theta-\theta_0) \ge \frac{1}{2}$, for $\theta - \theta_0 \in [-\frac{\pi}{3}, \frac{\pi}{3}] \mod 2\pi$

$$\mathcal{I} \ge \frac{1}{4}\int_0^{\frac{2\pi}{3}} d\theta \int_{\frac{1-\rho_0}{2\rho_0}|x-y|}^{1-\rho_0} d\rho\rho[\log(1+|x-y|\rho^{-1})]^2$$

$$
\begin{aligned}
&\geq \frac{\pi}{6}|x-y|^2 \int_{\frac{1-\rho_0}{2\rho_0}|x-y|}^{1-\rho_0} d\rho\rho^{-1}\left(1+\frac{|x-y|}{\rho}\right)^{-2} \\
&\geq \frac{\pi}{6}|x-y|^2 \int_{\frac{1-\rho_0}{2\rho_0}|x-y|}^{1-\rho_0} \rho^{-1} \\
&= \frac{\pi}{6}\log\left(\frac{2\rho_0}{|x-y|}\right)|x-y|^2,
\end{aligned}
\tag{2.29}
$$

where the second inequality is valid since $\log(1+\tau) \geq \tau/(\tau+1), \tau \geq 0$.

The lower bound in (2.20),together with (2.22) and (2.29) implies that there exist small positive constants $\tilde{c}_{2,1}(\rho_0), \tilde{c}_{2,2}(\rho_0)$ such that if $|x-y| \leq \tilde{c}_1$ then

$$
\mathfrak{d}_u^2(x,y) \geq \tilde{c}_{2,2}\log\left(\frac{2\rho_0}{|x-y|}\right)|x-y|^2.
$$

By (49) in Lemma 5.5 of [SSV18],

$$
\inf_{x,y\in B_{\rho_0}, |x-y|\geq \tilde{c}_{2,1}} \mathfrak{d}_u^2(x,y) = \tilde{c}_{2,3}(\rho_0) > 0.
$$

Since $1 \geq \max_{x,y\in B_{\rho_0}, |x-y|\geq \tilde{c}_1} \log(2\rho_0/|x-y|)\,|x-y|^2$, we deduce that the lower bound in (2.17) is valid for $c = \tilde{c}_{2,2} \wedge \tilde{c}_{2,3}$.

□

2.3 Further second order properties

In this section we prove further second order properties of the solution of the stochastic Poisson equation. Theorem 2.2 together with Proposition 2.2 will be used in the next section for proving the hitting probabilities result stated in Theorem 2.3.

Proposition 2.2. *Fix $\rho_0 \in (0,1)$ and let $(u(x), x \in B_1)$ be the stochastic process defined in (2.7).*

1. *For all $x, y \in B_{\rho_0}$,*
$$
|\sigma_x^2 - \sigma_y^2| \leq c(\eta,\rho_0,d)|x-y|^\eta, \tag{2.30}
$$
with $\eta = 1$, for $d = 1,2$, and $\eta \in (0,1)$ for $d = 3$.

2. *There exist constants $0 < c(d,\rho_0) < C(d,\rho_0)$ such that for any $x \in B_{\rho_0}$,*
$$
c \leq \sigma_x^2 \leq C. \tag{2.31}
$$

3. *For any $x, y \in B_{\rho_0}$ such that $x \neq y$, $\rho_{x,y} < 1$.*

4. $$
E(Var(u(x)|u(y))) \asymp \left[\log\left(\frac{2\rho_0}{|x-y|}\right)\right]^\beta |x-y|^{2\wedge(4-d)}, x,y \in B_{\rho_0}, \tag{2.32}
$$
where $\beta = 1$, if $d = 2$, and $\beta = 0$ if $d = 1,2$.

Proof. 1. *Case 1. d=1.* By (2.13),

$$|\sigma_x^2 - \sigma_y^2| = \frac{1}{6}|x-y||x+y|\left(2-(x^2+y^2)\right) \le \frac{2}{3}\rho_0|x-y|, x, y \in B_{\rho_0}.$$

Case 2. d=2. By (2.3) and (2.9),

$$\sigma_x^2 = \int_{B_1} dz \left[\log\left(\frac{|x-z||z|}{||z|^2 x - z|}\right)\right]^2$$

We claim that the map $x \mapsto \sigma_x^2$ has bounded partial derivatives in B_{ρ_0}. The mean value theorem will imply (2.30). Indeed, by the rule of the derivation under the integral sign,

$$\partial_{x_j}\sigma_x^2 = \int_{B_1} dz \log\left(\frac{|x-z||z|}{||z|^2x-z|}\right)\left[\frac{x_j - z_j}{|x-z|^2} - \frac{(|z|^2x_j - z_j)|z|^2}{||z|^2x-z|^2}\right], j = 1, 2.$$

By the lower bound in (2.4)

$$\frac{||z|^2x_j - z_j||z|^2}{||z|^2x-z|^2} \le \frac{||z|^2x_j - z_j|}{(1-|x||z|)^2} \le \frac{1+\rho_0}{(1-\rho_0)^2},$$

implying that

$$\begin{aligned}
&|\partial_{x_j}\sigma_x^2| \le (\mathcal{I} + \mathcal{J}),\\
&\quad \mathcal{I} := \int_{B_1} dz\,|\log(|x-z|)|\left[\frac{|x_j - z_j|}{|x-z|^2} + \frac{1+\rho_0}{(1-\rho_0)^2}\right],\\
&\quad \mathcal{J} := \int_{B_1} dz \left|\log\left(\frac{||z|^2x-z|}{|z|}\right)\right|\left[\frac{|x_j-z_j|}{|x-z|^2} + \frac{1+\rho_0}{(1-\rho_0)^2}\right].
\end{aligned}$$

Using the change of coordinates $z \mapsto x - z$

$$\begin{aligned}
\mathcal{I} &= \int_{B_1(x)} dz\,|\log(|z|)|\left[\frac{|z_j|}{|z|^2} + \frac{1+\rho_0}{(1-\rho_0)^2}\right]\\
&\le \int_{B_2} dz\,|\log(|z|)|\left[|z|^{-1} + \frac{1+\rho_0}{(1-\rho_0)^2}\right]\\
&= c(d)\int_0^2 d\rho|\log\rho|\left[1 + \frac{1+\rho_0}{(1-\rho_0)^2}\rho\right] = c(d,\rho_0) < \infty.
\end{aligned}$$

Following a similar procedure, but now applying (2.4), we deduce

$$\begin{aligned}
\mathcal{J} &\le \int_{B_2} dz \log\left(\frac{1+\rho_0}{1-\rho_0}\right)\left[|z|^{-1} + \frac{1+\rho_0}{(1-\rho_0)^2}\right]\\
&= c(d,\rho_0)\int_0^2 d\rho\left[1 + \frac{1+\rho_0}{(1-\rho_0)^2}\rho\right] = c(d,\rho_0) < \infty.
\end{aligned}$$

Case 3. d=3. This is Lemma 5.13 in [SSV18].

2. Due to (2.30) the map $x \mapsto \sigma_x^2$ is continuous in B_{ρ_0}, additionally by (2.9) is not hard to proof that $\sigma_x^2 > 0, x \in B_{\rho_0}$, implying (2.31).

3. This is Lemma 5.1 in [SSV18].

4. Due to (2.17), and 1. 2. 3. in this Proposition, the hypothesis of Proposition 1.1, are valid for $d = 2, 3$ implying (2.32).

If $d = 1$, condition 3. of Proposition 1.1 is not satisfied, hence we just deduce (1.51) i.e. the the upper bound in (2.32). In this case, by the isometry (2.2),

$$\begin{aligned}\sigma_{x,y} &= \frac{1}{4}\int_{-1}^{1} dz(|x-z| - |1-xz|)(|y-z| - |1-yz|)\\ &= \frac{1}{12}(1 - xy - |x-y|)\left(2(|x-y|+1) - \left(x^2+y^2\right)\right) \geq 0.\end{aligned}$$

This and (2.12) implies that,

$$\begin{aligned}\sigma_x\sigma_y - \sigma_{x,y} &= \frac{1}{12}|x-y|^2\,(1 - xy - |x-y|)\\ &\geq \frac{1-\rho_0^2}{12}|x-y|^2,\\ \sigma_x\sigma_y + \sigma_{x,y} \geq \sigma_x\sigma_y &\geq \frac{1}{6}(1-\rho_0^2)^2, \quad \sigma_y^2 \leq \frac{1}{6}.\end{aligned}$$

Thus, by (1.50)

$$\begin{aligned}E(\mathrm{Var}\,(u(x)|u(y)) &= \frac{(\sigma_x\sigma_y - \sigma_{x,y})(\sigma_x\sigma_y + \sigma_{x,y})}{\sigma_y^2})\\ &\geq \frac{(1-\rho_0^2)^3}{12}|x-y|^2,\end{aligned}$$

implying the lower bound in (2.32). □

2.4 Hitting probabilities

Consider the Gaussian random field

$$\boldsymbol{v} = (\boldsymbol{v}(x) = (v_1(x), \ldots, v_D(x)),\ x \in B_1)\,,$$

where $(v_j(x)),\ j = 1, \ldots, D$, are independent copies of the process $(v(x))$ defined in (2.6). The process $\boldsymbol{v}$ is the random field solution to the system of SPDEs

$$\begin{cases} -\Delta v_j(x) = \dot{W}_j(x), & x \in B_1,\\ \quad v_j(x) = v_0(x), & x \in \mathbb{S}^{d-1}, \end{cases} \tag{2.33}$$

$j = 1, ..., D$, where $(W_j, j = 1, ..., D)$ are independent copies of a white noise introduced at the beginning of section 2.1 and v_0 is such that the function $x \mapsto I_0(x)$ is continuous (see Remark 2.4 for sufficient conditions).

For $\tau \in \mathbb{R}_+$, let

$$g(\tau) = \left[\log\left(\frac{2\rho_0}{\tau}\right)\right]^{\frac{\beta}{2}} \tau^{1\wedge(2-d/2)}, q_g(\tau) = \tau^D\left(g^{-1}(\tau)\right)^{-d}, \tag{2.34}$$

where $d = 1, 2, 3$ and $\beta = 1_{d=2}$.

Let $D_0 = d[1 \wedge (2-d)]^{-1}$. According to Example 1.4 the function q_g satisfies the conditions required by the definition of the q_g-Hausdorff measure if and only if $D > D_0$. We are now able to prove the following result relative to hitting probabilities bounds for the D-dimensional random field $\boldsymbol{v}$.

Theorem 2.3. *Fix $\rho_0 \in (0,1)$, $N > 0$ and let $K = B_{\rho_0}$. Suppose that the function $K \ni x \mapsto I_0(x)$ satisfies condition (CX2). The D-dimensional random field $\boldsymbol{v}$ satisfy the following bounds:*

1. *Let $D > D_0$.*
   (a) *There exists a constant $C := C(d, D, K)$ such that for any Borel set $A \in \mathcal{B}(\mathbb{R}^D)$,*
   $$P(\boldsymbol{v}(K) \cap A \neq \emptyset) \leq C\mathcal{H}_{q_g}(A). \tag{2.35}$$
   (b) *Let $A \in \mathcal{B}(\mathbb{R}^D)$ be such that $A \subset B_N$. There exists a constant $c := c(d, D, K, N)$ such that*
   $$P(\boldsymbol{v}(K) \cap A \neq \emptyset) \geq c\,Cap_{(q_g)^{-1}}(A). \tag{2.36}$$

2. *Assume that $D < D_0$. Let $A \in \mathcal{B}(\mathbb{R}^D)$ be such that $A \subset B_N$. There exists a constant $c := c(d, D, K, N) > 0$ such that*
   $$P(\boldsymbol{v}(K) \cap A \neq \emptyset) \geq c[v_g(0)]^{-1} > 0, \tag{2.37}$$
   with v_g as defined in (1.54).

Proof. Theorem 2.2, Example 1.2 and the hypothesis on the (CX2) condition implies that $\boldsymbol{v}$ has a modification with a.s continuous paths on K.

We distinguish two cases:

Case 1. $D > D_0$. By Theorem 2.2 and hypothesis (CX2) v_1 is $\hat{g}$-Gaussian on $K^{(\eta)}$ (for $\eta > 0$ small enough). In Example 1.3 we proved that g satisfies the hypotheses of Lemma 1.1(2.). This facts together with 2. in Proposition 2.2 implies the validity of the hypotheses of Corollary 1.1 and then (2.35).

In Example 1.6, we verified (1.55). Additionally (CX1) and (CX3) are valid due to Proposition 2.2. (1.56) in Theorem 1.3 implies (2.36).

Case 2. $D < D_0$. According to Example 1.6 in this case (1.57) is valid instead of (1.55). Since the rest of the hypotheses of Theorem 1.3 remains valid, by (1.58) we deduce (2.37). □

Theorem 2.3 implies the following Corollary which identifies D_0 as the critical dimension for polarity of points.

Corollary 2.1. *If $D > D_0$, points are polar for $\boldsymbol{v}$ and are non polar if $D < D_0$.*

Proof. Assume first $D > D_0$. By the definition of the q_g measure we have $\mathcal{H}_{q_g}(\{z\}) = 0$. Hence the polarity of $\{z\}$ follows by (2.35).

If $D < D_0$, we apply (2.37) to $A = \{z\}$. In fact, if $D < D_0$ any bounded set A is non polar for $\boldsymbol{v}$. □

Chapter 3

$\boldsymbol{g}$-Gaussian random fields

This chapter addresses questions similar to those of Chapter 1 for a wider class of g-Gaussian processes: $\boldsymbol{g}$-Gaussian processes. By following the steps of the proofs of Theorems 1.2 and 1.3, together with some additional technical tools, we find upper and lower bounds for hitting probabilities in terms of the q-Hausdorff measure and $\mathfrak{g}$-capacity, respectively. These results can be found in Theorems 3.1 and 3.2, respectively.

$\boldsymbol{g}$-Gaussian processes are a generalization of anisotropic Gaussian processes. We finish this chapter by mentioning some examples of anisotropic Gaussian processes that are solutions of Stochastic Partial Differential Equations and whose hitting probabilities have been a subject of study in the past.

3.1 Definition

Fix I, J compact sets of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. Assume that

$$Y = \{Y(t,x) : (t,x) \in I \times J\}$$

is a Gaussian process with canonical metric

$$\mathfrak{d}((t,x),(\bar{t},\bar{x})) = \|Y(t,x) - Y(\bar{t},\bar{x})\|_{L^2(\Omega)}, (t,x),(\bar{t},\bar{x}) \in I \times J.$$

The functions $m_{t,x}, \sigma^2_{t,x}, (t,x) \in I \times J$ denote the mean and the variance of Y; and the functions $\sigma_{(t,x),(\bar{t},\bar{x})}, \rho_{(t,x),(\bar{t},\bar{x})}, (t,x),(\bar{t},\bar{x}) \in I \times J$ denote the covariance and the correlation of Y.

Definition 3.1. Let $\boldsymbol{g} = (g_1, g_2)$, with g_1, g_2 gauge functions. We say Y is $\boldsymbol{g}$-*Gaussian* on $I \times J$ if

$$\mathfrak{d}((t,x),(\bar{t},\bar{x})) \asymp g_1(|t-\bar{t}|) + g_2(|x-\bar{x}|), (t,x),(\bar{t},\bar{x}) \in I \times J, \tag{3.1}$$

where $\mathfrak{d}$ is the canonical metric of Y. If Y only satisfies the upper bound in (3.1) we say that it is $\hat{\boldsymbol{g}}$-*Gaussian* on $I \times J$.

Remark 3.1. The definition above can be extended to the case where we consider gauge functions $g_1, ..., g_n$ and compact sets $I_1, ..., I_n$. The results that we obtain in this chapter are easily generalized to this case. We avoid to write them in such a generality, since the applications we have in mind are solutions to Stochastic Partial Differential Equations which canonical metric satisfies (3.1).

Write

$$B_{\mathfrak{d},\varepsilon}(t,x) := \{(\bar{t},\bar{x}) \in I \times J : \mathfrak{d}_{(t,x),(\bar{t},\bar{x})} \leq \varepsilon\}$$

for the $\mathfrak{d}$ ball centered on $(t,x) \in I \times J$ and radius ε. We denote by $N_{\mathfrak{d}}(I \times J, \varepsilon) \equiv N(\varepsilon)$ the minimum number of such balls needed to cover $I \times J$.

If Y is a $\hat{\boldsymbol{g}}$-Gaussian process on $I \times J$, it is not hard to prove that

$$N(\varepsilon) \leq C \left(\frac{\varnothing_I}{g_1^{-1}(\varepsilon)}\right)^{d_1} \left(\frac{\varnothing_J}{g_2^{-1}(\varepsilon)}\right)^{d_2}, \varepsilon \in (0, g_1(\varnothing_I) \wedge g_2(\varnothing_J)]. \tag{3.2}$$

[AT07, Thm. 1.3.5] and (3.2) implies that there exists a positive random variable η such that

$$\sup_{\substack{(t,x),(\bar{t},\bar{x}) \in I \times J,\\ \mathfrak{d}(t,x),(\bar{t},\bar{x}) \leq \delta}} |Y(t,x) - Y(\bar{t},\bar{x})| \leq \tag{3.3}$$

$$C\left(\int_0^\delta d\varepsilon \sqrt{d_1 \log\left(\frac{\varnothing_I}{g_1^{-1}(\varepsilon)}\right)} + \int_0^\delta d\varepsilon \sqrt{\log\left(d_2 \frac{\varnothing_J}{g_2^{-1}(\varepsilon)}\right)}\right), \delta \in (0,\eta).$$

Similarly to Example 1.2, (3.3) implies the following criteria of sample path continuity for $\hat{\boldsymbol{g}}$-Gaussian processes.

Example 3.1. Let Y be a $\hat{\boldsymbol{g}}$-Gaussian process on $I \times J$, with I, J compact subsets of $\mathbb{R}^{d_1}, \mathbb{R}^{d_2}$ respectively. If $[g_1(\tau) \vee g_2(\tau)] \leq |\log(\tau)|^{-\nu}$, $\nu > \frac{1}{2}$, then Y has a modification with a.s. continuous sample paths.

Remark 3.2. By Example 3.1, we deduce that every $\hat{\boldsymbol{g}}$-Gaussian process with $\boldsymbol{g}$ having entries as the gauge functions in Example 3.2, has a modification with a.s. continuous sample paths.

3.2 Criteria for hitting probabilities

Let I, J be compact subsets of $\mathbb{R}^{d_1}$,$\mathbb{R}^{d_2}$, respectively, and

$$\boldsymbol{Y} = \{\boldsymbol{Y}(t,x) = (Y_1(t,x), \ldots, Y_D(t,x)),\ (t,x) \in I \times J\},$$

be a D-dimensional process with i.i.d. coordinates. The probability that the process $\boldsymbol{Y}$ hits A a Borel subset of $\mathbb{R}^D$ is

$$P(\boldsymbol{Y}(I \times J) \cap A \neq \emptyset). \tag{3.4}$$

We devote this section to estimate (3.4) when Y_1 is a $\boldsymbol{g}$-Gaussian process. The main results are Theorems 3.1 and 3.2 which yield upper and lower bounds in terms of the notions of q-Hausdorff measure and $\mathfrak{g}$-capacity, respectively.

3.2.1 Upper bounds for hitting probabilities

The following result is an extension of Lemma 3.1. We will apply it in the proof Theorem 3.1.

Lemma 3.1. *[HCSS21, Lem. 3.3] Let g_1, g_2 be differentiable gauge functions satisfying* (1.6), (1.8) *and* (1.9) *in Lemma 1.1. Let I and J be compact sets of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively. Fix $\eta > 0$ and let*

$$\boldsymbol{U} = \{\boldsymbol{U}(t,x) = (U_1(t,x), \dots, U_D(t,x)),\ (t,x) \in I^{(\eta)} \times J^{(\eta)}\},$$

be a D-dimensional stochastic process with a.s. continuous sample paths. Assume that there exists a positive constant $c_{3,1}$ that for any ε small enough,

$$\begin{aligned} E\left(\sup_{x\in J^{(\eta)}} \int_{B_\varepsilon(t)} ds \int_{B_\varepsilon(t)} d\bar{s}\ \exp\left(\frac{|\boldsymbol{U}(s,x) - \boldsymbol{U}(\bar{s},x)|}{g_1(|s-\bar{s}|)}\right)\right) &\le c_{3,1}\varepsilon^{2d_1}, \quad t \in I, \\ E\left(\sup_{t\in I^{(\eta)}} \int_{B_\varepsilon(x)} dy \int_{B_\varepsilon(x)} d\bar{y}\ \exp\left(\frac{|\boldsymbol{U}(t,y) - \boldsymbol{U}(t,\bar{y})|}{g_2(|y-\bar{y}|)}\right)\right) &\le c_{3,1}\varepsilon^{2d_2}, \quad x \in J. \end{aligned} \tag{3.5}$$

Let $S^1_\varepsilon(t) = B_{\frac{g_1^{-1}(\varepsilon)}{2}}(t)$, $S^2_\varepsilon(x) = B_{\frac{g_2^{-1}(\varepsilon)}{2}}(x)$ and $\tilde{S}_\varepsilon(t,x) = S^1_\varepsilon(t) \times S^2_\varepsilon(x)$. Then, for all $p \ge 1$, there exists a constant $c_{3,2}(c_{3,1}, p, d_1, d_2, \boldsymbol{g})$ such that, for all ε small enough and $(t,x) \in I \times J$,

$$E\left(\sup_{(s,y)\in\tilde{S}_\varepsilon(t,x)} |\boldsymbol{U}(s,y) - \boldsymbol{U}(t,x)|^p\right) \le c_{3,2}\ \varepsilon^p. \tag{3.6}$$

Proof. Consider the process $\boldsymbol{U}^{(t)} = \{\boldsymbol{U}(t,x), t \in I\}$. If in the proof of (1.10) in Lemma 1.1, we replace $\mathcal{C}_\varepsilon(\omega)$ by

$$\sup_{t\in I^{(\eta)}} \int_{S^2_\varepsilon(x)} dy \int_{S^2_\varepsilon(x)} d\bar{y}\ \exp\left(\frac{|\boldsymbol{U}^{(t)}(y,\omega) - \boldsymbol{U}^{(t)}(\bar{y},\omega)|}{g_2(|y-\bar{y}|)}\right),$$

$\sup_{y\in S_\varepsilon(x)} |\boldsymbol{M}(y) - \boldsymbol{M}(x)|$ by $\sup_{t\in I^{(\eta)}} \sup_{y\in S^2_\varepsilon(x)} |\boldsymbol{U}^{(t)}(y) - \boldsymbol{U}^{(t)}(x)|$, and (1.6) by (3.5), we deduce that

$$E\left(\sup_{t\in I^{(\eta)}} \sup_{y\in S^2_\varepsilon(x)} |U(t,y) - U(t,x)|^p\right) \le C(c_{3,1}, p, d_2, g_2)\varepsilon^p.$$

The same procedure, now with the process $\boldsymbol{U}^{(x)} = \{\boldsymbol{U}(t,x), x \in J\}$, leads to

$$E\left(\sup_{x\in J^{(\eta)}} \sup_{s\in S^1_\varepsilon(t)} |U(s,x) - U(t,x)|^p\right) \le C(c_{3,1}, p, d_1, g_1)\varepsilon^p.$$

(3.6) follows by the triangle inequality. □

Remark 3.3. Let $I^{(\eta)}, J^{(\eta)}$ be sets as in Lemma 3.1. Assume that U is a $\hat{\boldsymbol{g}}$-Gaussian process on $I^{(\eta)} \times J^{(\eta)}$ with continuous sample paths, we prove that it satisfies (3.5). By the continuity of the trajectories of U,

$$\begin{aligned} &E\left(\sup_{x\in J^{(\eta)}} \int_{B_\varepsilon(t)} ds \int_{B_\varepsilon(t)} d\bar{s} \exp\left(\frac{|U(s,x) - U(\bar{s},x)|}{[g_1(|s-\bar{s}|)]}\right)\right) \\ &\quad = \int_{B_\varepsilon(t)} ds \int_{B_\varepsilon(t)} d\bar{s} \exp E \sup_{x\in J^{(\eta)}} \frac{|U(s,x) - U(\bar{s},x)|}{g_1(|s-\bar{s}|)}. \end{aligned}$$

Since U is $\hat{\boldsymbol{g}}$-Gaussian, there exists a positive constant such that for all x in $I^{(\eta)}$,

$$E\left(\frac{|U(s,x)-U(\bar{s},x)|^2}{[g_1(|s-\bar{s}|)]^2}\right) \le C(g_1)E(Z^2),$$

where Z is a standard centered Gaussian random variable. Then by the Sudakov-Fernique inequality (see [Adl90, Thm. 2.9], for example),

$$E\left(\sup_{x\in J^{(\eta)}} \frac{|U(s,x)-U(\bar{s},x)|}{g_1(|s-\bar{s}|)}\right) \le C(g_1),$$

and we deduce the validity of the first inequality in (3.5). We prove the second inequality in (3.5) by similar arguments.

For $\varepsilon \in (0,1)$, $j=(j_1,\dots,j_{d_1},j_{d_1+1},\dots,j_{d_1+d_2}) \in \mathbb{Z}^{d_1+d_2}$, define

$$R_j^{\varepsilon,1} = \prod_{i=1}^{d_1}\left[\frac{g_1^{-1}(\varepsilon)}{\sqrt{d_1}}j_i, \frac{g_1^{-1}(\varepsilon)}{\sqrt{d_1}}(j_i+1)\right], \; R_j^{\varepsilon,2} = \prod_{i=d_1+1}^{d_1+d_2}\left[\frac{g_2^{-1}(\varepsilon)}{\sqrt{d_2}}j_i, \frac{g_2^{-1}(\varepsilon)}{\sqrt{d_2}}(j_i+1)\right],$$
$$\tilde{R}_j^{\varepsilon} = R_j^{\varepsilon,1}\times R_j^{\varepsilon,2}. \tag{3.7}$$

For $t\in R_j^{\varepsilon,1}$, and for $x \in R_j^{\varepsilon,2}$, let

$$t_j^{\varepsilon} = \left(\frac{g_1^{-1}(\varepsilon)}{\sqrt{d_1}}j_i\right)_{i=1,\dots,d_1}, x_j^{\varepsilon} = \left(\frac{g_2^{-1}(\varepsilon)}{\sqrt{d_2}}j_i\right)_{i=d_1+1,\dots,d_1+d_2}.$$

Fix $d_1,d_2\in\mathbb{N}_+$, for $\boldsymbol{g}=(g_1,g_2)$ with g_1,g_2 gauge functions we define

$$q_{\boldsymbol{g}}(\tau) = \frac{\tau^D}{\left(g_1^{-1}(\tau)\right)^{d_1}\left(g_2^{-1}(\tau)\right)^{d_2}}, \quad \tau\in\mathbb{R}_+. \tag{3.8}$$

Observe that if $g_1=g_2:=g$ then $q_{\boldsymbol{g}}=q_{g_1}$ with $d=d_1+d_2$ and q_{g_1} as defined in (1.31).

If we assume that g_1, g_2 are differentiable in $(0,\tau_0)$, then $q_{\boldsymbol{g}}$ is strictly increasing on $(0,\tau_0)$ if and only if

$$D > \tau\left(\frac{d_1}{g_1^{-1}(\tau)\dot{g}_1(g_1^{-1}(\tau))} + \frac{d_2}{g_2^{-1}(\tau)\dot{g}_2(g_2^{-1}(\tau))}\right), \quad \tau\in(0,\rho_0), \tag{3.9}$$

or equivalently,

$$D > g_2(\tau)\left(\frac{d_1}{g_1^{-1}(g_2(\tau))\dot{g}_1(g_1^{-1}(g_2(\tau)))} + \frac{g_2}{\tau\dot{g}_2(\tau)}\right), \tau\in(0,g_2^{-1}(\rho_0)). \tag{3.10}$$

i.e. that if (3.9) is satisfied then $\mathcal{H}_{q_{\boldsymbol{g}}}$ is a well defined Hausdorff measure. (See Definition 1.3 in Chapter 1 for the definition of q-Hausdorff measure).

Example 3.2. We study the map $q_{\boldsymbol{g}}$ in the case of the gauge functions in Example 1.1.

1. $g_i(\tau)=\tau^{\nu_i}$, $\tau\ge 0$, $\nu_i>0$, $i=1,2$. We have that $q_{\boldsymbol{g}}(\tau)=\tau^{D-\chi}$, $\chi = d_1/\nu_1+d_2/\nu_2$. Thus $q_{\boldsymbol{g}}$ is strictly increasing on $\mathbb{R}^+$ if and only if $D>\chi$. In this case we recover the γ-dimensional Hausdorff measure $\mathcal{H}_\gamma$, for $\gamma=D-\chi$.

2. $g_1(\tau) = \tau^{\nu_1}$, $g_2(\tau) = |\log\tau|^{\gamma}\,\tau^{\nu_2}$, $\tau \in [0, e^{-\frac{\gamma}{\nu_2}}]$, $\gamma, \nu_1, \nu_2 > 0$. In (1.37) from Example (3.2) we computed the inverse function of g_2 (with ν instead of ν_2), by writing this function in (3.8) we deduce,

$$q_{\boldsymbol{g}}(\tau) = \tau^{D-\frac{d_1}{\nu_1}} \exp\left[-\frac{d_2\nu_2}{\gamma}W_{-1}\left(-\frac{\nu_2}{\gamma}\tau^{\frac{1}{\gamma}}\right)\right],$$

with W_{-1} the lower real valued branch of the Lambert W function. Similarly to (1.39),

$$q_{\boldsymbol{g}}(\tau) \asymp \tau^{D-\chi} \exp\left(\gamma\frac{d_2}{\nu_2}\left|2\log\left(c\tau^{\frac{1}{\gamma}}\right)\right|^{\frac{1}{2}}\right), \chi = \frac{d_1}{\nu_1} + \frac{d_2}{\nu_2}, \tag{3.11}$$

with c depending on ν_2, γ. Thus $q_{\boldsymbol{g}}$ is strictly increasing on an interval around zero if and only if $D > \chi$.

Theorem 3.1. *[HCSS21, Thm. 3.3] Fix I, J compact sets of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d,2}$ and $\eta > 0$. Let*

$$\boldsymbol{Y} = \{\boldsymbol{Y}(t,x) = (Y_1(t,x), ..., Y_D(x)), (t,x) \in I^{(\eta)} \times J^{(\eta)}\}, \tag{3.12}$$

be a D-dimensional stochastic process with a.s. continuous sample paths and i.i.d. coordinates that are distributed as Gaussian random fields. Assume that $\sigma^2_{I^{(\eta)},J^{(\eta)}} := \inf_{(t,x)\in I^{(\eta)}\times J^{(\eta)}} \sigma^2_{t,x} > 0$, and that the process $\boldsymbol{Y}$ satisfies (3.5) for some gauge functions g_1, g_2. Suppose also that g_1, g_2 fulfills the hypothesis of Lemma 3.1(2.) and that the function $q_{\boldsymbol{g}}$ defined in (3.8) is strictly increasing on a small interval $(0, \rho_0)$.

Then there exists a constant $C(I, J, \sigma^2_{I^{(\eta)},J^{(\eta)}}, d_1, d_2, D)$ such that for any Borel set $A \subset \mathbb{R}^D$,

$$P\left(\boldsymbol{Y}(I\times J) \cap A \neq \emptyset\right) \leq C(I\times J, \sigma^2_{I^{(\eta)},J^{(\eta)}}, D, d_1, d_2)\mathcal{H}_{q_{\boldsymbol{g}}}(A). \tag{3.13}$$

Proof. Let $z \in \mathbb{R}^D$ and $\varepsilon > 0$ small enough. Since $\boldsymbol{Y}$ satisfies the conditions of Lemma 3.1, and $\tilde{R}^{\varepsilon}_j \subset \tilde{S}_{\varepsilon}(t^{\varepsilon}_j, x^{\varepsilon}_j)$, for all $p \geq 1$, there exists a constant $C(p, d_1, d_2)$ such that for all $\varepsilon > 0$ small enough and $(t,x) \in I \times J$,

$$E\Big(\sup_{(s,y)\in\tilde{R}^{\varepsilon}_j} \left|\boldsymbol{Y}(s,y) - \boldsymbol{Y}(t^{\varepsilon}_j, x^{\varepsilon}_j)\right|^p\Big) \leq C(p, d_1, d_2)\varepsilon^p. \tag{3.14}$$

The proof of Theorem 1.1 is also valid with $\boldsymbol{X}$ there replaced by $\boldsymbol{Y}$, R^{ε}_j by $\tilde{R}^{\varepsilon}_j$, and $I^{(\eta)}$ by $I^{(\eta)} \times J^{(\eta)}$. Thus, by (3.14) we deduce

$$P\left(\boldsymbol{Y}(\tilde{R}^{\varepsilon}_j) \cap B_{\varepsilon}(z) \neq \emptyset\right) \leq C(\sigma^2_{I^{(\eta)},J^{(\eta)}}, d_1, d_2, D)\varepsilon^D,$$

for some constant $C(\sigma^2_{I^{(\eta)},J^{(\eta)}}, d_1, d_2, D)$. Similarly as in the proof of Lemma 1.2, an argument based on total probabilities (see (1.44)) implies that,

$$P\left(\boldsymbol{Y}(I\times J) \cap B_{\varepsilon}(z) \neq \emptyset\right) \leq C(\sigma^2_{I^{(\eta)},J^{(\eta)}}, d_1, d_2, D)\; q_{\boldsymbol{g}}(\varepsilon). \tag{3.15}$$

We finish the proof of (3.13), using the same covering argument as in (1.41).

□

Remark 3.3 implies the following Corollary relative for upper bounds of hitting probabilities for $\hat{\boldsymbol{g}}$-Gaussian processes.

Corollary 3.1. *[HCSS21, Cor. 3.2] Consider a Gaussian process with continuous sample paths* $\boldsymbol{Y}$ *in* (3.12)*, and such that* $\sigma^2_{I^{(\eta)},J^{(\eta)}} := \inf_{(t,x)\in I^{(\eta)}\times J^{(\eta)}} \sigma^2_{t,x} > 0$*. Assume that process* Y_1 *is* $\hat{\boldsymbol{g}}$*- Gaussian* (3.5) *on* $I^{(\eta)} \times J^{(\eta)}$ *,with* g_1*,*g_2 *gauge functions satisfying the hypothesis of Lemma 3.1(2.) and such that the function* $q_{\boldsymbol{g}}$ *defined in* (3.8) *is strictly increasing on a small interval* $(0, \rho_0)$*. Then there exists a constant* $C(I, J, \sigma^2_{I^{(\eta)},J^{(\eta)}}, d_1, d_2, D)$ *such that for any Borel set* $A \subset \mathbb{R}^D$*,*

$$P\left(\boldsymbol{Y}(I \times J) \cap A \neq \emptyset\right) \leq C(I \times J, \sigma^2_{I^{(\eta)},J^{(\eta)}}, D, d_1, d_2)\mathcal{H}_{q_{\boldsymbol{g}}}(A). \tag{3.16}$$

Similarly as we did in Corollary 1.2 and with the same arguments, from (3.15) we derive the following result on polarity of singletons.

Corollary 3.2. *The hypotheses are those of Theorem 3.1. In addition assume that*

$$\lim_{\tau\downarrow 0} q_{\boldsymbol{g}}(\tau) := q_{\boldsymbol{g}}(0) = 0. \tag{3.17}$$

Then $P(\boldsymbol{Y}(I\times J)\cap\{z\} \neq \emptyset) = 0$*, that is, for the random field* $\boldsymbol{Y}$ *restricted to* $I \times J$*, any set* $\{z\} \subset \mathbb{R}^D$ *is polar.*

Example 3.3. We analyze condition (3.17) for the maps $q_{\boldsymbol{g}}$ in Example 3.2.

1. $g_i(\tau) = \tau^{\nu_i}$, $\tau \geq 0$, $\nu_i > 0$, $i = 1, 2$. $q_{\boldsymbol{g}}(\tau) = \tau^{D-\chi}$, $\chi = d_1/\nu_1 + d_2/\nu_2$. In this case

$$q_{\boldsymbol{g}}(0) = \begin{cases} 0 & \text{if } D > \chi, \\ 1 & \text{if } D = \chi, \\ \infty & \text{if } D < \chi. \end{cases} \tag{3.18}$$

   And (3.17) is satisfied if and only if $D > \chi$.

2. $g_1(\tau) = \tau^{\nu_1}$, $g_2(\tau) = |\log\tau|^{\gamma}\,\tau^{\nu_2}$, $\tau \geq 0$, $\gamma, \nu_1, \nu_2 > 0$.
   $q_{\boldsymbol{g}}(\tau) = \tau^{D-\frac{d_1}{\nu_1}} \exp\left[-\frac{d_2\nu_2}{\gamma} W_{-1}\left(-\frac{\nu_2}{\gamma}\tau^{\frac{1}{\gamma}}\right)\right]$. Let $\chi = d_1/\nu_1 + d_2/\nu_2$, (3.9) implies that

$$q_{\boldsymbol{g}}(0) = \begin{cases} 0 & \text{if } D > \chi, \\ \infty & \text{if } D \leq \chi. \end{cases} \tag{3.19}$$

   And (3.17) is satisfied if and only if $D > \chi$.

3.2.2 Lower bounds for hitting probabilities

The goal of this subsection is to extend Theorem 1.3 to $\hat{\boldsymbol{g}}$-Gaussian processes.

Let $Y = \{Y(t, x) : (t, x) \in I \times J\}$ be a Gaussian process in $I \times J$, with I,J compact subsets of $\mathbb{R}^{d_1}$,$\mathbb{R}^{d_2}$, respectively. Let $g_1 : [0, \varnothing_I] \to \mathbb{R}_+$, $g_2 : [0, \varnothing_J] \to \mathbb{R}_+$ be gauge functions. We will make use of the following conditions on Y:

(CY1) $\sigma^2_{t,x} \asymp 1$ for $(t, x) \in I \times J$.

(CY2) There exist a positive constant C such that for all $(t, x), (\bar{t}, \bar{x}) \in I \times J$

$$|m_{t,x} - m_{\bar{t},\bar{x}}| \leq C\left(g_1(|t - \bar{t}|) + g_2(|x - \bar{x}|)\right).$$

(CY3) There exists a positive constant c such that for all $(t, x), (\bar{t}, \bar{x}) \in I \times J$

$$E(\mathrm{Var}(Y(t, x)|Y(\bar{t}, \bar{x}))) \geq c\left(g_1(|t - \bar{t}|)^2 + g_2(|x - \bar{x}|)\right)^2.$$

The proof of the following Proposition which relates the canonical pseudometric and the conditional variance of a $\hat{\boldsymbol{g}}$-Gaussian process is analogous to that of Proposition 1.1.

Proposition 3.1. *Let Y be a Gaussian on process on $I \times J$ with I, J compact subsets of $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$ respectively, satisfying (CY1). Then*

$$E\,(\,Var(Y(t,x)|Y(\bar{t},\bar{x}))) \leq C\mathfrak{d}^2_{(t,x),(\bar{t},\bar{x})},\,(t,x),(\bar{t},\bar{x}) \in I \times J. \tag{3.20}$$

Additionally assume that

1. *Y is a $\hat{\boldsymbol{g}}$-Gaussian process in $I \times J$.*
2. $\lim_{\varepsilon \downarrow 0} \sup_{\substack{(t,x),(\bar{t},\bar{x}) \in I \times J \\ |(t-\bar{t},x-\bar{x})| \leq \varepsilon}} \frac{|\sigma_{t,x}-\sigma_{\bar{t},\bar{x}}|}{\mathfrak{d}_{(t,x),(\bar{t},\bar{x})}} = 0.$
3. *$\rho_{(t,x),(\bar{t},\bar{x})} < 1$ for all $(t,x),(\bar{t},\bar{x}) \in I \times J$, $(t,x) \neq (\bar{t},\bar{x})$.*

Then

$$E\,(\,Var(Y(t,x)|Y(\bar{t},\bar{x}))) \asymp \mathfrak{d}^2_{(t,x),(\bar{t},\bar{x})},\,(t,x),(\bar{t},\bar{x}) \in I \times J. \tag{3.21}$$

Set $\kappa_{I,J} = g_1(\varnothing_I) \vee g_2(\varnothing_J)$. Assume that g_1 and g_2 are differentiable, and define

$$v_{\boldsymbol{g}}(\tau) = \int_{\tau}^{\kappa_{I,J}} \rho^{-D+1}\left[g_1^{-1}(\rho)\right]^{d_1-1}\left[g_2^{-1}(\rho)\right]^{d_2-1}\left[\dot{g}_1(g_1^{-1}(\rho))\dot{g}_2(g_2^{-1}(\rho))\right]^{-1} d\rho, \tag{3.22}$$

for $\tau \in [0, \kappa_{I,J}]$.

To highlight the analogy between $v_{\boldsymbol{g}}$ and the function v_g defined in (1.54), we observe that if g in (1.54) is differentiable, by writing the change of variable $\rho \mapsto g(\rho)$,

$$v_g(\tau) = \int_{q^{-1}(\tau)}^{g(\varnothing_K)} \rho^{-D}\left[g^{-1}(\rho)\right]^{d-1}\left[\dot{g}(g^{-1}(\rho))\right]^{-1} d\rho.$$

Our purpose is to prove the following result which gives lower bounds for hitting probabilities in terms of the $(q_{\boldsymbol{g}})^{-1}$-capacity. (See Definition 1.4 in Chapter 1 for the definition).

Theorem 3.2. *Fix $I \subset \mathbb{R}^{d_1}$ and $J \subset \mathbb{R}^{d_2}$ compact sets of positive Lebesgue measure. Let*

$$\boldsymbol{Y} = \{\boldsymbol{Y}(t,x) = (Y_1(t,x),...,Y_D(t,x)),(t,x) \in I \times J\}$$

be a D-dimensional stochastic process with i.i.d. coordinates. Fix $N > 0$ and let $A \subset B_N \subset \mathbb{R}^D$ be a Borel set. Assume that Y_1 is a $\hat{\boldsymbol{g}}$-Gaussian process on $I \times J$ satisfying conditions (CY1)-(CY3) and that on $(0, \kappa_{I,J})$, the gauge functions g_i, $i = 1,2$, are differentiable.

1. *[HCSS21, Thm. 3.5] Suppose that the derivatives $\dot{g}_i$, $i = 1,2$ are non increasing on $(0, \kappa_{I,J})$ and*
$$\lim_{\tau \downarrow 0} v_{\boldsymbol{g}}(\tau/2)q_{\boldsymbol{g}}(\tau) \in (0,\infty), \tag{3.23}$$
where $q_{\boldsymbol{g}}$ is the function defined in (3.8). *Then there exists a constant $C := C(m,I,J,N,d_1,d_2,D) > 0$ such that*
$$P\,(\boldsymbol{Y}(I \times J) \cap A \neq \emptyset) \geq C\ \ Cap_{(q_{\boldsymbol{g}})^{-1}}(A). \tag{3.24}$$

2. Instead of (3.23)*, assume that*

$$v_{\boldsymbol{g}}(0) < \infty. \tag{3.25}$$

Then there exists a constant $C := C(m, I, J, N, d_1, d_2, D) > 0$ *such that*

$$P\left(\boldsymbol{Y}(I \times J) \cap A \neq \emptyset\right) \geq C[v_{\boldsymbol{g}}(0)]^{-1}. \tag{3.26}$$

Remark 3.4. Similarly to Theorem 1.3, (3.23) and (3.25) are excluding conditions: Assume that there exists gauge functions g_1, g_2 with non increasing derivatives on $(0, \kappa_{I,J})$ satisfying (3.23) and (3.25) for $\boldsymbol{g} = (g_1, g_2)$. This implies that $\lim_{\tau \downarrow 0} q_{\boldsymbol{g}}(\tau) \in (0, \infty)$ and in consequence

$$\begin{aligned} v_g(0) &= \int_0^{\kappa_{I,J}} \rho \left[q_{\boldsymbol{g}}(\rho) g_1^{-1}(\rho) \dot{g}_1(g_1^{-1}(\rho)) g_2^{-1}(\rho) \dot{g}_2(g_2^{-1}(\rho)) \right]^{-1} d\rho \\ &\geq c \int_0^{\kappa_{I,J}} \rho \left[g_1^{-1}(\rho) \dot{g}_1(g_1^{-1}(\rho)) g_2^{-1}(\rho) \dot{g}_2(g_2^{-1}(\rho)) \right]^{-1} d\rho \\ &= c \int_0^{g_1(\kappa_{I,J})} \rho^{-1} g_1(\rho) \left[g_2^{-1}(g_1(\rho)) \dot{g}_2(g_2^{-1}(g_1(\rho))) \right]^{-1} d\rho, \end{aligned} \tag{3.27}$$

where in the last line we apply the change of coordinates $\rho \mapsto g_1^{-1}(\rho)$. Since g_1, g_2 are continuous functions with $g_1(0) = g_2(0) = 0$,

$$\liminf_{\rho \downarrow 0} \frac{g_1(\rho)}{g_2^{-1}(g_1(\rho)) \dot{g}_2(g_2^{-1}(g_1(\rho)))} = \liminf_{\rho \downarrow 0} \frac{g_2(\rho)}{\rho \dot{g}_2(\rho)}. \tag{3.28}$$

By the fundamental theorem of calculus and since $\dot{g}_2$ is not increasing

$$\frac{g_2(\rho)}{\rho \dot{g}_2(\rho)} = \frac{\int_{\frac{\rho}{n}}^{\rho} \dot{g}_2(\tau) d\tau}{\rho \dot{g}_2(\rho)} \geq \frac{\dot{g}_2(\rho) \left(\rho - \frac{\rho}{n}\right)}{\dot{g}_2(\rho) \rho} = 1 - \frac{1}{n}, \rho \in (0, \kappa_{I,J}), n \geq 1. \tag{3.29}$$

By (3.27)-(3.29), it follows that

$$v_g(0) \geq c \int_0^{g_1(\kappa_{I,J})} \rho^{-1} d\rho = \infty,$$

which is a contradiction to (3.25).

Definition 3.2. Similar to the g-Gaussian case, when neither(1.55) nor (1.57) are satisfied, we say that we are on a *critical dimension* type case.

Proof. The approach to the proof is the similar as that of Theorem 1.3.

Proof of 1. For any $(t, x) \in I \times J$ and a probability measure μ on A, define

$$\begin{aligned} \bar{\nu}_n((t, x), \omega) &= \int_A (2\pi n)^{D/2} \exp\left(-\frac{n |\boldsymbol{Y}(t, x) - y|^2}{2} \right) \mu(dy) \\ &= \int_A \mu(dy) \int_{\mathbb{R}^D} d\xi \exp\left(-\frac{|\xi|^2}{2n} + i \langle \xi, \boldsymbol{Y}(t, x) - y \rangle \right), \quad n \geq 1, \end{aligned} \tag{3.30}$$

and let $\nu_n(I \times J)(\omega) = \int_{I \times J} \bar{\nu}_n((t, x), \omega)\, dt\, dx$. We aim to prove:

(i) There exists $C_1 > 0$ such that for any $n \geq 1$, $E\left(\nu_n(I \times J)\right) \geq \bar{C}_1$.

(ii) There exists $C_2 > 0$ such that for any $n \geq 1$, $E\left[(\nu_n(I \times J))^2\right] \leq C_2 \mathcal{E}_{(q_{\boldsymbol{g}})^{-1}}(\mu)$.

This implies (3.24) using the same argument based in the Paley-Zygmund inequality and the convergence of finite measures, which was explained at the beginning of the proof of 1. in Theorem 1.3.

Proof of (i). We deduce the validity of (i) by following the lines of the proof of 1.(i) in Theorem 1.3 and applying (CY1) instead of (CX1).

Proof of (ii). By following the computations used to derive (1.69) in Theorem 1.3, and using hypothesis (CY1)-(CY3) instead of (CX1)-(CX3), we can prove that

$$E\left[(\nu_n(I \times J))^2\right] \leq \tag{3.31}$$
$$C \int_{(I\times J)^2} dt\, dx\, d\bar{t}\, d\bar{x} \int_{A\times A} \mu(dy)\ \mu(d\bar{y})\ \frac{1}{\max\left([g_1(|t-\bar{t}|) + g_2(|x-\bar{x}|)]^D, |y-\bar{y}|^D\right)}.$$

For $h \geq 0$, set

$$\mathcal{I} := \int_{(I\times J)^2} dt\, dx\, d\bar{t}\, d\bar{x} \left[\max\left([g_1(|t-\bar{t}|) + g_2(|x-\bar{x}|)]^D, h^D\right)\right]^{-1}. \tag{3.32}$$

Apply the change of variables $(t, \bar{t}) \mapsto (t, t-\bar{t})$, $(x, \bar{x}) \mapsto (x, x-\bar{x})$, to deduce

$$\mathcal{I} \leq |I \times J| \int_{B_{\varnothing_I}} dt \int_{B_{\varnothing_J}} dx \left[\max\left([g_1(|t|) + g_2(|x|)]^D, h^D\right)\right]^{-1}, \tag{3.33}$$

where $|I \times J|$ denotes the Lebesgue measure of $I \times J$.

Let $\mathcal{I}_1$ denote the integral in (3.33) over the set of points (t, x) satisfying $g_1(|t|) + g_2(|x|) \leq h$. Changing to polar coordinates, we see that

$$\begin{aligned}
\mathcal{I}_1 &= h^{-D} \int_{B_{\varnothing_I}} dt \int_{B_{\varnothing_J}} dx\ 1_{\{g_1(|t|)+g_2(|x|)\leq h\}} \\
&\leq h^{-D} \left(\int_{B_{\varnothing_I}} dt\, 1_{\{g_1(|t|)\leq h\}}\right) \left(\int_{B_{\varnothing_J}} dx\ 1_{\{g_2(|x|)\leq h\}}\right) \\
&\leq C(d_1, d_2) h^{-D} \left(\int_0^{g_1^{-1}(h)} \rho^{d_1-1} d\rho\right) \left(\int_0^{g_2^{-1}(h)} \rho^{d_2-1} d\rho\right) = C(d_1, d_2) \left[q_{\boldsymbol{g}}(h)\right]^{-1}.
\end{aligned} \tag{3.34}$$

Next, we denote by $\mathcal{I}_2$ the integral in (3.33) over the set of points (r, z) such that $g_1(|t|) + g_2(|x|) > h$. Applying two changes of variables: first polar coordinates, $t \mapsto (\rho_1, \theta_1)$, $x \mapsto (\rho_2, \theta_2)$, and then $\rho_i \mapsto g_i(\rho_i)$, $i = 1, 2$, we obtain

$$\begin{aligned}
\mathcal{I}_2 &= C(d_1, d_2) \int_0^{\varnothing_I} d\rho_1 \int_0^{\varnothing_J} d\rho_2\ 1_{\{g_1(\rho_1)+g_2(\rho_2)>h\}} \left[g_1(\rho_1) + g_2(\rho_2)\right]^{-D} \rho_1^{d_1-1} \rho_2^{d_2-1} \\
&= C(d_1, d_2) \int_0^{g_1(\varnothing_I)} d\tau_1 \int_0^{g_2(\varnothing_J)} d\tau_2\ 1_{\{\tau_1+\tau_2>h\}} (\tau_1 + \tau_2)^{-D} \\
&\quad \times \left(g_1^{-1}(\tau_1)\right)^{d_1-1} \left(g_2^{-1}(\tau_2)\right)^{d_2-1} \left[\dot{g}_1(g_1^{-1}(\tau_1))\right]^{-1} \left[\dot{g}_2(g_2^{-1}(\tau_2))\right]^{-1} \\
&\leq C(d_1, d_2, D) \int_0^{g_1(\varnothing_I)} d\tau_1 \int_0^{g_2(\varnothing_J)} d\tau_2\ 1_{\{|(\tau_1,\tau_2)|>h/2\}} [|(\tau_1, \tau_2)|]^{-D}
\end{aligned}$$

$$\times \left(g_1^{-1}(\tau_1)\right)^{d_1-1} \left(g_2^{-1}(\tau_2)\right)^{d_2-1} \left[\dot g_1(g_1^{-1}(\tau_1))\right]^{-1} \left[\dot g_2(g_2^{-1}(\tau_2))\right]^{-1}, \tag{3.35}$$

where in the last inequality we have used $|(\tau_1,\tau_2)| \le \tau_1+\tau_2 \le 2|(\tau_1,\tau_2)|$. Because for $i=1,2$, g_i are increasing and $\dot g_i$ non increasing, we deduce

$$\begin{aligned}
\mathcal{I}_2 \le& \int_0^{g_1(\varnothing_I)} d\tau_1 \int_0^{g_2(\varnothing_J)} d\tau_2\ 1_{\{|(\tau_1,\tau_2)|>h/2\}}[|(\tau_1,\tau_2)|]^{-D}\\
&\times \left(g_1^{-1}(|(\tau_1,\tau_2)|)\right)^{d_1-1} \left(g_2^{-1}(|(\tau_1,\tau_2)|)\right)^{d_2-1}\\
&\times \left[\dot g_1(g_1^{-1}(|(\tau_1,\tau_2)|))\right]^{-1} \left[\dot g_2(g_2^{-1}(|(\tau_1,\tau_2)|))\right]^{-1}\\
=& C(d_1,d_2,D)\int_{h/2}^{\kappa_{I,J}} \rho^{-D+1}\left[g_1^{-1}(\rho)\right]^{d_1-1}\left[g_2^{-1}(\rho)\right]^{d_2-1}\\
&\times \left[\dot g_1(g_1^{-1}(\rho))\right]^{-1}\left[\dot g_2(g_2^{-1}(\rho))\right]^{-1}\ d\rho\\
=& C(d_1,d_2,D)v_{\boldsymbol g}(h/2) \le C(I,J)\left[q_{\boldsymbol g}(h)\right]^{-1},
\end{aligned} \tag{3.36}$$

where the last inequality holds because (3.23) is equivalent to

$$\sup_{\tau\in[0,\kappa_{I,J}]} v_{\boldsymbol g}(\tau/2)q_{\boldsymbol g}(\tau) \in (0,\infty).$$

Thus, from (3.31) by applying (3.34) and (3.36) with $h := |y-\bar y|$, we obtain

$$E\left[(\nu_n(I\times K))^2\right] \le C(I,J,d_1,d_2,D)\,\mathcal{E}_{(q_{\boldsymbol g})^{-1}}(\mu). \tag{3.37}$$

Finising the proof of (ii).

Proof of 2. The proof of (i) is still valid in this case. The proof of (ii) is valid until (3.31), which together with (3.25) implies that

$$\begin{aligned}
E\left[(\nu_n(I\times J))^2\right] &\le C\int_{(I\times J)^2} dtdxd\bar t d\bar x([g_1(|t-\bar t|)+g_2(|x-\bar x|)]^{-D}\\
&\le C(d_1,d_2,I,J)v_{\boldsymbol g}(0) < \infty.
\end{aligned}$$

Thus following the same arguments used in 1. we deduce (3.24) with $\mathrm{Cap}_{(q_{\boldsymbol g})^{-1}}(A)$ replaced by $[v_g(0)]^{-1}$.

□

Theorem 1.3 gives us a condition for non polarity of points:

Corollary 3.3. *Let* $\boldsymbol Y$ *be as in Theorem 1.2 and assume* (3.26). *Then, for any* $z\in\mathbb{R}^D$, $P(\boldsymbol Y(I\times J)\cap\{z\}\neq\emptyset)>0$, *that is* $\{z\}$ *is non polar for the process restricted to* $I\times J$.

Example 3.4. We compute $v_{\boldsymbol g}$ and analyze conditions (3.23) and (3.25) for the gauge functions in Example 3.2.

1. $g_i(\tau)=\tau^{\nu_i}, \tau>0, \nu_i>0, i=1,2.$

$$\begin{aligned}
v_{\boldsymbol g}(\tau) &= (\nu_1\nu_2)^{-1}\int_\tau^{\kappa_{I,J}} \rho^{-D+1}\rho^{(d_1-1)/\nu_1+(d_2-1)/\nu_2}\left[\rho^{2-(\nu_1^{-1}+\nu_2^{-1})}\right]^{-1} d\rho\\
&= \begin{cases} (\nu_1\nu_2(D-\chi))^{-1}\left[\tau^{-(D-\chi)}-\kappa_{I,J}^{-(D-\chi)}\right], & \text{if } D\neq\chi,\\ (\nu_1\nu_2)^{-1}\log\left(\frac{\kappa_{I,J}}{\tau}\right), & \text{if } D=\chi,\end{cases}
\end{aligned}$$

for $\chi = d_1/\nu_1 + d_2/\nu_2$.

Since $q_{\boldsymbol{g}}(\tau) = \tau^{D-\chi}$, we deduce

$$\lim_{\tau\downarrow 0} v_{\boldsymbol{g}}(\tau/2)q_{\boldsymbol{g}}(\tau) = \begin{cases} (2^{D-\chi}\nu_1\nu_2(D-\chi))^{-1}, & \text{if } D > \chi, \\ \infty, & \text{if } D \leq \chi. \end{cases}$$

Additionally,

$$v_{\boldsymbol{g}}(0) = \begin{cases} (\nu_1\nu_2(\chi - D))^{-1}\kappa_{I,J}^{\chi-D}, & \text{if } D < \chi, \\ \infty, & \text{if } D \geq \chi. \end{cases}$$

Thus (3.23), (3.25) is satisfied if and only if $D > \chi$, $D < \chi$ respectively.

2. $g_1(\tau) = \tau^{\nu_1}, g_2(\tau) = |\log\tau|^{\gamma}\tau^{\nu_2}, \tau > 0$, with $\gamma, \nu_1, \nu_2 > 0$. Due to (3.11)

$$\lim_{\tau\downarrow 0} q_g(\tau) = \begin{cases} \infty, & \text{if } D \leq \chi \\ 0, & \text{if } D > \chi, \end{cases} \tag{3.38}$$

for $\chi = d_1/\nu_1 + d_2/\nu_2$.

We have that

$$v_{\boldsymbol{g}}(\tau) = \nu_1^{-1}\int_{\tau}^{\kappa_{I,J}} \rho^{\frac{d_1}{\nu_1}-D}\left(g_2^{-1}(\rho)\right)^{d_2-1}\left[\dot{g}_2(g_2^{-1}(\rho))\right]^{-1}\, d\rho.$$

We claim that

$$v_{\boldsymbol{g}}(0) = \begin{cases} C(\nu_1,\nu_2,\gamma,d_1,d_2D), & \text{if either } D<\chi, \text{ or } D=\chi, \gamma(D-\frac{d_1}{\nu_1}) > 1 \\ \infty, & \text{if either } D>\chi, \text{ or } D=\chi, \gamma(D-\frac{d_1}{\nu_1}) \leq 1. \end{cases} \tag{3.39}$$

Indeed, since $g_2(0) = 0$, and by the change of coordinates $\rho \mapsto g_2^{-1}(\rho)$

$$\begin{aligned} v_{\boldsymbol{g}}(0) &= \nu_1^{-1}\int_0^{g_2^{-1}(\kappa_{I,J})} [g_2(\rho)]^{\frac{d_1}{\nu_1}-D}\rho^{d_2-1}\, d\rho \\ &= \nu_1^{-1}\int_0^{g_2^{-1}(\kappa_{I,J})} |\log\rho|^{\gamma\left(\frac{d_1}{\nu_1}-D\right)}\rho^{\nu_2(\chi-D)-1}\, d\rho. \end{aligned}$$

The rest of the proof of the claim is the same that (1.72) taking into consideration the values of the exponents.

Now assume that $D > \chi$. Computing the derivative of the reciproque of $q_{\boldsymbol{g}}$, we see that

$$\begin{aligned} \frac{d}{d\tau}\left((q_{\boldsymbol{g}}(\tau))^{-1}\right) = \tau^{\frac{d_1}{\nu_1}-D-1}\left(g_2^{-1}(\tau)\right)^{d_2-1}&\left[\left(\tfrac{d_1}{\nu_1}-D\right)g_2^{-1}(\tau)\right. \\ &\left.+d_2\tau\left(g_2^{-1}(\tau)\right)^{1-\nu_2}|\log g_2^{-1}(\tau)|^{1-\gamma}\left(\nu_2|\log g_2^{-1}(\tau)|-\gamma\right)^{-1}\right]. \end{aligned} \tag{3.40}$$

Apply the L'Hôpital's rule, which is valid in this case due to (3.38) and (3.39), to obtain

$$\lim_{\tau\downarrow 0}\left[v_{\boldsymbol g}(\tau)q_{\boldsymbol g}(\tau)\right]^{-1}=\lim_{\tau\downarrow 0}\frac{\frac{d}{d\tau}\left((q_{\boldsymbol g}(\tau))^{-1}\right)}{\frac{dv_{\boldsymbol g}}{d\tau}(\tau)}=\lim_{\tau\downarrow 0}\left(L_1(\tau)+L_2(\tau)\right),$$

where using (3.40), we have

$$L_1(\tau)=\frac{\tau^{\frac{d_1}{\nu_1}-D-1}\left(g_2^{-1}(\tau)\right)^{d_2}\left(\frac{d_1}{\nu_1}-D\right)}{\frac{dv_{\boldsymbol g}}{d\tau}(\tau)},$$

$$L_2(\tau)=\frac{d_2\tau^{\frac{d_1}{\nu_1}-D}\left(g_2^{-1}(\tau)\right)^{d_2-\nu_2}|\log g_2^{-1}(\tau)|^{1-\gamma}\left(\nu_2|\log g_2^{-1}(\tau)|-\gamma\right)^{-1}}{\frac{dv_{\boldsymbol g}}{d\tau}(\tau)}.$$

Since $\frac{1}{2}\nu_2|\log g_2^{-1}(\tau)|\le\nu_2|\log g_2^{-1}(\tau)|-\gamma\le\nu_2|\log g_2^{-1}(\tau)|$, as $\tau\downarrow 0$, we find:

$$\lim_{\tau\downarrow 0}L_1(\tau)=D\nu_1\nu_2-d_1\nu_2,\quad \lim_{\tau\downarrow 0}L_2(\tau)=-d_2\nu_1.$$

Consequently,

$$\lim_{\tau\downarrow 0}v_{\boldsymbol g}(\tau)q_{\boldsymbol g}(\tau)=(D\nu_1\nu_2-(d_1\nu_2+d_2\nu_1))^{-1}.$$

The last limit implies (3.23), since using (3.11) one can proof that $q_{\boldsymbol g}(\tau)\asymp q_{\boldsymbol g}(2\tau)$.

Resuming the discussion above, (3.23) holds if and only if $D>\chi$. And (3.25) is valid if and only if $D<\chi$ or $D=\chi$, $\gamma(D-d_1/\nu_1)=\gamma d_2/\nu_2>1$.

3.3 Anisotropic Gaussian processes

Definition 3.3. Fix $\eta>0$ and let $\boldsymbol Y$ be a D-dimensional Gaussian process with i.i.d coordinates,

$$\boldsymbol Y=\{\boldsymbol Y(t,x)=(Y_1(t,x),\dots,Y_D(t,x)),\ (t,x)\in I^{(\eta)}\times J^{(\eta)}\},$$

where I,J are compact subsets of $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$, respectively. Assume that X_1 is $\boldsymbol g$-Gaussian with $g_i(\tau)=\tau^{\nu_i},\nu_i>0$, $i=1,2$, this means,

$$\mathfrak{d}_{Y_1}((t,x),(\bar t,\bar x))\asymp|t-\bar t|^{\nu_1}+|x-\bar x|^{\nu_2},(t,x),(\bar t,\bar x)\in I^{(\eta)}\times K^{(\eta)}. \tag{3.41}$$

A process satisfying such kind of condition is called *anisotropic* Gaussian process. By Remark 3.2 the sample paths of $\boldsymbol Y$ are continuous a.s.

Let $\chi=d_1/\nu_1+d_2/\nu_2$. In Example 1.3(1.) we proved that g_1 and g_2 satisfies the hypothesis of Lemma 1.1(2.). Additionally, from Example 3.2 (1.), $q_{\boldsymbol g}(\tau)=\tau^{D-\chi}$ and $q_{\boldsymbol g}$ is increasing if and only if $D>\chi$. Thus Corollary 3.1, implies that if $\sigma^2_{I^{(\eta)},J^{(\eta)}}>0$ and $D>\chi$, for any Borel set $A\subset\mathbb{R}^D$,

$$P\left(\boldsymbol Y(I\times J)\cap A\neq\emptyset\right)\le C\,\mathcal{H}_{D-\chi}(A), \tag{3.42}$$

with $C:=C(I\times J,\sigma^2_{I^{(\eta)},J^{(\eta)}},d_1,d_2,D)$.

Now, suppose that Y_1 satisfies (CY1)-(CY3). Fix a bounded Borel set $A\subset B_N\subset\mathbb{R}^D$. According to Example 3.4, (3.23), (3.25) are satisfied if and only if $D>\chi$, $\chi>D$ respectively. Then, applying Theorem 3.2 we obtain,

1. If $D > \chi$,
$$P\left(\boldsymbol{Y}(I \times J) \cap A \neq \emptyset\right) \geq c\,\mathrm{Cap}_{D-\chi}(A), \tag{3.43}$$
with $c := c(I \times J, N, d_1, d_2, D)$.

2. If $\chi > D$,
$$P\left(\boldsymbol{Y}(I \times J) \cap A \neq \emptyset\right) \geq c > 0, \tag{3.44}$$
with $c := c(I \times J, N, d_1, d_2, D)$.

Remark 3.5. Fix $z \in \mathbb{R}^D$ and assume that (3.42), (3.43) and (3.44) are valid. Hence

$$P\left(\boldsymbol{Y}(I \times J) \cap \{z\} \neq \emptyset\right) = \begin{cases} \mathcal{H}_{D-\chi}(\{z\}) = 0, \text{ if } D > \chi, \\ P\left(\boldsymbol{Y}(I \times J) \cap \{z\} \neq \emptyset\right) \geq c > 0, \text{ if } D < \chi. \end{cases} \tag{3.45}$$

Thus, singletons $\{z\}$ are polar if $D > \chi$ and non polar if $D < \chi$.

Similarly to the case of isotropic Gaussian random fields we identify the value $D_0 = \chi$ as the *critical dimension* for polarity of points.

We present a selection of known results on hitting probabilities from examples of anisotropic Gaussian processes that are solutions of Stochastic Partial Differential Equations.

Example 3.5. *Stochastic heat equation.* Consider the linear stochastic heat equation

$$\frac{\partial v_i}{\partial t} - \Delta v_i = \dot{W}_i, \ (t, x) \in (0, T] \times \mathbb{R}, i = 1, ..., D,$$

with null initial conditions. The processes (W_i) are independent space-time white noises. The random field solution to this equation is the Gaussian stochastic process

$$u_i(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) W_i(ds, dy), \tag{3.46}$$

with $G(t, x) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{4t}\right) \mathbb{1}_{\{t \geq 0\}}$, this means that

$$\boldsymbol{u} = \{\boldsymbol{u}(t, x) = (u_1(t, x), ..., u_D(t, x)), (t, x) \in (0, T] \times \mathbb{R}^d\}$$

is a centered D-dimensional Gaussian process with i.i.d. coordinates and covariance

$$E(u_i(t, x) u_i(s, y)) = \int_0^{t \wedge s} dr \int_{\mathbb{R}^d} dz G(t-r, x-z) G(s-r, y-z).$$

This process was first introduced in [Fun83] on a more general context and it is known as Funaki's random string.

Let $u = \{u_1(t, x), (t, x) \in [0, T] \times \mathbb{R}\}$, it is a consequence of [MT02, Prop. 1] that

$$\mathfrak{d}_u(t, x) \asymp |t-s|^{\frac{1}{4}} + |x-y|^{\frac{1}{2}}, (t, x), (s, y) \in \mathbb{R}_+ \times \mathbb{R}.$$

so u is $\boldsymbol{g}$-Gaussian on any compact subset of $\mathbb{R}_+ \times \mathbb{R}$, with $\boldsymbol{g} = (g_1, g_2)$, $g_1(\tau) = \tau^{\frac{1}{4}}, g_2(\tau) = \tau^{\frac{1}{2}}$.

Fix $I \times J = [a_1, b_1] \times [a_2, b_2]$, $0 < a_1 < b_1 < \infty$,$-\infty < a_2 < b_2 < \infty$ a compact box. u satisfies that

- $\sigma_{t,x} \asymp 1$ for $(t,x) \in I \times J$.
- [MT02, (4.5), p.15] There exists a positive constant c such that

$$E(\mathrm{Var}(u(t,x)|u(\bar{t},\bar{x}))) \geq c\left(|t-\bar{t}|^{1/4} + |x-\bar{x}|^{1/2}\right)^2, (t,x),(\bar{t},\bar{x}) \in I \times J.$$

By (3.42), (3.43) and (3.44) we deduce the following hitting probabilities result relative to the solution to the stochastic heat equation in (3.46).

1. Let $D > 6$.
   (a) There exists a constant $C := C(I \times J, \sigma^2_{I(\eta),J(\eta)}, d_1, d_2, D)$ such that for any Borel set $A \in \mathcal{B}(\mathbb{R}^D)$,

$$P(\boldsymbol{u}(I \times J) \cap A \neq \emptyset) \leq C\mathcal{H}_{D-6}(A).$$

   (b) Fix $N > 0$. Let $A \in \mathcal{B}(\mathbb{R}^D)$ be such that $A \subset B_N$. There exists a constant $c := c(I \times J, d_1, d_2, D)$ such that

$$P(\boldsymbol{u}(I \times J) \cap A \neq \emptyset) \geq c\mathrm{Cap}_{D-6}(A).$$

2. Assume that $D < 6$. Let $A \in \mathcal{B}(\mathbb{R}^D)$ be such that $A \subset B_N$. There exists a constant $c := c(I \times J, d_1, d_2, D)) > 0$ such that

$$P(\boldsymbol{u}(K) \cap A \neq \emptyset) \geq c > 0.$$

And $D_0 = 6$ is the critical dimension.

Mueller and Tribe proved in [MT02, Thm. 1] that points are polar for $\boldsymbol{u}$ in the critical dimension, that is, if $D = 6$ for all $z \in \mathbb{R}^D$

$$P(\boldsymbol{u}(I \times J) \cap \{z\} \neq \emptyset) = 0.$$

Example 3.6. *Stochastic wave equation.* Consider the linear stochastic wave equation

$$\frac{\partial^2 v_i}{\partial t^2} - \Delta v_i = \dot{W}_i^\beta, \; (t,x) \in (0,T] \times \mathbb{R}, i = 1, ..., D,$$

with null initial conditions. Here (W_i^β) are independent space-time white-colored noises with covariance given by

$$E(\dot{W}_i^\beta(t,x)\dot{W}_i^\beta(s,y)) = \delta(t-s)|x-y|^{-\beta}, \beta \in (0, d \wedge 2).$$

The random field solution to this equation is the Gaussian stochastic process

$$u_i(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) W_i^\beta(ds,dy), \tag{3.47}$$

with G the fundamental solution of the wave equation, characterized by the property $\mathcal{F}G(t,*)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}$, this means that

$$\boldsymbol{u} = \{\boldsymbol{u}(t,x) = (u_1(t,x), ..., u_D(t,x)), (t,x) \in (0,T] \times \mathbb{R}^d\}$$

is a centered D-dimensional Gaussian process with i.i.d. coordinates and covariance

$$E(u_i(t,x)u_i(s,y)) = \int_0^{t\wedge s} dr \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dw G(t-r,x-z)G(s-r,y-w)|x-y|^{-\beta}.$$

This process was first introduced in [Dal99].

Let $u = \{u_1(t,x),(t,x) \in [0,T] \times \mathbb{R}\}$, and $I = [t_0,T]$, $J = [-M,M]^d$, for $t_0 \in (0,T]$, $M > 0$. [DSS10, Prop. 4.1] implies that

$$\mathfrak{d}_u(t,x) \asymp |t-s|^{1-\beta/2} + |x-y|^{1-\beta/2}, (t,x),(s,y) \in \mathbb{R}^+ \times \mathbb{R}.$$

so u is $\boldsymbol{g}$-Gaussian in $I \times J$, with $\boldsymbol{g} = (g_1,g_2)$, $g_1(\tau) = \tau^{1-\beta/2}, g_2(\tau) = \tau^{1-\beta/2}$. Additionally,

- $\sigma_{t,x} \asymp 1$ for $(t,x) \in I \times J$.
- By [DSS10, Prop. 4.1] the hypotheses of Proposition 3.1 are valid and in consequence there exists a positive constant c such that for all $(t,x),(\bar{t},\bar{x}) \in I \times J$
$$E(\mathrm{Var}(u(t,x)|u(\bar{t},\bar{x}))) \geq c\left(|t-\bar{t}|^{1-\beta/2} + |x-\bar{x}|^{1-\beta/2}\right)^2.$$

By (3.42), (3.43) and (3.44) we deduce similar hitting probabilites results that in Example 3.5 with the main difference that in the case of the stochastic wave equation the critical dimension is $D_0 = \chi = 2(d+1)/(2-d)$.

Dalang, Mueller and Xiao proved in [DMX17, Thm. 9.1] that points are polar for $\boldsymbol{u}$ in the critical dimension.

Chapter 4

The linear stochastic heat equation

We apply the results of Chapter 3 to the system of linear stochastic heat equations

$$\begin{cases} \frac{\partial v_j}{\partial t} - \Delta v_j = \dot{W}_j^{H,\alpha}, \ (t,x) \in (0,T] \times \mathbb{R}^d, \\ \quad v_j(0,x) = v_0(x), \ x \in \mathbb{R}^d. \end{cases}$$

$j = 1, ..., D$, with $(W_j^{H,\alpha}, j = 1, ..., D)$ independent copies of a fractional-colored noise with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $\alpha \in [0, d)$.

In Theorem 4.1 we find conditions in terms of the parameters of the noise implying that such equation has a solution. By Theorem 4.2 the the solution is a $\boldsymbol{g}$-Gaussian process. Section 4.3 is devoted to analyze additional second order properties related with the conditional variance of the solution.

In section 4.4, we prove the main result of this Chapter: Theorem 4.3 where we find upper and lower bounds for hitting probabilities.

4.1 The solution

Given $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions, its *Fourier transform* is given by the formula

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\varphi(x)dx.$$

Let f be the Fourier transform of a tempered non-negative measure μ on $\mathbb{R}^d$, i.e. a non-negative measure which satisfies:

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^l}\mu(d\xi) < \infty,$$

for some $l > 0$. Parseval's identity implies the following relation:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)f(x-y)\psi(y)dxdy = (2\pi)^{-d}\int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi), \tag{4.1}$$

for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$. We assume that for any non-negative measurable function h,

$$\int_{\mathbb{R}^d} h(\xi)\mu(d\xi) \asymp \int_{\mathbb{R}^d} h(\xi)|\xi|^{-\alpha}d\xi, \quad \text{for some } \alpha \in [0,d). \tag{4.2}$$

In an abridged form, we will write this property as $\mu(d\xi) \asymp |\xi|^{-\alpha}d\xi$.

Definition 4.1. Fix $\alpha \in [0, d)$, $H \in (1/2, 1)$ and let $\{W^{H,\alpha}(t, A), t \in [0, T], A \in \mathcal{B}(\mathbb{R}^d)\}$ be a centered Gaussian field with covariance

$$E(W^{H,\alpha}(t,A)W^{H,\alpha}(s,B)) = R_H(t,s)\int_A\int_B f(z-w)dzdw \qquad (4.3)$$
$$= \alpha_H \int_0^t\int_0^s\int_A\int_B |\tau-\sigma|^{2H-2} f(z-w)dzdwd\tau d\sigma,$$

where $R_H(t,s) := \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ is the covariance of a fractional Brownian motion with Hurst index H and $\alpha_H = H(2H-1)$.

If $\alpha > 0$, $W^{H,\alpha}$ is called a *fractional-colored* noise because it is a fractional Brownian motion in time and has a non trivial spatial covariance. Consider the particular case $f(x) = \delta_{\{0\}}(x)$. Then, $\mu(d\xi) = d\xi$ and (4.2) trivially holds with $\alpha = 0$. This corresponds to the *fractional-white* noise, whose covariance according to (4.3) is

$$E(W^{H,0}(t,A)W^{H,0}(s,B)) = R_H(t,s)|A \cap B|,$$

where $|\cdot|$ denotes the Lebesgue measure.

Example 4.1. The Riesz and the Bessel kernels are examples of functions f that satisfy the above assumptions

1. *Riesz kernels.* $f(x) = c_{\alpha,d}|x|^{\alpha-d}$, $0 < \alpha < d$, where $c_{\alpha,d} = 2^{d/2-\alpha}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$. The spectral measure is $\mu(d\xi) = |\xi|^{-\alpha}d\xi$.

2. *Bessel kernels.* $f(x) = \tilde{c}_\alpha \int_0^\infty w^{(\alpha-d)/2-1}e^{-w}e^{-|x|^2/(4w)}dw$, $\alpha > 0$, where $\tilde{\gamma}_\alpha = (\Gamma(\alpha/2)$. The spectral measure is $\mu(d\xi) = (1+|\xi|^2)^{-\alpha/2}d\xi$.

Now we construct the stochastic integral with respect to fractional colored noise. This integral was introduced in [BT08, Sec. 3] and it is inspired in the Dalang integral from the seminal paper [Dal99]. Let

$$\mathcal{E} = \left\{\varphi : \varphi = \sum_{i=1}^n a_i 1_{[0,t_i)\times A_i}, a_i \in \mathbb{R}, A_i \in B_b(\mathbb{R}^d), 0 \le t_i \le T\right\}$$

be the vector space of step functions in $\mathbb{R}^d \times [0,T]$. For $\varphi \in \mathcal{E}$, we define the stochastic integral with respect to $W^{H,\alpha}$ as

$$\int \varphi(s,y)W^{H,\alpha}(ds,dy) = \sum_{i=1}^n a_i W(t_i, A_i).$$

(4.3) implies that

$$E\left(\int \varphi(s,y)W^{H,\alpha}(ds,dy)\int \psi(s,y)W^{H,\alpha}(ds,dy)\right) = \langle\varphi,\psi\rangle_{\mathcal{HP}}, \varphi,\psi \in \mathcal{E}, \qquad (4.4)$$

where $\langle\cdot,\cdot\rangle_{\mathcal{HP}}$ is the inner product

$$\langle\varphi,\psi\rangle_{\mathcal{HP}} = \alpha_H \int_0^T d\tau \int_0^T d\sigma \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dw \varphi(\tau,z)\psi(\sigma,w)|\tau-\sigma|^{2H-2} f(z-w).$$

Due to Parseval's theorem and the convolution property of the Fourier transform

$$\langle \varphi, \psi \rangle_{\mathcal{HP}} = \frac{\alpha_H}{(2\pi)^d} \int_0^T d\tau \int_0^T d\sigma \int_{\mathbb{R}^d} \mu(d\xi) \mathcal{F}\varphi(\tau, *)(\xi) \overline{\mathcal{F}\psi(\sigma, *)(\xi)} |\tau - \sigma|^{2H-2}, \quad (4.5)$$

for any $\varphi, \psi \in \mathcal{E}$.

We define the Hilbert space $\mathcal{HP}$ as the closure of $\mathcal{E}$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{HP}}$. Since the map $\varphi \mapsto \int \varphi(t,x) W^{H,\alpha}(dt, dx)$ is an isometry between $\mathcal{E}$ and the Gaussian space of $W^{H,\alpha}$, it can be uniquely extended to $\mathcal{HP}$. We denote this extension by

$$\int_0^T \int_{\mathbb{R}^d} \varphi(t,x) W^{H,\alpha}(dt, dx), \varphi \in \mathcal{HP}.$$

The space $\mathcal{HP}$ may contain distributions, and it contains $|\mathcal{HP}|$ the space of measurable functions $\varphi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$|\varphi|^2_{|\mathcal{HP}|} = \alpha_H \int_0^T d\tau \int_0^T d\sigma \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dw |\varphi(\tau, z)\varphi(\sigma, w)| |\tau - \sigma|^{2H-2} f(z - w) < \infty.$$

The fundamental solution of the heat operator $\mathcal{L} = \frac{\partial v}{\partial t} - \Delta v$ on $\mathbb{R}_+ \times \mathbb{R}^d$ is given by

$$G(t,x) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{4t}\right) 1_{\{t \geq 0\}}.$$

Consider the linear stochastic heat equation

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta v_j = \dot{W}_j^{H,\alpha}, \ (t,x) \in (0,T] \times \mathbb{R}^d, \\ \quad v(0,x) = v_0(x), \ x \in \mathbb{R}^d. \end{cases} \quad (4.6)$$

where $W^{H,\alpha}$ is a fractional-colored noise and v_0 is a measurable function.

Definition 4.2. Assume that for any $(t,x) \in (0,T] \times \mathbb{R}^d$, $G(t - \cdot, x - *) \in \mathcal{HP}$. The random field solution to (4.6) is the Gaussian stochastic process

$$v(t,x) = I_0(t,x) + u(t,x), \quad (t,x) \in (0,T] \times \mathbb{R}^d, \quad (4.7)$$

where

$$I_0(t,x) = \int_{\mathbb{R}^d} G(t, x - y) v_0(y) dy, \quad u(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) W^{H,\alpha}(ds, dy). \quad (4.8)$$

Remark 4.1. Since $v_0 \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, the function I_0 is in $C^\infty([0,T] \times \mathbb{R}^d)$ and it is a solution to the heat equation

$$\begin{cases} \frac{\partial v}{\partial t} - \Delta v = 0, \ (t,x) \in (0,T] \times \mathbb{R}^d, \\ \quad v(0,x) = v_0(x), \ x \in \mathbb{R}^d. \end{cases}$$

(see [Eva98, Thm. 1, p.47]).

More generally, assume that $v_0 \in \mathcal{C}^\zeta(\mathbb{R}^d)$, for some $\zeta \in (0,1]$. Then the function

$$[0,T] \times \mathbb{R}^d \ni (t,x) \longrightarrow I_0(t,x) = \int_{\mathbb{R}^d} G(t, x - y) v_0(y) dy,$$

is globally Hölder continuous, jointly in (t,x), with exponents $(\zeta/2, \zeta)$ (see e.g. [DSS]).

Throughout this chapter, we will make use the following expression for the variance of $u(t,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}^d$

$$\begin{aligned}
\sigma_{t,x}^2 &:= E\left(|u(t,x)|^2\right) \\
&= \alpha_H \int_0^t d\tau \int_0^t d\sigma |\tau - \sigma|^{2H-2} \\
&\quad \times \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dw f(z-w) G(t-\tau, x-z) G(t-\sigma, x-w) \\
&= \frac{\alpha_H}{(2\pi)^d} \int_0^t d\tau \int_0^t d\sigma \ |\tau - \sigma|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-2(\tau+\sigma)|\xi|^2}, \qquad (4.9)
\end{aligned}$$

which follows from the isometry (4.4) and (4.5) since $\mathcal{F}(G(t,*))(\xi) = e^{-t|\xi|^2}$. From the second equality in (4.9), we see that u is stationary in x, i.e. that σ_{tx}^2 does not depend on x.

The next theorem establishes a necessary and sufficient condition for the integral in the r.h.s. of (4.8) to be well defined.

Theorem 4.1. *[Tud13, Thm. 2.5]* $(u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d)$ *given in* (4.8) *exists and satisfies*

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} E(u(t,x)^2) < \infty$$

if and only if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^{2H}} < \infty. \qquad (4.10)$$

Due to (4.2), (4.10) *holds if and only if* $0 < d - \alpha < 4H$.

Proof. Let

$$\mathcal{K}_t(\xi) := \int_0^t d\tau \int_0^t d\sigma \ |\tau - \sigma|^{2H-2} e^{-2(\tau+\sigma)|\xi|^2}.$$

We claim that for any $t > 0$, $\xi \in \mathbb{R}^d$,

$$c(H)(t^{2H} \wedge 1) \left(\frac{1}{1+|\xi|^2}\right)^{2H} \leq \mathcal{K}_t(\xi) \leq C(H) \left(\frac{t}{1+|\xi|^2}\right)^{2H}. \qquad (4.11)$$

The r.h.s of (4.9) together with (4.11) implies the theorem.

Upper bound. By the change of coordinates $(\sigma, \tau) \mapsto t^{-1}(\tau, \sigma)$,

$$\mathcal{K}_t(\xi) = t^{2H} \int_0^1 d\tau \int_0^1 d\sigma |\tau - \sigma|^{2H-2} e^{-2t(\tau+\sigma)|\xi|^2}. \qquad (4.12)$$

Suppose that $|\xi| \leq 1$, then by (4.12)

$$\begin{aligned}
\mathcal{K}_t(\xi) &\leq t^{2H} \int_0^1 d\tau \int_0^1 d\sigma |\tau - \sigma|^{2H-2} = c(H) t^{2H} \\
&\leq C(H) \left(\frac{t}{1+|\xi|^2)}\right)^{2H},
\end{aligned}$$

where in the last inequality we used that $\frac{1}{2} \leq \frac{1}{1+|\xi|^2}$ if $|\xi| \leq 1$.

Now assume that $|\xi| \geq 1$. Apply Hölder's inequality in (4.12), to get that

$$\begin{aligned}\mathcal{K}_t(\xi) &\leq t^{2H}\left(\int_0^1 d\tau e^{-\frac{2\tau|\xi|^2}{2H}}\right)^{2H}\left(\int_0^1 d\tau\left(\int_0^1 d\sigma\ |\tau-\sigma|^{2H-2}e^{-2\sigma|\xi|^2}\right)^{\frac{1}{1-2H}}\right)^{1-2H}\\ &= C(H)t^{2H}\left(\int_0^1 d\tau e^{-\frac{2\tau|\xi|^2}{2H}}\right)^{2H}\end{aligned}$$

Since in this case $\frac{1}{|\xi|^2} \leq \frac{2}{1+|\xi|^2}$,

$$\left(\int_0^1 d\tau e^{-\frac{2\tau|\xi|^2}{2H}}\right)^{2H} = \left(\frac{2H}{|\xi|^2}\right)^{2H}\left(1-e^{-\frac{2|\xi|^2}{2H}}\right)^{2H} \leq \left(\frac{4H}{1+|\xi|^2}\right)^{2H},$$

which finishes the proof of the upper bound.

Lower bound. Suppose that $t|\xi|^2 \leq \frac{1}{4}$. Using the fact that $e^{-\tau} \geq 1-\tau$ for all $\tau > 0$, we conclude that

$$e^{-2\tau|\xi|^2} \geq 1-2\tau|\xi|^2 \geq \frac{1}{2}, \text{ for all } \tau \in [0,t].$$

Hence

$$\mathcal{K}_t(\xi) \geq \left(\frac{1}{2}\right)^2 \int_0^t d\tau \int_0^t d\sigma|\tau-\sigma|^{2H-2}d\tau d\sigma = c(H)t^{2H} \geq c(H)\left(\frac{t}{1+|\xi|^2}\right)^{2H}.$$

Now assume that $t|\xi|^2 \geq \frac{1}{4}$. Using the change of coordinates $(\tau,\sigma) \mapsto (2\tau|\xi|^2, 2\sigma|\xi|^2)$ we obtain

$$\mathcal{K}_t(\xi) = \frac{c(H)}{|\xi|^{4H}}\int_0^{2t|\xi|^2}\int_0^{2t|\xi|^2} e^{-(\tau+\sigma)}|\tau-\sigma|^{2H-2}$$

Since the integrand is non negative,

$$\mathcal{K}_t(\xi) \geq \frac{c(H)}{|\xi|^{4H}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} e^{-(\tau+\sigma)}|\tau-\sigma|^{2H-2} = \frac{c(H)}{|\xi|^{4H}} \geq \frac{c(H)}{(1+|\xi|^2)^{2H}}.$$

Finishing the proof of the lower bound. □

In the remaining of the chapter we will assume the constraint $0 < d-\alpha < 4H$.

4.2 Equivalence for the canonical metric

The canonical metric associated with the process u in (4.8) is defined by

$$\mathfrak{d}_u((t,x),(s,y)) = \|u(t,x)-u(s,y)\|_{L^2(\Omega)}. \qquad (4.13)$$

This section is devoted to prove Theorem 4.2, which gives an equivalent pseudo-distance for $\mathfrak{d}_u$. This result implies that u is a $\boldsymbol{g}$-Gaussian process.

Let W^α be a centered Gaussian process with covariance

$$E(W^\alpha(t,A)W^\alpha(s,B)) = (t\wedge s)\int_A\int_B f(z-w)\ dzdw$$

$$= \int_{\mathbb{R}_+} \int_A \int_B 1_{[0,t]}(r) 1_{[0,s]}(r) f(z-w)\ dzdwdr.$$

This noise is called *white-colored* noise. Similarly to the fractional-colored noise, we can construct the stochastic integral with respect to this noise. In this case, the Hilbert space of integrands $\mathcal{WP}$ is given by the closure of $\mathcal{E}$ with respect to the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{WP}} = \int_0^T dr \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dw \varphi(r,z)\psi(r,w) f(z-w). \tag{4.14}$$

The stochastic integral in (4.8) can be written as an integral with respect to W^α(see e.g. [Tud13, (2.31)]):

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^d} G(t-r, x-z)\ W^{\alpha,H}(dr,dz) \\ &\quad = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} d\tau\ G(t-\tau, x-z)(\tau - r)_+^{H-\frac{3}{2}} \right) W^\alpha(dr,dz). \end{aligned} \tag{4.15}$$

Using this property, we prove the first of the following proposition.

Proposition 4.1. *1. [HCSS21, Prop. 4.1] There exist a positive constant $c_{4,1}$, which depend on α, d, H, and T, such that for any $t,s \in [0,T]$ and $x,y \in \mathbb{R}^d$*

$$\mathfrak{d}_u^2((t,x),(s,y)) \geq c_{4,1} |t-s|^{2H - \frac{d-\alpha}{2}}. \tag{4.16}$$

2. [Tud13, Thm. 2.6] There exist a positive constant $c_{4,2}$, which depend on α, d, H, and T, such that for any $t,s \in [0,T]$ and $x \in \mathbb{R}^d$

$$\mathfrak{d}_u^2((t,x),(s,x)) \leq c_{4,2} |t-s|^{2H - \frac{d-\alpha}{2}}. \tag{4.17}$$

Proof. Assume, without loss of generality, that $0 \leq s < t \leq T$.

1. From (4.15), the isometry with respect to inner product in (4.14), and Parseval's identity, we obtain,

$$\begin{aligned} &\mathfrak{d}_u^2((t,x),(s,y)) \\ &\quad = E\Big(\Big| \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \\ &\quad \times \Big(\int_{\mathbb{R}} d\tau\ \left[G(t-\tau, x-z) 1_{(\tau \leq t)} - G(s-\tau, y-z) 1_{(\tau \leq s)} \right] (\tau - r)_+^{H-\frac{3}{2}} \Big) \\ &\quad \times W^\alpha(dr,dz) \Big|^2 \Big) \\ &\quad = \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^d} dz \int_{\mathbb{R}^d} dw f(z-w) \\ &\quad \times \left(\int_{\mathbb{R}} d\tau\ \left[G(t-\tau, x-z) 1_{(\tau \leq t)} - G(s-\tau, y-z) 1_{(\tau \leq s)} \right] (\tau - r)_+^{H-\frac{3}{2}} \right) \\ &\quad \times \left(\int_{\mathbb{R}} d\tau\ \left[G(t-\tau, x-w) 1_{(\tau \leq t)} - G(s-\tau, y-w) 1_{(\tau \leq s)} \right] (\tau - r)_+^{H-\frac{3}{2}} \right) \\ &\quad = (2\pi)^{-d} \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^d} \mu(d\xi) \end{aligned}$$

$$\times \left| \mathcal{F}\left(\int_{\mathbb{R}} d\tau \ \left[G(t-\tau, x-\cdot) 1_{(\tau \le t)} - G(s-\tau, y-\cdot) 1_{(\tau \le s)} \right] (\tau - r)_+^{H-\frac{3}{2}} \right)(\xi) \right|^2$$
$$= (2\pi)^{-d} \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^d} \mu(d\xi) \left| \int_{\mathbb{R}} d\tau \ \left[e^{-2(t-\tau)|\xi|^2} 1_{(\tau \le t)} - e^{-2(s-\tau)|\xi|^2} 1_{(\tau \le s)} \right] \right.$$
$$\left. \times (\tau - r)_+^{H-\frac{3}{2}} \right|^2. \tag{4.18}$$

Split the domain of integration of the variable r into the subdomains $[s,t]$ and $[s,t]^c$, and observe that on $[s,t]$, the term $e^{-2(s-\tau)|\xi|^2} 1_{(\tau \le s)} (\tau - r)_+$ equals zero. Since the integrand is non negative, we have

$$\begin{aligned} \mathfrak{d}_u^2((t,x),(s,y)) &\ge (2\pi)^{-d} \int_s^t dr \int_{\mathbb{R}^d} \mu(d\xi) \left(\int_{\mathbb{R}} d\tau \ e^{-2(t-\tau)|\xi|^2} 1_{(\tau \le t)} (\tau - r)_+^{H-\frac{3}{2}} \right)^2 \\ &= (2\pi)^{-d} \int_s^t dr \int_{\mathbb{R}^d} \mu(d\xi) \left(\int_r^t d\tau \ e^{-2(t-\tau)|\xi|^2} 1_{(\tau \le t)} (\tau - r)^{H-\frac{3}{2}} \right)^2 \\ &\ge c(\alpha, d, H) |t-s|^{2H - \frac{d-\alpha}{2}}, \end{aligned} \tag{4.19}$$

where the last equality follows by explicit computations on the integral.

2. Similarly to (4.9)

$$\begin{aligned} &\mathfrak{d}_u^2((t,x),(s,x)) \\ &= \frac{\alpha_H}{(2\pi)^d} \int_0^t d\tau \int_0^t d\sigma \ |\tau - \sigma|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-2(2t-(\tau+\sigma))|\xi|^2} \\ &- \frac{2\alpha_H}{(2\pi)^d} \int_0^t d\tau \int_0^s d\sigma \ |\tau - \sigma|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-2(t+s-(\tau+\sigma))|\xi|^2} \\ &+ \frac{\alpha_H}{(2\pi)^d} \int_0^s d\tau \int_0^s d\sigma \ |\tau - \sigma|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-2(2s-(\tau+\sigma))|\xi|^2} \\ &= \frac{\alpha_H}{(2\pi)^d} \int_s^t d\tau \int_s^t d\sigma \ |\tau - \sigma|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi) e^{-2(2t-(\tau+\sigma))|\xi|^2} \\ &+ \frac{\alpha_H}{(2\pi)^d} \int_0^s d\tau \int_0^s d\sigma \ |\tau - \sigma|^{2H-2} \\ &\times \int_{\mathbb{R}^d} \mu(d\xi) \left(e^{-2(2t-(\tau+\sigma))|\xi|^2} - 2e^{-2(t+s-(\tau+\sigma))|\xi|^2} + e^{-2(2t-(\tau+\sigma))|\xi|^2} \right) \\ &+ \frac{2\alpha_H}{(2\pi)^d} \int_t^s d\tau \int_0^s d\sigma \ |\tau - \sigma|^{2H-2} \\ &\times \int_{\mathbb{R}^d} \mu(d\xi) \left(e^{-2(2t-(\tau+\sigma))|\xi|^2} - 2e^{-2(t+s-(\tau+\sigma))|\xi|^2} \right) \\ &:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned}$$

Let us first note that

$$\begin{aligned} \mathcal{I}_3 \le \frac{2\alpha_H}{(2\pi)^d} &\int_t^s d\tau \int_0^s d\sigma \ |\tau - \sigma|^{2H-2} \\ &\times \int_{\mathbb{R}^d} \mu(d\xi) e^{-2(t-\tau)|\xi|^2} \left(e^{-2(t-\sigma)|\xi|^2} - e^{-2(s-\sigma))|\xi|^2} \right) \le 0, \end{aligned}$$

and therefore this term can be neglected.

Concerning the first term above we can write, by the change of variables $(\tau,\sigma)\mapsto(\frac{\tau-s}{t-s},\frac{\sigma-s}{t-s})$ and using (4.9)

$$\mathcal{I}_1\leq\sigma_{1,0}^2C(d,H)|t-s|^{2H}.$$

It remains to analyze the term $\mathcal{I}_2$. Use the change of variables $(\tau,\sigma)\mapsto(\frac{\tau-s}{t-s},\frac{\sigma-s}{t-s})$ and then, the change of variables $\xi\mapsto(t-s)^{\frac{1}{2}}\xi$ along with (4.2) to obtain

$$\begin{aligned}\mathcal{I}_2&\leq C(d,H)|t-s|^{2H-\frac{d-\alpha}{2}}\int_0^{\frac{s}{t-s}}d\tau\int_0^{\frac{s}{t-s}}d\sigma\,|\tau-\sigma|^{2H-2}\\&\times\int_{\mathbb{R}^d}d\xi|\xi|^{-\alpha}\left(e^{-2(2+\tau+\sigma)|\xi|^2}-2e^{-2(1+\tau+\sigma))|\xi|^2}+e^{-2(\tau+\sigma)|\xi|^2}\right)\\&\leq C(d,H)|t-s|^{2H-\frac{d-\alpha}{2}}\int_0^{\infty}d\tau\int_0^{\infty}d\sigma\,|\tau-\sigma|^{2H-2}\\&\times\int_{\mathbb{R}^d}d\xi|\xi|^{-\alpha}\left(e^{-2(2+\tau+\sigma)|\xi|^2}-2e^{-2(1+\tau+\sigma))|\xi|^2}+e^{-2(\tau+\sigma)|\xi|^2}\right).\end{aligned}$$

Now, using the change of variables $\xi\mapsto(2+\tau+\sigma)\xi$, $\xi\mapsto(1+\tau+\sigma)\xi$,$\xi\mapsto(\tau+\sigma)\xi$, we can write

$$\begin{aligned}\mathcal{I}_2&\leq C(d,H)|t-s|^{2H-\frac{d-\alpha}{2}}\int_{\mathbb{R}^d}d\xi|\xi|^{-\alpha}e^{-2|\xi|^2}\int_0^{\infty}d\tau\int_0^{\infty}d\sigma\,|\tau-\sigma|^{2H-2}\\&\times\left[(2+u+v)^{-\frac{(d-\alpha)}{2}}-(1+u+v)^{-\frac{(d-\alpha)}{2}}+(u+v)^{-\frac{(d-\alpha)}{2}}\right]\\&:=C(d,H)|t-s|^{2H-\frac{d-\alpha}{2}}\times\mathcal{I}_{2,1}\times\mathcal{I}_{2,2}.\end{aligned}$$

$\mathcal{I}_{2,1}$ is finite since $d-\alpha>0$. We note that for u,v close to zero the integrand of $\mathcal{I}_{2,2}$ near zero is finite since $2H-2>0$ and that for u,v big enough

$$(2+u+v)^{-\frac{(d-\alpha)}{2}}-(1+u+v)^{-\frac{(d-\alpha)}{2}}+(u+v)^{-\frac{(d-\alpha)}{2}}\leq C(d)(u+v)^{-\frac{d-\alpha}{2}},$$

this together with the condition $4H-(d-\alpha)>0$ implies that $\mathcal{I}_2$ is finite. □

The proof of the next proposition is the same as that of [TX17, Thm. 4], where $\alpha\in(0,d)$. For the sake of completeness, we provide the details and see that the arguments can be adapted to cover the case $\alpha=0$.

Proposition 4.2. *[HCSS21, Prop. 4.2] Let $M>0$. There exists positive constants $c_{4,3},c_{4,4}$, that depend on α,d,H,M, such that for any $t>0$, $x,y\in[-M,M]^d$,*

$$\begin{aligned}&c_{4,3}(t^{2H}\wedge1)\left(\log\frac{2e\sqrt{d}M}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4H-(d-\alpha))}\\&\leq\mathfrak{d}_u^2((t,x),(t,y))\leq c_{4,4}(t^{2H}+1)\left(\log\frac{2e\sqrt{d}M}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4H-(d-\alpha))},\quad(4.20)\end{aligned}$$

where $\beta=1$, if $4H-(d-\alpha)=2$, and $\beta=0$, otherwise.

Proof. Similarly as in (4.9), using (4.5), we have

$$\mathfrak{d}_u^2((t,x),(t,y))$$

$$= \frac{\alpha_H}{(2\pi)^d} \int_0^t d\tau \int_0^t d\sigma\, |\tau-\sigma|^{2H-2} \int_{\mathbb{R}^d} \mu(d\xi)\, e^{-2(\tau+\sigma)|\xi|^2} \left(1-\cos[(x-y)\cdot\xi]\right). \tag{4.21}$$

Recall (4.2). After having applied the change of variables $\xi \mapsto \frac{\eta}{|x-y|}$, from (4.2), (4.21) and (4.11) we deduce

$$\begin{aligned} c_{1,d,H}(t^{2H}\wedge 1)|x-y|^{4H-d-\alpha} \int_{\mathbb{R}^d} d\eta \frac{\left(1-\cos\left[\left(\frac{x-y}{|x-y|}\right)\cdot\eta\right]\right)}{|\eta|^\alpha(|x-y|^2+|\eta|^2)^{2H}} &\le \mathfrak{d}_u^2((t,x),(t,y)) \\ \le c_{2,d,H}(t^{2H}+1)|x-y|^{4H-d-\alpha} \int_{\mathbb{R}^d} d\eta \frac{\left(1-\cos\left[\left(\frac{x-y}{|x-y|}\right)\cdot\eta\right]\right)}{|\eta|^\alpha(|x-y|^2+|\eta|^2)^{2H}}&, \end{aligned} \tag{4.22}$$

for all $t>0$ and $x,y\in\mathbb{R}^d$, with some positive and finite constants $c_{1,d,H}$, $c_{2,d,H}$.

Next we give lower and upper bounds for the terms on the left hand side and the right hand side of (4.22), respectively.

Lower bounds. By Schwartz's inequality, $\overline{B_1(0)} \subset \left\{\eta\in\mathbb{R}^d : \left|\frac{(x-y)}{|x-y|}\cdot\eta\right| \le 1\right\}$. Moreover, for $|\theta|\le 1$, $1-\cos\theta \ge \frac{\theta^2}{4}$. Consequently,

$$\mathcal{I} := \int_{\mathbb{R}^d} d\eta \frac{\left(1-\cos\left[\left(\frac{x-y}{|x-y|}\right)\cdot\eta\right]\right)}{|\eta|^\alpha(|x-y|^2+|\eta|^2)^{2H}} \ge \frac{1}{4}\int_{\overline{B_1(0)}} d\eta \frac{\left(\frac{x-y}{|x-y|}\cdot\eta\right)^2}{|\eta|^\alpha(|x-y|^2+|\eta|^2)^{2H}}. \tag{4.23}$$

Shrink the ball $\overline{B_1(0)}$ to the spherical sector defined by the constraint $\varphi\in[0,\pi/4]$ on the angle. Then, pass to spherical coordinates and, without loss of generality, suppose that $(x-y)/|x-y|$ is the unit vector $(1,0,\dots,0)$ in $\mathbb{R}^d$. Since $\frac{x-y}{|x-y|}\cdot\eta = |\eta|\cos\varphi$, where $\varphi\in[0,\pi/4]$ is the angle between $(x-y)/|x-y|$ and η, we obtain,

$$\mathcal{I} \ge C\int_0^1 d\rho \frac{\rho^{d-\alpha+1}}{(|x-y|^2+\rho^2)^{2H}}.$$

We estimate this integral by distinguishing three cases.

Case 1. $0<4H-(d-\alpha)<2$. Since $|x-y|^2+\rho^2 \le 4dM^2+1$,

$$\int_0^1 d\rho \frac{\rho^{d-\alpha+1}}{(|x-y|^2+\rho^2)^{2H}} \ge \int_0^1 d\rho \frac{\rho^{d-\alpha+1}}{(4dM^2+1)^{2H}} = \frac{1}{(d-\alpha+2)(4dM^2+1)^{2H}}.$$

Case 2. $4H-(d-\alpha)=2$. Because $|x-y|\le 2\sqrt{d}M$, we clearly have

$$\begin{aligned} \int_0^1 d\rho\, \frac{\rho^{d-\alpha+1}}{(|x-y|^2+\rho^2)^{2H}} &\ge c_{\alpha,d,H,M}\int_{\frac{|x-y|}{2e\sqrt{d}M}}^1 d\rho\, \rho^{d-\alpha-4H+1} \\ &= c_{\alpha,d,H,M}\log\left(\frac{2e\sqrt{d}M}{|x-y|}\right). \end{aligned}$$

Case 3. $4H-(d-\alpha)>2$. Using a similar argument as for case 2,

$$\int_0^1 d\rho \frac{\rho^{d-\alpha+1}}{(|x-y|^2+\rho^2)^{2H}} \ge c_{\alpha,d,H,M}\int_{\frac{|x-y|}{2e\sqrt{d}M}}^1 d\rho\rho^{d-\alpha+1-4H}$$

$$= c_{\alpha,d,H,M}|x-y|^{d-\alpha-4H+2}.$$

Upper bounds. Apply the inequality $1-\cos(\theta) \le 2\wedge\theta^2$ and then, use spherical coordinates to see that the integral $\mathcal{I}$ defined in (4.23) satisfies

$$\mathcal{I} \le \int_{\mathbb{R}^d} d\eta \frac{(2\wedge|\eta|)^2)}{|\eta|^\alpha(|x-y|^2+|\eta|^2)^{2H}} = c_d \int_0^\infty d\rho \frac{(1\wedge\rho^2)\rho^{d-\alpha-1}}{(|x-y|^2+\rho^2)^{2H}} := c_d\, \mathcal{J}. \tag{4.24}$$

We estimate $\mathcal{J}$ by considering three cases, as we did for the lower bounds.
Case 1. $0 < 4H-(d-\alpha) < 2$. Since $|x-y|^2+\rho^2 \ge \rho^2$, we have

$$\mathcal{J} \le \int_0^1 d\rho\ \rho^{d-\alpha-4H+1} + \int_1^\infty d\rho\ \rho^{d-\alpha-4H-1} = c_{\alpha,d,H}.$$

Case 2. $4H-(d-\alpha)=2$. Splitting the domain of integration of $\mathcal{J}$, we obtain

$$\mathcal{J} \le \int_0^{|x-y|} d\rho \frac{\rho^{d-\alpha+1}}{|x-y|^{4H}} + \int_{|x-y|}^{2e\sqrt{d}M} d\rho\ \rho^{d-\alpha-4H+1} + \int_{2e\sqrt{d}M}^\infty d\rho\ \rho^{d-\alpha-4H-1}$$
$$= \frac{1}{(d-\alpha+2)} + \log\left(\frac{2e\sqrt{d}M}{|x-y|}\right) + \frac{(2e\sqrt{d}M)^2}{2} \le c_{\alpha,d,H,M} \log\left(\frac{2e\sqrt{d}M}{|x-y|}\right).$$

Case 3. $4H-(d-\alpha)>2$. Using the inequalities $1/(|x-y|^2+\rho^2) \le 1/(|x-y|^2)$ and $1/(|x-y|^2+\rho^2) \le 1/\rho^2$, on $\{0\le\rho\le|x-y|\}$ and $\{|x-y|<\rho<\infty\}$, respectively, we have

$$\mathcal{J} \le |x-y|^{-4H}\int_0^{|x-y|} d\rho\ \rho^{d-\alpha+1} + \int_{|x-y|}^\infty d\rho\ \rho^{d-\alpha-4H+1} = c_{\alpha,d,H}|x-y|^{d-\alpha-4H+2}.$$

From (4.23), and using the lower and upper bounds obtained before, we deduce (4.20). □

We end this section by proving the equivalence for the canonical metric (4.13). It is a consequence of Propositions 4.1 and 4.2.

Theorem 4.2. *[HCSS21, Thm. 4.1] Fix $M>0$ and $t_0\in(0,T]$. There exists positive constants $c_{4,5}$, $c_{4,6}$ depending on α,d,t_0,H,M,T such that for any $t,s\in[t_0,T]$ and $x,y\in[-M,M]^d$,*

$$\mathfrak{d}_u^2((t,x),(s,y)) \asymp |t-s|^{2H-\frac{d-\alpha}{2}} + \left(\log\frac{2e\sqrt{d}M}{|x-y|}\right)^\beta |x-y|^{2\wedge(4H-(d-\alpha))}, \tag{4.25}$$

where $\beta=1$, if $4H-(d-\alpha)=2$, and $\beta=0$, otherwise.

The upper bound holds for any $t,s\in[0,T]$.

Proof. The estimate from above is a consequence of the upper bounds in (4.17) and (4.20), which hold for any $t,s\in[0,T]$.

We prove the estimates from below by distinguishing two cases.
Case 1. $|t-s|^{2H-\frac{d-\alpha}{2}} < \frac{c_{4,3}(t_o^{2H}\wedge 1)}{4c_{4,2}}\left(\log\frac{2e\sqrt{d}M}{|x-y|}\right)^\beta |x-y|^{2\wedge(4H-(d+\alpha))}$. Applying the triangle inequality and then, using the lower bound in (4.20) and the upper bound in (4.17), we obtain

$$\mathfrak{d}_u^2((t,x),(s,y)) \ge \frac{1}{2}\mathfrak{d}_u^2((t,x),(t,y)) - \mathfrak{d}_u^2((t,y),(s,y))$$

$$\geq \frac{c_{4,3}(t_0^{2H}\wedge 1)}{2}\left(\log\frac{2e\sqrt{d}M}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4H-(d-\alpha))}-c_{4,2}|t-s|^{2H-\frac{d-\alpha}{2}}$$
$$\geq \frac{c_{4,3}(t_0^{2H}\wedge 1)}{8}\left(\log\frac{2e\sqrt{d}M}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4H-(d-\alpha))}+\frac{c_{4,2}}{2}|t-s|^{2H-\frac{d-\alpha}{2}}.$$

Case 2. $|t-s|^{2H-\frac{d-\alpha}{2}} \geq \frac{c_{4,3}(t_0^{2H}\wedge 1)}{4c_{4,2}}\left(\log\frac{2e\sqrt{d}M}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4H-(d-\alpha))}$. By (4.16),

$$\mathfrak{d}_u^2((t,x),(s,y)) \geq c_{4,1}|t-s|^{2H-\frac{d-\alpha}{2}}$$
$$\geq \frac{c_{4,1}}{2}|t-s|^{2H-\frac{d-\alpha}{2}}+\frac{c_{4,3}(t_0^{2H}\wedge 1)}{8c_{4,2}}\left(\log\frac{2e\sqrt{d}M}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4H-(d-\alpha))}.$$

The proof is complete. □

4.3 Further second order properties

We prove Proposition 4.3 which together with Theorem 4.2, will be used in Section 4.4 to find hitting probabilities estimates for the solution of the stochastic heat equation.

Proposition 4.3. *[HCSS21, Lem. 4.1 & Lem.4.2] Fix $M > 0$ and $t_0 \in (0, T]$.*

1. *$\sigma_{t,x}^2 \asymp 1$ for $(t,x) \in [t_0,T]\times\mathbb{R}^d$,*
2. *For any $(t,x) \in (0,\infty)\times\mathbb{R}^d$, the mapping $t \mapsto \sigma_{t,x}^2$ is differentiable.*
3. *There exists $\eta > 0$ and $C > 0$, depending on α, d, t_0, H, M, T, such that, for all $s,t \in [t_0,T]$ and $x,y \in [-M,M]^d$,*
$$|\sigma_{t,x}^2-\sigma_{s,y}^2| \leq C\mathfrak{d}_u^{1+\eta}((t,x),(s,y)). \tag{4.26}$$
4. *For any $(t,x),(s,y) \in [t_0,T]\times\mathbb{R}^d$ such that $(t,x)\neq(s,y)$, $\rho_{(t,x),(s,y)} < 1$.*
5. *For any $(t,x),(s,y) \in [t_0,T]\times[-M,M]^d$,*
$$E(Var(u(t,x)\mid u(s,y))) \asymp |t-s|^{2H-\frac{d-\alpha}{2}}+\left(\log\frac{2e\sqrt{d}M}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4H-(d-\alpha))}, \tag{4.27}$$
where $\beta = 1$, if $4H-(d-\alpha) = 2$, and $\beta = 0$, otherwise.

Proof. 1. Use the change of variable $\xi \mapsto (\tau+\sigma)^{\frac{1}{2}}\xi$, in the last expression of the array (4.9) to get

$$\sigma_{t,x}^2 = \frac{\alpha_H}{(2\pi)^d}\int_0^t d\tau\int_0^t d\sigma\,\frac{|\tau-\sigma|^{2H-2}}{(\tau+\sigma)^{(d-\alpha)/2}}\int_{\mathbb{R}^d}\mu(d\xi)e^{-|\xi|^2}. \tag{4.28}$$

By (4.2)

$$\int_{\mathbb{R}^d}\mu(d\xi)e^{-|\xi|^2} \asymp \int_{\mathbb{R}}\rho^{d-\alpha-1}e^{-\rho^2} < \infty. \tag{4.29}$$

Apply the change of variables , $\tau \mapsto \frac{\tau}{t}$, $\sigma \mapsto \frac{\sigma}{t}$, in (4.28) together with (4.29) to see that $\sigma_{t,x}^2$ is bounded from below (respectively, from above) by

$$t^{2H-(d-\alpha)/2} c_{\alpha,d,H} \int_0^1 d\tau \int_0^1 d\sigma \, \frac{|\tau-\sigma|^{2H-2}}{(\tau+\sigma)^{(d-\alpha)/2}}. \tag{4.30}$$

Let $C_{\alpha,d,H} = \int_0^1 d\tau \int_0^1 d\sigma \frac{|\tau-\sigma|^{2H-2}}{(\tau+\sigma)^{\frac{d-\alpha}{2}}}$ and observe that, since $4H-(d-\alpha) > 0$, $C_{\alpha,d,H} < \infty$. From the above computations, we deduce that the lower inequality (respectively, the upper inequality) in the first claim holds with $c \le t_0^{2H-\frac{(d-\alpha)}{2}} C_{\alpha,d,H}$ (respectively, with $C \ge T^{2H-\frac{(d-\alpha)}{2}} C_{\alpha,d,H}$).

2. Using the same arguments that those in the proof of 1. we deduce that $\sigma_{t,x} = c_{\alpha,d,H} t^{H-(d-\alpha)/4}$ which clearly implies 2.

3. Assume, without loss of generality, that $0 < s \le t$. For all $x, y \in \mathbb{R}^d$, from (4.9) and similarly as in (4.30), it follows that

$$\begin{aligned}
&\left(\frac{\alpha_H}{(2\pi)^d}\right)^{-1} |\sigma_{t,x}^2 - \sigma_{s,y}^2| = \left(\frac{\alpha_H}{(2\pi)^d}\right)^{-1} (\sigma_{t,x}^2 - \sigma_{s,y}^2) \\
&= \int_{\mathbb{R}^d} \mu(d\xi) e^{-2(\tau+\sigma)|\xi|^2} \left(\int_0^t d\tau \int_0^t d\sigma \, |\tau-\sigma|^{2H-2} - \int_0^s d\tau \int_0^s d\sigma \, |\tau-\sigma|^{2H-2} \right) \\
&\le c_{\alpha,d,H} \left(\int_s^t d\tau \int_s^t d\sigma \, \frac{|\tau-\sigma|^{2H-2}}{(\tau+\sigma)^{\frac{d-\alpha}{2}}} + 2 \int_0^s d\tau \int_s^t d\sigma \, \frac{|\tau-\sigma|^{2H-2}}{(\tau+\sigma)^{\frac{d-\alpha}{2}}} \right).
\end{aligned} \tag{4.31}$$

Apply polar coordinates $(\tau, \sigma) \mapsto (\rho\cos\theta, \rho\sin\theta)$ and then, the mean value theorem, to see that

$$\begin{aligned}
&\int_s^t d\tau \int_s^t d\sigma \, \frac{|\tau-\sigma|^{2H-2}}{(\tau+\sigma)^{\frac{d-\alpha}{2}}} \\
&\le \int_{\sqrt{2}s}^{\sqrt{2}t} d\rho \, \rho^{2H-\frac{d-\alpha}{2}-1} \left(\int_0^{\frac{\pi}{2}} d\theta \frac{|\cos\theta - \sin\theta|^{2H-2}}{(\cos\theta+\sin\theta)^{\frac{d-\alpha}{2}}} \right) \\
&\le \frac{2^{H-\frac{(d-\alpha)}{4}} T^{2H-\frac{(d-\alpha)}{2}-1} (t-s)}{\left(2H - \frac{(d-\alpha)}{2}\right)^2} \int_0^{\frac{\pi}{2}} d\theta \, \frac{|\cos\theta - \sin\theta|^{2H-2}}{(\cos\theta+\sin\theta)^{\frac{d-\alpha}{2}}} \le C(\alpha, d, H, T)(t-s).
\end{aligned}$$

Since $0 < 2H - \frac{(d-\alpha)}{2} < 2$, we have $\eta_1 := \left(H - \frac{(d-\alpha)}{4}\right)^{-1} - 1 > 0$, and we deduce,

$$\int_s^t d\tau \int_s^t d\sigma \, \frac{|\tau-\sigma|^{2H-2}}{(\tau+\sigma)^{\frac{d-\alpha}{2}}} \le C(H, d, T)(t-s)^{\frac{4H-(d-\alpha)}{4}(1+\eta_1)}. \tag{4.32}$$

As for the second integral on the last line of (4.31), we have

$$\int_0^s d\tau \int_s^t d\sigma \, \frac{|\tau-\sigma|^{2H-2}}{(\tau+\sigma)^{\frac{d-\alpha}{2}}} \le \int_0^s d\tau \int_s^t d\sigma \, (\sigma-\tau)^{2H-\frac{(d-\alpha)}{2}-2}, \tag{4.33}$$

because $\tau \le \sigma$ implies $\tau + \sigma \ge \sigma - \tau$.

Our next goal is to obtain estimates from above on the right-hand side of (4.33) in terms of powers of $(t-s)$. For this, we consider three cases.

Case 1. $0 < 4H - (d-\alpha) < 2$.

$$\begin{aligned}\int_0^s d\tau \int_s^t d\sigma\ (\sigma-\tau)^{2H-\frac{(d-\alpha)}{2}-2} &= \frac{s^{2H-\frac{(d-\alpha)}{2}} + (t-s)^{2H-\frac{(d-\alpha)}{2}} - t^{2H-\frac{(d-\alpha)}{2}}}{(2H-\frac{(d-\alpha)}{2})(1+\frac{(d-\alpha)}{2}-2H)}\\ &\le \frac{(t-s)^{2H-\frac{(d-\alpha)}{2}}}{(2H-\frac{(d-\alpha)}{2})(1+\frac{(d-\alpha)}{2}-2H)} = C(\alpha,d,H)(t-s)^{2H-\frac{d-\alpha}{2}}\\ &= C(\alpha,d,H)(t-s)^{\frac{4H-(d-\alpha)}{4}(1+\eta_2)}, \end{aligned} \tag{4.34}$$

with $\eta_2 = 1$.
Case 2. $0 < 4H - (d-\alpha) = 2$.

$$\begin{aligned}\int_0^s d\tau \int_s^t d\sigma\ (\sigma-\tau)^{-1} &= t\log(t) - s\log(s) + (t-s)\log\left((t-s)^{-1}\right)\\ &\le 2[(t\log t - s\log s) \vee ((t-s)\log((t-s)^{-1}))]\\ &\le 2(t-s)[(\log T + 1) \vee \log((t-s)^{-1})],\end{aligned}$$

where in the last inequality we have applied the mean value theorem. This yields, for any $\gamma \in (0,1)$,

$$\begin{aligned}\int_0^s d\tau \int_s^t d\sigma\ (\sigma-\tau)^{-1} &\le 2(|\log T|+2)([(t-s)^\gamma \vee (t-s)] \le C(T)\ (t-s)^\gamma\\ &= C(T)(t-s)^{\frac{4H-(d-\alpha)}{4}(1+\eta_3)},\end{aligned} \tag{4.35}$$

with $\eta_3 = 2\gamma - 1$
Case 3. $2 < 4H - (d-\alpha) < 4$.

$$\begin{aligned}\int_0^s d\tau \int_s^t d\sigma\ (\sigma-\tau)^{2H-2-\frac{(d-\alpha)}{2}} &= \frac{t^{2H-\frac{(d-\alpha)}{2}} - s^{2H-\frac{(d-\alpha)}{2}} - (t-s)^{2H-\frac{(d-\alpha)}{2}}}{\left(2H-\frac{(d-\alpha)}{2}\right)\left(2H-1-\frac{(d-\alpha)}{2}\right)}\\ &\le \frac{t^{2H-\frac{(d-\alpha)}{2}} - s^{2H-\frac{(d-\alpha)}{2}}}{\left(2H-\frac{(d-\alpha)}{2}\right)\left(2H-1-\frac{(d-\alpha)}{2}\right)} \le \frac{T^{2H-1-\frac{(d-\alpha)}{2}}}{2H-1-\frac{(d-\alpha)}{2}}(t-s)\\ &\le C(\alpha,d,H,T)(t-s)^{\frac{4H-(d-\alpha)}{4}(1+\eta_4)},\end{aligned} \tag{4.36}$$

with $\eta_4 = \eta_1 = \left(H - \frac{(d-\alpha)}{4}\right)^{-1} - 1$.

Set $\eta = \min(\eta_i, i=1,2,3)$. Appealing to Theorem 4.2, and using (4.31), (4.32), (4.34), (4.35) and (4.36), we obtain

$$\begin{aligned}|\sigma^2_{t,x} - \sigma^2_{s,y}| &\le C(\alpha,d,H,T)(t-s)^{\left(H-\frac{d-\alpha}{4}\right)(1+\eta)}\\ &\le c_{4,5}^{-1} C(\alpha,d,H,T)\ \mathfrak{d}_u^{1+\eta}((t,x),(s,y)),\end{aligned}$$

with $c_{4,5}$ as in (4.25).

4. Assume that $\rho_{(t,x),(s,y)} = 1$ and hence, that there exists $\lambda \in \mathbb{R}$ such that

$$\|u(t,x) - \lambda u(s,y)\|_{L^2(\Omega)} = 0. \tag{4.37}$$

We will see that this assumption leads to a contradiction.

Case 1. $s < t$. Apply (4.18) with $u(s, y)$ replaced by $\lambda u(s, y)$ to obtain

$$\|u(t,x) - \lambda u(s,y)\|^2_{L^2(\Omega)} = (2\pi)^{-d} \int_{\mathbb{R}_+} dr \int_{\mathbb{R}^d} \mu(d\xi) \\ \times \left| \int_{\mathbb{R}} d\tau \ \left[e^{-2(t-\tau)|\xi|^2} 1_{(\tau \le t)} - \lambda e^{-2(s-\tau)|\xi|^2} 1_{(\tau \le s)} \right] (\tau - r)_+^{H-\frac{3}{2}} \right|^2 .$$

As in (4.19), this is bounded from below by a constant multiple of

$$\int_{\mathbb{R}^d} \mu(d\xi) \int_s^t dr \left(\int_r^t d\tau \ e^{-2(t-\tau)|\xi|^2} (\tau - r)^{H-\frac{3}{2}} \right)^2 .$$

A direct computation shows that $\int_s^t dr \left(\int_r^t d\tau \ e^{-2(t-\tau)|\xi|^2} (\tau - r)^{H-\frac{3}{2}} \right)^2 \neq 0$. Since we are assuming (4.37), we reach a contradiction.

We notice that, in the case under consideration, the arguments hold for any $(t, x), (s, y) \in [0, \infty) \times \mathbb{R}^d$.

Case 2. $s = t \in [t_0, T]$, $x \neq y$. Apply (4.21) with $u(t, y)$ replaced by $\lambda u(t, y)$ to see that

$$\|u(t,x) - \lambda u(s,y)\|^2_{L^2(\Omega)} = \frac{\alpha_H}{(2\pi)^d} \int_0^t d\tau \int_0^t d\sigma \ |\tau - \sigma|^{2H-2} \\ \times \int_{\mathbb{R}^d} \mu(d\xi) \ e^{-2(\tau+\sigma)|\xi|^2} \left(1 + \lambda^2 - 2\lambda \cos[(x-y)\cdot\xi]\right).$$

Using the lower bound estimates in (4.2) and (4.11), we deduce

$$\|u(t,x) - \lambda u(s,y)\|^2_{L^2(\Omega)} \ge C(\alpha, d, t_0, H) \int_{\mathbb{R}^d} \left(1 + \lambda^2 - 2\lambda \cos[(x-y)\cdot\xi]\right) \\ \times \frac{|\xi|^{-\alpha}}{(1 + |\xi|^2)^{2H}} \ d\xi.$$

By assumption, the integral on the right-hand side must be zero. However, this integral is bounded from below by the integral on the spherical sector of the ball B_1 where $2\cos[(x-y)\cdot\xi] \le \frac{1+\lambda^2}{2}$. Consequently,

$$0 = \|u(t,x) - \lambda u(t,y)\|^2_{L^2(\Omega)} \ge C(\alpha, d, t_0, H) \frac{1+\lambda^2}{2} \int_0^1 \frac{r^{d-\alpha-1}}{(1+r^2)^{2H}} dr,$$

which is a contradiction.

5. 1. and 3. implies that

$$\lim_{\varepsilon \downarrow 0} \sup_{\substack{(t,x),(\bar{t},\bar{x}) \in I \times J \\ |(t-\bar{t}, x-\bar{x})| \le \varepsilon}} \frac{|\sigma_{t,x} - \sigma_{\bar{t},\bar{x}}|}{\mathfrak{d}_{(t,x),(\bar{t},\bar{x})}} = 0.$$

By Theorem 4.2, 1. and 4., the rest of the hypothesis of Proposition 3.1 are also valid. Thus we deduce (3.21) which by (4.25) implies (4.27). □

4.4 Hitting probabilities

Consider the Gaussian random field

$$\boldsymbol{v} = \{\boldsymbol{v}(t,x) = (v_1(t,x), ..., v_D(t,x)),\ (t,x) \in [0,T] \times \mathbb{R}^d\},$$

where the components are independent copies of the process $v(t,x)$ defined in (4.7). The process $\boldsymbol{v}$ is the random field solution to the system of SPDEs

$$\begin{cases} \frac{\partial v_j}{\partial t}(t,x) = \Delta v_j(t,x) + \dot{W}_j^{H,\alpha},\ (t,x) \in (0,T] \times \mathbb{R}^d, \\ \quad v_j(0,x) = v_0(x), x \in \mathbb{R}^d, \end{cases}$$

$j = 1, \dots, D$, where $(W_j^{H,\alpha}, j = 1, \dots, D)$ are independent copies of the fractional-colored noise $W^{H,\alpha}$ introduced at the beginning of Section 4.1 and v_0 is such that the function $(t,x) \mapsto I_0(t,x)$ is continuous (see Remark 4.1 for sufficient conditions).

Throughout this section, we will consider the compact sets $I = [t_0, T]$ and $J = [-M, M]^d$, with $t_0 \in (0,T]$, $M > 0$, and the gauge functions defined in $\mathbb{R}_+$,

$$g_1(\tau) = \tau^{H - \frac{d-\alpha}{4}}, g_2(\tau) = \left(\log \frac{2e\sqrt{d}M}{\tau}\right)^{\frac{\beta}{2}} \tau^{1 \wedge \left(2H - \frac{d-\alpha}{2}\right)}, \beta = 1_{4H-(d-\alpha)=2}.$$

We remind the related function to $\boldsymbol{g} = (g_1, g_2)$ introduced in Chapter 3,

$$q_{\boldsymbol{g}}(\tau) = \tau^D (g_1^{-1}(\tau))^{-1} (g_2^{-1}(\tau))^{-d}.$$

Let $D_0 = [H - (d-\alpha)/4]^{-1} + d[1 \wedge (2H - (d-\alpha)/2]^{-1}$. According to Example 3.2 the function $q_{\boldsymbol{g}}$ satisfy the conditions required by the definition of the $q_{\boldsymbol{g}}$-Hausdorff measure if and only if $D > D_0$. We now give the main theorem on hitting probabilities for the process $\boldsymbol{v}$.

Theorem 4.3. *Suppose that the function $I \times J \ni (t,x) \mapsto I_0(t,x)$ satisfies the condition (CY2)*

1. *[HCSS21, Thm. 4.2] Let $D > D_0$.*

   (a) There exists a constant $C := C(I,J,D,d)$ such that for any Borel set $A \subset \mathbb{R}^D$,
   $$P(\boldsymbol{v}(I \times J) \cap A \neq \emptyset) \leq C\ \mathcal{H}_{q_{\boldsymbol{g}}}(A). \tag{4.38}$$

   (b) Fix $N > 0$ and let $A \subset B_N(0) \subset \mathbb{R}^D$ be a Borel set. There exists a constant $c := c(I,J,N,D,d)$ such that
   $$P(\boldsymbol{v}(I \times J) \cap A \neq \emptyset) \geq c\ Cap_{(q_{\boldsymbol{g}})^{-1}}(A). \tag{4.39}$$

2. *Assume that $D_0 < D$ or $D = D_0$, $\frac{\beta}{2} d[1 \wedge (2H - (d-\alpha)/2)]^{-1} > 1$. Fix $N > 0$ and let $A \subset B_N(0) \subset \mathbb{R}^D$ be a Borel set. There exists a constant $c := c(I,J,N,D,d)$ such that*
   $$P(\boldsymbol{v}(I \times J) \cap A \neq \emptyset) \geq c[v_{\boldsymbol{g}}(0)]^{-1} > 0. \tag{4.40}$$

Proof. Theorem 4.2, Example 3.1 and the hypothesis on the (CY2) condition implies that $\boldsymbol{v}$ has an a.s. continuous modification on $I \times J$.

Case 1. $D > D_0$. Proposition 4.3(1.) implies that $\sigma_{I^{(\eta)},J^{(\eta)}} > 0$ and by Theorem 4.2 u_1 is $\boldsymbol{g}$-Gaussian in $I^{(\eta)} \times J^{(\eta)}$ for $\eta > 0$ small enough. In Example 1.3 we proved that g_1 and g_2 satisfies the hypothesis of Lemma 1.1, and according to Example 3.2 $q_{\boldsymbol{g}}$ is strictly increasing on a small interval around zero. Thus the assumptions of Corollary 3.1 are valid and (3.16) is (4.38).

Additionally to the paragraph above, we observe that by Proposition 4.3, u_1 satisfies conditions (CY1)-(CY3) on $I \times J$. Since in Example 3.4, we verified the validity of (3.23), (3.24) in Theorem 1.2 implies (4.39).

Case 2. $D < D_0$ *or* $D = D_0$ *and* $\frac{\beta}{2}d[1 \wedge (2H - (d-\alpha)/2)]^{-1} > 1$. (4.40) follows by (3.26) in Theorem 3.2, since according to Example 3.4, in this case (3.25) is valid instead of (3.23).

□

Remark 4.2. Assume that $D = D_0$, $\frac{\beta}{2}d[1 \wedge (2H - (d-\alpha)/2)]^{-1} > 1$. This implies that $\beta = 1$ and forces to $4H - (d - \alpha) = 2$. Thus, we deduce that $D = 2 + d$ and $\frac{d}{2} > 1$. The assumptions are equivalent to the conditions $4H - (d - \alpha) = 2$, $D = d + 2$ and $d > 2$.

Theorem 4.3 implies the next corollary which identifies conditions $D = D_0$ and $\frac{\beta}{2}d[1 \wedge (2H - (d-\alpha)/2)]^{-1} \leq 1$ with the critical dimension for polarity of points of $\boldsymbol{v}$.

Corollary 4.1. *If* $D > D_0$, *points are polar for* $\boldsymbol{v}$. *If* $D < D_0$ *or* $D = D_0$, $\frac{\beta}{2}d[1 \wedge (2H - (d-\alpha)/2)]^{-1} > 1$, *points are non polar.*

Proof. Assume first $D > D_0$. By the definition of the $q_{\boldsymbol{g}}$ measure we have $\mathcal{H}_{q_g}(\{z\}) = 0$. Hence the polarity of $\{z\}$ follows by (4.38). The rest of the Corollary follows by (4.40). □

Chapter 5

The linear stochastic biharmonic heat equation

We apply the results of Chapter 3 to the linear stochastic biharmonic heat equation

$$\begin{cases} (\frac{\partial}{\partial t} + (-\Delta)^2)v_j = \dot{W}_j, (t,x) \in (0,T] \times \mathbb{T}^d, \\ \qquad v_j(0,x) = v_0(x), x \in \mathbb{T}^d. \end{cases}$$

$j = 1, ..., D$, with $(W_j, j = 1, ..., D)$ independent copies of a white noise.

We introduce some notation used throughout this chapter. As usually, $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, ...\}$; we set $\mathbb{Z}_2 = \{0, 1\}$, and for any integer $d \geq 1$, $\mathbb{N}^{d,*} = (\mathbb{N} \setminus \{0\})^d$. For any multiindex $k = (k_1, \dots, k_d) \in \mathbb{N}^d$, we set $|k| = (\sum_{j=1}^d k_j^2)^{1/2}$, and denote by $n(k)$ the number of null components of k. The results of this chapter are published in [HCSS22].

5.1 The solution

The *d-dimensional torus* $\mathbb{T}^d$ is the box $[0, 2\pi]^d$ with the identification $x \sim y$ if and only if $x_i \equiv y_i \mod 2\pi$. We denote by $\mathcal{B}(\mathbb{T}^d) = \{A \subset [0, 2\pi)^d : A \in \mathcal{B}(\mathbb{R}^d)\}$ the Borelians in the $\mathbb{T}^d$-torus and $L^2(\mathbb{T}^d)$ the set of square integrable functions in $\mathbb{T}^d$.

For $x \in [0, 2\pi)$, let $\varepsilon_{0,k}(x) = \pi^{-1/2} \sin(kx)$, $\varepsilon_{1,k}(x) = \pi^{-1/2} \cos(kx)$, $k \in \mathbb{N}^*$, and $\varepsilon_{1,0}(x) = (2\pi)^{-1/2}$. The set of functions $\mathbf{B}$ defined on $\mathbb{T}^d$ consisting of

$$\varepsilon_{i,k} := \varepsilon_{i_1,k_1} \otimes \cdots \otimes \varepsilon_{i_d,k_d}, \; i = (i_1, \dots, i_d) \in \mathbb{Z}_2^d,$$

with $k_j \in \mathbb{N}^*$ if $i_j = 0$, and $k_j \in \mathbb{N}$ if $i_j = 1$, is an orthonormal basis for $L^2(\mathbb{T}^d)$ (See [Gra14, Prop. 3.1.15 & Prop. 3.1.16], for example).

Define

$$(\mathbb{Z}_2 \times \mathbb{N})_+^d = \{(i,k) \in (\mathbb{Z}_2 \times \mathbb{N})^d : (i_j, k_j) \neq (0,0), \; \forall j = 1, \dots, d\}.$$

Notice that $\mathbf{B} = \{\varepsilon_{i,k} = \varepsilon_{i_1,k_1} \otimes \cdots \otimes \varepsilon_{i_d,k_d}, \; (i,k) \in (\mathbb{Z}_2 \times \mathbb{N})_+^d\}$.

Assume that $k_j > 0$, then

$$\begin{aligned} \sum_{i \in \mathbb{Z}_2} \varepsilon_{i,k_j}(x_j)\varepsilon_{i,k_j}(y) &= \frac{1}{\pi} (\sin(k_j x_j) \sin(k_j y_j) + \cos(k_j x_2) \sin(k_j y_j))) \\ &= \frac{1}{\pi} \cos(k_j(x_j - y_j)). \end{aligned}$$

The last equality together with the fact that $\varepsilon_{0,1}(x_j) = (2\pi)^{-1/2}$ implies that for any $x, y \in \mathbb{T}^d$,

$$\sum_{i \in \mathbb{Z}_2^d} \varepsilon_{i,k}(x)\varepsilon_{i,k}(y) = \frac{1}{2^{n(k)}\pi^d} \prod_{j=1}^{d} \cos(k_j(x_j - y_j)),\ k \in \mathbb{N}^d \text{ with } (i,k) \in (\mathbb{Z}_2 \times \mathbb{N})_+^d. \tag{5.1}$$

Let $(-\Delta)^2$ be the biharmonic operator (also called the *bilaplacian*) on $L^2(\mathbb{T}^d)$. The basis $\mathbf{B}$ is a set of eigenfunctions of $(-\Delta)^2$ with associated eigenvalues $\lambda_k = \sum_{j=1}^d k_j^4$, $k \in \mathbb{N}^d$. Observe that $d^{-1}|k|^4 \le \lambda_k \le |k|^4$, and $\inf_{k \in \mathbb{N}^{d,*}} \lambda_k = d$.

The semi-group G_t generated by $-\Delta^2$ is denoted by $G_t = e^{-t\Delta^2}$, that is for $f \in L^2(\mathbb{T}^d)$,

$$G_t(f) = \sum_{(i,k) \in (\mathbb{Z}_2 \times \mathbb{N})_+^d} e^{-\lambda_k^2 t} \langle f, \varepsilon_{i,k} \rangle_{L^2(\mathbb{T}^d)} \varepsilon_{i,k}. \tag{5.2}$$

This is a convolution semi-group with the Green function G defined by

$$G(t; x, y) = \sum_{(i,k) \in (\mathbb{Z}_2 \times \mathbb{N})_+^d} e^{-\lambda_k t} \varepsilon_{i,k}(x)\varepsilon_{i,k}(y) = \sum_{k \in \mathbb{N}^d} \frac{e^{-\lambda_k t}}{2^{n(k)}\pi^d} \prod_{j=1}^{d} \cos(k_j(x_j - y_j)), \tag{5.3}$$

the last equality being a consequence of (5.1).

Let $f \in L^1([0,T]) \times L^2(\mathbb{T}^D)$ and $v_0 \in L^2(\mathbb{T}^d)$, according to [DPZ14, Prop. A.6] there exist a unique weak solution(see [DPZ14, Def. A.5] for a definition) to the Cauchy problem

$$\begin{cases} (\frac{\partial}{\partial t} + (-\Delta)^2) v = f, (t,x) \in (0,T] \times \mathbb{T}^d, \\ \qquad v(0,x) = v(x), x \in \mathbb{T}^d, \end{cases}$$

given by

$$\begin{aligned} v(t,x) &= G_t(v_0(x)) + \int_0^t G_{t-s}(f(r,x))dr \\ &= \int_{\mathbb{T}^d} G(t; x, z) v_0(z) dz + \int_0^t dr \int_{\mathbb{T}^d} dz G(t-r; x, z) f(r, x). \end{aligned}$$

Definition 5.1. A centered Gaussian field $\{W(t,A), t \in (0,T], A \in \mathcal{B}(\mathbb{T}^d)\}$ with covariance

$$E(W(t,A)W(s,B)) = (t \wedge s)|A \cap B|$$

is called space-time *white noise*.

The construction of the stochastic integral with respect to white noise in $[0,T] \times \mathbb{T}^d$ is very similar to the case of white noise in $\mathbb{R}^d$ explained in section 2.1 of Chapter 2. As in (2.2)

$$\int_0^T \int_{\mathbb{T}^d} \varphi(r,z) W(dr, dz), \varphi \in L^2([0,T] \times \mathbb{T}^d),$$

turns out to be a centered Gaussian random field with covariance

$$E\left(\int_0^T \int_{\mathbb{T}^d} \varphi(r,z) W(dr,dz) \int_0^T \int_{\mathbb{T}^d} \psi(r,z) W(dr,dz) \right) = \langle \varphi, \psi \rangle_{L^2([0,T] \times \mathbb{T}^d)}. \tag{5.4}$$

Consider the linear stochastic biharmonic heat equation

$$\begin{cases} \left(\frac{\partial}{\partial t} + (-\Delta)^2\right) v(t,x) = \dot{W}(t,x), \ (t,x) \in (0,T] \times \mathbb{T}^d, \\ \qquad\qquad v(0,x) = v(x), \ x \in \mathbb{T}^d, \end{cases} \tag{5.5}$$

where W is a space-time white noise on $[0,T]\times\mathbb{T}^d$ and $v_0 : \mathbb{T}^d \longrightarrow \mathbb{R}$ is a measurable function.

Definition 5.2. Assume that for any $(t,x) \in (0,T]\times\mathbb{T}^d$, $G(t-\cdot;x,*) \in L^2([0,T]\times\mathbb{T}^d)$. The random field solution to (5.3), is the Gaussian stochastic process

$$v(t,x) = I_0(t,x) + u(t,x), (t,x) \in [0,T] \times \mathbb{T}^d,$$

where

$$I_0(t,x) = \int_{\mathbb{T}^d} G(t;x,z)v_0(z)dz, \ u(t,x) = \int_0^t \int_{\mathbb{T}^d} G(t-r;x,z)W(dr,dz), \tag{5.6}$$

where G given in (5.3), and the stochastic integral is a Wiener integral with respect to space-time white noise.

Remark 5.1. (5.6) is a linear version of the solution to the Cahn-Hilliard stochastic equation studied in [DPD96] and [CW01] but in a different spatial domain. Da Prato and Debussche originally considered a solution with boundary conditions in the spatial domain $[0,\pi]^d$.

We first find sufficient conditions on the initial condition v_0 implying the continuity of the map I_0.

Proposition 5.1. *[HCSS22, Prop. 5.1] Let $v_0 \in L^1(\mathbb{T}^d)$, $d = 1,2,3$. Then, the function $(t,x) \mapsto I_0(t,x)$ is jointly Lipschitz continuous.*

Proof. Increments in time. Fix $0 < s \le t \le T$, Using the definition of $G(t;x,z)$ given in (5.3), we see that for any $x \in \mathbb{T}^d$,

$$\begin{aligned}
&|I_0(t,x) - I_0(s,x)| \\
&= \left| \int_{\mathbb{T}^d} dz \ v_0(z) \sum_{k\in\mathbb{N}^{d,*}} \left(e^{-\lambda_k t} - e^{-\lambda_k s}\right) \sum_{\substack{i\in\mathbb{Z}_2^d \\ (i,k)\in(\mathbb{Z}_2\times\mathbb{N})^d_+}} \varepsilon_{i,k}(x)\varepsilon_{i,k}(z) \right| \\
&\le \int_{\mathbb{T}^d} dz|v_0(z)| \sum_{k\in\mathbb{N}^{d,*}} \frac{t-s}{\lambda_k} \frac{1}{2^{n(k)}\pi^d} \left| \prod_{j=1}^d \cos(k_j(x_j - z_j)) \right| \\
&\le C_d(t-s)\|v_0\|_{L^1(\mathbb{T}^d)} \sum_{k\in\mathbb{N}^{d,*}} \frac{1}{\lambda_k} \int_s^t d\rho e^{-\lambda_k \rho} \\
&\le C_d(t-s)\|v_0\|_{L^1(\mathbb{T}^d)} \sum_{k\in\mathbb{N}^{d,*}} \frac{1}{\lambda_k} = \left[\bar{C}_d \|v_0\|_{L^1(\mathbb{T}^d)}\right](t-s),
\end{aligned}$$

where the convergence of the series in the last equality is equivalent to to a harmonic series $\sum_{\substack{k\in\mathbb{N}^d \\ 0\le n(k)\le d-1}} |k|^{-4}$, which converges if and only if $d \le 3$.

Increments in space. Let $x, y \in \mathbb{T}^d$. Then, for any $t \in [0,T]$,

$$
\begin{aligned}
|I_0(t,x) - I_0(t,y)| &= \left| \int_{\mathbb{T}^d} dz\ v_0(z) \sum_{(i,k)\in(\mathbb{Z}_2\times\mathbb{N})^d_+} e^{-\lambda_k t}(\varepsilon_{i,k}(x) - \varepsilon_{i,k}(y))\varepsilon_{i,k}(z) \right| \\
&\le |x-y| \sum_{k\in\mathbb{N}^{d,*}} |k| e^{-\lambda_k t} \int_{\mathbb{T}^d} dz |v_0(z)|
\end{aligned}
$$

Up to a multiplicative constant depending on d, the series in the above expression is bounded by $\int_0^\infty \rho^d e^{-\frac{\rho^4}{2^{d-1}}} = C_d \Gamma_E\left(\frac{d+1}{4}\right)$, where Γ_E denotes the Euler Gamma function.

The proof of the proposition is complete. □

The following theorem gives necessary and sufficient conditions for the stochastic integral in (5.6) to be well defined.

Theorem 5.1. *[HCSS22, Thm. 2.1] The stochastic process* $(u(t,x),\ (t,x) \in [0,T]\times\mathbb{T}^d)$ *given in* (5.6) *is well-defined if and only if* $d = 1, 2, 3$. *In this case,*

$$
\sup_{(t,x)\in[0,T]\times\mathbb{T}^d} E(|u(t,x)|^2) < \infty. \tag{5.7}
$$

Proof. Fix $(t,x) \in (0,T] \times \mathbb{T}^d$. By (5.3) and applying Fubini's theorem, we have

$$
\begin{aligned}
\int_0^t dr \int_{\mathbb{T}^d} dz\ G^2(t-r;x,z) &= \sum_{(i,k)\in(\mathbb{Z}_2\times\mathbb{N})^d_+} \varepsilon^2_{i,k}(x) \left(\int_0^t dr\ e^{-2\lambda_k r} \right) \\
&= \sum_{k\in\mathbb{N}^d} \frac{1}{2^{n(k)}\pi^d} \int_0^t dr\ e^{-2\lambda_k r} \\
&= \frac{t}{(2\pi)^d} + \sum_{\substack{k\in\mathbb{N}^d \\ 0\le n(k)\le d-1}} \frac{1-e^{-2\lambda_k t}}{2^{n(k)+1}\pi^d \lambda_k}.
\end{aligned} \tag{5.8}
$$

Apply the inequalities $\frac{u}{1+u} \le 1 - e^{-u} \le 1$, valid for all $u \ge 0$, to see that the series in (5.8) is equivalent to a harmonic series $\sum_{\substack{k\in\mathbb{N}^d \\ 0\le n(k)\le d-1}} |k|^{-4}$, which converges if and only if $d \le 3$. Equivalently, the Wiener integral defining $u(t,x)$ is well-defined if and only if $d \le 3$. This finishes the proof of the first statement.

By the isometry property of the Wiener integral (5.4), $E((u(t,x))^2)$ is equal to the right-hand side of (5.8). Taking the supremum in (5.8), we have

$$
\begin{aligned}
\sup_{(t,x)\in[0,T]\times\mathbb{T}^d} E((u(t,x))^2) &\le \frac{T}{(2\pi)^d} + \sup_{t\in[0,T]} \sum_{k\in\mathbb{N}^d} \frac{1-e^{-2\lambda_k t}}{2^{n(k)+1}\pi^d\lambda_k} \\
&\le \frac{T}{(2\pi)^d} + \sum_{k\in\mathbb{N}^d} \frac{1}{2^{n(k)+1}\pi^d\lambda_k} \le C(T,d).
\end{aligned} \tag{5.9}
$$

□

In the rest of the chapter we will take $d \in \{1,2,3\}$.

5.2 Equivalence for the canonical metric

For the process u of Theorem 5.1, we define

$$\mathfrak{d}_u((t,x),(s,y)) = \|u(t,x) - u(s,y)\|_{L^2(\Omega)}.$$

This is the canonical metric associated with u. This section is devoted to establish an equivalent pseudo-distance for $\mathfrak{d}_u$.

Throughout the proofs, we will make frequent use of the identity

$$\begin{aligned}\mathfrak{d}_u((t,x),(s,y))^2 &= \frac{1}{2^{n(k)+1}\pi^d}\\ &\times \sum_{k\in\mathbb{N}^{d,*}} \frac{1-e^{-2\lambda_k s}}{\lambda_k}\left(e^{-2\lambda_k(t-s)} + 1 - 2e^{-\lambda_k(t-s)}\prod_{j=1}^{d}\cos(k_j(x_j-y_j))\right)\\ &+ \frac{1}{2^{n(k)+1}\pi^d}\sum_{k\in\mathbb{N}^{d,*}}\frac{1-e^{-2\lambda_k(t-s)}}{\lambda_k} + \frac{t-s}{(2\pi)^d},\end{aligned} \tag{5.10}$$

$0 \le s \le t$. This formula is proved using the Wiener isometry (5.4)

$$\begin{aligned}\mathfrak{d}_u((t,x),(s,y))^2 &= \int_0^t dr \int_{\mathbb{T}^d} dz\ (G(t-r;x,z) - G(s-r;y,z))^2\\ &= \int_0^s dr\int_{\mathbb{T}^d} dz\ (G(t-r;x,z)-G(s-r;y,z))^2 + \int_s^t dr\int_{\mathbb{T}^d} dz\ G^2(t-r;x,z),\end{aligned} \tag{5.11}$$

(the last equality holds because the Green's function $G(r;y,z)$ vanishes if $r<0$) and using the definition (5.3). The first (respectively, second) series term in (5.10) equals the first (respectively second) integral on the rignt-hand side of (5.11).

We start by analyzing the $L^2(\Omega)$-increments in the time variable of the process u.

Proposition 5.2. *[HCSS22, Prop. 3.1] 1. There exist constants $c_{5,1}(d,T)$ and $c_{5,2}(d)$ such that, for all $s,t\in[0,T]$, $x\in\mathbb{T}^d$,*

$$c_{5,1}(d,T)|t-s|^{1-d/4} \le \mathfrak{d}_u((t,x),(s,x))^2 \le c_{5,2}(d)|t-s|^{1-d/4}. \tag{5.12}$$

2. For any $(t,x),(s,y)\in[0,T]\times\mathbb{T}^d$,

$$\mathfrak{d}_u((t,x),(s,y))^2 \ge c_{5,1}(d,T)|t-s|^{1-d/4}, \tag{5.13}$$

where $c_{5,1}(d,T)$ is the same constant as in (5.12).

Proof. Without loss of generality, we suppose $0\le s<t\le T$. Use the first equality in (5.11) and then apply Lemma 5.1 bellow with $h := t-s$. This yields the second inequality in (5.12).

From (5.10), we have

$$\mathfrak{d}_u((t,x),(s,x))^2 \ge \frac{1}{2^{n(k)+1}\pi^d}\sum_{k\in\mathbb{N}^{d,*}}\frac{1-e^{-2\lambda_k(t-s)}}{\lambda_k}. \tag{5.14}$$

Let $r \geq d$. Applying the inequality $1 - e^{-u} \geq \frac{u}{1+u}$, $u \geq 0$, we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{N}^{d,*}} \frac{1-e^{-2\lambda_k(t-s)}}{\lambda_k} &\geq 2(t-s) \sum_{\substack{k\in\mathbb{N}^{d,*}\\ |k|>r}} \frac{1}{1+2\lambda_k(t-s)}\\
&\geq \frac{2(t-s)}{r^{-4}+2(t-s)} \sum_{\substack{k\in\mathbb{N}^{d,*}\\ |k|>r}} \frac{1}{|k|^4}\\
&\geq \frac{2(t-s)}{r^{-4}+2(t-s)} \int_{|z|>r}^{d,*} dz|z|^{-4}\\
&= c_d \frac{2(t-s)}{r^{-4}+2(t-s)} r^{d-4},
\end{aligned}$$

since $\lambda_k \leq |k|^4$. Choosing $r = \left(\frac{d^4 T}{t-s}\right)^{1/4}$, the inequality above yields

$$\mathfrak{d}_u((t,x),(s,x))^2 \geq c_{5,1}(d,T)(t-s)^{1-d/4},$$

with $c_{5,1}(d,T) = c_d \frac{2d^d T^{d/4}}{1+2d^4 T}$. This is the lower bound in (5.12).

Notice that from (5.10) we deduce

$$\mathfrak{d}_u((t,x),(s,y))^2 \geq \frac{1}{2^{n(k)+1}\pi^d} \sum_{k\in\mathbb{N}^{d,*}} \frac{1-e^{-2\lambda_k(t-s)}}{\lambda_k}.$$

Hence the proof above yields (5.13). □

For any $j = 1, \ldots, d$, fix real numbers $0 < c_{0,j} < 2\pi$ and define $J_j = [c_{0,j}, 2\pi - c_{0,j}]$ and $J = J_1 \times \ldots \times J_d \subsetneq \mathbb{T}^d$. The next statement deals with increments in space.

Proposition 5.3. *[HCSS22, Prop. 3.2] Let $(u(t,x),\ (t,x) \in [0,T] \times \mathbb{T}^d)$ be the stochastic process defined in Theorem 5.1 and let J be a compact set as described before. There exist positive constants $c(d)$, $C(d)$, $c_{5,3}(d)$ and $c_{5,4}(d)$ such that, for any $t > 0$, $x, y \in J$,*

$$\begin{aligned}
&c_{5,3}(d) C_t \left(\log \frac{c(d)}{|x-y|}\right)^{\beta} |x-y|^{2\wedge(4-d)}\\
&\qquad \leq \mathfrak{d}_u((t,x),(t,y))^2 \leq c_{5,4}(d) \left(\log \frac{C(d)}{|x-y|}\right)^{\beta} |x-y|^{2\wedge(4-d)}, \qquad (5.15)
\end{aligned}$$

where $C_t = (1 - e^{-2dt})$, and $\beta = 1_{\{d=2\}}$.

The upper bound holds for any $(t,x) \in [0,T] \times \mathbb{T}^d$. The lower bound holds for any $x, y \in \mathbb{T}^d$ if $|x-y|$ is small enough. For $t = 0$, the lower bound is non informative.

Proof. From (5.10) we deduce

$$\mathfrak{d}_u((t,x),(t,y))^2 = \frac{1}{2^{n(k)}\pi^d} \sum_{k\in\mathbb{N}^{d,*}} \frac{1-e^{-2\lambda_k t}}{\lambda_k} \left(1 - \prod_{j=1}^{d} \cos(k_j(x_j - y_j))\right). \qquad (5.16)$$

Upper bound. Because of (5.1) and the mean value theorem,

$$1 - \prod_{j=1}^{d} \cos(k_j(x_j - y_j)) = 2^{n(k)-1}\pi^d \sum_{i\in\mathbb{Z}_2^d} (\varepsilon_{i,k}(x) - \varepsilon_{i,k}(y))^2 \le \bar{C}(d)(1 \wedge (|k|\ |x-y|)^2). \tag{5.17}$$

for any $(i,k) \in (\mathbb{Z}_2 \times \mathbb{N})^d$.
Case $d = 1$. Since $\sum_{k\ge1} \frac{1}{k^2} < \infty$, from (5.16) and (5.17) we have

$$\mathfrak{d}_u((t,x),(t,y))^2 = \frac{1}{\pi} \sum_{k\ge1} \frac{1 - e^{-2\lambda_k t}}{\lambda_k}(1 - \cos(k(x-y))) \le C_d|x-y|^2. \tag{5.18}$$

Case $d = 2, 3$. For any $k \in \mathbb{N}^d$, let $I_k = [k_1, k_1+1) \times \cdots \times [k_d, k_d+1)$. Observe that for any d-dimensional vector $z \in I_k$, we have $|z| \le |k| + \sqrt{d}$. Fix $\rho_0 \ge \lfloor 3\sqrt{d} \rfloor + 1$ and let $\alpha > 0$. Then,

$$\begin{aligned} T_1(\alpha,\rho_0) := \sum_{\substack{k\in\mathbb{N}^d \\ |k|\ge\rho_0}} \frac{1}{|k|^\alpha} &\le \sum_{\substack{k\in\mathbb{N}^d \\ |k|\ge\rho_0}} \int_{I_k} \frac{dz}{(|z| - \sqrt{d})^\alpha} \\ &\le C_d \int_{\rho_0}^{\infty} \frac{dz}{(|z|-\sqrt{d})^\alpha} = C_d \int_{\rho_0}^{\infty} \frac{\rho^{d-1}\, d\rho}{(\rho - \sqrt{d})^\alpha} \\ &\le C_{d,\alpha} \int_{\rho_0}^{\infty} \rho^{d-1-\alpha}\, d\rho, \end{aligned} \tag{5.19}$$

where the last inequality holds because on $[\rho_0, \infty)$, $\rho - \sqrt{d} \ge 1/2\rho$.

Let ρ_0 be as above, $\rho_1 = \left\lfloor (3/2)\sqrt{d} \right\rfloor + 1$, and $\beta > 0$. By arguments similar to those used to obtain (5.19), we deduce

$$\begin{aligned} T_2(\beta,\rho_0) = \sum_{\substack{k\in\mathbb{N}^d \\ \rho_1\le|k|<\rho_0}} \frac{1}{|k|^\beta} \le \sum_{\substack{k\in\mathbb{N}^d \\ \rho_1\le|k|<\rho_0}} \int_{I_k} \frac{dz}{(|z|-\sqrt{d})^\beta} \le C_d \int_{\rho_1}^{\rho_0} \frac{\rho^{d-1}\, d\rho}{(\rho-\sqrt{d})^\beta} \\ \le C_{d,\beta} \int_{\rho_1}^{\rho_0} \rho^{d-1-\beta}\, d\rho, \end{aligned} \tag{5.20}$$

where in the last inequality, we have used that on $[\rho_1, \rho_0]$, $\rho - \sqrt{d} \ge (1/5)\rho$.

Set $h = |x-y|$ and $\rho_0 = \left\lfloor c_d h^{-\frac{2\wedge(4-d)}{4-d}} \right\rfloor + 1$, where $c_d = 3\sqrt{d}(2\pi\sqrt{d})^{\frac{2\wedge(4-d)}{4-d}}$. Notice that $\rho_0 \ge \lfloor 3\sqrt{d} \rfloor + 1$. Then, from (5.16), (5.17) and since x, y in $\mathbb{T}^d$,

$$\mathfrak{d}_u((t,x),(t,y))^2 \le C(d) \left[T_1(4,\rho_0) + h^2 \left(T_2(2,\rho_0) + \sum_{\substack{k\in\mathbb{N}^d \\ 1\le|k|<\rho_1}} \frac{1}{|k|^2} \right) \right]. \tag{5.21}$$

Using (5.19), with the choice of ρ_0 specified above, we see that $T_1(4,\rho_0) \le C_d h^{2\wedge(4-d)}$ and

$$T_2(2,\rho_0) \le C_d \times \begin{cases} \log\left(\frac{C}{h}\right), & d = 2, \\ h^{-1}, & d = 3. \end{cases}$$

Since $\sum_{k\in\mathbb{N}^d,\ 1\le|k|<\rho_1}\frac{1}{|k|^2}=\tilde{c}_d<\infty$, substituting the above estimates in the right-hand side of (5.21) we obtain the upper bound in (5.15).

Lower bound. Case $|x-y|$ small. We start from (5.16) to obtain

$$\mathfrak{d}_u((t,x),(t,y))^2\ge\frac{1-e^{-2t}}{2^{n(k)}\pi^d}\sum_{k\in\mathbb{N}^{d,*}}\frac{1-\prod_{j=1}^d\cos(k_j(x_j-y_j))}{|k|^4}. \tag{5.22}$$

Let $T(x,y)$ denote the series on the right-hand side of (5.22). Because for any $z\in[-\pi/2,\pi/2]$, we have $\cos z\le 1-(\frac{2}{\pi}z)^2$, we deduce

$$T(x,y)\ge\sum_{\substack{k\in\mathbb{N}^{d,*}\\ k_j|x_j-y_j|\le\pi/2}}\frac{1-\prod_{j=1}^d(1-[(2/\pi)k_j|x_j-y_j|]^2)}{|k|^4}. \tag{5.23}$$

Case $d=1$. Using (5.23), we obtain

$$\begin{aligned}T(x,y)&\ge(\tfrac{2}{\pi})^2|x-y|^2\sum_{\substack{k\in\mathbb{N}\setminus\{0\}\\ k|x-y|\le\pi/2}}\frac{1}{k^2}\ge(\tfrac{2}{\pi})^2|x-y|^2\int_1^{\frac{\pi}{2}|x-y|^{-1}}\rho^{-2}\,d\rho\\&=(\tfrac{2}{\pi})^2|x-y|^2\left(1-\frac{2}{\pi}|x-y|\right).\end{aligned}$$

Assume $|x-y|\le\frac{c_0\pi}{2}$, with $0<c_0<1$ arbitrarily close to 1. Then $1-\frac{2}{\pi}|x-y|\ge 1-c_0$ and, in this case,

$$\mathfrak{d}_u((t,x),(t,y))^2\ge 4(1-c_0)\frac{1-e^{-2t}}{\pi^3}|x-y|^2. \tag{5.24}$$

Case $d=2,3$. Consider the series on the right-hand side of (5.23) and apply the formula (5.31) of Lemma 5.2 below with $m:=d$ and $p_j=[(2/\pi)k_j|x_j-y_j|]^2$, to see that

$$T(x,y)\ge(2/\pi)^2\left[S_1(x,y)-(2/\pi)^2S_2(x,y)\right], \tag{5.25}$$

where

$$S_1(x,y)=\sum_{\substack{k\in\mathbb{N}^{d,*}\\ k_j|x_j-y_j|\le\pi/4}}\sum_{j=1}^d\frac{(k_j|x_j-y_j|)^2}{|k|^4},$$

$$S_2(x,y)=\sum_{\substack{k\in\mathbb{N}^{d,*}\\ k_j|x_j-y_j|\le\pi/4}}\sum_{\substack{j_1,j_2\in\{1,\dots,d\},\\ j_1<j_2}}\frac{(k_{j_1}|x_{j_1}-y_{j_1}|k_{j_2}|x_{j_2}-y_{j_2}|)^2}{|k|^4}.$$

Note that the condition $k_j|x_j-y_j|\le\pi/4$ implies $1-(2/\pi)^2(k_j|x_j-y_j|)^2\ge 3/4$. Hence, for $d=2$ we see that

$$\begin{aligned}&\sum_{j=1}^2(k_j|x_j-y_j|)^2-(2/\pi)^2(k_1|x_1-y_1|)^2(k_2|x_2-y_2|)^2\\&\quad=(k_1|x_1-y_1|)^2\left(1-(2/\pi)^2(k_2|x_2-y_2|)^2\right)+(k_2|x_2-y_2|)^2\end{aligned}$$

$$\geq \frac{3}{4}\sum_{j=1}^{2}(k_j|x_j - y_j|)^2.$$

Similarly, for $d = 3$ we have

$$\sum_{j=1}^{3}(k_j|x_j - y_j|)^2 \left(1 - (2/\pi)^2(k_{j+1}|x_{j+1} - y_{j+1}|)^2\right) \geq \frac{3}{4}\sum_{j=1}^{3}(k_j|x_j - y_j|)^2,$$

where in the sum above, we set $j + 1 = 1$ if $j = 3$.

Thus, in both dimensions $d = 2, 3$,

$$S_1(x, y) - (2/\pi)^2 S_2(x, y) \geq (3/4)S_1(x, y).$$

The next goal is to find a lower bound for $S_1(x, y)$. Without loss of generality we may and will assume $|x_1 - y_1| \leq |x_2 - x_2| \leq ... \leq |x_d - y_d|$. Set

$$\mathbb{N}^{d,*}_{\leq} := \{k \in \mathbb{N}^{d,*} : k_1 \leq k_2 \leq ... \leq k_d\}.$$

We claim that,

$$S_1(x, y) \geq \sum_{\substack{k\in\mathbb{N}^{d,*}_{\leq} \\ k_j|x_j - y_j|\leq\pi/4}} \sum_{j=1}^{d} \frac{(k_j|x_j - y_j|)^2}{|k|^4} \geq \frac{1}{\sqrt{2}d}\ |x - y|^2 \sum_{\substack{k\in\mathbb{N}^{d,*}_{\leq} \\ k_j|x_j - y_j|\leq\pi/4}} \frac{1}{|k|^2}. \quad (5.26)$$

Indeed, set $K = (k_j^2)_j$, $Z = (|x_j - y_j|^2)_j$ and let θ be the angle between the vectors K and Z. Because $\sum_{j=1}^{d}(k_j|x_j - y_j|)^2$ is the Euclidean scalar product between K and Z and $\theta \in [0, \pi/4]$,

$$\sum_{j=1}^{d}(k_j|x_j - y_j|)^2 \geq \cos(\pi/4)\left(\sum_{j=1}^{d} k_j^4\right)^{1/2}\left(\sum_{j=1}^{d}|x_j - y_j|^4\right)^{1/2} \geq \frac{1}{\sqrt{2}}\frac{|k|^2|x - y|^2}{d}.$$

Assume that $|x - y| \leq \frac{\pi}{5\sqrt{d}}$. The set $\{k \in \mathbb{N}^{d,*} : |k| \leq \frac{\pi}{4}|x - y|^{-1}\}$ is non empty and is included in $\{k \in \mathbb{N}^{d,*} : k_j \leq \frac{\pi}{4}|x_j - y_j|^{-1}, j = 1, \ldots, d\}$. Hence,

$$\sum_{\substack{k\in\mathbb{N}^{d,*}_{\leq} \\ k_j|x_j - y_j|\leq\pi/4}} \frac{1}{|k|^2} \geq \frac{1}{d!} \sum_{\substack{k\in\mathbb{N}^{d,*} \\ |k|\leq\frac{\pi}{4}|x-y|^{-1}}} \frac{1}{|k|^2} \geq C_d \int_{\sqrt{d}}^{\frac{\pi}{4}|x-y|^{-1}} \rho^{d-3}\ d\rho.$$

For $d = 2$, the last integral equals $\log\left(\frac{\pi}{4\sqrt{d}|x-y|}\right)$, while for $d = 3$, it is equal to $(\pi/4)|x - y|^{-1} - \sqrt{d}$. Observe that if $|x - y| \leq \frac{\pi}{5\sqrt{d}}$ this expression is bounded below by $(\pi/20)|x - y|^{-1}$.

Summarizing, from (5.26) and assuming $|x - y| \leq \frac{\pi}{5\sqrt{d}}$, the discussion above proves

$$S_1(x, y) \geq C_d \times \begin{cases} \log\left(\frac{\pi}{4\sqrt{d}|x-y|}\right) |x - y|^2, & d = 2, \\ |x - y|, & d = 3. \end{cases} \quad (5.27)$$

Therefore, for any $x, y \in \mathbb{T}^d$ such that $0 \le |x-y| \le \frac{\pi}{5\sqrt{d}}$, we have proved that the lower bound of (5.15) holds with the constant $c_3(d)$ depending only on d and $C_t = 1 - e^{-2t}$.

Lower bound. Case $|x-y|$ large. We recall a standard "continuity-compactness" argument that we will use to extend the validity of the lower bound established in the previous step, to every $x, y \in J$ satisfying $\frac{\pi}{5\sqrt{d}} < |x-y| < 2\pi$.

Consider the function

$$J^2 \ni (x,y) \mapsto \varphi_t(x,y) = \mathfrak{d}_u((t,x),(t,y))^2,$$

where $t > 0$ is fixed. Because of the upper bound in (5.15), this is a continuous function. Furthermore, from (5.16), we see that it is strictly positive. Thus, for any $c_0 > 0$, the minimun value m of φ_t over the compact set

$$\{\varphi_t(x,y); (x,y) \in J^2 : |x-y| \ge c_0\}$$

is achieved, and $m > 0$. Referring to the left hand-side of (5.15), let M be the maximum of the function

$$J^2 \ni (x,y) \mapsto \left(\log \frac{c(d)}{|x-y|}\right)^\beta |x-y|^{2\wedge(4-d)}, \quad \beta = 1_{\{d=2\}}.$$

Taking $c_0 = \frac{\pi}{5\sqrt{d}}$, we deduce,

$$\mathfrak{d}_u((t,x),(t,y))^2 \ge \frac{m}{M}\left(\log \frac{c(d)}{|x-y|}\right)^\beta |x-y|^{2\wedge(4-d)}, \quad \beta = 1_{\{d=2\}},$$

for any $x, y \in J$ such that $\frac{\pi}{5\sqrt{d}} < |x-y| < 2\pi$.

This ends the proof of the lower bound and of the Proposition. □

With Propositions 5.2 and 5.3 we obtain an equivalent expression of the canonical pseudo-distance (2.16), as stated in the next theorem.

Theorem 5.2. *[HCSS22, Thm. 3.1] Let $(u(t,x),\ (t,x) \in [0,T] \times \mathbb{T}^d)$ be the stochastic process defined in Theorem 5.1.*
1. There exist constants $c_{5,4}(d)$, $C(d)$ such that for any $(t,x),(s,y) \in [0,T] \times \mathbb{T}^d$,

$$\mathfrak{d}_u((t,x),(s,y))^2 \le c_{5,4}(d)\left(|t-s|^{1-d/4} + \left(\log \frac{C(d)}{|x-y|}\right)^\beta |x-y|^{2\wedge(4-d)}\right), \quad (5.28)$$

with $\beta = 1_{\{d=2\}}$.
2. Fix $t_0 \in (0,T]$ and let J be a compact subset of $\mathbb{T}^d$ as in Proposition 5.3. There exist constants $c_{5,6}(d,t_0,T)$ and $c(d)$ such that, for any $(t,x),(s,y) \in [t_0,T] \times J$,

$$\mathfrak{d}_u((t,x),(s,y))^{5,2} \ge c_{5,6}(d,t_0,T) \times \left(|t-s|^{1-d/4} + \left(\log \frac{c(d)}{|x-y|}\right)^\beta |x-y|^{2\wedge(4-d)}\right), \quad (5.29)$$

with $\beta = 1_{\{d=2\}}$.

Proof. The estimate from above follows by applying the triangle inequality and the upper bounds in (5.12) and (5.15), which hold for any $(t,x),(s,y)\in[0,T]\times\mathbb{T}^d$. The value of the multiplicative constant in the upper bound is $c_{5,5}(d)=2[c_{5,2}(d)+c_{5,4}(d)]$, where $c_{5,2}(d)$, $c_{5,4}(d)$ are given in (5.12), (5.15), respectively.

To prove the lower bound, we consider two cases (see Propositions 5.2 and 5.3 for the notations of the constants).

Case 1: $c_{5,2}(d)|t-s|^{1-d/4}\le\frac{c_{5,3}(d)C_{t_0}}{4}\left(\log\frac{c(d)}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4-d)}$, where $C_{t_0}=1-e^{-2t_0}$.

Applying the triangle inequality and then, using the lower bound in (5.15) and the upper bound in (5.12) we obtain,

$$\begin{aligned}\mathfrak{d}_u((t,x),(s,y))^2&\ge\frac{1}{2}\mathfrak{d}_u((t,x),(t,y))^2-\mathfrak{d}_u((t,y),(s,y))^2\\&\ge\frac{c_{5,3}(d)C_{t_0}}{2}\left(\log\frac{c(d)}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4-d)}-c_{5,2}(d)|t-s|^{1-d/4}\\&\ge\frac{c_{5,3}(d)C_{t_0}}{8}\left(\log\frac{c(d)}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4-d)}+\frac{c_{5,2}(d)}{2}|t-s|^{1-\frac{d}{4}}.\end{aligned}$$

Case 2: $c_{5,2}(d)|t-s|^{1-d/4}>\frac{c_{5,3}(d)C_{t_0}}{4}\left(\log\frac{c(d)}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4-d)}$.

By (5.13), we have

$$\begin{aligned}\mathfrak{d}_u((t,x),(s,y))^2\ge c_{5,1}(d,T)|t-s|^{1-d/4}&=\frac{c_1(d,T)}{c_{5,2}(d)}\left[c_{5,2}(d)|t-s|^{1-d/4}\right]\\\ge\frac{c_{5,1}(d,T)}{c_{5,2}(d)}&\left(\frac{c_{5,2}(d)}{2}|t-s|^{1-d/4}+\frac{c_3(d)C_{t_0}}{8}\left(\log\frac{c(d)}{|x-y|}\right)^{\beta}|x-y|^{2\wedge(4-d)}\right).\end{aligned}$$

The proof of the theorem is complete. □

We finish this section by proving the auxiliary lemmas used in the proofs of Propositions 5.2 and 5.3.

Lemma 5.1. *[HCSS22, Lem. 7.1] Let $d\in\{1,2,3\}$. There exists a constant C_d such that for any $h\ge0$ and $x\in\mathbb{T}^d$,*

$$\int_0^\infty dr\int_{\mathbb{T}^d}dz\ \left(G(r+h;x,z)-G(r;x,z)\right)^2\le C_dh^{1-d/4}.\qquad(5.30)$$

Proof. Using the expression (5.3),and the inequality $1-e^{-\tau}\le1\wedge\tau$, $\tau\ge0$, we see that

$$\begin{aligned}&\int_0^\infty dr\int_{\mathbb{T}^d}dz\ \left(G(r+h;x,z)-G(r;x,z)\right)^2\\&=\sum_{k\in\mathbb{N}^d}\frac{1}{2^{n(k)}\pi^d}\int_0^\infty dr\left(e^{-\lambda_k(r+h)}-e^{-\lambda_kr}\right)^2\\&=\sum_{\substack{k\in\mathbb{N}^d\\0\le n(k)\le d-1}}\frac{1}{2^{n(k)+1}\pi^d}\frac{\left(1-e^{-\lambda_kh}\right)^2}{\lambda_k}\le C_d\sum_{\substack{k\in\mathbb{N}^d\\0\le n(k)\le d-1}}\frac{\min(1,|k|^8h^2)}{|k|^4}\\&=C_d\sum_{\substack{k\in\mathbb{N}^d\\0\le n(k)\le d-1}}\min\left(|k|^{-4},|k|^4h^2\right):=C_d\,T(h).\end{aligned}$$

where we make use of the convergence of the series $\sum_{k\in\mathbb{N}^{d,*}} |k|^{-4}$.

Case 1. $h \geq 1$. We have that $\min(|k|^{-4}, |k|^4 h^2) = |k|^{-4}$. Thus, $T(h) = C < \infty$, which implies $T(h) \leq Ch$.

Case 2. $0 < h < 1$. Let $T(h) \leq T_1(h) + T_2(h)$, where

$$T_1(h) = \sum_{\substack{k\in\mathbb{N}^d,\ 0\leq n(k)\leq d-1 \\ |k|\leq \lfloor h^{-1/4}\rfloor}} \min\left(|k|^{-4}, |k|^4 h^2\right),$$

$$T_2(h) = \sum_{\substack{k\in\mathbb{N}^d\ 0\leq n(k)\leq d-1 \\ |k|> \lfloor h^{-1/4}\rfloor}} \min\left(|k|^{-4}, |k|^4 h^2\right).$$

For the first term, we have

$$T_1(h) \leq \sum_{\substack{k\in\mathbb{N}^d\ 0\leq n(k)\leq d-1 \\ |k|\leq \lfloor h^{-1/4}\rfloor}} |k|^4 h^2 \leq h \sum_{\substack{k\in\mathbb{N}^d\ 0\leq n(k)\leq d-1 \\ |k|\leq \lfloor h^{-1/4}\rfloor}} 1 \leq C_d\ h^{1-d/4}.$$

For the second term, we have

$$T_2(h) \leq \sum_{\substack{k\in\mathbb{N}^d\ 0\leq n(k)\leq d-1 \\ |k|> \lfloor h^{-1/4}\rfloor}} |k|^{-4} \leq C_d\ h^{1-d/4}.$$

Since $1-d/4 < 1$, the estimates obtained in the two instances of h imply (5.30). □

Lemma 5.2. *[HCSS22, Lem. 7.2] For* $p_j \in [0,1]$, $j = 1,\ldots,m$, *the following formula holds:*

$$1 - \prod_{j=1}^m (1-p_j) = \sum_{j=1}^m p_j - \sum_{\substack{i<j \\ 1\leq i,j\leq m}} p_i p_j + \sum_{\substack{i<j<k \\ 1\leq i,j,k\leq m}} p_i p_j p_k - \cdots + (-1)^{m-1} p_1 p_2 \cdots p_m. \tag{5.31}$$

Proof. On a probability space, consider independent events $(A_j)_{1\leq j\leq m}$ such that $p_j = P(A_j)$. Then,

$$1 - \prod_{j=1}^m (1-p_j) = 1 - P(A_1^c \cap \ldots \cap A_m^c)) = 1 - P(\cup_{j=1}^m A_j)^c = P(\cup_{j=1}^m A_j),$$

and (5.31) follows from the well-known inclusion-exclusion formula in probability theory. □

5.3 Further second order properties

We prove Proposition 5.4 which together with Theorem 5.2, will be used in Section 4.4 to find hitting probabilities estimates to the solution of the biharmonic heat equation.

Proposition 5.4. *[HCSS22, Lem. 4.1] Fix* $t_0 \in (0,T]$ *and let* J *be a compact subset of* $\mathbb{T}^d$ *as in Proposition 5.3.*

1. $\sigma^2_{t,x} \asymp 1$ *for* $(t,x) \in [t_0, T] \times \mathbb{T}^d$.

2. *There exists a constant* $C_{d,T}$ *such that for all* $s,t \in (0,T]$ *and* $x,y \in \mathbb{T}^d$,

$$|\sigma^2_{t,x} - \sigma^2_{s,y}| \le C_{d,T}\mathfrak{d}_u((t,x),(s,y))^2. \tag{5.32}$$

3. *For any* $(t,x),(s,y) \in [t_0,T] \times \mathbb{T}^d$ *such that* $(t,x) \neq (s,y)$, $\rho_{(t,x),(s,y)} < 1$.

4. *For any* $(t,x),(s,y) \in [t_0,T] \times J$,

$$E(Var(u(t,x) \mid u(s,y))) \asymp |t-s|^{1-\frac{d}{4}} + \left(\log \frac{C(d)}{|x-y|}\right)^{\beta} |x-y|^{2\wedge(4-d)}, \tag{5.33}$$

with $\beta = 1_{d=2}$.

Proof. 1. The upper bound is proved in (5.9). (5.8) implies that for any $t,x \in [t_0,T] \times \mathbb{T}^d$

$$\sigma^2_{t,x} \ge \frac{t}{(2\pi)^d} \ge \frac{t_0}{(2\pi)^d}.$$

2. Without loss of generality we may assume that $0 < s \le t$. Applying (5.8) yields

$$|\sigma^2_{t,x} - \sigma^2_{s,y}| = \frac{t-s}{(2\pi)^d} + \frac{1}{2^{n(k)+1}\pi^d} \sum_{\substack{k\in\mathbb{N}^d \\ 0\le n(k)\le d-1}} \frac{e^{-2\lambda_k s}\left(1-e^{-2\lambda_k(t-s)}\right)}{\lambda_k}.$$

Use the inequality (5.13) to get $\frac{t-s}{(2\pi)^d} \le \bar{c}_{d,T}\mathfrak{d}_u((t,x),(s,y))^2$. Since $e^{-2\lambda_k s} \le 1$ and because of (5.10), we see that the second term on the right-hand side of this equality is bounded above by $\mathfrak{d}_u((t,x),(s,y))^2$. This ends the proof of (5.32).

3. Assume that $\rho_{(t,x),(s,y)} = 1$. Then, there exist $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\|u(t,x) - \lambda u(s,y)\|_{L^2(\Omega)} = 0$. This leads to a contradiction. Indeed, consider first the case $0 < s < t$. By the isometry property of the Wiener integral,

$$\begin{aligned}\|u(t,x) - \lambda u(s,y)\|^2_{L^2(\Omega)} &= \int_0^s dr \int_{\mathbb{T}^d} dz (G(t-r;x,z) - \lambda G(s-r;y,z))^2 \\ &\quad + \int_s^t dr \int_{\mathbb{T}^d} dz\; G^2(t-r;x,z) \\ &\ge \int_0^{t-s} dr \int_{\mathbb{T}^d} dz\; G^2(r;x,z) > 0,\end{aligned} \tag{5.34}$$

by the properties of G.

Next, we assume $t = s$ and $x \neq y$. In this case by the first line in (5.34)

$$\int_0^t dr \int_{\mathbb{T}^d} dz (G(t-r;x,z) - \lambda G(t-r;y,z))^2 = 0.$$

This implies that for a.e. $z \in \mathbb{T}^d$, $G(t-r;x,z) = \lambda G(t-r;y,z)$, and that $\lambda > 0$. Since the expression of an element of $L^2(\mathbb{T}^d)$, in terms of the basis $\mathbf{B}$ is unique, by replacing the terms in the last equality with (5.3) we deduce that $\varepsilon_{i,k}(x) = \lambda\varepsilon_{i,k}(z)$, for every

$(i,k) \in (\mathbb{Z}_2 \times \mathbb{N})^d_+$. Thus in particular $\cos(x) = \lambda \cos(y)$ and $\sin(x) = \lambda \sin(y)$. Hence $\lambda = 1$ and as a consequence

$$\|u(t,x) - \lambda u(t,y)\|_{L^2(\Omega)} = \|u(t,x) - u(t,y)\|_{L^2(\Omega)} = 0$$

which is in contradiction with the lower bound in (5.15).

4. This fact follows by Proposition 3.1, which hypothesis are valid due to Theorem 5.2 together with the previous statements of this proposition. We skip the proof since it is analogous to 5. in Proposition 4.3. □

5.4 Hitting probabilities

Consider the Gaussian random field

$$\boldsymbol{v} = \left(\boldsymbol{v}(t,x) = \big(\boldsymbol{v}_1(t,x), \ldots, \boldsymbol{v}_D(t,x)\big),\ (t,x) \in [0,T] \times \mathbb{T}^d\right),$$

where $(\boldsymbol{v}_j(t,x)),\ j = 1, \ldots, D$, are independent copies of the process $(v(t,x))$ defined in (5.9).

For $\tau \in \mathbb{R}_+$, let

$$g_1(\tau) = \tau^{(4-d)/8}, \quad g_2(\tau) = \left(\log \frac{C(d)}{\tau}\right)^{\frac{\beta}{2}} \tau^{1\wedge((4-d)/2)}, \quad \beta = 1_{\{d=2\}},$$
$$q_{\boldsymbol{g}}(\tau) = \tau^D \left(g_1^{-1}(\tau)\right)^{-1} \left(g_2^{-1}(\tau)\right)^{-d}.$$

Let $D_0 = [(4-d)/8]^{-1} + d[1 \wedge ((4-d)/2)]^{-1}$. According to Example 3.2 the functions $q_{\boldsymbol{g}}$ satisfy the conditions required by the definitions of the $q_{\boldsymbol{g}}$-Hausdorff measure if and only if $D > D_0$.

In the next theorem, $I = [t_0, T]$ and $J = [0, M]^d$, where $0 < t_0 \leq T$ and $M \in (0, 2\pi)$.

Theorem 5.3. *[HCSS22, Thm. 6.1] Suppose that the function $I \times J \ni (t,x) \mapsto I_0(t,x)$ satisfies the condition (CY2). The hitting probabilities relative to the D-dimensional random field $\boldsymbol{v}$ satisfy the following bounds.*

1. *Let $D > D_0$.*
   (a) *There exists a constant $C := C(I,J,D,d)$ such that for any Borel set $A \in \mathcal{B}(\mathbb{R}^D)$,*
   $$P(\boldsymbol{v}(I \times J) \cap A \neq \emptyset)) \leq C\mathcal{H}_{q_{\boldsymbol{g}}}(A).$$
   (b) *Let $N > 0$ and $A \in \mathcal{B}(\mathbb{R}^D)$ be such that $A \subset B_N(0)$. There exists a constant $c := c(I,J,N,D,d)$ such that*
   $$P(\boldsymbol{v}(I \times J) \cap A \neq \emptyset)) \geq c\mathrm{Cap}_{(\bar{g}_q)^{-1}}(A).$$
2. *Assume that $D < D_0$ or $D = D_0$. Fix $N > 0$ and let $A \subset B_N \subset \mathbb{R}^D$ be a Borel set. There exists a constant $c := c(I,J,N,D,d) > 0$ such that*
   $$P(\boldsymbol{v}(I \times J) \cap A \neq \emptyset)) \geq c[v_{\boldsymbol{g}}(0)]^{-1} > 0.$$

Proof. As in Theorem 4.3, the proof of this Theorem is an application of Corollary 3.1 and Theorem 3.2, but now considering Theorem 5.2 and Proposition 5.4. We skip the details for avoiding repetitions. □

Theorem 5.3 implies the following Corollary which identifies $D = D_0$ with the critical dimension for polarity of points of $\boldsymbol{v}$.

Corollary 5.1. *If $D > D_0$, points are polar for $\boldsymbol{v}$. If $D < D_0$ points are non polar.*

Chapter 6

Future research

By a way of conclusion, we propose some open problems that arose during the development of this work and could be subject of future research.

Modulus of continuity

It follows from (1.3) in Chapter 1 that for any $\hat{g}$-Gaussian process on a compact subset K of $\mathbb{R}^d$, there exists a positive constant C depending on K that

$$\lim_{\delta \downarrow 0} \sup_{\substack{x,\bar{x}\in K,\\ \mathfrak{d}(x,\bar{x})\leq\delta}} |X(x) - X(\bar{x})| \left(\int_0^\delta d\varepsilon \sqrt{d \log\left(\frac{\varnothing_K}{g^{-1}(\varepsilon)} \right)} \right)^{-1} \leq C.$$

If the limit in the last line equals to a positive constant thus we would have computed the exact global modulus of continuity of X. General criteria for computing the exact global exact modulus of continuity of anisotropic Gaussian random fields can be found in [MWX13, Thm.4.1] and [LX21, Thm. 6.1].

An open problem is to find criteria for computing the exact global modulus of continuity of a g-Gaussian processes. Due to the simplicity of its covariance function a first step would be to compute it in the case of the family of γ-Gaussian processes, introduced in [MV05], defined by the Volterra representation

$$B^\gamma(t) := \int_0^t \sqrt{\left(\frac{d\gamma^2}{dt} \right)(t-s)} dW(s),$$

where W is a Brownian motion.

Polarity of points at the critical dimension

As we mentioned in Remark 1.6, it is an open problem to determine whether a g-Gaussian process hits or not points in the critical dimension.

Hausdorff dimension of level sets

Let K be a compact subset of $\mathbb{R}^d$, and

$$\boldsymbol{X} = \{\boldsymbol{X}(x) = (X_1(x), ..., X_D(x))), x \in K\}$$

a (d, D)-dimensional real valued stochastic process. For $z \in \mathbb{R}^d$ we denote the level set $\{x \in K : X(x) = x\}$ by L_z. In [Xia09, Thm. 7.1] the Hausdorff dimension of a level set is computed when X is an anisotropic Gaussian process.

We believe that the Hausdorff dimension of the level sets for g-Gaussian process is determined by the upper index of g (see [Xia07, (3.14) & Thm. 3.2.0]). Additionally, it is a question of interest if the dimension of the level sets can be given in terms of more general notions as the *critical parameter* of a *scale*, as defined in [Klo12].

Hitting probabilities for non-linear SPDEs

By following the line of [DKN13],we propose the study of hitting probabilities for the solution of the non linear system of stochastic partial differential equations given by

$$\begin{cases} \frac{\partial v_j}{\partial t}(t,x) - \Delta v_j(t,x) &= \sum_{i=1}^d \sigma_{i,j}(v_j(t,x))\dot{W}_i^{H,\alpha}(t,x) + b_i(v(t,x)), \\ v_j(0,x) &= v_0(x), \end{cases}$$

$(t, x) \in (0, T] \times \mathbb{R}^d$ with $W_j^{H,\alpha}$, $j = 1, ..., D$ independent copies of a fractional-colored noise with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $\alpha \in [0, d)$. When $b, \sigma \equiv 0$, we recover the linear stochastic heat equation studied in Chapter 4.

Since this solution is not a Gaussian process mainly two new technical difficulties arise:

- The necessity of extending the criteria for lower bounds for hitting probabilities in Theorem 3.2 to more general process. A good start point is to follow the density based hypothesis approach from Theorem 2.1 in [DSS10].

- The usual Malliavin calculus tools used for estimate the density of the solution in the case of [DKN13] does not immediately work in this new case since the driving noise is not white anymore. To develop a new technique for solutions driven by more general noises is an open problem.

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