UNIVERSITAT DE BARCELONA

Adams Representability in Triangulated Categories

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Ph.D. Thesis

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Contents

Resum

Els teoremes de representabilitat són fonamentals en teoria d'homotopia i en àlgebra homològica. El primer resultat d'aquest tipus va ser el teorema de representabilitat de Brown, del qual es desprèn que tota teoria de cohomologia additiva pot ser representada per un espectre. Aquest resultat va ser generalitzat al context de les categories triangulades per Neeman en els termes següents: tot functor cohomològic d'una categoria triangulada ben generada a la categoria de grups abelians que envia coproductes a productes és representable.

En aquesta tesi s'estudia el problema de representar functors cohomològics definits en subcategories de categories triangulades, de manera anàloga al teorema de representabilitat d'Adams per a functors definits en espectres finits. Més concretament, si $\mathcal{T}$ és una categoria triangulada amb coproductes i α és un cardinal regular, diem que $\mathcal{T}$ satisfà la *α-representabilitat d'Adams per a objectes* si tot functor cohomològic $H\colon \mathcal{T}^{\alpha} \to \mathrm{Ab}$ de la subcategoria plena d'objectes α-compactes $\mathcal{T}^{\alpha} \subset \mathcal{T}$ a la categoria dels grups abelians que envia coproductes de menys de α objectes a productes és de la forma $\mathcal{T}(-, X)|_{\mathcal{T}^{\alpha}}$ per a un objecte X de $\mathcal{T}$. La *α-representabilitat d'Adams per a morfismes* es defineix de manera semblant. En aquesta tesi s'imposen condicions a $\mathcal{T}$ que impliquen la validesa de la α-representabilitat d'Adams per a cardinals $\alpha > \aleph_0$ (on $\aleph_0$ és el primer cardinal infinit).

La importància de la validesa de la α-representabilitat en una categoria triangulada qualsevol ha estat posada de manifest recentment per Neeman i Rosický. Tot i així, s'han obtingut molt pocs resultats per a $\alpha > \aleph_0$. Nosaltres demostrem que si $\mathcal{T}$ és $\aleph_1$-compactament generada i $\mathcal{T}^{\aleph_1}$ té cardinalitat igual o inferior a $\aleph_1$, aleshores $\mathcal{T}$ satisfà la $\aleph_1$-representabilitat per a objectes (on $\aleph_1$ denota el cardinal successor de $\aleph_0$). Utilitzem aquest resultat per a donar exemples de categories triangulades que satisfan la $\aleph_1$-representabilitat d'Adams per a objectes, tals com la categoria derivada d'un anell de cardinalitat $\aleph_1$ o la categoria motívica estable d'un esquema noetherià amb un recobriment per oberts afins amb anells de cardinalitat inferior o igual a $\aleph_1$.

Més exactament, donem condicions necessàries i suficients per a la validesa de la α-representabilitat d'Adams en funció d'una noció de dimensió pura projectiva adient i estudiem detalladament el cas de les categories derivades d'anells. Utilitzem els nostres resultats juntament amb un resultat recent de Braun i Göbel per demostrar que la categoria derivada de $\mathbb{Z}$ no satisfà la α-representabilitat d'Adams per a morfismes si $\alpha > \aleph_0$. Aquest resultat dóna una resposta negativa a una pregunta de Rosický: és cert o no que per a tota categoria triangulada amb un model combinatori existeixen cardinals α arbitràriament grans per als quals $\mathcal{T}$ satisfà la α-representabilitat per a morfismes?

Els resultats d'aquesta tesi estan enunciats en un context molt general amb l'objectiu de fer-los útils per a altres aplicacions. Per aquest motiu, introduïm les nocions de *categoria α-Grothendieck* i *objecte α-pla*, com a anàlegs per a cardinals superiors de categoria de Grothendieck i objecte pla. Pel que fa a les aplicacions, el resultat més rellevant que demostrem és una generalització del Lema d'Auslander en el context de categories α-Grothendieck.

També donem resultats nous respecte a functors de Rosický, que van ser introduïts per Neeman com un formalisme abstracte per a estudiar la α-representabilitat d'Adams en categories triangulades.

Introduction

Summary

Representability theorems are fundamental in homotopy theory and homological algebra. The first result of this kind was Brown representability, according to which every additive cohomology theory can be represented by a spectrum. This result was generalized to the context of triangulated categories by Neeman, stating that every cohomological functor from a well generated triangulated category to the category of abelian groups that sends coproducts to products is representable.

In this thesis, we study the problem of representing cohomological functors defined on subcategories of triangulated categories, in the spirit of Adams representability of functors defined on finite spectra. More precisely, if $\mathcal{T}$ is a triangulated category with coproducts and α is a regular cardinal, we say that $\mathcal{T}$ satisfies *α-Adams representability for objects* if every cohomological functor $H\colon \mathcal{T}^{\alpha} \to \mathrm{Ab}$ from the full subcategory of α-compact objects $\mathcal{T}^{\alpha} \subset \mathcal{T}$ to the category of abelian groups that sends coproducts of less than α objects to products is of the form $\mathcal{T}(-,X)|_{\mathcal{T}^{\alpha}}$ for an object X in $\mathcal{T}$. One defines *α-Adams representability for morphisms* similarly. In this thesis, we give conditions on $\mathcal{T}$ ensuring that α-Adams representability holds for cardinals $\alpha > \aleph_0$ (where $\aleph_0$ denotes the first infinite cardinal).

The importance of the validity of α-Adams representability in triangulated categories has been pointed out recently by Neeman and Rosický. However, very few positive results have been obtained for $\alpha > \aleph_0$. We prove that if $\mathcal{T}$ is $\aleph_1$-compactly generated and $\mathcal{T}^{\aleph_1}$ has cardinality less than or equal to $\aleph_1$, then $\mathcal{T}$ satisfies $\aleph_1$-Adams representability for objects (where $\aleph_1$ denotes the successor cardinal of $\aleph_0$). We use this result to exhibit examples of triangulated categories that satisfy $\aleph_1$-Adams representability for objects, such as the derived category of a ring of cardinality $\aleph_1$ or the stable motivic homotopy category of a noetherian scheme with an affine open cover by rings of cardinality less than or equal to $\aleph_1$.

In fact, we provide necessary and sufficient conditions for α-Adams repre-

sentability to hold, in terms of a suitable notion of pure projective dimension, and study in detail the case of derived categories of rings. We show that our conditions can be used, together with a recent result by Braun and Göbel, to prove that derived category of $\mathbb{Z}$ does not satisfy α-Adams representability for morphisms if $\alpha > \aleph_0$. This result answers negatively a question raised by Rosický of whether for every triangulated category $\mathcal{T}$ with a combinatorial model there exist arbitrarily large regular cardinals α for which $\mathcal{T}$ satisfies α-Adams representability for morphisms.

The results in this thesis are presented in a very general context, in order to make them useful for various applications. For this reason, we introduce the notions of *α-Grothendieck categories* and *α-flat objects*, as higher-cardinal analogs of Grothendieck categories and flat objects. For our purposes, one of the most relevant results that we prove is a generalization of the Auslander Lemma to α-Grothendieck categories.

We also give new results about Rosický functors, which were introduced by Neeman as an abstract formalism to study α-Adams representability in triangulated categories.

Background and recent advances

Triangulated categories provide a suitable formalism to study stable homotopy and derived categories. Representability theorems appeared first in stable homotopy theory, but their generalizations to triangulated categories have been very fruitful in the context of derived categories.

Stable homotopy focuses on properties that become stable after suspending spaces sufficiently many times. A way to make this precise is to work in the *homotopy category of spectra* Ho(Sp) as introduced by Boardman [Vog70] and Adams [Ada74]. An object $\mathbb{X}$ in Ho(Sp) is a sequence of pointed topological spaces X_0, $X_1, \ldots$ together with structure maps $\sigma\colon \Sigma X_n \to X_{n+1}$ for $n \geq 0$. Morphisms are homotopy classes of maps between such objects.

Given a spectrum $\mathbb{E}$ such that the adjoint maps $E_n \to \Omega E_{n+1}$ are weak equivalences, one defines a generalized cohomology theory as $E^n(X) = \operatorname{colim}_i[\Sigma^i X, E_{n+i}]$, where $[-,-]$ denotes homotopy classes of maps of pointed spaces, for $n \in \mathbb{Z}$. The converse of this theorem is known as the *Brown Representability Theorem* [Bro62]. It states that every additive generalized cohomology theory E^* can be represented by a spectrum, in the sense that there exists a spectrum $\mathbb{E}$ such that $E^n(X) = \operatorname{colim}_i[\Sigma^i X, E_{n+i}]$ for every pointed space X and $n \in \mathbb{Z}$.

Whitehead [Whi62] showed that a spectrum $\mathbb{E}$ also defines a homology theory by taking $E_n(X) = \operatorname{colim}_i[S^{i+n}, X \wedge E_i]$ for $n \in \mathbb{Z}$, where S^{i+n} is the $(i+n)$-sphere. The converse is also true and in fact it is deduced, using Spanier–Whitehead duality [SW55], from the *Adams Representability Theorem* [Ada71]. Adams' theorem states that every generalized cohomology theory E^* defined only on finite CW-complexes can be represented by a spectrum, in the sense that there exists a spectrum $\mathbb{E}$ such that $E^n(X) = \mathrm{colim}_i[\Sigma^i X, E_{n+i}]$ for every finite CW-complex X and $n \in \mathbb{Z}$. The Adams Representability Theorem implies both Brown's theorem and the representability of generalized homology theories satisfying the limit axiom.

The theorems of Brown and Adams can be rephrased in a simpler form using the language of triangulated categories as introduced by Puppe in [Pup62] and Verdier in [Ver77] and [Ver96].

A *triangulated category* $\mathcal{T}$ is an additive category together with an equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$ and a class of sequences in $\mathcal{T}$ of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A,$$

called *triangles*, that satisfy a list of axioms. In the case of $\mathcal{T} = \mathrm{Ho(Sp)}$, the equivalence Σ is induced by the suspension functor $- \wedge S^1$ and the triangles are sequences inducing long exact sequences of abelian groups after applying the functors $[\mathbb{K}, -]$ and $[-, \mathbb{K}]$ for every spectrum $\mathbb{K}$.

An additive functor $H\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ is called *cohomological* if it sends every triangle

$$X \xrightarrow{\ u\ } Y \xrightarrow{\ v\ } Z \xrightarrow{\ w\ } \Sigma X$$

in $\mathcal{T}$ to a long exact sequence

$$\cdots \to\ H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z) \xrightarrow{H(w)} H(\Sigma X) \xrightarrow{H(\Sigma u)} H(\Sigma Y)\ \to \cdots$$

in the category Ab of abelian groups. We say that *Brown representability* holds for $\mathcal{T}$ if it has coproducts and, for every cohomological functor $H\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ that sends coproducts in $\mathcal{T}$ to products in Ab, there exists an object X in $\mathcal{T}$ and a natural isomorphism $H(Y) \cong \mathcal{T}(Y, X)$ for every Y in $\mathcal{T}$. The fact that Brown representability holds for Ho(Sp) is equivalent to the classical Brown Representability Theorem.

Brown representability does not hold for arbitrary triangulated categories, as shown by Casacuberta and Neeman in [CN09], but it is known to hold for many families of triangulated categories, such as derived categories of Grothendieck categories [Nee01a] and homotopy categories of combinatorial model categories [Ros05].

If $\mathcal{A}$ is an abelian category, its *derived category* $\mathrm{D}(\mathcal{A})$ has as objects the (cochain) complexes of objects in $\mathcal{A}$, and morphisms are obtained from morphisms of complexes by formally inverting quasi-isomorphisms. Two particular cases have been of historical importance. One is the derived category $\mathrm{D}(R)$ of a ring R, where $\mathcal{A}$ is then the category of R-modules, and the other one is the derived category $\mathrm{D}(qc/X)$ of quasi-coherent sheaves over a scheme X. Verdier introduced triangulated categories in [Ver77] and [Ver96], independently from Puppe, in order to give a general context to state and prove Grothendieck duality. He showed that if $f\colon X \to Y$ is a proper morphism of schemes, then, under suitable hypotheses on f, X and Y, the functor

$$Rf_*\colon \mathrm{D}(qc/X) \longrightarrow \mathrm{D}(qc/Y)$$

has a right adjoint $f^!$. Neeman noticed in [Nee96] that the existence of this adjunction is an easy consequence of the fact that Brown representability holds for $\mathrm{D}(qc/X)$.

It was Neeman [Nee01b] who first observed that the proofs of Brown and Adams were based on the fact that Ho(Sp) is generated by the suspensions of the sphere spectrum $\mathbb{S}$ in the following way: A set of objects S in an additive category $\mathcal{T}$ is said to *generate* $\mathcal{T}$ if for every non-zero object X in $\mathcal{T}$ there is a non-zero morphism $s \to X$ with $s \in S$.

A *compactly generated* triangulated category is a triangulated category generated by a set of compact objects, where an object X is called *compact* if the canonical morphism

$$\coprod_{i\in I} \mathcal{T}(X, Y_i) \longrightarrow \mathcal{T}(X, \coprod_{i\in I} Y_i)$$

is an isomorphism for every set of objects $\{Y_i\}_{i\in I}$ in $\mathcal{T}$. The full subcategory of compact objects is denoted by $\mathcal{T}^{\aleph_0}$. Examples of compactly generated triangulated categories include the stable homotopy category Ho(Sp), where the compact objects are the finite spectra, and the derived category of a ring $\mathrm{D}(R)$, where the compact objects are bounded complexes of finitely presented projective R-modules. Brown representability holds for every compactly generated triangulated category [Nee96].

It was also Neeman [Nee97] who introduced the analog of Adams representability for compactly generated triangulated categories, as follows. Let $\mathcal{T}$ be a compactly generated triangulated category.

1. $\aleph_0$-*Adams representability for objects* holds in $\mathcal{T}$ if, for every additive functor
$$H\colon \{\mathcal{T}^{\aleph_0}\}^{\mathrm{op}} \longrightarrow \mathrm{Ab}$$
that sends triangles to long exact sequences, there exists an object X in $\mathcal{T}$ and a natural isomorphism $H(Y) \cong \mathcal{T}(Y, X)$ for every Y in $\mathcal{T}^{\aleph_0}$.

2. $\aleph_0$-*Adams representability for morphisms* holds in $\mathcal{T}$ if, for every natural transformation $F\colon \mathcal{T}(-,X)|_{\mathcal{T}^{\aleph_0}} \to \mathcal{T}(-,Y)|_{\mathcal{T}^{\aleph_0}}$, there exists a morphism $f\colon X \to Y$ such that $F = \mathcal{T}(-,f)|_{\mathcal{T}^{\aleph_0}}$.

If $\mathcal{T}$ is compactly generated and $\mathcal{T}^{\aleph_0}$ is countable, then $\mathcal{T}$ satisfies $\aleph_0$-Adams representability for objects and for morphisms [Nee97]. In particular, we obtain the original Adams representability for Ho(Sp), but also for $\mathrm{D}(R)$ if the ring R is countable. However, in contrast with Brown representability, there are many examples of compactly generated triangulated categories that do not satisfy $\aleph_0$-Adams representability, as shown in [CKN01].

In the process of extending the proof of Brown representability for compactly generated triangulated categories to other triangulated categories, Neeman introduced in his book [Nee01b] the notion of *well generated* triangulated categories, which is the higher-cardinal analog of compactly generated ones. If α is a regular cardinal, one defines *α-compact objects* as higher-cardinal analogs of compact objects (in a precise sense that we do not explain in detail here). Each α-compact object X in $\mathcal{T}$ has the property that every morphism $X \to \coprod_{i\in I} Y_i$ factorizes through a morphism $X \to \coprod_{i\in J} Y_i$ with $J \subset I$ a set of cardinality less than α. The full subcategory of α-compact objects is denoted by $\mathcal{T}^\alpha$ and a triangulated category is called *α-compactly generated* if it is generated by a set of α-compact objects. A triangulated category is called *well generated* if it is α-compactly generated for some cardinal α. Neeman proved in [Nee01b] that Brown representability holds for every well generated triangulated category.

Well generated triangulated categories turn out to have other interesting properties than Brown representability. The most important one is a generalization of the Thomason Localization Theorem, stating that every Verdier localization of a well generated category is well generated. Using this result, Neeman proved that most interesting triangulated categories are well generated. For instance, the derived category of a Grothendieck category is well generated.

Since the publication of Neeman's book [Nee01b], many authors have published interesting articles about well generated triangulated categories. The starting point of this thesis was one of these articles: *Generalized Brown representability in homotopy categories* [Ros05] by Rosický. In this article, Rosický proved that the homotopy category of a stable combinatorial model category is well generated, and he also explored the idea of considering Adams representability in other subcategories than the subcategory of compact objects. This idea is implicit in Neeman's book [Nee01b] and it was further studied in [Nee09].

The articles [Nee09] and [Ros05] received much attention in the Research

Programme on Homotopy Theory and Higher Categories held in the CRM of Barcelona during the academic year 2007–2008. Neeman and Rosický attended this research programme and explained open questions and possible consequences of their articles. Those conversations motivated this thesis. Although the idea was already present in [Nee09] and [Ros05], the following definition, as a higher-cardinal analog of $\aleph_0$-Adams representability, is new in this thesis.

Definition 2.3.1. Let $\mathcal{T}$ be a triangulated category with coproducts. Let α be a regular cardinal.

1. *α-Adams representability for objects* holds for $\mathcal{T}$ if for every cohomological functor $H\colon \{\mathcal{T}^\alpha\}^{\mathrm{op}} \to \mathrm{Ab}$ that sends coproducts of less than α objects in $\mathcal{T}^\alpha$ to products in Ab there is a natural isomorphism $H(-) \cong \mathcal{T}(-,X)|_{\mathcal{T}^\alpha}$ for an object X in $\mathcal{T}$.

2. *α-Adams representability for morphisms* holds for $\mathcal{T}$ if every natural transformation $F\colon \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \to \mathcal{T}(-,Y)|_{\mathcal{T}^\alpha}$ is of the form $\mathcal{T}(-,f)|_{\mathcal{T}^\alpha}$ for a (not necessarily unique) morphism $f\colon X \to Y$ in $\mathcal{T}$.

Rosický [Ros05] and Neeman [Nee09] inferred strong consequences of α-Adams representability. For instance, Neeman proved in [Nee09] that, if $\mathcal{T}$ is a well generated triangulated category and satisfies α-Adams representability for morphisms for some α, then $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$ satisfy Brown representability. It was already known that every well generated triangulated category satisfies Brown representability, but the fact that if $\mathcal{T}$ is a well generated triangulated category then $\mathcal{T}^{\mathrm{op}}$ satisfies Brown representability is still a conjecture. The difficulty in proving this conjecture relies on the fact that the opposite of a well generated category is almost never well generated [Nee01b].

After [Nee01b] and [Ros05], it became clear that it is interesting to give conditions on a triangulated category in order to ensure that it satisfies α-Adams representability. However, the only known results at the time were for $\alpha = \aleph_0$. As we have already mentioned, a well generated triangulated category need not satisfy $\aleph_0$-Adams representability. Rosický [Ros05] raised the question of whether a well generated triangulated category satisfying some extra assumptions could satisfy α-Adams representability for a large enough regular cardinal α.

Question 2.3.34. *Let $\mathcal{T}$ be a well generated triangulated category with a combinatorial model. Is it true that there exist arbitrarily large regular cardinals α for which $\mathcal{T}$ satisfies α-Adams representability for morphisms?*

This question was the first target of this thesis. In order to address this question, we generalize some of the results in [Bel00b] and [Nee97] for higher cardinals using the following notation: If $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ denotes the category of contravariant additive functors from $\mathcal{T}^\alpha$ into Ab that send coproducts of less than α objects in $\mathcal{T}^\alpha$ into products, and

$$\begin{aligned} S_\alpha\colon\ \mathcal{T} &\longrightarrow \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha, \\ X &\longmapsto \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \end{aligned}$$

is the restricted Yoneda functor, then

1. $\mathcal{T}$ satisfies α-Adams representability for objects if and only if every cohomological functor in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ is in the essential image of S_α, and

2. $\mathcal{T}$ satisfies α-Adams representability for morphisms if and only if S_α is full.

The categories $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ play a very important role in this thesis. In contrast with the category $\mathrm{Mod}\text{-}\mathcal{T}^\alpha$ of additive functors from $\mathcal{T}^\alpha$ into Ab, $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ is usually not a Grothendieck category, because it need not have exact filtered colimits and even not enough injectives. We study the categories $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ in Chapter 4 using the abstract formalism of α-Grothendieck categories that we introduce as a higher-cardinal analog of Grothendieck categories.

The study of $\aleph_0$-Adams representability in [Nee97] and [Bel00b] relies in the notion of purity. The $\aleph_0$-*pure global dimension* of a triangulated category $\mathcal{T}$ is defined as

$$\mathrm{Pgldim}_{\aleph_0}(\mathcal{T}) = \sup\{\mathrm{pd}(H) \mid H\colon \mathcal{T}^{\aleph_0} \to \mathrm{Ab}\ \text{cohomological}\},$$

where $\mathrm{pd}(H)$ denotes the projective dimension of H in $\mathrm{Mod}\text{-}\mathcal{T}^{\aleph_0}$. It was proved in [Nee97] and [Bel00b] that

1. $\mathcal{T}$ satisfies $\aleph_0$-Adams representability for morphisms if and only if $\mathrm{Pgldim}_{\aleph_0}(\mathcal{T}) \leq 1$, and

2. $\mathcal{T}$ satisfies $\aleph_0$-Adams representability for objects if $\mathrm{Pgldim}_{\aleph_0}(\mathcal{T}) \leq 2$.

We have extended these results in Chapter 3 to uncountable cardinals.

After [Bel00b] and [Nee97], the study of $\aleph_0$-Adams representability continued in [CKN01] with the particular case of the derived category of a ring. Among other results, it was proved that, under certain circumstances, the $\aleph_0$-pure global dimension of the derived category of a ring is less than or equal to the classical global dimension of the ring. This result is an important source of examples of triangulated categories that do not satisfy

$\aleph_0$-Adams representability. We extend this result in Chapter 5 for uncountable cardinals.

The results in Chapter 3 and Chapter 5 led us to a reformulation of Question 2.3.34 in terms of higher purity of rings. Recall that a short exact sequence $0 \to A \to B \to C \to 0$ is called *α-pure exact* if for every α-presentable module P the induced sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(P, A) \longrightarrow \mathrm{Hom}(P, B) \longrightarrow \mathrm{Hom}(P, C) \longrightarrow 0$$

is exact. With this definition, we can define α-pure projective modules and the α-pure projective dimension of an R-module as usual. In February 2010, Braun and Göbel reported in [BG10] that $\mathrm{Pgldim}_\alpha(\mathbb{Z}) > 1$ for every regular cardinal $\alpha > \aleph_0$, which implies, using our reformulation, that the answer to Question 2.3.34 is negative. In this example, however, $\mathrm{Pgldim}_{\aleph_0}(\mathbb{Z}) = 1$. Recently, Bazzoni and Šťovíček [BŠ10] overcame this issue and proved that $\mathrm{Pgldim}_\alpha(k[X_1, \ldots, X_n]) > 1$ when $n \geq 2$ and k is an uncountable field, for all regular cardinals α.

Until now, all results about α-Adams representability for $\alpha > \aleph_0$ were negative (stating that representability failed in certain categories). In this thesis, we give examples of triangulated categories satisfying $\aleph_1$-Adams representability for objects in Chapter 6.

Summary of results

This thesis extends a number of known results about $\aleph_0$-Adams representability to α-Adams representability for regular cardinals $\alpha > \aleph_0$. As in the case of $\aleph_0$, if $\mathcal{T}$ is an α-compactly generated triangulated category, α-Adams representability is closely related to the properties of the restricted Yoneda functor

$$\begin{aligned} S_\alpha \colon\ \mathcal{T} &\longrightarrow \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha. \\ X &\longmapsto \mathcal{T}(-, X)|_{\mathcal{T}^\alpha} \end{aligned}$$

The category Mod_α-$\mathcal{T}^\alpha$ was extensively studied in [Nee01b]. In Section 2.3.1 we explain that most properties of Mod_α-$\mathcal{T}^\alpha$ do not depend on the triangulated structure, but just on the fact that $\mathcal{T}^\alpha$ is an additive category with coproducts of less than α objects.

If $\mathcal{C}$ is an essentially small additive category with coproducts of less than α objects, we denote by Mod_α-$\mathcal{C}$ the abelian category of additive contravariant functors from $\mathcal{C}$ to Ab that send coproducts of less than α objects to products. We prove in Section 2.3.1 that the category Mod_α-$\mathcal{C}$ is an exact subcategory of Mod-$\mathcal{C}$ with coproducts and exact products, and it is locally α-presentable.

A generating set of α-presentable projectives is given by $\{\mathcal{C}(-,X) \mid X \text{ in } \tilde{\mathcal{C}}\}$ where $\tilde{\mathcal{C}}$ is a set of representatives of the isomorphism classes of objects in $\mathcal{C}$.

The category Mod_α-$\mathcal{C}$ does not have exact filtered colimits in general, but only exact α-filtered colimits. A category is called *α-filtered* if every subcategory of cardinality less than α has a cocone. In analogy with the $\aleph_0$ case, we call an abelian category with a set of generators and with exact α-filtered colimits *α-Grothendieck*.

In Section 3.1 we prove a relation between the validity of α-Adams representability and the *α-pure global dimension* of the triangulated category,

$$\mathrm{Pgldim}_\alpha(\mathcal{T}) = \sup\{\mathrm{pd}(H) \mid H \text{ cohomological in } \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha\},$$

where $\mathrm{pd}(H)$ denotes the projective dimension of H in Mod_α-$\mathcal{T}^\alpha$. Our main result is the following:

Theorem 3.1.10. *Let α be a regular cardinal and let $\mathcal{T}$ be an α-compactly generated triangulated category.*

1. *If* $\mathrm{Pgldim}_\alpha(\mathcal{T}) \leq 2$, *then $\mathcal{T}$ satisfies α-Adams representability for objects.*

2. $\mathrm{Pgldim}_\alpha(\mathcal{T}) \leq 1$ *if and only if $\mathcal{T}$ satisfies α-Adams representability for objects and for morphisms.*

In the case $\alpha = \aleph_0$, this was proved by Neeman in [Nee97] and Beligiannis in [Bel00b]. This result translates α-Adams representability into a problem of homological algebra where it is possible to carry out computations. The most important part of the thesis is devoted to finding conditions on the triangulated category $\mathcal{T}$ in order to obtain an upper bound for its α-pure global dimension. For this reason, Chapter 4 studies cohomological functors and α-purity in Mod_α-$\mathcal{T}^\alpha$. Since most of the results depend only on the fact that Mod_α-$\mathcal{T}^\alpha$ is an α-Grothendieck category, we present our results in a very general context. One of our key results in Chapter 4 is the following generalization of the Auslander Lemma, a well-known result about Grothendieck categories.

Corollary 4.1.5. *Let $\mathcal{A}$ be an $\aleph_n$-Grothendieck category and let $M = \mathrm{colim}_{i<\gamma} M_i$ be such that $M_0 = 0$, $M_i \subset M_j$ if $i < j$ and $\gamma \geq \omega_n$. Assume that $\mathrm{colim}_{i<\beta} M_i = M_\beta$ for every limit ordinal β with $\omega_n \leq \beta \leq \gamma$. If $\mathrm{pd}(M_j/M_i) \leq d$ for all $i \leq j < i + \omega_n$, then $\mathrm{pd}(M) \leq n + d$.*

Notice that if $n = 0$ we obtain the classical Auslander Lemma and the projective dimension of the colimit is not greater than the projective dimension of the objects used to define it. The fact that for $n > 0$ the projective dimension increases is the ultimate reason that prevents us from obtaining examples that satisfy α-Adams representability for $\alpha > \aleph_1$.

In order to be able to use this generalization of the Auslander Lemma, we need to express every cohomological functor in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ as a colimit, as in the statement of the previous corollary. This will be done in Section 4.2 and Section 4.3.

For the study of α-Adams representability, we are interested in cohomological functors in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$. The natural analog for a general α-Grothendieck category (which need not be a category of functors) is that of an α-flat object.

Definition 4.2.1. Let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. An object in $\mathcal{A}$ is said to be *α-flat* if it is an α-filtered colimit of α-presentable projectives.

We prove in Lemma 4.2.2 that an object F is α-flat if the following equivalent definitions hold:

1. If $\mathcal{P} \subset \mathcal{A}$ is the full subcategory of α-presentable projective objects, the canonical diagram $(\mathcal{P} \downarrow F) \to \mathcal{A}$ is α-filtered.

2. Every morphism $N \to F$ where N is α-presentable factorizes through an α-presentable projective object.

In the case of categories of the form $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$, where $\mathcal{T}$ is an α-compactly generated triangulated category, Neeman [Nee01b] proved that the α-flat objects are precisely the cohomological functors. In the case of categories of the form $\mathrm{Mod}_\alpha\text{-}\mathcal{A}^\alpha$ where $\mathcal{A}$ is an abelian category with coproducts and $\mathcal{A}^\alpha$ is the full subcategory of α-presentable objects, we prove in Lemma 4.2.4 that the α-flat objects correspond precisely to the left exact functors.

In Section 4.3 we introduce α-purity in the general context of α-Grothendieck categories. A short exact sequence $0 \to A \to B \to C \to 0$ is called *α-pure exact* if for every α-presentable object P the induced sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(P, A) \longrightarrow \mathrm{Hom}(P, B) \longrightarrow \mathrm{Hom}(P, C) \longrightarrow 0$$

is exact or, equivalently, if it is an α-filtered colimit of split exact sequences. With this definition, we can define α-pure projective objects and the α-pure projective dimension of an object as usual.

The main result of Section 4.3 is the following corollary, that will be used to write cohomological functors in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ as colimits of the correct shape in order to apply our generalization of the Auslander Lemma.

Corollary 4.3.8. *Let α be a regular cardinal and let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. Every object M in $\mathcal{A}$ is the colimit of an α-filtered ascending chain of α-pure subobjects $M = \mathrm{colim}_{i<\gamma} N_i$ such that $\mathrm{colim}_{i<\beta} N_i = N_\beta$ for every limit ordinal $\beta \geq \alpha$ and, if $i \leq j \leq i + \alpha$, then N_j/N_i has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators.*

In Section 4.4 we introduce the notion of α-noetherianity in locally α-presentable α-Grothendieck categories. This property, when it holds, is very useful for the computation of α-pure global dimensions. An object X in an abelian category is called *α-noetherian* if for every set $\{A_i\}_{i<\gamma}$ of subobjects of X well ordered by inclusion, *i.e.* $A_i \subset A_j$ if $i < j$, there exists an ordinal β with $\#\beta < \alpha$ such that $A_i = A_j$ whenever $\beta \leq i, j < \gamma$. An abelian category is said to be *locally α-noetherian* if it has a generating set of α-presentable projective α-noetherian objects. In a locally α-noetherian abelian category, an object is α-generated if and only if it is α-presentable.

We also study the *α-flat dimension* of an object, which is defined as the minimal length of a resolution by α-flat objects. The *weak global dimension* $\mathrm{weakgldim}(\mathcal{A})$ of a locally α-presentable α-Grothendieck category $\mathcal{A}$ is then defined as the supremum of the α-flat dimensions of all the objects in $\mathcal{A}$. We prove in Lemma 4.4.7 that if $\mathcal{A}$ is locally α-noetherian, then

$$\mathrm{weakgldim}(\mathcal{A}) = \sup\{\mathrm{pd}(B) \mid B \ \alpha\text{-presentable}\}$$

and in the case $\alpha = \aleph_n$ we prove in Corollary 4.4.8 that

$$\mathrm{weakgldim}(\mathcal{A}) \leq \mathrm{projgldim}(\mathcal{A}) \leq \mathrm{weakgldim}(\mathcal{A}) + n.$$

Notice that if $\alpha = \aleph_0$ this implies that $\mathrm{weakgldim}(\mathcal{A}) = \mathrm{projgldim}(\mathcal{A})$. Brune [Bru83] used this fact to prove that, if $\mathcal{A}$ is locally noetherian, then every subobject of an $\aleph_0$-pure projective object is $\aleph_0$-pure projective. In particular, $\mathrm{Pgldim}_{\aleph_0}(\mathcal{A}) \leq 1$ and this is exploited in [BBL82] to compute the $\aleph_0$-pure global dimension of certain rings. These computations were used later in [CKN01] to provide counterexamples to $\aleph_0$-Adams representability. We generalize this result to the case $\alpha = \aleph_n$ in Theorem 4.4.10 and Proposition 4.4.11. These generalizations imply, in particular, that, if $\mathcal{T}$ is an $\aleph_n$-compactly generated triangulated category satisfying some extra hypotheses, then every object in $\mathrm{Mod}_{\aleph_n}\text{-}\mathcal{T}^{\aleph_n}$ has $\aleph_n$-pure projective dimension less than or equal to $n + 1$.

In Chapter 6 we use the results in the previous sections to give examples of triangulated categories satisfying $\aleph_1$-Adams representability for objects. They are consequences of the following general result.

Theorem 6.1.1. *Let $\mathcal{C}$ be a category with coproducts of less than $\aleph_n$ objects. If $\#\mathcal{C} \leq \aleph_n$, then every $\aleph_n$-flat object in $\mathrm{Mod}_{\aleph_n}$-$\mathcal{C}$ has projective dimension less than or equal to $n+1$.*

In the case $\alpha = \aleph_1$, as a consequence of Theorem 3.1.10, we obtain:

Corollary 6.1.3. *Let $\mathcal{T}$ be an $\aleph_1$-compactly generated triangulated category. If $\#\mathcal{T}^{\aleph_1} \leq \aleph_1$, then $\mathcal{T}$ satisfies $\aleph_1$-Adams representability for objects.*

In order to give concrete examples for which this corollary applies, we study in Chapter 5 the particular case of derived categories of rings. What we do is to relate the α-pure global dimension of the derived category of a ring with the α-pure global dimension of the ring. All the rings that we consider are associative with identity. In the first section, we review the definition of α-purity for rings as given in [JL89]. This notion is classical, but has not received much attention except for the case $\alpha = \aleph_0$. In the second section we prove the following result:

Corollary 5.2.3. *Let R be a ring and $\alpha > \aleph_0$ be a regular cardinal.*

1. *If R is α-coherent, then* $\mathrm{Pgldim}_\alpha(R) \leq \mathrm{Pgldim}_\alpha(\mathrm{D}(R))$.
2. *If R is hereditary, then* $\mathrm{Pgldim}_\alpha(R) = \mathrm{Pgldim}_\alpha(\mathrm{D}(R))$.

For the case $\alpha = \aleph_0$ this fact was proved by Christensen, Keller and Neeman in [CKN01], who used the computations in [BBL82] to prove that some derived categories of rings do not satisfy $\aleph_0$-Adams representability for morphisms. In the same spirit, we use Theorem 3.1.10 and Corollary 5.2.3 together with a recent computation by Braun and Göbel [BG10] stating that $\mathrm{Pgldim}_\alpha(\mathbb{Z}) > 1$ for every $\alpha > \aleph_0$ to prove the following result:

Proposition 5.2.9. *The category* $\mathrm{D}(\mathbb{Z})$ *does not satisfy α-Adams representability for morphisms if $\alpha > \aleph_0$.*

This result gives a negative answer to Question 2.3.34, because $\mathrm{D}(\mathbb{Z})$ is a compactly generated triangulated category with a combinatorial model, yet there is no regular cardinal $\alpha > \aleph_0$ for which α-Adams representability for morphisms holds. However, notice that $\mathrm{Pgldim}_{\aleph_0}(\mathbb{Z}) = 1$ and hence $\mathrm{D}(\mathbb{Z})$ satisfies $\aleph_0$-Adams representability for morphisms. After this example, it is natural to ask if there exists a triangulated category that does not satisfy α-Adams representability for any cardinal α. Recently, Bazzoni and Šťovíček [BŠ10] proved that there are rings R such that $\mathrm{Pgldim}_\alpha(R) > 1$ for every regular cardinal α. Specifically, after applying Theorem 3.1.10 and Corollary 5.2.3, we have the following result:

Proposition 5.2.11. *Let k be an uncountable field and let R be one of the following rings:*

1. $k[X_1, \ldots, X_n]$ *for* $n \geq 2$.

2. $\begin{pmatrix} k & V \\ 0 & k \end{pmatrix}$ *where V is a k-vector space with* $\dim_k(V) \geq 2$.

Then the triangulated category $\mathrm{D}(R)$ *does not satisfy α-Adams representability for morphisms for every regular cardinal α.*

It is worth mentioning that before the result of Bazzoni and Šťovíček was announced, Trlifaj [Trl10] studied the stable version of Question 2.3.34 and proved that for every regular cardinal α, there is a ring R such that $\mathrm{Pgldim}_\alpha(R) > 1$.

In Section 6.1 we exhibit examples that satisfy $\aleph_1$-Adams representability for objects. They all follow from Corollary 6.1.3 if we assume the Continuum Hypothesis (CH).

1. $\mathrm{D}(R)$ where R is a ring such that $\#R \leq \aleph_1$.

2. The stable homotopy category Ho(Sp).

3. The homotopy category of chain complexes of projective R-modules $\mathrm{K}(R\text{-Proj})$ where R is a ring such that $\#R \leq \aleph_1$.

4. The homotopy category of chain complexes of injective R-modules $\mathrm{K}(R\text{-Inj})$ where R is a noetherian ring such that $\#R \leq \aleph_1$.

5. The derived category of sheaves on a connected paracompact manifold $\mathrm{D}(Sh/M)$.

6. The stable motivic homotopy category $\mathcal{SH}(S)$ over a noetherian scheme of finite Krull dimension S with an affine open cover by rings of cardinality less than or equal to $\aleph_1$ (assuming that $2^{\aleph_1} = \aleph_2$).

In [CKN01] it is proved that the derived category of a ring R satisfies $\aleph_0$-Adams representability for objects and for morphisms if R is countable, but this need not be true if R has cardinality $\aleph_1$. Notice that, if we assume the Continuum Hypothesis, it follows from our results that the derived category of the ring $\mathbb{C}[X_1, \ldots, X_n]$ satisfies $\aleph_1$-representability for objects.

The triangulated categories $\mathrm{K}(R\text{-Proj})$ and $\mathrm{D}(Sh/M)$ need not be compactly generated. Hence, $\aleph_0$-Adams representability does not even make sense. Neeman proved in [Nee01a] and [Nee08] that they are $\aleph_1$-compactly generated, while the $\aleph_1$-Adams representability results are new in this thesis.

The first representability result for the stable motivic homotopy category $\mathcal{SH}(S)$ was obtained by Voevodsky [Voe98]. He proved that if the category of smooth S-schemes of finite type Sm/S is countable, then $\mathcal{SH}(S)$ satisfies $\aleph_0$-Adams representability for morphisms. In Section 6.2 we extend this result, following a proof given by Naumann and Spitzweck [NS09], to cover the case where the category Sm/S has cardinality $\aleph_1$. In particular, our result applies to schemes $S = \bigcup_I \operatorname{Spec}(R_i)$ with R_i rings of cardinality less than or equal to $\aleph_1$ for all $i \in I$. Hence, if we assume the Continuum Hypothesis, the rings R_i can be chosen as quotients of rings of polynomials over the complex numbers, which are of great interest in algebraic geometry.

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Chapter 1

Triangulated categories

In this chapter, we overview the main aspects of the theory of triangulated categories following Neeman's book [Nee01b]. This point of view is the most convenient for our purposes, although there are some differences with the foundational papers by Puppe [Pup62] and Verdier [Ver77], [Ver96], the most important being the treatment of the Octahedral Axiom.

We have divided this chapter into four sections. The first section contains the definitions and basic properties of triangulated categories. The second and third sections are devoted to examples, namely in the second section we describe the stable homotopy category and in the third section we discuss the derived category of an abelian category. Finally, the fourth section is devoted to localization, which is one of the most useful constructions in the theory of triangulated categories.

1.1 Triangulated categories

This section is a survey of the theory of triangulated categories. In particular, we set up the basic notation used in the thesis. Since everything in this section is well known, most results are stated without a proof. We give references for all results, mainly from Neeman's book [Nee01b]. Examples will be given in the following sections.

We work under the axiomatics of Zermelo and Fraenkel together with the Axiom of Choice (ZFC). All the categories that we consider have a proper class or a set of objects and, for every pair of objects, a set of morphisms between them. We will frequently use basic facts about categories, especially additive categories, for which we refer to [Mit65] or [Pop73].

A triangulated category is an additive category with extra structure, called distinguished triangles, that replaces short exact sequences (which are available in abelian categories, but not in additive categories in general). Before making this precise, we recall some terminology.

Definition 1.1.1. Let $\mathcal{C}$ be an additive category. A *representable (covariant) functor* from $\mathcal{C}$ to the category Ab of abelian groups is a functor of the form

$$\begin{aligned} \mathcal{C}(X,-)\colon\ \mathcal{C} &\longrightarrow \mathrm{Ab} \\ Y &\longmapsto \mathcal{C}(X,Y) \end{aligned}$$

for some object X in $\mathcal{C}$. Dually, a *representable (contravariant) functor* from $\mathcal{C}^{\mathrm{op}}$ to Ab is a functor of the form $\mathcal{C}(-,Y)$ for some object Y in $\mathcal{C}$.

Definition 1.1.2. Let $\mathcal{C}$ be an additive category and let $\Sigma\colon \mathcal{C} \to \mathcal{C}$ be an equivalence. A *candidate triangle* is a sequence of morphisms in $\mathcal{C}$ of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

such that the compositions $v \circ u$, $w \circ v$ and $\Sigma u \circ w$ vanish.

A *morphism of candidate triangles* is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd}$$

where the rows are candidate triangles. It is an *isomorphism of candidate triangles* if f, g and h are isomorphisms in $\mathcal{C}$.

A *homotopy* between two morphisms of candidate triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f'"] & Y \arrow[r, "v"] \arrow[d, "g'"] & Z \arrow[r, "w"] \arrow[d, "h'"] & \Sigma X \arrow[d, "\Sigma f'"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd}$$

consists of three morphisms $\Theta\colon Y \to X'$, $\Phi\colon Z \to Y'$ and $\Psi\colon \Sigma X \to Z'$ such that $f - f' = \Theta\, u + \Sigma^{-1}(w'\,\Psi)$, $g - g' = \Phi\, v + u'\,\Theta$ and $h - h' = \Psi\, w + v'\,\Phi$. If there exists a homotopy between two morphisms of candidate triangles, we say that these morphisms are *homotopic*.

Definition 1.1.3. Let $\mathcal{C}$ and Σ be as in the previous definition. Suppose that we are given a morphism of candidate triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'. \end{tikzcd}$$

We define the *mapping cone* of this morphism as the candidate triangle

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X'.$$

Up to isomorphism, the mapping cone of a morphism of triangles depends only on the homotopy class of the morphism [Nee01b, Lemma 1.3.3].

Definition 1.1.4. A *triangulated category* $\mathcal{T}$ is an additive category together with an equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$, which we call *suspension*, and a class of candidate triangles, called *distinguished triangles*, having the following properties.

TR0 The class of distinguished triangles is closed under isomorphisms and the candidate triangle

$$X \xrightarrow{id} X \longrightarrow 0 \longrightarrow \Sigma X$$

is distinguished for every object X in $\mathcal{T}$.

TR1 For any morphism $f\colon X \to Y$ in $\mathcal{T}$ there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X.$$

TR2 A candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is distinguished if and only if the candidate triangle

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

is distinguished.

TR3 For any commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & \Sigma X \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X', \end{tikzcd}$$

where the rows are distinguished triangles, there exists a morphism $h\colon Z \to Z'$ which makes the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd}$$

commutative.

TR4 The *Octahedral Axiom*. Let

$$X \xrightarrow{f} Y \xrightarrow{f'} C_f \xrightarrow{f''} \Sigma X,$$
$$Y \xrightarrow{g} Z \xrightarrow{g'} C_g \xrightarrow{g''} \Sigma Y \quad \text{and}$$
$$X \xrightarrow{g\,f} Z \xrightarrow{h} C_{gf} \xrightarrow{h'} \Sigma X$$

be three distinguished triangles in $\mathcal{T}$. Then we can construct a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "id"] & Y \arrow[r, "f'"] \arrow[d, "g"] & C_f \arrow[r, "f''"] \arrow[d] & \Sigma X \arrow[d, "id"] \\ X \arrow[r, "gf"] \arrow[d] & Z \arrow[r, "h"] \arrow[d, "g'"] & C_{gf} \arrow[r, "h'"] \arrow[d] & \Sigma X \arrow[d] \\ 0 \arrow[r] \arrow[d] & C_g \arrow[r, "id"] \arrow[d, "g''"] & C_g \arrow[r] \arrow[d] & 0 \arrow[d] \\ \Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \arrow[r, "\Sigma f'"] & \Sigma C_f \arrow[r, "-\Sigma f''"] & \Sigma^2 X \end{tikzcd}$$

where every row and column is a distinguished triangle.

Notice that, since every equivalence between additive categories is additive, Σ is an additive functor [Pop73, Ch. 3, Corollary 1.3]. It is also known that some parts of the definition are redundant [Nee01b, Remark 1.1.3]. It is not necessary to assume that the sequences

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in the class of distinguished triangles are such that the compositions $u \circ v$, $v \circ w$ and $w \circ \Sigma u$ vanish. It is also sufficient to assume that $\mathcal{T}$ is a category with a zero object enriched over abelian groups, *i.e.* the sets of morphisms from one object to another carry an abelian group structure and composition is bilinear. Finally, we also want to notice that it is not known if axiom [TR4] can be deduced from the other axioms. In fact, all known examples of categories satisfying [TR0], [TR1], [TR2] and [TR3] also satisfy [TR4].

From now on, when we say that $\mathcal{T}$ is a triangulated category we will assume that it is equipped with a triangulated structure, *i.e.* an equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$ and a class of distinguished triangles, which we call just *triangles.*

Remark 1.1.5. Let $\mathcal{T}$ be a triangulated category. Then its opposite category $\mathcal{T}^{\mathrm{op}}$ also has a triangulated structure with equivalence Σ^{-1} and triangles $X \to Y \to Z \to \Sigma X$ in $\mathcal{T}$ corresponding to triangles $Z \to Y \to X \to \Sigma^{-1} Z$ in $\mathcal{T}^{\mathrm{op}}$ [Nee01b, Remark 1.1.5].

Let $f\colon X \to Y$ be a morphism in a triangulated category. By [TR1] it can be completed to a triangle. Given two such completions

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X \quad \text{and}$$
$$X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow \Sigma X$$

it is not difficult to prove [Nee01b, Remark 1.1.21] using [TR3] that $Z \cong Z'$. Hence, Z is well defined up to non-canonical isomorphism. We will make an abuse of notation and call any object Z as in axiom [TR1] a *mapping cone* of f and denote it by C_f.

Lemma 1.1.6 ([Nee01b, Corollary 1.2.6]). *A mapping cone C_f of a morphism f is zero if and only if f is an isomorphism.*

In contrast with abelian categories, triangulated categories in general do not have kernels or cokernels. However, because of Lemma 1.1.6, we can think of C_f as a twisted version of kernel and cokernel.

The following proposition can be seen as an analog of the Five Lemma [Hat02, p. 129] or the Third Isomorphism Theorem [Fre64, p. 58]. In fact, it can be proved without using [TR4].

Proposition 1.1.7 ([Nee01b, Proposition 1.1.20]). *Let $\mathcal{T}$ be a triangulated category and let*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd}$$

be a morphism of triangles. Then, if f and g are isomorphisms, so is h.

Proposition 1.1.8 ([Nee01b, Section 1.2]). *Let $\mathcal{T}$ be a triangulated category. The class of triangles is closed under products, coproducts and direct summands.*

Products and coproducts need not exist in a triangulated category. Thus, Proposition 1.1.8 refers to those that exist.

For many of the results in this thesis, we will need that our triangulated categories have coproducts, or, at least, coproducts of certain cardinalities.

Recall that a cardinal α is called *regular* if every set of cardinality α is not the union of less than α subsets of cardinality less than α. For instance, $\aleph_n$ is regular for every integer $n \geq 0$ and every successor cardinal is regular. A cardinal that is not regular is called *singular*, *e.g.* $\aleph_\omega$. The following notation will be very convenient.

Definition 1.1.9. Let $\mathcal{T}$ be a triangulated category and let α be a regular cardinal. We define the following axiom.

TR5$_\alpha$ For any set of objects $\{X_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{T}$ such that $\#\Lambda < \alpha$, the coproduct $\coprod_{\lambda\in\Lambda} X_\lambda$ exists in $\mathcal{T}$.

Dually, we say that $\mathcal{T}$ satisfies [**TR5**$^*_\alpha$] if $\mathcal{T}^{\mathrm{op}}$ satisfies [TR5$_\alpha$], *i.e.* products of less than α objects exist in $\mathcal{T}$. We also define

TR5 All coproducts exist in $\mathcal{T}$, *i.e.* $\mathcal{T}$ satisfies [TR5$_\alpha$] for all α.

Dually, we say that $\mathcal{T}$ satisfies [**TR5***] if all products exist in $\mathcal{T}$.

Definition 1.1.10. Let $\mathcal{T}$ be a triangulated category and assume that it satisfies [TR5$_{\aleph_1}$]. Let

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_2 \xrightarrow{j_3} \dots$$

be a sequence of morphisms in $\mathcal{T}$. The *homotopy colimit* of the sequence is denoted by $\operatorname{Hocolim} X_i$ and is defined by the triangle

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{id-shift} \coprod_{i=0}^{\infty} X_i \longrightarrow \operatorname{Hocolim} X_i \longrightarrow \Sigma \coprod_{i=0}^{\infty} X_i$$

where the *shift* map is the direct sum of $j_{i+1}\colon X_i \to X_{i+1}$ for every $i \geq 0$ together with $0 \to X_0$.

It can be seen that the homotopy colimit of any strictly increasing subsequence is isomorphic to the homotopy colimit of the whole sequence; see [Nee01b, Lemma 1.7.1] for details.

The next proposition says that idempotents split in every triangulated category in which [TR5$_{\aleph_1}$] holds. The axiom [TR5$_{\aleph_1}$] is required because homotopy colimits are needed for the proof.

Proposition 1.1.11 ([Nee01b, Lemma 1.6.8]). *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5$_{\aleph_1}$]. *Let $e\colon X \to X$ be an idempotent morphism in $\mathcal{T}$,* i.e. *$e \circ e = e$. Then e splits,* i.e. *there are morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

such that $g \circ f = e$ and $f \circ g = id$.

1.1.1 Triangulated subcategories

Definition 1.1.12. Let $\mathcal{T}$ be a triangulated category. A full additive subcategory $\mathcal{S} \subset \mathcal{T}$ is called *triangulated* if $\mathcal{S}$ is closed under isomorphisms and suspension, and if for any triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in $\mathcal{T}$ such that X and Y are in $\mathcal{S}$, Z is also in $\mathcal{S}$.

Thus, the additive category $\mathcal{S}$ inherits a triangulated structure, where the triangles in $\mathcal{S}$ are just the triangles in $\mathcal{T}$ whose objects lie in $\mathcal{S}$.

Definition 1.1.13. Let $\mathcal{T}$ be a triangulated category satisfying [TR5] and let $\mathcal{S} \subset \mathcal{T}$ be a triangulated subcategory.

1. $\mathcal{S}$ is called *thick* if it contains all direct summands of its objects, *i.e.* if whenever $X \oplus Y$ is an object in $\mathcal{S}$, then so are X and Y.

2. $\mathcal{S}$ is called *α-localizing* for an infinite cardinal α if it is thick and closed under coproducts of less than α objects.

3. $\mathcal{S}$ is called *localizing* if it is closed under all coproducts, *i.e.* it is α-localizing for every infinite cardinal α.

Remark 1.1.14. If $\alpha > \aleph_0$, then every triangulated subcategory closed under coproducts of less than α objects is automatically thick by Proposition 1.1.11.

1.1.2 Exact functors

Definition 1.1.15. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be triangulated categories. An *exact functor*, also called *triangulated functor*, is an additive functor $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ equipped with a natural isomorphism

$$\phi_X\colon\ F(\Sigma X) \longrightarrow \Sigma F(X)$$

for every object X in $\mathcal{T}_1$, and such that, for any triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in $\mathcal{T}_1$, the candidate triangle

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\phi_X \circ F(w)} \Sigma F(X)$$

is a triangle in $\mathcal{T}_2$.

The inclusion functor of a triangulated subcategory $\mathcal{S} \hookrightarrow \mathcal{T}$ is exact with $\phi_X = id_{\Sigma X}$ for every object X in $\mathcal{S}$.

Definition 1.1.16. Let $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ be an exact functor. The *kernel* of F is the full subcategory $\ker(F)$ of $\mathcal{T}_1$ whose objects are those X such that $FX \cong 0$ in $\mathcal{T}_2$.

Proposition 1.1.17 ([Nee01b, Lemma 2.1.5])**.** *The kernel of an exact functor is a thick triangulated subcategory.*

1.1.3 Homological functors

Definition 1.1.18. Let $\mathcal{T}$ be a triangulated category. A functor from $\mathcal{T}$ to the category of abelian groups $H\colon \mathcal{T} \to \mathrm{Ab}$ is called *homological* if it sends triangles to long exact sequences, *i.e.* if for every triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

in $\mathcal{T}$ the sequence

$$\cdots \to H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z) \xrightarrow{H(w)} H(\Sigma X) \xrightarrow{H(\Sigma u)} H(\Sigma Y) \to \cdots$$

is exact in Ab. Dually, a functor $H\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ that takes triangles in $\mathcal{T}$ to long exact sequences is called *cohomological*.

The next proposition is one of the most important direct consequences of the axioms of a triangulated category.

Proposition 1.1.19 ([Nee01b, Lemma 1.1.10])**.** *Let $\mathcal{T}$ be a triangulated category.*

1. *For each object X, the functor $\mathcal{T}(X, -)\colon \mathcal{T} \to \mathrm{Ab}$ is homological and takes products in $\mathcal{T}$ to products in* Ab.

2. *For each object Y, the functor $\mathcal{T}(-,Y)\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ is cohomological and takes coproducts in $\mathcal{T}$ to products in* Ab*.*

The converse of the above proposition, stating that every homological functor taking products to products is representable, and similarly for cohomological functors, does not necessarily hold. When it does, we use the following terminology due to Neeman, although other authors name it differently.

Definition 1.1.20. Let $\mathcal{T}$ be a triangulated category satisfying [TR5].

1. We say that *Brown representability holds* for $\mathcal{T}$ if every cohomological functor $H\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{A}$ that sends coproducts in $\mathcal{T}$ to products in Ab is representable, *i.e.* there is a natural isomorphism $H(-) \cong \mathcal{T}(-,Y)$ for an object Y in $\mathcal{T}$.

2. We say that *the dual of Brown representability holds* for $\mathcal{T}$ if every homological functor $H\colon \mathcal{T} \to \mathcal{A}$ that sends products in $\mathcal{T}$ to products in Ab is representable, *i.e.* there is a natural isomorphism $H(-) \cong \mathcal{T}(X,-)$ for an object X in $\mathcal{T}$.

As we review in the following chapters, Brown representability is known to hold for a broad family of triangulated categories. It holds for well generated triangulated categories (Theorem 2.2.1). On the other hand, the dual of Brown representability is only known to hold under very restrictive hypotheses, although no counterexample has been given of a well generated triangulated category for which the dual of Brown representability fails to hold.

The following propositions are important consequences of Brown representability.

Proposition 1.1.21 ([Nee96, Theorem 4.1])**.** *Let $F\colon \mathcal{S} \to \mathcal{T}$ be an exact functor of triangulated categories that preserves coproducts. Assume that Brown representability holds for $\mathcal{S}$. Then F has a right adjoint.*

Proposition 1.1.22 ([Nee01b, Proposition 8.4.6])**.** *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5]*,* i.e. *having coproducts. Assume that Brown representability holds for $\mathcal{T}$. Then $\mathcal{T}$ satisfies* [TR5*]*,* i.e. *it has products.*

1.2 The stable homotopy category

The first representability results for a triangulated category were obtained for the homotopy category of spectra, also called *stable homotopy category*. We review these results in this section, since they were the motivating examples for other representability results. We mainly follow the book of Margolis [Mar83], but using simplicial sets instead of CW-complexes.

The notion of spectrum is central in homotopy theory. Spectra are the objects of the stable homotopy category, and can be introduced in many different ways. We can think of the stable homotopy category as a universal construction that inverts the suspension functor of spaces up to homotopy; see [Hov01] for details. There are different models for spectra yielding the same homotopy category. We will use simplicial spectra as defined by Bousfield and Friedlander [BF78]. Other important models are Adams CW-spectra [Ada74], S-modules [EKMM97] and symmetric spectra [HSS00].

In the whole section, when we say *space* we mean simplicial set. A good introduction to simplicial sets can be found in [Cur71]. We will denote by sSet_* the category of pointed simplicial sets. Recall that, given spaces X and Y, the *wedge sum* $X \vee Y$ is the quotient of the disjoint union $X \sqcup Y$ obtained by identifying the base points. The *smash product* $X \wedge Y$ is defined as $X \times Y / X \vee Y$. The *suspension functor* is defined as $\Sigma(-) = S^1 \wedge -$, where $S^1 = \Delta[1]/\partial\Delta[1]$ is the simplicial circle. It has a right adjoint $\Omega(-) = \mathrm{Hom}(S^1, -)$, where Hom denotes the mapping space in sSet_*.

Definition 1.2.1. A *spectrum* $\mathbb{X}$ is a sequence $X_0, X_1, \ldots, X_n, \ldots$ of pointed spaces together with a *structure map* $\sigma\colon \Sigma X_n \to X_{n+1}$ for every $n \geq 0$. A *map of spectra* $f\colon \mathbb{X} \to \mathbb{Y}$ is defined by a sequence of pointed maps $f_n\colon X_n \to Y_n$ such that the diagram

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\sigma"] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma Y_n \arrow[r, "\sigma"] & Y_{n+1} \end{tikzcd}$$

is commutative for every $n \geq 0$. We denote by Sp the category of spectra.

Given a pointed space X, we can consider the spectrum $\Sigma^\infty X$ given by the sequence of pointed spaces $X, \Sigma X, \ldots, \Sigma^n X, \ldots$ and identities as structure maps. The spectrum $\mathbb{S} = \Sigma^\infty S^0$ is called *sphere spectrum.* We define the homotopy groups of a spectrum $\mathbb{X}$ as $\pi_i(\mathbb{X}) = \operatorname{colim}_n \pi_{i+n}(X_n)$ for $i \in \mathbb{Z}$, where the colimit is that of the direct system

$$\ldots \longrightarrow \pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{\sigma_*} \pi_{i+n+1}(X_{n+1}) \longrightarrow \ldots$$

Definition 1.2.2. A map of spectra $f\colon \mathbb{X} \to \mathbb{Y}$ is a *stable weak equivalence* if $\pi_i(f)\colon \pi_i(\mathbb{X}) \to \pi_i(\mathbb{Y})$ is an isomorphism for every $i \in \mathbb{Z}$.

Bousfield and Friedlander [BF78] proved that there is a model category structure on the category of spectra with stable weak equivalences as weak equivalences. Hence, we can define the *homotopy category of spectra* Ho(Sp) by formally inverting stable weak equivalences. The set of homotopy classes of maps in this category is denoted by $[-,-]$.

An Ω*-spectrum* is a spectrum $\mathbb{X}$ such that the adjoints of the structure maps $X_n \to \Omega X_{n+1}$ are weak equivalences and X_n is fibrant for all $n \geq 0$.

Theorem 1.2.3 (Eilenberg and Steenrod [ES52])**.** *Let $\mathbb{E}$ be an Ω-spectrum. Then the functors*

$$\begin{aligned} H^n\colon\ \{\mathrm{sSet}_*\}^{\mathrm{op}} &\longrightarrow \mathrm{Ab} \\ X &\longmapsto [X, E_n] \end{aligned}$$

for every $n \in \mathbb{Z}$ form a reduced additive cohomology theory.

A definition of reduced additive cohomology and a proof of this theorem can be found in [ES52] or [Hat02, Section 3.1 and Theorem 4.58]. (For $n < 0$, define $E_n = \Omega^{-n} E_0$.)

The converse of this theorem was proved by Brown [Bro62] and is known as the *Brown Representability Theorem*. We will state it as Theorem 1.2.13 with a more convenient notation.

There is also a corresponding theorem for homology theories:

Theorem 1.2.4 (Whitehead [Whi62])**.** *Let $\mathbb{E}$ be a spectrum. Then the functors*

$$\begin{aligned} H_n\colon\ \mathrm{sSet}_* &\longrightarrow \mathrm{Ab} \\ X &\longmapsto \pi_n(X \wedge \mathbb{E}) \end{aligned}$$

for every $n \in \mathbb{Z}$ form a reduced additive homology theory, where $(X \wedge \mathbb{E})_k = X \wedge E_k$ for $k \geq 0$.

The definition of reduced additive homology and a proof of this theorem can be found in [Hat02, Section 2.3 and Theorem 4F.2].

The converse of this theorem was proved by Adams [Ada71] and is known as *Brown Representability for Homology*. We will state it as Theorem 1.2.16 with a more convenient notation.

1.2.1 Triangulated structure

It was proved by Puppe [Pup62] that the category Ho(Sp) has a triangulated category structure. We sketch in a series of lemmas how this can be done. Details can be found in [Ada74, Part III] or [Mar83, Ch. 2].

The category Ho(Sp) has an additive structure. The idea of the proof is simple. For any two spectra $\mathbb{X}$ and $\mathbb{Y}$, $[\Sigma\mathbb{X}, \mathbb{Y}]$ is a group and $[\Sigma^2\mathbb{X}, \mathbb{Y}]$ is an abelian group. The argument is the same that proves that the homotopy groups of a space $\pi_n(X)$ are abelian for $n \geq 2$ [Hat02, Sec. 4.1]. Since there is a bijection between $[\mathbb{X}, \mathbb{Y}]$ and $[\Sigma^2\mathbb{X}, \Sigma^2\mathbb{Y}]$, we have proved that $[\mathbb{X}, \mathbb{Y}]$ is an abelian group for all $\mathbb{X}$ and $\mathbb{Y}$ and it is easy to see that composition is bilinear.

We also have a functor $\Sigma\colon \mathrm{Ho(Sp)} \to \mathrm{Ho(Sp)}$ that is an equivalence of categories. This was constructed by Adams and then reformulated by Bousfield and Friedlander in the context of Quillen model categories [Hov01, Theorem 3.9].

Theorem 1.2.5 (Adams)**.** *The prolongation of the suspension functor to the category of spectra* $\Sigma\colon \mathrm{Sp} \longrightarrow \mathrm{Sp}$ *defined by* $(\Sigma\mathbb{X})_n = \Sigma(X_n)$ *induces an equivalence* $\Sigma\colon \mathrm{Ho(Sp)} \to \mathrm{Ho(Sp)}$.

It is worth mentioning that there is also a shift functor $s_+\colon \mathrm{Sp} \to \mathrm{Sp}$, where $(s_+\mathbb{X})_n = X_{n+1}$, that also induces an equivalence of categories in Ho(Sp) and is naturally isomorphic to Σ.

In order to see that every morphism in Ho(Sp) can be completed to a triangle, we use the mapping cone of spectra [Rud98, Ch. II, Definition 1.7]: A morphism $f\colon \mathbb{X} \to \mathbb{Y}$ in Ho(Sp) can always be represented by a map $f'\colon \mathbb{X}' \to \mathbb{Y}'$ in Sp such that $\mathbb{X}' \cong \mathbb{X}$ in Ho(Sp) with $\mathbb{X}'$ cofibrant in Sp, and $\mathbb{Y}' \cong \mathbb{Y}$ in Ho(Sp) with $\mathbb{Y}'$ fibrant in Sp. We define $\mathbb{C}(f')$, the *mapping cone of* f', as

$$C(f')_n = C(f'_n) = Y'_n \sqcup_{f'_n} C(X'_n),$$

where $C(X'_n) = X'_n \times I / X'_n \times \{0\}$ with structure maps

$$\Sigma C(f')_n = \Sigma Y'_n \sqcup_{\Sigma f'_n} C(\Sigma X'_n) \xrightarrow{\sigma_{Y'} \sqcup C(\sigma_{X'})} Y'_{n+1} \sqcup_{f'_{n+1}} C(X'_{n+1}) = C(f')_{n+1}.$$

This determines a triangle

$$\mathbb{X} \xrightarrow{f} \mathbb{Y} \longrightarrow \mathbb{C}(f') \longrightarrow \Sigma\mathbb{X},$$

unique up to natural isomorphism in Ho(Sp). This triangle is also called a *cofiber sequence of spectra*, or *Puppe sequence*, associated to f.

Definition 1.2.6. A sequence of maps $\mathbb{X} \to \mathbb{Y} \to \mathbb{Z} \to \Sigma\mathbb{X}$ in Ho(Sp) is defined to be a *triangle* if it is isomorphic to a cofiber sequence of spectra.

The fundamental property of cofiber sequences is the following.

Proposition 1.2.7 (Puppe [Ada74, Proposition 3.9 and 3.10]). *Let*

$$\mathbb{X} \xrightarrow{u} \mathbb{Y} \xrightarrow{v} \mathbb{Z} \xrightarrow{w} \Sigma\mathbb{X}$$

be a cofiber sequence in Ho(Sp)*. Then the sequences*

$$\ldots \longleftarrow [\Sigma^{-1}\mathbb{Z}, \mathbb{K}] \longleftarrow [\mathbb{X}, \mathbb{K}] \xleftarrow{u^*} [\mathbb{Y}, \mathbb{K}] \xleftarrow{v^*} [\mathbb{Z}, \mathbb{K}] \xleftarrow{w^*} [\Sigma\mathbb{X}, \mathbb{K}] \longleftarrow \ldots$$

and

$$\ldots \longrightarrow [\mathbb{K}, \Sigma^{-1}\mathbb{Z}] \longrightarrow [\mathbb{K}, \mathbb{X}] \xrightarrow{u_*} [\mathbb{K}, \mathbb{Y}] \xrightarrow{v_*} [\mathbb{K}, \mathbb{Z}] \xrightarrow{w_*} [\mathbb{K}, \Sigma\mathbb{X}] \longrightarrow \ldots$$

are exact for every spectrum $\mathbb{K}$*.*

Theorem 1.2.8 (Puppe [Pup62]). *The additive category* Ho(Sp) *together with the functor* Σ *and the class of triangles of Definition* 1.2.6 *satisfies the axioms of a triangulated category.*

The coproducts in Ho(Sp) can be constructed as levelwise wedge sums $(\mathbb{X} \vee \mathbb{Y})_n = X_n \vee Y_n$. With this definition, Ho(Sp) satisfies [TR5]. Even more, $\pi_n(\bigvee_{i \in I} X_i) \cong \bigoplus_{i \in I} \pi_n(X_i)$ for every index set I. The existence of arbitrary products will be a consequence of Brown representability; see [Ada74, Part III, Sec. 3], [Mar83, Ch. 2, Sec. 2] or [Wei94, Sec. 10.9] for details.

Remark 1.2.9. The triangulated category Ho(Sp) is generated by $\{\Sigma^i\mathbb{S}\}_{i \in \mathbb{Z}}$, in the sense that an object $\mathbb{X}$ in Ho(Sp) is zero if $[\Sigma^i\mathbb{S}, \mathbb{X}] = 0$ for every $i \in \mathbb{Z}$. The notion of generating set in a general triangulated category (Definition 2.1.1) will be extensively studied in Chapter 2.

Adams [Ada74, Part III, Sec. 4] defined a symmetric monoidal product $\wedge$ in Ho(Sp) called *smash product*. This is a bifunctor

$$- \wedge -\colon\ \mathrm{Ho(Sp)} \times \mathrm{Ho(Sp)} \longrightarrow \mathrm{Ho(Sp)}$$

that satisfies associative, commutative and identity laws and such that $- \wedge \mathbb{X}$ has a right adjoint $\mathrm{Hom}(\mathbb{X}, -)$ for every spectrum $\mathbb{X}$. The category Ho(Sp) together with the smash product is a (closed) symmetric monoidal category. However, this smash product does not come from a symmetric monoidal structure on Sp, because the coherence diagrams only commute up to homotopy. This was the motivation of [EKMM97] and [HSS00] to introduce the categories of *S-modules* and *symmetric spectra*, which have a model category structure with a strict smash product whose homotopy category is equivalent to Ho(Sp) as closed symmetric monoidal categories.

1.2.2 The category of finite spectra

The *Spanier–Whitehead category* SW_f is the category whose objects are pairs (X, n) where X is a pointed simplicial set with only finitely many non-degenerate simplices, $n \in \mathbb{Z}$, and, for every pair of objects (X, n) and (Y, m), the set of morphisms from (X, n) to (Y, m) is defined as $\{(X, n), (Y, m)\} = \operatorname{colim}_i [\Sigma^{n+i}X, \Sigma^{m+i}Y]$.

There is a functor $\Sigma^\infty \colon \mathrm{SW}_f \longrightarrow \mathrm{Ho}(\mathrm{Sp})$ that sends (X, n) to the spectrum $\mathbb{X}$ defined by $X_i = \Sigma^{i-n}X$ for $i \geq n$ and $X_i = *$ for $0 \leq i < n$ and with structure maps $\sigma_i \colon \Sigma(\Sigma^{i-n}X) = \Sigma^{i-n+1}X$ for $i \leq n$, $\sigma_{n-1} \colon * \to X$ and $\sigma_i \colon * \to *$ for $0 \leq i < n - 1$.

Definition 1.2.10. The *homotopy category of finite spectra* $\mathrm{Ho}(\mathrm{Sp}_f)$ is the full subcategory of $\mathrm{Ho}(\mathrm{Sp})$ whose class of objects is given by the essential image of SW_f under the functor Σ^∞. The objects in $\mathrm{Ho}(\mathrm{Sp}_f)$ are called *finite spectra* or *compact spectra*.

It is not difficult to see that $\mathrm{Ho}(\mathrm{Sp}_f)$ is a thick subcategory; see [Fre66, Corollary 5.2] for details. The following result says that the stable homotopy category is generated by $\mathrm{Ho}(\mathrm{Sp}_f)$ in the sense of Definition 2.1.1.

Proposition 1.2.11 ([Mar83, p. 24])**.** *The category* $\mathrm{Ho}(\mathrm{Sp})$ *is the smallest triangulated subcategory of* $\mathrm{Ho}(\mathrm{Sp})$ *containing* $\mathrm{Ho}(\mathrm{Sp}_f)$ *and closed by coproducts.*

One of the most important properties of finite spectra is that they satisfy the so-called *Spanier–Whitehead duality*, or *S-duality.*

Theorem 1.2.12 (Spanier and Whitehead [Mar83, Ch. 1, Theorem 11])**.** *There exists a functor* $D \colon \mathrm{Ho}(\mathrm{Sp}_f) \to \mathrm{Ho}(\mathrm{Sp}_f)^{\mathrm{op}}$*, unique up to natural equivalence, which is an equivalence of categories and such that* $D^2 \cong Id$*. Furthermore, there is a natural isomorphism* $[A \wedge B, C] \cong [A, DB \wedge C]$ *for all* A*,* B *and* C *in* $\mathrm{Ho}(\mathrm{Sp}_f)$*.*

1.2.3 Representability results

In this section we state the main representability results for $\mathrm{Ho}(\mathrm{Sp})$ in the language of triangulated categories.

Brown proved in [Bro62, Theorem II] that every additive generalized cohomology theory can be represented by an Ω-spectrum. We will state this theorem as in the book of Margolis [Mar83], since it takes advantage of the fact that $\mathrm{Ho}(\mathrm{Sp})$ is triangulated. For instance, the Mayer–Vietoris hypothesis in the original article by Brown is transformed into the assumption of being cohomological.

Theorem 1.2.13 (Brown [Mar83, Ch. 4, Theorem 11])**.** *For every additive functor* $H\colon \mathrm{Ho}(\mathrm{Sp})^{\mathrm{op}} \to \mathrm{Ab}$ *that sends coproducts in* $\mathrm{Ho}(\mathrm{Sp})$ *to products and sends triangles to long exact sequences, there is a natural isomorphism* $H(-) \cong [-, \mathbb{H}]$ *where* $\mathbb{H}$ *is a spectrum.*

A *stable natural transformation* of cohomology theories $\tau\colon H \to H'$ is a natural transformation such that the diagram

$$\begin{tikzcd} H(\Sigma\mathbb{X}) \arrow[r, "\tau_{\Sigma\mathbb{X}}"] \arrow[d, "\cong"] & H'(\Sigma\mathbb{X}) \arrow[d, "\cong"] \\ \Sigma H(\mathbb{X}) \arrow[r, "\Sigma\tau_{\mathbb{X}}"] & \Sigma H'(\mathbb{X}) \end{tikzcd}$$

commutes for every spectrum $\mathbb{X}$, where ΣH denotes the composite of H with $\Sigma\colon \mathrm{Ho}(\mathrm{Sp})^{\mathrm{op}} \to \mathrm{Ho}(\mathrm{Sp})^{\mathrm{op}}$. If H and H' are represented, respectively, by spectra $\mathbb{Y}$ and $\mathbb{Z}$, then it is a direct consequence of the Yoneda Lemma that there is a bijection between stable natural transformations from H into H' and $[\mathbb{Y}, \mathbb{Z}]$.

The following is a direct consequence of Proposition 1.1.22.

Corollary 1.2.14. *The triangulated category* $\mathrm{Ho}(\mathrm{Sp})$ *satisfies* [TR5*], i.e. *it has products.*

We want to point out that infinite products are not computed objectwise in general.

There is another representability result in Brown's paper [Bro62, Theorem II]. It was substantially improved by Adams in [Ada71, Theorem 1.6] in order to have representability results for cohomology theories that were only constructed for finite spectra, such as K-theory and some cohomology theories derived from Spanier–Whitehead duality (Theorem 1.2.12).

Theorem 1.2.15 (Adams [Mar83, Ch. 4, Theorem 13])**.** *For every additive functor* $H\colon \mathrm{Ho}(\mathrm{Sp}_f)^{\mathrm{op}} \to \mathrm{Ab}$ *that sends triangles to long exact sequences, there is a natural equivalence* $H(-) \cong [-, \mathbb{H}]|_{\mathrm{Ho}(\mathrm{Sp}_f)}$ *where* $\mathbb{H}$ *is in* $\mathrm{Ho}(\mathrm{Sp})$. *Furthermore, for any stable natural transformation between two cohomology theories* $\phi\colon H \to K$ *there is a map of spectra* f *such that* $\phi \cong [-, f]|_{\mathrm{Ho}(\mathrm{Sp}_f)}$.

As before, representability holds for stable natural transformations between cohomology theories. In this case, we stated it as part of the theorem because it does not follow from the Yoneda Lemma. In fact, there is no bijection between maps in $\mathrm{Ho}(\mathrm{Sp})$ and natural transformations of cohomology theories from $\mathrm{Ho}(\mathrm{Sp}_f)$, because there are nontrivial maps between spectra that induce the zero natural transformation between the corresponding cohomology theories. These are called *phantom maps*. After passing to the quotient of the homotopy classes of maps between spectra by the phantom maps, we do obtain a bijection; see [Mar83, Ch. 4, Theorem 15] for details.

One of the main interests of Adams' theorem is the next representability result for homology theories [Ada71, Theorem 1.9].

Theorem 1.2.16 (Adams [Mar83, Ch. 4, Theorem 16])**.** *For any additive functor* $H\colon \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ab}$ *that sends coproducts in* $\mathrm{Ho}(\mathrm{Sp})$ *to coproducts and sends triangles to long exact sequences, there is a natural isomorphism* $H(-) \cong [\mathbb{S}, - \wedge \mathbb{H}]$*, where* $\mathbb{H}$ *is a spectrum.*

This theorem is an immediate consequence of Theorem 1.2.15 together with Spanier–Whitehead duality (Theorem 1.2.12) and the following result.

Proposition 1.2.17 (Margolis [Mar83, Ch. 4, Theorem 6])**.** *For every spectrum* $\mathbb{X}$ *there exists a small subcategory* $\Lambda(\mathbb{X}) \subset \mathrm{Ho}(\mathrm{Sp}_f)$ *such that, for any additive functor* $H\colon \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ab}$ *that sends coproducts in* $\mathrm{Ho}(\mathrm{Sp})$ *to coproducts and sends triangles to long exact sequences, there is an isomorphism* $H(\mathbb{X}) \cong \mathrm{colim}_{\Lambda(\mathbb{X})} H(\mathbb{X}_\lambda)$ *where* $\mathbb{X}_\lambda$ *runs over all objects of* $\Lambda(\mathbb{X})$*, and* Λ *can be chosen functorially.*

Proof of Theorem 1.2.16. The theorem follows from the following composition of isomorphisms given by Theorem 1.2.15, Theorem 1.2.12 and Proposition 1.2.17:

$$\begin{aligned} H(\mathbb{X}) &\cong \mathrm{colim}_{\Lambda(\mathbb{X})} H(\mathbb{X}_\lambda) \cong \mathrm{colim}_{\Lambda(\mathbb{X})} H(D^2\mathbb{X}_\lambda) \\ &\cong \mathrm{colim}_{\Lambda(\mathbb{X})} (H \circ D) D\mathbb{X}_\lambda \cong \mathrm{colim}_{\Lambda(\mathbb{X})} [D\mathbb{X}_\lambda, \mathbb{D}] \\ &\cong \mathrm{colim}_{\Lambda(\mathbb{X})} [\mathbb{S}, \mathbb{X}_\lambda \wedge \mathbb{D}] \cong [\mathbb{S}, \mathbb{X} \wedge \mathbb{D}] \end{aligned}$$

where $\mathbb{D}$ is a spectrum such that $[-, \mathbb{D}]$ is naturally isomorphic to the cohomology theory $H \circ D\colon \mathrm{Ho}(\mathrm{Sp}_f)^{\mathrm{op}} \to \mathrm{Ab}$, that exists by Theorem 1.2.15. □

Remark 1.2.18. By [Boa70], the opposite of the stable homotopy category has no non-zero compact objects. Hence, we cannot dualize the original proof of the Brown Representability Theorem. For this reason, the dual of Brown representability was unsuspected until Neeman proved it, 36 years after Brown's theorem.

Theorem 1.2.19 (Neeman [Nee98a, Theorem 2.1])**.** *For every functor*

$$H\colon \mathrm{Ho}(\mathrm{Sp}) \longrightarrow \mathrm{Ab}$$

that sends products in $\mathrm{Ho}(\mathrm{Sp})$ *to products and sends triangles to long exact sequences, there is a natural isomorphism* $H(-) \cong [\mathbb{H}, -]$*, for some spectrum* $\mathbb{H}$*.*

1.3 The derived category of an abelian category

Derived categories were introduced by Grothendieck and his collaborators in the decade of 1960. The derived category of an abelian category $\mathcal{A}$ is a triangulated category that has as objects all (cochain) complexes of objects of $\mathcal{A}$ and its morphisms are obtained from morphisms of complexes by formally inverting all quasi-isomorphisms. We will summarize the theory of derived categories following [Kel96], [Kra07] and [Wei94].

1.3.1 The homotopy category of an additive category

Definition 1.3.1. Let $\mathcal{A}$ be an additive category. A *(cochain) complex* in $\mathcal{A}$ is a sequence of morphisms

$$\dots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \dots$$

such that $d_X^n \circ d_X^{n-1} = 0$ for all $n \in \mathbb{Z}$. The morphisms d_X^n are called *differentials* and we will omit the subscript if it is clear from the context.

A *morphism* $f\colon X \longrightarrow Y$ of complexes is a sequence of morphisms $f^n\colon X^n \longrightarrow Y^n$ such that the diagram

$$\begin{tikzcd}
\dots \arrow[r] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"'] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \dots \\
\dots \arrow[r] & Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

commutes for every $n \in \mathbb{Z}$. We denote by $\mathrm{Ch}(\mathcal{A})$ the *category of complexes* of $\mathcal{A}$.

A *homotopy* ϕ from $f\colon X \to Y$ to $g\colon X \to Y$ is a sequence of morphisms $\phi^n\colon X^n \to Y^{n-1}$ such that $d_Y^{n-1} \circ \phi^n + \phi^{n+1} \circ d_X^n = g^n - f^n$ for all n.

The objects in $\mathcal{A}$ are identified with the complexes concentrated in degree zero

$$\dots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \dots$$

This defines an additive functor $\mathcal{A} \hookrightarrow \mathrm{Ch}(\mathcal{A})$. If $\mathcal{A}$ is abelian, the category of complexes $\mathrm{Ch}(\mathcal{A})$ inherits the abelian structure of $\mathcal{A}$ [Wei94, Theorem 1.2.3].

A morphism of complexes f is said to be *null-homotopic* if it is homotopic to the zero morphism. The class of null-homotopic morphisms forms an ideal $\mathcal{I}$ in the sense of [Mit72, p. 18], *i.e.* for every pair of objects X and Y the subgroup $\mathcal{I}(X,Y) \subset \mathrm{Ch}(\mathcal{A})(X,Y)$ of null-homotopic maps from X into Y is such that a composition $f \circ g$ belongs to $\mathcal{I}$ if f or g belongs to $\mathcal{I}$.

Definition 1.3.2. Let $\mathcal{A}$ be an additive category. The *homotopy category* $\mathrm{K}(\mathcal{A})$ of $\mathcal{A}$ is defined as the quotient of $\mathrm{Ch}(\mathcal{A})$ with respect to $\mathcal{I}$, *i.e.* $\mathrm{K}(\mathcal{A})$ has the same objects as $\mathcal{A}$ and for every pair of objects X and Y we define $\mathrm{K}(\mathcal{A})(X,Y) = \mathrm{Ch}(\mathcal{A})(X,Y)/\mathcal{I}(X,Y)$.

From the fact that $\mathcal{I}$ is an ideal, it can be seen that $\mathrm{K}(\mathcal{A})$ is well defined as a category [Wei94, Sec. 1.4].

Definition 1.3.3. Let $\mathcal{A}$ be an additive category. The *shift functor*

$$\Sigma\colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$$

is defined by $(\Sigma X)^n = X^{n+1}$ with differentials $d^n_{\Sigma X} = -d^{n+1}_X$.

In the category of complexes there is a notion of *mapping complex*.

Definition 1.3.4. Let $\mathcal{A}$ be an additive category and let X and Y be a pair of complexes in $\mathrm{Ch}(\mathcal{A})$. We define the *mapping complex* $\mathrm{Hom}(X,Y)$ as the complex of abelian groups defined as follows. For every $n \in \mathbb{Z}$,

$$\mathrm{Hom}(X,Y)^n = \prod_{p\in\mathbb{Z}} \mathcal{A}(X^p, Y^{p+n})$$

and the differential is given by

$$d^n_{\mathrm{Hom}(X,Y)}((f_p)_{p\in\mathbb{Z}}) = (d^{p+n}_Y \circ f_p - (-1)^n f_{p+1} \circ d^p_X)_{p\in\mathbb{Z}}$$

for every $(f_p)_{p\in\mathbb{Z}}$ in $\mathrm{Hom}(X,Y)^n$.

Notice that $H^n(\mathrm{Hom}(X,Y)) \cong \mathrm{K}(\mathcal{A})(X,\Sigma^n Y)$, because $\ker(d^n_{\mathrm{Hom}(X,Y)})$ can be identified with $\mathrm{Ch}(\mathcal{A})(X,\Sigma^n Y)$ and $\mathrm{im}(d^{n-1}_{\mathrm{Hom}(X,Y)})$ is then identified with the ideal of null-homotopic maps $X \to \Sigma^n Y$; see [Kra07] for details.

Definition 1.3.5. Let $\mathcal{A}$ be an additive category and let $f\colon X \to Y$ be a morphism in $\mathrm{Ch}(\mathcal{A})$. We define the *mapping cone* of f as the complex C_f such that $(C_f)^n = X^{n+1} \oplus Y^n$ with differentials

$$X^{n+1} \oplus Y^n \xrightarrow{\left(\begin{smallmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{smallmatrix}\right)} X^{n+2} \oplus Y^{n+1}.$$

The mapping cone fits into a sequence of morphisms of complexes

$$X \to Y \to C_f \to \Sigma X,$$

where the morphisms in the sequence are given by

$$X^n \xrightarrow{f^n} Y^n \xrightarrow{\left(\begin{smallmatrix} 0 \\ id \end{smallmatrix}\right)} X^{n+1} \oplus Y^n \xrightarrow{(id \;\; 0)} X^{n+1}.$$

A sequence of morphisms in $\mathrm{K}(\mathcal{A})$ that is isomorphic to a sequence as above is called a *triangle of complexes*.

A proof of the following theorem can be found in [Wei94, Proposition 10.2.4].

Theorem 1.3.6 (Verdier [Ver96]). *Let $\mathcal{A}$ be an additive category. Then* $\mathrm{K}(\mathcal{A})$ *is a triangulated category with the shift functor* Σ *and the triangles of complexes.*

1.3.2 The derived category

The derived category is obtained from the homotopy category by formally inverting a certain class of morphisms. We explain this in detail.

Definition 1.3.7. Let $\mathcal{A}$ be an abelian category. The *cohomology* of a complex X in $\mathcal{A}$ is defined as $H^n(X) = \ker(d_X^n)/\mathrm{im}(d_X^{n-1})$ for every $n \in \mathbb{Z}$. It defines an additive functor $H^n \colon \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ab}$ for every $n \in \mathbb{Z}$.

Note that if two morphisms of complexes f and g are homotopic then $H^n(f) = H^n(g)$ for every $n \in \mathbb{Z}$ [Wei94, Lemma 10.1.1]. This says that cohomology of complexes defines a functor on $\mathrm{K}(\mathcal{A})$.

Definition 1.3.8. Let $\mathcal{A}$ be an abelian category. A morphism f in $\mathrm{Ch}(\mathcal{A})$ is said to be a *quasi-isomorphism* if $H^n(f)$ is an isomorphism for every $n \in \mathbb{Z}$.

The definition of a derived category is done by formally inverting the class of quasi-isomorphisms. This can be done using *calculus of fractions.* For a detailed explanation, we refer to [GZ67], [Kra07] or [Wei94, Sec. 10.3]. This construction can fail to produce a category, because there could be a proper class of morphisms between two objects after inverting a proper class of maps. However, we will see that, in many of the cases that we are interested in, this difficulty does not occur. Specifically, we need that the class of quasi-isomorphisms be *locally small*, in the sense defined below. We also want to notice that it is possible to work with categories with proper classes of morphisms between two objects by introducing the formalism of universes; see [DHKS04, Section VI.3.2] for details.

Definition 1.3.9. Let $\mathcal{C}$ be a category. A *multiplicative system* of morphisms in $\mathcal{C}$ is a class of morphisms S satisfying the following conditions.

1. S is closed under composition and contains all identity morphisms.

2. Let $f \colon X \to Y$ be a morphism in S. Then every pair of maps $g \colon Y' \to Y$ and $h \colon X \to X''$ in $\mathcal{C}$ can be completed to a pair of commutative

diagrams

$$\begin{tikzcd} X' \arrow[r, dotted] \arrow[d, dotted, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & X'' \arrow[d, dotted, "f''"] \\ Y \arrow[r, dotted] & Y'' \end{tikzcd}$$

such that f' and f'' are in S.

3. Let f, $g\colon X \to Y$ be a pair of morphisms in $\mathcal{C}$. Then there is a morphism $\alpha\colon X' \to X$ in S such that $f \circ \alpha = g \circ \alpha$ if and only if there is a morphism $\beta\colon Y \to Y'$ in S such that $\beta \circ f = \beta \circ g$.

A class S of morphisms in $\mathcal{C}$ is said to be *locally small* if for every object X in $\mathcal{C}$ there is a set of morphisms $S_X = \{X_i \to X\}_{i \in I}$ in S such that for every morphism $f\colon Y \to X$ in S there is a morphism $g\colon X_i \to Y$ in $\mathcal{C}$ for some $i \in I$ such that $f \circ g \in S_X$.

Theorem 1.3.10 (Gabriel and Zisman [GZ67])**.** *Let $\mathcal{C}$ be a category and let S be a locally small multiplicative system in $\mathcal{C}$. Then there exists a category $\mathcal{C}[S^{-1}]$ and a functor $\pi\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that $\pi(f)$ is an isomorphism for every $f \in S$ and π is universal with this property.*

Definition 1.3.11. Let $\mathcal{A}$ be an abelian category. The *derived category* $\mathrm{D}(\mathcal{A}) = \mathrm{K}(\mathcal{A})[Q^{-1}]$ of $\mathcal{A}$ is the localization of $\mathrm{K}(\mathcal{A})$ with respect to the multiplicative system defined by the class Q of quasi-isomorphisms. If the class of quasi-isomorphisms is locally small, then $\mathrm{D}(\mathcal{A})$ is a category with small hom-sets.

This construction will be reformulated in Section 1.4 as Example 1.4.2. In fact, this definition is what motivates Section 1.4, where we study localizations of triangulated categories.

Remark 1.3.12. Observe that we could have defined localization with respect to quasi-isomorphisms directly in the category $\mathrm{Ch}(\mathcal{A})$ and we would have obtained the same derived category; see [Kra07] for details. However, in many cases, the class of quasi-isomorphisms is locally small in $\mathrm{K}(\mathcal{A})$ but not in $\mathrm{Ch}(\mathcal{A})$. For instance, for $\mathcal{A}$ the category of modules over a ring or a category of sheaves over a scheme, localization is constructed using calculus of fractions in $\mathrm{K}(\mathcal{A})$.

Definition 1.3.13. An abelian category is *well powered* if the class of subobjects of every object is a set.

Theorem 1.3.14 (Gabber [Wei94, Sec. 10.4])**.** *Let $\mathcal{A}$ be a well powered abelian category satisfying* [AB5] *and with a set of generators. Then* $\mathrm{D}(\mathcal{A})$ *is a category with small hom-sets.*

The hypotheses of the theorem can be simplified due to the following fact: A category with a generator is well powered [Fre64, Proposition 3.35].

The following examples are particular cases of Theorem 1.3.14.

Example 1.3.15. Let R be a ring with identity and let R-Mod be the category of left R-modules. We denote $\mathrm{D}(R) = \mathrm{D}(R\text{-Mod})$.

Example 1.3.16. Let X be a topological space. Let $\mathrm{PSh}(X)$ be the category of presheaves of abelian groups on X, and let $\mathrm{Sh}(X)$ be the category of sheaves of abelian groups on X. Their derived categories are denoted by $\mathrm{D}(\mathrm{PSh}/X)$ and $\mathrm{D}(\mathrm{Sh}/X)$, respectively.

Now that we know that in many interesting cases the derived category is well defined, we see that it has a triangulated category structure.

Theorem 1.3.17 (Verdier [Ver96])**.** *Let $\mathcal{A}$ be an abelian category. Assume that $\mathrm{D}(\mathcal{A})$ is a category with small hom-sets. Then $\mathrm{D}(\mathcal{A})$ has a triangulated structure with the shift functor Σ and the triangles of complexes.*

Remark 1.3.18. The construction of the derived category of an abelian category was generalized to any *exact category* by Neeman [Nee90]. In particular, the homotopy category and the derived category construction can be applied to the full subcategories of free, projective, flat, injective or finitely generated modules, among others, and also to the full subcategory of coherent, quasi-coherent or perfect sheaves over a scheme. This is studied in detail in [Kel96].

Remark 1.3.19. Instead of considering all complexes, we could have considered the full subcategories of $\mathrm{Ch}(\mathcal{A})$ whose objects are the following ones:

$$\begin{aligned}
\mathrm{Ch}^-(\mathcal{A}) &= \{X \text{ in } \mathrm{Ch}(\mathcal{A}) \mid X^n = 0 \text{ for } n \gg 0\} \\
\mathrm{Ch}^+(\mathcal{A}) &= \{X \text{ in } \mathrm{Ch}(\mathcal{A}) \mid X^n = 0 \text{ for } n \ll 0\} \\
\mathrm{Ch}^b(\mathcal{A}) &= \{X \text{ in } \mathrm{Ch}(\mathcal{A}) \mid X^n = 0 \text{ for } |n| \gg 0\}.
\end{aligned}$$

These are called, respectively, the *bounded above*, *bounded below*, and *bounded* complexes. The corresponding homotopy categories are denoted by $\mathrm{K}^-(\mathcal{A})$, $\mathrm{K}^+(\mathcal{A})$, and $\mathrm{K}^b(\mathcal{A})$, and the corresponding derived categories obtained by localizing each homotopy category with respect to quasi-isomorphisms are denoted by $\mathrm{D}^-(\mathcal{A})$, $\mathrm{D}^+(\mathcal{A})$, and $\mathrm{D}^b(\mathcal{A})$.

The bounded below derived category of quasi-coherent sheaves over a noetherian scheme $\mathrm{D}^+(qc/X)$ is of historical importance, since Grothendieck needed it to formulate his generalization of Serre's duality. His student J.-L. Verdier worked out the construction in [Ver96] by formalizing the notion of triangulated category.

1.4 Localization of triangulated categories

The most important tool to construct new triangulated categories out of old ones is *localization*. It consists of formally inverting some of the morphisms of the original triangulated categories. However, in order to obtain well defined triangulated categories, the morphisms we want to invert have to be part of a localizing subcategory (Definition 1.1.13). We will review the basic aspects of localization theory, essentially due to Verdier [Ver77]. We follow [Kra08] and [Nee01b, Chapter 2].

Given a triangulated subcategory $\mathcal{S} \subset \mathcal{T}$, the construction of the Verdier quotient $\mathcal{T}/\mathcal{S}$ is done by localizing with respect to a multiplicative system $Mor_{\mathcal{S}}$ defined by the class of morphisms whose mapping cones belong to $\mathcal{S}$.

Theorem 1.4.1 (Verdier [Nee01b, Theorem 2.1.8]). *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory of $\mathcal{T}$. If the class of morphisms $Mor_{\mathcal{S}}$ is locally small, then there is an exact functor $F\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ with $\mathcal{T}/\mathcal{S}$ a triangulated category which is initial among all exact functors $G\colon \mathcal{T} \to \mathcal{D}$ which send all morphisms in $Mor_{\mathcal{S}}$ to invertible morphisms.*

The triangulated category $\mathcal{T}/\mathcal{S}$ is called the *Verdier quotient* of $\mathcal{T}$ by $\mathcal{S}$.

The conclusion of the theorem can be rephrased in terms of the kernel (Definition 1.1.16) as "there exists an exact functor $F\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ with $\mathcal{T}/\mathcal{S}$ a triangulated category such that $\ker(F)$ is the smallest thick subcategory containing $\mathcal{S}$"; see [Nee01b, Section 2.1] for details. We notice that if $\mathcal{S} \subset \mathcal{T}$ is a thick subcategory, then $\ker(F) = \mathcal{S}$ and F is initial among all functors $\mathcal{T} \to \mathcal{D}$ that send morphisms $f\colon X \to Y$ that can be completed to a triangle $X \to Y \to Z \to \Sigma X$ with Z in $\mathcal{S}$ to invertible morphisms.

Example 1.4.2. Let $\mathcal{A}$ be an abelian category. We can view $\mathrm{D}(\mathcal{A})$ as the Verdier quotient with respect to the thick subcategory of acyclic complexes $\mathcal{Q} \subset \mathrm{K}(\mathcal{A})$, *i.e.* $X \in \mathcal{Q}$ if and only if $H^n(X) = 0$ for every $n \in \mathbb{Z}$.

Proposition 1.4.3 ([Nee01b, Corollary 3.2.11]). *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5]*. Let $\mathcal{S} \subset \mathcal{T}$ be a localizing subcategory. Then $\mathcal{T}/\mathcal{S}$ satisfies* [TR5] *and the localization functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ preserves coproducts.*

Observe that the converse also holds, *i.e.* the kernel of a functor that preserves coproducts is localizing.

The fact that Verdier quotients may produce a category with a proper class of morphisms between two objects can be dealt with by assuming some extra hypotheses on the subcategory. The origin of the following theory goes back to Bousfield in the context of model categories, but the translation into triangulated categories is due to Neeman; see [Nee01b, Chapter 9] or

[Kra08]. We will use the original notation from homotopy theory. We will call *left Bousfield localization* what Neeman calls just Bousfield localization, and *right Bousfield localization* what Neeman calls Bousfield colocalization.

Definition 1.4.4. Let $\mathcal{T}$ be a triangulated category and let $\mathcal{S}$ be a thick subcategory of $\mathcal{T}$. We say that the localization functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ is a *left Bousfield localization* if it has a right adjoint.

Theorem 1.4.5 ([Nee01b, Theorem 9.1.16])**.** *Let $\mathcal{T}$ be a triangulated category and let $\mathcal{S}$ be a thick subcategory of $\mathcal{T}$. Let $\pi\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ be a left Bousfield localization. Then π induces an equivalence of categories between $\mathcal{T}/\mathcal{S}$ and the category of $\mathcal{S}$-*local objects, i.e. *the full subcategory of $\mathcal{T}$ whose objects x are such that $\mathcal{T}(s, x) = 0$ for every object s in $\mathcal{S}$.*

The following is a direct consequence of the definition of left Bousfield localization.

Corollary 1.4.6 ([Nee01b, Lemma 9.1.7])**.** *Suppose that $\pi\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ is a left Bousfield localization. Let $G\colon \mathcal{T}/\mathcal{S} \to \mathcal{T}$ be right adjoint to π. Then the unit of the adjunction $\eta\colon Id \to G \circ \pi$ is a natural isomorphism, or, equivalently, π is a fully faithful functor.*

Corollary 1.4.7 ([Nee01b, Example 8.4.5])**.** *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5] *and let $\mathcal{S} \subset \mathcal{T}$ be a localizing subcategory. Assume that Brown representability holds for $\mathcal{T}$ and that $\mathcal{T}/\mathcal{S}$ exists. Then the localization functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ is a left Bousfield localization.*

The following proposition will also be very useful and makes no use of Brown representability.

Proposition 1.4.8 ([Nee01b, Proposition 9.1.18])**.** *Let $\mathcal{T}$ be a triangulated category and let $\mathcal{S}$ be a thick triangulated subcategory of $\mathcal{T}$. The inclusion functor $\mathcal{S} \to \mathcal{T}$ has a right adjoint if and only if the localization functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ has a right adjoint,* i.e. *if it is a left Bousfield localization.*

The next corollary will simplify the use of localizations when working with well generated triangulated categories. It is, somehow, an improvement of Corollary 1.4.7, since we make no assumptions about the resulting quotient. Instead, we have to make an extra assumption on the localizing subcategory.

Corollary 1.4.9 ([Nee01b, Proposition 9.1.19])**.** *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5] *and let $\mathcal{S}$ be a localizing subcategory of $\mathcal{T}$. Assume that Brown representability holds for $\mathcal{S}$. Then the localization functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ is a left Bousfield localization.*

Proof. Since $\mathcal{S}$ is localizing, the inclusion functor $\mathcal{S} \to \mathcal{T}$ preserves coproducts and, by Proposition 1.1.21, it has a right adjoint, as a consequence of Brown representability. Finally, by Proposition 1.4.8, $\mathcal{T}/\mathcal{S}$ is a left Bousfield localization. $\square$

Definition 1.4.10. Let $\mathcal{T}$ be a triangulated category and let $\mathcal{S}$ be a thick subcategory of $\mathcal{T}$. We say that the localization functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ is a *right Bousfield localization* if it has a left adjoint.

If the localization functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ is both a left and right Bousfield localization, then, by the previous results, we can draw all pairs of adjoint functors in a diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r, shift left=2] & \mathcal{T} \arrow[l, shift left=4] \arrow[l, shift right=2] \arrow[r, shift left=2] & \mathcal{T}/\mathcal{S}. \arrow[l, shift left=4] \arrow[l, shift right=2] \end{tikzcd}$$

This situation is called the *six gluing functors* in [Nee01b, Definition 9.2.1] and it is also known under the name of *recollement*. It has been widely studied in algebraic geometry. For a treatment in general triangulated categories, see [Hei07] and [Kra08].

When we assume that Brown representability holds, we have the following interesting proposition, which was explained to us by Neeman.

Proposition 1.4.11. *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5] *and let $\mathcal{S} \subset \mathcal{T}$ be a localizing subcategory such that $\pi\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ is a right Bousfield localization. Assume that $\mathcal{T}$ satisfies the dual of Brown representability. Then $\mathcal{T}/\mathcal{S}$ satisfies the dual of Brown representability.*

Proof. Fix a homological functor $H\colon \mathcal{T}/\mathcal{S} \to \mathrm{Ab}$ that preserves products. By Proposition 1.1.22, $\mathcal{T}$ has products and, since the functor $\pi\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ is a right adjoint, it preserves products (it also preserves coproducts by Proposition 1.4.3). Then the composition $H \circ \pi$ preserves products and is homological. Since we are assuming that $\mathcal{T}$ satisfies the dual of Brown representability, we have a natural isomorphism $H \circ \pi \cong \mathcal{T}(t, -)$ for an object t in $\mathcal{T}$. Finally, denote by G the left adjoint to the localization π. Then we have $H(-) \cong (H \circ \pi \circ G)(-) \cong \mathcal{T}(t, G(-)) \cong (\mathcal{T}/\mathcal{S})(\pi(t), -)$. $\square$

Chapter 2

Well generated triangulated categories

Well generated triangulated categories were introduced by Neeman as a family of triangulated categories for which his proof of the Brown Representability Theorem could be carried out. Compactly generated triangulated categories, such as the stable homotopy category and all derived categories of rings, are, in particular, well generated. As we will see, many interesting triangulated categories are well generated due to the fact that every Verdier quotient of a compactly generated triangulated category is well generated. The converse of this fact has been proved independently by Heider [Hei07] and Porta [Por07] for certain families of well generated triangulated categories.

In this chapter, we review the definition of a well generated triangulated category and its basic properties. In particular, we state that Brown representability holds for every well generated triangulated category, as proved by Neeman, and some results about the dual of Brown representability, which is not known to hold for every well generated triangulated category. We also introduce some of the basic concepts that will be of central interest in the rest of the thesis, such as α-Adams representability and α-Grothendieck categories, that are generalizations of Adams representability and Grothendieck categories to cardinals α bigger than $\aleph_0$.

This chapter also contains original results. In Section 2.3.1 we prove some properties of the abelian category of contravariant functors from an additive category with coproducts of less than α objects into Ab that generalize results of Neeman [Nee01b] in the context of triangulated categories. In Section 2.3.2 we review the formalism of Rosický functors [Nee09] and study the relation with Adams representability. Finally, in the last section we give results about the cardinality of the category of α-compact objects that will be very useful to provide examples of categories satisfying $\aleph_1$-Adams representability for objects.

2.1 Well generated triangulated categories

A well generated triangulated category is a triangulated category generated by a set of α-compact objects where α is a regular cardinal. Recall that a cardinal α is called *regular* if no set of cardinality α is the union of less than α subsets of cardinality less than α. We begin by recalling the definition of a *generating set* in an additive category.

Definition 2.1.1. Let $\mathcal{C}$ be an additive category. A class of objects S in $\mathcal{C}$ is called a *class of generators* for $\mathcal{C}$ if, for every object X in $\mathcal{C}$, $X = 0$ if $\mathcal{C}(s, X) = 0$ for all $s \in S$.

If a class of generators consists of only one object, then this object is called a *generator* of $\mathcal{C}$. The dual notion of a generating class is called a *cogenerating class*.

The definition of α-compact objects is done in two steps. We first recall the definition of α-small objects.

Definition 2.1.2. Let $\mathcal{T}$ be a triangulated category satisfying [TR5], *i.e.* such that coproducts exist in $\mathcal{T}$. Let α be an infinite cardinal. An object s in $\mathcal{T}$ is *α-small* if every morphism $s \to \coprod_{i\in I} X_i$ in $\mathcal{T}$ factors through $\coprod_{i\in I'} X_i$, where $I' \subset I$ and $\#I' < \alpha$. We denote by $\mathcal{T}^{(\alpha)}$ the full subcategory of α-small objects in $\mathcal{T}$.

Remark 2.1.3. If α is a regular cardinal, then the category $\mathcal{T}^{(\alpha)}$ is an α-localizing triangulated subcategory of $\mathcal{T}$; see [Nee01b, Lemma 4.1.4 and Lemma 4.1.5].

In order to define α-compact objects, we need to introduce first the notion of an α-perfect class.

Definition 2.1.4. Let $\mathcal{T}$ be a triangulated category satisfying [TR5] and let α be a regular cardinal. A set S of α-small objects in $\mathcal{T}$ closed under coproducts of less than α objects is called *α-perfect* if, for every set of morphisms $f_i\colon X_i \to Y_i$, $i \in I$, in $\mathcal{T}$, the induced morphism

$$\mathcal{T}(s, \textstyle\coprod_{i\in I} X_i) \xrightarrow{\mathcal{T}(s, \coprod_{i\in I} f_i)} \mathcal{T}(s, \textstyle\coprod_{i\in I} Y_i)$$

is surjective for every $s \in S$ provided that the induced morphisms $\mathcal{T}(s, f_i)$ are surjective for every $i \in I$ and every $s \in S$.

There is another notion of α-perfect class defined by Neeman in [Nee01b, Definition 3.3.1]. Krause [Kra01, Lemma 4] proved that both definitions are equivalent for sets of α-small objects. We state Neeman's definition for completeness.

Definition 2.1.5. Let $\mathcal{T}$ be a triangulated category satisfying [TR5]. Let α be an infinite cardinal. A class of objects S in $\mathcal{T}$ is called *α-perfect* if it contains the zero object and, for every object $s \in S$ and every morphism

$$s \longrightarrow \coprod_{i \in I} X_i$$

in $\mathcal{T}$, where $\#I < \alpha$, there is a factorization

$$\begin{tikzcd} s \arrow[rr] \arrow[dr] & & \coprod_{i\in I} X_i \\ & \coprod_{i\in I} s_i \arrow[ur, "\coprod_{i\in I} f_i"'] & \end{tikzcd}$$

where $s_i \in S$ and $f_i \colon s_i \to X_i$ for every $i \in I$.

It is a result of Neeman [Nee01b, Corollary 3.3.10] that, given a triangulated subcategory $\mathcal{S} \subset \mathcal{T}$, there is a unique maximal α-perfect class in $\mathcal{S}$ for every infinite cardinal α. This enables us to give the following definition.

Definition 2.1.6. Let $\mathcal{T}$ be a triangulated category satisfying [TR5]. Let α be a regular cardinal. Let $\mathcal{T}^{\alpha}$ be the full subcategory whose class of objects is the maximal α-perfect class of α-small objects. Then the objects of $\mathcal{T}^{\alpha}$ are called *α-compact objects*.

Remark 2.1.7. In the special case $\alpha = \aleph_0$, since all classes are $\aleph_0$-perfect, the $\aleph_0$-compact objects are exactly the $\aleph_0$-small objects. In this case, they are just called *small* or *compact* and the categories $\mathcal{T}^{(\aleph_0)}$ and $\mathcal{T}^{\aleph_0}$ are denoted by $\mathcal{T}^c$.

Next we give an explicit description of α-compact objects in some triangulated categories.

Example 2.1.8. The compact objects in Ho(Sp) are precisely the finite spectra. That is, $\mathrm{Ho}(\mathrm{Sp})^c = \mathrm{Ho}(\mathrm{Sp}_f)$. This follows from the fact that

$$[\Sigma^i \mathbb{Y}, \bigvee_{j \in J} \mathbb{X}_j] \cong \bigoplus_{j \in J} [\Sigma^i \mathbb{Y}, \mathbb{X}_j]$$

for every finite spectrum $\mathbb{Y}$ and all i, for every index set J.

For derived categories of rings, we make a distinction between the characterizations of α-compact objects for $\alpha = \aleph_0$ and for $\alpha > \aleph_0$, as illustrated by the next two examples. All the rings we will consider are associative with identity and the R-modules will be on the right.

Example 2.1.9 ([AJS03, Lemma 4.3]). Let R be a ring. A complex in the derived category $\mathrm{D}(R)$ is compact if and only if it is quasi-isomorphic to a bounded complex of finitely generated projective R-modules. This can be deduced from the forthcoming Example 2.1.24.

Example 2.1.10 (Neeman [Mur09, Theorem 14]). Let R be a ring and $\alpha > \aleph_0$ a regular cardinal. First recall that a complex of R-modules P is said to be *K-projective* if the mapping complex $\mathrm{Hom}(P, X)$ is acyclic as a complex of abelian groups for every acyclic complex of R-modules X. A complex in $\mathrm{D}(R)$ is α-compact if and only if it is quasi-isomorphic to a K-projective complex of free R-modules with less than α generators.

If we impose conditions on the ring R, then we can give more explicit characterizations of α-compact objects in $\mathrm{D}(R)$, as follows. Recall that an R-module is *α-generated* if it has less than α generators and it is *α-presented* if it has less than α generators and less than α relations; see Definition 5.1.1 for the precise definition.

Definition 2.1.11 ([JL89, Chapter 7]). Let R be a ring. An R-module is called *α-coherent* if it is α-presented (Definition 5.1.1) and every α-generated submodule is α-presented. A ring will be called *α-coherent* if it is α-coherent as a left module over itself.

In the case $\alpha = \aleph_0$, this notion coincides with the usual notion of *coherence*; see [Pre09, Sec. 2.3.3].

Remark 2.1.12. Left noetherian rings and rings R such that $\#R < \alpha$ are α-coherent; see [Mur09, Lemma 18] for a proof. Also hereditary rings are α-coherent for all α. This can be easily seen from the definition. Recall that a ring R is *hereditary* if all its ideals are projective or, equivalently, if it has global projective dimension less than or equal to one. Let P be an α-generated ideal of R. Thus, there is an epimorphism $\phi\colon F \to P$ with F a free α-generated R-module. If R is hereditary, then P is projective and there is an isomorphism $F \cong P \oplus \ker(\phi)$. Since F is α-generated, so is $\ker(\phi)$. Hence, P is α-presented.

The next propositions describes α-compact objects in the derived category of an α-coherent ring. We first state the case $\alpha = \aleph_0$.

Proposition 2.1.13 ([CKN01, Lemma 1.1]). *Let R be a ring. Then the following are equivalent.*

1. *R is coherent and each finitely presented R-module is of finite projective dimension.*

2. *Each finitely presented R-module is compact as a complex in* $\mathrm{D}(R)$ *concentrated in degree zero.*

3. *A complex X is compact if and only if $H^n(X)$ is finitely presented for all $n \in \mathbb{Z}$ and $H^n(X) = 0$ for all but finitely many $n \in \mathbb{Z}$.*

For higher cardinals we have the following characterization.

Proposition 2.1.14 (Neeman [Mur09, Theorem 19])**.** *Let R be a ring and $\alpha > \aleph_0$ be a regular cardinal. Suppose that R is α-coherent. Let F be a complex of R-modules. Then the following are equivalent.*

1. *F is an α-compact object in* $\mathrm{D}(R)$*.*

2. *F is isomorphic, in* $\mathrm{D}(R)$*, to a complex of free R-modules with less than α generators.*

3. *$H^n(F)$ is an α-generated R-module for all $n \in \mathbb{Z}$.*

As a direct consequence of Proposition 2.1.14, if R is α-coherent for a cardinal $\alpha > \aleph_0$, then every α-presented R-module is α-compact as a complex concentrated in degree 0.

Now we can give the definition of an α-compactly generated triangulated category.

Definition 2.1.15. Let $\mathcal{T}$ be a triangulated category satisfying [TR5]. Let α be a regular cardinal. We say that $\mathcal{T}$ is *α-compactly generated* if there is a generating α-perfect set of α-small objects. We say that $\mathcal{T}$ is *well generated* if it is α-compactly generated for some regular cardinal α.

The $\aleph_0$-compactly generated triangulated categories are just called *compactly generated* triangulated categories.

Proposition 2.1.16 ([Nee01b, Lemma 4.2.4, Lemma 4.2.5 and Lemma 4.4.5])**.** *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5] *and α a regular cardinal. Then the category $\mathcal{T}^\alpha$ is α-localizing. If we assume further that $\mathcal{T}$ is α-compactly generated by a set S, then $\mathcal{T}^\alpha$ is the smallest α-localizing subcategory of $\mathcal{T}$ containing S.*

Remark 2.1.17. Let $\mathcal{T}$ be an α-compactly generated triangulated category and let $\beta > \alpha$ be a regular cardinal. Since every α-compact object is β-compact, *i.e.* $\mathcal{T}^\alpha \subset \mathcal{T}^\beta$, we infer that $\mathcal{T}$ is also a β-compactly generated triangulated category.

Example 2.1.18. Let Ho(Sp) be the stable homotopy category. Then $\{\Sigma^i \mathbb{S}\}_{i\in\mathbb{Z}}$ is a set of generators of Ho(Sp) as we observed in Remark 1.2.9, and they are compact since they are finite and $\mathrm{Ho(Sp)}^c = \mathrm{Ho}(\mathrm{Sp}_f)$.

Example 2.1.19. Let R be a ring. Then the derived category of R-modules $\mathrm{D}(R)$ is compactly generated.

Example 2.1.20 (Krause [Kra05]). Let R be a ring. Suppose that R is noetherian. The homotopy category of chain complexes of injective R-modules $\mathrm{K}(R\text{-Inj})$ is compactly generated and there is a natural equivalence between the subcategory of compact objects and the derived category of bounded complexes of finitely presented R-modules.

All the examples we have given so far are compactly generated. The following one is not.

Example 2.1.21 (Neeman [Nee08]). Let R be a ring. A complex in the homotopy category of chain complexes of projective R-modules $\mathrm{K}(R\text{-Proj})$ is compact if and only if it is isomorphic to a bounded below complex X of finitely generated projective R-modules such that $H^n(X) = 0$ for $n \ll 0$.

From this characterization it can be shown that, if R is a coherent ring, then $\mathrm{K}(R\text{-Proj})$ is compactly generated. However, $\mathrm{K}(R\text{-Proj})$ is not compactly generated if $R = k \oplus Q$ where k is a field and Q is an infinite dimensional vector space over k [Nee08, Example 7.16].

Neeman also gave a description of the α-compact objects in $\mathrm{K}(R)$-Proj) for $\alpha > \aleph_0$. A complex in $\mathrm{K}(R\text{-Proj})$ is α-compact for $\alpha > \aleph_0$ if and only if it is isomorphic to a chain complex of free R-modules with less than α generators. Using this description, it can be seen that $\mathrm{K}(R\text{-Proj})$ is $\aleph_1$-compactly generated for every ring R.

The following theorem describes the behavior of α-compact objects under Verdier localizations. It is the basic tool to see that the well generated triangulated categories behave well under Verdier localizations, as we will see in Proposition 2.2.5 and Corollary 2.2.6. This theorem was proved by Neeman, but he calls it "Thomason Localization Theorem" because it was proved by Thomason [TT90] in the particular case of the derived category of quasi-coherent sheaves over a noetherian scheme $\mathrm{D}(qc/X)$. From this fact, Thomason deduced important applications to K-theory. We formulate it as done in [Kra08, Theorem 7.2.1].

Theorem 2.1.22 (Neeman [Nee01b, Theorem 4.4.9]). *Let α be a regular cardinal. Let $\mathcal{T}$ be an α-compactly generated triangulated category and let $\mathcal{S} \subset \mathcal{T}$ be a localizing subcategory. Assume that the triangulated category obtained by passing to the Verdier quotient $\mathcal{T}/\mathcal{S}$ is defined. Assume further* *that $\mathcal{S}$ is also α-compactly generated and that the α-small objects generating $\mathcal{S}$ are also α-small in $\mathcal{T}$. If $\alpha > \aleph_0$, the natural functor*

$$\mathcal{T}^\alpha/\mathcal{S}^\alpha \longrightarrow \{\mathcal{T}/\mathcal{S}\}^\alpha$$

is an equivalence. If $\alpha = \aleph_0$, it is fully faithful and every object in in $\{\mathcal{T}/\mathcal{S}\}^c$ is a direct summand of an object in $\mathcal{T}^c/\mathcal{S}^c$.

This result is depicted in the following commutative diagram of categories:

$$\begin{tikzcd}
\mathcal{S}^\alpha \arrow[r] \arrow[dd, hook] & \mathcal{T}^\alpha \arrow[rr, dotted] \arrow[dr] \arrow[dd, hook] & & \{\mathcal{T}/\mathcal{S}\}^\alpha \arrow[dd, hook] \\
& & \mathcal{T}^\alpha/\mathcal{S}^\alpha \arrow[ur] & \\
\mathcal{S} \arrow[r] & \mathcal{T} \arrow[rr] & & \mathcal{T}/\mathcal{S}.
\end{tikzcd}$$

Remark 2.1.23. A way to ensure that the category $\mathcal{T}/\mathcal{S}$ has small hom-sets is to impose that a left Bousfield localization exists; see Theorem 1.4.5 for details. However, we will see in Proposition 2.2.5 and Corollary 2.2.6 that this extra condition is not necessary. This will be a consequence of the Brown Representability Theorem.

The Thomason Localization Theorem is the most important ingredient in the following example, which was observed by Neeman in order to prove that the Grothendieck Duality Theorem follows from the Brown Representability Theorem [Nee96].

Example 2.1.24. Let X be a quasi-compact, separated scheme. Then the derived category of quasi-coherent sheaves on X, which is denoted by $\mathrm{D}(qc/X)$, is compactly generated and $\mathrm{D}(qc/X)^c$ is the full subcategory of perfect complexes in $\mathrm{D}(qc/X)$. A complex is *perfect* if it is quasi-isomorphic to a bounded complex of vector bundles.

One of the bad properties of compactly generated triangulated categories is that a left Bousfield localization of compactly generated triangulated category with respect to a localizing subcategory can fail to be compactly generated even if the localizing category is generated by a set, as shown in [Nee01a]. However, it will be well generated, as we will see in Proposition 2.2.5. This is one of the motivations to introduce well generated triangulated categories for higher cardinals. We next give a concrete example.

Example 2.1.25 (Neeman [Nee01a]). Let M be a non-compact, connected manifold of dimension greater than or equal to 1. The derived category of sheaves of abelian groups over M is denoted by $\mathrm{D}(\mathrm{Sh}/M)$. It has no non-zero compact object. Hence, it cannot be compactly generated. However, by a general result of Alonso, Jeremías and Souto [AJS00], $\mathrm{D}(\mathrm{Sh}/M)$ is equivalent to a Verdier localization of the derived category of a ring. Because of this, Neeman deduces from Corollary 2.2.6 that $\mathrm{D}(\mathrm{Sh}/M)$ is well generated; see Example 2.2.7.

Next we give another example of a localization of a compactly generated triangulated category that is not compactly generated.

Example 2.1.26 (Neeman [Nee98b]). Let R be a commutative, noetherian, regular local ring of height 2. Let k be the residue field of R and let K be the field of fractions of R. Let $\mathcal{S}$ be the full subcategory of $\mathrm{D}(R)$ of objects E such that $E \otimes (k \oplus K) = 0$ where $\otimes$ is the derived tensor product in $\mathrm{D}(R)$. Then the Verdier localization $\mathrm{D}(R)/\mathcal{S}$ is not compactly generated.

As we already mentioned, one of the most important consequences of Theorem 2.1.22 is that the localization of a well generated triangulated category by a "nice enough" localizing subcategory is well generated. This will be explained in the next section, after we state the Brown Representability Theorem for well generated triangulated categories. However, we want to point out here that all known well generated triangulated categories appear as Verdier localizations of compactly generated ones, as we explained in [Nee09, Remark 1.4]. This is supported by two recent results by Porta and Heider that we describe in the following examples.

Example 2.1.27 (Keller and Porta [Kel06], [Por07]). *Algebraic triangulated categories* were introduced by Keller [Kel06] as a generalization of derived categories of abelian categories. They are triangulated categories that are equivalent to a full triangulated subcategory of the homotopy category of complexes of an additive category.

Porta proved, as part of his PhD thesis advised by Keller, that an algebraic triangulated category is well generated if and only if it is equivalent to the Verdier quotient of the derived category of a DG-category with respect to a localizing subcategory generated by a set of objects, which is always algebraic and compactly generated. Porta calls this theorem *Generalized Gabriel-Popescu Theorem*, due to the analogy with [Pop73, Ch. 3, Theorem 7.9].

Example 2.1.28 (Schwede and Heider [Sch08], [Hei07]). *Topological triangulated categories* were introduced by Schwede [Sch08] generalizing the construction of the stable homotopy category. A triangulated category is called *topological* if it is equivalent to a full subcategory of the homotopy category of a stable model category; see [Hov01] for details.

Heider proved, as part of his PhD thesis advised by Schwede, the analog of the Gabriel-Popescu Theorem in this case, *i.e.* a topological triangulated category is well generated if and only if it is equivalent to the Verdier localization of the derived category of a spectral category with respect to a localizing subcategory generated by a set of objects, which is always topological and compactly generated.

Example 2.1.29 (Muro, Schwede and Strickland [MSS07]). We also point out that there is only one known family of examples of triangulated categories that are neither algebraic nor topological. This family consists of the categories of finitely generated free R-modules $\mathcal{F}(R)$, for R a commutative local ring of characteristic 4. For instance, we can take R to be $\mathbb{Z}/4$. There exists a unique triangulated category structure on $\mathcal{F}(R)$ with $\Sigma = id_{\mathcal{F}(R)}$ and such that

$$R \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} R$$

is a triangle. The fact that it cannot be topological will not be explained here, but we can easily tell the reason why it cannot be algebraic. Given an object X in an algebraic triangulated category and a triangle

$$X \xrightarrow{2 \cdot id_X} X \longrightarrow C \longrightarrow \Sigma X,$$

the equation $2 \cdot id_C = 0$ holds [Kel06, 3.6] [Sch08]. Since R is of characteristic 4, $\mathcal{F}(R)$ cannot be algebraic.

Most of the examples mentioned so far, including algebraic and topological triangulated categories, are particular cases of Theorem 2.1.32 below. Before we state the result, we briefly recall the definition of a *stable combinatorial model category*; see [Dug01] or [Hov01] for details.

Definition 2.1.30. A *model category* is a category $\mathcal{K}$ with all limits and colimits together with three distinguished classes of maps called *weak equivalences*, *fibrations* and *cofibrations* satisfying the following axioms.

1. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be a pair of maps in $\mathcal{K}$. If two out of $\{f, g, g \circ f\}$ are weak equivalences, then so is the third.

2. Each of the classes of weak equivalences, fibrations and cofibrations is closed under retracts.

3. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

be a commutative diagram with i a cofibration and p a fibration. If i or p is a weak equivalence, then there exists a map $h\colon B \to X$ such that $f = h \circ i$ and $g = p \circ h$.

4. Every map $f\colon X \to Y$ can be factorized in the following two ways:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & X' \arrow[ur, "p"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i'"'] & & Y \\ & Y' \arrow[ur, "p'"'] & \end{tikzcd}$$

   where i is a cofibration and a weak equivalence, p is a fibration, i' is a cofibration and p' is a fibration and a weak equivalence.

Condition 2 in the definition is sometimes shortened by saying that i satisfies the *Left Lifting Property* (LLP) with respect to p, and that p has the *Right Lifting Property* (RLP) with respect to i.

Model categories provide a natural setting for homotopy theory. The homotopy category $\mathrm{Ho}(\mathcal{K})$ of a model category $\mathcal{K}$ is the category obtained by formally inverting the weak equivalences. It is a consequence of the axioms that the homotopy category is well defined.

Definition 2.1.31 ([Hov01, Definition 2.1.17])**.** A model category $\mathcal{K}$ is called *cofibrantly generated* if there are sets of maps I and J in $\mathcal{K}$ such that the following hold.

1. If A is a domain of a map in I, then $\mathcal{K}(A, -)$ commutes with λ-filtered colimits of sequences of maps that are transfinite compositions of pushouts of maps in I for some cardinal λ.

2. If X is a domain of a map in J, then $\mathcal{K}(X, -)$ commutes with λ-filtered colimits of sequences of maps that are transfinite compositions of pushouts of maps in J for some cardinal λ.

3. The class of fibrations is the class of maps with the RLP with respect to every map in J.

4. The class of maps that are both a fibration and a weak equivalence is the class of maps with the RLP with respect to every map in I.

A model category is called *combinatorial* if it is cofibrantly generated and the underlying category is locally λ-presentable (Definition 2.3.10) for some cardinal λ. In the homotopy category of every model category $\mathcal{K}$ with a zero object, it is possible to define a suspension functor $\Sigma\colon \mathrm{Ho}(\mathcal{K}) \to \mathrm{Ho}(\mathcal{K})$ [Hov01, Sec. 6.1]. We say that a model category with a zero object is *stable* if Σ is an equivalence. In this case, $\mathrm{Ho}(\mathcal{K})$ can be given a natural triangulated structure [Hov01, Ch. 6].

Theorem 2.1.32 (Rosický [Ros05, Proposition 6.10]). *Let $\mathcal{K}$ be a stable combinatorial model category. Then its homotopy category* $\mathrm{Ho}(\mathcal{K})$ *is well generated.*

Remark 2.1.33. The opposite category of a well generated triangulated category is almost never well generated. For instance, the opposite of the stable homotopy category has no non-zero compact objects by a theorem of Boardman [Boa70]. In fact, in [Nee01b, Appendix E.1], it was proved that the opposite of a compactly generated triangulated category cannot be well generated.

Neeman gave an example in [Nee01b, Appendix E.3] of a triangulated category $\mathcal{T}$ that is not well generated nor its opposite. It is the category $\mathrm{K}(\mathbb{Z})$, *i.e.* the homotopy category of abelian groups. In order to deal with this triangulated category and similar ones, Šťovíček [Šťo08] introduced *locally well generated triangulated categories*. These are triangulated categories such that, for every set of objects S in $\mathcal{T}$, the smallest localizing subcategory of $\mathcal{T}$ containing S is well generated. They include the homotopy category of essentially small additive categories. Hence, they can fail to satisfy Brown representability but they admit a theorem of existence of adjoints similar to Proposition 1.1.21.

2.2 Brown representability

The following result is the Brown Representability Theorem for well generated triangulated categories. It was one of the motivations for introducing well generated triangulated categories.

Theorem 2.2.1 (Neeman [Nee01b, Theorem 8.3.3]). *Let $\mathcal{T}$ be a well generated triangulated category. Then Brown representability holds for $\mathcal{T}$.*

Note that, by definition, well generated triangulated categories satisfy [TR5]. Observe also that, by the Yoneda Lemma, any natural transformation between cohomological functors from $\mathcal{T}$ into Ab that sends coproducts to products is represented by a unique morphism in $\mathcal{T}$.

There are many interesting consequences of this theorem apart from the existence of adjoints (Proposition 1.1.21) and the existence of products (Proposition 1.1.22).

The next lemma gives an equivalent notion of generating set for well generated categories.

Lemma 2.2.2 ([Nee01b, Proposition 8.4.1]). *Let $\mathcal{T}$ be an α-compactly generated triangulated category for a regular cardinal $\alpha > \aleph_0$. Let S be an α-perfect* *set of α-compact objects closed under suspensions. Then S is a generating set of $\mathcal{T}$ if and only if $\mathcal{T}$ is the smallest localizing subcategory of $\mathcal{T}$ containing S.*

The following corollaries of Theorem 2.2.1 are of major importance, partially because of their consequences in the study of Verdier localizations in well generated triangulated categories that we will state as Proposition 2.2.5 and Corollary 2.2.6.

Corollary 2.2.3 ([Kra02, Corollary p. 858])**.** *Let $\mathcal{T}$ be an α-compactly generated triangulated category for a regular cardinal α. Then $\mathcal{T}^\alpha$ has only a set of isomorphism classes of objects,* i.e. *it is essentially small, and $\mathcal{T}$ is the smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{T}^\alpha$.*

As a consequence of Corollary 2.2.3, if $\mathcal{T}$ is an α-compactly generated triangulated category, then a set of representatives of the isomorphism classes of objects in $\mathcal{T}^\alpha$ is an α-perfect generating set of $\mathcal{T}$.

Corollary 2.2.4 ([Nee01b, Proposition 8.4.2])**.** *Let $\mathcal{T}$ be a well generated triangulated category. Then the following hold.*

1. *For every regular cardinal α, the category $\mathcal{T}^\alpha$ is essentially small.*
2. $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$.

We would like to use this result to overcome the issue about smallness of hom-sets when working with Verdier quotients. This has been carried out with slight modifications in [Hei07], [Kra08] and [Por07] by different authors, all based on the original results by Neeman in [Nee01b].

The following results are direct consequences of Theorem 2.1.22 and Corollary 1.4.9. The main point is that we do not need to assume that the resulting Verdier quotient has small hom-sets as in Theorem 2.1.22.

Proposition 2.2.5 ([Kra08, Theorem 7.2.1])**.** *Let $\mathcal{T}$ be an α-compactly generated triangulated category for a regular cardinal α and let $\mathcal{S} \subset \mathcal{T}$ be a localizing subcategory which is also α-compactly generated. Then the functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ is a left Bousfield localization and $\mathcal{T}/\mathcal{S}$ is an α-compactly generated triangulated category.*

The following result is, essentially, what was used in [Nee01a] to prove that the derived category of sheaves of abelian groups over a manifold is well generated, not being compactly generated when the manifold is noncompact. It has also been used as a key result in [Por07] and [Hei07] to prove analogs of the Gabriel-Popescu Theorem for triangulated categories. We formulate it as in [Hei07, Proposition 3.3], [Kra08, Corollary 7.2.2] or [Por07, Corollary 3.12].

Corollary 2.2.6. *Let $\mathcal{T}$ be a well generated triangulated category and let $\mathcal{S} \subset \mathcal{T}$ be a localizing subcategory generated by a set of objects S. Then $\mathcal{S}$ and $\mathcal{T}/\mathcal{S}$ are well generated categories.*

Example 2.2.7 (Neeman [Nee01a]). Let $\mathcal{A}$ be a Grothendieck category. A result by Alonso, Jeremías and Souto [AJS00] says that there exists a ring R and a localizing subcategory $\mathcal{S}$ generated by a set of objects such that $\mathrm{D}(\mathcal{A})$ is equivalent to the Verdier localization $\mathrm{D}(R)/\mathcal{S}$. Hence, by Corollary 2.2.6, $\mathrm{D}(\mathcal{A})$ is well generated.

As a particular case, this proves that the derived category $\mathrm{D}(\mathrm{Sh}/X)$ of sheaves over a topological space X is a well generated triangulated category, although, as we observed in Example 2.1.25, it may fail to be compactly generated.

There are other interesting representability results that apply to well generated triangulated categories due to Franke and Krause. We review the relation between these results and Theorem 2.2.1, following [Kra02].

Theorem 2.2.8 (Krause [Kra02, Theorem A])**.** *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5]*. Then Brown representability holds for $\mathcal{T}$ if there exists a set S that satisfies the following conditions.*

1. *If t is an object in $\mathcal{T}$ and $\mathcal{T}(s,t) = 0$ for every $s \in S$, then $t = 0$.*
2. *For every set of morphisms $\{f_i \colon X_i \to Y_i\}_{i\in I}$ in $\mathcal{T}$ with $\#I = \aleph_0$, the induced morphism*

$$\mathcal{T}(s, \textstyle\coprod_{i\in I} X_i) \xrightarrow{\mathcal{T}(s,\coprod_{i\in I} f_i)} \mathcal{T}(s, \coprod_{i\in I} Y_i)$$

   is surjective for every $s \in S$ provided that the induced morphisms $\mathcal{T}(s, f_i)$ are surjective for every $i \in I$, and every $s \in S$.

By definition, all well generated categories satisfy the hypotheses of the above theorem.

Theorem 2.2.9 (Franke [Fra01, Theorem 2.4])**.** *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5]*. Then Brown representability holds for $\mathcal{T}$ if there is a set G of objects in $\mathcal{T}$ and a regular cardinal α satisfying the following conditions.*

1. *$\mathcal{T}$ is the smallest α^+-localizing full triangulated subcategory of $\mathcal{T}$ that contains all coproducts of objects in G, where α^+ denotes the successor cardinal of α.*

2. *There exist arbitrarily large regular cardinals κ and essentially small, full, κ-localizing triangulated subcategories $C(\kappa) \subset \mathcal{T}$ containing G and such that, for every object c in $C(\kappa)$ and $g \in G$, $\#C(g, c) < \kappa$.*

Krause proved in [Kra01, Theorem C] that, if there is a set S that satisfies the following conditions, then there is a set G and a cardinal α satisfying the hypotheses in Franke's Theorem.

1. If t is an object in $\mathcal{T}$ and $\mathcal{T}(s, t) = 0$ for every $s \in S$, then $t = 0$.

2. For every set of morphisms $\{f_i \colon X_i \to Y_i\}_{i \in I}$ in $\mathcal{T}$, the induced morphism
$$\mathcal{T}(s, \textstyle\coprod_{i \in I} X_i) \xrightarrow{\mathcal{T}(s, \coprod_{i \in I} f_i)} \mathcal{T}(s, \textstyle\coprod_{i \in I} Y_i)$$
is surjective for every $s \in S$ provided that the induced morphisms $\mathcal{T}(s, f_i)$ are surjective for every $i \in I$ and every $s \in S$.

Remark 2.2.10. Theorem 2.2.8 and Theorem 2.2.9 seem to be more general than the representability theorem that we stated as Theorem 2.2.1, but Neeman points out in [Nee09, Remark 1.4] that he does not know of any example of a category satisfying the hypotheses of the theorems by Krause and Franke that are neither well generated nor the opposite of a compactly generated triangulated category.

2.3 Adams representability for higher cardinals

The name "Adams Representability Theorem for higher cardinals" seems to be new. Rosický in [Ros09] calls it "λ-Brown" and Neeman in [Nee09] includes it as part of the existence of a Rosický functor. We have adopted this terminology because it was Adams who proved the theorem in the stable homotopy category for the cardinal $\aleph_0$.

Many of the results that we present in this chapter are well known when the cardinal is $\aleph_0$ and the category is compactly generated. The main references for this are [Bel00b], [CKN01] and [Nee97]. Even in this case, there is no standard terminology for the generalization of Adams' original representability theorem in the homotopy category of spectra. Neeman called it "Theorem of Brown and Adams" in [Nee97] and "Brown Representability Theorem" in [CKN01]. As we will see, some results that hold for compactly generated triangulated categories do not generalize to well generated triangulated categories.

We will first define Adams representability for higher cardinals and then explain the relationship with the previous work of Neeman [Nee09] and Rosický [Ros09]. Finally, we will state some open questions about this theorem.

Remember from Proposition 1.1.19 that, for every triangulated category $\mathcal{T}$ satisfying [TR5], the functors of the form $\mathcal{T}(-,X)$ are cohomological and take coproducts in $\mathcal{T}$ into products. Then, functors of the form $\mathcal{T}(-,X)|_{\mathcal{T}^\alpha}$ are also cohomological and send the coproducts that exist in $\mathcal{T}^\alpha$ into products. Since $\mathcal{T}^\alpha$ is α-localizing, it sends coproducts of less than α objects into products.

Definition 2.3.1. Let $\mathcal{T}$ be a triangulated category satisfying [TR5]. Let α be a regular cardinal.

1. We say that *α-Adams representability for objects* holds for $\mathcal{T}$ if for every cohomological functor $H\colon \{\mathcal{T}^\alpha\}^{\mathrm{op}} \to \mathrm{Ab}$ that sends coproducts of less than α objects in $\mathcal{T}^\alpha$ to products in Ab there is a natural isomorphism $H(-) \cong \mathcal{T}(-,X)|_{\mathcal{T}^\alpha}$ for an object X in $\mathcal{T}$.

2. We say that *α-Adams representability for morphisms* holds for $\mathcal{T}$ if every natural transformation $\phi\colon \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \to \mathcal{T}(-,Y)|_{\mathcal{T}^\alpha}$ is of the form $\phi = \mathcal{T}(-,f)|_{\mathcal{T}^\alpha}$ for a (not necessarily unique) morphism f in $\mathcal{T}$.

As in the case of the stable homotopy category, the non-uniqueness of the representability of morphisms is due to the existence of phantom morphisms; see Theorem 1.2.15.

As usual, we will write *Adams representability* instead of $\aleph_0$-Adams representability.

Definition 2.3.2. Let $\mathcal{T}$ be an α-compactly generated triangulated category. A morphism $f\colon X \to Y$ in $\mathcal{T}$ is called *α-phantom* if $\mathcal{T}(-,f)|_{\mathcal{T}^\alpha} = 0$.

We begin with the first positive result that appeared in the literature. Recall that the cardinality of an essentially small category $\mathcal{C}$ is defined by

$$\#\,\mathcal{C} = \#\coprod_{X,Y\in\tilde{\mathcal{C}}} \mathcal{C}(X,Y)$$

where $\tilde{\mathcal{C}}$ is a set of representatives of all isomorphism classes of objects in $\mathcal{C}$.

Theorem 2.3.3 (Neeman [Nee97])**.** *Let $\mathcal{T}$ be a compactly generated triangulated category. Assume that $\#\,\mathcal{T}^c \le \aleph_0$. Then $\mathcal{T}$ satisfies Adams representability for objects and for morphisms.*

This theorem includes the case of the stable homotopy category proved by Adams, since $\#\mathrm{Ho}(\mathrm{Sp}_f) = \aleph_0$ by Definition 1.2.10. However, the proof of Theorem 2.3.3 by Neeman is of a different nature than Adams' proof, because Neeman did not use the existence of models for the triangulated category, *i.e.* the triangulated category need not be the homotopy category of any model category.

Another example that satisfies the hypotheses of Theorem 2.3.3 is the derived category $\mathrm{D}(R)$ of a countable ring R. Remember from Example 2.1.9 that the compact objects in this case are isomorphic to bounded complexes of finitely generated projective R-modules. Hence, $\#\mathrm{D}(R)^c = \aleph_0$.

We also want to point out that it is not true that every compactly generated triangulated category satisfies Adams representability. We briefly sketch an example that will be reviewed in more detail as Example 5.2.7.

Example 2.3.4 ([CKN01]). Let k be a field such that $\#k = \aleph_t$, where $t \geq 0$, and let $R = k\langle X, Y\rangle$ be the ring of non-commutative polynomials with coefficients in k. Then $\mathrm{D}(R)$ satisfies Adams representability for morphisms if and only if $t = 0$.

As we already mentioned, the case $\alpha = \aleph_0$ has been studied in detail in [Bel00b], [CKN01] and [Nee97]. In contrast, almost nothing is known about α-Adams representability for $\alpha > \aleph_0$. We will explain in this section what makes the case $\alpha > \aleph_0$ more difficult. In Chapter 3 we will give new results about α-Adams representability by considering the notion of α-purity and in Chapter 5 we will study the case of derived categories of rings.

Remark 2.3.5. We first observe that it is not possible to extend Neeman's proof of Theorem 2.3.3. The problem is that in Neeman's proof [Nee97], and in all other proofs of similar results by Adams [Ada71], Beligiannis [Bel00b] or Margolis [Mar83], one uses the fact that a filtered colimit indexed by a category of cardinality $\aleph_0$ has a cofinal sequence [AR94, Theorem 1.5]. However, this is false for greater cardinals, as shown in [AR94, Example 1.8].

In fact, as a consequence of the results in Section 5, we will see that the analog of Theorem 2.3.3 for $\alpha > \aleph_0$ is not true in general.

2.3.1 Categories of additive functors

In order to study α-Adams representability we will need some results about the category of additive contravariant functors that send coproducts of less than α objects into products. We will study this here following the book of Neeman [Nee01b] and extending some results when necessary.

Definition 2.3.6. Let $\mathcal{T}$ be a triangulated category and let α be a regular cardinal. Let $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ be the category of additive contravariant functors from $\mathcal{T}^\alpha$ to Ab that send coproducts of less than α objects in $\mathcal{T}^\alpha$ to products. We define the *restricted Yoneda functor for* α as

$$\begin{aligned} S_\alpha \colon\ \mathcal{T} &\longrightarrow \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha, \\ X &\longmapsto \mathcal{T}(-,X)|_{\mathcal{T}^\alpha}. \end{aligned}$$

Observe that S_α is well defined since $\mathcal{T}^\alpha$ is α-localizing and essentially small, as we have seen in Proposition 2.1.16 and Corollary 2.2.4. With this notation, we can restate α-Adams representability as follows.

1. $\mathcal{T}$ satisfies α-Adams representability for objects if the essential image of S_α consists of all cohomological functors that send coproducts of less than α objects to products.

2. $\mathcal{T}$ satisfies α-Adams representability for morphisms if S_α is full.

The notation Mod_α-$\mathcal{T}^\alpha$ is motivated by the fact that, for a ring R, objects in the category Mod-R of right R-modules can be viewed as additive functors $R^{\mathrm{op}} \to \mathrm{Ab}$.

Neeman studied extensively the category Mod_α-$\mathcal{T}^\alpha$ in chapters 6 and 7 of his book [Nee01b]. We point out that he used the notation $\mathcal{E}x(\mathcal{T}^\alpha, \mathrm{Ab})$ instead of Mod_α-$\mathcal{T}^\alpha$. Some of Neeman's proofs are also valid if we take $\mathcal{C}$ an additive category with coproducts of less than α objects instead of $\mathcal{T}^\alpha$. We state these properties in this section and check that Neeman's proofs work in this more general setting. We will need this generality in Chapter 4, when we apply these results to locally α-presentable additive categories. In order to state these properties, we need to introduce some standard notation; see [Fre64], [Mit65] or [Pop73] for details.

We begin by recalling the definition of the Grothendieck axioms for abelian categories; see [Pop73, Section 2.8].

Definition 2.3.7. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ satisfies

AB3 if it has coproducts;

AB4 if it satisfies [AB3] and the coproduct of a set of monomorphisms is a monomorphism;

AB5 if it satisfies [AB3] and filtered colimits of exact sequences are exact.

Dually,

AB3* if it has products;

AB4* if it satisfies [AB3*] and the product of a set of epimorphisms is an epimorphism;

AB5* if it satisfies [AB3*] and filtered limits of exact sequences are exact.

A *Grothendieck category* is an abelian category satisfying [AB5] and having a generator.

We will also need the following standard terminology.

Definition 2.3.8. Let α be an infinite cardinal. A category I is called *α-filtered* if every subcategory $J \subset I$ such that $\#J < \alpha$ has a cocone in I. A colimit indexed by an α-filtered category is called an *α-filtered colimit*.

In the case $\alpha = \aleph_0$, α-filtered colimits are just called *filtered colimits*.
We next review some terminology form [AR94].

Definition 2.3.9. Let $\mathcal{C}$ be an additive category with colimits and X an object in $\mathcal{C}$.

1. X is called *α-presentable* if the functor $\mathcal{C}(X, -)$ commutes with α-filtered colimits.

2. X is called *α-generated* if the functor $\mathcal{C}(X, -)$ commutes with α-filtered colimits where all morphisms are monomorphisms.

Definition 2.3.10. Let $\mathcal{C}$ be an additive category with colimits. We say that $\mathcal{C}$ is *locally α-presentable* if it has a generating set of α-presentable objects, *i.e.* there is a set of α-presentable objects S in $\mathcal{C}$ such that, if $\mathcal{C}(s, X) = 0$ for every $s \in S$, then $X = 0$. The full subcategory of α-presentable objects in $\mathcal{C}$ will be denoted by $\mathcal{C}^\alpha$.

We will need the following standard facts about locally α-presentable categories. We follow the book of Adámek and Rosický [AR94]

Proposition 2.3.11 ([AR94, Proposition 1.16])**.** *Let $\mathcal{C}$ be a locally α-presentable category. Then a colimit of α-presentable objects indexed by a category of cardinality less than α is α-presentable.*

Theorem 2.3.12 ([AR94, Theorem 1.20])**.** *Let $\mathcal{C}$ be an additive category with colimits. Then $\mathcal{C}$ is locally α-presentable if and only if it satisfies the following conditions.*

1. *Every object in $\mathcal{C}$ is an α-filtered colimit of α-presentable objects.*

2. *There is, up to isomorphism, only a set of α-presentable objects.*

Definition 2.3.13. Let $\mathcal{C}$ be a category. For every subcategory $\mathcal{D} \subset \mathcal{C}$ and for every object X in $\mathcal{C}$, the forgetful functor $(\mathcal{D} \downarrow X) \to \mathcal{C}$ is called the *canonical diagram* of X with respect to $\mathcal{D}$ where $(\mathcal{D} \downarrow X)$ is the *comma-category* with respect to the inclusion $\mathcal{D} \subset \mathcal{C}$ associated to X. The objects of $(\mathcal{D} \downarrow X)$ are morphisms in $\mathcal{C}$ with domain in $\mathcal{D}$ and codomain X and, for every pair of objects $a\colon A \to X$ and $b\colon B \to X$ in $(\mathcal{D} \downarrow X)$, the set of morphisms form a into b is the set of morphisms $h\colon A \to B$ such that $a = b \circ h$.

Proposition 2.3.14 ([AR94, Proposition 1.22])**.** *Let $\mathcal{C}$ be a locally α-presentable category. Denote by $\mathcal{C}^\alpha \subset \mathcal{C}$ the subcategory of α-presentable objects. Then, for every object X in $\mathcal{C}$, the canonical diagram $(\mathcal{C}^\alpha \downarrow X) \to \mathcal{C}$ is α-filtered and its colimit is X.*

Notice that, if $\mathcal{C}$ is an abelian category with a generating set of α-presentable projective objects, a direct consequence of Definition 2.3.10 is that, for every object X in $\mathcal{A}$, there exists an epimorphism

$$\textstyle\coprod_{i \in I} P_i \longrightarrow X$$

where P_i is α-presentable projective object for every $i \in I$ [Fre64, Proposition 3.36]. The following theorem justifies the terminology of α-presentable and α-generated in this context.

Theorem 2.3.15 ([AR94, Proposition 1.69], [Pop73, Sec. 3.5])**.** *Let $\mathcal{A}$ be a locally α-presentable abelian category with a generating set of α-presentable projectives satisfying* [AB3]*, and let X be an object in $\mathcal{A}$.*

1. *X is α-presentable if and only if there is a short exact sequence*

$$\textstyle\coprod_{j \in J} Q_j \longrightarrow \coprod_{i \in I} P_i \longrightarrow X \longrightarrow 0$$

where P_i and Q_j are α-presentable projective for all $i \in I$, $j \in J$, and $\#I < \alpha$ and $\#J < \alpha$.

2. *X is α-generated if and only if there exists an epimorphism*

$$\textstyle\coprod_{i \in I} P_i \longrightarrow X$$

where P_i is α-presentable projective for all $i \in I$ and $\#I < \alpha$.

In the case of the category of R-modules α-presentable objects are commonly known as *α-presented modules* (Definition 5.1.1). For traditional reasons, we will use the terminology α-presented in the context of R-modules.

Remark 2.3.16. Let $\mathcal{A}$ be a locally α-presentable abelian category with a generating set of α-presentable projectives satisfying [AB3], and let X be an object in $\mathcal{A}$.

1. X is α-generated if and only if it is an epimorphic image of an α-generated object.

2. X is α-presentable if for every morphism $p\colon \coprod_{i\in I} P_i \to X$ with P_i α-presentable projective for all $i \in I$ and $\# I < \alpha$, $\ker(p)$ is also α-generated.

3. If X is α-generated and projective, then it is α-presentable.

Notation 2.3.17. Let α be an infinite cardinal and $\mathcal{C}$ an essentially small additive category with coproducts of less than α objects. As in the case of triangulated categories, Mod_α-$\mathcal{C}$ will denote the category of contravariant additive functors from $\mathcal{C}$ into Ab that take coproducts of less than α objects to products. We will prove in Proposition 2.3.19 that Mod_α-$\mathcal{C}$ is a full exact subcategory of the Grothendieck category Mod-$\mathcal{C}$ of all contravariant additive functors from $\mathcal{C}$ into Ab.

The following lemma generalizes a result of Neeman [Nee01b, Lemma A.1.3].

Lemma 2.3.18. *Let α be a regular cardinal and let $\mathcal{C}$ be an essentially small additive category with coproducts of less than α objects. Then the inclusion* Mod_α-$\mathcal{C} \subset$ Mod-$\mathcal{C}$ *preserves α-filtered colimits.*

Proof. Fix an α-filtered colimit $\mathrm{colim}_I F_i$ in Mod-$\mathcal{C}$ such that F_i is in Mod_α-$\mathcal{C}$ for all $i \in I$. Let $\{X_j\}_{j\in J}$ be a set of objects in $\mathcal{C}$ such that $\# J < \alpha$. We want to show that

$$(\mathrm{colim}_I F_i)\left(\coprod_{j\in J} X_j\right) \cong \prod_{j\in J} ((\mathrm{colim}_I F_i)(X_j))\,.$$

This will prove that $\mathrm{colim}_I F_i$ is also a colimit in Mod_α-$\mathcal{C}$ and hence the lemma. There always exists a morphism

$$(\mathrm{colim}_I F_i)\left(\textstyle\coprod_{j\in J} X_j\right) \xrightarrow{\phi} \textstyle\prod_{j\in J}((\mathrm{colim}_I F_i)(X_j))$$

induced by the inclusion of the factors of the coproduct.

We will first show that ϕ is an epimorphism. Let $a = (a_j)_{j\in J}$ be an element of $\prod_{j\in J}((\mathrm{colim}_I F_i)(X_j))$. Since colimits in Mod-$\mathcal{C}$ are defined objectwise, for every $j \in J$ there is an $i_j \in I$ such that a_j can be identified with an element in $F_{i_j}(X_j)$. Since $\# J < \alpha$ and I is α-filtered, there exists a cocone of $\{i_j\}_{j\in J}$ in I that we denote by i'. By construction, a can be identified with an element of $\prod_{j\in J} F_{i'}(X_j)$, but now $\prod_{j\in J} F_{i'}(X_j) \cong F_{i'}\left(\coprod_{j\in J} X_j\right)$ since $F_{i'}$ is in Mod_α-$\mathcal{C}$. If we consider the following commutative diagram

$$\begin{tikzcd}
(\mathrm{colim}_I F_i)\left(\coprod_{j\in J} X_j\right) \arrow[r, "\phi"] & \prod_{j\in J}((\mathrm{colim}_I F_i)(X_j)) \\
F_{i'}\left(\coprod_{j\in J} X_j\right) \arrow[u, "\pi"] \arrow[r, "\cong"] & \prod_{j\in J} F_{i'}(X_j), \arrow[u]
\end{tikzcd}$$

then $a = \phi(\pi(a))$. Hence, a is in the image of ϕ.

Next we show that ϕ is a monomorphism. Let a be an element of $(\mathrm{colim}_I F_i)\left(\coprod_{j\in J} X_j\right)$ such that $\phi(a) = 0$. Since colimits in Mod_α-$\mathcal{C}$ are defined objectwise, there exists an $i' \in I$, an element b of $F_{i'}\left(\coprod_{j\in J} X_j\right)$ and a map

$$F_{i'}\left(\coprod_{j\in J} X_j\right) \xrightarrow{\pi} (\mathrm{colim}_I F_i)\left(\coprod_{j\in J} X_j\right)$$

such that $\pi(b) = a$. If we consider the following commutative diagram

$$\begin{tikzcd}
(\mathrm{colim}_I F_i)\left(\coprod_{j\in J} X_j\right) \arrow[r, "\phi"] & \prod_{j\in J}((\mathrm{colim}_I F_i)(X_j)) \\
F_{i'}\left(\coprod_{j\in J} X_j\right) \arrow[u, "\pi"] \arrow[r, "\cong"] & \prod_{j\in J} F_{i'}(X_j), \arrow[u, "\psi"']
\end{tikzcd}$$

then $\phi(\pi(b)) = \phi(a) = 0$. Now we consider $b = (b_j)_{j\in J}$ as an object in $\prod_{j\in J} F_{i'}(X_j)$. Since the colimits in Mod_α-$\mathcal{C}$ are defined objectwise, for every $j \in J$ there exists an $i'_j \in I$, morphisms $j \to i'_j$ in I and a commutative diagram

$$\begin{tikzcd}
\prod_{j\in J} F_{i'}(X_j) \arrow[rr, "\psi"] \arrow[dr, "\iota"'] & & \prod_{j\in J}((\mathrm{colim}_I F_i)(X_j)) \\
& \prod_{j\in J} F_{i'_j}(X_j) \arrow[ur, "\psi'"'] &
\end{tikzcd}$$

such that $\iota(b) = 0$. Since $\# J < \alpha$ and I is α-filtered, there exists a cocone of $\{i'_j\}_{j\in J}$ in I that we denote by i''. Then there is a commutative diagram

$$\begin{tikzcd} \prod_{j\in J} F_{i'_j}(X_j) \arrow[rr, "\psi'"] \arrow[dr, "\iota'"'] & & \prod_{j\in J}((\mathrm{colim}_I F_i)(X_j)) \\ & \prod_{j\in J} F_{i''}(X_j) \arrow[ur, "\psi''"'] & \end{tikzcd}$$

such that $\iota' \circ \iota(b) = 0$. But now $\prod_{j\in J} F_{i''}(X_j) \cong F_{i''}\left(\coprod_{j\in J} X_j\right)$ since $F_{i''}$ is in Mod_α-$\mathcal{C}$ and, by construction, there is a commutative diagram

$$\begin{tikzcd} F_{i'}\left(\coprod_{j\in J} X_j\right) \arrow[rr, "\pi"] \arrow[dr, "\chi"'] & & (\mathrm{colim}_I F_i)\left(\coprod_{j\in J} X_j\right) \\ & F_{i''}\left(\coprod_{j\in J} X_j\right) \arrow[ur, "\pi'"'] & \end{tikzcd}$$

such that $a = \pi(b) = \pi' \circ \chi(b) = \pi'(0) = 0$. □

In the following proposition we summarize some of the properties of Mod_α-$\mathcal{C}$, which have been proved by Neeman [Nee01b] in the case $\mathcal{C} = \mathcal{T}^\alpha$.

Proposition 2.3.19. *Let α be an infinite cardinal and $\mathcal{C}$ an essentially small additive category with coproducts of less than α objects. The category* Mod_α-$\mathcal{C}$ *is an exact subcategory of* Mod-$\mathcal{C}$ *in the sense of [Fre64] and is a locally α-presentable satisfying* [AB3] *and* [AB4*]. *Even more, a generating set of α-presentable projectives is given by $\{\mathcal{C}(-,X) \mid X \text{ in } \tilde{\mathcal{C}}\}$ where $\tilde{\mathcal{C}}$ is a set of representatives of the isomorphism classes of objects in $\mathcal{C}$.*

Proof. Recall that the category Mod-$\mathcal{C}$ of additive contravariant functors from $\mathcal{C}$ into Ab inherits the abelian structure of Ab and also many of its properties [Pop73, Ch. 3, Theorem 4.2]. The short exact sequences are the objectwise exact sequences, *i.e.* a sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

is exact in Mod-$\mathcal{C}$ if and only if the sequence

$$0 \longrightarrow F(X) \longrightarrow G(X) \longrightarrow H(X) \longrightarrow 0$$

is exact in Ab for every object X in $\mathcal{C}$.

By [Fre64, Theorem 3.41], in order to see that Mod_α-$\mathcal{C} \subset$ Mod-$\mathcal{C}$ is exact it is enough to prove that, given a pair of objects A and B in Mod_α-$\mathcal{C}$, their coproduct in Mod-$\mathcal{C}$ lies in Mod$_\alpha$-$\mathcal{C}$ and that, given a morphism $f\colon M \to M'$ in Mod$_\alpha$-$\mathcal{C}$, its kernel and cokernel in Mod-$\mathcal{C}$ lie in Mod$_\alpha$-$\mathcal{C}$.

Let A and B be a pair of objects in Mod$_\alpha$-$\mathcal{C}$ and let $\{X_i\}_{i\in I}$ be a set of objects in $\mathcal{C}$ such that $\#I < \alpha$. Then

$$\begin{aligned}(A\oplus B)\left(\textstyle\coprod_{i\in I} X_i\right) &= A\left(\textstyle\coprod_{i\in I} X_i\right) \oplus B\left(\textstyle\coprod_{i\in I} X_i\right)\\ &= \left(\textstyle\prod_{i\in I} A(X_i)\right) \oplus \left(\textstyle\prod_{i\in I} B(X_i)\right)\\ &= \textstyle\prod_{i\in I}(A(X_i)\oplus B(X_i)) = \prod_{i\in I}(A\oplus B)(X_i).\end{aligned}$$

Let

$$0 \longrightarrow K \longrightarrow M \xrightarrow{f} M' \longrightarrow Q \longrightarrow 0$$

be a short exact sequence in Mod-$\mathcal{C}$ and let $\{X_i\}_{i\in I}$ be a set of objects in $\mathcal{C}$ with $\#I < \alpha$. Since exact sequences in Mod-$\mathcal{C}$ are objectwise exact and Ab satisfies [AB4*], we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K(\coprod_{i\in I} X_i) \arrow[r] \arrow[d, "f_1"] & M(\coprod_{i\in I} X_i) \arrow[r] \arrow[d, "f_2"] & M'(\coprod_{i\in I} X_i) \arrow[r] \arrow[d, "f_3"] & Q(\coprod_{i\in I} X_i) \arrow[r] \arrow[d, "f_4"] & 0\\
0 \arrow[r] & \prod_{i\in I} K(X_i) \arrow[r] & \prod_{i\in I} M(X_i) \arrow[r] & \prod_{i\in I} M'(X_i) \arrow[r] & \prod_{i\in I} Q(X_i) \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact and f_2 and f_3 are isomorphisms. Then, by diagram chase, it is easy to see that f_1 and f_4 are isomorphisms. This proves that Mod$_\alpha$-$\mathcal{C} \subset$ Mod-$\mathcal{C}$ is exact. In particular, Mod$_\alpha$-$\mathcal{C}$ is an abelian category and the exact sequences in Mod$_\alpha$-$\mathcal{C}$ are the objectwise exact sequences.

We will prove that Mod$_\alpha$-$\mathcal{C}$ satisfies [AB3*]. Recall that the product of $\{M_i\}_{j\in J}$ in Mod-$\mathcal{C}$ is computed as

$$\left(\prod_{j\in J} M_j\right)(X) = \prod_{j\in J}(M_j(X)).$$

If we assume that each M_j is in Mod$_\alpha$-$\mathcal{C}$, we claim that $\prod_{j\in J} M_j$ is also in Mod$_\alpha$-$\mathcal{C}$ and therefore it is also a product in this abelian category. Let $\{X_i\}_{i\in I}$ be a set of objects in $\mathcal{C}$ with $\# I < \alpha$, then we have the following sequence of isomorphisms:

$$\prod_{j\in J} M_j \left(\coprod_{i\in I} X_i\right) \cong \prod_{j\in J}\left(\prod_{i\in I} M_j(X_i)\right) \cong \prod_{i\in I}\left(\prod_{j\in J} M_j(X_i)\right).$$

This proves that Mod$_\alpha$-$\mathcal{C}$ satisfies [AB3*] and that products are computed objectwise. Since we have already proved that exact sequences in Mod$_\alpha$-$\mathcal{C}$

are computed objectwise, and Ab satisfies [AB4*], we infer that products are exact in Mod_α-$\mathcal{C}$.

The fact that $\{\mathcal{C}(-,X) \mid X \text{ in } \tilde{\mathcal{C}}\}$ is a generating set of projectives is a direct consequence of the Yoneda Lemma, stating that every morphism $\mathcal{C}(-,X) \to F$ in Mod_α-$\mathcal{C}$ corresponds bijectively to an element of $F(X)$. If F is such that $\mathcal{C}(-,X) \to F$ is zero for every X in $\tilde{\mathcal{C}}$, then $F(X) = 0$ for every X in $\tilde{\mathcal{C}}$. Hence, F is zero. This proves that $\{\mathcal{C}(-,X) \mid X \text{ in } \tilde{\mathcal{C}}\}$ generate. If $f\colon G \to F$ is an epimorphism in Mod_α-$\mathcal{C}$, then $f_X\colon G(X) \to F(X)$ is an epimorphism for every X in $\tilde{\mathcal{C}}$. Hence, every morphism $\mathcal{C}(-,X) \to F$ factorizes through G. This proves that the objects in $\{\mathcal{C}(-,X) \mid X \text{ in } \tilde{\mathcal{C}}\}$ are projective.

We will see that every object of the form $\mathcal{C}(-,X)$ is α-presentable. Consider an α-filtered colimit $\mathrm{colim}_I F_i$ of a diagram in Mod_α-$\mathcal{C}$. It is an object in Mod_α-$\mathcal{C}$ by Lemma 2.3.18. Then, by the Yoneda Lemma, there is a bijection between the set

$$\mathrm{Mod}_\alpha\text{-}\mathcal{C}(\mathcal{C}(-,X), \mathrm{colim}_I F_i)$$

and the set $(\mathrm{colim}_I F_i)(X)$ which is equal to $\mathrm{colim}_I(F_i(X))$ by Lemma 2.3.18. If we apply the Yoneda Lemma to each F_i, we obtain a bijection between the set $\mathrm{colim}_I(F_i(X))$ and

$$\mathrm{colim}_I \left(\mathrm{Mod}_\alpha\text{-}\mathcal{C}(\mathcal{C}(-,X), F_i)\right).$$

Hence, $\mathcal{C}(-,X)$ is α-presentable.

The fact that Mod_α-$\mathcal{C}$ satisfies [AB3] can be seen using the following abstract argument. Since the inclusion Mod_α-$\mathcal{C} \subset \mathrm{Mod}$-$\mathcal{C}$ is exact and preserves products, it preserves all limits [Pop73, Ch. 1, Theorem 4.1]. Then it has a left adjoint L and, in particular, it preserves coproducts [Fre64, Sec. 3 J]. Hence, a coproduct in Mod_α-$\mathcal{C}$ is the image under L of the corresponding coproduct in Mod-$\mathcal{C}$. □

From Proposition 2.3.19 it can be seen that Mod_α-$\mathcal{C}$ has enough projectives, and projective objects are precisely the direct summands of coproducts of the projectives in the generating set ([Nee01b, Remark 6.4.3]).

In general, coproducts in Mod-$\mathcal{C}$ of objects in Mod_α-$\mathcal{C}$ do not coincide with the coproducts computed in Mod_α-$\mathcal{C}$. In the case $\mathcal{C} = \mathcal{T}^\alpha$ for an α-compactly generated triangulated category $\mathcal{T}$, Neeman gave an explicit construction of the coproduct in Mod_α-$\mathcal{T}^\alpha$ and used this to prove that Mod_α-$\mathcal{T}^\alpha$ satisfies [AB4]; see [Nee01b, Proposition 6.1.15 and Lemma 6.3.2] for details.

Now we state some of the properties of the restricted Yoneda functor for α-compactly generated triangulated categories.

Proposition 2.3.20 (Neeman [Nee01b]). *Let α be a regular cardinal and let $\mathcal{T}$ be an α-compactly generated triangulated category. Then the restricted Yoneda functor*

$$\begin{aligned} S_\alpha\colon\ \mathcal{T} &\longrightarrow \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha. \\ X &\longmapsto \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \end{aligned}$$

is homological, respects products and coproducts and reflects isomorphisms.

Remark 2.3.21. In the case $\alpha = \aleph_0$, the category $\mathrm{Mod}_{\aleph_0}\text{-}\mathcal{T}^{\aleph_0}$ is equal to $\mathrm{Mod}\text{-}\mathcal{T}^c$ and it is a Grothendieck category, the representable functors being a set of projective generators. However, for $\alpha > \aleph_0$ this is not the case, since, in general, the category $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ does not satisfy [AB5]; *cf.* [Nee09, Remark 6.3.3]. Because of this, we cannot apply the standard procedure to prove that $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ has enough injectives. In fact, it does not have enough injectives in general. Neeman proved in [Nee01b, Section C.4] that if R is any discrete valuation ring and $\alpha = \aleph_1$, then $\mathrm{Mod}_\alpha\text{-D}(R)^\alpha$ does not have a cogenerator. And from this it can be seen [Nee01b, Lemma 6.4.6] that it does not have enough injectives. Neeman also observed in [Nee01b, Remark 6.4.7] that a slight modification of the argument also proves that $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ does not have enough injectives when $\mathcal{T}$ is $\mathrm{D}(\mathbb{Z})$ or Ho(Sp).

It is clear that the category $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ plays an important role in the study of Adams representability, but the fact that it is not Grothendieck makes the task more difficult. We have pointed out that not all colimits have to be exact. However, we will see in Proposition 2.3.25 that if $\mathcal{T}$ is an α-compactly generated triangulated category, then α-filtered colimits of exact sequences are exact in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$. Following [Nee01b, Definition B.2.1], we denote this property by [AB5$_\alpha$].

Definition 2.3.22. An abelian category is said to satisfy [AB5$_\alpha$] if it satisfies [AB3] and α-filtered colimits are exact. An α-*Grothendieck category* is an abelian category satisfying [AB5$_\alpha$] and with a generator.

In the case of Grothendieck categories, condition [AB5] can be formulated in terms of sums and intersections; see [Pop73, Ch. 3, Theorem 8.6] for details. We have an analogous description for [AB5$_\alpha$]. Before we give the proof, we recall the definition of sum and intersection.

Definition 2.3.23. Let $\mathcal{A}$ be an abelian category satisfying [AB3]. Let $\{X_i\}_{i\in I}$ be a set of subobjects of X. Then the image of the universal morphism $\coprod_{i\in I} X_i \to X$ is called the *sum* of $\{X_i\}_{i\in I}$ and it is denoted by $\sum_{i\in I} X_i$.

Let A and B be a pair of subobjects of X. The pullback of the inclusions into X is called the *intersection* of A and B and it is denoted by $A \cap B$.

Since finite products and finite coproducts coincide, we obtain the formula $X + Y = (X \oplus Y)/(X \cap Y)$.

Lemma 2.3.24. *Let $\mathcal{A}$ be an abelian category satisfying* [AB3]. *If $\mathcal{A}$ satisfies* [AB5$_\alpha$]*, then, for every object Y, and every well ordered α-filtered chain $\{X_i\}_{i\in I}$ of subobjects of Y such that $X_i \subset X_j$ if $i \leq j$, and every subobject X of Y,*

$$\sum_{i\in I}(X_i \cap X) = \left(\sum_{i\in I} X_i\right) \cap X.$$

Proof. We can identify I as the set of ordinals $\{i \mid i < \gamma\}$ where $\#I = \#\gamma$ and $\alpha \leq \gamma$. Since $\sum_{i<\gamma}(X_i \cap X) \subset \sum_{i<\gamma} X_i$ and $\sum_{i<\gamma}(X_i \cap X) = X$, there is an inclusion $\sum_{i<\gamma}(X_i \cap X) \subset (\sum_{i<\gamma} X_i) \cap X$. To see the other inclusion, we will compare two short exact sequences in $\mathcal{A}$.

For every $i < \gamma$ there is a short exact sequence

$$0 \to X \to X_i + X \to (X_i + X)/X \to 0$$

and, since our category is [AB5$_\alpha$], we obtain the following short exact sequence after taking the colimit

$$0 \to X \to \sum_{i<\gamma}(X_i + X) \to \mathrm{colim}_{i<\gamma}((X_i + X)/X) \to 0.$$

Hence,

$$\begin{aligned} \mathrm{colim}_{i<\gamma}((X_i + X)/X) &\cong \left(\textstyle\sum_{i<\gamma} X_i + X\right)/X \\ &\cong \left(\textstyle\sum_{i<\gamma} X_i\right) / \left(\left(\textstyle\sum_{i<\gamma} X_i\right) \cap X\right) \end{aligned}$$

For every $i < \gamma$, we have another short exact sequence

$$0 \to X_i \cap X \to X_i \to (X_i + X)/X \to 0$$

and taking the colimit we obtain a short exact sequence

$$0 \to \sum_{i<\gamma}(X_i \cap X) \to \sum_{i<\gamma} X_i \to \mathrm{colim}_{i<\gamma}((X_i + X)/X) \to 0.$$

In particular, $\mathrm{colim}_{i<\gamma}((X_i + X)/X) \cong \sum_{i<\gamma} X_i / \sum_{i<\gamma}(X_i \cap X)$.

Finally, if we put together the two isomorphisms we obtain

$$\left(\sum_{i<\gamma} X_i\right) \cap X \subset \sum_{i<\gamma}(X_i \cap X).$$

□

The following proposition is due to Neeman [Nee01b, Lemma A.1.3] in the case of triangulated categories, but, as in Proposition 2.3.19, it is true in a more general context and we will need this generality in Chapter 4.

Proposition 2.3.25. *Let α be a regular cardinal and let $\mathcal{C}$ be an essentially small additive category with coproducts of less than α objects. Then* Mod_α-$\mathcal{C}$ *is an α-Grothendieck category.*

Proof. Notice that Mod-$\mathcal{C}$ satisfies [AB5] since Ab does. On the other hand, α-filtered colimits coincide in Mod-$\mathcal{C}$ and Mod_α-$\mathcal{C}$ by Lemma 2.3.18. Therefore α-filtered colimits are exact in Mod_α-$\mathcal{C}$. □

In the case of triangulated categories, more is true. The next result, due to Neeman, says that Mod_α-$\mathcal{T}^\alpha$ satisfies a universal property with respect to abelian categories satisfying [AB5$_\alpha$].

Theorem 2.3.26 (Neeman [Nee01b, Theorem B.2.5])**.** *Let α be a regular cardinal. Let $\mathcal{T}$ be an α-compactly generated triangulated category. Then the restricted Yoneda functor $S_\alpha\colon \mathcal{T} \to \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ is universal among homological functors from $\mathcal{T}$ to abelian categories satisfying* [AB5$_\alpha$] *that respect products.*

The following proposition is a useful characterization of cohomological functors in Mod_α-$\mathcal{T}^\alpha$.

Proposition 2.3.27 ([Nee01b, Section 7.2])**.** *Let $\mathcal{T}$ be an α-compactly generated triangulated category. Let H be an object in* Mod_α-$\mathcal{T}^\alpha$*. Then the following conditions are equivalent.*

1. *H is cohomological,* i.e. *it sends triangles to long exact sequences.*
2. *H is an α-filtered colimit of representable functors in $\mathcal{T}^\alpha$,* i.e. *there is an isomorphism $H \cong \mathrm{colim}_{i\in I}\mathcal{T}^\alpha(-, X_i)$, where I is α-filtered.*

In the case $\alpha = \aleph_0$, Beligiannis [Bel00a, Remark 8.12] and Krause [Kra00, Lemma 2.7] proved independently that the conditions in Proposition 2.3.27 are also equivalent to the fact that F is flat in the Grothendieck category Mod-$\mathcal{T}^c$. Recall that, if $\mathcal{C}$ is an additive category, an object in Mod-$\mathcal{C}$ is *flat* if it is a filtered colimit of representable functors. We will use this characterization to define the analog of flat objects in the context of α-Grothendieck categories in Chapter 4. The proofs by Beligiannis and Krause use the existence of a tensor product in Grothendieck categories. However, since Mod_α-$\mathcal{T}^\alpha$ is not Grothendieck, we will have to prove it in a different way.

2.3.2 Rosický functors and a conjecture by Neeman

The notion of Rosický functor was introduced by Neeman in [Nee09]. He was motivated by the work of Rosický [Ros05], [Ros08], [Ros09] related to the restricted Yoneda functor (Definition 2.3.6). In fact, a Rosický functor is an abstraction of a restricted Yoneda functor together with some axioms, and its existence implies Brown representability and its dual. Rosický proposed a strategy to prove that for every triangulated category $\mathcal{T}$ with a combinatorial model the restricted Yoneda functor $S_\alpha \colon \mathcal{T} \to \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ is a Rosický functor for arbitrarily large cardinals α. However, as we will see in Chapter 5, this is not always possible.

We begin with the definition of a Rosický functor [Nee09, Definitions 1.19 and 1.20].

Definition 2.3.28. Let $\mathcal{T}$ be a triangulated category satisfying [TR5] and [TR5*]. Let $\mathcal{A}$ be an abelian category satisfying [AB3] and [AB3*]. A *Rosický functor* is a homological functor $H \colon \mathcal{T} \to \mathcal{A}$ together with a set of objects $\mathcal{P} \subset \mathcal{T}$, closed under suspension, having the following properties.

R1 H is full.

R2 H reflects isomorphisms.

R3 H preserves products and coproducts.

R4 The objects of the form $H(p)$ with $p \in \mathcal{P}$ are projective in $\mathcal{A}$ and generate it.

R5 For every object y in $\mathcal{T}$ and $p \in \mathcal{P}$, the natural map

$$\mathcal{T}(p, y) \longrightarrow \mathcal{A}(H(p), H(y))$$

is an isomorphism.

R6 There exists a regular cardinal α such that every object in $\mathcal{P}$ is α-small.

As we have observed in Proposition 2.3.19 and Proposition 2.3.20, if $\mathcal{T}$ is an α-compactly generated triangulated category then the restricted Yoneda functor $S_\alpha \colon \mathcal{T} \to \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ satisfies all the properties of a Rosický functor except [R1], and in this case [R1] amounts to say that $\mathcal{T}$ satisfies α-Adams representability for morphisms.

Example 2.3.29. Let $\mathcal{T}$ be a compactly generated triangulated category. Assume further that $\mathcal{T}^c$ is such that $\# \mathcal{T}^c \leq \aleph_0$. Then, by Theorem 2.3.3, $S_{\aleph_0} \colon \mathcal{T} \to \mathrm{Mod}\text{-}\mathcal{T}^c$ is a Rosický functor.

We will prove in this section that, if a triangulated category has a Rosický functor with $\mathcal{P} = \mathcal{T}^\alpha$, then S_α is a Rosický functor. The next example provides a family of Rosický functors not coming from a restricted Yoneda functor.

Example 2.3.30. Let R be a hereditary ring, *i.e.* a ring whose global dimension is less than or equal to 1. For any complex X in $\mathrm{D}(R)$ there is an isomorphism $X \cong \coprod_{n\in\mathbb{Z}} \Sigma^n H^{-n}(X)$ [Nee92, Lemma 6.7]. Because of this fact, it is easy to see that the cohomology functor

$$\begin{aligned} H\colon \ \mathrm{D}(R)^{\mathrm{op}} &\longrightarrow (R\text{-Mod})^{\mathbb{Z}}, \\ X &\longmapsto H^*(X) \end{aligned}$$

where $(R\text{ -Mod})^{\mathbb{Z}}$ is the category of graded R-modules, is a Rosický functor with $\mathcal{P} = \{\Sigma^n R \mid n \in \mathbb{Z}\}$ and $\alpha = \aleph_0$. On the other hand, H is different from the restricted Yoneda functor

$$\begin{aligned} S_{\aleph_0}\colon \ \mathrm{D}(R)^{\mathrm{op}} &\longrightarrow \mathrm{Mod}_{\aleph_0}\text{-}\mathrm{D}(R)^{\aleph_0}. \\ X &\longmapsto \mathrm{D}(R)(-,X)|_{\mathrm{D}(R)^{\aleph_0}} \end{aligned}$$

We give an example: If $R = k\langle X, Y\rangle$ is the hereditary ring of Example 2.3.4 with k a field such that $\#k = \aleph_t$, where $t \geq 0$, then $S_{\aleph_0}$ is full if and only $\#k = \aleph_0$. Hence, $S_{\aleph_0}$ can be a Rosický functor or not depending on the cardinality of the field k, whereas H is always Rosický. So the existence of a Rosický functor in the case $\mathcal{P} \neq \mathcal{T}^\alpha$ may have nothing to do with the fact that the restricted Yoneda functor is a Rosický functor. In particular, it may have nothing to do with Adams representability.

The most relevant consequence of having a Rosický functor is the following theorem, which follows from Theorem 1.11, Remark 1.12 and Theorem 1.17 in [Nee09].

Theorem 2.3.31 (Neeman [Nee09])**.** *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5] *and* [TR5*]. *If $\mathcal{T}$ has a Rosický functor, then $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$ satisfy Brown representability.*

Using the fact that an α-compactly generated triangulated category $\mathcal{T}$ satisfies α-Adams representability for morphisms if and only if the restricted Yoneda functor S_α is a Rosický functor, we have the following direct corollary.

Corollary 2.3.32. *Let $\mathcal{T}$ be an α-compactly generated triangulated category. Assume that there exists a regular cardinal $\beta \geq \alpha$ such that $\mathcal{T}$ satisfies β-Adams representability for morphisms. Then $\mathcal{T}$ satisfies Brown representability and its dual.*

Remark 2.3.33. We would like to comment in detail the results of Rosický [Ros05], [Ros08], [Ros09]. In order to do this, notice that his notation is different from ours.

Let $\mathcal{K}$ be a stable combinatorial model category as in Theorem 2.1.32. In particular, there is a regular cardinal α for which $\mathcal{K}$ is locally α-presentable.

We also need to introduce the following notation. Given a category $\mathcal{C}$ with colimits, we denote by $\mathrm{Ind}_\alpha(\mathcal{C})$ the closure of $\mathcal{C}$ by α-filtered colimits. It can be constructed as the full subcategory of the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of contravariant functors from $\mathcal{C}$ into Set, consisting of α-filtered colimits of representable functors. In particular, by the definition of locally α-presentable, $\mathcal{K} = \mathrm{Ind}_\alpha(\mathcal{K}_\alpha)$, where $\mathcal{K}_\alpha \subset \mathcal{K}$ is the full subcategory of α-presentable objects.

As we reviewed in Theorem 2.1.32, Rosický proved that $\mathrm{Ho}(\mathcal{K})$ is well generated. More precisely, there exist arbitrarily large regular cardinals α for which $\mathrm{Ho}(\mathcal{K}_\alpha)$ corresponds to the full subcategory of α-compact objects generating $\mathrm{Ho}(\mathcal{K})$. Hence, the restricted Yoneda functor, in Rosický's notation, is

$$E_\alpha \colon \mathrm{Ho}(\mathcal{K}) \longrightarrow \mathrm{Ind}_\alpha(\mathrm{Ho}(\mathcal{K}_\alpha)).$$

where $\mathrm{Ind}_\alpha(\mathrm{Ho}(\mathcal{K}_\alpha))$ corresponds, in our notation, to the homological functors in Mod_α-$\mathrm{Ho}(\mathcal{K}_\alpha)$ by Proposition 2.3.27. Furthermore, if we denote by $P\colon \mathcal{K} \to \mathrm{Ho}(\mathcal{K})$ the functor that sends $\mathcal{K}$ to its homotopy category, he proved that the diagram

$$\begin{tikzcd}
 & \mathrm{Ind}_\alpha(\mathcal{K}_\alpha) \arrow[d, equal] & & \\
\mathcal{K}_\alpha \arrow[r, hook] \arrow[d, "P|_{\mathcal{K}_\alpha}"'] & \mathcal{K} \arrow[r, "P"] \arrow[d, "\mathrm{Ind}_\alpha(P|_{\mathcal{K}_\alpha})"'] & \mathrm{Ho}(\mathcal{K}) \arrow[r, equal] \arrow[dl, "E_\alpha"] & \mathrm{Ho}(\mathrm{Ind}_\alpha(\mathcal{K}_\alpha)) \\
\mathrm{Ho}(\mathcal{K}_\alpha) \arrow[r, hook] & \mathrm{Ind}_\alpha(\mathrm{Ho}(\mathcal{K}_\alpha)) & &
\end{tikzcd}$$

commutes up to isomorphism. Hence, α-Adams representability for morphisms reduces to the fact that $\mathrm{Ind}_\alpha(P|_{\mathcal{K}_\alpha})$ is full. In the language of homotopy theory, this corresponds to a problem of strictifying a homotopy α-filtering colimit into an α-filtering colimit. In general, this cannot be done. Given an α-filtered colimit in $\mathrm{Ho}(\mathcal{K}_\alpha)$, Rosický managed to construct a weak colimit in $\mathcal{K}_\alpha$ with the same cocone, but it was not α-filtering any more.

The idea of Rosický was that, even if this strategy could not be carried out for α, there would always exist a regular cardinal $\beta \geq \alpha$, which could be taken to be arbitrarily large, such that the restricted Yoneda functor E_β (that is, S_β in our notation) would be full. We will see in Example 5.2.8 that this cannot be done in general either.

The following question is inspired by the previous remark.

Question 2.3.34. *Let $\mathcal{T}$ be a well generated triangulated category with a combinatorial model. Is it true that there exist arbitrarily large regular cardinals α for which $\mathcal{T}$ satisfies α-Adams representability for morphisms?*

If the answer to this question were affirmative, then, by Theorem 2.3.31, every well generated triangulated category with a combinatorial model would satisfy Brown representability and its dual. For Brown representability we already knew this, but for its dual it would be a remarkable achievement. As we will see in the next section, very little is known about the dual of Brown representability and an affirmative answer to this question would provide a proof for almost all known well generated triangulated categories, in particular for the derived category of any Grothendieck category.

We will see that the answer to Question 2.3.34 is negative in Example 5.2.8. Specifically, we will see that the compactly generated triangulated category $\mathrm{D}(\mathbb{Z})$, *i.e.* the derived category of chain complexes of abelian groups, does not satisfy α-Adams representability for morphisms when $\alpha > \aleph_0$. However, it does satisfy $\aleph_0$-Adams representability both for objects and for morphisms by Theorem 2.3.3.

According to Neeman [Nee09, Remark 1.25], the above question was the most important motivation to state the following more general conjecture.

Conjecture 2.3.35 (Neeman [Nee09, Conjecture 1.27])**.** *A triangulated category admits a Rosický functor if and only if it is well generated.*

Observe that the counterexample to Question 2.3.34 is not a counterexample to Neeman's conjecture, because the restricted Yoneda functor $S_{\aleph_0}$ for $\aleph_0$ is a Rosický functor.

The *only if* part of the theorem is easy. We prove it in the following proposition.

Proposition 2.3.36 (Rosický)**.** *Let $\mathcal{T}$ be a triangulated category satisfying both* [TR5] *and* [TR5*] *that admits a Rosický functor $H\colon \mathcal{T} \to \mathcal{A}$. Then $\mathcal{T}$ is α-compactly generated for some regular cardinal α. In fact, the cardinal α can be chosen to be the same as in the conditions of H being Rosický.*

Proof. By [R6], objects in $\mathcal{P}$ are α-small, and, by [R3], [R4] and [R5], $\mathcal{P}$ is α-perfect. Then we only have to see that $\mathcal{P}$ generates.

Let X be an object in $\mathcal{T}$ and assume that every morphism $p \to X$ is zero if $p \in \mathcal{P}$. By [R5], $p \to X$ is zero if and only if $H(p) \to H(X)$ is zero. Hence, $H(X) \cong 0$ by [R4] and this implies that $X \cong 0$ by [R2]. $\square$

The *if* part of Conjecture 2.3.35 states that for each well generated triangulated category there is a Rosický functor. Observe that this would be implied by a positive answer to Question 2.3.34 in the case of triangulated categories with a combinatorial model, but, in general, the actual relation between the two statements is subtle, as may be inferred from Example 2.3.30. The following theorem says that, under some extra conditions, the existence of a Rosický functor implies that the restricted Yoneda functor is in fact a Rosický functor.

Proposition 2.3.37. *Let $\mathcal{T}$ be a triangulated category satisfying both* [TR5] *and* [TR5*]. *Assume that $\mathcal{T}$ admits a Rosický functor $H\colon \mathcal{T} \to \mathcal{A}$ together with a set of objects $\mathcal{P} \subset \mathcal{T}^\alpha$. Then the restricted Yoneda functor S_α is a Rosický functor if $\mathcal{P}$ contains a representative of every isomorphism class of objects in $\mathcal{T}^\alpha$.*

Proof. By Proposition 2.3.36, $\mathcal{T}$ is α-compactly generated. Then, by Proposition 2.3.19 and Proposition 2.3.20, the restricted Yoneda functor S_α together with $\mathcal{P}$ satisfies conditions [R2] to [R6]. So we only have to see that S_α is full.

Since we are assuming that $\mathcal{P}$ contains a representative of every isomorphism class of objects in $\mathcal{T}^\alpha$, there are isomorphisms

$$\mathcal{A}(H(p), H(X)) \cong \mathcal{T}(p, X) \cong \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha(S_\alpha(p), S_\alpha(X))$$

for every p in $\mathcal{P}$ and for every object X in $\mathcal{T}$.

Let $F\colon S_\alpha(X) \to S_\alpha(Y)$ be a morphism in Mod_α-$\mathcal{T}^\alpha$. Since objects of the form $S_\alpha(p)$ with $p \in \mathcal{P}$ are a generating set of projectives, there is an epimorphism

$$\coprod_{i\in I} S_\alpha(p) \to S_\alpha(X)$$

where the coproduct is taken over the set of all morphisms $\{p \to X \mid p \in \mathcal{P}\}$. For every $p \in \mathcal{P}$, we consider the composition

$$\begin{tikzcd}
\coprod_{i\in I} S_\alpha(p) \arrow[r, "f'"] & S_\alpha(X) \arrow[r, "F"] & S_\alpha(Y) \\
S_\alpha(p). \arrow[u] \arrow[ur, "f'_i"] \arrow[urr, "g'_i"'] & &
\end{tikzcd}$$

Since $\mathcal{T}(p, X) \cong \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha(S_\alpha(p), S_\alpha(X))$, there is a bijective correspondence between the morphisms f'_i and g'_i and morphisms $f_i\colon p \to X$ and $g_i\colon p \to Y$ respectively. Then we have morphisms $f\colon \coprod_{i\in I} p \to X$ and $g\colon \coprod_{i\in I} p \to Y$

and they are such that $S_\alpha(f) = f'$ and $S_\alpha(g) = F \circ S_\alpha(f)$. We complete g into a triangle in $\mathcal{T}$

$$\coprod_{i\in I} p \xrightarrow{g} Y \longrightarrow Z \longrightarrow \Sigma \coprod_{i\in I} p.$$

If we apply H, we obtain the following long exact sequence

$$\cdots \longrightarrow H(\Sigma^{-1}Z) \longrightarrow H(\coprod_{i\in I} p) \longrightarrow H(Y) \longrightarrow H(Z) \longrightarrow H(\Sigma \coprod_{i\in I} p) \longrightarrow \cdots$$

But now, since H commutes with coproducts, and the objects of the form $H(p)$ with $p \in \mathcal{P}$ generate $\mathcal{A}$, and there is a bijection $\mathcal{A}(H(p), H(X)) \cong \mathcal{T}(p, X)$, the morphism $H(f)\colon H(\coprod_{i\in I} p) \to H(X)$ is an epimorphism. Hence, there exists a morphism $F'\colon H(X) \to H(Y)$ such that $F' \circ H(f) = H(g)$. Since H is full, there exists a morphism $h\colon X \to Y$ such that $H(h) = F'$. By the bijection $\mathcal{A}(H(p), H(X)) \cong \mathcal{T}(p, X)$, h is such that $h \circ f = g$. Then $S_\alpha(h) \circ S_\alpha(f) = S_\alpha(g) = F \circ S_\alpha(f)$. Since the objects of the form $S_\alpha(p)$ are a set of generators of $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$, we obtain that $S_\alpha(h) = F$. □

The converse of the above theorem is false. Example 2.3.30 gives a Rosický functor that is not the restricted Yoneda functor $S_{\aleph_0}$ and the class $\mathcal{P} = \{\Sigma^n R \mid n \in \mathbb{Z}\}$ does not contain representatives of all isomorphism classes of compact objects in $\mathrm{D}(R)$. Even more, as we already explained, depending on the ring R, the functor $S_{\aleph_0}$ will be full or not. Specifically, by Theorem 2.3.3, if $\#R \leq \aleph_0$ then $S_{\aleph_0}$ is full. However, according to Example 2.3.4, there are hereditary rings for which $\aleph_0$-Adams representability for morphisms does not hold, *i.e.* $S_{\aleph_0}$ is not full.

2.4 The dual of Brown representability

In the context of well generated triangulated categories, the dual of Brown representability is still a conjecture. Remember from Remark 2.1.33 that the opposite of a well generated triangulated category might not be well generated and so it is not possible to proceed as in the proof Theorem 2.2.1. But, surprisingly, the dual of Brown representability does hold for compactly generated triangulated categories. As we will see in this section, the only known results follow from quotients of compactly generated triangulated categories, or the possible different examples coming from Krause's Theorem 2.2.8, Franke's Theorem 2.2.9 or Neeman's Theorem 2.3.31. The first result in this direction was obtained by Neeman [Nee98a, Theorem 2.1] for the stable homotopy category. This theorem can be reformulated, in the spirit of Neeman's Theorem 2.3.31, to cover other examples. Specifically, if Adams representability for objects and for morphisms holds for a compactly generated triangulated category, then Brown representability and its dual also hold. Hence, the dual of Brown representability for compactly generated triangulated categories holds under the same hypotheses as Theorem 2.3.3. However, Krause extended in [Kra02] the result for all compactly generated triangulated categories. First of all, we need the following definition.

Definition 2.4.1. Let $\mathcal{T}$ be a triangulated category satisfying [TR5]. A set of objects S in $\mathcal{T}$ is said to be a set of *symmetric generators* for $\mathcal{T}$ if the following conditions hold.

1. For every object $s \in S$, $\mathcal{T}(s, -)$ commutes with coproducts.

2. There exists a set of objects R in $\mathcal{T}$ such that, for every morphism $f \colon X \to Y$ in $\mathcal{T}$, the induced morphism $\mathcal{T}(s, f) \colon \mathcal{T}(s, X) \to \mathcal{T}(s, Y)$ is surjective for every $s \in S$ if and only if the induced morphism $\mathcal{T}(f, r) \colon \mathcal{T}(Y, r) \to \mathcal{T}(X, r)$ is injective for every $r \in R$.

Theorem 2.4.2 (Krause [Kra02, Theorem B]). *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5]. *Assume that $\mathcal{T}$ has a set of symmetric generators. Then $\mathcal{T}$ has arbitrary products and satisfies the dual of Brown representability.*

If $\mathcal{T}$ is compactly generated and S is a set of small generators of a triangulated category $\mathcal{T}$, take as R the set of objects representing the functors

$$\begin{aligned} \mathcal{T}^{\mathrm{op}} &\longrightarrow \mathrm{Ab} \\ X &\longmapsto \mathrm{Ab}(\mathcal{T}(s, X), \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

for every $s \in S$. With this definition, $\mathcal{T}$ satisfies the hypotheses of Theorem 2.4.2. Hence, we have the following corollary.

Corollary 2.4.3. *Let $\mathcal{T}$ be a compactly generated triangulated category. Then the dual of Brown representability holds for $\mathcal{T}$,* i.e. *for any homological functor $H \colon \mathcal{T} \to \mathrm{Ab}$ that sends products to products there is a natural isomorphism $H(-) \cong \mathcal{T}(X, -)$ for some object X in $\mathcal{T}$.*

Observe that, by the Yoneda Lemma, we also have that any natural transformation between homological functors from $\mathcal{T}$ that send products to products is represented by a unique morphism in $\mathcal{T}$.

There is another result due to Neeman about the dual of Brown representability which has very restrictive hypotheses.

Theorem 2.4.4 (Neeman [Nee01b, Theorem 8.6.1])**.** *Let $\mathcal{T}$ be an α-compactly generated triangulated category for a regular cardinal α. Suppose that the abelian category Mod_α-$\mathcal{T}^\alpha$ of contravariant functors from $\mathcal{T}^\alpha$ to* Ab *taking coproducts of less than α objects to products has enough injectives. Then $\mathcal{T}$ has products and satisfies the dual of Brown representability.*

Observe that Theorem 2.4.4 is a consequence of Theorem 2.4.2. It was noticed in [Kra02, p. 859] that, if S is a perfect generating set for a triangulated category $\mathcal{T}$, then S is a symmetric set of generators if and only if Mod_α-$\mathcal{T}^\alpha$ has an injective cogenerator. On the other hand, no example is known that satisfies the hypotheses of Theorem 2.4.2 but not the hypotheses of Theorem 2.4.4.

We can also give another result about the dual of Brown representability based on localization of compactly generated triangulated categories. It is important to notice that the localization has to be taken on the right and not on the left. This is a direct consequence of Theorem 2.2.1, Corollary 2.2.6 and Proposition 1.4.11.

Corollary 2.4.5. *Let $\mathcal{T}$ be a compactly generated triangulated category. Then $\mathcal{T}$ satisfies the dual of Brown representability. Assume further that there is a localizing subcategory $\mathcal{S} \subset \mathcal{T}$ generated by a set of objects such that $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ is a right Bousfield localization. Then $\mathcal{T}/\mathcal{S}$ is well generated and satisfies the dual Brown representability.*

Remark 2.4.6. The class of categories satisfying Brown representability is strictly bigger than the class of well generated triangulated categories. For instance, consider the opposite of a compactly generated category. It is not well generated by Remark 2.1.33, but it satisfies Brown representability by Corollary 2.4.3. However, not all triangulated categories satisfy Brown representability or its dual. The only known example is due to Casacuberta and Neeman [CN09]. Their example is based on a result of Freyd [Fre64, Ch. 6, Exercice A] and is constructed as follows. Let $\mathcal{A}$ be the abelian category of $\mathbb{Z}[I]$-modules, where I is the class of all small ordinals. Casacuberta and Neeman proved that the full triangulated subcategory of acyclic complexes in the homotopy category $\mathrm{K}(\mathcal{A})$ does not satisfy Brown representability nor its dual. As a consequence, it cannot be well generated and its opposite cannot either.

As we already pointed out, Šťovíček proved in [Šťo08] that there are triangulated categories which are not well generated (nor their opposites), yet that can satisfy some of the consequences of Brown representability, like the theorem of existence of adjoints (Proposition 1.1.21). However, those categories are not known to satisfy Brown representability nor its dual. Šťovíček proved his results for the class of *locally well generated triangulated categories*, which includes the homotopy categories of skeletally small additive categories.

2.5 An upper bound for the cardinality of the category $\mathcal{T}^\alpha$

In this section we give upper bounds for the cardinality of the category of α-compact objects of a triangulated category. For $\alpha \leq \aleph_1$, the cardinality of the category of α-compact objects is related to α-Adams representability. For $\alpha = \aleph_0$ it was shown in [CKN01] and for $\alpha = \aleph_1$ it will be shown in Chapter 6. For the purposes of this thesis, we are mainly interested in the results that imply that $\#\mathcal{T}^{\aleph_1} \leq \aleph_1$, but we state our results for arbitrary regular cardinals. In particular, we show that, for any α-compactly generated triangulated category, there exists a cardinal β such that $\#\mathcal{T}^\beta \leq \beta$.

Remember that we are working under the ZFC axioms. The *Generalized Continuum Hypothesis* (GCH) states that $2^{\aleph_n} = \aleph_{n+1}$ for every ordinal n or, equivalently, that if X is a set and κ is a cardinal such that $\#X \leq \kappa \leq \#\mathcal{P}(X)$ then either $\#X = \kappa$ or $\#\mathcal{P}(X) = \kappa$. The GCH is independent of ZFC. For some results in this section, instead of assuming the GCH, we will only assume that $2^\lambda = \lambda^+$ for every cardinal λ greater or equal to a fixed cardinal. This is done because of the applications of Chapter 6.

We will also need basic results about cardinal exponentiation. We collect them in the following theorem. Remember that the *cofinality* of an ordinal α, denoted by $\operatorname{cf}\alpha$, is the least limit ordinal β such that there is an increasing sequence of ordinals of length β with limit α. A cardinal α is regular if and only if $\operatorname{cf}\alpha = \alpha$.

Recall that a cardinal α is a *strong limit cardinal* cardinal if $2^\lambda < \kappa$ for every $\lambda < \kappa$. Every strong limit cardinal is a limit cardinal and, if we assume the GCH, then every limit cardinal is a strong limit cardinal. A cardinal $\kappa > \aleph_0$ is called *inaccessible* if it is regular and strong limit.

Theorem 2.5.1 ([Jec03, Theorems 5.15 and Theorem 5.20])**.** *Let λ be an infinite cardinal. Then for an infinite cardinal κ, κ^λ is computed as follows.*

1. *If $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.*
2. *If there exists some $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \kappa^\mu$.*
3. *If $\kappa > \lambda$ and $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then*

    3.1. *if $\operatorname{cf}\kappa > \lambda$ then $\kappa^\lambda = \kappa$.*

3.2. *if* $\operatorname{cf}\kappa \leq \lambda$ *then* $\kappa^\lambda = \kappa^{\operatorname{cf}\kappa}$.

4. *If* κ *is inaccessible, then* $\lambda^\mu < \kappa$ *for all* λ, $\mu < \kappa$.

If we assume the GCH, *then* κ^λ *is computed as follows.*

1. *If* $\kappa \leq \lambda$, *then* $\kappa^\lambda = \lambda^+$.
2. *If* $\operatorname{cf}\kappa \leq \lambda < \kappa$, *then* $\kappa^\lambda = \kappa^+$.
3. *If* $\lambda < \operatorname{cf}\kappa$, *then* $\kappa^\lambda = \kappa$.

We will also need a variation on the cardinal exponentiation. Let $\kappa = \#A$ be a cardinal. If $\lambda \leq \kappa$, then κ^λ is the cardinality of the set of all subsets of A of cardinality λ ([Jec03, Lemma 5.7]). If, instead of this, we consider the cardinality of the set of all subsets of A of cardinality less than λ, then we obtain $\kappa^{<\lambda}$. We use the following terminology for this notion.

Definition 2.5.2. Let κ and λ be two cardinals. We define

$$\kappa^{<\lambda} = \sup\{\kappa^\mu \mid \mu \text{ is a cardinal and } \mu < \lambda\}.$$

The following theorem gives a way to compute the exponentiation that we just defined.

Theorem 2.5.3 ([Jec03, Ch. 5])**.** *Let* κ *be an infinite cardinal.*

1. *If* κ *is regular and a limit, then* $\kappa^{<\kappa} = 2^{<\kappa}$. *If* κ *is regular and a strong limit, then* $\kappa^{<\kappa} = \kappa$.
2. *If* κ *is singular and not a strong limit, then* $\kappa^{<\kappa} = 2^{<\kappa} > \kappa$.
3. *If* κ *is singular and a strong limit, then* $2^{<\kappa} = \kappa$ *and* $\kappa^{<\kappa} = \kappa^{\operatorname{cf}\kappa}$.
4. $\kappa^{<\aleph_0} = \kappa$.

Let λ *be an infinite cardinal. If we assume the* GCH, *then* $\kappa^{<\lambda}$ *is computed as follows.*

1. *If* $\kappa \leq \lambda$, *then* $\kappa^{<\lambda} = 2^{<\lambda} = \lambda$.
2. *If* $\operatorname{cf}\kappa \leq \mu < \kappa$ *for some* $\mu < \lambda$, *then* $\kappa^{<\lambda} = \kappa^+$.
3. *If* $\mu < \operatorname{cf}\kappa$ *for every* $\mu < \lambda$, *then* $\kappa^{<\lambda} = \kappa$.

Now we can begin with our first result in this subsection. The proof for the case $\alpha = \aleph_0$ is by Neeman and can be found in [MS06, Lemma 20.8.4], but we check that it also works for higher cardinals.

Proposition 2.5.4. *Let $\mathcal{T}$ be a triangulated category and let α be a regular cardinal. If $\mathcal{C} \subset \mathcal{T}$ is a full subcategory such that $\#\,\mathcal{C} \leq \alpha$, then the smallest thick subcategory $\mathcal{D} \subset \mathcal{T}$ containing $\mathcal{C}$ has cardinality less than or equal to α.*

Proof. Let X be an object in $\mathcal{T}$. We denote by S_X the full subcategory of $\mathcal{T}$ whose objects are the objects Y such that $\#\,\mathcal{T}(X,Y) \leq \alpha$ and $\#\,\mathcal{T}(Y,X) \leq \alpha$. The category S_X is clearly thick. If X is in $\mathcal{C}$, then $\mathcal{C} \subset S_X$ by assumption. Since $\mathcal{D}$ is the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{C}$, $\mathcal{D} \subset S_X$. This means that, for every object X in $\mathcal{C}$, $\#\,\mathcal{T}(X,Y) \leq \alpha$ and $\#\,\mathcal{T}(Y,X) \leq \alpha$ for every object Y in $\mathcal{D}$ or, what is the same, that for every object Y in $\mathcal{D}$, $\mathcal{C} \subset S_Y$ and then $\mathcal{D} \subset S_Y$. This proves that $\#\,\mathcal{D}(X,Y) \leq \alpha$ for every pair of objects X and Y in $\mathcal{D}$.

In order to finish the proof, we have to show that there is a set of isomorphism classes of objects in $\mathcal{D}$ of cardinality less than or equal to α. For this purpose, we will use that $\mathcal{D}$ can be inductively constructed by completing $\mathcal{C}$ with respect to direct summands and triangles; see [HPS97, Proposition 2.3.5] for details.

Let $\mathcal{D}_0 = \mathcal{C}$ and assume that we have constructed an ascending chain of subcategories $\mathcal{D}_0 \subset \cdots \subset \mathcal{D}_i$ such that $\#\,\mathcal{D}_j \leq \alpha$ for every $j \leq i$. We define $\mathcal{D}_{i+1}$ as the full subcategory of $\mathcal{T}$ whose objects are direct summands of objects C_f that occur in triangles

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow \Sigma X$$

where f is in $\mathcal{D}_i$. Since $\#\,\mathcal{D}_i \leq \alpha$, there is a set of less than or equal to α objects of the form C_f with f in $\mathcal{D}_i$. Now observe that each object of the form C_f is also in $\mathcal{D}$, since $\mathcal{D}$ is the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{C} = \mathcal{D}_0$. Then, by the previous part of the proof, $\#\,\mathcal{T}(C_f, C_f) \leq \alpha$ for every object of the form C_f. In particular, the set of idempotent morphisms $C_f \to C_f$ and then the set of direct summands of objects of the form C_f for f in $\mathcal{D}_i$ has cardinality less than or equal to α. This proves that the set of isomorphism classes of objects in $\mathcal{D}_{i+1}$ has cardinality less than or equal to α. But now, since $\mathcal{D}_{i+1}$ is a subcategory of $\mathcal{D}$, by the previous part of the proof, it has less than or equal to α morphisms between every pair of objects. This proves that $\#\,\mathcal{D}_{i+1} \leq \alpha$.

Finally, $\mathcal{D} = \bigcup_{i \in \mathbb{N}} \mathcal{D}_i$ has cardinality less than or equal to α. $\square$

If $\mathcal{T}$ a triangulated category and S is a generating set of compact objects, then the smallest thick subcategory containing all the objects in S is the category of compact objects of $\mathcal{T}$. Then we have the following direct corollary.

Corollary 2.5.5. *Let $\mathcal{T}$ be a triangulated category. If S is a generating set of compact objects such that the full subcategory $\mathcal{C} \subset \mathcal{T}$ with set of objects S has cardinality less than or equal to α, then $\#\mathcal{T}^c \leq \alpha$.*

Corollary 2.5.5 applies only to compactly generated triangulated categories. The next result applies to general α-compactly generated triangulated categories, but we need to assume an extra hypothesis on the triangulated category that might seem too strong. Nevertheless, it will be very useful in the proof of Theorem 2.5.10.

Proposition 2.5.6. *Let $\mathcal{T}$ be an α-compactly generated triangulated category for a regular cardinal α and let S be a generating set of α-compact objects. Let $\beta > \alpha$ be a regular cardinal and assume either that $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \beta$ or that β is inaccessible. Assume that $\#\mathcal{T}(s, Y) \leq \beta$ for every object s in $\mathcal{T}^\alpha$ and Y in $\mathcal{T}^\beta$. If $\#\,\mathcal{T}^\alpha \leq \beta$ for a regular cardinal $\beta \geq \alpha$, then $\#\,\mathcal{T}^\beta \leq \beta$.*

Proof. Recall from Proposition 2.1.16 that $\mathcal{T}^\beta$ is the smallest β-localizing subcategory containing S and, by Remark 2.1.17, we have an inclusion $\mathcal{T}^\alpha \subset \mathcal{T}^\beta$. We will construct $\mathcal{T}^\beta$ as a β-localizing subcategory $S_\beta \subset \mathcal{T}$ such that $S \subset S_\beta$ and $\#S_\beta \leq \beta$ following [Nee01b, Lemma 3.2.4 and Proposition 3.2.5]. The idea is to inductively complete S with respect to coproducts of less than β objects and triangles. We will prove that this construction does not increase cardinality. We first explain this construction for an arbitrary set of objects in $\mathcal{T}$.

Let R be a set of objects in $\mathcal{T}$. Following [Nee01b, Lemma 3.2.4], the smallest triangulated subcategory $T(R) \subset \mathcal{T}$ containing R can be constructed as $T(R) = \bigcup_{n=1}^{\infty} T_i(R)$ where $T_0(R)$ is the full subcategory of $\mathcal{T}$ with objects $S \cup \{0\}$ and $T_{i+1}(R)$ is the smallest full subcategory of $\mathcal{T}$ containing $T_i(R)$ and a class of representatives of objects C_f such that there exists a triangle

$$x \xrightarrow{f} y \longrightarrow C_f \longrightarrow \Sigma x$$

with f in $T_i(R)$.

Following [Nee01b, Proposition 3.2.5], we will construct by transfinite induction a category $S_\beta = \bigcup_{i<\beta} S_i$ and we will see that $\mathcal{T}^\beta = S_\beta$. We will use this construction to show that $\#\,\mathcal{T}^\beta = \#S_\beta \leq \beta$.

Let S' be the completion of S with respect to suspensions and notice that $\#S' = \max\{\#S, \aleph_0\}$. Let $S_0 = T(S')$. Since $S_0 \subset \mathcal{T}^\alpha$, we obtain that $\#S_0 \leq \#\,\mathcal{T}^\alpha \leq \beta$ and by hypothesis $\#\mathcal{T}(s, Y) \leq \beta$ for every object s in S_0 and Y in $\mathcal{T}^\beta$. Let $i < \beta$ be an ordinal and assume that we have constructed S_i such that $\#S_i \leq \beta$ and $\#\mathcal{T}(s, Y) \leq \beta$ for every object s in S_i and Y in

$\mathcal{T}^\beta$. If $i+1$ is a successor ordinal, we define the category S'_{i+1} as the full subcategory of $\mathcal{T}$ with objects a set of representatives of coproducts of less than β objects in S_i. Notice that $S'_{i+1} \subset \mathcal{T}^\beta$. Since we assume that $\# S_i \leq \beta$, then the set of objects in S'_{i+1} has cardinality less than or equal to $\beta \cdot \beta^{<\beta} = \beta$ by Theorem 2.5.3 and the assumption that $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \beta$ or the inaccessibility. For every object Y in $\mathcal{T}^\beta$ and every object $\coprod_{j \in J} X_j$ in S'_{i+1} with X_j in S_i and $\# J < \beta$, we have an isomorphism

$$\mathcal{T}\left(\coprod_{j \in J} X_j, Y\right) \cong \prod_{j \in J} \mathcal{T}(X_j, Y).$$

Then, by induction hypothesis, $\#\mathcal{T}(\coprod_{j \in J} X_j, Y) \leq \# \prod_{j \in J} \mathcal{T}(X_j, Y) \leq \beta^{\# J}$. But now $\beta^{\# J} \leq \beta$ by Theorem 2.5.1, since $\# J < \beta$ and we are assuming either that $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \beta$ or that β is inaccessible. Next we define $S_{i+1} = T(S'_{i+1})$. Since we have seen that $\# S'_{i+1} \leq \beta$, the set of objects in S_{i+1} has cardinality less than or equal to β. For the sets of morphisms, since S is a set of generators of $\mathcal{T}$, it is enough to prove that $\# \mathcal{T}(s, C_f) \leq \beta$ for every $s \in S$ and every C_f such that there exists a triangle

$$x \xrightarrow{f} y \longrightarrow C_f \xrightarrow{\phi} \Sigma x$$

with x and y in S'_{i+1}, but this is true by hypothesis since C_f is an object in $\mathcal{T}^\beta$. Thus, $\# S_{i+1} \leq \beta$.

If $i < \beta$ is a limit ordinal, we define $S_i = \bigcup_{j<i} S_j$. It is clear that if $\# S_j \leq \beta$ for all $j < i$, then $\# S_i \leq \beta$.

Finally, we define $S_\beta = \bigcup_{i<\beta} S_i$. By construction, $\# S_\beta \leq \beta$ and it is easy to see that S_β is triangulated and β-localizing; see [Nee01b, Proposition 3.2.5] for details. Since $\mathcal{T}^\beta$ is the smallest β-localizing subcategory of $\mathcal{T}$ containing S, $\mathcal{T}^\beta \subset S_\beta$. On the other hand, since each S_i is constructed by taking coproducts of less than β objects and mapping cones and $\mathcal{T}^\beta$ is closed under these constructions, $\mathcal{T}^\beta = S_\beta$ and then $\# \mathcal{T}^\beta \leq \# S_\beta \leq \beta$. □

Our next goal is to prove that for every well generated triangulated category $\mathcal{T}$ there exists a regular cardinal α such that $\# \mathcal{T}^\alpha \leq \alpha$. The first motivation for this improvement was an attempt to answer Question 2.3.34. The idea was to use the analogy with the case $\alpha = \aleph_0$ following [CKN01]. Remember that, if $\mathcal{T}$ is a compactly generated triangulated category and $\# \mathcal{T}^{\aleph_0} \leq \aleph_0$, then $\mathcal{T}$ satisfies $\aleph_0$-Adams representability for objects and for morphisms, by Theorem 2.3.3. However, the analog of the proof of Theorem 2.3.3 cannot be carried out for an arbitrary cardinal α, as we pointed out in

Remark 2.3.5. Thus, even if we know that for every well generated triangulated category $\mathcal{T}$ there is a regular cardinal α such that $\#\mathcal{T}^\alpha \leq \alpha$, this does not imply that α-Adams representability holds for $\mathcal{T}$ as in the case $\alpha = \aleph_0$. A counterexample to this will be given in Example 5.2.8. Nevertheless, for the applications to $\aleph_1$-Adams representability we will use this result to give an upper bound to $\#\mathcal{T}^{\aleph_1}$.

The first step for the proof is the following improvement of a theorem by Krause [Kra02, Theorem C].

Theorem 2.5.7. *Let $\mathcal{T}$ be an α-compactly generated triangulated category. Let β be a regular cardinal such that $\beta > \max\{\alpha^{<\alpha}, \#\mathcal{T}^\alpha\}$. Assume either that β is greater than or equal to an inaccessible cardinal greater than $\max\{\alpha^{<\alpha}, \#\mathcal{T}^\alpha\}$ or that $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \beta$ and β is not the successor cardinal γ^+ of any cardinal γ with cofinality $\mathrm{cf}(\gamma) < \alpha$. Let $\mathcal{S}_\beta$ be the full subcategory of objects X in $\mathcal{T}$ such that $\#\mathcal{T}(s, X) < \beta$ for all $s \in \mathcal{T}^\alpha$. Then*

1. *$\mathcal{S}_\beta$ is triangulated.*
2. *An object X is in $\mathcal{S}_\beta$ if and only if $X \cong \operatorname{Hocolim} X_n$ where $X_0 = 0$ and for every n there is a triangle*
$$\coprod_{i\in I_n} s_i \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \Sigma \coprod_{i\in I_n} s_i$$
where s_i is in $\mathcal{T}^\alpha$ and $\#I_n < \beta$.
3. *The category $\mathcal{S}_\beta$ is β-localizing,* i.e. *it is closed under coproducts of fewer than β summands.*

The theorem holds, in particular, for any $\beta > \max\{\alpha^{<\alpha}, \#\mathcal{T}^\alpha\}$ that is a double successor. This also includes the cardinals considered in [Kra02, Theorem C].

Before we go into the proof of Theorem 2.5.7, we need a lemma.

Lemma 2.5.8. *Assume the same hypotheses as in Theorem 2.5.7. Let $s, s_i \in \mathcal{T}^\alpha$ for all $i \in I$ and suppose that $\#I < \beta$. Then $\#\mathcal{T}(s, \coprod_{i\in I} s_i) < \beta$.*

Proof. Since s is α-small, every map $s \to \coprod_{i\in I} s_i$ factors through a coproduct of less than α objects $\coprod_{i\in J} s_i$, for $J \subset I$ such that $\#J < \alpha$. Since $\mathcal{T}^\alpha$ is closed under coproducts of less than α objects, we have that $\#\mathcal{T}(s, \coprod_{i\in J} s_i) \leq \#\mathcal{T}^\alpha$. By definition, $\#I < \beta$, and I has exactly $\#I^{<\alpha}$ subsets of cardinality less than α. Hence, $\#\mathcal{T}(s, \coprod_{i\in I} s_i) \leq \#I^{<\alpha} \cdot \#\mathcal{T}^\alpha$. Since $\beta > \max\{\alpha^{<\alpha}, \#\mathcal{T}^\alpha\}$, the proof will be finished if we show that $\#I^{<\alpha} < \beta$.

If we assume the hypothesis that β is greater than or equal to an inaccessible cardinal, then Theorem 2.5.3 implies that $\#I^{<\alpha} < \beta$. For the other hypothesis on β, we divide the proof into three different cases that cover all the possibilities in the hypotheses.

> Case $\#I \leq \alpha$. We have $\#I^{<\alpha} \leq \alpha^{<\alpha} < \beta$.
>
> Case $\#I > \alpha$ and β is a limit cardinal. In this case we always have $\#I < \#I^+ < \beta$. Since $\# I^+$ is regular and $\#I^+ > \alpha$, using that $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \beta$ and Theorem 2.5.3, we have the following inequalities: $\#I^{<\alpha} \leq (\#I^+)^{<\alpha} = \#I^+ < \beta$.
>
> Case $\#I > \alpha$ and $\beta = \gamma^+$. In particular, we have $\#I \leq \gamma$. Then $\#I^{<\alpha} \leq \gamma^{<\alpha}$ and this cardinal is less than or equal to $\gamma < \beta$, by the assumption that $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \beta$, Theorem 2.5.3 and our assumption that $\mathrm{cf}(\gamma) \geq \alpha$. Hence, $\#I^{<\alpha} < \beta$. □

Now we proceed to prove the theorem.

Proof of Theorem 2.5.7. For part 1, we only have to see that $\mathcal{S}_\beta$ is closed under the formation of triangles. Let $X \to Y \to Z \to \Sigma X$ be a triangle in $\mathcal{T}$ such that X and Y belong to $\mathcal{S}_\beta$ and for all $s \in \mathcal{T}^\alpha$ we have an exact sequence

$$\mathcal{T}(s, X) \to \mathcal{T}(s, Y) \to \mathcal{T}(s, Z) \to \mathcal{T}(s, \Sigma X)$$

where $\#\mathcal{T}(s, X) < \beta$, $\#\mathcal{T}(s, Y) < \beta$ and $\#\mathcal{T}(s, \Sigma X) < \beta$. Therefore, $\#\mathcal{T}(s, Z) < \beta$, since β is a regular cardinal.

For the *only if* part of 2, fix an object X in $\mathcal{S}_\beta$. We will construct the objects X_n of the statement recursively. In the case $n = 0$, we define $X_0 = 0$. Now suppose that for a fixed $n \geq 0$ we have constructed, for every $k \leq n$, a morphism $f_k \colon X_k \to X$ and a triangle

$$\textstyle\coprod_{i \in I_{k-1}} s_i \longrightarrow X_{k-1} \xrightarrow{h_k} X_k \longrightarrow \Sigma \coprod_{i \in I_{k-1}} s_i$$

where s_i is in $\mathcal{T}^\alpha$, $\#I_{k-1} < \beta$ and such that the diagram

$$\begin{tikzcd} X_{k-1} \arrow[d, "h_n"'] \arrow[dr, "f_{k-1}"] & \\ X_k \arrow[r, "f_k"] & X. \end{tikzcd}$$

commutes. Let

$$U_n = \coprod_{s \in S} \{x \in \mathcal{T}(s, X_n) \mid f_n \circ x = 0\}$$

where S is a set of representatives for the isomorphism classes of objects in $\mathcal{T}^\alpha$ and define a map

$$g_n \colon \coprod_{x \in U_n} s \xrightarrow{\{x\}} X_n.$$

Complete g_n into a triangle

$$\coprod_{i \in U_n} s_i \xrightarrow{g_n} X_n \xrightarrow{h_{n+1}} X_{n+1} \longrightarrow \Sigma \coprod_{i \in U_n} s_i.$$

Since $\mathcal{T}(-, X)$ is a cohomological functor and $f_n \circ g_n = 0$, there exists a map $f_{n+1} \colon X_{n+1} \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
X_n \arrow[d, "h_{n+1}"'] \arrow[dr, "f_n"] & \\
X_{n+1} \arrow[r, "f_{n+1}"] & X.
\end{tikzcd}$$

Observe that $\#U_n < \beta$ since X_n is in $\mathcal{S}_\beta$ and $\#S < \beta$. Following Definition 1.1.10, we construct the homotopy colimit

$$\coprod_{n=0}^{\infty} X_n \xrightarrow{id - shift} \coprod_{n=0}^{\infty} X_n \longrightarrow \operatorname{Hocolim} X_n \longrightarrow \Sigma \coprod_{n=0}^{\infty} X_n.$$

Let $\psi \colon \coprod_{n=0}^{\infty} X_n \to X$ be the morphism induced by the h_n for all $n \geq 0$. By [TR0] and [TR3], there is a morphism of triangles

$$\begin{tikzcd}
\coprod_{n=0}^{\infty} X_n \arrow[r, "id - shift"] \arrow[d] & \coprod_{n=0}^{\infty} X_n \arrow[r] \arrow[d, "\psi"] & \operatorname{Hocolim} X_n \arrow[r] \arrow[d, dotted, "\phi"] & \Sigma \coprod_{n=0}^{\infty} X_n \arrow[d] \\
0 \arrow[r] & X \arrow[r, "id_X"] & X \arrow[r] & 0.
\end{tikzcd}$$

We claim that $\mathcal{T}(-, \phi)|_{\mathcal{T}^\alpha} \colon \mathcal{T}(-, \operatorname{Hocolim} X_n)|_{\mathcal{T}^\alpha} \to \mathcal{T}(-, X)|_{\mathcal{T}^\alpha}$ is an isomorphism. Let $K_n = \ker(\mathcal{T}(-, f_n)|_{\mathcal{T}^\alpha})$ for every $n \geq 0$. Since the composition $K_n \to \mathcal{T}(-, X_n)|_{\mathcal{T}^\alpha} \to \mathcal{T}(-, X_{n+1})|_{\mathcal{T}^\alpha}$ is zero by construction, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r] \arrow[d, "0"] & \mathcal{T}(-, X_n)|_{\mathcal{T}^\alpha} \arrow[r, "\mathcal{T}(-,f_n)|_{\mathcal{T}^\alpha}"] \arrow[d, "\mathcal{T}(-,h_n)|_{\mathcal{T}^\alpha}"'] & \mathcal{T}(-, X)|_{\mathcal{T}^\alpha} \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & K_{n+1} \arrow[r] & \mathcal{T}(-, X_{n+1})|_{\mathcal{T}^\alpha} \arrow[r, "\mathcal{T}(-,f_{n+1})|_{\mathcal{T}^\alpha}"] & \mathcal{T}(-, X)|_{\mathcal{T}^\alpha} \arrow[r] & 0
\end{tikzcd}$$

with exact rows and such that $\mathcal{T}(-,h_n)|_{\mathcal{T}^\alpha}$ factorizes through $\mathcal{T}(-,X)|_{\mathcal{T}^\alpha}$ for every $n \geq 0$. We denote by $\rho_n \colon \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \to \mathcal{T}(-,X_{n+1})|_{\mathcal{T}^\alpha}$ this factorization. This implies the following equalities.

$$\mathcal{T}(-,f_{n+1})|_{\mathcal{T}^\alpha} \circ \rho_n \circ \mathcal{T}(-,f_n)|_{\mathcal{T}^\alpha} = \mathcal{T}(-,f_{n+1})|_{\mathcal{T}^\alpha} \circ \mathcal{T}(-,h_n)|_{\mathcal{T}^\alpha} = \mathcal{T}(-,f_n)|_{\mathcal{T}^\alpha}.$$

Since $\mathcal{T}(-,f_n)|_{\mathcal{T}^\alpha}$ is an epimorphism, $\mathcal{T}(-,f_{n+1})|_{\mathcal{T}^\alpha} \circ \rho_n = id$ and then the short exact sequence

$$0 \longrightarrow K_{n+1} \longrightarrow \mathcal{T}(-,X_{n+1})|_{\mathcal{T}^\alpha} \xrightarrow{\mathcal{T}(-,f_{n+1})|_{\mathcal{T}^\alpha}} \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \longrightarrow 0$$

splits for every $n \geq 0$. We display what we have obtained in the following diagram.

$$\begin{tikzcd}
\mathcal{T}(-,X_0)|_{\mathcal{T}^\alpha} \arrow[r, "\mathcal{T}(-,h_0)|_{\mathcal{T}^\alpha}"] \arrow[d, "\cong"] & \mathcal{T}(-,X_1)|_{\mathcal{T}^\alpha} \arrow[r, "\mathcal{T}(-,h_1)|_{\mathcal{T}^\alpha}"] \arrow[d, "\cong"] & \mathcal{T}(-,X_2)|_{\mathcal{T}^\alpha} \arrow[r] \arrow[d, "\cong"] & \dots \\
K_0 \oplus \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \arrow[r, "0\oplus id"] & K_1 \oplus \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \arrow[r, "0\oplus id"] & K_2 \oplus \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \arrow[r] & \dots
\end{tikzcd}$$

where all the vertical maps are isomorphisms. If we take the colimit we obtain a short exact sequence

$$0 \longrightarrow \coprod_{n=0}^{\infty} \mathcal{T}(-,X_n)|_{\mathcal{T}^\alpha} \xrightarrow{id-shift} \coprod_{n=0}^{\infty} \mathcal{T}(-,X_n)|_{\mathcal{T}^\alpha} \longrightarrow \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \longrightarrow 0.$$

In particular, we have seen that the morphism *id−shift* is a monomorphism. On the other hand, we have the triangle

$$\coprod_{n=0}^{\infty} X_n \xrightarrow{id-shift} \coprod_{n=0}^{\infty} X_n \longrightarrow \operatorname{Hocolim} X_n \longrightarrow \Sigma \coprod_{n=0}^{\infty} X_n.$$

Since the restricted Yoneda functor S_α is homological and preserves countable coproducts, we obtain a long exact sequence

$$\cdots \longrightarrow \coprod_{n=0}^{\infty} \mathcal{T}(-,X_n)|_{\mathcal{T}^\alpha} \xrightarrow{id-shift} \coprod_{n=0}^{\infty} \mathcal{T}(-,X_n)|_{\mathcal{T}^\alpha} \longrightarrow \mathcal{T}(-,\operatorname{Hocolim} X_n)|_{\mathcal{T}^\alpha}$$

$$\longrightarrow \coprod_{n=0}^{\infty} \mathcal{T}(-,\Sigma X_n)|_{\mathcal{T}^\alpha} \xrightarrow{id-shift} \coprod_{n=0}^{\infty} \mathcal{T}(-,\Sigma X_n)|_{\mathcal{T}^\alpha} \longrightarrow \cdots$$

But now we have seen that the two morphisms denoted by *id−shift* in the long exact sequence are monomorphisms. Hence, $\mathcal{T}(-,\operatorname{Hocolim} X_n)|_{\mathcal{T}^\alpha} \cong \mathcal{T}(-,X)|_{\mathcal{T}^\alpha}$. Since $\mathcal{T}^\alpha$ generates $\mathcal{T}$, we conclude that $\operatorname{Hocolim} X_n \cong X$.

For the converse of part 2, assume that $X \cong \operatorname{Hocolim} X_n$ as above. We want to see that $\#\,\mathcal{T}(s, \operatorname{Hocolim} X_n) < \beta$ for all $s \in \mathcal{T}^\alpha$. For every n we have a triangle

$$\coprod_{i \in I_n} s_i \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \Sigma \coprod_{i \in I_n} s_i$$

where $s_i \in \mathcal{T}^\alpha$ and $\#I_n < \beta$ and $X_0 = 0$. Since $\#\,\mathcal{T}(s, \coprod_{i\in I_n} s_i) < \beta$ by Lemma 2.5.8, one sees by induction that $\#\,\mathcal{T}(s, X_n) < \beta$ for all n. But now, since β is regular, $\#\,\coprod_{n=0}^{\infty} \mathcal{T}(s, X_n) < \beta$ and we have seen that there is a long exact sequence

$$\cdots \longrightarrow \coprod_{n=0}^{\infty} \mathcal{T}(s, X_n) \xrightarrow{id-shift} \coprod_{n=0}^{\infty} \mathcal{T}(s, X_n) \longrightarrow \mathcal{T}(s, \operatorname{Hocolim} X_n) \longrightarrow \cdots$$

This implies that $\#\,\mathcal{T}(s, \operatorname{Hocolim} X_n) < \beta$.

To prove 3, we use the characterization from part 2, although it can be proved directly from Lemma 2.5.8. Let K be a set such that $\#K < \beta$ and let $X_k \in \mathcal{S}_\beta$ for all $k \in K$. For every $k \in K$ there is a triangle

$$\coprod_{n=0}^{\infty} (X_k)_n \xrightarrow{id-shift} \coprod_{n=0}^{\infty} (X_k)_n \longrightarrow \operatorname{Hocolim}(X_k)_n \longrightarrow \Sigma \coprod_{n=0}^{\infty} (X_k)_n$$

and for every n there is a triangle

$$\coprod_{i \in (I_k)_n} s_i \longrightarrow (X_k)_n \longrightarrow (X_k)_{n+1} \longrightarrow \Sigma \coprod_{i \in (I_k)_n} s_i$$

where $s_i \in \mathcal{T}^\alpha$, $\#(I_k)_n < \beta$ and $(X_k)_0 = 0$.

Since coproducts of triangles are triangles, we can deduce that

$$\coprod_{k \in K} \operatorname{Hocolim}(X_k)_n \cong \operatorname{Hocolim} \coprod_{k \in K} (X_k)_n$$

and that for every n there are triangles

$$\coprod_{i \in (\bigsqcup_{k\in K} I_k)_n} s_i \longrightarrow \coprod_{k\in K} (X_k)_n \longrightarrow \coprod_{k \in K} (X_k)_{n+1} \longrightarrow \Sigma \coprod_{i \in (\bigsqcup_{k\in K} I_k)_n} s_i$$

where $s_i \in \mathcal{T}^\alpha$, $\#\bigsqcup_{k\in K}(I_k)_n < \beta \cdot \beta = \beta$ and $\coprod_{k \in K}(X_k)_0 = 0$. $\square$

Corollary 2.5.9. *Assume the same hypotheses as in Theorem 2.5.7. Then the category $\mathcal{S}_\beta$ coincides with the category $\mathcal{T}^\beta$ of β-compact objects.*

Proof. By part 2 of Theorem 2.5.7, any β-localizing subcategory of $\mathcal{T}$ containing $\mathcal{T}^\alpha$ contains $\mathcal{S}_\beta$. Therefore $\mathcal{S}_\beta$ is the smallest β-localizing subcategory of $\mathcal{T}$ containing S. $\square$

Theorem 2.5.10. *Let α be a regular cardinal and let $\mathcal{T}$ be an α-compactly generated triangulated category. Let β be a regular cardinal. Assume either that $\beta > \max\{\alpha^{<\alpha}, \#\mathcal{T}^\alpha\}$ is inaccessible or that $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \beta$ and $\beta \geq \max\{\alpha, \#\mathcal{T}^\alpha\}$. Then $\#\mathcal{T}^\beta \leq \beta$.*

Proof. If $\beta \geq \max\{\alpha, \#\mathcal{T}^\alpha\}$ and $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \beta$, by Theorem 2.5.7 and Corollary 2.5.9, $\#\mathcal{T}(s, Y) < \beta^+$ for every object s in $\mathcal{T}^\alpha$ and Y in $\mathcal{T}^{\beta^+}$. Hence, $\#\mathcal{T}(s, Y) \leq \beta$ for every object s in $\mathcal{T}^\alpha$ and Y in $\mathcal{T}^\beta$.

If $\beta > \max\{\alpha^{<\alpha}, \#\mathcal{T}^\alpha\}$ is inaccessible, by Theorem 2.5.7 and Corollary 2.5.9, $\#\mathcal{T}(s, Y) < \beta$ for every object s in $\mathcal{T}^\alpha$ and Y in $\mathcal{T}^\beta$.

In any case, we are under the hypothesis of Proposition 2.5.6 and this implies that $\#\mathcal{T}^\beta \leq \beta$. $\square$

Theorem 2.5.10 will be a key result in Section 6.2, where we will have to prove that $\#\mathcal{T}^{\aleph_1} \leq \aleph_1$ for $\mathcal{T}$ the stable motivic homotopy theory, in which we do not have a good description of $\aleph_1$-compact objects.

Finally, we notice that, in general, the bound in Theorem 2.5.10 is best possible, since the category of β-compact objects has to be closed under coproducts of less than β objects.

Chapter 3

Adams representability and purity for higher cardinals

In this chapter we define the α-pure global dimension of a triangulated category for a regular cardinal α. We will prove that it is closely related to α-Adams representability and to the existence of Rosický functors.

Many results in this chapter are known in the case $\alpha = \aleph_0$ and can be found in [Bel00b], [CKN01] and [Nee01b].

3.1 Purity for higher cardinals in triangulated categories

The notion of α-pure global dimension of a triangulated category was introduced by Neeman in [Nee97] for $\alpha = \aleph_0$ and widely studied by Beligiannis in [Bel00b]. The following definition is the analog for higher cardinals.

Definition 3.1.1. Let $\mathcal{T}$ be an α-compactly generated triangulated category. The *α-pure global dimension* of $\mathcal{T}$ is defined as

$$\mathrm{Pgldim}_{\alpha}(\mathcal{T}) = \sup\{\mathrm{pd}(H) \mid H \text{ cohomological in } \mathrm{Mod}_{\alpha}\text{-}\mathcal{T}^{\alpha}\},$$

where pd denotes the projective dimension of an object in the abelian category Mod_{α}-$\mathcal{T}^{\alpha}$.

Remark 3.1.2. Recall from Section 2.3 that the category Mod_{α}-$\mathcal{T}^{\alpha}$ of additive contravariant functors from $\mathcal{T}^{\alpha}$ to Ab that send coproducts of less than α objects in $\mathcal{T}^{\alpha}$ to products is an abelian category satisfying [AB4] and [AB4*]. We have also shown that each set of functors $\{\mathcal{T}^{\alpha}(-,X)\}$ indexed by a set of representatives of isomorphism classes of objects X in $\mathcal{T}^{\alpha}$ is a set of α-presentable projective generators of Mod_α-$\mathcal{T}^\alpha$. It then follows that, the abelian category Mod_α-$\mathcal{T}^\alpha$ has enough projectives and every projective object is a direct summand of a coproduct of objects from this generating set. Thus, the α-pure global dimension of $\mathcal{T}$ is well defined.

Definition 3.1.3. An object F in Mod_α-$\mathcal{T}^\alpha$ is said to be *free* if it is a coproduct of representables in $\mathcal{T}^\alpha$, *i.e.* $F = \coprod_{\lambda\in\Lambda} \mathcal{T}^\alpha(-,t_\lambda)$ where t_λ is in $\mathcal{T}^\alpha$ for all $\lambda \in \Lambda$.

We fix some notation. We denote by $\mathrm{Proj}(\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha)$ the full subcategory of projective objects of Mod_α-$\mathcal{T}^\alpha$. For a full subcategory $\mathcal{S} \subset \mathcal{T}$, we denote by $\mathrm{Add}(\mathcal{S})$ the completion of $\mathcal{S}$ by coproducts and direct summands in $\mathcal{T}$.

Proposition 3.1.5 below is essentially contained in [Bel00b, Proposition 8.4], although in a slightly different context. We give a proof for completeness. Before stating this proposition, we need the following lemma.

Lemma 3.1.4. *Let $\mathcal{T}$ be an α-compactly generated triangulated category and let X, Y be a pair of objects in $\mathcal{T}$. If X is an object in $\mathrm{Add}(\mathcal{T}^\alpha)$, then the restricted Yoneda functor*

$$\begin{aligned} S_\alpha\colon\ \mathcal{T} &\longrightarrow \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha. \\ X &\longmapsto \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \end{aligned}$$

induces a bijective correspondence between morphisms $X \to Y$ in $\mathcal{T}$ and morphisms $\mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \to \mathcal{T}(-,Y)|_{\mathcal{T}^\alpha}$ in Mod_α-$\mathcal{T}^\alpha$.

Proof. Assume first that $X = \coprod_{i\in I} R_i$, where R_i is an object in $\mathcal{T}^\alpha$ for every $i \in I$. Since S_α commutes with coproducts by Proposition 2.3.19, it is enough to see that a morphism $\mathcal{T}^\alpha(-,R_i) \to \mathcal{T}(-,Y)|_{\mathcal{T}^\alpha}$ is representable by a unique morphism $R_i \to Y$ for every $i \in I$ and this is the Yoneda Lemma.

For the general case, let $F\colon \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \to \mathcal{T}(-,Y)|_{\mathcal{T}^\alpha}$ be a morphism with $X \in \mathrm{Add}(\mathcal{T}^\alpha)$, *i.e.* there is an object X' in $\mathcal{T}$ such that $X \oplus X'$ is a coproduct of objects in $\mathcal{T}^\alpha$. Let $\pi_X\colon X \oplus X' \to X$ be the retraction into X and $\iota_X\colon X \to X \oplus X'$ the inclusion. They satisfy the equality $\pi_X \circ \iota_X = id_X$. Then, by the previous case, the composition

$$\begin{tikzcd} \mathcal{T}(-,X\oplus X')|_{\mathcal{T}^\alpha} \arrow[r, shift left, "\mathcal{T}(-,\pi_X)|_{\mathcal{T}^\alpha}"] & \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \arrow[l, bend left, "\mathcal{T}(-,\iota_X)|_{\mathcal{T}^\alpha}"] \arrow[r, "F"] & \mathcal{T}(-,Y)|_{\mathcal{T}^\alpha} \end{tikzcd}$$

is representable by a morphism $g\colon X \oplus X' \to Y$. Hence, $\mathcal{T}(-,g\circ\iota_X)|_{\mathcal{T}^\alpha} = \mathcal{T}(-,g)|_{\mathcal{T}^\alpha} \circ \mathcal{T}(-,\iota_X)|_{\mathcal{T}^\alpha} = F \circ \mathcal{T}(-,\pi_X)|_{\mathcal{T}^\alpha} \circ \mathcal{T}(-,\iota_X)|_{\mathcal{T}^\alpha} = F$.

In order to finish the proof, assume that there are two morphisms f, $f'\colon X \to Y$ such that $\mathcal{T}(-,f)|_{\mathcal{T}^\alpha} \cong \mathcal{T}(-,f')|_{\mathcal{T}^\alpha}$. By the first part of the proof, there is a bijective correspondence between the set of morphisms $\{\mathcal{T}(-, X \oplus X')|_{\mathcal{T}^\alpha} \to \mathcal{T}(-, Y)|_{\mathcal{T}^\alpha}\}$ and $\{X \oplus X' \to Y\}$. Since $\mathcal{T}(-, f \circ \pi_X)|_{\mathcal{T}^\alpha} = \mathcal{T}(-, f' \circ \pi_X)|_{\mathcal{T}^\alpha}$, then $f \circ \pi_X = f' \circ \pi_X$. Hence, $f = f'$ because π_X is an epimorphism. □

Proposition 3.1.5. *Let α be a regular cardinal. Let $\mathcal{T}$ be an α-compactly generated triangulated category. Then the restricted Yoneda functor*

$$\begin{aligned} S_\alpha|_{\mathrm{Add}(\mathcal{T}^\alpha)} \colon \ \mathrm{Add}(\mathcal{T}^\alpha) &\longrightarrow \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha \\ X &\longmapsto \mathcal{T}(-, X)|_{\mathcal{T}^\alpha} \end{aligned}$$

is fully faithful and induces an equivalence of categories between $\mathrm{Add}(\mathcal{T}^\alpha)$ *and* $\mathrm{Proj}(\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha)$.

Proof. By Lemma 3.1.4, it is enough to prove that the essential image of $S_\alpha|_{\mathrm{Add}(\mathcal{T}^\alpha)}$ is equal to $\mathrm{Proj}(\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha)$.

Fix an object Z in $\mathrm{Add}(T^\alpha)$, *i.e.* $\coprod_{i\in I} X_i = Z \oplus Y$ where every X_i is an object in $\mathcal{T}^\alpha$ for every $i \in I$. Then, by Proposition 2.3.20,

$$\coprod_{i\in I} \mathcal{T}^\alpha(-, X_i) = \mathcal{T}^\alpha(-, \coprod_{i\in I} X_i)|_{\mathcal{T}^\alpha} = S_\alpha(Z \oplus Y) = S_\alpha(Z) \oplus S_\alpha(Y)$$

and, by Proposition 2.3.19, every $S_\alpha(X_i) = \mathcal{T}^\alpha(-, X_i)$ is projective and hence so is $\coprod_{i\in I} \mathcal{T}^\alpha(-, X_i)$. Thus $S_\alpha(Z)$ is projective.

Now fix a projective object P in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$. In order to finish the proof, we have to see that it is the image of an object in $\mathrm{Add}(\mathcal{T}^\alpha)$. By Proposition 2.3.19, P is a direct summand of $\coprod_{i\in I} \mathcal{T}^\alpha(-, X_i) = \mathcal{T}(-, \coprod_{i\in I} X_i)|_{\mathcal{T}^\alpha} = S_\alpha(\coprod_{i\in I} X_i)$. Then there is an idempotent morphism

$$\mathcal{T}(-, \coprod_{i\in I} X_i) \longrightarrow \mathcal{T}(-, \coprod_{i\in I} X_i)$$

whose image is P and, by Lemma 3.1.4, it is the image of a morphism $\coprod_{i\in I} X_i \xrightarrow{e} \coprod_{i\in I} X_i$ such that e is idempotent in $\mathcal{T}$. Then, by Proposition 1.1.11, it splits and gives a decomposition $\coprod_{i\in I} X_i \cong F \oplus G$, with $S_\alpha(F) = P$. □

Since objects of the form $\mathcal{T}(-, X)|_{\mathcal{T}^\alpha}$ are cohomological in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ by Proposition 1.1.19, we have the following corollary.

Corollary 3.1.6. *Assume the same hypotheses as in Proposition* 3.1.5. *Then projective objects in* $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ *are cohomological.*

The next results are generalizations of results by Neeman [Nee97] and Beligiannis [Bel00b] for the case $\alpha = \aleph_0$. We start with the following lemma, which is a direct consequence of the Nine Lemma [Fre64, p. 58] for abelian categories using the fact that exact sequences in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ are objectwise exact, as we have seen in Proposition 2.3.19.

Lemma 3.1.7 ([Nee01b, Lemma 7.2.5])**.** *Let $\mathcal{T}$ be an α-compactly generated triangulated category. Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$. If two elements in $\{A, B, C\}$ are cohomological, then so is the third.

The following proposition generalizes a part of [Bel00b, Theorem 11.8 and Remark 11.12].

Proposition 3.1.8. *Let α be a regular cardinal and let $\mathcal{T}$ be an α-compactly generated triangulated category. If H is a functor in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ such that $\mathrm{pd}(H) \leq 2$, then there exists an object X in $\mathcal{T}$ and a natural isomorphism between H and $\mathcal{T}(-, X)|_{\mathcal{T}^\alpha}$.*

Proof. By Proposition 3.1.5, if $\mathrm{pd}(H) \leq 2$, then there is a projective resolution

$$0 \longrightarrow \mathcal{T}(-, R_2)|_{\mathcal{T}^\alpha} \xrightarrow{\psi} \mathcal{T}(-, R_1)|_{\mathcal{T}^\alpha} \xrightarrow{\phi} \mathcal{T}(-, R_0)|_{\mathcal{T}^\alpha} \longrightarrow H \longrightarrow 0$$

where R_0, R_1 and R_2 are direct summands of coproducts of α-compact objects. Even more, since $S_\alpha|_{\mathrm{Add}(\mathcal{T}^\alpha)}$ is fully faithful, the morphisms ϕ and ψ are of the form $\psi = \mathcal{T}(-, g)|_{\mathcal{T}^\alpha}$ and $\phi = \mathcal{T}(-, f)|_{\mathcal{T}^\alpha}$, where $f \circ g = 0$. Complete g to a triangle

$$R_2 \xrightarrow{g} R_1 \xrightarrow{\beta} C_g \longrightarrow \Sigma R_2.$$

Since $f \circ g = 0$, there exists an $h\colon C_g \to R_0$ such that $h \circ \beta = f$. On the other hand, since $\mathcal{T}(-, g)|_{\mathcal{T}^\alpha}$ is a monomorphism, we obtain a short exact sequence

$$0 \longrightarrow \mathcal{T}(-, R_2)|_{\mathcal{T}^\alpha} \longrightarrow \mathcal{T}(-, R_1)|_{\mathcal{T}^\alpha} \longrightarrow \mathcal{T}(-, C_g)|_{\mathcal{T}^\alpha} \longrightarrow 0.$$

Moreover, we have a short exact sequence

$$0 \longrightarrow \mathcal{T}(-, C_g)|_{\mathcal{T}^\alpha} \xrightarrow{\mathcal{T}(-,h)|_{\mathcal{T}^\alpha}} \mathcal{T}(-, R_0)|_{\mathcal{T}^\alpha} \longrightarrow H \longrightarrow 0.$$

Complete h to a triangle

$$C_g \xrightarrow{h} R_0 \longrightarrow C_h \longrightarrow \Sigma C_g.$$

Since $\mathcal{T}(-,h)|_{\mathcal{T}^\alpha}$ is a monomorphism, we obtain two short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{T}(-,C_g)|_{\mathcal{T}^\alpha} \arrow[r, "\rho"] \arrow[d, equal] & \mathcal{T}(-,R_0)|_{\mathcal{T}^\alpha} \arrow[r] \arrow[d, equal] & \mathcal{T}(-,C_h)|_{\mathcal{T}^\alpha} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{T}(-,C_g)|_{\mathcal{T}^\alpha} \arrow[r, "\rho"] & \mathcal{T}(-,R_0)|_{\mathcal{T}^\alpha} \arrow[r] & H \arrow[r] & 0, \end{tikzcd}$$

from which it follows that $H \cong \mathcal{T}(-,C_h)|_{\mathcal{T}^\alpha}$. □

The following result is a generalization of [Nee97, Proposition 4.11].

Proposition 3.1.9. *Let α be a regular cardinal. Let $\mathcal{T}$ be an α-compactly generated triangulated category and let $\phi\colon \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \to \mathcal{T}(-,Y)|_{\mathcal{T}^\alpha}$ be a morphism in* Mod_α-$\mathcal{T}^\alpha$. *Assume that* $\mathrm{pd}(\mathcal{T}(-,X)) \leq 1$ *and* $\mathrm{pd}(\mathcal{T}(-,Y)) \leq 1$. *Then there exists a morphism f in $\mathcal{T}$ such that $\phi = \mathcal{T}(-,f)|_{\mathcal{T}^\alpha}$.*

Proof. By Proposition 3.1.5, we can choose projective resolutions as follows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{T}(-,R_1)|_{\mathcal{T}^\alpha} \arrow[r, "\mathcal{T}(-,h_X)|_{\mathcal{T}^\alpha}"] \arrow[d, "\mathcal{T}(-,f_1)|_{\mathcal{T}^\alpha}"'] & \mathcal{T}(-,R_0)|_{\mathcal{T}^\alpha} \arrow[r, "\psi_X"] \arrow[d, "\mathcal{T}(-,f_0)|_{\mathcal{T}^\alpha}"] & \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & \mathcal{T}(-,T_1)|_{\mathcal{T}^\alpha} \arrow[r, "\mathcal{T}(-,h_Y)|_{\mathcal{T}^\alpha}"] & \mathcal{T}(-,T_0)|_{\mathcal{T}^\alpha} \arrow[r, "\psi_Y"] & \mathcal{T}(-,Y)|_{\mathcal{T}^\alpha} \arrow[r] & 0, \end{tikzcd}$$

where R_0, T_0, R_1 and T_1 are direct summands of coproducts of objects in $\mathcal{T}^\alpha$. By Lemma 3.1.4, we can assume that $\psi_X = \mathcal{T}(-,i_X)|_{\mathcal{T}^\alpha}$ and $\psi_Y = \mathcal{T}(-,i_Y)|_{\mathcal{T}^\alpha}$. If we denote by C_{h_X} and C_{h_Y} the mapping cones of h_X and h_Y respectively, then we can construct the following morphism of triangles in $\mathcal{T}$:

$$\begin{tikzcd} R_1 \arrow[r, "h_X"] \arrow[d, "f_1"] & R_0 \arrow[r, "i'_X"] \arrow[d, "f_0"] & C_{h_X} \arrow[r] \arrow[d, "g"] & \Sigma R_1 \arrow[d, "\Sigma f_1"] \\ T_1 \arrow[r, "h_Y"] & T_0 \arrow[r, "i'_Y"] & C_{h_Y} \arrow[r] & \Sigma T_1. \end{tikzcd}$$

and there are factorizations $i_X = j_X \circ i'_X$ and $i_Y = j_Y \circ i'_Y$. Since $\mathcal{T}(-,h_X)|_{\mathcal{T}^\alpha}$ and $\mathcal{T}(-,h_Y)|_{\mathcal{T}^\alpha}$ are monomorphisms, if we apply S_α to this morphism of triangles, we infer, by the long exact sequence associated to the triangles that $\mathcal{T}(-,j_X)|_{\mathcal{T}^\alpha}$ and $\mathcal{T}(-,j_Y)|_{\mathcal{T}^\alpha}$ are isomorphisms and we have a commutative diagram

$$\begin{tikzcd} \mathcal{T}(-,C_{h_X})|_{\mathcal{T}^\alpha} \arrow[r, "\mathcal{T}(-,j_X)|_{\mathcal{T}^\alpha}"] \arrow[d, "\mathcal{T}(-,g)|_{\mathcal{T}^\alpha}"'] & \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \arrow[d, "\phi"] \\ \mathcal{T}(-,C_{h_Y})|_{\mathcal{T}^\alpha} \arrow[r, "\mathcal{T}(-,j_Y)|_{\mathcal{T}^\alpha}"] & \mathcal{T}(-,Y)|_{\mathcal{T}^\alpha}. \end{tikzcd}$$

Finally, since S_α reflects isomorphisms, $\phi = S_\alpha(j_Y \circ g \circ j_X^{-1})$. $\square$

The following result is a crucial generalization of results by Neeman [Nee97, Propositions 4.10 and 4.11] and Beligiannis [Bel00b, Theorem 11.8].

Theorem 3.1.10. *Let α be a regular cardinal and let $\mathcal{T}$ be an α-compactly generated triangulated category.*

1. *If* $\operatorname{Pgldim}_\alpha(\mathcal{T}) \leq 2$*, then $\mathcal{T}$ satisfies α-Adams representability for objects.*

2. $\operatorname{Pgldim}_\alpha(\mathcal{T}) \leq 1$ *if and only if $\mathcal{T}$ satisfies α-Adams representability for objects and for morphisms.*

Proof. The first part corresponds to Proposition 3.1.8. The *only if* of the second part follows from Proposition 3.1.9. We next prove the converse.

Assume that $\mathcal{T}$ satisfies α-Adams representability for objects and for morphisms, and fix a cohomological functor H in $\operatorname{Mod}_\alpha\text{-}\mathcal{T}^\alpha$. By α-Adams representability for objects, $H \cong \mathcal{T}(-,X)|_{\mathcal{T}^\alpha}$. By Proposition 3.1.5, there is an exact sequence

$$\mathcal{T}(-,S)|_{\mathcal{T}^\alpha} \xrightarrow{\mathcal{T}(-,f)|_{\mathcal{T}^\alpha}} \mathcal{T}(-,R)|_{\mathcal{T}^\alpha} \xrightarrow{\ \phi\ } \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \longrightarrow 0$$

in $\operatorname{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ where S and R are summands of coproducts of α-compact objects. By Lemma 3.1.4, $\phi = \mathcal{T}(-,g)|_{\mathcal{T}^\alpha}$ for some $g\colon R \to X$. From this exact sequence we will construct a projective resolution of $H \cong \mathcal{T}(-,X)|_{\mathcal{T}^\alpha}$.

First complete $f\colon S \to R$ to a triangle $S \to R \to C_f \to \Sigma S$. Since S is in $\operatorname{Add}(\mathcal{T}^\alpha)$ and $\mathcal{T}(-,g)|_{\mathcal{T}^\alpha} \circ \mathcal{T}(-,f)|_{\mathcal{T}^\alpha}$ vanishes, $g \circ f$ also vanishes by Lemma 3.1.4 and we obtain the following factorization:

$$\begin{tikzcd} S \arrow[r, "f"] & R \arrow[rr, "g"] \arrow[dr, "g'"'] & & X. \\ & & C_f \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

On the other hand, by exactness, we have the following factorization of $\mathcal{T}(-,g')|_{\mathcal{T}^\alpha}$:

$$\begin{tikzcd} \mathcal{T}(-,S)|_{\mathcal{T}^\alpha} \arrow[r, "\mathcal{T}(-,f)|_{\mathcal{T}^\alpha}"] & \mathcal{T}(-,R)|_{\mathcal{T}^\alpha} \arrow[rr, "\mathcal{T}(-,g')"] \arrow[dr, "\mathcal{T}(-,g)|_{\mathcal{T}^\alpha}"'] & & \mathcal{T}(-,C_f)|_{\mathcal{T}^\alpha} \arrow[dl, dotted, bend left, "\mathcal{T}(-,h)|_{\mathcal{T}^\alpha}"] \\ & & \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \arrow[ur, "\beta"] \arrow[dr] & \\ & 0 \arrow[ur] & & 0 \end{tikzcd}$$

and, by α-Adams representability for morphisms, $\beta = \mathcal{T}(-,h')|_{\mathcal{T}^\alpha}$ for a morphism $h'\colon X \to C_f$. These two factorizations imply that

$$\mathcal{T}(-,h)|_{\mathcal{T}^\alpha} \circ \beta \circ \mathcal{T}(-,g)|_{\mathcal{T}^\alpha} = \mathcal{T}(-,h)|_{\mathcal{T}^\alpha} \circ \mathcal{T}(-,g')|_{\mathcal{T}^\alpha} = \mathcal{T}(-,g)|_{\mathcal{T}^\alpha}$$

and, since $\mathcal{T}(-,g)|_{\mathcal{T}^\alpha}$ is an epimorphism, we obtain that $\mathcal{T}(-,h)|_{\mathcal{T}^\alpha} \circ \beta = \mathcal{T}(-,h\circ h')|_{\mathcal{T}^\alpha} = \mathcal{T}(-,id_X)|_{\mathcal{T}^\alpha}$. Then $h \circ h'$ is an isomorphism in $\mathcal{T}$ by Proposition 2.3.20. Now complete h' to a triangle

$$\begin{tikzcd} X \arrow[r, "h'"] & C_f \arrow[l, bend left, "(h\circ h')^{-1}\circ h"] \arrow[r, "j"] & C_{h'} \arrow[r] & \Sigma X. \end{tikzcd}$$

Since idempotents split in $\mathcal{T}$ by Proposition 1.1.11, we have an isomorphism $C_f \cong C_{h'} \oplus X$. Since $j \circ h'$ vanishes, the composition $j \circ g' = j \circ (h' \circ g)$ also vanishes. Let $i\colon C_{h'} \to C_f$ be a section of j. There is a factorization of j through l:

$$\begin{tikzcd} & & & C_{h'} \arrow[dl, bend right, "i"'] & \\ S \arrow[r, "f"] & R \arrow[rr, "g'"] \arrow[dr, "g"'] & & C_f \arrow[u, "j"'] \arrow[rr, "l"] & & \Sigma S. \arrow[ull, dotted, "k"'] \\ & & X \arrow[ur, "h'"'] & & \end{tikzcd}$$

Notice that $id_{C_{h'}} = j \circ i = (k \circ l) \circ i\colon C_{h'} \to C_{h'}$. In particular, $l \circ i$ is a section of k. Hence, $\Sigma S \cong C_{h'} \oplus \Sigma^{-1} C_k$ where C_k is the cone of $k\colon \Sigma S \to C_{h'}$.

Summarizing, we have proved that the triangle $S \to R \to C_f \to \Sigma S$ is isomorphic to

$$\Sigma^{-1}(C_{h'} \oplus \Sigma^{-1}C_k) \longrightarrow R \longrightarrow C_{h'} \oplus X \longrightarrow C_{h'} \oplus \Sigma^{-1}C_k$$

and this triangle splits into two exact triangles $\Sigma^{-1}C_{h'} \to 0 \to C_{h'} \to C_{h'}$ and $\Sigma^{-2}C_k \to R \to X \to \Sigma^{-1}C_k$, and the last one gives the following exact sequence:

$$0 \longrightarrow \mathcal{T}(-,\Sigma^{-2}C_k)|_{\mathcal{T}^\alpha} \longrightarrow \mathcal{T}(-,R)|_{\mathcal{T}^\alpha} \xrightarrow{\mathcal{T}(-,g)} \mathcal{T}(-,X)|_{\mathcal{T}^\alpha} \longrightarrow 0.$$

Finally, since $\mathcal{T}(-,\Sigma^{-1}Y)|_{\mathcal{T}^\alpha}$ and $\mathcal{T}(-,R)|_{\mathcal{T}^\alpha}$ are projectives, this sequence is a projective resolution of $H \cong \mathcal{T}(-,X)|_{\mathcal{T}^\alpha}$. $\square$

In the case $\alpha = \aleph_0$, Neeman used this characterization in his proof of the fact that every compactly generated triangulated category $\mathcal{T}$ such that $\#\mathcal{T}^c \le \aleph_0$ satisfies Adams representability (Theorem 2.3.3). He proved that if $\#\mathcal{T}^c \le \aleph_0$, then $\mathrm{Pgldim}(\mathcal{T}) \le 1$ [Nee97, Theorem 5.1].

3.2 Rosický functors and purity

Remember from Section 2.3.2 that a Rosický functor is an abstraction of the restricted Yoneda functor, which was introduced by Neeman in [Nee01b]. Neeman stated Conjecture 2.3.35, claiming that every well generated triangulated category has a Rosický functor. We have already pointed out that there are Rosický functors that fail to be restricted Yoneda functors.

In this section we give a necessary and sufficient condition for the existence of a Rosický functor, not necessarily being a restricted Yoneda functor, and its relation with α-Adams representability. Most of the results contained in this section have analogs in the previous section, where we worked with the restricted Yoneda functor instead of an abstract Rosický functor. In that case, it is straightforward from the definition that if the restricted Yoneda functor is a Rosický functor, then the triangulated category satisfies α-Adams representability for morphisms. This is not always true for an abstract Rosický functor.

Our main interest in this thesis is α-Adams representability, and we have already observed in Section 2.3.2 that this is related to condition [R1], *i.e.* fullness of Rosický functors associated to a triangulated category. This motivates the following definition.

Definition 3.2.1. Let $\mathcal{T}$ be a triangulated category satisfying [TR5] and [TR5*], and let $\mathcal{A}$ be an abelian category satisfying [AB3] and [AB3*]. Let $H\colon \mathcal{T} \to \mathcal{A}$ be a homological functor satisfying all the conditions of a Rosický functor except for possibly [R1]. The *pure global dimension* associated to H is

$$\operatorname{Pgldim}_H(\mathcal{T}) = \sup\{\operatorname{pd}(H(X)) \text{ in } \mathcal{A} \mid X \text{ is an object in } \mathcal{T}\}.$$

Notice that, when H is the restricted Yoneda functor S_α, the inequality $\operatorname{Pgldim}_{S_\alpha}(-) \leq \operatorname{Pgldim}_\alpha(-) \mathit{holds}.$

Recall from Definition 2.3.28 that a Rosický functor is given by a functor $H\colon \mathcal{T} \to \mathcal{A}$ and a fixed set of objects, closed under suspension, that we denote by $\mathcal{P}$.

Proposition 3.2.2. *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5] *and* [TR5*]*, and let $\mathcal{A}$ be an abelian category satisfying* [AB3] *and* [AB3*]*. Let $H\colon \mathcal{T} \to \mathcal{A}$ be a homological functor and $\mathcal{P} \subset \mathcal{T}$ a set of objects satisfying all the conditions of a Rosický functor except for possibly* [R1]*, and let* $\operatorname{Add}(\mathcal{P})$ *be the completion of $\mathcal{P}$ under coproducts and direct summands in $\mathcal{T}$. Then H induces a bijection*

$$\mathcal{T}(p, Y) \longrightarrow \mathcal{A}(H(p), H(Y)).$$

Proof. Let $\coprod_{i\in I} p_i$ be a coproduct of objects in $\mathcal{P}$. By [R5], H induces a bijection

$$\mathcal{T}(p_i, Y) \longrightarrow \mathcal{A}(H(p_i), H(Y))$$

for all $i \in I$ and then it also induces a bijection

$$\textstyle\coprod_{i\in I} \mathcal{T}(p_i, Y) \longrightarrow \coprod_{i\in I} \mathcal{A}(H(p_i), H(Y)) .$$

But now, [R3] implies that $H(\coprod_{i\in I} p_i) \cong \coprod_{i\in I} H(p_i)$ and then there is a bijection

$$\textstyle\mathcal{T}(\coprod_{i\in I} p_i, Y) \longrightarrow \mathcal{A}(H(\coprod_{i\in I} p_i), H(Y)) .$$

Let p be an object in $\mathrm{Add}(\mathcal{P})$ such that $p \oplus q$ is a coproduct of objects in $\mathcal{P}$ for some object q in $\mathcal{T}$. Let $j\colon p \to p \oplus q$ be the inclusion and $r\colon p \oplus q \to p$ the retraction, so that $r \circ j = id_p$. We will prove that, for every object Y in $\mathcal{T}$, H induces a bijection

$$\mathcal{T}(p, Y) \longrightarrow \mathcal{A}(H(p), H(Y)).$$

We will first prove that it is surjective. Fix a morphism $\phi\colon H(p) \to H(Y)$ in $\mathcal{A}$. By the first part of the proof, there exists a morphism $\psi\colon p \oplus q \to Y$ such that $H(\psi) = \phi \circ H(r)$. Then $H(\psi) \circ H(j) = \phi \circ H(r) \circ H(j)$ and this implies that $H(\psi \circ j) = \phi$.

In order to finish the prove, we have to show that

$$\mathcal{T}(p, Y) \longrightarrow \mathcal{A}(H(p), H(Y))$$

is injective. Let f, $g\colon p \to Y$ be a pair of morphisms in $\mathcal{T}$ such that $H(f) = H(g)$. Then $H(f \circ r) = H(g \circ r)\colon H(p \oplus q) \to H(Y)$. Since $p \oplus q$ is a coproduct of objects in $\mathcal{P}$, by the first part of the proof, $f \circ r = g \circ r$ and, since r is an epimorphism, $f = g$. $\square$

Corollary 3.2.3. *Under the same hypotheses as in Proposition* 3.2.2*, H induces an equivalence of categories from* $\mathrm{Add}(\mathcal{P})$ *to* $\mathrm{Proj}(\mathcal{A})$*.*

Proof. Full faithfulness follows from Proposition 3.2.2. Since H is homological and respects coproducts by [R3], objects in the image of $H|_{\mathrm{Add}(\mathcal{P})}$ are summands of coproducts of objects in the image of $H|_{\mathcal{P}}$ that are projective by [R4]. Hence, objects in $\mathrm{Add}(\mathcal{P})$ map into projective objects. On the other hand, since objects of the form $H(p)$ generate $\mathcal{A}$ by [R4], for every projective object P in $\mathcal{A}$ there is an epimorphism $\pi\colon \coprod_{i\in I} H(p_i) \to P$ with $p_i \in \mathcal{P}$ for all $i \in I$ together with a section $s\colon P \to \coprod_{i\in I} H(p_i)$ such that $\pi \circ s = id_P$. Hence, P is a direct summand of $\coprod_{i\in I} H(p_i)$. By [R3],

$\coprod_{i\in I} H(p_i) \cong H(\coprod_{i\in I} p_i)$ and, by Proposition 3.2.2, there is morphism a $f\colon \coprod_{i\in I} p_i \to \coprod_{i\in I} p_i$ such that $H(f) = s\circ\pi$. Since $H(f)\circ H(f) = H(f)$, $H|_{\mathrm{Add}(\mathcal{P})}$ is faithful and idempotents split in $\mathcal{T}$ by Proposition 1.1.11, there is a decomposition $\coprod_{i\in I} p_i \cong q\oplus q'$ where $q = \mathrm{im}(f)$. Finally, since H is additive, we see that $H(q)\cong P$. $\square$

The following result is the analog of Proposition 3.1.8 in the context of Rosický functors. In fact, the proof is the same with a distinct notation. We include it for the sake of completeness.

Proposition 3.2.4. *Let $\mathcal{T}$ be a triangulated category satisfying* [TR5] *and* [TR5*]*, and let $\mathcal{A}$ be an abelian category satisfying* [AB3] *and* [AB3*]*. Let $H\colon \mathcal{T}\to\mathcal{A}$ be a homological functor satisfying all the conditions of a Rosický functor except for possibly* [R1]*. Let X be an object in $\mathcal{A}$ such that* $\mathrm{pd}(X)\le 2$*. Then X is in the essential image of H.*

Proof. Since $\mathrm{pd}(X)\le 2$ and using Corollary 3.2.3, we have a projective resolution of X in $\mathcal{A}$ of the form

$$0 \longrightarrow H(p_3) \xrightarrow{H(f_2)} H(p_2) \xrightarrow{H(f_1)} H(p_1) \xrightarrow{\phi} X \longrightarrow 0$$

with p_3, p_2 and p_1 in $\mathrm{Add}(\mathcal{P})$. Since $0 = H(f_1)\circ H(f_2) = H(f_1\circ f_2)$, it follows from Proposition 3.2.2 that $f_1\circ f_2 = 0$. Then, if we complete f_2 to a triangle

$$p_3 \xrightarrow{f_2} p_2 \xrightarrow{g} C_{f_2} \longrightarrow \Sigma p_2,$$

there exists a morphism $h\colon C_{f_2}\to p_1$ such that $h\circ g = f_1$. Since H is homological and $H(f_2)$ is a monomorphism, we obtain a pair of short exact sequences

$$0 \longrightarrow H(p_3) \xrightarrow{H(f_2)} H(p_2) \xrightarrow{H(g)} H(C_{f_2}) \longrightarrow 0,$$
$$0 \longrightarrow H(C_{f_2}) \xrightarrow{H(h)} H(p_1) \xrightarrow{\phi} X \longrightarrow 0.$$

Complete h to a triangle

$$C_{f_2} \xrightarrow{h} p_1 \longrightarrow C_h \longrightarrow \Sigma C_{f_2}.$$

Since H is homological and $H(h)$ is a monomorphism, we obtain a short exact sequence

$$0 \longrightarrow H(C_{f_2}) \xrightarrow{H(h)} H(p_1) \longrightarrow H(C_h) \longrightarrow 0.$$

Comparing short exact sequences we obtain that $H(C_h)\cong X$. $\square$

The following result is the analog of part 2 of Theorem 3.1.10 for a generic Rosický functor.

Theorem 3.2.5. *Assume the same hypotheses as in Proposition* 3.2.4. *Then* H *satisfies* [R1] *if and only if* $\mathrm{Pgldim}_H(\mathcal{T}) \leq 1$.

Proof. Assume that $\mathrm{Pgldim}_H(\mathcal{T}) \leq 1$ and fix a morphism $f\colon H(X) \to H(Y)$. Then, by [R2], [R4] and Corollary 3.2.3, there is a diagram

$$\begin{tikzcd} 0 \arrow[r] & H(r_1) \arrow[r, "H(h_X)"] \arrow[d, "H(f_1)"] & H(r_0) \arrow[r, "H(i_X)"] \arrow[d, "H(f_0)"] & H(X) \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & H(t_1) \arrow[r, "H(h_Y)"'] & H(t_0) \arrow[r, "H(i_Y)"'] & H(Y) \arrow[r] & 0 \end{tikzcd}$$

with exact rows and r_1, r_0, t_1 and t_0 objects in $\mathrm{Add}(\mathcal{P})$. Then we have a diagram in $\mathcal{T}$ that we can complete to the following morphism of triangles

$$\begin{tikzcd} r_1 \arrow[r, "h_X"] \arrow[d, "f_1"] & r_0 \arrow[r, "i'_p"] \arrow[d, "f_0"] & X' \arrow[r] \arrow[d, "g"] & \Sigma r_1 \arrow[d, "\Sigma f_1"] \\ t_1 \arrow[r, "h_Y"'] & t_0 \arrow[r, "i'_q"'] & Y' \arrow[r] & \Sigma t_1. \end{tikzcd}$$

Hence, there are factorizations $j_X \circ i'_X = i_X$ and $j_Y \circ i'_Y = i_Y$. But now, since $H(h_X)$ and $H(h_Y)$ are monomorphisms, $H(j_X)$ and $H(j_Y)$ are isomorphisms by the long exact sequences associated to the triangles and we have a commutative diagram

$$\begin{tikzcd} H(X') \arrow[r, "H(j_X)"] \arrow[d, "H(g)"'] & H(X) \arrow[d, "f"] \\ H(Y') \arrow[r, "H(j_Y)"] & H(Y). \end{tikzcd}$$

Hence, by condition [R2], j_X and j_Y are also isomorphisms and then $f = H(j_Y \circ g \circ j_X^{-1})$. This finishes the *if* part.

Now assume that H is a Rosický functor. Fix an object $H(x)$ in $\mathcal{A}$. We will construct a projective resolution of $H(x)$ of length 1. By [R4], objects of the form $H(p)$ with $p \in \mathcal{P}$ generate $\mathcal{A}$. Hence there is a short exact sequence

$$\textstyle\coprod_{i\in I} H(p_i) \longrightarrow \coprod_{j\in J} H(q_j) \longrightarrow H(x) \longrightarrow 0.$$

By [R3], H commutes with coproducts, and then, by Proposition 3.2.2, this sequence is the image under H of a sequence

$$\textstyle\coprod_{i\in I} p_i \xrightarrow{\psi} \coprod_{j\in J} q_j \xrightarrow{\phi} x$$

and the composition vanishes by [R2]. Now complete ψ to a triangle

$$\coprod_{i\in I} p_i \xrightarrow{\psi} \coprod_{j\in J} q_j \xrightarrow{\phi'} x' \xrightarrow{\chi} \Sigma(\coprod_{i\in I} p_i).$$

The morphism ϕ factors through ϕ' by a morphism $h\colon x' \to x$. On the other hand, since H is homological, $H(\phi')$ factors through $H(\phi)$ by a morphism $H(x) \to H(x')$ which is of the form $H(h')$ by [R1]. As a consequence, $H(h \circ h')$ is an isomorphism and, by [R2], $h \circ h'$ is an isomorphism. Now complete h' to a triangle

$$\begin{tikzcd} x \arrow[r, "h'"] & x' \arrow[l, bend left, "(h\circ h')^{-1}\circ h"] \arrow[r, "j"] & z \arrow[r] & \Sigma x. \end{tikzcd}$$

Since idempotents split in a triangulated category by Proposition 1.1.11, we infer that $x' \cong x \oplus z$.

Now observe that $j \circ \phi'$ vanishes. Using the same idea as before, we see that j factors through χ by a morphism $k\colon \Sigma(\coprod_{i\in I} p_i) \to z$ that has a section. In particular, $\Sigma(\coprod_{i\in I} p_i) \cong z \oplus \Sigma^{-1}y$, where y is the cone of k. Summarizing, we have proved that the triangle

$$\coprod_{i\in I} p_i \xrightarrow{\psi} \coprod_{j\in J} q_j \xrightarrow{\phi'} x' \xrightarrow{\chi} \Sigma(\coprod_{i\in I} p_i)$$

is isomorphic to

$$\Sigma^{-1}(z \oplus \Sigma^{-1}y) \longrightarrow \coprod_{j\in I} q_j \longrightarrow z \oplus x \longrightarrow z \oplus \Sigma^{-1}y.$$

This triangle splits into a coproduct of the triangles $\Sigma^{-1}z \to 0 \to z \to z$ and $\Sigma^{-2}y \to \coprod_{j\in J} q_j \to x \to \Sigma^{-1}y$, and the last one yields an exact sequence

$$0 \longrightarrow H(\Sigma^{-2}y) \longrightarrow H(\coprod_{j\in J} q_j) \longrightarrow H(x) \longrightarrow 0.$$

Finally, we claim that this is a projective resolution of $H(x)$. The object $H(\coprod_{j\in J} p_j)$ is projective by Proposition 3.2.2 and $H(\Sigma^{-2}y)$ is projective because $\Sigma^{-2}y$ is a direct summand of $\coprod_{i\in I} p_i$. □

Remark 3.2.6. We want to notice that the set $\mathcal{P}$ of generating α-small objects in the definition of a Rosický functor can fail to contain representatives of isomorphism classes of objects in $\mathcal{T}^\alpha$, as in Example 2.3.30. This fact makes the study of Rosický functors and, consequently, the study of Conjecture 2.3.35 more complicated. Specifically, Theorem 3.2.5 does not imply α-Adams representability in general, and the best that we can prove in this direction is Proposition 2.3.37.

Chapter 4

Homological algebra in α-Grothendieck categories

We have seen the relation between α-Adams representability for triangulated categories and the category $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ of additive contravariant functors from $\mathcal{T}^\alpha$ to abelian groups that send coproducts of less than α objects to products. We have observed that filtered colimits need not be exact in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$, although α-filtered colimits are exact. We have called the categories with this and some other standard properties *α-Grothendieck*. In this chapter, we study α-Grothendieck categories. Specifically, we study the analog of the Auslander Lemma, as well as analogs of flat objects and α-pure exact sequences, and we use them to find upper bounds for the α-pure projective dimension of a triangulated category. The most important consequence of the results in this chapter is that $\aleph_1$-Adams representability for objects holds in many categories. This consequence will be explained in Chapter 6.

4.1 Auslander Lemma for α-Grothendieck categories

We will need an analog of the Auslander Lemma [Aus55] in the context of α-Grothendieck categories. Recall that the classical Auslander Lemma states that if an R-module M is a union of a well-ordered continuous ascending chain of submodules, $M = \bigcup_{i<\gamma} M_i$, and $\mathrm{pd}(M_{i+1}/M_i) \leq n$ for every $i < \gamma$, then $\mathrm{pd}(M) \leq n$.

The proof of the Auslander Lemma for abelian categories [Aus55], [FS01, Ch. IV, Lemma 2.6], [Oso73, Theorem 2.18] and [Sim77, Proposition 2.6] uses the axiom [AB5], but in our case we only have [AB5$_\alpha$]. We will give three different statements, the last one being the most similar to the classical

Auslander Lemma.

Theorem 4.1.1. *Let $\mathcal{A}$ be an α-Grothendieck category for a regular cardinal α. Let $M = \mathrm{colim}_{i<\gamma} M_i$ with $M_0 = 0$, $M_i \subset M_j$ if $i < j$ and $\gamma \geq \alpha$. Assume that $\mathrm{colim}_{i<\beta} M_i = M_\beta$ for every limit ordinal β such that $\alpha \leq \beta < \gamma$. If $\mathrm{pd}(M_j/M_i) \leq n$ for every $i \leq j \leq i+\alpha$, then $\mathrm{pd}(M) \leq n$.*

Proof. We will proceed by induction on n. Recall that we identify α with the least ordinal of cardinality α. Suppose first that $n = 0$. For every $\alpha \leq i < \gamma$, we have an exact sequence

$$0 \longrightarrow M_i \xrightarrow{\iota_{i+1}} M_{i+1} \xrightarrow{\pi_{i+1}} M_{i+1}/M_i \longrightarrow 0$$

where M_{i+1}/M_i is projective since $\mathrm{pd}(M_{i+1}/M_i) = 0$. Hence, there exists a retraction $r_{i+1} \colon M_{i+1}/M_i \to M_{i+1}$ such that the morphism

$$M_i \oplus (M_{i+1}/M_i) \xrightarrow{(\iota_{i+1}, r_{i+1})} M_{i+1}$$

is an isomorphism. We will prove by transfinite induction on the ordinals $\alpha \leq i \leq \gamma$ that the morphism

$$M_\alpha \oplus \left(\coprod_{\alpha \leq j < i} M_{j+1}/M_j\right) \longrightarrow M_i$$

given by the inclusion of the first factor and composition of sections and inclusions in the other factors is an isomorphism. Notice that the coproduct $\coprod M_{j+1}/M_j$ can be written as $\coprod M_j/(\mathrm{colim}_{l<j} M_l)$ where $M_j/(\mathrm{colim}_{l<j} M_l) = 0$ if j is a limit ordinal. We claim that this implies that $M = \mathrm{colim}_{i<\gamma} M_i$ is projective. By assumption, $M_k = M_k/M_0$ is projective for every $0 \leq k \leq \alpha$. In particular, M_α is projective and, for every $\alpha \leq j < \gamma$, M_{j+1}/M_j is projective. Since every coproduct of projectives is projective, this implies that

$$M = \mathrm{colim}_{i<\gamma} M_i \cong M_\alpha \oplus \left(\coprod_{\alpha \leq j < \gamma} M_{j+1}/M_j\right)$$

is projective.

Now we proceed with transfinite induction. If $i = \alpha$ then there is nothing to prove. Fix an ordinal $\alpha < i < \gamma$ and assume that the result is true for every $\alpha \leq j < i$. If i is a successor ordinal, we have the following composition:

$$\left(M_\alpha \oplus \left(\coprod_{\alpha \leq j < i-1} M_{j+1}/M_j\right)\right) \oplus (M_i/M_{i-1}) \to\ M_{i-1} \oplus (M_i/M_{i-1}) \xrightarrow{(\iota_i, r_i)} M_i$$

where the first morphism is an isomorphism by the induction hypothesis, the second morphism is an isomorphism by the assumptions, and the composition is as indicated. If $\alpha < i < \gamma$ is a limit ordinal, $\mathrm{colim}_{j<i} M_j = M_i$ by assumption and then the colimit indexed by $\alpha \le j < i$ of the isomorphisms

$$M_\alpha \oplus \left(\coprod_{\alpha \le j' < j} M_{j'+1}/M_{j'}\right) \longrightarrow M_j$$

is the isomorphism indicated

$$M_\alpha \oplus \left(\coprod_{\alpha \le j < i} M_{j+1}/M_j\right) \longrightarrow M_i$$

as we wanted to prove. This finishes the case $n = 0$.

Assume that the result is true for $n - 1$. For every $i < \gamma$, let $P_i \to M_i$ be an epimorphism with P_i projective and let

$$0 \longrightarrow Q \longrightarrow \coprod_{i<\gamma} P_i \longrightarrow M \longrightarrow 0$$

be the short exact sequence induced by the morphisms $P_i \twoheadrightarrow M_i \hookrightarrow M$. Since $\coprod_{i<\gamma} P_i$ is projective, $\mathrm{pd}(M) = \mathrm{pd}(Q) + 1$. Hence, the proof will be finished if we prove that $\mathrm{pd}(Q) \le n - 1$. We define $Q_i = Q \cap (\coprod_{j \le i} P_j)$. Recall form Definition 2.3.23 that the intersection is defined to be the pullback of the inclusions $Q \hookrightarrow \coprod_{i<\gamma} P_i$ and $\coprod_{j\le i} P_j \hookrightarrow \coprod_{i<\gamma} P_i$. This gives an ascending chain of subobjects of Q such that $\mathrm{colim}_{i<\gamma} Q_i = Q$. We will check that Q satisfies the hypotheses of the theorem for $n - 1$. Notice that, by [Pop73, Ch. 2, Proposition 6.4], we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Q_i \arrow[r] \arrow[d, hook] & \coprod_{j\le i} P_j \arrow[r] \arrow[d, hook] & (\coprod_{j\le i} P_j)/Q_i \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & Q \arrow[r] \arrow[d, equal] & Q + \coprod_{j\le i} P_j \arrow[r] \arrow[d, hook] & (Q + \coprod_{j\le i} P_j)/Q \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & Q \arrow[r] & \coprod_{i<\gamma} P_i \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

with exact rows and where $Q + \coprod_{j\le i} P_j$ is the image of $Q \oplus \coprod_{j\le i} P_j$ in $\coprod_{i<\gamma} P_i$. As a consequence, $(\coprod_{j\le i} P_j)/Q_i$ is isomorphic to the image of the morphism $\coprod_{j\le i} P_j \to M$, which is equal to M_i by definition. Hence, we have a short exact sequence

$$0 \longrightarrow Q_i \longrightarrow \coprod_{j\le i} P_j \longrightarrow M_i \longrightarrow 0$$

for every $i < \gamma$.

For every limit ordinal $\alpha \leq \beta < \gamma$, $\mathrm{colim}_{i<\beta} M_i = M_\beta$. Since $\beta \geq \alpha$, the colimit is α-filtered, hence exact because $\mathcal{A}$ is α-Grothendieck. This implies that $\mathrm{colim}_{i<\beta} Q_i = Q_\beta$ for every limit ordinal $\alpha \leq \beta < \gamma$.

For every $i \leq j \leq i + \alpha$, we have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Q_i \arrow[r] \arrow[d] & \coprod_{k\leq i} P_k \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q_j \arrow[r] \arrow[d] & \coprod_{k\leq j} P_k \arrow[r] \arrow[d] & M_j \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q_j/Q_i \arrow[r] \arrow[d] & \coprod_{i<k\leq j} P_k \arrow[r] \arrow[d] & M_j/M_i \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $\mathrm{pd}(M_j/M_i) \leq n$. Then $\mathrm{pd}(Q_j/Q_i) \leq n-1$. Finally, we can apply the induction hypothesis on Q to obtain that $\mathrm{pd}(Q) \leq n-1$ and hence $\mathrm{pd}(M) = \mathrm{pd}(Q) + 1 \leq n$. □

For the proof of the second analog of the Auslander Lemma we will need some technical lemmas. The first one is very useful for transfinite induction arguments with respect to a filtered category.

Lemma 4.1.2 ([Jen72, Lemma 1.4] or [AR94, Lemma p. 15])**.** *Let I be a directed poset such that $\#I = \alpha$. Then I is a well ordered union of filtered categories of cardinality less than α. More precisely, the following conditions hold:*

1. $I = \bigcup_{\mu<\alpha} I_\mu$.
2. $\#I_\mu < \alpha$ *for every* $\mu < \alpha$.
3. $I_\mu \subset I_{\mu'}$ *if* $\mu < \mu'$.
4. $I_\mu = \bigcup_{\lambda<\mu} I_\lambda$ *for every limit ordinal* $\lambda < \alpha$.

We will also need a result about the vanishing of derived functors of limits in the category of abelian groups. This result is very close to [Jen72, Theorem 3.1], but our hypotheses are weaker. Nevertheless, the proof in [Jen72] can be adapted to our case, as follows.

Lemma 4.1.3. *Let $F\colon I^{\mathrm{op}} \to \mathrm{Ab}$ be an inverse sequence in the category* Ab *of abelian groups. Let $A_i = F(i)$. Assume that $\#I \leq \aleph_k$ and that every morphism $A_{i+1} \to A_i$ is surjective. Then $\lim_I^{(n)} A_i = 0$ for all $n \geq k+1$.*

Proof. We will prove this result by induction on k. We denote by $\{A_i\}_I$ the diagram in $\mathrm{Ab}^{I^{\mathrm{op}}}$. We can assume that I is a directed poset such that $\#I \leq \aleph_k$ by [AR94, Theorem 1.5].

Let $k = 0$. In this case, I is a directed poset of countable cardinality and the condition that $A_{i+1} \to A_i$ is surjective for every i is equivalent to the fact that $\{A_i\}_I$ is flasque ([Jen72, Proposition 2.1]), *i.e.* $A_i \to A_j$ is surjective for all $j < i$. Hence, we infer that $\lim_I^{(n)} A_i = 0$ for all $n \geq 1$.

Now assume that the result is true for $0 \leq h < k$. We will see that it is also true for k.

In order to compute $\lim_I{}^{(n)} A_i$ for $n \geq 0$, we fix an injective resolution of $\{A_i\}_I$ in $\mathrm{Ab}^{I^{\mathrm{op}}}$:

$$0 \longrightarrow \{A_i\}_I \longrightarrow \{F_i^0\}_I \longrightarrow \{F_i^1\}_I \longrightarrow \cdots$$

where $\{F_i^n\}_I$ are injectives in $\mathrm{Ab}^{I^{\mathrm{op}}}$ for all $n \geq 0$ and we have short exact sequences

$$0 \longrightarrow \{A_i\}_I \longrightarrow \{F_i^0\}_I \longrightarrow \{X_i^1\}_I \longrightarrow 0$$

$$0 \longrightarrow \{X_i^n\}_I \longrightarrow \{F_i^n\}_I \longrightarrow \{X_i^{n+1}\}_I \longrightarrow 0.$$

In this situation, $\lim_I{}^{(n)} A_i \cong \lim_I{}^{(1)} X_i^{n-1}$.

Since $\#I \leq \aleph_k$, we can write $I = \bigcup_{\mu<\gamma} I_\mu$, $\#I_\mu \leq \aleph_{k-1}$, as in Lemma 4.1.2. Observe that, for every $\mu < \gamma$, we can restrict the resolution of $\{A_i\}_I$ into I_μ and $\{A_i\}_{I_\mu}$ satisfies the hypotheses of the lemma with $\#I_\mu \leq \aleph_{k-1}$ for every $\mu < \gamma$. Then the induction hypothesis implies that $\lim_{I_\mu}^{(1)} X_i^n = 0$ for every $\mu < \gamma$ and $n \geq k$, and we want to prove that $\lim_I^{(1)} X_i^{n+1} = 0$. This can be restated as follows: We know that $\lim_{I_\mu} F_i^n \to \lim_{I_\mu} X_i^n$ is surjective for every $\mu < \gamma$ and $n \geq k$, and we want to prove that $\lim_I F_i^{n+1} \to \lim_I X_i^{n+1}$ is surjective. Let s be a global section of $\lim_I X_i^{n+1}$. For every ordinal $\nu < \gamma$, the restriction s_ν of s into $\lim_{I_\nu} X_i^{n+1}$ has a preimage t_ν in $\lim_{I_\nu} F_i^{n+1}$. We will prove by transfinite induction on $\nu < \gamma$ that we can choose t_ν for all $\nu < \gamma$ such that if $\mu < \nu$ the restriction of t_ν into I_μ coincides with t_μ. This will provide a preimage t of s into $\lim_I F_i^{n+1}$ and will finish the proof.

If ν is a limit cardinal, the section s_ν is determined by the preimages of all s_μ for $\mu < \nu$ given by the induction hypothesis.

If $\nu = \mu + 1$, the result follows by diagram chase on the following.

$$
\begin{tikzcd}
 & & & & & & \lim_I X_i^{n+1} \arrow[dd, dotted] & \\
 & & & & \lim_I F_i^{n+1} \arrow[urr] \arrow[dd, dotted] & & & 0 \\
 & & & & & & \lim_{I_{\mu+1}} X_i^{n+1} \arrow[ur] \arrow[dd, "f^{n+1}"] & \\
\lim_{I_{\mu+1}} F_i^n \arrow[drr, "\chi'_{\mu+1}"] \arrow[ddd, "f^n"] & & & & \lim_{I_{\mu+1}} F_i^{n+1} \arrow[urr, "\chi_{\mu+1}"] \arrow[ddd, "g^{n+1}"] & & & 0 \\
 & & \lim_{I_{\mu+1}} X_i^n \arrow[urr, "\psi_{\mu+1}"] \arrow[dr] \arrow[ddd, "h^n"] & & & & \lim_{I_\mu} X_i^{n+1} \arrow[ur] & \\
 & 0 \arrow[ur] & & 0 & & & & \\
\lim_{I_\mu} F_i^n \arrow[d] \arrow[drr, "\chi'_\mu"] & & & & \lim_{I_\mu} F_i^{n+1} \arrow[uurr, "\chi_\mu"] \arrow[d] & & & \\
0 & & \lim_{I_\mu} X_i^n \arrow[urr, "\psi_\mu"] \arrow[drr] & & 0 & & & \\
0 \arrow[urr] & & & & 0. & & &
\end{tikzcd}
$$

We check how this is done. Let s be a global section in $\lim_I X_i^{n+1}$ and let $s_{\mu+1}$ and s_μ be its restrictions to $\lim_{I_{\mu+1}} X_i^{n+1}$ and $\lim_{I_\mu} X_i^{n+1}$ respectively. The inductive hypothesis says that there exists a preimage of s_μ in $\lim_{I_\mu} F_i^{n+1}$, that we denote by t_μ, such that its restriction into $\lim_{I_\nu} F_i^{n+1}$ is a preimage of s_ν for every $\nu \leq \mu$. Now let $t_{\mu+1}$ be a preimage of $s_{\mu+1}$. Its restriction t'_μ into $\lim_{I_\mu} F_i^{n+1}$ can be different from t_μ, but $\chi_\mu(t_\mu) = s_\mu = \chi_\mu(t'_\mu)$. By exactness, there is an element v_μ in $\lim_{I_\mu} X_i^n$ such that $\psi_\mu(v_\mu) = t_\mu - t'_\mu$. Also by exactness, there is an element u_μ in $\lim_{I_\mu} F_i^n$ such that $\chi'_\mu(u_\mu) = v_\mu$. But now $\{F_i\}_I$ is flasque; hence, u_μ is the restriction of an element $u_{\mu+1}$ in $\lim_{I_{\mu+1}} F_i^n$. Let $t''_{\mu+1} = \psi_{\mu+1} \circ \chi'_{\mu+1}(u_{\mu+1})$. By construction, $t''_{\mu+1}$ restricts into $t_\mu - t'_\mu$ in $\lim_{I_\mu} F_i^{n+1}$ and, by exactness, $\chi_{\mu+1}(t''_{\mu+1}) = 0$. Then the element $t_{\mu+1} + t''_{\mu+1}$ is a preimage of $s_{\mu+1}$ and its restriction into $\lim_{I_\mu} F_i^{n+1}$ is $t'_\mu + (t_\mu - t'_\mu) = t_\mu$. $\square$

Before we state the next result, we want to notice that some well know results about the vanishing of limits in Grothendieck categories can be false in our more general context. For instance, Neeman proved in [Nee02] and

[Nee01b, Section A.5] that an inverse sequence of epimorphisms indexed by $\mathbb{N}$ can have non-vanishing derived limits if the category does not have enough injectives. In the presence of enough injectives, the result is well known; see [Jen72, Proposition 2.1] or [Wei94, Lemma 3.5.3].

Recall that ω_n denotes the least ordinal with cardinality $\aleph_n$. Notice that it is a limit ordinal.

Theorem 4.1.4. *Let $\mathcal{A}$ be an $\aleph_n$-Grothendieck category and let $M = \operatorname{colim}_{i<\omega_n} M_i$ where $M_0 = 0$ and $M_i \subset M_j$ if $i < j$. If $\operatorname{pd}(M_i/M_j) \leq d$ for every pair $j < i < \omega_n$, then $\operatorname{pd}(M) \leq n + d$.*

Proof. We will proceed by induction on d. If $d = 0$, we have to prove that $\operatorname{Ext}^m(\operatorname{colim}_{i<\omega_n} M_i, -) = 0$ for all $m > n$. By assumption, $M_i = M_i/M_0$ is projective for every $i < \omega_n$. Hence, $\mathcal{A}(M_i, -)$ is exact. On the other hand, the colimit $\operatorname{colim}_{i<\omega_n} M_i$ is $\aleph_n$-filtered. Since $\mathcal{A}$ is $\aleph_n$-Grothendieck, this colimit is exact and, by the Grothendieck spectral sequence [Wei94, Corollary 5.8.4], $\operatorname{Ext}^m(\operatorname{colim}_{i<\omega_n} M_i, -) = \lim^{(m)}_{i<\omega_n} \mathcal{A}(M_i, -)$. We claim that the morphism $\mathcal{A}(M_{i+1}, -) \to \mathcal{A}(M_i, -)$ induced by the inclusion $M_i \hookrightarrow M_{i+1}$ is an epimorphism. Since M_{i+1}/M_i is projective, $\operatorname{Ext}^1(M_{i+1}/M_i, -) = 0$. Hence, the long exact sequence associated to the functors Ext^n induced by the short exact sequence

$$0 \longrightarrow M_i \longrightarrow M_{i+1} \longrightarrow M_{i+1}/M_i \longrightarrow 0$$

gives a short exact sequence

$$\begin{tikzcd} 0 \arrow[d, equal] & & \mathcal{A}(M_i,-) \arrow[ll] & \mathcal{A}(M_{i+1},-) \arrow[l] & \mathcal{A}(M_{i+1}/M_i,-) \arrow[l] & 0. \arrow[l] \\ \operatorname{Ext}^1(M_{i+1}/M_i,-) \end{tikzcd}$$

This proves that the sequence defined by $\mathcal{A}(M_{i+1}, -) \to \mathcal{A}(M_i, -)$ satisfies the hypotheses of Lemma 4.1.3. Hence, $\lim^{(m)}_{i<\omega_n} \mathcal{A}(M_i, -) = 0$ for every $m > n$.

Assume that the result is true for $d - 1$. For every $i < \omega_n$, let $P_i \to M_i$ be an epimorphism with P_i projective and let

$$0 \longrightarrow Q \longrightarrow \textstyle\coprod_{i<\omega_n} P_i \longrightarrow M \longrightarrow 0$$

be the short exact sequence induced by the morphisms $P_i \twoheadrightarrow M_i \hookrightarrow M$. Since $\coprod_{i<\omega} P_i$ is projective, $\operatorname{pd}(M) = \operatorname{pd}(Q) + 1$. Hence, the proof will be finished if we prove that $\operatorname{pd}(Q) \leq n + d - 1$. We proceed as in the proof of Theorem 4.1.1 and define $Q_i = Q \cap (\coprod_{j\leq i} P_j)$. Thus, we have an ascending chain of subobjects of Q such that $\mathrm{colim}_{i<\omega_n} Q_i = Q$ together with short exact sequences

$$0 \longrightarrow Q_i \longrightarrow \coprod_{j\leq i} P_j \longrightarrow M_i \longrightarrow 0.$$

We will check that Q satisfies the hypotheses of the theorem for $n-1$. For every $j < i < \omega_n$, we have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Q_j \arrow[r] \arrow[d] & \coprod_{k\leq j} P_k \arrow[r] \arrow[d] & M_j \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q_i \arrow[r] \arrow[d] & \coprod_{k\leq i} P_k \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q_i/Q_j \arrow[r] \arrow[d] & \coprod_{j<k\leq i} P_k \arrow[r] \arrow[d] & M_i/M_j \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $\mathrm{pd}(M_i/M_j) \leq d$. Then $\mathrm{pd}(Q_i/Q_j) \leq d-1$. Finally, we can apply the induction hypothesis on Q to obtain that $\mathrm{pd}(Q) \leq n+d-1$. As we have already observed, this implies that $\mathrm{pd}(M) \leq n+d$. $\square$

Corollary 4.1.5. *Let $\mathcal{A}$ be an $\aleph_n$-Grothendieck category and let $M = \mathrm{colim}_{i<\gamma} M_i$ where $M_0 = 0$, $M_i \subset M_j$ if $i < j$ and $\gamma \geq \omega_n$. Assume that $\mathrm{colim}_{i<\beta} M_i = M_\beta$ for every limit ordinal β such that $\omega_n \leq \beta < \gamma$. If $\mathrm{pd}(M_j/M_i) \leq d$ for every $i \leq j < i+\omega_n$, then $\mathrm{pd}(M) \leq n+d$.*

We notice that, if $n = 0$, then the hypotheses in Corollary 4.1.5 can be reduced to "$M = \mathrm{colim}_{i<\gamma} M_i$ is a continuous ascending chain such that $\mathrm{pd}(M_{i+1}/M_i) \leq d$ for every $i < \gamma$" as in the classical Auslander Lemma [Aus55].

Proof. We will first prove that, if $i + \omega_n \leq \gamma$, then $\mathrm{pd}(M_{i+\omega_n}/M_i) \leq n+d$. Notice that $M_{i+\omega_n}/M_i = \mathrm{colim}_{i\leq j<i+\omega_n}(M_j/M_i)$ by assumption and this is an $\aleph_n$-filtered colimit of inclusions where $\mathrm{pd}(M_j/M_i) \leq d$ for every $i \leq j < i+\omega_n$. Now we are under the hypotheses of Theorem 4.1.4 and it follows that $\mathrm{pd}(M_{i+\omega_n}/M_i) \leq n+d$.

We have proved, in particular, that $\mathrm{pd}(M_j/M_i) \leq n+d$ for every $i \leq j \leq i+\omega_n$. By Theorem 4.1.1, $\mathrm{pd}(M) \leq n+d$. $\square$

4.2 Flat objects in α-Grothendieck categories

We have seen that α-Adams representability is closely related to the projective dimension of cohomological functors in categories of the form $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$. For the case $\alpha = \aleph_0$, $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ is a Grothendieck category and cohomological functors correspond to flat objects in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ by [Bel00a, Remark 8.12] or [Kra00, Lemma 2.7]. For the general case, we have seen in Proposition 2.3.27 that a functor in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ is cohomological if and only if it is an α-filtered colimit of representables. This motivates the following definition.

Definition 4.2.1. Let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. An object in $\mathcal{A}$ is said to be α-*flat* if it is an α-filtered colimit of α-presentable projectives.

Lemma 4.2.2. *Let α be a regular cardinal and let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. Let F be an object in $\mathcal{A}$. Then the following are equivalent.*

1. *F is α-flat,* i.e. *it is an α-filtered colimit of α-presentable projective objects.*

2. *Let $\mathcal{P} \subset \mathcal{A}$ be the full subcategory of α-presentable projective objects of $\mathcal{A}$. The canonical diagram $(\mathcal{P} \downarrow F) \to \mathcal{A}$ is α-filtered.*

3. *Every morphism $N \to F$ where N is α-presentable factorizes through an α-presentable projective object.*

Proof.

$2 \Rightarrow 1$. By assumption, the canonical diagram $D\colon (\mathcal{P} \downarrow F) \to \mathcal{A}$ is an α-filtered colimit of α-presentable projective objects. Hence, we only have to prove that the colimit of the canonical diagram of F in $\mathcal{P}$ is F and this is not difficult to proved using standard arguments; see [AR94, Proposition 1.22] for details.

$1 \Rightarrow 3$. Let $f\colon N \to F$ be a morphism where N is α-presentable. By hypothesis, $F = \mathrm{colim}_J Q_j$ where J is α-filtered and Q_j is α-presentable projective for every $j \in J$. Since N is α-presentable and J is α-filtered, the canonical morphism

$$\mathrm{colim}_J \mathcal{A}(N, Q_j) \longrightarrow \mathcal{A}(N, F)$$

is an isomorphism. In particular, $f\colon N \to F$ factorizes through some Q_j, which is α-presentable projective.

$3 \Rightarrow 2.$ Let $D\colon (\mathcal{P} \downarrow F) \to \mathcal{A}$ be the canonical diagram of F indexed by the comma-category of α-presentable projectives mapping into F. We have to prove that $(\mathcal{P} \downarrow F)$ is α-filtered. Fix a subcategory $\mathcal{R}$ of $(\mathcal{P} \downarrow F)$ of cardinality less than α. Since F is a cocone of $D|_{\mathcal{R}}$ in $\mathcal{A}$, there is a map $\phi\colon \mathrm{colim} D|_{\mathcal{R}} \to F$. By Proposition 2.3.11, since $\mathcal{R}$ has cardinality less than α, $\mathrm{colim} D|_{\mathcal{R}}$ is α-presentable. By hypothesis, $\phi\colon \mathrm{colim} D|_{\mathcal{R}} \to F$ factorizes through an α-presentable projective object Q. In particular, there is a morphism in $(\mathcal{P} \downarrow F)$

$$\begin{tikzcd} \mathrm{colim} D|_{\mathcal{R}} \arrow[rr] \arrow[dr] & & Q \arrow[dl] \\ & F & \end{tikzcd}$$

given by the factorization. Notice that $Q \to F$ is an object in $(\mathcal{P} \downarrow F)$. This proves that $(\mathcal{P} \downarrow F)$ is α-filtered. □

Recall from Section 2.3.1 that, if $\mathcal{C}$ is an essentially small additive category with coproducts of less than α objects, then the category Mod_α-$\mathcal{C}$ of contravariant additive functors from $\mathcal{C}$ into Ab that take coproducts of less than α objects to products is a locally α-presentable α-Grothendieck category with a set of α-presentable projective generators given by a set of functors $\{\mathcal{C}(-, X)\}$ indexed by a set of representatives of isomorphism classes of objects in $\mathcal{C}$.

The following result is a direct consequence of Lemma 4.2.2.

Lemma 4.2.3. *Let α be an infinite cardinal and $\mathcal{C}$ an essentially small additive category with coproducts of less than α objects. Then an α-presentable object in Mod_α-$\mathcal{C}$ is α-flat if and only if it is projective.*

Proof. By definition, every α-presentable projective object is α-flat. To see the converse, let F be an α-presentable α-flat object in Mod_α-$\mathcal{C}$. By Lemma 4.2.2, there is a factorization of the identity map

$$\begin{tikzcd} F \arrow[rr, "id_F"] \arrow[dr] & & F \\ & \mathcal{C}(-, P). \arrow[ur] & \end{tikzcd}$$

In particular, F is a retract of $\mathcal{C}(-, P)$ and hence projective. □

In the previous chapters, we have applied these results to categories of the form Mod_α-$\mathcal{T}^\alpha$ for $\mathcal{T}$ an α-compactly generated triangulated category. Notice that if $\mathcal{A}$ is an abelian category satisfying [AB3], then the subcategory of α-presentable objects $\mathcal{A}^\alpha \subset \mathcal{A}$ has coproducts of less than α objects. Hence, $\mathrm{Mod}_\alpha\text{-}\mathcal{A}^\alpha$ is a locally α-presentable α-Grothendieck and we can define the *restricted Yoneda functor*

$$\begin{aligned} S_\alpha\colon\ \mathcal{A} &\longrightarrow \mathrm{Mod}_\alpha\text{-}\mathcal{A}^\alpha \\ X &\longmapsto \mathcal{A}(-,X)|_{\mathcal{A}^\alpha}. \end{aligned}$$

Since the short exact sequences in $\mathrm{Mod}_\alpha\text{-}\mathcal{A}^\alpha$ are the objectwise short exact sequences, we infer that S_α is a left exact functor.

The following lemma is a very useful characterization of α-flat objects in $\mathrm{Mod}_\alpha\text{-}\mathcal{A}^\alpha$. For locally finitely presentable abelian categories, the result may be found in [Bru83], [JL89, Theorem B.10] or [Sim77, Therorem 2.1].

Lemma 4.2.4. *Let $\mathcal{A}$ be an abelian category satisfying* [AB3]*. An object in* $\mathrm{Mod}_\alpha\text{-}\mathcal{A}^\alpha$ *is α-flat if and only if it is left exact as a functor.*

Proof. Let $F = \mathrm{colim}_I \mathcal{A}^\alpha(-,P_i)$ be an α-flat object in $\mathrm{Mod}_\alpha\text{-}\mathcal{A}^\alpha$ with P_i α-presentable for every $i \in I$ and I α-filtered, and let $A \to B \to C \to 0$ be an exact sequence in $\mathcal{A}^\alpha$. For every $i \in I$ we have a short exact sequence

$$\mathcal{A}^\alpha(A,P_i) \longleftarrow \mathcal{A}^\alpha(B,P_i) \longleftarrow \mathcal{A}^\alpha(C,P_i) \longleftarrow 0.$$

Recall that the category of abelian groups satisfies [AB5]. Thus, by taking the colimit of the above exact sequences over I we obtain an exact sequence

$$\mathrm{colim}_I \mathcal{A}^\alpha(A,P_i) \longleftarrow \mathrm{colim}_I \mathcal{A}^\alpha(B,P_i) \longleftarrow \mathrm{colim}_I \mathcal{A}^\alpha(C,P_i) \longleftarrow 0$$

and, since I is α-filtered, this exact sequence coincides with the image of $A \to B \to C \to 0$ through $\mathrm{colim}_I \mathcal{A}^\alpha(-,P_i)$ by Lemma 2.3.18. This proves that the functor $F = \mathrm{colim}_I \mathcal{A}^\alpha(-,P_i)$ is left exact.

To see the converse, suppose that F is a left exact functor in $\mathrm{Mod}_\alpha\text{-}\mathcal{A}^\alpha$. We will prove that F is α-flat using the characterization 3 of Lemma 4.2.2. Let $h\colon N \to F$ be a morphism where N is α-presentable. There is an exact sequence

$$\mathcal{A}^\alpha(-,Q) \xrightarrow{\mathcal{A}^\alpha(-,f)} \mathcal{A}^\alpha(-,P) \xrightarrow{\quad g \quad} N \longrightarrow 0$$

in $\mathrm{Mod}_\alpha\text{-}\mathcal{A}^\alpha$. If we denote by C the cokernel of f in $\mathcal{A}^\alpha$, we have an exact sequence

$$Q \xrightarrow{\ f\ } P \xrightarrow{\ p\ } C \longrightarrow 0.$$

This induces a complex

$$\begin{tikzcd}
\mathcal{A}^\alpha(-,Q) \arrow[r, "\mathcal{A}^\alpha(-,f)"] & \mathcal{A}^\alpha(-,P) \arrow[r, "\mathcal{A}^\alpha(-,p)"] \arrow[d, "g"] & \mathcal{A}^\alpha(-,\mathcal{C}) \\
& N \arrow[ur, dotted, "\phi"'] &
\end{tikzcd}$$

Since g is an epimorphism, there exists a morphism $\phi\colon N \to \mathcal{A}^\alpha(-,C)$ such that $\phi \circ g = \mathcal{A}^\alpha(-,p)$. Since F is a left exact functor the following sequence

$$F(Q) \xleftarrow{F(f)} F(P) \xleftarrow{F(p)} F(C) \longleftarrow 0$$

is exact and, using the Yoneda Lemma, we see that there is a morphism $\psi\colon \mathcal{A}^\alpha(-,C) \to F$ such that the following diagram

$$\begin{tikzcd}
\mathcal{A}^\alpha(-,Q) \arrow[r, "\mathcal{A}^\alpha(-,f)"] & \mathcal{A}^\alpha(-,P) \arrow[r, "\mathcal{A}^\alpha(-,p)"] \arrow[d, "g"'] & \mathcal{A}^\alpha(-,\mathcal{C}) \arrow[ddl, "\psi"] \\
& N \arrow[d, "h"'] & \\
& F. &
\end{tikzcd}$$

commutes. This implies that $h \circ g = \psi \circ \mathcal{A}^\alpha(-,p) = \psi \circ \phi \circ g$. Finally, since g is an epimorphism we obtain the desired factorization $h = \phi \circ \psi$. $\square$

The analog of Lemma 4.2.4 for triangulated categories also holds by a result of Neeman [Nee01b, Lemma 7.2.4] (which we stated as Proposition 2.3.27). This result says that, if $\mathcal{T}$ is an α-compactly generated triangulated category, then an object in $\mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ is α-flat if and only if it is cohomological as a functor from $\mathcal{T}^\alpha$ to Ab.

Proposition 4.2.5. *Let $\mathcal{A}$ be an abelian category satisfying* [AB3]. *Then the restricted Yoneda functor $S_\alpha\colon \mathcal{A} \to \mathrm{Mod}_\alpha\text{-}\mathcal{A}^\alpha$ induces an equivalence between $\mathcal{A}$ and the category of α-flat objects in* $\mathrm{Mod}_\alpha\text{-}\mathcal{A}^\alpha$.

Proof. Notice that every functor of the form $S_\alpha(A) = \mathcal{A}(-,A)|_{\mathcal{A}^\alpha}$ is left exact. Then it is α-flat, by Lemma 4.2.4. Now we will prove the converse, *i.e.* that every α-flat object is in the image of S_α. Let $F = \mathrm{colim}_I \mathcal{A}^\alpha(-,Y_i)$ where Y_i is α-presentable for every $i \in I$ and I is α-filtered. By the universal property of the colimit, there is a canonical map

$$\mathrm{colim}_I \mathcal{A}^\alpha(-,Y_i) \to \mathcal{A}^\alpha(-,\mathrm{colim}_I Y_i)|_{\mathcal{A}^\alpha}.$$

Since Mod_α-$\mathcal{A}^\alpha$ is locally α-presentable, it will be enough to see that for every object A in $\mathcal{A}^\alpha$ the following morphism is an isomorphism:

$$\mathrm{Hom}(\mathcal{A}^\alpha(-,A), \mathrm{colim}_I \mathcal{A}^\alpha(-,Y_i)) \to \mathrm{Hom}(\mathcal{A}^\alpha(-,A), \mathcal{A}^\alpha(-, \mathrm{colim}_I Y_i)|_{\mathcal{A}^\alpha}).$$

We first use that $\mathcal{A}^\alpha(-,A)$ is α-presentable to see that

$$\mathrm{Hom}(\mathcal{A}^\alpha(-,A), \mathrm{colim}_I \mathcal{A}^\alpha(-,Y_i)) = \mathrm{colim}_I \mathrm{Hom}(\mathcal{A}^\alpha(-,A), \mathcal{A}^\alpha(-,Y_i)).$$

By the Yoneda Lemma, the last term is isomorphic to $\mathrm{colim}_I \mathcal{A}^\alpha(A, Y_i)$, which is isomorphic to $\mathcal{A}(A, \mathrm{colim}_I Y_i)|_{\mathcal{A}^\alpha}$ since A is α-presentable. Finally, by the Yoneda Lemma, it is isomorphic to $\mathrm{Hom}(\mathcal{A}^\alpha(-,A), \mathcal{A}^\alpha(-, \mathrm{colim}_I Y_i)|_{\mathcal{A}^\alpha})$ and the previous morphism is the composition of these isomorphisms. □

We notice that, by Definition 2.3.1 and Proposition 2.3.27, this result is true for a triangulated category $\mathcal{T}$ instead of an abelian category $\mathcal{A}$ if and only if α-Adams representability holds for $\mathcal{T}$.

We will need more properties of the category of α-flat objects. If $\mathcal{T}$ is an α-compactly generated triangulated category, Lemma 3.1.7 says that the category of cohomological functors in Mod_α-$\mathcal{T}^\alpha$ is an exact category. The following lemma is its analog for abelian categories.

Lemma 4.2.6. *Let $\mathcal{A}$ be an abelian category satisfying* [AB3]. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence in Mod_α-$\mathcal{A}^\alpha$. *If two of A, B or C are left exact, then so is the third.*

Proof. The proof follows by diagram chase after we notice that a short exact sequence of functors is objectwise exact. □

4.3 α-purity in α-Grothendieck categories

We next introduce the notion of purity in a general locally α-presentable α-Grothendieck category. Most of the results that we present in this section are well known when $\alpha = \aleph_0$, but for higher cardinals the proofs are more delicate.

Definition 4.3.1. Let α be a regular cardinal and let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category.

1. A short exact sequence $0 \to A \to B \to C \to 0$ is called *α-pure exact* if for every α-presentable object P the induced sequence of abelian groups
$$0 \longrightarrow \mathrm{Hom}(P, A) \longrightarrow \mathrm{Hom}(P, B) \longrightarrow \mathrm{Hom}(P, C) \longrightarrow 0$$
is exact.

2. An object P in $\mathcal{A}$ is called *α-pure projective* if for every α-pure exact sequence
$$0 \to A \to B \to C \to 0$$
the induced sequence
$$0 \longrightarrow \mathcal{A}(P, A) \longrightarrow \mathcal{A}(P, B) \longrightarrow \mathcal{A}(P, C) \longrightarrow 0$$
is exact. Equivalently, P is α-pure projective if every α-pure exact sequence $0 \to Y \to X \to P \to 0$ splits.

As in the case of modules, *pure exactness* coincides with *$\aleph_0$-pure exactness.*

Lemma 4.3.2. *Let α be a regular cardinal and let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. Let*
$$0 \longrightarrow N \xrightarrow{\;f\;} M \xrightarrow{\;g\;} M/N \longrightarrow 0$$
be an α-pure exact sequence. If M is α-flat, then so are N and M/N.

Proof. We will first prove that N is α-flat using the characterizations of α-flatness given in Lemma 4.2.2. Specifically, we will prove that if $\phi\colon Q \to N$ is a morphism from an α-presentable object Q, then it factorizes through an α-presentable projective object. Since M is α-flat, $f \circ \phi\colon Q \to M$ factorizes through an α-presentable projective object P, by Lemma 4.2.2. Hence there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "f"] & M \arrow[r, "g"] & M/N \arrow[r] & 0 \\
& Q \arrow[u, "\phi"] \arrow[r, "f'"] & P \arrow[u, "\psi"] \arrow[r, "g'"] & P/Q \arrow[u, "\chi"] \arrow[r] & 0.
\end{tikzcd}$$

Since the upper exact sequence is α-pure exact and P/Q is α-presentable, there exists a morphism $b\colon P/Q \to M$ such that $g \circ b = \chi$. Hence,
$$g \circ (\psi - b \circ g') = \chi \circ g' - \chi \circ g' = 0.$$

By exactness, there is a morphism $d\colon P \to N$ such that $\psi - b \circ g' = f \circ d$. Then $f \circ \phi = \psi \circ f' = (f \circ d + b \circ g') \circ f' = f \circ d \circ f' + 0$. Since f is a monomorphism, we obtain $\phi = d \circ f'$, which is a factorization of ϕ by the α-presentable projective object P.

Next we will prove that M/N is α-flat. Let $q\colon Q \to M/N$ be a morphism with Q α-presentable. By α-purity, there exists a morphism $q'\colon Q \to M$ such that $g \circ q' = q$. Since M is α-flat by hypothesis, there exists a factorization of q' through an α-presentable projective object P by Lemma 4.2.2. We have the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] & M \arrow[r, "g"] & M/N \arrow[r] & 0 \\ & & P \arrow[u, "p"] & Q \arrow[l, "c"] \arrow[ul, "q'"] \arrow[u, "q"'] & \end{tikzcd}$$

Then $q = g \circ p \circ c$ factorizes through the α-presentable projective P. Finally, Lemma 4.2.2 implies that M/N is α-flat. $\square$

The next two lemmas give characterizations of α-pure exact sequences. The first lemma is more general than the version stated here, and a proof can be found in [AR94, Proposition 2.30]. We give an argument adapted to our terminology.

Lemma 4.3.3. *Let α be a regular cardinal and let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. A short exact sequence is α-pure if and only if it is an α-filtered colimit of short split exact sequences.*

Proof. Let $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ be an α-pure exact sequence. Since $\mathcal{A}$ is locally α-presentable, $C \cong \operatorname{colim}_I C_i$ where the colimit is α-filtered and C_i is α-presentable for all $i \in I$. We denote by $f_i\colon C_i \to \operatorname{colim}_I C_i$ the canonical morphism and by B_i the pullback of the morphisms f_i and v for all $i \in I$. Then we have the following commutative diagram:

$$\begin{tikzcd} & & B_i \arrow[r, "v_i"] \arrow[d, "g_i"] & C_i \arrow[r] \arrow[d, "f_i"] & 0 \\ 0 \arrow[r] & A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r] & 0. \end{tikzcd}$$

By the universal property of the pullback, there is a morphism $u_i\colon A \to B_i$ such that $v_i \circ u_i = 0$ and $g_i \circ u_i = u$. Hence, we obtain a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "u_i"] \arrow[d, equal] & B_i \arrow[r, "v_i"] \arrow[d, "g_i"] & C_i \arrow[r] \arrow[d, "f_i"] & 0 \\ 0 \arrow[r] & A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r] & 0. \end{tikzcd}$$

If we take the colimit of the upper short exact sequence, since $C = \mathrm{colim}_I C_i$ and $A = \mathrm{colim}_I A$, we obtain that $\mathrm{colim}_I B_i \cong B$. Finally, we have to prove that $0 \longrightarrow A \xrightarrow{u_i} B_i \xrightarrow{v_i} C_i \longrightarrow 0$ splits for all $i \in I$. Since each C_i is α-presentable, every $f_i \colon C_i \to C$ factorizes through B. But now the universal property of the pullback implies that there exists a morphism $p_i \colon C_i \to B_i$ such that $v_i \circ p_i = id_C$.

In order to see the converse, assume that $0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0$ is an α-filtered colimit of split short exact sequences $0 \to A_i \xrightarrow{u_i} B_i \xrightarrow{v_i} C_i \to 0$. We denote by $g_i \colon B_i \to B$ and $f_i \colon C_i \to C$ the canonical morphisms and $p_i \colon C_i \to B_i$ a morphism such that $v_i \circ p_i = id_C$ for all $i \in I$. Fix an α-presentable object F and a morphism $\phi \colon F \to C$. We have to prove that ϕ can be factorized through v. Since $C = \mathrm{colim}_i C_i$ and the colimit is α-filtered, the universal morphism

$$\mathrm{colim}_{i\in I}\mathcal{A}(F, C_i) \longrightarrow \mathcal{A}(F, \mathrm{colim}_{i\in I} C_i)$$

is an isomorphism. Hence, there is a factorization of ϕ through a morphism $\phi_i \colon F \to C_i$ for some $i \in I$. If we define $\psi = g_i \circ p_i \circ \phi_i$, we have the following commutative diagram:

$$\begin{tikzcd} & F \arrow[d, "\phi_i"] \arrow[dl, bend right, "\psi"'] \arrow[ddr, bend left, "\phi"] & \\ B_i \arrow[r, shift left, "u_i"] \arrow[d, "g_i"] & C_i \arrow[l, shift left, "p_i"] \arrow[d, "f_i"] & \\ B \arrow[r, "u"] & C. & \end{tikzcd}$$

Hence, $u \circ \psi = u \circ g_i \circ p_i \circ \phi_i = f_i \circ u_i \circ p_i \circ \phi_i = f_i \circ \phi_i = \phi$. □

Before stating the following lemma, recall that an object X in a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives is *α-generated* if there exists an epimorphism

$$\textstyle\coprod_{i\in I} P_i \longrightarrow X$$

where P_i is α-presentable projective for all $i \in I$ and $\#I < \alpha$.

Lemma 4.3.4. *Let α be a regular cardinal and let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. A short exact sequence $0 \to A \to B \to C \to 0$ is α-pure if and only if for every commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "f'"] \arrow[d, "\phi"] & F \arrow[d, "\psi"] & & \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

with exact rows, where F is α-presentable projective and H is α-generated, there exists a map $\sigma\colon F \to A$ such that $\sigma \circ f' = \phi$.

Proof. We will first prove the *only if* part. Assume that we are given a diagram as in the statement. It can be completed to

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "f'"] \arrow[d, "\phi"] & F \arrow[r, "g'"] \arrow[d, "\psi"] & M \arrow[r] \arrow[d, "\chi"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

with exact rows. In particular, M is also α-presentable. By α-purity, there exists a morphism $\rho\colon M \to B$ such that $g \circ \rho = \chi$. Then $g \circ (\psi - \rho \circ g') = \chi \circ g' - \chi \circ g' = 0$. By exactness, there exists a morphism $\sigma\colon F \to A$ such that $f \circ \sigma = \psi - \rho \circ g'$. Hence, $f \circ \sigma \circ f' = (\psi - \rho \circ g') \circ f' = f \circ \phi - 0$. Since f is a monomorphism, $\sigma \circ f' = \phi$.

We will prove the *if* part. Let $\chi\colon M \to C$ be a morphism with M α-presentable. We have to prove that there is a morphism $\rho\colon M \to B$ such that $g \circ \rho = \chi$. By assumption, there is an epimorphism $g'\colon F \to M$ with F α-presentable projective. Since F is projective, there is a morphism $\psi\colon F \to B$ such that $\chi \circ g' = g \circ \psi$. Since M is α-presentable, the kernel of g', that we will denote by $f'\colon H \to F$, is α-generated and, by exactness, there is a morphism $\phi\colon H \to A$ such that $f \circ \phi = \psi \circ f'$. By hypothesis, there exists a morphism $\sigma\colon F \to A$ such that $\sigma \circ f' = \phi$. Then $(\psi - f \circ \sigma) \circ f' = f \circ \phi - f \circ \phi = 0$. By exactness, there exists a morphism $\rho\colon M \to B$ such that $\rho \circ g' = \psi - f \circ \sigma$. Hence, $g \circ \rho \circ g' = g \circ (\psi - f \circ \sigma) = \chi \circ g' - 0$. Since g' is an epimorphism, we conclude that $g \circ \rho = \chi$. □

The following lemma summarizes some of the elementary properties of α-pure monomorphisms.

Lemma 4.3.5. *Let α be a regular cardinal and let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. Let*

$$A \overset{f}{\hookrightarrow} B \overset{g}{\hookrightarrow} C$$

be a pair of monomorphisms. Then the following holds.

1. *If f and g are α-pure monomorphisms, then so is $g \circ f$.*

2. *If $g \circ f$ is an α-pure monomorphism, then so is f.*

3. *If g is an α-pure monomorphism, then so is $g'\colon B/A \to C/A$.*

4. *If $g \circ f$ and $g'\colon B/A \to C/A$ are α-pure monomorphisms, then so is g.*

Proof. We start by proving item 1. By Lemma 4.3.4, it is enough to show that for every commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[rr, "f'"] \arrow[d, "\phi"] & & F \arrow[d, "\psi"] \\ & A \arrow[r, hook, "f"] & B \arrow[r, hook, "g"] & C \end{tikzcd}$$

with f' a monomorphism, F an α-presentable projective object and H an α-generated object, there exists a map $\sigma\colon F \to A$ such that $\sigma \circ f' = \phi$. Since g is an α-pure monomorphism, there is a morphism $\rho\colon F \to B$ such that $\rho \circ f' = f \circ \phi$. But now, since f is an α-pure monomorphism, there is a morphism σ such that $\sigma \circ f' = \phi$.

Next we prove item 2. By Lemma 4.3.4, it is enough to show that for every commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "f'"] \arrow[d, "\phi"] & F \arrow[d, "\psi"] & \\ & A \arrow[r, hook, "f"] & B \arrow[r, hook, "g"] & C \end{tikzcd}$$

with f' a monomorphism, F an α-presentable projective object and H an α-generated object, there exists a map $\sigma\colon F \to A$ such that $\sigma \circ f' = \phi$. Since we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[rr, "f'"] \arrow[d, "\phi"] & & F \arrow[d, "g\circ\psi"] \\ & A \arrow[r, hook, "f"] & B \arrow[r, hook, "g"] & C \end{tikzcd}$$

and $g \circ f$ is an α-pure monomorphism, there exists a morphism $\sigma\colon F \to A$ such that $\sigma \circ f' = \phi$.

For the proof of items 3 and 4, we will use the following commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & B/A \arrow[r, "g'"] \arrow[u] & C/A \arrow[r, "k"] \arrow[u] & (C/A)/(B/A) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & B \arrow[r, "g"] \arrow[u, "h"] & C \arrow[r, "j"] \arrow[u, "i"] & C/B \arrow[r] \arrow[u, "l", "\cong"'] & 0 \\
0 \arrow[r] & A \arrow[r, equal] \arrow[u, "f"] & A \arrow[r] \arrow[u, "g\circ f"] & 0 \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

where the rows and columns are exact.

For item 3, it is enough to prove that the short exact sequence

$$0 \longrightarrow B/A \xrightarrow{g'} C/A \xrightarrow{k} (C/A)/(B/A) \longrightarrow 0$$

is α-pure. We fix an α-presentable object Q and $\phi\colon Q \to (C/A)/(B/A)$. In order to finish the proof, it is enough to factorize ϕ through k. Since g is an α-pure monomorphism by hypothesis, there is a morphism $\psi\colon Q \to C$ such that $l^{-1} \circ \phi = j \circ \psi$. Then $k \circ (i \circ \psi) = l \circ j \circ \psi = l \circ l^{-1} \circ \phi = \phi$.

For item 4, it is enough to prove that the short exact sequence

$$0 \longrightarrow B \xrightarrow{g} C \xrightarrow{j} C/B \longrightarrow 0$$

is α-pure. We fix an α-presentable object Q and $\phi\colon Q \to C/B$. In order to finish the proof, it is enough to factorize ϕ through j. Since g' is an α-pure monomorphism by hypothesis, there is a morphism $\psi\colon Q \to C/A$ such that $k \circ \psi = l \circ \phi$. But now, since $g \circ f$ is also an α-pure monomorphism there exists a morphism $\chi\colon Q \to C$ such that $i \circ \chi = \psi$. Then $l \circ j \circ \chi = k \circ i \circ \chi = k \circ \psi = l \circ \phi$. Since l is an isomorphism, $j \circ \chi = \phi$. □

We want to prove that, in a locally α-presentable α-Grothendieck category, every object is the colimit of an ascending chain of α-pure subobjects. We will prove this statement after giving two preparatory lemmas.

Lemma 4.3.6. *Let α be a regular cardinal and let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. The colimit of an ascending chain of α-pure subobjects indexed by an ordinal of cardinality greater than or equal to α is an α-pure subobject.*

Proof. Let $f_i\colon A_i \to M$ be an ascending chain of α-pure monomorphisms indexed by an ordinal γ of cardinality greater than or equal to α. Hence, $\mathrm{colim}_{i<\gamma} A_i$ is an α-filtered colimit. Since $\mathcal{A}$ is an α-Grothendieck category, this colimit is exact, $f = \mathrm{colim}_{i<\gamma} f_i\colon \mathrm{colim}_{i<\gamma} A_i \to M$ is a monomorphism and $\mathrm{colim}_{i<\gamma} M/A_i \cong M/(\mathrm{colim}_{i<\gamma} A_i)$. Hence, we have to prove that the exact sequence

$$0 \longrightarrow \mathrm{colim}_{i<\gamma} A_i \xrightarrow{f} M \longrightarrow \mathrm{colim}_{i<\gamma} M/A_i \longrightarrow 0$$

is α-pure exact. Fix a morphism $g\colon B \to \mathrm{colim}_{i<\gamma} M/A_i$ with B α-presentable. Since the colimit is α-filtered, the canonical morphism

$$\mathrm{colim}_{i<\gamma} \mathcal{A}(B, M/A_i) \longrightarrow \mathcal{A}(B, \mathrm{colim}_{i<\gamma} M/A_i)$$

is an isomorphism. By its surjectivity, there is a factorization of f as

$$B \longrightarrow M/A_i \xrightarrow{\iota} \mathrm{colim}_{i<\gamma} M/A_i,$$

where ι is the canonical map, for some $i < \gamma$. Hence, we have the following commutative diagram:

$$\begin{tikzcd}
 & & & & B \arrow[dll, "g_i"'] \arrow[d, "g"] & \\
0 \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d] & M \arrow[r, "p_i"] \arrow[d, equal] & M/A_i \arrow[rr] \arrow[dr, "\iota"] & & 0 \\
0 \arrow[r] & \mathrm{colim}_{i<\gamma} A_i \arrow[r, "f"] & M \arrow[rr, "p"] & & \mathrm{colim}_{i<\gamma} M/A_i \arrow[r] & 0
\end{tikzcd}$$

where the upper exact sequence is α-pure by assumption. Hence, there exists a morphism $h\colon B \to M$ such that $p_i \circ h = g_i$. Then $p \circ h = \iota \circ g_i = g$. This proves that f is an α-pure monomorphism. □

The following lemma is a slightly more detailed variant of [AR94, Theorem 2.33]. A self-contained proof is included for completeness.

Lemma 4.3.7. *Let α be a regular cardinal and let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. Every monomorphism $\phi\colon N \hookrightarrow M$, where N is λ^+-generated, factorizes as*

$$\begin{tikzcd}
N \arrow[rr, hook, "\phi"] \arrow[dr, hook] & & M \\
 & N' \arrow[ur, hook] &
\end{tikzcd}$$

where $N \hookrightarrow N'$ is a monomorphism and $N' \hookrightarrow M$ is an α-pure monomorphism and N' has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha, \lambda\}$ *generators.*

Proof. The object N' will be constructed recursively using the characterization of α-pure monomorphism given in Lemma 4.3.4.

For every morphism $g\colon A \to N$ with A α-generated and every morphism $f\colon A \to B$ with B α-presentable projective, we choose, in case it exists, one and only one map $B \to M$ inducing a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ N \arrow[r, hook, "\phi"] & M. \end{tikzcd}$$

Denote by $\mathcal{U}_1 = \{h\colon B \to M\}$ the set of all these morphisms. We claim that $\mathcal{U}_1$ has cardinality less than or equal to $\max\{\#\mathcal{A}^\alpha, \lambda\}$.

Since $\mathcal{A}$ is generated by a set of α-presentable projectives, a pair of morphisms $\theta_1, \theta_2\colon A \to B$ are different if and only if there exists an α-presentable projective object P together with a morphism $\eta\colon P \to A$ such that $\theta_1 \circ \eta$ and $\theta_2 \circ \eta$ are different. Hence, the cardinality of the set of morphisms $A \to B$ with B α-presentable projective and A α-generated is less than or equal to the cardinality of the set of morphisms $P \to A \to B$ with P α-presentable projective, and this cardinality is bounded by $\#\mathcal{A}^\alpha$.

In order to finish the proof of our claim, we have to show that the cardinality of the set of morphisms $A \to N$ with A α-generated is bounded by $\max\{\#\mathcal{A}^\alpha, \lambda\}$. By the same reason as before, it is enough to prove it for the set of morphisms $P \to A \to N$ with P α-presentable projective. Since N is generated by less than or equal to λ generators, there is an epimorphism

$$\psi\colon \textstyle\coprod_{i<\lambda} Q_i \twoheadrightarrow N$$

with Q_i α-presentable for every $i < \lambda$. Every morphism $P \to N$ with P α-presentable projective factors through ψ. For this reason, our claim will be proved if we show that $\#\mathcal{A}(P, \coprod_{i<\lambda} Q_i) \leq \max\{\#\mathcal{A}^\alpha, \lambda\}$ where P and Q_i are α-presentable objects. We will prove this by transfinite induction on λ. If $\lambda < \alpha$, then $\coprod_{i<\lambda} Q_i$ is α-presentable and $\#\mathcal{A}(P, \coprod_{i<\lambda} Q_i) \leq \#\mathcal{A}^\alpha$. If $\lambda \geq \alpha$, then $\coprod_{i<\lambda} Q_i = \mathrm{colim}_{i<\lambda} \coprod_{j<i} Q_j$ is an α-filtered colimit. Since P is α-presentable, the canonical morphism

$$\mathrm{colim}_{i<\lambda}\mathcal{A}(P, \coprod_{j<i} Q_i) \longrightarrow \mathcal{A}(P, \coprod_{i<\lambda} Q_i)$$

is an isomorphism. The induction hypothesis says that $\#\mathcal{A}(P, \coprod_{j<i} Q_i) \leq \max\{\#\mathcal{A}^\alpha, \lambda\}$. Since the canonical map given by the coproduct

$$\coprod_{i<\lambda} \mathcal{A}(P, \coprod_{j<i} Q_i) \longrightarrow \mathrm{colim}_{i<\lambda}\mathcal{A}(P, \coprod_{j<i} Q_i)$$

is always an epimorphism,

$$\#\mathrm{colim}_{i<\lambda}\mathcal{A}(P,\coprod_{j<i}Q_i) \leq \#\coprod_{i<\lambda}\mathcal{A}(P,\coprod_{j<i}Q_i) \leq \max\{\#\mathcal{A}^\alpha,\lambda\}.$$

This finishes the proof of our claim.

Define N_1 as the image of the morphism $\coprod_{\mathcal{U}_1} B \to M$. Notice that it contains N. This is because $\mathcal{A}$ is locally α-presentable and then there is an epimorphism $g\colon \coprod_{k\in K} Q_k \to N$ with Q_k α-presentable for every $k \in K$. Since the morphisms $\phi \circ g_k \colon Q_k \to N \to M$ are in $\mathcal{U}_1$, we have that $N \subset N_1$. This gives a commutative diagram

$$\begin{tikzcd}
\coprod_{\mathcal{U}_1} B \arrow[rr] \arrow[d] \arrow[dr, two heads] & & M \\
N \arrow[r, hook] & N_1. \arrow[ur, hook] &
\end{tikzcd}$$

Fix an ordinal $\gamma < \alpha$ and assume that we have constructed an object N_i for every $i < \gamma$ which is generated by less than or equal to $\max\{\alpha,\#\mathcal{A}^\alpha,\lambda\}$ generators, *i.e.* there is an epimorphism $\coprod_{j\in J_i} Q_j \to N_i$ where Q_j is α-presentable for every $j \in J_i$, $\#J_i \leq \max\{\alpha,\#\mathcal{A}^\alpha,\lambda\}$, and $N \subset N_0 \subset \cdots N_i \subset \cdots \subset M$. If γ is a limit ordinal, we define $N_\gamma = \mathrm{colim}_{i<\gamma} N_i$. Then there is an epimorphism $\coprod_{j\in J_\gamma} Q_j \to N_\gamma$ where $J_\gamma = \bigsqcup_{i<\gamma} J_i$ and so N_γ is generated by less than or equal to $\max\{\alpha,\#\mathcal{A}^\alpha,\lambda\}$ generators. If $\gamma = \kappa^+$ then we proceed exactly as in the case of N_1 and construct $N_\kappa \subset N_\gamma \subset M$ where N_γ is generated by less than or equal to $\max\{\alpha,\#\mathcal{A}^\alpha,\lambda\}$ generators. With this procedure we construct a chain of inclusions

$$N \subset N_1 \subset \cdots \subset N_\gamma \subset \cdots \subset M$$

for every ordinal $\gamma < \alpha$. We define $N' = \mathrm{colim}_{\gamma<\alpha} N_\gamma \subset M$. Since this is a colimit of inclusions indexed by a set of cardinality α, it is α-filtered and there is an epimorphism

$$\coprod_{j\in J} Q_j \to N'$$

where $J = \bigsqcup_{\gamma<\alpha} J_\gamma$ and there are epimorphisms $\coprod_{j\in J_\gamma} Q_j \to N_\gamma$ with Q_j α-presentable objects and $\#J_\gamma \leq \max\{\alpha,\#\mathcal{A}^\alpha,\lambda\}$. This implies that $\#J \leq \max\{\alpha,\#\mathcal{A}^\alpha,\lambda\}$ and so N' is generated by less than or equal to $\max\{\alpha,\#\mathcal{A}^\alpha,\lambda\}$ generators.

In order to finish the proof, we have to check that the map $\phi\colon N' \hookrightarrow M$

is an α-pure monomorphism. Let

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ N' \arrow[r, hook, "\phi"] & M \end{tikzcd}$$

be a commutative diagram where f is a monomorphism, A is α-generated and B is α-presentable projective. By Lemma 4.3.4, the proof will be finished if there exists a map $\rho\colon B \to N'$ such that $\rho \circ f = g$. Since A is α-generated, there exists an epimorphism $\chi\colon P \to A$ where P is α-presentable and, since $N' = \mathrm{colim}_{\gamma<\alpha} N_\gamma$ is α-filtered, the universal morphism

$$\mathrm{colim}_{\gamma<\alpha}\mathcal{A}(P, N_\gamma) \to \mathcal{A}(P, N')$$

is an isomorphism. Then the map $g \circ \chi\colon P \to N'$ factorizes through some N_γ. But now, by definition, there exists a morphism $h'\colon B \to M$ in $\mathcal{U}_\gamma$ giving a commutative diagram

$$\begin{tikzcd} P \arrow[rr, "f\circ\chi"] \arrow[d, two heads, "\chi"'] & & B \arrow[ddd, "h'"] \\ A \arrow[urr, dotted, "f"'] \arrow[d, "g_\gamma"'] \arrow[ddr, dotted, "g"] & & \\ N_\gamma \arrow[d, hook, "\mu_\gamma"'] & & \\ N_{\gamma+1} \arrow[r, hook, "\iota"] & N' \arrow[r, hook, "\phi"] & M \end{tikzcd}$$

with ι the canonical map to the colimit. By construction, there is a map $\sigma\colon B \to N_{\gamma+1}$ such that $\sigma \circ (f \circ \chi) = \mu_\gamma \circ g_\gamma \circ \chi$. Define $\rho = \iota \circ \sigma$. Then $\rho \circ f \circ \chi = \iota \circ \sigma \circ f \circ \chi = \iota \circ \mu_\gamma \circ g_\gamma \circ \chi = g \circ \chi$. Since χ is an epimorphism, we obtain $\rho \circ f = g$. $\square$

Corollary 4.3.8. *Let α be a regular cardinal and let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. Every object M in $\mathcal{A}$ is the colimit of an α-filtered ascending chain of α-pure subobjects $M = \mathrm{colim}_{i<\gamma} N_i$ such that $\mathrm{colim}_{i<\beta} N_i = N_\beta$ for every limit ordinal $\beta \geq \alpha$ and, if $i \leq j \leq i+\alpha$, then N_j/N_i has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators.*

Proof. Since $\mathcal{A}$ is locally α-presentable every object can be written as $M = \mathrm{colim}_{i\in I} M_i$ where the colimit is α-filtered, M_i is α-presentable for every $i \in I$, and $M_0 = 0$. Choose a well ordering in I and let γ be the smallest ordinal with cardinality $\#I$. We identify the objects $i \in I$ with the set of ordinals $i < \gamma$ We denote by $\iota_i \colon M_i \to M$ the canonical morphisms of the colimit.

We will define an ascending chain of α-pure monomorphisms $N_i \to M$ by transfinite induction on $i < \gamma$. Let $N_0 = 0$ and let $N'_\alpha = \sum_{i<\alpha} \iota_i(M_i)$ (see Definition 2.3.23). Notice that N'_α has less than or equal to α generators and, since $\mathcal{A}$ is an α-Grothendieck category, it is a subobject of M. By Lemma 4.3.7, there is a factorization

$$\begin{tikzcd} N'_\alpha \arrow[rr, hook] \arrow[dr, hook] & & M \\ & N_\alpha \arrow[ur, hook, "\psi"'] & \end{tikzcd}$$

where ψ is an α-pure monomorphism and N_α has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators. Let $N_i = N_\alpha$ for every $0 < i < \alpha$.

Fix an ordinal $\alpha < i < \gamma$ and assume that we have constructed an ascending chain of α-pure subobjects $N_j \hookrightarrow M$ as in the statement for every $j \leq i$. Then there is an inclusion $N_i + \iota_i(M_i) \hookrightarrow M$. By Lemma 4.3.7, there is a factorization

$$\begin{tikzcd} (N_i + \iota_i(M_i))/N_i \arrow[rr, hook] \arrow[dr, hook] & & M/N_i \\ & N'_{i+1} \arrow[ur, hook, "\phi'"'] & \end{tikzcd}$$

where ϕ' is an α-pure monomorphism and N'_{i+1} has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators. We define N_{i+1} as the following pullback

$$\begin{tikzcd} N_{i+1} \arrow[r, hook, "\phi"] \arrow[d, two heads] & M \arrow[d, two heads, "\pi"] \\ N'_{i+1} \arrow[r, hook, "\phi'"] & M/N_i. \end{tikzcd}$$

Notice that, by standard arguments in abelian categories, $N_i \subset N_{i+1}$ and $N'_{i+1} \cong N_{i+1}/N_i$ and hence N_{i+1} has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators. We have to prove that ϕ is an α-pure monomorphism. But this follows from item 4 of Lemma 4.3.5, since $\phi' \colon N_{i+1}/N_i \to M/N_i$ and $N_i \hookrightarrow N_{i+1} \hookrightarrow M$ are α-pure monomorphisms.

Now let $\alpha < i < \gamma$ be a limit ordinal and assume that we have constructed an ascending chain of α-pure subobjects $N_j \hookrightarrow M$ as in the statement for every $j < i$. We define $N_i = \mathrm{colim}_{j<i} N_j$. It is an α-pure subobject of M by Lemma 4.3.6 and the fact that it is an α-filtered colimit in an α-Grothendieck category.

By construction, $\iota_i(M_i) \subset N_{i+1} \subset M$ for every $i < \gamma$. Then $M = \mathrm{colim}_{i\in I} M_i = \mathrm{colim}_{i<\gamma} N_i$.

Finally, we have to prove that, if $i \leq j \leq i+\alpha$, then N_j/N_i has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators. For $i = 0$ this is clear by definition. Now we fix $0 < i < \gamma$. We will proceed by transfinite induction on j. In the case that j is an ordinal such that $i \leq j+1 \leq i+\alpha$, assume that we have proved that N_j/N_i has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators. Then there is an epimorphism $f_j \colon P_j \twoheadrightarrow N_j/N_i$ from a projective object P_j with less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators. By construction, N_{j+1}/N_j has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators and hence there exists an epimorphism $f_{j+1} \colon P_{j+1} \twoheadrightarrow N_{j+1}/N_j$ from a projective object with less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators. Since we have a short exact sequence

$$0 \longrightarrow N_j/N_i \longrightarrow N_{j+1}/N_i \longrightarrow N_{j+1}/N_j \longrightarrow 0,$$

there is an epimorphism $P_j \coprod P_{j+1} \twoheadrightarrow N_{j+1}/N_i$ and this proves that N_{j+1}/N_i has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators. Finally, in the case that j is a limit ordinal such that $i \leq j \leq i+\alpha$, assume that we have proved that N_k/N_i has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators for every $i < k < j$. If $j \leq \alpha$, then $N_j/N_i = 0$. If $\alpha < j$, then there exists an epimorphism $f_k \colon P_k \twoheadrightarrow N_k/N_i$ from a projective object P_k with less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators for every $i \leq k < j$. Then there exists an epimorphism $\coprod_{i<k<j} P_k \twoheadrightarrow \mathrm{colim}_{k<j}(N_k/N_i) \cong N_j/N_i$ since the colimit is α-filtered and $\mathcal{A}$ is an α-Grothendieck category. This proves that N_j/N_i has less than or equal to $\max\{\alpha, \#\mathcal{A}^\alpha\}$ generators. □

The next lemma shows the relation between α-purity and α-flatness. In the case of Grothendieck categories and $\alpha = \aleph_0$, a proof of this fact can be found in [Pop73, Ch. 3.8, Exercise 9].

Lemma 4.3.9. *Let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives.*

1. *An object F in $\mathcal{A}$ is α-flat if and only if every short exact sequence*

$$0 \to A \to B \to F \to 0$$

   is α-pure.

2. *Every projective object is α-flat.*

3. *A retract of an α-flat object is α-flat.*

Proof.

1. Assume that F is α-flat. Let $0 \to A \to B \to F \to 0$ be a short exact sequence and let $F = \mathrm{colim}_I P_i$, where P_i is an α-presentable projective for every $i \in I$ and I is α-filtered. Let B_i be the pullback of B along $P_i \to F$ for every $i \in I$. There is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, dotted] \arrow[d, equal] & B_i \arrow[r, dotted] \arrow[d, dotted] & P_i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & F \arrow[r] & 0, \end{tikzcd}$$

where the two rows are exact. Since P_i is projective, the upper exact sequence in the diagram splits for every $i \in I$. This implies that the short exact sequence $0 \to A \to B \to F \to 0$ is an α-filtered colimit of split exact sequences, hence α-pure exact by Lemma 4.3.3.

Conversely, suppose that F is such that every short exact sequence of the form $0 \to A \to B \to F \to 0$ is α-pure exact. We will see that F is α-flat via the characterization 3 of Lemma 4.2.2. Let $f\colon M \to F$ be a morphism where M is α-presentable and let

$$0 \longrightarrow K \longrightarrow \textstyle\coprod_{i\in I} P_i \xrightarrow{\phi} F \longrightarrow 0$$

be a short α-pure exact sequence with P_i α-presentable projective for all $i \in I$. By α-pure exactness, there is a morphism $f'\colon M \to \coprod_{i\in I} P_i$ such that $\phi \circ f' = f$. But, since M is α-presentable, f' factorizes through a coproduct of less than α objects and we obtain the following commutative diagram

$$\begin{tikzcd} & & \coprod_{i\in J} P_i \arrow[d] & M \arrow[l] \arrow[dl, "f'"'] \arrow[d, "f"] & \\ 0 \arrow[r] & K \arrow[r] & \coprod_{i\in I} P_i \arrow[r, "\phi"] & F \arrow[r] & 0 \end{tikzcd}$$

where $J \subset I$ and $\#J < \alpha$. This gives a factorization of f through $\coprod_{i\in J} P_i$ that is α-presentable projective and this proves that F is α-flat.

2. It is a direct consequence of the part 1 and Lemma 4.3.3.

3. Let F be an α-flat object and let $r\colon F \twoheadrightarrow X$ be a retraction. There is a morphism $i\colon X \to F$ such that $r \circ i = id_X$. Let $f\colon M \to X$ be a morphism with M α-presentable. By Lemma 4.2.2, there is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[d, "i"'] \\ P \arrow[r, "f'"] & F \arrow[u, bend right, "r"'] \end{tikzcd}$$

with P α-presentable projective. Then $r \circ j \circ f' = r \circ i \circ f = f$ and this proves that X is α-flat using the characterization of Lemma 4.2.2. $\square$

4.4 Homological dimensions in α-Grothendieck categories

In this section we prove some relations between the weak global dimension and the projective global dimension of locally α-presentable α-Grothendieck categories. We also give conditions for the existence of an upper bound for the projective global dimension. For the case $\alpha = \aleph_0$, some of the results that we present in this section are classical and have been studied in [Bru83].

We begin with a definition that generalizes the concept of noetherianity in abelian categories [Pop73, Section 5.7].

Definition 4.4.1. Let $\mathcal{A}$ be an abelian category and let X be an object in $\mathcal{A}$. We say that X is *α-noetherian* if for every set $\{A_i\}_{i<\gamma}$ of subobjects of X well ordered by inclusion, where $\alpha \leq \gamma$, there exists an ordinal β with $\#\beta < \alpha$ such that $A_i = A_j$ for every $\beta \leq i, j < \gamma$. An abelian category is said to be *locally α-noetherian* if it has a generating set of α-presentable projective α-noetherian objects.

The next proposition will be very useful when working with α-noetherian objects.

Proposition 4.4.2. *Let $\mathcal{A}$ be a locally α-presentable category. The following are equivalent.*

1. *Every α-presentable projective object in $\mathcal{A}$ is α-noetherian.*
2. *Every subobject of an α-presentable projective object is α-generated.*
3. *Every subobject of an α-generated object is α-generated.*
4. *An object is α-generated if and only if it is α-presentable.*
5. *Every subobject of an α-presentable projective object is α-presentable.*

Proof. We will prove the implications $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$ and $2 \Leftrightarrow 5$.

$1 \Rightarrow 2$. Let $f\colon X \hookrightarrow P$ be a subobject of an α-presentable projective object. Since $\mathcal{A}$ is generated by a set of α-presentable projectives, there is an epimorphism

$$\textstyle\coprod_{i<\gamma} Q_i \xrightarrow{r} X \longrightarrow 0$$

Then $\{f \circ r(\coprod_{i<\xi} Q_i)\}_{\xi<\gamma}$ is a set of subobjects of P well ordered by inclusion. By hypothesis, there is an ordinal β with $\#\beta < \alpha$ such that $f \circ r(\coprod_{i<\xi} Q_i) = f \circ r(\coprod_{i<\xi'} Q_i)$ for every $\beta \leq \xi, \xi' < \gamma$. But now, since f is a monomorphism

$$\textstyle\coprod_{i<\beta} Q_i \xrightarrow{r} X$$

is an epimorphism and X is α-generated.

$2 \Rightarrow 1.$ Let $\{Q_i\}_{i<\gamma}$ be a set of subobjects of an α-presentable projective object P well ordered by inclusion with $\gamma \geq \alpha$. In particular, $\mathrm{colim}_{i<\gamma} Q_i$ is also a subobject of P. Hence, it is α-presentable by hypothesis. Let

$$T \longrightarrow \mathrm{colim}_{i<\gamma} Q_i$$

be an epimorphism from an α-presentable projective object T. Then, by definition of being α-presentable, this morphism factorizes through $T \to \mathrm{colim}_{i<\beta} Q_i$ with β an ordinal such that $\#\beta < \alpha$. This implies that $Q_i = Q_j$ for every $\beta \leq i, j < \gamma$, and so P is α-noetherian.

$2 \Rightarrow 3.$ Let M be an α-generated object and N a subobject of M. We consider the pullback diagram

$$\begin{tikzcd} N \arrow[r, hook, "f"] & M \\ X \arrow[u, dotted, "g'"] \arrow[r, dotted, "f'"'] & P \arrow[u, "g"'] \end{tikzcd}$$

where P is α-presentable projective, g is an epimorphism and f is a monomorphism. Observe that the diagram is also a pushout diagram and then g' is an epimorphism and f' is a monomorphism [Pop73, Ch. 2, Corollary 4.3]. In particular, X is a subobject of an α-presentable projective object. By hypothesis it is α-generated and then so is N.

$3 \Rightarrow 2.$ Observe that 2 is a particular case of 3.

$3 \Rightarrow 4.$ First notice that every α-presentable object is α-generated. Now assume that X is α-generated. Then there exists a short exact sequence

$$0 \longrightarrow \ker(r) \hookrightarrow \textstyle\coprod_{i<\alpha} Q_i \xrightarrow{r} X \longrightarrow 0$$

with Q_i α-presentable projective. Then $\ker(r)$ is α-generated by 2 and this implies that X is α-presentable.

$4 \Rightarrow 3$. Let M be an α-generated object and N a subobject of M. Then there is a short exact sequence

$$0 \longrightarrow N \overset{i}{\hookrightarrow} M \twoheadrightarrow \operatorname{coker}(i) \longrightarrow 0$$

and $\operatorname{coker}(i)$ is α-generated. By 4, $\operatorname{coker}(i)$ is α-presentable and then N is α-generated.

$2 \Rightarrow 5$. We have proved that $2 \Rightarrow 4$. Then 2 and 4 imply that every submodule of an α-presentable projective object is α-presentable.

$5 \Rightarrow 2$. This follows from the fact that every α-presentable object is α-generated. $\square$

The following lemma says the category Mod_α-$\mathcal{C}$ is locally α^+-noetherian if $\#\mathcal{C} \leq \alpha$.

Lemma 4.4.3. *Let α be a regular cardinal and let $\mathcal{C}$ be an additive category with coproducts of less than α objects. If $\#\mathcal{C} \leq \alpha$, then every α^+-generated object in Mod_α-$\mathcal{C}$ is α^+-presentable.*

Since Mod_α-$\mathcal{C}$ is also locally α-presentable, Lemma 4.4.3 together with Proposition 4.4.2 implies that Mod_α-$\mathcal{C}$ is locally α^+-noetherian.

Proof. If M is an α^+-generated object in Mod_α-$\mathcal{C}$, then there is a short exact sequence

$$0 \longrightarrow K \longrightarrow \textstyle\coprod_{i\in I} \mathcal{C}(-, X_i) \xrightarrow{p} M \longrightarrow 0$$

where $\#I \leq \alpha$. Recall that the kernel in Mod_α-$\mathcal{C}$ is defined objectwise, *i.e.* for every Y in $\mathcal{C}$ we have $K(Y) = \ker(p(Y))$. Hence, by the Yoneda Lemma, an element in $K(Y)$ corresponds to $y \in \mathrm{Mod}_\alpha\text{-}\mathcal{C}(\mathcal{C}(-,Y), \coprod_{i\in I} \mathcal{C}(-, X_i))$ such that $p(y) = 0$. If $\#I < \alpha$, then, for every object Y in $\mathcal{C}$,

$$\mathrm{Mod}_\alpha\text{-}\mathcal{C}\left(\mathcal{C}(-,Y), \coprod_{i\in I} \mathcal{C}(-, X_i)\right) = \mathcal{C}\left(Y, \coprod_{i\in I} X_i\right)$$

which has cardinality less than or equal to α because $\#\mathcal{C} \leq \alpha$. If $\#I = \alpha$, we can identify I with the set of ordinals of cardinality less than α. Then $\coprod_{\gamma<\alpha} \mathcal{C}(-, X_i) = \operatorname{colim}_{\gamma<\alpha} \left(\coprod_{i\leq\gamma} \mathcal{C}(-, X_i)\right)$ where the colimit is α-filtered, being a chain of inclusions of cardinality α. On the other hand, every object $\coprod_{i\leq\gamma} \mathcal{C}(-, X_i)$ is a coproduct of cardinality less than α. As a consequence,

$$\mathrm{Mod}_\alpha\text{-}\mathcal{C}\left(\mathcal{C}(-,Y), \coprod_{i\in I} \mathcal{C}(-, X_i)\right) = \operatorname{colim}_{\gamma<\alpha}\left(\mathcal{C}\left(Y, \coprod_{i\leq\gamma} X_i\right)\right).$$

By hypothesis, for every object Y in $\mathcal{C}$, $\coprod_{i\leq\gamma}\mathcal{C}(Y,X_i)$ has cardinality less than or equal to α and the colimit is indexed by a poset of cardinality α, then $\#\operatorname{colim}_{\gamma<\alpha}(\mathcal{C}(Y,\coprod_{i<\gamma}X_i))\leq\alpha$. In conclusion, we have proved that

$$\#\mathrm{Mod}_\alpha\text{-}\mathcal{C}\left(\mathcal{C}(-,Y),\coprod_{i\in I}\mathcal{C}(-,X_i)\right)\leq\alpha$$

for every Y in $\mathcal{C}$. Since $K(Y)\subset\mathrm{Mod}_\alpha\text{-}\mathcal{C}(\mathcal{C}(-,Y),\coprod_{i\in I}\mathcal{C}(-,X_i))$ and there are, essentially, less than or equal to α objects Y in $\mathcal{C}$, K is α^+-generated and then M is α^+-presentable. □

The next result is a generalization of [Bru83, Lemma 2.2].

Lemma 4.4.4. *Let $\mathcal{A}$ be a locally α-noetherian abelian category with a generating set of α-presentable projectives. Suppose that $U=\coprod_{i<\gamma}U_i$ is a subobject of M in $\mathcal{A}$ and M/U is α-generated. Then $M\cong F\oplus S$ where F is α-presentable and $S\cong\coprod_{\beta\leq i<\gamma}U_i$ for an ordinal β such that $\#\beta<\alpha$.*

Proof. We have the following short exact sequence:

$$0\longrightarrow U\xrightarrow{f}M\xrightarrow{g}M/U\longrightarrow 0.$$

Since M/U is α-generated, there exists an epimorphism $h\colon P\to M/U$ where P is an α-generated projective object. Since P is projective, there exists a morphism $\phi\colon P\to M$ such that $g\circ\phi=h$. If $V=\phi(P)$, then $M=U+V=(U\oplus V)/(U\cap V)$. For every cardinal $\xi<\gamma$, we can consider the following subobject of V:

$$X_\xi=(\textstyle\coprod_{i<\xi}U_i)\cap V\overset{\psi}{\hookrightarrow}V.$$

This gives a well ordered set of subobjects of V indexed by γ. Since V is α-generated, by noetherianity there exists a cardinal $\beta<\alpha$ such that $X_i=X_j$ for every $\beta\leq i,j<\gamma$. This implies that $(\coprod_{i<\beta}U_i)\cap V=(\coprod_{i<\gamma}U_i)\cap V=U\cap V$.

Finally, since $M=(U\oplus V)/(U\cap V)$ and

$$U\oplus V=((\textstyle\coprod_{i<\beta}U_i)\oplus V)\oplus(\coprod_{\beta\leq i<\gamma}U_i),$$

we conclude that $M=((\coprod_{i<\beta}U_i)+V)\oplus(\coprod_{\beta\leq i<\gamma}U_i)$. □

Definition 4.4.5. Let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. An *α-flat resolution* of an object A in $\mathcal{A}$ is an exact sequence

$$\cdots \to F_k \to F_{k-1} \to \cdots \to F_0 \to A \to 0,$$

where every F_k is α-flat for every $k \geq 0$. The *α-flat dimension* of an object A in $\mathcal{A}$ is the minimal length of an α-flat resolution of A.

The *weak global dimension* of $\mathcal{A}$, denoted by $\mathrm{weakgldim}(\mathcal{A})$, is the supremum of the α-flat dimensions of all the objects in $\mathcal{A}$.

As a direct consequence of Lemma 4.3.9, every projective resolution is an α-flat resolution, and hence we have the following corollary.

Corollary 4.4.6. *The weak global dimension is less than or equal to the projective global dimension.*

Lemma 4.4.7. *Let $\mathcal{A}$ be a locally α-presentable α-Grothendieck category with a generating set of α-presentable projectives. Assume that $\mathcal{A}$ is locally α-noetherian. Then*

$$\mathrm{weakgldim}(\mathcal{A}) = \sup\{\mathrm{pd}(B) \mid B \ \alpha\text{-}presentable\}.$$

Proof. Let $m = \sup\{\mathrm{pd}(B) \mid B \ \alpha\text{-presentable}\}$ and fix an object X in $\mathcal{A}$. We will prove that the α-flat dimension of X is less than or equal to m. Since $\mathcal{A}$ is locally α-presentable, $X = \mathrm{colim}_I X_i$ where X_i is α-presentable for all $i \in I$ and I is α-filtered. We can consider ([Jen72]) a projective resolution of $\{X_i\}_I$ in $\mathcal{A}^I$

$$0 \to \{P_i^r\}_I \to \{P_i^{r-1}\}_I \to \cdots \to \{P_i^0\}_I \to \{X_i\}_I \to 0$$

such that, for every $i \in I$,

$$0 \to P_i^r \to P_i^{r-1} \to \cdots \to P_i^0 \to X_i \to 0$$

is a projective resolution of X_i with P_i^j α-presentable for every $0 \leq j \leq r$. By assumption, $\mathrm{pd}(X_i) \leq m$ for every $i \in I$, hence we have projective resolutions

$$0 \to Y_i \to P_i^{m-1} \to \cdots \to P_i^0 \to X_i \to 0$$

and Y_i is α-presentable since every P_i^j is α-presentable for every $0 \leq j \leq r$. Since $\mathcal{A}$ satisfies [AB5$_\alpha$], by taking the colimit over I we obtain an exact sequence

$$0 \to \mathrm{colim}_I Y_i \to \mathrm{colim}_I P_i^{m-1} \to \cdots \to \mathrm{colim}_I P_i^0 \to \mathrm{colim}_I X_i \cong X \to 0.$$

This gives an α-flat resolution of X of length less than or equal to m.

To see the other inequality, impose that $\mathrm{weakgldim}(\mathcal{A}) = n$. Fix an α-presentable object A in $\mathcal{A}$. Since $\mathcal{A}$ is a locally α-presentable abelian category, A has a resolution by α-presentable projective objects

$$\cdots \to P_1 \to P_0 \to A \to 0.$$

Since projective objects are α-flat by Lemma 4.3.9, it is also an α-flat resolution. Since A has weak dimension less than or equal than n, there is an α-flat resolution

$$0 \to Y \to P_{k-1} \to \cdots \to P_0 \to A \to 0.$$

Now, Y is a subobject of an α-presentable projective object, and hence, by noetherianity, it is also α-presentable. Finally, by Lemma 4.2.3, Y is also projective and the above sequence gives a projective resolution of A of length $k+1 \leq n$. □

In the case $\alpha = \aleph_0$, $\mathcal{A}$ is a Grothendieck category and we obtain that $\mathrm{weakgldim}(\mathcal{A}) = \mathrm{projgldim}(\mathcal{A})$; see [Bru83] for details. In our general context, since every projective object is α-flat, we only have that the projective global dimension is a lower bound for the α-flat one: $\mathrm{weakgldim}(\mathcal{A}) \leq \mathrm{projgldim}(\mathcal{A})$. Our next goal is to give an upper bound. With this aim, we will use our generalization of the Auslander Lemma (Corollary 4.1.5).

Corollary 4.4.8. *Let $\mathcal{A}$ be a locally $\aleph_n$-presentable $\aleph_n$-Grothendieck category with a generating set of $\aleph_n$-presentable projectives. Assume that $\mathcal{A}$ is locally $\aleph_n$-noetherian. Then*

$$\mathrm{weakgldim}(\mathcal{A}) \leq \mathrm{projgldim}(\mathcal{A}) \leq \mathrm{weakgldim}(\mathcal{A}) + n.$$

Proof. The first inequality follows from the fact that every projective object is $\aleph_n$-flat. For the second one, let $m = \sup\{\mathrm{pd}(B) \mid B\ \aleph_n\text{-presentable}\}$. By Lemma 4.4.7, $\mathrm{weakgldim}(\mathcal{A}) = m$. Fix an object A in $\mathcal{A}$. Since $\mathcal{A}$ has a generating set of $\aleph_n$-presentable projectives, there is an epimorphism $\phi\colon \coprod_{i\in I} P_i \to A$ with P_i $\aleph_n$-presentable projective for all $i \in I$. If A is $\aleph_n$-presentable, then $\mathrm{pd}(A) \leq m$. If A is not $\aleph_n$-presentable, then $\#I \geq \aleph_n$. Choose a well ordering in I and let γ be the smallest ordinal with cardinality $\#I$. We define $Q_i = \phi(\coprod_{j<i} P_j)$ for every $i < \gamma$. Then A is an $\aleph_n$-filtered colimit of a continuous ascending chain of subobjects $A = \mathrm{colim}_{i<\gamma} Q_i$. If $i \leq j < i + \omega_n$, then Q_j/Q_i is $\aleph_n$-generated, since there is an epimorphism $(\coprod_{k<j} P_k)/(\coprod_{k<i} P_k) \cong \coprod_{i\leq k<j} P_k \twoheadrightarrow Q_j/Q_i$. By noetherianity Q_j/Q_i is $\aleph_n$-presentable and then $\mathrm{pd}(Q_j/Q_i) \leq m$ for ever j such that $i \leq j < i + \omega_n$. Now Corollary 4.1.5 implies that $\mathrm{pd}(A) \leq n + m$. Hence, $\mathrm{projgldim}(\mathcal{A}) \leq m + n$. □

Since the weak global dimension of an abelian category is defined by α-flat resolutions, and α-flat objects in Mod_α-$\mathcal{A}^\alpha$ are exactly the left exact functors by Lemma 4.2.4, we have the following direct consequence of Lemma 4.2.6.

Corollary 4.4.9. *Let $\mathcal{A}$ be an abelian category satisfying* [AB3]. *Then the weak global dimension of* Mod_α-$\mathcal{A}^\alpha$ *is less than or equal to* 2.

Proof. Let X be an object in Mod_α-$\mathcal{A}^\alpha$. We construct an α-flat resolution of X

$$\cdots \longrightarrow F_1 \xrightarrow{f} F_0 \longrightarrow X \longrightarrow 0.$$

By Lemma 4.2.6, the kernel of f is also an α-flat object. Then the α-flat dimension of X is less than or equal to 2. □

The next result is a generalization of a result by Brune [Bru83, Theorem 2.1] for arbitrary cardinals.

Theorem 4.4.10. *Let $\mathcal{A}$ be a locally $\aleph_n$-presentable $\aleph_n$-Grothendieck category with a generating set of $\aleph_n$-presentable projectives. Assume that $\mathcal{A}$ is locally $\aleph_n$-noetherian. Let $\mathcal{A}^{\aleph_n}$ be the full subcategory of $\aleph_n$-presentable objects in $\mathcal{A}$. If* $\mathrm{Mod}_{\aleph_n}$-$\mathcal{A}^{\aleph_n}$ *is locally $\aleph_n$-noetherian, then every subobject of an $\aleph_n$-flat object in* $\mathrm{Mod}_{\aleph_n}$-$\mathcal{A}^{\aleph_n}$ *has projective dimension less than or equal to* $n+1$.

Proof. We begin by proving the following particular case: Every subobject U of an $\aleph_n$-presentable projective object P in $\mathrm{Mod}_{\aleph_n}$-$\mathcal{A}^{\aleph_n}$ has projective dimension less than or equal to 1. By Lemma 4.4.7, $\mathrm{weakgldim}(\mathrm{Mod}_{\aleph_n}\text{-}\mathcal{A}^{\aleph_n})$ is equal to $\sup\{\mathrm{pd}(B) \mid B\ \aleph_n\text{-presentable in } \mathrm{Mod}_{\aleph_n}\text{-}\mathcal{A}^{\aleph_n}\}$, and, by Corollary 4.4.9, $2 \geq \mathrm{weakgldim}(\mathrm{Mod}_{\aleph_n}\text{-}\mathcal{A}^{\aleph_n})$. Then there is a projective resolution of P/U of length 2

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P \longrightarrow P/U \longrightarrow 0.$$

In particular, we obtain a projective resolution of U of length 1.

Now we will prove the general case. Let U be a subobject of an $\aleph_n$-flat object F in $\mathrm{Mod}_{\aleph_n}$-$\mathcal{A}^{\aleph_n}$. We have seen in Proposition 4.2.5 that $\aleph_n$-flat objects are of the form $F = \mathcal{A}(-, M)|_{\mathcal{A}^{\aleph_n}}$ for M an object in $\mathcal{A}$. Remember that, since $\mathcal{A}$ is locally $\aleph_n$-noetherian, an object is $\aleph_n$-generated if and only if it is $\aleph_n$-presentable. If M is $\aleph_n$-presentable, we are in the situation of the previous case since $\mathcal{A}(-, M)|_{\mathcal{A}^{\aleph_n}} = \mathcal{A}^{\aleph_n}(-, M)$ would be $\aleph_n$-presentable projective. If M is not $\aleph_n$-presentable, then there is an epimorphism $h\colon \coprod_{i<\gamma} Q_i \to M$ where $\gamma \geq \aleph_n$ and Q_i is $\aleph_n$-presentable projective for every $i < \gamma$. We define $M_i = \coprod_{j<i} h(Q_j)$ for every $i < \gamma$. This defines a continuous ascending chain

$\{M_i\}_{i<\gamma}$ of subobjects of M such that $\mathrm{colim}_{i<\gamma} M_i = M$ is an $\aleph_n$-filtered colimit and M_j/M_i is $\aleph_n$-presentable for every $i \leq j < i + \omega_n$ and $M_0 = 0$. Since the colimit is $\aleph_n$-filtered, $\mathcal{A}(-, M)|_{\mathcal{A}^{\aleph_n}} = \mathrm{colim}_{i<\gamma}\mathcal{A}(-, M_i)|_{\mathcal{A}^{\aleph_n}}$ in $\mathrm{Mod}_{\aleph_n}$-$\mathcal{A}^{\aleph_n}$.

Let $U_i = U \cap \mathcal{A}(-, M_i)|_{\mathcal{A}^{\aleph_n}}$. Since we are assuming that $\mathcal{A}$ satisfies [AB5$_{\aleph_n}$] and $\{U_i\}_{i<\gamma}$ define a continuous well ordered chain of subobjects of U that is $\aleph_n$-filtered, Lemma 2.3.24 implies that

$$\begin{aligned}\mathrm{colim}_{i<\gamma} U_i &= \mathrm{colim}_{i<\gamma}(U \cap \mathcal{A}(-, M_i)|_{\mathcal{A}^{\aleph_n}}) \\ &= U \cap (\mathrm{colim}_{i<\gamma}\mathcal{A}(-, M_i)|_{\mathcal{A}^{\aleph_n}}) \\ &= U \cap \mathcal{A}(-, M)|_{\mathcal{A}^{\aleph_n}} = U.\end{aligned}$$

Next we show that $\{U_i\}_{i<\gamma}$ satisfies the hypotheses of the generalization of the Auslander Lemma (Corollary 4.1.5). First observe that $U_0 = 0$ and that, for every $i < \gamma$, $U_i \subset \mathcal{A}(-, M_i)|_{\mathcal{A}^{\aleph_n}}$. Fix a pair of ordinals $i \leq j < i + \omega_n$. We can rewrite U_j/U_i as

$$(U \cap \mathcal{A}(-, M_j)|_{\mathcal{A}^{\aleph_n}}) \,/\, ((U \cap \mathcal{A}(-, M_j)|_{\mathcal{A}^{\aleph_n}}) \cap \mathcal{A}(-, M_i)|_{\mathcal{A}^{\aleph_n}})$$

and this is equal to

$$((U \cap \mathcal{A}(-, M_j)|_{\mathcal{A}^{\aleph_n}}) + \mathcal{A}(-, M_i)|_{\mathcal{A}^{\aleph_n}}) \,/ \mathcal{A}(-, M_i)|_{\mathcal{A}^{\aleph_n}}.$$

Hence, $U_j/U_i \subset \mathcal{A}(-, M_j)|_{\mathcal{A}^{\aleph_n}}/\mathcal{A}(-, M_i)|_{\mathcal{A}^{\aleph_n}}$. But, since $S_{\aleph_n}$ is left exact, we obtain that $U_j/U_i \subset \mathcal{A}(-, M_j/M_i)|_{\mathcal{A}^{\aleph_n}}$ and now, since M_j/M_i is $\aleph_n$-presentable, $\mathcal{A}(-, M_j/M_i)|_{\mathcal{A}^{\aleph_n}}$ is $\aleph_n$-presentable projective in $\mathrm{Mod}_{\aleph_n}$-$\mathcal{A}^{\aleph_n}$ and, by the first part of the proof, U_j/U_i has projective dimension less than or equal to 1 for every $i \leq j < i + \omega_n$. Finally, by Corollary 4.1.5, U has projective dimension less than or equal to $n + 1$. □

The conclusion of Theorem 4.4.10 can be reformulated in different ways, as shown in the following proposition.

Proposition 4.4.11. *Let $\mathcal{A}$ be a locally $\aleph_n$-presentable $\aleph_n$-Grothendieck category with a generating set of $\aleph_n$-presentable projectives. Assume that $\mathcal{A}$ is locally $\aleph_n$-noetherian. Then the following are equivalent:*

1. *Every subobject of an $\aleph_n$-flat object in* $\mathrm{Mod}_{\aleph_n}$-$\mathcal{A}^{\aleph_n}$ *has projective dimension less than or equal to $n + 1$.*

2. *In $\mathcal{A}$ every subobject of an $\aleph_n$-pure projective object has $\aleph_n$-pure projective dimension less than or equal to n.*

3. $\operatorname{projgldim}(\operatorname{Mod}_{\aleph_n}\text{-}\mathcal{A}^{\aleph_n}) \leq n+2$.

In the case $n=0$ this result is also part of Brune's [Bru83, Theorem 2.1] and item 2 is known as the *Kulikov property*.

Proof.

$1 \Rightarrow 2$. Let $0 \to Q \xrightarrow{f} P \to X \to 0$ be a short exact sequence in $\mathcal{A}$ such that P is $\aleph_n$-pure projective. We will prove that Q has $\aleph_n$-pure projective dimension less than or equal to n. If we apply $S_{\aleph_n}$ to the sequence, we obtain an exact sequence $0 \to S_{\aleph_n}(Q) \to S_{\aleph_n}(P) \to S_{\aleph_n}(X)$, where every object is $\aleph_n$-flat and $S_{\aleph_n}(P)$ is projective in $\operatorname{Mod}_{\aleph_n}\text{-}\mathcal{A}^{\aleph_n}$. But now $\operatorname{coker}(S_{\aleph_n}(f)) \subset S_{\aleph_n}(X)$. Then, by hypothesis, $\operatorname{pd}(\operatorname{coker}(S_{\aleph_n}(f)) \leq n+1$. Hence, $\operatorname{pd}(S_{\aleph_n}(Q)) \leq n$ and so Q has $\aleph_n$-pure projective dimension less than or equal to n.

$2 \Rightarrow 3$. Let X be an object in $\operatorname{Mod}_{\aleph_n}\text{-}\mathcal{A}^{\aleph_n}$ and let

$$\cdots \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \longrightarrow X \longrightarrow 0$$

be a projective resolution of X in $\operatorname{Mod}_{\aleph_n}\text{-}\mathcal{A}^{\aleph_n}$. By Corollary 4.4.9, the $\aleph_n$-flat dimension of X is less than or equal to 2. In particular, $F = \ker(f_2)$ is $\aleph_n$-flat and

$$0 \longrightarrow F \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \longrightarrow X \longrightarrow 0$$

is an $\aleph_n$-flat resolution of X. But now, by assumption, F has projective dimension less than or equal to n in $\operatorname{Mod}_{\aleph_n}\text{-}\mathcal{A}^{\aleph_n}$. Therefore X has projective dimension less than or equal to $n+2$.

$3 \Rightarrow 1$. If U is a subobject of a projective object P in $\operatorname{Mod}_{\aleph_n}\text{-}\mathcal{A}^{\aleph_n}$, then, by assumption, there is a projective resolution of P/U of length $n+2$

$$0 \longrightarrow P_{n+2} \longrightarrow P_{n+1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P \longrightarrow P/U \longrightarrow 0.$$

In particular, we obtain a projective resolution of U of length $n+1$. Now we can proceed exactly as in the proof of Theorem 4.4.10 to see that the same conclusion is true if U is a subobject of an $\aleph_n$-flat object F in $\operatorname{Mod}_{\aleph_n}\text{-}\mathcal{A}^{\aleph_n}$. We sketch how this is done. First we write $F = \mathcal{A}(-,M)|_{\mathcal{A}^{\aleph_n}}$ where $M = \operatorname{colim}_{i<\gamma} M_i = M$ is an $\aleph_n$-filtered colimit of a continuous ascending chain of subobjects of M such that M_j/M_i is $\aleph_n$-presentable for every $i \leq j < i+\omega_n$ and $M_0 = 0$. Since the colimit is

$\aleph_n$-filtered, $F = \mathcal{A}(-,M)|_{\mathcal{A}^{\aleph_n}} = \mathrm{colim}_{i<\gamma}\mathcal{A}(-,M_i)|_{\mathcal{A}^{\aleph_n}}$ in $\mathrm{Mod}_{\aleph_n}$-$\mathcal{A}^{\aleph_n}$. By [AB5$_{\aleph_n}$] we can write $U = \mathrm{colim}_{i<\gamma}U_i$ where $U_i = U \cap \mathcal{A}(-,M_i)|_{\mathcal{A}^{\aleph_n}}$. Finally, it is easy to show that $\{U_i\}_{i\in I}$ satisfies the hypothesis of the Auslander Lemma (Corollary 4.1.5) and it implies that U has projective dimension less than or equal to $n+1$. □

Finally, we want to notice that, in the case $n = 0$, the converse of Theorem 4.4.10 also holds as a consequence of Lemma 4.4.4. In our general context, however, we can only prove the following result.

Proposition 4.4.12. *Let $\mathcal{A}$ be a locally $\aleph_n$-presentable $\aleph_n$-Grothendieck category with a generating set of $\aleph_n$-presentable projectives. Assume that $\mathcal{A}$ is locally $\aleph_n$-noetherian. If we assume that* $\mathrm{projgldim}(\mathrm{Mod}_{\aleph_n}\text{-}\mathcal{A}^{\aleph_n}) \leq 2$, *then* $\mathrm{Mod}_{\aleph_n}$-$\mathcal{A}^{\aleph_n}$ *is locally $\aleph_n$-noetherian.*

Proof. By Proposition 4.4.2, in order to see that $\mathrm{Mod}_{\aleph_n}$-$\mathcal{A}^{\aleph_n}$ is locally $\aleph_n$-noetherian it is enough to prove the following: If U is a subobject of $\mathcal{A}^{\aleph_n}(-,X)$ in the category $\mathrm{Mod}_{\aleph_n}$-$\mathcal{A}^{\aleph_n}$, then U is $\aleph_n$-generated.

By assumption, $\mathrm{pd}(\mathcal{A}^{\aleph_n}(-,X)/U) \leq 2$. Then there is a projective resolution of $\mathcal{A}^{\aleph_n}(-,X)/U$ of the form

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}(-,Q)|_{\mathcal{A}^{\aleph_n}} \arrow[r, "\phi"] & \mathcal{A}(-,P)|_{\mathcal{A}^{\aleph_n}} \arrow[rr, "\psi"] \arrow[dr] & & \mathcal{A}^{\aleph_n}(-,X) \arrow[r] & \mathcal{A}^{\aleph_n}(-,X)/U \arrow[r] & 0 \\
& & & U \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where $Q = \coprod_{i\in I} Q_i$ and $P = \coprod_{j\in J} P_j$ with Q_i and P_j $\aleph_n$-presentable for all $i \in I$ and $j \in I$. By the Yoneda Lemma, we can assume that $\phi = \mathcal{A}(-,f)|_{\mathcal{A}^{\aleph_n}}$ for $f\colon Q \to P$ and $\psi = \mathcal{A}(-,g)|_{\mathcal{A}^{\aleph_n}}$ for $g\colon P \to X$. Then we have the following sequence:

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r, "f"] & P \arrow[rr] \arrow[dr, "\pi"'] & & X \\
& & & P/Q \arrow[ur] \arrow[dr] & \\
& & 0 \arrow[ur] & & 0
\end{tikzcd}$$

and P/Q is a subobject of X which is $\aleph_n$-presentable. Since $\mathcal{A}$ is locally $\aleph_n$-noetherian, P/Q is also $\aleph_n$-presentable and we are under the hypotheses of Lemma 4.4.4. Hence, there exists a subset $I' \subset I$ such that $\#(I \setminus I') < \aleph_n$, and $S = \coprod_{i\in I'} Q_i$ is a direct subobject of $P \cong S \oplus F$ with F $\aleph_n$-presentable. Since $\pi(S) = 0$, $\mathrm{im}(\mathcal{A}^{\aleph_n}(-,g|_F)) = \mathrm{im}(\mathcal{A}(-,g)|_{\mathcal{A}^{\aleph_n}}) = \mathrm{im}(\psi) = U$. Then there is an epimorphism

$$\mathcal{A}^{\aleph_n}(-,F) \longrightarrow U.$$

and U is $\aleph_n$-generated. □

Chapter 5

Adams representability for derived categories of rings

In this chapter we study derived categories of rings in the spirit of [CKN01], although our treatment is valid for uncountable cardinals. We will finish the chapter showing how a recent result by Braun and Göbel implies that $\mathrm{D}(\mathbb{Z})$ does not satisfy α-Adams representability for morphisms for any uncountable regular cardinal α. This example gives a negative answer to Question 2.3.34.

In the first section we recall the notion of α-purity for rings following [JL89]. In contrast to classical purity, that we will review in Appendix A, the notion of α-purity has not received much attention, and few results are known about it.

In the second section we study the relation between the α-pure global dimension of a ring and the α-pure global dimension of its derived category. Many results in this section are generalizations of results in [CKN01].

5.1 Higher purity for rings

In this section we define α-purity for a ring. The case $\alpha = \aleph_0$ is classical and we review it in Appendix A. In the case $\alpha > \aleph_0$, the definition can be found in Fuchs' book [Fuc70, Section 31]. However, it has not been studied as much as the case $\alpha = \aleph_0$. The first results that we present in this section are taken from [JL89, p. 137–138] and are formal consequences of the definitions.

Throughout the chapter, all the rings R will be associative with identity and R-Mod will denote the category of left R-modules.

Definition 5.1.1. Let R be a ring. An R-module M is called *α-generated* if it can be generated as an R-module by a subset of cardinality less than α

or, equivalently, if there is an epimorphism

$$\coprod_{i\in I} R \longrightarrow M \to 0$$

where $\#I < \alpha$. An R-module M is called *α-presented* if there exists a short exact sequence

$$0 \longrightarrow G \longrightarrow \coprod_{i\in I} R \longrightarrow M \longrightarrow 0$$

where $\#I < \alpha$ and G is α-generated.

Definition 5.1.2. Let R be a ring and α be a regular cardinal. A short exact sequence of R-modules $0 \to A \to B \to C \to 0$ is called *α-pure exact* if for every α-presented R-module P the induced sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}(P, A) \longrightarrow \mathrm{Hom}(P, B) \longrightarrow \mathrm{Hom}(P, C) \longrightarrow 0$$

is exact.

In the case $\alpha = \aleph_0$ there are many equivalent definitions of α-pure exact sequences, as we will explain in Appendix A. For higher cardinals, not so many equivalent definitions are available.

Lemma 5.1.3 ([GU71])**.** *Let R be a ring and α be a regular cardinal. Then every R-module is an α-filtered colimit of α-presented R-modules.*

Proposition 5.1.4 ([AR94, Proposition 2.30], [JL89, Proposition 7.16])**.** *Let R be a ring and α be a regular cardinal. A short exact sequence of R-modules is α-pure exact if and only if it is an α-filtered colimit of split short exact sequences. In particular, α-pure exact sequences are closed under α-filtered colimits.*

Now we can define what is a projective object relative to α-pure exact sequences.

Definition 5.1.5. Let R be a ring and let α be a regular cardinal. An R-module P is called *α-pure projective* if for every α-pure exact sequence

$$0 \to A \to B \to C \to 0$$

the induced sequence

$$0 \longrightarrow \mathrm{Hom}(P, A) \longrightarrow \mathrm{Hom}(P, B) \longrightarrow \mathrm{Hom}(P, C) \longrightarrow 0$$

is exact.

Lemma 5.1.6 ([JL89, Proposition 7.16]). *Let R be a ring and α be a regular cardinal. An R-module P is α-pure projective if and only if every α-pure exact sequence* $0 \to Y \to X \to P \to 0$ *splits.*

Remark 5.1.7. One can think of α-purity as a theory of homological algebra where one forces α-presented modules to be projective. Since every R-module M admits an α-pure exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is a coproduct of α-presented modules, the category of R-modules has enough α-pure projectives and it follows that α-pure projectives are precisely summands of coproducts of α-presented modules.

Definition 5.1.8. Let R be a ring and α be a regular cardinal. An *α-pure projective resolution* of an R-module M is a long exact sequence

$$\ldots \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

where P_n is α-pure projective and $0 \longrightarrow \ker(f_n) \longrightarrow P_n \longrightarrow \ker(f_{n-1}) \longrightarrow 0$ is α-pure exact for all $n \geq 0$. The *α-pure projective dimension* of an R-module M is less than or equal to n, denoted by $\mathrm{Ppd}_\alpha(M) \leq n$, if there exists an α-pure projective resolution of length n of M. The *α-pure global dimension* of R is defined as

$$\mathrm{Pgldim}_\alpha(R) = \sup\{\mathrm{Ppd}_\alpha(M) \mid M \text{ an } R\text{-module}\}.$$

If we take $\alpha = \aleph_0$ then we obtain the classical notions of purity that can be found in Appendix A and that will be denoted by Pgldim.

5.2 Higher purity for the derived category of a ring

In this section we prove a relation between the α-pure global dimension of a ring and the α-pure global dimension of its derived category. Our results in this section are strongly influenced by results of Christensen, Keller and Neeman [CKN01] for the case $\alpha = \aleph_0$.

Remember that we have already given a characterization of α-compact objects in the derived category of a ring. For $\alpha = \aleph_0$ we have done it in Example 2.1.9 and Proposition 2.1.13, and for $\alpha > \aleph_0$ in Example 2.1.10 and Proposition 2.1.14.

In Proposition 2.1.14 we have considered the R-modules as complexes in $\mathrm{D}(R)$ concentrated in degree 0. This can be used to define a functor

$$\begin{tikzcd}
\text{Mod-}R \arrow[rr] \arrow[dr, hook] & & \text{Mod}_\alpha\text{-}\mathcal{T}^\alpha \\
& \mathrm{D}(R) = \mathcal{T} \arrow[ur, "S_\alpha"'] &
\end{tikzcd}$$

where S_α is the restricted Yoneda functor. More surprising is the fact that there are functors in the other direction: The cohomology functors of chain complexes $H^n \colon \mathrm{D}(R) \to \text{Mod-}R$ can be extended to functors

$$\begin{tikzcd}
\text{Mod}_\alpha\text{-}\mathcal{T}^\alpha \arrow[rr, "H^n"] & & \text{Mod-}R \\
& \mathcal{T} = \mathrm{D}(R) \arrow[ul, "S_\alpha"] \arrow[ur, "H^n"'] &
\end{tikzcd}$$

by defining $H^n(F) = F(\Sigma^{-n}R)$ for every $F \in \text{Mod}_\alpha\text{-}\mathcal{T}^\alpha$. These two functors between Mod-R and $\text{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ will be used in the following lemma to compare α-purity for rings and α-purity for their derived categories. For the case $\alpha = \aleph_0$, this result has been proved in [CKN01, Lemma 1.3]. We will use α-pure exact sequences in the category $\text{Mod}_\alpha\text{-}\mathcal{T}^\alpha$. Recall that they have been defined in Definition 4.3.1.

Lemma 5.2.1. *Let R be a ring. Assume that R is α-coherent for $\alpha > \aleph_0$. Then:*

1. *The functor*

$$\begin{tikzcd}
\text{Mod-}R \arrow[rr] \arrow[dr, hook, "\iota"'] & & \text{Mod}_\alpha\text{-}\mathcal{T}^\alpha \\
& \mathrm{D}(R) = \mathcal{T} \arrow[ur, "S_\alpha"'] &
\end{tikzcd}$$

   commutes with α-filtered colimits; it takes α-pure projective R-modules to projective objects in $\text{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ and it takes α-pure exact sequences of R-modules to α-pure exact sequences in $\text{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ (hence it is exact).

2. *For every $n \in \mathbb{Z}$, the functor*

$$\begin{tikzcd}
\text{Mod}_\alpha\text{-}\mathcal{T}^\alpha \arrow[rr, "H^n"] & & \text{Mod-}R \\
& \mathcal{T} = \mathrm{D}(R) \arrow[ul, "S_\alpha"] \arrow[ur, "H^n"'] &
\end{tikzcd}$$

   defined by $H^n(F) = F(\Sigma^{-n}R)$ commutes with α-filtered colimits; it takes projective objects in $\text{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ to α-pure projective R-modules and takes α-pure exact sequences in $\text{Mod}_\alpha\text{-}\mathcal{T}^\alpha$ to α-pure exact sequences of R-modules.

Proof. The first part follows from results in [Ros05]; see Remark 2.3.33. We next give an alternative argument, which is useful towards the second part. Let $M = \mathrm{colim}_I M_i$ be an α-filtered colimit in Mod-R. By the universal property of the colimit, we have a morphism

$$\phi\colon \mathrm{colim}_I \mathcal{T}(-,\iota(M_i))|_{\mathcal{T}^\alpha} \longrightarrow \mathcal{T}(-,\iota(\mathrm{colim}_I M_i))$$

in Mod_α-$\mathcal{T}^\alpha$. We will prove that $\phi(P)$ is an isomorphism for every object P in $\mathcal{T}^\alpha$. Recall that $\{\Sigma^n\iota(R) \mid n \in \mathbb{Z}\}$ is a generating set of compact objects in $\mathrm{D}(R)$ and that every object in $\mathcal{T}^\alpha$ can be constructed recursively from the set $\{\Sigma^n\iota(R) \mid n \in \mathbb{Z}\}$ by coproducts of less than α objects and triangles; see [Nee01b, Propostion 3.2.5] for details. Let $P = \Sigma^n\iota(R)$. Since α-filtered colimits are exact in R-Mod, we have the following sequence of isomorphisms: $\mathrm{colim}_I \mathcal{T}(\Sigma^n\iota(R),\iota(M_i)) \cong \mathrm{colim}_I H^{-n}(M_i) \cong H^{-n}(\mathrm{colim}_I M_i) \cong \mathcal{T}(R,\iota(\mathrm{colim}_I M_i))$ and so $\phi(P)$ is an isomorphism. Now let $P = \coprod_{j\in J} P_j$ be a coproduct in $\mathcal{T}^\alpha$ and assume that $\phi(P)$ is an isomorphism for every $j \in J$. Then we have the following sequence of isomorphisms:

$$\begin{aligned}\mathrm{colim}_I \mathcal{T}(\textstyle\coprod_{j\in J} P_j,\iota(M_i)) &\cong \textstyle\prod_{j\in J} \mathrm{colim}_I \mathcal{T}(P_j,\iota(M_i))\\ &\cong \textstyle\prod_{j\in J} \mathcal{T}(P_j,\iota(\mathrm{colim}_I M_i))\\ &\cong \mathcal{T}(\textstyle\coprod_{j\in J} P_j,\iota(\mathrm{colim}_I M_i)).\end{aligned}$$

Finally, let P be an object in $\mathcal{T}^\alpha$ such that there exists a triangle

$$P_1 \longrightarrow P_2 \longrightarrow P \longrightarrow \Sigma P_1$$

where $\phi(P_1)$ and $\phi(P_2)$ are isomorphisms. Since representable functors are cohomological and α-filtered colimits are exact in Ab, we obtain that both functors $\mathrm{colim}_I \mathcal{T}(-,\iota(M_i))|_{\mathcal{T}^\alpha}$ and $\mathcal{T}(-,\iota(\mathrm{colim}_I M_i))$ are cohomological. Then, if we apply ϕ to the previous triangle, we obtain a morphism between long exact sequences where $\phi(\Sigma^n P_1)$ and $\phi(\Sigma^n P_2)$ are isomorphisms for every $n \in \mathbb{Z}$. Hence, $\phi(P)$ is also an isomorphism by the Five Lemma.

Let M be an α-pure projective R-module. Then, as we observed in Remark 5.1.7, M is a direct summand of a coproduct of α-presented modules: $\coprod_{i\in I} P_i = M \oplus Q$ where P_i is α-presented for every $i \in I$. Since S_α commutes with coproducts, $\coprod_{i\in I} S_\alpha(\iota(P_i)) \cong S_\alpha(\iota(M)) \oplus S_\alpha(\iota(Q))$. Since R is α-coherent, Proposition 2.1.14 implies that $\iota(P_i)$ is α-compact for every $i \in I$. Hence, $S_\alpha(P_i) = \mathcal{T}^\alpha(-,P_i)$ is projective by Proposition 2.3.19. Since $S_\alpha(\iota(M))$ is a summand of a coproduct of projectives, it is itself projective.

By Proposition 5.1.4, every α-pure exact sequence of R-modules is an α-filtered colimit of split exact sequences. Since S_α preserves α-filtered colimits and is homological by Proposition 2.3.20, it sends every α-pure exact sequence to an α-filtered colimit of split exact sequences, and α-filtered colimits of split exact sequences are α-pure exact in Mod_α-$\mathcal{T}^\alpha$ by Lemma 4.3.3.

Next we prove the second part. Since H^n is defined as evaluation on $\Sigma^{-n}R$, it preserves α-filtered colimits by Lemma 2.3.18.

Let X be a projective object in Mod_α-$\mathcal{T}^\alpha$. By Proposition 2.3.19, there exists an object Y in Mod_α-$\mathcal{T}^\alpha$ such that $X \oplus Y = \coprod_{i\in I} \mathcal{T}^\alpha(-, P_i)$. Again by Proposition 2.3.19 we know that $H^n \colon \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha \to \mathrm{Mod}\text{-}R$ preserves finite coproducts, and by Proposition 2.3.20 we know that S_α respects all coproducts. Hence we have the following sequence of isomorphisms: $H^n(X) \oplus H^n(Y) \cong H^n(X \oplus Y) = H^n(\coprod_{i\in I} \mathcal{T}^\alpha(-, P_i)) \cong H^n(\mathcal{T}(-, \coprod_{i\in I} P_i)|_{\mathcal{T}^\alpha}) \cong H^n(\coprod_{i\in I} P_i) \cong \coprod_{i\in I} H^n(P_i)$. So it is enough to prove that $H^n(P_i)$ is α-pure projective for every $i \in I$. By Example 2.1.10, P_i is quasi-isomorphic to a complex of free R-modules with less than α generators and therefore, since R is α-coherent, $H^n(P_i)$ is α-presentable for every $n \in \mathbb{Z}$. Hence, $H^n(X)$ is α-pure projective.

Finally, by Lemma 4.3.3, every α-pure exact sequence in Mod_α-$\mathcal{T}^\alpha$ is an α-filtered colimit of split exact sequences. But we have already proved that $H^n \colon \mathrm{Mod}_\alpha\text{-}\mathcal{T}^\alpha \to \mathrm{Mod}\text{-}R$ commutes with α-filtered colimits and it is exact because it is defined as evaluation on $\Sigma^{-n}R$. Hence, H^n sends every α-pure exact sequence to an α-filtered colimit of split exact sequence in Mod-R, which is α-pure exact by Lemma 5.1.4. □

Theorem 5.2.2. *Let R be a ring. Assume that R is α-coherent for $\alpha > \aleph_0$. Then, for any R-module M,*

$$\mathrm{Ppd}_\alpha(M) = \mathrm{pd}(S_\alpha(M))$$

where the left-hand side is the α-pure projective dimension of the module M and the right-hand side is its projective dimension as an object of Mod_α-$\mathrm{D}(R)^\alpha$ *by considering M as a chain complex concentrated in degree* 0.

Proof. By Lemma 5.2.1, the restricted Yoneda functor Mod_α-$\mathrm{D}(R)^\alpha$, where the R-modules are considered as objects in $\mathrm{D}(R)$ concentrated in degree 0, takes α-pure projective resolutions to projective resolutions. This implies the inequality $\mathrm{Ppd}_\alpha(M) \geq \mathrm{pd}(S_\alpha(M))$.

To prove the converse, fix a projective resolution of $\mathrm{D}(R)(-, M)|_{\mathrm{D}(R)^\alpha}$ in the category Mod_α-$\mathrm{D}(R)^\alpha$. If M is α-presented, since we are assuming that R is α-coherent, Proposition 2.1.14 implies that M is α-compact as a complex concentrated in degree 0 in $\mathrm{D}(R)$, hence $\mathrm{D}(R)^\alpha(-, M)$ is projective by Proposition 2.3.19 and the projective resolution is null-homotopic. Now let M be an arbitrary R-module. Lemma 5.1.3 implies that it is an α-filtered colimit of α-presented R-modules. Then our fixed projective resolution of

$\mathrm{D}(R)(-,M)|_{\mathrm{D}(R)^\alpha}$ is an α-filtered colimit of null-homotopic complexes in $\mathrm{Mod}_\alpha\text{-}\mathrm{D}(R)^\alpha$ and then it is an α-pure exact resolution. By the second part of Lemma 5.2.1, its image under H^0 is an α-pure projective resolution of $H^0(S_\alpha(M)) = M$. This implies the inequality $\mathrm{Ppd}_\alpha(M) \leq \mathrm{pd}(S_\alpha(M))$. □

Recall that a ring is *hereditary* if it has projective global dimension less than or equal to 1 or, equivalently, if every submodule of a projective R-module is projective. For the basic properties of hereditary rings we refer to [Rot09, Ch. 4]. The ring of non-commutative polynomials over a field and all commutative principal ideal domains are examples of hereditary rings. We have seen in Remark 2.1.12 that hereditary rings are α-coherent for every α.

Corollary 5.2.3. *Let R be a ring and $\alpha > \aleph_0$ be a regular cardinal.*

1. *If R is α-coherent, then* $\mathrm{Pgldim}_\alpha(R) \leq \mathrm{Pgldim}_\alpha(\mathrm{D}(R))$.
2. *If R is hereditary, then* $\mathrm{Pgldim}_\alpha(R) = \mathrm{Pgldim}_\alpha(\mathrm{D}(R))$.

Proof. For the first part, observe that we are under the conditions of Proposition 2.1.14 and Theorem 5.2.2. This implies that, for every R-module M, $\mathrm{Ppd}_\alpha(M) = \mathrm{pd}(S_\alpha(M))$. Hence, $\mathrm{Pgldim}_\alpha(R) \leq \mathrm{Pgldim}_\alpha(\mathrm{D}(R))$.

For the second part, remember from Example 2.3.30 that every object in $\mathrm{D}(R)$ is of the form $X \cong \coprod_{n\in\mathbb{Z}} \Sigma^n H^{-n}(X)$. Then the projective dimension of $S_\alpha(X)$ in $\mathrm{Mod}_\alpha\text{-}\mathrm{D}(R)^\alpha$ is not greater than the supremum of the projective dimensions of $S_\alpha(H^n(X))$ and $\mathrm{Ppd}_\alpha(H^n(X)) = \mathrm{pd}(S_\alpha(H^n(X)))$ by Theorem 5.2.2. But, by definition, $\mathrm{Ppd}_\alpha(H^n(X)) \leq \mathrm{Pgldim}_\alpha(R)$. Hence, $\mathrm{Pgldim}_\alpha(R) \geq \mathrm{Pgldim}_\alpha(\mathrm{D}(R))$. □

The analog of this result in the case $\alpha = \aleph_0$ was proved by Christensen, Keller and Neeman in [CKN01, Proposition 1.4]. However, for $\alpha = \aleph_0$, we have to add the hypothesis that "every finitely presented R-module is of finite projective dimension" in the statement of item 1. This difference follows from the characterizations of α-compact objects that we explained in Proposition 2.1.13 and Proposition 2.1.14.

In Example 5.2.8 we will use Corollary 5.2.3 together with an example of Braun and Göbel showing that $\mathrm{Pgldim}_\alpha(\mathbb{Z}) > 1$ to prove that $\mathrm{D}(\mathbb{Z})$ does not satisfy α-Adams representability for morphisms for any $\alpha > \aleph_0$.

We begin by recalling the examples provided in [CKN01] for the case $\alpha = \aleph_0$, which use the classical results of purity for rings that we explain in Appendix A.

Example 5.2.4 ([CKN01]). Let k be a field such that $\#k = \aleph_t$, where $t \geq 0$, and let $R = k[X,Y]$. It is coherent and $\mathrm{Pgldim}_{\aleph_0}(R) = t+1$ [BL82]; see Appendix A for details. Then Adams representability holds for $\mathrm{D}(R)$ if and only if $\#k \leq \aleph_0$.

This result can be extended in the following way.

Example 5.2.5 ([CKN01]). Let R be a finite dimensional hereditary algebra over an algebraically closed field of uncountable cardinality $\aleph_t$, $t > 0$. Then, by Theorem A.0.22, $\mathrm{Pgldim}_{\aleph_0}(R) \leq 1$ if and only if R has finite representation type. Hence R satisfies Adams representability for morphisms if and only if R has finite representation type.

Giving a counterexample to Adams representability for objects is more involved, because we cannot use Theorem 3.1.10 as in the case of Adams representability for morphisms. We will use the following result.

Theorem 5.2.6 ([CKN01, Theorem 2.13])**.** *Let R be a hereditary ring. Then:*

1. *Adams representability for morphisms holds in* $\mathrm{D}(R)$ *if and only if* $\mathrm{Pgldim}_{\aleph_0}(R) \leq 1$.

2. *Adams representability for objects holds in* $\mathrm{D}(R)$ *if and only if* $\mathrm{Pgldim}_{\aleph_0}(R) \leq 2$.

All the implications follow from Theorem 3.1.10 except for the *only if* part of 2.

Observe that, in order to use Theorem 5.2.6, we need a hereditary ring. Hence, we cannot use $k[X, Y]$ as before. For this reason, we will use the ring of non-commutative polynomials over a field $k\langle X, Y\rangle$, which is hereditary and has the same pure global dimension as $k[X, Y]$; see Appendix A for details.

Example 5.2.7 ([CKN01]). Let k be a field such that $\#k = \aleph_t$ and let $R = k\langle X, Y\rangle$. Then R is hereditary and $\mathrm{Pgldim}_{\aleph_0}(k\langle X, Y\rangle) = t+1$. By Theorem 5.2.6, $\mathrm{D}(R)$ satisfies Adams representability for objects if and only if $\#k \leq \aleph_1$. In particular, if $\#k = \aleph_1$, then $\mathrm{D}(R)$ satisfies Adams representability for objects but not for morphisms.

Let $k = \mathbb{C}$. Curiously enough, the pure global dimension of $k\langle X, Y\rangle$ depends on the set-theoretical axioms that we are considering. Notice that the equality $\#\,\mathbb{C} = \aleph_t$ is undecidable with ZFC for $t > 0$. Then, by Theorem 5.2.6, Adams representability for objects in $\mathrm{D}(\mathbb{C}\langle X, Y\rangle)$ is undecidable using ZFC. This argument follows from the original work of Osofsky on homological dimension of modules [Oso73].

The next example, due to Braun and Göbel, together with the results in this chapter and the previous chapter, will provide a negative answer to Question 2.3.34.

Example 5.2.8 (Braun and Göbel [BG10]). For every regular cardinal $\alpha > \aleph_0$, there exists a cotorsion abelian group A such that $\mathrm{Ppd}_\alpha(A) > 1$.

Consider the category $\mathrm{D}(\mathbb{Z})$. It is compactly generated, as every derived category of a ring. On the one hand, since $\mathbb{Z}$ is hereditary, every complex X in $\mathrm{D}(\mathbb{Z})$ is isomorphic to $\coprod_{n\in\mathbb{Z}}\Sigma^n H^{-n}(X)$, as we have seen in Example 2.3.30. On the other hand, by condition 3 of Proposition 2.1.13, a complex is compact in $\mathrm{D}(\mathbb{Z})$ if and only if $H^n(X)$ is finitely presented for all $n\in\mathbb{Z}$ and $H^n(X)\cong 0$ for all but finitely many $n\in\mathbb{Z}$. Hence, since $\#\mathbb{Z}=\aleph_0$, we have $\#(\mathrm{D}(\mathbb{Z})^{\aleph_0})\leq\aleph_0$. By Neeman's Theorem 2.3.3, $\mathrm{D}(\mathbb{Z})$ satisfies $\aleph_0$-Adams representability for objects and for morphisms and, consequently, $\mathrm{D}(\mathbb{Z})$ and $\mathrm{D}(\mathbb{Z})^{\mathrm{op}}$ satisfy Brown representability.

Since $\mathbb{Z}$ is hereditary and noetherian, $\mathrm{Pgldim}_\alpha(\mathbb{Z})=\mathrm{Pgldim}_\alpha(\mathrm{D}(\mathbb{Z}))$ by Corollary 5.2.3. But we know by Example 5.2.8 that $\mathrm{Pgldim}_\alpha(\mathbb{Z})>1$ for every $\alpha>\aleph_0$. Then, as a direct consequence of Theorem 3.1.10, we obtain the following result.

Proposition 5.2.9. *The triangulated category* $\mathrm{D}(\mathbb{Z})$ *does not satisfy α-Adams representability for morphisms if $\alpha>\aleph_0$.*

This gives a negative answer to Question 2.3.34. However, this result does not refute Neeman's Conjecture 2.3.35, because $\mathrm{Pgldim}_{\aleph_0}(\mathbb{Z})=1$ and hence $\mathrm{D}(\mathbb{Z})$ satisfies $\aleph_0$-Adams representability both for morphisms and for objects and $S_{\aleph_0}$ is a Rosický functor. Furthermore, there is at least another essentially different Rosický functor, as we have explained in Example 2.3.30.

In [BG10], Braun and Göbel stated another natural question related to Rosický's question: "Does there exist a ring R such that $\mathrm{Pgldim}_\alpha(R)>1$ for every regular cardinal α?". Recently, Bazzoni and Šťovíček [BŠ10] proved that the following examples provide a positive answer to this question.

Examples 5.2.10 (Bazzoni and Šťovíček [BŠ10])*.* Let k be an uncountable field. Then $\mathrm{Pgldim}_\alpha(R)>1$ for every α if R is one of the following rings:

1. $k[X_1,\ldots,X_n]$ for $n\geq 2$.

2. $\begin{pmatrix} k & V \\ 0 & k\end{pmatrix}$ where V is a k-vector space with $\dim_k(V)\geq 2$.

Notice that these rings are α-coherent for every regular cardinal α. Then, as a direct consequence of Theorem 3.1.10 and Corollary 5.2.3, we have the following result.

Proposition 5.2.11. *Let R be one of the rings of Examples 5.2.10. Then the triangulated category* $\mathrm{D}(R)$ *does not satisfy α-Adams representability for morphisms for any regular cardinal α.*

Chapter 6

$\aleph_1$-Adams representability for objects

In this final chapter we use some of our previous results to prove that $\aleph_1$-Adams representability for objects holds for certain kinds of triangulated categories. As far as we know, the results in this section are the first genuine positive results about α-Adams representability for $\alpha > \aleph_0$.

6.1 $\aleph_1$-Adams representability

In this section, we prove $\aleph_1$-Adams representability for objects for $\aleph_1$-compactly generated triangulated categories $\mathcal{T}$ such that $\#\mathcal{T}^{\aleph_1} \leq \aleph_1$. The main ingredient is the following general result.

Theorem 6.1.1. *Let $\mathcal{C}$ be a category with coproducts of less than $\aleph_n$ objects. If $\#\mathcal{C} \leq \aleph_n$, then every $\aleph_n$-flat object in $\mathrm{Mod}_{\aleph_n}$-$\mathcal{C}$ has projective dimension less than or equal to $n+1$.*

Proof. By Proposition 2.3.19, $\mathrm{Mod}_{\aleph_n}$-$\mathcal{C}$ is locally $\aleph_n$-presentable and $\aleph_n$-Grothendieck. Since $\#\mathcal{C} \leq \aleph_n$, we infer from Lemma 4.4.3 that $\mathrm{Mod}_{\aleph_n}$-$\mathcal{C}$ is locally $\aleph_{n+1}$-noetherian.

Let H be an $\aleph_n$-flat object in $\mathrm{Mod}_{\aleph_n}$-$\mathcal{C}$. By Corollary 4.3.8, we can write H as an ascending chain of subobjects, say $H \cong \mathrm{colim}_{i<\gamma} H_i$ with $H_0 = 0$ and H_j/H_i $\aleph_{n+1}$-generated if $i \leq j \leq i + \omega_n < \gamma$, and such that $\mathrm{colim}_{i<\beta} H_i = H_\beta$ for every limit ordinal $\beta \geq \omega_n$ and $H_i \hookrightarrow H$ is an $\aleph_n$-pure monomorphism for every $i < \gamma$.

Notice that, since $H_i \hookrightarrow H$ is an $\aleph_n$-pure monomorphism, so is $H_i \hookrightarrow H_j$ for every $i < j$, by Lemma 4.3.5. Then, by Lemma 4.3.2, H_j/H_i is also $\aleph_n$-flat.

Since $\mathrm{Mod}_{\aleph_n}\text{-}\mathcal{C}$ is locally $\aleph_{n+1}$-noetherian, H_j/H_i is $\aleph_{n+1}$-presentable for every $i < j$. Then, whenever $i < j \leq i + \omega_n$, we have

$$H_j/H_i \cong \mathrm{colim}_{k<\omega_n}\mathcal{C}(-, X_k).$$

This colimit is exact, since $\mathrm{Mod}_{\aleph_n}\text{-}\mathcal{C}$ is $\aleph_n$-Grothendieck and the indexing poset has cardinality $\aleph_n$. On the other hand, $\mathcal{C}(-, X_k)$ is projective for every $k < \omega_n$. By the Grothendieck Spectral Sequence [Wei94, Corollary 5.8.4],

$$\mathrm{Ext}^m(H_j/H_i, -) \cong \lim{}^{(m)}_{k<\omega_n}\mathrm{Mod}_{\aleph_n}\text{-}\mathcal{C}(\mathcal{C}(-, X_k), -)$$

and the last term is zero if $m > n+1$ by [Jen72, Theorem 3.1]. In particular, we have proved that $\mathrm{pd}(H_j/H_i) \leq n+1$ assuming $i \leq j \leq i + \omega_n$. Hence, we have checked that H satisfies all the hypotheses of Theorem 4.1.1 and therefore $\mathrm{pd}(H) \leq n+1$. □

We infer the following direct consequence.

Corollary 6.1.2. *Let $\mathcal{T}$ be an $\aleph_n$-compactly generated triangulated category. If $\#\mathcal{T}^{\aleph_n} \leq \aleph_n$, then* $\mathrm{Pgldim}_{\aleph_n}(\mathcal{T}) \leq n+1$.

Proof. Recall that

$$\mathrm{Pgldim}_{\aleph_n}(\mathcal{T}) = \sup\{\mathrm{pd}(H) \mid H \text{ cohomological in } \mathrm{Mod}_{\aleph_n}\text{-}\mathcal{T}^{\aleph_n}\}.$$

By Proposition 2.3.27, every cohomological functor in $\mathrm{Mod}_{\aleph_n}\text{-}\mathcal{T}^{\aleph_n}$ is $\aleph_n$-flat. Hence, Theorem 6.1.1 implies that $\mathrm{Pgldim}_{\aleph_n}(\mathcal{T}) \leq n+1$. □

Corollary 6.1.3. *Let $\mathcal{T}$ be an $\aleph_1$-compactly generated triangulated category. If $\#\mathcal{T}^{\aleph_1} \leq \aleph_1$, then $\mathcal{T}$ satisfies $\aleph_1$-Adams representability for objects.*

Proof. By Corollary 6.1.2, $\mathrm{Pgldim}_{\aleph_1}(\mathcal{T}) \leq 2$ and, by Theorem 3.1.10, this implies $\aleph_1$-Adams representability for objects. □

Corollary 6.1.3 seems to be the first known situation where α-Adams representability for objects holds for $\alpha > \aleph_0$. However, our results provide no information about $\aleph_1$-Adams representability for morphisms.

We next give examples where Corollary 6.1.3 applies. The abbreviation CH means Continuum Hypothesis, stating that $2^{\aleph_0} = \aleph_1$.

Proposition 6.1.4. *Assume the* CH. *The stable homotopy category* $\mathrm{Ho}(\mathrm{Sp})$ *is such that* $\#\mathrm{Ho}(\mathrm{Sp})^{\aleph_1} \leq \aleph_1$ *and satisfies $\aleph_1$-Adams representability for objects.*

Proof. Since Ho(Sp) is compactly generated and we know that $\#\mathrm{Ho(Sp)}^{\aleph_0} \leq \aleph_0$, Theorem 2.5.10 implies that $\#\mathrm{Ho(Sp)}^{\aleph_1} \leq \aleph_1$. Hence, by Corollary 6.1.3, Ho(Sp) satisfies $\aleph_1$-Adams representability for objects. $\square$

In the case of derived categories of rings, we can use Neeman's characterization of $\aleph_1$-compact objects that we described in Example 2.1.10 to prove the following result.

Proposition 6.1.5. *Assume the* CH. *Let R be a ring such that $\#R \leq \aleph_1$ and let* $\mathrm{D}(R)$ *denote its derived category. Then* $\#\mathrm{D}(R)^{\aleph_1} \leq \aleph_1$ *and* $\mathrm{D}(R)$ *satisfies $\aleph_1$-Adams representability for objects.*

Proof. We have explained in Example 2.1.10 that a complex in $\mathrm{D}(R)$ is $\aleph_1$-compact if and only if it is quasi-isomorphic to a K-projective complex of free R-modules with less than $\aleph_1$ generators. First notice that there are less than or equal to $\aleph_1$ free R-modules with less than $\aleph_1$ generators. Then the category $\mathrm{D}(R)^{\aleph_1}$ has less than or equal to $\aleph_1$ objects. In order to check the condition for morphisms, it is enough to prove that $\#\mathrm{Hom}(\bigoplus_{i\in I} R, \bigoplus_{j\in J} R) \leq \aleph_1$ for every I and J such that $\#I$, $\#J < \aleph_1$. But now

$$\mathrm{Hom}\left(\bigoplus_{i\in I} R, \bigoplus_{j\in J} R\right) \cong \prod_{i\in I} \mathrm{Hom}\left(R, \bigoplus_{j\in J} R\right)$$

and, since $\#R \leq \aleph_1$, $\#\mathrm{Hom}(R, \bigoplus_{j\in J} R) \leq \aleph_1 \cdot \#J = \aleph_1$. Then, since $\#I < \aleph_1$ and we are assuming the CH, $\#\mathrm{Hom}(\bigoplus_{i\in I} R, \bigoplus_{j\in J} R) \leq \aleph_1^{\#I} = \aleph_1$. This proves that $\#\mathrm{D}(R)^{\aleph_1} \leq \aleph_1$. Finally, by Corollary 6.1.3, $\mathrm{D}(R)$ satisfies $\aleph_1$-Adams representability for objects. $\square$

If we assume the CH, Proposition 6.1.5 applies to rings of polynomials over the complex numbers $\mathbb{C}[X_1, \ldots, X_n]$ and also to rings of the form $\mathbb{C}[X_1, \ldots, X_n]/I$ where I is an ideal of $\mathbb{C}[X_1, \ldots, X_n]$.

Using the results in this section together with the results of Braun and Göbel (Example 5.2.8) and Bazzoni and Šťovíček (Example 5.2.10), we give examples of rings with $\aleph_1$-pure global dimension exactly equal to 2.

Proposition 6.1.6. *Assume the* CH. *Let R be one of the following rings:*

1. *The ring of integers.*

2. *$k[X_1, \ldots, X_n]$ for $n \geq 2$ where k is a field and $\#k = \aleph_1$.*

3. $\begin{pmatrix} k & V \\ 0 & k \end{pmatrix}$ *where V is a k-vector space with* $\dim_k(V) \geq 2$ *and $\#k = \aleph_1$.*

Then $\mathrm{Pgldim}_{\aleph_1}(R) = \mathrm{Pgldim}_{\aleph_1}(\mathrm{D}(R)) = 2$.

Proof. Since R is α-coherent in each case, $\mathrm{Pgldim}_{\aleph_1}(R) \leq \mathrm{Pgldim}_{\aleph_1}(\mathrm{D}(R))$ by Corollary 5.2.3, and $\mathrm{Pgldim}_{\aleph_1}(R) > 1$ by Example 5.2.8 or Example 5.2.10. On the other hand, since we are assuming the CH, we have $\#\mathrm{D}(R)^{\aleph_1} \leq \aleph_1$ by Proposition 6.1.5 and then $\mathrm{Pgldim}_{\aleph_1}(\mathrm{D}(R)) \leq 2$ by Corollary 6.1.2. □

All the applications that we have considered so far involve compactly generated triangulated categories. The next result is about triangulated categories of the form $\mathrm{K}(R\text{-Proj})$. Such categories have been widely studied by Neeman in [Nee08] as we have pointed out in Example 2.1.21. Among other things, he proved that $\mathrm{K}(R\text{-Proj})$ is always $\aleph_1$-compactly generated, although it need not be compactly generated.

Proposition 6.1.7. *Assume the* CH. *Let R be a ring such that $\#R \leq \aleph_1$. The homotopy category of chain complexes of projective R-modules* $\mathrm{K}(R\text{-Proj})$ *satisfies $\aleph_1$-Adams representability for objects.*

Proof. By [Nee08, Theorem 5.9], $\mathrm{K}(R\text{-Proj})$ is $\aleph_1$-compactly generated and a complex of projective R-modules is $\aleph_1$-compact if and only if it is isomorphic in $\mathrm{K}(R\text{-Proj})$ to a complex of free R-modules with less than $\aleph_1$ generators. Since we are assuming the CH and $\#R \leq \aleph_1$, we can proceed exactly as in the proof of Proposition 6.1.5 to conclude that $\#\mathrm{K}(R\text{-Proj})^{\aleph_1} \leq \aleph_1$. Hence, by Corollary 6.1.3, $\mathrm{K}(R\text{-Proj})$ satisfies $\aleph_1$-Adams representability for objects. □

The same result holds for the category $\mathrm{K}(R\text{-Inj})$, as follows.

Proposition 6.1.8. *Assume the* CH. *Let R be a noetherian ring such that $\#R \leq \aleph_1$. The homotopy category of chain complexes of injective R-modules* $\mathrm{K}(R\text{-Inj})$ *satisfies $\aleph_1$-Adams representability for objects.*

Proof. By [Kra05], we know that $\mathrm{K}(R\text{-Inj})$ is compactly generated and there is a natural equivalence between the category of compact objects $\mathrm{K}(R\text{-Inj})^{\aleph_0}$ and the derived category of bounded complexes of finitely presented R-modules $\mathrm{D}^b(R\text{-mod})$. We want to prove that $\#\mathrm{D}^b(R\text{-mod}) \leq \aleph_1$. Since $\#R \leq \aleph_1$, there are less than or equal to $\aleph_1$ finitely presented R-modules and, using the CH, there are less than or equal to $\aleph_1$ morphisms between them. Then the category $\mathrm{D}^b(R\text{-mod})$ has less than or equal to $\aleph_1$ objects. In order to check the condition for morphisms, we have to prove that $\#\mathrm{D}^b(R\text{-mod})(X, Y) \leq \aleph_1$ for every pair of bounded chain complexes of finitely presented R-modules X and Y. Since R is noetherian, there is a projective resolution $\tilde{X}$ of X by finitely presented free R-modules. Then $\mathrm{D}^b(R\text{-mod})(X, Y) = \mathrm{K}(R)(\tilde{X}, Y)$.

Since $\#R \leq \aleph_1$, we can proceed as in Proposition 6.1.5 to show that $\#\mathrm{K}(R)(\tilde{X}, Y) \leq \aleph_1$. Finally, since we are assuming the CH, Theorem 2.5.10 implies that $\#\mathrm{K}(R\text{-Inj})^{\aleph_1} \leq \aleph_1$. Hence, by Corollary 6.1.3, $\mathrm{K}(R\text{-Inj})$ satisfies $\aleph_1$-Adams representability for objects. □

We next describe applications to derived categories of Grothendieck categories. As we have seen in Example 2.2.7, such categories are always well generated, but for the applications in this section we have to be more precise about how this is proved. The key step in the proof is the following result.

Theorem 6.1.9 (Alonso, Jeremías and Souto [AJS00, Corollary 5.2]). *Let $\mathcal{A}$ be a Grothendieck category and let U be a generator of $\mathcal{A}$. Let $R = \mathrm{End}_{\mathcal{A}}(U)$. Then there exists an exact functor $a\colon \mathrm{D}(R) \to \mathrm{D}(\mathcal{A})$ with a right adjoint. Furthermore, let $\beta = \max\{\#R, \aleph_0\}$ and L be the set of complexes Y in $\mathrm{D}(R)$ that satisfy the following properties:*

1. $Y^j = 0$ *if* $j > 0$.
2. $Y^0 = R$.
3. $Y^j = \coprod_{I_j} R$, *where* $\#I_j \leq \beta$ *if* $j < 0$.
4. $a(H^j(Y)) = 0$ *for every* $j \in \mathbb{Z}$.

If we define $\mathcal{L}$ to be the smallest localizing subcategory of $\mathrm{D}(R)$ containing L, then there is an equivalence of categories between $\mathrm{D}(\mathcal{A})$ and $\mathrm{D}(R)/\mathcal{L}$.

Corollary 6.1.10 (Neeman [Nee01a, Proposition 2.1]). *Let $\mathcal{A}$ be a Grothendieck category and let U be a generator of $\mathcal{A}$. Let $R = \mathrm{End}_{\mathcal{A}}(U)$ and let $\beta = \max\{\#R, \aleph_0\}$. Then $\mathrm{D}(\mathcal{A})$ is β^+-compactly generated.*

Since in the original statement by Neeman the cardinal β is not made explicit, we supply a proof.

Proof. By Theorem 6.1.9, there is an equivalence between $\mathrm{D}(\mathcal{A})$ and $\mathrm{D}(R)/\mathcal{L}$, where $\mathcal{L}$ is generated by a set of objects L. Since $\#R \leq \beta$, it follows from Remark 2.1.12 that R is β^+-coherent and then all the objects in L are β^+-compact by Proposition 2.1.14. Let $\mathcal{L}^{\beta^+}$ be the smallest β^+-localizing category containing L. Since $\mathrm{D}(R)$ is compactly generated (by R), $\mathrm{D}(R)^{\beta^+}$ is the smallest β^+-localizing subcategory containing R. But now we are under the hypotheses of Theorem 2.1.22, which implies that $\mathrm{D}(R)/\mathcal{L}$ is a β^+-compactly generated triangulated category. □

In the same article where Corollary 6.1.10 was proved, Neeman proved that if M is a non-compact, connected manifold of dimension greater than or equal to 1, then the derived category $\mathrm{D}(\mathrm{Sh}/M)$ of sheaves of abelian groups over M has no non-zero compact object. Hence, $\mathrm{D}(\mathrm{Sh}/M)$ cannot be compactly generated and, in particular, it does not satisfy classical Adams representability. We will show in the next result that $\mathrm{D}(\mathrm{Sh}/M)$ is $\aleph_1$-compactly generated under suitable assumptions on M and hence satisfies $\aleph_1$-Adams representability for objects.

Corollary 6.1.11. *Assume the* CH. *Let M be a connected paracompact manifold. Then* $\mathrm{D}(\mathrm{Sh}/M)$ *is an $\aleph_1$-compactly generated triangulated category and satisfies $\aleph_1$-Adams representability for objects.*

Proof. Let $\{U_i\}_{i\in I}$ be a countable base of of open sets of M (that exists because M is paracompact). By [Gro57, Section 1.9], a countable set of generators of Sh/M is given by $\{F_{U_i}\}_{i\in I}$, where F_{U_i} denotes the sheaf that sends every open set V to the abelian group of continuous functions $\{U_i \cap V \to \mathbb{Z}\}$ where $\mathbb{Z}$ is endowed with the discrete topology. Let $\mathcal{R}$ be the full subcategory of Sh/M that has as set of objects $\{F_{U_i}\}_{i\in I}$. This is a ring with several objects, *i.e.* a small category enriched over Ab. This ring with several objects is countable and hence the many-object version of Corollary 6.1.10 implies that $\mathrm{D}(\mathrm{Sh}/M)$ is an $\aleph_1$-compactly generated triangulated category.

Since $\#\mathcal{R} \leq \aleph_0$ and we are assuming the CH, the many-object version of Proposition 6.1.5 implies that $\#\mathrm{D}(\mathcal{R})^{\aleph_1} \leq \aleph_1$. On the other hand, by Theorem 2.1.22 and Theorem 6.1.9, $\mathrm{D}(\mathrm{Sh}/M)^{\aleph_1} = \mathrm{D}(\mathcal{R})^{\aleph_1}/\mathcal{L}^{\aleph_1}$ for a localizing subcategory $\mathcal{L}$. Now recall from Theorem 1.4.1 that the objects in $\mathrm{D}(\mathcal{R})^{\aleph_1}/\mathcal{L}^{\aleph_1}$ coincide with the objects in $\mathrm{D}(\mathcal{R})^{\aleph_1}$ and the morphisms are equivalence classes of diagrams in $\mathrm{D}(\mathcal{R})^{\aleph_1}$ of the form

$$X \xleftarrow{\;s\;} X' \longrightarrow Y$$

where s is such that its mapping cone C_s is in $\mathcal{L}^{\aleph_1}$. Hence, $\mathrm{D}(\mathrm{Sh}/M)^{\aleph_1} \leq \aleph_1$ and then Corollary 6.1.3 implies that $\mathrm{D}(\mathrm{Sh}/M)$ satisfies $\aleph_1$-Adams representability for objects. □

6.2 $\aleph_1$-Adams representability in $\mathbb{A}^1$-homotopy theory

The stable motivic homotopy category or stable $\mathbb{A}^1$-homotopy category was introduced by Voevodsky and his collaborators as a category whose objects represent cohomology theories for schemes, in analogy with the category of spectra, whose objects represent generalized cohomology theories for spaces. Motivic homotopy theory was the main tool for the proof of the Milnor Conjecture and later the Bloch–Kato Conjecture. This is because the stable motivic homotopy category provides a way to compare different cohomology theories for schemes, namely, in the case of the Milnor Conjecture, étale cohomology and Milnor K-theory.

We will recall how to construct the stable motivic homotopy category following [Voe98] and [Isa05]. Through this section S will denote a noetherian scheme of finite Krull dimension and Sm/S will denote the category of smooth schemes of finite type over S. We will always assume that the objects in Sm/S are bounded above by a fixed inaccessible cardinal, and therefore that the category Sm/S is essentially small.

The motivation for constructing the (unstable) motivic homotopy category is to provide a setting for doing abstract homotopy theory in the category Sm/S. The first problem for doing so is that Sm/S does not have all colimits. The standard way of formally adding all colimits is to consider the category $\mathrm{PSh}(Sm/S)$ of presheaves on Sm/S. In order to describe weak equivalences between schemes, we embed $\mathrm{PSh}(Sm/S)$ into the category $\mathrm{sPSh}(Sm/S)_*$ of its pointed simplicial objects, *i.e.* the category of contravariant functors from Sm/S into sSet_*. Notice that the category sSet_* also embeds into $\mathrm{sPSh}(Sm/S)_*$ by sending every simplicial set to its constant simplicial presheaf and that $\mathrm{sPSh}(Sm/S)_*$ is enriched over simplicial sets. Its simplicial mapping space is defined as

$$\mathrm{Map}_*(X,Y)_n = \mathrm{sPSh}(Sm/S)_*(X \times \Delta^n, Y).$$

The reason for considering pointed spaces is because it is necessary for the definition of motivic spectra. If U is an object in Sm/S, we denote by U_+ the object $U \coprod S$ in $\mathrm{sPSh}(Sm/S)_*$ with base point S.

The *pointed motivic homotopy category* $\mathcal{H}(S)_*$ is the homotopy category of a suitable model category structure on $\mathrm{sPSh}(Sm/S)_*$. There are different model category structures giving the same homotopy category. We will use the one described by Isaksen in [Isa05] since it has the best properties for our purposes. This model category is described using two left Bousfield localizations on the objectwise flasque model structure on $\mathrm{sPSh}(Sm/S)_*$.

Recall that we gave the definition of a cofibrantly generated model category in Definition 2.1.31. We will also need the following notion. Let $f\colon X \to Y$ be a morphism in $\mathrm{sPSh}(Sm/S)$ and let $g\colon K \to L$ be a morphism in sSet. Then the *pushout product* of f and g in $\mathrm{sPSh}(Sm/S)_*$ is the morphism

$$G_+ \wedge K_+ \coprod_{F_+ \wedge K_+} F_+ \wedge L_+ \longrightarrow G_+ \wedge L_+.$$

Let S be a noetherian scheme with finite Krull dimension. For every object X in Sm/S and every finite collection of monomorphisms $\{U_i \to X\}_{i\in I}$ in Sm/S, let $\bigcup_{i\in I} U_i$ be the coequalizer of

$$\coprod_{(i,j)\in I\times I} U_i \times U_j \overset{\phi}{\underset{\psi}{\rightrightarrows}} \coprod_{k\in I} U_i$$

where ϕ is the projection onto the first factor and ψ is the projection onto the second factor.

1. Let I be the set of morphisms in $\mathrm{sPSh}(Sm/S)_*$ obtained by the pushout product of morphisms of the form $f\colon \bigcup_{i\in I} U_i \to X$ and morphisms of the form $\partial\Delta^n \to \Delta^n$.

2. Let J be the set of morphisms in $\mathrm{sPSh}(Sm/S)_*$ obtained by the pushout product of morphisms of the form $f\colon \bigcup_{i\in I} U_i \to X$ and morphisms of the form $\Lambda^n_k \to \Delta^n$.

Theorem 6.2.1 (Isaksen [Isa05])**.** *Let S be a noetherian scheme with finite Krull dimension. There is a cofibrantly generated simplicial model category structure on* $\mathrm{sPSh}(Sm/S)_*$ *with generating cofibrations and generating trivial cofibrations given by I and J as defined above and the following three classes of maps:*

1. *A map f in* $\mathrm{sPSh}(Sm/S)_*$ *is called an* objectwise weak equivalence *if* $\mathrm{Map}_*(X_+, f)$ *is a weak equivalence of simplicial sets for every object X in Sm/S.*

2. *A map in* $\mathrm{sPSh}(Sm/S)_*$ *is called a* flasque fibration *if it has the RLP with respect to the morphisms in J.*

3. *A map in* $\mathrm{sPSh}(Sm/S)_*$ *is called a* flasque cofibration *if it has the LLP with respect to maps that are objectwise weak equivalences and flasque fibrations.*

This model structure is called the *objectwise flasque model structure*.

Until now we did not need a Grothendieck topology on the category Sm/S. From now on we will use the Nisnevich topology, which is the one that suits best for motivic homotopy theory. Recall that an *elementary distinguished Nisnevich square* in Sm/S is a pushout square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

such that p is an étale morphism, i is an open embedding and the morphism $p^{-1}(X-U) \to X-U$ is an isomorphism.

Definition 6.2.2 ([Isa05])**.** Let S be a noetherian scheme with finite Krull dimension. The *local flasque model structure* on $\mathrm{sPSh}(Sm/S)_*$ is the left Bousfield localization of the objectwise flasque model structure with respect to the set of maps of the form

$$\Big(U \coprod_{U\times_X V} V\Big)_+ \longrightarrow X_+$$

indexed by all elementary Nisnevich squares.

We still have to make the essential step to obtain the motivic homotopy category. This is based on the idea that it is possible to do homotopy theory of schemes in such a way that the affine line $\mathbb{A}^1$ plays the role of an interval.

Theorem 6.2.3 ([Isa05])**.** *Let S be a noetherian scheme with finite Krull dimension. The* motivic flasque model structure *is the left Bousfield localization of the local flasque model structure with respect to the set of maps $X_+ \to (X \times_S \mathbb{A}^1)_+$ induced by the inclusion of the origin into the affine line $0 \to \mathbb{A}^1$ for X in Sm/S.*

The homotopy category of this model category is the *pointed motivic homotopy theory* of smooth schemes over S as defined by Voevodsky [Voe98] and will be denoted by $\mathcal{H}(S)_*$.

Theorem 6.2.4 ([Isa05] [NS09])**.** *The motivic flasque model structure on* $\mathrm{sPSh}(Sm/S)_*$ *is proper, simplicial, monoidal and almost finitely generated.*

The term *almost finitely generated* is defined in Definition [Hov01, Definition 4.1].

We denote by $\mathbb{S}^1$ the pointed simplicial circle $\Delta^1/\partial\Delta^1$ with base point the class of $\partial\Delta^1$, and by $\mathbb{G}_m^1$ the *Tate circle* $\mathbb{A}^1 - 0$ pointed at 1. We will use the notation $\mathbb{S}^n = (\mathbb{S}^1)^{\wedge n}$ and $\mathbb{G}_m^n = (\mathbb{G}_m)^{\wedge n}$. The existence of these two kinds of spheres is one of the features of motivic homotopy theory. There is a canonical isomorphism $\mathbb{S}^1 \wedge \mathbb{G}_m^1 \cong \mathbb{P}^1$ in $\mathcal{H}(S)_*$, where $\mathbb{P}^1$ has ∞ as base point.

In order to construct the stable motivic homotopy theory, Voevodsky [Voe98] followed the same construction that we explained in Section 1.2, but for $\mathrm{sPSh}(Sm/S)_*$ and the equivalence $-\wedge \mathbb{P}^1$ instead of sSet_* and $-\wedge \mathbb{S}^1$. This construction was later formalized by Hovey [Hov01] in a more abstract setting. With Hovey's procedure we can construct the category of motivic spectra $\mathrm{Sp}(\mathrm{sPSh}(Sm/S)_*, - \wedge \mathbb{P}^1)$ together with a model structure called the *stable motivic model structure*. The homotopy category of this model category is the *stable motivic homotopy category*, that we will denote by $\mathcal{SH}(S)$. As in the case of classical spectra, there is a functor

$$\Sigma^\infty_{\mathbb{P}^1} \colon \mathcal{H}(S)_* \longrightarrow \mathcal{SH}(S).$$

Since $\mathcal{SH}(S)$ is the homotopy category of a stable model category (Example 2.1.28), it has a triangulated category structure defined analogously as in the case of Ho(Sp). Voevodsky proved that $\mathcal{SH}(S)$ has a compactly generated triangulated structure [Voe98, Section 5]. In order to give a concrete set of compact generators, we need some more terminology.

Definition 6.2.5. Let S be a noetherian scheme with finite Krull dimension. The category of *pointed motivic spaces of finite type* is the smallest full subcategory $\mathrm{sPSh}(Sm/S)^f_* \subset \mathrm{sPSh}(Sm/S)_*$ such that

1. $Sm/S \subset \mathrm{sPSh}(Sm/S)^f_*$.
2. For every pushout square

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

   where i is a monomorphism and A, B and C are in $\mathrm{sPSh}(Sm/S)^f_*$, D is in $\mathrm{sPSh}(Sm/S)^f_*$.

Theorem 6.2.6 (Voevodsky [Voe98, Proposition 5.2], Naumann and Spitzweck [NS09, Theorem 10])**.** *Let S be a noetherian scheme of finite Krull dimension. The image of a pointed motivic space of finite type under $\Sigma^\infty_{\mathbb{P}^1}$ is compact in $\mathcal{SH}(S)$. Furthermore, a set of compact generators of $\mathcal{SH}(S)$ is given by*

$$\mathcal{G} = \{\Sigma^{p,q}\Sigma^\infty_{\mathbb{P}^1} U_+ \mid U \text{ in } Sm/S \text{ and } p, q \in \mathbb{Z}\}$$

where $\Sigma^{p,q} = \mathbb{S}^{p-q} \wedge \mathbb{G}_m^q \wedge (-)$.

Theorem 6.2.7 (Voevodsky [Voe98, Proposition 5.5], Naumann and Spitzweck [NS09, Theorem 13])**.** *Let S be a noetherian scheme of finite Krull dimension. If the category Sm/S is countable, then so is $\mathcal{SH}(S)^c$.*

Thanks to this result, we may infer that Adams representability holds in $\mathcal{SH}(S)$ by Neeman's Theorem 2.3.3. This result is used in [NSØ09] in order to represent cohomology theories that are only defined on the subcategory of compact objects by a motivic spectrum.

In this section, we extend Voevodsky's result (Theorem 6.2.7) following the argument given by Naumann and Spitzweck [NS09]. We prove that, under the assumption that $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \alpha$, if $\# Sm/S \leq \alpha$, then $\# \mathcal{SH}(S)^\alpha \leq \alpha$ for every regular cardinal α. Hence, by Corollary 6.1.3, we will be able to infer that if Sm/S has cardinality less than or equal to $\aleph_1$, then $\mathcal{SH}(S)$ satisfies $\aleph_1$-Adams representability for objects. In particular, we will be able to extend the representability result to the case where S has an affine open cover $S = \bigcup_I \operatorname{Spec}(R_i)$ with R_i rings of cardinality less than or equal to $\aleph_1$ for all $i \in I$. If we assume the Continuum Hypothesis, the result will be true for rings of polynomials over $\mathbb{C}$.

For the countable case, the proof in [NS09] is based on a series of propositions. The proof of these can be extended to arbitrary regular cardinals α. Some of the arguments are straightforward generalizations of results in [NS09]. For the rest we give details in this section.

Our main theorem is the following.

Theorem 6.2.8. *Assume that $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \alpha$. Let S be a noetherian scheme of finite Krull dimension. Let α be a regular cardinal. If the category Sm/S of smooth S-schemes of finite type has cardinality less than or equal to α, then so does $\mathcal{SH}(S)^\alpha$.*

We will give a proof of Theorem 6.2.8 after stating a series of preparatory results. Before doing this, we state a direct consequence that follows using Corollary 6.1.3.

Corollary 6.2.9. *Assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Let S be a noetherian scheme of finite Krull dimension. If the category Sm/S of smooth S-schemes of finite type has cardinality less than or equal to $\aleph_1$, then so does $\mathcal{SH}(S)^{\aleph_1}$, and $\mathcal{SH}(S)$ satisfies $\aleph_1$-Adams representability for objects.*

We begin the preparatory results with the technical proposition stated below, that is a generalization of [NS09, Proposition 5] and provides a convenient way to construct cofibrant replacements. We will need some results about localizations of cofibrantly generated model categories, for which we refer to [Hir03]. We say that a pointed simplicial presheaf X in $\mathrm{sPSh}(\mathcal{C})_*$ is *sectionwise of cardinality less than or equal to α* if, for every object U in $\mathcal{C}$, the set of non-degenerate simplices of $X(U)$ has cardinality less than or equal to α.

Proposition 6.2.10. *Let $\mathcal{C}$ be a small category and $\mathrm{sPSh}(\mathcal{C})_*$ be the category of pointed simplicial presheaves on $\mathcal{C}$. Let α be a regular cardinal and let J' be a set of morphisms in $\mathrm{sPSh}(\mathcal{C})_*$ such that*

i. $\#J' \leq \alpha$.

ii. *For every* $f \in J'$, *the domain* $\mathrm{dom}(f)$ *is* α-presentable, i.e. *the functor*

$$\mathrm{sPSh}(\mathcal{C})_*(\mathrm{dom}(f), -)$$

commutes with α*-filtered colimits.*

iii. *For every* $f \in J'$ *and every object* G *in* $\mathrm{sPSh}(\mathcal{C})_*$ *sectionwise of cardinality less than or equal to* α, $\#\mathrm{sPSh}(\mathcal{C})_*(\mathrm{dom}(f), G) \leq \alpha$.

iv. *For every* $f \in J'$ *and every object* U *in* $\mathcal{C}$,

$$\#\mathrm{sPSh}(\mathcal{C})_*(U_+, \mathrm{codom}(f)) \leq \alpha,$$

where $\mathrm{codom}(f)$ *is the codomain of* f.

Then every map $f\colon F \to G$ *in* $\mathrm{sPSh}(\mathcal{C})_*$ *can be functorially factorized as*

$$F \xrightarrow{\iota} F' \xrightarrow{\pi} G\ ,$$

where

1. ι *is a relative* J'*-cell complex.*
2. π *has the right lifting property with respect to* J'.
3. *If* $G = *$ *is the terminal object of* $\mathrm{sPSh}(\mathcal{C})_*$ *and* F *is sectionwise of cardinality less than or equal to* α, *then so is* F'.

Proof. The proof is based on the small object argument [Hir03, Proposition 10.5.16]. Recall ([Hir03, Definition 10.5.8]) that a map is a *relative* J'*-cell complex* if it is a transfinite composition of pushouts of morphisms in J'. We construct the factorization of a map $f\colon F \to G$ as a colimit defined by a sequence

$$\begin{tikzcd}
F = F_0 \arrow[r, "\iota_0"] \arrow[drr, "\pi_0"'] & F_1 \arrow[r, "\iota_1"] \arrow[dr, "\pi_1"] & F_2 \arrow[r, "\iota_2"] \arrow[d, "\pi_2"] & \cdots \arrow[r] & F_\beta \arrow[r] \arrow[dll, "\pi_\beta"] & \cdots \\
 & & G & & &
\end{tikzcd}$$

indexed by all ordinals $\beta < \alpha$, where the objects F_β are defined in the following way, starting from $F_0 = F$. Let β be an ordinal and assume that F_β has already been defined. Let $\mathcal{D}_\beta$ be the set of all commutative squares

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & F_\beta \arrow[d, "\pi_\beta"] \\
Y \arrow[r] & G
\end{tikzcd}$$

with $f \in J'$. We define $F_{\beta+1}$ as the pushout

$$\begin{tikzcd} \coprod_{\mathcal{D}_\beta} X \arrow[r] \arrow[d] & F_\beta \arrow[d, "\iota_\beta"] \\ \coprod_{\mathcal{D}_\beta} Y \arrow[r] & F_{\beta+1}. \end{tikzcd}$$

By the universal property of the pushout, there is a map $\pi_{\beta+1}\colon F_{\beta+1} \to G$ such that $\pi_\beta = \pi_{\beta+1} \circ \iota_\beta$. If β is a limit ordinal, we define $F_\beta = \mathrm{colim}_{i<\beta} F_i$. The desired factorization is given by

$$F \xrightarrow{\iota} F' = \mathrm{colim}_{\beta<\alpha} F_\beta \xrightarrow{\pi} G$$

where $\iota = \iota_0 \circ \iota_1 \circ \cdots$ and π is the morphism defined by the universal property of the pushout. By construction, ι is a relative J'-cell complex.

We will now prove item 2. Let $f\colon X \to Y$ be a morphism in J' and let

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & F' \arrow[d, "\pi"] \\ Y \arrow[r, "h"] & G \end{tikzcd}$$

be a commutative diagram. Since $F' = \mathrm{colim}_{\beta<\alpha} F_\beta$ is an α-filtered colimit and X is α-presentable, by condition ii, the canonical morphism

$$\mathrm{colim}_{\beta<\alpha}\, \mathrm{sPSh}(\mathcal{C})_*(X, F_\beta) \to \mathrm{sPSh}(\mathcal{C})_*(X, F')$$

is an isomorphism. Thus, the map g factors through a map $F_\beta \to F'$ for some $\beta < \alpha$ and, by construction, we have a lifting as in the following diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[rrr, bend left, "g"] & F_\beta \arrow[r] & F_{\beta+1} \arrow[r] & F' \arrow[d, "\pi"] \\ Y \arrow[rru, dotted] \arrow[rrr, "h"] & & & G. \end{tikzcd}$$

This proves item 2.

Finally, if F is sectionwise of cardinality less than or equal to α and $G = *$, conditions i and iii guarantee that $\#\mathcal{D}_\beta \leq \alpha$ for every $\beta < \alpha$ and condition iv implies that F_β is sectionwise of cardinality less than or equal to α for every $\beta < \alpha$. By condition ii, F' is an α-filtered colimit of objects that are sectionwise of cardinality less than or equal to α. Hence F' is sectionwise of cardinality less than or equal to α. $\square$

The following result is a generalization of [NS09, Proposition 6].

Proposition 6.2.11. *Let S be a noetherian scheme of finite Krull dimension such that $\#Sm/S \leq \alpha$ for a regular cardinal α. Let F be an object in* $\mathrm{sPSh}(Sm/S)_*$ *which is sectionwise of cardinality less than or equal to α. Then there is a trivial cofibration $F \to F'$ such that F' is fibrant and sectionwise of cardinality less than or equal to α in the motivic flasque model structure on* $\mathrm{sPSh}(Sm/S)_*$.

Proof. Recall from Theorem 6.2.1 that J is a generating set of trivial cofibrations for the objectwise flasque model structure on $\mathrm{sPSh}(Sm/S)_*$. The motivic flasque model structure on $\mathrm{sPSh}(Sm/S)_*$ was described in Definition 6.2.2 and Theorem 6.2.3 by two consecutive left Bousfield localizations on the objectwise flasque model structure. We define $\mathcal{N}$ to be the union of the set of morphisms with respect to which we localize in Definition 6.2.2 and Theorem 6.2.3. By [Hir03, Proposition 3.3.16 and Proposition 4.2.4], a morphism in $\mathrm{sPSh}(Sm/S)_*$ is fibrant in the motivic flasque model structure if and only if it has the RLP with respect to the set of morphisms $J' = J \bigcup \Lambda(\mathcal{N})$, where

$$\Lambda(\mathcal{N}) = \{X \wedge \Delta^n_+ \coprod_{X \wedge \partial\Delta^n_+} Y \wedge \partial\Delta^n_+ \to Y \wedge \Delta^n_+ \mid X \to Y \in \mathcal{N}, n \geq 0\}.$$

Next we check conditions i to iv of Proposition 6.2.10 with $\mathcal{C} = Sm/S$:

i. $\#J' \leq \alpha$ since $\#Sm/S \leq \alpha$.

ii. The domains in J are $\aleph_0$-presentable by [Isa05, Lemma 3.1]. The domains in $\mathcal{N}$ are $\aleph_0$-presentable because they are in Sm/S; see [AGV72, I, Remarque 9.11.3] for details. Since finite simplicial sets are also $\aleph_0$-presentable by [Hov01, Lemma 3.1.2], we deduce that the objects in $J' = J \bigcup \Lambda(\mathcal{N})$ are $\aleph_0$-presentable. Then the domains of morphisms in J are α-presentable for every regular cardinal $\alpha \geq \aleph_0$.

iii. Let $f \in J'$ and G be an object in $\mathrm{sPSh}(Sm/S)_*$ sectionwise of cardinality less than or equal to α. If $f \in J$, then $\mathrm{dom}(f)$ is of the form

$$(\textstyle\bigcup_{i \in I} U_i)_+ \wedge \Delta^n_+ \coprod_{(\bigcup_{i \in I} U_i)_+ \wedge \Lambda^n_{k+}} (X_+ \wedge \Lambda^n_{k+})$$

for some finite collection of monomorphisms $\{U_i \to X\}_{i \in I}$ in Sm/S. Since $\bigcup_{i \in I} U_i$ is constructed as a coequalizer of finite coproducts of objects in Sm/S, it is enough to prove that $\#\mathrm{sPSh}(Sm/S)_*(U_i, G) =$

$\#G(U_i) \leq \alpha$ and this is true because G is sectionwise of cardinality less than or equal to α. If $f \in \Lambda(\mathcal{N})$, then $\mathrm{dom}(f)$ is of the form

$$X \wedge \Delta^n_+ \coprod_{X \wedge \partial\Delta^n_+} Y \wedge \partial\Delta^n_+$$

where X and Y are constructed with a finite number of pushouts with respect to objects in Sm/S. Since G is sectionwise of cardinality less than or equal to α, $\#\mathrm{sPSh}(Sm/S)_*(X, G) \leq \alpha$ and this implies that $\#\mathrm{sPSh}(Sm/S)_*(\mathrm{dom}(f), G) \leq \alpha$.

iv. Fix an object U in Sm/S and $f \in J'$. Notice that, for every object Y in $\mathrm{sPSh}(Sm/S)_*$, $\mathrm{sPSh}(Sm/S)_*(U, Y) = Y(U)_0$ is a simplicial set with all non-degenerate simplices in degree 0.

If $f \in J$, then $\mathrm{codom}(f) = X_+ \wedge \Delta^n_+$ for some X in Sm/S, hence $\#\mathrm{sPSh}(Sm/S)_*(U_+, \mathrm{codom}(f)) = \#(X_+(U_+) \wedge \Delta^n_+)_0 \leq \#Sm/S(U, X)$ and this is less than or equal to α since $\#Sm/S \leq \alpha$. If $f \in \Lambda(\mathcal{N})$, then $\mathrm{codom}(f) = X \wedge \Delta^n_+$ where $X = \mathrm{codom}(g)$ with $g \in \mathcal{N}$. By definition, X is constructed as a finite number of pushouts with respect to objects in Sm/S, since $\#Sm/S \leq \alpha$, this implies that

$$\#\mathrm{sPSh}(Sm/S)_*(U_+, \mathrm{codom}(f)) \leq \alpha.$$

Now we can apply Proposition 6.2.10 to obtain a factorization

$$F \xrightarrow{\iota} F' \xrightarrow{\pi} *$$

in $\mathrm{sPSh}(Sm/S)_*$ such that ι is a relative J'-cell complex and π has the RLP with respect to J'. In order to finish the proof, we have to check that ι is a trivial cofibration and π is a fibration in the motivic flasque model structure of Theorem 6.2.3.

The morphisms in J are trivial cofibrations in the objectwise flasque model structure. Hence, they are also trivial cofibrations in the motivic flasque model structure, since left Bousfield localizations preserve cofibrations and weak equivalences. The morphisms in $\mathcal{N}$ and the morphisms of the form $\partial\Delta^n \to \Delta^n$ are cofibrations between cofibrant objects in the objectwise flasque model structure. Then the morphisms

$$X_+ \wedge \Delta^n_+ \coprod_{X_+ \wedge \partial\Delta^n_+} Y_+ \wedge \partial\Delta^n_+ \longrightarrow Y_+ \wedge \Delta^n_+$$

in $\Lambda(\mathcal{N})$ are cofibrations, since $\mathrm{sPSh}(Sm/S)_*$ is a simplicial model category; see [Hir03, Propostion 9.3.7] for details. Now they become trivial cofibrations in the motivic flasque model structure, by [Hir03, Proposition 4.2.4]. We have seen that the morphisms in J' are all trivial cofibrations in the motivic flasque model structure. Since ι is a relative J'-cell complex, it is a trivial cofibration.

Since π has the RLP with respect to $J' = J \bigcup \Lambda(\mathcal{N})$, as we pointed out at the beginning of the proof, it is a fibration and hence F' is fibrant. $\square$

Recall that there are mapping spaces $\mathrm{Map}_*(-,-)$ given by the simplicial enrichment in $\mathrm{sPSh}(Sm/S)_*$. We will denote by $\mathrm{RMap}_*(-,-)$ their derived analogs inducing a simplicial enrichment in the homotopy category $\mathcal{H}(S)_*$.

The following result is a generalization of the case $\alpha = \aleph_0$ in [NS09, Theorem 9].

Theorem 6.2.12. *Let S be a noetherian scheme of finite Krull dimension such that $\#Sm/S \leq \alpha$ for a regular cardinal α. Let F be an object in* $\mathrm{sPSh}(Sm/S)_*$ *sectionwise of cardinality less than or equal to α and X an object in* $\mathrm{sPSh}(Sm/S)_*^f$. *Then $\#\,\pi_n\mathrm{RMap}_*(X,F) \leq \alpha$ for all $n \geq 0$.*

Proof. If X is in Sm/S and $F \to F'$ is the fibrant replacement of Proposition 6.2.11, then

$$\mathrm{RMap}_*(X,F) = \mathrm{Map}_*(X,F') = F'(X),$$

which has cardinality less than or equal to α, since F is sectionwise of cardinality less than or equal to α.

The objects in $\mathrm{sPSh}(Sm/S)_*^f$ are constructed from objects in Sm/S by finitely many pushouts in $\mathrm{sPSh}(Sm/S)_*$. Since the identity functor in $\mathrm{sPSh}(Sm/S)_*$ defines a simplicial (left) Quillen equivalence from the motivic flasque model structre to the motivic local injective model structure (where cofibrations are defined levelwise [Isa05, Theorem 4.2]), we can assume that the objects in $\mathrm{sPSh}(Sm/S)_*^f$ are constructed from objects in Sm/S by finitely many homotopy pushouts in the motivic flasque model structure on $\mathrm{sPSh}(Sm/S)_*$. Hence the proof will be finished if we prove that the class

$$\{X \in \mathrm{sPSh}(Sm/S)_* \mid \#\pi_n(X,F) \leq \alpha,\, n \geq 0\}$$

is closed under homotopy pushouts taken with respect to the motivic flasque model structure. We will prove something stronger: If

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

is a homotopy pullback of simplicial sets where $\pi_n(B)$, $\pi_n(C)$ and $\pi_n(D)$ have cardinality less than or equal to α for $n \geq 0$, we claim that $\#\pi_n(A) \leq \alpha$ for

$n \geq 0$. We can assume that B, C and D are Kan complexes and that f and g are fibrations. By [May67, Section 9], there are strong deformation retracts $B' \subset B$, $C' \subset C$ and $D' \subset D$ such that B', C' and D' are minimal Kan complexes and $\#B$, $\#C$ and $\#D \leq \alpha$. Hence A is weakly equivalent to the Kan complex $B' \times_{D'} C'$ and $\#B' \times_{D'} C' \leq \alpha$. □

The following result is the analog of [NS09, Theorem 12].

Theorem 6.2.13. *Assume that $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \alpha$. Let S be a noetherian scheme of finite Krull dimension such that $\#Sm/S \leq \alpha$ for a regular cardinal α. Let F be an object in $\mathcal{SH}(S)^\alpha$ and E a motivic spectrum such that E_n is sectionwise of cardinality less than of equal to α for all $n \geq 0$. Then $\#\mathcal{SH}(S)(F, \Sigma^{p,q}E) \leq \alpha$.*

Proof. We claim that the full subcategory $\mathcal{R} \subset \mathcal{SH}(S)$ of objects X such that $\#\mathcal{SH}(S)(X, \Sigma^{p,q}E) \leq \alpha$ for all p, $q \in \mathbb{Z}$ is an α-localizing subcategory. We will first prove that it is a triangulated subcategory. It is clearly closed under suspension and given a morphism $f\colon X \to Y$ with X and Y in $\mathcal{R}$ we have a long exact sequence

$$\cdots \leftarrow \mathcal{SH}(S)(\Sigma X, \Sigma^{p,q}E) \leftarrow \mathcal{SH}(S)(C_f, \Sigma^{p,q}E) \leftarrow \mathcal{SH}(S)(Y, \Sigma^{p,q}E) \leftarrow \cdots$$

for all p, $q \in \mathbb{Z}$. Since $\#\mathcal{SH}(S)(Y, \Sigma^{p,q}E) \leq \alpha$ and $\#\mathcal{SH}(S)(\Sigma X, \Sigma^{p,q}E) \leq \alpha$ we deduce that $\#\mathcal{SH}(S)(C_f, \Sigma^{p,q}E) \leq \alpha$ and then C_f is in $\mathcal{R}$. We next show that it is closed by coproducts of less than α objects. Let $\{X_i\}_{i\in I}$ be a set of objects in $\mathcal{R}$ such that $\#I < \alpha$. Then $\#\mathcal{SH}(S)(\coprod_{i\in I} X_i, \Sigma^{p,q}E) = \#\prod_{i\in I} \mathcal{SH}(S)(X_i, \Sigma^{p,q}E) \leq \alpha^\kappa$, where $\kappa < \alpha$. Since we are assuming that $2^\lambda = \lambda^+$ for every cardinal $\lambda \leq \alpha$, Theorem 2.5.1 implies that

$$\#\mathcal{SH}(S)(\coprod_{i\in I} X_i, \Sigma^{p,q}E) \leq \alpha.$$

By Proposition 2.1.16, $\mathcal{SH}(S)^\alpha$ is the smallest α-localizing subcategory generated by $\mathcal{G}$. Hence, we can assume that $F = \Sigma^\infty_{\mathbb{P}^1} U_+$ for some object U in Sm/S.

Fix p, $q \in \mathbb{Z}$ and choose an integer $k \geq 0$ such that $q' = q + k \geq 0$ and $p' = p - q + k$. Then

$$\mathcal{SH}(S)(F, \Sigma^{-p,-q}E) = \mathcal{SH}(S)(\Sigma^\infty_{\mathbb{P}^1}(U_+ \wedge \mathbb{S}^{p'} \wedge \mathbb{G}_m^{q'}), (\mathbb{P}^1)^{\wedge^L k}(E))$$

where $\mathbb{P}^1 \wedge^L -$ is the left derived functor of $\mathbb{P}^1 \wedge -$. Since $U_+ \wedge \mathbb{S}^{p'} \wedge \mathbb{G}_m^{q'}$ is isomorphic in $\mathcal{H}_*(S)$ to an object in $\mathrm{sPSh}(Sm/S)^f_*$, [NS09, Theorem 10 and Lemma 11] imply that

$$\mathcal{SH}(S)(F, \Sigma^{-p,-q}E) = \mathrm{colim}_n \mathcal{H}_*(S)(\Sigma^n_{\mathbb{P}^1}(U_+ \wedge \mathbb{S}^{p'} \wedge \mathbb{G}_m^{q'}), \Sigma^k_{\mathbb{P}^1}E).$$

Finally, by Theorem 6.2.12,

$$\#\mathcal{H}_*(S)(\Sigma^n_{\mathbb{P}^1}(U_+ \wedge \mathbb{S}^{p'} \wedge \mathbb{G}_m^{q'}), \Sigma^k_{\mathbb{P}^1} E) \leq \alpha.$$

Hence, $\#\mathcal{SH}(S)(F, \Sigma^{-p,-q}E) \leq \alpha$. □

Finally we can give our proof of Theorem 6.2.8.

Proof of Theorem 6.2.8. We will prove that the full subcategory $\mathcal{T}_0 \subset \mathcal{T}$ with objects $\mathcal{G}$ has cardinality less than or equal to α. By Corollary 2.5.5, this implies that $\#\mathcal{SH}(S)^c \leq \alpha$. Now, Theorem 2.5.10 implies that $\#\mathcal{SH}(S)^\alpha \leq \alpha$.

Since $\#\mathcal{G} \leq \#Sm/S \leq \alpha$, the cardinality of a set of isomorphism classes of objects in $\mathcal{T}_0$ is less than or equal to α. Hence, it is enough to prove that

$$\#\mathcal{SH}(S)(\Sigma^\infty_{\mathbb{P}^1} U_+, \Sigma^{p,q}\Sigma^\infty_{\mathbb{P}^1} V_+) \leq \alpha$$

for all p, $q \in \mathbb{Z}$ and U, $V \in Sm/S$. Fix an object V in Sm/S and denote by E the motivic spectrum defined by $E_n = (\Sigma^\infty_{\mathbb{P}^1} V)_n = (\mathbb{P}^1)^{\wedge n} \wedge V$. For every object W in Sm/S, the simplicial set $E_n(W)$ is a quotient of the simplicial set

$$\big((Sm/S)(W, \mathbb{P}^1)\big)^n \times (Sm/S)(W, V) = \mathbb{P}^1(W)^n \times V(W),$$

which has cardinality less than or equal to $\#Sm/S \leq \alpha$. This proves that E is sectionwise of cardinality less than or equal to α. By Theorem 6.2.13, $\#\mathcal{SH}(S)(\Sigma^\infty_{\mathbb{P}^1} U_+, \Sigma^{p,q}E) = \#\mathcal{SH}(S)(\Sigma^\infty_{\mathbb{P}^1} U_+, \Sigma^{p,q}\Sigma^\infty_{\mathbb{P}^1} V_+) \leq \alpha$. □

To conclude, we want to stress the importance of Theorem 2.5.10 in the proof of Theorem 6.2.8, in comparison with the proof in the case $\alpha = \aleph_0$ in [NS09, Theorem 13]. Theorem 2.5.10 is used in order to control the cardinality of sets of the form $\mathcal{SH}(S)^\alpha(X, \coprod_{i\in I} Y_i)$ with $\#I < \alpha$. In the case $\alpha = \aleph_0$, there is no need of this, since $\mathcal{SH}(S)^c(X, \coprod_{i\in I} Y_i) = \coprod_{i\in I} \mathcal{SH}(S)^c(X, Y_i)$.

Appendix A

Purity for rings

Purity is intimately related to Adams representability, as exhibited by Christensen, Keller and Neeman [CKN01]. In this chapter we review the basic aspects of purity for rings, as explained in [FS01, Ch. I, Sec. 8] and [JL89, Appendix A].

Throughout this chapter, R will be an associative ring with identity and R-Mod will denote the category of left R-modules. A submodule A of an R-module B is *pure* if every finite system of equations over A,

$$r_{i1}x_1 + \cdots + r_{im}x_m = a_i \in A \text{ for } 1 \leq i \leq n,$$

with coefficients $r_{ij} \in R$ and unknowns $x_1, \ldots, x_m$, has a solution in A whenever it is solvable in B.

Definition A.0.14. Let R be a ring. A short exact sequence of R-modules $0 \to A \to B \to C \to 0$ is called *pure exact* if one of the following equivalent conditions holds.

1. A is a pure subobject of B.

2. For every finitely presentable R-module P, the induced sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(P, A) \longrightarrow \mathrm{Hom}_R(P, B) \longrightarrow \mathrm{Hom}_R(P, C) \longrightarrow 0$$

is exact.

3. For every right R-module P, the induced sequence

$$0 \longrightarrow P \otimes_R A \longrightarrow P \otimes_R B \longrightarrow P \otimes_R C \longrightarrow 0$$

is exact.

4. For every finitely presentable R-module P, the induced sequence
$$0 \longrightarrow P \otimes_R A \longrightarrow P \otimes_R B \longrightarrow P \otimes_R C \longrightarrow 0$$
is exact.
5. The dual sequence
$$0 \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}) \longleftarrow 0$$
is a splitting short exact sequence of R-modules.
6. The dual sequence
$$0 \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z}) \longleftarrow 0$$
is a pure exact sequence of R-modules.
7. The sequence $0 \to A \to B \to C \to 0$ is a filtered colimit of split exact sequences.

An R-module P is called *pure projective* if for every short pure exact sequence $0 \to A \to B \to C \to 0$ the induced sequence of abelian groups
$$0 \longrightarrow \mathrm{Hom}(P, A) \longrightarrow \mathrm{Hom}(P, B) \longrightarrow \mathrm{Hom}(P, C) \longrightarrow 0$$
is exact. A *pure projective resolution* of an R-module M is a long exact sequence
$$\ldots \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$
where P_n is pure projective and $0 \longrightarrow \ker(f_n) \longrightarrow P_n \longrightarrow \mathrm{coker}(f_{n+1}) \longrightarrow 0$ is pure exact for all $n \geq 0$. Now we can define the *pure projective dimension* of an R-module M, denoted by $\mathrm{Ppd}(M)$, as the least length of the pure projective resolutions of M; see [JL89, Appendix A], [FS01, Ch. VI, Sec. 12] or [Pre09, Sec. 2.2] for details. The *pure global dimension* of a ring, denoted by $\mathrm{Pgldim}(-)$, is the supremum of the pure projective dimension of its modules.

Dually, we can define *pure injective modules*. They are also known as *algebraically compact modules* and have been widely studied; see [Fuc70, Chapter VII], [JL89, Chapter 7] or [FS01, Chapter XIII, Sec. 3] for details.

Observe that if $0 \to A \to B \to C \to 0$ is a short exact sequence, then C is flat if and only if the sequence is pure exact [FS01, Lemma 9.1]. On the other hand, if C and B are flat modules then so is A. Because of these facts, projective resolutions of flat modules are also pure projective resolutions. For this reason, the following proposition is useful in order to give lower bounds for the pure projective global dimension of a ring.

Proposition A.0.15. *Let R be a ring with identity.*

1. *A finitely presentable flat R-module X is projective,* i.e. $\mathrm{pd}(X) = 0$.

2. *A countably presentable flat R-module X has* $\mathrm{pd}(X) \leq 1$ [Jen72].

3. *An $\aleph_n$-presentable flat R-module X has* $\mathrm{pd}(X) \leq n + 1$ [Oso73].

We state some known results about pure global dimensions of rings.

Theorem A.0.16 ([KS75, Theorem 2.1])**.** *Let R be a ring with identity. Then* $\mathrm{Pgldim}(R) = 0$ *if and only if R is a pure quasi-Frobenius ring. This is equivalent to claiming that R is an Artinian principal ideal domain. These are algebras of finite representation type, which include the finite dimensional ones.*

Example A.0.17. By a result of Kulikov ([Fuc70, Theorem 18.1]), $\mathbb{Z}$ and $k[X]$ for any field k have pure global dimension 1.

The following theorem says that the pure global dimension of a ring is somehow bounded above by its cardinality.

Theorem A.0.18 ([GJ73], [KS75, Theorem 2.2])**.** *Let R be a ring with identity and assume that* $\max\{\aleph_0, \#R\} = \aleph_t$*. Then* $\mathrm{Pgldim}(R) \leq t + 1$.

Now we will exhibit some examples for which the upper bound that we just found is reached. Remember that $k\langle X_1, \ldots, X_n\rangle$ is the ring of non-commutative polynomials over the field k. This example will be of interest in the thesis because it is a hereditary ring, whereas $k[X_1, \ldots, X_n]$ is not.

Theorem A.0.19 ([BL82, Theorem 3.2])**.** *Assume that k is a field such that $\#k = \aleph_t$. Let $R = k[X_1, \ldots, X_n]$ and $S = k\langle X_1, \ldots, X_n\rangle$. Then* $\mathrm{Pgldim}(R) = \mathrm{Pgldim}(S) \geq t + 1$.

We sketch the proof of this theorem, because it contains several important ideas: Let Q be the quotient field of R. Since Q is flat, $\mathrm{Ppd}(Q) = \mathrm{pd}(Q)$ and hence it will be enough to see that $\mathrm{pd}(Q) \leq t + 1$.

The first step in the proof is the following result [Oso73, Theorem 2.59]: $\mathrm{pd}(Q) = \min\{n, t + 1\}$. The fact that $\mathrm{pd}(Q) \leq n$ is part of Hilbert's Syzygy Theorem. The fact that $\mathrm{pd}(Q) \leq t + 1$ is seen by induction, applying the Auslander Lemma and a well known lemma by Kaplanski ([Oso73, Lemma 2.25]). The details of this step of the proof can be found in the original references [Oso73, 2.43] and [Jen72], but it also appears in [KS75, Proposition 1.3] and [FS01, Theorem 2.8].

The next important observation is [BL82, Lemma 3.1]: If $n \geq 2$, then $\mathrm{Pgldim}(k[X_1, \dots, X_n]) = \mathrm{Pgldim}(k\langle X_1, \dots, X_n\rangle)$ does not depend on n and then $\mathrm{Pgldim}(R) = \mathrm{Pgldim}(S) \geq t + 1$ for all $n \geq 2$.

Theorem A.0.19 is also true for polynomial rings with an infinite set of variables. In fact, this was proved before. In the statement we will use the notation $\aleph_{-1}$ to denote any finite cardinal.

Theorem A.0.20 ([KS75, Theorem 2.3])**.** *Let $R[Y]$ be a polynomial ring such that R is a commutative integral domain with $\#R = \aleph_t$ and $\#Y = \aleph_m$ for $t \geq -1$ and $m \geq 0$. Then* $\mathrm{Pgldim}(R[Y]) = \max(t, m) + 1$*. If $\#k$ or $\#Y$ are greater than or equal to $\aleph_\omega$, then* $\mathrm{Pgldim}(R[Y]) = \infty$*.*

In view of the applications to Adams representability, we will be interested in hereditary rings, *i.e.* rings of global dimension less than or equal to 1. As we already said, rings of non-commutative polynomials over a field are hereditary. The following theorem by Baer, Brune and Lenzing collects various results about the pure global dimension of finite-dimension hereditary algebras. Before we state the result, we recall some basic definitions; see [Pre09] for details.

Definition A.0.21. Let R be a finite dimensional associative algebra over an algebraically closed field k. Then the following holds.

1. R is of *finite representation type* if has only a finite number of isomorphism classes of indecomposable modules of finite length.

2. R is of *wild representation type* if there is a representation embedding of $k\langle X, Y\rangle$-Mod into R-Mod.

3. R is of *tame representation type* if it is not of wild representation type.

Theorem A.0.22 (Brune, Baer and Lenzing [BBL82, Theorem 3.4])**.** *Let R be a finite dimensional algebra over an algebraically closed field of uncountable cardinality $\aleph_t$ with $t > 0$. Suppose that R is hereditary. Then:*

1. $\mathrm{Pgldim}(R) = 0$ *if R is of finite representation type.*

2. $\mathrm{Pgldim}(R) = 2$ *if R is of tame representation type.*

3. $\mathrm{Pgldim}(R) = t + 1$ *if R is of wild representation type.*

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Index

Oriol Raventós Morera

Adams Representability in Triangulated Categories

This thesis contains new results about the representability of cohomological functors defined on a subcategory of compact objects (with respect to a fixed cardinal) of a well generated triangulated category. Classical theorems of Adams for the stable homotopy category and Neeman for compactly generated triangulated categories are extended to the first uncountable cardinal. The case of derived categories of rings and the stable motivic category are studied in detail. These results contribute to answering negatively a question raised by Rosický of whether all cohomological functors defined on a subcategory of compact objects with respect to a large enough cardinal are representable. Some of the findings in this thesis are based on new results about abelian categories, the most relevant being a generalization of the Auslander Lemma for non Grothendieck categories.

Representabilitat d'Adams en categories triangulades

En aquesta tesi s'obtenen resultats nous sobre la representabilitat de functors cohomològics definits en subcategories d'objectes compactes (respecte a un cardinal fixat) d'una categoria triangulada ben generada. S'estenen al primer cardinal no numerable teoremes antics d'Adams per a la categoria d'homotopia estable i de Neeman per a categories triangulades compactament generades. S'estudien en detall els casos de la categoria derivada d'un anell i la categoria motívica estable. Aquests resultats contribueixen a respondre negativament una pregunta de Rosický sobre si tots els functors cohomològics definits en una subcategoria d'objectes compactes respecte a un cardinal suficientment gran són representables. Alguns dels avenços d'aquesta tesi es basen en nous resultats sobre categories abelianes, el més rellevant dels quals és una generalització del lema d'Auslander per a categories que no són de Grothendieck.