Universitat Politècnica de Catalunya

Doctoral thesis

Homoclinic and chaotic phenomena to L_3 in the Restricted 3-Body Problem

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Als meu pares i les meves germanes.

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To the student reading this thesis: Lastly, I want to dedicate this thesis to the PhD student that will be assigned to read and/or reproduce some of the results found in here. I'm sorry, I know, the formulas are horrible and there are too many bounds. Let me know if you need any help or have any questions, it will be great to help.

Mar Giralt Miron.

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UNIVERSITAT POLITÈCNICA DE CATALUNYA

Abstract

Departament de Matemàtiques
Facultat de Matemàtiques i Estadística

Doctoral degree in Applied Mathematics

Homoclinic and chaotic phenomena to L_3 in the Restricted 3-Body Problem

by Mar GIRALT MIRON

The Restricted 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies, called the primaries. If the primaries perform circular motions and the massless body is coplanar with them, one has the Restricted Planar Circular 3-Body Problem (RPC3BP). In synodic coordinates, it is a two degrees of freedom autonomous Hamiltonian system with five critical points, L_1,..,L_5, called the Lagrange points.

The Lagrange point L_3 is a saddle-center critical point which is collinear with the primaries and is located beyond the largest one. This thesis focuses on the study of the one dimensional unstable and stable manifolds associated to L_3 and the analysis of different homoclinic and chaotic phenomena surrounding them. We assume that the ratio between the masses of the primaries is small.

First, we obtain an asymptotic formula for the distance between the unstable and stable manifolds of L_3. When the ratio between the masses of the primaries is small the eigenvalues associated with L_3 have different scales, with the modulus of the hyperbolic eigenvalues smaller than the elliptic ones. Due to this rapidly rotating dynamics, the invariant manifolds of L_3 are exponentially close to each other with respect to the mass ratio and, therefore, the classical perturbative techniques (i.e. the Poincaré-Melnikov method) cannot be applied. In fact, the formula for the distance between the unstable and stable manifolds of L_3 relies on a Stokes constant which is given by the inner equation. This constant can not be computed analytically but numerical evidences show that is different from zero. Then, one infers that there do not exist 1-round homoclinic orbits, i.e. homoclinic connections that approach the critical point only once.

The second result of the thesis concerns the existence of 2-round homoclinic orbits to L_3, i.e. connections that approach the critical point twice. More concretely, we prove that there exist 2-round connections for a specific sequence of values of the mass ratio parameters. We also obtain an asymptotic expression for this sequence.

In addition, we prove that there exists a set of Lyapunov periodic orbits whose two dimensional unstable and stable manifolds intersect transversally. The family of Lyapunov periodic orbits of L_3 has Hamiltonian energy level exponentially close to that of the critical point L_3. Then, by the Smale-Birkhoff homoclinic theorem, this implies the existence of chaotic motions (Smale horseshoe) in a neighborhood exponentially close to L_3 and its invariant manifolds.

In addition, we also prove the existence of a generic unfolding of a quadratic homoclinic tangency between the unstable and stable manifolds of a specific Lyapunov periodic orbit, also with Hamiltonian energy level exponentially close to that of L_3.

UNIVERSITAT POLITÈCNICA DE CATALUNYA

Ressenya

Departament de Matemàtiques
Facultat de Matemàtiques i Estadística

Programa de Doctorat en Matemàtica Aplicada

Fenòmens homoclínics i caòtics a L_3 en el problema restringit dels 3 cossos

per Mar Giralt Miron

El problema restringit dels 3 cossos modela el moviment d'un cos de massa negligible que es troba sota la influència gravitatòria de dos cossos massius anomenats primaris. Si els primaris realitzen moviments circulars i el cos sense massa és coplanar amb ells, es té el problema restringit planar i circular dels 3 cossos (RPC3BP). En coordenades sinòdiques, aquest és un sistema Hamiltonià autònom de dos graus de llibertat i té cinc punts crítics, L_1,..,L_5, anomenats punts de Lagrange.

El punt de Lagrange L_3 és un punt crític de tipus centre-sella, colineal amb els primaris i que es troba al cantó oposat del primari petit respecte del gran. Aquesta tesi estudia les varietats unidimensionals inestable i estable associades a L_3 i analitza alguns dels diferents fenòmens homoclínics i caòtics que les envolten. A més, suposarem que la ràtio entre les masses dels primaris és petita.

Primerament, obtenim una fórmula asimptòtica per a la distància entre les varietats inestable i estable de L_3. Quan la ràtio entre les masses dels primaris és petita, els valors propis associats a L_3 tenen escales diferents; és a dir, el mòdul dels valors propis hiperbòlics és més petit que el dels el·líptics. Degut a aquesta dinàmica de rotació ràpida, les varietats invariants de L_3 es troben exponencialment properes l'una de l'altre respecte a la ràtio de masses i, per tant, les tècniques pertorbatives clàssiques (és a dir, el mètode de Poincaré-Melnikov) no apliquen. És més, la fórmula per a la distància entre les varietats inestable i estable de L_3 ve donada per una constant de Stokes obtinguda mitjançant l'anomenada equació inner. Aquesta constant no es pot calcular analíticament, tot i així, evidències numèriques mostren que és diferent de zero. D'aquest resultat és pot inferir que no existeixen òrbites homoclíniques de 1 volta, és a dir, connexions homoclíniques que s'apropen al punt crític només una vegada.

El segon resultat de la tesi estudia l'existència d'òrbites homoclíniques a L_3 de 2 voltes, és a dir, connexions que s'acosten dues vegades al punt crític. Més concretament, demostrem que existeixen connexions de 2 voltes per a una successió específica de valors de la ràtio de masses tendint a zero i obtenim una expressió asimptòtica per a aquesta successió.

Endemés, demostrem que existeix un conjunt d'òrbites periòdiques de Lyapunov les varietats inestables i estables bidimensionals de les quals es tallen transversalment. Aquest conjunt d'òrbites periòdiques de Lyapunov de L_3 té un nivell d'energia Hamiltonià exponencialment proper al del punt crític L_3. Per tant, segons el teorema homoclínic de Smale-Birkhoff, això implica l'existència de moviments caòtics (és a dir, d'una ferradura de Smale) en un entorn exponencialment proper de L_3 i les seves varietats invariants.

A més, també demostrem l'existència del desplegament genèric d'una tangència quadràtica homoclínica entre les varietats inestable i estable associades a una òrbita periòdica de Lyapunov concreta, també amb un nivell d'energia Hamiltonià exponencialment proper al de L_3.

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Introduction

This introduction is devoted to present an overview of the problem and the main results obtained in this thesis. The rest of the chapters are structured in three different parts, Part I, II and III. Each part is independent of each other (and from this introduction) and self-contained, therefore they can be read independently if desired.

1 Statement of the problem

Throughout history, the study of the motion of celestial objects has been of wide scientific interest. The desire to understand the motions of the Sun, Moon, planets and visible stars has been an important problem for different cultures and thinkers through time. Celestial Mechanics is the branch of astronomy dedicated to the analysis of the movement of objects in outer space. Notably, some of the greatest mathematicians of the last centuries, like Newton, Lagrange, Laplace and Poincaré, have devoted their life to the study of this discipline.

The origin of modern analytic Celestial Mechanics has its roots in the law of universal gravitation formulated by Isaac Newton in 1678. Indeed, considering no additional forces like drag forces, celestial motions are mainly governed by the attraction forces generated by the masses of the objects considered. A model for this system is the N-body problem, which aims to predict the individual motions of a group of N bodies interacting with each other gravitationally.

1.1 The N-Body Problem

The N-body problem considers N point masses m_i for $i = 1, .., N$, moving under the influence of mutual gravitational attraction. Let us denote by $\mathbf{r}_i \in \mathbb{R}^3$ the position vector of the i-th body and G the universal gravitational constant. Then, Newton's gravitational law indicates that the force exerted by mass m_i to mass m_j is

$$F_{ij} = G m_i m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{\|\mathbf{r}_j - \mathbf{r}_i\|^3}.$$

By Newton's second law and considering all the possible interactions, we obtain the equations of motion of the N-body problem

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \sum_{\substack{j=1\\ j\neq i}}^{N} G m_i m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{\|\mathbf{r}_j - \mathbf{r}_i\|^3}, \qquad i = 1, .., N,$$

where t is the time of the system. This second order differential system of equations can be written using the Hamiltonian formulation. Indeed, denoting the momenta as $\mathbf{p}_i = m_i \frac{d\mathbf{r}_i}{dt} \in \mathbb{R}^3$, the Hamilton's equation of the motion becomes

$$\frac{d\mathbf{r}_i}{dt} = \frac{\partial H_N}{\partial \mathbf{p}_i}, \qquad \frac{d\mathbf{p}_i}{dt} = -\frac{\partial H_N}{\partial \mathbf{r}_i}, \qquad i = 1, ..., N,$$

with Hamiltonian

$$H_N(\mathbf{r}_1, .., \mathbf{r}_N, \mathbf{p}_1, .., \mathbf{p}_N) = \sum_{i=1}^{N} \frac{\|\mathbf{p}_i\|^2}{2m_i} - \sum_{1\leq i<j\leq N} \frac{Gm_im_j}{\|\mathbf{r}_j - \mathbf{r}_i\|}.$$

Notice that the N-body problem is a system of $6N$ first-order differential equations.

For $N = 2$, the 2-Body Problem is integrable, that is it can be solved by means of first integrals. On the contrary, for $N \geq 3$, the N-Body Problem has no general closed-form solution and one should expect it has chaotic behaviors. In particular, the 3-Body Problem has been of deep interest in the last centuries and a major source of development in the fields of analysis and dynamical systems.

At the end of the nineteenth century Henri Poincaré published pioneering works on the qualitative study of non-linear dynamical systems, with special focus in the 3-Body Problem (see [Che15] for an overview). In these works, Poincaré developed geometric and topological methods to understand the complex behaviors that non-linear systems could exhibit. Such methods have become the foundation for a major part of the modern qualitative theory of dynamical systems. From then, it has been established that one of the fundamental problems is to understand how the invariant manifolds of the different invariant objects (fixed points, periodic orbits, invariant tori) structure the global dynamics.

1.2 Stability of the Solar System

One of the oldest questions in dynamical systems concerns the stability of the Solar System. The most used model is the N-Body Problem in the planetary regime; that is, one massive body (the Sun) and $N - 1$ small bodies (the planets) performing approximated ellipses. In his early works, Poincaré already established the problem of stability to be one of the central questions of the 3-Body Problem.

In the last century this question has been a focus on the study of dynamical systems. In particular, one of the fundamental questions has been to understand the measure and "distribution" of the wandering[1] and non-wandering sets. Indeed, in [Her98], Michael Herman finishes its survey on important open questions in dynamical systems with two questions in the N- Body Problem, one in the general regime and the other in the planetary one. Roughly speaking, these questions are: "Are the non-wandering sets of the N-Body Problem nowhere dense?" and "In the planetary setting, is it possible to find wandering domains close to the orbits of the planets?".

Thanks to Arnol'd-Herman-Féjoz KAM Theorem, we know that many of the configurations in the planetary regime are stable, that is, the phase space has abundance of invariant tori, see [Arn63; LR95; Rob95; Féj04; CP11]. However, in the phase space the gaps left by the invariant tori leave room for instability. For instance, one could expect the appearance of instabilities close to mean-motion resonances, see [FGK+16].

The aim of this thesis is to study some of the instability and homoclinic phenomena arising in a specific mean-motion resonance of a particular case of the 3-Body Problem: the Restricted Planar Circular 3-Body Problem (RPC3BP).

[1] A point is *wandering* if some neighborhood of it, when propagated by the flow of the differential equation, never comes back to intersect itself.

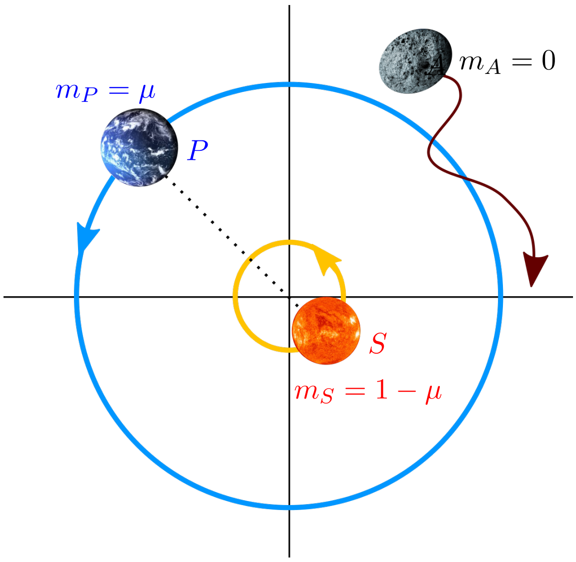


FIGURE 1: Representation of the position plane of the Restricted Planar Circular 3-Body Problem (RPC3BP).

1.3 The Restricted Planar Circular 3-Body Problem

The 3-Body Problem is a very complex system consisting in 18 first-order differential equations. Therefore, it is convenient to consider a simplified model; the Restricted 3-Body Problem which considers one of the bodies (say the third) to have negligible mass. That is, it is assumed that the two massive bodies, which we call the primaries, are not influenced by the massless body and, as a result, their motions are given by solutions of a 2-Body Problem (i.e. governed by the classical Kepler laws). On the contrary, the motion of the third body is affected by the movement of the two primaries, in the style of the 3-Body Problem. This model is of practical interest in the case of spacecrafts or small asteroids under the gravitational influence of two massive bodies, like a double star or a star-planet system.

Under the force of gravity, the motion of 2 bodies is given by planar conic sections (circles, ellipses, parabolas or hyperbolas on a plane). The setting known as the Restricted Planar Circular 3-Body Problem (RPC3BP), see Figure 1, is when the primaries perform circular motions and the third body is coplanar with them.

Let us name the two primaries S (star) and P (planet) and the third body A (asteroid). Normalizing their masses, we can assume that $m_S = 1 - \mu$ and $m_P = \mu$, with $\mu \in \left(0, \frac{1}{2}\right]$, and $m_A = 0$. In addition, we scale the units so that the gravity constant becomes $G = 1$. Let $\mathbf{r}_S, \mathbf{r}_P, \mathbf{r}_A \in \mathbb{R}^2$ denote the position of the three bodies as a function of time in the plane of motion. Since the primaries follow circular orbits one can assume that

$$\mathbf{r}_S(t) = \mu(\cos t, \sin t)^T, \qquad \mathbf{r}_P(t) = (\mu - 1)(\cos t, \sin t)^T.$$

Then, by Newton's law of universal gravitation, the massless body satisfies

$$\frac{d^2 \mathbf{r}_A}{dt^2} = -(1-\mu)\frac{\mathbf{r}_A - \mathbf{r}_S(t)}{\|\mathbf{r}_A - \mathbf{r}_S(t)\|^3} - \mu\frac{\mathbf{r}_A - \mathbf{r}_P(t)}{\|\mathbf{r}_A - \mathbf{r}_P(t)\|^3}. \tag{1}$$

Notice that the problem has been reduced to a system of 4 non-autonomous first-order differential equations.

The classic approach is to consider a rotating framework that fixes the position of the primaries in time, usually referred to as a synodic framework, see [Sze67] for example. One of the advantages of this new setting is that (1) becomes independent of time. Indeed, let us impose that the primaries are fixed at positions $\mathbf{q}_S = (\mu, 0)^T$ and $\mathbf{q}_P = (\mu - 1, 0)^T$ and denote by $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^2 \times \mathbb{R}^2$ the position and momenta of the third body given by the change of coordinates

$$\mathbf{q} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mathbf{r}_A, \qquad \mathbf{p} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \frac{d\,\mathbf{r}_A}{dt}.$$

Then, the RPC3BP becomes

$$\begin{aligned} \frac{d\,\mathbf{q}}{dt} &= \mathbf{p} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{q}, \\ \frac{d\,\mathbf{p}}{dt} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{p} - (1-\mu)\frac{\mathbf{q} - (\mu, 0)}{\|\mathbf{q} - (\mu, 0)\|^3} - \mu\frac{\mathbf{q} - (\mu - 1, 0)}{\|\mathbf{q} - (\mu - 1, 0)\|^3}. \end{aligned} \tag{2}$$

This is a 2-degrees of freedom autonomous Hamiltonian system with respect to the Hamiltonian

$$h(\mathbf{q}, \mathbf{p}; \mu) = \frac{\|\mathbf{p}\|^2}{2} - \mathbf{q}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{p} - \frac{(1-\mu)}{\|\mathbf{q} - (\mu, 0)\|} - \frac{\mu}{\|\mathbf{q} - (\mu - 1, 0)\|}. \tag{3}$$

Notice that this Hamiltonian is analytic away from $\mathbf{q} = (\mu, 0)$ and $\mathbf{q} = (\mu - 1, 0)$, which correspond to collision with the primaries S and P, respectively.

1.4 The Lagrange points

By analyzing the system of equations given by Hamiltonian h in (3), one can obtain the equilibrium points of the RPC3BP in synodic coordinates. We remark that, in a non-rotating framework, the equilibrium points will correspond to periodic orbits with the same period as the two primaries, i.e. in $1:1$ resonance.

For $\mu = 0$, the system has a circle of critical points $(\mathbf{q}, \mathbf{p}) = (q_1, q_2, p_1, p_2)$ with $\|\mathbf{q}\| = 1$ and $(p_1, p_2) = (-q_2, q_1)$. By contrast, for $\mu > 0$, it is a classical result that h has five equilibrium points: L_1, L_2, L_3, L_4 and L_5, called Lagrange points (see for instance [Sze67]). A representation of the location of the points can be found in Figure 2.

The Lagrange points L_4 and L_5 lie on the vertex of an equilibrium triangle between them and the two primaries. In inertial (non-rotating) coordinates, L_4 moves ahead of the primary P and L_5 behind. For $\mu > 0$ small, they are of center-center type, i.e. the linearization of (2) around them has two pairs of purely imaginary eigenvalues. Due to its stability, it is common to find objects orbiting around these points (for instance the Trojan and Greek Asteroids associated to the pair Sun-Jupiter, see [GDF+89; CG90; RG06]).

The Lagrange points, L_1, L_2 and L_3 are collinear with the primaries and are all of center-saddle type. That is, the linearization of (2) around them has a pair of real eigenvalues (saddle) and a pair of purely imaginary ones (center). This inherits three different behaviors near the equilibrium points: an expanding and contracting ones, given by the saddle, and a rotating one, given by the center.

The L_1 and L_2 equilibrium points are situated at each side of the primary P. Due to its interest in astrodynamics, a lot of attention has been paid to the study of their invariant manifolds (see [KLM+00; GLM+01; CGM+04]).

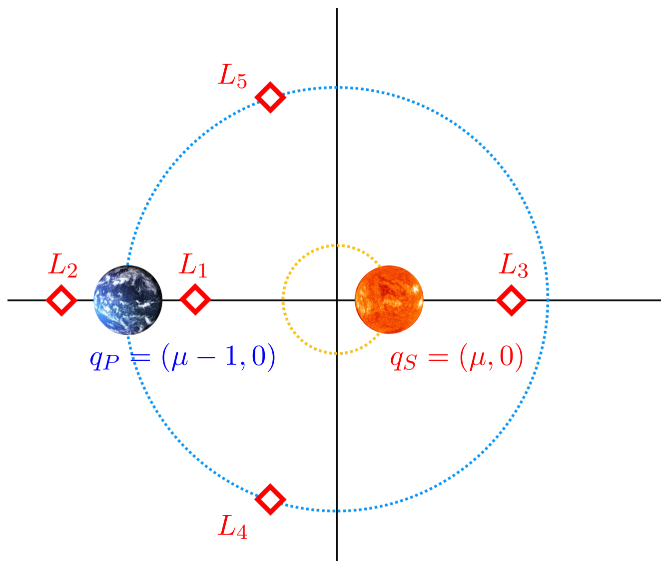


FIGURE 2: Projection onto the $\mathbf{q}$-plane of the equilibrium points for the RPC3BP on rotating coordinates.

In this work, we focus on the study of the Lagrange point L_3. The L_3 point is located opposite to the small primary P with respect to the massive primary S. Due to its situation and its non-stable behavior, it has received somewhat less attention. However, the associated invariant manifolds (more precisely its center-stable and center-unstable invariant manifolds) play an important role in structuring the dynamics of the RPC3BP. For example, one can see that they act as boundaries of effective stability of the stability domains around L_4 and L_5 (see [GJM+01; SST13]). Moreover, the invariant manifolds of L_3 play also a fundamental role in creating transfer orbits from the small primary to L_3 in the RPC3BP (see [HTL07; TFR+10]) or between primaries in the Bicircular 4-Body Problem (see [JN20; JN21]).

Over the past years, one of the main focus of the study of the dynamics "close" to L_3 and its invariant manifolds has been the so called "horseshoe-shaped orbits", first considered in [Bro11]. These are quasi-periodic orbits that encompass the critical points L_4, L_3 and L_5 and the interest on them arise when modeling the motion of co-orbital satellites, the most famous being Saturn's satellites Janus and Epimetheus, and near Earth asteroids. Recently, in [NPR20], the authors have proved the existence of 2-dimensional elliptic invariant tori on which the trajectories mimic the motions followed by Janus and Epimetheus (see also [DM81a; DM81b; LO01; CH03; BM05; BO06; BFP13; CPY19]).

Rather than looking at stable motions "close to" L_3 as [NPR20], the goal of this thesis is rather different: its objective is to study the chaotic and homoclinic phenomena around L_3 and its invariant manifolds.

1.5 Perturbative approach

In this work, we will consider the perturbative case of the RPC3BP. This means that we assume the mass ratio parameter $\mu > 0$ to be small, which implies that the primary S is much bigger than the primary P. This instance is consistent with a star-planet system, for example the Sun-Earth system. In this setting we can split the Hamiltonian h (see (3)) in an unperturbed Hamiltonian plus a perturbation:

$$h(\mathbf{q}, \mathbf{p}; \mu) = h_0(\mathbf{q}, \mathbf{p}) + \mu h_1(\mathbf{q}; \mu), \tag{4}$$

where

$$h_0(\mathbf{q},\mathbf{p}) = \frac{\|\mathbf{p}\|^2}{2} - \mathbf{q}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{p} - \frac{1}{\|\mathbf{q}\|},$$
$$\mu h_1(\mathbf{q};\mu) = \frac{1}{\|\mathbf{q}\|} - \frac{(1-\mu)}{\|\mathbf{q}-(\mu,0)\|} - \frac{\mu}{\|\mathbf{q}-(\mu-1,0)\|}.$$

The Hamiltonian h_0 corresponds to a 2-Body Problem between the primary S and the massless body. Therefore, h_0 is integrable and follows the classical Kepler problem. Moreover, the perturbation μh_1 can be considered a small perturbation as long as the body is far from collision with the primaries.

The critical point L_3 (see [Sze67] for the details) satisfies that, as $\mu \to 0$,

$$(\mathbf{q},\mathbf{p}) = (d_\mu, 0, 0, d_\mu), \qquad \text{with} \quad d_\mu = 1 + \frac{5}{12}\mu + \mathcal{O}(\mu^3). \tag{5}$$

The eigenvalues of the linearization with respect to the Lagrange point L_3 can be also expressed perturbatively, $\mu \to 0$, as follows

$$\mathrm{Spec}_{L_3} = \{\pm\rho_{\mathrm{eig}}(\mu), \pm i\,\omega_{\mathrm{eig}}(\mu)\}, \quad \text{with} \quad \begin{cases} \rho_{\mathrm{eig}}(\mu) = \sqrt{\frac{21}{8}\mu} + \mathcal{O}(\mu^{\frac{3}{2}}), \\ \omega_{\mathrm{eig}}(\mu) = 1 + \frac{7}{8}\mu + \mathcal{O}(\mu^2). \end{cases} \tag{6}$$

Since the ratio between the eigenvalues is $\mathcal{O}(\sqrt{\mu})$, the system posseses two time scales which translates to rapidly rotating dynamics coupled with a slow hyperbolic behavior around the critical point L_3. This setting is known as an *a priori stable* setting, since the hyperbolicity of the equilibrium point is created by the $\mathcal{O}(\mu)$ perturbation and cannot be detected in the unperturbed system (notice that $\rho_{\mathrm{eig}}(0) = 0$). Indeed, if one takes the limit $\mu \to 0$ in (4) one obtains the classical integrable Kepler problem in the elliptic regime (i.e. negative energy), where no hyperbolicity is present. However, when $\mu > 0$, L_3 possesses one dimensional unstable and stable manifolds.

Let us recall that, on an inertial system of coordinates, the Lagrange points correspond to periodic orbits on a $1:1$ resonance with the primaries. Then, being far from collision, the dynamics close to the Lagrange point L_3 and its invariant manifolds for small μ are rather similar to that of other mean motion resonances which play an important role in creating instabilities in the Solar system, see [FGK+16].

This thesis studies some of the instabilities found in the mean-motion $1:1$ resonance region of L_3 in the RPC3BP.

1.6 Chaotic dynamics

Poincaré, while trying to integrate the 3-Body Problem, realized that one of the obstructions for integration was given by the existence of transverse intersections between the unstable and stable manifolds of periodic orbits, see [Poi90]. These transverse intersections create a complex tangle between the unstable and stable manifolds which fold and stretch more and more when approaching the periodic orbit (see Figure 3). A dynamical system exhibiting this type of behavior is now said to display *chaotic dynamics*.

The complexity of this homoclinic (or heteroclinic) tangle was analyzed by Stephen Smale, see [Sma65; Sma67]. In these works, Smale introduced the *horseshoe map* (see [AP90] for instance) which became a core example of a dynamical system exhibiting chaotic motions. Since then, one of the classical methods to prove the existence of

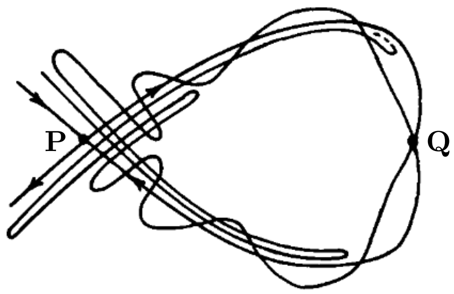


FIGURE 3: Illustration of the homoclinic tangle occurring at a hyperbolic saddle point for a discrete dynamical system. Source [GH83, Figure 5.2.7].

chaotic dynamics has been the Smale-Birkhoff homoclinic theorem (see [Sma67] or [GH83; KH95] for a modern exposition).

Smale-Birkhoff homoclinic Theorem. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with an hyperbolic point* **P** *and a transverse homoclinic point* **Q**. *Then, on a small neighborhood of* **P** *one can build an* **Smale's horseshoe map**. *That is, there exists an invariant set $X \subset U$ homeomorphic to $\{0,1\}^{\mathbb{Z}}$ such that $f|_X$ is conjugated to the shift map*

$$\begin{aligned} \sigma : \{0,1\}^{\mathbb{Z}} &\to \{0,1\}^{\mathbb{Z}} \\ (\omega_k)_{k\in\mathbb{Z}} &\mapsto (\omega_{k+1})_{k\in\mathbb{Z}}. \end{aligned}$$

As a consequence, f exhibits **chaotic behavior**.

For many physically relevant models, to prove the existence of chaotic motions is usually a remarkably difficult problem. In particular, this is the case of many Celestial Mechanics models. Most of the known results have been found in *a priori unstable* settings, i.e. nearly integrable regimes where there is an unperturbed problem which already presents some form of "hyperbolicity". This is the case in the vicinity of collision orbits (see for example [Moe89; BM06; Bol06; Moe07]) or close to parabolic orbits (which allows to construct chaotic/oscillatory motions), see [Sit60; Ale76; Lli80; Mos01; GMS16; GSM$^+$17; GPS$^+$21; GMP$^+$22]. There are also several results in regimes far from integrable which rely on computer assisted proofs [Ari02; WZ03; Cap12; GZ19]. The problem tackled in this work is different since we are considering an *a priori stable* setting, that is the unperturbed system has no hyperbolicity.

Motivated by the Smale-Birkhoff homoclinic Theorem, this thesis proves the existence of transverse homoclinic orbits in a tubular neighborhood of the unstable and stable manifolds of L_3 contained in the $1:1$ resonance region.

1.7 The homoclinic phenomena in a bifurcation scenario

Let us recall that the (weak) hyperbolicity of the Lagrange point L_3 is created by the $\mathcal{O}(\mu)$ perturbation. Indeed, for $\mu = 0$, the equilibrium point L_3 degenerates and the spectrum of its linear part consists in a pair of purely imaginary and a double 0 eigenvalues, (see (6)). This bifurcation scenario is known as the $0^2 i\omega$ resonance or Hamiltonian Hopf-Zero bifurcation.

Most of the studies in homoclinic phenomena around a saddle-center equilibrium are focused on the non-degenerate case, namely the equilibrium point is hyperbolic, see [Ler91; MHO92; Rag97a; Rag97b; BRS03]. However, for the resonance $0^2 i\omega$ cases, the results are more rare. In [GG10], the authors study this singularity combining numerical and analytic techniques. The reversible case is considered in [Lom99; Lom00] where the author proves the existence of transverse homoclinic orbits for every periodic orbit exponentially close to the origin, except the origin itself. In [JBL16], the authors show the existence of homoclinic connections with several loops for every periodic orbit close to the equilibrium point.

By contrast, this thesis deals with the existence of one dimensional homoclinic connections for the equilibrium point and the existence of transverse homoclinic orbits associated to periodic orbits (exponentially) close to the equilibrium point. In the case of the (non-Hamiltonian) Hopf-zero singularity, we remark the similar work [BIS20]. Also, in [GGS+21], the authors use similar techniques to analyze breather solutions for the nonlinear Klein-Gordon partial differential equation.

2 Main results

Since L_3 is a center-saddle critical point, it possesses 1-dimensional unstable and stable manifolds and a 2-dimensional center manifold. We denote as $W^{\mathrm{u}}(L_3)$ and $W^{\mathrm{s}}(L_3)$ the unstable and stable manifolds, respectively.

The first result obtained in this thesis analyzes the distance between the unstable and stable manifolds of L_3, see Section 2.1. The second result concerns the existence of 2-round homoclinic orbits to L_3, i.e. 1-dimensional homoclinic connections that approach the equilibrium point twice, see Section 2.2. The third and last result of the thesis studies the existence of chaotic motions in a neighborhood close to L_3 and its invariant manifolds by means of the existence of transversal homoclinic orbits, see Section 2.3.

2.1 Distance between the invariant manifolds of L_3

The aim of this section is to give an asymptotic formula for the distance between the invariant manifolds $W^{\mathrm{u}}(L_3)$ and $W^{\mathrm{s}}(L_3)$, for small values of the parameter μ, in an appropriate transverse section.

The invariant manifolds $W^{\mathrm{u}}(L_3)$ and $W^{\mathrm{s}}(L_3)$ have two branches each, see Figure 4. We denote by $W^{\mathrm{u},+}(L_3)$ and $W^{\mathrm{s},+}(L_3)$ the pair that circumvents L_5 whereas the ones that circumvent L_4 are denoted by $W^{\mathrm{u},-}(L_3)$ and $W^{\mathrm{s},-}(L_3)$. These branches are symmetric. Indeed, one can see that the Hamiltonian system associated to $h(\mathbf{q},\mathbf{p};\mu)$ in (4) is reversible with respect to the involution

$$\Psi(\mathbf{q},\mathbf{p}) = \Psi(q_1,q_2,p_1,p_2) = (q_1,-q_2,-p_1,p_2). \tag{7}$$

Therefore, by (5), $L_3 = (d_\mu, 0, 0, d_\mu)$ belongs to the symmetry axis of Ψ and the $+$ branches of the invariant manifolds of L_3 are symmetric to the $-$ with a reverse time. Thus, to compute the distance between the manifolds, one can restrict the study to the first ones, $W^{\mathrm{u},+}(L_3)$ and $W^{\mathrm{s},+}(L_3)$.

We perform the classical symplectic polar change of coordinates

$$\mathbf{q} = r\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \qquad \mathbf{p} = R\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} - \frac{G}{r}\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix},$$

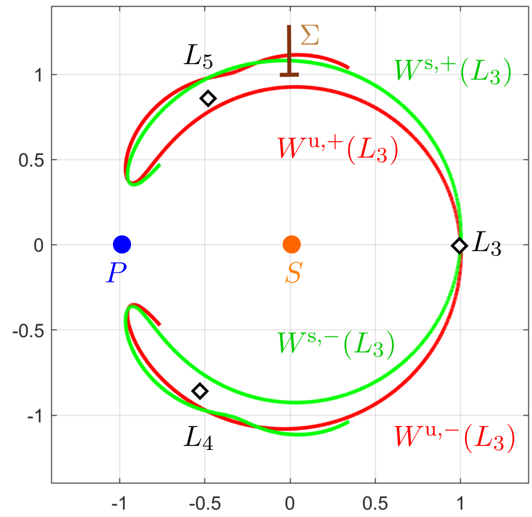


FIGURE 4: Projection onto the $\mathbf{q}$-plane of the unstable (red) and stable (green) manifolds of L_3, for $\mu = 0.0028$.

where R is the radial linear momentum and G is the angular momentum. We consider as well the 3-dimensional section

$$\Sigma = \left\{ (r, \theta, R, G) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}^2 \ : \ r > 1, \, \theta = \frac{\pi}{2} \right\} \tag{8}$$

and denote by $(r_*^{\mathrm{u}}, \frac{\pi}{2}, R_*^{\mathrm{u}}, G_*^{\mathrm{u}})$ and $(r_*^{\mathrm{s}}, \frac{\pi}{2}, R_*^{\mathrm{s}}, G_*^{\mathrm{s}})$ the first crossing of the invariant manifolds with this section (see Figure 4). The next theorem measures the distance between these points for $0 < \mu \ll 1$. Its proof can be found in Parts I and II of the thesis; see Section 3.2 for the details in the strategy followed.

Theorem A. (Distance between the unstable and stable manifolds of L_3). *There exists $\mu_0 > 0$ such that, for $\mu \in (0, \mu_0)$,*

$$\|(r_*^{\mathrm{u}}, R_*^{\mathrm{u}}, G_*^{\mathrm{u}}) - (r_*^{\mathrm{s}}, R_*^{\mathrm{s}}, G_*^{\mathrm{s}})\| = \sqrt[3]{4}\, \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right],$$

where:

- *The constant $A > 0$ is given by the real-valued integral*

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}} dx \approx 0.177744. \tag{9}$$

- *The constant $\Theta \in \mathbb{C}$ is the Stokes constant associated to the inner equation analyzed in Theorem I.2.7 in Part I.*

In Theorem A, due to the rapidly rotating dynamics of the system (see (6)), the distance between the stable and unstable manifolds of L_3 is exponentially small with respect to $\sqrt{\mu}$. This is usually known as a *beyond all orders phenomenon*, since the difference between the manifolds cannot be detected by expanding the manifolds in series of powers of μ. Due to this phenomenon, the classical Melnikov Method (see for instance [GH83]) cannot be applied to obtain Theorem A.

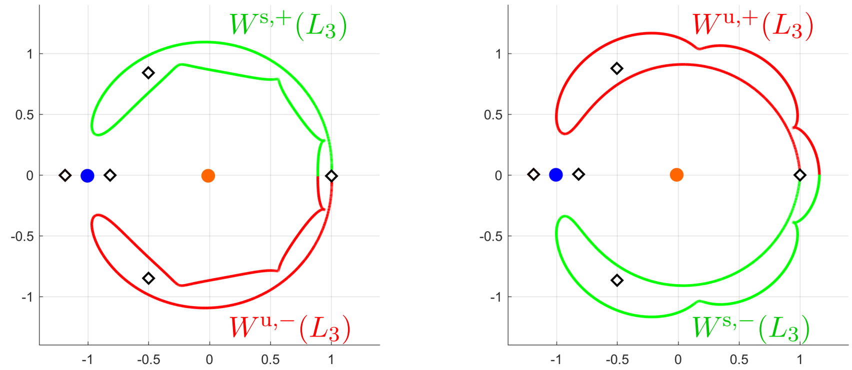


FIGURE 5: Projection onto the **q**-plane for examples of 2-round homoclinic connection to L_3. (Left) $\mu = 0.012144$, (right) $\mu = 0.004192$.

Due to the symmetry in (7), an analogous result to Theorem A holds for the opposite branches $W^{u,-}(L_3)$ and $W^{s,-}(L_3)$. Moreover, a more general result can be proved for sections $\Sigma(\theta_*) = \{r > 1, \theta = \theta_*\}$, see Remark II.1.2.

Notice that, since the unstable and stable manifolds of L_3 are 1-dimensional and the system is autonomous, the manifolds either coincide in the section Σ or not, but there is no possible transverse intersection between them. Therefore, it is not possible to apply the Smale-Birkhoff homoclinic Theorem and the breakdown of the invariant manifolds of L_3 does not lead to the existence of chaotic orbits.

The validity of Theorem A as an asymptotic formula relies in proving that $\Theta \neq 0$. Unfortunately, there is not an analytic proof of this fact. However, by numerical computation one obtains $|\Theta| \approx 1.63$, see Remark I.2.8 in Part I. The corresponding code can be found at [Gir22]. As a result, we consider the following ansatz.

Ansatz A. *The constant Θ given in Theorem A satisfies that $\Theta \neq 0$.*

We expect that, by means of a computer assisted proof, it would be possible in the future to obtain rigorous estimates and verify $\Theta \neq 0$, following the strategy in [BCG+22].

2.2 Homoclinic phenomena to L_3

Theorem A and Ansatz A imply that the invariant manifolds of L_3 do not meet the first time they cross section Σ. Certainly this do not prevent the existence of multi-round homoclinic connections. This section is devoted to study the existence of such homoclinic connections for certain values of the mass parameter μ.

To state it, we first classify the homoclinic orbits by how many "rounds" they take before returning to L_3. In particular, we say that an homoclinic connection to L_3 is *k-round* if, on a μ-neighborhood of this critical point, the closure of the homoclinic orbit has k connected components, (see Figure 5 for examples of 2-round connections).

According to this definition, Theorem A and Ansatz A imply the following.

Corollary A. **(1-round homoclinic connections).** *Assume Ansatz A. There exists $\mu_0 > 0$ such that, for $\mu \in (0, \mu_0)$, there do not exist 1-round homoclinic connections to L_3.*

E. Barrabés, J.M. Mondelo and M. Ollé in [BMO09] analyze the existence of multi-round homoclinic connections to L_3 in the RPC3BP. In particular, they conjectured

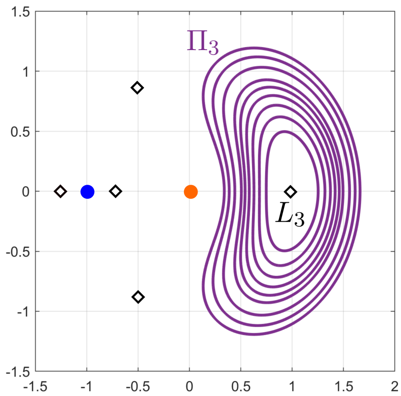


FIGURE 6: Projection onto the **q**-plane of the family of Lyapunov periodic orbits Π_3 (purple) for $\mu = 0.0028$.

the existence of 2-round homoclinic orbits for a sequence of mass ratios $\{\mu_n\}_{n\in\mathbb{N}}$ satisfying $\mu_n \to 0$ as $n \to \infty$ and supported their claim with numeric computations. In the following result, we prove this conjecture.

Theorem B. (2-round homoclinic connections). *Assume Ansatz A and consider $\rho_{\mathrm{eig}}(\mu)$ given in (6) and $A > 0$ given in Theorem A. Then, there exists a sequence $\{\mu_n\}_{n\geq N_0}$ with N_0 big enough, of the form*

$$\mu_n = \frac{A}{n\pi\rho_{\mathrm{eig}}(0)}\left(1 + \mathcal{O}\left(\frac{1}{\log n}\right)\right), \qquad \text{for } n \gg 1,$$

such that, the Hamiltonian system (3) has a 2-round homoclinic connection to the equilibrium point L_3 between $W^{\mathrm{u},+}(L_3)$ and $W^{\mathrm{s},-}(L_3)$.

Theorem B is proved in Part III, see Section 3.3 for the details in the strategy followed. Using the same tools, one can obtain an analogous result for the homoclinic connections between $W^{\mathrm{u},-}(L_3)$ and $W^{\mathrm{s},+}(L_3)$.

Remark B. (Multi-round homoclinic connections). *In [BMO09], the authors also conjectured the existence of k-round homoclinic connections for $k > 2$ for different sequences of the mass parameter μ. We believe that our strategy can be also applied for proving the existence of k-round homoclinic symmetric connections.*

2.3 Chaotic phenomena associated to L_3

Let us recall that do not exist transverse intersections between the stable and unstable manifolds of L_3. In order to look for chaotic dynamics by means of the Smale-Birkhoff homoclinic Theorem, we study the unstable and stable manifolds of nearby periodic orbits.

The Lyapunov Center Theorem (see for instance [MO17]) is a classical result that ensures the existence of a family of periodic orbits emanating from a saddle-center equilibrium point. Moreover, close to the equilibrium point, the periodic orbits are hyperbolic and have 2-dimensional unstable and stable manifolds.

Proposition C. (Lyapunov periodic orbits to L_3). *There exist $\mu_0, \varrho_0 > 0$ small enough such that, for $\mu \in (0, \mu_0)$, the Hamiltonian system with Hamiltonian (3) has a family of hyperbolic periodic orbits*

$$\Pi_3 = \left\{P_{3,\varrho} \text{ periodic orbit } : \ h(P_{3,\varrho}) = \varrho^2 + h(L_3),\ \varrho \in (0, \varrho_0)\right\},$$

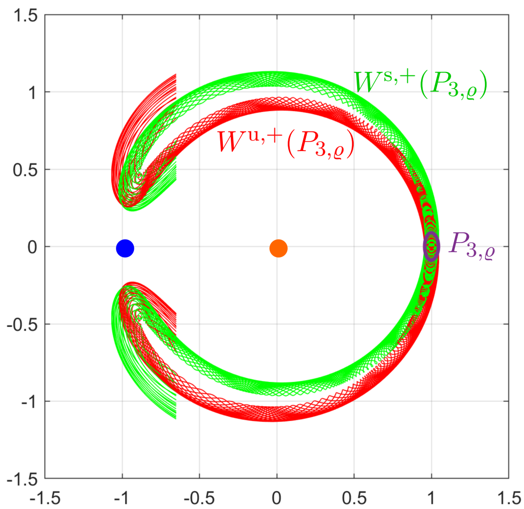


FIGURE 7: Projection onto the $\mathbf{q}$-plane of the unstable (red) and stable (green) manifolds of the Lyapunov periodic orbit $P_{3,\varrho}$ (purple), for $\mu = 0.003$.

which depend regularly on $\varrho \in (0, \varrho_0)$ *and satisfy that* $\mathrm{dist}(P_{3,\varrho}, L_3) \to 0$ *as* $\varrho \to 0$ *in the sense of Hausdorff distance.*

We denote by $W^{\mathrm{u}}(P_{3,\varrho})$ and $W^{\mathrm{s}}(P_{3,\varrho})$ the 2-dimensional unstable and stable invariant manifolds of the Lyapunov periodic orbit $P_{3,\varrho}$. Analogously to the L_3 case, the invariant manifolds have two branches each which we denote by $W^{\mathrm{u},+}(P_{3,\varrho})$ and $W^{\mathrm{s},+}(P_{3,\varrho})$ the ones that circumvent L_5 and, by $W^{\mathrm{u},-}(P_{3,\varrho})$ and $W^{\mathrm{s},-}(P_{3,\varrho})$, the ones that surround L_4 (see Figure 7). By the Smale-Birkhoff homoclinic theorem, proving the existence of transverse intersections between $W^{\mathrm{u},+}(P_{3,\varrho})$ and $W^{\mathrm{s},+}(P_{3,\varrho})$ implies the existence of chaotic motions on a neighborhood of L_3 and its invariant manifolds. More specifically, we prove the following result.

Theorem C. (Chaotic motions). *Let the constants* $A > 0$ *and* Θ *be as given in Theorem A and* ϱ_0 *in Proposition C. Assume Ansatz A. Then, there exist* $\mu_0 > 0$ *and functions* $\varrho_{\min}, \varrho_{\max} : (0, \mu_0) \to [0, \varrho_0]$ *of the form*

$$\varrho_{\min}(\mu) = \frac{\sqrt[6]{2}}{2} |\Theta| \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[1 + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right],$$
$$\varrho_{\max}(\mu) = \frac{\sqrt[6]{2}}{2} |\Theta| \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[2 + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right],$$

such that, for $\mu \in (0, \mu_0)$ *and* $\varrho \in (\varrho_{\min}(\mu), \varrho_{\max}(\mu)]$, *the invariant manifolds* $W^{\mathrm{u},+}(P_{3,\varrho})$ *and* $W^{\mathrm{s},+}(P_{3,\varrho})$ *intersect transversally.*

In particular, there exists a tubular neighborhood around the invariant manifolds $W^{\mathrm{u}}(L_3)$ *and* $W^{\mathrm{s}}(L_3)$ *with the boundary at the energy level* $h = h(L_3) + \mathcal{O}(\mu^{\frac{2}{3}} e^{-\frac{2A}{\sqrt{\mu}}})$ *where one can construct a Smale's horseshoe map for a suitable Poincaré map induced by the flow of the Hamiltonian* h *in* (3).

Theorem C is proved in Part III, see Section 3.3 for the details in the strategy followed. Notice that, due to the symmetry in (7), an analogous result holds for the transverse intersections of branches $W^{\mathrm{u},-}(P_{3,\varrho})$ and $W^{\mathrm{s},-}(P_{3,\varrho})$. Moreover, by restricting μ one can take $\varrho_{\max}(\mu)$ bigger.

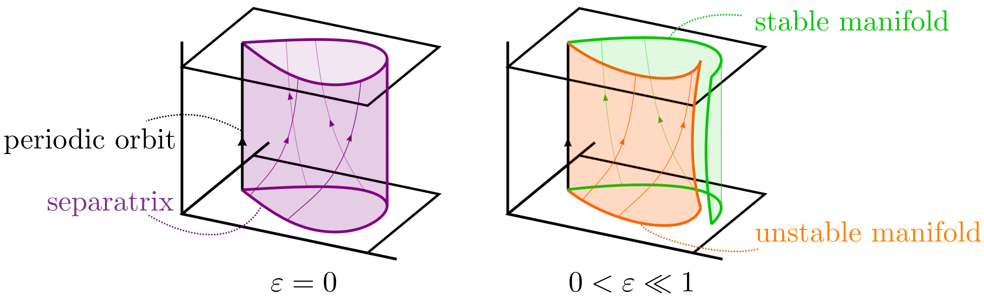


FIGURE 8: Representation in $\mathbb{R}^3$ of the perturbative setting of the splitting of separatrices phenomenon of a periodic orbit, with perturbative parameter ε.

Following the same ideas behind Theorem C, one can see that, besides transversal intersections, one can find a quadratic homoclinic tangency at the energy level $\varrho = \varrho_{\min}(\mu)$. The following theorem is proved together with Theorem C in Part III.

Theorem D. (Quadratic homoclinic tangency). *Assume Ansatz A and denote by f_ϱ the flow of the Hamiltonian system given in (3) restricted to the energy level $h = \varrho^2 + h(L_3)$. Let $\varrho_0, \mu_0 > 0$ and $\varrho_{\min}(\mu) : (0, \mu_0) \to [0, \varrho_0]$ be as given in Theorem C. Then, for a fixed μ and ϱ close to $\varrho_{\min}(\mu)$, the flow f_ϱ unfolds generically an homoclinic quadratic tangency between $W^{\mathrm{u},+}(P_{3,\varrho_{\min}(\mu)})$ and $W^{\mathrm{s},+}(P_{3,\varrho_{\min}(\mu)})$.*

We use the definition of generic unfolding given in [Dua08] for area preserving diffeomorphisms (see Part III for more details). Theorem D should lead to prove the existence of a Newhouse domain. We do not enter in details in this thesis, but one should expect that the unfolding of the quadratic tangency would lead to the existence of infinitely many elliptic islands and of Smale's horseshoes of maximal Hausdorff dimension.

3 Strategy of the proofs of the main results

In this section we review the techniques and strategies used to prove the main results presented in Section 2: Theorems A, B, C and D.

3.1 Exponentially small splitting of separatrices

The work found in this thesis it is rooted on the ideas and techniques developed in the last century to deal with the *splitting of separatrices* phenomenon. This phenomenon was discovered by H. Poincaré and described in his celebrated memoir "*Sur le problème des trois corps et les équations de la dynamique*", see [Poi90], which became a turning point in the study of the 3-Body Problem.

Let us consider a perturbative problem such that, in the unperturbed case, there exists an hyperbolic invariant object with an unstable and stable invariant manifolds that coincide forming an homoclinic connection or *separatrix*. The splitting of separatrices phenomenon occurs when, in the perturbed case, the homoclinic connection is destroyed and the unstable and stable manifolds "split" (see Figure 8).

In [Poi90], Poincaré developed a perturbative method to measure the size of the splitting. Seventy years later it was rediscovered by V.I Arnold and V. K. Melnikov, see [Mel63; Arn64], and is now the standard theory to analyze the breakdown of homoclinic and heteroclinic connections, known as the Poincaré-Melnikov Theory (see [GH83] for a modern exposition). By means of this theory, one obtains an asymptotic formula for the distance between the unstable and stable manifolds. The leading term is given by the so-called *Melnikov function*. Studying this asymptotic formula, one obtains information about the possible intersections between the manifolds.

However, as Poincaré already realized, this theory cannot be applied to singular problems where the distance between manifolds is exponentially small, as in Theorem A. Indeed, the breakdown of homoclinic connections to L_3 fits into what is usually referred to as *exponentially small splitting of separatrices* problems. It is, what is usually called, a *beyond all orders* phenomenon.

The first obtention of an asymptotic formula for an exponentially small splitting of separatrices did not appear until the 1980's with the pioneering work by V. F. Lazutkin for the standard map [Laz84; Laz05]. Even if Lazutkin's work was not complete (the complete proof was achieved by V. G. Gelfreich in [Gel99]), the ideas he developed have been very influential and have been the basis of many of the works in the field (and in particular of this work). Other methods to deal with exponentially small splitting of separatrices are Treschev's continuous averaging (see [Tre97]) or "direct" series methods (see [GGM99]).

Usually, the exponentially small splitting of separatrices problems are classified as regular or singular.

In the *regular cases*, even if Poincaré-Melnikov theory cannot be straightforwardly applied, the Melnikov function gives the leading term for the distance between the perturbed invariant manifolds. That is, Melnikov theory provides the first order for the distance but leads to too rough estimates for the higher order terms. After Lazutkin's ideas, there were some studies on upper bounds of the splitting of the invariant manifolds, see [HMS88; FS90a; FS90b]. Concerning asymptotic formulas, the first results were for rapidly forced periodic perturbations of 1-degree of freedom Hamiltonian systems, see [DS92; DS97; Gel97b; Gel00; BF04; BF05; GGS20], for close to the identity area preserving maps, see [DR98], and for Hamiltonian systems with two or more degrees of freedom which have hyperbolic tori with fast quasiperiodic dynamics [Sau01]. In particular, [GMS16] provides the first prove of exponentially small splitting of separatrices in a Celestial Mechanics problem and later [GSM+17; GPS+21; GMP+22].

In the *singular cases*, the exponentially small first order for the distance between the invariant manifolds is no longer given by the Melnikov function. Instead, one has to consider an auxiliary equation, usually called *inner equation*, which does not depend on the perturbative parameter and provides the first order for the distance. The model studied by Lazutkin in [Laz84] falls under the singular case. In [Gel97a], the author studied the inner equation of certain second order equations periodically perturbed. In [GS01; OSS03], resurgence theory was applied to rigorously study the inner equation of certain examples. Some other results on inner equations can be found on [Bal06; BS08; BM12].

There are few works providing proofs in the singular cases, see for example [Lom00; MSS11]. In [GOS10] the splitting of separatrices for a perturbed pendulum is studied. The most general result of splitting of separatrices in both regular and singular cases is given in [Bal06; BFG+12; Gua13] for Hamiltonian systems with a periodic perturbation in time. In [GG10], the Hamiltonian-Hopf bifurcation is studied both numerically and analytically. In [BS08; BCS13; BCS18a; BCS18b] the authors performed a detailed analysis of the breakdown of the invariant manifolds of the Hopf-zero singularity in a non-Hamiltonian setting.

Due to the extreme sensitivity of the exponentially small splitting of separatrices phenomenon on the features of each particular model, most of the available results apply under quite restrictive hypothesis and, therefore, cannot be applied to analyze the invariant manifolds of L_3.

We highlight the work by J. Font i Arjó who, in his PhD thesis in 1993, see [Fon93a], performed a numerical study of the breakdown of the manifolds of L_3 before the majority of the current analytic techniques to deal with the exponentially small splitting of separatrices phenomenon were established.

3.2 Strategy for the proof of Theorem A

Let us recall that, for the limit problem h in (3) with $\mu = 0$, the five Lagrange point disappear into a circle of (degenerate) critical points. As a consequence, the one-dimensional invariant manifolds of L_3 disappear when $\mu = 0$ and there do not exist a separatrix for the unperturbed problem. For this reason, the first step of the proof of Theorem A (see Step A in the list below) is to perform a singular change of coordinates to obtain a "new first order" Hamiltonian with a center-saddle equilibrium point (close to L_3) with stable and unstable manifolds that coincide along a separatrix. To perform the change of coordinates we use the Poincaré planar elements (see [MO17]) plus a singular (with respect to μ) scaling. The $1:1$ averaged Hamiltonian has been also studied to obtain "good" approximations for the global dynamics in the $1:1$ resonant zone, see for example [RNP16; PA21] and the references therein.

The constant A in (9) is given by the height of the maximal strip of analyticity of the time-parametrization of the unperturbed separatrix. Therefore, to obtain its value, one has to compute the imaginary part of the singularities of the separatrix which are closer to the real line (see Step B in the list below). On all the previous mentioned works on splitting of separatrices, either the separatrix of the unperturbed model has an analytic expression (see for example [DS92; GG10; GOS10; BCS13; GMS16]) or otherwise certain properties of its analytic continuation are given as assumptions (see [DS97; Gel97a; BF04; BFG+12]). In this case, we do not have an explicit expression for the time-parameterization of the separatrix and, to obtain its complex singularities, we need to rely on techniques of analytical continuation to analyze them (see Section I.2.2). In particular, we describe the parametrization of the separatrix in terms of a multivalued function involving a complex integral (see Theorem I.2.2 below). The value we obtain in (9) agrees with the numerical computations of the distance between the invariant manifolds given in [Fon93a; SST13].

The breakdown of invariant manifolds of L_3 falls under the category of singular exponentially splitting of separatrices. Therefore, the constant Θ in Theorem A is not correctly given by the Melnikov function but by the analysis of the inner equation of the system (see Steps C and D below). In particular, Θ corresponds to a Stokes constant that depends on the full jet of the Hamiltonian and, as a result, it does not have a closed formula.

To prove Theorem A, we follow similar strategies of those in [BFG+12; BCS13]. We split the proof in two parts, which can be read independently. In the following list, we present the main steps of our strategy.

In Part I, we complete the following steps:

A. We perform a change of coordinates which captures the slow-fast dynamics of the system. The new Hamiltonian becomes a (fast) oscillator weakly coupled to a 1-degree of freedom Hamiltonian with a saddle point and a separatrix associated to it.

B. We analyze the analytical continuation of a time-parametrization of the separatrix. In particular, we obtain its maximal strip of analyticity (centered at the real line). We also describe the character and location of the singularities at the boundaries of this region.

C. We derive the inner equation, which gives the first order of the original system close to the singularities of the separatrix described in Step B. This equation is independent of the perturbative parameter μ.

D. We study two special solutions of the inner equation which are approximations of the perturbed invariant manifolds near the singularities. Moreover, we provide an asymptotic formula for the difference between these two solutions of the inner equation. We follow the approach presented in [BS08].

In Part II we perform the remaining steps necessary to complete the proof of Theorem A:

E We prove the existence of the analytic continuation of the parametrizations of the invariant manifolds of L_3, $W^{\mathrm{u},+}(L_3)$ and $W^{\mathrm{s},+}(L_3)$, in an appropriate complex domain called boomerang domain. This domain contains a segment of the real line and intersects a sufficiently small neighborhood of the singularities of the unperturbed separatrix.

F. By using complex matching techniques, we show that, close to the singularities of the unperturbed separatrix, the solutions of the inner equation obtained in Step D are "good approximations" of the parameterizations of the perturbed invariant manifolds obtained in Step E.

G. We obtain an asymptotic formula for the difference between the perturbed invariant manifolds by proving that the dominant term comes from the difference between the solutions of the inner equation.

3.3 Strategy for the proof of Theorems B, C and D

The proof of Theorems B, C and D rely on the the asymptotic formula for the distance of the invariant manifolds of L_3 obtained in Theorem A under Ansatz A, i.e. $\Theta \neq 0$. Their proofs can be found in Part III.

To prove the existence of 2-round homoclinic orbits between the branches $W^{\mathrm{u},+}(L_3)$ and $W^{\mathrm{s},-}(L_3)$ of the invariant manifolds (i.e. Theorem B), we take advantage of the fact that the Hamiltonian h is reversible with respect to the involution $\Psi(q_1, q_2, p_1, p_2) = (q_1, -q_2, -p_1, p_2)$ (see (7)). Therefore, $W^{\mathrm{u},+}(L_3)$ and $W^{\mathrm{s},-}(L_3)$ are symmetric with respect to the symmetry axis $\{q_2 = 0, p_1 = 0\}$ and, by symmetry, it is only necessary to prove that there exists a sequence of the parameters μ for which the invariant manifold $W^{\mathrm{u},+}(L_3)$ intersects the symmetry axis. To obtain this result, we need to extend $W^{\mathrm{u},+}(L_3)$ from section Σ (as given in Theorem A) to a neighborhood of the equilibrium point L_3.

To study the invariant manifolds near L_3, we use a normal form result for Hamiltonian systems in a neighborhood of saddle-center critical points. Note that, the classical normal form result by J. Moser in [Mos58] is not enough for our purposes. Indeed, we need to control that the radius of convergence of the normal form does not goes to zero when $\mu \to 0$. For that reason, we rely on a more quantitative normal form obtained by T. Jézéquel, P. Bernard and E. Lombardi in [JBL16] that ensures that the normalization does not blow up when $\mu \to 0$.

To prove the existence of transverse intersections and quadratic tangencies between the 2-dimensional manifolds $W^{\mathrm{u}}(P_{3,\varrho})$ and $W^{\mathrm{s}}(P_{3,\varrho})$ (i.e Theorems C and D)

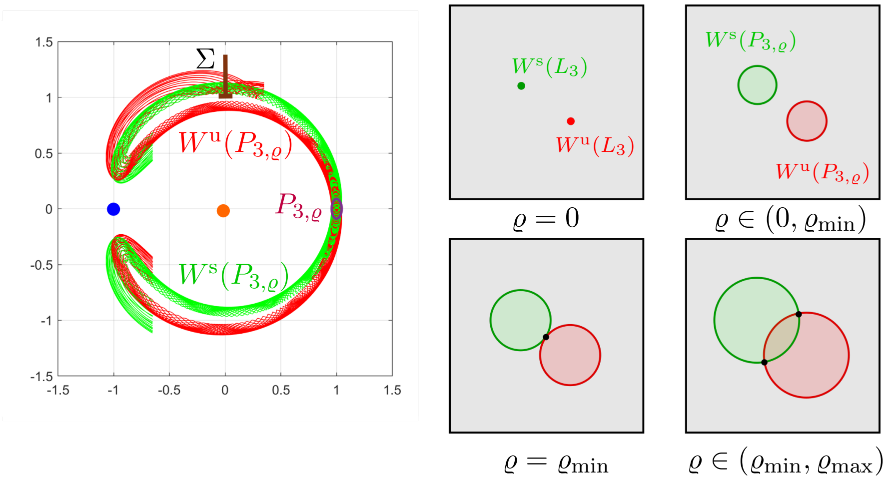


FIGURE 9: Left: Projection onto the **q**-plane of the intersection of the unstable and stable manifolds $W^{\mathrm{u},+}(P_{3,\varrho})$ and $W^{\mathrm{s},+}(P_{3,\varrho})$ with section Σ. Right: Representation in section Σ of the different possibilities given in Corollary A and Theorems C and D.

we study their intersections with the section Σ (see (8)) and compare them with the results for the 1-dimensional manifolds of L_3 obtained in Theorem A, see Figure 9. Note that, since $W^{\mathrm{u}}(L_3)$ and $W^{\mathrm{s}}(L_3)$ are exponentially close to each other with respect to $\sqrt{\mu}$, the energy levels where chaotic motions are found are also exponentially close to the energy level of L_3.

Part I

Complex singularities and the inner equation

Abstract

The Restricted 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies, called the primaries. If the primaries perform circular motions and the massless body is coplanar with them, one has the Restricted Planar Circular 3-Body Problem (RPC3BP). In synodic coordinates, it is a two degrees of freedom Hamiltonian system with five critical points, $L_1, .., L_5$, called the Lagrange points.

The Lagrange point L_3 is a saddle-center critical point which is collinear with the primaries and is located beyond the largest of the two. Between Part I and II, we provide an asymptotic formula for the distance between the one dimensional stable and unstable invariant manifolds of L_3 when the ratio between the masses of the primaries μ is small. It implies that L_3 cannot have one-round homoclinic orbits.

If the mass ratio μ is small, the hyperbolic eigenvalues are weaker than the elliptic ones by factor of order $\sqrt{\mu}$. This implies that the distance between the invariant manifolds is exponentially small with respect to μ and, therefore, the classical Poincaré–Melnikov method cannot be applied.

In this part, we approximate the RPC3BP by an averaged integrable Hamiltonian system which possesses a saddle center with a homoclinic orbit and we analyze the complex singularities of its time parameterization. We also derive and study the inner equation associated to the original perturbed problem. The difference between certain solutions of the inner equation gives the leading term of the distance between the stable and unstable manifolds of L_3.

Chapter I.1

Introduction

The understanding of the motions of the 3-Body Problem has been of deep interest in the last centuries. Since Poincaré, see [Poi90], one of the fundamental problems is to understand how the invariant manifolds of its different invariant objects (periodic orbits, invariant tori) structure its global dynamics. Assume that one of the bodies (say the third) has mass zero. Then, one has the Restricted 3-Body Problem. In this model, the two first bodies, called the primaries, are not influenced by the massless one. As a result, their motions are governed by the classical Kepler laws. If one further assumes that the primaries perform circular motion and that the third body is coplanar with them, one has the Restricted Planar Circular 3-Body Problem (RPC3BP).

Let us name the two primaries S (star) and P (planet). Normalizing their masses, we can assume that $m_S = 1-\mu$ and $m_P = \mu$, with $\mu \in \left(0, \frac{1}{2}\right]$. Since the primaries follow circular orbits, in rotating (usually also called synodic) coordinates, their positions can be fixed at $q_S = (\mu, 0)$ and $q_P = (\mu-1, 0)$. Then, denoting by $(q,p) \in \mathbb{R}^2 \times \mathbb{R}^2$ the position and momenta of the third body and taking appropriate units, the RPC3BP is a 2-degrees of freedom Hamiltonian system with respect to

$$h(q,p;\mu) = \frac{||p||^2}{2} - q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{(1-\mu)}{||q-(\mu,0)||} - \frac{\mu}{||q-(\mu-1,0)||}. \tag{I.1.1}$$

For $\mu > 0$, it is a well known fact that h has five equilibrium points: L_1, L_2, L_3, L_4 and L_5, called Lagrange points[1] (see Figure I.1.1(a)). On an inertial (non-rotating) system of coordinates, the Lagrange points correspond to periodic orbits with the same period as the two primaries, i.e they lie on a 1:1 mean motion resonance. The three collinear points with the primaries, L_1, L_2 and L_3, are of center-saddle type and, for small μ, the triangular ones, L_4 and L_5, are of center-center type (see for instance [Sze67]).

Since the points L_1 and L_2 are rather close to the small primary, their invariant manifolds have been widely studied for their interest in astrodynamics applications, (see [KLM⁺00; GLM⁺01; CGM⁺04]). The dynamics around the points L_4 and L_5 have also been considerably studied since, due to its stability, it is common to find objects orbiting around these points (for instance the Trojan and Greek Asteroids associated to the pair Sun-Jupiter, see [GDF⁺89; CG90; RG06]).

On the contrary, the invariant manifolds of the Lagrange point L_3 have received somewhat less attention. Still, they structure the dynamics in regions of the phase space of the RPC3BP. In particular, the horseshoe-shaped orbits that explain the orbits of Saturn satellites Janus and Epimetheus lie "close" to the invariant manifolds of L_3 (see [NPR20]). Moreover, the invariant manifolds of L_3 (more precisely

[1] For $\mu = 0$, the system has a circle of critical points (q,p) with $\|q\| = 1$ and $p = (p_1, p_2) = (-q_2, q_1)$.

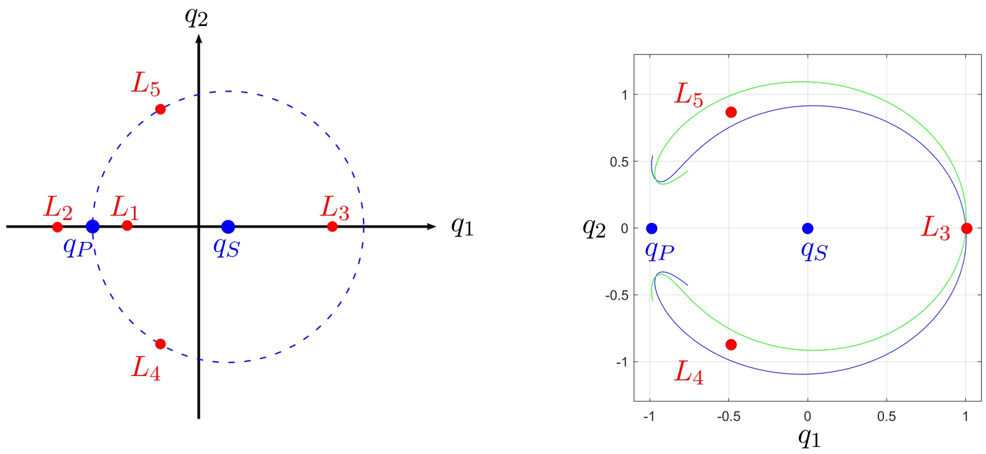


FIGURE I.1.1: (a) Projection onto the q-plane of the equilibrium points for the RPC3BP on rotating coordinates for $\mu > 0$. (b) Plot of the stable (green) and unstable (blue) manifolds of L_3 projected onto the q-plane for $\mu = 0.003$.

its center-stable and center-unstable manifolds) act as effective boundaries of the stability domains around $L_{4,5}$ (see in [SST13]). See Chapter II.1 in Part II for more references about the dynamics of the RPC3BP in a neighborhood of L_3 and its invariant manifolds.

The purpose of Part I and II is to study the invariant manifolds of L_3 and, particularly, show that they do not intersect for $0 < \mu \ll 1$ (at their first round).

I.1.1 The unstable and stable invariant manifolds of L_3

The eigenvalues of the the Lagrange point L_3 satisfy that

$$\mathrm{Spec} = \{\pm\sqrt{\mu}\,\rho(\mu), \pm i\,\omega(\mu)\}\,, \quad \text{with} \quad \begin{cases} \rho(\mu) = \sqrt{\frac{21}{8}} + \mathcal{O}(\mu), \\ \omega(\mu) = 1 + \frac{7}{8}\mu + \mathcal{O}(\mu^2), \end{cases} \tag{I.1.2}$$

as $\mu \to 0$ (see [Sze67]). Notice that, due to the different size in the eigenvalues, the system posseses two time scales, which translates to rapidly rotating dynamics coupled with a slow hyperbolic behavior around the critical point L_3.

The one dimensional unstable and stable invariant manifolds have two branches each (see Figure I.1.1(b)). One pair, which we denote by $W^{\mathrm{u},+}(\mu)$ and $W^{\mathrm{s},+}(\mu)$ circumvents L_5 whereas the other, denoted as $W^{\mathrm{u},-}(\mu)$ and $W^{\mathrm{s},-}(\mu)$, circumvents L_4. Since the Hamiltonian system associated to h in (I.1.1) is reversible with respect to the involution

$$\Phi(q,p) = (q_1, -q_2, -p_1, p_2), \tag{I.1.3}$$

the $+$ branches are symmetric to the $-$ ones. Thus, one can restrict the study to the first ones.

As already mentioned, the aim of the work in Parts I and II is to give an asymptotic formula for the distance between $W^{\mathrm{u},+}(\mu)$ and $W^{\mathrm{s},+}(\mu)$, for $0 < \mu \ll 1$ (in an appropriate transverse section). However, due to the rapidly rotating dynamics of the system (see (I.1.2)), the stable and unstable manifolds of L_3 are exponentially close to each other with respect to $\sqrt{\mu}$. This implies that the classical Melnikov Theory [Mel63] cannot be applied and that obtaining the asymptotic formula is a rather involved problem.

The precise statement for the asymptotic formula for the distance is properly stated in Part II. Let us give here a more informal statement. Consider the classical symplectic polar coordinates (r, θ, R, G), where R is the radial momentum and G the angular momentum, and the section $\Sigma = \{\theta = \pi/2, r > 1\}$. Then, if one denotes by P^{u} and P^{s} the first intersections of the invariant manifolds $W^{\mathrm{u},+}(\mu)$,$W^{\mathrm{s},+}(\mu)$ with Σ, the distance between these points, is given by

$$\mathrm{dist}_{\Sigma}(P^{\mathrm{u}}, P^{\mathrm{s}}) = \sqrt[3]{4}\,\mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log \mu|}\right)\right], \tag{I.1.4}$$

for $0 < \mu \ll 1$ and certain constants $A > 0$ and $\Theta \in \mathbb{C}$. The proof of this asymptotic formula spans both Parts I and II. In this first part, we perform the first steps towards the proof (see Section I.1.3 below). In particular, we obtain and describe the constants A and Θ appearing in (I.1.4).

A fundamental problem in dynamical systems is to prove that a model has chaotic dynamics (for instance a Smale Horseshoe). For many physically relevant problems, like those in Celestial Mechanics, this is usually a remarkably difficult problem. Certainly, the fact that the invariant manifolds of L_3 do not coincide does not lead to chaotic dynamics. However, one should expect the existence of Lyapunov periodic orbits which are exponentially close (with respect to $\sqrt{\mu}$) to L_3 and whose stable and unstable invariant manifolds intersect transversally. If so, the Smale-Birkhoff Theorem would imply the existence of a hyperbolic set whose dynamics is conjugated to that of the Bernoulli shift (in particular, with positive topological entropy) exponentially close to the invariant manifolds of L_3.

I.1.2 Exponentially small splitting of separatrices

Even though there is a standard theory to analyze the breakdown of homoclinic and heteroclinic connections, the so called Poincaré-Melnikov Theory (see [Mel63] and [GH83] for a more modern exposition), it cannot be applied to obtain (I.1.4) due to its exponential smallness. Indeed, the breakdown of homoclinic connections to L_3 fits into what is usually referred to as exponentially small splitting of separatrices problems. This *beyond all orders* phenomenon was first detected by Poincaré in [Poi90] when he studied the non integrability of the 3-Body Problem.

The first obtention of an asymptotic formula for an exponentially small splitting of separatrices did not appear until the 1980's with the pioneering work by Lazutkin for the *standard map* [Laz84; Laz05]. Even if Lazutkin's work was not complete (the complete proof was achieved by Gelfreich in [Gel99]), the ideas he developed have been very influential and have been the basis of many of the works in the field (and in particular of this work). Other methods to deal with exponentially small splitting of separatrices are Treschev's *continuous averaging* (see [Tre97]) or "direct" series methods (see [GGM99]).

Usually, the exponentially small splitting of separatrices problems are classified as *regular* or *singular*.

In the regular cases, even if Melnikov theory cannot be straightforwardly applied, the Melnikov function gives the leading term for the distance between the perturbed invariant manifolds. That is, Melnikov theory provides the first order for the distance but leads to too crude estimates for the higher order terms. This phenomenon has been studied in rapidly forced periodic or quasi-periodic perturbations of 1-degree of freedom Hamiltonian systems, see [HMS88; DS92; DS97; Gel97b; Gel00; BF04; BF05; GGS20], in close to the identity area preserving maps, see [DR98], and in Hamiltonian systems with two or more degrees of freedom which have hyperbolic tori with fast quasiperiodic dynamics [Sau01]. In particular, [GMS16] provides the first prove of exponentially small splitting of separatrices in a Celestial Mechanics problem (see also [GSM+17; GPS+21]).

In the singular cases, the exponentially small first order for the distance between the invariant manifolds is no longer given by the Melnikov function. Instead, one has to consider an auxiliary equation, usually called *inner equation*, which does not depend on the perturbative parameter and provides the first order for the distance. Some results on inner equations can be found on [Gel97a; GS01; OSS03; Bal06; BS08; BM12] and the application of the inner equation analysis to the original problem can be found in [BFG+12; Lom00; GOS10; GG10; MSS11; BCS13; BCS18a; BCS18b].

Due to the extreme sensitivity of the exponentially small splitting of separatrices phenomenon on the features of each particular model, most of the available results apply under quite restrictive hypothesis and, therefore, cannot be applied to analyze the invariant manifolds of L_3.

I.1.3 Strategy to obtain an asymptotic formula for the breakdown of the invariant manifolds of L_3

For the limit problem h in (I.1.1) with $\mu = 0$, the five Lagrange point "disappear" into the circle of (degenerate) critical points $\|q\| = 1$ and $p = (p_1, p_2) = (-q_2, q_1)$. As a consequence, the one-dimensional invariant manifolds of L_3 disappear when $\mu = 0$ too. For this reason, to analyze perturbatively these invariant manifolds, the first step is to perform a singular change of coordinates to obtain a "new first order" Hamiltonian which has a center saddle equilibrium point (close to L_3) with stable and unstable manifolds that coincide along a separatrix. To perform the change of coordinates we use the Poincaré planar elements (see [MO17]) plus a singular (with respect to μ) scaling.

In the following list, we present the main steps of our strategy to prove formula (I.1.4). We split the list in two. First we explain the steps performed in this part and later those carried out in Part II.

In this part, we complete the following steps:

A. We perform a change of coordinates which captures the slow-fast dynamics of the system. The new Hamiltonian becomes a (fast) oscillator weakly coupled to a 1-degree of freedom Hamiltonian with a saddle point and a separatrix associated to it.

B. We analyze the analytical continuation of a time-parametrization of the separatrix. In particular, we obtain its maximal strip of analyticity (centered at the real line). We also describe the character and location of the singularities at the boundaries of this region.

C. We derive the inner equation, which gives the first order of the original system close to the singularities of the separatrix described in Step B. This equation is independent of the perturbative parameter μ.

D. We study two special solutions of the inner equation which are approximations of the perturbed invariant manifolds near the singularities. Moreover, we provide an asymptotic formula for the difference between these two solutions of the inner equation. We follow the approach presented in [BS08].

In Part II we complete the following steps:

E. We prove the existence of the analytic continuation of suitable parametrizations of $W^{\mathrm{u},+}(\delta)$ and $W^{\mathrm{s},+}(\delta)$ in appropriate complex domains (and as graphs). These domains contain a segment of the real line and intersect a neighborhood sufficiently close to the singularities of the separatrix.

F. By using complex matching techniques, we compare the solutions of the inner equation with the graph parametrizations of the perturbed invariant manifolds.

G. Finally, we prove that the dominant term of the difference between manifolds is given by the term obtained from the difference of the solutions of the inner equation.

The structure of this part goes as follows. In Chapter I.2, we present the main results for the Steps A to D and introduce some heuristics to contextualize them. Chapters I.3-I.5 are devoted to the proof of the results in Chapter I.2.

The constants in the asymptotic formula for the distance. The constant A in (I.1.4) is given by the height of the maximal strip of analyticity of the unperturbed separatrix (see Step B). Therefore, to obtain its value, one has to compute the imaginary part of the singularities of the separatrix which are closer to the real line.

On all the previous mentioned works on splitting of separatrices, either the separatrix of the unperturbed model has an analytic expression (see for example [DS92; GG10; GOS10; BCS13; GMS16]) or otherwise certain properties of its analytic continuation are given as assumptions (see [DS97; Gel97a; BF04; BFG+12]). In this case, we do not have an explicit expression for the time-parameterization of the separatrix and, to obtain its complex singularities, we need to rely on techniques of analytical continuation to analyze them (see Section I.2.2). In particular, we describe the parametrization of the separatrix in terms of a multivalued function involving a complex integral and (see Theorem I.2.2 below) we obtain

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x}\sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}dx \approx 0.177744.$$

This value agrees with the numerical computations of the distance between the invariant manifolds given in [Fon93a; SST13].

Since we are in a singular case, the constant Θ in (I.1.4) is not correctly given by the Melnikov function but by the analysis of the inner equation of the system (see Step C above and also Sections I.2.3 and I.2.4). In particular, Θ corresponds to a Stokes constant and does not have a closed formula. By a numerical computation, we see that $|\Theta| \approx 1.63$ (see Remark I.2.8). We expect that, by means of a computer assisted proof, it would be possible to obtain rigorous estimates and verify that $|\Theta| \neq 0$, see [BCG+22].

Chapter I.2

Main results

We devote this chapter to state the main results concerning the Steps A, B, C and D explained in Section I.1.3. First, in Section I.2.1, we present the changes of coordinates involved to rewrite the Hamiltonian h in (I.1.1) as a singular perturbation problem given by a fast oscillator weakly coupled with a one degree of freedom Hamiltonian with a saddle point and a separatrix (Step A). In Section I.2.2, we consider the time-parametrization of the separatrix and analyze the properties of its analytical continuation (Step B). In Section I.2.3 we give some heuristic ideas regarding the parametrization of the perturbed manifolds on certain complex domains (Step E) and deduce the singular change of variables which leads to the inner equation (Step C). Finally, in Section I.2.4, we present the study of certain solutions of the inner equation and give an asymptotic formula for their difference (Step D).

I.2.1 A singular perturbation formulation of the problem

When studying a close to integrable Hamiltonian system at a resonance, it is usual to “blow-up” the “resonant zone” to capture the slow-fast time scales. In this section we present the singular change of coordinates which transforms the Hamiltonian h in (I.1.1) into a pendulum-like Hamiltonian plus a fast oscillator with a small coupling, namely

$$H(\lambda, \Lambda, x, y) = -\frac{3}{2}\Lambda^2 + V(\lambda) + \frac{xy}{\sqrt{\mu}} + o(1),$$

with respect to the symplectic form $d\lambda \wedge d\Lambda + i dx \wedge dy$. In these coordinates, the first order of the Hamiltonian has a saddle in the (λ, Λ)–plane and a center in the (x, y)–plane. Notice that the system possesses two time scales (~ 1 and $\sim 1/\sqrt{\mu}$). Recall that this two time scales are also present in the eigenvalues of L_3 in (I.1.2).

We consider Poincaré coordinates for the RPC3BP (see (I.1.1)) in order to write the system as a close to integrable Hamiltonian system and decouple (at first order) the saddle and the center behaviour. To this end, we first consider the symplectic polar and Delaunay coordinates.

Polar Coordinates. Let us consider the change of coordinates:

$$\phi_{\text{pol}} : (r, \theta, R, G) \mapsto (q, p),$$

where r is the radius, θ the argument of q, R the linear momentum in the r direction and G is the angular momentum. Then, the Hamiltonian (I.1.1), becomes

$$H^{\text{pol}} = H_0^{\text{pol}} + \mu H_1^{\text{pol}}, \tag{I.2.1}$$

where

$$\begin{aligned} H_0^{\mathrm{pol}}(r,R,G) =& \frac{1}{2}\left(R^2+\frac{G^2}{r^2}\right)-\frac{1}{r}-G, \\ \mu H_1^{\mathrm{pol}}(r,\theta;\mu) =& \frac{1}{r}-\frac{1-\mu}{\sqrt{r^2-2\mu r\cos\theta+\mu^2}} \\ & -\frac{\mu}{\sqrt{r^2+2(1-\mu)r\cos\theta+(1-\mu)^2}}. \end{aligned} \tag{I.2.2}$$

The critical point L_3 (see [Sze67] for the details) satisfies that, as $\mu \to 0$,

$$(r,\theta,R,G)=(d_\mu,0,0,d_\mu^2), \qquad \text{with} \quad d_\mu = 1+\frac{5}{12}\mu+\mathcal{O}(\mu^3). \tag{I.2.3}$$

Delaunay coordinates. The Delaunay elements, denoted $(\ell, L, \hat{g}, G)$, are action–angle variables for the 2-Body Problem (for negative energy) in non-rotating coordinates. The variable ℓ is the mean anomaly, $\hat{g}$ is the argument of the pericenter, L is the square root of the semi major axis and G is the angular momentum, (see [MO17]).

Let us introduce some formulae to describe these elements from the non-rotating polar coordinates $(r,\hat{\theta},R,G)$, namely $\hat{\theta}=\theta+t$. The action L is defined by

$$-\frac{1}{2L^2}=\frac{1}{2}\left(R^2+\frac{G^2}{r^2}\right)-\frac{1}{r},$$

and the (osculating) eccentricity of the body is expressed as

$$e=\sqrt{1-\frac{G^2}{L^2}}=\frac{\sqrt{(L-G)(L+G)}}{L}. \tag{I.2.4}$$

Let us recall now the "anomalies": the three angular parameters that define a position at the (osculating) ellipse. These are the mean anomaly ℓ, the eccentric anomaly u, and the true anomaly f, which satisfy

$$r=L^2(1-e\cos u) \qquad \text{and} \qquad \hat{\theta}=f+\hat{g}. \tag{I.2.5}$$

To use these elements in a rotating frame, we consider rotating Delaunay coordinates (ℓ,L,g,G), where the new angle is defined as $g=\hat{g}-t$ (the argument of the pericenter with respect to the line defined by the primaries S and J). As a result,

$$\theta=f+g, \tag{I.2.6}$$

and the unperturbed Hamiltonian H_0^{pol} becomes

$$H_0^{\mathrm{pol}}=-\frac{1}{2L^2}-G.$$

The eccentric and true anomalies are related by

$$\cos f=\frac{\cos u-e}{1-e\cos u}, \qquad \sin f=\frac{\sqrt{1-e^2}\sin u}{1-e\cos u}, \tag{I.2.7}$$

and the mean anomaly u is given by Kepler's equation

$$u-e\sin u=\ell. \tag{I.2.8}$$

The critical point L_3 (see (I.2.3)) satisfies $\theta = \ell + g = 0$ and

$$L = \sqrt{\frac{d_\mu}{2 - d_\mu^3}} = 1 + \mathcal{O}(\mu), \qquad G = d_\mu^2 = 1 + \mathcal{O}(\mu), \qquad L - G = \mathcal{O}(\mu^2). \tag{I.2.9}$$

Note that the Delaunay coordinates are not well defined for circular orbits ($e = 0$), since the pericenter, and as a consequence the angle g, are not well defined.

Poincaré coordinates. To "blow-down" the singularity of the Delaunay coordinates at circular motions, we use the classical Poincaré coordinates, which can be expressed by means of (rotating) Delaunay variables. Let us define

$$\phi_{\mathrm{Poi}} : (\lambda, L, \eta, \xi) \mapsto (r, \theta, R, G), \tag{I.2.10}$$

given by

$$\lambda = \ell + g, \qquad \eta = \sqrt{L - G}e^{ig}, \qquad \xi = \sqrt{L - G}e^{-ig}. \tag{I.2.11}$$

These coordinates are symplectic and analytic. Moreover, even though they are defined through the Delaunay variables, they are also analytic when the eccentricity tends to zero (i.e at $L = G$), see [MO17; Féj13].

The Hamiltonian equation associated to (I.2.1), expressed in Poincaré coordinates, defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge dL + i\, d\eta \wedge d\xi$ and

$$H^{\mathrm{Poi}} = H_0^{\mathrm{Poi}} + \mu H_1^{\mathrm{Poi}},$$

where

$$H_0^{\mathrm{Poi}}(L, \eta, \xi) = -\frac{1}{2L^2} - L + \eta\xi \qquad \text{and} \qquad H_1^{\mathrm{Poi}} = H_1^{\mathrm{pol}} \circ \phi_{\mathrm{Poi}}. \tag{I.2.12}$$

In Poincaré coordinates, the critical point L_3, as given in (I.2.9), satisfies

$$\lambda = 0, \qquad (L, \eta, \xi) = (1, 0, 0) + \mathcal{O}(\mu).$$

The linearized part of the vector field associated to this point has, at first order, an uncoupled nilpotent and center blocks,

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} + \mathcal{O}(\mu).$$

The center is found on the projection to coordinates (η, ξ) and the degenerate behavior on the projection to (λ, L).

The perturbative term μH_1^{Poi} is not explicit. We overcome this problem by computing the first terms of the series of μH_1^{Poi} in powers of (η, ξ), (see Lemma I.4.1).

Notice that, on the original coordinates, Hamiltonian h (see (I.1.1)) is analytic at points away from collision with the primaries. However, the collisions are not as easily defined in Poincaré coordinates.

A singular scaling. We consider the parameter

$$\delta = \mu^{\frac{1}{4}}$$

and we define the symplectic scaling

$$\phi_{\mathrm{sc}} : (\lambda, \Lambda, x, y) \mapsto (\lambda, L, \eta, \xi), \qquad L = 1 + \delta^2\Lambda, \quad \eta = \delta x, \quad \xi = \delta y, \tag{I.2.13}$$

and the time reparameterization $t = \delta^{-2}\tau$. The transformed equations are Hamiltonian with respect to the Hamiltonian

$$\delta^{-4}\left(H_0^{\mathrm{Poi}} \circ \phi_{\mathrm{sc}}\right) + \left(H_1^{\mathrm{Poi}} \circ \phi_{\mathrm{sc}}\right)$$

and the symplectic form $d\lambda \wedge d\Lambda + i dx \wedge dy$. The Hamiltonian (up to a constant) satisfies

$$\begin{aligned} \delta^{-4}\left(H_0^{\mathrm{Poi}} \circ \phi_{\mathrm{sc}}\right) &= -\frac{3}{2}\Lambda^2 + \frac{1}{\delta^4} F_{\mathrm{p}}(\delta^2\Lambda) + \frac{xy}{\delta^2}, \\ \left(H_1^{\mathrm{Poi}} \circ \phi_{\mathrm{sc}}\right) &= V(\lambda) + \mathcal{O}(\delta), \end{aligned}$$

with

$$\begin{aligned} V(\lambda) &= H_1^{\mathrm{Poi}}(\lambda, 1, 0, 0; 0), \\ F_{\mathrm{p}}(z) &= \left(-\frac{1}{2(1+z)^2} - (1+z)\right) + \frac{3}{2} + \frac{3}{2}z^2 = \mathcal{O}(z^3). \end{aligned} \tag{I.2.14}$$

The function $V(\lambda)$, which we call the potential, has an explicit formula:

$$V(\lambda) = H_1^{\mathrm{pol}}(1, \lambda; 0) = 1 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}}, \tag{I.2.15}$$

where H_1^{pol} is defined in (I.2.2). Indeed, taking $\delta = 0$ on the change of coordinates (I.2.13), we have that $(\lambda, L, \eta, \xi) = (\lambda, 1, 0, 0)$. These coordinates, correspond with a circular orbit, $e = 0$, and applying (I.2.5) and (I.2.6), we obtain that $(r, \theta) = (1, \lambda)$.

We summarize the previous results in the following theorem.

Theorem I.2.1. *The Hamiltonian system given by h in (I.1.1) expressed in coordinates (λ, Λ, x, y) defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge d\Lambda + i dx \wedge dy$ and the Hamiltonian*

$$H = H_{\mathrm{p}} + H_{\mathrm{osc}} + H_1, \tag{I.2.16}$$

with

$$H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + V(\lambda), \qquad H_{\mathrm{osc}}(x, y; \delta) = \frac{xy}{\delta^2}, \tag{I.2.17}$$

$$H_1(\lambda, \Lambda, x, y; \delta) = \left(H_1^{\mathrm{Poi}} \circ \phi_{\mathrm{sc}}\right) - V(\lambda) + \frac{1}{\delta^4} F_{\mathrm{p}}(\delta^2\Lambda), \tag{I.2.18}$$

and H_1^{Poi}, F_{p} and V defined in (I.2.12), (I.2.14) and (I.2.15), respectively. Moreover, the Hamiltonian H is real-analytic[1] *away from collision with the primaries.*

Moreover, for $\delta > 0$ small enough:

- *The critical point L_3 (see (I.2.3)) expressed in coordinates (λ, Λ, x, y) is given by*

$$\mathfrak{L}(\delta) = \left(0, \delta^2\mathfrak{L}_\Lambda(\delta), \delta^3\mathfrak{L}_x(\delta), \delta^3\mathfrak{L}_y(\delta)\right), \tag{I.2.19}$$

[1]Real-analytic in the sense of $\overline{H(\lambda, \Lambda, x, y; \delta)} = H(\overline{\lambda}, \overline{\Lambda}, y, x; \overline{\delta})$.

with $|\mathfrak{L}_\Lambda(\delta)|$, $|\mathfrak{L}_x(\delta)|$, $|\mathfrak{L}_y(\delta)| \leq C$, for some constant $C > 0$ independent of δ.

- *The point $\mathfrak{L}(\delta)$ is a saddle-center equilibrium point and its linear part is*

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ -\frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix} + \mathcal{O}(\delta).$$

Therefore, it possesses one-dimensional stable and unstable manifolds and a two- dimensional center manifold.

The proof of Theorem I.2.1 follows from the results obtained through Section I.2.1.

Since the original Hamiltonian is symmetric with respect to the involution Φ in (I.1.3), the Hamiltonian H is reversible with respect to the involution

$$\widetilde{\Phi}(\lambda, \Lambda, x, y) = (-\lambda, \Lambda, y, x). \tag{I.2.20}$$

From now on, we consider as "new" unperturbed Hamiltonian

$$H_0(\lambda, \Lambda, x, y; \delta) = H_{\mathrm{p}}(\lambda, \Lambda) + H_{\mathrm{osc}}(x, y; \delta), \tag{I.2.21}$$

which corresponds to an uncoupled pendulum-like Hamiltonian H_{p} and an oscillator H_{osc}, and we refer to H_1 as the perturbation.

I.2.2 The Hamiltonian H_{p} and its separatrices

In this section we analyze the 1-degree of freedom Hamiltonian $H_{\mathrm{p}}(\lambda, \Lambda)$ introduced in (I.2.17),

$$H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + V(\lambda), \qquad V(\lambda) = 1 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}},$$

and the associated Hamiltonian system

$$\dot{\lambda} = -3\Lambda, \qquad \dot{\Lambda} = -\sin\lambda \left(1 - \frac{1}{(2 + 2\cos\lambda)^{\frac{3}{2}}}\right), \tag{I.2.22}$$

This Hamiltonian system has a singularity at $\lambda = \pi$, which corresponds to the collision with the small primary P, and a saddle at $(\lambda, \Lambda) = (0, 0)$ with two homoclinic connections or separatrices, see Figure I.2.1. From now on, we only consider the separatrix on the right; by symmetry (see (I.2.20)), the results obtained below are analogous for the separatrix on the left.

We consider the real-analytic time parametrization of the separatrix,

$$\begin{aligned} \sigma : \mathbb{R} &\to \mathbb{T} \times \mathbb{R} \\ t &\mapsto \sigma(t) = (\lambda_h(t), \Lambda_h(t)), \end{aligned} \tag{I.2.23}$$

with initial condition

$$\sigma(0) = (\lambda_0, 0) \quad \text{where} \quad \lambda_0 \in \left(\frac{2}{3}\pi, \pi\right).$$

Theorem I.2.2. *The real-analytic time parametrization σ in (I.2.23) satisfies:*

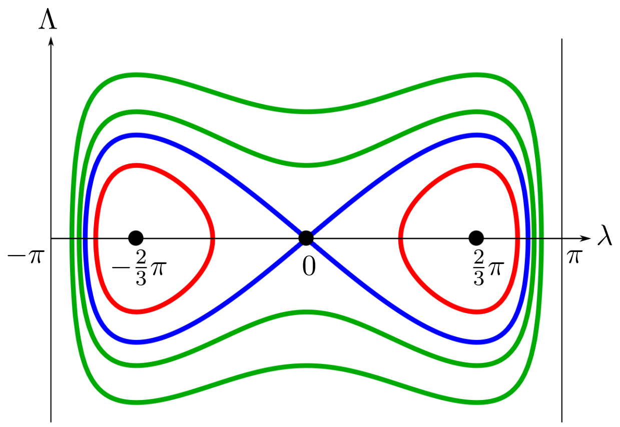


FIGURE I.2.1: Phase portrait of equation (I.2.22). On blue the two separatrices.

- *It extends analytically to the strip*

$$\Pi_A = \{t \in \mathbb{C} \ : \ |\mathrm{Im}\, t| < A\}, \tag{I.2.24}$$

where

$$A = \int_0^{a_+} \frac{1}{1-x} \sqrt{\frac{x}{3(x+1)(a_+ - x)(x - a_-)}} dx \approx 0.177744, \tag{I.2.25}$$

with $a_\pm = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$.

- *It has only two singularities in* $\partial \Pi_A$ *at* $t = \pm iA$.

- *There exists* $\nu > 0$ *such that, for* $t \in \mathbb{C}$ *with* $|t - iA| < \nu$ *and* $\arg(t - iA) \in (-\frac{3\pi}{2}, \frac{\pi}{2})$, $\sigma(t) = (\lambda_h(t), \Lambda_h(t))$ *can be expressed as*

$$\begin{aligned} \lambda_h(t) &= \pi + 3\alpha_+ (t - iA)^{\frac{2}{3}} + \mathcal{O}(t - iA)^{\frac{4}{3}}, \\ \Lambda_h(t) &= -\frac{2\alpha_+}{3} \frac{1}{(t - iA)^{\frac{1}{3}}} + \mathcal{O}(t - iA)^{\frac{1}{3}}, \end{aligned} \tag{I.2.26}$$

with $\alpha_+ \in \mathbb{C}$ *such that* $\alpha_+^3 = \frac{1}{2}$.
An analogous result holds for $|t + iA| < \nu$, $\arg(t + iA) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ *and* $\alpha_- = \overline{\alpha_+}$.

We can also describe the zeroes of $\Lambda_h(t)$ in $\overline{\Pi_A}$.

Proposition I.2.3. *Consider the real-analytic time parametrization* $\sigma(t) = (\lambda_h(t), \Lambda_h(t))$ *and the domain* Π_A *defined in* (I.2.23) *and* (I.2.24) *respectively. Then,* $\Lambda_h(t)$ *has only one zero in* $\overline{\Pi_A}$ *at* $t = 0$.

We can expand the region of analyticity of the time parametrization σ.

Corollary I.2.4. *There exists* $0 < \beta < \frac{\pi}{2}$ *such that the real-analytic time parametrization* $\sigma(t)$ *extends analytically to*

$$\begin{aligned} \Pi_{A,\beta}^{\mathrm{ext}} = &\{t \in \mathbb{C} \ : \ |\mathrm{Im}\, t| < \tan\beta \,\mathrm{Re}\, t + A\} \cup \\ &\{t \in \mathbb{C} \ : \ |\mathrm{Im}\, t| < -\tan\beta \,\mathrm{Re}\, t + A\}\,. \end{aligned} \tag{I.2.27}$$

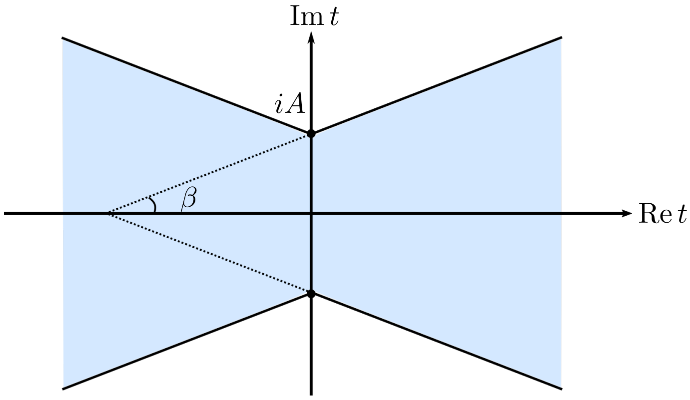


FIGURE I.2.2: Representation of the domain $\Pi^{\text{ext}}_{A,\beta}$ in (I.2.27).

(See Figure I.2.2). Moreover,

1. *σ has only two singularities on $\partial\Pi^{\text{ext}}_{A,\beta}$ at $t = \pm iA$.*

2. *Λ_h has only one zero in the closure of $\Pi^{\text{ext}}_{A,\beta}$ at $t = 0$.*

Proof. By [Fon95], there exists $T > 0$ such that $\sigma(t)$ is analytic in $\{|\mathrm{Re}\, t| > T\}$. Moreover, applying Theorem I.2.2, $\sigma(t)$ has two branching points at $t = \pm iA$ and can be expressed as in (I.2.26) in the domains

$$D_1 = \left\{|t - iA| < \nu,\ \arg(t - iA) \in \left(-\tfrac{3\pi}{2}, \tfrac{\pi}{2}\right)\right\} \bigcup \left\{|t + iA| < \nu,\ \arg(t + iA) \in \left(-\tfrac{\pi}{2}, \tfrac{3\pi}{2}\right)\right\},$$

for some $\nu > 0$. This implies that the only singularities in D_1 are at $t = \pm iA$.

Thus, we only need to check Item 1 in

$$\left(\Pi^{\text{ext}}_{A,\beta} \cap \{|\mathrm{Re}\, t| \leq T\}\right) \setminus D_1.$$

To this end, note that, by Theorem I.2.2, $\sigma(t)$ is analytic in the compact set $D_2 = \left(\overline{\Pi_A} \cap \{|\mathrm{Re}\, t| \leq T\}\right) \setminus D_1$. Therefore, there exists a cover of ∂D_2 by open balls centered in ∂D_2 where $\sigma(t)$ is analytic. Moreover, since ∂D_2 is compact, it has a finite subcover. This implies that there exists $\eta > 0$ such that we can extend the analyticity domain of $\sigma(t)$ to $(\Pi_{A+\eta} \cap \{|\mathrm{Re}\, t| \leq T\}) \setminus D_1$. In particular, taking $\beta = \arctan(\eta/T)$, $\sigma(t)$ is analytic in $(\Pi^{\text{ext}}_{A,\beta} \cap \{|\mathrm{Re}\, t| \leq T\}) \setminus D_1$.

The prove of Item 2 follows the same lines. □

I.2.3 Derivation of the inner equation

The inner equation associated to the Hamiltonian H in (I.2.16) describes the dominant behavior of suitable complex parametrizations of the invariant manifolds close to (one of) the singularities $\pm iA$ of the unperturbed separatrix. Let us explain how this equation arises from the Hamiltonian H.

First, we consider the translation of the equilibrium point $\mathfrak{L}(\delta)$ to the origin,

$$\phi_{\text{eq}} : (\lambda, \Lambda, x, y) \mapsto (\lambda, \Lambda, x, y) + \mathfrak{L}(\delta). \tag{I.2.28}$$

Second, to measure the distance of the stable and unstable manifolds, we parameterize them as graphs. In the unperturbed case, we know that the invariant manifolds coincide along the separatrix $(\lambda_h(t), \Lambda_h(t), 0, 0)$. Since we need to involve, in some sense, the time; we consider as a new independent variable u such that $\lambda_h(u) = \lambda$. Notice that $\dot{u} = 1$ for the unperturbed system. To this end, we consider the symplectic change of coordinates

$$\phi_{\rm sep} : (u, w, x, y) \to (\lambda, \Lambda, x, y), \qquad \lambda = \lambda_h(u), \quad \Lambda = \Lambda_h(u) - \frac{w}{3\Lambda_h(u)}, \tag{I.2.29}$$

where $\sigma = (\lambda_h, \Lambda_h)$ is the parametrization of the separatrix studied in Theorem I.2.2. Notice that, except for $u = 0$ (see Proposition I.2.3), the perturbed manifolds can be expressed as a graph and the change (I.2.29) is well defined.

The Hamiltonian H, written in these coordinates and after the translation $\phi_{\rm eq}$ in (I.2.28), becomes

$$H^{\rm sep} = H_0^{\rm sep} + H_1^{\rm sep}, \tag{I.2.30}$$

with

$$H_0^{\rm sep}(w, x, y) = w + \frac{xy}{\delta^2}, \qquad H_1^{\rm sep} = H \circ (\phi_{\rm eq} \circ \phi_{\rm sep}) - H_0^{\rm sep}.$$

Since we look for the perturbed manifolds as graphs with respect to u, we consider parametrizations

$$z^\diamond(u) = \left(w^\diamond(u), x^\diamond(u), y^\diamond(u)\right)^T, \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s},$$

such that the unstable and stable invariant manifolds of H associated to $\mathfrak{L}(\delta)$ can be expressed as

$$W^\diamond = \left\{ \left(\lambda_h(u), \Lambda_h(u) - \frac{w^\diamond(u)}{3\Lambda_h(u)}, x^\diamond(u), y^\diamond(u) \right) + \mathfrak{L}(\delta) \right\}, \quad \text{for } \diamond = \mathrm{u}, \mathrm{s},$$

with u belonging to appropriate domains. The proof of existence of $z^{\rm u}$ and $z^{\rm s}$ defined in appropriate (complex) domains requires a significant amount of technicalities. We present this result in Part II, (see Section II.3.2).

Due to the slow-fast character of the system, to capture the asymptotic first order of the difference $\Delta z = z^{\rm u} - z^{\rm s}$, we need to give the main terms of this difference close to the singularities, concretely, up to distance of order δ^2. To this end, we derive the inner equation, see [Bal06; BS08], which contains the first order of the Hamiltonian $H^{\rm sep}$ (see (I.2.30)) close to (one of) the singularities and is independent of the small parameter δ. That is, we look, for instance, for a Hamiltonian which is a good approximations of $H^{\rm sep}$ in a neighborhood of $u = iA$. Here, we focus on a domain near the singularity $u = iA$, but a similar analysys can be done near $u = -iA$.

Since we need to control the difference up to distance of order δ^2 of the singularity $u = iA$, we consider U such that

$$u - iA = \delta^2 U.$$

Notice that we can take $|U| \gg 1$ independent of δ. Close to the singularity $u = iA$, the homoclinic connection is not the dominant term of the perturbed invariant manifolds anymore. Let us be more precise, take $\Lambda = \Lambda_h(u) - \frac{w}{3\Lambda_h(u)}$, and recall that, by Theorem I.2.2, we have

$$\Lambda_h(u) \sim -\frac{2\alpha_+}{3}(u - iA)^{-\frac{1}{3}}, \quad \text{for} \quad |u - iA| \le \nu,$$

or equivalently,

$$\Lambda_h(iA+\delta^2 U) \sim -\frac{2\alpha_+}{3\delta^{\frac{2}{3}}U^{\frac{1}{3}}} \sim \mathcal{O}\left(\frac{1}{\delta^{\frac{2}{3}}U^{\frac{1}{3}}}\right).$$

Then,

$$w^\diamond(iA+\delta^2 U) \sim 3\Lambda_h^2(iA+\delta^2 U) \sim \mathcal{O}\left(\frac{1}{\delta^{\frac{4}{3}}U^{\frac{2}{3}}}\right). \tag{I.2.31}$$

In addition, the unperturbed Hamiltonian must have all of its terms of the same order. Therefore,

$$\frac{x^\diamond(iA+\delta^2 U)\, y^\diamond(iA+\delta^2 U)}{\delta^2} \sim \mathcal{O}\left(\frac{1}{\delta^{\frac{4}{3}}U^{\frac{2}{3}}}\right). \tag{I.2.32}$$

By symmetry, $x^\diamond(iA+\delta^2 U)$, $y^\diamond(iA+\delta^2 U) \sim \mathcal{O}(\delta^{\frac{1}{3}}U^{-\frac{1}{3}})$.

To avoid the dependence on the inner equation with respect to α_+ (see Theorem I.2.2) and to keep the symplectic character, we perform the scaling

$$\phi_{\mathrm{in}} : (U,W,X,Y) \to (u,w,x,y), \tag{I.2.33}$$

given by

$$U = \frac{u-iA}{\delta^2}, \qquad W = \delta^{\frac{4}{3}}\frac{w}{2\alpha_+^2}, \qquad X = \frac{x}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_+}, \qquad Y = \frac{y}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_+},$$

and the time scaling $\tau = \delta^2 t$. The heuristics above lead us to assume that $(U,W,X,Y) = \mathcal{O}(1)$ when $u - iA = \mathcal{O}(\delta^2)$. In the following proposition, by applying the change of coordinates ϕ_{in}, we obtain the inner equation of the Hamiltonian H^{sep}.

Proposition I.2.5. *The Hamiltonian equations associated to (I.2.30) expressed in inner coordinates (see (I.2.33)) are Hamiltonian with respect to*

$$H^{\mathrm{in}} = \mathcal{H} + H_1^{\mathrm{in}},$$

where

$$\mathcal{H}(U,W,X,Y) = H^{\mathrm{in}}(U,W,X,Y;\delta)\big|_{\delta=0} = W + XY + \mathcal{K}(U,W,X,Y), \tag{I.2.34}$$

with

$$\mathcal{K}(U,W,X,Y) = -\frac{3}{4}U^{\frac{2}{3}}W^2 - \frac{1}{3U^{\frac{2}{3}}}\left(\frac{1}{\sqrt{1+\mathcal{J}(U,W,X,Y)}} - 1\right) \tag{I.2.35}$$

and

$$\begin{aligned}\mathcal{J}(U,W,X,Y) = {}& \frac{4W^2}{9U^{\frac{2}{3}}} - \frac{16W}{27U^{\frac{4}{3}}} + \frac{16}{81U^2} + \frac{4(X+Y)}{9U}\left(W - \frac{2}{3U^{\frac{2}{3}}}\right) \\ & - \frac{4i(X-Y)}{3U^{\frac{2}{3}}} - \frac{X^2+Y^2}{3U^{\frac{4}{3}}} + \frac{10XY}{9U^{\frac{4}{3}}}.\end{aligned} \tag{I.2.36}$$

Moreover, if $c_1^{-1} \le |U| \le c_1$ and $|(W,X,Y)| \le c_2$ for some $c_1 > 1$ and $0 < c_2 < 1$, we have that there exist $b_0, \gamma_1, \gamma_2 > 0$ independent of δ, c_1, c_2 such that

$$\left|H_1^{\mathrm{in}}(U,W,X,Y;\delta)\right| \le b_0 c_1^{\gamma_1} c_2^{\gamma_2} \delta^{\frac{4}{3}}.$$

Remark I.2.6. *The change of coordinates* (I.2.33) *allows us to study an approximation of the invariant manifolds* $z^{\mathrm{u,s}}(u)$ *near the singularity* $u = iA$. *To obtain an approximation near* $u = -iA$, *one can proceed analogously by*

$$U = \frac{u + iA}{\delta^2}, \qquad W = \delta^{\frac{4}{3}} \frac{w}{2\alpha_-^2}, \qquad X = \frac{x}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_-}, \qquad Y = \frac{y}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_-},$$

where $\alpha_- = \overline{\alpha_+}$, *(see Theorem I.2.2).*

I.2.4 The solutions of the inner equation and their difference

We devote this section to study two special solutions of the inner equation given by the Hamiltonian $\mathcal{H}$ in (I.2.34). We introduce $Z = (W, X, Y)$ and the matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}. \tag{I.2.37}$$

Then, the equation associated to the Hamiltonian $\mathcal{H}$ can be written as

$$\left\{ \begin{array}{l} \dot{U} = 1 + g(U, Z), \\ \dot{Z} = \mathcal{A}Z + f(U, Z), \end{array} \right. \tag{I.2.38}$$

where $f = (-\partial_U \mathcal{K}, i\partial_Y \mathcal{K}, -i\partial_X \mathcal{K})^T$ and $g = \partial_W \mathcal{K}$.

We look for solutions of this equation parametrized as graphs with respect to U, namely we look for functions

$$Z_0^\diamond(U) = \left(W_0^\diamond(U), X_0^\diamond(U), Y_0^\diamond(U)\right)^T, \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s},$$

satisfying the invariance condition given by (I.2.38), that is

$$\partial_U Z_0^\diamond = \mathcal{A} Z_0^\diamond + \mathcal{R}[Z_0^\diamond], \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s}, \tag{I.2.39}$$

where

$$\mathcal{R}[\varphi](U) = \frac{f(U, \varphi) - g(U, \varphi)\mathcal{A}\varphi}{1 + g(U, \varphi)}. \tag{I.2.40}$$

In order to "select" the solutions we are interested in, we point out that, since we need some uniformity with respect to δ and $U = \delta^{-2}(u - iA)$, then $\operatorname{Re} U \to \pm\infty$ as $\delta \to 0$, depending on the sign of $\operatorname{Re} U$. Then, according to (I.2.31) and (I.2.32), we deduce that $(W, X, Y) \to 0$ as $\operatorname{Re} U \to \pm\infty$. For that reason, we look for $Z_0^\diamond$ satisfying the asymptotic conditions

$$\lim_{\operatorname{Re} U \to -\infty} Z_0^{\mathrm{u}}(U) = 0, \qquad \lim_{\operatorname{Re} U \to +\infty} Z_0^{\mathrm{s}}(U) = 0. \tag{I.2.41}$$

In fact, for a fixed $\beta_0 \in \left(0, \frac{\pi}{2}\right)$, we look for functions Z_0^{u} and Z_0^{s} satisfying (I.2.39), (I.2.41) defined in the domains

$$\mathcal{D}_\kappa^{\mathrm{u}} = \left\{ U \in \mathbb{C} : \, |\operatorname{Im} U| \geq \tan\beta_0 \operatorname{Re} U + \kappa \right\}, \qquad \mathcal{D}_\kappa^{\mathrm{s}} = -\mathcal{D}_\kappa^{\mathrm{u}}, \tag{I.2.42}$$

respectively, for some $\kappa > 0$ big enough (see Figure I.2.3). We analyze the the

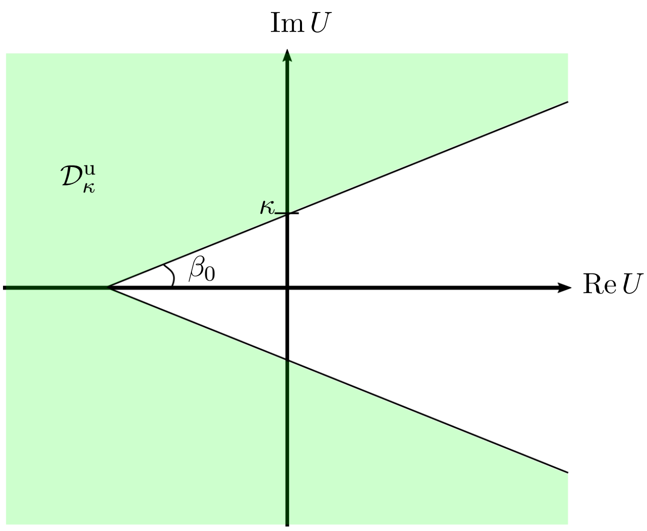


FIGURE I.2.3: The inner domain, $\mathcal{D}^{\mathrm{u}}_\kappa$, for the unstable case.

difference $\Delta Z_0 = Z_0^{\mathrm{u}} - Z_0^{\mathrm{s}}$ in the overlapping domain

$$\mathcal{E}_\kappa = \mathcal{D}^{\mathrm{u}}_\kappa \cap \mathcal{D}^{\mathrm{s}}_\kappa \cap \{U \in \mathbb{C} \,:\, \operatorname{Im} U < 0\}\,. \tag{I.2.43}$$

Theorem I.2.7. *There exist $\kappa_0, b_1 > 0$ such that for any $\kappa \geq \kappa_0$, the equation (I.2.39) has analytic solutions $Z_0^\diamond(U) = (W_0^\diamond(U), X_0^\diamond(U), Y_0^\diamond(U))^T$, for $U \in \mathcal{D}^\diamond_\kappa$, $\diamond = \mathrm{u}, \mathrm{s}$, satisfying*

$$|U^{\frac{8}{3}} W_0^\diamond(U)| \leq b_1, \qquad |U^{\frac{4}{3}} X_0^\diamond(U)| \leq b_1, \qquad |U^{\frac{4}{3}} Y_0^\diamond(U)| \leq b_1. \tag{I.2.44}$$

In addition, there exist $\Theta \in \mathbb{C}$ and $b_2 > 0$ independent of κ, and a function $\chi = (\chi_1, \chi_2, \chi_3)^T$ such that

$$\Delta Z_0(U) = Z_0^{\mathrm{u}}(U) - Z_0^{\mathrm{s}}(U) = \Theta e^{-iU}\Big((0,0,1)^T + \chi(U)\Big), \tag{I.2.45}$$

and, for $U \in \mathcal{E}_\kappa$,

$$|U^{\frac{7}{3}} \chi_1(U)| \leq b_2, \qquad |U^2 \chi_2(U)| \leq b_2, \qquad |U \chi_3(U)| \leq b_2.$$

Remark I.2.8. *This theorem implies that $\Theta = \lim_{\operatorname{Im} U \to -\infty} \Delta Y_0(U) e^{iU}$. Thus, we can obtain a numerical approximation of the constant Θ. Indeed, for $\rho > \kappa_0$, we can define*

$$\Theta_\rho = |\Delta Y_0(-i\rho)|\, e^{\rho}, \tag{I.2.46}$$

which, for ρ big enough, satisfies $\Theta_\rho \approx |\Theta|$.

To compute $\Delta Y_0(-i\rho) = Y_0^{\mathrm{u}}(-i\rho) - Y_0^{\mathrm{s}}(-i\rho)$, we first look for good approximations of $Z_0^{\mathrm{u}}(U)$ for $\operatorname{Re} U \ll -1$ and of $Z_0^{\mathrm{s}}(U)$ for $\operatorname{Re} U \gg 1$, as power series in $U^{-\frac{1}{3}}$. One can easily check that $Z_0^{\mathrm{u}}(U)$ as $\operatorname{Re} U \to -\infty$ and $Z_0^{\mathrm{s}}(U)$ as $\operatorname{Re} U \to +\infty$ have the *same asymptotics expansion:*

$$\begin{aligned} W_0^\diamond(U) &= \frac{4}{243\,U^{\frac{8}{3}}} - \frac{172}{2187\,U^{\frac{14}{3}}} + \mathcal{O}\left(U^{-\frac{20}{3}}\right), \\ X_0^\diamond(U) &= -\frac{2i}{9\,U^{\frac{4}{3}}} + \frac{28}{81\,U^{\frac{7}{3}}} + \frac{20i}{27\,U^{\frac{10}{3}}} - \frac{16424}{6561\,U^{\frac{13}{3}}} + \mathcal{O}\left(U^{-\frac{16}{3}}\right), \\ Y_0^\diamond(U) &= \frac{2i}{9\,U^{\frac{4}{3}}} + \frac{28}{81\,U^{\frac{7}{3}}} - \frac{20i}{27\,U^{\frac{10}{3}}} - \frac{16424}{6561\,U^{\frac{13}{3}}} + \mathcal{O}\left(U^{-\frac{16}{3}}\right). \end{aligned}$$

We use these expressions to set up the initial conditions for the numerical integration for computing $\Delta Y_0(-i\rho)$. We take as initial points the value of the truncated power series at order $U^{-\frac{13}{3}}$ at $U = 1000 - i\rho$ (for $\diamond = \mathrm{s}$) and $U = -1000 - i\rho$ (for $\diamond = \mathrm{u}$). (See Table I.2.1). We perform the numerical integration for different values of $\rho \leq 23$ and an integration solver with tolerance 10^{-12}.

Table I.2.1 shows that the constant Θ is approximately 1.63 which indicates that it is not zero. We expect that this computation method can be implemented rigorously [BCG+22].

ρ	$\lvert\Delta Y_0(-i\rho)\rvert$	e^ρ	Θ_ρ
13	$3.7 \cdot 10^{-6}$	$4.4 \cdot 10^{5}$	1.6373
14	$1.4 \cdot 10^{-6}$	$1.2 \cdot 10^{6}$	1.6361
15	$5.0 \cdot 10^{-7}$	$3.3 \cdot 10^{6}$	1.6351
16	$1.8 \cdot 10^{-7}$	$8.9 \cdot 10^{6}$	1.6341
17	$3.7 \cdot 10^{-8}$	$2.4 \cdot 10^{7}$	1.6333
18	$6.8 \cdot 10^{-8}$	$6.6 \cdot 10^{7}$	1.6326
19	$9.1 \cdot 10^{-9}$	$1.8 \cdot 10^{8}$	1.6320
20	$3.4 \cdot 10^{-9}$	$4.9 \cdot 10^{8}$	1.6315
21	$1.2 \cdot 10^{-9}$	$1.3 \cdot 10^{9}$	1.6312
22	$4.6 \cdot 10^{-10}$	$3.6 \cdot 10^{9}$	1.6313
23	$1.7 \cdot 10^{-10}$	$9.7 \cdot 10^{9}$	1.6323

TABLE I.2.1: Computation of Θ_ρ, as defined in (I.2.46), for different values of $\rho \leq 23$.

Chapter I.3

Analytic continuation of the separatrix

In this chapter we prove Theorem I.2.2 and Proposition I.2.3, which deal with the study of the complex singularities and zeroes of the analytic extension of the time-parametrization $\sigma(t) = (\lambda_h(t), \Lambda_h(t))$ of the homoclinic connection given in (I.2.23).

Let us recall that $\sigma(t)$ is a solution of the Hamiltonian system H_{p} in (I.2.17) and it is found at the energy level $H_{\mathrm{p}} = -\frac{1}{2}$. Therefore,

$$(\dot{\lambda})^2 = 15 - 12\cos^2\left(\frac{\lambda}{2}\right) - \frac{3}{\cos\left(\frac{\lambda}{2}\right)}. \tag{I.3.1}$$

Equation (I.3.1) can be solved as $t = F(\lambda)$, where F is a function defined by means of an integral. Prove Theorem I.2.2 and Proposition I.2.3 boils down to studying the analytic continuation of F^{-1}.

We divide the proof of Theorem I.2.2 into three main steps. First, in Section I.3.1, we perform the change of variables $q = \cos(\frac{\lambda}{2})$ and rephrase Theorem I.2.2 in terms of $q(t)$ (Theorem I.3.1). Then, in Section I.3.2, we analyze all the possibles types of singularities that $q(t)$ may have (Proposition I.3.2), which turn out to be poles or branching points. In addition we prove that all the singularities have to be given by integrals along suitable complex paths. Finally, in Section I.3.3, taking into account all complex paths leading to singularities, we prove that the singularities of $q(t)$ with smaller imaginary part (in the first Riemann sheet of $q(t)$) are $t = \pm iA$.

Finally, in Section I.3.4, we use the results obtained in the previous sections, to analyze the zeroes of $\Lambda_h(t)$ in the strip of analyticity Π_A (see (I.2.24)), thus proving Proposition I.2.3.

In order to simplify the notation, through the rest of the chapters we denote by C any positive constant independent of t.

I.3.1 Reformulation of Theorem I.2.2

To prove Theorem I.2.2, it is more convenient to work with the variable $q = \cos\left(\frac{\lambda}{2}\right)$ instead of λ. Notice that this change of coordinates, when restricted to $\lambda \in (0, \pi)$, is a diffeomorphism.

Theorem I.3.1. *Consider the real-analytic time parametrization $\sigma(t) = (\lambda_h(t), \Lambda_h(t))$ introduced in (I.2.23) and denote $a_\pm = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$. Then, $q(t) = \cos\left(\frac{\lambda_h(t)}{2}\right)$ satisfies*

$$q(t) \in [a_+, 1) \quad \text{for} \quad t \in \mathbb{R}, \qquad q(0) = a_+, \tag{I.3.2}$$

and the differential equation

$$\dot{q}^2 = \frac{3}{q}(q-1)^2(q+1)(q-a_-)(q-a_+). \tag{I.3.3}$$

Moreover, we have that:

- *The function $q(t)$ extends analytically to the strip Π_A defined in (I.2.24).*
- *The function $q(t)$ has only two singularities on $\partial\Pi_A$ at $t = \pm iA$.*
- *There exists $\nu > 0$ such that, for $t \in \mathbb{C}$ with $|t - iA| < \nu$ and $\arg(t - iA) \in (-\frac{3\pi}{2}, \frac{\pi}{2})$, we have*

  $$q(t) = -\frac{3\alpha_+}{2}(t-iA)^{\frac{2}{3}} + \mathcal{O}(t-iA)^{\frac{4}{3}}, \tag{I.3.4}$$

  with $\alpha_+ \in \mathbb{C}$ such that $\alpha_+^3 = \frac{1}{2}$.

  An analogous result holds for $|t + iA| < \nu$ and $\arg(t + iA) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ with $\alpha_- = \overline{\alpha_+}$.

Theorem I.2.2 is a corollary of Theorem I.3.1.

Proof of Theorem I.2.2. To obtain Theorem I.2.2 from Theorem I.3.1 it is enough to prove that $\Lambda_h(t)$ has no singularities in $\overline{\Pi_A} \setminus \{\pm iA\}$ and that $(\lambda_h(t), \Lambda_h(t))$ can be expressed as in (I.2.26) close to $t = \pm iA$.

Since $\dot{\lambda}_h = -3\Lambda_h$ and using the change of coordinates $q = \cos(\frac{\lambda}{2})$, we have that

$$\Lambda_h^2(t) = \frac{4}{9}\frac{\dot{q}^2(t)}{(1-q^2(t))}. \tag{I.3.5}$$

We claim that, if $\Lambda_h(t)$ has a singularity at $t = t^*$, then $\Lambda_h^2(t)$ has a singularity at $t = t^*$ as well. Indeed, the only case when the previous affirmation could be false is if t^* is a branching point of order $\frac{k}{2}$ with $k \geq 1$ an odd natural number. In this case,

$$\lambda_h(t) = \lambda_h(t^*) + C(t-t^*)^{\frac{k}{2}-1}\left(1 + \mathcal{O}(t-t^*)^{\beta}\right), \qquad \text{when } 0 < |t - t^*| \ll 1,$$

for some $\beta > 0$. Replacing this expression in (I.3.1) and comparing orders we see that this case is not possible.

Thus, we proceed to prove that $\Lambda_h(t)$ has no singularities in $\overline{\Pi_A} \setminus \{\pm iA\}$. Let us assume it has. That is, there exists $t^* \in \overline{\Pi_A} \setminus \{\pm iA\}$ such that $\Lambda_h(t)$ is singular at $t = t^*$. Note that Theorem I.3.1 implies that $q(t)$ and $\dot{q}(t)$ are analytic in a neighborhood of $t^* \in \overline{\Pi_A} \setminus \{\pm iA\}$.

1. If $q^2(t^*) \neq 1$, $1/(1-q^2(t))$ is analytic for $0 < |t - t^*| \ll 1$. Since $\dot{q}(t)$ is also analytic in this neighborhood, (I.3.5) implies that $\Lambda_h^2(t)$ has no singularity at $t = t^*$ and we reach a contradiction.

2. If $q^2(t^*) = 1$, by (I.3.3) and (I.3.5), we deduce that

   $$\Lambda_h^2 = \frac{4}{3q}(1-q)(q-a_+)(q-a_-). \tag{I.3.6}$$

   Since by Theorem I.3.1 q is analytic in $\overline{\Pi_A} \setminus \{\pm iA\}$, then Λ_h^2 must be as well.

Finally, we notice that, by equations (I.3.4) and (I.3.5), we have

$$\Lambda_h^2(t) = \frac{4}{9}\alpha_\pm^2(t \mp iA)^{-\frac{2}{3}} + \mathcal{O}(1), \qquad \text{when } 0 < |t \mp iA| \ll 1.$$

Therefore, $\Lambda_h(t)$ has branching points of order $-\frac{1}{3}$ in $t = \pm iA$. Moreover, integrating the expression for $\Lambda_h(t)$ and applying that $q(t) = \cos(\frac{\lambda_h(t)}{2})$ (and (I.3.4)), it is immediate to see that $\lambda_h(t)$ has branching points of order $\frac{2}{3}$ at $t = \pm iA$ and can be expressed as in (I.2.26) close to $t = \pm iA$. □

We devote Sections I.3.2 and I.3.3 to prove Theorem I.3.1. The statements (I.3.2) and (I.3.3) are straightforward by applying the change of coordinates $q = \cos(\frac{\lambda}{2})$ to equation (I.3.1).

We divide the rest of the proof of Theorem I.3.1 into two parts. In Section I.3.2 we classify the singularities of $q(t)$ and introduce a way to compute them using integration in complex paths. Finally, in Section I.3.3 we prove that the singularities of $q(t)$ with smallest imaginary part are $t = \pm iA$ and are branching points of order $\frac{2}{3}$.

I.3.2 Classification of the singularities of $q(t)$

Equation (I.3.3) with initial condition $q(0) = a_+ = -\frac{1}{2} + \frac{\sqrt{2}}{2}$ is equivalent to

$$t = \int_{a_+}^{q(t)} f(s)ds, \qquad \text{for } t \in \mathbb{R},$$

where

$$f(q) = \frac{1}{q-1}\sqrt{\frac{q}{3(q+1)(q-a_+)(q-a_-)}},$$

is defined in $\mathbb{R} \setminus \{[a_-, -1] \cup (0, a_+] \cup \{1\}\}$ with $a_\pm = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$. From [Fon95], we know that there exist $\upsilon > 0$ such that $q(t)$ can be extended to the open complex strip

$$\Pi_\upsilon = \{t \in \mathbb{C} \ : \ |\mathrm{Im}\, t| < \upsilon\},$$

and $q(t)$ has singularities in $\partial\Pi_\upsilon$. Namely,

$$t = \int_{a_+}^{q(t)} f(q)dq, \qquad \text{for } t \in \Pi_\upsilon. \tag{I.3.7}$$

Since f is a multi-valued function in the complex plane, in order to analyze the possible values of $\int_{a_+}^{q(t)} f(s)ds$, we consider its complete analytic continuation. That is,

$$\begin{aligned} \hat{f} : \mathscr{R}_f &\to \mathbb{C} \\ \big(q; \arg \mathtt{g}(q)\big) &\mapsto \frac{\sqrt{\mathtt{g}(q)}}{q-1}, \end{aligned} \qquad \text{where} \quad \mathtt{g}(q) = \frac{q}{3(q+1)(q-a_+)(q-a_-)}, \tag{I.3.8}$$

and $\mathscr{R}_f$ is the Riemann surface associated to f. We define $\mathtt{p} : \mathscr{R}_f \to \mathbb{C}$ as the projection to the complex plane. We choose the first Riemann sheet to correspond to $\arg \mathtt{g}(q) \in (-\pi, \pi]$. Accordingly the second Riemann sheet corresponds to $(\pi, 3\pi]$.

To integrate $\hat{f}$ along a path $\gamma \subset \mathscr{R}_f$, we introduce the notation

$$\int_\gamma \hat{f}(q)dq = \int_{q_0}^{q_{\text{end}}} \hat{f}d\gamma = \int_{s_0}^{s_{\text{end}}} \hat{f}(\gamma(s))\gamma'(s)ds,$$

such that $\gamma : (s_0, s_{\text{end}}) \to \mathscr{R}_f$ where $\lim_{s\to s_0} \mathrm{p}\gamma(s) = q_0$ and $\lim_{s\to s_{\text{end}}} \mathrm{p}\gamma(s) = q_{\text{end}}$. Moreover, we assume that the paths $\gamma \subset \mathscr{R}_f$ are $\mathcal{C}^0$ and $\mathcal{C}^1$-piecewise. Therefore, by (I.3.7), we have that

$$t = \int_{a_+}^{q(t)} \hat{f}d\gamma, \qquad \text{for } t \in \Pi_\upsilon. \tag{I.3.9}$$

Now, for $q \in \mathscr{R}_f$ and an integration path $\gamma \subset \mathscr{R}_f$, we define the function G as the right hand side of (I.3.9),

$$G(q) = \int_{a_+}^{q} \hat{f}\, d\gamma.$$

Notice that for a given $q \in \mathscr{R}_f$, $G(q)$ may depend on the integration path, and therefore, G may be multi-valued on $\mathscr{R}_f$. However, by (I.3.9), G is single-valued when

$$t = G(q(t)), \qquad \text{for } t \in \Pi_\upsilon.$$

We use G to characterize and locate the singularities of $q(t)$. Indeed, if function $G(q)$ is biholomorphic at $q = q^*$, then $q(t)$ is analytic at a neighborhood of all values of t such that $q(t) = q^*$. Therefore, $q(t)$ may have singularities at $t = t^*$ when the hypothesis of the Inverse Function Theorem are not satisfied for G. That is, for $q(t^*) = q^*$ such that either

$$G'(q^*) = 0, \qquad G \notin \mathcal{C}^1 \text{ at } q = q^*, \qquad \text{or} \qquad |q^*| \to \infty. \tag{I.3.10}$$

Namely, when there exist q^* and $\gamma \subset \mathscr{R}_f$ satisfying (I.3.10), such that

$$t^* = G(q^*) = \int_{a_+}^{q^*} \hat{f}\, d\gamma. \tag{I.3.11}$$

Since G is a multivalued function, the values of t^* can, and in fact will, depend on the integration path on $\gamma \subset \mathscr{R}_f$. From (I.3.8) and (I.3.10), one deduces that the singularities may take place only if $q(t^*) = q^*$ with

$$q^* = 0, 1, -1, a_+, a_- \qquad \text{and} \qquad |q^*| \to \infty. \tag{I.3.12}$$

The following proposition proves that we only need to consider $|q^*| \to \infty$ and $q^* = 0$.

Proposition I.3.2. *Let $q(t)$ be a solution of equation (I.3.3) with initial condition $q(0) = a_+$. Then, the singularities $t^* \in \mathbb{C}$ of the analytic extension of $q(t)$ are characterized by either*

$$t^* = \int_{a_+}^{0} \hat{f}\, d\gamma, \qquad \text{or} \qquad t^* = \int_{a_+}^{\infty} \hat{f}\, d\gamma,$$

for some path $\gamma \subset \mathscr{R}_f$.

Moreover:

- *If* $t^* = \int_{a_+}^{0} \hat{f} d\gamma$ *and* $\operatorname{Im} t^* > 0$ *with* $\arg(t - t^*) \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$, *then*

$$q(t) = -\frac{3\alpha}{2}(t-t^*)^{\frac{2}{3}} + \mathcal{O}(t-t^*)^{\frac{4}{3}}, \qquad \text{for } 0 < |t-t^*| \ll 1, \tag{I.3.13}$$

where $\alpha \in \mathbb{C}$ *satisfies* $\alpha^3 = \frac{1}{2}$. *If* $\operatorname{Im} t^* < 0$ *and* $\arg(t-t^*) \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$, *the same holds true.*

- *If* $t^* = \int_{a_+}^{\infty} \hat{f} d\gamma$, *then*

$$q(t) = -\frac{1}{\sqrt{3}(t-t^*)}\left(1 + \mathcal{O}(t-t^*)\right), \qquad \text{for } 0 < |t-t^*| \ll 1. \tag{I.3.14}$$

Proof. To prove this result first we need to analyze all the possible values of q^* that may lead to singularities, (see (I.3.12)). We will use the expressions of t and t^* given in (I.3.9) and (I.3.11), respectively.

1. If $|q^*| \to \infty$, we have

$$t - t^* = \int_{\infty}^{q(t)} \hat{f}\, d\gamma.$$

Then, since

$$\hat{f}(q) = \frac{1}{\sqrt{3}q^2} + \mathcal{O}\left(\frac{1}{q^3}\right), \qquad \text{for} \quad |q| \gg 1,$$

we obtain

$$t - t^* = -\frac{1}{\sqrt{3}q} + \mathcal{O}\left(\frac{1}{q^2}\right),$$

which implies (I.3.14).

2. If $q^* = a_+$, we have that

$$t - t^* = \int_{a_+}^{q} \hat{f} d\gamma. \tag{I.3.15}$$

The function $\hat{f}$ can be written as

$$\hat{f}(q) = \frac{h_{a_+}(q)}{\sqrt{q - a_+}},$$

for some function h_{a_+} which is analytic and non-zero in a neighborhood of a_+. Then, h_{a_+} can be written as $h_{a_+}(q) = \sum_{k=0}^{\infty} c_k (q-a_+)^k$, with $c_0 \neq 0$ and, for $0 < |q - a_+| \ll 1$, we obtain from (I.3.15)

$$t - t^* = \sqrt{q - a_+} \sum_{k=0}^{\infty} \frac{c_k}{k + \frac{1}{2}} (q - a_+)^k,$$

which implies $(t-t^*)^2 = g_{a_+}(q)$ with

$$g_{a_+}(q) = (q - a_+)\left(\sum_{k=0}^{\infty} \frac{c_k}{k+\frac{1}{2}}(q-a_+)^k\right)^2.$$

The function $g_{a_+}(q)$ is analytic on a neighborhood of a_+ and satisfies $g_{a_+}(a_+) = 0$, $g'_{a_+}(a_+) = 4c_0^2 \neq 0$. Thus, applying the Inverse Function Theorem,

$$q(t) = g_{a_+}^{-1}\left((t-t^*)^2\right), \qquad \text{for} \quad 0 < |t - t^*| \ll 1. \tag{I.3.16}$$

Therefore, $q(t)$ is analytic for $|t - t^*| \ll 1$. One can analogously prove that the same happens at $q^* = a_-$ and $q^* = -1$.

3. The value $q^* = 1$ corresponds to the saddle point $(\lambda, \Lambda) = (0, 0)$ of H_{p} (see (I.2.17)). Indeed,
$$\int_1^q \hat{f} d\gamma \quad \text{is divergent.} \tag{I.3.17}$$
This implies that $q(t) \neq 1$ for any complex t.

4. If $q^* = 0$,
$$t - t^* = \int_0^q \hat{f}\, d\gamma.$$
We can introduce $h_0(q) = \frac{1}{\sqrt{q}} \hat{f}(q)$ which is analytic and of the form $h_0(q) = \sum_{k=0}^{\infty} c_k q^k$ with $c_0 \neq 0$. Then, for $0 < |q| \ll 1$, we obtain
$$(t - t^*)^{\frac{2}{3}} = g_0(q) = q \left(\sum_{k=0}^{\infty} \frac{c_k}{k + \frac{3}{2}} q^k \right)^{\frac{2}{3}} = q \left(\frac{2c_0}{3} + \mathcal{O}(q) \right)^{\frac{2}{3}},$$
where $g_0(q)$ is analytic in a neighborhood of $q = 0$ and satisfies $g_0(0) = 0$, $g_0'(0) = \left(\frac{2}{3} c_0\right)^{\frac{2}{3}} \neq 0$. Thus, applying the Inverse Function Theorem,
$$q(t) = g_0^{-1}\left((t - t^*)^{\frac{2}{3}} \right) = \sum_{k=1}^{\infty} C_k (t - t^*)^{\frac{2k}{3}}, \quad \text{for } 0 < |t - t^*| <\ll 1, \tag{I.3.18}$$
for some $C_k \in \mathbb{C}$ and choosing the Riemann sheet $\arg\,(t - t^*) \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$ for $\operatorname{Im} t^* > 0$ and $\arg\,(t - t^*) \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ for $\operatorname{Im} t^* < 0$. Replacing (I.3.18) in equation (I.3.3) we obtain $C_1 = -\frac{3\alpha}{2}$, where $\alpha \in \mathbb{C}$ satisfies $\alpha^3 = \frac{1}{2}$, which implies (I.3.13).

$\square$

I.3.3 Singularities closest to the real axis

Proposition I.3.2 provides the type of singularities that $q(t)$ may posses in its first Riemann sheet. Then, to prove Theorem I.3.1, we look for those singularities which are closest to the real axis. To do so, we analyze
$$\int_{a_+}^0 \hat{f} d\gamma, \qquad \int_{a_+}^{\infty} \hat{f} d\gamma, \tag{I.3.19}$$
along all paths $\gamma \subset \mathscr{R}_f$ with such endpoints and prove that, for all possible paths, the only singularities in the complex strip $\overline{\Pi_A}$ (see (I.2.24)) are $t = \pm iA$.

We introduce the following paths
$$\mathcal{P}_0 = \left\{ \gamma : (s_0, s_{\mathrm{end}}) \to \mathscr{R}_f \ : \ \lim_{s \to s_0} (\mathtt{p}\gamma(s), \arg \mathtt{g}(\gamma(s))) = (a_+, 0), \ \lim_{s \to s_{\mathrm{end}}} \mathtt{p}\gamma(s) = 0 \right\},$$
$$\mathcal{P}_\infty = \left\{ \gamma : (s_0, s_{\mathrm{end}}) \to \mathscr{R}_f \ : \ \lim_{s \to s_0} (\mathtt{p}\gamma(s), \arg \mathtt{g}(\gamma(s))) = (a_+, 0), \ \lim_{s \to s_{\mathrm{end}}} |\mathtt{p}\gamma(s)| = \infty \right\},$$
with the natural projection $\mathtt{p} : \mathscr{R}_f \to \mathbb{C}$.

We have chosen $\theta_0 := \lim_{s\to s_0} \arg \mathsf{g}(\gamma(s)) = 0$ without loss of generality. Indeed, since $q(t) \in [a_+, 1)$ for $t \in \mathbb{R}$ (see (I.3.2)), it could be either 0 or 2π. The paths with asymptotic argument 2π can be analyzed analogously and lead to singularities with opposed signed with respect to those given by paths in $\mathcal{P}_0$, $\mathcal{P}_\infty$.

Furthermore, the asymptotic argument of the paths at its endpoints is not specified since it is given by the path itself.

For a given path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$, we can define a path $T[\gamma] : [s_0, s_{\text{end}}) \to \mathbb{C}$ in the t-plane (or, more precisely, on the Riemann surface of $q(t)$) as

$$T[\gamma](s) = \int_{s_0}^{s} \hat{f}(\gamma(\tau))\gamma'(\tau)d\tau, \qquad \text{for } s \in [s_0, s_{\text{end}}). \tag{I.3.20}$$

Note that then the value of the integrals in (I.3.19) is just

$$t^*(\gamma) = \lim_{s\to s_{\text{end}}} T[\gamma](s) = \int_\gamma \hat{f}(q)dq. \tag{I.3.21}$$

Since we are interested in the singularities of $q(t)$ on its first Riemann sheet, we only consider the paths $T[\gamma]$ which belong to the complex strip Π_A (Except, of course, the endpoint of the path $t^*(\gamma) \in \partial\Pi_A$). See Figure I.3.1.

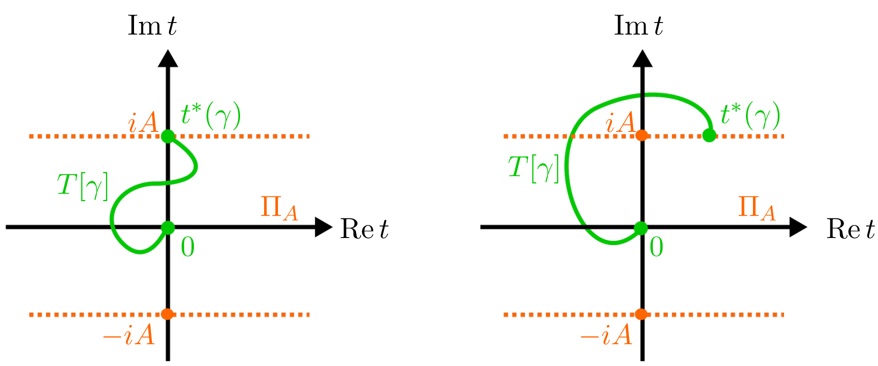


FIGURE I.3.1: Example of paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$. Left: $T[\gamma] \subset \Pi_A$ and $t^*(\gamma) = iA$. Right: $T[\gamma] \not\subset \Pi_A$.

The following definition characterizes the paths that we consider.

Definition I.3.3. *We say that a singularity t^* of $q(t)$ is* visible *if there exists a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ such that*

- $t^* = t^*(\gamma)$,
- $T[\gamma](s) \in \Pi_A$, *for* $s \in [s_0, s_{\text{end}})$.

Remark I.3.4. *In [GS08], the authors use a different definition of visible singularity: $t^* \in \mathbb{C}$ is considered a visible singularity if $q(t)$ can be continued from the real axis and then along the vertical line with a path of the form*

$$\zeta(t) = \operatorname{Re} t^* + it, \qquad \textit{for} \quad t \in [0, \operatorname{Im} t^*).$$

This condition on the paths is more restrictive than merely imposing that $T[\gamma] \subset \Pi_\upsilon$ for $\upsilon = \operatorname{Im} t^*$. However, to compute $t^*(\gamma)$, they are equivalent since both paths belong to Π_υ.

Theorem I.3.1 is a consequence of Proposition I.3.2 and the following result.

Proposition I.3.5. *There exist two paths $\gamma_\pm \in \mathcal{P}_0$ yielding the visible singularities $t^*(\gamma_\pm) = \pm iA$. Moreover, these are the only two visible singularities of $q(t)$.*

We devote the rest of this section to prove Proposition I.3.5. Let us introduce some tools and considerations to simplify the analysis of the integrals in (I.3.19).

- If $\gamma \subset \mathscr{R}_f$ is an integration path then, defining $\eta = \overline{\gamma}$[1] , we have that

$$\int_\eta \hat{f}(q)dq = \overline{\int_\gamma \hat{f}(q)dq}. \tag{I.3.22}$$

- Notice that the paths considered cannot contain the singularities of $\hat{f}$ (except in their endpoints) since $0, a_\pm, \pm 1 \notin \mathscr{R}_f$.
- When saying that a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ crosses $\mathbb{R}$, we refer to the two lines whose complex projection onto $\mathscr{R}_f$ coincide with $\mathbb{R}$. Analogously for any other interval.
- Instead of detailing the paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$, we only describe their projections $\mathtt{p}\gamma \subset \mathbb{C}$. This omission makes sense since paths γ are continuous on $\mathscr{R}_f$ and, as a result, $\arg \mathtt{g}(\gamma)$ must be continuous as well (see (I.3.8)). Therefore, we can let the natural arguments of $\mathtt{p}\gamma$ and the initial point $(a_+, 0)$ of the path define $\arg \mathtt{g}(\gamma)$.

To prove Proposition I.3.5, we classify the paths as follows.

A. Paths not crossing $\mathbb{R}$:

 A.1. Paths in $\mathcal{P}_0$ not crossing $\mathbb{R}$: Lemma I.3.6.

 A.2. Paths in $\mathcal{P}_\infty$ not crossing $\mathbb{R}$: Lemma I.3.7.

B. Paths first crossing the real axis at $\mathbb{R} \setminus [0, 1]$:

 B.1. First crossing of $\mathbb{R}$ at $(1, +\infty)$: Lemma I.3.10.

 B.2. First crossing of $\mathbb{R}$ at $(-\infty, a_-)$: Lemma I.3.11.

 B.3. First crossing of $\mathbb{R}$ at $(-1, 0)$: Lemma I.3.12.

 B.4. First crossing of $\mathbb{R}$ at $(a_-, -1)$: Lemma I.3.13.

C. Paths first crossing the real axis at $(0, 1)$:

 C.1. Paths in $\mathcal{P}_0$ only crossing $\mathbb{R}$ at $(0, 1)$: Lemma I.3.14.

 C.2. Paths in $\mathcal{P}_\infty$ only crossing $\mathbb{R}$ at $(0, 1)$: Lemma I.3.15.

 C.3. Paths also crossing $\mathbb{R} \setminus [0, 1]$: Lemma I.3.16.

I.3.3.1 Paths not crossing the real axis

In this section, we check the singularities resulting from the paths A.1 and A.2.

Lemma I.3.6. *There exist only two singularities, $t^*_{1,\pm}$, given by the paths $\gamma \in \mathcal{P}_0$ not crossing the real axis. These singularities are visible and*

$$t^*_{1,\pm} = \mp iA,$$

with A defined in (I.2.25) and satisfying $A \in \left[\frac{3}{50}, \frac{3}{10}\right]$.

[1]We define the conjugation on a Riemann surface as the natural continuation of the conjugation in the complex plane. That is, for $z = (x, \theta) \in \mathscr{R}_f$, its conjugated is $\overline{z} = (\overline{x}, -\theta) \in \mathscr{R}_f$.

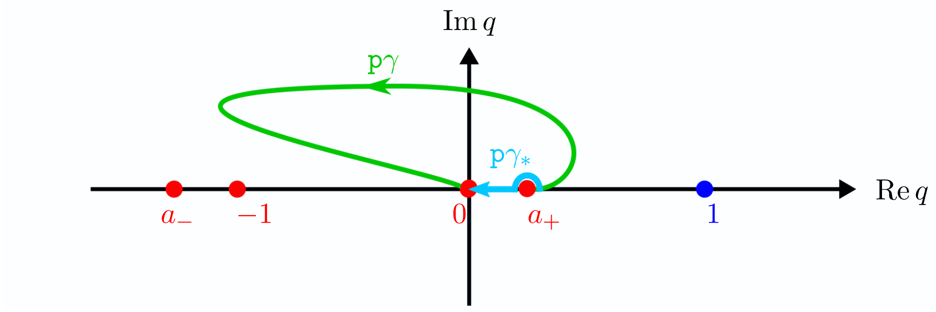


FIGURE I.3.2: Example of a path $\gamma \in \mathcal{P}_0$ such that $\mathrm{p}\gamma \subset \mathbb{C}^+$. Path γ_* as defined in (I.3.23).

Proof. Let us consider paths $\gamma \in \mathcal{P}_0$ such that $\mathrm{p}\gamma \subset \mathbb{C}^+ = \{\operatorname{Im} z > 0\}$. Notice that, since γ does not cross the real axis, it does not encircle any singularity of $\hat{f}$ and, by Cauchy's Integral Theorem, all the paths considered generate the same singularity $t^*_{1,+}$. The singularity $t^*_{1,-}$ is given by the conjugated paths (see (I.3.22)).

Let us consider the path $\gamma_* = \gamma^1_* \vee \gamma^2_* \vee \gamma^3_*$ with

$$\begin{cases} \gamma^1_*(q) = (q, 0) & \text{with } q \in (a_+, a_+ + \varepsilon], \\ \mathrm{p}\gamma^2_*(\phi) = a_+ + \varepsilon e^{i\phi} & \text{with } \phi \in [0, \pi], \\ \mathrm{p}\gamma^3_*(q) = q & \text{with } q \in [a_+ - \varepsilon, 0), \end{cases} \tag{I.3.23}$$

for $\varepsilon > 0$ small enough, (see Figure I.3.2). Then, the resulting singularity is

$$t^*_{1,+} = t^*(\gamma_*) = \int_{\gamma_*} \hat{f}(q)dq = \sum_{j=1}^{3} \int_{\gamma^j_*} \hat{f}(q)dq.$$

Since $\int_{\gamma^j_*} \hat{f}(q)dq = \mathcal{O}(\sqrt{\varepsilon})$ for $j = 1, 2$, taking the limit $\varepsilon \to 0$, we have

$$t^*_{1,+} = \lim_{\varepsilon \to 0} \int_{\gamma^3_*} \hat{f}(q)dq.$$

Then, by following the natural arguments of the path γ_*, we obtain

$$\arg(\gamma^3_*) = 0, \quad \arg(\gamma^3_* - a_+) = \pi, \quad \arg(\gamma^3_* + 1) = 0, \quad \arg(\gamma^3_* - a_-) = 0,$$

and, as a consequence, by the definition of A in (I.2.25), we have

$$t^*_{1,+} = \lim_{\varepsilon \to 0} \int_{\gamma^3_*} \hat{f}(q)dq = \int_{a_+}^{0} \frac{1}{x-1} \sqrt{\frac{x}{3(x+1)\,|x-a_+|\,e^{i\pi}(x-a_-)}}dq = -iA. \tag{I.3.24}$$

Moreover,

$$\begin{aligned} A &\leq \frac{\sqrt{a_+}}{(1-a_+)\sqrt{3\,|a_-|}} \int_0^{a_+} \frac{dx}{\sqrt{a_+ - x}} = \frac{4\sqrt{3}}{3} \frac{a_+^{3/2}}{1-a_+} \leq \frac{3}{10}, \\ A &\geq \frac{1}{\sqrt{3a_+(a_+ + 1)(a_+ - a_-)}} \int_0^{a_+} \sqrt{x}\, dx = \frac{2\sqrt[4]{74}}{9} a_+^{3/2} \geq \frac{3}{50}. \end{aligned}$$

Now, it just remains to see that $t^*_{1,+}$ is visible, namely, we check that $T[\gamma_*] \subset \Pi_A$. Indeed, for $s \in (0, a_+]$, we have that

$$|\mathrm{Im}\, T[\gamma_*](s)| = \int_s^{a_+} \frac{1}{1-x}\sqrt{\frac{x}{3(x+1)(a_+-x)(x-a_-)}}dx < A.$$

□

Lemma I.3.7. *There exist only two singularities, $t^*_{2,\pm}$, resulting from the paths $\gamma \in \mathcal{P}_\infty$ not crossing the real axis. These singularities are not visible and have imaginary part*

$$\mathrm{Im}\,(t^*_{2,\pm}) = \mp\pi\sqrt{\frac{2}{21}}.$$

Proof. Let us consider paths $\gamma \in \mathcal{P}_\infty$ such that $\mathtt{p}\gamma \subset \mathbb{C}^+$. Then, since $\hat{f}(q)$ decays with a rate of $|q|^{-2}$ as $|q| \to \infty$, all paths considered generate the same singularity, $t^*_{2,+}$. The singularity $t^*_{2,-}$ is given by the conjugated paths.

Let us consider the path $\gamma_* = \gamma^1_* \vee \gamma^2_* \vee \gamma^3_*$ where

$$\begin{cases} \gamma^1_*(q) = (q,0) & \text{with } q \in (a_+, 1-\varepsilon], \\ \mathtt{p}\gamma^2_*(\phi) = 1+\varepsilon e^{i\phi} & \text{with } \phi \in [\pi, 0], \\ \mathtt{p}\gamma^3_*(q) = q & \text{with } q \in [1+\varepsilon, +\infty), \end{cases} \tag{I.3.25}$$

for any small enough $\varepsilon > 0$. (See Figure I.3.3). Then, the resulting singularity is

$$t^*_{2,+} = t^*(\gamma_*) = \sum_{j=1}^{3}\int_{\gamma^j_*}\hat{f}(q)dq.$$

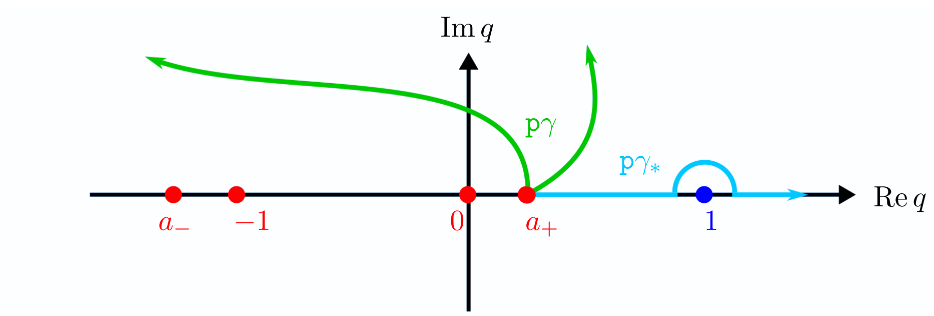


FIGURE I.3.3: Example of a path $\gamma \in \mathcal{P}_\infty$ such that $\mathtt{p}\gamma \subset \mathbb{C}^+$. Path γ_* as defined in (I.3.25).

Since $\hat{f}(q) \in \mathbb{R}$ when $\mathtt{p}(q) \in (a_+, 1) \cup (1, +\infty) \subset \mathbb{R}$, the integrals on γ^1_* and γ^3_* take real values. Therefore, γ^2_* is the only path that contributes to the imaginary part to the singularity. Notice that the path γ^2_* partially encircles the pole $q = (1,0)$ of $\hat{f}(q)$. Then, one has

$$\mathrm{Im}\,(t^*_{2,+}) = \mathrm{Im}\int_{\gamma^2_*}\hat{f}(q)dq = -\pi\mathrm{Res}\left(\hat{f}, (1,0)\right) = -\pi\sqrt{\frac{2}{21}}.$$

Since, by Lemma I.3.6, $|\mathrm{Im}\,(t^*_{2,+})| > A$, the singularity $t^*_{2,+}$ is not visible. □

Remark I.3.8. *Using mathematical software, one can see that the singularities $t^*_{2,\pm}$ in Lemma I.3.7 satisfy $t^*_{2,\pm} \approx -0.086697 \mp 0.969516i$.*

I.3.3.2 Paths first crossing the real axis at $\mathbb{R} \setminus [0,1]$

In this section, we continue the proof of Proposition I.3.5 by checking that the singularities generated by paths B.1 to B.4 (see the list in Section I.3.3) are not visible.

First, we introduce some concepts. Let us consider a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$. We define the *parameter of the first crossing* of the real line as

$$s_1(\gamma) = \inf \{s \in (s_0, s_{\text{end}}) \ : \ \operatorname{Im} \mathrm{p}\gamma(s) = 0\},$$

the *location of the first crossing* as

$$q_1(\gamma) = \mathrm{p}\gamma(s_1(\gamma)) \in \mathbb{R} \setminus \{a_-, -1, 0, a_+, 1\},$$

the *piece of the path* before the first crossing of the real line as

$$\gamma_1(\gamma) = \{\gamma(s) \ : \ s \in (s_0, s_1(\gamma))\},$$

and the *time of the first crossing* as

$$t_1(\gamma) = \int_{\gamma_1(\gamma)} \hat{f}(q) dq.$$

In the following lemmas, we focus on the paths that stay in $\mathbb{C}^+$ until the first crossing of the real line, that is $\mathrm{p}\gamma_1(\gamma) \subset \mathbb{C}^+$ (see (I.3.22) for the conjugate paths, i.e. $\mathrm{p}\gamma_1(\gamma) \subset \mathbb{C}^-$).

Remark I.3.9. *To prove that a singularity $t^*(\gamma)$ is not visible (see Definition I.3.3) it is sufficient to check that $|\operatorname{Im} t_1(\gamma)| \geq A$.*

Lemma I.3.10. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ such that $q_1(\gamma) \in (1, +\infty)$ are not visible.*

Proof. Consider a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ with $q_1 = q_1(\gamma) \in (1, +\infty)$ and $\mathrm{p}\gamma_1(\gamma) \subset \mathbb{C}^+$. Integrating the function $\hat{f}$ along the path $\gamma_1(\gamma)$ is equivalent to integrate $\hat{f}$ along the path $\eta = \eta^1 \vee \eta^2 \vee \eta^3$ where

$$\begin{cases} \eta^1(q) = (q, 0) & \text{with } q \in (a_+, 1-\varepsilon], \\ \mathrm{p}\eta^2(\phi) = 1 + \varepsilon e^{i\phi} & \text{with } \phi \in [\pi, 0], \\ \mathrm{p}\eta^3(q) = q & \text{with } q \in [1+\varepsilon, q_1), \end{cases} \tag{I.3.26}$$

for $\varepsilon > 0$ small enough, (see Figure I.3.4). Then,

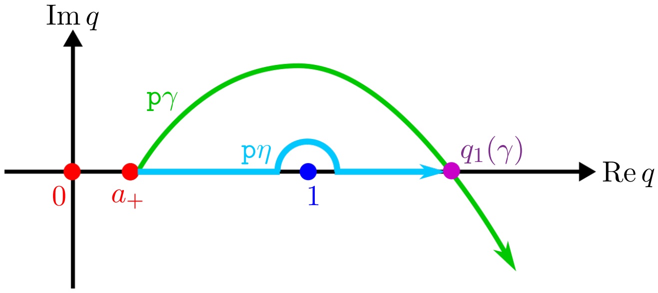


FIGURE I.3.4: Example of a path $\gamma \in \mathcal{H}_{q,\infty}$ such that $q_1(\gamma) \in (1, +\infty)$ and $\mathrm{p}\gamma_1(\gamma) \subset \mathbb{C}^+$. The path η has been defined in (I.3.26).

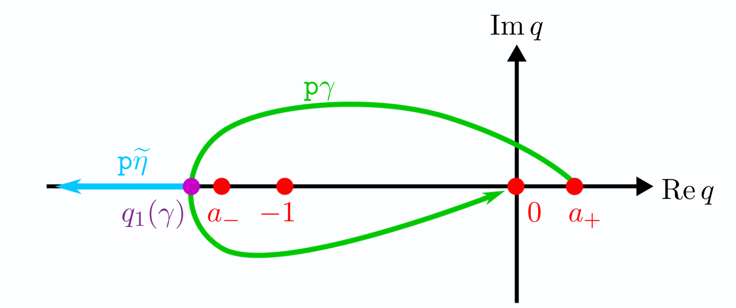


FIGURE I.3.5: Example of a path $\gamma \in \mathcal{P}_0$ such that $q_1(\gamma) \in (-\infty, a_-)$ and $\mathtt{p}\gamma_1(\gamma) \subset \mathbb{C}^+$. The path $\widetilde{\eta}$ has been defined in (I.3.27).

$$t_1(\gamma) = \int_\eta \hat{f}(q)dq = \sum_{j=1}^{3} \int_{\eta^j} \hat{f}(q)dq,$$

and the integrals on η^1 and η^3 take real values since $\hat{f}(q) \in \mathbb{R}$ when $\mathtt{p}(q) \in (a_+, 1) \cup (1, +\infty) \subset \mathbb{R}$. Analogously to the proof of Lemma I.3.7, we have that

$$|\mathrm{Im}\, t_1(\gamma)| = \left|\mathrm{Im} \int_{\eta^2} \hat{f}(q)dq\right| = \pi \left|\mathrm{Res}\left(\hat{f}, (1,0)\right)\right| = \pi\sqrt{\frac{2}{21}} > A.$$

Therefore, $t^*(\gamma)$ is not visible (see Remark I.3.9). $\square$

Lemma I.3.11. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ such that $q_1(\gamma) \in (-\infty, a_-)$ are not visible.*

Proof. Consider a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ with $q_1 = q_1(\gamma) \in (-\infty, a_-)$ and $\mathtt{p}\gamma_1(\gamma) \subset \mathbb{C}^+$. To compute $t_1(\gamma)$ we introduce the auxiliary path $\eta_\infty = \gamma_1(\gamma) \vee \widetilde{\eta}$, where

$$\mathtt{p}\widetilde{\eta}(q) = q \qquad \text{with } q \in [q_1, -\infty), \tag{I.3.27}$$

(see Figure I.3.5). Then, taking into account that $\hat{f}|_{\widetilde{\eta}} \subset \mathbb{R}$,

$$|\mathrm{Im}\, t_1(\gamma)| = \left|\mathrm{Im} \int_{\eta_\infty} \hat{f}(q)dq\right| = \left|\mathrm{Im} \int_{\gamma_*} \hat{f}(q)dq\right| = \pi\sqrt{\frac{2}{21}} > A,$$

where γ^* is the path defined in (I.3.25). Therefore $t^*(\gamma)$ is not visible. $\square$

Lemma I.3.12. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ such that $q_1(\gamma) \in (-1, 0)$ are not visible.*

Proof. Let $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ be a path such that $q_1 = q_1(\gamma) \in (-1, 0)$ and $\mathtt{p}\gamma_1(\gamma) \subset \mathbb{C}^+$. Integrating the function $\hat{f}$ along the path $\gamma_1(\gamma)$ is equivalent to integrating $\hat{f}$ along $\eta = \eta^1 \vee \eta^2 \vee \eta^3 \vee \eta^4 \vee \eta^5$ where

$$\begin{cases} \eta^1(q) = (q, 0) & \text{with } q \in (a_+, a_+ + \varepsilon], \\ \mathtt{p}\eta^2(\phi) = a_+ + \varepsilon e^{i\phi} & \text{with } \phi \in [0, \pi], \\ \mathtt{p}\eta^3(q) = q & \text{with } q \in [a_+ - \varepsilon, \varepsilon], \\ \mathtt{p}\eta^4(\phi) = \varepsilon e^{i\phi} & \text{with } \phi \in [0, \pi], \\ \mathtt{p}\eta^5(q) = q & \text{with } q \in [-\varepsilon, q_1), \end{cases} \tag{I.3.28}$$

for $\varepsilon > 0$ small enough, (see Figure I.3.6). Using that $\int_{\eta^j} \hat{f}(q)dq = \mathcal{O}(\sqrt{\varepsilon})$ for

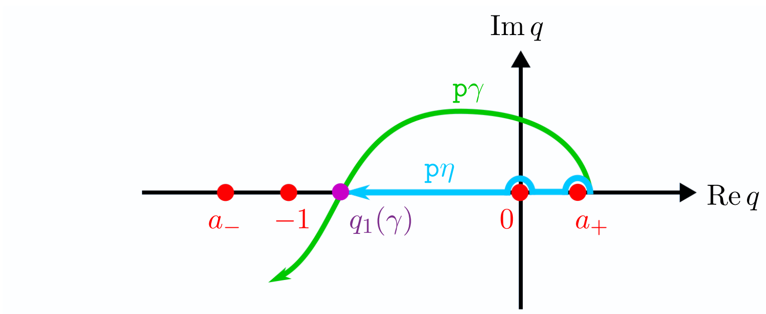


FIGURE I.3.6: Example of a path $\gamma \in \mathcal{P}_\infty$ such that $q_1(\gamma) \in (-1, 0)$ and $\mathrm{p}\gamma_1(\gamma) \subset \mathbb{C}^+$. The path η has been defined in (I.3.28).

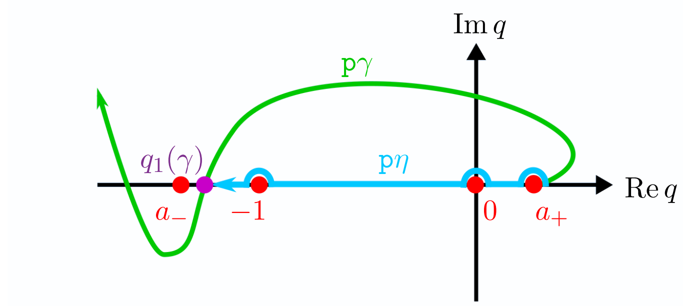


FIGURE I.3.7: Example of a path $\gamma \in \mathcal{P}_\infty$ such that $q_1(\gamma) \in (a_-, -1)$ and $\mathrm{p}\gamma_1(\gamma) \subset \mathbb{C}^+$. Path η as defined in (I.3.29).

$j = 1, 2, 4$, that $\hat{f}|_{\eta^5} \subset \mathbb{R}$, and (I.3.24), one has that $t^*(\gamma)$ is not visible since

$$|\mathrm{Im}\, t_1(\gamma)| = \lim_{\varepsilon \to 0} \left| \int_{\eta^3} \hat{f}(q) dq \right| = A.$$

□

Lemma I.3.13. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ such that $q_1(\gamma) \in (a_-, -1)$ are not visible.*

Proof. Take a path $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ with $q_1 = q_1(\gamma) \in (a_-, -1)$ and $\mathrm{p}\gamma_1(\gamma) \subset \mathbb{C}^+$. The integral of the function $\hat{f}$ along the path $\gamma_1(\gamma)$ coincides with the integral along the path

$$\eta = \bigvee_{j=1}^{7} \eta^j, \tag{I.3.29}$$

where the paths η^j, $j = 1, 2, 3, 4$, are defined in (I.3.28) and

$$\begin{cases} \mathrm{p}\eta^5(q) = q & \text{with } q \in [-\varepsilon, -1 + \varepsilon], \\ \mathrm{p}\eta^6(\phi) = -1 + \varepsilon e^{i\phi} & \text{with } \phi \in [0, \pi], \\ \mathrm{p}\eta^7(q) = q & \text{with } q \in [-1 - \varepsilon, q_1), \end{cases}$$

for small enough $\varepsilon > 0$, (see Figure I.3.7). Then, proceeding analogously to the proof of Lemma I.3.12,

$$\mathrm{Im}\, t_1(\gamma) = \lim_{\varepsilon \to 0} \mathrm{Im} \int_\eta \hat{f}(q) dq = -A + \sum_{j=5}^{7} \lim_{\varepsilon \to 0} \mathrm{Im} \int_{\eta^j} \hat{f}(q) dq. \tag{I.3.30}$$

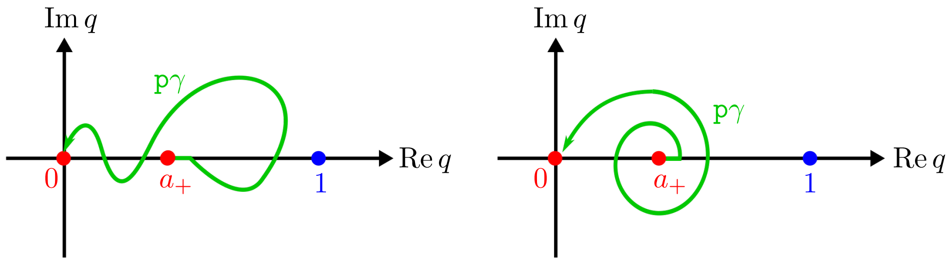


FIGURE I.3.8: Example of paths $\gamma \in \mathcal{P}_0$ only crossing $\mathbb{R}$ at $(0,1)$. Left: $\theta_{\text{end}}(\gamma) = \pi$. Right: $\theta_{\text{end}}(\gamma) = 3\pi$.

Since $\hat{f}|_{\eta^5} \subset \mathbb{R}$ and $\int_{\eta^6} \hat{f}(q)dq = \mathcal{O}(\sqrt{\varepsilon})$, it only remains to compute the integral on η^7. Following the natural arguments of the path η, we obtain

$$\arg(\eta^7) = \pi, \quad \arg(\eta^7 - a_+) = \pi, \quad \arg(\eta^7 + 1) = \pi, \quad \arg(\eta^7 - a_-) = 0$$

and, as a consequence,

$$\lim_{\varepsilon \to 0} \int_{\eta^7} \hat{f}(q)dq = \int_{-1}^{q_1} \frac{1}{q-1} \sqrt{\frac{|q|e^{i\pi}}{3|q+1|e^{i\pi}|q-a_+|e^{i\pi}(q-a_-)}} dq = -iB(q_1),$$

where $B(q_1)$ a real-valued, positive and strictly decreasing function for $q_1 \in (a_-, -1)$. Then, $t^*(\gamma)$ is not visible since, by (I.3.30), $|\operatorname{Im} t_1(\gamma)| = A + B(q_1) > A$. $\square$

I.3.3.3 Paths first crossing the real axis at $[0,1]$

In this section, we check that the singularities generated by paths C.1 to C.3 (see the list in Section I.3.3) are either not visible or $\pm iA$.

Lemma I.3.14. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0$ only crossing $\mathbb{R}$ at $(0,1)$ are either $t^*(\gamma) = iA$ or $t^*(\gamma) = -iA$.*

Proof. The paths considered in this lemma can turn around the branching point $q = a_+$, but not around the other branching points nor the pole. Therefore, we classify these paths depending on how many turns they perform around $q = a_+$. In order to do so, we define

$$\theta_{\text{end}}(\gamma) = \lim_{s \to s_{\text{end}}} \arg(\gamma(s) - a_+). \tag{I.3.31}$$

The considered paths satisfy $\theta_{\text{end}}(\gamma) = (2k+1)\pi$ for some $k \in \mathbb{Z}$ (see Figure I.3.8). Integrating the function $\hat{f}$ along the path γ is equivalent to integrating along $\eta = \eta^1 \vee \eta^2 \vee \eta^3$ with

$$\begin{cases} \eta^1(q) = (q, 0) & \text{with } q \in (a_+, a_+ - \varepsilon], \\ \mathrm{p}\eta^2(\phi) = a_+ + \varepsilon e^{i\phi} & \text{with } \phi \in [\pi, (2k+1)\pi], \\ \mathrm{p}\eta^3(q) = q & \text{with } q \in [a_+ - \varepsilon, 0), \end{cases} \tag{I.3.32}$$

for small enough $\varepsilon > 0$. Since $\int_{\eta^j} \hat{f}(q)dq = \mathcal{O}(\sqrt{\varepsilon})$ for $j = 1, 2$,

$$t^*(\gamma) = \lim_{\varepsilon \to 0} \int_{\eta^3} \hat{f}(q)dq = \int_{a_+}^{0} \frac{1}{q-1} \sqrt{\frac{q}{3(q+1)\,|q-a_+|\,e^{i(2k+1)\pi}(q-a_-)}} dq = (-1)^{k+1} iA.$$

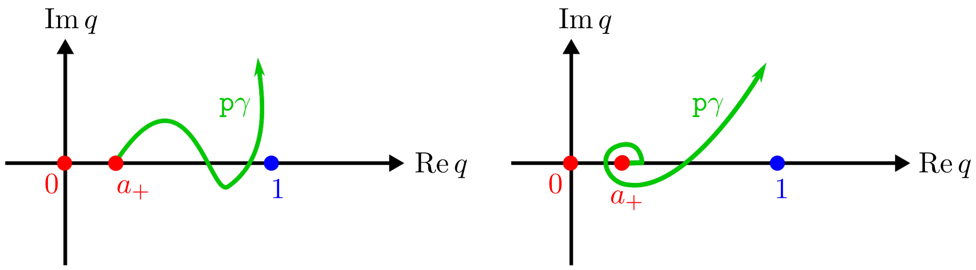


FIGURE I.3.9: Example of paths $\gamma \in \mathcal{P}_\infty$ only crossing $\mathbb{R}$ at $(0,1)$ such that $\mathrm{p}\gamma$ ends on the positive complex plane. Left: $\theta_{\text{end}}(\gamma) \in (0,\pi)$. Right: $\theta_{\text{end}}(\gamma) \in (2\pi, 3\pi)$.

□

Lemma I.3.15. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_\infty$ only crossing $\mathbb{R}$ at $(0,1)$ are not visible.*

Proof. We analyze the paths $\gamma \in \mathcal{P}_\infty$ such that $\mathrm{p}\gamma$ goes to infinity on $\mathbb{C}^+$ (by (I.3.22), the paths on $\mathbb{C}^-$ give conjugated results). Following the proof of Lemma I.3.14, we classify the paths γ depending on $\theta_{\text{end}}(\gamma)$, the final argument with respect to a_+ (see (I.3.31) and Figure I.3.9). The paths considered satisfy

$$\theta_{\text{end}}(\gamma) \in \big(2\pi k, (2k+1)\pi\big), \qquad \text{for some } k \in \mathbb{Z}.$$

We compute $t^*(\gamma)$ using the path $\eta = \eta^1 \vee \eta^2 \vee \eta^3 \vee \eta^4 \vee \eta^5$ where the paths η^1, η^2 are defined in (I.3.32) and

$$\begin{cases} \mathrm{p}\eta^3(q) = q & \text{with } q \in [a_+ + \varepsilon, 1-\varepsilon], \\ \mathrm{p}\eta^4(\phi) = 1 + \varepsilon e^{i\phi} & \text{with } \phi \in [\pi, 0], \\ \mathrm{p}\eta^5(q) = q & \text{with } q \in [1+\varepsilon, +\infty), \end{cases}$$

for small enough $\varepsilon > 0$. Since the integrals on η^3 and η^5 take real values and applying the results in Lemma I.3.14 for η^1 and η^2, we obtain

$$\operatorname{Im} t^*(\gamma) = \operatorname{Im} \int_{\eta^4} \hat{f}(q)dq.$$

Proceeding as in the proof of Lemma I.3.7 and following the natural arguments of the path η, one deduces that

$$\operatorname{Im} t^*(\gamma) = -\pi \operatorname{Res}\left(\hat{f}, (1, 2\pi k)\right) = (-1)^{k+1}\pi\sqrt{\frac{2}{21}}.$$

Therefore, since $|\operatorname{Im} t^*(\gamma)| > A$, the singularity is not visible. □

Lemma I.3.16. *The singularities $t^*(\gamma)$ given by paths $\gamma \in \mathcal{P}_0 \cup \mathcal{P}_\infty$ both crossing $(0,1)$ and $\mathbb{R} \setminus [0,1]$ are not visible.*

Proof. Let us define the parameter of the first crossing at $\mathbb{R} \setminus [0,1]$ as

$$s_2(\gamma) = \inf\left\{s \in (s_0, s_{\text{end}}) \;:\; \operatorname{Im} \mathrm{p}\gamma(s) = 0,\, \operatorname{Re} \mathrm{p}\gamma(s) \notin [0,1]\right\}$$

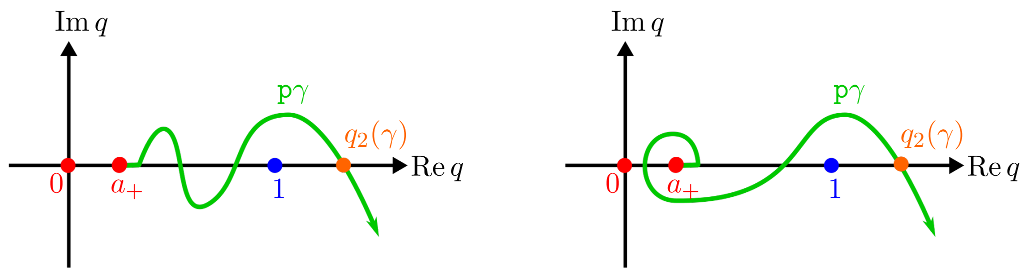


FIGURE I.3.10: Example of paths $\gamma \in \mathcal{P}_\infty$ crossing both $(0,1)$ and $\mathbb{R}\setminus[0,1]$ with $q_2(\gamma) \in (1,+\infty)$ and such that $\mathtt{p}\gamma$ approaches $q_2(\gamma)$ from $\mathbb{C}^+$. Left: $\theta_2(\gamma) = 0$. Right: $\theta_2(\gamma) = 2\pi$.

and the corresponding point

$$q_2(\gamma) = \mathtt{p}\gamma(s_2(\gamma)) \in \mathbb{R} \setminus \{[0,1], a_-, -1\}\,.$$

We consider paths γ with $q_2 = q_2(\gamma) \in (1,+\infty)$ and such that $\mathtt{p}\gamma$ approaches $q_2(\gamma)$ from $\mathbb{C}^+$ (see I.3.22). The cases $q_2 \in (-\infty, a_-)$, $q_2 \in (a_-, -1)$ and $q_2 \in (-1, 0)$ are proved analogously.

The strategy is to classify the paths γ depending on how many turns they perform around $q = a_+$ before crossing $\mathbb{R}\setminus[0,1]$. To this end, we define $\theta_2(\gamma) = \arg(q_2(\gamma) - a_+)$ (see Figure I.3.10). The paths we are considering satisfy $\theta_2(\gamma) = 2\pi k$ for some $k \in \mathbb{Z}$. We also define the piece of path before the crossing as $\gamma_2(\gamma) = \{\gamma(s) \;:\; s \in (s_0, s_2(\gamma))\}$ and the corresponding time

$$t_2(\gamma) = \int_{\gamma_2(\gamma)} \hat{f}(q)dq.$$

To prove that a singularity $t^*(\gamma)$ is not visible, it is sufficient to check that $|\mathrm{Im}\, t_2(\gamma)| \geq A$.

1. Consider $\theta_2(\gamma) = 2\pi k$ with k an even number. Let us consider the path η as defined in (I.3.26) replacing q_1 by q_2 in its definition. This path η lies entirely on the first Riemann sheet, that is $\arg \mathtt{g}(\eta) \in (-\pi, \pi]$. Integrating the function $\hat{f}$ along $\gamma_2(\gamma)$ is equivalent to integrating it along

   $$\xi = \widetilde{\eta}^1 \vee \widetilde{\eta}^2 \vee \widetilde{\eta}^3 \vee \eta,$$

   where

   $$\begin{cases} \widetilde{\eta}^1(q) = (q, 0) & \text{with } q \in (a_+, a_+ + \varepsilon], \\ \mathtt{p}\widetilde{\eta}^2(\phi) = a_+ + \varepsilon e^{i\phi} & \text{with } \phi \in [0, 2\pi k], \\ \mathtt{p}\widetilde{\eta}^3(q) = q & \text{with } q \in [a_+ + \varepsilon, a_+), \end{cases} \tag{I.3.33}$$

   for $\varepsilon > 0$ small enough. Notice that this construction makes sense since the path $\widetilde{\eta}^3$ has argument $\arg \mathtt{g}(\widetilde{\eta}^3) = 2\pi k$ (which belongs to the first Riemann sheet).

   Then, since $\int_{\widetilde{\eta}^j} \hat{f}(q)dq = \mathcal{O}(\sqrt{\varepsilon})$ for $j = 1,2,3$ and applying Lemma I.3.10, we have

   $$|\mathrm{Im}\, t_2(\gamma)| = \lim_{\varepsilon \to 0} \left|\mathrm{Im} \int_\xi \hat{f}(q)dq\right| = \left|\mathrm{Im} \int_\eta \hat{f}(q)dq\right| > A.$$

2. Consider $\theta_2(k) = 2\pi k$ with k an odd integer. We define the path η^*, lying on the second Riemann sheet, as

$$\eta^* = (\mathrm{p}\eta, \arg \mathtt{g}(\eta) + 2\pi) \in \mathbb{C} \times (\pi, 3\pi],$$

where η is the path introduced in (I.3.26) (replacing q_1 by q_2 in its definition). Note that the path η lies on the first Riemann sheet ($\arg \mathtt{g}(\eta) \in (-\pi, \pi]$).

It can be easily checked that switching the Riemann sheet implies a change in sign. That is,

$$\int_{\eta^*} \hat{f}(q)dq = -\int_{\eta} \hat{f}(q)dq. \tag{I.3.34}$$

Then, integrating the function $\hat{f}$ along the path $\gamma_2(\gamma)$ is equivalent to integrating it over

$$\xi = \widetilde{\eta}^1 \vee \widetilde{\eta}^2 \vee \widetilde{\eta}^3 \vee \eta^*,$$

where paths $\widetilde{\eta}^j$ for $j = 1, 2, 3$, are defined on (I.3.33). This construction makes sense since the path $\widetilde{\eta}^3$ has argument $\arg \mathtt{g}(\widetilde{\eta}^3) = 2\pi k$ (which belongs to the second Riemann sheet).

Then, since $\int_{\widetilde{\eta}^j} \hat{f}(q)dq = \mathcal{O}(\sqrt{\varepsilon})$ for $j = 1, 2, 3$ and applying Lemma I.3.10 and formula (I.3.34), we have

$$|\mathrm{Im}\, t_2(\gamma)| = \lim_{\varepsilon \to 0} \left| \mathrm{Im} \int_{\xi} \hat{f}(q)dq \right| = \left| \mathrm{Im} \int_{\eta} \hat{f}(q)dq \right| > A.$$

□

I.3.4 Proof of Proposition I.2.3

For $t \in \mathbb{R}$, $\Lambda_h(t)$ satisfies $\Lambda_h(t) = 0$ if and only if $t = 0$ (see Figure I.2.1). To prove Proposition I.2.3, we follow the same techniques used in the proof of Theorem I.2.2.

Let us consider $q(t) = \cos(\frac{\Lambda_h(t)}{2})$ as introduced in Theorem I.3.1. Then, by (I.3.6),

$$\Lambda_h^2(t) = \frac{4}{3q(t)}\big(1 - q(t)\big)\big(q(t) - a_+\big)\big(q(t) - a_-\big).$$

Let $\Lambda_h(t^*) = 0$ for a given t^*. Then, defining $q^* = q(t^*)$, we have three options:

$$q^* = 1, a_+, a_-.$$

We have seen that $q^* = 1$ corresponds to the saddle equilibrium point, namely $|t^*| \to \infty$, (see (I.3.17)). Therefore, it cannot lead to zeroes of $\Lambda_h(t)$. On the contrary, $q^* = a_\pm$ leads to zeroes of $\Lambda_h(t)$, since we have seen that $q(t)$ is well defined and analytic in a neighborhood of such t^* (see (I.3.16)).

To prove Proposition I.2.3 it only remains to compute all possible values of $t^* \in \overline{\Pi_A}$ such that $q^* = q(t^*)$ with $q^* = a_+, a_-$. To do so, we use the techniques and results presented in Section I.3.3.

From now on, we consider integration paths $\gamma : (s_0, s_{\mathrm{end}}) \to \mathscr{R}_f$ with initial point $\lim_{s \to s_0} \gamma(s) = (a_+, 0)$ and endpoint $\lim_{s \to s_0} \mathrm{p}\gamma(s) = q^* = a_\pm$. Moreover, we say that a zero t^* of Λ_h is *visible* if there exist a path γ such that $t^* = t^*(\gamma) \in \overline{\Pi_A}$ and $T[\gamma](s) \in \Pi_A$ for $s \in [s_0, s_{\mathrm{end}})$, (see (I.3.20) and (I.3.21)).

First, we recall some of the results obtained in Sections I.3.3.2 and I.3.3.3.

- Consider $q_1 \in (-\infty, a_-) \cup (a_-, -1) \cup (1, +\infty)$. In the proofs of Lemmas I.3.10, I.3.11, I.3.13 and I.3.16 we have seen that

$$\left|\operatorname{Im} \int_{a_+}^{q_1} \hat{f} d\gamma\right| > A. \tag{I.3.35}$$

- Consider $q_1 \in (-1, 0)$. In the proofs of Lemmas I.3.12 and I.3.16, we have seen that

$$\left|\operatorname{Im} \int_{a_+}^{q_1} \hat{f} d\gamma\right| = A. \tag{I.3.36}$$

Now, we classify the paths depending on its endpoint q^*.

1. Consider $q^* = a_-$. Analogously to the proof of (I.3.35), it can be seen that

$$|\operatorname{Im} t^*(\gamma)| = \left|\operatorname{Im} \int_{a_+}^{a_-} \hat{f} d\gamma\right| > A.$$

   Therefore, $q^* = a_-$ does not lead to any visible zero.

2. Consider $q^* = a_+$. Notice that, by (I.3.35) and (I.3.36), any path crossing $\mathbb{R} \setminus [0, 1]$ leads to non-visible zeroes. Therefore, we only consider paths γ either crossing $(0, 1)$ or not crossing $\mathbb{R}$.

   Since in $(0, 1)$ the only singularity of $\hat{f}(q)$ is the branching point $q = a_+$, there exists a homotopic path $\eta = \eta^1 \vee \eta^2 \vee \eta^3$ defined by

$$\begin{cases} \eta^1(q) = (q, 0) & \text{with } q \in (a_+, a_+ + \varepsilon], \\ \mathrm{p}\eta^2(\phi) = a_+ + \varepsilon e^{i\phi} & \text{with } \phi \in [0, 2\pi k], \\ \mathrm{p}\eta^3(q) = q & \text{with } q \in [a_+ + \varepsilon, a_+), \end{cases}$$

   for some $k \in \mathbb{Z}$ and $\varepsilon > 0$ small enough. Then,

$$t^*(\gamma) = \int_{a_+}^{a_+} \hat{f} d\gamma = \lim_{\varepsilon \to 0} \int_\eta \hat{f}(q) dq = 0.$$

   Therefore, these paths lead to the only visible zero $t^* = 0$.

□

Chapter I.4

The inner system of coordinates

This chapter is devoted to prove Proposition I.2.5. That is we perform the suitable changes of coordinates, described in Section I.2.3, to Hamiltonian H obtained in Theorem I.2.1 (see (I.2.16)) to obtain the inner Hamiltonian $\mathcal{H}$. However, recall that the Hamiltonian H is defined by means of H_1^{Poi} (see (I.2.12)) which does not have a closed form. For this reason, a preliminary step to to prove Proposition I.2.5 is to provide suitable expansions for H_1^{Poi} in an appropriate domain. This is done in Section I.4.1. Then, in Section I.4.2, we apply the changes of coordinates introduced in Section I.2.3 to conclude the proof of the proposition.

I.4.1 The Hamiltonian in Poincaré variables

First, we give some formulae to translate the Delaunay variables and other orbital elements into Poincaré coordinates (see (I.2.10)).

- **Eccentricity** e (see (I.2.4)): It can be written as

$$e = 2\,\tilde{e}(L,\eta,\xi)\sqrt{\eta\xi}, \quad \text{where} \quad \tilde{e}(L,\eta,\xi) = \frac{\sqrt{2L-\eta\xi}}{2L} = \frac{1}{\sqrt{2L}} + \mathcal{O}(\eta\xi). \tag{I.4.1}$$

  Notice that $\tilde{e}$ is analytic for $(L,\eta,\xi) \sim (1,0,0)$[1].

- **Argument of the perihelion** g**:** From the expression of η and ξ in (I.2.11),

$$\cos g = \frac{\eta+\xi}{2\sqrt{\eta\xi}}, \qquad \sin g = -i\frac{\eta-\xi}{2\sqrt{\eta\xi}}. \tag{I.4.2}$$

- **Mean anomaly** ℓ**:** Since $\lambda = \ell + g$, we have that

$$\cos\ell = \frac{1}{2\sqrt{\eta\xi}}\left(e^{-i\lambda}\eta + e^{i\lambda}\xi\right), \qquad \sin\ell = \frac{i}{2\sqrt{\eta\xi}}\left(e^{-i\lambda}\eta - e^{i\lambda}\xi\right).$$

  These expressions are not analytic at $(\eta,\xi)=(0,0)$. However, by (I.4.1),

$$e\cos\ell = \tilde{e}(L,\eta,\xi)\left(e^{-i\lambda}\eta + e^{i\lambda}\xi\right), \quad e\sin\ell = i\tilde{e}(L,\eta,\xi)\left(e^{-i\lambda}\eta - e^{i\lambda}\xi\right), \tag{I.4.3}$$

  are analytic for $(L,\eta,\xi) \sim (1,0,0)$.

[1]This expansion is valid as long as $L \neq 0$. However, since our analysis focuses on $L \sim 1$, to simplify notation we use this more restrictive domain.

– **Eccentric anomaly u:** It can be implicitly defined by $u = \ell + e \sin u$ (see (I.2.8)), which implies

$$u = \ell + e \sin \ell + e^2 \cos \ell \sin \ell + \mathcal{O}(e \sin \ell, e \cos \ell)^3.$$

Then, by (I.4.1) and (I.4.3),

$$\begin{aligned} e \cos u &= \frac{1}{\sqrt{2L}} \left(e^{-i\lambda}\eta + e^{i\lambda}\xi \right) + \frac{1}{2L} \left(e^{-i\lambda}\eta - e^{i\lambda}\xi \right)^2 + \mathcal{O}(e^{-i\lambda}\eta, e^{i\lambda}\xi)^3, \\ e \sin u &= \frac{i}{\sqrt{2L}} \left(e^{-i\lambda}\eta - e^{i\lambda}\xi \right) + \frac{i}{2L} \left(e^{-2i\lambda}\eta^2 - e^{2i\lambda}\xi^2 \right) + \mathcal{O}(e^{-i\lambda}\eta, e^{i\lambda}\xi)^3, \end{aligned} \tag{I.4.4}$$

which are also analytic for $(L, \eta, \xi) \sim (1, 0, 0)$.

For any $\zeta \in [-1, 1]$, we define the function

$$D[\zeta] = \left(r^2 - 2\zeta r \cos \theta + \zeta^2 \right) \circ \phi_{\mathrm{Poi}}. \tag{I.4.5}$$

By the definition of μH_1^{Poi} in (I.2.12), we have that

$$\mu H_1^{\mathrm{Poi}} = \frac{1}{\sqrt{D[0]}} - \frac{1-\mu}{\sqrt{D[\mu]}} - \frac{\mu}{\sqrt{D[\mu - 1]}}. \tag{I.4.6}$$

Lemma I.4.1. *For $|(L-1, \eta, \xi)| \ll 1$ and any $\zeta \in [-1, 1]$, one can split $D[\zeta]$ as*

$$D[\zeta] = D_0[\zeta] + D_1[\zeta] + D_2[\zeta] + D_{\geq 3}[\zeta],$$

where

$$\begin{aligned} D_0[\zeta](\lambda, L) =& L^4 - 2\zeta L^2 \cos \lambda + \zeta^2, \\ D_1[\zeta](\lambda, L, \eta, \xi) =& \eta \frac{\sqrt{2L^3}}{2} \left(3\zeta - 2L^2 e^{-i\lambda} - \zeta e^{-2i\lambda} \right) \\ &+ \xi \frac{\sqrt{2L^3}}{2} \left(3\zeta - 2L^2 e^{i\lambda} - \zeta e^{2i\lambda} \right), \\ D_2[\zeta](\lambda, L, \xi, \eta) =& -\eta^2 \frac{L e^{-i\lambda}}{4} \left(\zeta + 2L^2 e^{-i\lambda} + 3\zeta e^{-2i\lambda} \right) \\ &- \xi^2 \frac{L e^{i\lambda}}{4} \left(\zeta + 2L^2 e^{i\lambda} + 3\zeta e^{2i\lambda} \right) + \eta\xi L \left(3L^2 + 2\zeta \cos \lambda \right). \end{aligned}$$

Fix $\varrho \geq 0$. Then, for $|\operatorname{Im} \lambda| \leq \varrho$, the function $D_{\geq 3}[\zeta]$ is analytic and satisfies

$$|D_{\geq 3}[\zeta](\lambda, L, \eta, \xi)| \leq C \left| (\eta, \xi) \right|^3, \tag{I.4.7}$$

with $C = C(\varrho)$ a positive constant independent of $\zeta \in [-1, 1]$.

Proof of Lemma I.4.1. In view of the definition of $D[\zeta]$ in (I.4.5), we look for expansions for r^2 and $r \cos \theta$ (expressed in Poincaré coordinates) in powers of (η, ξ).

Let us consider first r^2. Taking into account that $r = L^2(1 - e \cos u)$ (see (I.2.5)) and the expansions in (I.4.4) we obtain

$$\begin{aligned} r^2 =& L^4 - L^3 \sqrt{2L} e^{-i\lambda}\eta - L^3 \sqrt{2L} e^{i\lambda}\xi + 3L^3 \eta\xi \\ &- \frac{L^3}{2} e^{-2i\lambda}\eta^2 - \frac{L^3}{2} e^{2i\lambda}\xi^2 + \mathcal{O}(e^{-i\lambda}\eta, e^{i\lambda}\xi)^3. \end{aligned} \tag{I.4.8}$$

Now, we compute an expansion for $r\cos\theta$. Taking into account (I.2.6) and (I.2.7),

$$r\cos\theta = L^2\left(\cos(g+u) - e\cos g - \left(\sqrt{1-e^2}-1\right)\sin u\sin g\right).$$

Notice that, since $\lambda = \ell+g$ and $u = \ell+e\sin u$, we have that $\cos(g+u) = \cos(\lambda+e\sin u)$ is analytic at $(\eta,\xi) = (0,0)$. Then, using (I.4.1), (I.4.2) and (I.4.4), we deduce

$$\begin{aligned} r\cos\theta = {} & L^2\cos\lambda - \eta\frac{\sqrt{2L^3}}{2}\left(1+ie^{-i\lambda}\sin\lambda\right) - \xi\frac{\sqrt{2L^3}}{2}\left(1-ie^{i\lambda}\sin\lambda\right) \\ & - \eta\xi L\cos\lambda + \frac{L}{4}\eta^2\left(e^{-i\lambda}+e^{-2i\lambda}\cos\lambda - 2ie^{-2i\lambda}\sin\lambda\right) \\ & + \xi^2\frac{L}{4}\left(e^{i\lambda}+e^{2i\lambda}\cos\lambda+2ie^{2i\lambda}\sin\lambda\right) + \mathcal{O}(e^{i\lambda}\eta, e^{-i\lambda}\xi)^3. \end{aligned} \tag{I.4.9}$$

Then, joinig the results in (I.4.8) and (I.4.9) with the definition of $D[\zeta]$ in (I.4.5), we obtain its expansion in (η,ξ). Moreover, since $D[\zeta]$ is analytic for $(L,\eta,\xi)\sim(1,0,0)$ and $|\mathrm{Im}\,\lambda|\leq\varrho$, the terms of order 3 satisfy the estimate in (I.4.7). □

Remark I.4.2. *Observe that the Hamiltonian $H^{\mathrm{Poi}} = H_0^{\mathrm{Poi}}+\mu H_1^{\mathrm{Poi}}$ in (I.2.12) is analytic away from collision with the primaries. By the decomposition of μH_1^{Poi} in (I.4.6), collisions with the primary S are given by the zeroes of the function $D[\mu]$ and collisions with P are given by the zeroes of $D[\mu-1]$.*

Since our analysis is performed for $|(L-1,\eta,\xi)|\leq\varepsilon\ll 1$ and $0<\mu\ll 1$, by Lemma I.4.1, one has

$$D[\mu] = 1+\mathcal{O}(\mu,\varepsilon), \qquad D[\mu-1] = 2+2\cos\lambda+\mathcal{O}(\mu,\varepsilon).$$

That is, collisions with S are not possible whereas collisions with P may take place when $\lambda\sim\pi$.

I.4.2 Proof of Proposition I.2.5

To prove Proposition I.2.5, we analyze the Hamiltonian H^{in} which is given (up to a constant) by

$$\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2}\left(H\circ\phi_{\mathrm{eq}}\circ\phi_{\mathrm{sep}}\circ\phi_{\mathrm{in}}\right),$$

where the changes ϕ_{eq}, ϕ_{sep} and ϕ_{in} are defined in (I.2.28), (I.2.29) and (I.2.33), and $H = H_0+H_1$ (see (I.2.21) and H_1 in (I.2.18)).

In the rest of the section, when performing changes of coordinates, to simplify notation, we omit the constant terms in the Hamiltonians.

Using the formulas for H_1^{Poi} in (I.4.6) and Lemma I.4.1, we split H_1^{Poi} into two terms: one for the perturbation coming from the massive primary (S) and the other coming from the small primary (P),

$$H_1^{\mathrm{Poi}} = H_1^{\mathrm{Poi},S} + H_1^{\mathrm{Poi},P},$$

which, recalling that $\mu=\delta^4$, are defined as

$$H_1^{\mathrm{Poi},S} = \frac{1}{\delta^4}\left(\frac{1}{\sqrt{D[0]}} - \frac{1-\delta^4}{\sqrt{D[\delta^4]}}\right) \quad \text{and} \quad H_1^{\mathrm{Poi},P} = -\frac{1}{\sqrt{D[\delta^4-1]}}. \tag{I.4.10}$$

We also define the Hamiltonian $H^{\mathrm{eq}} = H \circ \phi_{\mathrm{eq}}$, which can be split as

$$H^{\mathrm{eq}} = H_0 + R^{\mathrm{eq}} + H_1^{\mathrm{eq},P} + H_1^{\mathrm{eq},S},$$

with

$$\begin{aligned} H_1^{\mathrm{eq},*}(\lambda, \Lambda, x, y; \delta) &= H_1^{\mathrm{Poi},*} \circ \phi_{\mathrm{sc}} \circ \phi_{\mathrm{eq}}, \qquad \text{for } * = S, P, \\ R^{\mathrm{eq}}(\lambda, \Lambda, x, y; \delta) &= -V(\lambda) + \frac{1}{\delta^4} F_{\mathrm{p}}(\delta^2\Lambda + \delta^4 \mathfrak{L}_\Lambda(\delta)) \\ &\quad - 3\delta^2 \Lambda \mathfrak{L}_\Lambda(\delta) + \delta(x\mathfrak{L}_y(\delta) + y\mathfrak{L}_x(\delta)). \end{aligned} \tag{I.4.11}$$

We recall that ϕ_{sc} is the scaling given in (I.2.13), V is the potential in (I.2.15), F_{p} is the function (I.2.14) and $(\mathfrak{L}_\Lambda, \mathfrak{L}_x, \mathfrak{L}_y)$ are introduced in (I.2.19).

Then, the Hamiltonian H^{in} can be written as

$$H^{\mathrm{in}} = \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} \left(H_0 \circ \Psi + R^{\mathrm{eq}} \circ \Psi + H_1^{\mathrm{Poi},P} \circ \Phi + H_1^{\mathrm{Poi},S} \circ \Phi \right), \tag{I.4.12}$$

where

$$\Psi = \phi_{\mathrm{sep}} \circ \phi_{\mathrm{in}} \quad \text{and} \quad \Phi = \phi_{\mathrm{sc}} \circ \phi_{\mathrm{eq}} \circ \phi_{\mathrm{sep}} \circ \phi_{\mathrm{in}}.$$

In the following lemmas, we introduce expressions for the changes Ψ and Φ.

Lemma I.4.3. *The change of coordinates $\Psi = (\Psi_\lambda, \Psi_\Lambda, \Psi_x, \Psi_y)$ satisfies*

$$\begin{aligned} \Psi_\lambda(U) &= \pi + 3\alpha_+ \delta^{\frac{4}{3}} U^{\frac{2}{3}} \left(1 + g_\lambda(\delta^2 U)\right), \\ \Psi_\Lambda(U, W) &= -\frac{2\alpha_+}{3\delta^{\frac{2}{3}} U^{\frac{1}{3}}} \left(1 + g_\Lambda(\delta^2 U)\right) + \frac{\alpha_+ U^{\frac{1}{3}} W}{\delta^{\frac{2}{3}}} \left(1 + \widetilde{g}_\Lambda(\delta^2 U)\right), \\ \Psi_x(X) &= \delta^{\frac{1}{3}} \sqrt{2}\alpha_+ X, \\ \Psi_y(Y) &= \delta^{\frac{1}{3}} \sqrt{2}\alpha_+ Y, \end{aligned}$$

where $g_\lambda(z)$, $g_\Lambda(z)$, $\widetilde{g}_\Lambda(z) \sim \mathcal{O}(z^{\frac{2}{3}})$. Moreover, taking into account the time-parametrization of the separatrix (λ_h, Λ_h) given in (I.2.23)*, we have that*

$$\Lambda_h \circ \phi_{\mathrm{in}} = -\frac{2\alpha_+}{3\delta^{\frac{2}{3}} U^{\frac{1}{3}}} \left(1 + g_\Lambda(\delta^2 U)\right). \tag{I.4.13}$$

Lemma I.4.4. *The change of coordinates $\Phi = (\Phi_\lambda, \Phi_L, \Phi_\eta, \Phi_\xi)$ satisfies*

$$\begin{aligned} \Phi_\lambda(U) &= \Psi_\lambda(U), & \Phi_L(U, W) &= 1 + \delta^2 \Psi_\Lambda(U, W) + \delta^4 \mathfrak{L}_\Lambda(\delta), \\ \Phi_\eta(X) &= \delta \Psi_x(X) + \delta^4 \mathfrak{L}_x(\delta), & \Phi_\xi(Y) &= \delta \Psi_y(Y) + \delta^4 \mathfrak{L}_y(\delta), \end{aligned}$$

where $\Psi = (\Psi_\lambda, \Psi_\Lambda, \Psi_x, \Psi_y)$ is the change of coordinates given in Lemma I.4.3.

We omit the proofs of these lemmas since they are a straightforward consequence of Theorem I.2.2 and the definitions of the changes of coordinates (see (I.2.13), (I.2.28), (I.2.29) and (I.2.33)).

End of the proof of Proposition I.2.5. We analyze each component of (I.4.12).

We denote by $C(c_1, c_2) > 0$ any constant satisfying that there exist $b_0, \gamma_1, \gamma_2 > 0$ independent of c_1, c_2, δ such that $C(c_1, c_2) \leq b_0 c_1^{\gamma_1} c_2^{\gamma_2}$.

1. We compute the first term of the Hamiltonian H^{in} in (I.4.12). Since $H_{\mathrm{p}}(\lambda_h, \Lambda_h) = H_{\mathrm{p}}(0,0) = -\frac{1}{2}$ and taking into account (I.4.13), we have

$$\begin{aligned}\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} H_0 \circ \Psi =& \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2}\left(w - \frac{w^2}{6\Lambda_h^2(u)} + \frac{xy}{\delta^2}\right) \circ \phi_{\mathrm{in}} \\ =& W + XY - \frac{3}{4}U^{\frac{2}{3}}W^2\left(\frac{1}{1+g_\Lambda(\delta^2 U)}\right)^2 \\ =& W + XY - \frac{3}{4}U^{\frac{2}{3}}W^2 + \mathcal{O}\left(\delta^{\frac{4}{3}}U^{\frac{2}{3}}W^2\right).\end{aligned}$$

Since $|U| \leq c_1$ and $|W| \leq c_2$, the error term $\mathcal{O}(\delta^{\frac{4}{3}}U^{\frac{2}{3}}W^2)$ can be bounded by $C(c_1, c_2)\delta^{\frac{4}{3}}$.

For the other terms in (I.4.12), to simplify the notation, we are not specifying the dependence of the error terms on the variables (U, W, X, Y). Moreover, when referring to error terms of order $\mathcal{O}(\delta^a)$, we mean that they can be bounded by $C(c_1, c_2)\delta^a$.

2. For the second term of the Hamiltonian H^{in} in (I.4.12) (see by (I.4.11)) we have

$$\begin{aligned}\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} R^{\mathrm{eq}} \circ \Psi =& -\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} V(\Psi_\lambda) + \frac{1}{2\alpha_+^2\delta^{\frac{8}{3}}} F_{\mathrm{p}}\left(\delta^2\Psi_\Lambda + \delta^4 \mathfrak{L}_\Lambda(\delta)\right) \\ & - \frac{3\delta^{\frac{10}{3}}\mathfrak{L}_\Lambda(\delta)}{2\alpha_+^2}\Psi_\Lambda + \frac{\delta^{\frac{7}{3}}\mathfrak{L}_y(\delta)}{\sqrt{2}\alpha_+}\Psi_x + \frac{\delta^{\frac{7}{3}}\mathfrak{L}_x(\delta)}{\sqrt{2}\alpha_+}\Psi_y,\end{aligned} \tag{I.4.14}$$

where $F_{\mathrm{p}}(z) = \mathcal{O}(z^3)$ (see (I.2.14)) and $V(\lambda)$ is the potential given in (I.2.15).

First we analyze the potential term. By Lemma I.4.3, we have that

$$\begin{aligned}-\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} V(\Psi_\lambda(U)) =& \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2}\frac{1}{\sqrt{2+2\cos\Psi_\lambda(U)}} + \mathcal{O}(\delta^{\frac{4}{3}}) \\ =& \frac{\delta^{\frac{4}{3}}}{2\alpha_+^2}\left(9\alpha_+^2\delta^{\frac{8}{3}}U^{\frac{4}{3}}\left(1+g_\lambda(\delta^2 U)\right)^2 + \mathcal{O}\left(\delta^{\frac{16}{3}}U^{\frac{8}{3}}\right)\right)^{-\frac{1}{2}} + \mathcal{O}(\delta^{\frac{4}{3}}) \\ =& \frac{1}{3U^{\frac{2}{3}}} + \mathcal{O}(\delta^{\frac{4}{3}}).\end{aligned}$$

Then, since $c_1^{-1} \leq |U| \leq c_1$ and $|(W, X, Y)| \leq c_2$, by (I.4.14) and Lemma I.4.3,

$$\left|\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} R^{\mathrm{eq}} \circ \Psi - \frac{1}{3U^{\frac{2}{3}}}\right| \leq C(c_1, c_2)\delta^{\frac{4}{3}}.$$

3. We deal with the third term of the Hamiltonian H^{in} (see (I.4.10)). Since

$$|\Phi(U, W, X, Y) - (\pi, 1, 0, 0)| \leq C(c_1, c_2)\delta^{\frac{4}{3}} \quad \text{and} \quad |\mathrm{Im}\,\Phi_\lambda(U)| \leq C(c_1)\delta^{\frac{4}{3}},$$

the hypotheses of Lemma I.4.1 hold and therefore:

$$\begin{aligned}\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2}H_1^{\mathrm{Poi},P}\circ\Phi &= -\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2}\frac{1}{\sqrt{D[\delta^4-1]\circ\Phi}}\\ &= -\left(\frac{2\alpha_+}{\delta^{\frac{8}{3}}}\left(D_0+D_1+D_2+D_{\geq 3}\right)[\delta^4-1]\circ\Phi\right)^{-\frac{1}{2}}.\end{aligned}$$

We compute every term $D_j[\delta^4-1]$, $j=0,1,2,\geq 3$.

a) The term $D_0[\delta^4-1]$ satisfies

$$\begin{aligned}D_0[\delta^4-1](\lambda,L;\delta) =& L^4+2(1-\delta^4)L^2\cos\lambda+(1-\delta^4)^2\\ =& 2(1+\cos\lambda)+4(L-1)(1+\cos\lambda)+2(L-1)^2(3+\cos\lambda)\\ &+4(L-1)^3+(L-1)^4-2\delta^4(1+\cos\lambda)\\ &-4\delta^4(L-1)\cos\lambda-2\delta^4(L-1)^2\cos\lambda+\delta^8.\end{aligned}$$

Performing the change Φ, by Lemma I.4.4, we have that

$$\begin{aligned}\frac{2\alpha_+}{\delta^{\frac{8}{3}}}D_0[\delta^4-1]\circ\Phi &= \frac{2\alpha_+}{\delta^{\frac{8}{3}}}\left(2(1+\cos\Phi_\lambda)+4(\Phi_L-1)^2+\mathcal{O}(\delta^{\frac{4}{3}})\right)\\ &= 9U^{\frac{4}{3}}+4U^{\frac{2}{3}}W^2-\frac{16}{3}W+\frac{16}{9U^{\frac{2}{3}}}+\mathcal{O}(\delta^{\frac{4}{3}}).\end{aligned}$$

b) Analgously the term $D_1[\delta^4-1]$ satisfies

$$\begin{aligned}D_1[\delta^4-1] =& \eta\frac{\sqrt{2L^3}}{2}\Big[(-3-2e^{-i\lambda}+e^{-2i\lambda})-4(L-1)e^{-i\lambda}\\ &\qquad -2(L-1)^2e^{-i\lambda}+\delta^4(3-e^{-2i\lambda})\Big]\\ &+\xi\frac{\sqrt{2L^3}}{2}\Big[(-3-2e^{i\lambda}+e^{2i\lambda})-4(L-1)e^{i\lambda}\\ &\qquad -2(L-1)^2e^{i\lambda}+\delta^4(3-e^{2i\lambda})\Big]\end{aligned}$$

and therefore

$$\begin{aligned}\frac{2\alpha_+}{\delta^{\frac{8}{3}}}D_1[\delta^4-1]\circ\Phi =& \frac{2\alpha_+^2}{\delta^{\frac{4}{3}}}X\left(-4i\left(\Phi_\lambda-\pi\right)-4(\Phi_L-1)+\mathcal{O}(\delta^{\frac{4}{3}})\right)\\ &+\frac{2\alpha_+^2}{\delta^{\frac{4}{3}}}Y\left(4i\left(\Phi_\lambda-\pi\right)-4(\Phi_L-1)+\mathcal{O}(\delta^{\frac{4}{3}})\right)\\ =& X\left(-12iU^{\frac{2}{3}}+4U^{\frac{1}{3}}-\frac{8}{3U^{\frac{1}{3}}}\right)\\ &+Y\left(12iU^{\frac{2}{3}}+4U^{\frac{1}{3}}-\frac{8}{3U^{\frac{1}{3}}}\right)+\mathcal{O}(\delta^{\frac{4}{3}}).\end{aligned}$$

c) The term $D_2[\delta^4-1]$ satisfies

$$
\begin{aligned}
D_2[\delta^4-1] = &-\eta^2 \frac{Le^{-i\lambda}}{4}\left(-1+2L^2e^{-i\lambda}-3e^{-2i\lambda}+\delta^4\left(1+3e^{-2i\lambda}\right)\right)\\
&-\xi^2 \frac{Le^{-i\lambda}}{4}\left(-1+2L^2e^{i\lambda}-3e^{2i\lambda}+\delta^4\left(1+3e^{2i\lambda}\right)\right)\\
&+\eta\xi L\left(3L^2-2\cos\lambda+\delta^4 2\cos\lambda\right)
\end{aligned}
$$

and

$$\frac{2\alpha_+}{\delta^{\frac{8}{3}}} D_2[\delta^4-1]\circ\Phi = -3X^2-3Y^2+5XY+\mathcal{O}(\delta^{\frac{4}{3}}).$$

d) By the estimates of $D_{\geq 3}[\delta^4-1]$ in (I.4.7) and Lemma I.4.4,

$$\left|\frac{2\alpha_+}{\delta^{\frac{8}{3}}} D_3[\delta^4-1]\circ\Phi\right| \leq C(c_1,c_2)\delta^{\frac{4}{3}}.$$

Collecting these results, we conclude that

$$\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} H_1^{\mathrm{Poi},P}\circ\Phi = -\frac{1}{3U^{\frac{2}{3}}}\frac{1}{\sqrt{1+\mathcal{J}(U,W,X,Y)}} + \mathcal{O}(\delta^{\frac{4}{3}}),$$

where the function $\mathcal{J}$ is given in (I.2.36).

4. Proceeding analogously as for $H_1^{\mathrm{Poi},P}$, it can be checked that

$$\left|\frac{\delta^{\frac{4}{3}}}{2\alpha_+^2} H_1^{\mathrm{Poi},S}\circ\Phi\right| \leq C(c_1,c_2)\delta^{\frac{4}{3}}.$$

□

Chapter I.5

Analysis of the inner equation

We split the proof of Theorem I.2.7 into two parts. In Section I.5.1 we prove the existence of the solutions Z_0^{u} and Z_0^{s} and the estimates in (I.2.44). In Section I.5.2, we provide the asymptotic formula for the difference $\Delta Z_0 = Z_0^{\mathrm{u}} - Z_0^{\mathrm{s}}$ given in (I.2.45). For both parts, we follow the approach given in [BS08].

Throughout this chapter, we fix the $\beta_0 \in (0, \frac{\pi}{2})$ appearing in the definition of the domains $\mathcal{D}_\kappa^{\mathrm{u}}$ and $\mathcal{D}_\kappa^{\mathrm{s}}$ in (I.2.42) and $\mathcal{E}_\kappa$ in (I.2.43). We denote the components of all the functions and operators by a numerical sub-index $f = (f_1, f_2, f_3)^T$, unless stated otherwise. In order to simplify the notation, we denote by C any positive constant independent of κ.

I.5.1 Existence of suitable solutions of the inner equation

From now on we deal only with the analysis for Z_0^{u}. The analysis for Z_0^{s} is analogous.

I.5.1.1 Preliminaries and set up

The invariance equation (I.2.39), that is $\partial_U Z_0^{\mathrm{u}} = \mathcal{A} Z_0^{\mathrm{u}} + \mathcal{R}[Z_0^{\mathrm{u}}]$, can be written as $\mathcal{L} Z_0^{\mathrm{u}} = \mathcal{R}[Z_0^{\mathrm{u}}]$ where $\mathcal{L}$ is the linear operator

$$\mathcal{L}\varphi = (\partial_U - \mathcal{A})\varphi. \tag{I.5.1}$$

Notice that if we can construct a left-inverse of $\mathcal{L}$ in an appropriate Banach space, we can write (I.2.39) as a fixed point equation to be able apply the Banah fixed point theorem.

Given $\nu \in \mathbb{R}$ and $\kappa > 0$, we define the norm

$$\|\varphi\|_\nu = \sup_{U \in \mathcal{D}_\kappa^{\mathrm{u}}} |U^\nu \varphi(U)|,$$

where the domain $\mathcal{D}_\kappa^{\mathrm{u}}$ is given in (I.2.42), and we introduce the Banach space

$$\mathcal{X}_\nu = \{\varphi : \mathcal{D}_\kappa^{\mathrm{u}} \to \mathbb{C} \ : \ \varphi \text{ analytic}, \ \|\varphi\|_\nu < +\infty\}.$$

Next lemma, proven in [Bal06], gives some properties of these Banach spaces. We use this lemma throughout the chapter without mentioning it.

Lemma I.5.1. *Let $\kappa > 0$ and $\nu, \eta \in \mathbb{R}$. The following statements hold:*

1. *If $\nu > \eta$, then $\mathcal{X}_\nu \subset \mathcal{X}_\eta$ and $\|\varphi\|_\eta \leq (\kappa \cos\beta_0)^{\eta-\nu} \|\varphi\|_\nu$.*
2. *If $\varphi \in \mathcal{X}_\nu$ and $\zeta \in \mathcal{X}_\eta$, then the product $\varphi\zeta \in \mathcal{X}_{\nu+\eta}$ and $\|\varphi\zeta\|_{\nu+\eta} \leq \|\varphi\|_\nu \|\zeta\|_\eta$.*

In the following lemma, we introduce a left-inverse of the operator $\mathcal{L}$ in (I.5.1).

Lemma I.5.2. *Consider the operator*

$$\mathcal{G}[\varphi](U) = \left(\int_{-\infty}^{U} \varphi_1(S)dS, \int_{-\infty}^{U} e^{-i(S-U)}\varphi_2(S)dS, \int_{-\infty}^{U} e^{i(S-U)}\varphi_3(S)dS \right)^T.$$

Fix $\eta > 1$, $\nu > 0$ and $\kappa \geq 1$. Then, $\mathcal{G} : \mathcal{X}_\eta \times \mathcal{X}_\nu \times \mathcal{X}_\nu \to \mathcal{X}_{\eta-1} \times \mathcal{X}_\nu \times \mathcal{X}_\nu$ is a continuous linear operator and is a left-inverse of $\mathcal{L}$.

Moreover, there exist a constant $C > 0$ such that

1. *If $\varphi \in \mathcal{X}_\eta$, then $\mathcal{G}_1[\varphi] \in \mathcal{X}_{\eta-1}$ and $\|\mathcal{G}_1[\varphi]\|_{\eta-1} \leq C \|\varphi\|_\eta$.*
2. *If $\varphi \in \mathcal{X}_\nu$ and $j = 2, 3$, then $\mathcal{G}_j[\varphi] \in \mathcal{X}_\nu$ and $\|\mathcal{G}_j[\varphi]\|_\nu \leq C \|\varphi\|_\nu$.*

Proof. It follows the same lines as the proof of Lemma 4.6 in [Bal06]. □

Let us then define the fixed point operator

$$\mathcal{F} = \mathcal{G} \circ \mathcal{R}. \tag{I.5.2}$$

A solution of $Z_0^{\mathrm{u}} = \mathcal{F}[Z_0^{\mathrm{u}}]$ belonging to $\mathcal{X}_\eta \times \mathcal{X}_\nu \times \mathcal{X}_\nu$ with $\eta, \nu > 0$ satisfies equation (I.2.39) and the asymptotic condition (I.2.41). Therefore, to prove the first part of Theorem I.2.7 and the asymptotic estimates in (I.2.44), we look for a fixed point of the operator $\mathcal{F}$ in the Banach space

$$\mathcal{X}_\times = \mathcal{X}_{\frac{8}{3}} \times \mathcal{X}_{\frac{4}{3}} \times \mathcal{X}_{\frac{4}{3}},$$

endowed with the norm

$$\|\varphi\|_\times = \|\varphi_1\|_{\frac{8}{3}} + \|\varphi_2\|_{\frac{4}{3}} + \|\varphi_3\|_{\frac{4}{3}}.$$

Proposition I.5.3. *There exists $\kappa_0 > 0$ such that for any $\kappa \geq \kappa_0$, the fixed point equation $Z_0^{\mathrm{u}} = \mathcal{F}[Z_0^{\mathrm{u}}]$ has a solution $Z_0^{\mathrm{u}} \in \mathcal{X}_\times$. Moreover, there exists a constant $b_3 > 0$, independent of κ, such that*

$$\|Z_0^{\mathrm{u}}\|_\times \leq b_3.$$

Remark I.5.4. *Notice that $\mathcal{D}_\kappa^{\mathrm{u}} \subseteq \mathcal{D}_{\kappa_0}^{\mathrm{u}}$ when $\kappa \geq \kappa_0$ (see (I.2.42)). Then, for some $\nu \in \mathbb{R}$, if $\zeta \in \mathcal{X}_\nu$ (defined for κ) then $\zeta \in \mathcal{X}_\nu$ (defined for κ_0). This allows us to take κ as big as we need.*

I.5.1.2 Proof of Proposition I.5.3

We first state a technical lemma whose proof is postponed until Section I.5.3.1. For $\varrho > 0$, we define the closed ball

$$B(\varrho) = \left\{ \varphi \in \mathcal{X}_\times \ : \ \|\varphi\|_\times \leq \varrho \right\}.$$

Lemma I.5.5. *Let $\mathcal{R}$ be the operator defined in (I.2.40). Then, for $\varrho > 0$ and for $\kappa > 0$ big enough, there exists a constant $C > 0$ such that, for any $Z_0 \in B(\varrho)$,*

$$\|\mathcal{R}_1[Z_0]\|_{\frac{11}{3}} \leq C, \qquad \|\mathcal{R}_j[Z_0]\|_{\frac{4}{3}} \leq C, \qquad j = 2, 3,$$

and

$$\|\partial_W \mathcal{R}_1[Z_0]\|_3 \le C, \qquad \|\partial_X \mathcal{R}_1[Z_0]\|_{\frac{7}{3}} \le C, \qquad \|\partial_Y \mathcal{R}_1[Z_0]\|_{\frac{7}{3}} \le C,$$
$$\|\partial_W \mathcal{R}_j[Z_0]\|_{\frac{2}{3}} \le C, \qquad \|\partial_X \mathcal{R}_j[Z_0]\|_2 \le C, \qquad \|\partial_Y \mathcal{R}_j[Z_0]\|_2 \le C, \quad j = 2,3.$$

The next lemma gives properties of the operator $\mathcal{F}$.

Lemma I.5.6. *Let $\mathcal{F}$ be the operator defined in (I.5.2). Then, for $\kappa > 0$ big enough, there exists a constant $b_4 > 0$ independent of κ such that*

$$\|\mathcal{F}[0]\|_\times \le b_4.$$

Moreover, for $\varrho > 0$ and $\kappa > 0$ big enough, there exists a constant $b_5 > 0$ independent of κ such that, for any $Z_0 = (W_0, X_0, Y_0)^T$, $\widetilde{Z}_0 = (\widetilde{W}_0, \widetilde{X}_0, \widetilde{Y}_0)^T \in B(\varrho) \subset \mathcal{X}_\times$,

$$\|\mathcal{F}_1[Z_0] - \mathcal{F}_1[\widetilde{Z}_0]\|_{\frac{8}{3}} \le b_5 \left(\frac{1}{\kappa^2} \|W_0 - \widetilde{W}_0\|_{\frac{8}{3}} + \|X_0 - \widetilde{X}_0\|_{\frac{4}{3}} + \|Y_0 - \widetilde{Y}_0\|_{\frac{4}{3}} \right),$$
$$\|\mathcal{F}_j[Z_0] - \mathcal{F}_j[\widetilde{Z}_0]\|_{\frac{4}{3}} \le \frac{b_5}{\kappa^2} \|Z_0 - \widetilde{Z}_0\|_\times, \qquad j = 2,3.$$

Proof. The estimate for $\mathcal{F}[0]$ is a direct consequence of Lemmas I.5.2 and I.5.5.

To estimate the Lipschitz constant, we first estimate each component $\mathcal{R}_j[Z_0] - \mathcal{R}_j[\widetilde{Z}_0]$ separately for $j = 1,2,3$. By the mean value theorem we have

$$\mathcal{R}_j[Z_0] - \mathcal{R}_j[\widetilde{Z}_0] = \left[\int_0^1 D\mathcal{R}_j[sZ_0 + (1-s)\widetilde{Z}_0] ds \right] (Z_0 - \widetilde{Z}_0).$$

Then, for $j = 2,3$, we have

$$\begin{aligned}
\|\mathcal{R}_1[Z_0] - \mathcal{R}_1[\widetilde{Z}_0]\|_{\frac{11}{3}} \le{}& \|W_0 - \widetilde{W}_0\|_{\frac{8}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_W \mathcal{R}_1[\varphi]\|_1 \\
&+ \|X_0 - \widetilde{X}_0\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_X \mathcal{R}_1[\varphi]\|_{\frac{7}{3}} + \|Y_0 - \widetilde{Y}_0\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_Y \mathcal{R}_1[\varphi]\|_{\frac{7}{3}}, \\
\|\mathcal{R}_j[Z_0] - \mathcal{R}_j[\widetilde{Z}_0]\|_{\frac{4}{3}} \le{}& \|W_0 - \widetilde{W}_0\|_{\frac{8}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_W \mathcal{R}_j[\varphi]\|_{-\frac{4}{3}} \\
&+ \|X_0 - \widetilde{X}_0\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_X \mathcal{R}_j[\varphi]\|_0 + \|Y_0 - \widetilde{Y}_0\|_{\frac{4}{3}} \sup_{\varphi \in B(\varrho)} \|\partial_Y \mathcal{R}_j[\varphi]\|_0.
\end{aligned}$$

Applying Lemma I.5.5, we obtain

$$\|\mathcal{R}_1[Z_0] - \mathcal{R}_1[\widetilde{Z}_0]\|_{\frac{11}{3}} \le C \left(\frac{1}{\kappa^2} \|W_0 - \widetilde{W}_0\|_{\frac{8}{3}} + \|X_0 - \widetilde{X}_0\|_{\frac{4}{3}} + \|Y_0 - \widetilde{Y}_0\|_{\frac{4}{3}} \right),$$
$$\|\mathcal{R}_j[Z_0] - \mathcal{R}_j[\widetilde{Z}_0]\|_{\frac{4}{3}} \le \frac{C}{\kappa^2} \|Z_0 - \widetilde{Z}_0\|_\times, \quad j = 2,3.$$

Finally, applying Lemma I.5.2, we obtain the estimates in the lemma. □

Lemma I.5.6 shows that, by assuming $\kappa > 0$ big enough, the operators $\mathcal{F}_2$ and $\mathcal{F}_3$ have Lipschitz constant less than 1. However, this is not the case for $\mathcal{F}_1$. To overcome this problem, we apply a Gauss-Seidel argument and define a new operator

$$\widetilde{\mathcal{F}}[Z_0] = \widetilde{\mathcal{F}}[(W_0, X_0, Y_0)] = \begin{pmatrix} \mathcal{F}_1[W_0, \mathcal{F}_2[Z_0], \mathcal{F}_3[Z_0]] \\ \mathcal{F}_2[Z_0] \\ \mathcal{F}_3[Z_0] \end{pmatrix},$$

which has the same fixed points as $\mathcal{F}$ and turns out to be contractive in a suitable ball.

End of the proof of Proposition I.5.3. We first obtain an estimate for $\|\widetilde{\mathcal{F}}[0]\|_\times$. Notice that

$$\widetilde{\mathcal{F}}[0] = \mathcal{F}[0] + \big(\widetilde{\mathcal{F}}[0] - \mathcal{F}[0]\big) = \mathcal{F}[0] + \big(\mathcal{F}_1\left[0, \mathcal{F}_2[0], \mathcal{F}_3[0]\right] - \mathcal{F}_1[0], 0, 0\big)^T.$$

Then, by Lemma I.5.6, $(0, \mathcal{F}_2[0], \mathcal{F}_3[0])^T \in \mathcal{X}_\times$ and

$$\begin{aligned}
\|\widetilde{\mathcal{F}}[0]\|_\times &\leq \|\mathcal{F}[0]\|_\times + \|\mathcal{F}_1[0, \mathcal{F}_2[0], \mathcal{F}_3[0]] - \mathcal{F}_1[0]\|_{\frac{8}{3}} \\
&\leq \|\mathcal{F}[0]\|_\times + C\,\|\mathcal{F}_2[0]\|_{\frac{4}{3}} + C\,\|\mathcal{F}_3[0]\|_{\frac{4}{3}} \leq C\,\|\mathcal{F}[0]\|_\times .
\end{aligned}$$

Thus, we can fix $\varrho > 0$ such that

$$\|\widetilde{\mathcal{F}}[0]\|_\times \leq \frac{\varrho}{2}.$$

Now, we prove that the operator $\widetilde{\mathcal{F}}$ is contractive in $B(\varrho) \subset \mathcal{X}_\times$. By Lemma I.5.6 and assuming $\kappa > 0$ big enough, we have that for $Z_0, \widetilde{Z}_0 \in B(\varrho)$,

$$\begin{aligned}
\|\widetilde{\mathcal{F}}_1[Z_0] - \widetilde{\mathcal{F}}_1[\widetilde{Z}_0]\|_{\frac{8}{3}} \leq& \frac{C}{\kappa^2}\|W_0 - \widetilde{W}_0\|_{\frac{8}{3}} + \|\mathcal{F}_2[Z_0] - \mathcal{F}_2[\widetilde{Z}_0]\|_{\frac{4}{3}} + \|\mathcal{F}_3[Z_0] - \mathcal{F}_3[\widetilde{Z}_0]\|_{\frac{4}{3}} \\
\leq& \frac{C}{\kappa^2}\|W_0 - \widetilde{W}_0\|_{\frac{8}{3}} + \frac{C}{\kappa^2}\|Z_0 - \widetilde{Z}_0\|_\times \leq \frac{C}{\kappa^2}\|Z_0 - \widetilde{Z}_0\|_\times, \\
\|\widetilde{\mathcal{F}}_j[Z_0] - \widetilde{\mathcal{F}}_j[\widetilde{Z}_0]\|_{\frac{4}{3}} \leq& \frac{C}{\kappa^2}\|Z_0 - \widetilde{Z}_0\|_\times, \qquad \text{for } j = 2, 3.
\end{aligned}$$

Then, there exists $\kappa_0 > 0$ such that for $\kappa \geq \kappa_0$, the operator $\widetilde{\mathcal{F}} : B(\varrho) \to B(\varrho)$ is well defined and contractive. Therefore $\widetilde{\mathcal{F}}$ has a fixed point $Z_0^{\mathrm{u}} \in B(\varrho) \subset \mathcal{X}_\times$. □

I.5.2 Asymptotic formula for the difference

The strategy to prove the second part of Theorem I.2.7 is divided in three steps. In Section I.5.2.1 we characterize $\Delta Z_0 = Z_0^{\mathrm{u}} - Z_0^{\mathrm{s}}$ as a solution of a linear homogeneous equation. In Section I.5.2.2, we prove that ΔZ_0 is in fact the unique solution of this linear equation in a suitable Banach space. Finally, in Section I.5.2.3, we introduce a Banach subspace of the previous one (with exponential weights) to obtain exponentially small estimates for ΔZ_0.

I.5.2.1 A homogeneous linear equation for ΔZ_0

By Theorem I.2.7, the difference $\Delta Z_0(U) = Z_0^{\mathrm{u}}(U) - Z_0^{\mathrm{s}}(U)$ is well defined for $U \in \mathcal{E}_\kappa$ (see (I.2.43)). Since Z_0^{u}, Z_0^{s} satisfy the same invariance equation (I.2.39), their difference ΔZ_0 satisfies

$$\partial_U \Delta Z_0 = \mathcal{A}\Delta Z_0 + R(U)\Delta Z_0,$$

where

$$R(U) = \int_0^1 D_Z\mathcal{R}[sZ_0^{\mathrm{u}} + (1-s)Z_0^{\mathrm{s}}](U)ds, \tag{I.5.3}$$

and $\mathcal{A}$ and $\mathcal{R}$ are given in (I.2.37) and (I.2.40), respectively. We denote by R_1, R_2 and R_3, the rows of the matrix R.

By the method of variation of parameters, there exists $c = (c_w, c_x, c_y)^T \in \mathbb{C}^3$ such that

$$\Delta Z_0(U) = e^{\mathcal{A}U}\left(c + \int_{U_0}^{U} e^{-\mathcal{A}S} R(S)\Delta Z_0(S)dS\right).$$

By Proposition I.5.3, $\Delta Z_0 = Z_0^{\mathrm{u}} - Z_0^{\mathrm{s}}$ satisfies $\lim_{\operatorname{Im} U \to -\infty} \Delta Z_0(U) = 0$. Therefore ΔZ_0 satisfies

$$\Delta Z_0 = \Delta Z_{\mathrm{init}} + \mathcal{I}[\Delta Z_0], \tag{I.5.4}$$

where $\mathcal{I}$ is the linear operator

$$\mathcal{I}[\varphi](U) = \begin{pmatrix} \displaystyle\int_{-i\infty}^{U} \langle R_1(S), \varphi(S)\rangle dS \\ \displaystyle e^{iU}\int_{-i\infty}^{U} e^{-iS}\langle R_2(S), \varphi(S)\rangle dS \\ \displaystyle e^{-iU}\int_{-i\kappa}^{U} e^{iS}\langle R_3(S), \varphi(S)\rangle dS \end{pmatrix}, \tag{I.5.5}$$

and ΔZ_{init} is the function

$$\Delta Z_{\mathrm{init}}(U) = (0, 0, c_y e^{-iU})^T = (0, 0, e^{\kappa}\Delta Y_0(-i\kappa)e^{-iU})^T.$$

I.5.2.2 Characterization of ΔZ_0 as a fixed point

Given $\nu \in \mathbb{R}$ and $\kappa > 0$, we define the norm

$$\|\varphi\|_{\nu} = \sup_{U \in \mathcal{E}_\kappa} |U^{\nu}\varphi(U)|,$$

where the domain $\mathcal{E}_\kappa$ is given in (I.2.43), and we introduce the Banach space

$$\mathcal{Y}_{\nu} = \{\varphi : \mathcal{E}_\kappa \to \mathbb{C} \ : \ \varphi \text{ analytic }, \|\varphi\|_{\nu} < +\infty\}.$$

Note that $\mathcal{Y}_\nu$ satisfy analogous properties as the ones in Lemma I.5.1. In this section, we use this lemma without mentioning it.

We state a technical lemma, whose proof is postponed to Section I.5.3.2.

Lemma I.5.7. *Let $\mathcal{I}$ be the operator defined in (I.5.5). Then, for $\kappa > 0$ big enough, there exists a constant $b_6 > 0$ independent of κ such that, for $\Psi \in \mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$,*

$$\begin{aligned}
\|\mathcal{I}_1[\Psi]\|_{\frac{8}{3}} &\leq b_6\left(\frac{1}{\kappa^2}\|\Psi_1\|_{\frac{8}{3}} + \|\Psi_2\|_{\frac{4}{3}} + \|\Psi_3\|_{\frac{4}{3}}\right), \\
\|\mathcal{I}_j[\Psi]\|_{\frac{4}{3}} &\leq \frac{b_6}{\kappa^2}\left(\|\Psi_1\|_{\frac{8}{3}} + \|\Psi_2\|_{\frac{4}{3}} + \|\Psi_3\|_{\frac{4}{3}}\right), \qquad j = 2, 3.
\end{aligned}$$

These estimates characterize ΔZ_0 as the unique solution of (I.5.4) in $\mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$.

Lemma I.5.8. *For $\kappa > 0$ big enough, ΔZ_0 is the unique solution of equation (I.5.4) belonging to $\mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$. In particular,*

$$\Delta Z_0 = \sum_{n \geq 0} \mathcal{I}^n[\Delta Z_{\mathrm{init}}].$$

Proof. By Theorem I.2.7, for $\kappa > 0$ big enough, ΔZ_0 is a solution of equation (I.5.4) which satisfies $\Delta Z_0 = Z_0^{\mathrm{u}} - Z_0^{\mathrm{s}} \in \mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$. Then, it only remains to prove that equation (I.5.4) has a unique solution in $\mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$. To this end, it is enough to show that the operator $\mathcal{I}$ is contractive with a suitable norm in $\mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$. Taking

$$\|\Psi\|_{\times} = \|\Psi_1\|_{\frac{8}{3}} + \kappa \|\Psi_2\|_{\frac{4}{3}} + \kappa \|\Psi_3\|_{\frac{4}{3}} ,$$

Lemma I.5.7 implies

$$\|\mathcal{I}[\Psi]\|_{\times} \le \frac{C}{\kappa} \|\Psi\|_{\times}$$

and, taking κ big enough, the result is proven. □

I.5.2.3 Exponentially small estimates for ΔZ_0

Once we have proved that ΔZ_0 is the unique solution of (I.5.4) in $\mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$, we use this equation to obtain exponentially small estimates for ΔZ_0.

For any $\nu \in \mathbb{R}$, we consider the norm

$$[\![\varphi]\!]_{\nu} = \sup_{U \in \mathcal{E}_\kappa} \left| U^{\nu} e^{iU} \varphi(U) \right| ,$$

and the associated Banach space

$$\mathcal{Z}_{\nu} = \{\varphi : \mathcal{E}_\kappa \to \mathbb{C} \ : \ \varphi \text{ analytic}, \ [\![\varphi]\!]_{\nu} < +\infty\} .$$

Moreover, for $\nu_1, \nu_2, \nu_3 \in \mathbb{R}$, we consider the product space

$$\mathcal{Z}_{\nu_1,\nu_2,\nu_3} = \mathcal{Z}_{\nu_1} \times \mathcal{Z}_{\nu_2} \times \mathcal{Z}_{\nu_3}, \quad \text{with} \quad [\![\varphi]\!]_{\nu_1,\nu_2,\nu_3} = \textstyle\sum_{j=1}^{3} [\![\varphi_j]\!]_{\nu_j} .$$

Next lemma, gives some properties of these Banach spaces. It follows the same lines as Lemma I.5.1.

Lemma I.5.9. *Let $\kappa > 0$ and $\nu, \eta \in \mathbb{R}$. The following statements hold:*

1. *If $\nu > \eta$, then $\mathcal{Z}_\nu \subset \mathcal{Z}_\eta$ and $[\![\varphi]\!]_{\eta} \le (\kappa \cos \beta_0)^{\eta-\nu} [\![\varphi]\!]_{\nu}$.*
2. *If $\varphi \in \mathcal{Z}_\nu$ and $\zeta \in \mathcal{Y}_\eta$, then the product $\varphi\zeta \in \mathcal{Z}_{\nu+\eta}$ and $[\![\varphi\zeta]\!]_{\nu+\eta} \le [\![\varphi]\!]_{\nu} \|\zeta\|_{\eta}$.*
3. *If $\varphi \in \mathcal{Z}_\nu$ then $e^{iU}\varphi \in \mathcal{Y}_\nu$ and $\|e^{iU}\varphi\|_{\nu} = [\![\varphi]\!]_{\nu}$.*

The next lemma analyzes how the operator $\mathcal{I}$ acts on the space $\mathcal{Z}_{\frac{4}{3},0,0}$. Its proof is postponed to Section I.5.3.2.

Lemma I.5.10. *Let $\mathcal{I}$ be the operator defined in (I.5.5). For $\kappa > 0$ big enough, there exists a constant $b_7 > 0$ independent of κ such that, for $\Psi \in \mathcal{Z}_{\frac{4}{3},0,0}$,*

$$[\![\mathcal{I}_1[\Psi]]\!]_{\frac{7}{3}} \le b_7 [\![\Psi]\!]_{\frac{4}{3},0,0} , \quad [\![\mathcal{I}_2[\Psi]]\!]_{2} \le b_7 [\![\Psi]\!]_{\frac{4}{3},0,0} , \quad [\![\mathcal{I}_3[\Psi]]\!]_{0} \le \frac{b_7}{\kappa} [\![\Psi]\!]_{\frac{4}{3},0,0} .$$

Moreover, there exists $\widetilde{\Theta}(\kappa) \in \mathbb{C}$ (depending on Ψ), such that

$$\mathcal{I}_3[\Psi] - e^{-iU}\widetilde{\Theta}(\kappa) \in \mathcal{Z}_1 .$$

End of the proof of the second part of Theorem I.2.7. Lemma I.5.10 implies that operator $\mathcal{I} : \mathcal{Z}_{\frac{4}{3},0,0} \to \mathcal{Z}_{\frac{4}{3},0,0}$ is well defined and contractive. Indeed, taking $\kappa > 0$ big enough and $\Psi \in \mathcal{Z}_{\frac{4}{3},0,0}$,

$$\begin{aligned} [\![\mathcal{I}[\Psi]]\!]_{\frac{4}{3},0,0} &= [\![\mathcal{I}_1[\Psi]]\!]_{\frac{4}{3}} + [\![\mathcal{I}_2[\Psi]]\!]_0 + [\![\mathcal{I}_3[\Psi]]\!]_0 \\ &\leq \frac{C}{\kappa} [\![\mathcal{I}_1[\Psi]]\!]_{\frac{7}{3}} + \frac{C}{\kappa^2} [\![\mathcal{I}_2[\Psi]]\!]_2 + [\![\mathcal{I}_3[\Psi]]\!]_0 \leq \frac{C}{\kappa} [\![\Psi]]\!]_{\frac{4}{3},0,0} . \end{aligned} \tag{I.5.6}$$

Therefore, since $\Delta Z_{\text{init}} = (0,0,c_y e^{-iU})^T \in \mathcal{Z}_{\frac{4}{3},0,0}$, Lemma I.5.8 and (I.5.6) imply that

$$\Delta Z_0 = (\Delta W_0, \Delta X_0, \Delta Y_0)^T = \sum_{n\geq 0} \mathcal{I}^n[\Delta Z_{\text{init}}] \in \mathcal{Z}_{\frac{4}{3},0,0}.$$

Lemma I.5.10 implies $\mathcal{I} : \mathcal{Z}_{\frac{4}{3},0,0} \to \mathcal{Z}_{\frac{7}{3},2,0} \subset \mathcal{Z}_{\frac{4}{3},0,0}$, which allows to give better estimates for ΔZ_0. Indeed, we have that $\Delta Z_0 - \Delta Z_{\text{init}} = \mathcal{I}[\Delta Z_0] \in \mathcal{Z}_{\frac{7}{3},2,0}$, which implies

$$\Delta W_0 = \mathcal{I}_1[\Delta Z_0] \in \mathcal{Z}_{\frac{7}{3}}, \qquad \Delta X_0 = \mathcal{I}_2[\Delta Z_0] \in \mathcal{Z}_2,$$

Moreover, by the second statement in Lemma I.5.10, there exists $\widetilde{\Theta}(\kappa)$ such that

$$\Delta Y_0 - c_y e^{-iU} - \widetilde{\Theta}(\kappa) e^{-iU} \in \mathcal{Z}_1.$$

Calling $\Theta = c_y + \widetilde{\Theta}(\kappa)$ we have that $\Delta Y_0(U) - \Theta e^{-iU} \in \mathcal{Z}_1$, and, therefore $\Theta = \lim_{\operatorname{Im} U \to -\infty} \Delta Y_0(U) e^{iU}$, which is independent of κ. Then, ΔZ_0 is of the form

$$\Delta Z_0(U) = e^{-iU}\Big((0,0,\Theta)^T + \chi(U)\Big), \quad \text{with} \quad \chi \in \mathcal{Y}_{\frac{7}{3}} \times \mathcal{Y}_2 \times \mathcal{Y}_1.$$

Now we prove that, if there exists $U_0 \in \mathcal{E}_\kappa$ such that $\Delta Z_0(U_0) \neq 0$, then $\Theta \neq 0$. This implies $\Delta Z_0(U) \neq 0$ for all $U \in \mathcal{E}_\kappa$, since ΔZ_0 is a solution of an homogeneous linear differential equation. Therefore $c_y \neq 0$. Indeed, $c_y = 0$ would imply $\Delta Z_{\text{init}} = 0$ and, by Lemma I.5.8, one could conclude $\Delta Z_0 \equiv 0$.

Thus, it only remains to prove that $c_y \neq 0$ implies $\Theta \neq 0$. By Lemma I.5.8,

$$\Delta Z_0 - \Delta Z_{\text{init}} = \sum_{n\geq 1} \mathcal{I}^n[\Delta Z_{\text{init}}].$$

In addition, by the estimate (I.5.6), $[\![\mathcal{I}_3]\!]_{\frac{4}{3}} \leq \frac{1}{4}$ if $\kappa > 0$ is big enough. Since $\Delta Y_{\text{init}} = c_y e^{-iU}$, we deduce that

$$[\![\Delta Y_0 - \Delta Y_{\text{init}}]\!]_0 \leq \sum_{n\geq 1} \frac{1}{4^n} [\![\Delta Y_{\text{init}}]\!]_0 = \frac{1}{3} |c_y| ,$$

and, by the definition of the norm $[\![\cdot]\!]_0$, for any $U \in \mathcal{E}_\kappa$,

$$\left| e^{iU} \Delta Y_{\text{init}}(U) \right| - \left| e^{iU} \Delta Y_0(U) \right| \leq \frac{1}{3} |c_y| .$$

Hence, using that $e^{iU} \Delta Y_0 = \Theta + \chi_3(U)$ with $\chi_3 \in \mathcal{Y}_1$ and $e^{iU} \Delta Y_{\text{init}}(U) = c_y$, we have that for all $U \in \mathcal{E}_\kappa$,

$$\left| e^{iU} \Delta Y_0(U) \right| = |\Theta + \chi_3(U)| \geq \left| e^{iU} \Delta Y_{\text{init}}(U) \right| - \frac{1}{3} |c_y| = \frac{2}{3} |c_y| .$$

Finally, taking $\operatorname{Im}(U) \to -\infty$, we obtain that $|\Theta| \geq \frac{2}{3} |c_y| > 0$. $\square$

I.5.3 Proof of the technical lemmas

We devote this section to prove Lemma I.5.5 of Section I.5.3.1 and Lemmas I.5.7 and I.5.10 of Section I.5.3.2.

I.5.3.1 Proof of Lemma I.5.5

Fix $\varrho > 0$ and take $Z_0 = (W_0, X_0, Y_0)^T \in B(\varrho) \subset \mathcal{X}_\times$. By the definition (I.2.40) of $\mathcal{R}$,

$$\mathcal{R}[Z_0](U) = \left(\frac{f_1(U,Z_0)}{1+g(U,Z_0)}, \frac{\widetilde{f}_2(U,Z_0)}{1+g(U,Z_0)}, \frac{\widetilde{f}_3(U,Z_0)}{1+g(U,Z_0)} \right), \tag{I.5.7}$$

where

$$\widetilde{f}_2(U,Z_0) = f_2(U,Z_0) - iX_0 g(U,Z_0), \qquad \widetilde{f}_3(U,Z_0) = f_3(U,Z_0) + iY_0 g(U,Z_0),$$

with $g = \partial_W \mathcal{K}$, $f = (-\partial_U \mathcal{K}, i\partial_Y \mathcal{K}, -i\partial_X \mathcal{K})^T$ and $\mathcal{K}$ is the Hamiltonian given in (I.2.35) in terms of the function $\mathcal{J}$ (see (I.2.36)).

We first estimate $\mathcal{J}$ and its derivatives. For $\kappa > 0$ big enough, we have

$$|\mathcal{J}(U,Z_0)| \leq \frac{C}{U^2}, \qquad |1+\mathcal{J}(U,Z_0)| \geq 1 - \frac{C}{\kappa^2} \geq \frac{1}{2}.$$

Moreover, its derivatives satisfy

$$|\partial_U \mathcal{J}(U,Z_0)| \leq \frac{C}{|U|^3}, \quad |\partial_W \mathcal{J}(U,Z_0)| \leq \frac{C}{|U|^{\frac{4}{3}}}, \quad |\partial_X \mathcal{J}(U,Z_0)|, |\partial_Y \mathcal{J}(U,Z_0)| \leq \frac{C}{|U|^{\frac{2}{3}}},$$

and

$$\begin{aligned}
|\partial_{UW} \mathcal{J}(U,Z_0)| &\leq \frac{C}{|U|^{\frac{7}{3}}}, & |\partial_{UX} \mathcal{J}(U,Z_0)| &\leq \frac{C}{|U|^{\frac{5}{3}}}, & |\partial_{UY} \mathcal{J}(U,Z_0)| &\leq \frac{C}{|U|^{\frac{5}{3}}}, \\
\left|\partial_W^2 \mathcal{J}(U,Z_0)\right| &\leq \frac{C}{|U|^{\frac{2}{3}}}, & |\partial_{WX} \mathcal{J}(U,Z_0)| &\leq \frac{C}{|U|}, & |\partial_{WY} \mathcal{J}(U,Z_0)| &\leq \frac{C}{|U|}, \\
\left|\partial_X^2 \mathcal{J}(U,Z_0)\right| &\leq \frac{C}{|U|^{\frac{4}{3}}}, & |\partial_{XY} \mathcal{J}(U,Z_0)| &\leq \frac{C}{|U|^{\frac{4}{3}}}, & \left|\partial_Y^2 \mathcal{J}(U,Z_0)\right| &\leq \frac{C}{|U|^{\frac{4}{3}}}.
\end{aligned}$$

Using these estimates, we obtain the following bounds for $g, f_1, \widetilde{f}_2$ and $\widetilde{f}_3$,

$$\begin{aligned}
|g(U,Z_0)| &= \left| -\frac{3}{2} U^{\frac{2}{3}} W_0 + \frac{1}{6U^{\frac{2}{3}}} \frac{\partial_W \mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} \right| \leq \frac{C}{|U|^2}, \\
|f_1(U,Z_0)| &= \left| \frac{W_0^2}{2U^{\frac{1}{3}}} - \frac{2}{9U^{\frac{5}{3}}} \frac{\mathcal{J}}{\sqrt{1+\mathcal{J}}(1+\sqrt{1+\mathcal{J}})} - \frac{1}{6U^{\frac{2}{3}}} \frac{\partial_U \mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} \right| \leq \frac{C}{|U|^{\frac{11}{3}}}, \\
\left|\widetilde{f}_2(U,Z_0)\right| &= \left| \frac{i}{6U^{\frac{2}{3}}} \frac{\partial_Y \mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} - iX_0 g(U,Z_0) \right| \leq \frac{C}{|U|^{\frac{4}{3}}}, \\
\left|\widetilde{f}_3(U,Z_0)\right| &= \left| -\frac{i}{6U^{\frac{2}{3}}} \frac{\partial_X \mathcal{J}}{(1+\mathcal{J})^{\frac{3}{2}}} + iY_0 g(U,Z_0) \right| \leq \frac{C}{|U|^{\frac{4}{3}}}.
\end{aligned}$$

Analogously, we obtain estimates for the derivatives,

$$
\begin{aligned}
&|\partial_W g(U,Z_0)| \leq C\,|U|^{\frac{2}{3}}, && |\partial_X g(U,Z_0)| \leq \frac{C}{|U|^{\frac{5}{3}}}, && |\partial_Y g(U,Z_0)| \leq \frac{C}{|U|^{\frac{5}{3}}},\\
&|\partial_W f_1(U,Z_0)| \leq \frac{C}{|U|^{3}}, && |\partial_X f_1(U,Z_0)| \leq \frac{C}{|U|^{\frac{7}{3}}}, && |\partial_Y f_1(U,Z_0)| \leq \frac{C}{|U|^{\frac{7}{3}}},\\
&\left|\partial_W \widetilde{f}_2(U,Z_0)\right| \leq \frac{C}{|U|^{\frac{2}{3}}}, && \left|\partial_X \widetilde{f}_2(U,Z_0)\right| \leq \frac{C}{|U|^{2}}, && \left|\partial_Y \widetilde{f}_2(U,Z_0)\right| \leq \frac{C}{|U|^{2}},\\
&\left|\partial_W \widetilde{f}_3(U,Z_0)\right| \leq \frac{C}{|U|^{\frac{2}{3}}}, && \left|\partial_X \widetilde{f}_3(U,Z_0)\right| \leq \frac{C}{|U|^{2}}, && \left|\partial_Y \widetilde{f}_3(U,Z_0)\right| \leq \frac{C}{|U|^{2}}.
\end{aligned}
$$

Using these results we estimate the components of $\mathcal{R}$ in (I.5.7),

$$
\|\mathcal{R}_1[Z_0]\|_{\frac{11}{3}} = \left\| \frac{f_1(\cdot,Z_0)}{1+g(\cdot,Z_0)} \right\|_{\frac{11}{3}} \leq C, \qquad \left\|\mathcal{R}_j^{\mathrm{in}}[Z_0]\right\|_{\frac{4}{3}} = \left\| \frac{\widetilde{f}_j(\cdot,Z_0)}{1+g(\cdot,Z_0)} \right\|_{\frac{4}{3}} \leq C,
$$

for $j=2,3$. Moreover,

$$
\begin{aligned}
&\|\partial_W \mathcal{R}_1[Z_0]\|_3 = \left\| \frac{\partial_W f_1}{1+g} - \frac{f_1 \partial_W g}{(1+g)^2} \right\|_3 \leq C, \quad \|\partial_X \mathcal{R}_1[Z_0]\|_{\frac{7}{3}} = \left\| \frac{\partial_X f_1}{1+g} - \frac{f_1 \partial_X g}{(1+g)^2} \right\|_{\frac{7}{3}} \leq C,\\
&\|\partial_W \mathcal{R}_2[Z_0]\|_{\frac{2}{3}} = \left\| \frac{\partial_W \widetilde{f}_2}{1+g} - \frac{\widetilde{f}_2 \partial_W g}{(1+g)^2} \right\|_{\frac{2}{3}} \leq C, \quad \|\partial_X \mathcal{R}_2[Z_0]\|_{2} = \left\| \frac{\partial_X \widetilde{f}_2}{1+g} - \frac{\widetilde{f}_2 \partial_X g}{(1+g)^2} \right\|_{2} \leq C.
\end{aligned}
$$

Analogously, we obtain the rest of the estimates,

$$
\|\partial_Y \mathcal{R}_1[Z_0]\|_{\frac{7}{3}}, \ \|\partial_Y \mathcal{R}_2[Z_0]\|_{2}, \ \|\partial_W \mathcal{R}_3[Z_0]\|_{\frac{2}{3}}, \ \|\partial_X \mathcal{R}_3[Z_0]\|_{2}, \ \left\|\partial_Y \mathcal{R}_3^{\mathrm{in}}[Z_0]\right\|_{2} \leq C.
$$

□

I.5.3.2 Proof of Lemmas I.5.7 and I.5.10

Let us introduce, for $\kappa > 0$ and $\alpha \geq 0$, the following linear operators,

$$
\mathcal{B}_\alpha[\Psi](U) = e^{i\alpha U} \int_{-i\infty}^{U} e^{-i\alpha S} \Psi(S) dS, \quad \widetilde{\mathcal{B}}[\Psi](U) = e^{-iU} \int_{-i\kappa}^{U} e^{iS} \Psi(S) dS. \tag{I.5.8}
$$

The following lemma is proven in [Bal06].

Lemma I.5.11. *Fix $\eta > 1$, $\nu > 0$, $\alpha > 0$ and $\kappa > 1$. Then, the following operators are well defined*

$$
\mathcal{B}_0 : \mathcal{Y}_\eta \to \mathcal{Y}_{\eta-1}, \qquad \mathcal{B}_\alpha : \mathcal{Y}_\nu \to \mathcal{Y}_\nu, \qquad \widetilde{\mathcal{B}} : \mathcal{Y}_\nu \to \mathcal{Y}_\nu
$$

and there exists a constant $C > 0$ such that

$$
\|\mathcal{B}_0[\Psi]\|_{\eta-1} \leq C\,\|\Psi\|_\eta, \qquad \|\mathcal{B}_\alpha[\Psi]\|_\nu \leq C\,\|\Psi\|_\nu, \qquad \|\widetilde{\mathcal{B}}[\Psi]\|_\nu \leq C\,\|\Psi\|_\nu.
$$

It is clear that $\mathcal{I}_1[\Psi] = \mathcal{B}_0[\langle R_1, \Psi\rangle]$, $\mathcal{I}_2[\Psi] = \mathcal{B}_1[\langle R_2, \Psi\rangle]$ and $\mathcal{I}_3[\Psi] = \widetilde{\mathcal{B}}[\langle R_3, \Psi\rangle]$ (see (I.5.3) and (I.5.5)). Thus, we use this lemma to prove Lemmas I.5.7 and I.5.10.

Proof of Lemma I.5.7. By the definition of the operator R and Lemma I.5.5, we have that

$$\begin{aligned} &\|R_{1,1}\|_3 \leq C, \qquad \|R_{1,2}\|_{\frac{7}{3}} \leq C, \qquad \|R_{1,3}\|_{\frac{7}{3}} \leq C, \\ &\|R_{j,1}\|_{\frac{2}{3}} \leq C, \qquad \|R_{j,2}\|_2 \leq C, \qquad \|R_{j,3}\|_2 \leq C, \qquad \text{for } j = 2,3. \end{aligned} \tag{I.5.9}$$

Then, by Lemma I.5.11, for κ big enough and $\Psi \in \mathcal{Y}_{\frac{8}{3}} \times \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_{\frac{4}{3}}$, we have that

$$\begin{aligned} \|\mathcal{I}_1[\Psi]\|_{\frac{8}{3}} &= \|\mathcal{B}_0[\langle R_1, \Psi\rangle]\|_{\frac{8}{3}} \leq C \|\langle R_1, \Psi\rangle\|_{\frac{11}{3}} \\ &\leq C \left(\|R_{1,1}\|_1 \|\Psi_1\|_{\frac{8}{3}} + \|R_{1,2}\|_{\frac{7}{3}} \|\Psi_2\|_{\frac{4}{3}} + \|R_{1,3}\|_{\frac{7}{3}} \|\Psi_3\|_{\frac{4}{3}} \right) \\ &\leq C \left(\frac{1}{\kappa^2} \|\Psi_1\|_{\frac{8}{3}} + \|\Psi_2\|_{\frac{4}{3}} + \|\Psi_3\|_{\frac{4}{3}} \right), \end{aligned}$$

which gives the first estimate of the lemma. Analogously, by Lemma I.5.11,

$$\begin{aligned} \|\mathcal{I}_2[\Psi]\|_{\frac{4}{3}} &= \|\mathcal{B}_1[\langle R_2, \Psi\rangle]\|_{\frac{4}{3}} \leq C \|\langle R_2, \Psi\rangle\|_{\frac{4}{3}}, \\ \|\mathcal{I}_3[\Psi]\|_{\frac{4}{3}} &= \|\widetilde{\mathcal{B}}[\langle R_3, \Psi\rangle]\|_{\frac{4}{3}} \leq C \|\langle R_3, \Psi\rangle\|_{\frac{4}{3}}, \end{aligned}$$

and applying (I.5.9), for $j = 2,3$, we have

$$\begin{aligned} \|\langle R_j, \Psi\rangle\|_{\frac{4}{3}} &\leq \|R_{j,1}\|_{-\frac{4}{3}} \|\Psi_1\|_{\frac{8}{3}} + \|R_{j,2}\|_0 \|\Psi_2\|_{\frac{4}{3}} + \|R_{j,3}\|_0 \|\Psi_3\|_{\frac{4}{3}} \\ &\leq \frac{C}{\kappa^2} \left(\|\Psi_1\|_{\frac{8}{3}} + \|\Psi_2\|_{\frac{4}{3}} + \|\Psi_3\|_{\frac{4}{3}} \right), \end{aligned}$$

which gives the second and third estimates of the lemma. □

Proof of Lemma I.5.10. Let us consider $\Psi \in \mathcal{Z}_{\frac{4}{3},0,0}$ and define

$$\Phi(U) = e^{iU} R(U) \Psi(U),$$

in such a way that, by the definition of the operator $\mathcal{B}_\alpha$ in (I.5.8),

$$\begin{aligned} e^{iU}\mathcal{I}_1[\Psi](U) &= e^{iU} \int_{-i\infty}^{U} e^{-iS} \Phi_1(S) dS = \mathcal{B}_1[\Phi_1], \\ e^{iU}\mathcal{I}_2[\Psi](U) &= e^{i2U} \int_{-i\infty}^{U} e^{-i2S} \Phi_2(S) dS = \mathcal{B}_2[\Phi_2], \\ e^{iU}\mathcal{I}_3[\Psi](U) &= \int_{-i\kappa}^{U} \Phi_3(S) dS. \end{aligned} \tag{I.5.10}$$

Since $e^{iU}\Psi \in \mathcal{Y}_{\frac{4}{3}} \times \mathcal{Y}_0 \times \mathcal{Y}_0$, by the estimates in (I.5.9), we have that, for $j = 2,3$,

$$\begin{aligned} \|\Phi_1\|_{\frac{7}{3}} &\leq \|R_{1,1}\|_1 \left\|e^{iU}\Psi_1\right\|_{\frac{4}{3}} + \sum_{k=2,3} \|R_{1,k}\|_{\frac{7}{3}} \left\|e^{iU}\Psi_k\right\|_0 \leq C \left[\!\left[\Psi\right]\!\right]_{\frac{4}{3},0,0}, \\ \|\Phi_j\|_2 &\leq \|R_{j,1}\|_{\frac{2}{3}} \left\|e^{iU}\Psi_1\right\|_{\frac{4}{3}} + \sum_{k=2,3} \|R_{j,k}\|_2 \left\|e^{iU}\Psi_k\right\|_0 \leq C \left[\!\left[\Psi\right]\!\right]_{\frac{4}{3},0,0}. \end{aligned} \tag{I.5.11}$$

Therefore, Lemma I.5.11 and (I.5.10) imply

$$\begin{aligned}\llbracket \mathcal{I}_1[\Psi] \rrbracket_{\frac{7}{3}} &= \|\mathcal{B}_1[\Phi_1]\|_{\frac{7}{3}} \leq C \|\Phi_1\|_{\frac{7}{3}} \leq C \llbracket \Psi \rrbracket_{\frac{4}{3},0,0}, \\ \llbracket \mathcal{I}_2[\Psi] \rrbracket_2 &= \|\mathcal{B}_2[\Phi_2]\|_2 \leq C \|\Phi_2\|_2 \leq C \llbracket \Psi \rrbracket_{\frac{4}{3},0,0}.\end{aligned}$$

Now, we deal with operator $\mathcal{I}_3$. Notice that, by the definition of the operator $\mathcal{B}_\alpha$ in (I.5.8) and (I.5.10), we have that

$$e^{iU}\mathcal{I}_3[\Psi](U) = \int_{-i\kappa}^{-i\infty} \Phi_3(S)dS + \int_{-i\infty}^{U} \Phi_3(S)dS = -\mathcal{B}_0[\Phi_3](-i\kappa) + \mathcal{B}_0[\Phi_3](U).$$

Then, by Lemma I.5.11 and using the estimates (I.5.11), we obtain

$$\begin{aligned}\llbracket \mathcal{I}_3[\Psi] \rrbracket_0 &\leq |\mathcal{B}_0[\Phi_3](-i\kappa)| + \|\mathcal{B}_0[\Phi_3]\|_0 \leq 2 \|\mathcal{B}_0[\Phi_3]\|_0 \\ &\leq \frac{C}{\kappa} \|\mathcal{B}_0[\Phi_3]\|_1 \leq \frac{C}{\kappa} \|\Phi_3\|_2 \leq \frac{C}{\kappa} \llbracket \Psi \rrbracket_{\frac{4}{3},0,0}.\end{aligned}$$

Finally, taking $\widetilde{\Theta}(\kappa) = -\mathcal{B}_0[\Phi_3](-i\kappa)$, we conclude

$$\left\llbracket \mathcal{I}_3[\Psi](U) - e^{-iU}\widetilde{\Theta}(\kappa) \right\rrbracket_1 = \|\mathcal{B}_0[\Phi_3]\|_1 \leq C \|\Phi_3\|_2 \leq C \llbracket \Psi \rrbracket_{\frac{4}{3},0,0}.$$

$\square$

Part II

Breakdown of homoclinic orbits to L_3

Abstract

The Restricted 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies called the primaries. If one assumes that the primaries perform circular motions and that all three bodies are coplanar, one has the Restricted Planar Circular 3-Body Problem (RPC3BP). In rotating coordinates, it can be modeled by a two degrees of freedom Hamiltonian, which has five critical points called the Lagrange points $L_1, \ldots, L_5$.

The Lagrange point L_3 is a saddle-center critical point which is collinear with the primaries and beyond the largest of the two. In this part, we obtain an asymptotic formula for the distance between the stable and unstable manifolds of L_3 for small values of the mass ratio $0 < \mu \ll 1$. In particular we show that L_3 cannot have (one round) homoclinic orbits.

If the ratio between the masses of the primaries μ is small, the hyperbolic eigenvalues of L_3 are weaker, by a factor of order $\sqrt{\mu}$, than the elliptic ones. This rapidly rotating dynamics makes the distance between manifolds exponentially small with respect to $\sqrt{\mu}$. Thus, classical perturbative methods (i.e the Melnikov-Poincaré method) can not be applied.

The obtention of this asymptotic formula relies on the results obtained in Part I on the complex singularities of the homoclinic of a certain averaged equation and on the associated inner equation.

In this part, we relate the solutions of the inner equation to the analytic continuation of the parameterizations of the invariant manifolds of L_3 via complex matching techniques. We complete the proof of the asymptotic formula for their distance showing that its dominant term is the one given by the analysis of the inner equation.

Chapter II.1

Introduction

The Restricted Circular 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies, called the primaries, which perform a circular motion. If one also assumes that the massless body moves on the same plane as the primaries one has the Restricted Planar Circular 3-Body Problem (RPC3BP).

Let us name the two primaries S (star) and P (planet) and normalize their masses so that $m_S = 1-\mu$ and $m_P = \mu$, with $\mu \in \left(0, \frac{1}{2}\right]$. Choosing a suitable rotating coordinate system, the positions of the primaries can be fixed at $q_S = (\mu, 0)$ and $q_P = (\mu - 1, 0)$. Then, the position and momenta of the third body, $(q,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, are governed by the Hamiltonian system associated to the Hamiltonian

$$h(q,p;\mu) = \frac{||p||^2}{2} - q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{(1-\mu)}{||q-(\mu,0)||} - \frac{\mu}{||q-(\mu-1,0)||}. \tag{II.1.1}$$

Note that this Hamiltonian is autonomous. The conservation of h corresponds to the preservation of the classical Jacobi constant.

For $\mu > 0$, it is a well known fact that (II.1.1) has five critical points, usually called Lagrange points (see Figure II.1.1). On an inertial (non-rotating) system of coordinates, the Lagrange points correspond to periodic dynamics with the same period as the two primaries, i.e on a 1:1 mean motion resonance. The three collinear Lagrange points, L_1, L_2 and L_3, are of center-saddle type whereas, for small μ, the triangular ones, L_4 and L_5, are of center-center type (see, for instance, [Sze67]).

Due to its interest in astrodynamics, a lot of attention has been paid to the study of the invariant manifolds associated to the points L_1 and L_2 (see [KLM+00; GLM+01; CGM+04]). The dynamics around the points L_4 and L_5 has also been heavily studied since, due to its stability, it is common to find objects orbiting around these points (for instance the Trojan and Greek Asteroids associated to the pair Sun-Jupiter, see [GDF+89; CG90; RG06]). Since the point L_3 is located "at the other side" of the massive primary, it has received somewhat less attention. However, the associated invariant manifolds (more precisely its center-stable and center-unstable invariant manifolds) play an important role in the dynamics of the RPC3BP since they act as boundaries of *effective stability* of the stability domains around L_4 and L_5 (see [GJM+01; SST13]). The invariant manifolds of L_3 play also a fundamental role in creating transfer orbits from the small primary to L_3 in the RPC3BP (see [HTL07; TFR+10]) or between primaries in the Bicircular 4-Body Problem (see [JN20; JN21]).

Moreover, being far from collision, the dynamics close to the Lagrange point L_3 and its invariant manifolds for small μ are rather similar to that of other mean motion resonances which play an important role in creating instabilities in the Solar system, see [FGK+16]. On the contrary, since the points L_1 and L_2 are close to collision for small μ, the analysis of the associated dynamics is quite different.

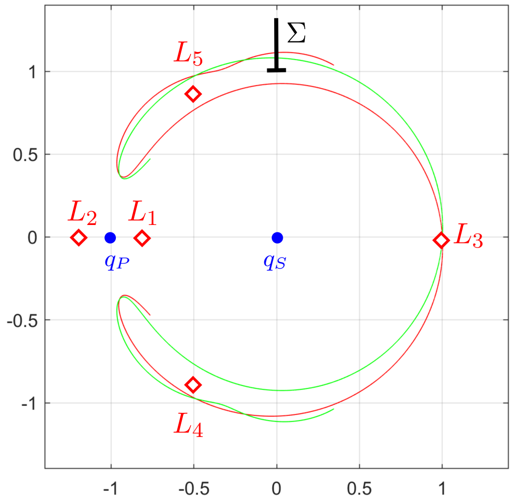


FIGURE II.1.1: Projection onto the q-plane of the Lagrange points (red) and the unstable (blue) and stable (green) manifolds of L_3, for $\mu = 0.0028$.

Over the past years, one of the main focus of study of the dynamics "close" to L_3 and its invariant manifolds has been the so called "horseshoe-shaped orbits", first considered in [Bro11], which are quasi-periodic orbits that encompass the critical points L_4, L_3 and L_5. The interest on these types of orbits arise when modeling the motion of co-orbital satellites, the most famous being Saturn's satellites Janus and Epimetheus, and near Earth asteroids. Recently, in [NPR20], the authors have proved the existence of 2-dimensional elliptic invariant tori on which the trajectories mimic the motions followed by Janus and Epimetheus (see also [DM81a; DM81b; LO01; CH03; BM05; BO06; BFP13; CPY19]).

Rather than looking at stable motions "close to" L_3 as [NPR20], the goal of this work is rather different: its objective is to prove the breakdown of homoclinic connections to L_3. Indeed, since L_3 is a center-saddle critical point, it possesses 1-dimensional unstable and stable manifolds, which we denote by $W^{\mathrm{u}}(\mu)$ and $W^{\mathrm{s}}(\mu)$, respectively, and a 2-dimensional center manifold. Theorem II.1.1 below gives an asymptotic formula for the distance between the stable and unstable invariant manifolds (at a suitable transverse section) for mass ratio $\mu > 0$ small enough.

II.1.1 The distance between the invariant manifolds of L_3

The one dimensional unstable and stable invariant manifolds of L_3 have two branches each (see Figure II.1.1). One pair circumvents L_5, which we denote by $W^{\mathrm{u},+}(\mu)$ and $W^{\mathrm{s},+}(\mu)$, and the other, $W^{\mathrm{u},-}(\mu)$ and $W^{\mathrm{s},-}(\mu)$, circumvents L_4. Since the Hamiltonian system associated to the Hamiltonian h is reversible with respect to the involution

$$\Phi(q, p; t) = (q_1, -q_2, -p_1, p_2; -t),$$

the $+$ branches of the invariant manifolds are symmetric with respect to the $-$ branches. Thus, we restrict our analysis to the positive branches.

To measure the distance between $W^{\mathrm{u/s},+}(\mu)$, we consider the symplectic polar change of coordinates

$$q = r\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, \qquad p = R\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} - \frac{G}{r}\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}, \tag{II.1.2}$$

where R is the radial linear momentum and G is the angular momentum.

We consider the 3-dimensional section

$$\Sigma = \left\{(r,\theta,R,G) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R}^2 \,:\, r>1,\, \theta = \frac{\pi}{2}\right\}$$

and denote by $(r_*^{\mathrm{u}}, \frac{\pi}{2}, R_*^{\mathrm{u}}, G_*^{\mathrm{u}})$ and $(r_*^{\mathrm{s}}, \frac{\pi}{2}, R_*^{\mathrm{s}}, G_*^{\mathrm{s}})$ the first crossing of the invariant manifolds with this section.

The next theorem measures the distance between these points for $0<\mu\ll 1$.

Theorem II.1.1. *There exists $\mu_0>0$ such that, for $\mu\in(0,\mu_0)$,*

$$\|(r_*^{\mathrm{u}}, R_*^{\mathrm{u}}, G_*^{\mathrm{u}}) - (r_*^{\mathrm{s}}, R_*^{\mathrm{s}}, G_*^{\mathrm{s}})\| = \sqrt[3]{4}\,\mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}}\left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log\mu|}\right)\right],$$

where:

- *The constant $A>0$ is the real-valued integral*

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x}\sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}\,dx \approx 0.177744. \tag{II.1.3}$$

- *The constant $\Theta\in\mathbb{C}$ is the Stokes constant associated to the inner equation analyzed in Theorem I.2.7 in Part I and in Theorem II.3.13 below.*

Remark II.1.2. *We can prove the same result for any section*

$$\Sigma(\theta_*) = \left\{(r,\theta,R,G) \in \mathbb{R}\times\mathbb{T}\times\mathbb{R}^2 \,:\, r>1,\, \theta = \theta_*\right\},$$

with $\theta_*\in(0,\theta_0)$ and $\theta_0 = \arccos\left(\frac{1}{2}-\sqrt{2}\right)$ (the value of μ_0 depends on how close to the endpoints of the interval θ_* is). The section $\theta=\theta_0$ is close to the "turning point" of the invariant manifolds (see Figure II.1.1).

The constant A in (II.1.3) is derived from the values of the complex singularities of the separatrix of certain integrable averaged system studied in detail in Theorem I.2.2 in Part I. The results obtained about this separatrix are summarized in Theorem II.3.1 below.

The origin of the constant Θ appearing in Theorem II.1.1 is explained in Theorem I.2.7, which analyzes the so-called inner equation. Moreover, in Remark I.2.8 it is seen, by a numerical computation, that $|\Theta|\approx 1.63$. We expect that one should be able to prove that $|\Theta|\neq 0$ by means of rigorous computer computations (see [BCG+22]). Note that $|\Theta|\neq 0$ implies that there are not primary (i.e. one round) homoclinic orbits to L_3.

A fundamental problem in dynamical systems is to prove whether a given model has chaotic dynamics (for instance a Smale horseshoe). For many physically relevant models this is usually remarkably difficult. This is the case of many Celestial Mechanics models, where most of the known chaotic motions have been found in nearly integrable regimes where there is an unperturbed problem which already presents some form of "hyperbolicity". This is the case in the vicinity of collision orbits (see for example [Moe89; BM06; Bol06; Moe07]) or close to parabolic orbits (which allows to construct chaotic/oscillatory motions), see [Sit60; Ale76; Lli80; Mos01; GMS16; GSM+17; GPS+21]. There are also several results in regimes far from integrable which rely on computer assisted proofs [Ari02; WZ03; Cap12; GZ19]. The problem tackled in this work is radically different. Indeed, if one takes the limit $\mu \to 0$ in (II.1.1) one obtains the classical integrable Kepler problem in the elliptic regime, where no hyperbolicity is present. Instead, the (weak) hyperbolicity is created by the $\mathcal{O}(\mu)$ perturbation, which can be captured considering an integrable averaged Hamiltonian along the $1:1$ mean motion resonance[1].

One of the classical methods to construct chaotic dynamics is the Smale-Birkhoff homoclinic theorem by proving the existence of transverse homoclinic orbits to invariant objects, most commonly, periodic orbits. Certainly the breakdown of homoclinic orbits to the critical point L_3 given by Theorem II.1.1 does not lead to the existence of chaotic orbits. However, one should expect that Theorem II.1.1 implies that there exist Lyapunov periodic orbits exponentially close to L_3 whose stable and unstable invariant manifolds intersect transversally. This would create chaotic motions "exponentially close" to L_3 and its invariant manifolds (see Part III).

As already mentioned, Theorem II.1.1 rules out the existence of primary homoclinic connections to L_3 in the RPC3BP for $0 < \mu \ll 1$. However, it does not prevent the existence of multiround homoclinic orbits, that is homoclinic orbits which pass close to L_3 multiple times. It has been conjectured (see for instance [BMO09], where the authors analyze this problem numerically) that multi-round homoclinic connections to L_3 should exist for a sequence of values $\{\mu_k\}_{k\in\mathbb{N}}$ satisfying $\mu_k \to 0$ as $k \to \infty$.

A first step towards proving Arnold diffusion along the $1:1$ mean motion resonance in the 3-Body Problem? Consider the 3-Body Problem in the planetary regime, that is one massive body (the Sun) and two small bodies (the planets) performing approximate ellipses (including the "Restricted limit" when one of planets has mass zero). A fundamental problem is to assert whether such configuration is stable (i.e. is the Solar system stable?). Thanks to Arnold-Herman-Féjoz KAM Theorem, many of such configurations are stable, see [Arn63; Féj04]. However, it is widely expected that there should be strong instabilities created by Arnold diffusion mechanisms (as conjectured by Arnold in [Arn64]). In particular, it is widely believed that one of the main sources of such instabilities dynamics are the mean motion resonances, where the period of the two planets is resonant (i.e. rationally dependent) [FGK+16].

The RPC3BP has too low dimension (2 degrees of freedom) to possess Arnold diffusion. However, since it can be seen as a first order for higher dimensional models, the analysis performed in this work can be seen as a humble first step towards constructing Arnold diffusion in the $1:1$ mean motion resonance. In this resonance, the RPC3BP has a normally hyperbolic invariant manifold given by the center manifold of the Lagrange point L_3. This normally hyperbolic invariant manifold is foliated by the classical Lyapunov periodic orbits. One should expect that the techniques developed in the present work would allow to prove that the invariant manifolds of these periodic orbits intersect transversally within the corresponding energy level of (II.1.1). Still, this is a much harder problem than the one considered in this work and the technicalities involved would be considerable.

[1]The $1:1$ averaged Hamiltonian has been also studied to obtain "good" approximations for the global dynamics in the $1:1$ resonant zone, see for example [RNP16; PA21] and the references therein.

This transversality would not lead to Arnold diffusion due to the low dimension of the RPC3BP. However, if one considers either the Restricted Spatial Circular 3-Body Problem with small $\mu > 0$ which has three degrees of freedom, the Restricted Planar Elliptic 3-Body Problem with small $\mu > 0$ and eccentricity of the primaries $e_0 > 0$, which has two and a half degrees of freedom, or the "full" planar 3-Body Problem (i.e. all three masses positive, two small) which has three degrees of freedom (after the symplectic reduction by the classical first integrals) one should be able to construct orbits with a drastic change in angular momentum (or inclination in the spatial setting).

In the Restricted Planar Elliptic 3-Body Problem the change of angular momentum would imply the transition of the zero mass body orbit from a close to circular ellipse to a more eccentric one. In the full 3BP, due to total angular momentum conservation, the angular momentum would be transferred from one body to the other changing both osculating ellipses. This behavior would be analogous to that of [FGK+16] for the $3:1$ and $1:7$ resonances. In that paper, the transversality between the invariant manifolds of the normally hyperbolic invariant manifold was checked numerically for the realistic Sun-Jupiter mass ratio $\mu = 10^{-3}$. Arnold diffusion instabilities have been analyzed numerically for the Restricted Spatial Circular 3-Body Problem in [TSS14].

II.1.2 The strategy to prove Theorem II.1.1

The main difficulty in proving Theorem II.1.1 is that the distance between the stable and unstable manifolds of L_3 is exponentially small with respect to $\sqrt{\mu}$ (this is also usually known as a *beyond all orders* phenomenon). This implies that the classical Melnikov Method [GH83] to detect the breakdown of homoclinics cannot be applied.

To prove Theorem II.1.1, we follow the strategy of exponentially small splitting of separatrices (already outlined in Part I) which goes back to the seminal work by Lazutkin [Laz84; Laz05]. See Chapter I.1 for a list of references on the recent developments in the field of exponentially small splitting of separatrices. In particular, we follow similar strategies of those in [BFG+12; BCS13].

In the present work the first order of the difference between manifolds is not given by the Melnikov function. Instead, we must derive and analyze an inner equation which provides the dominant term of this distance. As a consequence, we need to "match" (i.e. compare) certain solutions of the inner equation with the parameterizations of the perturbed invariant manifolds.

The first part of the proof, that was completed in Part I, dealt with the following steps:

A. We perform a change of coordinates to capture the slow-fast dynamics of the system. The first order of the new Hamiltonian has a saddle point with an homoclinic connection (also known as separatrix) and a fast harmonic oscillator.

B. We study the analytical continuation of the time-parametrization of the separatrix of this first order. In particular, we obtain its maximal strip of analyticity and the singularities at the boundary of this strip.

C. We derive the inner equation.

D. We study two special solutions which will be "good approximation" of the perturbed invariant manifolds near the singularities of the unperturbed separatrix (see Step F below).

The remaining steps necessary to complete the proof of Theorem II.1.1 are the following:

E We prove the existence of the analytic continuation of the parametrizations of the invariant manifolds of L_3, $W^{\mathrm{u},+}(\delta)$ and $W^{\mathrm{s},+}(\delta)$, in an appropriate complex domain called boomerang domain. This domain contains a segment of the real line and intersects a sufficiently small neighborhood of the singularities of the unperturbed separatrix.

F. By using complex matching techniques, we show that, close to the singularities of the unperturbed separatrix, the solutions of the inner equation obtained in Step D are "good approximations" of the parameterizations of the perturbed invariant manifolds obtained in Step E.

G. We obtain an asymptotic formula for the difference between the perturbed invariant manifolds by proving that the dominant term comes from the difference between the solutions of the inner equation.

The structure of this part goes as follows. In Chapter II.2 we perform the change of coordinates introduced in Step A and state Theorem II.2.2, which is a reformulation of Theorem II.1.1 in this new set of variables. Then, in Chapter II.3, we state the results concerning Steps B, C and D above (which are proven in Part I) and we carry out Steps E, F and G. These steps lead to the proof of Theorem II.2.2. Chapters II.4 and II.5 are devoted to proving the results in Chapter II.3 which concern Steps E and F.

Chapter II.2

A singular formulation of the problem

The Lagrange point L_3 is a centre-saddle equilibrium point of the Hamiltonian h in (II.1.1) whose eigenvalues, as $\mu \to 0$, satisfy

$$\mathrm{Spec} = \left\{ \pm\sqrt{\mu}\,\rho(\mu), \pm i\,\omega(\mu) \right\}, \quad \text{with} \quad \begin{cases} \rho(\mu) = \sqrt{\frac{21}{8}} + \mathcal{O}(\mu), \\ \omega(\mu) = 1 + \frac{7}{8}\mu + \mathcal{O}(\mu^2). \end{cases}$$

The center and saddle eigenvalues are found at different time-scales. Moreover, when $\mu = 0$, the unstable and stable manifolds of L_3 "collapse" to a circle of critical points. Applying a suitable singular change of coordinates, the Hamiltonian h can be written as a perturbation of a pendulum-like Hamiltonian weakly coupled with a fast oscillator. The construction of this change of variables is presented in detail in Section I.2.1. In this chapter, we summarize the most important properties of this set of coordinates.

The Hamiltonian h expressed in the classical (rotating) Poincaré coordinates, $\phi_{\mathrm{Poi}} : (\lambda, L, \eta, \xi) \to (q, p)$, defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge dL + i\, d\eta \wedge d\xi$ and the Hamiltonian

$$H^{\mathrm{Poi}} = H_0^{\mathrm{Poi}} + \mu H_1^{\mathrm{Poi}}, \tag{II.2.1}$$

with

$$H_0^{\mathrm{Poi}}(L, \eta, \xi) = -\frac{1}{2L^2} - L + \eta\xi \qquad \text{and} \qquad H_1^{\mathrm{Poi}} = h_1 \circ \phi_{\mathrm{Poi}}. \tag{II.2.2}$$

Moreover, the critical point L_3 satisfies

$$\lambda = 0, \qquad (L, \eta, \xi) = (1, 0, 0) + \mathcal{O}(\mu)$$

and the linearization of the vector field at this point has, at first order, an uncoupled nilpotent and center blocks,

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} + \mathcal{O}(\mu).$$

Since ϕ_{Poi} is an implicit change of coordinates, there is no explicit expression for H_1^{Poi}. However, it is possible to obtain series expansion in powers of $(L-1, \eta, \xi)$, (see Lemma I.4.1 and also Appendix II.A).

To capture the slow-fast dynamics of the system, renaming

$$\delta = \mu^{\frac{1}{4}},$$

we perform the singular symplectic scaling

$$\phi_{\mathrm{sc}} : (\lambda, \Lambda, x, y) \mapsto (\lambda, L, \eta, \xi), \qquad L = 1 + \delta^2 \Lambda, \quad \eta = \delta x, \quad \xi = \delta y \tag{II.2.3}$$

and the time reparametrization $t = \delta^{-2}\tau$. Defining the potential

$$V(\lambda) = H_1^{\mathrm{Poi}}(\lambda, 1, 0, 0; 0) = 1 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}}, \tag{II.2.4}$$

the Hamiltonian system associated to H^{Poi}, expressed in scaled coordinates, defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge d\Lambda + i dx \wedge dy$ and the Hamiltonian

$$H = H_{\mathrm{p}} + H_{\mathrm{osc}} + H_1, \tag{II.2.5}$$

where

$$H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + V(\lambda), \qquad H_{\mathrm{osc}}(x, y; \delta) = \frac{xy}{\delta^2}, \tag{II.2.6}$$

$$H_1(\lambda, \Lambda, x, y; \delta) = H_1^{\mathrm{Poi}}(\lambda, 1 + \delta^2\Lambda, \delta x, \delta y; \delta^4) - V(\lambda) + \frac{1}{\delta^4} F_{\mathrm{p}}(\delta^2 \Lambda) \tag{II.2.7}$$

and

$$F_{\mathrm{p}}(z) = \left(-\frac{1}{2(1+z)^2} - (1+z) \right) + \frac{3}{2} + \frac{3}{2} z^2 = \mathcal{O}(z^3). \tag{II.2.8}$$

Therefore, we can define the "new" first order

$$H_0 = H_{\mathrm{p}} + H_{\mathrm{osc}}. \tag{II.2.9}$$

From now on, we refer to H_0 as the unperturbed Hamiltonian and we identify H_1 as the perturbation.

The next proposition, proven in Section I.2.1 in Theorem I.2.1, gives some properties of the Hamiltonian H.

Proposition II.2.1. *The Hamiltonian H, away from collision with the primaries, is real-analytic in the sense of $\overline{H(\lambda, \Lambda, x, y; \delta)} = H(\overline{\lambda}, \overline{\Lambda}, y, x; \overline{\delta})$.*

Moreover, for $\delta > 0$ small enough,

- *The critical point L_3 expressed in coordinates (λ, Λ, x, y) is given by*

$$\mathfrak{L}(\delta) = \left(0, \delta^2 \mathfrak{L}_\Lambda(\delta), \delta^3 \mathfrak{L}_x(\delta), \delta^3 \mathfrak{L}_y(\delta)\right), \tag{II.2.10}$$

  with $|\mathfrak{L}_\Lambda(\delta)|$, $|\mathfrak{L}_x(\delta)|$, $|\mathfrak{L}_y(\delta)| \leq C$, for some constant $C > 0$ independent of δ.

- *The point $\mathfrak{L}(\delta)$ is a saddle-center equilibrium point and its linearization is*

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ -\frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix} + \mathcal{O}(\delta).$$

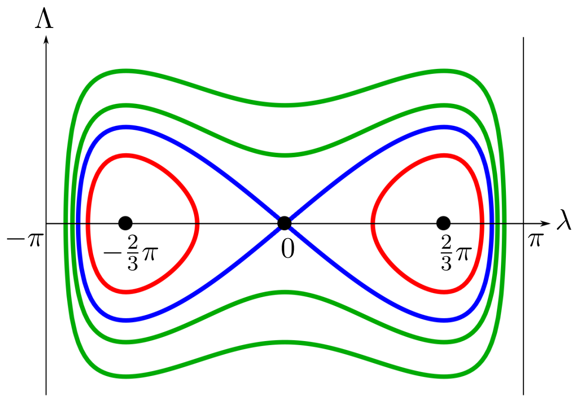


FIGURE II.2.1: Phase portrait of the system given by Hamiltonian $H_{\mathrm{p}}(\lambda, \Lambda)$ on (II.2.6). On blue the two separatrices.

Therefore, it possesses a one-dimensional unstable and stable manifolds, $\mathcal{W}^{\mathrm{u}}(\delta)$ and $\mathcal{W}^{\mathrm{s}}(\delta)$.

The unperturbed system given by H_0 in (II.2.9) has two homoclinic connections in the (λ, Λ)-plane associated to the saddle point $(0,0)$ and described by the energy level $H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{1}{2}$ (see Figure II.2.1). We define

$$\lambda_0 = \arccos\left(\frac{1}{2} - \sqrt{2}\right), \tag{II.2.11}$$

which satisfies $H_{\mathrm{p}}(\lambda_0, 0) = -\frac{1}{2}$ so that, for the unperturbed system, λ_0 is the "turning point" in the (λ, Λ) variables. We will see that, in our regime, $\theta \approx \lambda$ and thus the value of θ_0 introduced in Remark II.1.2 is indeed close to the "turning point" of the invariant manifolds (see Figure II.1.1).

We rewrite Theorem II.1.1, in fact the more general result in Remark II.1.2, in the set of coordinates (λ, Λ, x, y). For $\lambda_* \in (0, \lambda_0)$, we consider the 3-dimensional section

$$\mathcal{S}(\lambda_*) = \left\{(\lambda, \Lambda, x, y) \in \mathbb{R}^2 \times \mathbb{C}^2 \ : \ \lambda = \lambda_*, \, \Lambda > 0, \, x = \overline{y}\right\},$$

which is transverse to the flow of H, and we define the first crossings of the invariant manifolds $\mathcal{W}^{\mathrm{u,s}}(\delta)$ with this section as $(\lambda_*, \Lambda_*^{\mathrm{u}}, x_*^{\mathrm{u}}, y_*^{\mathrm{u}})$ and $(\lambda_*, \Lambda_*^{\mathrm{s}}, x_*^{\mathrm{s}}, y_*^{\mathrm{s}})$.

Theorem II.2.2. *Fix an interval $[\lambda_1, \lambda_2] \subset (0, \lambda_0)$ with λ_0 as given in* (II.2.11)*. Then, there exists $\delta_0 > 0$ and $b_0 > 0$ such that, for $\delta \in (0, \delta_0)$ and $\lambda_* \in [\lambda_1, \lambda_2]$, the first crossings are analytic with respect to λ^* and*

$$|\Lambda_*^{\diamond}| \leq b_0, \qquad |x_*^{\diamond}|, |y_*^{\diamond}| \leq b_0 \delta^3, \qquad \diamond = \mathrm{u}, \mathrm{s}. \tag{II.2.12}$$

Moreover,

$$|x_*^{\mathrm{u}} - x_*^{\mathrm{s}}| = |y_*^{\mathrm{u}} - y_*^{\mathrm{s}}| = \sqrt[6]{2}\, \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log \delta|}\right)\right],$$
$$|\Lambda_*^{\mathrm{u}} - \Lambda_*^{\mathrm{s}}| = \mathcal{O}(\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}),$$

where A and Θ are the constants introduced in Theorem II.1.1.

II.2.1 Proof of Theorem II.1.1

To prove Theorem II.1.1 (and Remark II.1.2) from Theorem II.2.2 we need to "undo" the changes of coordinates $\phi_{\rm Poi}$ and $\phi_{\rm sc}$ and adjust the section from $\lambda = $ constant to $\theta = $ constant.

First, we consider the change $\phi_{\rm sc}$ given by $(\lambda, L, \eta, \xi) = (\lambda, 1 + \delta^2\Lambda, \delta x, \delta y)$, (see (II.2.3)). For $\lambda_* \in [\lambda_1, \lambda_2]$ we define

$$L^\diamond(\lambda_*;\delta) = 1 + \delta^2\Lambda_*^\diamond, \qquad \eta^\diamond(\lambda_*;\delta) = \delta x_*^\diamond, \qquad \xi^\diamond(\lambda_*;\delta) = \delta y_*^\diamond, \quad \text{for } \diamond = \mathrm{u}, \mathrm{s}. \tag{II.2.13}$$

Then, by Theorem II.2.2, one has

$$\begin{aligned} |\Delta L(\lambda_*;\delta)| &= |L^{\rm u}(\lambda_*;\delta) - L^{\rm s}(\lambda_*;\delta)| = \mathcal{O}\left(\delta^{\frac{10}{3}} e^{-\frac{A}{\delta^2}}\right), \\ |\Delta\eta(\lambda_*;\delta)| &= |\eta^{\rm u}(\lambda_*;\delta) - \eta^{\rm s}(\lambda_*;\delta)| = \sqrt[6]{2}\,\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}\left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right], \\ \Delta\xi(\lambda_*;\delta) &= \overline{\Delta\eta(\lambda_*;\delta)}. \end{aligned} \tag{II.2.14}$$

Next, we study the change $\phi_{\rm Poi}$. In the following result, we give a series expression of the polar coordinates with respect to the Poincaré elements. Its proof is a direct consequence of the definition of the Poincaré variables (see, for instance, Section I.4.1).

Lemma II.2.3. *Fix $\varrho > 0$. Then, for $|(L-1,\eta,\xi)| \ll 1$ and $|\mathrm{Im}\,\lambda| \leq \varrho$, the polar coordinates (r,θ,R,G) introduced in (II.1.2) satisfy*

$$\begin{aligned} r &= 1 + 2(L-1) - \frac{e^{-i\lambda}}{\sqrt{2}}\eta - \frac{e^{i\lambda}}{\sqrt{2}}\xi + \mathcal{O}(L-1,\eta,\xi)^2, \\ \theta &= \lambda + i\sqrt{2}e^{-i\lambda}\eta - i\sqrt{2}e^{i\lambda}\xi + \mathcal{O}(L-1,\eta,\xi)^2, \\ R &= \frac{ie^{-i\lambda}}{\sqrt{2}}\eta - \frac{ie^{i\lambda}}{\sqrt{2}}\xi + \mathcal{O}(L-1,\eta,\xi)^2, \qquad G = L - \eta\xi. \end{aligned}$$

Since in Theorem II.2.2 the distance is measured in the section $\lambda = \lambda_*$ whereas the Theorem II.1.1, and more generally Remark II.1.2, measures it in the section $\theta = \theta^*$, we must "translate" the estimates in (II.2.14) to the new section. By Lemma II.2.3, let g_θ be the function such that $\theta = \lambda + g_\theta(\lambda, L, \eta, \xi)$. Then, for $\diamond = \mathrm{u}, \mathrm{s}$, we consider

$$F^\diamond(\lambda,\theta,\delta) = \theta - \lambda + g_\theta\left(\lambda, L^\diamond(\lambda;\delta), \eta^\diamond(\lambda;\delta), \xi^\diamond(\lambda;\delta)\right).$$

Applying the Implicit Function Theorem, Lemma II.2.3 and that, by (II.2.13), $L^\diamond(\lambda;0) = 1$ and $\eta^\diamond(\lambda;0) = \xi^\diamond(\lambda;0) = 0$, then there exist function $\widehat{\lambda}^\diamond(\theta;\delta)$ such that $F^\diamond(\widehat{\lambda}^\diamond(\theta;\delta),\theta,\delta) = 0$ and

$$\begin{aligned} \widehat{\lambda}^\diamond(\theta;\delta) = {} & \theta - i\sqrt{2}e^{-i\theta}\widehat{\eta}^\diamond(\theta;\delta) + i\sqrt{2}e^{i\theta}\widehat{\xi}^\diamond(\theta;\delta) \\ & + \mathcal{O}\left(\widehat{L}^\diamond(\theta;\delta) - 1, \widehat{\eta}^\diamond(\theta;\delta), \widehat{\xi}^\diamond(\theta;\delta)\right)^2, \end{aligned} \tag{II.2.15}$$

with $\widehat{\eta}^\diamond(\theta;\delta) = \eta^\diamond(\widehat{\lambda}^\diamond(\theta;\delta);\delta)$, $\widehat{\xi}^\diamond(\theta;\delta) = \xi^\diamond(\widehat{\lambda}^\diamond(\theta;\delta);\delta)$ and $\widehat{L}^\diamond(\theta;\delta) = L^\diamond(\widehat{\lambda}^\diamond(\theta;\delta);\delta)$. Notice that, by (II.2.12) (plus Cauchy estimates for their derivatives) and (II.2.13),

$$\widehat{\lambda}^\diamond(\theta;\delta) = \theta + \mathcal{O}(\delta^4).$$

Thus, for any $[\theta_1,\theta_2]\subset(0,\lambda_0)$ and δ small enough, there exists $[\lambda_1,\lambda_2]\subset(0,\lambda_0)$ such that, for $\theta\in[\theta_1,\theta_2]$ one has $\widehat{\lambda}^{\mathrm{u,s}}(\theta;\delta)\in[\lambda_1,\lambda_2]$. In addition,

$$\begin{aligned}\widehat{L}^{\diamond}(\theta;\delta)&=L^{\diamond}(\theta;\delta)+\mathcal{O}(\delta^6)=1+\mathcal{O}(\delta^2),\\ \widehat{\eta}^{\diamond}(\theta;\delta)&=\eta^{\diamond}(\theta;\delta)+\mathcal{O}(\delta^8)=\mathcal{O}(\delta^4),\\ \widehat{\xi}^{\diamond}(\theta;\delta)&=\xi^{\diamond}(\theta;\delta)+\mathcal{O}(\delta^8)=\mathcal{O}(\delta^4).\end{aligned}\tag{II.2.16}$$

Then, since $\Lambda_*^{\mathrm{u,s}}>0$, by (II.2.13) one has that $\widehat{L}^{\mathrm{u,s}}(\theta;\delta)>1$ for $\theta\in[\theta_1,\theta_2]$. Moreover, by Lemma II.2.3 and taking δ small enough, one has $r^{\mathrm{u,s}}(\theta)-1>0$.

The difference between the invariant manifolds in a section of fixed $\theta\in[\theta_1,\theta_2]$ is given by

$$\begin{aligned}\Delta\widehat{\lambda}(\theta;\delta)&=\widehat{\lambda}^{\mathrm{u}}(\theta;\delta)-\widehat{\lambda}^{\mathrm{s}}(\theta;\delta), &\qquad \Delta\widehat{L}(\theta;\delta)&=\widehat{L}^{\mathrm{u}}(\theta;\delta)-\widehat{L}^{\mathrm{s}}(\theta;\delta),\\ \Delta\widehat{\eta}(\theta;\delta)&=\widehat{\eta}^{\mathrm{u}}(\theta;\delta)-\widehat{\eta}^{\mathrm{s}}(\theta;\delta), & \Delta\widehat{\xi}(\theta;\delta)&=\widehat{\xi}^{\mathrm{u}}(\theta;\delta)-\widehat{\xi}^{\mathrm{s}}(\theta;\delta).\end{aligned}$$

Then, by (II.2.15) and (II.2.16), one has that

$$\Delta\widehat{\lambda}(\theta;\delta)=-i\sqrt{2}e^{-i\theta}\Delta\widehat{\eta}(\theta;\delta)+i\sqrt{2}e^{i\theta}\Delta\widehat{\xi}(\theta;\delta)+\mathcal{O}\left(\delta^2\Delta\widehat{L}(\theta;\delta),\delta^4\Delta\widehat{\eta}(\theta;\delta),\delta^4\Delta\widehat{\xi}(\theta;\delta)\right).$$

Moreover, by the mean value theorem, (II.2.14) and (II.2.16),

$$\begin{aligned}\Delta\widehat{L}(\theta;\delta)&=\Delta L(\widehat{\lambda}^{\mathrm{u}}(\theta;\delta);\delta)+\widehat{L}^{\mathrm{s}}(\widehat{\lambda}^{\mathrm{u}}(\theta;\delta);\delta)-\widehat{L}^{\mathrm{s}}(\widehat{\lambda}^{\mathrm{s}}(\theta;\delta);\delta)\\ &=\mathcal{O}(\delta^{\frac{10}{3}}e^{-\frac{A}{\delta^2}})+\delta^2\mathcal{O}\left(\Delta\widehat{\lambda}(\theta;\delta)\right).\end{aligned}$$

Analogously,

$$\begin{aligned}\Delta\widehat{\eta}(\theta;\delta)&=\Delta\eta(\widehat{\lambda}^{\mathrm{u}}(\theta;\delta);\delta)+\delta^4\mathcal{O}\left(\Delta\widehat{\lambda}(\theta;\delta)\right),\\ \Delta\widehat{\xi}(\theta;\delta)&=\overline{\Delta\eta(\lambda^{\mathrm{u}}(\theta;\delta);\delta)}+\delta^4\mathcal{O}\left(\Delta\widehat{\lambda}(\theta;\delta)\right).\end{aligned}$$

Therefore, using (II.2.14), one can conclude that

$$\begin{aligned}|\Delta\widehat{\lambda}(\theta;\delta)|&=\mathcal{O}\left(\delta^{\frac{4}{3}}e^{-\frac{A}{\delta^2}}\right), &\qquad |\Delta\widehat{\eta}(\theta;\delta)|&=\sqrt[6]{2}\,\delta^{\frac{4}{3}}e^{-\frac{A}{\delta^2}}\left[|\Theta|+\mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right],\\ |\Delta\widehat{L}(\theta;\delta)|&=\mathcal{O}\left(\delta^{\frac{10}{3}}e^{-\frac{A}{\delta^2}}\right), & \Delta\widehat{\xi}(\theta;\delta)&=\overline{\Delta\widehat{\eta}(\theta;\delta)}.\end{aligned}$$

Once we have adjusted the transverse section, it only remains to apply Lemma II.2.3 to translate these differences to polar coordinates. That is,

$$\begin{aligned}r^{\mathrm{u}}-r^{\mathrm{s}}&=-\sqrt{2}\cos\theta\,\mathrm{Re}\,\Delta\widehat{\eta}(\theta;\delta)-\sqrt{2}\sin\theta\,\mathrm{Im}\,\Delta\widehat{\eta}(\theta;\delta)+\mathcal{O}(\delta^{\frac{10}{3}}e^{-\frac{A}{\delta^2}}),\\ R^{\mathrm{u}}-R^{\mathrm{s}}&=-\sqrt{2}\cos\theta\,\mathrm{Im}\,\Delta\widehat{\eta}(\theta;\delta)+\sqrt{2}\sin\theta\,\mathrm{Re}\,\Delta\widehat{\eta}(\theta;\delta)+\mathcal{O}(\delta^{\frac{16}{3}}e^{-\frac{A}{\delta^2}}),\\ G^{\mathrm{u}}-G^{\mathrm{s}}&=\mathcal{O}(\delta^{\frac{10}{3}}e^{-\frac{A}{\delta^2}}),\end{aligned}$$

which implies

$$\begin{aligned}\|(r^{\mathrm{u}},R^{\mathrm{u}},G^{\mathrm{u}})-(r^{\mathrm{s}},R^{\mathrm{s}},G^{\mathrm{s}})\|&=\sqrt{2}\,|\Delta\widehat{\eta}(\theta;\delta)|+\mathcal{O}(\delta^{\frac{10}{3}}e^{-\frac{A}{\delta^2}})\\ &=\sqrt[3]{4}\,\delta^{\frac{4}{3}}e^{-\frac{A}{\delta^2}}\left[|\Theta|+\mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right].\end{aligned}$$

To conclude the proof of Theorem II.1.1, it is enough to recall that $\delta = \mu^{\frac{1}{4}}$.

Chapter II.3

Proof of Theorem II.2.2

In this chpater, we present the main steps necessary to prove Theorem II.2.2 (see the list in Section II.1.2) and complete its proof. In Section II.3.1 we summarize the results concerning the analysis of the separatrix of the unperturbed Hamiltonian H_{p} (see (II.2.6)) done in Section I.2.2 (Step B). In Section II.3.2, we prove the existence of parametrizations of the perturbed invariant manifolds in suitable complexs domains (Step E). In Section II.3.3, we study the difference between the perturbed manifolds near the singularities of the perturbed separatrix. In particular, in Section II.3.3.1, we summarize the results concerning the derivation (Step C) and analysis (Step D) of the inner equation obtained in Sections I.2.3 and I.2.4, in Section II.3.3.2, we compare certain solutions of the inner equation with the parametrizations of the perturbed manifolds by means of complex matching techniques (Step F). Finally, in Section II.3.4, we combine all the previous results to obtain the dominant term of the difference between the invariant manifolds and prove Theorem II.2.2 (Step G).

II.3.1 Analytical continuation of the unperturbed separatrix

The unperturbed Hamiltonian

$$H_0(\lambda, \Lambda, x, y) = H_{\mathrm{p}}(\lambda, \Lambda) + H_{\mathrm{osc}}(x, y)$$

(see (II.2.9)) possesses a saddle with two separatrices in the (λ, Λ)-plane (see Figure II.2.1). Let us consider the real-analytic time parametrization of the separatrix with $\lambda \in (0, \pi)$,

$$\begin{aligned} \sigma : \mathbb{R} &\to \mathbb{T} \times \mathbb{R} \\ t &\mapsto \sigma(t) = (\lambda_h(t), \Lambda_h(t)), \end{aligned} \tag{II.3.1}$$

with initial condition $\sigma(0) = (\lambda_0, 0)$ where $\lambda_0 = \arccos\left(\frac{1}{2} - \sqrt{2}\right) \in \left(\frac{2}{3}\pi, \pi\right)$.

The following result (which encompass Theorem I.2.2, Proposition I.2.3 and Corollary I.2.4) gives the properties of the analytic extension of $\sigma(t)$ to the domain

$$\begin{aligned} \Pi^{\mathrm{ext}}_{A,\beta} = &\{t \in \mathbb{C} \ : \ |\mathrm{Im}\, t| < \tan\beta \,\mathrm{Re}\, t + A\} \cup \\ &\{t \in \mathbb{C} \ : \ |\mathrm{Im}\, t| < -\tan\beta \,\mathrm{Re}\, t + A\}, \end{aligned} \tag{II.3.2}$$

with A as given in (II.1.3) (see Figure II.3.1).

Theorem II.3.1. *The real-analytic time parametrization σ defined in (II.3.1) satisfies:*

- *There exists $0 < \beta_0 < \frac{\pi}{2}$ such that $\sigma(t)$ extends analytically to Π_{A,β_0}.*

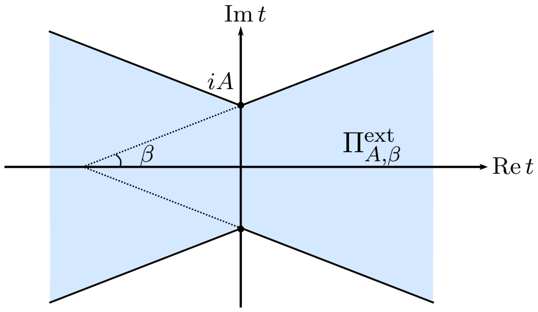


FIGURE II.3.1: Representation of the domain $\Pi^{\mathrm{ext}}_{A,\beta}$ in (II.3.2).

- *$\sigma(t)$ has only two singularities on $\partial\Pi^{\mathrm{ext}}_{A,\beta_0}$ at $t = \pm iA$.*
- *There exists $\upsilon > 0$ such that, for $t \in \mathbb{C}$ with $|t - iA| < \upsilon$ and $\arg(t - iA) \in (-\frac{3\pi}{2}, \frac{\pi}{2})$,*

$$\lambda_h(t) = \pi + 3\alpha_+(t - iA)^{\frac{2}{3}} + \mathcal{O}(t - iA)^{\frac{4}{3}},$$

$$\Lambda_h(t) = -\frac{2\alpha_+}{3}\frac{1}{(t - iA)^{\frac{1}{3}}} + \mathcal{O}(t - iA)^{\frac{1}{3}},$$

with $\alpha_+ \in \mathbb{C}$ such that $\alpha_+^3 = \frac{1}{2}$.
An analogous result holds for $|t + iA| < \upsilon$, $\arg(t+iA) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ and $\alpha_- = \overline{\alpha_+}$.

- *$\Lambda_h(t)$ has only one zero in $\overline{\Pi^{\mathrm{ext}}_{A,\beta_0}}$ at $t = 0$.*

II.3.2 The perturbed invariant manifolds

In this section, following the approach described in [BFG+12; BCS13; GMS16], we study the analytic continuation of the parametrizations of the perturbed one-dimensional stable and unstable manifolds, $\mathcal{W}^{\mathrm{u}}(\delta)$ and $\mathcal{W}^{\mathrm{s}}(\delta)$.

Since we measure the distance between the invariant manifolds in the section $\lambda = \lambda_*$ (see Theorem II.2.2), we parameterize them as graphs with respect to λ (whenever is possible) or, more conveniently, with respect to the independent variable u defined by $\lambda = \lambda_h(u)$.

To define these suitable parameterizations we first translate the equilibrium point $\mathfrak{L}(\delta)$ to $\mathbf{0}$ by the change of coordinates

$$\phi_{\mathrm{eq}} : (\lambda, \Lambda, x, y) \mapsto (\lambda, \Lambda, x, y) + \mathfrak{L}(\delta). \tag{II.3.3}$$

Second, we consider the symplectic change of coordinates

$$\phi_{\mathrm{sep}} : (u, w, x, y) \to (\lambda, \Lambda, x, y), \quad \lambda = \lambda_h(u), \quad \Lambda = \Lambda_h(u) - \frac{w}{3\Lambda_h(u)}. \tag{II.3.4}$$

We refer to (u, w, x, y) as the *separatrix coordinates.*

Let us remark that ϕ_{sep} is not defined for $u = 0$ since $\Lambda_h(0) = 0$ (see Theorem II.3.1). We deal with this fact later when considering the domain of definition for u.

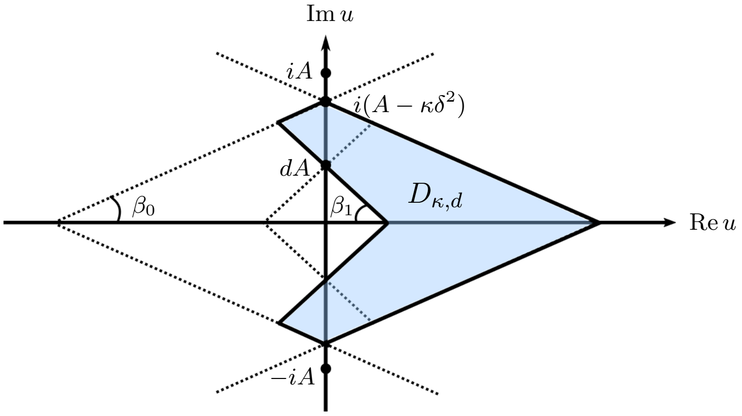


FIGURE II.3.2: The boomerang domain $D_{\kappa,d}$ defined in (II.3.7).

After these changes of variables, we look for the perturbed invariant manifolds as a graph with respect to u. In other words, we look for functions

$$z^\diamond(u) = (w^\diamond(u), x^\diamond(u), y^\diamond(u))^T, \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s},$$

such that the invariant manifolds given in Proposition II.2.1 can be expressed as

$$\mathcal{W}^\diamond(\delta) = \left\{ \left(\lambda_h(u), \Lambda_h(u) - \frac{w^\diamond(u)}{3\Lambda_h(u)}, x^\diamond(u), y^\diamond(u) \right) + \mathfrak{L}(\delta) \right\}, \quad \text{for } \diamond = \mathrm{u}, \mathrm{s}, \quad \text{(II.3.5)}$$

with u belonging to an appropriate domain contained in $\Pi^{\mathrm{ext}}_{A,\beta_0}$ (see (II.3.2)). The graphs z^{u} and z^{s} must satisfy the asymptotic conditions

$$\lim_{\mathrm{Re}\, u \to -\infty} \left(\frac{w^{\mathrm{u}}(u)}{\Lambda_h(u)}, x^{\mathrm{u}}(u), y^{\mathrm{u}}(u) \right) = \lim_{\mathrm{Re}\, u \to +\infty} \left(\frac{w^{\mathrm{s}}(u)}{\Lambda_h(u)}, x^{\mathrm{s}}(u), y^{\mathrm{s}}(u) \right) = 0. \qquad \text{(II.3.6)}$$

Remark II.3.2. *Since the Hamiltonian H is real-analytic in the sense of $\overline{H(\lambda, \Lambda, x, y; \delta)} = H(\overline{\lambda}, \overline{\Lambda}, y, x; \overline{\delta})$ (see Proposition II.2.1), then we say that $z(u) = (w(u), x(u), y(u))^T$ is real-analytic if it satisfies*

$$w(\overline{u}) = \overline{w(u)}, \qquad x(\overline{u}) = y(u), \qquad y(\overline{u}) = x(u).$$

The classical way to study exponentially small splitting of separatrices, in this setting, is to look for solutions z^{u} and z^{s} in a certain complex common domain containing a segment of the real line and intersecting a $\mathcal{O}(\delta^2)$ neighborhood of the singularities $u = \pm iA$ of the separatrix.

Recall that the invariant manifolds can not be expressed as a graph in a neighborhood of $u = 0$. To overcome this technical problem, we find solutions z^{u} and z^{s} defined in a complex domain, which we call *boomerang domain* due to its shape (see Figure II.3.2). Namely,

$$\begin{aligned} D_{\kappa,d} = \{ u \in \mathbb{C} \ : \ & |\mathrm{Im}\, u| < A - \kappa\delta^2 + \tan\beta_0 \mathrm{Re}\, u, \\ & |\mathrm{Im}\, u| < A - \kappa\delta^2 - \tan\beta_0 \mathrm{Re}\, u, \\ & |\mathrm{Im}\, u| > dA - \tan\beta_1 \mathrm{Re}\, u \}, \end{aligned} \qquad \text{(II.3.7)}$$

where $\kappa > 0$ is such that $A - \kappa\delta^2 > 0$, β_0 is the constant given in Theorem II.3.1 and $\beta_1 \in [\beta_0, \frac{\pi}{2})$ and $d \in (\frac{1}{4}, \frac{1}{2})$ are independent of δ.

Theorem II.3.3. *Fix a constant $d \in (\frac{1}{4}, \frac{1}{2})$. Then, there exists $\delta_0, \kappa_0 > 0$ such that, for $\delta \in (0, \delta_0)$, $\kappa \geq \kappa_0$, the graph parameterizations z^{u} and z^{s} introduced in (II.3.5) can be extended real-analytically to the domain $D_{\kappa,d}$.*

Moreover, there exists a real constant $b_1 > 0$ independent of δ and κ such that, for $u \in D_{\kappa,d}$ we have that

$$|w^\diamond(u)| \leq \frac{b_1\delta^2}{|u^2+A^2|} + \frac{b_1\delta^4}{|u^2+A^2|^{\frac{8}{3}}}, \quad |x^\diamond(u)| \leq \frac{b_1\delta^3}{|u^2+A^2|^{\frac{4}{3}}}, \quad |y^\diamond(u)| \leq \frac{b_1\delta^3}{|u^2+A^2|^{\frac{4}{3}}}.$$

Notice that the asymptotic conditions (II.3.6) do not have any meaning in the domain $D_{\kappa,d}$ since it is bounded. Therefore, to prove the existence of z^{u} and z^{s} in $D_{\kappa,d}$ one has to start with different domains where these asymptotic conditions make sense and then find a way to extend them real-analytically to $D_{\kappa,d}$. We describe the details of these process in the following Sections II.3.2.1 and II.3.2.2.

II.3.2.1 Analytic extension of the stable and unstable manifolds

The Hamiltonian H written in separatrix coordinates (see (II.3.3) and (II.3.4)) becomes

$$H^{\mathrm{sep}} = H_0^{\mathrm{sep}} + H_1^{\mathrm{sep}}, \tag{II.3.8}$$

with

$$H_0^{\mathrm{sep}} = w + \frac{xy}{\delta^2}, \qquad H_1^{\mathrm{sep}} = H \circ (\phi_{\mathrm{eq}} \circ \phi_{\mathrm{sep}}) - H_0^{\mathrm{sep}}. \tag{II.3.9}$$

Introducing the notation $z = (w, x, y)^T$ and defining

$$\mathcal{A}^{\mathrm{sep}} = \frac{i}{\delta^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{II.3.10}$$

the equations associated to the Hamiltonian H^{sep} can be written as

$$\left\{ \begin{array}{l} \dot{u} = 1 + g^{\mathrm{sep}}(u,z), \\ \dot{z} = \mathcal{A}^{\mathrm{sep}} z + f^{\mathrm{sep}}(u,z), \end{array} \right. \tag{II.3.11}$$

where $g^{\mathrm{sep}} = \partial_w H_1^{\mathrm{sep}}$ and $f^{\mathrm{sep}} = (-\partial_u H_1^{\mathrm{sep}}, i\partial_y H_1^{\mathrm{sep}}, -i\partial_x H_1^{\mathrm{sep}})^T$. Consequently, the parameterizations $z^{\mathrm{u}}(u)$ and $z^{\mathrm{s}}(u)$ given in (II.3.5) satisfy the invariance equation

$$\partial_u z^\diamond = \mathcal{A}^{\mathrm{sep}} z^\diamond + \mathcal{R}^{\mathrm{sep}}[z^\diamond], \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s}, \tag{II.3.12}$$

with

$$\mathcal{R}^{\mathrm{sep}}[\varphi](u) = \frac{f^{\mathrm{sep}}(u,\varphi) - g^{\mathrm{sep}}(u,\varphi)\mathcal{A}^{\mathrm{sep}}\varphi}{1 + g^{\mathrm{sep}}(u,\varphi)}. \tag{II.3.13}$$

Remark II.3.4. *Note that one can use this invariance equation whenever*

$$1 + g^{\mathrm{sep}}(u,\varphi) = 1 + \partial_w H_1^{\mathrm{sep}}(u,\varphi) \neq 0$$

This condition is satisfied in the different domains that are considered in this section and in the forthcoming ones and it is checked in Appendix II.A (see (II.A.16) and (II.A.32)). This fact is also used later in Section II.3.3.

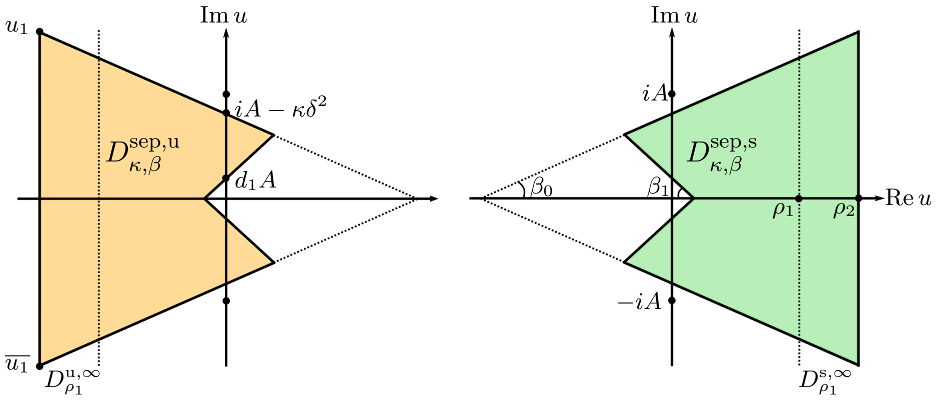


FIGURE II.3.3: The outer domains $D^{\mathrm{sep,u}}_{\kappa,\beta}$ and $D^{\mathrm{sep,s}}_{\kappa,\beta}$ defined in (II.3.15)

The first step is to look for solutions of this equation in the domains

$$D^{\mathrm{u},\infty}_{\rho_1} = \{u \in \mathbb{C} \,:\, \mathrm{Re}\, u < -\rho_1\}\,, \qquad D^{\mathrm{s},\infty}_{\rho_1} = \{u \in \mathbb{C} \,:\, \mathrm{Re}\, u > \rho_1\}\,, \tag{II.3.14}$$

for some $\rho_1 > 0$, which allows us to take into account the asymptotic conditions (II.3.6).

Proposition II.3.5. *Fix $\rho_1 > 0$. Then, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, the equation (II.3.12) has a unique real-analytic solution $z^\diamond = (w^\diamond, x^\diamond, y^\diamond)^T$ in $D^{\diamond,\infty}_{\rho_1}$ (for $\diamond = \mathrm{u}, \mathrm{s}$) satisfying the corresponding asymptotic condition (II.3.6).*

Moreover, there exists $b_2 > 0$ independent of δ such that, for $u \in D^{\diamond,\infty}_{\rho_1}$,

$$|w^\diamond(u)e^{-2\rho u}| \leq b_2\delta^2, \qquad |x^\diamond(u)e^{-\rho u}| \leq b_2\delta^3, \qquad |y^\diamond(u)e^{-\rho u}| \leq b_2\delta^3.$$

with $\nu = \sqrt{\frac{21}{8}}$ for $\diamond = \mathrm{u}$ and $\nu = -\sqrt{\frac{21}{8}}$ for $\diamond = \mathrm{s}$.

This proposition is proved in Section II.4.1.

To extend analytically the invariant manifolds to reach the boomerang domain $D_{\kappa,d}$ we have to face the problem that these parameterizations become undefined at $u = 0$. To overcome it, first we extend the solutions z^{u} and z^{s} of Proposition II.3.5 to the *outer domains* (see Figure II.3.3)

$$\begin{aligned} D^{\mathrm{sep,u}}_{\kappa,\beta} = \{u \in \mathbb{C} \,:\, &|\mathrm{Im}\, u| < A - \kappa\delta^2 - \tan\beta_0 \mathrm{Re}\, u, \\ &|\mathrm{Im}\, u| > d_1 A + \tan\beta_1 \mathrm{Re}\, u,\ \mathrm{Re}\, u > -\rho_2\}\,, \\ D^{\mathrm{sep,s}}_{\kappa,\beta} = \big\{u \in \mathbb{C} \,:\, &-u \in D^{\mathrm{sep,u}}_{\kappa,\beta}\big\}, \end{aligned} \tag{II.3.15}$$

where $d_1 \in (\frac{1}{4}, \frac{1}{2})$ and $\rho_2 > \rho_1$ are fixed independent of δ, and $\kappa > 0$ is such that $A - \kappa\delta^2 > 0$.

Proposition II.3.6. *Consider the functions z^{u}, z^{s} and the constant $\rho_1 > 0$ obtained in Proposition II.3.5. Fix constants $\rho_2 > \rho_1$ and $d_1 \in (\frac{1}{4}, \frac{1}{2})$. Then, there exist $\delta_0, \kappa_1 > 0$ such that, for $\delta \in (0, \delta_0)$, $\kappa \geq \kappa_1$, the functions $z^\diamond = (w^\diamond, x^\diamond, y^\diamond)^T$, $\diamond = \mathrm{u}, \mathrm{s}$, can be extended analytically to the domain $D^{\mathrm{sep},\diamond}_{\kappa,\beta}$.*

Moreover, there exists $b_3 > 0$ independent of δ and κ such that, for $u \in D^{\mathrm{sep},\diamond}_{\kappa,\beta}$,

$$|w^\diamond(u)| \leq \frac{b_3\delta^2}{|u^2 + A^2|} + \frac{b_3\delta^4}{|u^2 + A^2|^{\frac{8}{3}}}, \quad |x^\diamond(u)| \leq \frac{b_3\delta^3}{|u^2 + A^2|^{\frac{4}{3}}}, \quad |y^\diamond(u)| \leq \frac{b_3\delta^3}{|u^2 + A^2|^{\frac{4}{3}}}.$$

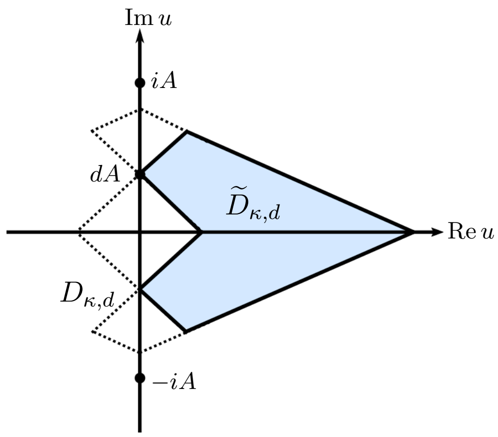


FIGURE II.3.4: The domain $\widetilde{D}_{\kappa,d}$ defined in (II.3.16).

This proposition is proved in Section II.4.2.

Notice that taking ρ_2 big enough, $d_1 \leq d$ and $\kappa_1 \leq \kappa_0$ we have $D_{\kappa_0,d} \subset D^{\mathrm{s,out}}_{\kappa_1,d_1,\rho_2}$. Therefore, for the stable manifold z^{s}, Proposition II.3.6 implies Theorem II.3.3. However, we still need to extend further z^{u} in order to reach $D_{\kappa_0,d}$.

II.3.2.2 Further analytic extension of the unstable manifold

Since by Proposition II.3.6 the unstable solution z^{u} is defined in $D^{\mathrm{u,out}}_{\kappa_1,d_1,\rho_2}$, To prove Theorem II.3.3 it only remains to extend it to the points in the boomerang domain $D_{\kappa_0,d}$ which do not belong to the outer unstable domain. Namely, we extend z^{u} to

$$\begin{aligned} \widetilde{D}_{\kappa,d} = \{u \in \mathbb{C} \ : \ & |\mathrm{Im}\, u| < A - \kappa\delta^2 - \tan\beta_0 \mathrm{Re}\, u, \\ & |\mathrm{Im}\, u| < dA + \tan\beta_1 \mathrm{Re}\, u, \ |\mathrm{Im}\, u| > dA - \tan\beta_1 \mathrm{Re}\, u\}, \end{aligned} \tag{II.3.16}$$

for suitable κ and d (see Figure II.3.4). Notice that $\widetilde{D}_{\kappa,d} \subset D_{\kappa,d}$ and that $\widetilde{D}_{\kappa,d}$ only contains points at distance of $u = \pm iA$ of order 1 with respect to δ.

As we have mentioned, to measure the difference between the invariant manifolds $\mathcal{W}^{\mathrm{u}}(\delta)$ and $\mathcal{W}^{\mathrm{s}}(\delta)$ it is convenient to parameterize them as graphs (see (II.3.5)). However, these graph parametrizations are not defined at $u = 0$. Moreover, since all the fixed point arguments that we apply to obtain the graph parameterizations rely on complex path integration, we are not able to extend them to domains which are not simply connected. Therefore, to reach $\widetilde{D}_{\kappa,d}$ from $D^{\mathrm{sep,u}}_{\kappa,\beta}$, we need to switch to a different parametrization that is well defined at $u = 0$.

The auxiliary parametrization we consider is the classical time-parametrization which is associated to the Hamiltonian H in (II.2.5). (Recall that the graph parametrization z^{u} was associated to the Hamiltonian $H^{\mathrm{sep}} = H \circ \phi_{\mathrm{eq}} \circ \phi_{\mathrm{sep}}$).

This analytic extension procedure has three steps:

1. We consider the *outer transition domain* (see Figure II.3.5)

$$\begin{aligned} \widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3} = \{v \in \mathbb{C} \ : \ & |\mathrm{Im}\, v| < A - \kappa_2\delta^2 - \tan\beta_0 \mathrm{Re}\, v, \\ & |\mathrm{Im}\, v| > d_2 A + \tan\beta_1 \mathrm{Re}\, v, \\ & |\mathrm{Im}\, v| < d_3 A + \tan\beta_1 \mathrm{Re}\, v\}, \end{aligned} \tag{II.3.17}$$

   where $d_1 < d_2 < d_3 < \frac{1}{2}$ are independent of δ and $\kappa_2 > \kappa_1$ is such that $A - \kappa_2\delta^2 > 0$. Notice that $\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3} \subset D^{\mathrm{u,out}}_{\kappa_1,d_1,\rho_2}$.

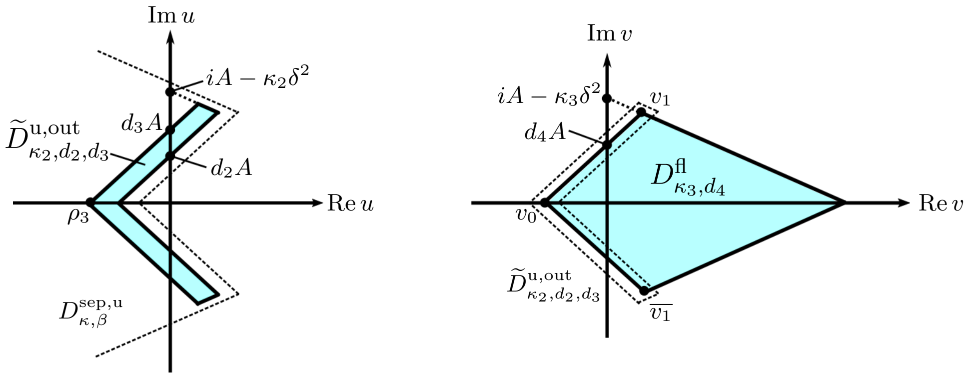


FIGURE II.3.5: The domain $\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$ given in (II.3.17) (left) and $D^{\mathrm{fl}}_{\kappa_3,d_4}$ in (II.3.19) (right).

Since $\dot{u}=1+o(1)$ (see (II.3.11)), we look for a real-analytic and close to the identity change of coordinates $u=v+\mathcal{U}(v)$ defined in $\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$ such that the time-parametrization

$$\Gamma^{\mathrm{u}}(v)=\phi_{\mathrm{eq}}\circ\phi_{\mathrm{sep}}(v+\mathcal{U}(v),z^{\mathrm{u}}(v+\mathcal{U}(v))) \tag{II.3.18}$$

is a solution of the Hamiltonian H in (II.2.5). That is, $\dot{v}=1$ and $\Gamma^{\mathrm{u}}(v)\in\mathcal{W}^{\mathrm{u}}(\delta)$ for $v\in\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$. See the details in Proposition II.3.7 and Corollary II.3.8 below.

2. We extend analytically the time-parametrization $\Gamma^{\mathrm{u}}(v)$ to reach the domain $\widetilde{D}_{\kappa,d}$. In particular, we extend Γ^{u} to the *flow domain*

$$\begin{aligned} D^{\mathrm{fl}}_{\kappa_3,d_4}=\{v\in\mathbb{C}\ :\ &|\mathrm{Im}\,v|<A-\kappa\delta^2-\tan\beta_0\mathrm{Re}\,v,\\ &|\mathrm{Im}\,v|<d_4A+\tan\beta_1\mathrm{Re}\,v\}\,, \end{aligned} \tag{II.3.19}$$

where $d_4\in(d_2,d_3)$ is independent of δ and $\kappa_3>\kappa_2$ is such that $A-\kappa_3\delta^2>0$. Notice that,

$$\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}\cap D^{\mathrm{fl}}_{\kappa_3,d_4}\neq\emptyset,\qquad\text{and}\qquad\widetilde{D}_{\kappa_4,d_5}\subset D^{\mathrm{fl}}_{\kappa_3,d_4},$$

for $d_5\in(d_1,d_4)$ and $\kappa_4>\kappa_3$. See the details in Proposition II.3.9.

3. We prove that there exists a real-analytic close to the identity change of variables of the form $v=u+\mathcal{V}(u)$, $u\in\widetilde{D}_{\kappa_4,d_5}$, such that the function $z^{\mathrm{u}}(u)$ defined by

$$(u,z^{\mathrm{u}}(u))=(\phi_{\mathrm{eq}}\circ\phi_{\mathrm{sep}})^{-1}\Big(\Gamma^{\mathrm{u}}(u+\mathcal{V}(u))\Big) \tag{II.3.20}$$

gives an invariant graph of H^{sep} in (II.3.8). See the details in Proposition II.3.10 and Corollary II.3.11 below.

As a consequence, we have extended analytically z^{u} to $\widetilde{D}_{\kappa_4,d_5}$.

For the first step, we look for a function $\mathcal{U}$ such that $(v+\mathcal{U}(v),z^{\mathrm{u}}(v+\mathcal{U}(v)))$ is a solution of the differential equations given by the Hamiltonian H^{sep} in (II.3.8). Therefore, $\mathcal{U}$ satisfies

$$\partial_v\mathcal{U}(v)=\partial_w H_1^{\mathrm{sep}}\left(v+\mathcal{U}(v),z^{\mathrm{u}}(v+\mathcal{U}(v))\right). \tag{II.3.21}$$

The next proposition ensures that $\mathcal{U}$ exists and it is well defined for $v\in\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$.

Proposition II.3.7. *Let the function z^{u} and the constants ρ_2, d_1 and κ_1 be as obtained in Proposition II.3.6 and consider constants $d_2, d_3 \in (d_1, \frac{1}{2})$ such that $d_2 < d_3$ and $\kappa_2 > \kappa_1$. Then, there exists δ_0 such that, for $\delta \in (0, \delta_0)$, the equation* (II.3.21) *has a real-analytic solution $\mathcal{U} : \widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3} \to \mathbb{C}$.*

Moreover, for some constant $b_4 > 0$ independent of δ and for $v \in \widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$, $\mathcal{U}$ satisfies

$$|\mathcal{U}(v)| \leq b_4\delta^2 \qquad \text{and} \qquad v + \mathcal{U}(v) \in D^{\mathrm{u,out}}_{\kappa_1,d_1,\rho_2}.$$

This proposition is proved in Section II.4.3. Together with Proposition II.3.7 implies the following corollary.

Corollary II.3.8. *Under the hypothesis of Proposition II.3.7, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, the function Γ^{u} in* (II.3.18) *is well defined and real-analytic in $\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$.*

On the following, we use without mention that $\Gamma^{\mathrm{u}}(v)$ can be split as

$$\Gamma^{\mathrm{u}}(v) = \Gamma_h(v) + \widehat{\Gamma}(v), \quad \text{with} \quad \begin{cases} \Gamma_h = (\lambda_h, \Lambda_h, 0, 0)^T, \\ \widehat{\Gamma} = (\hat{\lambda}, \hat{\Lambda}, x, y)^T. \end{cases} \tag{II.3.22}$$

The next proposition extends the parametrization Γ^{u} to the domain $D^{\mathrm{fl}}_{\kappa_3,d_4}$ (see (II.3.19)).

Proposition II.3.9. *Let the function Γ^{u} and the constants d_2, d_3 and κ_2 be as obtained in Corollary II.3.8 and Proposition II.3.7 and fix $d_4 \in (d_2, d_3)$ and $\kappa_3 > \kappa_2$. Then, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, Γ^{u} can be real-analytically extended to $D^{\mathrm{fl}}_{\kappa_3,d_4}$.*

Moreover, there exists a constant $b_5 > 0$ independent of δ such that, for $v \in D^{\mathrm{fl}}_{\kappa_3,d_4}$,

$$|\hat{\lambda}(v)| \leq b_5\delta^2, \qquad |\hat{\Lambda}(v)| \leq b_5\delta^2, \qquad |x(v)| \leq b_5\delta^3, \qquad |y(v)| \leq b_5\delta^3.$$

This proposition is proved in Section II.4.4.

For the third step, we "go back" to the graph parametrization $z^{\mathrm{u}}(u)$ by looking for a change $v = u + \mathcal{V}(u)$ for $u \in \widetilde{D}_{\kappa,d}$. Notice that, in order to satisfy equation (II.3.20) and recalling (II.2.10), $\mathcal{V}$ must be a solution of

$$\hat{\lambda}(u + \mathcal{V}(u)) = \lambda_h(u) - \lambda_h(u + \mathcal{V}(u)). \tag{II.3.23}$$

Then, one can easily recover the graph parametrization $(w^{\mathrm{u}}(u), x^{\mathrm{u}}(u), y^{\mathrm{u}}(u))$ using the equations

$$\begin{aligned} \Lambda_h(u) - \Lambda_h(u + \mathcal{V}(u)) - \frac{w^{\mathrm{u}}(u)}{3\Lambda_h(u)} + \delta^2\mathfrak{L}_\Lambda(\delta) &= \hat{\Lambda}(u + \mathcal{V}(u)), \\ x^{\mathrm{u}}(u) + \delta^3\mathfrak{L}_x(\delta) &= x(u + \mathcal{V}(u)), \\ y^{\mathrm{u}}(u) + \delta^3\mathfrak{L}_y(\delta) &= y(u + \mathcal{V}(u)). \end{aligned} \tag{II.3.24}$$

The next proposition ensures that $\mathcal{V}$ exists and it is well defined in $\widetilde{D}_{\kappa,d}$ (see (II.3.16)).

Proposition II.3.10. *Let the function Γ^{u} and the constants d_4 and κ_3 be as obtained in Proposition II.3.9 and the constant d_1 as obtained in Proposition II.3.6. Let us consider constants $d_5 \in (d_1, d_4)$ and $\kappa_4 > \kappa_3$. Then, there exists $\delta_0 > 0$ such that, for*

$\delta \in (0, \delta_0)$, equation (II.3.23) has a real-analytic solution $\mathcal{V} : \widetilde{D}_{\kappa_4,d_5} \to \mathbb{C}$ satisfying

$$|\mathcal{V}(u)| \leq b_6 \delta^2 \qquad \textit{and} \qquad u + \mathcal{V}(u) \in D^{\mathrm{fl}}_{\kappa_3,d_4}.$$

for some constant $b_6 > 0$ independent of δ and $u \in \widetilde{D}_{\kappa_4,d_5}$.

Proposition II.3.10 is proved in Section II.4.5. Summarizing all the previous results we obtain the following result.

Corollary II.3.11. *Let the function $\mathcal{V}$ and the constants d_5 and κ_4 be as obtained in Proposition II.3.10. Then, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, equation (II.3.24) has a unique solution $z^{\mathrm{u}} = (w^{\mathrm{u}}, x^{\mathrm{u}}, y^{\mathrm{u}})^T : \widetilde{D}_{\kappa_4,d_5} \to \mathbb{C}^3$.*

Moreover, there exists a constant $b_7 > 0$ independent of δ such that, for $u \in \widetilde{D}_{\kappa_4,d_5}$,

$$|w^{\mathrm{u}}(u)| \leq b_7 \delta^2, \qquad |x^{\mathrm{u}}(u)| \leq b_7 \delta^3, \qquad |y^{\mathrm{u}}(u)| \leq b_7 \delta^3.$$

To finish this section, notice that, taking ρ_2 big enough, $d \geq d_5$ and $\kappa_0 \geq \kappa_4$ we have that

$$D_{\kappa_0,d} \subset D^{\mathrm{u,out}}_{\kappa_1,d_1,\rho_2} \cup \widetilde{D}_{\kappa_4,d_5}, \qquad \text{with} \qquad D^{\mathrm{u,out}}_{\kappa_1,d_1,\rho_2} \cap \widetilde{D}_{\kappa_4,d_5} \neq \emptyset,$$

and then, Corollary II.3.11 and Proposition II.3.6 imply the statements of Theorem II.3.3 referring to the unstable manifold z^{u}.

II.3.3 A first order of the invariant manifolds near the singularities

Let us consider the difference

$$\Delta z = (\Delta w, \Delta x, \Delta y)^T = z^{\mathrm{u}} - z^{\mathrm{s}},$$

where z^{u} and z^{s} are the perturbed invariant graphs given in Theorem II.3.3. Since z^{u} and z^{s} satisfy the invariance equation (II.3.12), the difference Δz satisfies the linear equation

$$\partial_u \Delta z(u) = \mathcal{A}^{\mathrm{sep}} \Delta z(u) + \widetilde{\mathcal{B}}^{\mathrm{spl}}(u) \Delta z(u), \tag{II.3.25}$$

where $\mathcal{A}^{\mathrm{sep}}$ is as given in (II.3.10) and

$$\widetilde{\mathcal{B}}^{\mathrm{spl}}(u) = \int_0^1 D_z \mathcal{R}^{\mathrm{sep}}[\sigma z^{\mathrm{u}} + (1-\sigma) z^{\mathrm{s}}](u) d\sigma. \tag{II.3.26}$$

Since z^{u} and z^{s} are already defined in $D_{\kappa,d}$, $\widetilde{\mathcal{B}}^{\mathrm{spl}}(u)$ can be considered as a "known" function.

In addition, since the graphs of z^{u} and z^{s} belong to the same energy level of H^{sep} (see (II.3.8)), we have that

$$H^{\mathrm{sep}}(u, z^{\mathrm{u}}(u); \delta) - H^{\mathrm{sep}}(u, z^{\mathrm{s}}(u); \delta) = 0, \quad \text{for } u \in D_{\kappa,d}.$$

Therefore, we can reduce (II.3.25) to a two dimensional equation. Indeed, defining $\Upsilon = (\Upsilon_1, \Upsilon_2, \Upsilon_3)$ such that

$$\Upsilon(u) = \int_0^1 D_z H^{\mathrm{sep}} \left(u, \sigma z^{\mathrm{u}}(u) + (1-\sigma) z^{\mathrm{s}}(u)\right) d\sigma, \tag{II.3.27}$$

and applying the mean value theorem we have that

$$\Upsilon_1(u)\Delta w(u) + \Upsilon_2(u)\Delta x(u) + \Upsilon_3(u)\Delta y(u) = 0.$$

Notice that $\Upsilon_1(u) = 1+\int_0^1 \partial_w H_1^{\mathrm{sep}}\left(u, \sigma z^{\mathrm{u}}(u) + (1-\sigma)z^{\mathrm{s}}(u)\right) d\sigma$ and therefore $\Upsilon_1(u) \neq 0$ for $u \in D_{\kappa,d}$ (see Remark II.3.4). Therefore, writing

$$\Delta w(u) = -\frac{\Upsilon_2(u)}{\Upsilon_1(u)}\Delta x(u) - \frac{\Upsilon_3(u)}{\Upsilon_1(u)}\Delta y(u) \tag{II.3.28}$$

and defining $\Delta\Phi = (\Delta x, \Delta y)^T$, the last two components of (II.3.25) are equivalent to

$$\partial_u \Delta\Phi(u) = \mathcal{A}^{\mathrm{spl}}(u)\Delta\Phi(u) + \mathcal{B}^{\mathrm{spl}}(u)\Delta\Phi(u), \tag{II.3.29}$$

where

$$\begin{aligned} \mathcal{A}^{\mathrm{spl}} &= \begin{pmatrix} \frac{i}{\delta^2} + \widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,2} & 0 \\ 0 & -\frac{i}{\delta^2} + \widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,3} \end{pmatrix}, \\ \mathcal{B}^{\mathrm{spl}} &= \begin{pmatrix} -\frac{\Upsilon_2}{\Upsilon_1}\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,1} & \widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,3} - \frac{\Upsilon_3}{\Upsilon_1}\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,1} \\ \widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,2} - \frac{\Upsilon_2}{\Upsilon_1}\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,1} & -\frac{\Upsilon_3}{\Upsilon_1}\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,1} \end{pmatrix}. \end{aligned} \tag{II.3.30}$$

Next, we give an heuristic idea of how to obtain an exponentially small bound for $\Delta y(u)$ for $u \in D_{\kappa,d}$. The case for Δx is analogous. If we omit the influence of $\widetilde{\mathcal{B}}^{\mathrm{spl}}$, then there exists $c_y \in \mathbb{C}$ such that Δy is of the form

$$\Delta y(u) = c_y\, e^{-\frac{i}{\delta^2}u}.$$

Evaluating this function at the points

$$u_+ = i(A-\kappa\delta^2), \qquad u_- = -i(A-\kappa\delta^2),$$

one has $\Delta y(u_+) \sim c_y e^{\frac{A}{\delta^2}-\kappa}$. Then, since $\Delta y(u_+) \sim 1$, it implies that $c_y \sim e^{-\frac{A}{\delta^2}+\kappa}$ and, as a consequence, Δy is exponentially small for $u \in \mathbb{R}$. However, we are not interested in an upper bound of Δy but in an asymptotic formula. Thus we have to find the constant c_y, or more precisely a good approximation of it.

To this end, we need to give the main terms of Δy at $u = u_+$. Likewise we need to analyze $\Delta x(u) \sim c_x\, e^{\frac{i}{\delta^2}u}$ at $u = u_-$. To perform this analysis we proceed as follows:

1. We provide suitable solutions $Z_0^{\mathrm{u,s}}(U)$ of the so-called inner equation. The inner equation, see [Bal06; BS08], describes the dominant behavior of the functions z^{u} and z^{s} close to (one of) the singularities $u = \pm iA$. In particular, it involves the first order of the Hamiltonian H^{sep} close to a singularity and it is independent of the small parameter δ. See Section II.3.3.1.

2. We check how well $z^{\mathrm{u,s}}(u)$ are approximated by $Z_0^{\mathrm{u,s}}(U)$ around the singularities $u = \pm iA$ by means of a complex matching procedure. See Section II.3.3.2.

II.3.3.1 The inner equation

In this section we summarize the results on the derivation and study of the inner equation obtained in Sections I.2.3 and I.2.4. We focus on the inner equation around the singularity $u = iA$, but analogous results hold near $u = -iA$.

To derive the inner equation, we look for a new Hamiltonian which is a good approximation of H^{sep}, given in (II.3.8), in a suitable neighborhood of $u = iA$. First, we scale the variables (u, w, x, y) so that the graphs $z^{\mathrm{u,s}}(u)$ become $\mathcal{O}(1)$-functions when $u - iA = \mathcal{O}(\delta^2)$. Since, by Theorem II.3.3, we have that

$$w^{\diamond}(u) = \mathcal{O}(\delta^{-\frac{4}{3}}), \qquad x^{\diamond}(u) = \mathcal{O}(\delta^{\frac{1}{3}}), \qquad y^{\diamond}(u) = \mathcal{O}(\delta^{\frac{1}{3}}), \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s},$$

we consider the symplectic scaling $\phi_{\mathrm{in}} : (U, W, X, Y) \to (u, w, x, y)$, given by

$$U = \frac{u - iA}{\delta^2}, \qquad W = \delta^{\frac{4}{3}} \frac{w}{2\alpha_+^2}, \qquad X = \frac{x}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_+}, \qquad Y = \frac{y}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_+}, \tag{II.3.31}$$

where $\alpha_+ \in \mathbb{C}$ is the constant given by in Theorem II.3.1, which is added to avoid the dependence of the inner equation on it. Moreover, we also perform the time scaling $\tau = \delta^2 T$. We refer to (U, W, X, Y) as the *inner coordinates*.

Proposition II.3.12. *The Hamiltonian system associated to (II.3.8) expressed in the inner coordinates is Hamiltonian with respect to the symplectic form $dU \wedge dW + idX \wedge dY$ and*

$$H^{\mathrm{in}} = \mathcal{H} + H_1^{\mathrm{in}}, \tag{II.3.32}$$

where

$$\mathcal{H}(U, W, X, Y) = H^{\mathrm{in}}(U, W, X, Y; \delta)|_{\delta=0} = W + XY + \mathcal{K}(U, W, X, Y),$$

with

$$\begin{aligned}
\mathcal{K}(U, W, X, Y) &= -\frac{3}{4}U^{\frac{2}{3}}W^2 - \frac{1}{3U^{\frac{2}{3}}}\left(\frac{1}{\sqrt{1 + \mathcal{J}(U, W, X, Y)}} - 1\right), \\
\mathcal{J}(U, W, X, Y) &= \frac{4W^2}{9U^{\frac{2}{3}}} - \frac{16W}{27U^{\frac{4}{3}}} + \frac{16}{81U^2} + \frac{4(X+Y)}{9U}\left(W - \frac{2}{3U^{\frac{2}{3}}}\right) \\
&\quad - \frac{4i(X-Y)}{3U^{\frac{2}{3}}} - \frac{X^2 + Y^2}{3U^{\frac{4}{3}}} + \frac{10XY}{9U^{\frac{4}{3}}}.
\end{aligned}$$

Moreover, if $c_1^{-1} \leq |U| \leq c_1$ and $|(W, X, Y)| \leq c_2$ for some $c_1 > 1$ and $0 < c_2 < 1$, there exist $b_0, \gamma_1, \gamma_2 > 0$ independent of δ, c_1, c_2 such that

$$|H_1^{\mathrm{in}}(U, W, X, Y; \delta)| \leq b_0 c_1^{\gamma_1} c_2^{\gamma_2} \delta^{\frac{4}{3}}. \tag{II.3.33}$$

This result is also stated in Proposition I.2.5 and proven in Chapter I.4.

Now, we present the study of the inner Hamiltonian $\mathcal{H}$. Denoting $Z = (W, X, Y)^T$, the equations associated to the Hamiltonian $\mathcal{H}$, can be written as

$$\begin{cases} \dot{U} = 1 + g^{\mathrm{in}}(U, Z), \\ \dot{Z} = \mathcal{A}^{\mathrm{in}} Z + f^{\mathrm{in}}(U, Z), \end{cases}$$

where

$$\mathcal{A}^{\mathrm{in}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \tag{II.3.34}$$

and $f^{\mathrm{in}} = (-\partial_U \mathcal{K}, i\partial_Y \mathcal{K}, -i\partial_X \mathcal{K})^T$ and $g^{\mathrm{in}} = \partial_W \mathcal{K}$. We look for invariant graphs $Z = Z_0^{\mathrm{u}}(U)$ and $Z = Z_0^{\mathrm{s}}(U)$ of this equation, that satisfy the invariance equation also

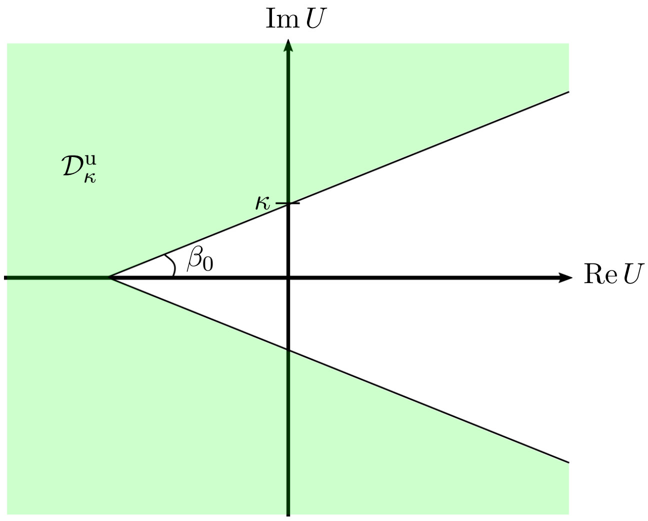


FIGURE II.3.6: The inner domain $\mathcal{D}^{\rm u}_\kappa$ for the unstable case.

called *inner equation*,

$$\partial_U Z_0^\diamond(U) = \mathcal{A}^{\rm in} Z_0^\diamond + \mathcal{R}^{\rm in}[Z_0^\diamond](U), \qquad \text{for } \diamond = \rm u, s, \tag{II.3.35}$$

with

$$\mathcal{R}^{\rm in}[\varphi](U) = \frac{f^{\rm in}(U,\varphi) - g^{\rm in}(U,\varphi)\mathcal{A}^{\rm in}\varphi}{1+g^{\rm in}(U,\varphi)}. \tag{II.3.36}$$

These functions $Z_0^{\rm u}$ and $Z_0^{\rm s}$ will be defined in the domains

$$\mathcal{D}^{\rm u}_\kappa = \{U \in \mathbb{C} : |\mathrm{Im}\, U| \geq \tan\beta_0 \mathrm{Re}\, U + \kappa\}, \qquad \mathcal{D}^{\rm s}_\kappa = -\mathcal{D}^{\rm u}_\kappa,$$

respectively, for some $\kappa > 0$ and with β_0 as given in Theorem II.3.3 (see Figure II.3.6). Moreover, we analyze the difference $\Delta Z_0 = Z_0^{\rm u} - Z_0^{\rm s}$ in the overlapping domain

$$\mathcal{E}_\kappa = \mathcal{D}^{\rm u}_\kappa \cap \mathcal{D}^{\rm s}_\kappa \cap \{U \in \mathbb{C} \,:\, \mathrm{Im}\, U < 0\}. \tag{II.3.37}$$

Theorem II.3.13. *There exist $\kappa_5, b_1 > 0$ such that for $\kappa \geq \kappa_5$, the equation (II.3.35) has analytic solutions $Z_0^\diamond(U) = (W_0^\diamond(U), X_0^\diamond(U), Y_0^\diamond(U))^T$, for $U \in \mathcal{D}^\diamond_\kappa$, $\diamond = \rm u, s$, satisfying*

$$|U^{\frac{8}{3}} W_0^\diamond(U)| \leq b_1, \qquad |U^{\frac{4}{3}} X_0^\diamond(U)| \leq b_1, \qquad |U^{\frac{4}{3}} Y_0^\diamond(U)| \leq b_1.$$

In addition, there exist $\Theta \in \mathbb{C}$, $b_2 > 0$ independent of κ, and an analytic function $\chi = (\chi_1, \chi_2, \chi_3)^T$ such that, for $U \in \mathcal{E}_\kappa$,

$$\Delta Z_0(U) = Z_0^{\rm u}(U) - Z_0^{\rm s}(U) = \Theta e^{-iU}\Big((0,0,1)^T + \chi(U)\Big),$$

with $|(U^{\frac{7}{3}}\chi_1(U), U^2\chi_2(U), U\chi_3(U))| \leq b_2$.

This result is also stated in Theorem I.2.7 and proven in Chapter I.5.

Remark II.3.14. *To obtain the analogous result to Theorem II.3.13 near the singularity $u = -iA$, one must perform the change of coordinates*

$$V = \frac{u + iA}{\delta^2}, \qquad \widehat{W} = \delta^{\frac{4}{3}} \frac{w}{2\alpha_-^2}, \qquad \widehat{X} = \frac{x}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_-}, \qquad \widehat{Y} = \frac{y}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_-},$$

where $\alpha_- \in \mathbb{C}$ is $\alpha_- = \overline{\alpha_+}$ (see Theorem II.3.1). Then, for $V \in \overline{\mathcal{D}^{\diamond}_{\kappa}}$, one can prove the existence of the corresponding solutions

$$\widehat{Z}^{\diamond}_0(V) = (\widehat{W}^{\diamond}_0(V), \widehat{X}^{\diamond}_0(V), \widehat{Y}^{\diamond}_0(V))^T, \qquad \text{where } \diamond = \mathrm{u}, \mathrm{s}.$$

Due to the real-analyticity of the problem (see Remark II.3.2) we have that $\widehat{X}^{\diamond}(V) = \overline{Y^{\diamond}(U)}$. Therefore, the difference $\Delta\widehat{Z}_0 = \widehat{Z}^{\mathrm{u}}_0 - \widehat{Z}^{\mathrm{s}}_0$, is given asymptotically for $U \in \overline{\mathcal{E}_\kappa}$ by

$$\Delta\widehat{Z}_0(V) = \overline{\Theta} e^{iV}\Big((0,1,0)^T + \zeta(V)\Big),$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3)^T$ satisfies $|(V^{\frac{7}{3}}\zeta_1(V), V\zeta_2(V), V^2\zeta_3(V)| \leq C$, for a constant C independent of κ.

II.3.3.2 Complex matching estimates

We now study how well the solutions of the inner equation approximate the solutions of the original system given by by Proposition II.3.6 in an appropriate domain. As in the previous section, we focus on the singularity $u = iA$, but analogous results can be proven for $u = -iA$ (see Remark II.3.14). Let us recall that the functions $z^{\mathrm{u,s}}$ are expressed in the separatrix coordinates (see (II.3.4)) while the functions $Z_0^{\mathrm{u,s}}$ are expressed in inner coordinates (see (II.3.31)).

We first define the matching domains in separatrix coordinates and, later, we translate them to the inner coordinates. Let us consider β_2, β_3, and γ independent of δ and κ, such that

$$0 < \beta_2 < \beta_0 < \beta_3 < \frac{\pi}{2}, \qquad \text{and} \qquad \gamma \in \left[\frac{3}{5}, 1\right),$$

with β_0 as given in Theorem II.3.1. Then, we define $u_j \in \mathbb{C}$ $j = 2, 3$ (see Figure II.3.7), as the points satisfying:

- $\operatorname{Im} u_j = -\tan\beta_j \operatorname{Re} u_j + A - \kappa\delta^2$.
- $|u_j - u_+| = \delta^{2\gamma}$, where $u_+ = i(A - \kappa\delta^2)$.
- $\operatorname{Re} u_2 < 0$ and $\operatorname{Re} u_3 > 0$.

We define the matching domains in the separatrix coordinates as the triangular domains

$$D^{\mathrm{mch,u}}_\kappa = \widehat{u_+\, u_2\, u_3}, \qquad D^{\mathrm{mch,s}}_\kappa = \widehat{u_+\,(-\overline{u_2})\,(-\overline{u_3})}.$$

Let d_1, ρ_2 and κ_1 be as given in Proposition II.3.6. Then, for $\kappa \geq \kappa_1$ and $\delta > 0$ small enough, the matching domains satisfy

$$D^{\mathrm{mch,u}}_\kappa \subset D^{\mathrm{sep,u}}_{\kappa,\beta} \qquad \text{and} \qquad D^{\mathrm{mch,s}}_\kappa \subset D^{\mathrm{sep,s}}_{\kappa,\beta}, \tag{II.3.38}$$

and, as a result, z^{u} and z^{s} are well defined in $D^{\mathrm{mch,u}}_\kappa$ and $D^{\mathrm{mch,s}}_\kappa$, respectively.

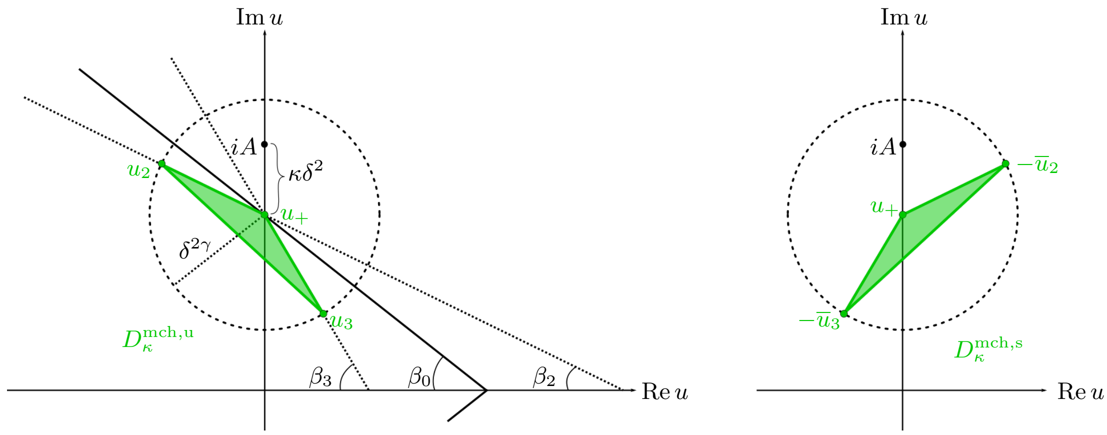


FIGURE II.3.7: The matching domains $D_\kappa^{\mathrm{mch,u}}$ and $D_\kappa^{\mathrm{mch,s}}$ in the outer variables.

The matching domains in inner variables are defined by

$$\mathcal{D}_\kappa^{\mathrm{mch},\diamond} = \left\{ U \in \mathbb{C} \ : \ \delta^2 U + iA \in D_\kappa^{\mathrm{mch},\diamond} \right\}, \qquad \text{for } \diamond = \mathrm{u},\mathrm{s}, \tag{II.3.39}$$

with

$$U_j = \frac{u_j - iA}{\delta^2}, \qquad \text{for } j = 2, 3. \tag{II.3.40}$$

Therefore, for $U \in \mathcal{D}_\kappa^{\mathrm{mch},\diamond}$,

$$\kappa \cos\beta_2 \leq |U| \leq \frac{C}{\delta^{2(1-\gamma)}}.$$

By definition,

$$\mathcal{D}_\kappa^{\mathrm{mch,u}} \subset \mathcal{D}_\kappa^{\mathrm{u}} \qquad \text{and} \qquad \mathcal{D}_\kappa^{\mathrm{mch,s}} \subset \mathcal{D}_\kappa^{\mathrm{s}},$$

for $\kappa \geq \kappa_5$ (see Theorem II.3.13). Thus, $Z_0^{\mathrm{u,s}}$ is well defined in $\mathcal{D}_\kappa^{\mathrm{mch,u,s}}$.

In order to compare $z^{\mathrm{u,s}}(u)$ and $Z_0^{\mathrm{u,s}}(U)$, we translate $z^{\mathrm{u,s}}$ to inner coordinates

$$Z^\diamond(U) = \left(W^\diamond, X^\diamond, Y^\diamond\right)^T(U) = \left(\delta^{\frac{4}{3}} \frac{w^\diamond}{2\alpha_+^2}, \frac{x^\diamond}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_+}, \frac{y^\diamond}{\delta^{\frac{1}{3}}\sqrt{2}\alpha_+} \right)^T (\delta^2 U + iA), \tag{II.3.41}$$

with $\diamond = \mathrm{u},\mathrm{s}$ and $z^\diamond = (w^\diamond, x^\diamond, y^\diamond)^T$ are given in Proposition II.3.6. Therefore, by (II.3.38), $Z^\diamond$ is well defined in the matching domain $\mathcal{D}_\kappa^{\mathrm{mch},\diamond}$ (which is expressed in inner variables).

Next theorem gives estimates for $Z^{\mathrm{u,s}} - Z_0^{\mathrm{u,s}}$.

Theorem II.3.15. *Consider κ_1 and κ_5 as obtained in Proposition II.3.6 and Theorem II.3.13, respectively. Then, there exist $\gamma^* \in [\frac{3}{5}, 1)$, $\kappa_6 \geq \max\{\kappa_1, \kappa_5\}$ and $\delta_0 > 0$ such that, for $\gamma \in (\gamma^*, 1)$, there exists $b_{11} > 0$ satisfying that, for $U \in \mathcal{D}_\kappa^{\mathrm{mch},\diamond}$, $\kappa \geq \kappa_6$ and $\delta \in (0, \delta_0)$,*

$$|U^{\frac{4}{3}} W_1^\diamond(U)| \leq b_{11}\delta^{\frac{2}{3}(1-\gamma)}, \qquad |U\, X_1^\diamond(U)| \leq b_{11}\delta^{\frac{2}{3}(1-\gamma)}, \qquad |U\, Y_1^\diamond(U)| \leq b_{11}\delta^{\frac{2}{3}(1-\gamma)},$$

with $(W_1^\diamond, X_1^\diamond, Y_1^\diamond)^T = Z_1^\diamond = Z^\diamond - Z_0^\diamond$ and $\diamond = \mathrm{u},\mathrm{s}$.

This theorem is proven in Chapter II.5.

II.3.4 The asymptotic formula for the difference

We look for an asymptotic expression for the difference

$$\Delta\Phi = (\Delta x, \Delta y)^T = (x^{\mathrm{u}} - x^{\mathrm{s}}, y^{\mathrm{u}} - y^{\mathrm{s}})^T,$$

where $(x^{\mathrm{u}}, y^{\mathrm{u}})$ and $(x^{\mathrm{s}}, y^{\mathrm{s}})$ are components of the perturbed invariant graphs given in Theorem II.3.3. Recall that, by (II.3.29), $\Delta\Phi$ satisfies

$$\partial_u \Delta\Phi(u) = \mathcal{A}^{\mathrm{spl}}(u)\Delta\Phi(u) + \mathcal{B}^{\mathrm{spl}}(u)\Delta\Phi(u), \tag{II.3.42}$$

with $\mathcal{A}^{\mathrm{spl}}$ and $\mathcal{B}^{\mathrm{spl}}$ as given in (II.3.30). The equation is split as a dominant part, given by the matrix $\mathcal{A}^{\mathrm{spl}}$ and a small perturbation corresponding to the the matrix $\mathcal{B}^{\mathrm{spl}}$. Therefore, it makes sense to look for $\Delta\Phi$ as $\Delta\Phi = \Delta\Phi_0 + h.o.t$ with a suitable dominant term $\Delta\Phi_0 = (\Delta x_0, \Delta y_0)^T$ satisfying

$$\partial_u \Delta\Phi_0(u) = \mathcal{A}^{\mathrm{spl}}(u)\Delta\Phi_0(u). \tag{II.3.43}$$

A fundamental matrix of (II.3.43), for $u \in D_{\kappa,d}$, is given by

$$\mathcal{M}(u) = \begin{pmatrix} m_x(u) & 0 \\ 0 & m_y(u) \end{pmatrix}, \tag{II.3.44}$$

with

$$\begin{aligned} m_x(u) &= e^{\frac{i}{\delta^2}u} B_x(u), \quad & B_x(u) &= \exp\left(\int_{u_*}^{u} \widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,2}(s)ds\right), \\ m_y(u) &= e^{-\frac{i}{\delta^2}u} B_y(u), \quad & B_y(u) &= \exp\left(\int_{u_*}^{u} \widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,3}(s)ds\right), \end{aligned} \tag{II.3.45}$$

and a fixed $u_* \in D_{\kappa,d} \cap \mathbb{R}$. Then, $\Delta\Phi_0$ must be of form

$$\Delta\Phi_0(u) = \begin{pmatrix} \Delta x_0(u) \\ \Delta y_0(u) \end{pmatrix} = \begin{pmatrix} c_x^0 m_x(u) \\ c_y^0 m_y(u) \end{pmatrix}, \tag{II.3.46}$$

for suitable constants $c_x^0, c_y^0 \in \mathbb{C}$ which we now determine.

By Theorems II.3.13 and II.3.15 and using the inner change of coordinates in (II.3.31), we have a good approximation of $\Delta y(u)$ near the singularity $u = iA$ given by

$$\Delta y(u) \approx \sqrt{2}\alpha_+ \delta^{\frac{1}{3}} \Delta Y_0\left(\frac{u - iA}{\delta^2}\right).$$

Then, taking $u = u_+ = i(A - \kappa\delta^2)$, we have that

$$\Delta y(u_+) \approx \Delta y_0(u_+) \approx \sqrt{2}\alpha_+ \delta^{\frac{1}{3}} \Delta Y_0\left(\frac{u_+ - iA}{\delta^2}\right) = \sqrt{2}\alpha_+ \delta^{\frac{1}{3}} e^{-\kappa}\Theta(1 + \chi_3(-i\kappa)).$$

Then, using that $\Delta y(u_+) \approx \Delta y_0(u_+) = c_y^0 m_y(u_+)$, and proceeding analogously for the component Δx at the point $u_- = -i(A - \kappa\delta^2)$ (see Remark II.3.14), we take

$$c_x^0 = \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \overline{\Theta}\, \sqrt{2}\alpha_- B_x^{-1}(u_-) \qquad \text{and} \qquad c_y^0 = \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \Theta \sqrt{2}\alpha_+ B_y^{-1}(u_+). \tag{II.3.47}$$

To prove Theorem II.2.2, we check that $\Delta\Phi_0(u)$ is the leading term of $\Delta\Phi(u)$, for $u \in \mathbb{R} \cap D_{\kappa,d}$, by estimating the remainder $\Delta\Phi_1 = \Delta\Phi - \Delta\Phi_0$.

In order to simplify the notation, throughout the rest of the document, we denote by C any positive constant independent of δ and κ to state estimates.

II.3.4.1 End of the proof of Theorem II.2.2

We look for $\Delta\Phi_1$ as the unique solution of an integral equation. Since $\Delta\Phi$ satisfies (II.3.42), by the variations of constants formula

$$\Delta\Phi(u) = \begin{pmatrix} c_x m_x(u) \\ c_y m_y(u) \end{pmatrix} + \begin{pmatrix} m_x(u) \displaystyle\int_{u_-}^{u} m_x^{-1}(s)\, \pi_1 \left(\mathcal{B}^{\mathrm{spl}}(s)\Delta\Phi(s)\right) ds \\ m_y(u) \displaystyle\int_{u_+}^{u} m_y^{-1}(s)\, \pi_2 \left(\mathcal{B}^{\mathrm{spl}}(s)\Delta\Phi(s)\right) ds \end{pmatrix}, \tag{II.3.48}$$

where $\mathcal{M}(u)$ is the fundamental matrix (II.3.44), s belongs to some integration path in $D_{\kappa,d}$ and c_x and c_y are defined as

$$c_x = \Delta x(u_-) m_x^{-1}(u_-), \qquad c_y = \Delta y(u_+) m_y^{-1}(u_+). \tag{II.3.49}$$

For $k_1, k_2 \in \mathbb{C}$, we define

$$\mathcal{I}[k_1, k_2](u) = \big(k_1\, m_x(u), k_2\, m_y(u)\big)^T, \tag{II.3.50}$$

and the operator

$$\mathcal{E}[\varphi](u) = \begin{pmatrix} m_x(u) \displaystyle\int_{u_-}^{u} m_x^{-1}(s)\, \pi_1 \left(\mathcal{B}^{\mathrm{spl}}(s)\varphi(s)\right) ds \\ m_y(u) \displaystyle\int_{u_+}^{u} m_y^{-1}(s)\, \pi_2 \left(\mathcal{B}^{\mathrm{spl}}(s)\varphi(s)\right) ds \end{pmatrix}. \tag{II.3.51}$$

Then, with this notation, $\Delta\Phi_0 = \mathcal{I}[c_x^0, c_y^0]$ (see (II.3.47)) and equation (II.3.48) is equivalent to $\Delta\Phi = \mathcal{I}[c_x, c_y] + \mathcal{E}[\Delta\Phi]$. Since $\mathcal{E}$ is a linear operator, $\Delta\Phi_1 = \Delta\Phi - \Delta\Phi_0$ satisfies

$$\Delta\Phi_1(u) = \mathcal{I}[c_x - c_x^0, c_y - c_y^0](u) + \mathcal{E}[\Delta\Phi_0](u) + \mathcal{E}[\Delta\Phi_1](u). \tag{II.3.52}$$

To obtain estimates for $\Delta\Phi_1$, we first prove that $\mathrm{Id} - \mathcal{E}$ is invertible in the Banach space $\mathcal{X}_\times^{\mathrm{spl}} = \mathcal{X}^{\mathrm{spl}} \times \mathcal{X}^{\mathrm{spl}}$, with

$$\mathcal{X}^{\mathrm{spl}} = \left\{ \varphi : D_{\kappa,d} \to \mathbb{C} \;:\; \|\varphi\|^{\mathrm{spl}} = \sup_{u \in D_{\kappa,d}} \left| e^{\frac{A - |\mathrm{Im}\, u|}{\delta^2}} \varphi(u) \right| < +\infty \right\},$$

endowed with the norm

$$\|\varphi\|_\times^{\mathrm{spl}} = \|\varphi_1\|^{\mathrm{spl}} + \|\varphi_2\|^{\mathrm{spl}}, \tag{II.3.53}$$

for $\varphi = (\varphi_1, \varphi_2)$. Therefore, to prove Theorem II.2.2 it is enough to see that $\Delta\Phi_1$ satisfies that $\|\Delta\Phi_1\|_\times^{\mathrm{spl}} \leq C\delta^{\frac{1}{3}} |\log\delta|^{-1}$.

First, we state a lemma whose proof is postponed to Appendix II.B.1.

Lemma II.3.16. *Let κ_0, δ_0 be the constants given in Theorem II.3.3. Then, there exists a constant $C > 0$ such that, for $\kappa \geq \kappa_0$, $\delta \in (0, \delta_0)$ and $u \in D_{\kappa,d}$, the function Υ in (II.3.27), the matrix $\mathcal{B}^{\mathrm{spl}}$ in (II.3.30) and the functions B_x, B_y in (II.3.45) satisfy* *for* $\kappa \geq \kappa_0$, $\delta \in (0, \delta_0)$ *and* $u \in D_{\kappa,d}$,

$$|\Upsilon_1(u) - 1| \leq \frac{C}{\kappa^2}, \qquad |\Upsilon_2(u)| \leq \frac{C\delta}{|u^2 + A^2|^{\frac{4}{3}}}, \qquad |\Upsilon_3(u)| \leq \frac{C\delta}{|u^2 + A^2|^{\frac{4}{3}}}, \tag{II.3.54}$$

$$C^{-1} \leq |B_*(u)| \leq C, \quad * = x, y, \quad \text{and} \quad |\mathcal{B}^{\text{spl}}_{i,j}(u)| \leq \frac{C\,\delta^2}{|u^2 + A^2|^2}, \quad i, j \in \{1, 2\}\,.$$

In the next lemma we obtain estimates for the linear operator $\mathcal{E}$ (see (II.3.51)).

Lemma II.3.17. *Let* κ_0, δ_0 *be the constants as given in Theorem II.3.3. There exists* $b_{12} > 0$ *such that for* $\delta \in (0, \delta_0)$ *and* $\kappa \geq \kappa_0$, *the operator* $\mathcal{E} : \mathcal{X}^{\text{spl}}_\times \to \mathcal{X}^{\text{spl}}_\times$ *in* (II.3.51) *is well defined and satisfies that, for* $\varphi \in \mathcal{X}^{\text{spl}}_\times$,

$$\|\mathcal{E}[\varphi]\|^{\text{spl}}_\times \leq \frac{b_{12}}{\kappa}\, \|\varphi\|^{\text{spl}}_\times\,.$$

In particular, $\mathrm{Id} - \mathcal{E}$ *is invertible and*

$$\left\|(\mathrm{Id} - \mathcal{E})^{-1}[\varphi]\right\|^{\text{spl}}_\times \leq 2\, \|\varphi\|^{\text{spl}}_\times\,.$$

Proof. Let us consider $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2)^T$, $\varphi \in \mathcal{X}^{\text{spl}}_\times$ and $u \in D_{\kappa,d}$. We only prove the estimate for $\mathcal{E}_2[\varphi](u)$. The corresponding one for $\mathcal{E}_1[\varphi](u)$ follows analogously.

By the definition of m_y in (II.3.45) and Lemma II.3.16, we have that

$$\begin{aligned}
|\mathcal{E}_2[\varphi](u)| &\leq C\delta^2 e^{\frac{\operatorname{Im} u}{\delta^2}} \left| \int_{u_+}^{u} e^{-\frac{\operatorname{Im} s}{\delta^2}} \frac{|\varphi_1(s)| + |\varphi_2(s)|}{|s^2 + A^2|^2} ds \right| \\
&\leq C\delta^2 e^{\frac{\operatorname{Im} u - A}{\delta^2}}\, \|\varphi\|^{\text{spl}}_\times \left| \int_{u_+}^{u} e^{\frac{|\operatorname{Im} s| - \operatorname{Im} s}{\delta^2}} \frac{ds}{|s^2 + A^2|^2} \right|.
\end{aligned}$$

Let us consider the case $\operatorname{Im} u < 0$. Then, for a fixed $u_0 \in \mathbb{R} \cap D_{\kappa,d}$, we define the integration path $\rho_t \subset D_{\kappa,d}$ as

$$\rho_t = \begin{cases} u_+ + 2t(u_0 - u_+) & \text{for } t \in (0, \frac{1}{2}), \\ u_0 + (2t - 1)(u - u_0) & \text{for } t \in [\frac{1}{2}, 1). \end{cases}$$

Then,

$$|\mathcal{E}_2[\varphi](u)| \leq C\delta^2 e^{-\frac{|\operatorname{Im} u| + A}{\delta^2}}\, \|\varphi\|^{\text{spl}}_\times \left| \int_0^{\frac{1}{2}} \frac{dt}{|\rho_t - iA|^2} + \int_{\frac{1}{2}}^{1} \frac{e^{\frac{2|\operatorname{Im} \rho_t|}{\delta^2}}}{|\rho_t + iA|^2} dt \right| \leq \frac{C}{\kappa} e^{\frac{|\operatorname{Im} u| - A}{\delta^2}}\, \|\varphi\|^{\text{spl}}_\times\,.$$

If $\operatorname{Im} u \geq 0$, we consider the integration path $\rho_t = u_+ + t(u - u_+)$ for $t \in [0, 1]$ and we obtain

$$|\mathcal{E}_2[\varphi](u)| \leq C\delta^2 e^{\frac{|\operatorname{Im} u| - A}{\delta^2}}\, \|\varphi\|^{\text{spl}}_\times \left| \int_0^1 \frac{|u - u_+|}{|\rho_t - iA|^2} dt \right| \leq \frac{C}{\kappa} e^{\frac{|\operatorname{Im} u| - A}{\delta^2}}\, \|\varphi\|^{\text{spl}}_\times\,.$$

Therefore, $\|\mathcal{E}_2[\varphi]\|^{\text{spl}} \leq \frac{C}{\kappa}\, \|\varphi\|^{\text{spl}}_\times$. □

Notice that, by (II.3.52), $\Delta\Phi_1$ satisfies

$$(\mathrm{Id} - \mathcal{E})\Delta\Phi_1(u) = \mathcal{I}[c_x - c_x^0, c_y - c_y^0](u) + \mathcal{E}[\Delta\Phi_0](u). \tag{II.3.55}$$

Since, by Lemma II.3.17, $\mathrm{Id} - \mathcal{E}$ is invertible in $\mathcal{X}_\times^{\mathrm{spl}}$ we have an explicit formula for $\Delta\Phi_1$. Nevertheless, we still need good estimates for the right hand side with respect to the norm (II.3.53).

Lemma II.3.18. *There exist $\kappa_*, \delta_0, b_{13} > 0$ such that, for $\kappa = \kappa_* |\log \delta|$ and $\delta \in (0, \delta_0)$,*

$$\left\| \mathcal{I}[c_x - c_x^0, c_y - c_y^0] \right\|_\times^{\mathrm{spl}} \leq \frac{b_{13}\, \delta^{\frac{1}{3}}}{|\log \delta|} \qquad \text{and} \qquad \| \mathcal{E}[\Delta\Phi_0](u) \|_\times^{\mathrm{spl}} \leq \frac{b_{13}\, \delta^{\frac{1}{3}}}{|\log \delta|},$$

with $\mathcal{I}$, (c_x^0, c_y^0), (c_x, c_y), $\mathcal{E}$ and $\Delta\Phi_0$ defined in (II.3.50), (II.3.47), (II.3.49), (II.3.51) and (II.3.46), respectively.

Proof. By the definition of the function $\mathcal{I}$,

$$\left\| \mathcal{I}[c_x - c_x^0, c_y - c_y^0] \right\|_\times^{\mathrm{spl}} = \left| c_x - c_x^0 \right| \| m_x \|^{\mathrm{spl}} + \left| c_y - c_y^0 \right| \| m_y \|^{\mathrm{spl}},$$

where m_x and m_y are given in (II.3.45). Then, by Lemma II.3.16,

$$\| m_x \|^{\mathrm{spl}} = e^{\frac{A}{\delta^2}} \sup_{u \in D_{\kappa,d}} \left[e^{-\frac{\mathrm{Im}\, u + |\mathrm{Im}\, u|}{\delta^2}} |B_x(u)| \right] \leq C e^{\frac{A}{\delta^2}}, \qquad \| m_y \|^{\mathrm{spl}} \leq C e^{\frac{A}{\delta^2}},$$

and, as a result,

$$\left\| \mathcal{I}[c_x - c_x^0, c_y - c_y^0] \right\|_\times^{\mathrm{spl}} \leq C e^{\frac{A}{\delta^2}} \left(|c_x - c_x^0| + |c_y - c_y^0| \right). \tag{II.3.56}$$

We now obtain an estimate for $|c_y - c_y^0|$. The estimate for $|c_x - c_x^0|$ follows analogously.

By the definition of m_y (see (II.3.45)), one has

$$\left| c_y - c_y^0 \right| = e^{-\frac{A}{\delta^2} + \kappa} \left| B_y^{-1}(u_+) \right| |\Delta y(u_+) - \Delta y_0(u_+)|. \tag{II.3.57}$$

Let us denote $\Delta Y = Y^{\mathrm{u}} - Y^{\mathrm{s}}$ where $Y^{\mathrm{u,s}}$ are given on (II.3.41). Recall that $Y^{\mathrm{u,s}} = Y_0^{\mathrm{u,s}} + Y_1^{\mathrm{u,s}}$ where $Y_0^{\mathrm{u,s}}$ is the third component of $Z_0^{\mathrm{u,s}}$, the solutions of the inner equation (see Theorems II.3.13 and II.3.15). We write,

$$\Delta y(u_+) = \sqrt{2}\alpha_+ \delta^{\frac{1}{3}} \Delta Y \left(\frac{u_+ - iA}{\delta^2} \right) = \sqrt{2}\alpha_+ \delta^{\frac{1}{3}} \left[\Delta Y_0 (-i\kappa) + Y_1^{\mathrm{u}} (-i\kappa) - Y_1^{\mathrm{s}} (-i\kappa) \right].$$

By the definition of Δy_0 in (II.3.46) (see also (II.3.47)), we have $\Delta y_0(u_+) = \sqrt{2}\alpha_+ \delta^{\frac{1}{3}} \Theta e^{-\kappa}$. Then, by (II.3.57) and Lemma II.3.16,

$$\left| c_y - c_y^0 \right| \leq C \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2} + \kappa} \Big[\left| \Delta Y_0 (-i\kappa) - \Theta e^{-\kappa} \right| + |Y_1^{\mathrm{u}} (-i\kappa)| + |Y_1^{\mathrm{s}} (-i\kappa)| \Big],$$

and, applying Theorems II.3.13 and II.3.15, we obtain

$$\left| c_y - c_y^0 \right| \leq C \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2} + \kappa} \left[\left| \chi_3(-i\kappa) e^{-\kappa} \right| + \frac{C}{\kappa} \delta^{\frac{2}{3}(1-\gamma)} \right] \leq \frac{C}{\kappa} \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left(1 + \delta^{\frac{2}{3}(1-\gamma)} e^{\kappa} \right),$$

where $\gamma \in (\gamma^*, 1)$ with $\gamma^* \in [\frac{3}{5}, 1)$ given in Theorem II.3.15. Taking $\kappa = \kappa_* |\log \delta|$ with $0 < \kappa_* < \frac{2}{3}(1-\gamma)$, we obtain

$$\left| c_y - c_y^0 \right| \leq \frac{C \delta^{\frac{1}{3}}}{|\log \delta|} e^{-\frac{A}{\delta^2}} \left(1 + \delta^{\frac{2}{3}(1-\gamma) - \kappa_*} \right) \leq \frac{C \delta^{\frac{1}{3}}}{|\log \delta|} e^{-\frac{A}{\delta^2}}.$$

This bound and (II.3.56) prove the first estimate of the lemma.

For the second estimate, it only remains to bound $\Delta\Phi_0$ and apply Lemma II.3.17. Indeed, by the definition of $\Delta\Phi_0$ in (II.3.47), Lemma II.3.16 and (II.3.56), we have that

$$\|\Delta\Phi_0\|_{\times}^{\mathrm{spl}} = \left\|\mathcal{I}[c_x^0, c_y^0]\right\|_{\times}^{\mathrm{spl}} \leq Ce^{\frac{A}{\delta^2}} \left(\left|c_x^0\right| + \left|c_y^0\right|\right) \leq C\delta^{\frac{1}{3}}.$$

Since $\kappa = \kappa_* |\log \delta|$ with $0 < \kappa_* < \frac{2}{3}(1-\gamma)$, Lemma II.3.17 implies $\|\mathcal{E}[\Delta\Phi_0]\|_{\times}^{\mathrm{spl}} \leq \frac{C\delta^{\frac{1}{3}}}{|\log\delta|}$. □

With this lemma, we can give sharp estimates for $\Delta\Phi_1$ by using equation (II.3.55). Indeed, since the right hand side of this equation belongs to $\mathcal{X}_{\times}^{\mathrm{spl}}$, by Lemma II.3.17,

$$\Delta\Phi_1(u) = (\mathrm{Id} - \mathcal{E})^{-1} \left(\mathcal{I}[c_x - c_x^0, c_y - c_y^0](u) + \mathcal{E}[\Delta\Phi_0](u)\right).$$

Then, Lemmas II.3.17 and II.3.18 imply

$$\|\Delta\Phi_1\|_{\times}^{\mathrm{spl}} \leq \frac{C\delta^{\frac{1}{3}}}{|\log\delta|}. \tag{II.3.58}$$

To prove Theorem II.2.2, it only remains to analyze $B_x(u_-)$ and $B_y(u_+)$.

Lemma II.3.19. *Let κ_* be as given in Lemma II.3.18. Then, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$ and $\kappa = \kappa_* |\log\delta|$, the functions B_x, B_y defined in (II.3.45) satisfy*

$$\begin{aligned} B_x^{-1}(u_-) &= e^{-\frac{4i}{9}(\pi - \lambda_h(u_*))} \left(1 + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right), \\ B_y^{-1}(u_+) &= e^{\frac{4i}{9}(\pi - \lambda_h(u_*))} \left(1 + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right), \end{aligned}$$

where $u_\pm = \pm i(A - \kappa\delta^2)$.

This lemma is proven in Appendix II.B.2.

Let $u_* \in D_{\kappa,d} \cap \mathbb{R}$. We compute the first order of $\Delta\Phi_0(u_*) = (\Delta x_0(u_*), \Delta y_0(u_*))^T$. Since, by Theorem II.3.1, $(\alpha_+)^3 = (\alpha_-)^3 = \frac{1}{2}$, and applying Lemma II.3.19 and (II.3.47), we obtain

$$|\Delta x_0(u_*)| = |\Delta y_0(u_*)| = \sqrt[6]{2}\, |\Theta|\, \delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left(1 + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right).$$

Moreover, by (II.3.58),

$$|\Delta x(u_*) - \Delta x_0(u_*)|\,, |\Delta y(u_*) - \Delta y_0(u_*)| \leq \frac{C\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}}}{|\log\delta|}.$$

Finally, notice that the section $u = u_* \in D_{\kappa,d} \cap \mathbb{R}$ translates to $\lambda = \lambda^* := \lambda_h(u_*)$ (see (II.3.4)). Moreover, since $\dot{\lambda}_h = -3\Lambda_h$ (see (II.3.1)), one deduces that $\Lambda_h(u) > 0$ for $u > 0$. Therefore, by the change of coordinates (II.3.4), Theorem II.3.3 and taking δ small enough,

$$\Lambda_*^\diamond = \Lambda_h(u_*) - \frac{w^\diamond(u_*)}{3\Lambda_h(u_*)} = \Lambda_h(u_*) + \mathcal{O}(\delta^2) > 0, \qquad \text{with } \diamond = \mathrm{u}, \mathrm{s},$$

and, therefore using formula (II.3.28) for Δw and Lemma II.3.16, we obtain that

$$|\Lambda_*^{\mathrm{u}} - \Lambda_*^{\mathrm{s}}| \leq C\,|\Delta w(u_*)| \leq C\delta\,|\Delta x(u_*)| + C\delta\,|\Delta y(u_*)| \leq C\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}.$$

Chapter II.4

The perturbed invariant manifolds

In this chapter, we prove Theorem II.3.3 by following the scheme detailed in Sections II.3.2.1 and II.3.2.2.

Throughout this chapter and the following ones, we denote the components of all the functions and operators by a numerical sub-index $f = (f_1, f_2, f_3)^T$, unless stated otherwise.

II.4.1 The invariant manifolds in the infinity domain

The first step is to prove Proposition II.3.5, which deals with the proof of the existence of parameterizations z^{u} and z^{s} satisfying the invariance equation (II.3.12) and the asymptotic conditions (II.3.6). We only consider the $-$u$-$ case, being the $-$s$-$ case analogous.

Consider the invariance equation (II.3.12), $\partial_u z^{\mathrm{u}} = \mathcal{A}^{\mathrm{sep}} z^{\mathrm{u}} + \mathcal{R}^{\mathrm{sep}}[z^{\mathrm{u}}]$, with $\mathcal{A}^{\mathrm{sep}}$ and $\mathcal{R}^{\mathrm{sep}}$ defined in (II.3.10) and (II.3.13), respectively. This equation can be written as

$$\mathcal{L} z^{\mathrm{u}} = \mathcal{R}^{\mathrm{sep}}[z^{\mathrm{u}}], \qquad \text{with} \qquad \mathcal{L}\varphi = (\partial_u - \mathcal{A}^{\mathrm{sep}})\varphi. \tag{II.4.1}$$

In order to obtain a fixed point equation from (II.4.1), we look for a left inverse of $\mathcal{L}$ in a suitable Banach space. To this end, for a fixed $\rho_1 > 0$ and a given $\alpha \in \mathbb{R}$, we introduce

$$\mathcal{X}_\alpha^\infty = \left\{ \varphi : D_{\rho_1}^{\mathrm{u},\infty} \to \mathbb{C} \ : \ \varphi \text{ real-analytic}, \ \|\varphi\|_\alpha^\infty := \sup_{u \in D_{\rho_1}^{\mathrm{u},\infty}} |e^{-\alpha u} \varphi(u)| < \infty \right\},$$

and the product space $\mathcal{X}_\times^\infty = \mathcal{X}_{2\rho}^\infty \times \mathcal{X}_\rho^\infty \times \mathcal{X}_\rho^\infty$, with $\nu = \sqrt{\frac{21}{8}}$ endowed with the weighted product norm

$$\|\varphi\|_\times^\infty = \delta \|\varphi_1\|_{2\rho}^\infty + \|\varphi_2\|_\rho^\infty + \|\varphi_3\|_\rho^\infty .$$

Next lemmas, proven in [BFG+12], give some properties of these Banach spaces and provide a left inverse operator of $\mathcal{L}$.

Lemma II.4.1. *Let $\alpha, \beta \in \mathbb{R}$. Then, the following statements hold:*

1. *If $\alpha > \beta \geq 0$, then $\mathcal{X}_\alpha^\infty \subset \mathcal{X}_\beta^\infty$. Moreover $\|\varphi\|_\beta^\infty \leq \|\varphi\|_\alpha^\infty$.*
2. *If $\varphi \in \mathcal{X}_\alpha^\infty$ and $\zeta \in \mathcal{X}_\beta^\infty$, then $\varphi\zeta \in \mathcal{X}_{\alpha+\beta}^\infty$ and $\|\varphi\zeta\|_{\alpha+\beta}^\infty \leq \|\varphi\|_\alpha^\infty \|\zeta\|_\beta^\infty$.*

Lemma II.4.2. *The linear operator $\mathcal{G} : \mathcal{X}_\times^\infty \to \mathcal{X}_\times^\infty$ given by*

$$\mathcal{G}[\varphi](u) = \left(\int_{-\infty}^{u} \varphi_1(s)ds, \int_{-\infty}^{u} e^{-\frac{i}{\delta^2}(s-u)}\varphi_2(s)ds, \int_{-\infty}^{u} e^{\frac{i}{\delta^2}(s-u)}\varphi_3(s)ds \right)^T$$

is continuous, injective and is a left inverse of the operator $\mathcal{L}$.

Moreover, there exists a constant C independent of δ and ρ_1 such that, for $\varphi \in \mathcal{X}_\times^\infty$,

$$\|\mathcal{G}[\varphi]\|_\times^\infty \leq C \left(\|\varphi_1\|_{2\rho}^\infty + \delta^2 \|\varphi_2\|_\rho^\infty + \delta^2 \|\varphi_3\|_\rho^\infty \right).$$

Notice that the eigenvalues of the saddle point $(0,0)$ of $H_\mathrm{p}(\lambda,\Lambda)$ (see (II.2.6)) are $\pm\sqrt{\frac{21}{8}}$. Then, the parametrization of the separatrix $\sigma = (\lambda_h, \Lambda_h)$ (see (II.3.1)) satisfies

$$\lambda_h \in \mathcal{X}_\rho^\infty \quad \text{and} \quad \Lambda_h \in \mathcal{X}_\rho^\infty. \tag{II.4.2}$$

Therefore, z^u is a solution of (II.4.1) satisfying the asymptotic conditions (II.3.6) if and only if $z^\mathrm{u} \in \mathcal{X}_\times^\infty$ and satisfies the fixed point equation

$$\varphi = \mathcal{F}[\varphi] = \mathcal{G} \circ \mathcal{R}^{\mathrm{sep}}[\varphi].$$

Thus, Proposition II.3.5 is a straightforward consequence of the following proposition.

Proposition II.4.3. *There exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, equation $\varphi = \mathcal{F}[\varphi]$ has a solution $z^\mathrm{u} \in \mathcal{X}_\times^\infty$. Moreover, there exists a real constant $b_{14} > 0$ independent of δ such that $\|z^\mathrm{u}\|_\times^\infty \leq b_{14}\delta^3$.*

To see that $\mathcal{F}$ is a contractive operator, we have to pay attention to the nonlinear terms $\mathcal{R}^{\mathrm{sep}}$.

Lemma II.4.4. *Fix $\varrho > 0$ and let $\mathcal{R}^{\mathrm{sep}}$ be the operator defined in (II.3.13). Then, for $\delta > 0$ small enough*[1] *and $\|\varphi\|_\times^\infty \leq \varrho\delta^3$, there exists a constant $C > 0$ such that*

$$\|\mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_{2\rho}^\infty \leq C\delta^2, \qquad \|\mathcal{R}_j^{\mathrm{sep}}[\varphi]\|_\rho^\infty \leq C\delta, \qquad j = 2,3,$$

and

$$\begin{aligned}
&\|\partial_w \mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_0^\infty \leq C\delta^2, \quad &&\|\partial_x \mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_\rho^\infty \leq C\delta, \quad &&\|\partial_y \mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_\rho^\infty \leq C\delta, \\
&\|\partial_w \mathcal{R}_j^{\mathrm{sep}}[\varphi]\|_{-\rho}^\infty \leq C\delta, \quad &&\|\partial_x \mathcal{R}_j^{\mathrm{sep}}[\varphi]\|_0^\infty \leq C, \quad &&\|\partial_y \mathcal{R}_j^{\mathrm{sep}}[\varphi]\|_0^\infty \leq C, \quad j = 2,3.
\end{aligned}$$

The proof of this lemma is postponed to Appendix II.A.1.

Proof of Proposition II.4.3. Consider the closed ball

$$B(\varrho) = \left\{ \varphi \in \mathcal{X}_\times^\infty \ : \ \|\varphi\|_\times^\infty \leq \varrho \right\}.$$

First, we obtain an estimate for $\mathcal{F}[0]$. By Lemmas II.4.2 and II.4.4, if δ is small enough,

$$\|\mathcal{F}[0]\|_\times^\infty \leq C\delta \|\mathcal{R}_1^{\mathrm{sep}}[0]\|_{2\nu}^\infty + C\delta^2 \|\mathcal{R}_2^{\mathrm{sep}}[0]\|_\nu^\infty + C\delta^2 \|\mathcal{R}_3^{\mathrm{sep}}[0]\|_\nu^\infty \leq \frac{1}{2} b_{14}\delta^3, \tag{II.4.3}$$

[1] To simplify the exposition, in this lemma and in the technical lemmas from now on, we avoid referring to the existence of δ_0 and just mention that δ must be small enough. We follow the same convention for κ whenever is needed.

for some $b_{14} > 0$.

Then, it only remains to check that the operator $\mathcal{F}$ is contractive in $B(b_{14}\delta^3)$. Let $\varphi, \tilde{\varphi} \in B(b_{14}\delta^3)$. Then, by the mean value theorem,

$$\mathcal{R}_j^{\text{sep}}[\varphi] - \mathcal{R}_j^{\text{sep}}[\widetilde{\varphi}] = \left[\int_0^1 D\mathcal{R}_j^{\text{sep}}[s\varphi + (1-s)\widetilde{\varphi}]ds\right](\varphi - \widetilde{\varphi}), \qquad j = 1, 2, 3.$$

Applying Lemmas II.4.1 and II.4.4 and the above equality, we obtain

$$\begin{aligned}
\|\mathcal{R}_1^{\text{sep}}[\varphi] - \mathcal{R}_1^{\text{sep}}[\widetilde{\varphi}]\|_{2\rho}^{\infty} \leq & \sup_{\zeta \in B(b_{14}\delta^3)} \Big[\|\varphi_1 - \widetilde{\varphi}_1\|_{2\rho}^{\infty} \|\partial_w \mathcal{R}_1^{\text{sep}}[\zeta]\|_0^{\infty} \\
& + \|\varphi_2 - \widetilde{\varphi}_2\|_{\rho}^{\infty} \|\partial_x \mathcal{R}_1^{\text{sep}}[\zeta]\|_{\rho}^{\infty} + \|\varphi_3 - \widetilde{\varphi}_3\|_{\rho}^{\infty} \|\partial_y \mathcal{R}_1^{\text{sep}}[\zeta]\|_{\rho}^{\infty} \Big] \leq C\delta \|\varphi - \widetilde{\varphi}\|_{\times}^{\infty}, \\
\|\mathcal{R}_j^{\text{sep}}[\varphi] - \mathcal{R}_j^{\text{sep}}[\widetilde{\varphi}]\|_{\rho}^{\infty} \leq & \sup_{\zeta \in B(b_{14}\delta^3)} \Big[\|\varphi_1 - \widetilde{\varphi}_1\|_{2\rho}^{\infty} \|\partial_w \mathcal{R}_j^{\text{sep}}[\zeta]\|_{-\rho}^{\infty} \\
& + \|\varphi_2 - \widetilde{\varphi}_2\|_{\rho}^{\infty} \|\partial_x \mathcal{R}_j^{\text{sep}}[\zeta]\|_0^{\infty} + \|\varphi_3 - \widetilde{\varphi}_3\|_{\rho}^{\infty} \|\partial_y \mathcal{R}_j^{\text{sep}}[\zeta]\|_0^{\infty} \Big] \leq C \|\varphi - \widetilde{\varphi}\|_{\times}^{\infty}.
\end{aligned}$$

for $j = 2, 3$. Then, by Lemma II.4.2 and taking δ small enough,

$$\begin{aligned}
\|\mathcal{F}[\varphi] - \mathcal{F}[\widetilde{\varphi}]\|_{\times}^{\infty} \leq & C\delta \|\mathcal{R}_1^{\text{sep}}[\varphi] - \mathcal{R}_1^{\text{sep}}[\widetilde{\varphi}]\|_{2\rho}^{\infty} + C\delta^2 \sum_{j=2}^{3} \|\mathcal{R}_j^{\text{sep}}[\varphi] - \mathcal{R}_j^{\text{sep}}[\widetilde{\varphi}]\|_{\rho}^{\infty} \\
\leq & C\delta^2 \|\varphi - \widetilde{\varphi}\|_{\times}^{\infty} \leq \frac{1}{2} \|\varphi - \widetilde{\varphi}\|_{\times}^{\infty}.
\end{aligned} \tag{II.4.4}$$

Then, by the definition of ϱ in (II.4.3) and (II.4.4), $\mathcal{F} : B(b_{14}\delta^3) \to B(b_{14}\delta^3)$ is well defined and contractive. Therefore, $\mathcal{F}$ has a fixed point $z^{\text{u}} \in B(b_{14}\delta^3)$. □

II.4.2 The invariant manifolds in the outer domain

To prove Proposition II.3.6, we must extend analytically the parameterizations z^{u} and z^{s} given in Proposition II.3.5 to the outer domains, $D_{\kappa,\beta}^{\text{sep,u}}$ and $D_{\kappa,\beta}^{\text{sep,s}}$, respectively. Again, we only deal with the unstable -u- case, being the -s- case analogous. We prove the existence of z^{u} by means of a fixed point argument in a suitable Banach space.

Given $\alpha, \beta \in \mathbb{R}$, we consider the norm

$$\|\varphi\|_{\alpha,\beta}^{\text{out}} = \sup_{u \in D_{\kappa,\beta}^{\text{sep,u}}} \left| g_\delta^{-\alpha}(u) \left(u^2 + A^2\right)^{\beta} \varphi(u) \right|, \quad g_\delta(u) = \frac{1}{|u^2 + A^2|} + \frac{\delta^2}{|u^2 + A^2|^{\frac{8}{3}}},$$

and the associated Banach space

$$\mathcal{X}_{\alpha,\beta}^{\text{out}} = \left\{ \varphi : D_{\kappa,\beta}^{\text{sep,u}} \to \mathbb{C} \; : \; \varphi \text{ real-analytic}, \; \|\varphi\|_{\alpha,\beta}^{\text{out}} < \infty \right\}. \tag{II.4.5}$$

These Banach spaces have the following properties, which we use without mentioning along the chapter. Their proof follows the same lines as the proof of Lemma 7.1 in [BFG+12].

Lemma II.4.5. *The following statements hold:*

1. *If* $\varphi \in \mathcal{X}^{\text{out}}_{\alpha,\beta_1}$, *then* $\varphi \in \mathcal{X}^{\text{out}}_{\alpha,\beta_2}$ *for any* $\beta_2 \in \mathbb{R}$ *and*

$$\begin{cases} \|\varphi\|^{\text{out}}_{\alpha,\beta_2} \leq C \|\varphi\|^{\text{out}}_{\alpha,\beta_1}, & \text{for } \beta_2 - \beta_1 > 0, \\ \|\varphi\|^{\text{out}}_{\alpha,\beta_2} \leq C(\kappa\delta^2)^{\beta_2-\beta_1} \|\varphi\|^{\text{out}}_{\alpha,\beta_1}, & \text{for } \beta_2 - \beta_1 \leq 0. \end{cases}$$

2. *If* $\varphi \in \mathcal{X}^{\text{out}}_{\alpha,\beta_1}$, *then* $\varphi \in \mathcal{X}^{\text{out}}_{\alpha-1,\beta_2}$ *for any* $\beta_2 \in \mathbb{R}$ *and*

$$\begin{cases} \|\varphi\|^{\text{out}}_{\alpha-1,\beta_2} \leq C \|\varphi\|^{\text{out}}_{\alpha,\beta_1}, & \text{for } \beta_2 - \beta_1 > \frac{5}{3}, \\ \|\varphi\|^{\text{out}}_{\alpha-1,\beta_2} \leq C\delta^2(\kappa\delta^2)^{(\beta_2-\beta_1)-\frac{8}{3}} \|\varphi\|^{\text{out}}_{\alpha,\beta_1}, & \text{for } \beta_2 - \beta_1 \leq \frac{5}{3}. \end{cases}$$

3. *If* $\varphi \in \mathcal{X}^{\text{out}}_{\alpha_1,\beta_1}$ *and* $\zeta \in \mathcal{X}^{\text{out}}_{\alpha_2,\beta_2}$, *then* $\varphi\zeta \in \mathcal{X}^{\text{out}}_{\alpha_1+\alpha_2,\beta_1+\beta_2}$ *and*

$$\|\varphi\zeta\|^{\text{out}}_{\alpha_1+\alpha_2,\beta_1+\beta_2} \leq \|\varphi\|^{\text{out}}_{\alpha_1,\beta_1} \|\zeta\|^{\text{out}}_{\alpha_2,\beta_2}.$$

4. *If* $\varphi \in \mathcal{X}^{\text{out}}_{0,\beta+1}$ *and* $\zeta \in \mathcal{X}^{\text{out}}_{0,\beta+\frac{8}{3}}$, *then* $\varphi + \delta^2\zeta \in \mathcal{X}^{\text{out}}_{1,\beta}$ *and*

$$\|\varphi + \delta^2\zeta\|^{\text{out}}_{1,\beta} \leq \|\varphi\|^{\text{out}}_{0,\beta+1} + \|\zeta\|^{\text{out}}_{0,\beta+\frac{8}{3}}.$$

Let us recall that, by Proposition II.3.5, the invariance equation (II.3.12) has a unique solution z^{u} in the domain $D^{\text{u},\infty}_{\rho_1}$ satisfying the asymptotic condition (II.3.6). Our objective is to extend analytically z^{u} to the outer domain $D^{\text{sep,u}}_{\kappa,\beta}$. Notice that, since $\rho_1 < \rho_2$, $D^{\text{u},\infty}_{\rho_1} \cap D^{\text{sep,u}}_{\kappa,\beta} \neq \emptyset$ (see definitions (II.3.14) and (II.3.15) of $D^{\text{u},\infty}_{\rho_1}$ and $D^{\text{sep,u}}_{\kappa,\beta}$).

As explained in Section II.4.1, equation (II.3.12) is equivalent to $\mathcal{L}z^{\text{u}} = \mathcal{R}^{\text{sep}}[z^{\text{u}}]$ with $\mathcal{L}\varphi = (\partial_u - \mathcal{A}^{\text{sep}})\varphi$ and $\mathcal{R}^{\text{sep}}$ given in (II.3.13). In the following lemma we introduce a right-inverse operator of $\mathcal{L}$ defined on $\mathcal{X}^{\text{out}}_{\alpha,\beta}$.

Lemma II.4.6. *Let us consider the operator* $\mathcal{G}[\varphi] = (\mathcal{G}_1[\varphi_1], \mathcal{G}_2[\varphi_2], \mathcal{G}_3[\varphi_3])^T$, *such that*

$$\mathcal{G}[\varphi](u) = \left(\int_{-\rho_2}^{u} \varphi_1(s)ds, \int_{\overline{u}_1}^{u} e^{-\frac{i}{\delta^2}(s-u)}\varphi_2(s)ds, \int_{u_1}^{u} e^{\frac{i}{\delta^2}(s-u)}\varphi_3(s)ds \right)^T,$$

where u_1 *and* $\overline{u}_1$ *are the vertices of the domain* $D^{\text{sep,u}}_{\kappa,\beta}$ *(see Figure II.3.3). Fix* $\beta > 0$. *There exists a constant* C *such that:*

1. *If* $\varphi \in \mathcal{X}^{\text{out}}_{1,\beta}$, *then* $\mathcal{G}_1[\varphi] \in \mathcal{X}^{\text{out}}_{1,\beta-1}$ *and* $\|\mathcal{G}_1[\varphi]\|^{\text{out}}_{1,\beta-1} \leq C \|\varphi\|^{\text{out}}_{1,\beta}$.
2. *If* $\varphi \in \mathcal{X}^{\text{out}}_{0,\beta}$, *then* $\mathcal{G}_j[\varphi] \in \mathcal{X}^{\text{out}}_{0,\beta}$, $j = 2, 3$, *and* $\|\mathcal{G}_j[\varphi]\|^{\text{out}}_{0,\beta} \leq C\delta^2 \|\varphi\|^{\text{out}}_{0,\beta}$.

The proof of this lemma follows the same lines as the proof of Lemma 7.3 in [BFG+12].

Consider u_1 and $\overline{u_1}$ as in Figure II.3.3 and the function

$$F^0(u) = \left(w^{\text{u}}(-\rho_2),\, x^{\text{u}}(\overline{u}_1)e^{-\frac{i}{\delta^2}(\overline{u}_1-u)},\, y^{\text{u}}(u_1)e^{\frac{i}{\delta^2}(u_1-u)} \right)^T.$$

Notice that, since $0 < \rho_1 < \rho_2$, we have $\{-\rho_2, u_1, \overline{u}_1\} \in D^{\text{u},\infty}_{\rho_1}$. Therefore, by Proposition II.3.5, z^{u} is already defined at these points. We define the fixed point operator

$$\mathcal{F}[\varphi] = F^0 + \mathcal{G} \circ \mathcal{R}^{\text{sep}}[\varphi], \tag{II.4.6}$$

where the operator $\mathcal{R}^{\mathrm{sep}}$ is given in (II.3.13). Since $\mathcal{L}(F^0)=0$, by Lemma II.4.6, a solution $z^{\mathrm{u}}=\mathcal{F}[z^{\mathrm{u}}]$ satisfies $\mathcal{L}z^{\mathrm{u}}=\mathcal{R}^{\mathrm{sep}}[z^{\mathrm{u}}]$ and by construction is the real-analytic continuation of the function z^{u} obtained in Proposition II.3.5.

We rewrite Proposition II.3.6 in terms of the operator $\mathcal{F}$ defined in the Banach space

$$\mathcal{X}_{\times}^{\mathrm{out}}=\mathcal{X}_{1,0}^{\mathrm{out}}\times\mathcal{X}_{0,\frac{4}{3}}^{\mathrm{out}}\times\mathcal{X}_{0,\frac{4}{3}}^{\mathrm{out}},$$

endowed with the norm

$$\|\varphi\|_{\times}^{\mathrm{out}}=\delta\,\|\varphi_1\|_{1,0}^{\mathrm{out}}+\|\varphi_2\|_{0,\frac{4}{3}}^{\mathrm{out}}+\|\varphi_3\|_{0,\frac{4}{3}}^{\mathrm{out}}.$$

Proposition II.4.7. *There exist $\delta_0,\kappa_1>0$ such that, for $\delta\in(0,\delta_0)$ and $\kappa\geq\kappa_1$, the fixed point equation $z^{\mathrm{u}}=\mathcal{F}[z^{\mathrm{u}}]$ has a unique solution $z^{\mathrm{u}}\in\mathcal{X}_{\times}^{\mathrm{out}}$. Moreover, there exists a real constant $b_{15}>0$ independent of δ and κ such that $\|z^{\mathrm{u}}\|_{\times}^{\mathrm{out}}\leq b_{15}\delta^3$.*

We prove this proposition through a fixed point argument. First, we state a technical lemma, whose proof is postponed until Appendix II.A.2. Fix $\varrho>0$ and define

$$B(\varrho)=\left\{\varphi\in\mathcal{X}_{\times}^{\mathrm{out}}\ :\ \|\varphi\|_{\times}^{\mathrm{out}}\leq\varrho\right\}.$$

Lemma II.4.8. *Fix $\varrho>0$ and let $\mathcal{R}^{\mathrm{sep}}$ be the operator defined in (II.3.13). For $\delta>0$ small enough and $\kappa>0$ big enough, there exists a constant $C>0$ such that, for $\varphi\in B(\varrho\delta^3)$,*

$$\|\mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_{1,1}^{\mathrm{out}}\leq C\delta^2,\qquad\|\mathcal{R}_j^{\mathrm{sep}}[\varphi]\|_{0,\frac{4}{3}}^{\mathrm{out}}\leq C\delta,\qquad j=2,3,$$

and,

$$\begin{aligned}
&\|\partial_w\mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_{1,\frac{1}{3}}^{\mathrm{out}}\leq C\delta^2, && \|\partial_x\mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_{0,\frac{7}{3}}^{\mathrm{out}}\leq C\delta, && \|\partial_y\mathcal{R}_1^{\mathrm{sep}}[\varphi]\|_{0,\frac{7}{3}}^{\mathrm{out}}\leq C\delta,\\
&\|\partial_w\mathcal{R}_2^{\mathrm{sep}}[\varphi]\|_{0,\frac{2}{3}}^{\mathrm{out}}\leq C\delta, && \|\partial_x\mathcal{R}_2^{\mathrm{sep}}[\varphi]\|_{1,-\frac{2}{3}}^{\mathrm{out}}\leq C, && \|\partial_y\mathcal{R}_2^{\mathrm{sep}}[\varphi]\|_{0,2}^{\mathrm{out}}\leq C\delta^2,\\
&\|\partial_w\mathcal{R}_3^{\mathrm{sep}}[\varphi]\|_{0,\frac{2}{3}}^{\mathrm{out}}\leq C\delta, && \|\partial_x\mathcal{R}_3^{\mathrm{sep}}[\varphi]\|_{0,2}^{\mathrm{out}}\leq C\delta^2, && \|\partial_y\mathcal{R}_3^{\mathrm{sep}}[\varphi]\|_{1,-\frac{2}{3}}^{\mathrm{out}}\leq C.
\end{aligned}$$

The next lemma gives properties of the operator $\mathcal{F}$.

Lemma II.4.9. *Fix $\varrho>0$ and let $\mathcal{F}$ be the operator defined in (II.4.6). Then, for $\delta>0$ small enough and $\kappa>0$ big enough, there exist constants $b_{16},b_{17}>0$ independent of δ and κ such that*

$$\|\mathcal{F}[0]\|_{\times}^{\mathrm{out}}\leq b_{16}\delta^3.$$

Moreover, for $\varphi,\tilde{\varphi}\in B(\varrho\delta^3)$,

$$\begin{aligned}
\delta\,\|\mathcal{F}_1[\varphi]-\mathcal{F}_1[\tilde\varphi]\|_{1,0}^{\mathrm{out}}&\leq b_{17}\left(\frac{\delta}{\kappa^2}\,\|\varphi_1-\tilde\varphi_1\|_{1,0}^{\mathrm{out}}+\|\varphi_2-\tilde\varphi_2\|_{0,\frac{4}{3}}^{\mathrm{out}}+\|\varphi_3-\tilde\varphi_3\|_{0,\frac{4}{3}}^{\mathrm{out}}\right),\\
\|\mathcal{F}_j[\varphi]-\mathcal{F}_j[\tilde\varphi]\|_{0,\frac{4}{3}}^{\mathrm{out}}&\leq\frac{b_{17}}{\kappa^2}\,\|\varphi-\tilde\varphi\|_{\times}^{\mathrm{out}},\qquad \textit{for}\quad j=2,3.
\end{aligned}$$

Proof. First, we obtain the estimates for $\mathcal{F}[0]$. By Proposition II.3.5, we have that

$$|w^{\mathrm{u}}(-\rho_2)|\leq C\delta^2,\quad|x^{\mathrm{u}}(\overline{u_1})|\leq C\delta^3,\quad|y^{\mathrm{u}}(u_1)|\leq C\delta^3,$$

and, as a result, $\|F^0\|_\times^{\text{out}} \leq C\delta^3$. Then, applying Lemmas II.4.6 and II.4.8, we obtain

$$\|\mathcal{F}[0]\|_\times^{\text{out}} \leq \left\|F^0\right\|_\times^{\text{out}} + C\delta \left\|\mathcal{R}_1^{\text{sep}}[0]\right\|_{1,1}^{\text{out}} + C\delta^2 \textstyle\sum_{j=2}^3 \|\mathcal{R}_j^{\text{sep}}[0]\|_{0,\frac{4}{3}}^{\text{out}} \leq C\delta^3.$$

For the second statement, since $\mathcal{F} = F^0 + \mathcal{G} \circ \mathcal{R}^{\text{sep}}$ and $\mathcal{G}$ is linear, we need to compute estimates for $\mathcal{R}^{\text{sep}}[\varphi] - \mathcal{R}^{\text{sep}}[\tilde{\varphi}]$. Then, by the mean value theorem,

$$\mathcal{R}_j^{\text{sep}}[\varphi] - \mathcal{R}_j^{\text{sep}}[\tilde{\varphi}] = \left[\int_0^1 D\mathcal{R}_j^{\text{sep}}[s\varphi + (1-s)\tilde{\varphi}]ds\right](\varphi - \tilde{\varphi}), \qquad j = 1, 2, 3.$$

In addition, by Lemmas II.4.5 and II.4.8, for $j = 2, 3$, we have the estimates

$$\begin{aligned}
&\|\partial_w \mathcal{R}_1^{\text{sep}}[\varphi]\|_{0,1}^{\text{out}} \leq \frac{C}{\kappa^2}, \quad &&\|\partial_x \mathcal{R}_1^{\text{sep}}[\varphi]\|_{1,-\frac{1}{3}}^{\text{out}} \leq \frac{C}{\delta}, \quad &&\|\partial_y \mathcal{R}_1^{\text{sep}}[\varphi]\|_{1,-\frac{1}{3}}^{\text{out}} \leq \frac{C}{\delta}, \\
&\|\partial_w \mathcal{R}_j^{\text{sep}}[\varphi]\|_{-1,\frac{4}{3}}^{\text{out}} \leq \frac{C}{\kappa^2\delta}, \quad &&\|\partial_x \mathcal{R}_j^{\text{sep}}[\varphi]\|_{0,0}^{\text{out}} \leq \frac{C}{\kappa^2\delta^2}, \quad &&\|\partial_y \mathcal{R}_j^{\text{sep}}[\varphi]\|_{0,0}^{\text{out}} \leq \frac{C}{\kappa^2\delta^2}.
\end{aligned}$$

We estimate each component separately. For $j = 1$, we have that

$$\begin{aligned}
\delta\|\mathcal{R}_1^{\text{sep}}[\varphi] - \mathcal{R}_1^{\text{sep}}[\tilde{\varphi}]\|_{1,1}^{\text{out}} \leq \sup_{\zeta \in B(\varrho\delta^3)} \delta\Big[&\|\varphi_1 - \tilde{\varphi}_1\|_{1,0}^{\text{out}} \|\partial_w \mathcal{R}_1^{\text{sep}}[\zeta]\|_{0,1}^{\text{out}} \\
&+ \|\varphi_2 - \tilde{\varphi}_2\|_{0,\frac{4}{3}}^{\text{out}} \|\partial_x \mathcal{R}_1^{\text{sep}}[\zeta]\|_{1,-\frac{1}{3}}^{\text{out}} + \|\varphi_3 - \tilde{\varphi}_3\|_{0,\frac{4}{3}}^{\text{out}} \|\partial_y \mathcal{R}_1^{\text{sep}}[\zeta]\|_{1,-\frac{1}{3}}^{\text{out}} \Big] \\
\leq \frac{C\delta}{\kappa^2} &\|\varphi_1 - \tilde{\varphi}_1\|_{1,0}^{\text{out}} + C\|\varphi_2 - \tilde{\varphi}_2\|_{0,\frac{4}{3}}^{\text{out}} + C\|\varphi_3 - \tilde{\varphi}_3\|_{0,\frac{4}{3}}^{\text{out}}.
\end{aligned}$$

Analogously, for $j = 2, 3$, we obtain

$$\begin{aligned}
\|\mathcal{R}_j^{\text{sep}}[\varphi] - \mathcal{R}_j^{\text{sep}}[\tilde{\varphi}]\|_{0,\frac{4}{3}}^{\text{out}} \leq \sup_{\zeta \in B(\varrho\delta^3)} \Big[&\|\varphi_1 - \tilde{\varphi}_1\|_{1,0}^{\text{out}} \|\partial_w \mathcal{R}_j^{\text{sep}}[\zeta]\|_{-1,\frac{4}{3}}^{\text{out}} \\
&+ \|\varphi_2 - \tilde{\varphi}_2\|_{0,\frac{4}{3}}^{\text{out}} \|\partial_x \mathcal{R}_j^{\text{sep}}[\zeta]\|_{0,0}^{\text{out}} + \|\varphi_3 - \tilde{\varphi}_3\|_{0,\frac{4}{3}}^{\text{out}} \|\partial_y \mathcal{R}_j^{\text{sep}}[\zeta]\|_{0,0}^{\text{out}} \Big] \\
\leq \frac{C}{\kappa^2\delta^2} &\|\varphi - \tilde{\varphi}\|_\times^{\text{out}},
\end{aligned}$$

and, using Lemma II.4.6, we obtain the estimates for the second statement. ☐

Lemma II.4.9 shows that, by assuming κ big enough, operators $\mathcal{F}_2$ and $\mathcal{F}_3$ have Lipschitz constant less than 1 with the norm in $\mathcal{X}_\times^{\text{out}}$. However, we are not able to control the Lipschitz constant of $\mathcal{F}_1$. To overcome this problem, we apply a Gauss-Seidel argument to define a new operator

$$\widetilde{\mathcal{F}}[z] = \widetilde{\mathcal{F}}[(w, x, y)] = \begin{pmatrix} \mathcal{F}_1[w, \mathcal{F}_2[z], \mathcal{F}_3[z]] \\ \mathcal{F}_2[z] \\ \mathcal{F}_3[z] \end{pmatrix},$$

which turns out to be contractive in a suitable ball and has the same fixed points as the operator $\mathcal{F}$.

End of the proof of Proposition II.4.7. We look for a fixed point of $\widetilde{\mathcal{F}}$. First, we obtain an estimate for $\|\widetilde{\mathcal{F}}[0]\|_\times^{\text{out}}$. We rewrite it as

$$\widetilde{\mathcal{F}}[0] = \mathcal{F}[0] + \Big(\mathcal{F}_1[0, \mathcal{F}_2[0], \mathcal{F}_3[0]] - \mathcal{F}_1[0], 0, 0\Big)^T,$$

and we notice that, by Lemma II.4.9, $\|(0,\mathcal{F}_2[0],\mathcal{F}_3[0])\|_\times^{\mathrm{out}} \leq \|\mathcal{F}[0]\|_\times^{\mathrm{out}} \leq C\delta^3$. Then, applying Lemma II.4.9, there exists constant $b_{15}>0$ such that

$$\begin{aligned}\|\widetilde{\mathcal{F}}[0]\|_\times^{\mathrm{out}} &\leq \|\mathcal{F}[0]\|_\times^{\mathrm{out}} + \|\mathcal{F}_1[0,\mathcal{F}_2[0],\mathcal{F}_3[0]] - \mathcal{F}_1[0]\|_{1,0}^{\mathrm{out}} \\ &\leq \|\mathcal{F}[0]\|_\times^{\mathrm{out}} + C\,\|\mathcal{F}_2[0]\|_{0,\frac{4}{3}}^{\mathrm{out}} + C\,\|\mathcal{F}_3[0]\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq \frac{1}{2}b_{15}\delta^3.\end{aligned} \tag{II.4.7}$$

Now, we prove that the operator $\widetilde{\mathcal{F}}$ is contractive in $B(b_{15}\delta^3)$. Indeed, by Lemma II.4.9, we have that, for $\varphi,\tilde{\varphi}\in B(b_{15}\delta^3)$,

$$\begin{aligned}\delta\|\widetilde{\mathcal{F}}_1[\varphi]-\widetilde{\mathcal{F}}_1[\tilde\varphi]\|_{1,0}^{\mathrm{out}} &\leq C\left(\frac{\delta}{\kappa^2}\,\|\varphi_1-\tilde\varphi_1\|_{1,0}^{\mathrm{out}} + \|\mathcal{F}_2[\varphi]-\mathcal{F}_2[\tilde\varphi]\|_{0,\frac{4}{3}}^{\mathrm{out}} + \|\mathcal{F}_3[\varphi]-\mathcal{F}_3[\tilde\varphi]\|_{0,\frac{4}{3}}^{\mathrm{out}}\right) \\ &\leq \frac{C\delta}{\kappa^2}\,\|\varphi_1-\tilde\varphi_1\|_{1,0}^{\mathrm{out}} + \frac{2C}{\kappa^2}\,\|\varphi-\tilde\varphi\|_\times^{\mathrm{out}} \leq \frac{C}{\kappa^2}\,\|\varphi-\tilde\varphi\|_\times^{\mathrm{out}}, \\ \|\widetilde{\mathcal{F}}_j[\varphi]-\widetilde{\mathcal{F}}_j[\tilde\varphi]\|_{0,\frac{4}{3}}^{\mathrm{out}} &= \|\mathcal{F}_j[\varphi]-\mathcal{F}_j[\tilde\varphi]\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq \frac{C}{\kappa^2}\,\|\varphi-\tilde\varphi\|_\times^{\mathrm{out}}, \qquad \text{for } j=2,3.\end{aligned}$$

Then, for $\kappa>0$ big enough, we have that $\|\widetilde{\mathcal{F}}[\varphi]-\widetilde{\mathcal{F}}[\tilde\varphi]\|_\times^{\mathrm{out}} \leq \frac{1}{2}\,\|\varphi-\tilde\varphi\|_\times^{\mathrm{out}}$. Together with (II.4.7), this implies that $\widetilde{\mathcal{F}}: B(b_{15}\delta^3)\to B(b_{15}\delta^3)$ is well defined and contractive. Therefore, $\widetilde{\mathcal{F}}$ has a fixed point $z^{\mathrm{u}}\in B(b_{15}\delta^3)$. □

II.4.3 Switching to the time-parametrization

In this section, by means of a fixed point argument, we prove Proposition II.3.7. That is, we obtain a change of variables $\mathcal{U}$ satisfying (II.3.21), that is

$$\partial_v\mathcal{U} = R[\mathcal{U}] \qquad \text{where} \qquad R[\mathcal{U}] = \partial_w H_1^{\mathrm{sep}}\left(v+\mathcal{U}(v), z^{\mathrm{u}}(v+\mathcal{U}(v))\right). \tag{II.4.8}$$

To this end, we consider the Banach space

$$\mathcal{Y}^{\mathrm{out}} = \left\{\varphi: \widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}\to\mathbb{C} \;:\; \varphi \text{ real-analytic}, \|\varphi\|_{\sup} := \sup_{v\in\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}} |\mathcal{U}(v)| < \infty\right\}. \tag{II.4.9}$$

First, we state a technical lemma. Its proof is a direct consequence of the proof of Lemma II.4.8 (see also Remark II.A.7 in Appendix II.A.2).

Lemma II.4.10. *Fix $\varrho>0$. For $\delta>0$ small enough and $\varphi\in\mathcal{Y}^{\mathrm{out}}$ such that $\|\varphi\|_{\sup}\leq\varrho\delta^2$, there exists a constant $C>0$ such that $\|R[\varphi]\|_{\sup}\leq C\delta^2$ and $\|DR[\varphi]\|_{\sup}\leq C\delta^2$.*

Let us define the operators

$$G[\varphi](v) = \int_{\rho_3}^{v}\varphi(s)ds, \qquad \text{and} \qquad F = G\circ R, \tag{II.4.10}$$

where $\rho_3\in\mathbb{R}$ is the rightmost vertex of the domain $\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$ (see Figure II.3.5). Then, a solution $\mathcal{U}=F[\mathcal{U}]$ satisfies equation (II.4.8) and the initial condition $\mathcal{U}(\rho_3)=0$.

Proof of Proposition II.3.7. The operator G in (II.4.10) satisfies that, for $\varphi\in\mathcal{Y}^{\mathrm{out}}$,

$$\|G[\varphi]\|_{\sup}\leq C\|\varphi\|_{\sup}. \tag{II.4.11}$$

Then, by Lemma II.4.10, there exists $b_4>0$ independent of δ such that

$$\|F[0]\|_{\sup} \leq C\|R[0]\|_{\sup} \leq \frac{1}{2} b_4\delta^2. \tag{II.4.12}$$

Moreover, for $\varphi,\tilde{\varphi} \in B(b_4\delta^2) = \left\{\varphi \in \mathcal{Y}^{\mathrm{out}} : \|\varphi\|_{\sup} \leq b_4\delta^2\right\}$, by the mean value theorem and Lemma II.4.10,

$$\|R[\varphi]-R[\tilde{\varphi}]\|_{\sup} = \left\|\int_0^1 DR[s\varphi+(1-s)\tilde{\varphi}]ds\right\|_{\sup} \|\varphi-\tilde{\varphi}\|_{\sup} \leq C\delta^2\|\varphi-\tilde{\varphi}\|_{\sup}.$$

Then, by Lemma II.4.10, (II.4.11), (II.4.12) and taking δ small enough, F is well defined and contractive in $B(b_4\delta^2)$ and, as a result, has a fixed point $\mathcal{U} \in B(b_4\delta^2)$.

It only remains to check that $v+\mathcal{U}(v) \in D^{\mathrm{u,out}}_{\kappa_1,d_1,\rho_2}$ for $v \in \widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$. Indeed, since $\|\mathcal{U}\|_{\sup} \leq b_4\delta^2$ and $\widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3} \subset D^{\mathrm{sep,u}}_{\kappa,\beta}$, taking δ small enough the statement is proved. □

II.4.4 Extending the time-parametrization

In this section, we extend analytically the parametrization Γ^{u} given in Corollary II.3.8 from the transition domain $D^{\mathrm{sep,u}}_{\kappa,\beta}$ to the flow domain $D^{\mathrm{fl}}_{\kappa_3,d_4}$ (see (II.3.19)).

Since Γ^{u} satisfies the equations given by H in (II.2.5), $\widehat{\Gamma}=\Gamma^{\mathrm{u}}-\Gamma_h$ (see (II.3.22)) satisfies

$$\begin{cases} \partial_v\hat{\lambda} = -3\hat{\Lambda} + \partial_\Lambda H_1(\Gamma_h+\widehat{\Gamma};\delta), \\ \partial_v\hat{\Lambda} = -V'(\lambda_h+\hat{\lambda}) + V'(\lambda_h) - \partial_\lambda H_1(\Gamma_h+\widehat{\Gamma};\delta), \\ \partial_v x = i\dfrac{x}{\delta^2} + i\partial_y H_1(\Gamma_h+\widehat{\Gamma};\delta), \\ \partial_v y = -i\dfrac{y}{\delta^2} - i\partial_x H_1(\Gamma_h+\widehat{\Gamma};\delta), \end{cases}$$

which can be rewritten as $\mathcal{L}^{\mathrm{fl}}\widehat{\Gamma} = \mathcal{R}^{\mathrm{fl}}[\widehat{\Gamma}]$, where

$$\mathcal{L}^{\mathrm{fl}}\varphi = \Big(\partial_v - \mathcal{A}^{\mathrm{fl}}(v)\Big)\varphi, \qquad \mathcal{A}^{\mathrm{fl}}(v) = \begin{pmatrix} 0 & -3 & 0 & 0 \\ -V''(\lambda_h(v)) & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix}, \tag{II.4.13}$$

and

$$\mathcal{R}^{\mathrm{fl}}[\varphi](v) = \begin{pmatrix} \partial_\Lambda H_1(\Gamma_h(v)+\varphi(v);\delta) \\ T[\varphi_1](v) - \partial_\lambda H_1(\Gamma_h(v)+\varphi(v);\delta) \\ i\partial_y H_1(\Gamma_h(v)+\varphi(v);\delta) \\ -i\partial_x H_1(\Gamma_h(v)+\varphi(v);\delta) \end{pmatrix}, \tag{II.4.14}$$

with $T[\varphi_1] = -V'(\lambda_h+\varphi_1)+V'(\lambda_h)+V''(\lambda_h)\varphi_1$.

We look for $\widehat{\Gamma}$ through fixed point argument in the Banach space $\mathcal{X}^{\mathrm{fl}}_\times = \left(\mathcal{X}^{\mathrm{fl}}\right)^4$, where

$$\mathcal{X}^{\mathrm{fl}} = \left\{\varphi : D^{\mathrm{fl}}_{\kappa_3,d_4} \to \mathbb{C} \;:\; \varphi \text{ real-analytic}, \; \|\varphi\|^{\mathrm{fl}} := \sup_{v\in D^{\mathrm{fl}}_{\kappa_3,d_4}} |\varphi(v)| < \infty\right\},$$

endowed with the norm

$$\|\varphi\|_{\times}^{\mathrm{fl}} = \delta\|\varphi_1\|^{\mathrm{fl}} + \delta\|\varphi_2\|^{\mathrm{fl}} + \|\varphi_3\|^{\mathrm{fl}} + \|\varphi_4\|^{\mathrm{fl}}.$$

A fundamental matrix of the linear equation $\dot{\xi} = \mathcal{A}^{\mathrm{fl}}(v)\xi$ is

$$\Phi(v) = \begin{pmatrix} 3\Lambda_h(v) & 3f_h(v) & 0 & 0 \\ -\dot{\Lambda}_h(v) & -\dot{f}_h(v) & 0 & 0 \\ 0 & 0 & e^{\frac{i}{\delta^2}v} & 0 \\ 0 & 0 & 0 & e^{-\frac{i}{\delta^2}v} \end{pmatrix} \text{ with } f_h(v) = \Lambda_h(v)\int_{v_0}^{v} \frac{1}{\Lambda_h^2(s)} ds.$$

Note that $f_h(v)$ is analytic at $v = 0$.

To look for a right inverse of operator $\mathcal{L}^{\mathrm{fl}}$ in (II.4.13), let us consider the linear operator

$$\mathcal{G}^{\mathrm{fl}}[\varphi](v) = \left(\int_{v_0}^{v} \varphi_1(s)ds, \int_{v_0}^{v} \varphi_2(s)ds, \int_{\overline{v}_1}^{v} \varphi_3(s)ds, \int_{v_1}^{v} \varphi_4(s)ds\right)^T,$$

where v_0, v_1 and $\overline{v}_1$ are the vertexs of the domain $D^{\mathrm{fl}}_{\kappa_3,d_4}$ (see Figure II.3.5). Then, the linear operator $\widehat{\mathcal{G}}[\varphi] = \Phi\mathcal{G}[\Phi^{-1}\varphi]$ is a right inverse of the operator $\mathcal{L}^{\mathrm{fl}}$, and, for $\varphi \in \mathcal{X}^{\mathrm{fl}}_{\times}$, satisfies

$$\|\widehat{\mathcal{G}}_1[\varphi]\|^{\mathrm{fl}} + \|\widehat{\mathcal{G}}_2[\varphi]\|^{\mathrm{fl}} \leq C\left(\|\varphi_1\|^{\mathrm{fl}} + \|\varphi_2\|^{\mathrm{fl}}\right), \quad \|\widehat{\mathcal{G}}_j[\varphi]\|^{\mathrm{fl}} \leq C\delta^2\|\varphi_j\|^{\mathrm{fl}} \text{ for } j = 3, 4. \tag{II.4.15}$$

Next, we state a technical lemma providing estimates for $\mathcal{R}^{\mathrm{fl}}$. Its proof is a direct consequence of the definition of the operator in (II.4.14) and Corollary II.A.2, which gives estimates for H_1^{Poi} in (II.2.1) (see also the change of coordinates (II.2.3) which relates H_1^{Poi} and H_1).

Lemma II.4.11. *Fix $\varrho > 0$ and consider $\varphi \in \mathcal{X}^{\mathrm{fl}}_{\times}$ with $\|\varphi\|^{\mathrm{fl}}_{\times} \leq \varrho\delta^3$. Then, for $\delta > 0$ small enough , there exists a constant $C > 0$ such that the operator $\mathcal{R}^{\mathrm{fl}}$ in (II.4.14) satisfies*

$$\begin{gathered} \|\mathcal{R}_1^{\mathrm{fl}}[\varphi]\|^{\mathrm{fl}}, \|\mathcal{R}_2^{\mathrm{fl}}[\varphi]\|^{\mathrm{fl}} \leq C\delta^2, \qquad \|\mathcal{R}_3^{\mathrm{fl}}[\varphi]\|^{\mathrm{fl}}, \|\mathcal{R}_4^{\mathrm{fl}}[\varphi]\|^{\mathrm{fl}} \leq C\delta, \\ \|D_j\mathcal{R}_l^{\mathrm{fl}}[\varphi]\|^{\mathrm{fl}} \leq C\delta, \qquad j, l \in \{1,2,3,4\}. \end{gathered}$$

Denote by $\mathbf{e}_j$, $j = 1,2,3,4$, the canonical basis in $\mathbb{R}^4$. Noticing that, by Corollary II.3.8, the function $\widehat{\Gamma} = (\hat{\lambda}, \hat{\Lambda}, x, y)$ is already defined at $\{v_0, v_1, \overline{v}_1\} \in \widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$, we can consider the function

$$F^0(v) = \Phi(v)\left[\Phi^{-1}(v_0)\left(\hat{\lambda}(v_0)\mathbf{e}_1 + \hat{\Lambda}(v_0)\mathbf{e}_2\right) + x(\overline{v}_1)\Phi^{-1}(\overline{v}_1)\mathbf{e}_3 + y(v_1)\Phi^{-1}(v_1)\mathbf{e}_4\right].$$

Then, since $\widehat{\mathcal{G}}(F^0) = 0$, it only remains to check that $\mathcal{F} = F^0 + \widehat{\mathcal{G}} \circ \mathcal{R}^{\mathrm{fl}}$ is contractive in a suitable ball of $\mathcal{X}^{\mathrm{fl}}_{\times}$.

End of proof of Proposition II.3.9. First, we obtain a suitable estimate for $\mathcal{F}[0]$. Applying Propositions II.3.6 and II.3.7 and using (II.3.18) we obtain that, for $v \in \widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$,

$$|\hat{\lambda}(v)| \leq C\delta^2, \qquad |\hat{\Lambda}(v)| \leq C\delta^2, \qquad |x(v)| \leq C\delta^3, \qquad |y(v)| \leq C\delta^3.$$

Therefore, since $\{v_0, v_1, \overline{v}_1\} \in \widetilde{D}^{\mathrm{u,out}}_{\kappa_2,d_2,d_3}$,

$$\|F^0\|^{\mathrm{fl}}_{\times} \leq C\delta|\hat{\lambda}(v_0)| + C\delta|\hat{\Lambda}(v_0)| + C\,|x(\overline{v}_1)| + C\,|y(v_1)| \leq C\delta^3,$$

and, applying (II.4.15) and Lemma II.4.11, there exists $b_5 > 0$ independent of δ such that

$$\|\mathcal{F}[0]\|^{\mathrm{fl}}_{\times} \leq \|F^0\|^{\mathrm{fl}}_{\times} + \|\mathcal{G} \circ \mathcal{R}^{\mathrm{fl}}[0]\|^{\mathrm{fl}}_{\times} \leq \frac{1}{2} b_5 \delta^3. \tag{II.4.16}$$

Let us define $B(b_5\delta^3) = \left\{\varphi \in \mathcal{X}^{\mathrm{fl}}_{\times} \,:\, \|\varphi\|^{\mathrm{fl}}_{\times} \leq b_5\delta^3\right\}$. By the mean value theorem and Lemma II.4.11, for $\varphi, \tilde{\varphi} \in B(b_5\delta^3)$ and $j = 1, .., 4$, we obtain

$$\|\mathcal{R}^{\mathrm{fl}}_j[\varphi] - \mathcal{R}^{\mathrm{fl}}_j[\tilde{\varphi}]\|^{\mathrm{fl}} \leq \sum_{l=1}^{4} \left[\sup_{\zeta \in B(b_5\delta^3)} \left\{\|D_l \mathcal{R}^{\mathrm{fl}}_j[\zeta]\|^{\mathrm{fl}}\right\} \|\varphi_l - \tilde{\varphi}_l\|^{\mathrm{fl}}\right] \leq C\|\varphi - \tilde{\varphi}\|^{\mathrm{fl}}_{\times}.$$

Then, by (II.4.15) and taking δ small enough,

$$\begin{aligned}\|\mathcal{F}[\varphi] - \mathcal{F}[\tilde{\varphi}]\|^{\mathrm{fl}}_{\times} \leq{}& C\delta \left[\sum_{j=1}^{2} \|\mathcal{R}^{\mathrm{fl}}_j[\varphi] - \mathcal{R}^{\mathrm{fl}}_j[\tilde{\varphi}]\|^{\mathrm{fl}}\right] + C\delta^2 \left[\sum_{l=3}^{4} \|\mathcal{R}^{\mathrm{fl}}_l[\varphi] - \mathcal{R}^{\mathrm{fl}}_l[\tilde{\varphi}]\|^{\mathrm{fl}}\right] \\ \leq{}& C\delta\|\varphi - \tilde{\varphi}\|^{\mathrm{fl}}_{\times} \leq \frac{1}{2}\|\varphi - \tilde{\varphi}\|^{\mathrm{fl}}_{\times}.\end{aligned} \tag{II.4.17}$$

Therefore, by (II.4.16) and (II.4.17), $\mathcal{F}$ is well defined and contractive in $B(b_5\delta^3)$ and, as a result, has a fixed point $\widehat{\Gamma} \in B(b_5\delta^3)$. □

II.4.5 Back to a graph parametrization

Now we prove Proposition II.3.10 by obtaining the change of variables $\mathcal{V} : \widetilde{D}_{\kappa_4,d_5} \to \mathbb{C}$ as a solution of equation (II.3.23). This equation is equivalent to $\mathcal{V} = \mathcal{N}[\mathcal{V}]$ with

$$\mathcal{N}[\varphi](u) = \frac{1}{3\Lambda_h(u)} \left[\hat{\lambda}(u + \varphi(u)) + \lambda_h(u + \varphi(u)) - \lambda_h(u) + 3\Lambda_h(u)\varphi(u)\right].$$

We obtain $\mathcal{V}$ by means of a fixed point argument in the Banach space

$$\widetilde{\mathcal{Y}} = \left\{\varphi : \widetilde{D}_{\kappa_4,d_5} \to \mathbb{C} \,:\, \varphi \text{ real-analytic}, \|\varphi\|_{\sup} := \sup_{u \in \widetilde{D}_{\kappa_4,d_5}} |\varphi(u)| < \infty\right\}.$$

Proof of Proposition II.3.10. Let us first notice that, by Theorem II.3.1,

$$C^{-1} \leq \|\Lambda_h\|_{\sup} \leq C. \tag{II.4.18}$$

Since $d_5 < d_4$ and $\kappa_4 > \kappa_3$, we have that $\widetilde{D}_{\kappa_4,d_5} \subset D^{\mathrm{fl}}_{\kappa_3,d_4}$, (see (II.3.16) and (II.3.19)). Then, applying Proposition II.3.9, there exists $b_6 > 0$ independent of δ such that

$$\|\mathcal{N}[0]\|_{\sup} \leq \frac{1}{3}\|(\Lambda_h)^{-1}\|_{\sup}\|\hat{\lambda}\|_{\sup} \leq \frac{1}{2} b_6 \delta^2.$$

Next, we compute the Lipschitz constant of $\mathcal{N}$ in $B(b_6\delta^2) = \{\varphi \in \widetilde{\mathcal{Y}} \,:\, \|\varphi\|_{\sup} \leq b_6\delta^2\}$. By the mean value theorem, for $\varphi, \tilde{\varphi} \in B(b_6\delta^2)$ and $\varphi_s = (1-s)\varphi + s\tilde{\varphi}$, we have that

$$\|\mathcal{N}[\varphi] - \mathcal{N}[\tilde{\varphi}]\|_{\sup} \leq \sup_{u \in \widetilde{D}_{\kappa_4,d_5}} \left| \int_0^1 D\mathcal{N}[\varphi_s](u) ds \right| \|\varphi - \tilde{\varphi}\|_{\sup}.$$

For $u \in \widetilde{D}_{\kappa_4,d_5}$ and δ small enough, we have that $u + \varphi_s(u) \in D^{\mathrm{fl}}_{\kappa_3,d_4}$. Therefore, by Proposition II.3.9, (II.4.18) and recalling that $\dot{\lambda}_h = -3\Lambda_h$,

$$|D\mathcal{N}[\varphi_s](u)| \leq \frac{1}{3\,|\Lambda_h(u)|} \left\{ |\partial_v \hat{\lambda}(u + \varphi_s(u))| + |\Lambda_h(u + \varphi_s(u)) - \Lambda_h(u)| \right\} \leq C\delta^2,$$

and, taking δ small enough, $\|\mathcal{N}[\varphi] - \mathcal{N}[\tilde{\varphi}]\|_{\sup} \leq \frac{1}{2}\|\varphi - \tilde{\varphi}\|_{\sup}$. Therefore, the operator $\mathcal{N}$ is well defined and contractive in $B(b_6\delta^2)$ and, as a result, has a fixed point $\mathcal{V} \in B(b_6\delta^2)$.

Besides, since $\widetilde{D}_{\kappa_4,d_5} \subset D^{\mathrm{fl}}_{\kappa_3,d_4}$, we obtain that $u + \mathcal{V}(u) \in D^{\mathrm{fl}}_{\kappa_3,d_4}$ for $u \in \widetilde{D}_{\kappa_4,d_5}$ and δ small enough.

□

Chapter II.5

Complex matching estimates

This chapter is devoted to prove Theorem II.3.15 which provides estimates for $Z_1^{\mathrm{u,s}} = Z^{\mathrm{u,s}} - Z_0^{\mathrm{u,s}}$ in the matching domains $\mathcal{D}_\kappa^{\mathrm{mch,u}}$ and $\mathcal{D}_\kappa^{\mathrm{mch,s}}$, given in (II.3.39). We only prove the theorem for Z_1^{u}, being the proof for Z_1^{s} analogous.

II.5.1 Preliminaries and set up

Proposition II.3.12 shows that the Hamiltonian H^{sep} expressed in inner coordinates, that is H^{in} as given in (II.3.32), is of the form $H^{\mathrm{in}} = W + XY + \mathcal{K} + H_1^{\mathrm{in}}$. Then, the equation associated to H^{in} can be written as

$$\left\{ \begin{array}{l} \dot{U} = 1 + g^{\mathrm{in}}(U,Z) + g^{\mathrm{mch}}(U,Z), \\ \dot{Z} = \mathcal{A}^{\mathrm{in}} Z + f^{\mathrm{in}}(U,Z) + f^{\mathrm{mch}}(U,Z), \end{array} \right. \tag{II.5.1}$$

where $\mathcal{A}^{\mathrm{in}}$ is given in (II.3.34) and

$$\begin{aligned} f^{\mathrm{in}} &= (-\partial_U \mathcal{K}, i\partial_Y \mathcal{K}, -i\partial_X \mathcal{K})^T, & g^{\mathrm{in}} &= \partial_W \mathcal{K}, \\ f^{\mathrm{mch}} &= \left(-\partial_U H_1^{\mathrm{in}}, i\partial_Y H_1^{\mathrm{in}}, -i\partial_X H_1^{\mathrm{in}}\right)^T, & g^{\mathrm{mch}} &= \partial_W H_1^{\mathrm{in}}. \end{aligned} \tag{II.5.2}$$

Notice that, since $(u, z^{\mathrm{u}}(u)) = \phi_{\mathrm{in}}(U, Z^{\mathrm{u}}(U))$ (see (II.3.41)), $(U, Z^{\mathrm{u}}(U))$ is an invariant graph of equation (II.5.1). Therefore, Z^{u} satisfies the invariance equation

$$\partial_U Z^{\mathrm{u}} = \mathcal{A}^{\mathrm{in}} Z^{\mathrm{u}} + \mathcal{R}^{\mathrm{in}}[Z^{\mathrm{u}}] + \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}],$$

with $\mathcal{R}^{\mathrm{in}}$ as defined in (II.3.36) and

$$\mathcal{R}^{\mathrm{mch}}[\varphi] = \frac{\mathcal{A}^{\mathrm{in}}\varphi + f^{\mathrm{in}}(U,\varphi) + f^{\mathrm{mch}}(U,\varphi)}{1 + g^{\mathrm{in}}(U,\varphi) + g^{\mathrm{mch}}(U,\varphi)} - \mathcal{A}^{\mathrm{in}}\varphi - \mathcal{R}^{\mathrm{in}}[\varphi]. \tag{II.5.3}$$

Similarly Z_0^{u} satisfies the invariance equation $\partial_U Z_0^{\mathrm{u}} = \mathcal{A}^{\mathrm{in}} Z_0^{\mathrm{u}} + \mathcal{R}^{\mathrm{in}}[Z_0^{\mathrm{u}}]$ (see Theorem II.3.13) and, therefore, the difference $Z_1^{\mathrm{u}} = Z^{\mathrm{u}} - Z_0^{\mathrm{u}}$ must be a solution of

$$\partial_U Z_1^{\mathrm{u}} = \mathcal{A}^{\mathrm{in}} Z_1^{\mathrm{u}} + \mathcal{B}(U) Z_1^{\mathrm{u}} + \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}], \tag{II.5.4}$$

with

$$\mathcal{B}(U) = \int_0^1 D_Z \mathcal{R}^{\mathrm{in}}[(1-s)Z_0^{\mathrm{u}} + sZ^{\mathrm{u}}](U) ds. \tag{II.5.5}$$

The key point is that, since the existence of both Z_0^{u} and Z^{u} is already been proven, we can think of $\mathcal{B}(U)$ and $\mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}](U)$ as known functions. Therefore, equation (II.5.4) can be understood as a non homogeneous linear equation with independent term

$\mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}](U)$. Moreover, defining the linear operator $\mathcal{L}^{\mathrm{in}}\varphi = (\partial_U - \mathcal{A}^{\mathrm{in}})\varphi$, equation (II.5.4) is equivalent to

$$\mathcal{L}^{\mathrm{in}} Z_1^{\mathrm{u}}(U) = \mathcal{B}(U) Z_1^{\mathrm{u}}(U) + \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}](U). \tag{II.5.6}$$

We prove Theorem II.3.15 by solving this equation (with suitable initial conditions). To this end, we define the Banach space $\mathcal{X}_\times^{\mathrm{mch}} = \mathcal{X}_{\frac{4}{3}}^{\mathrm{mch}} \times \mathcal{X}_1^{\mathrm{mch}} \times \mathcal{X}_1^{\mathrm{mch}}$ with

$$\mathcal{X}_\alpha^{\mathrm{mch}} = \left\{ \varphi : \mathcal{D}_\kappa^{\mathrm{mch,u}} \to \mathbb{C} \ : \ \varphi \text{ real-analytic}, \ \|\varphi\|_\alpha^{\mathrm{mch}} = \sup_{U \in \mathcal{D}_\kappa^{\mathrm{mch,u}}} |U^\alpha \varphi(U)| < \infty \right\},$$

endowed with the product norm $\|\varphi\|_\times^{\mathrm{mch}} = \|\varphi_1\|_{\frac{4}{3}}^{\mathrm{mch}} + \|\varphi_2\|_1^{\mathrm{mch}} + \|\varphi_3\|_1^{\mathrm{mch}}$.

Next lemma gives some properties of these Banach spaces.

Lemma II.5.1. *Let $\gamma \in [\frac{3}{5}, 1)$ and $\alpha, \beta \in \mathbb{R}$. The following statements hold:*

1. *If $\varphi \in \mathcal{X}_\alpha^{\mathrm{mch}}$, then $\varphi \in \mathcal{X}_\beta^{\mathrm{mch}}$ for any $\beta \in \mathbb{R}$. Moreover,*

$$\begin{cases} \|\varphi\|_\beta^{\mathrm{mch}} \leq C\kappa^{\beta-\alpha} \|\varphi\|_\alpha^{\mathrm{mch}}, & \text{for } \alpha > \beta, \\ \|\varphi\|_\beta^{\mathrm{mch}} \leq C\delta^{2(\alpha-\beta)(1-\gamma)} \|\varphi\|_\alpha^{\mathrm{mch}}, & \text{for } \alpha < \beta. \end{cases}$$

2. *If $\varphi \in \mathcal{X}_\alpha^{\mathrm{mch}}$ and $\zeta \in \mathcal{X}_\beta^{\mathrm{mch}}$, then $\varphi\zeta \in \mathcal{X}_{\alpha+\beta}^{\mathrm{mch}}$ and $\|\varphi\zeta\|_{\alpha+\beta}^{\mathrm{mch}} \leq \|\varphi\|_\alpha^{\mathrm{mch}} \|\zeta\|_\beta^{\mathrm{mch}}$.*

This lemma is a direct consequence of the fact that, as explained in Section II.3.3.2, U satisfies

$$\kappa \cos\beta_2 \leq |U| \leq \frac{C}{\delta^{2(1-\gamma)}}. \tag{II.5.7}$$

Now, we present the main result of this chapter, which implies Theorem II.3.15.

Proposition II.5.2. *There exist $\gamma^* \in [\frac{3}{5}, 1)$, $\kappa_6 \geq \max\{\kappa_1, \kappa_5\}$, $\delta_0 > 0$ and $b_{18} > 0$ such that, for $\gamma \in (\gamma^*, 1)$, $\kappa \geq \kappa_6$ and $\delta \in (0, \delta_0)$, Z_1^{u} satisfies $\|Z_1^{\mathrm{u}}\|_\times^{\mathrm{mch}} \leq b_{18}\, \delta^{\frac{2}{3}(1-\gamma)}$.*

II.5.2 An integral equation formulation

To prove Proposition II.5.2, we first introduce a right-inverse of $\mathcal{L}^{\mathrm{in}} = \partial_U - \mathcal{A}^{\mathrm{in}}$.

Lemma II.5.3. *The operator $\mathcal{G}^{\mathrm{in}}[\varphi] = \left(\mathcal{G}_1^{\mathrm{in}}[\varphi_1], \mathcal{G}_2^{\mathrm{in}}[\varphi_2], \mathcal{G}_3^{\mathrm{in}}[\varphi_3]\right)^T$ defined as*

$$\mathcal{G}^{\mathrm{in}}[\varphi](U) = \left(\int_{U_3}^{U} \varphi_1(S) dS, \int_{U_3}^{U} e^{-i(S-U)} \varphi_2(S) dS, \int_{U_2}^{U} e^{i(S-U)} \varphi_3(S) dS \right)^T, \tag{II.5.8}$$

where U_2 and U_3 are introduced in (II.3.40), is a right inverse of $\mathcal{L}^{\mathrm{in}}$.

Moreover, there exists a constant $C > 0$ such that:

1. *Let $\alpha > 1$. If $\varphi \in \mathcal{X}_\alpha^{\mathrm{mch}}$, then $\mathcal{G}_1^{\mathrm{in}}[\varphi] \in \mathcal{X}_{\alpha-1}^{\mathrm{mch}}$ and $\left\|\mathcal{G}_1^{\mathrm{in}}[\varphi]\right\|_{\alpha-1}^{\mathrm{mch}} \leq C \|\varphi\|_\alpha^{\mathrm{mch}}$.*
2. *Let $\alpha > 0$, $j = 2, 3$. If $\varphi \in \mathcal{X}_\alpha^{\mathrm{mch}}$, then $\mathcal{G}_j^{\mathrm{in}}[\varphi] \in \mathcal{X}_\alpha^{\mathrm{mch}}$ and $\|\mathcal{G}_j^{\mathrm{in}}[\varphi]\|_\alpha^{\mathrm{mch}} \leq C \|\varphi\|_\alpha^{\mathrm{mch}}$.*

The proof of this lemma follows the same lines as the proof of Lemma 20 in [BCS13]. Using the operator $\mathcal{G}^{\mathrm{in}}$, equation (II.5.6) is equivalent to

$$Z_1(U) = C^{\mathrm{mch}} e^{\mathcal{A}^{\mathrm{in}} U} + \mathcal{G}^{\mathrm{in}}\left[\mathcal{B} \cdot Z_1\right](U) + \left(\mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z]\right)(U),$$

where $C^{\mathrm{mch}} = (C_W^{\mathrm{mch}}, C_X^{\mathrm{mch}}, C_Y^{\mathrm{mch}})^T$ is defined as

$$C_W^{\mathrm{mch}} = W_1(U_3), \qquad C_X^{\mathrm{mch}} = e^{-iU_3} X_1(U_3), \qquad C_Y^{\mathrm{mch}} = e^{iU_2} Y_1(U_2).$$

Then, defining the operator $\mathcal{T}[\varphi](U) = \mathcal{G}^{\mathrm{in}}\left[\mathcal{B}\cdot\varphi\right](U)$, this equation is equivalent to

$$(\mathrm{Id} - \mathcal{T})Z_1^{\mathrm{u}} = C^{\mathrm{mch}} e^{\mathcal{A}^{\mathrm{in}}U} + \left(\mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]\right) \tag{II.5.9}$$

and therefore, to estimate Z_1^{u}, we need to prove that $\mathrm{Id} - \mathcal{T}$ is invertible in $\mathcal{X}_\times^{\mathrm{mch,u}}$.

Lemma II.5.4. *Let us consider operators $\mathcal{B}$ and $\mathcal{G}^{\mathrm{in}}$ as given in (II.5.5) and (II.5.8). Then, for $\gamma \in [\frac{3}{5}, 1)$, $\kappa > 0$ big enough and $\delta > 0$ small enough, for $\varphi \in \mathcal{X}_\times^{\mathrm{mch}}$,*

$$\|\mathcal{T}[\varphi]\|_\times^{\mathrm{mch}} = \left\|\mathcal{G}^{\mathrm{in}}[\mathcal{B}\cdot\varphi]\right\|_\times^{\mathrm{mch}} \leq \frac{1}{2}\|\varphi\|_\times^{\mathrm{mch}}$$

and therefore

$$\left\|(\mathrm{Id} - \mathcal{T})^{-1}[\varphi]\right\|_\times^{\mathrm{mch}} \leq 2\|\varphi\|_\times^{\mathrm{mch}}.$$

To prove this lemma, we use the following estimates, whose proof is a direct result of Lemma I.5.5 in Part I.

Lemma II.5.5. *Fix $\varrho > 0$ and take $\kappa > 0$ big enough. Then, there exists a constant C (depending on ϱ but independent of κ) such that, for $\varphi \in \mathcal{X}_\times^{\mathrm{mch}}$ with $\|\varphi\|_\times^{\mathrm{mch}} \leq \varrho$, the functions g^{in} and f^{in} in (II.3.36) and the operator $\mathcal{R}^{\mathrm{in}}$ in (II.5.2) satisfy*

$$\left\|g^{\mathrm{in}}(\cdot,\varphi)\right\|_2^{\mathrm{mch}} \leq C, \qquad \left\|f_1^{\mathrm{in}}(\cdot,\varphi)\right\|_{\frac{11}{3}}^{\mathrm{mch}} \leq C, \qquad \left\|f_j^{\mathrm{in}}(\cdot,\varphi)\right\|_{\frac{4}{3}}^{\mathrm{mch}} \leq C, \quad j = 2,3$$

and

$$\begin{aligned}
&\left\|\partial_W \mathcal{R}_1^{\mathrm{in}}[\varphi]\right\|_3^{\mathrm{mch}} \leq C, \quad \left\|\partial_X \mathcal{R}_1^{\mathrm{in}}[\varphi]\right\|_{\frac{7}{3}}^{\mathrm{mch}} \leq C, \quad \left\|\partial_Y \mathcal{R}_1^{\mathrm{in}}[\varphi]\right\|_{\frac{7}{3}}^{\mathrm{mch}} \leq C,\\
&\left\|\partial_W \mathcal{R}_j^{\mathrm{in}}[\varphi]\right\|_{\frac{2}{3}}^{\mathrm{mch}} \leq C, \quad \left\|\partial_X \mathcal{R}_j^{\mathrm{in}}[\varphi]\right\|_2^{\mathrm{mch}} \leq C, \quad \left\|\partial_Y \mathcal{R}_j^{\mathrm{in}}[\varphi]\right\|_2^{\mathrm{mch}} \leq C, \quad j = 2,3.
\end{aligned}$$

Proof of Lemma II.5.4. Let Z^{u} be as given in (II.3.41). Then, by Proposition II.3.6, estimates (II.5.7) and taking $\gamma \in [\frac{3}{5}, 1)$, we have that, for $U \in \mathcal{D}_\kappa^{\mathrm{mch,u}}$,

$$|W^{\mathrm{u}}(U)| \leq \frac{C}{|U|^{\frac{8}{3}}} + \frac{C\delta^{\frac{4}{3}}}{|U|} \leq \frac{C}{|U|^{\frac{8}{3}}}, \qquad \|X^{\mathrm{u}}\|_{\frac{4}{3}}^{\mathrm{mch}} \leq C, \qquad \|Y^{\mathrm{u}}\|_{\frac{4}{3}}^{\mathrm{mch}} \leq C. \tag{II.5.10}$$

Then, using also Theorem II.3.13, we obtain that $(1-s)Z_0^{\mathrm{u}} + sZ^{\mathrm{u}} \in \mathcal{X}_\times^{\mathrm{mch}}$ for $s \in [0,1]$ and $\gamma \in [\frac{3}{5}, 1)$ and $\|(1-s)Z_0^{\mathrm{u}} + sZ^{\mathrm{u}}\|_\times^{\mathrm{mch}} \leq C$. As a result, using the definition of $\mathcal{B}$ in (II.5.5) and Lemma II.5.5,

$$\begin{aligned}
&\|\mathcal{B}_{1,1}\|_3^{\mathrm{mch}} \leq C, \quad \|\mathcal{B}_{1,2}\|_{\frac{7}{3}}^{\mathrm{mch}} \leq C, \quad \|\mathcal{B}_{1,3}\|_{\frac{7}{3}}^{\mathrm{mch}} \leq C,\\
&\|\mathcal{B}_{j,1}\|_{\frac{2}{3}}^{\mathrm{mch}} \leq C, \quad \|\mathcal{B}_{j,2}\|_2^{\mathrm{mch}} \leq C, \quad \|\mathcal{B}_{j,3}\|_2^{\mathrm{mch}} \leq C, \quad \text{for } j = 2,3.
\end{aligned} \tag{II.5.11}$$

Therefore, by Lemmas II.5.3 and II.5.1 and (II.5.11), we obtain

$$\begin{aligned}
\|\mathcal{T}_1[\varphi]\|_{\frac{4}{3}}^{\mathrm{mch}} &\leq C\,\|\pi_1(\mathcal{B}\varphi)\|_{\frac{7}{3}}^{\mathrm{mch}} \\
&\leq C\left[\|\mathcal{B}_{1,1}\|_1^{\mathrm{mch}}\,\|\varphi_1\|_{\frac{4}{3}}^{\mathrm{mch}} + \|\mathcal{B}_{1,2}\|_{\frac{4}{3}}^{\mathrm{mch}}\,\|\varphi_2\|_1^{\mathrm{mch}} + \|\mathcal{B}_{1,3}\|_{\frac{4}{3}}^{\mathrm{mch}}\,\|\varphi_3\|_1^{\mathrm{mch}}\right] \\
&\leq \frac{C}{\kappa^2}\,\|\varphi_1\|_{\frac{4}{3}}^{\mathrm{mch}} + \frac{C}{\kappa}\,\|\varphi_2\|_1^{\mathrm{mch}} + \frac{C}{\kappa}\,\|\varphi_3\|_1^{\mathrm{mch}} \leq \frac{C}{\kappa}\,\|\varphi\|_{\times}^{\mathrm{mch}}.
\end{aligned}$$

Proceeding analogously, for $j = 2, 3$, we have

$$\|\mathcal{T}_j[\varphi]\|_1^{\mathrm{mch}} \leq C\left[\|\mathcal{B}_{j,1}\|_{-\frac{1}{3}}^{\mathrm{mch}}\,\|\varphi_1\|_{\frac{4}{3}}^{\mathrm{mch}} + \sum_{l=2}^{3}\|\mathcal{B}_{j,l}\|_0^{\mathrm{mch}}\,\|\varphi_l\|_1^{\mathrm{mch}}\right] \leq \frac{C}{\kappa}\,\|\varphi\|_{\times}^{\mathrm{mch}}.$$

Taking $\kappa > 0$ big enough, we obtain the statement of the lemma. □

II.5.3 End of the proof of Proposition II.5.2

To complete the proof of Proposition II.5.2, we study the right-hand side of equation (II.5.9).

First, we deal with the term $C^{\mathrm{mch}}e^{\mathcal{A}^{\mathrm{in}}U}$. Recall that U_2 and U_3 in (II.3.40) satisfy

$$\frac{C^{-1}}{\delta^{2(1-\gamma)}} \leq |U_j| \leq \frac{C}{\delta^{2(1-\gamma)}}, \qquad \text{for } j = 2, 3.$$

Then, taking into account that $W_1^{\mathrm{u}} = W^{\mathrm{u}} - W_0^{\mathrm{u}}$, (II.5.10) and Theorem II.3.13 imply

$$|C_W^{\mathrm{mch}}| = |W_1^{\mathrm{u}}(U_3)| \leq |W^{\mathrm{u}}(U_3)| + |W_0^{\mathrm{u}}(U_3)| \leq \frac{C}{|U_3|^{\frac{8}{3}}} \leq C\delta^{\frac{16}{3}(1-\gamma)}$$

and, as a result, by Lemma II.5.1, $\|C_W^{\mathrm{mch}}\|_{\frac{4}{3}}^{\mathrm{mch}} \leq C\delta^{\frac{8}{3}(1-\gamma)}$. Analogously, for $U \in \mathcal{D}_\kappa^{\mathrm{mch,u}}$,

$$|C_X^{\mathrm{mch}}e^{iU}| = |e^{i(U-U_3)}X_1^{\mathrm{u}}(U_3)| \leq \frac{Ce^{-\mathrm{Im}\,(U-U_3)}}{|U_3|^{\frac{4}{3}}} \leq C\delta^{\frac{8}{3}(1-\gamma)}$$

and then $\|C_X^{\mathrm{mch}}e^{iU}\|_1^{\mathrm{mch}} \leq C\delta^{\frac{2}{3}(1-\gamma)}$. An analogous result holds for $C_Y^{\mathrm{mch}}e^{-iU}$. Therefore,

$$\|C^{\mathrm{mch}}e^{\mathcal{A}^{\mathrm{in}}U}\|_{\times}^{\mathrm{mch}} \leq C\delta^{\frac{2}{3}(1-\gamma)}. \tag{II.5.12}$$

Now, we estimate the norm of $\mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]$. The operator $\mathcal{R}^{\mathrm{mch}}$ in (II.5.3) can be rewritten as

$$\mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}] = \frac{f^{\mathrm{mch}}(1+g^{\mathrm{in}}) - g^{\mathrm{mch}}(\mathcal{A}^{\mathrm{in}}Z^{\mathrm{u}} + f^{\mathrm{in}})}{(1+g^{\mathrm{in}})(1+g^{\mathrm{in}}+g^{\mathrm{mch}})}.$$

Then by (II.5.10), Lemmas II.5.1 and II.5.5 and taking κ big enough, we obtain

$$\begin{aligned}
&\left\|g^{\mathrm{in}}(\cdot, Z^{\mathrm{u}})\right\|_0^{\mathrm{mch}} \leq \frac{C}{\kappa^2} \leq \frac{1}{2}, &&\left\|iX^{\mathrm{u}} + f_2^{\mathrm{in}}(\cdot, Z^{\mathrm{u}})\right\|_0^{\mathrm{mch}} \leq C, \\
&\left\|f_1^{\mathrm{in}}(\cdot, Z^{\mathrm{u}})\right\|_0^{\mathrm{mch}} \leq C, &&\left\|-iY^{\mathrm{u}} + f_3^{\mathrm{in}}(\cdot, Z^{\mathrm{u}})\right\|_0^{\mathrm{mch}} \leq C.
\end{aligned} \tag{II.5.13}$$

To analyze f^{mch} and g^{mch} (see (II.5.2)) we rely on the estimates for H_1^{in} in (II.3.33) and its derivatives, which can be easily obtained by Cauchy estimates. Indeed, they can be applied since $U \in \mathcal{D}_\kappa^{\mathrm{mch,u}}$ and, by (II.5.10),

$$|W^{\mathrm{u}}(U)|\,, |X^{\mathrm{u}}(U)|\,, |Y^{\mathrm{u}}(U)| \leq C.$$

Then, there exists $m > 0$ such that

$$|g^{\mathrm{mch}}(U, Z^{\mathrm{u}})| \leq C\delta^{\frac{4}{3}-2m(1-\gamma)}, \quad |f_j^{\mathrm{mch}}(U, Z^{\mathrm{u}})| \leq C\delta^{\frac{4}{3}-2m(1-\gamma)}, \text{ for } j = 1, 2, 3. \tag{II.5.14}$$

We note that, for $\gamma \in (\gamma_0^*, 1)$ with $\gamma_0^* = \max\{\frac{3}{5}, \frac{3m-2}{3m}\}$, we have that $\frac{4}{3} - 2m(1-\gamma) > 0$. Then, for $\gamma \in (\gamma_0^*, 1)$, δ small enough and κ big enough, using (II.5.13) and (II.5.14) we obtain

$$|\mathcal{R}_j^{\mathrm{mch}}[Z^{\mathrm{u}}](U)| \leq C\delta^{\frac{4}{3}-2m(1-\gamma)}, \qquad \text{for } j = 1, 2, 3.$$

Then, by Lemmas II.5.1 and II.5.3,

$$\begin{aligned}\|\mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_\times^{\mathrm{mch}} &= \|\mathcal{G}_1^{\mathrm{in}} \circ \mathcal{R}_1^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_{\frac{4}{3}}^{\mathrm{mch}} + \textstyle\sum_{j=2}^3 \|\mathcal{G}_j^{\mathrm{in}} \circ \mathcal{R}_j^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_1^{\mathrm{mch}} \\ &\leq C\|\mathcal{R}_1^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_{\frac{7}{3}}^{\mathrm{mch}} + \textstyle\sum_{j=2}^3 C\|\mathcal{R}_j^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_1^{\mathrm{mch}} \leq C\delta^{\frac{4}{3}-2\left(m+\frac{7}{3}\right)(1-\gamma)}.\end{aligned}$$

If we take $\gamma^* = \max\left\{\frac{3}{5}, \gamma_0^*, \gamma_1^*\right\}$ with $\gamma_1^* = \frac{3m+5}{3m+7}$, and $\gamma \in (\gamma^*, 1)$,

$$\|\mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]\|_\times^{\mathrm{mch}} \leq C\delta^{\frac{2}{3}(1-\gamma)}. \tag{II.5.15}$$

To complete the proof of Proposition II.5.2, we consider equation (II.5.9). By Lemma II.5.4, $(\mathrm{Id} - \mathcal{T})$ is invertible in $\mathcal{X}_\times^{\mathrm{mch}}$ and moreover

$$\begin{aligned}\|Z_1^{\mathrm{u}}\|_\times^{\mathrm{mch}} &= \left\|(\mathrm{Id} - \mathcal{T})^{-1}\left(C^{\mathrm{mch}} e^{\mathcal{A}^{\mathrm{in}} U} + \mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]\right)\right\|_\times^{\mathrm{mch}} \\ &\leq 2\left\|C^{\mathrm{mch}} e^{\mathcal{A}^{\mathrm{in}} U} + \mathcal{G}^{\mathrm{in}} \circ \mathcal{R}^{\mathrm{mch}}[Z^{\mathrm{u}}]\right\|_\times^{\mathrm{mch}}.\end{aligned}$$

Then, it is enough to apply (II.5.12) and (II.5.15).

$\square$

Appendix II.A

Estimates for the invariant manifolds

In this appendix we prove the technical Lemmas II.4.4 and II.4.8. All these results involve, in some sense, estimates for the first and second derivatives of the Hamiltonian H_1^{sep} in (II.3.9). However, to obtain estimates for H_1^{sep}, we first obtain some properties of H_1^{Poi} (see (II.2.2)), which can be written as

$$H_1^{\mathrm{Poi}} = \frac{1}{\mu}\mathcal{P}[0] - \frac{1-\mu}{\mu}\mathcal{P}[\mu] - \mathcal{P}[\mu-1], \tag{II.A.1}$$

where

$$\mathcal{P}[\zeta](\lambda, L, \eta, \xi) = \left(\|q - (\zeta, 0)\|^{-1}\right) \circ \phi_{\mathrm{Poi}}. \tag{II.A.2}$$

In Section I.4.1, we computed the series expansion of $\mathcal{P}[\zeta]$ in powers of (η, ξ). In particular, $\mathcal{P}[\zeta]$ can be written as

$$\mathcal{P}[\zeta](\lambda, L, \eta, \xi) = \frac{1}{\sqrt{A[\zeta](\lambda) + B[\zeta](\lambda, L, \eta, \xi)}} \tag{II.A.3}$$

where A and B are of the form

$$A[\zeta](\lambda) = 1 - 2\zeta\cos\lambda + \zeta^2, \tag{II.A.4}$$

$$\begin{aligned} B[\zeta](\lambda, L, \eta, \xi) = {} & 4(L-1)(1-\zeta\cos\lambda) + \frac{\eta}{\sqrt{2}}\left(3\zeta - 2e^{-i\lambda} - \zeta e^{-2i\lambda}\right) \\ & + \frac{\xi}{\sqrt{2}}\left(3\zeta - 2e^{i\lambda} - \zeta e^{2i\lambda}\right) + R[\zeta](\lambda, L, \eta, \xi), \end{aligned} \tag{II.A.5}$$

and, for fixed $\varrho > 0$, R is analytic and satisfies that

$$|R[\zeta](\lambda, L, \eta, \xi)| \leq K(\varrho)\, |(L-1, \eta, \xi)|^2, \tag{II.A.6}$$

for $|\mathrm{Im}\,\lambda| \leq \varrho$, $|(L-1, \eta, \xi)| \ll 1$ and $\zeta \in [-1, 1]$.

Then, wherever $|A[\zeta](\lambda)| > |B[\zeta](\lambda, L, \eta, \xi)|$, $\mathcal{P}[\zeta](\lambda, L, \eta, \xi)$ can be written as

$$\mathcal{P}[\zeta](\lambda, L, \eta, \xi) = \frac{1}{\sqrt{A[\zeta]}} + \sum_{n=1}^{+\infty} \binom{-\frac{1}{2}}{n} \frac{(B[\zeta])^n}{(A[\zeta])^{n+\frac{1}{2}}}. \tag{II.A.7}$$

Remark II.A.1. *The Hamiltonian $H^{\mathrm{Poi}} = H_0^{\mathrm{Poi}} + \mu H_1^{\mathrm{Poi}}$ (see (II.2.1) and (II.A.1)) is analytic away from the collisions with the primaries, that is zeroes of the denominators of $\mathcal{P}[\mu]$ and $\mathcal{P}[\mu-1]$. For $0 < \mu \ll 1$, one has*

$$A[\mu] = 1 + \mathcal{O}(\mu), \qquad A[\mu-1] = 2 + 2\cos\lambda + \mathcal{O}(\mu).$$

Therefore, in the regime that we consider, collisions with the primary S are not possible but collisions with P may take place at $\lambda \sim \pi$.

We now obtain estimates for H_1^{Poi} in domains "far" from $\lambda = \pi$.

Lemma II.A.2. *Fix $\lambda_0 \in (0, \pi)$ and $\mu_0 \in (0, \frac{1}{2})$ and consider the Hamiltonian H_1^{Poi} and the potential V introduced in (II.A.1) and (II.2.4), respectively. Then, for for $|\lambda| < \lambda_0$, $|(L-1, \eta, \xi)| \ll 1$ and $\mu \in (0, \mu_0)$, the Hamiltonian H_1^{Poi} can be written as*

$$H_1^{\mathrm{Poi}}(\lambda, L, \eta, \xi; \mu) - V(\lambda) = D_0(\mu, \lambda) + D_1(\mu, \lambda)\big((L-1), \eta, \xi\big) + D_2(\lambda, L, \eta, \xi; \mu),$$

such that, for $j = 1, 2, 3$,

$$|D_0(\mu, \lambda)| \leq K\mu, \qquad |(D_1(\mu, \lambda))_j| \leq K, \qquad |D_2(\lambda, L, \eta, \xi; \mu)| \leq K\, |(L-1, \eta, \xi)|^2,$$

with K a positive constant independent of λ and μ.

II.A.1 Estimates in the infinity domain

To prove Lemma II.4.4, we need to obtain estimates for $\mathcal{R}^{\mathrm{sep}}$ and its derivatives. Let us recall that, by its definition in (II.3.13), for $z = (w, x, y)$ we have

$$\mathcal{R}^{\mathrm{sep}}[z] = \left(\frac{f_1^{\mathrm{sep}}(\cdot, z)}{1 + g^{\mathrm{sep}}(\cdot, z)}, \frac{f_2{}^{\mathrm{sep}}(\cdot, z) - \frac{ix}{\delta^2}\, g^{\mathrm{sep}}(\cdot, z)}{1 + g^{\mathrm{sep}}(\cdot, z)}, \frac{f_3{}^{\mathrm{sep}}(\cdot, z) + \frac{iy}{\delta^2}\, g^{\mathrm{sep}}(\cdot, z)}{1 + g^{\mathrm{sep}}(\cdot, z)} \right)^T, \tag{II.A.8}$$

where $g^{\mathrm{sep}} = \partial_w H_1^{\mathrm{sep}}$ and $f^{\mathrm{sep}} = (-\partial_u H_1^{\mathrm{sep}}, i\partial_y H_1^{\mathrm{sep}}, -i\partial_x H_1^{\mathrm{sep}})^T$.

Therefore, we need to obtain first estimates for the first and second derivatives of H_1^{sep}, introduced in (II.3.9), that is

$$H_1^{\mathrm{sep}} = H \circ (\phi_{\mathrm{eq}} \circ \phi_{\mathrm{sep}}) - \left(w + \frac{xy}{\delta^2} \right), \tag{II.A.9}$$

where $H = H_0 + H_1$ with $H_0 = H_{\mathrm{p}} + H_{\mathrm{osc}}$ (see (II.2.5), (II.2.9)).

Since (λ_h, Λ_h) is a solution of the Hamiltonian H_{p} and belongs to the energy level $H_{\mathrm{p}} = -\frac{1}{2}$,

$$H_0 \circ \phi_{\mathrm{sep}} = H_{\mathrm{p}} \left(\lambda_h(u), \Lambda_h(u) - \frac{w}{3\Lambda_h(u)} \right) + H_{\mathrm{osc}}(x, y; \delta) = -\frac{1}{2} + w - \frac{w^2}{6\Lambda_h^2(u)} + \frac{xy}{\delta^2}.$$

Therefore, by (II.A.9), the Hamiltonian H_1^{sep} can be expressed (up to a constant) as

$$H_1^{\mathrm{sep}} = M \circ \phi_{\mathrm{sep}} - \frac{w^2}{6\Lambda_h^2(u)}, \tag{II.A.10}$$

where

$$M(\lambda, \Lambda, x, y; \delta) = (H \circ \phi_{\mathrm{eq}})(\lambda, \Lambda, x, y; \delta) - H_0(\lambda, \Lambda, x, y).$$

In the following lemma we give properties of M.

Lemma II.A.3. *Fix constants $\varrho > 0$ and $\lambda_0 \in (0, \pi)$. Then, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, $|\lambda| < \lambda_0$, $|\Lambda| < \varrho$ and $|(x, y)| < \varrho\delta$, the function M satisfies*

$$
\begin{aligned}
&|\partial_\lambda M| \leq C\delta^2 |(\lambda, \Lambda)| + C\delta |(x, y)| , && |\partial_x M| \leq C\delta |(\lambda, \Lambda, x, y)| ,\\
&|\partial_\Lambda M| \leq C\delta^2 |(\lambda, \Lambda)| + C\delta |(x, y)| , && |\partial_y M| \leq C\delta |(\lambda, \Lambda, x, y)| ,
\end{aligned}
$$

and

$$
\left|\partial_\lambda^2 M\right| , |\partial_{\lambda\Lambda} M| , \left|\partial_\Lambda^2 M\right| \leq C\delta^2, \qquad |\partial_{ij} M| \leq C\delta, \quad \text{for } i, j \in \{\lambda, \Lambda, x, y\} .
$$

Proof. Applying $\phi_{\rm eq}$ (see (II.3.3)) to the Hamiltonian $H = H_0 + H_1$, we have that

$$
\begin{aligned}
M &= (H_0 \circ \phi_{\rm eq} - H_0) + H_1 \circ \phi_{\rm eq}\\
&= \delta(x\mathfrak{L}_y + y\mathfrak{L}_x) + 3\delta^2\Lambda\mathfrak{L}_\Lambda + \delta^4 \left(-\frac{3}{2}\mathfrak{L}_\Lambda^2 + \mathfrak{L}_x\mathfrak{L}_y\right) + H_1 \circ \phi_{\rm eq}.
\end{aligned}
\tag{II.A.11}
$$

Then,

$$
|\partial_{ij} M| \leq \left|\partial_{ij} H_1(\lambda, \Lambda + \delta^2\mathfrak{L}_\Lambda, x + \delta\mathfrak{L}_x, y + \delta\mathfrak{L}_y; \delta)\right| , \quad \text{for } i, j \in \{\lambda, \Lambda, x, y\} .
\tag{II.A.12}
$$

Since $|\Lambda| < \varrho$ and $|(x, y)| < \varrho\delta$, then $\left|\Lambda + \delta^2\mathfrak{L}_\Lambda\right| < 2\varrho$ and $\left|(x + \delta^3\mathfrak{L}_x, y + \delta^3\mathfrak{L}_y)\right| < 2\varrho\delta$, for δ small. By the definition of H_1 in (II.2.7) we have that,

$$
H_1(\lambda, \Lambda, x, y; \delta) = H_1^{\rm Poi}\left(\lambda, 1 + \delta^2\Lambda, \delta x, \delta y; \delta^4\right) - V(\lambda) + \frac{1}{\delta^4} F_{\rm p}(\delta^2\Lambda),
$$

where $H_1^{\rm Poi}$ is given in (II.2.2) (see also (II.2.3)), V is given (II.2.4) and $F_{\rm p}$ is given (II.2.8) and satisfies $F_{\rm p}(s) = \mathcal{O}(s^3)$. Since $\left|(\delta^2\Lambda, \delta x, \delta y)\right| < 2\varrho\delta^2 \ll 1$, we apply Lemma II.A.2 (recall that $\delta = \mu^{\frac{1}{4}}$) and Cauchy estimates to obtain

$$
\left|\partial_\lambda^2 H_1\right| |\partial_{\lambda\Lambda} H_1| , \left|\partial_\Lambda^2 H_1\right| \leq C\delta^2, \qquad |\partial_{ij} H_1| \leq C\delta, \quad \text{for } i, j \in \{\lambda, \Lambda, x, y\} .
\tag{II.A.13}
$$

Then, (II.A.12) and (II.A.13) give the estimates for the second derivatives of M.

For the first derivatives of M, let us take into account that, by Theorem II.3.1, 0 is a critical point of both Hamiltonians $(H \circ \phi_{\rm eq})$ and H_0 and, therefore, also of $M = (H \circ \phi_{\rm eq}) - H_0$. This fact and the estimates of the second derivatives, together with the mean value theorem, gives the estimates for the first derivatives of M. □

End of the proof of Lemma II.4.4. Let us consider $\varphi = (\varphi_w, \varphi_x, \varphi_y)^T \in \mathcal{X}_\times^\infty$ such that $\|\varphi\|_\times^\infty \leq \varrho\delta^3$. We estimate the first and second derivatives of $H_1^{\rm sep}$ evaluated at $(u, \varphi(u))$ (recall (II.A.8)), given by

$$
H_1^{\rm sep}(u, \varphi(u); \delta) = M\left(\lambda_h(u), \Lambda_h(u) - \frac{\varphi_w(u)}{3\Lambda_h(u)}, \varphi_x(u), \varphi_y(u); \delta\right) - \frac{\varphi_w^2(u)}{6\Lambda_h^2(u)}.
\tag{II.A.14}
$$

First, let us define

$$
\varphi_\lambda(u) = \lambda_h(u), \qquad \varphi_\Lambda(u) = \Lambda_h(u) - \frac{\varphi_w(u)}{3\Lambda_h(u)} \quad \text{and} \quad \Phi = (\varphi_\lambda, \varphi_\Lambda, \varphi_x, \varphi_y).
$$

Since $\|\varphi\|_{\times}^{\infty} \le \varrho\delta^3$ and $\lambda_h, \Lambda_h \in \mathcal{X}_\rho^\infty$ (see (II.4.2)),

$$\|\varphi_w\|_{2\rho}^{\infty} \le C\delta^2, \qquad \|\varphi_x\|_{\rho}^{\infty}, \|\varphi_y\|_{\rho}^{\infty} \le C\delta^3, \qquad \|\varphi_\lambda\|_{\rho}^{\infty}, \|\varphi_\Lambda\|_{\rho}^{\infty} \le C. \tag{II.A.15}$$

Moreover since, by Theorem II.3.1, $\lambda_h(u) \neq \pi$ for $u \in D_{\rho_1}^{\mathrm{u},\infty}$, we have that

$$|\varphi_\lambda(u)| = |\lambda_h(u)| < \pi, \quad |\varphi_\Lambda(u)| \le Ce^{-\rho\rho_1} \le C, \quad |(\varphi_x(u), \varphi_y(u))| \le C\delta^3 e^{-\rho\rho_1} \le C\delta^3$$

and, therefore, we can apply Lemma II.A.3 to (II.A.14). In the following computations, we use generously Lemma II.4.1 without mentioning it.

1. First, we consider $g^{\mathrm{sep}} = \partial_w H_1^{\mathrm{sep}}$. By (II.A.14), we have that

$$g^{\mathrm{sep}}(u, \varphi(u)) = -\frac{\partial_\Lambda M \circ \Phi(u)}{3\Lambda_h(u)} - \frac{\varphi_w(u)}{3\Lambda_h^2(u)}.$$

   Notice that, by Theorem II.3.1, $|\Lambda_h(u)| \ge C$ for $u \in D_{\rho_1}^{\mathrm{u},\infty}$. Then, $\|\Lambda_h^{-1}\|_{-\rho}^{\infty} \le C$. Therefore, by Lemma II.A.3 and estimates (II.A.15), we have that

$$\begin{aligned} \|g^{\mathrm{sep}}(\cdot, \varphi)\|_0^{\infty} \le & \, C\delta \left[\delta \|\varphi_\lambda\|_\rho^\infty + \delta \|\varphi_\Lambda\|_\rho^\infty + \|\varphi_x\|_\rho^\infty + \|\varphi_y\|_\rho^\infty\right] \\ & + C\|\varphi_w\|_{2\rho}^\infty \le C\delta^2. \end{aligned} \tag{II.A.16}$$

   To compute its derivative with respect to w, by (II.A.14), we have that

$$\partial_w g^{\mathrm{sep}}(u, \varphi(u)) = \frac{\partial_\Lambda^2 M \circ \Phi(u)}{9\Lambda_h^2(u)} - \frac{1}{3\Lambda_h^2(u)},$$

   and, by Lemma II.A.3 and estimates (II.A.15), $\|\partial_w g^{\mathrm{sep}}(\cdot, \varphi)\|_{-2\rho}^{\infty} \le C$. Following a similar procedure, we obtain $\|\partial_x g^{\mathrm{sep}}(\cdot, \varphi)\|_{-\rho}^{\infty} \le C\delta$ and $\|\partial_y g^{\mathrm{sep}}(\cdot, \varphi)\|_{-\rho}^{\infty} \le C\delta$.

2. Now, we obtain estimates for $f_1^{\mathrm{sep}} = -\partial_u H_1^{\mathrm{sep}}$. By (II.A.14), we have that

$$\begin{aligned} f_1^{\mathrm{sep}}(u, \varphi(u)) = & - \dot\lambda_h(u)\partial_\lambda M \circ \Phi(u) - \frac{\dot\Lambda_h(u)}{3\Lambda_h^3(u)}\varphi_w^2(u) \\ & - \left(\dot\Lambda_h(u) + \frac{\dot\Lambda_h(u)}{3\Lambda_h^2(u)}\varphi_w(u)\right)\partial_\Lambda M \circ \Phi(u). \end{aligned}$$

   Then, since $\dot\lambda_h, \dot\Lambda_h \in \mathcal{X}_\rho^\infty$, by Lemma II.A.3 and estimates (II.A.15), we have that $\|f_1^{\mathrm{sep}}(\cdot, \varphi)\|_{2\rho}^{\infty} \le C\delta^2$. To compute its derivative with respect to x, by (II.A.14),

$$\partial_x f_1^{\mathrm{sep}}(u, \varphi(u)) = - \dot\lambda_h(u)\partial_{x\lambda} M \circ \Phi(u) - \left(\dot\Lambda_h(u) + \frac{\dot\Lambda_h(u)}{3\Lambda_h^2(u)}\varphi_w(u)\right)\partial_{x\Lambda} M \circ \Phi(u)$$

   and, therefore, $\|\partial_x f_1^{\mathrm{sep}}(\cdot, \varphi)\|_{\rho}^{\infty} \le C\delta$. Similarly one can obtain $\|\partial_w f_1^{\mathrm{sep}}(\cdot, \varphi)\|_{0}^{\infty} \le C\delta^2$ and $\|\partial_y f_1^{\mathrm{sep}}(\cdot, \varphi)\|_{\rho}^{\infty} \le C\delta$.

3. Analogously to the previous estimates, we can obtain bounds for $f_2^{\mathrm{sep}} = i\partial_y H_1^{\mathrm{sep}}$ and $f_3^{\mathrm{sep}} = -i\partial_x H_1^{\mathrm{sep}}$. Then, for $j = 2, 3$, it can be seen that $\|f_j^{\mathrm{sep}}(\cdot, \varphi)\|_{\rho}^{\infty} \le C\delta$, and differentiating we obtain $\|\partial_w f_j^{\mathrm{sep}}(\cdot, \varphi)\|_{-\rho}^{\infty} \le C\delta$, $\|\partial_x f_j^{\mathrm{sep}}(\cdot, \varphi)\|_{0}^{\infty} \le C\delta$ and $\|\partial_y f_j^{\mathrm{sep}}(\cdot, \varphi)\|_{0}^{\infty} \le C\delta$.

Then, by the definition of $\mathcal{R}^{\rm sep}$ in (II.A.8) and the just obtained estimates, we complete the proof of the lemma. $\square$

II.A.2 Estimates in the outer domain

To obtain estimates of $\mathcal{R}^{\rm sep}$, we write $H_1^{\rm sep}$ in (II.3.9) (up to a constant) as

$$H_1^{\rm sep} = H_1 \circ \phi_{\rm eq} \circ \phi_{\rm sep} - \frac{w^2}{6\Lambda_h^2(u)} + \delta(x\mathfrak{L}_y + y\mathfrak{L}_x) + 3\delta^2\mathfrak{L}_\Lambda\left(\Lambda_h(u) - \frac{w}{3\Lambda_h(u)}\right),$$

(see (II.A.10) and (II.A.11)). Then, by the definition of H_1 in (II.2.7), we obtain

$$\begin{aligned} H_1^{\rm sep} = &(H_1^{\rm Poi} - V) \circ \phi_{\rm sc} \circ \phi_{\rm eq} \circ \phi_{\rm sep} + \frac{1}{\delta^4}F_{\rm p}\left(\delta^2\Lambda_h(u) - \frac{\delta^2 w}{3\Lambda_h(u)} + \delta^4\mathfrak{L}_\Lambda\right) \\ &- \frac{w^2}{6\Lambda_h^2(u)} + \delta(x\mathfrak{L}_y + y\mathfrak{L}_x) + 3\delta^2\mathfrak{L}_\Lambda\left(\Lambda_h(u) - \frac{w}{3\Lambda_h(u)}\right), \end{aligned}$$

where $H_1^{\rm Poi}$ is given in (II.A.1), the potential V in (II.2.4) and $F_{\rm p}$ in (II.2.8). The changes of coordinates $\phi_{\rm sc}$, $\phi_{\rm eq}$ and $\phi_{\rm sep}$ are given in (II.2.3), (II.3.3) and (II.3.4), respectively.

Considering $z = (w, x, y)$, we denote the composition of change of coordinates as

$$(\lambda, L, \eta, \xi) = \Theta(u, z) = (\phi_{\rm sc} \circ \phi_{\rm eq} \circ \phi_{\rm sep})(u, z). \tag{II.A.17}$$

Then, since $\mu = \delta^4$, the Hamiltonian $H_1^{\rm sep}$ can be split (up to a constant) as

$$H_1^{\rm sep} = M_P + M_S + M_R, \tag{II.A.18}$$

where

$$M_P(u, z; \delta) = -\left(\mathcal{P}[\delta^4 - 1] - \frac{1}{\sqrt{2 + 2\cos\lambda}}\right) \circ \Theta(u, z), \tag{II.A.19}$$

$$M_S(u, z; \delta) = \left(\frac{1}{\delta^4}\mathcal{P}[0] - \frac{1-\delta^4}{\delta^4}\mathcal{P}[\delta^4] - 1 + \cos\lambda\right) \circ \Theta(u, z), \tag{II.A.20}$$

$$\begin{aligned} M_R(u, z; \delta) = &-\frac{w^2}{6\Lambda_h^2(u)} + \delta^2\mathfrak{L}_\Lambda\left(3\Lambda_h(u) - \frac{w}{\Lambda_h(u)}\right) + \delta(x\mathfrak{L}_y + y\mathfrak{L}_x), \\ &+ \frac{1}{\delta^4}F_{\rm p}\left(\delta^2\Lambda_h(u) - \frac{\delta^2 w}{3\Lambda_h(u)} + \delta^4\mathfrak{L}_\Lambda\right), \end{aligned} \tag{II.A.21}$$

and $\mathcal{P}$ is the function given in (II.A.2).

To obtain estimates for the derivatives of M_P, M_S and M_R, we first analyze the change of coordinates Θ in (II.A.17). It can be expressed as

$$\Theta(u, z) = \Big(\pi + \Theta_\lambda(u), 1 + \Theta_L(u, w), \Theta_\eta(x), \Theta_\xi(y)\Big), \tag{II.A.22}$$

where

$$\begin{aligned} \Theta_\lambda(u) &= \lambda_h(u) - \pi, & \Theta_\eta(x) &= \delta x + \delta^4\mathfrak{L}_x(\delta), \\ \Theta_L(u, w) &= \delta^2\Lambda_h(u) - \frac{\delta^2 w}{3\Lambda_h(u)} + \delta^4\mathfrak{L}_\Lambda(\delta), & \Theta_\xi(x) &= \delta y + \delta^4\mathfrak{L}_y(\delta). \end{aligned}$$

Next lemma, which is a direct consequence of Theorem II.3.1, gives estimates for this change of coordinates.

Lemma II.A.4. *Fix $\varrho > 0$ and $\delta > 0$ small enough. Then, for $\varphi \in B(\varrho\delta^3) \subset \mathcal{X}_\times^{\mathrm{out}}$,*

$$\|\Theta_\lambda\|_{0,-\frac{2}{3}}^{\mathrm{out}} \leq C, \qquad \|\Theta_L(\cdot,\varphi)\|_{0,\frac{1}{3}}^{\mathrm{out}} \leq C\delta^2, \qquad \|\Theta_\eta(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq C\delta^4,$$
$$\left\|\Theta_\lambda^{-1}\right\|_{0,\frac{2}{3}}^{\mathrm{out}} \leq C, \qquad \|1+\Theta_L(\cdot,\varphi)\|_{0,0}^{\mathrm{out}} \leq C, \qquad \|\Theta_\xi(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq C\delta^4.$$

Moreover, its derivatives satisfy

$$\|\partial_u\Theta_\lambda\|_{0,\frac{1}{3}}^{\mathrm{out}} \leq C, \qquad \|\partial_u\Theta_L(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq C\delta^2, \qquad \|\partial_w\Theta_L(\cdot,\varphi)\|_{0,-\frac{1}{3}}^{\mathrm{out}} \leq C\delta^2,$$
$$\|\partial_{uw}\Theta_L(\cdot,\varphi)\|_{0,\frac{2}{3}}^{\mathrm{out}} \leq C\delta^2, \qquad \partial_x\Theta_\eta, \partial_y\Theta_\xi \equiv \delta, \qquad \partial_w^2\Theta_L, \partial_x^2\Theta_\eta, \partial_y^2\Theta_\xi \equiv 0.$$

In the next lemma we obtain estimates for the derivatives of M_P.

Lemma II.A.5. *Fix $\varrho > 0$, $\delta > 0$ small enough and $\kappa > 0$ big enough. Then, for $\varphi \in B(\varrho\delta^3)$ and $* = x, y$,*

$$\|\partial_u M_P(\cdot,\varphi)\|_{1,1}^{\mathrm{out}} \leq C\delta^2, \quad \|\partial_w M_P(\cdot,\varphi)\|_{1,-\frac{2}{3}}^{\mathrm{out}} \leq C\delta^2, \quad \|\partial_* M_P(\cdot,\varphi)\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq C\delta,$$
$$\|\partial_{uw} M_P(\cdot,\varphi)\|_{1,\frac{1}{3}}^{\mathrm{out}} \leq C\delta^2, \quad \|\partial_{u*} M_P(\cdot,\varphi)\|_{0,\frac{7}{3}}^{\mathrm{out}} \leq C\delta, \quad \left\|\partial_w^2 M_P(\cdot,\varphi)\right\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq C\delta^4,$$
$$\|\partial_{w*} M_P(\cdot,\varphi)\|_{0,\frac{5}{3}}^{\mathrm{out}} \leq C\delta^3, \quad \left\|\partial_*^2 M_P(\cdot,\varphi)\right\|_{0,2}^{\mathrm{out}} \leq C\delta^2, \quad \|\partial_{xy} M_P(\cdot,\varphi)\|_{0,2}^{\mathrm{out}} \leq C\delta^2.$$

Proof. We consider $\varphi \in B(\varrho\delta^3) \subset \mathcal{X}_\times^{\mathrm{out}}$ and we estimate the derivatives of $\mathcal{P}[\delta^4 - 1] \circ \Theta(u, \varphi(u))$. We first obtain bounds for $A[\delta^4 - 1]$ and $B[\delta^4 - 1]$ (see (II.A.4) and (II.A.5)). To simplify the notation, we define

$$\widetilde{A}(u) = A[\delta^4 - 1](\pi + \Theta_\lambda(u)), \qquad \widetilde{B}(u,z) = B[\delta^4 - 1] \circ \Theta(u,z). \tag{II.A.23}$$

In the following computations we use extensively the results in Lemma II.4.5 without mentioning them.

1. Estimates of $\widetilde{A}(u)$: Defining $\hat{\lambda} = \lambda - \pi$, by (II.A.4),

$$A[\delta^4 - 1](\hat{\lambda} + \pi) = 2(1 - \cos\hat{\lambda}) - 2\delta^4(1 - \cos\hat{\lambda}) + \delta^8.$$

Then, applying Lemma II.A.4,

$$\|\sin\Theta_\lambda\|_{0,-\frac{2}{3}}^{\mathrm{out}} \leq C\,\|\Theta_\lambda\|_{0,-\frac{2}{3}}^{\mathrm{out}} \leq C, \qquad \left\|(1-\cos\Theta_\lambda)^{-1}\right\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq C\left\|\Theta_\lambda^{-2}\right\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq C$$

and, as a result,

$$\begin{aligned} \|\widetilde{A}^{-1}\|_{0,\frac{4}{3}}^{\mathrm{out}} &\leq C\left\|(1-\cos\Theta_\lambda)^{-1}\right\|_{0,\frac{4}{3}}^{\mathrm{out}} \leq C, \\ \|\partial_u\widetilde{A}\|_{0,-\frac{1}{3}}^{\mathrm{out}} &\leq C\,\|\sin\Theta_\lambda\|_{0,-\frac{2}{3}}^{\mathrm{out}}\,\|\partial_u\Theta_\lambda\|_{0,\frac{1}{3}}^{\mathrm{out}} \leq C. \end{aligned} \tag{II.A.24}$$

2. Estimates of $\widetilde{B}(u,\varphi(u))$: Considering the auxiliary variables $(\hat{\lambda},\hat{\Lambda}) = (\lambda-\pi, L-1)$, we have that

$$\begin{aligned} B[\delta^4-1](\pi+\hat{\lambda},1+\hat{\Lambda},\eta,\xi) = & 4\hat{\Lambda}(1-\cos\hat{\lambda}+\delta^4\cos\hat{\lambda}) \\ & + \frac{\eta}{\sqrt{2}}(-3+2e^{-i\hat{\lambda}}+e^{-2i\hat{\lambda}}+\delta^4(3+e^{-2i\hat{\lambda}})) \\ & + \frac{\xi}{\sqrt{2}}(-3+2e^{i\hat{\lambda}}+e^{2i\hat{\lambda}}+\delta^4(3+e^{2i\hat{\lambda}})) \\ & + R[\delta^4-1](\pi+\hat{\lambda},1+\hat{\Lambda},\eta,\xi). \end{aligned} \tag{II.A.25}$$

Then, by the estimates in (II.A.6) and Lemma II.A.4,

$$\begin{aligned} \|\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{1,-2} \leq & C\left\|\Theta_L(\cdot,\varphi)\Theta_\lambda^2\right\|^{\mathrm{out}}_{0,-1} + \frac{C}{\delta^2}\left\|\Theta_\eta(\cdot,\varphi)\Theta_\lambda\right\|^{\mathrm{out}}_{0,\frac{2}{3}} \\ & + \frac{C}{\delta^2}\left\|\Theta_\xi(\cdot,\varphi)\Theta_\lambda\right\|^{\mathrm{out}}_{0,\frac{2}{3}} + \frac{C}{\delta^2}\left\|(\Theta_L,\Theta_\eta,\Theta_\xi)^2\right\|^{\mathrm{out}}_{0,\frac{2}{3}} \leq C\delta^2. \end{aligned} \tag{II.A.26}$$

Now, we look for estimates of the first derivatives of $\widetilde{B}(u,\varphi(u))$. By its definition in (II.A.23) and the expression of Θ in (II.A.22), we have that

$$\begin{aligned} \partial_u\widetilde{B} &= \left[\partial_\lambda B[\delta^4-1]\circ\Theta\right]\partial_u\Theta_\lambda + \left[\partial_L B[\delta^4-1]\circ\Theta\right]\partial_u\Theta_L, \\ \partial_w\widetilde{B} &= \left[\partial_L B[\delta^4-1]\circ\Theta\right]\partial_w\Theta_L, \\ \partial_x\widetilde{B} &= \left[\partial_\eta B[\delta^4-1]\circ\Theta\right]\partial_x\Theta_\eta, \qquad \partial_y\widetilde{B} = \left[\partial_\xi B[\delta^4-1]\circ\Theta\right]\partial_y\Theta_\xi. \end{aligned} \tag{II.A.27}$$

Differentiating (II.A.25) and applying Lemma II.A.4,

$$\begin{aligned} \left\|\partial_\lambda B[\delta^4-1]\circ\Theta(\cdot,\varphi)\right\|^{\mathrm{out}}_{1,-\frac{4}{3}} \leq & C\left\|\Theta_L(\cdot,\varphi)\Theta_\lambda\right\|^{\mathrm{out}}_{-\frac{1}{3}} + \frac{C}{\delta^2}\left\|\Theta_\eta(\cdot,\varphi)\right\|^{\mathrm{out}}_{0,\frac{4}{3}} \\ & + \frac{C}{\delta^2}\left\|\Theta_\xi(\cdot,\varphi)\right\|^{\mathrm{out}}_{0,\frac{4}{3}} + C\delta^2 \leq C\delta^2, \\ \left\|\partial_L B[\delta^4-1]\circ\Theta(\cdot,\varphi)\right\|^{\mathrm{out}}_{1,-\frac{7}{3}} \leq & C\left\|\Theta_\lambda^2\right\|^{\mathrm{out}}_{0,-\frac{4}{3}} + \frac{C}{\delta^2}\left\|\Theta_L(\cdot,\varphi)\right\|^{\mathrm{out}}_{0,\frac{1}{3}} + \frac{C}{\kappa} \leq C, \\ \left\|\partial_* B[\delta^4-1]\circ\Theta(\cdot,\varphi)\right\|^{\mathrm{out}}_{0,-\frac{2}{3}} \leq & C\left\|\Theta_\lambda\right\|^{\mathrm{out}}_{0,-\frac{2}{3}} + \frac{C}{\kappa} \leq C, \quad \text{for } *=\eta,\xi. \end{aligned}$$

Then, using also (II.A.27) and taking $*=x,y$,

$$\|\partial_u\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{1,-1} \leq C\delta^2, \quad \|\partial_w\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{1,-\frac{8}{3}} \leq C\delta^2, \quad \|\partial_*\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{0,-\frac{2}{3}} \leq C\delta. \tag{II.A.28}$$

Analogously, for the second derivatives, one can obtain the estimates

$$\begin{aligned} &\|\partial_{uw}\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{1,-\frac{5}{3}} \leq C\delta^2, \ \|\partial_w^2\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{0,\frac{2}{3}} \leq C\delta^4, \ \|\partial_{u*}\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{0,\frac{1}{3}} \leq C\delta, \\ &\|\partial_{w*}\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{0,-\frac{1}{3}} \leq C\delta^3, \ \|\partial_*^2\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{0,0} \leq C\delta^2, \ \|\partial_{xy}\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{0,0} \leq C\delta^2. \end{aligned} \tag{II.A.29}$$

Now, we are ready to obtain estimates for $M_P(u,\varphi(u))$ by using the series expansion (II.A.7). First, we check that it is convergent. Indeed, by (II.A.24) and (II.A.26), for

$u \in D^{\mathrm{sep,u}}_{\kappa,\beta}$ and taking κ big enough we have that

$$\left|\frac{\widetilde{B}(u,\varphi(u))}{\widetilde{A}(u)}\right| \leq \|\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{0,-\frac{4}{3}}\|\widetilde{A}^{-1}\|^{\mathrm{out}}_{0,\frac{4}{3}} \leq \frac{C}{\kappa^2\delta^2}\|\widetilde{B}(\cdot,\varphi)\|^{\mathrm{out}}_{1,-2} \leq \frac{C}{\kappa^2} \ll 1.$$

Therefore, by (II.A.3) and (II.A.19),

$$|M_P(u,\varphi(u))| \leq \left|\frac{1}{\sqrt{A[\delta^4-1](\lambda_h(u))}} - \frac{1}{\sqrt{2+2\cos\lambda_h(u)}}\right| + C\frac{|\widetilde{B}(u,\varphi(u))|}{|\widetilde{A}(u)|^{\frac{3}{2}}}. \tag{II.A.30}$$

Then, to estimate M_P and its derivatives, it only remains to analyze the u-derivative of its first term. Indeed, by the definition of $A[\delta^4-1]$ in (II.A.4).

$$\left\|\partial_u\left(\frac{1}{\sqrt{A[\delta^4-1](\lambda_h(u))}} - \frac{1}{\sqrt{2+2\cos\lambda_h(u)}}\right)\right\|^{\mathrm{out}}_{0,\frac{4}{3}} \leq C\delta^4. \tag{II.A.31}$$

Therefore, applying estimates (II.A.24), (II.A.26), (II.A.28), (II.A.29) and (II.A.31), to the derivatives of M_P and using (II.A.30), we obtain the statement of the lemma. □

Analogously to Lemma II.A.5, we obtain estimates for the first and second derivatives of M_S and M_R (see (II.A.20) and (II.A.21)).

Lemma II.A.6. *Fix $\varrho > 0$, $\delta > 0$ small enough and $\kappa > 0$ big enough. Then, for $\varphi \in B(\varrho\delta^3)$ and $* = x, y$, we have*

$$\begin{aligned}
&\|\partial_u M_S(\cdot,\varphi)\|^{\mathrm{out}}_{0,\frac{4}{3}} \leq C\delta^2, && \|\partial_w M_S(\cdot,\varphi)\|^{\mathrm{out}}_{0,-\frac{1}{3}} \leq C\delta^2, && \|\partial_* M_S(\cdot,\varphi)\|^{\mathrm{out}}_{0,0} \leq C\delta,\\
&\|\partial_{uw} M_S(\cdot,\varphi)\|^{\mathrm{out}}_{0,\frac{2}{3}} \leq C\delta^2, && \|\partial_{u*} M_S(\cdot,\varphi)\|^{\mathrm{out}}_{0,\frac{1}{3}} \leq C\delta, && \left\|\partial^2_w M_S(\cdot,\varphi)\right\|^{\mathrm{out}}_{0,-\frac{2}{3}} \leq C\delta^4,\\
&\|\partial_{w*} M_S(\cdot,\varphi)\|^{\mathrm{out}}_{0,-\frac{1}{3}} \leq C\delta^3, && \left\|\partial^2_* M_S(\cdot,\varphi)\right\|^{\mathrm{out}}_{0,0} \leq C\delta^2, && \|\partial_{xy} M_S(\cdot,\varphi)\|^{\mathrm{out}}_{0,0} \leq C\delta^2.
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_u M_R(\cdot,\varphi)\|^{\mathrm{out}}_{1,1} \leq C\delta^2, && \|\partial_w M_R(\cdot,\varphi)\|^{\mathrm{out}}_{1,-\frac{2}{3}} \leq C\delta^2, && \|\partial_* M_R(\cdot,\varphi)\|^{\mathrm{out}}_{0,0} \leq C\delta,\\
&\|\partial_{uw} M_R(\cdot,\varphi)\|^{\mathrm{out}}_{1,\frac{1}{3}} \leq C\delta^2, && \partial_{u*} M_R(\cdot,\varphi) \equiv 0, && \left\|\partial^2_w M_R(\cdot,\varphi)\right\|^{\mathrm{out}}_{0,-\frac{2}{3}} \leq C,\\
&\partial_{w*} M_R(\cdot,\varphi) \equiv 0, && \partial^2_* M_R(\cdot,\varphi) \equiv 0, && \partial_{xy} M_R(\cdot,\varphi) \equiv 0.
\end{aligned}$$

End of the proof of Lemma II.4.8. We start by estimating the first and second derivatives of $H_1^{\mathrm{sep}}(u,\varphi(u);\delta)$ in suitable norms. Recall that by (II.A.18), $H_1^{\mathrm{sep}} = M_P + M_S + M_R$. Therefore, taking $\varphi \in B(\varrho\delta^3) \subset \mathcal{X}^{\mathrm{out}}_{\times}$ and applying Lemmas II.A.5 and II.A.6:

1. For $g^{\mathrm{sep}} = \partial_w H_1^{\mathrm{sep}}$ one has

$$\begin{aligned}
\|g^{\mathrm{sep}}(\cdot,\varphi)\|^{\mathrm{out}}_{1,-\frac{2}{3}} \leq& \|\partial_w M_P(\cdot,\varphi)\|^{\mathrm{out}}_{1,-\frac{2}{3}} + C\,\|\partial_w M_S(\cdot,\varphi)\|^{\mathrm{out}}_{0,-\frac{1}{3}} + \|\partial_w M_R(\cdot,\varphi)\|^{\mathrm{out}}_{1,-\frac{2}{3}}\\
\leq& C\delta^2
\end{aligned}$$

and, in particular, for κ big enough

$$\|g^{\mathrm{sep}}(\cdot,\varphi)\|^{\mathrm{out}}_{0,0} \leq C\kappa^{-2} \ll 1. \tag{II.A.32}$$

Analogously, $\|\partial_w g^{\rm sep}(\cdot,\varphi)\|^{\rm out}_{0,-\frac{2}{3}} \leq C$ and $\|\partial_* g^{\rm sep}(\cdot,\varphi)\|^{\rm out}_{0,\frac{5}{3}} \leq C\delta^3$, for $* = x, y$.

2. For $f_1^{\rm sep} = -\partial_u H_1^{\rm sep}$, one has that

$$\|f_1^{\rm sep}(\cdot,\varphi)\|^{\rm out}_{1,1} \leq \|\partial_u M_P(\cdot,\varphi)\|^{\rm out}_{1,1} + C\,\|\partial_u M_S(\cdot,\varphi)\|^{\rm out}_{0,\frac{4}{3}} + \|\partial_u M_R(\cdot,\varphi)\|^{\rm out}_{1,1} \leq C\delta^2,$$

$$\|\partial_w f_1^{\rm sep}(\cdot,\varphi)\|^{\rm out}_{1,\frac{1}{3}} \leq C\delta^2 \text{ and } \|\partial_* f_1^{\rm sep}(\cdot,\varphi)\|^{\rm out}_{0,\frac{7}{3}} \leq C\delta, \quad \text{for } * = x, y.$$

3. For $f_2^{\rm sep} = i\partial_y H_1^{\rm sep}$ and $f_3^{\rm sep} = -i\partial_x H_1^{\rm sep}$, we can obtain the estimates

$$\begin{aligned} \|f_2(\cdot,\varphi)\|^{\rm out}_{0,\frac{4}{3}} &\leq \|\partial_y M_P(\cdot,\varphi)\|^{\rm out}_{0,\frac{4}{3}} + C\,\|\partial_y M_S(\cdot,\varphi) + \partial_y M_R(\cdot,\varphi)\|^{\rm out}_{0,0} \leq C\delta, \\ \|f_3(\cdot,\varphi)\|^{\rm out}_{0,\frac{4}{3}} &\leq \|\partial_x M_P(\cdot,\varphi)\|^{\rm out}_{0,\frac{4}{3}} + C\,\|\partial_x M_S(\cdot,\varphi) + \partial_x M_R(\cdot,\varphi)\|^{\rm out}_{0,0} \leq C\delta. \end{aligned} \tag{II.A.33}$$

Analogously, we have that $\|\partial_w f_j^{\rm sep}(\cdot,\varphi)\|^{\rm out}_{0,\frac{5}{3}} \leq C\delta^3$ and $\|\partial_* f_j^{\rm sep}(\cdot,\varphi)\|^{\rm out}_{0,2} \leq C\delta^2$, for $j = 2, 3$ and $* = x, y$.

Joining these estimates and taking κ big enough, we complete the proof of the lemma. □

Remark II.A.7. *Note that that $\widetilde{D}^{\rm u,out}_{\kappa_2,d_2,d_3} \subset D^{\rm sep,u}_{\kappa,\beta}$ and $\mathcal{Y}^{\rm out} \subset \mathcal{X}^{\rm out}_{0,0}$ (see* (II.4.9) *and* (II.4.5)*). Then, the proof of Lemma II.4.10 is a direct consequence of the estimates for $g^{\rm sep}$ and its derivatives in Item 1 above and the fact that, by* (II.3.11) *and* (II.4.8)*,*

$$R[\mathcal{U}](v) = \partial_w H_1^{\rm sep}\left(v + \mathcal{U}(v), z^{\rm u}(v + \mathcal{U}(v))\right) = g^{\rm sep}\left(v + \mathcal{U}(v), z^{\rm u}(v + \mathcal{U}(v))\right).$$

Appendix II.B

Estimates for the difference

In this appendix we prove Lemmas II.3.16 and II.3.19.

II.B.1 Proof of Lemma II.3.16

First, we prove the estimates for the operator Υ given in (II.3.27). For $\sigma \in [0,1]$, we define $z_\sigma = \sigma z^{\mathrm{u}} + (1-\sigma) z^{\mathrm{s}}$ with $z_\sigma = (w_\sigma, x_\sigma, y_\sigma)^T$. Then, by Theorem II.3.3, for $u \in D_{\kappa,d}$, we have that

$$|w_\sigma(u)| \leq \frac{C\delta^2}{|u^2+A^2|} + \frac{C\delta^4}{|u^2+A^2|^{\frac{8}{3}}}, \qquad |x_\sigma(u)|, |y_\sigma(u)| \leq \frac{C\delta^3}{|u^2+A^2|^{\frac{4}{3}}}. \tag{II.B.1}$$

Recalling that $H^{\mathrm{sep}} = w + \frac{xy}{\delta^2} + H_1^{\mathrm{sep}}$ (see (II.3.8)), one has

$$\begin{aligned}
|\Upsilon_1(u) - 1| &\leq \sup_{\sigma\in[0,1]} \left|\partial_w H_1^{\mathrm{sep}}(u, z_\sigma(u))\right|, \\
|\Upsilon_2(u)| &\leq \frac{|y_\sigma(u)|}{\delta^2} + \sup_{\sigma\in[0,1]} \left|\partial_x H_1^{\mathrm{sep}}(u, z_\sigma(u))\right|, \\
|\Upsilon_3(u)| &\leq \frac{|x_\sigma(u)|}{\delta^2} + \sup_{\sigma\in[0,1]} \left|\partial_y H_1^{\mathrm{sep}}(u, z_\sigma(u))\right|.
\end{aligned}$$

Then, by (II.B.1) and applying (II.A.32) and (II.A.33) in the proof of Lemma II.4.8 we obtain the estimates for Υ_1, Υ_2 and Υ_3.

We also need estimates for the matrix $\widetilde{\mathcal{B}}^{\mathrm{spl}}$ given in (II.3.26), which satisfies

$$|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{i,j}(u)| \leq \sup_{\sigma\in[0,1]} \left|(D_z \mathcal{R}^{\mathrm{sep}}[z_\sigma](u))_{i,j}\right|,$$

for $z_\sigma = \sigma z^{\mathrm{u}} + (1-\sigma) z^{\mathrm{s}}$. Then, by (II.B.1) and applying Lemma II.4.8, for $u \in D_{\kappa,d}$,

$$\begin{aligned}
\left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,1}(u)\right| &\leq \frac{C\delta}{|u^2+A^2|^{\frac{2}{3}}}, & \left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,1}(u)\right| &\leq \frac{C\delta}{|u^2+A^2|^{\frac{2}{3}}}, \\
\left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,2}(u)\right| &\leq \frac{C}{|u^2+A^2|^{\frac{1}{3}}} + \frac{C\delta^2}{|u^2+A^2|^{2}}, & \left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,2}(u)\right| &\leq \frac{C\delta^2}{|u^2+A^2|^{2}}, \\
\left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,3}(u)\right| &\leq \frac{C\delta^2}{|u^2+A^2|^{2}}, & \left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,3}(u)\right| &\leq \frac{C}{|u^2+A^2|^{\frac{1}{3}}} + \frac{C\delta^2}{|u^2+A^2|^{2}}.
\end{aligned} \tag{II.B.2}$$

Then, by (II.3.54) and taking κ big enough,

$$\left|\mathcal{B}^{\mathrm{spl}}_{1,1}(u)\right| \leq \frac{|\Upsilon_2(u)|}{|\Upsilon_1(u)|}\left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,1}(u)\right| \leq \frac{C\delta^2}{|u^2+A^2|^2},$$
$$\left|\mathcal{B}^{\mathrm{spl}}_{1,2}(u)\right| \leq \left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,3}(u)\right| + \frac{|\Upsilon_3(u)|}{|\Upsilon_1(u)|}\left|\widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,1}(u)\right| \leq \frac{C\delta^2}{|u^2+A^2|^2},$$

and analogous estimates hold for $\mathcal{B}^{\mathrm{spl}}_{2,1}$ and $\mathcal{B}^{\mathrm{spl}}_{2,2}$.

Finally, we compute estimates for $B_y(u)$ (see (II.3.45)) and $u \in D_{\kappa,d}$. The estimates for $B_x(u)$ can be computed analogously. Let us consider the integration path $\rho_t = u_* + (u-u_*)t$, for $t \in [0,1]$. Then

$$B_y(u) = \exp\left(\int_0^1 \widetilde{\mathcal{B}}^{\mathrm{spl}}_{2,2}(\rho_t)(u-u_*)dt\right).$$

Using the bounds in (II.B.2), we have that

$$|\log B_y(u)| \leq C\,|u-u_*|\left|\int_0^1 \frac{1}{\left|\rho_t^2+A^2\right|^{\frac{1}{3}}} + \frac{\delta^2}{\left|\rho_t^2+A^2\right|^2}dt\right| \leq C,$$

which implies $C^{-1} \leq |B_y(u)| \leq C$.

II.B.2 Proof of Lemma II.3.19

We only give an expression for $B_y(u_+)$. The result for $B_x(u_-)$ is analogous. First, we analyze $\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,3}$.

Lemma II.B.1. *For $\delta > 0$ small enough, $\kappa > 0$ large enough and $u \in D_{\kappa,d}$, the function $\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,3}$ defined in (II.3.26) is of the form*

$$\widetilde{\mathcal{B}}^{\mathrm{spl}}_{3,3}(u) = -\frac{4i}{3}\Lambda_h(u) + \delta^2 m(u;\delta),$$

for some function m satisfying

$$|m(u;\delta)| \leq \frac{C}{|u^2+A^2|^2}.$$

Proof. Let us define $z_\tau = \tau z^{\mathrm{u}} + (1-\tau)z^{\mathrm{s}}$ and recall that, for $u \in D_{\kappa,d}$,

$$\widetilde{\mathcal{B}}_{3,3}(u) = \int_0^1 \partial_y \mathcal{R}^{\mathrm{sep}}_3[z_\tau](u)d\tau. \tag{II.B.3}$$

Then, by the expression of $\mathcal{R}^{\mathrm{sep}}_3$ in (II.A.8), the estimates in the proof of Lemma II.4.8 (see Appendix II.A.2) and Theorem II.3.3, we have that

$$\partial_y \mathcal{R}^{\mathrm{sep}}_3[z_\tau](u) = \frac{i}{\delta^2} g^{\mathrm{sep}}(u, z_\tau(u)) + \delta^2 \widetilde{m}(u;\delta),$$

where $|\widetilde{m}(u;\delta)| \leq \frac{C}{|u^2+A^2|^2}$. In the following, to simplify notation, we denote by $\widetilde{m}(u;\delta)$ any function satisfying the previous estimate. Since $g^{\mathrm{sep}} = \partial_w H^{\mathrm{sep}}_1$, by (II.A.18) one has

$$g^{\text{sep}}(u, z_\tau(u)) = \partial_w M_P(u, z_\tau(u); \delta) + \partial_w M_S(u, z_\tau(u); \delta) + \partial_w M_R(u, z_\tau(u); \delta),$$

with M_P, M_S and M_R as given in (II.A.19), (II.A.20) and (II.A.21), respectively. Then, taking into account that $F_{\mathrm{p}}(s) = 2z^3 + \mathcal{O}(z^4)$ (see (I.2.14)) and following the proofs of Lemmas II.A.5 and II.A.6, it is a tedious but an easy computation to see that,

$$\begin{aligned} g^{\text{sep}}(u, z_\tau(u)) = &\, \partial_w M_P(u, 0, 0, 0; \delta) + \partial_w M_S(u, 0, 0, 0; \delta) \\ & - \frac{w_\tau(u)}{3\Lambda_h^2(u)} - \frac{\delta^2 \mathfrak{L}_\Lambda(\delta)}{\Lambda_h(u)} - 2\delta^2 \Lambda_h(u) + \delta^4 \widetilde{m}(u; \delta), \end{aligned}$$

and, by (II.B.3),

$$\begin{aligned} \widetilde{\mathcal{B}}_{3,3}(u) = &\, \frac{i}{\delta^2} \left[\partial_w M_P(u, 0, 0, 0; \delta) + \partial_w M_S(u, 0, 0, 0; \delta) \right] \\ & - i \frac{w^{\mathrm{u}}(u) + w^{\mathrm{s}}(u)}{6\delta^2 \Lambda_h^2(u)} - i \frac{\mathfrak{L}_\Lambda(\delta)}{\Lambda_h(u)} - 2i\Lambda_h(u) + \delta^2 \widetilde{m}(u; \delta). \end{aligned} \tag{II.B.4}$$

Next, we study the terms $w^{\mathrm{u,s}}(u)$. Since $H^{\text{sep}} = w + \frac{xy}{\delta^2} + M_P + M_S + M_R$ (see (II.3.8) and (II.A.18)), one can see that

$$H^{\text{sep}}(u, z^{\mathrm{u}}(u); \delta) = H^{\text{sep}}(u, z^{\mathrm{s}}(u); \delta) = \lim_{\operatorname{Re} u \to \pm\infty} H^{\text{sep}}(u, 0, 0, 0; \delta) = \delta^4 K(\delta),$$

with $|K(\delta)| \leq C$, for δ small enough. Then, by Theorem II.3.3, for $\diamond = \mathrm{u}, \mathrm{s}$,

$$|w^\diamond(u) + M_P(u, z^\diamond(u); \delta) + M_S(u, z^\diamond(u); \delta) + M_R(u, z^\diamond(u); \delta)| \leq \frac{C\delta^4}{|u^2 + A^2|^{\frac{8}{3}}}.$$

Again, following the proofs of Lemmas II.A.5 and II.A.6, one obtains

$$\left| w^\diamond(u) + M_P(u, 0, 0, 0; \delta) + M_S(u, 0, 0, 0; \delta) + \delta^2 \Lambda_h(u)(3\mathfrak{L}_\Lambda + 2\Lambda_h^2(u)) \right| \leq \frac{C\delta^4}{|u^2 + A^2|^{\frac{8}{3}}},$$

and, by (II.B.4),

$$\begin{aligned} \widetilde{\mathcal{B}}_{3,3}(u) = &\, -\frac{4i}{3}\Lambda_h(u) + \frac{i}{\delta^2} \left[\partial_w M_P(u, 0, 0, 0; \delta) + \frac{M_P(u, 0, 0, 0; \delta)}{3\Lambda_h^2(u)} \right] \\ & + \frac{i}{\delta^2} \left[\partial_w M_S(u, 0, 0, 0; \delta) + \frac{M_S(u, 0, 0, 0; \delta)}{3\Lambda_h^2(u)} \right] + \delta^2 \widetilde{m}(u; \delta). \end{aligned}$$

Therefore, it only remains to check that

$$\left| \partial_w M_{P,S}(u, 0, 0, 0; \delta) + \frac{M_{P,S}(u, 0, 0, 0; \delta)}{3\Lambda_h^2(u)} \right| \leq \frac{C\delta^4}{|u^2 + A^2|^2}.$$

Indeed, by (II.A.7) and the definition (II.A.19) of M_P, one has

$$M_P(u, w, 0, 0; \delta) = \mathcal{M}_P \left(u, \delta^2 \Lambda_h(u) - \frac{\delta^2 w}{3\Lambda_h(u)} + \delta^4 \mathfrak{L}_\Lambda(\delta) \right),$$

where $\mathcal{M}_P(u,\Lambda)$ is an analytic function for $u \in D_{\kappa,d}$ and $|\Lambda| \ll 1$. Moreover, following the proof of Lemma II.A.5, there exist a_0 and a_1 such that

$$|\mathcal{M}_P(u,\Lambda) - a_0(u;\delta) - a_1(u;\delta)\Lambda| \leq \frac{C\Lambda^2}{|u^2+A^2|^2},$$

with

$$|a_0(u;\delta)| \leq \frac{C\delta^4}{|u^2+A^2|^{\frac{2}{3}}}, \qquad |a_1(u;\delta)| \leq \frac{C}{|u^2+A^2|^{\frac{2}{3}}}.$$

Therefore,

$$\begin{aligned}\left|\partial_w M_P(u,0,0,0;\delta) + \frac{M_P(u,0,0,0;\delta)}{3\Lambda_h^2(u)}\right| &\leq \frac{|a_0(u)|}{3\Lambda_h^2(u)} + \frac{\delta^4\mathfrak{L}_\Lambda(\delta)\,|a_1(u)|}{3\Lambda_h^2(u)} + \frac{C\delta^4}{|u^2+A^2|^2}\\ &\leq \frac{C\delta^4}{|u^2+A^2|^2}.\end{aligned}$$

An analogous estimate holds for M_S. $\square$

End of the proof of Lemma II.3.19. By Lemma II.B.1 and recalling that $u_+ = iA - \kappa\delta^2$,

$$\begin{aligned}\log B_y(u_+) = \int_{u_*}^{u_+} \widetilde{\mathcal{B}}_{3,3}^{\mathrm{spl}}(u)du &= -\frac{4i}{3}\int_{u^*}^{iA}\Lambda_h(u)du\\ &\quad + \frac{4i}{3}\int_{u_+}^{iA}\Lambda_h(u)du + \delta^2\int_{u^*}^{u_+} m(u;\delta).\end{aligned} \tag{II.B.5}$$

Then, by Theorem II.3.1 and taking into account that $\kappa = \kappa_* |\log\delta|$ (see Lemma II.3.18), we obtain

$$\left|\log B_y(u_+) + \frac{4i}{3}\int_{u^*}^{iA}\Lambda_h(u)du\right| \leq \frac{C}{\kappa} + C\kappa^{\frac{2}{3}}\delta^{\frac{4}{3}} + \frac{C\delta^2}{|u_* - iA|} \leq \frac{C}{|\log\delta|}.$$

Finally, recalling that $\dot{\lambda}_h = -3\Lambda_h$, applying the change of coordinates $\lambda = \lambda_h(u)$ and using that $\lambda_h(iA) = \pi$, we have that

$$\frac{4i}{3}\int_{u^*}^{iA}\Lambda_h(u)du = -\frac{4i}{9}\int_{\lambda_h(u_*)}^{\pi} d\lambda = -\frac{4i}{9}\left(\pi - \lambda_h(u_*)\right).$$

Joining the last statements with (II.B.5), we obtain the statement of the lemma. $\square$

Part III

Homoclinic reconnections and chaotic dynamics

Abstract

The Restricted 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies, called the primaries. If the primaries perform circular motions and the massless body is coplanar with them, one has the so called Restricted Planar Circular 3-Body Problem (RPC3BP). In synodic coordinates, can be modeled by a two degrees of freedom Hamiltonian system with five critical points, $L_1, .., L_5$, called the Lagrange points.

The Lagrange point L_3 is a saddle-center critical point, which is collinear with the primaries and beyond the largest one. When the ratio between the masses of the primaries μ is small, the modulus of the hyperbolic eigenvalues are weaker, by a factor of order $\sqrt{\mu}$, than the elliptic ones. Due to the rapidly rotating dynamics, the 1-dimensional unstable and stable manifold of L_3 are exponentially close to each other with respect to $\sqrt{\mu}$. In Parts I and II the authors provided an asymptotic formula for the distance between these invariant manifolds for small ratios of the mass parameter. This result relies on a Stokes constant which we assume that is non zero. In this part and under this assumption, we study different chaotic and homoclinic phenomena occurring in a neighborhood of L_3 and its invariant manifolds.

The first result concerns the existence of 2-round homoclinic connections to L_3, i.e. homoclinic orbits that approach the critical point 2-times. More concretely, we prove the existence of 2-round homoclinic orbits for a specific sequence of mass ratio parameters. The second result studies the family of Lyapunov periodic orbits of L_3 with Hamiltonian energy level exponentially close to that of L_3. In particular, we show that there exists a set of periodic orbits whose unstable and stable manifolds intersect transversally. By the Smale-Birkhoff homoclinic theorem, this implies the existence of chaotic motions (Smale horseshoe) exponentially close to L_3 and its invariant manifolds. In addition, we also show the existence of a generic unfolding of a quadratic homoclinic tangency.

Chapter III.1

Introduction

The understanding of the planetary motions, and in particular of its stability or instability, has been a fundamental field of study in the last centuries. A significant model to approximate and understand the motions of different celestial bodies is the 3-Body Problem and its various simplified models. Indeed, since Poincaré (see [Poi90]), one of the cornerstone problems of dynamical systems has been to understand how the invariant manifolds of the different invariant objects (periodic orbits, invariant tori) structure the global dynamics of the 3-Body Problem.

The Restricted Circular 3-Body Problem models the motion of a body of negligible mass under the gravitational influence of two massive bodies, called the primaries, which perform a circular motion. If one also assumes that the massless body moves on the same plane as the primaries one has the Restricted Planar Circular 3-Body Problem (RPC3BP). Let us name the two primaries S (star) and P (planet) and normalize their masses so that $m_S = 1 - \mu$ and $m_P = \mu$, with $\mu \in \left(0, \frac{1}{2}\right]$. Choosing a suitable rotating coordinate system, the positions of the primaries can be fixed at $q_S = (\mu, 0)$ and $q_P = (\mu - 1, 0)$ and then, the position and momenta of the third body, $(q, p) \in \mathbb{R}^2 \times \mathbb{R}^2$, are governed by the Hamiltonian system associated to the two degrees of freedom Hamiltonian

$$h(q, p; \mu) = h_0(q, p) + \mu h_1(q; \mu) \tag{III.1.1}$$

where

$$\begin{aligned} h_0(q, p) &= \frac{||p||^2}{2} - q^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{1}{||q||}, \\ \mu h_1(q; \mu) &= \frac{1}{||q||} - \frac{(1-\mu)}{||q - (\mu, 0)||} - \frac{\mu}{||q - (\mu - 1, 0)||}. \end{aligned} \tag{III.1.2}$$

Note that this Hamiltonian is autonomous and the conservation of h corresponds to the preservation of the classical Jacobi constant.

For $0 < \mu \leq \frac{1}{2}$, it is a well known fact that (III.1.1) has five critical points, usually called Lagrange points. On an inertial (non-rotating) system of coordinates, the Lagrange points correspond to periodic dynamics with the same period as the two primaries, i.e on a 1:1 mean motion resonance. The three collinear Lagrange points, L_1, L_2 and L_3, are of center-saddle type whereas, for small μ, the triangular ones, L_4 and L_5, are of center-center type (see, for instance, [Sze67]).

Due to its interest in astrodynamics, a lot of attention has been paid to the study of the invariant manifolds associated to the points L_1 and L_2 (see [KLM+00; GLM+01; CGM+04]). The dynamics around the points L_4 and L_5 has also been heavily studied since, due to its stability, it is common to find objects orbiting around these points (for instance the Trojan and Greek Asteroids associated to the pair Sun-Jupiter, see [GDF+89; CG90; RG06]). Since the point L_3 is located "at the other

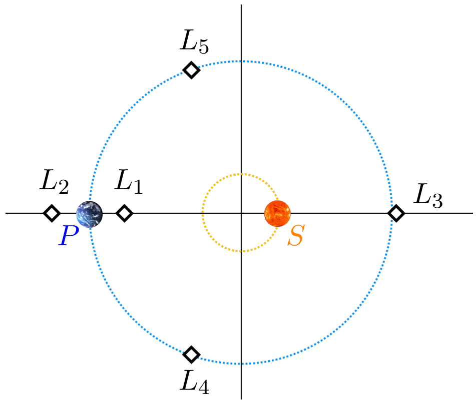


FIGURE III.1.1: Projection onto the q-plane of the Lagrange equilibrium points for the RPC3BP on rotating coordinates.

side" of the massive primary, it has received somewhat less attention. However, the associated invariant manifolds (more precisely its center-stable and center-unstable invariant manifolds) play an important role in the dynamics of the RPC3BP since they act as boundaries of *effective stability* of the stability domains around L_4 and L_5 (see [GJM+01; SST13]). The invariant manifolds of L_3 are also relevant in creating transfer orbits from the small primary to L_3 in the RPC3BP (see [HTL07; TFR+10]) or between primaries in the Bicircular 4-Body Problem (see [JN20; JN21]).

Over the past years, one of the main focus of the study of the dynamics "close" to L_3 and its invariant manifolds has been the so called "horseshoe-shaped orbits", first considered in [Bro11], which are quasi-periodic orbits that encompass the critical points L_4, L_3 and L_5. The interest on these types of orbits arise when modeling the motion of co-orbital satellites, the most famous being Saturn's satellites Janus and Epimetheus, and near Earth asteroids. Recently, in [NPR20], the authors have proved the existence of 2-dimensional elliptic invariant tori on which the trajectories mimic the motions followed by Janus and Epimetheus (see also [DM81a; DM81b; LO01; CH03; BM05; BO06; BFP13; CPY19]).

The aim of this work is different, rather than looking at stable motions "close to" L_3 as [NPR20], the goal is to look for homoclinic and chaotic phenomena arising from L_3 and its invariant manifolds. Being far from collision, these dynamics are rather similar to that of other mean motion resonances which play an important role in creating instabilities in the Solar system, see for instance [FGK+16]. On the contrary, even though the points L_1 and L_2 are saddle-center, the analysis of their associated dynamics is quite different since they are close to collision with the primary P for small values of μ.

The results presented here (see Section III.1.2) heavily rely on the previous works in Parts I and II. Their objective was to prove the breakdown of homoclinic connections to L_3 for small values of the mass ratio $\mu > 0$. In particular, to give an asymptotic formula for the distance between the 1-dimensional stable and unstable invariant manifolds (at a first crossing with a suitable transverse section). Before presenting the main results of this part, in the next section, we state the main result given in Parts I and II, (see Theorem III.1.1 below).

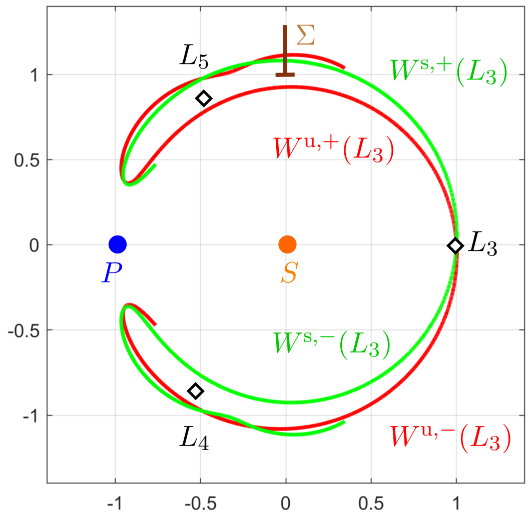


FIGURE III.1.2: Projection onto the q-plane of the unstable (red) and stable (green) manifolds of L_3, for $\mu = 0.0028$.

III.1.1 The distance between the invariant manifolds of L_3

The critical point L_3 (see [Sze67] for the details) satisfies that, as $\mu \to 0$,

$$(q_1, q_2, p_1, p_2) = (d_\mu, 0, 0, d_\mu), \qquad \text{with} \quad d_\mu = 1 + \frac{5}{12}\mu + \mathcal{O}(\mu^3). \tag{III.1.3}$$

Moreover, the eigenvalues of the Lagrange point L_3 satisfy that

$$\mathrm{Spec} = \left\{ \pm\sqrt{\mu}\,\rho_{\mathrm{eig}}(\mu), \pm i\,\omega_{\mathrm{eig}}(\mu) \right\}, \quad \text{with} \quad \left\{ \begin{array}{l} \rho_{\mathrm{eig}}(\mu) = \sqrt{\frac{21}{8}} + \mathcal{O}(\mu), \\ \omega_{\mathrm{eig}}(\mu) = 1 + \frac{7}{8}\mu + \mathcal{O}(\mu^2), \end{array} \right. \tag{III.1.4}$$

as $\mu \to 0$ (see [Sze67]). Therefore, L_3 has a one-dimensional unstable and stable manifolds, which we denote as $W^{\mathrm{u}}(L_3)$ and $W^{\mathrm{s}}(L_3)$. Notice that, due to the different size in the eigenvalues, the system posseses two time scales which translates to rapidly rotating dynamics coupled with a slow hyperbolic behavior around the critical point L_3.

The manifolds $W^{\mathrm{u}}(L_3)$ and $W^{\mathrm{s}}(L_3)$ have two branches each. One pair, which we denote by $W^{\mathrm{u},+}(L_3)$ and $W^{\mathrm{s},+}(L_3)$ circumvents L_5 whereas the other circumvents L_4 and it is denoted as $W^{\mathrm{u},-}(L_3)$ and $W^{\mathrm{s},-}(L_3)$, see Figure III.1.2. Notice that the Hamiltonian system associated to h in (III.1.1) is reversible with respect to the involution

$$\Psi(q, p) = (q_1, -q_2, -p_1, p_2). \tag{III.1.5}$$

Therefore, by (III.1.3), $L_3 = (d_\mu, 0, 0, d_\mu)$ belongs to the symmetry axis given by Ψ and the $+$ branches of the invariant manifolds of L_3 are symmetric to the $-$ ones. Thus, to compute the distance between the manifolds, one can restrict the study to the first ones, $W^{\mathrm{u},+}(L_3)$ and $W^{\mathrm{s},+}(L_3)$.

We perform the classical symplectic polar change of coordinates

$$q = r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad p = R \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} - \frac{G}{r} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}, \tag{III.1.6}$$

where R is the radial linear momentum and G is the angular momentum. We consider as well the 3-dimensional section

$$\Sigma = \left\{ (r,\theta,R,G) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}^2 \,:\, r > 1,\, \theta = \frac{\pi}{2} \right\} \tag{III.1.7}$$

and denote by $(r_*^{\mathrm{u}}, \frac{\pi}{2}, R_*^{\mathrm{u}}, G_*^{\mathrm{u}})$ and $(r_*^{\mathrm{s}}, \frac{\pi}{2}, R_*^{\mathrm{s}}, G_*^{\mathrm{s}})$ the first crossing of the invariant manifolds with this section (see Figure III.1.2). The next theorem measures the distance between these points for $0 < \mu \ll 1$. Its proof is given in Theorem II.1.1 in Part II.

Theorem III.1.1. (Distance between the unstable and stable manifolds of L_3). *There exists $\mu_0 > 0$ such that, for $\mu \in (0, \mu_0)$,*

$$\|(r_*^{\mathrm{u}}, R_*^{\mathrm{u}}, G_*^{\mathrm{u}}) - (r_*^{\mathrm{s}}, R_*^{\mathrm{s}}, G_*^{\mathrm{s}})\| = \sqrt[3]{4}\, \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log\mu|} \right) \right],$$

where:

- *The constant $A > 0$ is given by the real-valued integral*

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}}\, dx \approx 0.177744. \tag{III.1.8}$$

- *The constant $\Theta \in \mathbb{C}$ is the Stokes constant associated to the inner equation analyzed in Theorem I.2.7 in Part I.*

For the limit problem h in (III.1.1) with $\mu = 0$, the five Lagrange point disappear into the circle of (degenerate) critical points $\|q\| = 1$ and $p = (p_1, p_2) = (-q_2, q_1)$. As a consequence, the one-dimensional invariant manifolds of L_3 disappear when $\mu = 0$ too. For this reason, in Section I.2.1 in Part I, it was performed a singular change of coordinates to obtain a new first order Hamiltonian with a center-saddle equilibrium point (close to L_3). We reintroduce this change of coordinates in Section III.2.1 and, in Theorem III.2.5 we reproduce Theorem III.1.1 in this set of variables.

Notice that, in Theorem III.1.1, due to the rapidly rotating dynamics of the system (see (III.1.4)), the distance between the stable and unstable manifolds of L_3 is exponentially small with respect to $\sqrt{\mu}$. Due to the symmetry in (III.1.5), an analogous result holds for the opposite branches. In addition, a more general result can be proved for sections $\Sigma(\theta_*) = \{r > 1,\, \theta = \theta_*\}$, (see Theorem III.2.5 below or Part II for more details).

By numerical computation one obtains $|\Theta| \approx 1.63$ (see Remark I.2.8 in Part I. The corresponding code can be found at [Gir22]). We expect that, by means of a computer assisted proof, it would be possible to obtain rigorous estimates and verify $\Theta \neq 0$, see [BCG+22]. As a result, we consider the following ansatz.

Ansatz III.1.2. *The constant Θ as given in Theorem III.1.1 satisfies that $\Theta \neq 0$.*

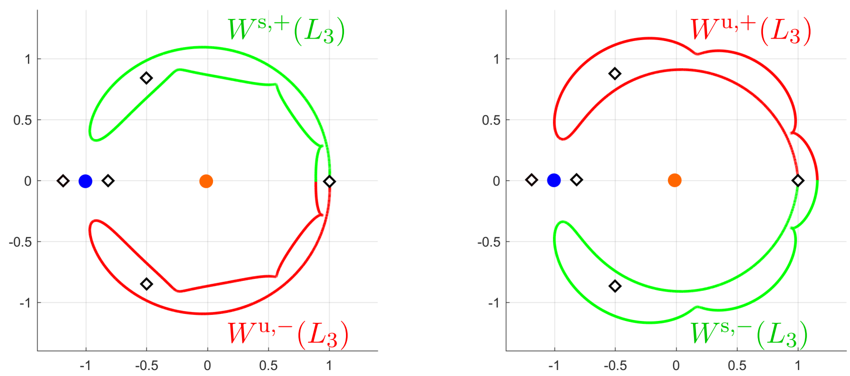


FIGURE III.1.3: Projection onto the q-plane for examples of 2-round homoclinic connection to L_3. (Left) $\mu = 0.012144$, (right) $\mu = 0.004192$.

III.1.2 Main results

Theorem III.1.1 and Ansatz III.1.2 imply that the invariant manifolds of L_3 do not meet the first time they meet the section Σ. Certainly this do not prevent the existence of multi-round homoclinic orbits. The first result of this part is the existence of such homoclinic connections for certain values of the mass parameter μ.

To state it, we first classify the types of homoclinic orbits by how many "rounds" they take before returning to L_3. In particular, we say that an homoclinic connection to L_3 is *k-round* if, on a δ-neighborhood of this critical point, the closure of the homoclinic orbit has k connected components, (see Figure III.1.3 for examples of 2-round connections).

According to this definition, Theorem III.1.1 and Ansatz III.1.2 imply the following.

Corollary III.1.3. (1-round homoclinic connections). *Assume Ansatz III.1.2. There exists $\mu_0 > 0$ such that, for $\mu \in (0, \mu_0)$, there do not exist 1-round homoclinic connections to L_3.*

E. Barrabés, J.M. Mondelo and M. Ollé in [BMO09] analyze the existence of multi-round homoclinic connections to L_3 in the RPC3BP. In particular, they conjectured the existence of 2-round homoclinic orbits for a sequence of mass ratios $\{\mu_n\}_{n\in\mathbb{N}}$ satisfying $\mu_n \to 0$ as $n \to \infty$ and supported their claim with numeric computations. In the following result, we prove this conjecture.

Theorem III.1.4. (2-round homoclinic connections). *Assume Ansatz III.1.2 and consider $\rho_{\mathrm{eig}}(\mu)$ given in (III.1.4) and $A > 0$ given in Theorem III.1.1. Then, there exists a sequence $\{\mu_n\}_{n\geq N_0}$ with N_0 big enough, of the form*

$$\mu_n = \frac{A}{n\pi\rho_{\mathrm{eig}}(0)}\left(1 + \mathcal{O}\left(\frac{1}{\log n}\right)\right), \qquad \text{for } n \gg 1,$$

such that, the Hamiltonian system (III.1.1) has a 2-round homoclinic connection to the equilibrium point L_3 between $W^{u,+}(L_3)$ and $W^{s,-}(L_3)$.

This theorem is a consequence of Theorem III.2.8, which is stated in a different system of coordinates and it is proved in Chapter III.3. Using the same tools, one can obtain an analogous result for the homoclinic connections between $W^{u,-}(L_3)$ and $W^{s,+}(L_3)$.

Remark III.1.5. (Multi-round homoclinic connections). *In [BMO09], the authors also conjectured the existence of k-round homoclinic connections for $k > 2$ for different sequences of the mass parameter μ. We believe that our strategy can be applied also for proving the existence of k-round homoclinic symmetric connections.*

Next we study the existence of chaotic phenomena associated to L_3 and its invariant manifolds. In particular, we prove the existence of a Smale horseshoe close to the invariant manifolds of L_3 by means of the Smale-Birkhoff homoclinic theorem. This classical result (see [Sma67; KH95]) states that, by proving the existence of transverse homoclinic orbits to periodic orbits (for flows) one can construct symbolic dynamics.

The Lyapunov Center Theorem (see for instance [MO17]) ensures the existence of a family of periodic orbits emanating from a saddle-center which, close to the equilibrium point, are hyperbolic. Let us denote by Π_3 the Lyapunov family of hyperbolic periodic orbits of L_3. This family can be parametrized by the energy level given by the Hamiltonian h in (III.1.1).

Proposition III.1.6. (Lyapunov periodic orbits to L_3). *There exist $\mu_0, \varrho_0 > 0$ small enough such that, for $\mu \in (0, \mu_0)$, system (III.1.1) has a family of hyperbolic periodic orbits*

$$\Pi_3 = \left\{ P_{3,\varrho} \text{ periodic orbit } : \ h(P_{3,\varrho}) = \varrho^2 + h(L_3),\ \varrho \in [0, \varrho_0] \right\}.$$

In Proposition III.2.4 we state this result in a different set of coordinates and provide estimates for the periodic orbits. Its proof can be found in Appendix III.A.

We denote by $W^{\mathrm{u}}(P_{3,\varrho})$ and $W^{\mathrm{s}}(P_{3,\varrho})$ the 2-dimensional unstable and stable invariant manifolds of the Lyapunov periodic orbit $P_{3,\varrho}$. Analogously to the L_3 case, the invariant manifolds have two branches each which we denote by $W^{\mathrm{u},+}(P_{3,\varrho})$ and $W^{\mathrm{s},+}(P_{3,\varrho})$ the ones that circumvent L_5 and, by $W^{\mathrm{u},-}(P_{3,\varrho})$ and $W^{\mathrm{s},-}(P_{3,\varrho})$, the ones that surround L_4 (see Figure III.1.4). By the Smale-Birkhoff homoclinic theorem, proving the existence of transverse intersections between $W^{\mathrm{u},+}(P_{3,\varrho})$ and $W^{\mathrm{s},+}(P_{3,\varrho})$ implies the existence of chaotic motions on a neighborhood of L_3 and its invariant manifolds. More specifically, we prove the following result.

Theorem III.1.7. (Chaotic motions). *Let the constants $A > 0$ and Θ be as given in Theorem III.1.1 and ϱ_0 in Proposition III.1.6. Assume Ansatz III.1.2. Then, there exist $\mu_0 > 0$ and functions $\varrho_{\min}, \varrho_{\max} : (0, \mu_0) \to [0, \varrho_0]$ of the form*

$$\varrho_{\min}(\mu) = \frac{\sqrt[6]{2}}{2} |\Theta| \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[1 + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right],$$
$$\varrho_{\max}(\mu) = \frac{\sqrt[6]{2}}{2} |\Theta| \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left[2 + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right],$$

such that, for $\mu \in (0, \mu_0)$ and $\varrho \in (\varrho_{\min}(\mu), \varrho_{\max}(\mu)]$, the invariant manifolds $W^{\mathrm{u},+}(P_{3,\varrho})$ and $W^{\mathrm{s},+}(P_{3,\varrho})$ intersect transversally.

Consider the section $\widehat{\Sigma}_\varrho = \Sigma \cap \left\{ h = \varrho^2 + h(L_3) \right\}$ with Σ as given in (III.1.7) and the induced Poincaré map $\mathcal{P} : \widehat{\Sigma}_\varrho \to \widehat{\Sigma}_\varrho$. Then, there exists $M > 0$ such that $\mathcal{P}^M$ has an invariant set $\mathcal{X}$, homeomorphic to $\mathbb{Z}^{\mathbb{N}}$, such that $\mathcal{P}^M|_{\mathcal{X}}$ is topologically conjugated to the shift.

Notice that, due to the symmetry in (III.1.5), an analogous result holds for the transverse intersections of branches $W^{\mathrm{u},-}(P_{3,\varrho})$ and $W^{\mathrm{s},-}(P_{3,\varrho})$. To prove Theorem III.1.7, we use the asymptotic formula for the distance of the invariant manifolds of L_3 obtained in Theorem III.1.1. Since $W^{\mathrm{u}}(L_3)$ and $W^{\mathrm{s}}(L_3)$ are exponentially close

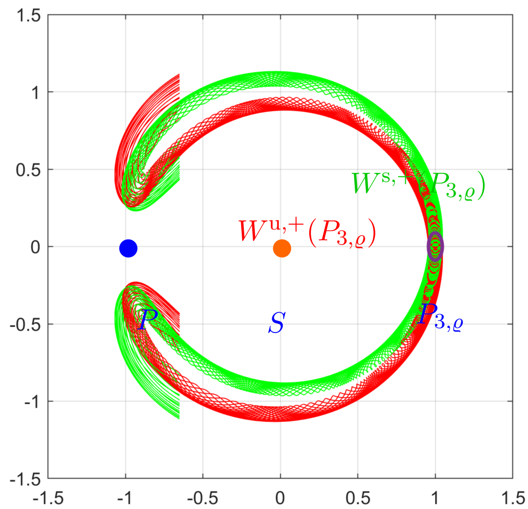


FIGURE III.1.4: Projection onto the q-plane of the unstable (red) and stable (green) manifolds of the Lyapunov periodic orbit $P_{3,\varrho}$ (blue), for $\mu = 0.003$.

to each other with respect to $\sqrt{\mu}$, the energy levels where chaotic motions are found is also exponentially close to the energy level of L_3. In addition, by restricting μ one can take $\varrho_{\max}(\mu)$ bigger, (see Theorem III.2.8).

Theorem III.1.8. (Homoclinic tangencies). *Assume Ansatz III.1.2 and denote by f_ϱ the flow of the Hamiltonian system given in* (III.1.1) *restricted to the energy level $h = \varrho^2 + h(L_3)$. Let $\varrho_0, \mu_0 > 0$ and $\varrho_{\min}(\mu) : (0, \mu_0) \to [0, \varrho_0]$ be as given in Theorem III.1.7. Then, for a fixed μ and ϱ close to $\varrho_{\min}(\mu)$, the flow f_ϱ unfolds generically an homoclinic quadratic tangency between $W^{u,+}(P_{3,\varrho_{\min}(\mu)})$ and $W^{s,+}(P_{3,\varrho_{\min}(\mu)})$.*

Both Theorems III.1.7 and III.1.8 are stated in the previously mentioned set of coordinates in Theorem III.2.8 and proved in Chapter III.4.

Remark III.1.9. *We use the definition of generic unfolding given in [Dua08] for area preserving diffeomorphisms (see Theorem III.2.8 for more details). In particular, Theorem III.1.8 should lead to prove the existence of a Newhouse domain.*

III.1.3 State of the art

A fundamental problem in dynamical systems is to prove whether a given model has chaotic dynamics. For many physically relevant models this is usually remarkably difficult. This is the case of many Celestial Mechanics models, where most of the known chaotic motions have been found in nearly integrable regimes where there is an unperturbed problem which already presents some form of "hyperbolicity". This is the case in the vicinity of collision orbits (see for example [Moe89; BM06; Bol06; Moe07]) or close to parabolic orbits (which allows to construct chaotic/oscillatory motions), see [Sit60; Ale76; Lli80; Mos01; GMS16; GSM+17; GPS+21]. There are also several results in regimes far from integrable which rely on computer assisted proofs [Ari02; WZ03; Cap12; GZ19].

The problem tackled in this work is radically different. Indeed, if one takes the limit $\mu \to 0$ in (III.1.1) one obtains the classical integrable Kepler problem in the elliptic regime, where no hyperbolicity is present. Instead, the (weak) hyperbolicity is created by the $\mathcal{O}(\mu)$ perturbation. The bifurcation scenario we are dealing with is the so called $0^2 i\omega$ resonance or Hamiltonian Hopf-Zero bifurcation. Indeed, for $\mu > 0$ the Hamitlonian system given by h in (III.1.1) has a saddle-center equilibrium point at L_3. However, for $\mu = 0$, the equilibrium point degenerates and the spectrum of its linear part consists in a pair of purely imaginary and a double 0 eigenvalues, (see (III.1.4)).

Most of the studies in homoclinic phenomena around a saddle-center equilibrium are focused on the non-degenerate case, see [Ler91; MHO92; Rag97a; Rag97b; BRS03]. However, for the resonance $0^2 i\omega$ cases, to the best of authors knowledge, the results are more rare. In [GG10], the authors study this singularity combining numerical and analytic techniques. The reversible case is considered in [Lom99; Lom00] where the author proves the existence of transverse homoclinic connections for every periodic orbit exponentially close to the origin, except the origin itself. In [JBL16], the authors show the existence of homoclinic connections with several loops for every periodic orbit close to the equilibrium point.

On the contrary, the work here presented shows the existence of homoclinic connections for both the equilibrium point and periodic orbits (exponentially) close to the equilibrium point. In the case of the (non-Hamiltonian) Hopf-zero singularity, we remark the similar work [BIS20]. Also, in [GGS+21], the authors use similar techniques to analyze breather solutions for the nonlinear Klein-Gordon partial differential equation.

A first step towards proving Arnold diffusion? Consider the 3-Body Problem in the planetary regime, that is one massive body (the Sun) and two small bodies (the planets) performing approximate ellipses (including the "restricted case" when one of the planets has mass zero). A fundamental problem is to assert whether such configuration is stable (i.e. is the Solar system stable?). Thanks to Arnold-Herman-Féjoz KAM Theorem, many of such configurations are stable, see [Arn63; Féj04]. However, it is widely expected that there should be strong instabilities created by Arnold diffusion mechanisms (as conjectured by Arnold in [Arn64]). In particular, it is widely believed that one of the main sources of such instabilities dynamics are the mean motion resonances, where the period of the two planets is resonant (i.e. rationally dependent) [FGK+16].

The RPC3BP has too low dimension (2 degrees of freedom) to possess Arnold diffusion. However, since it can be seen as a first order for higher dimensional models, the analysis performed in this part can be seen as a humble first step towards constructing Arnold diffusion along the $1:1$ mean motion resonance. In this resonance, the RPC3BP has a normally hyperbolic invariant manifold given by the center manifold of the Lagrange point L_3. This normally hyperbolic invariant manifold is foliated by the classical Lyapunov periodic orbits. One should expect that the techniques developed in this work would allow to prove that the invariant manifolds of these periodic orbits intersect transversally for a non exponentially small range of energy levels. Still, this is a much harder problem than the one considered in this work and the technicalities involved would be considerable.

This transversality would not lead to Arnold diffusion due to the low dimension of the RPC3BP. However, it is a first order for other instances of the 3-body problem where one should be able to construct orbits with a drastic change in angular momentum (or inclination in the spatial setting). For example:

(i) The Restricted Spatial Circular 3-Body Problem with small $\mu > 0$ which has three degrees of freedom.

(ii) The Restricted Planar Elliptic 3-Body Problem with small $\mu > 0$ and eccentricity of the primaries $e_0 > 0$, which has two and a half degrees of freedom.

(iii) The "full" planar 3-Body Problem (i.e. all three masses positive, two small) which has three degrees of freedom (after the symplectic reduction by the classical first integrals).

This part is organized as follows. Chapter III.2 is devoted to reformulate the main results of the part (Theorems III.1.1, III.1.4, III.1.7 and III.1.8) by performing the singular change of coordinates introduced in Section I.2.1 in Part I. Finally, Chapter III.3 to III.5 are devoted to the proofs of the results in Chapter III.2.

Chapter III.2

Reformulation of the problem

Recall that, for the unperturbed problem h in (III.1.1) with $\mu = 0$, the five Lagrange point disappear into a circle of degenerate critical points. For this reason, in Section I.2.1 in Part I, we introduced a singular change of coordinates to obtain a new first order Hamiltonian which has a center saddle equilibrium point (close to L_3) with stable and unstable manifolds that coincide along a separatrix.

First, in Section III.2.1, we introduce the main features of this change of coordinates, its relation to L_3 and the Lyapunov family of periodic orbits surrounding L_3. Then, in Section III.2.2, we state Theorems III.2.5, III.2.8 and III.2.10, which are reformulations of Theorems III.1.1, III.1.4 and together Theorems III.1.7 and III.1.8 in the new set of coordinates.

III.2.1 Change of coordinates and Lyapunov periodic orbits

Applying a suitable singular change of coordinates, the Hamiltonian h can be written as a perturbation of a pendulum-like Hamiltonian weakly coupled with a fast oscillator. We summarize the most important properties of this set of coordinates, which was studied in detail in Section I.2.1.

The Hamiltonian h expressed in the classical (rotating) Poincaré coordinates, $\phi^{\mathrm{Poi}} : (\lambda, L, \eta, \xi) \to (q, p)$, defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge dL + i\, d\eta \wedge d\xi$ and Hamiltonian

$$H^{\mathrm{Poi}} = H_0^{\mathrm{Poi}} + \mu H_1^{\mathrm{Poi}}, \tag{III.2.1}$$

with

$$H_0^{\mathrm{Poi}}(L, \eta, \xi) = -\frac{1}{2L^2} - L + \eta\xi \qquad \text{and} \qquad H_1^{\mathrm{Poi}} = h_1 \circ \phi_{\mathrm{Poi}}. \tag{III.2.2}$$

Moreover, the critical point L_3 satisfies

$$\lambda = 0, \qquad (L, \eta, \xi) = (1, 0, 0) + \mathcal{O}(\mu), \tag{III.2.3}$$

and the linearization of the vector field at this point has, at first order, an uncoupled nilpotent and center blocks. Since ϕ^{Poi} is an implicit change of coordinates, there is no explicit expression for H_1^{Poi}. However, since H_1^{Poi} is analytic for $|(L, \eta, \xi)| \ll 1$, it is possible to obtain series expansion in powers of $(L - 1, \eta, \xi)$ (see Lemma I.4.1).

In addition, since the original Hamiltonian h is reversible with respect to the involution Ψ in (III.1.5), the Hamiltonian H^{Poi} is reversible with respect to the involution

$$\Phi_{\mathrm{Poi}}(\lambda, L, \eta, \xi) = (-\lambda, L, \xi, \eta). \tag{III.2.4}$$

To capture the slow-fast dynamics of the system, renaming

$$\delta = \mu^{\frac{1}{4}}, \tag{III.2.5}$$

we perform the singular symplectic scaling

$$\phi_\delta : (\lambda, \Lambda, x, y) \mapsto (\lambda, L, \eta, \xi), \qquad L = 1 + \delta^2\Lambda, \quad \eta = \delta x, \quad \xi = \delta y, \tag{III.2.6}$$

and the time reparametrization $t = \delta^{-2}t'$. Defining the potential

$$V(\lambda) = H_1^{\mathrm{Poi}}(\lambda, 1, 0, 0; 0) = 1 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}}, \tag{III.2.7}$$

the Hamiltonian system associated to H^{Poi}, expressed in scaled coordinates, defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge d\Lambda + i dx \wedge dy$ and the Hamiltonian

$$H = H_{\mathrm{p}} + H_{\mathrm{osc}} + H_1, \tag{III.2.8}$$

where

$$H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + V(\lambda), \qquad H_{\mathrm{osc}}(x, y; \delta) = \frac{xy}{\delta^2}, \tag{III.2.9}$$

$$H_1(\lambda, \Lambda, x, y; \delta) = H_1^{\mathrm{Poi}}(\lambda, 1 + \delta^2\Lambda, \delta x, \delta y; \delta^4) - V(\lambda) + \frac{1}{\delta^4} F_{\mathrm{p}}(\delta^2\Lambda), \tag{III.2.10}$$

and

$$F_{\mathrm{p}}(z) = \left(-\frac{1}{2(1+z)^2} - (1+z) \right) + \frac{3}{2} + \frac{3}{2}z^2 = \mathcal{O}(z^3). \tag{III.2.11}$$

We introduce a suitable neighborhood where the coordinates (λ, Λ, x, y) are defined. For $c_0 > 0$ we denote

$$U_{\mathbb{R}}(c_0, c_1) = \left\{ (\lambda, \Lambda, x, \overline{x}) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} \times \mathbb{C}^2 : |\pi - \lambda| > c_0, |(\Lambda, x)| < c_1 \right\}. \tag{III.2.12}$$

Notice that, the domain is still 4-dimensional. For technical reasons, we consider some of the objects of the system in an analytical extension of the domain $U_{\mathbb{R}}$. In particular we employ the domain

$$\begin{aligned} U_{\mathbb{C}}(c_0, c_1) = \big\{ &(\lambda, \Lambda, x, y) \in \mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}^3 : \\ &|\pi - \operatorname{Re}\lambda| > c_0, |(\operatorname{Im}\lambda, \Lambda, x, y)| < c_1 \big\}. \end{aligned} \tag{III.2.13}$$

The next proposition states some properties of the Hamiltonian H.

Proposition III.2.1. *Fix $c_0, c_1 > 0$. Then, there exists $\delta_0 = \delta_0(c_0, c_1) > 0$ such that, for $\delta \in (0, \delta_0)$, one has that*

- *The Hamiltonian H as given in (III.2.8) is real-analytic in the sense of $\overline{H(\lambda, \Lambda, x, y; \delta)} = H(\overline{\lambda}, \overline{\Lambda}, y, x; \overline{\delta})$ in the domain $U_{\mathbb{C}}(c_0, c_1)$.*
- *There exists $b_0 > 0$ independent of δ such that, for $(\lambda, \Lambda, x, y) \in U_{\mathbb{C}}(c_0, c_1)$, the second derivatives of the Hamiltonian H_1 given in (III.2.10) satisfy*

$$\begin{aligned} &\left|\partial_\lambda^2 H_1\right|, \left|\partial_{\lambda x} H_1\right|, \left|\partial_{\lambda y} H_1\right| \leq b_0\delta, && \left|\partial_{\lambda\Lambda} H_1\right|, \left|\partial_\Lambda^2 H_1\right| \leq b_0\delta^2, \\ &\left|\partial_x^2 H_1\right|, \left|\partial_{xy} H_1\right|, \left|\partial_y^2 H_1\right| \leq b_0\delta^2, && \left|\partial_{\Lambda x} H_1\right| \left|\partial_{\Lambda y} H_1\right| \leq b_0\delta^3. \end{aligned}$$

Moreover[1],

$$|\partial_{\alpha_1,\alpha_2,\alpha_3} H_1| \leq b_0\delta, \qquad \textit{with} \qquad \alpha_1,\alpha_2,\alpha_3 \in \{\lambda,\Lambda,x,y\}\,.$$

Proof. The first statement follows from Theorem I.2.1 in Part I. The second statement is a consequence of Lemma II.A.3 in Part II. □

Remark III.2.2. *Consider $M \subseteq \mathbb{C}^k$ a symmetric subset with respect to the real line. We say that a function $\zeta = (\zeta_\lambda, \zeta_\Lambda, \zeta_x, \zeta_y) : M \to U_{\mathbb{C}}(c_0, c_1)$ is real-analytic if, for $m \in M$, $\zeta_\lambda(\overline{m}) = \overline{\zeta_\lambda(m)}$, $\zeta_\Lambda(\overline{m}) = \overline{\zeta_\Lambda(m)}$, $\zeta_x(\overline{m}) = \overline{\zeta_y(m)}$ and $\zeta_y(\overline{m}) = \overline{\zeta_x(m)}$. Notice that, as a consequence, $\zeta(m) \in U_{\mathbb{R}}(c_0)$, for $m \in M \cap \mathbb{R}^k$.*

Notice that, by (III.2.4), the Hamiltonian H is reversible with respect to the involution

$$\Phi(\lambda,\Lambda,x,y) = (-\lambda,\Lambda,y,x). \tag{III.2.14}$$

with corresponding symmetry axis

$$\mathcal{S} = \{\lambda = 0,\, x = y\}\,. \tag{III.2.15}$$

In the next proposition, proven in Theorem I.2.1 in Part I, we obtain an expression and suitable estimates for the equilibrium point L_3.

Proposition III.2.3. *There exist $\delta_0 > 0$ and $b_1 > 0$ such that, for $\delta \in (0,\delta_0)$, the critical point L_3 expressed in coordinates (λ,Λ,x,y) is of the form*

$$\mathfrak{L}(\delta) = \left(0, \delta^2\mathfrak{L}_\Lambda(\delta), \delta^3\mathfrak{L}_x(\delta), \delta^3\mathfrak{L}_y(\delta)\right)^T \in \mathcal{S}, \tag{III.2.16}$$

with $|\mathfrak{L}_\Lambda(\delta)|, |\mathfrak{L}_x(\delta)|, |\mathfrak{L}_y(\delta)| \leq b_1$ and $\mathcal{S}$ as given in (III.2.15).

The linearization of $\mathfrak{L}(\delta)$ is given by

$$\begin{pmatrix} 0 & -3 & 0 & 0 \\ -\frac{7}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix} + \mathcal{O}(\delta).$$

This analysis leads us to define a "new" first order for the Hamiltonian H in (III.2.8) as

$$H_0(\lambda,\Lambda,x,y;\delta) = H_{\mathrm{p}}(\lambda,\Lambda) + H_{\mathrm{osc}}(x,y;\delta), \tag{III.2.17}$$

and we refer to H_0 as the unperturbed Hamiltonian and to H_1 (see (III.2.10)) as the perturbation.

Notice that the unperturbed Hamiltonian is uncoupled. In the (x,y)-plane, it displays a fast oscillator of velocity $\frac{1}{\delta^2}$ whereas, in the (λ,Λ)-plane, it possesses a saddle at $(0,0)$ with two homoclinic connections or separatrices at the energy level $H_{\mathrm{p}}(\lambda,\Lambda) = -\frac{1}{2}$, (see Figure III.2.1). We define

$$\lambda_0 = \arccos\left(\frac{1}{2} - \sqrt{2}\right), \tag{III.2.18}$$

which satisfies $H_{\mathrm{p}}(\lambda_0,0) = H_{\mathrm{p}}(0,0) = -\frac{1}{2}$ and corresponds with the crossing point of the right separatrix with the axis $\{\Lambda = 0\}$. (See Section III.4.1 for more details).

[1] One can obtain more precise estimates for the third derivatives of H_1. However, these rough estimates are sufficient for the proofs of this part.

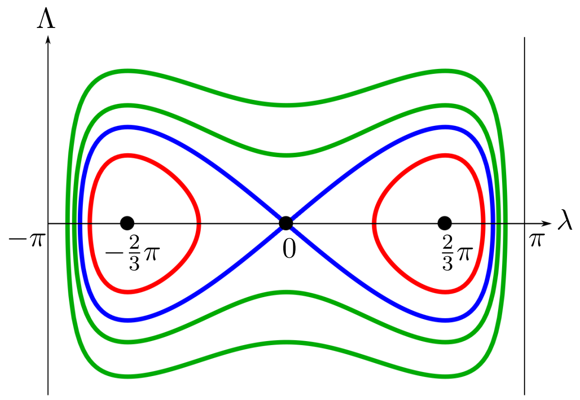


FIGURE III.2.1: Phase portrait of the system given by Hamiltonian $H_{\mathrm{p}}(\lambda,\Lambda)$ on (III.2.9). On blue the two separatrices.

The Lyapunov Center Theorem (see for instance [MO17]) ensures the existence of a family of periodic orbits emanating from a saddle-center equilibrium point. In our setting, this family will correspond to perturbed orbits of the fast oscillator, centered at $\mathfrak{L}(\delta)$. Since we need quantitative estimates for the parametrization of this family, we present a more thorough result in the following proposition. Its proof can be found in Appendix III.A.

For $d>0$, we denote

$$\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}, \qquad \mathbb{T}_d=\{\tau\in\mathbb{C}/2\pi\mathbb{Z} \,:\, |\mathrm{Im}\,\tau|<d\}\,. \tag{III.2.19}$$

Proposition III.2.4. *Let $d,c_0,c_1>0$. There exist $\rho_0,\delta_0>0$ such that, for $\delta\in(0,\delta_0)$, there exists a family of periodic orbits $\{\mathfrak{P}_\rho(\tau;\delta) \,:\, \tau\in\mathbb{T}_d\}_{\rho\in[0,\rho_0]}$, where $\mathfrak{P}_\rho:\mathbb{T}_d\to U_{\mathbb{C}}(c_0,c_1)$ are real-analytic functions satisfying that*

$$H(\mathfrak{P}_\rho(\tau;\delta))=\frac{\rho^2}{\delta^2}+H(\mathfrak{L}(\delta)).$$

Furthermore, there exist $\omega_{\rho,\delta}>0$ and a constant $b_2>0$, independent of ρ and δ, such that the parametrization of the periodic orbit satisfies

$$\dot\tau=\frac{\omega_{\rho,\delta}}{\delta^2} \qquad \text{with} \qquad |\omega_{\rho,\delta}-1|\le b_2\delta^4.$$

In addition, the parametrization can be written as

$$\mathfrak{P}_\rho(\tau;\delta)=\mathfrak{L}(\delta)+\rho\cdot\left(0,0,e^{-i\tau},e^{i\tau}\right)^T+\delta\rho\cdot\left(\lambda_{\mathfrak{P}},\Lambda_{\mathfrak{P}},x_{\mathfrak{P}},y_{\mathfrak{P}}\right)^T(\tau), \tag{III.2.20}$$

where $|\lambda_{\mathfrak{P}}(\tau)|,|\Lambda_{\mathfrak{P}}(\tau)|\le b_2$, and $|x_{\mathfrak{P}}(\tau)|,|y_{\mathfrak{P}}(\tau)|\le b_2\delta^3$.

III.2.2 Main results in scaled Poincaré coordinates

To prove the results in Theorems III.1.4, III.1.7 and III.1.8, we analyze the unstable and stable manifolds of both, the critical point $\mathfrak{L}(\delta)$ and the family of periodic orbits $\mathfrak{P}_\rho(\cdot,\delta)$, for small values of δ. Recall that there exists two symmetric branches for each stable and unstable manifold (see Figure III.1.2) with respect to the involution $\Phi(\lambda,\Lambda,x,y)=(-\lambda,\Lambda,x,y)$ as given in (III.2.14).

For $\delta>0$, we denote by $\mathcal{W}^{\mathrm{u}}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s}}(\mathfrak{L})$ the 1-dimensional unstable and stable manifolds of $\mathfrak{L}(\delta)$. In addition, as done in Chapter III.1, we consider each branch independently. Let ψ_t be the flow given by the Hamiltonian H and $\mathbf{e_1} = (1,0,0,0)^T$. We denote

$$\mathcal{W}^{\mathrm{u},+}(\mathfrak{L}) = \left\{ z \in \mathcal{W}^{\mathrm{u}}(\mathfrak{L}) \;:\; \lim_{t\to-\infty} \langle \psi_t(z), \mathbf{e_1} \rangle = 0^+ \right\}, \qquad \mathcal{W}^{\mathrm{s},-}(\mathfrak{L}) = \Phi\left(\mathcal{W}^{\mathrm{u},+}\right),$$
$$\mathcal{W}^{\mathrm{s},+}(\mathfrak{L}) = \left\{ z \in \mathcal{W}^{\mathrm{s}}(\mathfrak{L}) \;:\; \lim_{t\to+\infty} \langle \psi_t(z), \mathbf{e_1} \rangle = 0^+ \right\}, \qquad \mathcal{W}^{\mathrm{u},-}(\mathfrak{L}) = \Phi\left(\mathcal{W}^{\mathrm{s},+}\right),$$

the branches of $\mathcal{W}^{\diamond}(\mathfrak{L})$, for $\diamond = \mathrm{u}, \mathrm{s}$.

Next result, proven in Theorem II.2.2 in Part II, gives an asymptotic formula for the distance between the first intersection of the one dimensional manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ on a suitable section. In particular, Theorem III.1.1 is a consequence of this result.

Theorem III.2.5. *Fix an interval $[\lambda_1, \lambda_2] \subset (0, \lambda_0)$ with λ_0 as given in (III.2.18). There exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$ and $\lambda_* \in [\lambda_1, \lambda_2]$, the invariant manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ intersect the section $\{\lambda = \lambda_*, \Lambda > 0\}$. Denote by $(\lambda_*, \Lambda_\delta^{\mathrm{u}}, x_\delta^{\mathrm{u}}, y_\delta^{\mathrm{u}})$ and $(\lambda_*, \Lambda_\delta^{\mathrm{s}}, x_\delta^{\mathrm{s}}, y_\delta^{\mathrm{s}})$ the first intersection points of the unstable and stable manifolds with this section, respectively. They satisfy*

$$y_\delta^{\mathrm{u}} - y_\delta^{\mathrm{s}} = \sqrt[6]{2}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[\Theta + \mathcal{O}\left(\frac{1}{|\log \delta|} \right) \right], \qquad x_\delta^{\mathrm{u}} - x_\delta^{\mathrm{s}} = \overline{y_\delta^{\mathrm{u}} - y_\delta^{\mathrm{s}}},$$
$$\Lambda_\delta^{\mathrm{u}} - \Lambda_\delta^{\mathrm{s}} = \mathcal{O}\left(\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}} \right),$$

where $A > 0$ and $\Theta \in \mathbb{C}$ are the constants described in Theorem III.1.1.

Now, we consider the case of homoclinic connections between the 1-dimensional branches, $\mathcal{W}^{\mathrm{u},\pm}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},\pm}(\mathfrak{L})$, of the unstable and stable manifolds.

Definition III.2.6. *Let $\Gamma(t)$ be an an homoclinic orbit to the critical point $\mathfrak{L}(\delta)$ and B_δ a ball centered at $\mathfrak{L}(\delta)$ of radius δ. Then, we say $\Gamma(t)$ is k-round if*

$$\overline{\bigcup_{t\in\mathbb{R}} \Gamma(t)} \setminus B_\delta \qquad \text{has } k \text{ connected components.}$$

Let us recall that, by Ansatz III.1.2, one has that $\Theta \neq 0$. Then, Theorem III.2.5 imply that they do not exist homoclinic connections between the "+" branches of the stable and unstable manifolds of $\mathfrak{L}(\delta)$ at its first intersection with the section $\{\lambda = \lambda_*, \Lambda > 0\}$. Moreover, due to the symmetry of the system (see (III.2.14)), an analogous result holds for $\mathcal{W}^{\mathrm{u},-}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},-}(\mathfrak{L})$. Therefore, one obtains the following corollary, which is equivalent to Corollary III.1.3.

Corollary III.2.7. *Assume Ansatz III.1.2. Then, there exists $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0)$, there do not exist 1-round homoclinic connections to $\mathfrak{L}(\delta)$.*

In the next result, we study the existence of 2-round homoclinic connections to $\mathfrak{L}(\delta)$ for certain values of the parameter δ. Theorem III.1.4 is a direct consequence of it.

Theorem III.2.8. *Assume Ansatz III.1.2. Then, there exist $N_0 > 0$ and a sequence $\{\delta_n\}_{n \geq N_0}$ satisfying*

$$\delta_n = \sqrt[8]{\frac{8}{21}} \sqrt[4]{\frac{A}{n\pi}} \left(1 + \mathcal{O}\left(\frac{1}{\log n} \right) \right), \qquad \text{for } n \geq N_0.$$

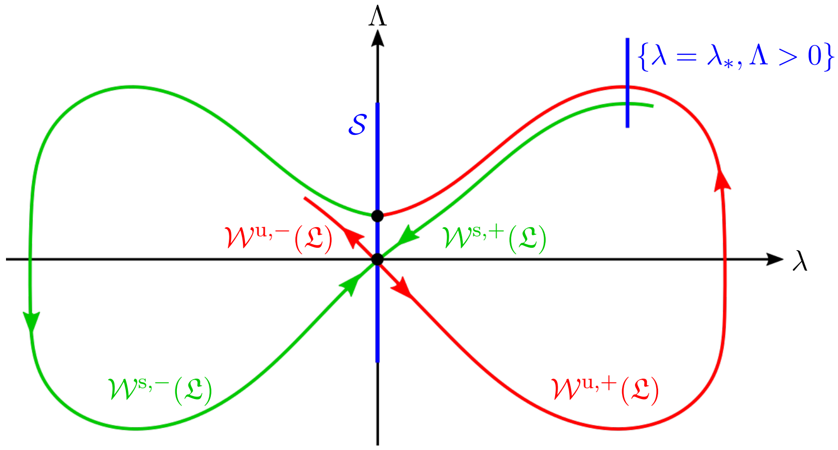


FIGURE III.2.2: Projection into the (λ, Λ)-plane of the unstable and stable manifolds and its intersections with the symmetry axis and section $\{\lambda = \lambda_*, \Lambda > 0\}$.

such that, for each $n \geq N_0$, there exist a 2-round homoclinic connection to the equilibrium point $\mathfrak{L}(\delta_n)$ between $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},-}(\mathfrak{L})$.

To prove Theorem III.2.8, we take advantage of the fact that the Hamiltonian H is reversible with respect to the axis $\mathcal{S} = \{\lambda = 0,\, x = y\}$, (see (III.2.15)). Therefore, by symmetry, it is only necessary to see that there exists a sequence of δ such that $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ intersects the symmetry axis $\mathcal{S}$, see Figure III.2.2.

In particular, we extend the manifold $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ from the section $\{\lambda = \lambda_*, \Lambda > 0\}$, studied in Theorem III.2.5, to a neighborhood of the critical point $\mathfrak{L}(\delta)$ and look for intersections with $\mathcal{S}$. To study the invariant manifolds near $\mathfrak{L}(\delta)$, we use a normal form result for systems in a neighborhood of a saddle-center critical point. Note that, the classical normal form result by Moser in [Mos58] is not enough for our purposes. Indeed, we need to control that the radius of convergence of the normal form does not goes to zero when $\delta \to 0$. For that reason, we apply a more quantitative normal form obtained by T. Jézéquel, P. Bernard and E. Lombardi in [JBL16] that ensures that the normalization does not blow up when $\delta \to 0$. The complete proof of Theorem III.2.8 is postponed to Chapter III.3.

Next, we focus on the study of the intersections between the unstable and stable manifolds of the family of periodic orbits $\{\mathfrak{P}_\rho(\cdot, \delta)\}_{\rho \in [0,\rho_0]}$ in Proposition III.2.4. Let us denote by $\mathcal{W}^{\mathrm{u}}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s}}(\mathfrak{P}_\rho)$ the 2-dimensional unstable and stable manifolds of the periodic orbit $\mathfrak{P}_\rho(\cdot, \delta)$. Analogously to the invariant manifolds of $\mathfrak{L}(\delta)$, for $\diamond \in \{\mathrm{u}, \mathrm{s}\}$, we denote each branch as $\mathcal{W}^{\diamond,+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\diamond,-}(\mathfrak{P}_\rho)$, (see Figure III.1.4).

To prove Theorem III.1.7 and III.1.8, we focus on the study of the "+" invariant manifolds. By symmetry, there exist analogous results for the "−" invariant manifolds. In particular, for $c_0, c_1 > 0$, we look for intersections between $\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho)$ in a suitable 2-dimensional section of the domain $U_{\mathbb{R}}(c_0, c_1)$ (see (III.2.12)) within a fixed energy level. In particular, for technical reasons, we choose such section as

$$\Sigma_\rho = \Big\{ (\lambda, \Lambda, x, y) \in U_{\mathbb{R}}(c_0, c_1) : \\ \Lambda = \delta^2 \mathfrak{L}_\Lambda(\delta),\, H(\lambda, \Lambda, x, y) = \frac{\rho^2}{\delta^2} + H(\mathfrak{L}(\delta)) \Big\}, \tag{III.2.21}$$

where $\mathfrak{L} = (0, \delta^2 \mathfrak{L}_\Lambda, \delta^3 \mathfrak{L}_x, \delta^3 \mathfrak{L}_y)^T$as given in Proposition III.2.3.

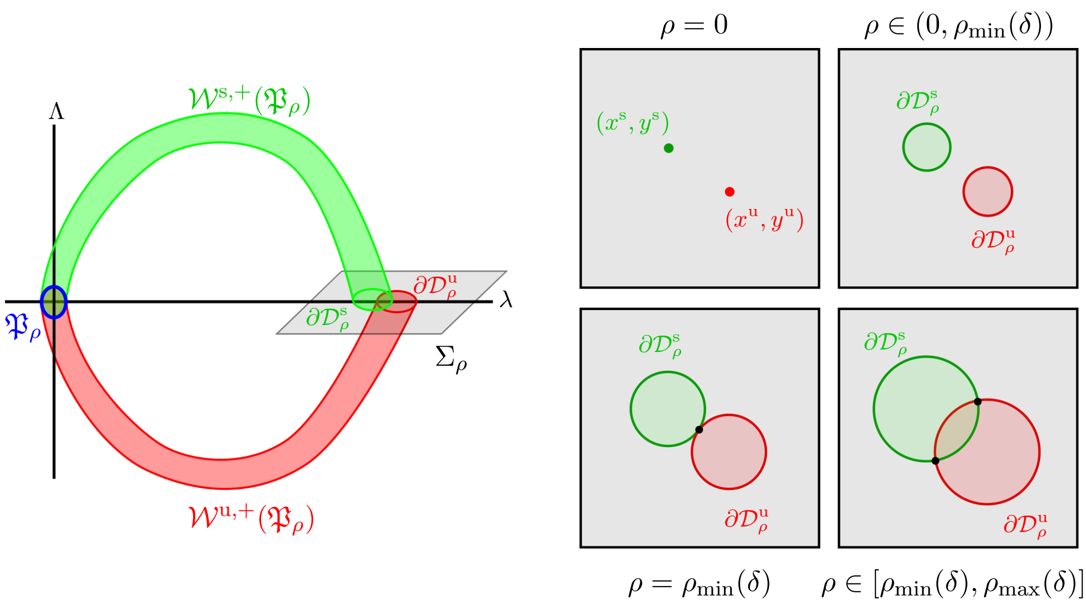


FIGURE III.2.3: Left: Intersection of the unstable and stable manifolds $\mathcal{W}^{u,+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{s,+}(\mathfrak{P}_\rho)$ with section Σ_ρ. Right: different possibilities given in Corollary III.2.9 and Theorem III.2.10.

We notice that, by Proposition III.2.4, the periodic orbit $\mathfrak{P}_\rho$ belongs to the energy level $H = \frac{\rho^2}{\delta^2} + H(\mathfrak{L}(\delta))$ where Σ_ρ is included. In addition, for $\rho = 0$, one has that $\mathfrak{P}_0 = \mathfrak{L}$. For this case, Theorem III.2.10 can be adapted to give an asymptotic formula for the distance between the first intersection of $\mathcal{W}^{u,+}(\mathfrak{L})$ and $\mathcal{W}^{s,+}(\mathfrak{L})$ with the section Σ_0. It is an almost direct consequence of Theorem III.2.10, see Appendix III.B for the details.

Corollary III.2.9. *There exists $\delta_0 > 0$ such that, for every $\delta \in (0, \delta_0)$, the invariant manifolds $\mathcal{W}^{u,+}(\mathfrak{L})$ and $\mathcal{W}^{s,+}(\mathfrak{L})$ intersect the section Σ_0. Denote by $(\lambda^u_\delta, \delta^2\mathfrak{L}_\Lambda, x^u_\delta, y^u_\delta)$ and $(\lambda^s_\delta, \delta^2\mathfrak{L}_\Lambda, x^s_\delta, y^s_\delta)$ the first intersection points of the unstable and stable manifolds, respectively, with the section. Then, they satisfy*

$$|x^u_\delta - x^s_\delta| = |y^u_\delta - y^s_\delta| = \sqrt[6]{2}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[|\Theta| + \mathcal{O}\left(\frac{1}{|\log \delta|} \right) \right],$$

where $A > 0$ and $\Theta \in \mathbb{C}$ are the constants described in Theorem III.1.1.

In the next result, we see that the 2-dimensional invariant manifolds $\mathcal{W}^{u,+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{s,+}(\mathfrak{P}_\rho)$ intersect in the section Σ_ρ for certain values of ρ, (see Figure III.2.3). Its proof is postponed to Chapter III.4.

Theorem III.2.10. *Assume Ansatz III.1.2. Let ρ_0 and $\mathfrak{P}_\rho$, for $\rho \in [0, \rho_0]$, be as given in Proposition III.2.4. Then, the following is satisfied.*

- *There exists $\delta_0 > 0$ such that, for every $\rho \in [0, \rho_0]$ and $\delta \in (0, \delta_0)$, the invariant manifolds $\mathcal{W}^{u,+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{s,+}(\mathfrak{P}_\rho)$ intersect the section Σ_ρ. The first intersection is given by closed curves, which we denote by $\partial\mathcal{D}^u_\rho$ and $\partial\mathcal{D}^s_\rho$.*

- *Let $R > 1$. There exists $\delta_R > 0$, satisfying $\lim_{R\to\infty} \delta_R = 0$, and functions $\rho_{\min}, \rho_{\max} : (0, \delta_R) \to [0, \rho_0]$ such that, for $\delta \in (0, \delta_R)$ and $\rho \in [\rho_{\min}(\delta), \rho_{\max}(\delta)]$,* *the curves $\partial\mathcal{D}^{\mathrm{u}}_\rho$ and $\partial\mathcal{D}^{\mathrm{s}}_\rho$ intersect. Moreover,*

$$\rho_{\min}(\delta) = \frac{\sqrt[6]{2}}{2}|\Theta|\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}\left[1+\mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right],$$
$$\rho_{\max}(\delta) = \frac{\sqrt[6]{2}}{2}|\Theta|\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}\left[R+\mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right].$$

- *For $\rho \in (\rho_{\min}(\delta), \rho_{\max}(\delta)]$, the curves $\partial\mathcal{D}^{\mathrm{u}}_\rho$ and $\partial\mathcal{D}^{\mathrm{s}}_\rho$ intersect transversally at least twice.*

- *For $\rho = \rho_{\min}(\delta)$, the curves $\partial\mathcal{D}^{\mathrm{u}}_\rho$ and $\partial\mathcal{D}^{\mathrm{s}}_\rho$ have at least one quadratic tangency at a point $Q_0 \in \partial\mathcal{D}^{\mathrm{u}}_\rho \cap \partial\mathcal{D}^{\mathrm{s}}_\rho$.*

- *Fix $\delta \in (0,\delta_0)$ and let ζ be any smooth curve transverse to $\partial\mathcal{D}^{\mathrm{u}}_{\rho_{\min}}$ and $\partial\mathcal{D}^{\mathrm{s}}_{\rho_{\min}}$ within $\Sigma_{\rho_{\min}}$ at Q_0. Then, for ρ close to $\rho_{\min}$, the local intersections of $\partial\mathcal{D}^{\mathrm{u}}_\rho$ and $\partial\mathcal{D}^{\mathrm{s}}_\rho$ with the curve ζ cross each other with relative non zero velocity at $(Q_0, \rho_{\min})$.*

Theorem III.2.10 implies in particular that, for small values of δ, there exist transverse intersections between some unstable and stable manifolds of Lyapunov periodic orbits of $\mathfrak{L}(\delta)$. Thus, applying the Smale-Birkhoff homoclinic theorem (see [Sma67; KH95]), it is possible to construct a Smale horseshoe map in a tubular neighborhood of the invariant manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ of size $\mathcal{O}(\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}})$. By symmetry, an analogous result holds for $\mathcal{W}^{\mathrm{u},-}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},-}(\mathfrak{L})$. This proves Theorem III.1.7.

Moreover, the last two statements of Theorem III.2.10 imply the existence of a generic unfolding of a quadratic tangency between $\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho)$. (We follow the definition of generic unfolding given in [Dua08]). Indeed, denoting by f_ϱ to the flow of h in (III.1.1) restricted to the energy level $h = \varrho + h(L_3)$, for $\delta \in (0,\delta_0)$, one has that f_ϱ unfolds generically an homoclinic quadratic tangency. Finally, noticing that the energy level $H(\lambda,\Lambda,x,y;\delta) = \frac{\rho^2}{\delta^2} + H(\mathfrak{L})$ corresponds to $h(q,p;\mu) = \sqrt{\mu}\rho^2 + h(L_3)$ (see (III.2.5) and (III.2.6)), one proves Theorem III.1.8. In particular, from this result one should be able to prove the existence of a Newhouse domain (see Remark III.1.9).

Chapter III.3

Proof of Theorem III.2.8

To prove Theorem III.2.8, we perform a detailed analysis of the system close to the equilibrium point $\mathfrak{L}(\delta)$ by means of a normal form. In the next proposition we introduce the normal form result given in [JBL16] adapted to Hamiltonian H in (III.2.8). Then, in Proposition III.3.2, we translate the results in Theorem III.2.5 and the symmetry axis $\mathcal{S}$ in (III.2.15) into the new set of coordinates. The proof of both results is postponed to Appendix III.C.

Proposition III.3.1. *There exist $\delta_0, \varrho_0, c_0, c_1 > 0$ and a family of analytic canonical changes of coordinates*

$$\begin{aligned}\mathcal{F}_\delta : B(\varrho_0) = \left\{\mathbf{z} \in \mathbb{R}^4 : |\mathbf{z}| < \varrho_0\right\} &\to U_{\mathbb{R}}(c_0, c_1)\\ (v_1, w_1, v_2, w_2) &\mapsto (\lambda, \Lambda, x, y),\end{aligned}$$

with respect to the form $dv_1 \wedge dw_1 + dv_2 \wedge dw_2$ defined for $\delta \in (0, \delta_0)$ such that $\mathcal{F}_\delta(0) = \mathfrak{L}(\delta)$ and the Hamiltonian H (see (III.2.8)) in the new coordinates reads

$$\begin{aligned}\mathcal{H}(v_1, w_1, v_2, w_2; \delta) &= H(\mathcal{F}_\delta(v_1, w_1, v_2, w_2); \delta) - H(\mathfrak{L}(\delta); \delta)\\ &= v_1 w_1 + \frac{\alpha(\delta)}{2\delta^2}\left(v_2^2 + w_2^2\right) + \mathcal{R}(v_1 w_1, v_2^2 + w_2^2; \delta),\end{aligned}$$

where $\alpha(\delta)$ is a $\mathcal{C}^1$-function in δ satisfying that $\alpha(\delta) = \sqrt{\frac{8}{21}} + \mathcal{O}(\delta^4)$ and $\mathcal{R}$ satisfies that

$$\left|\mathcal{R}(v_1 w_1, v_2^2 + w_2^2; \delta)\right| \leq C|(v_1 w_1, v_2^2 + w_2^2)|^2,$$

for $(v_1, w_1, v_2, w_2) \in B(\varrho_0)$ and $C > 0$ a constant independent of δ.

The system of equations given by Hamiltonian $\mathcal{H}$ in Proposition III.3.1 is of the form

$$\begin{aligned}\dot{v_1} &= v_1\left(1 + \partial_1 \mathcal{R}(v_1 w_1, v_2^2 + w_2^2; \delta)\right),\\ \dot{w_1} &= -w_1\left(1 + \partial_1 \mathcal{R}(v_1 w_1, v_2^2 + w_2^2; \delta)\right),\\ \dot{v_2} &= w_2\left(\frac{\alpha(\delta)}{\delta^2} + \partial_2 \mathcal{R}(v_1 w_1, v_2^2 + w_2^2; \delta)\right),\\ \dot{w_2} &= -v_2\left(\frac{\alpha(\delta)}{\delta^2} + \partial_2 \mathcal{R}(v_1 w_1, v_2^2 + w_2^2; \delta)\right).\end{aligned} \tag{III.3.1}$$

Since this system has two conserved quantities: $v_1 w_1$ and $v_2^2 + w_2^2$. Then

$$\begin{aligned} v_1(t) &= v_1(0) e^{t\nu_1(\delta)}, \\ w_1(t) &= w_1(0) e^{-t\nu_1(\delta)}, \\ \begin{pmatrix} v_2(t) \\ w_2(t) \end{pmatrix} &= \mathrm{Rot}(t\nu_2(\delta)) \begin{pmatrix} v_2(0) \\ w_2(0) \end{pmatrix}, \qquad \mathrm{Rot}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \end{aligned} \tag{III.3.2}$$

where, for $(v_1(0), w_1(0), v_2(0), w_2(0)) \in B(\varrho_0)$,

$$\begin{aligned} \nu_1(\delta) &= 1 + \partial_1 \mathcal{R}\left(v_1(0)w_1(0), v_2^2(0) + w_2^2(0); \delta\right) > 0, \\ \nu_2(\delta) &= \frac{\alpha(\delta)}{\delta^2} + \partial_2 \mathcal{R}\left(v_1(0)w_1(0), v_2^2(0) + w_2^2(0); \delta\right) > 0. \end{aligned} \tag{III.3.3}$$

Notice that, by (III.3.2), the local unstable and stable manifolds are given by $\{w_1 = v_2 = w_2 = 0\}$ and $\{v_1 = v_2 = w_2 = 0\}$, respectively.

Proposition III.3.2. *Assume the setting given in Proposition III.3.1. Then,*

1. *Fix an interval $[\lambda_1, \lambda_2] \subset (0, \lambda_0)$ with λ_0 as given in (III.2.18) and denote $\mathbf{z}_\delta^{\mathrm{u}}(\lambda_*)$ and $\mathbf{z}_\delta^{\mathrm{s}}(\lambda_*)$ to the first intersection of the invariant manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ with the section $\{\lambda = \lambda_*, \Lambda > 0\}$ with $\lambda_* \in [\lambda_1, \lambda_2]$, respectively (see Theorem III.2.5).*

   There exist $0 < \varrho_1 < \varrho_2 < \varrho_0$ such that, for $\varrho \in [\varrho_1, \varrho_2]$ and $\delta \in (0, \delta_0)$, there exist $\lambda_*(\varrho) \in [\lambda_1, \lambda_2]$ and $(v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}}), (v_1^{\mathrm{s}}, w_1^{\mathrm{s}}, v_2^{\mathrm{s}}, w_2^{\mathrm{s}}) \in B(\varrho_0)$ such that

$$(v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}}) = \mathcal{F}_\delta\left(\mathbf{z}_\delta^{\mathrm{u}}(\lambda_*(\varrho))\right), \qquad (v_1^{\mathrm{s}}, w_1^{\mathrm{s}}, v_2^{\mathrm{s}}, w_2^{\mathrm{s}}) = \mathcal{F}_\delta\left(\mathbf{z}_\delta^{\mathrm{s}}(\lambda_*(\varrho))\right), \tag{III.3.4}$$

   and

$$\begin{aligned} v_1^{\mathrm{u}} &= -\frac{\sqrt[3]{2}}{\varrho}\delta^{-\frac{4}{3}} e^{-\frac{2A}{\delta^2}} \left[|\Theta|^2 + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right], & v_1^{\mathrm{s}} &= 0, \\ w_1^{\mathrm{u}} &= \varrho + \mathcal{O}\left(\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}\right), & w_1^{\mathrm{s}} &= \varrho, \\ v_2^{\mathrm{u}} &= \sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[\mathrm{Re}\,\Theta + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right], & v_2^{\mathrm{s}} &= 0, \\ w_2^{\mathrm{u}} &= \sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[-\mathrm{Im}\,\Theta + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right], & w_2^{\mathrm{s}} &= 0. \end{aligned}$$

2. *Let $\mathcal{S} = \{\lambda = 0, x = y\}$ be the symmetry axis of Hamiltonian H given in (III.2.15).*

   There exist real-analytic functions $\Psi_1, \Psi_2 : B(\varrho_0) \times (0, \delta_0) \to \mathbb{R}$ and constants $C_1, C_2 > 0$ such that the curve

$$\mathcal{S}_{\mathrm{loc}} = \left\{v_1 + w_1 = \Psi_1(v_1, w_1, v_2, w_2; \delta),\ w_2 = \Psi_2(v_1, w_1, v_2, w_2; \delta)\right\} \tag{III.3.5}$$

   satisfies that $\mathcal{F}_\delta(\mathcal{S}_{\mathrm{loc}}) \subset \mathcal{S}$ and, for $(v_1, w_1, v_2, w_2; \delta) \in B(\varrho_0) \times (0, \delta_0)$,

   (a) $|\Psi_1(v_1, w_1, v_2, w_2; \delta)| \leq C_1 \delta\, |(v_1, w_1, v_2, w_2)| + C_2\, |(v_1, w_1)|^2$.

   (b) $|\Psi_2(v_1, w_1, v_2, w_2; \delta)| \leq C_1 \delta\, |(v_1, w_1, v_2, w_2)|$.

From now on, we work in the set of local coordinates $(v_1, w_1, v_2, w_2) \in B(\varrho_0)$ given in Proposition III.3.1. Then, to prove Theorem III.2.8, it remains to extend the

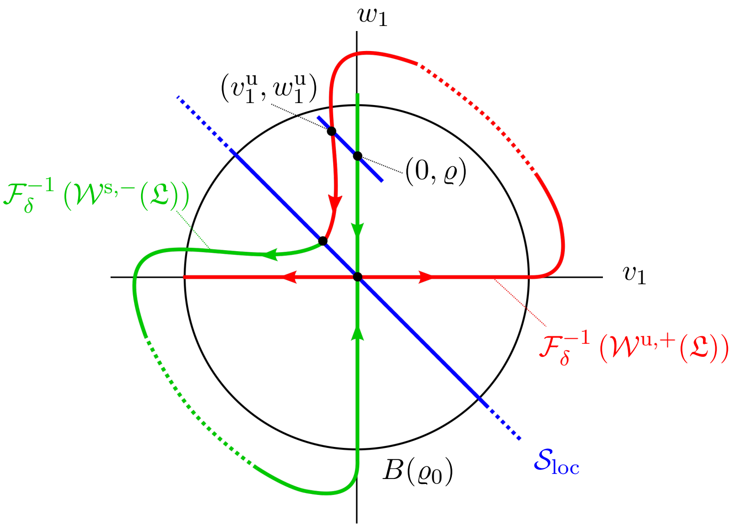


FIGURE III.3.1: Representation of the unstable and stable manifolds in local coordinates (v_1, w_1, v_2, w_2) given in Propositions III.3.1 and III.3.2.

unstable manifold from the point $(v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}})$ given in (III.3.10) and to analyze for which values of the parameter $\delta > 0$ it intersects with the symmetry curve $\mathcal{S}_{\mathrm{loc}}$ given in (III.3.5), (see Figure III.3.1).

To give an intuition of the proof, in the next lemma, we consider the intersection of the unstable manifold with a convenient "first order" of the symmetry curve $\mathcal{S}_{\mathrm{loc}}$. From now on, we denote by C any positive constant independent of δ.

Lemma III.3.3. *Assume Ansatz III.1.2. Let $\Phi^{\mathrm{u}}(t;\delta)$ be the trajectory of the Hamiltonian system given by Hamiltonian $\mathcal{H}$ in Proposition III.3.1 with initial condition $(v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}})$ as given in Proposition III.3.2. Then, there exist $N_0 > 0$ and sequences $\{\widehat{T}_n\}_{n\geq N_0}$ and $\{\widehat{\delta}_n\}_{n\geq N_0}$ such that , for $n \geq N_0$,*

$$\Phi^{\mathrm{u}}(\widehat{T}_n; \widehat{\delta}_n) \in \{v_1 + w_1 = 0,\ w_2 = 0\}.$$

Moreover,

$$\widehat{\delta}_n = \sqrt[8]{\frac{8}{21}}\sqrt[4]{\frac{A}{n\pi}}\left(1 + \mathcal{O}\left(\frac{1}{\log n}\right)\right), \qquad \textit{for } n \geq N_0,$$

where $A > 0$ is the constant described in Theorem III.1.1.

Proof. Let $(v_1(t), w_1(t), v_2(t), w_2(t))$ be a trajectory of the Hamiltonian system given by $\mathcal{H}(v_1, w_1, v_2, w_2; \delta)$. We want $\delta > 0$ to be such that there exists $T_\delta^0 > 0$ satisfying

$$\begin{gathered}(v_1(0), w_1(0), v_2(0), w_2(0)) = (v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}}),\\ \left(v_1(T_\delta^0), w_1(T_\delta^0), v_2(T_\delta^0), w_2(T_\delta^0)\right) \in \{v_1 + w_1 = 0,\ w_2 = 0\}.\end{gathered}$$

In other words, using (III.3.2),

$$v_1^{\mathrm{u}} e^{\nu_1(\delta)T_\delta^0} + w_1^{\mathrm{u}} e^{-\nu_1(\delta)T_\delta^0} = 0, \tag{III.3.6}$$

$$\cos(\nu_2(\delta)T_\delta^0) w_2^{\mathrm{u}} - \sin(\nu_2(\delta)T_\delta^0) v_2^{\mathrm{u}} = 0, \tag{III.3.7}$$

where, by its definition in (III.3.3) and Proposition III.3.2, one has that

$$\nu_1(\delta) = 1 + \mathcal{O}\left(\delta^{-\frac{8}{3}} e^{-\frac{4A}{\delta^2}}\right), \qquad \nu_2(\delta) = \frac{1}{\delta^2}\sqrt{\frac{8}{21}} + \mathcal{O}\left(\delta^2\right). \tag{III.3.8}$$

Equation (III.3.6) is solved by

$$\begin{aligned} T_\delta^0 &= -\frac{1}{2\nu_1(\delta)} \ln\left(-\frac{v_1^{\mathrm{u}}}{w_1^{\mathrm{u}}}\right) \\ &= \frac{A}{\delta^2} + \frac{2}{3}\log\delta - \log\left(\sqrt[6]{2}\,|\Theta|\,\varrho^{-1}\right) + \mathcal{O}\left(\frac{1}{|\log\delta|}\right) \\ &= \frac{A}{\delta^2}\left(1 + \mathcal{O}(\delta^2\,|\log\delta|)\right). \end{aligned} \tag{III.3.9}$$

Next, we study equation (III.3.7). Let us denote $\theta = \arg\Theta$. From Proposition III.3.2,

$$\begin{aligned} v_2^{\mathrm{u}} &= \sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}}\left[|\Theta|\cos\theta + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right], \\ w_2^{\mathrm{u}} &= -\sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}}\left[|\Theta|\sin\theta + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right]. \end{aligned}$$

By Ansatz III.1.2, one has that $\Theta \neq 0$. Then, (III.3.7) is equivalent to

$$\cos(\nu_2(\delta)T_\delta^0)\sin\theta + \sin(\nu_2(\delta)T_\delta^0)\cos\theta = \sin\left(\theta + \nu_2(\delta)T_\delta^0\right) = g_0(\delta),$$

where $g_0(\delta)$ contains the higher order terms, that is,

$$\begin{aligned} g_0(\delta) = &- \cos(\nu_2(\delta)T_\delta^0)\left(\sqrt[4]{\frac{8}{21}}\frac{e^{\frac{A}{\delta^2}}\delta^{-\frac{1}{3}}}{\sqrt[3]{4}\,|\Theta|} w_2^{\mathrm{u}} + \sin\theta\right) \\ &- \sin(\nu_2(\delta)T_\delta^0)\left(\sqrt[4]{\frac{8}{21}}\frac{e^{\frac{A}{\delta^2}}\delta^{-\frac{1}{3}}}{\sqrt[3]{4}\,|\Theta|} v_2^{\mathrm{u}} - \cos\theta\right). \end{aligned}$$

Note that $g_0(\delta)$ satisfies that $g_0(\delta) = \mathcal{O}(|\log\delta|^{-1})$. So, we deduce that, for $n \in \mathbb{Z}$,

$$\nu_2(\delta)T_\delta^0 + \theta = n\pi - \arcsin g_0(\delta).$$

Using the asymptotic expressions of $k_2(\delta)$ and T_δ^0 in (III.3.8) and (III.3.9), we have that δ has to satisfy

$$\frac{A}{\delta^4}\sqrt{\frac{8}{21}}\left(1 + g_1(\delta)\right) = \pi n,$$

where $g_1(\delta) = \mathcal{O}(\delta^2\,|\log\delta|)$. Then, by the Implicit Function Theorem, there exists $N_0 > 0$ and a sequence $\{\widehat{\delta}_n\}_{n\geq N_0} \subset (0,\delta_0)$ satisfying the previous equation and the asymptotic expression of the lemma. Finally, one has that $\widehat{T}_n = T_\delta^0$ for $\delta = \widehat{\delta}_n$. $\square$

End of the proof of Theorem III.2.8. We proceed analogously to the proof of Lemma III.3.3. Let us consider the expressions of $(v_1(t), w_1(t), v_2(t), w_2(t))$ given in (III.3.2) and $T_\delta > 0$, such that

$$\begin{aligned} (v_1(0), w_1(0), v_2(0), w_2(0)) &= (v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}}), \\ (v_1(T_\delta), w_1(T_\delta), v_2(T_\delta), w_2(T_\delta)) &\in \mathcal{S}_{\mathrm{loc}}, \end{aligned}$$

with $\mathcal{S}_{\mathrm{loc}} = \{v_1 + w_1 = \Psi_1,\ w_2 = \Psi_2\}$ as given in Proposition III.3.2.

First, we deal with the equation $v_1 + w_1 = \Psi_1$. Then, T_δ must satisfy

$$v_1(T_\delta) + w_1(T_\delta) = \Psi_1\left(v_1(T_\delta), w_1(T_\delta), v_2(T_\delta), w_2(T_\delta)\right). \tag{III.3.10}$$

Let us denote $\tau(\delta) = T_\delta - T_\delta^0$, with T_δ^0 satisfying $v_1(T_\delta^0) + w_1(T_\delta^0) = 0$ (see equation (III.3.6) and (III.3.9)). Then, by (III.3.2), τ has to satisfy

$$\begin{aligned} v_1(T_\delta^0)e^{\nu_1(\delta)\tau} + w_1(T_\delta^0)e^{-\nu_1(\delta)\tau} &= w_1(T_\delta^0)(e^{-\nu_1(\delta)\tau} - e^{\nu_1(\delta)\tau}) \\ &= \Psi_1(v_1(T_\delta^0+\tau), w_1(T_\delta^0+\tau), v_2(T_\delta^0+\tau), w_2(T_\delta^0+\tau)). \end{aligned}$$

Namely $\tau(\delta) = F[\tau](\delta)$ with

$$F[\tau] = \frac{e^{-\nu_1(\delta)\tau} - e^{\nu_1(\delta)\tau} + 2\tau\nu_1(\delta)}{2\nu_1(\delta)} - \frac{\Psi_1(v_1(t), w_1(t), v_2(t), w_2(t))|_{t=T_\delta^0+\tau}}{2\nu_1(\delta)w_1(T_\delta^0)}.$$

First we obtain estimates for $F[0]$. Indeed, by Proposition III.3.2 and (III.3.8),

$$\begin{aligned} |F[0](\delta)| &\le \frac{\left|\Psi_1(v_1(T_\delta^0), w_1(T_\delta^0), v_2(T_\delta^0), w_2(T_\delta^0))\right|}{2\left|\nu_1(\delta)w_1(T_\delta^0)\right|} \\ &\le C\frac{\delta\left|(v(T_\delta^0), w(T_\delta^0))\right| + \left|(v_1(T_\delta^0), w_1(T_\delta^0))\right|^2}{\left|w_1(T_\delta^0)\right|}. \end{aligned}$$

Let us recall that, by (III.3.9), we have an asymptotic expression for T_δ^0. Then, by (III.3.2), (III.3.8) and Proposition III.3.2,

$$\begin{aligned} w_1(T_\delta^0) &= w_1^{\mathrm{u}} e^{-\nu_1(\delta)T_\delta^0} = \sqrt[6]{2}\,|\Theta|\,\delta^{-\frac{2}{3}}e^{-\frac{A}{\delta^2}}\left[1 + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right], \\ v_2(T_\delta^0) &= \cos(\nu_2(\delta)T_\delta^0)v_2^{\mathrm{u}} + \sin(\nu_2(\delta)T_\delta^0)w_2^{\mathrm{u}} = \mathcal{O}\left(\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}\right), \\ w_2(T_\delta^0) &= -\sin(\nu_2(\delta)T_\delta^0)v_2^{\mathrm{u}} + \cos(\nu_2(\delta)T_\delta^0)w_2^{\mathrm{u}} = \mathcal{O}\left(\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}\right). \end{aligned} \tag{III.3.11}$$

Since $v_1(T_\delta^0) = -w_1(T_\delta^0)$, one has that $|F[0](\delta)| \le C\delta$. Next, we study the Lipschitz constant of the operator F. Let us consider continuous functions $\tau_0, \tau_1 : (0, \delta_0) \to \mathbb{R}$ such that $|\tau_0(\delta)|, |\tau_1(\delta)| \le C\delta$ and the function $\tau_\sigma = \sigma\tau_1 + (1-\sigma)\tau_0$. Then, by the Mean Value Theorem,

$$\begin{aligned} &|F[\tau_1] - F[\tau_0]| \le C\,|\tau_1 - \tau_0| \cdot \\ &\qquad \sup_{\sigma\in[0,1]}\left\{ |\tau_\sigma|^2 + \delta^{\frac{2}{3}}e^{\frac{A}{\delta^2}}\left|D\Psi_1(v_1, w_1, v_2, w_2)\cdot(\dot v_1, \dot w_1, \dot v_2, \dot w_2)^T\right|_{t=T_\delta^0+\tau_\sigma} \right\}. \end{aligned}$$

Since Ψ_1 is a real-analytic change of coordinates, by Proposition III.3.2, one can obtain $|D\Psi_1| \le C\delta + C\,|(v_1, w_1)|$. Moreover, using (III.3.1), one can obtain estimates for the derivatives $(\dot v_1, \dot w_1, \dot v_2, \dot w_2)$. Then,

$$|F[\tau_1](\delta) - F[\tau_0](\delta)| \le C\delta\,|\tau_1(\delta) - \tau_0(\delta)|.$$

This implies that, taking $\delta > 0$ small enough, $|F[\tau_1] - F[\tau_0]| \le \frac{1}{2}|\tau_1 - \tau_0|$ and, as a result, F is well defined and contractive. Hence, F has a fixed point $\tau(\delta)$ such that

$|\tau(\delta)| \leq C\delta$. Therefore, there exists T_δ satisfying equation (III.3.10) such that

$$T_\delta = T_\delta^0 + \tau(\delta) = \frac{A}{\delta^2}(1 + \mathcal{O}(\delta^2 |\log \delta|)). \tag{III.3.12}$$

Next, we study the equation $w_2 = \Psi_2$. Indeed, one has that $\delta > 0$ must satisfy

$$w_2(T_\delta) = \Psi_2\left(v(T_\delta), w(T_\delta)\right). \tag{III.3.13}$$

Ansatz III.1.2 implies that $\Theta \neq 0$. Then, by (III.3.2), δ has to satisfy

$$\cos(\nu_2(\delta)T_\delta)\sin\theta + \sin(\nu_2(\delta)T_\delta)\cos\theta = \sin\left(\theta + \nu_2(\delta)T_\delta\right) = \widehat{g_0}(\delta),$$

where

$$\begin{aligned}\widehat{g_0}(\delta) = \Psi_2\left(v(T_\delta), w(T_\delta)\right) &- \cos(\nu_2(\delta)T_\delta)\left(\sqrt[4]{\frac{8}{21}}\frac{e^{\frac{A}{\delta^2}}\delta^{-\frac{1}{3}}}{\sqrt[3]{4}\,|\Theta|}w_2^{\mathrm{u}} + \sin\theta\right)\\ &- \sin(\nu_2(\delta)T_\delta)\left(\sqrt[4]{\frac{8}{21}}\frac{e^{\frac{A}{\delta^2}}\delta^{-\frac{1}{3}}}{\sqrt[3]{4}\,|\Theta|}v_2^{\mathrm{u}} - \cos\theta\right).\end{aligned}$$

So, we deduce that, for $n \in \mathbb{Z}$,

$$\nu_2(\delta)T_\delta + \theta = n\pi - \arcsin\widehat{g_0}(\delta).$$

By Proposition III.3.2 and using the asymptotic expressions in (III.3.11) and (III.3.12),

$$|\widehat{g_0}(\delta)| \leq C\delta\,|(v(T_\delta), w(T_\delta))| + \frac{C}{|\log\delta|} \leq C\delta\left|(v(T_\delta^0), w(T_\delta^0))\right| + \frac{C}{|\log\delta|} \leq \frac{C}{|\log\delta|}.$$

Then, δ has to satisfy

$$\frac{A}{\delta^4}\sqrt{\frac{8}{21}}\,(1 + \widehat{g_1}(\delta)) = \pi n,$$

where $\widehat{g_1}(\delta) = \mathcal{O}(\delta^2 |\log\delta|)$. Then, by the Implicit Function Theorem, there exists $N_0 > 0$ and a sequence $\{\delta_n\}_{n \geq N_0} \subset (0, \delta_0)$ satisfying the statement of the Theorem and that

$$\delta_n = \sqrt[8]{\frac{8}{21}}\sqrt[4]{\frac{A}{n\pi}}\left(1 + \mathcal{O}\left(\frac{1}{\log n}\right)\right), \qquad \text{for } n \geq N_0.$$

□

Chapter III.4

Proof of Theorem III.2.10

This section is devoted to prove Theorem III.2.10. First, in Section III.4.1 we sum up the results concerning the unperturbed separatrix of the Hamiltonian H_{p} in (III.2.9) presented in Part I. Next, in Section III.4.2, we obtain and analyze parametrizations of the unstable and stable manifold of the Lyapunov periodic orbits given in Proposition III.2.4. Last, in Section III.4.3, we analyze the intersections between the manifolds to finish the prove of Theorem III.2.10.

Throughout this section and the following ones, we denote the components of all the functions and operators by a numerical sub-index $f = (f_1, f_2, f_3, f_4)^T$, unless stated otherwise. In addition, we denote the canonical base of $\mathbb{C}^4$ by $\{\mathbf{e_j}\}_{j=1..4}$.

III.4.1 The unperturbed separatrix

Let us consider the unperturbed system H_0 as given in (III.2.17). Notice that the plane $\{x = y = 0\}$ is invariant for H_0 and the dynamics on it are described by

$$H_{\mathrm{p}}(\lambda, \Lambda) = -\frac{3}{2}\Lambda^2 + V(\lambda), \qquad V(\lambda) = 1 - \cos\lambda - \frac{1}{\sqrt{2 + 2\cos\lambda}},$$

(see (III.2.9)). The origin $(\lambda, \Lambda) = (0, 0)$ is a saddle point with two separatrices associated to it (see Figure III.2.1). In Part I, we studied a real-analytic time-parametrization of such separatrices. The following result summarizes Theorem I.2.2 and Corollary I.2.4.

Proposition III.4.1. *Let $\lambda_0 > 0$ be as given in (III.2.18). There exists $0 < \beta_0 < \frac{\pi}{2}$ such that the time-parametrization $(\lambda_{\mathrm{p}}(u), \Lambda_{\mathrm{p}}(u))$ of the right separatrix (i.e, $\lambda_{\mathrm{p}}(u) \in$*

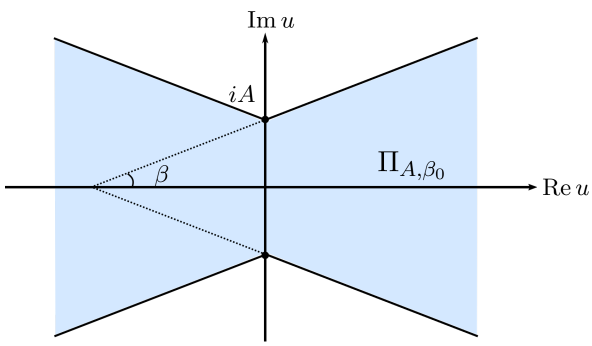


FIGURE III.4.1: Representation of the domain Π_{A,β_0} in (III.4.1).

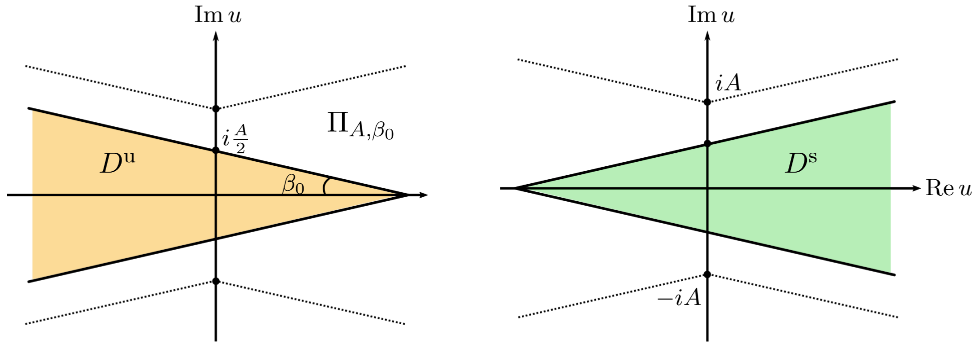


FIGURE III.4.2: Representation of the domains D^{u} and D^{s} in (III.4.3).

$(0,\pi)$) of H_{p} with $(\lambda_{\mathrm{p}}(0), \Lambda_{\mathrm{p}}(0)) = (\lambda_0, 0)$ extends analytically to

$$\begin{aligned} \Pi_{A,\beta_0} = &\{u \in \mathbb{C} \ : \ |\mathrm{Im}\, u| < \tan\beta \,\mathrm{Re}\, u + A\} \cup \\ &\{u \in \mathbb{C} \ : \ |\mathrm{Im}\, u| < -\tan\beta \,\mathrm{Re}\, u + A\}, \end{aligned} \tag{III.4.1}$$

with $A > 0$ as given in (III.1.8), (see Figure III.4.1). Moreover,

- *For $|u| \gg 1$, there exists $C > 0$ such that $|\lambda_{\mathrm{p}}(u)|, |\Lambda_{\mathrm{p}}(u)| \leq Ce^{-\sqrt{\frac{21}{8}}|\mathrm{Re}\, u|}$.*
- *For $u \in \overline{\Pi_{A,\beta_0}}$, $\lambda_{\mathrm{p}}(u) = \pi$ if and only if $u = \pm iA$.*
- *For $u \in \overline{\Pi_{A,\beta_0}}$, $\Lambda_{\mathrm{p}}(u) = 0$ if and only if $u = 0$.*

Next result establishes a suitable domain for the time-parametrization of the unperturbed separatrix, which we denote as

$$\sigma_{\mathrm{p}}(u) = (\lambda_{\mathrm{p}}(u), \Lambda_{\mathrm{p}}(u), 0, 0)^T. \tag{III.4.2}$$

III.4.2 Existence of the perturbed invariant manifolds

We devote this section to obtain and analyze parametrizations of the 2-dimensional branches of the manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho)$, where $\{\mathfrak{P}_\rho\}_{\rho\in(0,\rho_0]}$ is the family of periodic orbits given in Proposition III.2.4.

We find parametrizations of the manifolds through a Perron-like method. In particular, following the ideas in [BFG+12], we write the perturbed manifolds as functions of τ, which parametrizes the Lyapunov periodic orbit $\mathfrak{P}_\rho(\tau;\delta)$, and u, which parametrizes the unperturbed homoclinic orbit $\sigma_{\mathrm{p}}(u)$ (see (III.4.2)).

Let us consider the following complex domains (see Figure III.4.2),

$$D^{\mathrm{u}} = \left\{u \in \mathbb{C} \ : \ |\mathrm{Im}\, u| < \tfrac{A}{2} - \tan\beta_0 \,\mathrm{Re}\, u\right\}, \qquad D^{\mathrm{s}} = \{u \in \mathbb{C} \ : \ -u \in D^{\mathrm{u}}\}. \tag{III.4.3}$$

Then, for $\diamond \in \{\mathrm{u},\mathrm{s}\}$, we consider the parametrizations $Z^\diamond(u,\tau)$ satisfying that

$$\{Z^\diamond(u,\tau) \ : \ (u,\tau) \in D^\diamond \times \mathbb{T}_d\} \subseteq \mathcal{W}^{\diamond,+}(\mathfrak{P}_\rho).$$

Notice that, for the unperturbed problem, since $\sigma_{\mathrm{p}}(u)$ is a time-parametrization it satisfies $\dot{u} = 1$. In addition, by Proposition III.2.4, the dynamics for $\mathfrak{P}_\rho(\tau;\delta)$ satisfy $\dot{\tau} = \frac{\omega_{\rho,\delta}}{\delta^2}$. Therefore, we impose that the dynamics on the perturbed parametrizations

$Z^\diamond$ are given by

$$\dot{u} = 1, \qquad \dot{\tau} = \frac{\omega_{\rho,\delta}}{\delta^2}.$$

Hence, the parametrizations satisfy the system given by the Hamiltonian H in (III.2.8):

$$\partial_u Z^\diamond(u,\tau) + \frac{\omega_{\rho,\delta}}{\delta^2}\partial_\tau Z^\diamond(u,\tau) = \begin{pmatrix} \mathbf{J} & 0 \\ 0 & i\mathbf{J} \end{pmatrix} DH(Z^\diamond(u,\tau);\delta) \text{ with } \mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{III.4.4}$$

and we impose the asymptotic conditions

$$\lim_{\operatorname{Re} u \to -\infty} Z^{\mathrm{u}}(u,\tau) = \lim_{\operatorname{Re} u \to +\infty} Z^{\mathrm{s}}(u,\tau) = \mathfrak{P}_\rho(\tau;\delta), \quad \text{for all} \quad \tau \in \mathbb{T}_d. \tag{III.4.5}$$

In the next theorem we prove the existence and certain properties of the parametrization $Z^\diamond$, for $\diamond \in \{\mathrm{u},\mathrm{s}\}$. In order to do so, we consider the decomposition

$$Z^\diamond(u,\tau) = \mathfrak{P}_\rho(\tau;\delta) + \sigma_{\mathrm{p}}(u) + Z_1^\diamond(u,\tau), \tag{III.4.6}$$

with σ_{p} as given in (III.4.2). Notice that, since $D^{\mathrm{u}} \subset \Pi_{\frac{A}{2},\beta_0}$, σ_{p} is well defined in D^{u}. The proof of the following result is deferred to Section III.5.1.

Proposition III.4.2. *Fix $d > 0$ and $\diamond \in \{\mathrm{u},\mathrm{s}\}$. Let $\rho_0 > 0$ be the constant given in Proposition III.2.4. There exist $c_0, c_1, \delta_0, b_3 > 0$ such that, for $\rho \in [0,\rho_0]$ and $\delta \in (0,\delta_0)$, equation (III.4.4) has a unique real-analytic solution $Z^\diamond : D^\diamond \times \mathbb{T}_d \to U_{\mathbb{C}}(c_0,c_1)$ that can be decomposed as in (III.4.6) and satisfies*

$$\langle Z_1^\diamond(0,\tau), \mathbf{e_2}\rangle = 0, \qquad \textit{for all } \tau \in \mathbb{T}_d,$$

and the corresponding asymptotic condition in (III.4.5). In addition, for $\nu = \frac{1}{2}\sqrt{\frac{21}{8}}$,

$$|Z_1^\diamond(u,\tau)| \le b_3 \delta e^{-\nu|\operatorname{Re} u|}, \qquad \textit{for } (u,\tau) \in D^\diamond \times \mathbb{T}_d.$$

Notice that, by Proposition III.2.4 when $\rho = 0$, $\mathfrak{P}_0(\tau;\delta) \equiv \mathfrak{L}(\delta)$ is a fixed point and that, $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ are 1-dimensional invariant manifolds. Then, for $\diamond \in \{\mathrm{u},\mathrm{s}\}$, Proposition III.4.2 provides parametrizations $z_1^\diamond$ independent of τ satisfying

$$\{z^\diamond(u) \ : \ u \in D^\diamond\} \subseteq \mathcal{W}^{\diamond,+}(\mathfrak{L}),$$

that can be decomposed as

$$z^\diamond(u) = \mathfrak{L} + \sigma_{\mathrm{p}}(u) + z_1^\diamond(u). \tag{III.4.7}$$

Corollary III.4.3. *Let $\diamond \in \{\mathrm{u},\mathrm{s}\}$. There exist $c_0, c_1, \delta_0, b_3 > 0$ such that, for $\delta \in (0,\delta_0)$ and $\rho = 0$, equation (III.4.4) has a unique real-analytic solution $z^\diamond : D^\diamond \to U_{\mathbb{C}}(c_0,c_1)$ that can be decomposed as in (III.4.7) and satisfies $\langle z_1^\diamond(0), \mathbf{e_2}\rangle = 0$ and the corresponding asymptotic condition in (III.4.5). In addition, for $\nu = \frac{1}{2}\sqrt{\frac{21}{8}}$,*

$$|z_1^\diamond(u)| \le b_3 \delta e^{-\nu|\operatorname{Re} u|}, \qquad \textit{for } u \in D^\diamond.$$

Finally, for $\diamond \in \{\mathrm{u},\mathrm{s}\}$, we can measure how well the 1-dimensional manifolds $\mathcal{W}^{\diamond,+}(\mathfrak{L})$ approximate the 2-dimensional manifolds $\mathcal{W}^{\diamond,+}(\mathfrak{P}_\rho)$.

Proposition III.4.4. *Fix $d > 0$ and $\diamond \in \{\mathrm{u},\mathrm{s}\}$. Let ρ_0 be the constant in Proposition III.2.4 and $Z_1^\diamond$ and $z_1^\diamond$ be the parametrizations given in Proposition III.4.2 and*

Corollary III.4.3, respectively. Then, there exists $\delta_0 > 0$ and a constant $b_4 > 0$ such that, for $\rho \in [0, \rho_0]$ and $\delta \in (0, \delta_0)$,

$$|Z_1^\diamond(u,\tau) - z_1^\diamond(u)| \leq b_4 \delta\rho, \qquad \text{for} \quad (u,\tau) \in D^\diamond \times \mathbb{T}_d.$$

The proof of this proposition is postponed to Section III.5.2.

III.4.3 End of the proof of Theorem III.2.10

To prove the first statement of Theorem III.2.10, in the next lemma we study the intersections between the section Σ_ρ (see (III.2.21)) and the unstable and stable manifolds of $\mathfrak{P}_\rho$ parametrized by Z^{u} and Z^{s}, respectively.

Lemma III.4.5. *Fix $\diamond \in \{\mathrm{u}, \mathrm{s}\}$. Let ρ_0 and $\mathfrak{P}_\rho$ be as given in Proposition III.2.4, $Z^\diamond$ be the parametrization given in (III.4.6) and Proposition III.4.2 and Σ_ρ the section given in (III.2.21). Then, there exists $\delta_0 > 0$ and a real-analytic function $\mathcal{U}_\rho^\diamond : \mathbb{T}_d \to D^\diamond$ such that, for $\rho \in [0, \rho_0]$ and $\delta \in (0, \delta_0)$,*

$$Z^\diamond(\mathcal{U}_\rho^\diamond(\tau), \tau) \in \Sigma_\rho, \qquad \text{for } \tau \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

Moreover, there exists $C > 0$ independent of ρ and δ such that, for $\tau \in \mathbb{T}_d$,

$$\mathcal{U}_0^\diamond \equiv 0, \qquad |\mathcal{U}_\rho^\diamond(\tau)| \leq C\delta\rho.$$

Proof. Since the parametrization $Z^\diamond$ is real-analytic (see Remark III.2.2), one has that

$$Z^\diamond(u,\tau) \in U_{\mathbb{R}}(c_0, c_1) \qquad \text{for } (u,\tau) \in (D^\diamond \cap \mathbb{R}) \times \mathbb{T}.$$

In addition, by Propositions III.2.4 and III.5.4, one has that

$$H(Z^\diamond(u,\tau)) = H(\mathfrak{P}_\rho(\tau;\delta)) = \frac{\rho^2}{\delta^2} + H(\mathfrak{L}(\delta)), \qquad \text{for } (u,\tau) \in (D^\diamond \cap \mathbb{R}) \times \mathbb{T}.$$

Therefore, it is only necessary to find a function $\mathcal{U}_\rho^\diamond(\tau)$ satisfying that $\langle Z^\diamond(\mathcal{U}_\rho^\diamond(\tau), \tau), \mathbf{e}_2 \rangle = \delta^2 \mathfrak{L}_\Lambda(\delta)$ for all $\tau \in \mathbb{T}$. Then, by the decomposition (III.4.6) of $Z^\diamond$ and Proposition III.2.4,

$$\delta\rho\Lambda_{\mathfrak{P}}(\tau) + \Lambda_{\mathrm{p}}(\mathcal{U}_\rho^\diamond(\tau)) + \langle Z_1^\diamond(\mathcal{U}_\rho^\diamond(\tau), \tau), \mathbf{e}_2 \rangle = 0.$$

By Proposition III.4.1, one has that $\Lambda_{\mathrm{p}}(u) = \dot\Lambda_{\mathrm{p}}(0)u + \mathcal{O}(u^2)$ with $\dot\Lambda_{\mathrm{p}}(0) = -V'(\lambda_0) \neq 0$. Then, $\mathcal{U}_\rho^\diamond$ is a solution of the fixed point equation given by the operator

$$F[\mathcal{U}_\rho^\diamond](\tau) = -\frac{1}{\dot\Lambda_{\mathrm{p}}(0)} \left[\delta\rho\Lambda_{\mathfrak{P}}(\tau) + \left(\Lambda_{\mathrm{p}}(\mathcal{U}_\rho^\diamond(\tau)) - \dot\Lambda_{\mathrm{p}}(0)\mathcal{U}_\rho^\diamond \right) + \langle Z_1^\diamond(\mathcal{U}_\rho^\diamond(\tau), \tau), \mathbf{e}_2 \rangle \right].$$

Notice that, by Propositions III.2.4 and III.4.2,

$$|F[0](\tau)| = \delta\rho \frac{|\Lambda_{\mathfrak{P}}(\tau)|}{|\dot\Lambda_{\mathrm{p}}(0)|} \leq C\delta\rho.$$

Moreover, for real-analytic functions $\mathcal{U}, \mathcal{V} : \mathbb{T}_d \to D^\diamond$ satisfying that $|\mathcal{U}|, |\mathcal{V}| \leq C\delta\rho$ and applying the Mean Value Theorem, Proposition III.4.2, the operator satisfies that, if δ small enough,

$$|F[\mathcal{U}]-F[\mathcal{V}]|\leq C|\mathcal{U}^2-\mathcal{V}^2|+|\mathcal{U}-\mathcal{V}|\sup_{s\in[0,1]}|\langle\partial_u Z_1^\diamond(s\mathcal{U}+(1-s)\mathcal{V},\tau),\mathbf{e_2}\rangle|$$
$$\leq C\delta\rho\,|\mathcal{U}-\mathcal{V}|\leq\frac{1}{2}\,|\mathcal{U}-\mathcal{V}|\,,$$

where we have used that $\langle Z_1^\diamond(0,\tau),\mathbf{e_2}\rangle=0$. Hence, F has a fixed point $\mathcal{U}_\rho^\diamond$ satisfying that $\left|\mathcal{U}_\rho^\diamond(\tau)\right|\leq C\delta\rho$, for $\tau\in\mathbb{T}_d$. □

Then, the first statement of Theorem III.2.10 is a direct consequence of Lemma III.4.5. Moreover, if we denote by $\partial\mathcal{D}_\rho^{\mathrm{u}}$ and $\partial\mathcal{D}_\rho^{\mathrm{s}}$ the first intersection of the manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho)$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho)$ with the section Σ_ρ, respectively, one has that

$$\begin{aligned}\partial\mathcal{D}_\rho^{\mathrm{u}}&=\left\{Z^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau),\tau)\ :\ \tau\in\mathbb{T}\right\}\subset\Sigma_\rho\cap\mathcal{W}^{\mathrm{u},+}(\mathfrak{P}_\rho),\\ \partial\mathcal{D}_\rho^{\mathrm{s}}&=\left\{Z^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau),\tau)\ :\ \tau\in\mathbb{T}\right\}\subset\Sigma_\rho\cap\mathcal{W}^{\mathrm{s},+}(\mathfrak{P}_\rho).\end{aligned}\tag{III.4.8}$$

In particular, the first intersection of the manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ with the section Σ_0 correspond to the points $\partial\mathcal{D}_0^{\mathrm{u}}=\{z^{\mathrm{u}}(0)\}$ and $\partial\mathcal{D}_0^{\mathrm{s}}=\{z^{\mathrm{s}}(0)\}$.

To prove the rest of statements, we study the difference between the parametrizations Z^{u} and Z^{s} at the points considered in (III.4.8). Since $\Sigma_\rho\subset U_{\mathbb{R}}(c_0,c_1)$ (see (III.2.21)), by the definition of the domain in (III.2.12), for $\tau^{\mathrm{u}},\tau^{\mathrm{s}}\in\mathbb{T}$ one has that

$$\begin{aligned}&\langle Z^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}}),\tau^{\mathrm{u}})-Z^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau^{\mathrm{s}}),\tau^{\mathrm{s}}),\mathbf{e_2}\rangle=0,\\ &\langle Z^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{s}}),\tau^{\mathrm{u}})-Z^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau^{\mathrm{s}}),\tau^{\mathrm{s}}),\mathbf{e_4}\rangle=\overline{\langle Z^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}}),\tau^{\mathrm{u}})-Z^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau),\tau^{\mathrm{s}}),\mathbf{e_3}\rangle},\end{aligned}$$

and $\langle Z^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}}),\tau^{\mathrm{u}})-Z^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau),\tau^{\mathrm{s}}),\mathbf{e_1}\rangle$ can be recovered by the conservation of energy $H=\frac{\rho^2}{\delta^2}+H(\mathfrak{L})$. Therefore, it suffices to study the zeroes of the function

$$\Delta(\tau^{\mathrm{u}},\tau^{\mathrm{s}}):=\langle Z^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}}),\tau^{\mathrm{u}})-Z^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau^{\mathrm{s}}),\tau^{\mathrm{s}}),\mathbf{e_4}\rangle.$$

Let us recall that, by Proposition III.4.4, the difference $\Delta(\tau^{\mathrm{u}},\tau^{\mathrm{s}})$ is given at first order, by the difference $z^{\mathrm{u}}-z^{\mathrm{s}}$. Therefore, by (III.4.6) and (III.4.7), for $\diamond\in\{\mathrm{u},\mathrm{s}\}$, we consider the decomposition

$$Z^\diamond(\mathcal{U}^\diamond(\tau),\tau)=\mathfrak{P}_\rho(\tau)+\sigma_{\mathrm{p}}(\mathcal{U}^\diamond(\tau))+z_1^\diamond(\mathcal{U}^\diamond(\tau))+\left(Z_1^\diamond(\mathcal{U}^\diamond(\tau),\tau)-z_1^\diamond(\mathcal{U}^\diamond(\tau))\right),$$

where $Z_1^\diamond$ and $z_1^\diamond$ are given in Proposition III.4.2 and Corollary III.4.3, respectively. Recall that, $\sigma_{\mathrm{p}}=(\lambda_{\mathrm{p}},\Lambda_{\mathrm{p}},0,0)$ (see (III.4.2)) and, by Proposition III.2.4, $\mathfrak{P}_\rho=\mathfrak{L}+\rho(0,0,e^{i\tau},e^{-i\tau})+\delta\rho(\lambda_{\mathfrak{P}},\Lambda_{\mathfrak{P}},x_{\mathfrak{P}},y_{\mathfrak{P}})$. Therefore, we look for τ^{u} and τ^{s} such that

$$\Delta(\tau^{\mathrm{u}},\tau^{\mathrm{s}})=\ \rho(e^{-i\tau^{\mathrm{u}}}-e^{-i\tau^{\mathrm{s}}})+\sqrt[6]{2}\,\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}\,|\Theta|\,e^{i\theta}+M(\delta)+R(\tau^{\mathrm{u}},\tau^{\mathrm{s}},\delta,\rho)=0,\tag{III.4.9}$$

where

$$\begin{aligned}\theta&=\arg\langle z_1^{\mathrm{u}}(0)-z_1^{\mathrm{s}}(0),\mathbf{e_4}\rangle,\\ M(\delta)&=\langle z_1^{\mathrm{u}}(0)-z_1^{\mathrm{s}}(0),\mathbf{e_4}\rangle-\sqrt[6]{2}\,\delta^{\frac{1}{3}}e^{-\frac{A}{\delta^2}}\,|\Theta|\,e^{i\theta},\\ R(\tau^{\mathrm{u}},\tau^{\mathrm{s}},\delta,\rho)&=\langle Z_1^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}}),\tau^{\mathrm{u}})-z_1^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}})),\mathbf{e_4}\rangle-\langle Z_1^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau^{\mathrm{s}}),\tau^{\mathrm{s}})-z_1^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau^{\mathrm{s}})),\mathbf{e_4}\rangle\\ &\quad+\langle z_1^{\mathrm{u}}(\mathcal{U}_\rho^{\mathrm{u}}(\tau^{\mathrm{u}}))-z_1^{\mathrm{u}}(0),\mathbf{e_4}\rangle-\langle z_1^{\mathrm{s}}(\mathcal{U}_\rho^{\mathrm{s}}(\tau^{\mathrm{s}}))-z_1^{\mathrm{s}}(0),\mathbf{e_4}\rangle\\ &\quad+\delta\rho(y_{\mathfrak{P}}(\tau^{\mathrm{u}})-y_{\mathfrak{P}}(\tau^{\mathrm{s}})).\end{aligned}$$

Notice that, by Corollary III.2.9, Propositions III.2.4 and III.4.4 and Lemma III.4.5,

$$M(\delta) = \mathcal{O}\left(\frac{\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}}}{|\log \delta|}\right), \qquad R(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, \delta, \rho) = \mathcal{O}(\delta\rho).$$

Since, by Ansatz III.1.2, $|\Theta| \neq 0$, we can consider the auxiliary parameter $r \in (0, r_0]$,

$$r = \frac{2e^{\frac{A}{\delta^2}}}{\sqrt[6]{2}\,\delta^{\frac{1}{3}}|\Theta|}\rho, \qquad \text{and} \qquad r_0 = \frac{2e^{\frac{A}{\delta^2}}}{\sqrt[6]{2}\,\delta^{\frac{1}{3}}|\Theta|}\rho_0. \tag{III.4.10}$$

Then, equation (III.4.9) is equivalent to

$$r(e^{-i(\tau^{\mathrm{u}}+\theta)} - e^{-i(\tau^{\mathrm{s}}+\theta)}) + 2 + g(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, \delta) = 0, \tag{III.4.11}$$

where,

$$\begin{aligned} g(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, \delta) &= \frac{2e^{\frac{A}{\delta^2}} e^{-i\theta}}{\sqrt[6]{2}\,\delta^{\frac{1}{3}}|\Theta|}\left(M(\delta) + R\left(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, \delta, \frac{\sqrt[6]{2}}{2}\delta^{\frac{1}{3}}|\Theta|e^{-\frac{A}{\delta^2}} r\right)\right) \\ &= \mathcal{O}\left(\frac{1}{|\log \delta|}\right) + \mathcal{O}\left(\delta r\right). \end{aligned}$$

By introducing $G = (G_1, G_2) : \mathbb{T}^2 \times [0, r_0] \times [0, \delta_0) \to \mathbb{R}^2$, as

$$\begin{aligned} G_1(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, \delta) &= r\left(\cos(\tau^{\mathrm{u}} + \theta) - \cos(\tau^{\mathrm{s}} + \theta)\right) + 2 + \operatorname{Re} g(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, \delta), \\ G_2(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, \delta) &= r\left(\sin(\tau^{\mathrm{u}} + \theta) - \sin(\tau^{\mathrm{s}} + \theta)\right) + \operatorname{Im} g(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, \delta), \end{aligned} \tag{III.4.12}$$

equations (III.4.11) are equivalent to $G(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, \delta) = (0, 0)$. Next result characterizes the solutions of this equation (see also Figure III.4.3).

Lemma III.4.6. *Fix $\gamma \in (0, \frac{\pi}{2})$ and consider $I_\gamma = [-\theta - \gamma, -\theta + \gamma]$. There exists δ_γ satisfying $\lim_{\gamma \to \pi/2} \delta_\gamma = 0$ and functions*

$$(\tau^{\mathrm{u}}_*, r_*) : I_\gamma \times (0, \delta_\gamma) \to \mathbb{T} \times \mathbb{R},$$

such that $G(\tau^{\mathrm{u}}_*(\tau^{\mathrm{s}}, \delta), \tau^{\mathrm{s}}, r_*(\tau^{\mathrm{s}}, \delta), \delta) = (0, 0)$ and

$$\begin{aligned} \tau^{\mathrm{u}}_*(\tau^{\mathrm{s}}, \delta) &= \pi - \tau^{\mathrm{s}} - 2\theta + \mathcal{O}\left(\frac{1}{|\log \delta|}\right), \\ r_*(\tau^{\mathrm{s}}, \delta) &= \frac{1}{\cos(\tau^{\mathrm{s}} + \theta)} + \mathcal{O}\left(\frac{1}{|\log \delta|}\right). \end{aligned}$$

Proof. For $r \geq 1$, the equation $G(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, 0) = (0, 0)$ has a family of solutions given by

$$S_\alpha = \left\{(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, 0) = \left(\pi - \alpha - \theta, \alpha - \theta, \frac{1}{\cos \alpha}, 0\right)\right\} \quad \text{with } \alpha \in [-\gamma, \gamma] \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Therefore, for $\delta > 0$, it only remains to find zeroes of function G using the Implicit Function Theorem around every solution of this family. □

To prove the second statement of Theorem III.2.10, we assume the setting given by Lemma III.4.6. In particular, for $R > 1$, we take $\gamma = \arccos(\frac{1}{R}) \in (0, \frac{\pi}{2})$.

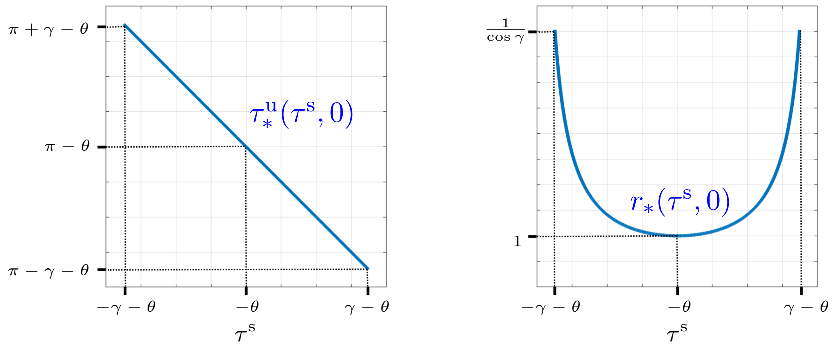


FIGURE III.4.3: Plot in τ^{s} of functions $\tau_*^{\mathrm{u}}(\tau^{\mathrm{s}}, 0)$ and $r_*(\tau^{\mathrm{s}}, 0)$ as given in Lemma III.4.6.

First consider equation $G(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, 0) = (0, 0)$ and notice that it has solutions for $r \in [1, R]$, see Figure III.4.3. Moreover, the minimum and maximum points, for $\delta = 0$, are given at $r_*(-\theta, 0) = 1$ and $r_*(\pm\gamma - \theta) = R$. Therefore, for $\delta \in (0, \delta_\gamma)$, there exist $r_{\min}(\delta)$ and $r_{\max}(\delta)$ such that, for $r \in [r_{\min}(\delta), r_{\max}(\delta)]$, equation $G(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, \delta) = (0, 0)$ has at least one solution and

$$r_{\min}(\delta) = 1 + \mathcal{O}\left(\frac{1}{|\log \delta|}\right), \qquad r_{\max}(\delta) = R + \mathcal{O}\left(\frac{1}{|\log \delta|}\right).$$

In addition, there exists $\tau_{\min}^{\mathrm{s}}(\delta)$, such that

$$\tau_{\min}^{\mathrm{s}}(\delta) = -\theta + \mathcal{O}\left(\frac{1}{|\log \delta|}\right), \quad r_{\min}(\delta) = r_*(\tau_{\min}^{\mathrm{s}}(\delta), \delta), \quad \partial_{\tau^{\mathrm{s}}} r_*(\tau_{\min}^{\mathrm{s}}(\delta), \delta) = 0. \tag{III.4.13}$$

Taking into account (III.4.10), we define

$$\rho_{\min}(\delta) = \frac{\sqrt[6]{2}}{2}\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} |\Theta| r_{\min}(\delta), \qquad \rho_{\max}(\delta) = \frac{\sqrt[6]{2}}{2}\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} |\Theta| r_{\max}(\delta),$$

and assume $\delta > 0$ small enough such that $\rho_{\max}(\delta) < \rho_0$. Then, for $\rho \in [\rho_{\min}(\delta), \rho_{\max}(\delta)]$, the closed curves $\partial\mathcal{D}_\rho^{\mathrm{u}}$ and $\partial\mathcal{D}_\rho^{\mathrm{s}}$ (see (III.4.8)) intersect at least once. See Figure III.4.4 for a representation of the case $\delta = 0$.

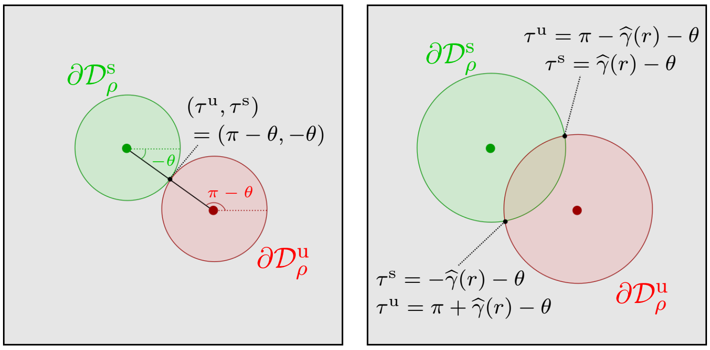
Finally, we prove the third and fourth statement of Theorem III.2.10. Let us denote the solutions of equation $G(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, \delta) = (0, 0)$ given in Lemma III.4.6 as

$$P(\tau^{\mathrm{s}}, \delta) = (\tau_*^{\mathrm{u}}(\tau^{\mathrm{s}}, \delta), \tau^{\mathrm{s}}, r_*(\tau^{\mathrm{s}}, \delta), \delta)$$

and consider the function

$$\widetilde{G}(\tau^{\mathrm{s}}, \delta) = \det\left(\frac{\partial G}{\partial(\tau^{\mathrm{u}}, \tau^{\mathrm{s}})}(P(\tau^{\mathrm{s}}, \delta))\right), \qquad (\tau^{\mathrm{s}}, \delta) \in I_\gamma \times (0, \delta_\gamma).$$

Then, the values such that $\widetilde{G} \neq 0$ correspond to transverse intersections of the closed curves $\partial\mathcal{D}_\rho^{\mathrm{u}}$ and $\partial\mathcal{D}_\rho^{\mathrm{s}}$. Therefore, by (III.4.13), we need to see that $(\tau_{\min}^{\mathrm{s}}(\delta), \delta)$ is a simple zero of $\widetilde{G}$ and otherwise $\widetilde{G} \neq 0$, for $\tau^{\mathrm{s}} \neq \tau_{\min}^{\mathrm{s}}(\delta)$. Likewise, the values such that $\widetilde{G} = 0$ and $\partial_{\tau^{\mathrm{s}}}\widetilde{G} \neq 0$ correspond to quadratic tangencies.



(a) $r = 1$ (b) $r \in (1, R]$

$\widehat{\gamma}(r) = \arccos(\frac{1}{r})$

FIGURE III.4.4: Representation of solutions of equation $G(\tau^{\mathrm{u}}, \tau^{\mathrm{s}}, r, 0) = (0, 0)$ (see (III.4.12)) in function of coordinate r.

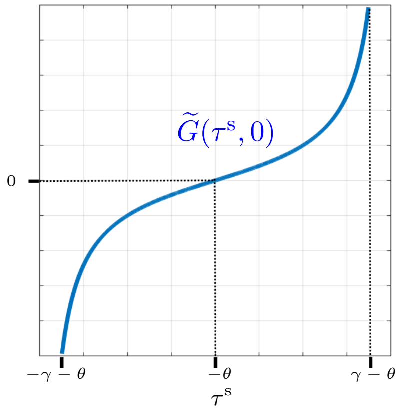


FIGURE III.4.5: Plot in τ^{s} of function $\widetilde{G}(\tau^{\mathrm{s}}, 0)$ as given in (III.4.12).

By the definition of function G in (III.4.12) and Lemma III.4.6, for $(\tau^{\mathrm{s}}, \delta) \in I_\gamma \times (0, \delta_\gamma)$, one has that

$$\widetilde{G}(\tau^{\mathrm{s}}, \delta) = 2\tan(\tau^{\mathrm{s}} + \theta) + \mathcal{O}\left(\frac{1}{|\log \delta|}\right), \quad \partial_{\tau^{\mathrm{s}}}\widetilde{G}(\tau^{\mathrm{s}}, \delta) = \frac{2}{\cos^2(\tau^{\mathrm{s}} + \theta)} + \mathcal{O}\left(\frac{1}{|\log \delta|}\right),$$

(see Figure III.4.5). Notice that, for δ small enough,

$$\partial_{\tau^{\mathrm{s}}}\widetilde{G}(\tau^{\mathrm{s}}, \delta) \geq 2 + \mathcal{O}\left(\frac{1}{|\log \delta|}\right) > 0.$$

Therefore, $\widetilde{G}$ is a strictly increasing function in τ^{s} and can only have one simple zero. Moreover, this zero corresponds to $\tau^{\mathrm{s}} = \tau^{\mathrm{s}}_{\mathrm{min}}(\delta)$. Indeed, since $G(P(\tau^{\mathrm{s}}_{\mathrm{min}}(\delta), \delta)) = (0, 0)$ and $\partial_{\tau^{\mathrm{s}}} r_*(\tau^{\mathrm{s}}_{\mathrm{min}}(\delta), \delta) = 0$ (see (III.4.13)), taking the derivatives one has that

$$\partial_{\tau^{\mathrm{s}}} G(P(\tau^{\mathrm{s}}_{\mathrm{min}}(\delta), \delta)) + \partial_{\tau^{\mathrm{u}}} G(P(\tau^{\mathrm{s}}_{\mathrm{min}}(\delta), \delta))\partial_{\tau^{\mathrm{s}}}\tau^{\mathrm{u}}_*(\tau^{\mathrm{s}}_{\mathrm{min}}(\delta), \delta) = (0, 0),$$

and, as a result, the vectors $\partial_{\tau^{\mathrm{s}}} G$ and $\partial_{\tau^{\mathrm{u}}} G$ at $P(\tau^{\mathrm{s}}_{\mathrm{min}}(\delta), \delta)$ are linearly dependent and, therefore, $\widetilde{G}(\tau^{\mathrm{s}}_{\mathrm{min}}, \delta) = 0$. Hence, there exists at least one quadratic tangency at $r = r_{\mathrm{min}}(\delta)$ and at least two transverse intersection for each $r \in (r_{\mathrm{min}}(\delta), r_{\mathrm{max}}(\delta)]$.

Chapter III.5

The perturbed invariant manifolds

In this section, we prove Propositions III.4.2 and III.4.4. We only prove these results for the unstable manifold, the proof for the stable manifold is analogous.

From now on, we consider a fixed $d > 0$ and the corresponding complex torus $\mathbb{T}_d$ (see (III.2.19)). We also set ρ_0 satisfying the conditions in Proposition III.2.4 and $\rho \in [0, \rho_0]$. To avoid cumbersome notations, throughout the rest of the document, we omit the dependence in the parameter δ unless necessary and denote by C any positive constant independent of δ and ρ to state estimates.

III.5.1 Proof of Proposition III.4.2

We look for parametrizations of the invariant manifold $\mathcal{W}^{u,+}(\mathfrak{P}_\rho)$ of the form (see (III.4.6))

$$Z^{\mathrm{u}}(u,\tau) = \mathfrak{P}_\rho(\tau) + \sigma_{\mathrm{p}}(u) + Z_1^{\mathrm{u}}(u,\tau), \qquad (u,\tau) \in D^{\mathrm{u}} \times \mathbb{T}_d,$$

satisfying the equation (III.4.4) and the asymptotic condition given in (III.4.5).

Let us recall that $H = H_{\mathrm{p}} + H_{\mathrm{osc}} + H_1$ (see (III.2.8)). Since $\sigma_{\mathrm{p}} = (\lambda_{\mathrm{p}}, \Lambda_{\mathrm{p}}, 0, 0)$ is a solution of the unperturbed system $H_{\mathrm{p}} + H_{\mathrm{osc}}$, it satisfies equation (III.4.4) for the unperturbed Hamiltonian (see Proposition III.4.1). By Proposition III.2.4, $\mathfrak{P}_\rho$ satisfies (III.4.4). Then, the parametrization $Z_1^{\diamond}$ satisfies

$$\mathcal{L}_\rho Z_1^{\mathrm{u}} = \mathcal{R}_\rho[Z_1^{\mathrm{u}}], \tag{III.5.1}$$

where

$$\mathcal{L}_\rho \zeta = \left(\partial_u + \frac{\omega_{\rho,\delta}}{\delta^2}\partial_\tau - \mathcal{A}(u)\right)\zeta, \quad \mathcal{A} = \begin{pmatrix} 0 & -3 & 0 & 0 \\ -V''(\lambda_{\mathrm{p}}(u)) & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix} \tag{III.5.2}$$

and

$$\mathcal{R}_\rho[\zeta] = \begin{pmatrix} \partial_\Lambda H_1(\mathfrak{P}_\rho + \sigma_{\mathrm{p}} + \zeta) - \partial_\Lambda H_1(\mathfrak{P}_\rho) \\ -T_\rho[\zeta_1] - \partial_\lambda H_1(\mathfrak{P}_\rho + \sigma_{\mathrm{p}} + \zeta) + \partial_\lambda H_1(\mathfrak{P}_\rho) \\ i\partial_y H_1(\mathfrak{P}_\rho + \sigma_{\mathrm{p}} + \zeta) - i\partial_y H_1(\mathfrak{P}_\rho) \\ -i\partial_x H_1(\mathfrak{P}_\rho + \sigma_{\mathrm{p}} + \zeta) + i\partial_x H_1(\mathfrak{P}_\rho) \end{pmatrix}, \tag{III.5.3}$$

with

$$T_\rho[\zeta_1] = V'(\lambda_{\mathrm{p}} + \mathfrak{P}_{\rho,1} + \zeta_1) - V'(\lambda_{\mathrm{p}}) - V'(\mathfrak{P}_{\rho,1}) - V''(\lambda_{\mathrm{p}})\zeta_1. \tag{III.5.4}$$

We solve equation (III.5.1) by means of a fixed point scheme on a suitable Banach space. For $\alpha \geq 0$, we consider the Banach space

$$\mathcal{Y}_\alpha = \left\{ \zeta : D^{\mathrm{u}} \times \mathbb{T}_d \to \mathbb{C} \ : \ \zeta \text{ real-analytic, } \|\zeta\|_\alpha := \sup_{(u,\tau)\in D^{\mathrm{u}}\times\mathbb{T}_d} \left| e^{-\alpha u}\zeta(u,\tau)\right| < +\infty \right\},$$

where D^{u} is the domain introduced in (III.4.3). We also consider the product Banach space $\mathcal{Y}_\alpha^4 = \mathcal{Y}_\alpha \times ... \times \mathcal{Y}_\alpha$ endowed with the norm

$$\|\zeta\|_\alpha^\times = \sum_{j=1}^4 \|\zeta_j\|_\alpha.$$

In the next lemma, we state some properties of these Banach spaces. We will use them without special mention throughout the section.

Lemma III.5.1. *The following statements hold.*

1. *If $\alpha \geq \beta \geq 0$, then $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_\beta$. Moreover, for $\zeta \in \mathcal{Y}_\alpha$, $\|\zeta\|_\beta \leq C\|\zeta\|_\alpha$.*
2. *If $\zeta \in \mathcal{Y}_\alpha$ and $\eta \in \mathcal{Y}_\beta$, then $\zeta\eta \in \mathcal{Y}_{\alpha+\beta}$ and $\|\zeta\eta\|_{\alpha+\beta} \leq \|\zeta\|_\alpha\|\eta\|_\beta$.*

Next, we obtain and analyze a suitable right-inverse of operator $\mathcal{L}_\rho = \partial_u + \frac{\omega_{\rho,\delta}}{\delta^2}\partial_\tau - \mathcal{A}$, introduced in (III.5.2).

Lemma III.5.2. *Fix $u_0 \in \mathbb{R}\setminus\{0\}$ and consider the linear differential equation $\dot\zeta = \mathcal{A}(u)\zeta$, with $\mathcal{A}$ as given in (III.5.2). Then, a real-analytic fundamental matrix of this equation is*

$$\Phi(u) = \begin{pmatrix} 3f_\Phi(u) & 3g_\Phi(u) & 0 & 0 \\ -\dot f_\Phi(u) & -\dot g_\Phi(u) & 0 & 0 \\ 0 & 0 & e^{\frac{i}{\delta^2}u} & 0 \\ 0 & 0 & 0 & e^{-\frac{i}{\delta^2}u} \end{pmatrix},$$

with

$$f_\Phi(u) = \frac{1}{3\xi(0)}\left(\xi(u) - \frac{\dot\xi(0)}{\dot\Lambda_{\mathrm{p}}(0)}\dot\Lambda_{\mathrm{p}}(u)\right), \quad g_\Phi(u) = -\frac{\Lambda_{\mathrm{p}}(u)}{\dot\Lambda_{\mathrm{p}}(0)}, \quad \xi(u) = \Lambda_{\mathrm{p}}(u)\int_{u_0}^u \frac{dv}{\Lambda_{\mathrm{p}}^2(v)},$$

where the integration path in D^{u} corresponds to the straight line if $u \in \mathbb{C}\setminus\mathbb{R}$ and a path avoiding $u = 0$ when $u \in \mathbb{R}$.

Moreover, $\Phi(u)$ satisfies that $\det\Phi(u) = 1$, $\Phi(0) = \mathbf{Id}$ and that there exists a constant $C > 0$ such that, denoting $\nu = \frac{1}{2}\sqrt{\frac{21}{8}}$,

$$\|g_\Phi\|_{2\nu} \leq C, \quad \|\dot g_\Phi\|_{2\nu} \leq C, \quad \|f_\Phi\|_{-2\nu} \leq C, \quad \|\dot f_\Phi\|_{-2\nu} \leq C.$$

Proof. Let us recall that, by Proposition III.4.1, the time-parametrization of the separatrix satisfies that $\dot\lambda_{\mathrm{p}}(u) = -3\Lambda_{\mathrm{p}}(u)$ and $\dot\Lambda_{\mathrm{p}}(u) = -V'(\lambda_{\mathrm{p}}(u))$, for $u \in \Pi_{A,\beta_0}$. Then, a fundamental matrix of the equation $\dot\zeta = \mathcal{A}(u)\zeta$ is given by

$$\phi(u) = \begin{pmatrix} 3\xi(u) & 3\Lambda_{\mathrm{p}}(u) & 0 & 0 \\ -\dot\xi(u) & -\dot\Lambda_{\mathrm{p}}(u) & 0 & 0 \\ 0 & 0 & e^{\frac{i}{\delta^2}u} & 0 \\ 0 & 0 & 0 & e^{-\frac{i}{\delta^2}u} \end{pmatrix},$$

We stress that ξ is real-analytic in $D^{\mathrm{u}} \subset \Pi_{A,\beta_0}$. Indeed, one has that $u = 0$ is the only zero of $\Lambda_{\mathrm{p}}(u)$ (see Proposition III.4.1), that $\dot{\Lambda}_{\mathrm{p}}(0) = -V'(\lambda_{\mathrm{p}}(0)) \neq 0$ and $\ddot{\Lambda}_{\mathrm{p}}(0) = 0$. Thus, $\Lambda_{\mathrm{p}}(u) = \dot{\Lambda}_{\mathrm{p}}(0)u + \mathcal{O}(u^3)$. That implies that the integral appearing on ξ does not depend on the path of integration since its residue is zero. As a consequence, $\xi(u) \in \mathbb{R}$ for $u \in \mathbb{R}$.

In addition, since $\xi(0) = -\dot{\Lambda}_{\mathrm{p}}^{-1}(0) \neq 0$, we can perform a linear transformation to $\phi(u)$ to obtain the fundamental matrix $\Phi(u)$ satisfying $\Phi(0) = \mathrm{Id}$ and $\det \Phi(u) = 1$. Lastly, recalling that, by Proposition III.4.1, $\|\lambda_{\mathrm{p}}\|_{2\nu} \leq C$ and $\|\Lambda_{\mathrm{p}}\|_{2\nu} \leq C$, we obtain the corresponding estimates for f_Φ and g_Φ. □

We construct now a right-inverse of the operator $\mathcal{L}_\rho = \partial_u + \frac{\omega_{\rho,\delta}}{\delta^2}\partial_\tau - \mathcal{A}(u)$. For $\zeta \in \mathcal{Y}_\nu^4$, we consider the operator

$$\mathcal{G}_\rho[\zeta](u,\tau) = \sum_{j=1}^{4} \mathcal{G}_{\rho,j}[\zeta](u,\tau)\mathbf{e_j}, \tag{III.5.5}$$

given by

$$\begin{pmatrix} \mathcal{G}_{\rho,1}[\zeta](u,\tau) \\ \mathcal{G}_{\rho,2}[\zeta](u,\tau) \end{pmatrix} = \begin{pmatrix} 3f_\Phi(u) & 3g_\Phi(u) \\ -\dot{f}_\Phi(u) & -\dot{g}_\Phi(u) \end{pmatrix} \begin{pmatrix} \displaystyle\int_{-\infty}^{0} \mathcal{I}_1[\zeta_1,\zeta_2]\left(u+t, \tau + \frac{\omega_{\rho,\delta}}{\delta^2}t\right) dt \\ \displaystyle\int_{-u}^{0} \mathcal{I}_2[\zeta_1,\zeta_2]\left(u+t, \tau + \frac{\omega_{\rho,\delta}}{\delta^2}t\right) dt \end{pmatrix}$$

and

$$\begin{aligned} \mathcal{G}_{\rho,3}[\zeta](u,\tau) &= \int_{-\infty}^{0} e^{-\frac{i}{\delta^2}t}\zeta_3\left(u+t, \tau + \frac{\omega_{\rho,\delta}}{\delta^2}t\right) dt, \\ \mathcal{G}_{\rho,4}[\zeta](u,\tau) &= \int_{-\infty}^{0} e^{\frac{i}{\delta^2}t}\zeta_4\left(u+t, \tau + \frac{\omega_{\rho,\delta}}{\delta^2}t\right) dt, \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}_1[\zeta_1,\zeta_2](u,\tau) &= -\dot{g}_\Phi(u)\zeta_1(u,\tau) - 3g_\Phi(u)\zeta_2(u,\tau), \\ \mathcal{I}_2[\zeta_1,\zeta_2](u,\tau) &= \dot{f}_\Phi(u)\zeta_1(u,\tau) + 3f_\Phi(u)\zeta_2(u,\tau). \end{aligned}$$

Lemma III.5.3. *For $\rho \in [0,\rho_0]$ and $\delta \in (0,1)$, the operator $\mathcal{G}_\rho : \mathcal{Y}_\nu^4 \to \mathcal{Y}_\nu^4$ is well defined and is a right-inverse of the operator $\mathcal{L}_\rho$ given in (III.5.2). Moreover, $\mathcal{G}_{\rho,2}[\zeta](0,\cdot) \equiv 0$ and there exists a constant $C > 0$ independent of ρ and δ such that*

$$\|\mathcal{G}_\rho[\zeta]\|_\nu^\times \leq C\|\zeta\|_\nu^\times.$$

In addition, if $\partial_\tau\zeta \equiv 0$, one has that $\mathcal{G}_\rho[\zeta] = \mathcal{G}_{\tilde{\rho}}[\zeta]$ for $\rho, \tilde{\rho} \in [0,\rho_0]$.

Proof. The fact that $\mathcal{G}_\rho$ is a right inverse of $\mathcal{L}_\rho$ is straightforward. We show how to obtain estimates for $\mathcal{G}_{\rho,1}$. The estimates for $\mathcal{G}_{\rho,2}$, $\mathcal{G}_{\rho,3}$ and $\mathcal{G}_{\rho,4}$ are analogous.

Let $\zeta_1, \zeta_2 \in \mathcal{Y}_\nu$. By the estimates in Lemma III.5.2, for $(u,\tau) \in D^{\mathrm{u}} \times \mathbb{T}_d$ one has

$$\begin{aligned} |\mathcal{I}_1[\zeta_1,\zeta_2](u,\tau)| &\leq C|e^{3\nu u}| \left(\|\zeta_1\|_\nu + \|\zeta_2\|_\nu\right), \\ |\mathcal{I}_2[\zeta_1,\zeta_2](u,\tau)| &\leq C|e^{-\nu u}| \left(\|\zeta_1\|_\nu + \|\zeta_2\|_\nu\right). \end{aligned}$$

Then,

$$\begin{aligned}
\left|\mathcal{G}_{\rho,1}(u,\tau)e^{-\nu u}\right| \leq & C\left|e^{-3\nu u}\right|\left|\int_{-\infty}^{0}\mathcal{I}_1[\zeta_1,\zeta_2]\left(u+t,\tau+\frac{\omega_{\rho,\delta}}{\delta^2}t\right)dt\right| \\
& + C\left|e^{\nu u}\right|\left|\int_{-u}^{0}\mathcal{I}_2[\zeta_1,\zeta_2]\left(u+t,\tau+\frac{\omega_{\rho,\delta}}{\delta^2}t\right)dt\right| \\
\leq & C\left(\|\zeta_1\|_\nu+\|\zeta_2\|_\nu\right).
\end{aligned}$$

□

We introduce the fixed point operator

$$\mathcal{F}_\rho=\mathcal{G}_\rho\circ\mathcal{R}_\rho, \tag{III.5.6}$$

with $\mathcal{R}_\rho$ and $\mathcal{G}_\rho$ as given in (III.5.3) and (III.5.5), respectively. Then, equation (III.5.1) can be expressed as $Z_1^{\mathrm{u}}=\mathcal{F}_\rho[Z_1^{\mathrm{u}}]$.

proving Proposition III.4.2 is equivalent to prove the following result.

Proposition III.5.4. *Let $\rho_0>0$ be the constant given in Proposition III.2.4. There exists $\delta_0>0$ and $b_3>0$ such that, for $\rho\in[0,\rho_0]$ and $\delta\in(0,\delta_0)$, equation $Z_1^{\mathrm{u}}=\mathcal{F}[Z_1^{\mathrm{u}}]$ has a unique solution $Z_1^{\mathrm{u}}\in\mathcal{Y}_\nu^4$ satisfying*

$$\|Z_1^{\mathrm{u}}\|_\nu^\times\leq b_3\delta.$$

Proof. For $\varsigma>0$, let us consider $B(\varsigma)=\left\{\zeta\in\mathcal{Y}_\nu^4\ :\ \|\zeta\|_\nu^\times\leq\varsigma\right\}$. We will check that $\mathcal{F}_\rho:B(\varsigma)\to B(\varsigma)$ is a contraction for a suitable ς.

We first claim that there exist $\delta_0>0$ such that, for $\rho\in[0,\rho_0]$ and $\delta\in(0,\delta_0)$,

$$\|\mathcal{R}_\rho[\zeta]\|_\nu^\times\leq C\delta,\qquad\|\partial_j\mathcal{R}_\rho[\zeta]\|_0^\times\leq C\delta, \tag{III.5.7}$$

for $\zeta\in B(\varsigma\delta)$ and $j=1,..,4$. Indeed, we obtain the estimates for $\mathcal{R}_{\rho,2}[\zeta]$, the other cases are proven analogously. For the derivatives it is enough to apply Cauchy estimates.

We recall the definitions

$$\begin{aligned}
\sigma_{\mathrm{p}}&=(\lambda_{\mathrm{p}},\Lambda_{\mathrm{p}},0,0)^T,\\
\mathfrak{P}_\rho&=(0,\delta^2\mathfrak{L}_\Lambda,\delta^3\mathfrak{L}_x,\delta^3\mathfrak{L}_y)^T+\rho(0,0,e^{i\tau},e^{-i\tau})^T+\delta\rho\big(\lambda_{\mathfrak{P}},\Lambda_{\mathfrak{P}},x_{\mathfrak{P}},y_{\mathfrak{P}}\big)^T,\\
\mathcal{R}_{\rho,2}[\zeta]&=-\partial_\lambda H_1(\mathfrak{P}_\rho+\sigma_{\mathrm{p}}+\zeta)+\partial_\lambda H_1(\mathfrak{P}_\rho)-T_\rho[\zeta_1],\\
T_\rho[\zeta_1]&=V'(\lambda_{\mathrm{p}}+\delta\rho\lambda_{\mathfrak{P}}+\zeta_1)-V'(\lambda_{\mathrm{p}})-V'(\delta\rho\lambda_{\mathfrak{P}})-V''(\lambda_{\mathrm{p}})\zeta_1,
\end{aligned} \tag{III.5.8}$$

where V is the potential given in (III.2.7). Then, by the Mean Value Theorem,

$$\begin{aligned}
\mathcal{R}_{\rho,2}[\zeta](u,\tau)=&-\int_0^1 D\partial_\lambda H_1(s\sigma_{\mathrm{p}}(u)+s\zeta(u,\tau)+\mathfrak{P}_\rho(\tau))ds\,(\sigma_{\mathrm{p}}(u)+\zeta(u,\tau))\\
&-\zeta_1(u,\tau)\left[V''(\lambda_{\mathrm{p}}(u)+\delta\rho\lambda_{\mathfrak{P}}(\tau))-V'''(\lambda_{\mathrm{p}}(u))\right]+\mathcal{O}\left(\zeta_1(u,\tau)\right)^2\\
&-\delta\rho\lambda_{\mathfrak{P}}(\tau)\lambda_{\mathrm{p}}(u)V'''(0)+\mathcal{O}\left(\delta\rho\lambda_{\mathfrak{P}}(\tau)\lambda_{\mathrm{p}}(u)\right)^2.
\end{aligned}$$

From Proposition III.2.4 and Proposition III.4.1, one easily checks that

$$\|s\sigma_{\mathrm{p}}+s\zeta+\mathfrak{P}_\rho\|_0^\times\leq C,\qquad\text{for }s\in[0,1].$$

Thus, applying the estimates in Proposition III.2.1 and using that $\lambda_{\rm p}, \Lambda_{\rm p} \in \mathcal{Y}_{2\nu}$

$$\begin{aligned}\|\mathcal{R}_{\rho,2}[\zeta]\|_\nu \leq& C\delta\|\lambda_{\rm p}+\zeta_1\|_\nu + C\delta^2\|\Lambda_{\rm p}+\zeta_2\|_\nu + C\delta\|\zeta_3\|_\nu + C\delta\|\zeta_4\|_\nu \\ &+ C\|\zeta_1\|_\nu + C\delta\rho\|\lambda_{\rm p}\|_\nu \leq C\delta.\end{aligned}$$

and (III.5.7) is proven.

As a consequence of (III.5.7) and using lemmas III.5.3, there exists a constant $b_3 > 0$ such

$$\|\mathcal{F}_\rho[0]\|_\nu^\times \leq C\|\mathcal{R}_\rho[0]\|_\nu^\times \leq \frac{1}{2}b_3\delta. \tag{III.5.9}$$

In addition, for $\zeta, \tilde{\zeta} \in B(b_3\delta)$ and by the Mean Value Theorem,

$$\mathcal{R}_\rho[\zeta] - \mathcal{R}_\rho[\tilde{\zeta}] = \left[\int_0^1 D\mathcal{R}_\rho[s\zeta + (1-s)\tilde{\zeta}]ds\right](\zeta - \tilde{\zeta}).$$

Then, from Lemma III.5.3 and (III.5.7), we deduce that

$$\begin{aligned}&\|\mathcal{F}_\rho[\zeta] - \mathcal{F}_\rho[\tilde{\zeta}]\|_\nu^\times \leq C\|\mathcal{R}_\rho[\zeta] - \mathcal{R}_\rho[\tilde{\zeta}]\|_\nu^\times \\ &\quad \leq \sup_{s\in[0,1]}\sum_{k=1}^4 \|\partial_k\mathcal{R}_\rho[s\zeta + (1-s)\tilde{\zeta}]\|_0\|\zeta_k - \tilde{\zeta}_k\|_\nu \leq C\delta\|\zeta - \tilde{\zeta}\|_\nu^\times.\end{aligned} \tag{III.5.10}$$

This implies that, taking δ small enough, $\|\mathcal{F}_\rho[\zeta]-\mathcal{F}_\rho[\tilde{\zeta}]\|_\nu^\times \leq \frac{1}{2}\|\zeta-\tilde{\zeta}\|_\nu^\times$ and, therefore, $\mathcal{F}_\rho : B(b_3\delta) \to B(b_3\delta)$ is well defined and contractive. Hence, $\mathcal{F}_\rho$ has a fixed point $Z_1^{\rm u} \in B(b_3\delta)$. Since $\mathcal{G}_{\rho,2}[\zeta](0,\cdot) \equiv 0$ (see (III.5.5)) and $\mathcal{F}_\rho = \mathcal{G}_\rho \circ \mathcal{R}_\rho$, this solution satisfies that

$$\langle Z_1^{\rm u}(0,\tau), \mathbf{e_2}\rangle = 0, \qquad \text{for all} \quad \tau \in \mathbb{T}_d.$$

$\square$

III.5.2 Proof of Proposition III.4.4

Let us consider the parametrizations $Z_1^{\rm u}(u,\tau)$ and $z_1^{\rm u}(u)$ given in Proposition III.4.2 and Corollary III.4.3, respectively.

Let us recall that, by Proposition III.5.4, $Z_1^{\rm u}$ satisfies $Z_1^{\rm u} = (\mathcal{G}_\rho \circ \mathcal{R}_\rho)[Z_1^{\rm u}]$ and, as a result, $z_1^{\rm u} = (\mathcal{G}_0 \circ \mathcal{R}_0)[z_1^{\rm u}]$. By Lemma III.5.3, since $z_1^{\rm u}$ does not depend on τ, one has that

$$z_1^{\rm u} = \mathcal{G}_\rho \circ \mathcal{R}_0[z_1^{\rm u}], \qquad \text{for any } \rho \in [0,\rho_0]. \tag{III.5.11}$$

Then, by Proposition III.5.4,

$$\begin{aligned}Z_1^{\rm u} - z_1^{\rm u} &= \mathcal{F}_\rho[Z_1^{\rm u}] - \mathcal{G}_\rho \circ \mathcal{R}_0[z_1^{\rm u}] \\ &= \mathcal{F}_\rho[Z_1^{\rm u}] - \mathcal{F}_\rho[z_1^{\rm u}] + \mathcal{G}_\rho\left(\mathcal{R}_\rho[z_1^{\rm u}] - \mathcal{R}_0[z_1^{\rm u}]\right),\end{aligned} \tag{III.5.12}$$

where we recall that $\mathcal{F}_\rho = \mathcal{G}_\rho \circ \mathcal{R}_\rho$ (see (III.5.6)).

Let us consider the constant b_3 as given in Proposition III.4.2. It is clear that,

$$Z_1^{\rm u}, z_1^{\rm u} \in B(b_3\delta) := \left\{\zeta \in \mathcal{Y}_\nu^4 \,:\, \|\zeta\|_\nu^\times \leq b_3\delta\right\}.$$

Since $\mathcal{F}_\rho$ is contractive with Lipschitz constant $\mathrm{Lip}(\mathcal{F}_\rho) \leq C\delta$ (see (III.5.10)), for δ small enough, one has that

$$\|\mathcal{F}_\rho[Z_1^{\mathrm{u}}] - \mathcal{F}_\rho[z_1^{\mathrm{u}}]\|_\nu^\times \leq C\delta\|Z_1^{\mathrm{u}} - z_1^{\mathrm{u}}\|_\nu^\times \leq \frac{1}{2}\|Z_1^{\mathrm{u}} - z_1^{\mathrm{u}}\|_\nu^\times.$$

Thus, by (III.5.12) and Lemma III.5.3,

$$\|Z_1^{\mathrm{u}} - z_1^{\mathrm{u}}\|_\nu^\times \leq \frac{1}{2}\|Z_1^{\mathrm{u}} - z_1^{\mathrm{u}}\|_\nu^\times + C\|\mathcal{R}_\rho[z_1^{\mathrm{u}}] - \mathcal{R}_0[z_1^{\mathrm{u}}]\|_\nu^\times, \tag{III.5.13}$$

We claim that, for $\rho \in [0, \rho_0]$ and $\delta > 0$ small enough,

$$\|\mathcal{R}_\rho[z_1^{\mathrm{u}}] - \mathcal{R}_0[z_1^{\mathrm{u}}]\|_\nu^\times \leq C\delta\rho. \tag{III.5.14}$$

Indeed, first we consider estimates for $\mathcal{R}_{\rho,1}$ as given in (III.5.3). One has that

$$\begin{aligned}\mathcal{R}_{\rho,1}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,1}[z_1^{\mathrm{u}}] = &\big(\partial_\Lambda H_1(\sigma_{\mathrm{p}} + \mathfrak{P}_\rho + z_1^{\mathrm{u}}) - \partial_\Lambda H_1(\mathfrak{P}_\rho)\big)\\ &- \big(\partial_\Lambda H_1(\sigma_{\mathrm{p}} + \mathfrak{P}_0 + z_1^{\mathrm{u}}) - \partial_\Lambda H_1(\mathfrak{P}_0)\big).\end{aligned}$$

Denoting $\mathfrak{P}^s = (1-s)\mathfrak{P}_0 + s\mathfrak{P}_\rho$, by the Mean Value Theorem,

$$\mathcal{R}_{\rho,1}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,1}[z_1^{\mathrm{u}}] = (\mathfrak{P}_\rho - \mathfrak{P}_0)^T\left[\int_{[0,1]^2} D^2\partial_\Lambda H_1(r(\sigma_{\mathrm{p}} + z_1^{\mathrm{u}}) + \mathfrak{P}^s)drds\right](\sigma_{\mathrm{p}} + z_1^{\mathrm{u}}).$$

Then, using Lemma III.5.1 and for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\lambda, \Lambda, x, y)$, one sees that

$$\begin{aligned}\|\mathcal{R}_{\rho,1}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,1}[z_1^{\mathrm{u}}]\|_\nu \leq \sum_{j=1}^4\sum_{k=1}^4 \sup_{s\in[0,1]}\sup_{r\in[0,1]} &\|\partial_{\alpha_j\alpha_k\Lambda} H_1(r(\sigma_{\mathrm{p}} + z_1^{\mathrm{u}}) + \mathfrak{P}^s)\|_0\\ &\cdot \|\sigma_{\mathrm{p}} + z_1^{\mathrm{u}}\|_\nu^\times \|\mathfrak{P}_\rho - \mathfrak{P}_0\|_0^\times.\end{aligned}$$

Notice that, Proposition III.2.4 implies that $\|\mathfrak{P}_\rho - \mathfrak{P}_0\|_0^\times \leq C\rho$ and Proposition III.4.1 and Corollary III.4.3 imply that $\|\sigma_{\mathrm{p}} + z_1^{\mathrm{u}}\|_\nu^\times \leq C$. These estimates and those of Proposition III.2.1, which bound $\|\partial_{\alpha_j\alpha_k\Lambda} H_1\|_0$, imply that

$$\|\mathcal{R}_{\rho,1}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,1}[z_1^{\mathrm{u}}]\|_\nu \leq C\delta\rho.$$

Analogously, it can be seen that

$$\begin{aligned}&\|\mathcal{R}_{\rho,2}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,2}[z_1^{\mathrm{u}}]\|_\nu \leq C\delta\rho + \|T_\rho[z_1^{\mathrm{u}}] - T_0[z_1^{\mathrm{u}}]\|_\nu,\\ &\|\mathcal{R}_{\rho,3}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,3}[z_1^{\mathrm{u}}]\|_\nu \leq C\delta\rho,\\ &\|\mathcal{R}_{\rho,4}[z_1^{\mathrm{u}}] - \mathcal{R}_{0,4}[z_1^{\mathrm{u}}]\|_\nu \leq C\delta\rho,\end{aligned}$$

with T_ρ defined in (III.5.4). Therefore, it only remains to analyze $T_\rho[z_1^{\mathrm{u}}] - T_0[z_1^{\mathrm{u}}]$. Indeed, applying the Mean Value Theorem one sees that

$$\begin{aligned}T_\rho[z_1^{\mathrm{u}}] - T_0[z_1^{\mathrm{u}}] &= V'(\lambda_{\mathrm{p}} + \mathfrak{P}_{\rho,1} + z_1^{\mathrm{u}}) - V'(\mathfrak{P}_{\rho,1}) - V'(\lambda_{\mathrm{p}} + z_1^{\mathrm{u}}) + V'(0)\\ &= \mathfrak{P}_{\rho,1}\,(\lambda_{\mathrm{p}} + z_1^{\mathrm{u}})\int_{[0,1]^2} V'''(s\lambda_{\mathrm{p}} + r\mathfrak{P}_{\rho,1} + sz_1^{\mathrm{u}})drds.\end{aligned}$$

Then, since $\lambda_{\mathrm{p}} \in \mathcal{Y}_{2\nu}$ and taking into account that $\mathfrak{P}_{\rho,1}(\tau) = \delta\rho\lambda_{\mathfrak{P}}(\tau)$ with $\|\lambda_{\mathfrak{P}}\|_0 \leq C$ (see Proposition III.2.4), one has that $\|T_\rho[z_1^{\mathrm{u}}] - T_0[z_1^{\mathrm{u}}]\|_\nu \leq C\delta\rho$. This proves (III.5.14) and, by (III.5.13), Proposition III.4.4 holds.

Appendix III.A

Lyapunov periodic orbits

In this section we prove Proposition III.2.4. Let us recall that the equilibrium point L_3 in the set of coordinates (λ, Λ, x, y) (see (III.2.6)), by Proposition III.2.3 and for $\delta > 0$ small enough, is given by

$$\mathfrak{L}(\delta) = \left(0, \delta^2 \mathfrak{L}_\Lambda(\delta), \delta^3 \mathfrak{L}_x(\delta), \delta^3 \mathfrak{L}_y(\delta)\right)^T,$$

with $|\mathfrak{L}_\Lambda(\delta)|, |\mathfrak{L}_x(\delta)| \, |\mathfrak{L}_y(\delta)| \leq b_1$. We observe that $\mathfrak{L}(\delta)$ is an equilibrium point of the Hamiltonian system given by H, using that $H = H_0 + H_1$, we have that

$$\begin{aligned} &\partial_\lambda H_1(\mathfrak{L}(\delta); \delta) = 0, && \partial_\Lambda H_1(\mathfrak{L}(\delta); \delta) = 3\delta^2 \mathfrak{L}_\Lambda(\delta), \\ &\partial_x H_1(\mathfrak{L}(\delta); \delta) = -\delta \mathfrak{L}_y(\delta), && \partial_y H_1(\mathfrak{L}(\delta); \delta) = -\delta \mathfrak{L}_x(\delta). \end{aligned} \tag{III.A.1}$$

In addition, one can easily check that

$$H(\mathfrak{L}(\delta); \delta) = -\frac{1}{2} - \frac{3}{2}\delta^4 \mathfrak{L}_\Lambda^2(\delta) + \delta^4 \mathfrak{L}_x(\delta)\mathfrak{L}_y(\delta) + H_1(\mathfrak{L}(\delta); \delta). \tag{III.A.2}$$

For $\rho > 0$, we consider a polar symplectic change of coordinates $\phi_{\mathrm{Lya}} : (\lambda, J, \varphi, I) \to (\lambda, \Lambda, x, y)$ given by

$$\Lambda = J + \delta^2 \mathfrak{L}_\Lambda(\delta), \quad x = \sqrt{\rho^2 + I} e^{-i\varphi} + \delta^3 \mathfrak{L}_x(\delta), \quad y = \sqrt{\rho^2 + I} e^{i\varphi} + \delta^3 \mathfrak{L}_y(\delta). \tag{III.A.3}$$

The Hamiltonian system associated to H expressed in the coordinates (λ, J, φ, I) is a Hamiltonian system with respect to the symplectic form $d\lambda \wedge dJ + d\varphi \wedge dI$ and $H^{\mathrm{Lya}} = H \circ \phi_{\mathrm{Lya}}$ is

$$\begin{aligned} H^{\mathrm{Lya}}(\lambda, J, \varphi, I; \rho, \delta) = &-\frac{3}{2}J^2 + V(\lambda) + \frac{\rho^2 + I}{\delta^2} + H_1(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, I); \delta) - 3\delta^2 J \mathfrak{L}_\Lambda \\ &+ \delta\sqrt{\rho^2 + I}\left(e^{-i\varphi}\mathfrak{L}_y + e^{i\varphi}\mathfrak{L}_x\right) - \frac{3}{2}\delta^4 \mathfrak{L}_\Lambda + \delta^4 \mathfrak{L}_x \mathfrak{L}_y, \end{aligned}$$

which, using (III.A.1) and (III.A.2), can be rewritten as

$$\begin{aligned} H^{\mathrm{Lya}}(\lambda, J, \varphi, I; \rho, \delta) = &-\frac{3}{2}J^2 + V(\lambda) + \frac{1}{2} + \frac{I}{\delta^2} + H_1(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, I); \delta) \\ &- H_1(\mathfrak{L}; \delta) - DH_1(\mathfrak{L}; \delta) \cdot (\phi_{\mathrm{Lya}}(\lambda, J, \varphi, I) - \mathfrak{L})^T \\ &+ \frac{\rho^2}{\delta^2} + H(\mathfrak{L}; \delta). \end{aligned} \tag{III.A.4}$$

We are interested in proving the existence of a periodic orbit in the energy level $H^{\mathrm{Lya}} = \frac{\rho^2}{\delta^2} + H(\mathfrak{L}; \delta)$. To this end, in the following lemma, we first obtain an expression of I with respect to the other coordinates, by means of restricting the energy level.

Let us denote by $B(\varsigma)=\{z\in\mathbb{C}\,:\,|z|<\varsigma\}$, the open ball of radius ς.

Lemma III.A.1. *Fix $d,\lambda_0,J_0,\rho_0>0$. There exists $\delta_0>0$ such that, for all $\rho\in(0,\rho_0]$ and $\delta\in(0,\delta_0)$, there exists a function*

$$\widehat{I}_{\rho,\delta}:B(\delta\rho\lambda_0)\times B(\delta\rho J_0)\times\mathbb{T}_d\to\mathbb{C},$$

such that $H^{\mathrm{Lya}}(\lambda,J,\varphi,\widehat{I}_{\rho,\delta}(\lambda,J,\varphi);\rho,\delta)=\frac{\rho^2}{\delta^2}+H(\mathfrak{L};\delta)$.

Moreover, there exists a constant $C>0$ independent of ρ and δ such that

$$\begin{aligned}
|\widehat{I}_{\rho,\delta}(\lambda,J,\varphi;\delta)|&\leq C\delta^4\rho^2, &\qquad |\partial_\lambda\widehat{I}_{\rho,\delta}(\lambda,J,\varphi;\delta)|&\leq C\delta^3\rho,\\
|\partial_J\widehat{I}_{\rho,\delta}(\lambda,J,\varphi;\delta)|&\leq C\delta^3\rho, &\qquad |\partial_\varphi\widehat{I}_{\rho,\delta}(\lambda,J,\varphi;\delta)|&\leq C\delta^4\rho^2.
\end{aligned}$$

Proof. One has that the function $\widehat{I}_{\rho,\delta}$ should satisfy the equation $\widehat{I}_{\rho,\delta}=F[\widehat{I}_{\rho,\delta}]$ with

$$\begin{aligned}
F[I](\lambda,J,\varphi)=&\,\delta^2H^{\mathrm{Lya}}(\lambda,J,\varphi,I;\rho,\delta)-I-\rho^2-\delta^2H(\mathfrak{L};\delta)\\
=&\,\delta^2\left[\frac{3}{2}J^2+V(\lambda)+\frac{1}{2}+H_1(\phi_{\mathrm{Lya}}(\lambda,J,\varphi,I);\delta)\right.\\
&\left.-H_1(\mathfrak{L};\delta)-DH_1(\mathfrak{L};\delta)\cdot(\phi_{\mathrm{Lya}}(\lambda,J,\varphi,I)-\mathfrak{L})^T\right].
\end{aligned}$$

Let $(\lambda,J,\varphi)\in B(\delta\rho\lambda_0)\times B(\delta\rho J_0)\times\mathbb{T}_d$. Then, using the estimates of D^2H_1 in Proposition III.2.1, one has that

$$|F[0](\lambda,J,\varphi)|\leq C\delta^2\rho^2.$$

In addition, for functions $\iota_1,\iota_2:B(\delta\rho\lambda_0)\times B(\delta\rho J_0)\times\mathbb{T}_d\to\mathbb{C}$ such that $|\iota_1|,|\iota_2|\leq C\delta^2\rho^2$, by the estimates of the third derivatives of H_1 in Proposition III.2.1 and the mean value theorem, one has that

$$|F[\iota_1](\lambda,J,\varphi)-F[\iota_2](\lambda,J,\varphi)|\leq C\delta^3\rho\,|\iota_1-\iota_2|\leq C\delta_0^3\rho_0\,|\iota_1-\iota_2|\,.$$

Then, taking δ_0 small enough and applying the fixed point theorem, one obtains the existence of function $\widehat{I}_{\rho,\delta}$ and its corresponding bounds. The estimates for the derivatives of $\widehat{I}_{\rho,\delta}$ are straightforward from Cauchy's integral theorem. □

By Lemma III.A.1, the Hamiltonian system on the energy level $H^{\mathrm{Lya}}=\frac{\rho^2}{\delta^2}+H(\mathfrak{L};\delta)$ is of the form

$$\dot{\lambda}=-3J+f_1(\lambda,J,\varphi),\qquad \dot{J}=-\frac{7}{8}\lambda+f_2(\lambda,J,\varphi),\qquad \dot{\varphi}=\frac{1}{\delta^2}+g(\lambda,J,\varphi),\tag{III.A.5}$$

where, denoting $\widehat{I}_{\rho,\delta} = \widehat{I}_{\rho,\delta}(\lambda, J, \varphi)$ and using the expression of H^{Lya} in (III.A.4) and that $V''(0) = -\frac{7}{8}$,

$$
\begin{aligned}
f_1(\lambda, J, \varphi) &= \partial_\Lambda H_1\left(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, \widehat{I}_{\rho,\delta}); \delta\right) - \partial_\Lambda H_1(\mathfrak{L}(\delta); \delta), \\
f_2(\lambda, J, \varphi) &= -V'(\lambda) + V''(0)\lambda - \partial_\lambda H_1\left(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, \widehat{I}_{\rho,\delta}); \delta\right) + \partial_\lambda H_1(\mathfrak{L}(\delta); \delta), \\
g(\lambda, J, \varphi) &= \frac{e^{-i\varphi}}{2\sqrt{\rho^2 + \widehat{I}_{\rho,\delta}}}\left(\partial_x H_1\left(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, \widehat{I}_{\rho,\delta}); \delta\right) - \partial_x H_1(\mathfrak{L}(\delta); \delta)\right) \\
&\quad + \frac{e^{i\varphi}}{2\sqrt{\rho^2 + \widehat{I}_{\rho,\delta}}}\left(\partial_y H_1\left(\phi_{\mathrm{Lya}}(\lambda, J, \varphi, \widehat{I}_{\rho,\delta}); \delta\right) - \partial_y H_1(\mathfrak{L}(\delta); \delta)\right).
\end{aligned}
\tag{III.A.6}
$$

Since $\dot{\varphi} \neq 0$, we look for the periodic orbit of the system (III.A.5) as a graph over φ. In other words, we look for periodic functions

$$
w = (w_\lambda, w_J) : \mathbb{T}_d \to \mathbb{C}^2, \qquad w = w(\varphi),
$$

satisfying the invariance equation $\mathcal{L}w = \mathcal{R}[w]$, with

$$
\begin{aligned}
\mathcal{L}w &= (\partial_\varphi - \delta^2 \mathcal{A})w, \qquad \mathcal{A} = \begin{pmatrix} 0 & -3 \\ -\frac{7}{8} & 0 \end{pmatrix}, \\
\mathcal{R}[w](\varphi) &= \delta^2 \left(\frac{\mathcal{A}w + f(w_\lambda(\varphi), w_J(\varphi), \varphi)}{1 + \delta^2 g(w_\lambda(\varphi), w_J(\varphi), \varphi)} - \mathcal{A}w \right),
\end{aligned}
\tag{III.A.7}
$$

where the functions $f = (f_1, f_2)$ and g are given in (III.A.6).

Let us consider the Banach space

$$
\mathcal{Z} = \left\{ h : \mathbb{T}_d \to \mathbb{C} \; : \; h \text{ analytic}, \; \|h\| := \sup_{\varphi \in \mathbb{T}_d} |h(\varphi)| < +\infty \right\},
$$

and the space $\mathcal{Z}^2$ endowed with the product norm $\|h\|_\times = \|h_1\| + \|h_2\|$.

Proposition III.A.2. *There exist $\rho_0, \delta_0, b_6 > 0$ such that, for $\rho \in (0, \rho_0]$ and $\delta \in (0, \delta_0)$, there exists a solution of $\mathcal{L}w = \mathcal{R}[w]$ belonging to $\mathcal{Z}^2$ satisfying*

$$
\|w\|_\times \leq b_6 \delta \rho.
$$

To prove Proposition III.A.2 we first study the right-inverse of operator $\mathcal{L} = \partial_\varphi - \delta^2 \mathcal{A}$ in $\mathcal{Z}^2$. First, notice that

$$
\mathcal{A} = \mathcal{P}\mathcal{D}\mathcal{P}^{-1} \quad \text{where} \quad \mathcal{D} = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 3 & 3 \\ -\nu & \nu \end{pmatrix}, \quad \nu = \sqrt{\frac{21}{8}}.
$$

Lemma III.A.3. *We consider the linear operator*

$$
\begin{aligned}
\mathcal{G}[h](\varphi) = {} & \mathcal{P} e^{\varphi \delta^2 \mathcal{D}} (e^{-2\pi \delta^2 \mathcal{D}} - \mathrm{Id})^{-1} \int_0^{2\pi} e^{-\theta \delta^2 \mathcal{D}} \mathcal{P}^{-1} h(\theta) d\theta \\
& + \mathcal{P} e^{\varphi \delta^2 \mathcal{D}} \int_0^{\varphi} e^{-\theta \delta^2 \mathcal{D}} \mathcal{P}^{-1} h(\theta) d\theta.
\end{aligned}
\tag{III.A.8}
$$

The operator $\mathcal{G} : \mathcal{Z}^2 \to \mathcal{Z}^2$ is a right-inverse of the operator $\mathcal{L}$ given in (III.A.7). In addition, there exists $C > 0$ such that, for $\delta \in (0,1)$,

$$\|\mathcal{G}[h]\|^\times \leq \frac{C}{\delta^2}\|h\|^\times, \qquad \text{for } h \in \mathcal{Z}^2.$$

Proof. If w is a solution of $\mathcal{L}[w] = h$, it must exist $K_0 \in \mathbb{R}^2$ such that

$$w(\varphi) = \mathcal{P}e^{\varphi\delta^2\mathcal{D}}\left[K_0 + \int_0^\varphi e^{-\theta\delta^2\mathcal{D}}\mathcal{P}^{-1}h(\theta)d\theta\right].$$

Then, imposing that w has to be 2π-periodic, one obtains (III.A.8). The estimates for the operator are straightforward from

$$\|(e^{-2\pi\delta^2\mathcal{D}} - \mathrm{Id})^{-1}\| \leq (1 - \|e^{-2\pi\delta^2\mathcal{D}} - \mathrm{Id}\|)^{-1} \leq \frac{C}{\delta^2}.$$

□

For $\varsigma > 0$, we denote $\mathcal{B}(\varsigma) = \left\{h \in \mathcal{Z}^2 \, : \, \|h\|^\times \leq \varsigma\right\}$.

Lemma III.A.4. *Let $\mathcal{R}$ be as given in (III.A.7) and fix constants $\rho_0, \varsigma > 0$. Then, there exist $\delta_0, C > 0$ such that, for $\rho \in (0, \rho_0]$, $\delta \in (0, \delta_0)$ and $h \in \mathcal{B}(\varsigma\delta\rho)$,*

$$\|\mathcal{R}_1[h]\| \leq C\delta^5\rho, \qquad \|\mathcal{R}_2[h]\| \leq C\delta^3\rho.$$

and

$$\|\partial_1\mathcal{R}_1[h]\| \leq C\delta^4, \quad \|\partial_2\mathcal{R}_1[h]\| \leq C\delta^4, \quad \|\partial_1\mathcal{R}_2[h]\| \leq C\delta^3, \quad \|\partial_2\mathcal{R}_2[h]\| \leq C\delta^4.$$

Proof. Let $h = (h_1, h_2) \in \mathcal{B}(\varsigma\delta\rho)$ and $\varphi \in \mathbb{T}_d$. For $s \in [0,1]$, we denote

$$z^s(\varphi) = s\,\phi_{\mathrm{Lya}}\left(h_1(\varphi), h_2(\varphi), \varphi, \widehat{I}_{\rho,\delta}(h(\varphi))\right) + (1-s)\mathfrak{L}(\delta).$$

We notice that, by the definition in (III.A.3) of ϕ_{Lya},

$$z^1(\varphi) - z^0(\varphi) = \left(h_1(\varphi), h_2(\varphi), \sqrt{\rho^2 + \widehat{I}_{\rho,\delta}(h(\varphi))}e^{-i\varphi}, \sqrt{\rho^2 + \widehat{I}_{\rho,\delta}(h(\varphi))}e^{i\varphi}\right)^T.$$

We recall that $f_1 = \partial_\Lambda H_1(\phi_{\mathrm{Lya}}) - \partial_\Lambda H_1(\mathfrak{L})$ (see (III.A.6)) and then, by the mean value theorem and the estimates in Proposition III.2.1 and Lemma III.A.1,

$$\begin{aligned} |f_1(h(\varphi),\varphi)| \leq \sup_{s\in[0,1]} \Big\{ &|\partial_{\Lambda\lambda}H_1\left(z^s(\varphi)\right)| \, |h_1(\varphi)| + \left|\partial_\Lambda^2 H_1\left(z^s(\varphi)\right)\right| |h_2(\varphi)| \\ &+ \big(\, |\partial_{\Lambda x}H_1\left(z^s(\varphi)\right)| + |\partial_{\Lambda y}H_1\left(z^s(\varphi)\right)| \,\big) \, |\rho^2 + \widehat{I}_{\rho,\delta}(h(\varphi))|^{\frac{1}{2}} \Big\} \leq C\delta^3\rho. \end{aligned} \tag{III.A.9}$$

Analogously,

$$|f_2(h(\varphi),\varphi)| \leq C\delta\rho, \qquad |g(h(\varphi),\varphi)| \leq C\delta^2. \tag{III.A.10}$$

To obtain estimates for the derivatives of f_1, f_2 and g, we compute them by

$$\partial_\lambda f_1(h(\varphi),\varphi) = \partial_{\lambda\Lambda} H_1\left(z^1(\varphi)\right) + \frac{\partial_\lambda \widehat{I}_{\rho,\delta}}{2\sqrt{\rho^2+\widehat{I}_{\rho,\delta}}}\left[e^{-i\varphi}\partial_{\Lambda x} H_1\left(z^1(\varphi)\right) + e^{i\varphi}\partial_{\Lambda y} H_1\left(z^1(\varphi)\right)\right],$$

$$\partial_J f_1(h(\varphi),\varphi) = \partial^2_{\Lambda} H_1\left(z^1(\varphi)\right) + \frac{\partial_J \widehat{I}_{\rho,\delta}}{2\sqrt{\rho^2+\widehat{I}_{\rho,\delta}}}\left[e^{-i\varphi}\partial_{\Lambda x} H_1\left(z^1(\varphi)\right) + e^{i\varphi}\partial_{\Lambda y} H_1\left(z^1(\varphi)\right)\right],$$

where $\widehat{I}_{\rho,\delta} = \widehat{I}_{\rho,\delta}(h(\varphi))$. Then, using the estimates in Proposition III.2.1 and Lemma III.A.1,

$$|\partial_\lambda f_1(h(\varphi),\varphi)| \leq C\delta^2, \qquad |\partial_J f_1(h(\varphi),\varphi)| \leq C\delta^2. \tag{III.A.11}$$

Analogously,

$$\begin{aligned} |\partial_\lambda f_2(h(\varphi),\varphi)| &\leq C\delta, & |\partial_J f_2(h(\varphi),\varphi)| &\leq C\delta^2, \\ |\partial_\lambda g(h(\varphi),\varphi)| &\leq \frac{C\delta}{\rho}, & |\partial_J g(h(\varphi),\varphi)| &\leq \frac{C\delta^3}{\rho}. \end{aligned} \tag{III.A.12}$$

Lastly, joining the previous bounds in (III.A.9), (III.A.10), (III.A.11), (III.A.12) with the definition of the operator $\mathcal{R}$ in (III.A.7), we obtain the statement of the lemma. □

Proof of Proposition III.A.2. A fixed point of $w = \mathcal{F}[w]$ with $\mathcal{F} = \mathcal{G}\circ\mathcal{R}$ is a periodic solution of $\mathcal{L}w = \mathcal{R}[w]$. By Lemmas III.A.3 and III.A.4, there exists $b_6 > 0$ such that

$$\|\mathcal{F}[0]\|^\times \leq \frac{C}{\delta^2}\left(\|\mathcal{R}_1[0]\| + \|\mathcal{R}_2[0]\|\right) \leq \frac{b_6}{2}\delta\rho. \tag{III.A.13}$$

Moreover, for $h, \hat{h} \in \mathcal{B}(b_6\delta\rho)$, by the mean value theorem,

$$\|\mathcal{R}[h] - \mathcal{R}[\hat{h}]\|^\times \leq \sup_{s\in[0,1]}\left[\|D\mathcal{R}[(1-s)h + s\hat{h}](h-\hat{h})\|^\times\right].$$

Thus, by Lemmas III.A.3 and III.A.4,

$$\|\mathcal{F}[h] - \mathcal{F}[\hat{h}]\|^\times \leq \frac{C}{\delta^2}\|\mathcal{R}[h] - \mathcal{R}[\hat{h}]\|^\times \leq C\delta\|h-\hat{h}\|^\times. \tag{III.A.14}$$

Then, if δ small enough, the operator $\mathcal{F} : \mathcal{B}(b_6\delta\rho) \to \mathcal{B}(b_6\delta\rho)$ is well defined and contractive and, as a consequence, it has a fixed point $w \in \mathcal{B}(b_6\delta\rho)$. □

End of the proof of Proposition III.2.4. Let $w(\varphi) = (w_\lambda(\varphi), w_J(\varphi))$ be the solution of $\mathcal{L}w = \mathcal{R}[w]$ in Proposition III.A.2 and introduce $w_I(\varphi) = \widehat{I}_{\rho,\delta}(w(\varphi),\varphi)$ as given in Lemma III.A.1. Then, the curve $(w_\lambda(\varphi), w_J(\varphi), \varphi, w_I(\varphi))$ is a parametrization of a periodic solution given in the energy level $H^{\mathrm{Lya}} = \frac{\rho^2}{\delta^2} + H(\mathfrak{L})$. However, $\dot{\varphi} = \partial_t\varphi = \frac{1}{\delta^2} + g(w(\varphi),\varphi)$ and then, in order to prove Proposition III.2.4, we look for a reparametrization $\varphi = \widehat{\varphi}(\tau)$ and $\omega_{\rho,\delta}$ such that $\dot{\tau} = \frac{\omega_{\rho,\delta}}{\delta^2}$. Moreover, we impose $\varphi(t)|_{t=0} = 0$ and therefore $\widehat{\varphi}(2\pi) = 2\pi$. Then, $\widehat{\varphi}$ must satisfy that

$$\partial_\tau\widehat{\varphi} = \frac{1+\delta^2 g(w(\widehat{\varphi}),\widehat{\varphi})}{\omega_{\rho,\delta}} \qquad \text{and} \qquad \widehat{\varphi}(2\pi) = 2\pi.$$

Notice that, by (III.A.10) and for δ small enough, one has that $\partial_\tau \widehat{\varphi} \neq 0$. Then, its inverse $\tau \equiv \widehat{\tau}(\varphi)$ satisfies that

$$\partial_\varphi \widehat{\tau} = \frac{\omega_{\rho,\delta}}{1+\delta^2 g(w(\varphi),\varphi)} \qquad \text{and} \qquad \widehat{\tau}(2\pi) = 2\pi.$$

These conditions give definitions for the functions $\widehat{\tau}(\varphi)$ and $\omega_{\rho,\delta}$,

$$\widehat{\tau}(\varphi) = \omega_{\rho,\delta} \int_0^\varphi \frac{d\eta}{1+\delta^2 g(w(\eta),\eta)} \qquad \text{and} \qquad \omega_{\rho,\delta} = \frac{2\pi}{\int_0^{2\pi} \frac{d\varphi}{1+\delta^2 g(w(\varphi),\varphi)}}.$$

We notice that $\widehat{\tau}(\varphi+2\pi) = 2\pi + \widehat{\tau}(\varphi)$. By the estimate for g in (III.A.10), we obtain

$$|\omega_{\rho,\delta} - 1| \leq C\delta^4, \qquad |\widehat{\tau}(\varphi) - \varphi| \leq C\delta^4, \qquad |\widehat{\varphi}(\tau) - \tau| \leq C\delta^4. \tag{III.A.15}$$

Then, for $\tau \in \mathbb{T}_d$, the curve

$$\mathfrak{P}_\rho(\tau;\delta) = \phi_{\text{Lya}}\big(w_\lambda(\widehat{\varphi}(\tau)), w_J(\widehat{\varphi}(\tau)), \widehat{\varphi}(\tau), w_I(\widehat{\varphi}(\tau))\big)$$

is a real-analytic and 2π-periodic solution of the Hamiltonian system given by the Hamiltonian H in (III.2.8) and int belongs to the energy level $H = \frac{\rho^2}{\delta^2} + H(\mathfrak{L})$. In addition, the functions in (III.2.20),

$$\begin{aligned}
\lambda_{\mathfrak{P}}(\tau) &= \frac{w_\lambda(\widehat{\varphi}(\tau))}{\delta\rho}, \qquad & x_{\mathfrak{P}}(\tau) &= \frac{\sqrt{\rho^2 + w_I(\widehat{\varphi}(\tau))}e^{-i\widehat{\varphi}(\tau)} - \rho e^{-i\tau}}{\delta\rho}, \\
\Lambda_{\mathfrak{P}}(\tau) &= \frac{w_J(\widehat{\varphi}(\tau))}{\delta\rho}, \qquad & y_{\mathfrak{P}}(\tau) &= \frac{\sqrt{\rho^2 + w_I(\widehat{\varphi}(\tau))}e^{i\widehat{\varphi}(\tau)} - \rho e^{i\tau}}{\delta\rho},
\end{aligned}$$

satisfy, by Lemma III.A.1, Proposition III.A.2 and (III.A.15), that $|\lambda_{\mathfrak{P}}(\tau)|, |\Lambda_{\mathfrak{P}}(\tau)| \leq C$ and $|x_{\mathfrak{P}}(\tau)|, |y_{\mathfrak{P}}(\tau)| \leq C\delta^3$. $\square$

Appendix III.B

Difference between the invariant manifolds of L_3

In this section we prove Corollary III.2.9 relying on the results stated in Sections III.4.2 and III.5.1.

Let us consider the real-analytic time parametrizations z^{u} and z^{s} of the unstable and stable manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ given in (III.4.7) and Corollary III.4.3. Notice that, for $u \in D^{\mathrm{u}} \cap D^{\mathrm{s}}$ (see (III.4.3)), they satisfy

$$\left|z^{\mathrm{u}}(u) - \sigma_{\mathrm{p}}(u) - \delta^2 \mathfrak{L}_\Lambda\right| \lesssim \delta, \qquad \left|z^{\mathrm{s}}(u) - \sigma_{\mathrm{p}}(u) - \delta^2 \mathfrak{L}_\Lambda\right| \lesssim \delta, \tag{III.B.1}$$

where $\sigma_{\mathrm{p}} = (\lambda_{\mathrm{p}}, \Lambda_{\mathrm{p}}, 0, 0)^T$ is given in (III.4.2). Moreover, $z^{\mathrm{u}}(0), z^{\mathrm{s}}(0) \in \left\{\Lambda = \delta^2 \mathfrak{L}_\Lambda\right\}$ and, since z^{u} and z^{s} satisfy equation (III.4.4) and are independent of τ, for $z^\diamond = (\lambda^\diamond, \Lambda^\diamond, x^\diamond, y^\diamond)$, $\diamond = \mathrm{u}, \mathrm{s}$, one has that

$$\begin{aligned} \frac{d\lambda^\diamond}{du} &= -3\Lambda^\diamond + \partial_\Lambda H_1(z^{\mathrm{u}}; \delta), & \frac{dx^\diamond}{du} &= \frac{i}{\delta^2} x^\diamond + i\partial_y H_1(z^{\mathrm{u}}; \delta) \\ \frac{d\Lambda^\diamond}{du} &= -V'(\lambda^\diamond) - \partial_\lambda H_1(z^{\mathrm{u}}; \delta), & \frac{dy^\diamond}{du} &= -\frac{i}{\delta^2} y^\diamond - i\partial_x H_1(z^{\mathrm{u}}; \delta). \end{aligned} \tag{III.B.2}$$

Fix $\lambda_* \in \left(\frac{2\pi}{3}, \lambda_0\right)$, with λ_0 as given in (III.2.18). By Proposition III.4.1, there exists $u_* > 0$ such that $\lambda_* = \lambda_{\mathrm{p}}(u_*)$. Therefore, by (III.B.1) and for $\delta > 0$ small enough, there exist $T^{\mathrm{u}}, T^{\mathrm{s}} = u_* + \mathcal{O}(\delta)$ such that $z^{\mathrm{u}}(T^{\mathrm{u}}), z^{\mathrm{s}}(T^{\mathrm{s}}) \in \{\lambda = \lambda_*, \Lambda > 0\}$. Moreover, by Theorem III.2.5,

$$z^{\mathrm{u}}(T^{\mathrm{u}}) - z^{\mathrm{s}}(T^{\mathrm{s}}) = \sqrt[6]{2}\,\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[\left(0, 0, \overline{\Theta}, \Theta\right)^T + \mathcal{O}_\delta\right], \tag{III.B.3}$$

where $\mathcal{O}_\delta = (0, \mathcal{O}(\delta), \mathcal{O}(|\log \delta|^{-1}), \mathcal{O}(|\log \delta|^{-1}))^T$. Therefore, to prove Corollary III.2.9, we need to deduce the difference $z^{\mathrm{u}}(0) - z^{\mathrm{s}}(0)$ from (III.B.3).

We first define $\Delta(u) = z^{\mathrm{u}}(u) - z^{\mathrm{s}}(u)$, for $u \in [0, T^{\mathrm{u}}]$. It is clear that, by (III.B.2) function $\Delta(u)$ satisfies the linear equation

$$\frac{d}{du}\Delta(u) = (M_0(u) + M_1(u))\Delta(u),$$

with

$$M_0(u) = \begin{pmatrix} 0 & -3 & 0 & 0 \\ -V''(\lambda_{\mathrm{p}}(u)) & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\delta^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{\delta^2} \end{pmatrix},$$

$$M_1(u) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ m(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \int_0^1 \mathbf{J} D^2 H_1 \left(\varsigma z^{\mathrm{u}}(u) + (1-\varsigma) z^{\mathrm{s}}(u)\right) d\varsigma,$$

$$m(u) = V''(\lambda_{\mathrm{p}}(u)) - \int_0^1 V''\left(\varsigma \lambda^{\mathrm{u}}(u) + (1-\varsigma)\lambda^{\mathrm{s}}(u)\right) d\varsigma,$$

where $\mathbf{J}$ is the symplectic matrix associated with the form $d\lambda \wedge d\lambda + i dx \wedge dy$. Moreover, from Proposition III.2.1 and Corollary III.4.3, we deduce that $|M_1(u)| \leq C\delta$, for $u \in [0, T^{\mathrm{u}}]$. Let $\Phi(u)$ be the fundamental matrix of the differential equation $\frac{d}{du}\Phi(u) = M_0 \Phi(u)$ given in Lemma III.5.2, which satisfies $\Phi(0) = \mathrm{Id}$. Then,

$$\Delta(u) = \Phi(u) \left[\Phi^{-1}(T^{\mathrm{u}}) \Delta(T^{\mathrm{u}}) + \int_{T^{\mathrm{u}}}^{u} \Phi^{-1}(\sigma) M_1(\sigma) \Delta(\sigma) d\sigma \right].$$

On one hand, using Gronwall's Lemma, one has that $|\Delta(u)| \leq C |\Delta(T^{\mathrm{u}})|$ for $u \in [0, T^{\mathrm{u}}]$ and, on the other hand

$$\left| \Delta(0) - \Phi^{-1}(T^{\mathrm{u}}) \Delta(T^{\mathrm{u}}) \right| \leq C\delta T^{\mathrm{u}} \left| \Delta(T^{\mathrm{u}}) \right|. \tag{III.B.4}$$

Therefore, to obtain an asymptotic formula for $\Delta(0)$, we need to compute $\Delta(T^{\mathrm{u}})$. We write

$$\Delta(T^{\mathrm{u}}) = z^{\mathrm{u}}(T^{\mathrm{u}}) - z^{\mathrm{s}}(T^{\mathrm{s}}) + z^{\mathrm{s}}(T^{\mathrm{s}}) - z^{\mathrm{s}}(T^{\mathrm{u}}). \tag{III.B.5}$$

Since the difference $z^{\mathrm{u}}(T^{\mathrm{u}}) - z^{\mathrm{s}}(T^{\mathrm{s}})$ is given by (III.B.3), we only need to analyze the component $z^{\mathrm{s}}(T^{\mathrm{s}}) - z^{\mathrm{s}}(T^{\mathrm{u}})$. Indeed, one has that

$$z^{\mathrm{s}}(T^{\mathrm{s}}) - z^{\mathrm{s}}(T^{\mathrm{u}}) = (T^{\mathrm{s}} - T^{\mathrm{u}}) \int_0^1 \partial_u z^{\mathrm{s}}(\varsigma T^{\mathrm{u}} + (1-\varsigma) T^{\mathrm{s}}) d\varsigma.$$

In addition, since $z^{\mathrm{s}} = (\lambda^{\mathrm{s}}, \Lambda^{\mathrm{s}}, x^{\mathrm{s}}, y^{\mathrm{s}})$ satisfies equation (III.B.2) and using the Mean Value Theorem, we obtain that

$$T^{\mathrm{u}} - T^{\mathrm{s}} = \frac{\Lambda^{\mathrm{u}}(T^{\mathrm{s}}) - \Lambda^{\mathrm{u}}(T^{\mathrm{u}})}{V'(\lambda_{\mathrm{p}}(u_*)) + \beta(T^{\mathrm{u}}, T^{\mathrm{s}})},$$

where, denoting $T(\varsigma) = \varsigma T^{\mathrm{u}} + (1-\varsigma) T^{\mathrm{s}}$, the function β satisfies

$$\beta(T^{\mathrm{u}}, T^{\mathrm{s}}) = \int_0^1 \left[V'(\lambda^{\mathrm{u}}(T(\varsigma))) - V'(\lambda_{\mathrm{p}}(u_*)) \right] d\varsigma + \int_0^1 \partial_\lambda H_1(z^{\mathrm{u}}(T(\varsigma))) d\varsigma.$$

Notice that $V'(\lambda_{\mathrm{p}}(u_*)) = V'(\lambda_*) \neq 0$ (see (III.2.7)). In addition, since $T^{\mathrm{u}}, T^{\mathrm{s}} = u_* + \mathcal{O}(\delta)$, by (III.B.1) and the estimates in Proposition III.2.1, one can see that $|\beta(T^{\mathrm{u}}, T^{\mathrm{s}})| \leq C\delta$. Therefore, one has that $|T^{\mathrm{u}} - T^{\mathrm{s}}| \leq C\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}$ and, as a result,

$$z^{\mathrm{s}}(T^{\mathrm{s}}) - z^{\mathrm{s}}(T^{\mathrm{u}}) = \mathcal{O}\left(\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}} \right).$$

Therefore, by (III.B.3) and (III.B.5)

$$\Delta(T^{\mathrm{u}}) = \sqrt[6]{2}\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[\left(0, 0, \overline{\Theta}, \Theta\right)^T + \widetilde{\mathcal{O}_\delta} \right], \tag{III.B.6}$$

where $\widetilde{\mathcal{O}_\delta} = (\mathcal{O}(\delta), \mathcal{O}(\delta), \mathcal{O}(|\log \delta|^{-1}), \mathcal{O}(|\log \delta|^{-1}))^T$. Lastly, joining the results in (III.B.4) and (III.B.6), we obtain

$$|\Delta(0)| = \sqrt[6]{2}\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[|\Phi^{-1}(T^{\mathrm{u}}) \left(0, 0, \overline{\Theta}, \Theta\right)^T | + \widetilde{\mathcal{O}_\delta} \right].$$

Then, applying the expression of the fundamental matrix Φ given in Lemma III.5.2, we obtain the statement of the corollary.

Appendix III.C

Normal form in a neighborhood of L_3

We devote this section to prove Propositions III.3.1 and III.3.2.

III.C.1 Proof of Proposition III.3.1

First, in the following lemma, we introduce a series of linear changes of coordinates in order to put the Hamiltonian $H(\lambda, \Lambda, x, y; \delta)$ in (III.2.8) in the form considered in [JBL16].

Lemma III.C.1. *Fix $c_0, c_1 > 0$. There exists $\delta_0, \widehat{\varrho}_0 > 0$ and a family of affine transformations*

$$\begin{aligned}\widehat{\phi}_\delta : B(\widehat{\varrho}_0) = \left\{\mathbf{z} \in \mathbb{R}^4 : |\mathbf{z}| < \widehat{\varrho}_0\right\} &\to U_{\mathbb{R}}(c_0, c_1)\\ (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2) &\mapsto (\lambda, \Lambda, x, y),\end{aligned}$$

defined for $\delta \in (0, \delta_0)$, with $\mathcal{C}^1$-functions of δ as coefficients and such that $\widehat{\phi}_\delta(0) = \mathfrak{L}(\delta)$ and

$$D\widehat{\phi}_0 = \begin{pmatrix} \frac{2}{\sqrt{7}} & \frac{2}{\sqrt{7}} & 0 & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & \sqrt[4]{\frac{2}{21}} & i\sqrt[4]{\frac{2}{21}} \\ 0 & 0 & \sqrt[4]{\frac{2}{21}} & -i\sqrt[4]{\frac{2}{21}}, \end{pmatrix}, \quad D\widehat{\phi}_0^{-1} = \begin{pmatrix} \frac{\sqrt{7}}{4} & -\sqrt{3} & 0 & 0 \\ \frac{\sqrt{7}}{4} & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt[4]{\frac{21}{32}} & \sqrt[4]{\frac{21}{32}} \\ 0 & 0 & -i\sqrt[4]{\frac{21}{32}} & i\sqrt[4]{\frac{21}{32}} \end{pmatrix}.$$

Moreover, $\widehat{\phi}_\delta$ is a symplectic scaling with respect to the form $d\widehat{v}_1 \wedge d\widehat{w}_1 + d\widehat{v}_2 \wedge \widehat{w}_2$. Then, the Hamiltonian system given by H (see (III.2.8)) in the new coordinates and after a scaling in time is Hamiltonian with respect to

$$\begin{aligned}\widehat{H}(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2; \delta) &= H\left(\widehat{\phi}_\delta(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2); \delta\right) - H(\mathfrak{L}(\delta); \delta)\\ &= \widehat{v}_1\widehat{w}_1 + \frac{\alpha(\delta)}{2\delta^2}\left(\widehat{v}_2^2 + \widehat{w}_2^2\right) + \widehat{K}(\widehat{v}_1, \widehat{w}_1)\\ &\quad + \delta\widehat{H}_1(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2; \delta),\end{aligned} \tag{III.C.1}$$

where $\alpha(\delta)$ is a $\mathcal{C}^1$-function in δ satisfying $\alpha(\delta) = \sqrt{\frac{8}{21}} + \mathcal{O}(\delta^4)$ and, for $(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2) \in B(\widehat{\varrho}_0)$, there exists a constant $C > 0$ independent of δ such that

$$|\widehat{K}(\widehat{v}_1, \widehat{w}_1)| \leq C\,|\widehat{v}_1 + \widehat{w}_1|^3, \qquad |\widehat{H}_1(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2; \delta)| \leq C\,|(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2)|^3.$$

Proof. For technical reasons, we consider first the Poincaré Hamiltonian $H^{\mathrm{Poi}}(\lambda, L, \eta, \xi; \mu)$ introduced in (III.2.1) instead of the scaled version H defined in (III.2.8). Let us denote the point L_3 in Poincaré coordinates (λ, L, η, ξ) as $L_3^{\mathrm{Poi}} = (\phi^{\mathrm{Poi}})^{-1}(L_3)$. Therefore, L_3^{Poi} is a saddle-center equilibrium point of the system given by H^{Poi} and, by (III.2.3), it satisfies that

$$\lambda = 0, \qquad (L, \eta, \xi) = (1, 0, 0) + \mathcal{O}(\mu).$$

We perform several changes of coordinates.

1. Translation of the equilibrium point: We translate L_3^{Poi} to the origin by means of the translation $\phi^{\mathrm{eq}} : (\lambda, \widetilde{L}, \widetilde{\eta}, \widetilde{\xi}) \to (\lambda, L, \eta, \xi)$ such that $\phi^{\mathrm{eq}}(0) = L_3^{\mathrm{Poi}}$. The Hamiltonian system associated to H^{Poi} after this translation defines a Hamiltonian system with respect to the symplectic form $d\lambda \wedge d\widetilde{L} + i d\widetilde{\eta} \wedge d\widetilde{\xi}$ and Hamiltonian

$$H^{\mathrm{eq}} = H^{\mathrm{Poi}} \circ \phi^{\mathrm{eq}} - H^{\mathrm{Poi}}(L_3^{\mathrm{Poi}}).$$

Denoting $\widetilde{\mathbf{z}} = (\lambda, \widetilde{L}, \widetilde{\eta}, \widetilde{\xi})$, $H^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu)$ can be written as

$$H^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu) = H_0^{\mathrm{eq}}(\widetilde{\mathbf{z}}) + R_2^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu) + R_3^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu),$$

with

$$\begin{aligned} H_0^{\mathrm{eq}}(\widetilde{\mathbf{z}}) &= \frac{1}{2} D^2 H^{\mathrm{Poi}}(L_3^{\mathrm{Poi}}; 0)[\widetilde{\mathbf{z}}, \widetilde{\mathbf{z}}] = -\frac{3}{2}\widetilde{L}^2 + \widetilde{\eta}\widetilde{\xi}, \\ R_2^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu) &= \frac{1}{2} D^2 H^{\mathrm{Poi}}(L_3^{\mathrm{Poi}}; \mu)[\widetilde{\mathbf{z}}, \widetilde{\mathbf{z}}] - H_0^{\mathrm{eq}}(\widetilde{\mathbf{z}}) = \mathcal{O}(\mu\, |\widetilde{\mathbf{z}}|^2), \\ R_3^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu) &= (H^{\mathrm{Poi}} \circ \phi^{\mathrm{eq}})(\widetilde{\mathbf{z}}; \mu) - H_0^{\mathrm{eq}}(\widetilde{\mathbf{z}}) - R_2^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu) - H^{\mathrm{Poi}}(L_3^{\mathrm{Poi}}; \mu) \\ &= \mathcal{O}(\widetilde{L}^3) + \mathcal{O}(\mu\, |\widetilde{\mathbf{z}}|^3), \end{aligned} \tag{III.C.2}$$

where we have used that $\widetilde{L} = L - 1 + \mathcal{O}(\mu)$, $\widetilde{\eta} = \eta + \mathcal{O}(\mu)$ and $\widetilde{\xi} = \xi + \mathcal{O}(\mu)$. Notice that as a result, for $\mu > 0$, $\widetilde{\mathbf{z}} = 0$ is saddle-center point of the system given by the Hamiltonian $H^{\mathrm{eq}}(\widetilde{\mathbf{z}}; \mu)$.

2. Reduction of the terms order 2: Following the proof of [JBL16, Theorem 1.3] one obtains that, for $\mu \geq 0$, there exists a family $\phi_\mu^{\mathrm{red}} : \mathbf{x} = (x_\lambda, x_L, x_\eta, x_\xi) \mapsto \widetilde{\mathbf{z}} = (\lambda, \widetilde{L}, \widetilde{\eta}, \widetilde{\xi})$ of real-analytic linear diffeomorphisms satisfying that $D\phi_0^{\mathrm{red}}(0) = \mathbf{Id}$ and that

$$H^{\mathrm{red}}(\mathbf{x}; \mu) = (H^{\mathrm{eq}} \circ \phi_\mu^{\mathrm{red}})(\mathbf{x}; \mu) = H_0^{\mathrm{eq}}(\mathbf{x}) + R_2^{\mathrm{red}}(\mathbf{x}; \mu) + R_3^{\mathrm{red}}(\mathbf{x}; \mu),$$

where $R_2^{\mathrm{red}}(\mathbf{x}; \mu)$ is a real polynomial of degree 2 in $\mathbf{x}$ with $\mathcal{C}^1$-functions of μ as coefficients and

$$R_2^{\mathrm{red}}(\mathbf{x}; \mu) = \mathcal{O}(\mu\, |\mathbf{x}|^2), \qquad \left\{ H_0^{\mathrm{eq}} \circ \mathbf{J}, R_2^{\mathrm{red}} \right\} = 0, \qquad R_3^{\mathrm{red}}(\mathbf{x}; \mu) = \mathcal{O}(|\mathbf{x}|^3),$$

where $\mathbf{J}$ is the matrix associated to the symplectic form $dx_\lambda \wedge dx_L + i dx_\eta \wedge dx_\xi$. The fact that $\left\{ H_0^{\mathrm{eq}} \circ \mathbf{J}, R_2^{\mathrm{red}} \right\} = 0$ and that R_2^{red} is a polynomial of order 2 and $\mathcal{O}(\mu|\mathbf{x}|^2)$ imply that there exist $\mathcal{C}^1$-functions $\sigma_1(\mu), \sigma_2(\mu) = \mathcal{O}(1)$ such that

$$R_2^{\mathrm{red}}(x_\lambda, x_L, x_\eta, x_\xi; \mu) = \mu\sigma_1(\mu)\frac{x_\lambda^2}{2} + \mu\sigma_2(\mu) x_\eta x_\xi.$$

Since ϕ_μ^{red} is linear and taking into account that $D\phi_0^{\mathrm{red}}(0) = \mathbf{Id}$ and the definition of the potential $V(\lambda)$ in (III.2.7), one has that

$$\sigma_1(0) = \frac{1}{\mu}\partial_\lambda^2 H^{\mathrm{Poi}}(L_3^{\mathrm{Poi}};\mu)\Big|_{\mu=0} = \partial_\lambda^2 H_1^{\mathrm{Poi}}(0,1,0,0;0) = V''(0) = \frac{7}{8}. \quad \text{(III.C.3)}$$

Therefore, by (III.C.2), one has that

$$H^{\mathrm{red}}(\mathbf{x};\mu) = -\frac{3}{2}x_L^2 + \mu\sigma_1(\mu)\frac{x_\lambda^2}{2} + (1+\mu\sigma_2(\mu))x_\eta x_\xi + R_3^{\mathrm{red}}(\mathbf{x};\mu).$$

In addition, since the terms of order 3 and higher of H^{eq} are of the form $\mathcal{O}(\widetilde{L}^3)+\mathcal{O}(\mu\,|\widetilde{\mathbf{x}}|^3)$ (see (III.C.2)), one has that

$$R_3^{\mathrm{red}}(\mathbf{x};\mu) = \mathcal{O}(x_L^3) + \mathcal{O}(\mu\,|\mathbf{x}|^3).$$

3. Scaling: We rename the perturbative parameter $\delta = \mu^{\frac{1}{4}}$ (see (III.2.5)) and, similarly to (III.2.6), we consider $\phi^{\mathrm{sca}} : \mathbf{y} = (y_\lambda, y_L, y_\eta, y_\xi) \mapsto \mathbf{x} = (x_\lambda, x_L, x_\eta, x_\xi)$ such that

$$x_\lambda = \frac{1}{\sqrt{\sigma_1(\delta^4)}}y_\lambda, \quad x_L = \frac{\delta^2}{\sqrt{3}}y_L, \quad x_\eta = \frac{\delta}{\sqrt[4]{3\sigma_1(\delta^4)}}y_\eta, \quad x_\xi = \frac{\delta}{\sqrt[4]{3\sigma_1(\delta^4)}}y_\xi,$$

and a scaling in time by a factor of $\delta^2\sqrt{3\sigma_1(\mu)}$. The Hamiltonian system expressed in these coordinates defines a system associated with the form $dy_\lambda \wedge dy_L + i dy_\eta \wedge dy_\xi$ and Hamiltonian

$$H^{\mathrm{sca}}(y;\delta) = \frac{1}{2}\left(y_\lambda^2 - y_L^2\right) + \alpha(\delta)\frac{y_\eta y_\xi}{\delta^2} + K^{\mathrm{sca}}(y_\lambda) + \delta H_1^{\mathrm{sca}}(\mathbf{y};\delta), \quad \text{(III.C.4)}$$

where

$$\begin{aligned}
\alpha(\delta) &= \frac{1+\delta^4\sigma_2(\delta^4)}{\sqrt{3\sigma_1(\delta^4)}} = \sqrt{\frac{8}{21}} + \mathcal{O}(\delta^4),\\
K^{\mathrm{sca}}(y_\lambda) &= \frac{1}{\delta^4}R_3^{\mathrm{red}}\left(\frac{y_\lambda}{\sqrt{\sigma_1(0)}},0,0,0;0\right) = \mathcal{O}(y_\lambda^3),\\
\delta H_1^{\mathrm{sca}}(\mathbf{y};\delta) &= \frac{1}{\delta^4}R_3^{\mathrm{red}}\left(\phi^{\mathrm{sca}}(\mathbf{y});\delta^4\right) - K^{\mathrm{sc}}(y_\lambda) = \mathcal{O}(\delta\,|\mathbf{y}|^3),
\end{aligned}$$

where we have used Cauchy estimates to bound DR_3^{red}.

4. Linearization: Consider the symplectic change of coordinates $\phi^{\mathrm{lin}} : (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2) \mapsto \mathbf{y} = (y_\lambda, y_L, y_\eta, y_\xi)$

$$\begin{pmatrix} y_\lambda \\ y_L \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \widehat{v}_1 \\ \widehat{w}_1 \end{pmatrix}, \qquad \begin{pmatrix} y_\eta \\ y_\xi \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} \widehat{v}_2 \\ \widehat{w}_2 \end{pmatrix}.$$

Then, the Hamiltonian system associated to (III.C.4) expressed in these coordinates, defines a Hamiltonian system with respect to the form $d\widehat{v}_1 \wedge d\widehat{w}_1 + d\widehat{v}_2 \wedge d\widehat{w}_2$ and the Hamiltonian

$$\begin{aligned}
\widehat{H}(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2;\delta) =& \widehat{v}_1\widehat{w}_1 + \frac{\alpha(\delta)}{2\delta^2}\left(\widehat{v}_2^2 + \widehat{w}_2^2\right) + \widehat{K}(\widehat{v}_1, \widehat{w}_1)\\
&+ \delta\widehat{H}_1(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2;\delta),
\end{aligned} \quad \text{(III.C.5)}$$

where

$$\widehat{K}(\widehat{v}_1,\widehat{w}_1)=K^{\mathrm{sca}}\left(\frac{\widehat{v}_1+\widehat{w}_1}{\sqrt{2}}\right)=\mathcal{O}\left(|\widehat{v}_1+\widehat{w}_1|^3\right),\qquad \widehat{H}_1=H_1^{\mathrm{sca}}\circ\phi^{\mathrm{lin}}.$$

□

Next proposition introduces a normal form expression in a neighborhood of the saddle-center equilibrium point. It is a direct consequence of [JBL16, Proposition C.1]. We consider an artificial parameter $\nu>0$ and rewrite $\widehat{H}$ in (III.C.1) as

$$\begin{aligned}\widehat{H}(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2;\delta,\nu)=&\widehat{v}_1\widehat{w}_1+\frac{\alpha(\delta)}{2\delta^2}\left(\widehat{v}_2^2+\widehat{w}_2^2\right)+K(\widehat{v}_1,\widehat{w}_1)\\&+\nu\widehat{H}_1(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2;\delta).\end{aligned}\tag{III.C.6}$$

We are interested in the case $\nu=\delta$, but in order to apply [JBL16] we are forced to use this artificial parameter.

Proposition III.C.2. *There exist $\delta_0,\varrho_0>0$ and a family of analytical canonical change of coordinates*

$$\begin{aligned}\widehat{\mathcal{F}}_{\delta,\nu}=(\varphi_{1,\nu},\psi_{1,\nu},\varphi_{2,\nu},\psi_{2,\nu}):B(\varrho_0)&\to B(\widehat{\varrho}_0)\subset\mathbb{R}^4\\(v_1,w_1,v_2,w_2)&\mapsto(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2),\end{aligned}$$

with respect to the form $dv_1\wedge dw_1+dv_2\wedge dw_2$ defined for $\nu\in[0,\delta_0)$ and $\delta\in(0,\delta_0)$ such that the Hamiltonian $\widehat{H}$ in (III.C.6) in the new coordinates reads

$$\begin{aligned}\mathcal{H}(v_1,w_1,v_2,w_2;\delta,\nu)&=\widehat{H}\left(\widehat{\mathcal{F}}_{\delta,\nu}(v_1,w_1,v_2,w_2);\delta,\nu\right)\\&=v_1w_1+\frac{\alpha(\delta)}{2\delta^2}\left(v_2^2+w_2^2\right)+\mathcal{R}(v_2w_2,v_2^2+w_2^2)\end{aligned}$$

and there exists $C>0$ independent of δ and ν such that, for $(v_1,w_1,v_2,w_2)\in B(\varrho_0)$,

$$\left|\mathcal{R}(v_1w_1,v_2^2+w_2^2;\delta)\right|\leq C|(v_1w_1,v_2^2+w_2^2)|^2.$$

In addition, for all $(v_1,w_1,v_2,w_2)\in B(\varrho_0)$ and all $(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)\in B(\widehat{\varrho}_0)$, the individual components of the change of coordinates satisfy

(1) $|\varphi_{1,\nu}(v_1,w_1,v_2,w_2)-v_1|\leq C\left\{|(v_1,w_1)|^2+\nu\,|(v_1,w_1,v_2,w_2)|^2\right\}$,
$|\varphi_{1,\nu}^{-1}(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)-\widehat{v}_1|\leq C\left\{|(\widehat{v}_1,\widehat{w}_1)|^2+\nu\,|(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)|^2\right\}$,

(2) $|\psi_{1,\nu}(v_1,w_1,v_2,w_2)-w_1|\leq C\left\{|(v_1,w_1)|^2+\nu\,|(v_1,w_1,v_2,w_2)|^2\right\}$,
$|\psi_{1,\nu}^{-1}(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)-\widehat{w}_1|\leq C\left\{|(\widehat{v}_1,\widehat{w}_1)|^2+\nu\,|(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)|^2\right\}$,

(3) $|\varphi_{2,\nu}(v_1,w_1,v_2,w_2)-v_2|\leq C\nu\,|(v_1,w_1,v_2,w_2)|^2$,
$|\varphi_{2,\nu}^{-1}(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)-\widehat{v}_2|\leq C\nu\,|(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)|^2$,

(4) $|\psi_{2,\nu}(v_1,w_1,v_2,w_2)-w_2|\leq C\nu\,|(v_1,w_1,v_2,w_2)|^2$,
$|\psi_{2,\nu}^{-1}(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)-\widehat{w}_2|\leq C\nu\,|(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)|^2$.

Notice that Proposition III.3.1 is a direct consequence of Lemma III.C.1 and Proposition III.C.2.

III.C.2 Proof of Proposition III.3.2

To prove Proposition III.3.2 we translate the results in Theorem III.2.5 (Statement 1) and the axis of symmetry $\mathcal{S}$ (Statement 2) into the set of coordinates (v_1, w_1, v_2, w_2) given in Proposition III.3.1.

Recall that in the proof of Proposition III.3.1 we have used the "intermediate" system of coordinates $(\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2)$. Therefore, we translate first the results via the change of coordinates $\widehat{\phi}_\delta : (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2) \to (\lambda, \Lambda, x, y)$, given by Lemma III.C.1. Then, we apply the second change of coordinates $\mathcal{F}_{\delta,\nu} : (v_1, w_1, v_2, w_2) \to (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2)$ with $\nu = \delta$, given by Proposition III.C.2.

Statement 1: Let $\lambda_* \in [\lambda_1, \lambda_2] \subset (0, \lambda_0)$ to be chosen later and consider the section $\Sigma(\lambda_*) = \{\lambda = \lambda_*, \Lambda > 0\}$. Let $\mathbf{z}^{\mathrm{u}}_\delta(\lambda_*)$ and $\mathbf{z}^{\mathrm{s}}_\delta(\lambda_*)$ be the first intersections of the invariant manifolds $\mathcal{W}^{\mathrm{u},+}(\mathfrak{L})$ and $\mathcal{W}^{\mathrm{s},+}(\mathfrak{L})$ with the section $\Sigma(\lambda_*)$, respectively.

Let us recall that, by Proposition III.2.3, the critical point $\mathfrak{L}(\delta)$ in (λ, Λ, x, y) coordinates is of the form $\mathfrak{L}(\delta) = \big(0, \delta^2\mathfrak{L}_\Lambda(\delta), \delta^3\mathfrak{L}_x(\delta), \delta^3\mathfrak{L}_y(\delta)\big)^T$, with $\mathfrak{L}_\Lambda, \mathfrak{L}_x, \mathfrak{L}_y = \mathcal{O}(1)$. Then, applying the change of coordinates $\widehat{\phi}_\delta$ given in Lemma III.C.1, there exist $\mathcal{C}^1$ functions $\gamma_1, \gamma_2 : (0, \delta_0) \to \mathbb{R}^4$ satisfying $\gamma_1, \gamma_2 = \mathcal{O}(1)$ such that

$$
\begin{aligned}
\widehat{\Sigma}(\lambda_*, \delta) = \widehat{\phi}_\delta\left(\Sigma(\lambda_*)\right) = \Big\{ & \widehat{v}_1 + \widehat{w}_1 + \delta\langle\gamma_1(\delta), (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2)\rangle = \tfrac{\sqrt{7}}{2}\lambda_*, \\
& \widehat{w}_1 - \widehat{v}_1 + \delta\langle\gamma_2(\delta), (\widehat{v}_1, \widehat{w}_1, \widehat{v}_2, \widehat{w}_2)\rangle + \delta^2\sqrt{6}\mathfrak{L}_\Lambda(\delta) > 0 \Big\}.
\end{aligned}
\tag{III.C.7}
$$

Notice that, on a first order, it corresponds to $\widehat{\Sigma}(\lambda_*, 0) = \{\widehat{v}_1 + \widehat{w}_1 = \frac{\sqrt{7}}{2}\lambda_*,\ \widehat{w}_1 > \widehat{v}_1\}$. Moreover, we denote

$$
\begin{aligned}
(\widehat{v}^{\mathrm{u}}_1, \widehat{w}^{\mathrm{u}}_1, \widehat{v}^{\mathrm{u}}_2, \widehat{w}^{\mathrm{u}}_2) &= \widehat{\phi}_\delta^{-1}\left(\mathbf{z}^{\mathrm{u}}_\delta(\lambda_*)\right) \in \widehat{\Sigma}(\lambda_*, \delta), \\
(\widehat{v}^{\mathrm{s}}_1, \widehat{w}^{\mathrm{s}}_1, \widehat{v}^{\mathrm{s}}_2, \widehat{w}^{\mathrm{s}}_2) &= \widehat{\phi}_\delta^{-1}\left(\mathbf{z}^{\mathrm{s}}_\delta(\lambda_*)\right) \in \widehat{\Sigma}(\lambda_*, \delta).
\end{aligned}
$$

Since $\widehat{\phi}_\delta$ is an affine transformation, by Theorem III.2.5 and Lemma III.C.1, one has that

$$
\begin{aligned}
\widehat{v}^{\mathrm{u}}_1 - \widehat{v}^{\mathrm{s}}_1 &= \left[\left(\frac{\sqrt{7}}{4}, -\sqrt{3}, 0, 0\right) + \mathcal{O}(\delta)\right] \cdot \left(\mathbf{z}^{\mathrm{u}}_\delta(\lambda_*) - \mathbf{z}^{\mathrm{s}}_\delta(\lambda_*)\right)^T = \mathcal{O}\left(\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}\right), \\
\widehat{w}^{\mathrm{u}}_1 - \widehat{w}^{\mathrm{s}}_1 &= \left[\left(\frac{\sqrt{7}}{4}, \sqrt{3}, 0, 0\right) + \mathcal{O}(\delta)\right] \cdot \left(\mathbf{z}^{\mathrm{u}}_\delta(\lambda_*) - \mathbf{z}^{\mathrm{s}}_\delta(\lambda_*)\right)^T = \mathcal{O}\left(\delta^{\frac{4}{3}} e^{-\frac{A}{\delta^2}}\right), \\
\widehat{v}^{\mathrm{u}}_2 - \widehat{v}^{\mathrm{s}}_2 &= \left[\left(0, 0, \sqrt[4]{\frac{21}{32}}, \sqrt[4]{\frac{21}{32}}\right) + \mathcal{O}(\delta)\right] \cdot \left(\mathbf{z}^{\mathrm{u}}_\delta(\lambda_*) - \mathbf{z}^{\mathrm{s}}_\delta(\lambda_*)\right)^T \\
&= \sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[\operatorname{Re}\Theta + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right], \\
\widehat{w}^{\mathrm{u}}_2 - \widehat{w}^{\mathrm{s}}_2 &= \left[\left(0, 0, -i\sqrt[4]{\frac{21}{32}}, i\sqrt[4]{\frac{21}{32}}\right) + \mathcal{O}(\delta)\right] \cdot \left(\mathbf{z}^{\mathrm{u}}_\delta(\lambda_*) - \mathbf{z}^{\mathrm{s}}_\delta(\lambda_*)\right)^T \\
&= -\sqrt[3]{4}\sqrt[4]{\frac{21}{8}}\delta^{\frac{1}{3}} e^{-\frac{A}{\delta^2}} \left[\operatorname{Im}\Theta + \mathcal{O}\left(\frac{1}{|\log\delta|}\right)\right].
\end{aligned}
\tag{III.C.8}
$$

Next, we consider the change of coordinates $\widehat{\mathcal{F}}_{\delta,\nu}$ with $\nu=\delta$ given in Proposition III.C.2. Let us denote

$$(v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}}) = \widehat{\mathcal{F}}_{\delta,\delta}^{-1}(\widehat{v}_1^{\mathrm{u}}, \widehat{w}_1^{\mathrm{u}}, \widehat{v}_2^{\mathrm{u}}, \widehat{w}_2^{\mathrm{u}}), \quad (v_1^{\mathrm{s}}, w_1^{\mathrm{s}}, v_2^{\mathrm{s}}, w_2^{\mathrm{s}}) = \widehat{\mathcal{F}}_{\delta,\delta}^{-1}(\widehat{v}_1^{\mathrm{s}}, \widehat{w}_1^{\mathrm{s}}, \widehat{v}_2^{\mathrm{s}}, \widehat{w}_2^{\mathrm{s}}). \tag{III.C.9}$$

Since, the local stable manifold is given by $\{v_1=v_2=w_2=0\}$ (see (III.3.2)), one has that $v_1^{\mathrm{s}}=v_2^{\mathrm{s}}=w_2^{\mathrm{s}}=0$ and we call $\varrho=w_1^{\mathrm{s}}$, (see Figure III.4.4). Since $\widehat{\mathcal{F}}_{\delta,\delta}(0,\varrho,0,0) = (\widehat{v}_1^{\mathrm{s}}, \widehat{w}_1^{\mathrm{s}}, \widehat{v}_2^{\mathrm{s}}, \widehat{w}_2^{\mathrm{s}}) \in \widehat{\Sigma}(\lambda_*,\delta)$ for $\lambda\in[\lambda_1,\lambda_2]$, by (III.C.7), the value λ_* must satisfy

$$\lambda_* = \sqrt{\frac{7}{4}}\left[\varphi_{1,\delta}(0,\varrho,0,0)+\psi_{1,\delta}(0,\varrho,0,0)+\delta^4\langle\gamma_1(\delta),\widehat{\mathcal{F}}_{\delta,\delta}(0,\varrho,0,0)\rangle\right],$$

where $\widehat{\mathcal{F}}_{\delta,\delta}=(\varphi_{1,\delta},\psi_{1,\delta},\varphi_{2,\delta},\psi_{2,\delta})$. Then, by Proposition III.C.2, one has that

$$\lambda_* = \sqrt{\frac{7}{4}}\varrho\left(1+\mathcal{O}(\varrho,\delta)\right)$$

and there exists $[\varrho_1,\varrho_2]\subset(0,\varrho_0)$ such that, for $\varrho\in[\varrho_1,\varrho_2]$ and $\delta>0$ small enough, one has that $\lambda_*\in[\lambda_1,\lambda_2]$.

Next, we consider the values for the extension of the unstable manifold $(v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}})$. Since $(v_1^{\mathrm{s}}, w_1^{\mathrm{s}}, v_2^{\mathrm{s}}, w_2^{\mathrm{s}})=(0,\varrho,0,0)$ and (III.C.9), one has that

$$\begin{aligned}
w_1^{\mathrm{u}}-\varrho &= \psi_{1,\delta}^{-1}(\widehat{v}_1^{\mathrm{u}}, \widehat{w}_1^{\mathrm{u}}, \widehat{v}_2^{\mathrm{u}}, \widehat{w}_2^{\mathrm{u}}) - \psi_{1,\delta}^{-1}(\widehat{v}_1^{\mathrm{s}}, \widehat{w}_1^{\mathrm{s}}, \widehat{v}_2^{\mathrm{s}}, \widehat{w}_2^{\mathrm{s}}),\\
v_2^{\mathrm{u}} &= \varphi_{2,\delta}^{-1}(\widehat{v}_1^{\mathrm{u}}, \widehat{w}_1^{\mathrm{u}}, \widehat{v}_2^{\mathrm{u}}, \widehat{w}_2^{\mathrm{u}}) - \varphi_{2,\delta}^{-1}(\widehat{v}_1^{\mathrm{s}}, \widehat{w}_1^{\mathrm{s}}, \widehat{v}_2^{\mathrm{s}}, \widehat{w}_2^{\mathrm{s}}),\\
w_2^{\mathrm{u}} &= \psi_{2,\delta}^{-1}(\widehat{v}_1^{\mathrm{u}}, \widehat{w}_1^{\mathrm{u}}, \widehat{v}_2^{\mathrm{u}}, \widehat{w}_2^{\mathrm{u}}) - \psi_{2,\delta}^{-1}(\widehat{v}_1^{\mathrm{s}}, \widehat{w}_1^{\mathrm{s}}, \widehat{v}_2^{\mathrm{s}}, \widehat{w}_2^{\mathrm{s}}),
\end{aligned}$$

For w_1^{u}, by the Mean Value Theorem, Proposition III.C.2 and (III.C.8), one obtains

$$|w_1^{\mathrm{u}}-\varrho| \leq C\,|\widehat{v}_1^{\mathrm{u}}-\widehat{v}_1^{\mathrm{s}}| + C\,|\widehat{w}_1^{\mathrm{u}}-\widehat{w}_1^{\mathrm{s}}| + C\delta\,|\widehat{v}_2^{\mathrm{u}}-\widehat{v}_2^{\mathrm{s}}| + C\delta\,|\widehat{w}_2^{\mathrm{u}}-\widehat{w}_2^{\mathrm{s}}| \leq C\delta^{\frac{4}{3}}e^{-\frac{A}{\delta^2}}.$$

Analogously, for v_2^{u}, one has that

$$|v_2^{\mathrm{u}}-(\widehat{v}_2^{\mathrm{u}}-\widehat{v}_2^{\mathrm{s}})| \leq C\delta\,|(\widehat{v}_1^{\mathrm{u}}-\widehat{v}_1^{\mathrm{s}}, \widehat{w}_1^{\mathrm{u}}-\widehat{w}_1^{\mathrm{s}}, \widehat{v}_2^{\mathrm{u}}-\widehat{v}_2^{\mathrm{s}}, \widehat{w}_2^{\mathrm{u}}-\widehat{w}_2^{\mathrm{s}})| \leq C\delta^{\frac{4}{3}}e^{-\frac{A}{\delta^2}}$$

and, by (III.C.8), one obtains the expression of the statement for v_2^{u}. An analogous estimate holds for w_2^{u}. Lastly, by the expression of Hamiltonian $\mathcal{H}$ in Proposition III.C.2, one sees that $\mathcal{H}(v_1^{\mathrm{u}}, w_1^{\mathrm{u}}, v_2^{\mathrm{u}}, w_2^{\mathrm{u}}) = \mathcal{H}(0,0,0,0)=0$ and obtains the expression for v_1^{u}.

Statement 2: Let us consider the symmetry axis $\mathcal{S}=\{\lambda=0, x=y\}$ given in (III.2.15). Notice that, by Proposition III.2.3, one has that $\phi_\delta(0)=\mathfrak{L}(\delta)\in\mathcal{S}$. Then, applying the affine transformation $\widehat{\phi}_\delta$ given in Lemma III.C.1, there exist functions $\gamma_3,\gamma_4:(0,\delta_0)\to\mathbb{R}$ satisfying $\gamma_3,\gamma_4=\mathcal{O}(1)$ such that

$$\begin{aligned}
\widehat{\phi}_\delta(\mathcal{S}) = \big\{\widehat{v}_1+\widehat{w}_1+\delta\langle\gamma_3(\delta),(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)\rangle &=0,\\
\widehat{w}_2+\delta\langle\gamma_4(\delta),(\widehat{v}_1,\widehat{w}_1,\widehat{v}_2,\widehat{w}_2)\rangle &=0\big\}.
\end{aligned}$$

Then, applying the change of coordinates $\widehat{\mathcal{F}}_{\delta,\delta}$, one has that

$$\mathcal{S}_{\mathrm{loc}} = \{v_1+w_1=\Psi_1(v_1,w_1,v_2,w_2;\delta),\ w_2=\Psi_2(v_1,w_1,v_2,w_2;\delta)\},$$

where

$$\Psi_1 = (\varphi_{1,\delta} - v_1) + (\psi_{1,\delta} - w_1) + \delta\langle\gamma_3(\delta), \mathcal{F}_{\delta,\delta}\rangle,$$
$$\Psi_2 = (\psi_{2,\delta} - w_2) + \delta\langle\gamma_4(\delta), \mathcal{F}_{\delta,\delta}\rangle.$$

Then, by Proposition III.C.2 and for $(v_1, w_1, v_2, w_2) \in B(\varrho_0)$ and $\delta > 0$ small enough,

$$|\Psi_1(v_1, w_1, v_2, w_2; \delta)| \leq C\delta\,|(v_1, w_1, v_2, w_2)| + C\,|(v_1, w_1)|^2\,,$$
$$|\Psi_2(v_1, w_1, v_2, w_2; \delta)| \leq C\delta\,|(v_1, w_1, v_2, w_2)|\,.$$

Conclusions and future work

The present thesis has been devoted to the rigorous study of the Lagrange point L_3 and its invariant manifolds, in particular on the study of homoclinic and chaotic phenomena. The setting considered is that of the Restricted Circular Planar 3-Body Problem. This thesis was performed from September of 2018 to July of 2022 by Mar Giralt Miron and was supervised by Professors Inma Baldomá Barraca and Marcel Guàrdia Munárriz.

The results here presented can also be found in the articles [BGG22a], [BGG21] and [BGG22b]. At the current date, the first is already published, the second one is under revisions and the third one is being prepared.

To conclude the thesis, we introduce different future work that has not been possible to tackle on the development of this thesis and different open problems for which this thesis could be seen as a first step.

Future work

Stokes constant. The coincidence or not of the unstable and stable invariant manifolds of L_3, for small values of μ, depends on the value of the Stokes constant $\Theta \in \mathbb{C}$ given in Theorem A. In Ansatz A it was assumed that $\Theta \neq 0$. Following the approach in [BCG+22], it should be possible to obtain a rigorous estimates and verify the ansatz by means of a computer assisted proof.

Multi-round homoclinic orbits. We say that an homoclinic connection to L_3 is k-round if, on a μ-neighborhood of this equilibrium point, the closure of the homoclinic orbit has k connected components. In Corollary A and Theorem B we analyze the existence of 1-round and 2-round, respectively. Following the strategy applied to prove Theorem B, one should be able to prove the existence of k-round homoclinic symmetric connections for $k > 2$, as conjectured in [BMO09].

Newhouse domain In Theorem D it was seen that there exists a generic unfolding of an homoclinic quadratic tangency between the invariant manifolds of a certain Lyapunov periodic orbit of L_3. It is expected that this fact would lead to the existence of a Newhouse domain as established in [Dua08] for area-preserving surface diffeomorphisms. More research is necessary to completely understand the scope of this result.

Numerical study To the best of our knowledge, the last complete numerical study of the breakdown of the homoclinic orbits to L_3 was performed in 1993 in [Fon93b]. It would be valuable to perform with current techniques a complete numerical study of the splitting of separatrices of L_3 and to numerically check the extend of the validity of the results obtained in this thesis.

Open problems

Singularities of a general separatrix The size of the exponentially small term in the splitting of separatrices phenomenon is given by the height of the maximal strip of analyticity of the time-parametrization of the separatrix. In the case of L_3, the separatrix did not have an explicit expression for its time-parameterization and, to obtain its complex singularities, it was necessary to rely on techniques of analytical continuation to analyze them (see Chapter I.3). This method could be generalized to be applied to integrable one degree of freedom Hamiltonians with more general expressions.

Higher dimensional models The RPC3BP is a convenient simplification of 2 degrees of freedom of the planetary 3-Body Problem. Moreover, it can be taken as a first step in order to study more complex models of higher dimension. Indeed, one can consider either the Restricted Spatial Circular 3-Body Problem with small $\mu > 0$ which has 3 degrees of freedom, the Restricted Planar Elliptic 3-Body Problem with small $\mu > 0$ and eccentricity of the primaries $e_0 > 0$, which has 2 and a half degrees of freedom, or the full planetary planar 3-Body Problem (i.e. all three masses positive, two small) which has three degrees of freedom (after the symplectic reduction by the classical first integrals).

Arnol'd diffusion Arnol'd diffusion is the name given to a mechanism of strong instability found in nearly integrable Hamiltonian systems. The first result on this phenomenon was published by V. I. Arnol'd in 1964, see [Arn64], and refers to the existence of solutions that exhibit a significant change in the action variables. In a footnote in [Arn64], V. I. Arnol'd conjectured the existence of this type of instabilities for the 3-Body Problem and it is widely expected to be true. In particular, it is believed that one of the main sources of such instabilities dynamics are the mean motion resonances, where the period of the two planets is resonant (i.e. rationally dependent), see [FGK+16].

The RPC3BP has too low dimension (2 degrees of freedom) to possess Arnol'd diffusion. However, since it can be seen as a first order for higher dimensional models (see the paragraph above), the analysis performed in this thesis can be seen as a humble first step towards constructing Arnol'd diffusion in the $1:1$ mean motion resonance. Indeed, in the Restricted Planar Elliptic 3-Body Problem the change of angular momentum would imply the transition of the zero mass body orbit from a close to circular ellipse to a more eccentric one. In the full 3-Body Problem, due to total angular momentum conservation, the angular momentum would be transferred from one body to the other changing both osculating ellipses. This behavior would be analogous to that of [FGK+16] for the $3:1$ and $1:7$ resonances. In that paper, the transversality between the invariant manifolds of the normally hyperbolic invariant manifold was checked numerically for the realistic Sun-Jupiter mass ratio $\mu = 10^{-3}$. Arnol'd diffusion instabilities have been analyzed numerically for the Restricted Spatial Circular 3-Body Problem in [TSS14].

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